Functions

Dylan Yu

primeri.org

Contents

§1 Introduction

Functions are a staple of mathematics, both in normal school mathematics and contest mathematics. Knowing how to deal with functions is a key part of dealing with math competition algebra problems in general.

§1.1 Definitions

Function

A ***function*** f from a set X to a set Y is a relation that assigns to each element in set X **exactly one element** in set Y.

Domain

The ***domain*** is the set of X (a.k.a. the **input**).

Range

The ***range*** is the set of Y (a.k.a. the **output**).

§1.2 Existence of a Function

Theorem 1.4 (Vertical Line Test)

If you can draw a vertical line that passes through more than one point of a relation on a graph, it is not a function. If you cannot, it is a function.

Of course, assume that you aren't bound to physical constraints, like getting tired from drawing a long line.

Example 1.5

What are the domain and range of the function $f(x) = \sqrt{16 - x^2}$?

Solution. Note that if $a < 0$, then $\sqrt{a}$ is undefined for reals. Thus, $16 - x^2 \geq 0 \implies \boxed{-4 \leq x \leq 4}$. Since $x^2 \geq 0$, we have that $0 \leq 16 - x^2 \leq 16$, so the range is $\boxed{0 \leq y \leq 4}$. □

Example 1.6

What is the domain of the function $f(x) = \frac{\sqrt{x-10}}{x-15}$?

Solution. The numerator tells us that $x \geq 10$ and the denominator tells us $x \neq 15$. Thus, in **interval notation** we have $\boxed{[10, 15) \cup (15, \infty)}$. □

Example 1.7

What is the range of the function $f(x) = 3|x - 5| - 4$?

Solution. We know that $|x - 5| \geq 0$, so $3|x - 5| - 4 \geq 3 \cdot 0 - 4 = -4$. Thus, $\boxed{[-4, \infty)}$ is our range. □

§2 Combinations of Functions

Theorem 2.1 (Common Function Combinations)
The following are some common combinations of functions:

1. **Sum:** $(f+g)(x) = f(x) + g(x)$
2. **Difference:** $(f-g)(x) = f(x) - g(x)$
3. **Product:** $(fg)(x) = f(x) \cdot g(x)$
4. **Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$
5. **Composition:** $(f \circ g)(x) = f(g(x))$

Example 2.2
If $f(x) = 2x+3$ and $g(x) = 2x-3$, then what is $(fg)(4)$?

Solution. Note that $(fg)(x) = (2x+3)(2x-3)$. Thus, $(fg)(4) = (11)(5) = \boxed{55}$. □

§2.1 Domain and Range of a Composite Function

Composite Function
A ***composite function*** is a function within another function, e.g. $f(g(x))$.

Domain of Composite Function
The ***domain of a composite function*** is the intersection of domains of the starting and final function.

Range of Composite Function
The ***range of a composite function*** is the range of the final function restricted by the starting function.

Theorem 2.6 (Horizontal Asymptote of Rational Functions)
Let $f(x), g(x)$ be polynomials, $\deg f(x) = a, \deg g(x) = b$, and the leading term of $f(x), g(x)$ has coefficient c, d, respectively.

1. $a > b$. Then there is no horizontal asymptote – instead, there is a slant asymptote, which is the quotient of $\frac{f(x)}{g(x)}$ (i.e. $q(x)$, where $f(x) = g(x)q(x) + r(x)$, $\deg r(x) < \deg q(x)$).
2. $a = b$. The horizontal asymptote is $y = \frac{c}{d}$.
3. $a < b$. The horizontal asymptote is $y = 0$.

Example 2.7
Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x}{x-3}$. Then $g(x)$ is the starting function and $f(g(x))$ is the final function. Find the domain of $f(g(x))$.

Solution. Note that
$$f(g(x)) = \frac{1}{\frac{x}{x-3} + 2} = \frac{x-3}{3(x-2)},$$
so $x \neq 2$, but $g(x) = \frac{x}{x-3}$, so $x \neq 3$, implying the domain is $\boxed{x \neq 2, 3}$. □

Example 2.8
Using the same functions as the example above, find the asymptotes and range of $f(g(x))$.

Solution. Since $f(g(x)) = \frac{x-3}{3(x-2)}$, the **vertical asymptote** is $\boxed{x = 2}$. The **horizontal asymptote** is $\boxed{y = \dfrac{1}{3}}$ as per the rules above. Since $\frac{x-3}{3(x-2)}$ passes through $(3, 0)$, but x cannot be 3 since it is undefined for $g(x)$, and nothing else passes through the x-axis, $y = 0$ is not in the range. The horizontal asymptote tells us that it does not pass through $y = \frac{1}{3}$, and all other values work, so the range of $f(g(x))$ is $\boxed{(-\infty, 0) \cup \left(0, \dfrac{1}{3}\right) \cup \left(\dfrac{1}{3}, \infty\right)}$. □

Example 2.9
If $f(x) = \sqrt{x}$ and $g(x) = x - 1$, what is the domain and range of $(g \circ f)(x)$?

Solution. Note that $(g \circ f)(x) = \sqrt{x} - 1$. It is obvious that $x \geq 0$, and all other values work, so the domain is $\boxed{[0, \infty)}$. Since $\sqrt{x} \geq 0$, we have $(g \circ f)(x) \geq -1$, with no other restrictions, so the range is $\boxed{[-1, \infty)}$. □

Example 2.10
If $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x-1}$, what is the domain and range of $(g \circ f)(x)$?

Solution. Note that $(g \circ f)(x) = \frac{1}{\frac{1}{x} - 1} = \frac{x}{1-x}$. Note that x cannot be 0, since $f(0)$ is undefined, and x cannot be 1, since $g(f(1))$ is undefined. Thus, the domain is $\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$. The vertical asymptote is $x = 1$ and the horizontal asymptote is $y = -1$, and $x \neq 0$ so $y \neq 0$, implying the range is $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$. □

§3 Types of Functions

§3.1 Piecewise-Defined Function

Piecewise Function
A ***piecewise function*** is a function that is defined by two or more equations over a specified domain.

Example 3.2 (Absolute Value is Piecewise)
Let $f(x) = |x|$. Then

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Let's try a few examples.

Example 3.3
What are the domain and range of the piecewise function as follows?

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ x - 1 & x \geq 0 \end{cases}.$$

Solution. The domain includes $x < 0$ and $x \geq 0$, which is all values, so the domain is $\boxed{(-\infty, \infty)}$. For $x \geq 0$, we have $f(x) = x - 1$, so the range there is $y \geq -1$. For $x < 0$, we have $f(x) = x^2 + 1$, so $x^2 > 0$, implying the range there is $y > 1$. Thus, the range together is $\boxed{[-1, \infty)}$. □

§4 Properties of Functions

§4.1 Odd and Even Functions

Even
A function f is ***even*** if $f(x) = f(-x)$.

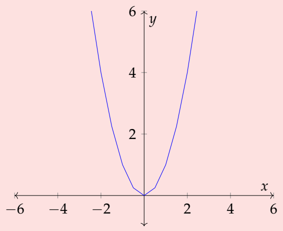


Figure 1: Graph of an even function.

Odd
A function f is ***odd*** if $f(x) = -f(-x)$.

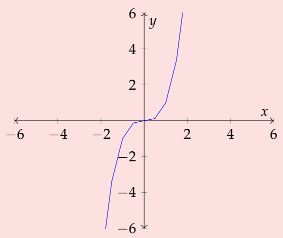


Figure 2: Graph of an odd function.

Note that:

- reflecting an even function across the y-axis yields the same function, and
- rotating an odd function across the x-axis also yields the same function.

Theorem 4.3 (Parity of Functions Comes From Its Components)
If $h(x) = f(x) + g(x)$, then $h(x)$ is even if $f(x)$ and $g(x)$ are both even, and $h(x)$ is odd if $f(x)$ and $g(x)$ are both odd.

Example 4.4
Is $f(x) = x^3 - 2x$ odd, even, or neither?

Solution. Note that $f(-x) = -x^3 + 2x = -f(x)$, implying it is $\boxed{\text{odd}}$. □

§4.2 Periodic Functions

Periodic
A function f is ***periodic*** if there exists a number p such that

$$f(x + p) = f(x)$$

for all numbers x.

Fundamental Period
The smallest period is called the ***fundamental period*** of the function.

Theorem 4.7 (Periodic Function Transformations)
If a periodic function f has period p, then $y = cf(x)$ still has period p, and $y = f(cx)$ has period $\frac{p}{c}$.

Note that the **smallest period** is simply called the ***period***.

Theorem 4.8 (Period of Trigonometric Functions)
The period of $f(x) = a\sin(bx + c) + d$, $f(x) = a\cos(bx + c) + d$, $a\csc(bx + c) + d$, and $a\sec(bx + c) + d$ are all $\frac{2\pi}{b}$. Furthermore, the period of $a\tan(bx + c) + d$ and $a\cot(bx + c) + d$ are both $\frac{\pi}{b}$.

Corollary 4.9
The periods of $\sin(x), \cos(x), \csc(x), \sec(x)$ are all 2π. The periods of $\tan(x), \cot(x)$ are both π.

Example 4.10
If a function $f(x) = \sin x$ has period 2π, then what is the period of the function $f(x) = -3\sin 3x$?

Solution. The period is $\boxed{\frac{2\pi}{3}}$. □

§5 Inverse Functions

Inverse Function
An ***inverse function*** is a function that reverses function f.

If f is a function mapping x to y, then the inverse function of f maps y back to x. The inverse function of f is usually denoted by f^{-1}. **Do not confuse it with** $\frac{1}{f}$.

Fact 5.2. If f^{-1} exists, then $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

Fact 5.3. The graph of the inverse is the graph of the function reflected across $y = x$.

Example 5.4
If $f(x) = x + 5 : \{(1,6),(2,7),(3,8),(4,9)\}$, then $f^{-1}(x) = \{(6,1),(7,2),(8,3),(9,4)\}$.

Taking the inverse is not particularly hard, but there are some things to pay attention to.

Example 5.5
What is the inverse function of $f(x) = \frac{3x-5}{2}$?

Solution. If we switch x and y we get $x = \frac{3y-5}{2} \implies \boxed{f^{-1}(x) = \frac{2x+5}{3}}$. □

Example 5.6
If $f(4) = 35$, then what is $f^{-1}(35)$?

Solution. The x- and y-values have been switched, so the answer is $\boxed{4}$. □

§5.1 Existence of an Inverse Function

One-to-One
If a function satisfies the property that each x-value corresponds to one y-value, and each y-value corresponds to one x-value, then the function is ***one-to-one***.

Theorem 5.8 (Inverse Function Criterion)
If a function f is one-to-one, then its inverse is a function. More specifically, f is one-to-one if f is increasing/decreasing on its entire domain.

Theorem 5.9 (Horizontal Line Test)
If you can draw a horizontal line passing through more than one point of a function on a graph, its inverse is not a function. If you cannot, it is a function.

This makes a lot of sense, since **the graph of the inverse is just the function flipped across the $y = x$ line**. This also makes it a lot easier to draw graphs.

Example 5.10
Does the function $f(x) = \sqrt{x-2}+3$ have an inverse function?

Solution. It passes the Horizontal Line Test, as seen below:

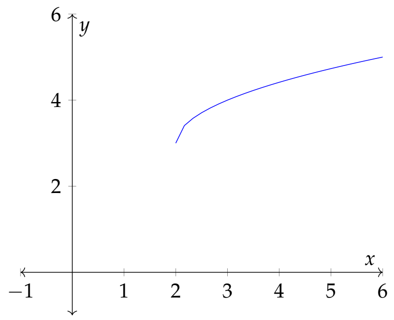


Figure 3: $f(x) = \sqrt{x-2}+3$ passes the Horizontal Line Test.

□

Example 5.11
Does the function $g(x) = |x+3|$ have an inverse function?

Solution. It does not pass the Horizontal Line Test, as seen below:

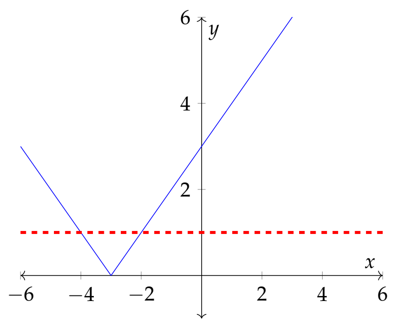


Figure 4: $g(x) = |x+3|$ does not pass the Horizontal Line Test.

□

§6 Common Problem Types

§6.1 Recursive Functions

Recursive Function
A ***recursive function*** is a function that relies on previous values of the function.

Usually, these have a domain of $\mathbb{N}$. There are a few types of these problems:

1. **Engineer's Induction**: there is usually a pattern to find. Specifically, there could be an obvious period of the function (this is indicated when the goal is to compute some arbitrary $f(a)$ where a is large), or it could just be a well-known function

(i.e. $2^x, x^2, ax+b$, etc.). **Warning**: engineer's induction is known to sometimes fail. You can usually trust after 5-6 steps, and it has to make sense with the context of the problem (e.g. if the graph of weight of tomatoes vs. cost of tomatoes is quintic by engineer's, there's probably something weird going on).

2. **Symmetric Functions**: there are rigid properties given about the function and the task is again to find some arbitrary value. The idea is to take advantage of symmetry, and furthermore, see if stuff telescopes. It is important to note $\sum_{k=a}^{b}(f(k+1)-f(k)) = f(b+1)-f(a)$.

The trick for both of these to try small cases.

§6.2 Functional Equations

These overlap heavily with recursive functions. The main difference is that they care less about the values and more about the overall function itself. Again, there a few types:

1. **Baby FEs**: similar to symmetric functions. For example,

$$a(f(x))+b\left(f\left(\frac{k}{x}\right)\right)=g(x),$$

where a, b, k are constants, and $f(x), g(x)$ are functions. Note that $g(x)$ is usually given. The idea is to plug in values that give us as little (distinct) variables as possible. In this specific case, the solution is

$$f(x)=\frac{a(g(x))-b(g(\frac{k}{x}))}{a^2-b^2}.$$

2. **Teen FEs**[1]: this is best explained using an example.

Find all function in the domain or reals, such that for all real x and y we have

$$yf(2x)-xf(2y)=8xy(x^2-y^2).$$

The idea here is to rewrite this in a form where

$$g(a)=g(b)$$

for some g. In this case, $g(x)=\frac{f(x)}{x}-x^2$. By getting that g is constant, we can then derive f.

§7 An Instructive Example

Now that we've gone over two major problem types, let's do an example before we move on to the problem set. This is a really cool example because it allows us to try both types of problems (recursive functions and functional equations).

[1] Also known as easier olympiad FEs.

Example 7.1 (AMC 10B 2018/20)
A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

Let's start by trying to solve the problem using the methods we've highlighted for "recursive functions" as we are explicitly told the function f is defined recursively.

Solution. Let's try to think of this using Engineer's Induction. Let's start by listing out the first few values of the function. Starting at $n = 1$ and going upward, we have:

$$1, 1, 3, 6, 8, 8, 7, 7, 9, 12, 14, 14, 13, 13, 15$$

A pattern that seems to appear is that for odd multiple of 3 n, $f(n) = n$, and after this n, $f(n+1) = n+3$, $f(n+2) = n+2$, $f(n+3) = n$, $f(n+4) = n-1$, and $f(n+5) = n$. Applying this logic for 2018 yields:

$$f(2018) = \boxed{2017}$$

□

The second way to solve this problem invokes the second time of common functions problem, which is functional equations.

Solution. Using a technique highlighted in the past section, let's start by getting some related equations and then adding them to cancel out terms:

$$f(n) = f(n-1) - f(n-2) + n$$

$$f(n-1) = f(n-2) - f(n-3) + n - 1$$

These add to:

$$f(n) = 2n - 1 - f(n-3)$$

Using this to create more related equations to add yields:

$$f(n) + f(n-3) = 2n - 1$$

$$f(n+3) + f(n) = 2n + 5$$

Subtracting the first equation from the second equation gives:

$$f(n+3) - f(n-3) = 6$$

We can repeat this strategy yet again (by now the strategy here seems to be clear: get equations, cancel things out) to get:

$$f(2018) - f(2012) = 6$$

$$f(2012) - f(2006) = 6$$

$$\dots$$

$$f(8) - f(2) = 6$$

Adding all of this yields:

$$f(2018) - f(2)=2016$$

We know that $f(2) = 1$, so:

$$f(2018) = \boxed{2017}$$

□

It's fairly clear why this is an illustrative example, because it shows both of our main problem solving techniques for functions related problems. This is a fairly good example of applying the techniques from both types of problems in different ways to solve the problem.

§8 Exercises

Exercise 8.1 (Harold Reiter). Let

$$g(x) = \begin{cases} |x| - 2 & \text{if } x \leq 0 \\ x - 3 & \text{if } 0 < x < 4 \\ 3 - x & \text{if } 4 \leq x \end{cases}$$

Find a number x such that $g(x) = -4$.

Exercise 8.2 (Harold Reiter). Consider the function $F : \mathbb{N} \to \mathbb{N}$ defined by

$$F(n) = \begin{cases} \frac{n}{3} & \text{if } n \text{ is a multiple of } 3 \\ 2n+1 & \text{if otherwise} \end{cases}$$

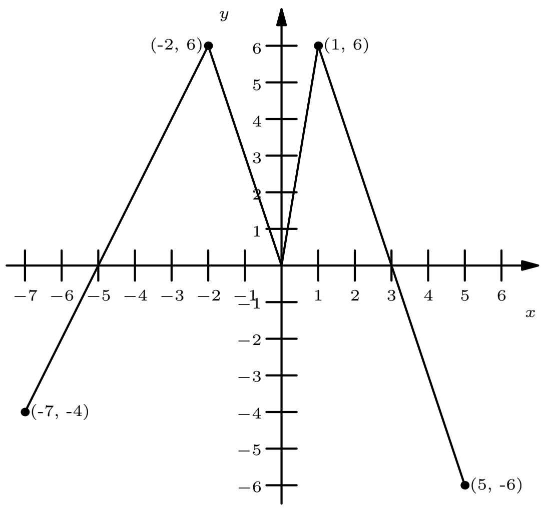
For how many positive integers k is it true that $F(F(k)) = k$?

Exercise 8.3 (Harold Reiter). Suppose $f(0) = 3$ and $f(n) = f(n-1)+2$. Let $T = f(f(f(f(5))))$. What is the sum of the digits of T?

Exercise 8.4. Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. What is $f(21)$?

Exercise 8.5 (AMC12 2000). Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

Exercise 8.6 (AMC12A 2002). The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 6$ have?



Exercise 8.7 (AMC12A 2006). The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x.$$

What is the largest set of real numbers that can be in the domain of f?

Exercise 8.8 (AMC12B 2010). Monic quadratic polynomials $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeros at $x = -23, -21, -17,$ and -15, and $Q(P(x))$ has zeros at $x = -59, -57, -51$ and -49. What is the sum of the minimum values of $P(x)$ and $Q(x)$?

Exercise 8.9 (AIME 1984). The function f is defined on the set of integers and satisfies $f(n) = \begin{cases} n-3 & \text{if } n \geq 1000 \\ f(f(n+5)) & \text{if } n < 1000 \end{cases}$

Exercise 8.10 (AIME 1988). The function f, defined on the set of ordered pairs of positive integers, satisfies the following properties:

$$\begin{aligned} f(x,x) &= x, \\ f(x,y) &= f(y,x), \quad \text{and} \\ (x+y)f(x,y) &= yf(x,x+y). \end{aligned}$$

Calculate $f(14, 52)$.

Exercise 8.11 (AIME 1994). The function f has the property that, for each real number x,

$$f(x) + f(x-1) = x^2.$$

If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

Exercise 8.12 (AIME 1997). The function f defined by $f(x) = \frac{ax+b}{cx+d}$, where a,b,c and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$ and $f(f(x)) = x$ for all values except $\frac{-d}{c}$. Find the unique number that is not in the range of f.