Introduction to Calculus: Limits

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This handout is the first in our Calculus series, and it aims to introduce one of the core concepts of calculus, limits. The style of the handout will be more theoretical than a typical calculus course taught in high school; this is done to help transition interested students from the style of math that is taught in school to more sophisticated undergraduate styles of teaching mathematics, making this course a good gateway to learning pure/academia mathematics.

Further examples are provided at the end of the handout.

§1 Defining a Limit, Intuitively

First, to build intuition, we provide a rough graphical description.

Definition 1.1 (A limit, intuitively)**.** Given a function f defined around a point $x = a$, f approaches a limit l if $f(x)$ gets closer and closer to l as we get closer to, or *approach* a, from both the positive and negative sides.

Note that a limit only describes the behavior of a function around a certain point, not necessarily at it. Next, let's look at some examples of limits.

Definition 1.2 (Notation for a limit)**.** Given a function $f(x)$ and an x-value a, we write the limit l as $f(x)$ approaches a as

$$\lim_{x\to a} f(x) = l.$$

We also write $\lim_{x\to a^-} f(x)$ for limits approaching from the left-hand, or negative side, and $\lim_{x\to a^+} f(x)$ for the right-hand side.

We'll start with a really simple example of a limit.

Example 1.3

Let $f(x) = x^4 + x^2 - 1$. What is the limit of $f(x)$ as x approaches 2?

We write this as $\lim_{x\to 2} f(x)$. Don't be intimidated by the fancy notation - all this is asking is the value that $f(x)$ approaches as we get closer to 2 from the positive and negative side.

We have to first compute the limits from the left and right-hand sides, or $\lim_{x\to 2^-} f(x)$, and

$\lim_{x\to 2^+} f(x)$. Both of these values are equal to 19, which can be checked by graphing or testing points close to $x = 2$.

Because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 19$, $\lim_{x\to 2} f(x) = 19$. Graphically, we can also see that as we approach $x = 2$ from both sides of the point, the value of $f(x)$ approaches 19.

For this example, this makes sense, because $f(2) = 19$, but remember that the limit at a certain x-value is not necessarily equal to $f(x)$. Let's look at a few examples using similar functions where $f(2)$ is not defined or is not equal to the limit of f at $x = 2$.

Example 1.4

Let

$$f(x) = \begin{cases} 0 & x = 2 \\ x^4 + x^2 - 1 & x \neq 2 \end{cases}.$$

What is $\lim_{x\to 2} f(x)$?

Graphing this function, we obtain the following graph.

Because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 19$, we still have $\lim_{x\to 2} f(x) = 19$. However, note that $f(x) \neq 19$.

We now see that a limit does *not* depend on the value of $f(a)$; rather, it depends on the function's behavior *around* $x = a$.

Example 1.5

$$f(x) = \begin{cases} x^2 - 4 & x < 3 \\ 4 & x = 3 \\ x + 3 & x > 3 \end{cases}.$$

What is $\lim_{x\to 3} f(x)$?

Let's calculate the limits from the left-hand and right-hand sides.

$$\lim_{x\to 3^-} f(x) = 5$$

and

$$\lim_{x\to 3^+} f(x) = 6,$$

so since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, the limit does not exist.

§2 Defining a limit, rigorously

So far, the working definition of a limit that we've provided is a bit wishy-washy; it leaves a lot up to interpretation. So, we'll revise our definition of limits to use *epsilons* and *deltas*.

Formally, here is the definition of a limit:

Definition 2.1. To say that $\lim_{x \to a} f(x) = l$ means that given any $\epsilon > 0$, there exists a $\delta > 0$ such that all x for which $|x - a| < \delta$ and $x \neq a$, $|f(x) - l| < \epsilon$.

First, we'll interpret this. Colloquially, we can say that we can zoom in as close to $x = a$ as we want, and we will be able to bound the value of $f(x)$ from both the top and the bottom within this interval.

Think of this as drawing a box with width 2δ and with height 2ϵ. We can then say that $f(x)$ must be inside this box for all x-values in the interval.

Another interesting and important component of this definition is our requirement that $x \neq a$. As we mentioned in the previous section, the value of a limit does *not* depend on the value of the function at a point; rather it depends only on the behavior of the function around the point.

Let's demonstrate this through another example.

Example 2.2

Show, using ϵ and δ, that $\lim_{x \to 4}(3x + 4) = 16$.

For this proof, we must show that for every value of ϵ, we can assign a value of δ such that for all $|x - 4| < \delta$, $|f(x) - 16| < \epsilon$.

Working backwards, we can solve for a value of δ in terms of any ϵ. We have

$$|3x + 4 - 16| < \epsilon,$$

and rearranging terms, we see

$$3|x - 4| < \epsilon,$$

and

$$|x - 4| < \frac{\epsilon}{4}.$$

Therefore, a value of δ that satisfies $|f(x) - L| < \epsilon$ for $|x - 4| < \delta$ is $\delta = \frac{\epsilon}{3}$.
We are now ready for the proof.

Proof. Suppose we are given a value of $\epsilon > 0$. Then, we define $\delta = \frac{\epsilon}{3}$, and since $\epsilon > 0$, we also have $\delta > 0$. Substituting our value of δ into the original definition, we must have $|x - 4| < \frac{\epsilon}{3}$, and therefore

$$3|x - 4| < \epsilon.$$

Repeating our steps from before in reverse, we see that $|3x + 4 - 16| < \epsilon$, and therefore $|f(x) - 16| < \epsilon$ for $|x - 4| < \delta$, meaning that $\lim_{x \to 4}(3x + 4) = 16$. □

§3 Limit Properties

Here are various properties of a limit that can be useful while evaluating them.

Given that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and c is a constant:

- $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$
- $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, given that $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} f(x)^c = \left[\lim_{x\to a} f(x)\right]^n$

§4 Continuity

A closely related concept to limits is continuity.

Definition 4.1. A function f is continuous on a certain interval or point, if, for every point c on that interval, or for the point c itself, $f(c)$ exists and $\lim_{x\to c} = f(c)$.

Then, it also follows that if a function $f(x)$ is continuous on its domain, then for all points c on the domain, $\lim_{x\to c} = f(x)$.

Also, if the limit does not exist at a point, we say that the function has a discontinuity at that point.

Example 4.2
Let

$$f(x) = \begin{cases} x+k & x < 2 \\ x^2 - 1 & x \geq 2 \end{cases}.$$

Find a value of k such that $f(x)$ is continuous at $x = 2$.

We have two conditions to fulfill here: that $f(2)$ exists and that $\lim_{x\to 2} f(x) = f(2)$.

Therefore we must have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, as usual. Let's first evaluate $\lim_{x\to 2^+} f(x)$. We have

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 - 1) = 3.$$

We also must have $\lim_{x\to 2^-} x + k = 3$, since the limit must exist, and the value of k that satisfies this is $k = 1$. Since then $\lim_{x\to 2} = f(2) = 3$, we must have $k = 1$.

§5 Limits at Infinity

The epsilon-delta definition of a limit at infinity is quite similar to that of a limit at a point.

Definition 5.1. $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there is a value M such that for all $x > M$, $|f(x) - L| < \epsilon$.

Similarly, for limits at $-\infty$, we have

$$\lim_{x\to -\infty} f(x) = L$$

if for every ϵ there exists a value M such that for every $x < M$, $|f(x) - L| < \epsilon$.

Example 5.2

Find $\lim\limits_{x\to\infty} \dfrac{3x+4}{x-3}$.

We can divide the numerator and the denominator of $f(x)$ by the highest power of x (in this case, just x) and use limit rules to evaluate the limit. We obtain

$$\lim_{x\to\infty} \frac{3+\frac{4}{x}}{1-\frac{3}{x}} = \frac{\lim\limits_{x\to\infty} 3+\dfrac{4}{x}}{\lim\limits_{x\to\infty} 1-\dfrac{3}{x}} = \lim_{x\to\infty} \frac{3+0}{1+0} = 3.$$

Note that the concepts of a limit at infinity and of a horizontal asymptote are closely connected; if $\lim\limits_{x\to\infty} f(x) = L$, then $f(x)$ has a horizontal asymptote $y = L$.

Another important concept to note is that of indeterminate forms; for example, $\infty - \infty$ is not always 0 and $\frac{\infty}{\infty}$ is not always 1. Above, in Example 5.2, we saw an example of where the limits of the numerator and denominator were both ∞, but the limit was equal to 3.

Example 5.3 (Indeterminate form $\infty - \infty$)

Evaluate $\lim\limits_{x\to\infty} (\sqrt{x^2-x} - x)$.

At first glance, this limit seems as if it evaluates to $\infty - \infty = 0$, but we have to be careful. Let's try to write this limit in another form that isn't indeterminate. We multiply by $\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}+x}$ and try to simplify. We obtain

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{x^2-x} - x) &= \lim_{x\to\infty} (\sqrt{x^2-x} - x) \cdot \frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}+x} \\
&= \lim_{x\to\infty} \frac{x^2-x-x^2}{\sqrt{x^2-x}+x} \\
&= \lim_{x\to\infty} \frac{-x}{\sqrt{x^2-x}+x} \\
&= \lim_{x\to\infty} \frac{-x}{x\left(\sqrt{1-\frac{1}{x}}+1\right)} \\
&= \lim_{x\to\infty} \frac{-1}{\sqrt{1-\frac{1}{x}}+1} \\
&= -\frac{1}{2}
\end{aligned}$$

As we can see, this is *not* equal to 0.

§6 More Examples

Example 6.1

Evaluate $\lim\limits_{x\to 0} \dfrac{x}{3-\sqrt{x+9}}$.

This limit is pretty simple to evaluate: we first multiply the numerator and denominator by the conjugate of the denominator, $3+\sqrt{x+9}$, to get

$$\lim_{x\to 0}\frac{x(3+\sqrt{x+9})}{9-(x+9)}=\frac{x(3+\sqrt{x+9})}{-x}=-(3+\sqrt{x+9}).$$

Plugging in $x=0$, we obtain -6.

Example 6.2

Evaluate $\lim\limits_{x\to-\infty} x^4+4x^2+5$.

Note that $\lim\limits_{x\to-\infty}(x^4+4x^2+5)=\lim\limits_{x\to-\infty}x^4+\lim\limits_{x\to-\infty}(4x^2+5)$. Both of these limits evaluate to ∞; since $\infty+\infty$ is not an indeterminate form, the value of the limit is just ∞.

Example 6.3

Evaluate $\lim\limits_{x\to\infty}\sqrt{\dfrac{x^3+7x}{4x^3+5}}$.

Because of the limit properties described in Section 3, we can write

$$\lim_{x\to\infty}\sqrt{\frac{x^3+7x}{4x^3+5}}=\sqrt{\lim_{x\to\infty}\frac{x^3+7x}{4x^3+5}}.$$

Dividing both numerator and denominator by x^3, we obtain $\sqrt{\lim\limits_{x\to\infty}\dfrac{1+\frac{7}{x^2}}{4+\frac{5}{x^3}}}$, which simplifies to $\sqrt{\dfrac{1}{4}}=\dfrac{1}{2}$.

§7 Exercises

These exercises are mostly taken from textbooks, and a few will be taken from competitions. They have sources so you can find the solutions after as well.

Exercise 7.1 (Spivak Exercise 5.1)**.** Evaluate the following limits.

- $\lim\limits_{x\to 1}\dfrac{x^2-1}{x+1}$
- $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$
- $\lim\limits_{x\to y}\dfrac{x^n-y^n}{x-y}$
- $\lim\limits_{h\to 0}\dfrac{\sqrt{a+h}-\sqrt{a}}{h}$

Exercise 7.2 (Spivak Exercise 5.33)**.** Evaluate the following limits.

- $\lim\limits_{x\to\infty}\dfrac{x+\sin^3(x)}{5x+6}$

- $\lim_{x\to\infty} \frac{x\sin(x)}{x^2+5}$
- $\lim_{x\to\infty} \frac{x^2(1+\sin^2(x))}{(x+\sin(x))^2}$

Exercise 7.3 (Kline Exercise 25.4.4)**.** Given that $\lim_{x\to 0} \frac{\sin(mx)}{x} = m$ and $\lim_{x\to 0} \frac{x}{\sin(nx)} = \frac{1}{n}$, what is $\lim_{x\to 0} \frac{\sin(mx)}{\sin(nx)}$?

Exercise 7.4 (Spivak Exercise 5.34)**.** Prove that $\lim_{x\to 0^+} f(1/x) = \lim_{x\to\infty} f(x)$.

Exercise 7.5 (Kline (Proof of Theorem 1, Ch 25))**.** Prove that, if $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$, $\lim_{x\to a} f(x) + g(x) = b + c$.

Exercise 7.6 (Spivak Exercise 5.9)**.** Show that $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$.

Exercise 7.7 (Spivak Exercise 5.32)**.** Show that, given 2 polynomials $f(x)$ and $g(x)$ with degree m and n respectively, $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists if and only if $m \geq n$.

Exercise 7.8 (Kline Exercise 25.3.7)**.** Show, using ϵ and δ, that 5 is *not* the limit of $f(x) = 4x^2$ as x approaches 1.

Exercise 7.9 (BMT 2018 Calculus #2)**.** Let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \frac{1}{\frac{1}{a_n}+\frac{1}{b_n}}$, with $a_0 = 13$ and $b_0 = 29$. What is $\lim_{n\to\infty} a_n b_n$?

Exercise 7.10 (Kline Exercise 25.5)**.** Prove that if $\lim_{x\to a} f(x) = b$, then b is unique; that is, there cannot be another limit, say c, in addition to b.

Exercise 7.11 (BMT 2018 Calculus #7)**.** What is the following limit:

$$\lim_{n\to 0} \frac{\tan(3x)\sin(4x) + \sin(5x)\tan(2x)}{\tan(6x)\sin(7x)\cos(8x)}$$

Hint: use the approximations $\sin(nx) = nx$ and $\tan(nx) = nx$ near $x = 0$.

[1] [2]

References

[1] Morris Kline. *Calculus: An Intuitive and Physical Approach*. Dover Books on Mathematics. Dover Publications, 1998.

[2] Michael Spivak. *Calculus*. Fourth. Publish or Perish, 2008.