Combinatorial Arguments and Identities

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Contents

§1 Introduction

Remark 1.1 (Disclaimer). In this handout, when we refer to basic counting, we are referring to the counting techniques highlighted in Primeri's "Basic Counting" handout.

Combinatorial identities are a very powerful technique when it comes to dealing with math competition counting problems. Common concepts, like Stars and Bars, allow us to simply solve situations where we have to find the number of ways to choose things.

§1.1 "n choose k"

Combination
The function with inputs n, k written as $\binom{n}{k}$ denotes the number of ways that k objects can be chosen from n distinct objects. Hence the name, n choose k.

Theorem 1.2
The expression $\binom{n}{k}$ can be expanded out as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can use some ideas of basic counting to prove the above statement. You can motivate the choice to use counting techniques by noting the abundance of factorials in the expression. Multiplying and dividing by factorials is something that is characteristic of counting permutations. Now that we're thinking of the theorem statement in terms of permutations, we can continue like this:

Proof. The number of ways to choose k objects from n objects is the same as reordering n objects where k of them are the identical "chosen" marker and $n-k$ of them are the identical "not chosen" marker. In this situation, the positions at which the "chosen" markers are placed would correspond to the object at that index being chosen. Note that the $n!$ refers to permuting the n objects and division by $k!$ and $(n-k)!$ take care of the identical groups of markers. Thus, it must be true that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\square$

Remark 1.3. For those of you who are unfamiliar with combinatorics, this is what one refers to as a "combinatorial argument." A ***combinatorial argument*** is an argument in which a quantity is counted in two different ways, with the equality between the two resulting in an equation. For example, by definition $\binom{n}{k}$ counts the number of ways to choose k objects from n objects and we used a combinatorial argument to show that because $\frac{n!}{k!(n-k)!}$ does the same, they are equivalent.

Exercise 1.4. What is the number of ways to choose 8 objects from 10 objects?

Exercise 1.5. Use a combinatorial argument to prove that

$$\binom{n}{k} = \binom{n}{n-k}$$

The above is a well-known result that can make simplifying expressions significantly easier when solving combinatorics and counting problems.

§1.2 The Binomial Theorem

Theorem 1.6 (Binomial Theorem)
For some positive integer n:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Proof. Think of it like this. If we want to create an $a^{n-r}b^r$ term, we get this by taking the a from $n-r$ of the $(a+b)$ terms and the b from the remaining r. So, we want to choose the $n-r$ of the n $a+b$ terms in order to create an $a^{n-r}b^r$ term, meaning that there will be a total of $\binom{n}{n-r}$ $a^{n-r}b^r$ terms. Note that this is equivalent to the above statement of the binomial theorem, so we are done. □

> **Binomial Coefficient**
> Because the concept of "n choose k" can be applied to the process of expanding out binomials, it is more formally referred to as the ***binomial coefficient***.

From now on, this handout will always refer to $\binom{n}{k}$ by its formal name, the binomial coefficient, rather than just "n choose k."

Example 1.7
What is the coefficient of the x^3y^4 term when $(x+y)^7$ is expanded out?

Solution. By the binomial theorem, it is easy to see that the coefficient of x^3y^4 will be:

$$\binom{7}{3} = \boxed{35}$$

□

The below example is a bit more complex than the one above. Instead, it expands on the same idea and applies it to three variables. The idea that the coefficient is the number of ways to choose specific terms from a product of different monomials, binomials, trinomials, etc... can be generalized.

Example 1.8
What is the coefficient of the $x^5y^9z^3$ term when $(x+y+z)^{17}$ is expanded out?

Solution. We can expand on the idea of the Binomial Theorem (the multinomial coefficient) to note that this is equal to the number of ways to choose x from 5 of the $x+y+z$ terms, y from 9 of them and z from 3 of them. This can be written as:

$$\boxed{\frac{17!}{5! \cdot 9! \cdot 3!}}$$

3

Note that we'll just leave it in this form because this is a rather large number. Actually computing this number is left as an exercise to the occasional reader who finds bashing out large multiplications and divisions to be fun. □

> *Remark 1.9.* Monomials, binomials, trinomials, etc... which are expressions of one, two, three,... variables respectively are generally referred to as **multinomials**.

Theorem 1.10 (Multinomial Coefficient)

Just like we have the binomial coefficient, we have the multinomial coefficient. The multinomial coefficient for m variables (a_1 to a_m) is equal to:

$$\frac{(a_1 + \ldots + a_m)!}{(a_1)! \ldots (a_m)!}$$

We'll leave the proof of the above as an exercise to the reader, although it is almost identical to combining the proofs of two things we have already proved in this handout.

§2 Some Important Techniques

§2.1 Grid-Walking

Before we get into discussing how to solve problems related to "grid-walking," it might help to give an example of a generic grid-walking problem to explain what kinds of problems we are referring to.

Example 2.1

John is attempting to travel on the Cartesian plane. He is in a square grid with vertices $(0,0), (0,5), (5,0), (5,5)$. If he starts at $(0,0)$ and can only move in the positive x or y directions to reach his destination of $(5,5)$, how many successful paths exist?

The types of problems we are referring to are the problems where we are given point A and point B and we are asked how many ways we can go from point A to point B.

Theorem 2.2

The number of ways to go from point A to point B on a grid, with coordinates $(0,0)$ and (a,b) for $a, b > 0$, if you can only go in the positive x and y directions is equal to $\binom{a+b}{a}$.

Proof. Note that since each movement increases the x or y coordinate by 1 and the total different in the x and y coordinates between $(0,0)$ and (a,b) is $a + b$, there will be exactly $a + b$ moves. a of these moves will be in the positive x direction and b will be in the positive y direction.

Once again, we are going to make a combinatorial argument. Finding the number of these paths on a grid is equivalent to permuting U...UR...R where U represents a positive y movement and R represents a positive x movement such that there are a R's and b U's. This is equivalent to choosing a of the $a + b$ places to place an R in, which is written as $\boxed{\dbinom{a+b}{a}}$. □

Now that we've dealt with this, we can move back to Example 2.1 and solve it instantly using the theorem we just proved:

Solution. Note that this equivalent to the situation in the theorem with a and b both equal to 5. Thus, our answer is:

$$\text{number of ways} = \binom{5+5}{5} = \boxed{252}$$

$\square$

Unfortunately, grid-walking situations are not normally this simple. Let's try an example that practices a similar idea but there's now a twist: there are restricted areas and whatever is walking along the grid has to avoid them:

> **Example 2.3**
> John has to go from the same starting point, $(0,0)$, to the same ending point, $(5,5)$, by moving only in the positive x, y directions but this time he is unable to pass through the construction zone at $(3,3)$. How many successful paths exist for John?

Once again, there is a rather well-known way to approach these grid-walking problems with the twist of a "restricted" zone.

Solution. Just before we start, let's get a diagram so that we can easily visualize what is going on:
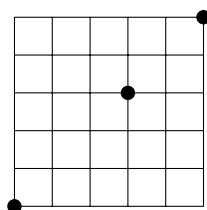
The standard way to deal with these types of problems is to use the idea of complementary counting, where you first find the number of total ways (both good and bad) and then subtract out the bad ones. From Example 2.1, we know that the total number of ways is equal to 252. However, we now need to find the number of paths that go from $(0,0)$ to $(5,5)$ while passing through $(3,3)$ so that we can subtract them out.

Note that this is equal to the number of ways to go from $(0,0)$ to $(3,3)$ multiplied by the number of ways to go from $(3,3)$ to $(5,5)$ as you can choose a way to go from $(0,0)$ to $(3,3)$ and then multiply that by the number of choices of ways to finish the path from there. Applying Theorem 2.2 yields:

$$\text{number of "bad" paths} = \binom{6}{3} \cdot \binom{4}{2} = 120$$

Note that we can generate the second binomial coefficient because we can simply shift the origin from Theorem 2.2 to the point that we need it to be at. Now, we can just finish off the problem by subtracting the number of bad ways from the total number of ways and we get:

$$\text{number of successful paths} = 252 - 120 = \boxed{132}$$

$\square$

Remark 2.4. When you see that a point or a path has to be avoided, this **IS SCREAMING** for an application of complementary counting.

Note that we can generalize this:

> **Theorem 2.5**
>
> If we have a rectangular grid and we would like to go from $(0,0)$ to (a,b) by only moving one unit up or right at a time without passing through some point (m,n), the number of successful paths is equal to:
>
> $$N = \binom{a+b}{a} - \binom{m+n}{m} \cdot \binom{a+b-m-n}{m-a}$$

The proof of the above is similar to our previous reasoning and is left to the reader.

The last grid-walking situation is when some path is blocked. Luckily, it's a similar combination of Theorem 2.2 and complementary counting.

§2.2 Committee Forming

Another common situation that makes an appearance quite frequently in combinatorial arguments is the concept of forming committees fitting some specifications out of a group of people. I'm not going to go through this in too much detail right now but I'm going to give an example of how committee forming arguments are used when proving combinatorial identities. These arguments will show up again in a bit when we get to the identities.

> **Example 2.6**
> Prove that for positive integer n,
>
> $$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof. Note that $\binom{n}{k}$ refers to the number of ways to choose a committee of k people from a group of n people. If you look at this from the angle of forming committees, you can note that the sum on the left hand side of the above expression just represents all possible committees with number of people from 0 to n inclusive.

Note that this must be equal to the right hand side because for each of the n people, they are either in the committee or not which presents 2 choices, meaning that it comes out to 2^n. □

Although this is just a short example, it demonstrates how "committee-forming" arguments are quite powerful when it comes to dealing with binomial coefficients. We'll see more uses of this when proving other identities like Vandermonde's later in this handout. Before we move on, we'll just do one more example of a problem that requires some form of "committee-forming" argument:

> **Example 2.7 (AIME I 2020/7)**
> A club consisting of 11 men and 12 women needs to choose a committee from among
> its members so that the number of women on the committee is one more than the
> number of men on the committee. The committee could have as few as 1 member or
> as many as 23 members. Let N be the number of such committees that can be formed.
> Find N.

Solution. In this situation, the two varying quantities are the number of women and the
number of men, and the latter can be expressed in terms of the former. So, let's start by
letting the number of women be w, meaning that there are also $w - 1$ men selected.

Before we attempt the problem, let's just take a step back and look at our situation.
We want to elect a committee with w women, $w - 1$ men and in this situation, $12 - w$
women and $12 - w$ men are not chosen. Note that the sum of the number of women
chosen and the number of men not chosen is constant at 12. If we choose any 12 of the
23 people, we can make a committee based on these people because the men in this 12
are the men not chosen and the women are the women chosen. So, the number of ways
to make a committee is the number of ways to choose 12 of the 23 people regardless of
gender. So:

$$N = \boxed{\binom{23}{12}}$$

$\square$

§3 Pascal's Triangle

§3.1 The Triangle

> **Pascal's Triangle**
> Pascal's triangle is a triangle where the top row has 1 number and each successive
> row has 1 more number than the last. All numbers on the left and right edges are 1
> and all other numbers are the sum of the two numbers above them. Note that the
> triangle expands downward infinitely.

See the below picture for a diagram of the first few rows of Pascal's Triangle:

When dealing with Pascal's Triangle, there are a few conventions that you will want to remember:

1. The topmost row that contains only the single 1 is referred to as the row with index 0. So, the rows from top to bottom go $0, 1, 2, \ldots$.

2. The leftmost 1 in each row has index 0, so the numbers in each row are also counted as $0, 1, 2, \ldots$.

> **Theorem 3.1**
> The kth number in the nth row of Pascal's triangle is equal to $\binom{n}{k}$.

Proof. Take a close look at Pascal's triangle. If you think about it, if each number represented a point on a grid, each number actually represents the number of ways to go from the top 1 to that number while only moving downward diagonally. So, this all comes down to a grid-walking argument!

If you turn your head a little, note that you can create a "rectangular grid" placing the topmost 1 in Pascal's triangle at $(0, 0)$ and the kth number in the nth row at $(n, n - k)$ because there need to be k "down and right" movements and $n - k$ "down and left movements." So, the number of paths by Theorem 2.2 is just $\binom{n}{k}$. $\qquad\square$

Also, if you take a look at Pascal's Triangle, there is also an instance of a rather cool property that we already proved:

> **Fact 3.2.** The sum of the kth row of Pascal's Triangle is 2^k.

Note that we already proved this in Example 2.6 and now follows after Theorem 3.1.

§3.2 Pascal's Identity

Pascal's Identity is just an extension of the ideas from the last section:

> **Theorem 3.3 (Pascal's Identity)**
> $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$

Note that although this is obvious from the facts we have already proved about Pascal's Triangle, we will still go over two different proofs of this identity. One will be a combinatorial proof (making use of a committee forming argument) whereas the other will be an algebraic proof. First up, we have the combinatorial argument:

Committee-Forming Proof. Assume we have n people and we want to form a committee with k people in it. It is clear that the right hand side refers to the number of ways we can form such a committee.

Let's pick an arbitrary person of the n people, and assume they are named person A. Note that there are $\binom{n-1}{k-1}$ ways to choose a committee with person A in it because then you have to choose $k - 1$ people from the remaining $n - 1$. Similarly, there are $\binom{n-1}{k}$ ways to choose a committee that does not have person A in it because then you have to choose k people from the remaining $n - 1$. So, together these make up the two cases and it follows that:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

$\qquad\square$

Remark 3.4. We used something called a ***bijection***, where a quantity is mapped to another quantity we know how to compute. Above, we used a bijection mapping $\binom{n}{k}$ to choosing k people from n people and showed that the left and right hand sides are counting the same thing.

We can also prove the same identity through an algebraic proof, as seen here:

Algebraic Proof. Let LHS denote the left hand side:

$$\text{LHS} = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$$

This simplifies to:

$$\text{LHS} = \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

$\square$

Example 3.5

Simplify $\binom{37}{7} + 2\binom{37}{8} + \binom{37}{9}$ into only a single binomial coefficient.

Solution. Note that we have:

$$\binom{37}{7} + \binom{37}{8} + \binom{37}{8} + \binom{37}{9} = \binom{38}{8} + \binom{38}{9} = \boxed{\binom{39}{9}}$$

$\square$

§4 Common Identities

§4.1 Stars and Bars

Theorem 4.1 (Stars and Bars)

The number of ways to put n indistinguishable objects into k distinct categories is:

$$N = \binom{n+k-1}{k-1}$$

There is a clever combinatorial argument to prove this:

Proof. Think of a backyard. If I want to split my backyard into n rectangular sections with equal width, I can do this by placing $n-1$ fences in my backyard. Similarly, to split the n objects into k categories, I can line up the n objects in a line with $k-1$ dividers and find the ways to permute these.

So, the numbers of ways to put n indistinguishable objects into k distinct categories is equivalent to the number of ways to rearrange a sequence containing n identical objects and $k-1$ identical dividers.

As there are a total of $n+k-1$ places in the line and $k-1$ of those must be chosen to place dividers in, there are $\binom{n+k-1}{k-1}$ ways to do this.. $\square$

Stars and Bars is a very powerful technique that is used in many AMC and AIME problems. This is probably the identity you will use the most.

> **Example 4.2 (AMC 10B/12B 2020)**
> Let $D(n)$ denote the number of ways of writing the positive integer n as a product
> $$n = f_1 \cdot f_2 \cdots f_k,$$
> where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

Solution. Note that $96 = 3 \cdot 2^5$. So, we can consider the 3 and the powers of 2 separately. If there are x different factors in a factorization of 96, there are x ways to choose in which factor the 3 goes. There must also be a power of 2 in each of the other factors at the very least to ensure that each factor is strictly greater than 1.

There are now $5 - (x - 1) = 6 - x$ powers of 2 remaining and x factors to place them in. The number of ways to do this can be found with Stars and Bars. This is:

$$\binom{6 - x + x - 1}{6 - x} = \binom{5}{6 - x}$$

So, if 96 is being factored into x factors, there are $x\binom{5}{6-x}$ ways to do this. So, the answer is:

$$\sum_{x=1}^{6} x\binom{5}{6 - x} = 1 + 10 + 30 + 40 + 25 + 6 = \boxed{112}$$

$\square$

§4.2 Hockey-Stick Identity

> **Theorem 4.3 (Hockey-Stick Identity)**
> For integer k and $n \geq k$,
> $$\sum_{i=k}^{n} \binom{i}{k} = \binom{n + 1}{k + 1}$$

Below is an algebraic proof of the identity:

Proof. Let $n = k + x$. Let's just expand everything out:

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{k}{k} + \binom{k + 1}{k} + \ldots + \binom{k + x}{x} = \binom{k + 1}{k + 1} + \binom{k + 1}{k} + \ldots + \binom{k + x}{x}$$

This looks suspiciously like Pascal's Identity. Let's apply Pascal's Identity repeatedly going from the left to the right:

$$\binom{k + 1}{k + 1} + \binom{k + 1}{k} + \ldots + \binom{k + x}{x} = \binom{k + x + 1}{k + 1} = \binom{n + 1}{k + 1}$$

As the left hand side is equal to the right hand side now, we are done. $\square$

Exercise 4.4. Find the relationship between the statement of the Hockey-Stick Identity and Pascal's Triangle (hint: it looks like a hockey stick).

Before moving on to the last identity we will cover in this handout, let's do an example. Although this example is not solved with only Hockey-Stick Identity, all of the content covered in this problem has been covered already in this handout:

> **Example 4.5 (AIME 1986)**
> The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17}$, where $y = x + 1$ and that a_i's are constants. Find the value of a_2.

Solution. Let $f(x) = 1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$. Note that $f(x) = (-1)^k (x + 1 - 1)^k = (-1)^k (y - 1)^k = (1 - y)^k$. By the Binomial Theorem, the coefficient of the y^a term must be $\binom{a}{2}$ for $a \geq 2$. For $a = 0, 1$, the coefficients are 1 and -1 so they cancel out, meaning we don't have to worry about them. So, our answer is equal to:

$$\sum_{i=2}^{17} \binom{i}{2} = \binom{2}{2} + \cdots + \binom{17}{2}$$

By the Hockey-Stick Identity, this must be equal to $\binom{18}{3} = \boxed{816}$. $\square$

§4.3 Vandermonde's Identity

> **Theorem 4.6 (Vandermonde's Identity)**
> For integers m, n, k, r:
> $$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

This identity is quite easy to prove using a combinatorial argument:

Proof. Looking at the statement, it appears that we are choosing r total out of a pool of $m + n$ things with k of those r coming from one section and the rest coming from the other. So, let's set it up so that our total pool of people is split into groups of m and n with some being chosen from one group and the rest from the other:

Let's say that I want to choose a committee of r people out of $m + n$ people such that this group of people includes m girls and n boys. Note that if I wanted the committee to have x girls and $r - x$ boys, the ways to create the committee would be:

$$N = \binom{m}{x} \binom{n}{r-x}$$

So, this means that the summation in the left hand side of Vandermonde's Identity cycles over the ways to choose k girls and $r - k$ boys for each possible value of k. In total, this is just the total number of ways to make the committee while disregarding the number by gender, which is just $\binom{m+n}{r}$ $\square$

Exercise 4.7. Prove $\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k}$ using Vandermonde's Identity.

For the next example, we're going to revisit a problem we already looked at in the committee-forming section but we will solve it with a different approach using Vandermonde's Identity.

> **Example 4.8 (2020 AIME I 7)**
> A club consisting of 11 men and 12 women needs to choose a committee from among
> its members so that the number of women on the committee is one more than the
> number of men on the committee. The committee could have as few as 1 member or
> as many as 23 members. Let N be the number of such committees that can be formed.
> Find the sum of the prime numbers that divide N.

Solution. Essentially, the problem is asking us for $\sum_{n=1}^{11}\binom{12}{n}\binom{11}{n-1}$. Also, note that $\binom{n}{k} = \binom{n}{n-k}$ because the number of ways to choose k from n is also the number of ways to choose $n-k$ not to choose. Using this:

$$\sum_{n=1}^{11}\binom{12}{n}\binom{11}{n-1} = \sum_{n=1}^{11}\binom{12}{n}\binom{11}{12-n}$$

Note that we can directly apply Vandermonde's Identity here to get:

$$\sum_{n=1}^{11}\binom{12}{n}\binom{11}{12-n} = \binom{23}{12}$$

After prime factorizing this, it is easy to see that the sum of the prime numbers that
divide N is $\boxed{81}$. $\qquad\square$

§5 A Quick Tangent: Catalan Numbers

> *Remark 5.1.* While the Catalan numbers are not a topic that will usually be covered on the
> AMC or any major computational contest, this section is for those of you who are interested
> in learning about a challenging result of grid-walking techniques.

> **Catalan Numbers**
> The nth Catalan number is the number of paths from $(0,0)$ to (n,n) that don't go
> above the line $y = x$.

> **Theorem 5.2 (Catalan Explicit Formula)**
> The nth Catalan number C_n can be expressed as:
> $$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

Proof. We use complementary counting. As in Theorem 2.2, there are $\binom{2n}{n}$ ways to go
from $(0,0)$ to (n,n) in total. Now, we have to count the number of paths from $(0,0)$ to
(n,n) that go above the line $y = x$. Since this condition appears difficult to count, we can
use a transformation on the paths to make them simpler to count but keeps the number
of paths the same. Specifically, consider the transformation where we take a path going
above the line $y = x$, and reflect the entire part of the path after the first point where
the path goes above $y = x$ over the line $y = x + 1$. For example, the path formed by the
green and red parts will be sent to the path formed by the green and blue parts in this
image. We can always do this transformation because the condition guarantees the path
goes above the line.
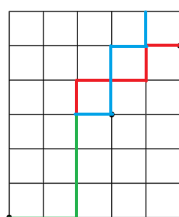
We see that each path is sent to a path from $(0, 0)$ to $(n-1, n+1)$, but we also know that each path from $(0, 0)$ must go over the line $y = x$ to get to $(n-1, n+1)$, so we are able to reverse this transformation. Since the transformation is reversible, it is a bijection, which guarantees that the number of paths stays the same after the transformation. However, we now know how to count these paths: this is just $\binom{2n}{n-1}$ from theorem 2.2! Thus, since we were doing complementary counting, we get $\binom{2n}{n} - \binom{2n}{n-1}$, which we can manipulate to get

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{2n!}{n! \cdot n!} - \frac{2n!}{(n-1)!(n+1)!} = \frac{2n!}{n! \cdot n!}\left(1 - \frac{n}{n+1}\right) = \boxed{\frac{\binom{2n}{n}}{n+1}}.$$

$\square$

The Catalan numbers are an interesting sequence of numbers with a recurrence relation:

> **Fact 5.3.** The $n+1$'th Catalan number C_{n+1} can be expressed as:
>
> $$C_{n+1} = C_0 C_1 + \ldots + C_{n-1} C_n + C_n C_0 = \sum_{k=0}^{n} C_k C_{n-k}$$

§6 Exercises

Exercise 6.1 (AMC 8 2019). Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

Exercise 6.2 (AMC 10 2003). Pat is to select six cookies from a tray containing only chocolate chip, oatmeal and peanut butter cookies. There are at least six each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Exercise 6.3 (AMC10/12B 2019). How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Exercise 6.4 (2007 HMMT Feb). Kevin has four red marbles and eight blue marbles. He arranges these twelve marbles randomly, in a ring. Determine the probability that no two red marbles are adjacent.

Exercise 6.5 (2020 A(N)IME). The garden of Gardenia has 15 identical tulips, 16 identical roses, and 23 identical sunflowers. David wants to use the flowers in the garden to make a bouquet. However, the Council of Elders has a law that any bouquet using Gardenia's flowers must have more roses than tulips and more sunflowers than roses. For example, one possible bouquet could

have 0 tulips, 3 roses, and 8 sunflowers. If N is the number of ways David can make his bouquet, what is the remainder when N is divided by 1000?

Exercise 6.6 (AIME II 2013). Melinda has three empty boxes and 12 textbooks, three of which are mathematics textbooks. One box will hold any three of her textbooks, one will hold any four of her textbooks, and one will hold any five of her textbooks. If Melinda packs her textbooks into these boxes in random order, the probability that all three mathematics textbooks end up in the same box can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 6.7 (AIME 1986). In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence HHTTHHHHTHHTTTT of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

Exercise 6.8 (2008 AIME II). There are two distinguishable flagpoles, and there are 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. Let N be the number of distinguishable arrangements using all of the flags in which each flagpole has at least one flag and no two green flags on either pole are adjacent. Find the remainder when N is divided by 1000.

Exercise 6.9 (AIME I 2015). Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Exercise 6.10 (2006 AIME II). Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded a point and the loser gets 0 points. The total points are accumilated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 6.11 (2016 CMIMC). 1007 distinct potatoes are chosen independently and randomly from a box of 2016 potatoes numbered $1, 2, \ldots, 2016$, with p being the smallest chosen potato. Then, potatoes are drawn one at a time from the remaining 1009 until the first one with value $q < p$ is drawn. If no such q exists, let $S = 1$. Otherwise, let $S = pq$. Then given that the expected value of S can be expressed as simplified fraction $\frac{m}{n}$, find $m + n$.

Exercise 6.12 (2014 HMMT February). An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_z), \ldots, (x_k, y_k)$ of points in $\mathbb{R}^2$ such that $(a, b) = (x_1, y_1), (c, d) = (x_k, y_k)$, and for each $1 \leq i < k$ we have that either $(x_{i+1}, y_{y+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$. Two up-right paths are said to intersect if they share any point. Find the number of pairs (A, B) where A is an up-right path from $(0, 0)$ to $(4, 4)$, B is an up-right path from $(2, 0)$ to $(6, 4)$, and A and B do not intersect.

Exercise 6.13 (2021 CMIMC). How many four-digit positive integers $\overline{a_1 a_2 a_3 a_4}$ have only nonzero digits and have the property that $|a_i - a_j| \neq 1$ for all $1 \leq i < j \leq 4$?