Basic Counting

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Basic counting is quite a large and broad topic on AMC-level competitions. This handout intends to cover the things you NEED to know. It's necessary to remark that in many problems, the main ideas presented here are used in conjunction - don't feel restricted to just using one method per problem (and this should apply to any math problem in general).

Contents

§1 The Basics of Counting

This section is probably the first topic students go through when starting competition math. Nevertheless, it's a very important tool and you should always keep it at heart. Let's start with a basic example:

Example 1.1
Suppose you roll a single fair six-sided die 3 times. How many possible outcomes are there for the sequence of results?

Solution. There are 6 possibilities for each result. Additionally, each result is independent of the others, so we just do $6^3 = \boxed{216}$. □

This is a poster example of the "fundamental principle of counting". There are also spin-offs of this that take some logical variation - take a look at this next example:

Example 1.2
6 people want to have a White Elephant exchange with their cars (there are 6 of them). In order they choose exactly 1 car that has not been picked yet. If the 6 cars are all distinct, in how many distinct ways can this be done?

Solution. Unfortunately we can't just do 6^6, because it's not true that each person has 6 options independent each other. However, we can still think about it iteratively. The first person has 6 possible options for their car; the next has 5, because one was already picked; the next has 4 since two were already picked; then it continues all the way down to the last person, who evidently only has 1 possible option. Then the answer is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{720}$. □

This example is also a classic example of using a factorial.

These are the two most basic examples you'll see in combinatorics; we'll provide some generalizations in the next section. Alas, counting problems are generally not this easy - they require some insight. We'll delve into an example which is tacklable with basic counting knowledge, but at the same time is not completely fundamental:

Example 1.3
How many 5 letter words are there such that the third and fifth letters are vowels and no letters are repeated (we're using the standard alphabet)?

Solution. Perhaps we just tried counting from the beginning - there's 26 options for the first letter, 25 for the second letter, 5 for the third letter ... wait, there might not be 5 for the third letter! It's possible that it was already used in the first 2 letters! We need to work around this.

So naturally, we'll pick the third letter first (and the fifth too). There are 5 options for the third and 4 for the fifth (since one possible vowel was already used for the third letter). Then we can fill out the rest easily - there are 24 options for the first letter since 2 were already used by the vowels, 23 for the second, and 22 for the fourth. In all, this is $5 \cdot 4 \cdot 24 \cdot 23 \cdot 22 = \boxed{242880}$ words. □

As demonstrated by the first example, this idea of counting can already start to get insane.

Moral
It's important to keep thinking and not blindly multiply numbers, however, and these should be able to help you tackle such problems.

§2 Fundamental Functions

This section mainly covers the common functions that are absolutely necessary in counting. Their definitions themselves are just extensions of Example 1.1 and Example 1.2.

The most familiar one:

> The number of ways to order n people in a row is given by $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1$.

Example 1.2 is just a corollary of this. There is a variant worth going into:

Example 2.2
Instead of ordering in a row, in how many ways can the people be ordered in a circle, if rotations don't matter?

Solution. Let's start with n people ordered in a row first. Join the ends to form a circle. This means there are $n!$ ways. Actually, there aren't! We're overcounting. Essentially we could've taken a cyclic shift of that row and made the same circle, since order doesn't matter. More clearly, check the diagram. Those arrangements are counted the same - and thus ABC and BCA are counted the same. We're just shifting the letters (and the end goes to the beginning). Going back to the original, there are n possible shifted rows from the original inclusive (n possible options for the first person in the row, and the rest of the row is then determined). Thus, the answer just becomes $\frac{n!}{n} = \boxed{(n-1)!}$. $\square$

Another approach is to fix a person in the circle, then order the rest in $(n-1)!$ ways because rotations will be taken care of. Additionally, if reflections don't matter, you can just divide by 2 (work out the details yourself :P).

We also have the following "brothers", so to speak:

The number of ways to choose k people from a group of n and then order them in a row is given by
$$^nP_k = \frac{n!}{(n-k)!}.$$
Note that $^nP_0 = n!$.

The number of ways to choose k people from a group of n is given by
$$^nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Don't freak out if this seems foreign - let's break it down.

Look at ^nP_k first. We can preset the row and then just fill it out - there are n possible people for the first seat, $n-1$ for the second, $\cdots$, all the way to $n-k+1$ possible people for the kth. Thus we just multiply these, and using a clever factorial trick gives

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

Then ^nC_k follows quite naturally. Note that any choice of k people is counted $k!$ times in ^nP_k. For example, the set $\{A, B, C\}$ is represented $3! = 6$ times, since there's $3! = 6$ possible orderings - ABC, ACB, BAC, BCA, CAB, CBA. So we can simply divide:

$$^nC_k = \frac{^nP_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Let's try one more example that's similar in flavor:

Example 2.5 (General Form)
How many possible n-letter permutations are there that compose of exactly a_1 copies of one letter, exactly a_2 copies of the second, $\cdots$, and exactly a_k copies of the kth? In particular, $a_1 + a_2 + \cdots + a_k = n$.

Solution. Being optimistic, let's strategically overcount and then hope that we can divide by some convenient expressions like we did in previous examples. For demonstration, set $n = 6$, where there are 3 copies of A, 2 copies of B, and 1 copy of C. Suppose we look at the permutation $ABABAC$. We want this to count only once. However, if we just counted with 6!, then $ABABAC$ would be counted multiple times, since identical As are treated different, for example. To account for this, we'll divide by 3!, because there are 3! ways to order those As while keeping those positions. In more detail, I'm saying that $ABABAC$ is counted 3! times when As are treated different. Now, using the exact same logic, we can divide out for B and C as well - we're dividing by 2! and 1!, respectively. Thus, the count becomes

$$\frac{6!}{3!2!1!}.$$

This logic is generalizable. Our answer is thus

$$\boxed{\frac{n!}{a_1!a_2!\cdots a_k!}}.$$

□

With these foundations quickly laid, we can continue.

§3 Casework

Unexpectedly, "casework" is considered "work with cases." Whether you realize it or not, casework is used in 90% of combinatorial problems. Most problems aren't solved with just one application of a formula - you need to dig deeper. Often times you'll need to divide into smaller cases.

There's really no theory to be covered in this section. Rather, we'll give some examples. Generally, getting good at casework takes experience - you'll be able to divide into more meaningful cases, less cases, etc.

Example 3.1 (AIME II 2004/2)
A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Given that the probability that they get the same color combination, irrespective of order, is m/n, where m and n are relatively prime positive integers, find $m+n$.

Solution. We have three cases:

- Both pick two red candies.
- Both pick two blue candies.
- Both pick two different candies.

The probability that Terry picks two red candies is $\frac{10}{20}\cdot\frac{9}{19}=\frac{9}{38}$, and the probability that Mary picks two red candies after Terry chooses two red candies is $\frac{8}{18}\cdot\frac{7}{17}=\frac{28}{153}$. So the probability that they both pick two red candies is $\frac{9}{38}\cdot\frac{28}{153}=\frac{14}{323}$.

Using the same logic, the probability that they both pick two blue candies is also $\frac{14}{323}$.

Now, the probability that Terry picks two different candies is $\frac{20}{20}\cdot\frac{9}{19}=\frac{9}{19}$, and the probability that Mary picks two different candies after Terry chooses two different candies is $\frac{8}{18}\cdot\frac{7}{17}=\frac{28}{153}$. So the probability that they both pick two red candies is $\frac{9}{19}\cdot\frac{28}{153}=\frac{28}{323}$.

Therefore, the probability that they get the same color combination, irrespective of order, is $\frac{56}{323}$, and our answer is $56+323=\boxed{379}$. □

Example 3.2 (AMC 10A 2005/14)
How many three-digit numbers satisfy the property that the middle digit is the

average of the first and the last digits?

Solution. If the middle digit is the average of the first and the last digits, then twice the middle digit is the sum of the first and the last digits. We have numerous cases:

- The middle digit is 1.
- The middle digit is 2.
- The middle digit is 3.
- The middle digit is 4.
- The middle digit is 5.
- The middle digit is 6.
- The middle digit is 7.
- The middle digit is 8.
- The middle digit is 9.

If the middle digit is 1, then there are only two three-digit numbers that satisfy the given property (111, 210).

If the middle digit is 2, then there are only four three-digit numbers that satisfy the given property (123, 222, 321, 420).

If the middle digit is 3, then there are only six three-digit numbers that satisfy the given property (135, 234, 333, 432, 531, 630).

If the middle digit is 4, then there are only eight three-digit numbers that satisfy the given property (147, 246, 345, 444, 543, 642, 741, 840).

If the middle digit is 5, then there are only nine three-digit numbers that satisfy the given property (159, 258, 357, 456, 555, 654, 753, 852, 951).

If the middle digit is 6, then there are only seven three-digit numbers that satisfy the given property (369, 468, 567, 666, 765, 864, 963).

If the middle digit is 7, then there are only five three-digit numbers that satisfy the given property (579, 678, 777, 876, 975).

If the middle digit is 8, then there are only three three-digit numbers that satisfy the given property (789, 888, 987).

If the middle digit is 9, then there is only one three-digit number that satisfies the given property (999).

So the total number of three-digit numbers that satisfy the property is $2+4+6+8+9+7+5+3+1=\boxed{45}$. □

Example 3.3 (AIME I 2014/2)
An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

Solution. We have two cases:

- Both marbles are green.
- Both marbles are blue.

The probability both marbles are green is $\frac{4}{10} \cdot \frac{16}{N+16} = \frac{32}{5N+80}$.

The probability both marbles are blue is $\frac{6}{10} \cdot \frac{N}{N+16} = \frac{3N}{5N+80}$.

Therefore,

$$\frac{32+3N}{5N+80} = \frac{58}{10},$$

and cross-multiplying and solving for N results in $N = \boxed{144}$. □

§4 Constructive Counting

Constructive counting is sort of hard to describe, since technically you're utilizing some aspect of it in most combinatorical problems. However, we can agree that constructive counting problems involve counting objects that satisfy a property by "constructing" those objects. For example, we've already had a glimpse of this in the Fundamental Functions section.

Example 4.1 (AIME I 2003/9)
An integer between 1000 and 9999, inclusive, is called balanced if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there?

Solution. We will do casework based on the sum (notice you cannot simply choose the first three digits and fill in the last digit, since we could have them be $9, 8, 0$ which would require that the last digit be 17, which is invalid).

If the sum s is between 1 and 9, then there are s ways to choose the first two digits since the first digit cannot be 0 (namely $s, 0$ through $1, s-1$) and $s+1$ ways to choose the last two digits ($s, 0$ through $0, s$), giving $s(s+1)$ ways. If the sum s is between 10 and 18 we get $19-s$ ways to do both the first two and last two digits. Thus, we just need to evaluate

$$\sum_{i=1}^{9} i(i+1) + \sum_{i=10}^{18} i^2 = 615,$$

which can be done by hand or using the identity $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. □

Example 4.2 (AIME II 2003/3)
Define a *good word* as a sequence of letters that consists only of the letters A, B, and C - some of these letters may not appear in the sequence - and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?

Solution. After we pick the first letter, every other letter has exactly two choices regardless of what we picked before it. Since the first letter has 3 possibilities and there are 6 other letters we get $3 \cdot 2^6 = 192$. □

Example 4.3 (AIME I 2004/6)
An integer is called snakelike if its decimal representation $a_1a_2a_3 \cdots a_k$ satisfies $a_i < a_{i+1}$ if i is odd and $a_i > a_{i+1}$ if i is even. How many snakelike integers between 1000 and 9999 have four distinct digits?

Solution. First, suppose that the digits in order are $b_1 < b_2 < b_3 < b_4$. Then, we have that there are 5 possible ways to rearrange these four digits into a snakelike integer (we can list them as $b_1b_4b_2b_3, b_1b_3b_2b_4, b_2b_4b_1b_3, b_2b_3b_1b_4, b_3b_4b_1b_2$). Since there are $\binom{10}{4}$ ways to pick the b values, we get $\binom{10}{4} \cdot 5$ numbers. However, we cannot have the first integer be 0, so we need to subtract out the number of snakelike numbers with the first digit as 0. If the three nonzero digits are $b_1 < b_2 < b_3$, then we have 2 possible snakelike numbers we can form (namely $0b_2b_1b_3, 0b_3b_1b_2$), and there are $\binom{9}{3}$ ways to pick these three numbers, giving a final answer of

$$\binom{10}{4} \cdot 5 - \binom{9}{3} \cdot 2 = 882.$$

□

§5 Complementary Counting

Often times, it's easier to find the ***complement***, or the opposite of what we want. Some of the most important cue phrases suggesting to use complementary counting are "at least" and "at most".

Let's walk through a basic example.

Example 5.1
A group of 2 monsters is to be created from 3 zombies and 3 skeletons (all distinguishable). How many possible groups with at least 1 zombie can be created?

Solution. We should think about how the total number of groups is composed in terms of how many zombies there are. There are the groups of exactly 0 zombies, the groups of exactly 1 zombies, the groups of exactly 2 zombies, and the groups of exactly 3 zombies. These four categories make up the total number of groups.

Now, the number of groups with at least 1 zombie is precisely the sum of the numbers of groups from the 1-zombie case, 2-zombie case, and the 3-zombie case (even though this one isn't possible). An easy way to count this is by subtracting the number of 0-zombie groups from the total number of groups. To count the number of 0-zombie groups, we note that any such group has only skeletons. Thus, we simply need to choose 2 skeletons from 3, which is given by $\binom{3}{2} = 3$. To count the total number of groups, we simply need to choose 2 monsters from 6, which is given by $\binom{6}{2} = 15$.

Now we simply subtract. The final answer becomes $15 - 3 = \boxed{12}$. □

Do you see why this is useful? We could've instead computed the numbers from each category and summed them, which wouldn't be too slow, but the complementary counting method was faster and eliminates the need to count several cases.

Let's take a look at a couple more examples before we go.

Example 5.2 (AMC 12B 2008/22)
A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

Solution. First, let's logically simplify this. We want 2 adjacent spaces to be open, out of the $16 - 12 = 4$ open spaces left. We could go into several cases on which spaces are left, but we can think about it in another way - in what situation is Auntie Em not able to park? Remark that this is really just equivalent to choosing 4 spaces from the 16 for which no two are adjacent. In a more intuitive way, we want 4 bars to divide 12 stars such that none of the bars are next to each other.

If you've read our Stars and Bars handout, you can try tackling this yourself. If not, we'll go through it here anyway. Imagine the 4 numbers dividing the 16 numbers into 5 groups. To prevent consecutive numbers, the middle 3 groups should have at least one number in them. We can give one to each group, so that now all 5 groups have no extra requirement of numbers. We have 9 stars left and 4 bars, so now the number of such arrangements is $\binom{13}{4}$.

Now we can extract the answer. There are $\binom{16}{4}$ total possible parking arrangements (we just choose 4 spaces that are empty), $\binom{13}{4}$ of which don't allow Auntie Em to park. Thus, the answer is

$$1 - \frac{\binom{13}{4}}{\binom{16}{4}} = 1 - \frac{11}{28} - \boxed{\frac{17}{28}}.$$

□

Example 5.3 (AIME I 2002/1)
Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. One possible approach to finding the count is adding the number of sequences for which the three-letter arrangement is a palindrome with the number of sequences for which the three-digit arrangement is a palindrome, and then subtracting the overcount (in other words, PIE).

Another approach is simply subtracting the number of sequences in which neither the three-letter arrangement nor three-digit arrangement is a palindrome from the total. We'll use this approach.

Note that the three-letter arrangement not being a palindrome is equivalent to the first and third letters being different. Now we use our basic counting skills - there are 26 options for the first letter, 26 for the second, and 25 for the third (we subtracted just that 1 since it can't be the same as the first). Thus there are $26 \cdot 26 \cdot 25$ such three-letter arrangements that are not palindromes. Similarly, there are $10 \cdot 10 \cdot 9$ three-digit arrangements that are not palindromes.

Now keep in mind that the three-letter arrangement and the three-digit arrangement are independent of each other. Thus, the number of sequences in which neither the three-letter arrangement nor three-digit arrangement is a palindrome can be obtained by multiplying these two counts, which is $26 \cdot 26 \cdot 25 \cdot 10 \cdot 10 \cdot 9$. The total number of sequences is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$. We want the complement, which is

$$1 - \frac{26 \cdot 26 \cdot 25 \cdot 10 \cdot 10 \cdot 9}{26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10} = 1 - \frac{45}{52} = \boxed{\frac{7}{52}}.$$

□

That about sums it up for complementary counting. You want to use it to your advantage when the thing you DON'T want to count is actually easy to count. Always be on the lookout for cue words. Finally, please remember:

Moral
Make sure that you understand how different cases make up the total count when complementary counting. The cases shouldn't overlap! Furthermore, you may need to subtract multiple cases from the total. In summary, don't do things blindly!

§6 Exercises

Exercise 6.1 (AMC 10B 2019). A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

Exercise 6.2 (AIME 2019). Jenn randomly chooses a number J from $1, 2, 3, \ldots, 19, 20$. Bela then randomly chooses a number B from $1, 2, 3, \ldots, 19, 20$ distinct from J. The value of $B - J$ is at least 2 with a probability that can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Exercise 6.3 (HMMO 2020). Jody has 6 distinguishable balls and 6 distinguishable sticks, all of the same length. How many ways are there to use the sticks to connect the balls so that two disjoint non-interlocking triangles are formed? Consider rotations and reflections of the same arrangement to be indistinguishable.

Exercise 6.4 (AIME 2018). Kathy has 5 red cards and 5 green cards. She shuffles the 10 cards and lays out 5 of the cards in a row in a random order. She will be happy if and only if all the red cards laid out are adjacent and all the green cards laid out are adjacent. For example, card orders $RRGGG$, $GGGGR$, or $RRRRR$ will make Kathy happy, but $RRRGR$ will not. The probability that Kathy will be happy is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Exercise 6.5 (HMMO 2020). Nine fair coins are flipped independently and placed in the cells of a 3 by 3 square grid. Let p be the probability that no row has all its coins showing heads and no column has all its coins showing tails. If $p = \frac{a}{b}$ for relatively prime positive integers a and b, compute $a+b$.

Exercise 6.6 (HMMO 2020). How many ways are there to arrange the numbers $\{1,2,3,4,5,6,7,8\}$ in a circle so that every two adjacent elements are relatively prime? Consider rotations and reflections of the same arrangement to be indistinguishable.