Logarithms

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Contents

§1 Definitions

Natural Base e

The ***natural base***, or e, is defined as

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \text{ or } e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}.$$

e is approximately 2.717281828.

§1.1 Exponential Functions

Exponential Function

An ***exponential function*** f with base a must satisfy the property that $f(x) = a^x$, where $a > 0, a \neq 1$, and x is any real number.

If $a > 1$, then the graph is roughly:

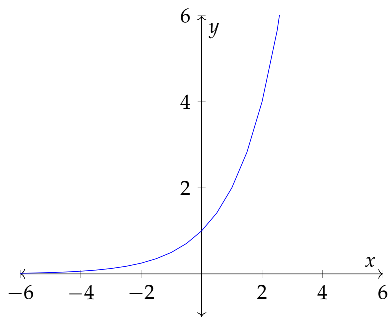

Figure 1: The graph of an exponential function when $a > 1$.

If $0 < a < 1$, then the graph is roughly:

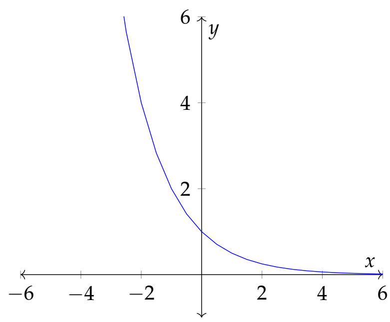

Figure 2: The graph of an exponential function when $0 < a < 1$.

Note that the graph of $y = a^x$ is the reflection of $a^{-x} = \left(\frac{1}{a}\right)^x$.

§1.2 Logarithmic Functions

Logarithmic Function

A ***logarithmic function*** f with base a must satisfy the property that $f(x) = \log_a x$, where $x > 0, a > 0$, and $a \neq 1$.

If $a > 1$, then the graph is roughly:

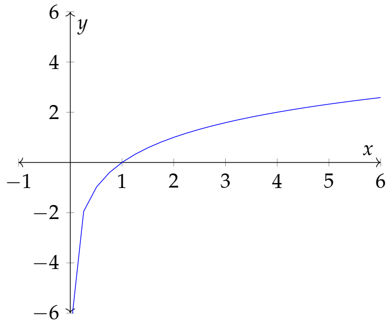

Figure 3: The graph of a logarithmic function when $a > 1$.

If $0 < a < 1$, then the graph is roughly:

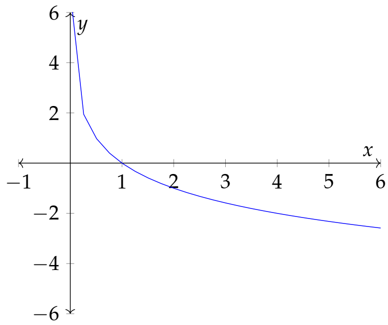

Figure 4: The graph of a logarithmic function when $0 < a < 1$.

The graph of $y = \log_{\frac{1}{a}} x$ is the reflection in the x-axis of the graph of $y = \log_a x$.

§2 Properties

If a, b, c are positive integers, then:

1. (Multiplication Law) $x^a \cdot x^b = x^{a+b}$
2. (Division Law) $x^a \div x^b = x^{a-b}$
3. (Power Law) $(x^a)^b = x^{ab}$
4. (Power of a Product Law) $(xy)^a = x^a \cdot y^a$
5. (Power of a Quotient Law) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
6. (Zero Exponent) $x^0 = 1$ $(x \neq 0)$
7. (Negative Exponent) $x^{-a} = \frac{1}{x^a} = \left(\frac{1}{x}\right)^a$ $(x \neq 0)$
8. (Fractional Exponent) $x^{\frac{a}{b}} = \sqrt[b]{x^a} = (\sqrt[b]{x})^a$

If a is a positive integer, $a \neq 1$, and $x > 0$, then:

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x \log_a a = x$
4. $a^{\log_a x} = x$
5. If $\log_a x = \log_a y$, then $x = y$.
6. (Common Logarithms) $\log_{10} x = \log x$
7. (Natural Logarithms) $\log_e x = \ln x$
8. (Change of Base) $\log_a x = \frac{\log_b x}{\log_b a}$
9. (Product Property) $\log_a (xy) = \log_a x + \log_b y$
10. (Quotient Property) $\log_a \frac{x}{y} = \log_a x - \log_a y$
11. (Power Property) $\log_a x^n = n \log_a x$
12. (All Real n Except 0) $\log_a x = \log_{a^n} x^n$
13. (Reciprocal Property) $\log_a x = \frac{1}{\log_x a}$
14. (Chain Rule) $\log_a b \cdot \log_c d = \log_a d \cdot \log_c b$

§3 Domain and Range

Theorem 3.1 (Domain and Range of Exponential Function)
If $f(x) = a^x$ then the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$, with a horizontal asymptote of $y = 0$.

Theorem 3.2 (Domain and Range of Logarithmic Function)
If $f(x) = \log_a x$, then the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

It is important to keep track of the domain and range. For example, if $e^x = -1$, either there are no solutions or something has gone wrong.

Remark 3.3. Questions regarding domain and range are not likely to show up on the AMC 10 or AIME. . . rather, they are conditions that can aid in solving the problem. For example, there are often problems of the form

$$f(x)^{g(x)} = 1,$$

and knowing what f and g can be is important.

§4 Applications

A large application of these functions is money, or more specifically, ***interest***.

Theorem 4.1 (Compound Interest Formula)
Say we are compounding a set amount of times in a time period. Let P be the initial principle balance, r be the interest rate, n be the number of times interest is applied per time period, and t is the number of time periods elapsed. Then the final amount is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Usually, the time period is one year.

Theorem 4.2 (Continuously Compounded Formula)
Say we are compounding continuously. Let the principal balance be P, interest rate be r, and number of time periods be t. Then the final amount A is

$$A = Pe^{rt}.$$

Theorem 4.3 (Half-Life Formula)
Let N_0 be the initial quantity of a substance, t be the time elapsed, and $t_{\frac{1}{2}}$. Then the amount $N(t)$ remaining is

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}.$$

> *Remark 4.4.* If I am being honest, interest formulas are just things you should know **in general**. They don't always pertain to competitions, but I would be remiss if I didn't talk about them.

§5 Examples

§5.1 Review

Example 5.1
If $f(x) = \log_3(x-3)$ and $g(x) = \log_3(x+3)$, then what is $(f+g)(6)$?

Solution. Note that $(f+g)(x) = \log_3(x-3) + \log_3(x+3) = \log_3(x^2-9)$, so $(f+g)(6) = \log_3(6^2-9) = \boxed{3}$. □

Example 5.2
If $f(x) = \log_2(x^2-3x+2)$ and $g(x) = \log_2(x-2)$, then what is $(f-g)(9)$?

Solution. Note that $(f-g)(x) = \log_2(x^2-3x+2) - \log_2(x-2) = \log_2(x-1)$, so $(f-g)(9) = \log_2(9-1) = \boxed{3}$. □

Example 5.3
If $f(x) = e^x$ and $g(x) = 3\ln(x-3)$, then what is $f(g(5))$?

Solution. Note that $f(g(x)) = e^{3\ln(x-3)} = (x-3)^3$, so $f(g(5)) = 2^3 = \boxed{8}$. □

Example 5.4
If $\ln(x-2) + \ln(2x-3) = 2\ln x$, then what is x?

Solution. Using the properties above, we get that $\ln(2x^2-7x+6) = \ln x^2 \implies x^2 - 7x + 6 = 0 \implies (x-1)(x-6) = 0$, so $x = 1, 6$, but $\ln(x-2) = \ln(-1)$ if $x = 1$, which is impossible, so only $x = \boxed{6}$ works. □

Example 5.5
\$10000 is invented in an account is an interest rate of 5% compounded continuously. How long will it take the balance to double?

Solution. Using the formula, we get $20000 = 10000e^{0.05t} \implies t = \frac{\ln 2}{0.05} = \boxed{13.86}$. ☐

Example 5.6
If $\log_3 x + \log_3(x-8) = 2$, then what is x?

Solution. Using the properties above, we get $\log_3(x^2 - 8x) = 2 \implies x^2 - 8x - 9 = 0 \implies (x-9)(x+1) = 0$, so $x = 9, -1$, but $\log_3(-1)$ is undefined, so $x = \boxed{9}$. ☐

Example 5.7
If $y = 3e^x - 2$, what is its inverse function?

Solution. If we replace x- and y-values, we get $x = 3e^y - 2 \implies f^{-1}(x) = \boxed{\ln\left(\frac{x+2}{3}\right)}$. ☐

Example 5.8
If the total of \$10000 is invested at an annual interest rate of 5%, compounded annually, what is the balance in the account after 5 years?

Solution. Using the formula, we get $10000(1+5\%)^5 = \boxed{12762.82}$. ☐

Example 5.9
If \$5000 is invested an annual interest rate of 6%, compounded quarterly, what is the amount of the balance after 5 years?

Solution. Using the formula, we get $5000\left(1 + \frac{0.06}{4}\right)^{4(5)} = \boxed{6734.28}$. ☐

Example 5.10
If $3^{2x} - 3^x - 72 = 0$, what is the value of x?

Solution. If we let $3^x = y$, we get $(y-9)(y+8) = 0$, Thus, $3^x = 9, -8$, which has only one solution of $x = \boxed{2}$. ☐

Example 5.11
If $\frac{2^{x^2+1}}{2^{x-1}} = 16$, then what is x?

Solution. Since $\frac{2^{x^2+1}}{2^{x-1}} = 2^{x^2-x+2} = 16 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0$, so $x = \boxed{-1, 2}$. ☐

§5.2 Competition Examples

Example 5.12 (AIME II 2009/2)
Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Solution. Note that

$$(\log_3 7)^2 = \log_3 7 \cdot \log_a 27 = \log_3 27 \cdot \log_a 7 = 3\log_a 7,$$

and similarly

$$(\log_7 11)^2 = 2\log_b 11,$$

$$(\log_{11} 25)^2 = \frac{1}{2}\log_c 25.$$

Thus,

$$7^3 + 11^2 + 25^{\frac{1}{2}} = \boxed{469}.$$

□

Example 5.13 (AIME II 2013/2)
Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

Solution. Taking each side to the power of 2, we get

$$\log_{2^a}(\log_{2^b}(2^{1000})) = 1,$$

and doing it again gives us

$$\log_{2^b}(2^{1000}) = 2^a,$$

and once more gives us

$$2^{1000} = (2^b)^{2^a} = 2^{b\cdot 2^a},$$

$$1000 = b \cdot 2^a.$$

Since a, b are positive integers, $a \leq 3$. Let's test all the cases:

- $a = 3, b = 125$
- $a = 2, b = 250$
- $a = 1, b = 500$

Thus, the answer is $(3 + 125) + (2 + 250) + (1 + 500) = \boxed{881}$. □

Example 5.14 (AIME II 2020/3)
The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. We can rewrite each side as

$$\log_2 3^{\frac{20}{x}} = \log_2 3^{\frac{2020}{x+3}},$$

$$\frac{20}{x} = \frac{2020}{x+3},$$

$$20x + 60 = 2020x,$$

$$x = \frac{60}{2000} = \frac{3}{100}.$$

Thus, the answer is $\boxed{103}$. □

§6 Problem Solving Strategies

1. **Applications**: these are just questions that apply interest or growth formulas. The idea for all of these are just to apply the formula.
2. **Wrapped Logs**: basically questions of the form $y = \log(\log(\ldots(\ldots)\ldots))$. One idea is to eliminate the logs by using the fact $a = b \implies c^a = c^b$, but other algebraic manipulations apply here too.
3. **Composition of Logs**: you can deal with these the same way you deal with composition of logs. These are similar to wrapped logs, but usually there is some symmetry to take advantage of.

More often than not, logarithm problems are secretly just algebraic manipulation problems, because the property $\log a + \log b = \log ab$ is extremely useful for constructing weird expressions that can be split into easy parts. In other words, take advantage of as many properties as possible. A few other tips:

- Exponential functions can sometimes become polynomials if we change $\log_c x = \log_x c$ for some constant c and variable x.
- Change everything to the same base, especially if the differing bases are just powers of one another.
- There exist number theoretic logarithm problems. They rarely show up on AMC or AIME. However, they are important to keep in mind. For example, the function

$$f(n) = n - \left(\left\lfloor \frac{n}{10^{\lfloor \log_{10} n \rfloor - 1}} \right\rfloor (10^{n-1} - 1)\right)$$

  takes the first digit and brings it to the back. For example, $f(1234) = 2341$. Note how $\log_{10} n$ is used in the process of finding the number of digits of n.

§7 Exercises

Exercise 7.1. If $f(x) = 3x - 4$ and $g(x) = 2^x$, then what is $f(g(2))$?

Exercise 7.2 (Harold Reiter). An amount of \$2000 is invested at r% interest compounded continuously. After four years, the account has grown to \$2800. Assuming that it continues to grow at this rate for 16 more years, how much will be in the account?

Exercise 7.3 (NC-SMC 2016/7). Let a, b, and c be three consecutive terms of a geometric progression (in the given order). Assume that these three terms are each greater than 1, and the common ratio of the geometric progression is greater than 1. Then what is the value of $\frac{\log_b 3(\log_{a^2} c - \log_c \sqrt{a})}{\log_a 9 - 2\log_c 3}$?

Exercise 7.4 (AMC 12B 2008/23). The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

Exercise 7.5 (AMC 12A 2019/23). Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)} \qquad \text{and} \qquad a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

Exercise 7.6 (AMC12B 2002). For all integers n greater than 1, define $a_n = \frac{1}{\log_n 2002}$. Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. Then what is the value of $b - c$?

Exercise 7.7 (AMC12B 2010). For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40?$$

Exercise 7.8 (AMC12A 2014). The domain of the function

$$f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$$

is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Exercise 7.9 (AMC12B 2011). Let $f(x) = 10^{10x}, g(x) = \log_{10}\left(\frac{x}{10}\right), h_1(x) = g(f(x))$, and $h_n(x) = h_1(h_{n-1}(x))$ for integers $n \geq 2$. What is the sum of the digits of $h_{2011}(1)$?

Exercise 7.10 (AHSME 1997). For any positive integer n, let

$$f(n) = \begin{cases} \log_8 n, & \text{if } \log_8 n \text{ is rational,} \\ 0, & \text{otherwise.} \end{cases}$$

What is $\sum_{n=1}^{1997} f(n)$?

Exercise 7.11 (AIME II 2013). Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

Exercise 7.12 (AIME II 2015). In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p + q$.

Exercise 7.13 (AIME II 2016). Let x, y, and z be real numbers satisfying the system $\log_2(xyz - 3 + \log_5 x) = 5$, $\log_3(xyz - 3 + \log_5 y) = 4$, $\log_4(xyz - 3 + \log_5 z) = 4$, Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

Exercise 7.14 (AIME I 2020). There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 7.15 (AIME II 2010). Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.