Combinatorial Identities

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§1 The Identities

Note that each of these identities assumes familiarity with the binomial coefficient $\binom{n}{k}$, also known as 'n choose k.' If you are not familiar with the binomial coefficient, you can check out either the Wolfram or Brilliant.org pages on the topic.

In this handout, I'll first define and prove all of the needed identities and then have a section purely consisting of examples. There will also be some exercises given at the end. Solutions will not be given for the exercises but sources will be given so you can find them on your own.

§1.1 Pascal's Identity

Theorem 1.1 (Pascal's Identity)

For any integers n, k:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

must be true.

Proof. Notice that there are two expressions in the left hand side where things are being chosen from a set of $n-1$ objects. Immediately, this tells us to try and think of choosing from n objects as two cases in terms of choosing from $n-1$ objects.

The idea comes from the fact that we are choosing $k-1$ and k objects in each of these two cases. Choose one of the n objects and fix that object as object 1. Note that there are $\binom{n-1}{k-1}$ ways to choose $k-1$ objects from the $n-1$ objects other than object 1. So, there are $\binom{n-1}{k-1}$ ways to choose k objects from n objects such that object 1 is one of those k objects.

Now, let's deal with choosing k objects such that object 1 is not one of those k objects. The way to do this is to choose k objects from the $n-1$ objects not including object 1. So, this is $\binom{n-1}{k}$. We've addressed both of the cases for choosing k objects from n objects, so this means that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. □

Note that the above is an example of a combinatorial argument. Alternatively, identities can be proved algebraically by expanding out the binomial coefficients and solving.

Exercise 1.2. Find the algebraic proof for Pascal's Identity.

§1.2 Hockey-Stick Identity

Theorem 1.3 (Hockey-Stick Identity)

For integer k and $n \geq k$,

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Below is an algebraic proof of the identity:

Proof. Let $n = k + x$. Let's just expand everything out:

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{k+x}{x} = \binom{k+1}{k+1} + \binom{k+1}{k} + \ldots + \binom{k+x}{x}$$

This looks suspiciously like Pascal's Identity. Let's apply Pascal's Identity repeatedly going from the left to the right:

$$\binom{k+1}{k+1} + \binom{k+1}{k} + \ldots + \binom{k+x}{x} = \binom{k+x+1}{k+1} = \binom{n+1}{k+1}$$

As the left hand side is equal to the right hand side now, we are done. □

A combinatorial proof of the identity exists and will be left to the reader as an exercise (see Exercises section).

§1.3 Vandermonde's Identity

Theorem 1.4 (Vandermonde's Identity)

For integers m, n, k, r:

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

This identity is quite easy to prove using a combinatorial argument:

Proof. Looking at the statement, it appears that we are choosing r total out of a pool of $m + n$ things with k of those r coming from one section and the rest coming from the other. So, let's set it up so that our total pool of people is split into groups of m and n with some being chosen from one group and the rest from the other:

Let's say that I want to choose a committee of r people out of $m + n$ people such that this group of people includes m girls and n boys. Note that if I wanted the committee to have x girls and $r - x$ boys, the ways to create the committee would be:

$$N = \binom{m}{x}\binom{n}{r-x}$$

So, this means that the summation in the left hand side of Vandermonde's Identity cycles over the ways to choose k girls and $r - k$ boys for each possible value of k. In total, this is just the total number of ways to make the committee while disregarding the number by gender, which is just $\binom{m+n}{r}$ □

While the proof may look simple, this problem has applications in many hard problems, some of which we will go over in the examples section.

Exercise 1.5. Prove $\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k}$ using Vandermonde's Identity.

§1.4 Stars and Bars

Theorem 1.6 (Stars and Bars)
The number of ways to put n indistinguishable objects into k distinct categories is:

$$N = \binom{n+k-1}{k-1}$$

There is a clever combinatorial argument to prove this:

Proof. Think of a backyard. If I want to split my backyard into n rectangular sections with equal width, I can do this by placing $n-1$ fences in my backyard. Similarly, to split the n objects into k categories, I can line up the n objects in a line with $k-1$ dividers and find the ways to permute these.

So, the numbers of ways to put n indistinguishable objects into k distinct categories is equivalent to the number of ways to rearrange a sequence containing n identical objects and $k-1$ identical dividers.

As there are a total of $n+k-1$ places in the line and $k-1$ of those must be chosen to place dividers in, there are $\binom{n+k-1}{k-1}$ ways to do this.. $\square$

Stars and Bars is a very powerful technique that is used in many AMC and AIME problems. This is probably the identity you will use the most.

§1.5 Binomial Theorem

Theorem 1.7 (Binomial Theorem)
For integer positive integer n:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Proof. Note that in order to create a term with $a^{n-r}b^r$, you must choose a b from r of the $(a+b)$ terms being multiplied and a from the remaining ones. So, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r}$. $\square$

This explains the name of the binomial coefficient, as it is the coefficient in the binomial theorem.

Example 1.8
What is the coefficient of the x^3y^4 term when $(x+y)^7$ is expanded out?

Using the binomial theorem, the coefficient of x^3y^4 is equal to:

$$\binom{7}{3} = \boxed{35}$$

The binomial theorem doesn't just apply to two variable equations. It can also be applied to larger expressions.

Example 1.9

What is the coefficient of the $x^5y^9z^3$ term when $(x+y+z)^{17}$ is expanded out?

This is the number of ways to choose 5 x terms, 9 y terms and 3 z terms from 17 expressions. This is equal to $\frac{17!}{(5!)(9!)(3!)}$ as the coefficient of $x^5y^9z^3$.

§2 Examples

Example 2.1 (AIME 1986)

The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1y + a_2y^2 + \cdots + a_{16}y^{16} + a_{17}y^{17}$, where $y = x+1$ and that a_i's are constants. Find the value of a_2.

Let $f(x) = 1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$. Note that $f(x) = (-1)^k(x+1-1)^k = (-1)^k(y-1)^k = (1-y)^k$. By the Binomial Theorem, the coefficient of the y^a term must be $\binom{a}{2}$ for $a \geq 2$. For $a = 0, 1$, the coefficients are 1 and -1 so they cancel out, meaning we don't have to worry about them. So, our answer is equal to:

$$\sum_{i=2}^{17} \binom{i}{2} = \binom{2}{2} + \cdots + \binom{17}{2}$$

By the Hockey-Stick Identity, this must be equal to $\binom{18}{3} = \boxed{816}$.

Example 2.2 (AIME I 2020)

A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

Essentially, the problem is asking us for $\sum_{n=1}^{11} \binom{12}{n}\binom{11}{n-1}$. Also, note that $\binom{n}{k} = \binom{n}{n-k}$ because the number of ways to choose k from n is also the number of ways to choose $n-k$ not to choose. Using this:

$$\sum_{n=1}^{11} \binom{12}{n}\binom{11}{n-1} = \sum_{n=1}^{11} \binom{12}{n}\binom{11}{12-n}$$

Note that we can directly apply Vandermonde's Identity here to get:

$$\sum_{n=1}^{11} \binom{12}{n}\binom{11}{12-n} = \binom{23}{12}$$

After prime factorizing this, it is easy to see that the sum of the prime numbers that divide N is $\boxed{81}$.

Example 2.3 (AMC10/12B 2020)
Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

(A) 112 **(B)** 128 **(C)** 144 **(D)** 172 **(E)** 184

Note that $96 = 3 \cdot 2^5$. So, we can consider the 3 and the powers of 2 separately. If there are x different factors in a factorization of 96, there are x ways to choose in which factor the 3 goes. There must also be a power of 2 in each of the other factors at the very least to ensure that each factor is strictly greater than 1.

There are now $5 - (x - 1) = 6 - x$ powers of 2 remaining and x factors to place them in. The number of ways to do this can be found with Stars and Bars. This is:

$$\binom{6 - x + x - 1}{6 - x} = \binom{5}{6 - x}$$

So, if 96 is being factored into x factors, there are $x\binom{5}{6-x}$ ways to do this. So, the answer is:

$$\sum_{x=1}^{6} x\binom{5}{6 - x} = 1 + 10 + 30 + 40 + 25 + 6 = \boxed{112}$$

which is choice A.

§3 Exercises

Now, here are some exercises you can do for practice. They are separated into subsections based on difficulty.

§3.1 Easy Exercises

Exercise 3.1 (AMC10A 2018)**.** When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

(A) 42 **(B)** 49 **(C)** 56 **(D)** 63 **(E)** 84

Exercise 3.2 (AMC8 2018)**.** From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?

(A) $\frac{2}{7}$ **(B)** $\frac{5}{42}$ **(C)** $\frac{11}{14}$ **(D)** $\frac{5}{7}$ **(E)** $\frac{6}{7}$

Exercise 3.3 (AMC8 2019)**.** Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the people has at least 2 apples?

(A) 105 **(B)** 114 **(C)** 190 **(D)** 210 **(E)** 380

Exercise 3.4 (AMC10A 2003)**.** Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

(A) 22 **(B)** 25 **(C)** 27 **(D)** 28 **(E)** 29

Exercise 3.5 (AIME I 2013)**.** Melinda has three empty boxes and 12 textbooks, three of which are mathematics textbooks. One box will hold any three of her textbooks, one will hold any four of her textbooks, and one will hold any five of her textbooks. If Melinda packs her textbooks into these boxes in random order, the probability that all three mathematics textbooks end up in the same box can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

§3.2 Harder Exercises

Exercise 3.6 (2020 A(N)IME (AoPS Mock Contest))**.** The garden of Gardenia has 15 identical tulips, 16 identical roses, and 23 identical sunflowers. David wants to use the flowers in the garden to make a bouquet. However, the Council of Elders has a law that any bouquet using Gardenia's flowers must have more roses than tulips and more sunflowers than roses. For example, one possible bouquet could have 0 tulips, 3 roses, and 8 sunflowers. If N is the number of ways David can make his bouquet, what is the remainder when N is divided by 1000?

Exercise 3.7 (AIME II 2000)**.** Given that

$$\frac{1}{2!17!}+\frac{1}{3!16!}+\frac{1}{4!15!}+\frac{1}{5!14!}+\frac{1}{6!13!}+\frac{1}{7!12!}+\frac{1}{8!11!}+\frac{1}{9!10!}=\frac{N}{1!18!}$$

find the greatest integer that is less than $\frac{N}{100}$.

Exercise 3.8 (AIME 1986)**.** In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence HHTTHHHHTHHTTTT of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

Exercise 3.9. Prove the Hockey-Stick Identity using a combinatorial argument.

Exercise 3.10 (AIME I 2015)**.** Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Exercise 3.11 (AMC10/12B 2019)**.** How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

(A) 55 **(B)** 60 **(C)** 65 **(D)** 70 **(E)** 75