# Manipulating Polynomials

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## Contents

# §1 Introduction

Factorizations, substitutions, and other manipulations are going to be denoted under one list. Quite a few are taken from here, but I took some from the list that I thought could possibly show up on an AMC or AIME-level contest (so not those two contests specifically, but something similar).

## §1.1 Elementary Results

Some results appear so much that they are basically mandatory. These are applied in different ways, but they should always be considered if they do appear.

**1 (Difference of Powers)** $x^n - y^n = (x-y)\left(\sum_{i=0}^{n-1} x^{n-1-i}y^i\right)$.

**2 (Sum of Odd Powers)** $x^{2n+1} + y^{2n+1} = (x+y)\left(\sum_{i=0}^{2n} (-1)^i (x^{2n-i}y^i)\right)$.

Difference of squares and cubes are special cases of these two.

**3 (Sum of Squares)** $x^2 + y^2 = (x+y)^2 - 2xy = (x-y)^2 + 2xy$.

**4** $x^2 + y^2 + z^2 \pm (xy + yz + zx) = \frac{(x\pm y)^2 + (y \pm z)^2 + (z \pm x)^2}{2}$.

**5 (Square of Sums)** $x^2 + y^2 + z^2 + 2(xy + yz + zx) = (x+y+z)^2$.

**6** $x^2 + y^2 + z^2 + 3(xy + yz + zx) = (x+y)(y+z) + (y+z)(z+x) + (z+x)(x+y)$.

**7** $xy + yz + zx - (x^2 + y^2 + z^2) = (x-y)(y-z) + (y-z)(z-x) + (z-x)(x-y)$.

**8** $x^2 + y^2 \pm xy = \frac{x^2 + y^2 + (x \pm y)^2}{2}$.

**9** $4(x^2 + xy + y^2) = 3(x+y)^2 + (x-y)^2$.

**10** $3(x^2 - xy + y^2) = (x^2 + xy + y^2) + 2(x-y)^2$.

**11** $x^2 + y^2 + z^2 - (xy + yz + zx) = (x-y)^2 + (x-z)(y-z)$.

**12** $x^2 + y^2 + z^2 - (xy + yz + zx) = \frac{1}{6}[(2x - y - z)^2 + (2y - z - x)^2 + (2z - x - y)^2]$.

**13** $(xy + yz + zx)(x + y + z) = (x+y)(y+z)(z+x) + xyz$.

**14** $3(x+y)(y+z)(z+x) = (x+y+z)^3 - (x^3 + y^3 + z^3)$.

The above result has helped me in Mathcounts and AMCs time and time again.

**15 (Titu)** $x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)[(x-y)^2 + (y-z)^2 + (z-x)^2]$.

The above is often used in solving inequalities.

**16 (Method of Lagrange)** $\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$.

**17 (Sophie Germain's Identity)** $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

The above is very important! We'll use it in the problems below.

**18** $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$.

## §1.2 Simon's Favorite Factoring Trick

I'm going to highlight this in particular, because it is important for basically any contest.

**19 (SFFT)** $xy + bx + ay + ab = (x + a)(y + b)$.

This doesn't seem that useful, but it's other name might help you understand it: ***completing the rectangle***. Basically, if we are lacking the $ab$ term on the LHS, we can add it to both sides to "complete the rectangle". This is rarely useful if $x, y, a, b$ aren't all integers.

**Example 1.1 (Alcumus)**
How many distinct ordered pairs of positive integers $(m, n)$ are there so that the sum of the reciprocals of $m$ and $n$ is $\frac{1}{4}$?

*Solution.* We are given that

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{4}$$

Multiplying by $4mn$ yields

$$4n + 4m = mn \implies mn - 4n - 4m = 0$$

We can factor this by SFFT which yields

$$(m - 4)(n - 4) = 16$$

If we let $x = m - 4$ and $y = n - 4$ we have the following values of $m$ and $n$

| $x$ | $y$ | $m = x + 4$ | $n = y + 4$ |
|---|---|---|---|
| $-1$ | $-16$ | 3 | $-12$ |
| $-16$ | $-1$ | $-14$ | 3 |
| $-2$ | $-8$ | 2 | $-4$ |
| $-8$ | $-2$ | $-4$ | 2 |
| $-4$ | $-4$ | 0 | 0 |
| 1 | 16 | 5 | 20 |
| 16 | 1 | 20 | 5 |
| 2 | 8 | 6 | 12 |
| 8 | 2 | 12 | 6 |
| 4 | 4 | 8 | 8 |

There are only $\boxed{5}$ pairs for which both $m$ and $n$ are positive integers. □

## §2 Classifying Factorizations

When there is ***asymmetry***, the idea is just to make it symmetrical or simply factor. It is more interesting to consider *full symmetry* and *partial symmetry*. I consider ***full symmetry*** to be when all $n$ variables are "the same". More rigorously, a function $f(x_1, x_2, \ldots, x_n)$ is fully symmetric if any permutation $\pi(x_1, x_2, \ldots, x_n)$ gives us the same function:

$$f(x_1, x_2, \ldots, x_n) = f(\pi(x_1, x_2, \ldots, x_n)).$$

***Partial symmetry*** is when if we take some subset of these $n$ variables, rearrange them, and the function is the same. For example, if we switch $x_1$ and $x_2$, and get the same function, it is partially symmetric.

> *Remark 2.1.* These are definitions I made up. You (probably) won't see this anywhere else.

There are so many examples of fully symmetric factorizations:

**20** $(x+y+z)^3 - (y+z-x)^3 - (z+x-y)^3 - (x+y-z)^3 = 24xyz.$

**21** $(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$

**22 (Lagrange's Identity)** $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2.$

It is pretty obvious what we are going to take advantage of in these factorizations. Less obvious is how partial symmetry is used. Let's use the following to illustrate this:

**23** If a polynomial $P(a,b,c)$ satisfies: $P(a,b,c) = P(b,a,c)$ and $P(a,a,c) = 0$, then $(a-b)^2$ is a factor of $P(a,b,c)$.

**Example 2.2 (ELMO SL 2010)**
Let $a, b, c$ be positive reals. Prove that

$$\frac{(a-b)(a-c)}{2a^2+(b+c)^2} + \frac{(b-c)(b-a)}{2b^2+(c+a)^2} + \frac{(c-a)(c-b)}{2c^2+(a+b)^2} \geq 0.$$

I'm not going to solve this here, but you can read the solution here. For now, just take a look at the first two terms on the LHS. It is obvious that if we let the sum of those two fractions be $P(a,b,c)$, we have

1. $P(a,b,c) = P(b,a,c)$, and
2. $P(a,a,c) = 0$.

Look familiar?

## §3 Examples

**Example 3.1**
Determine all pairs of positive integers $(x,y)$ such that $2xy$ is a perfect square and $x^2+y^2$ is a prime number.

This problem screams parity, for two reasons:

1. $2xy$ being a perfect square means there's a factor of 2 inside $xy$
2. $x^2 + y^2$ being a prime number means it's probably not 2, but if it is, that's pretty great too.

*Solution.* If $x^2 + y^2$ is even, it must be 2. Thus, $x = y = 1$, but this is not a solution. Thus, $x^2 + y^2$ is odd, implying one of $x, y$ is even and the other is odd. WLOG, let $x$ be even and $y$ be odd, i.e. there exists integers $a, b$ such that $x = 2a, y = 2b + 1$. Then

$$4a(2b+1)$$

is a perfect square, and since $a$ and $2b + 1$ can't have any common factors (otherwise $x^2 + y^2 = (2a)^2 + (2b + 1)^2$ wouldn't be a prime), we must have that $a$ and $2b + 1$ are squares, say $m^2$ and $n^2$, respectively. Thus,

$$x^2 + y^2 = 4m^4 + n^4,$$

and using Sophie Germain we get

$$(n^2 + 2m^2 + 2mn)(n^2 + 2m^2 - 2mn),$$

which we can easily show is never prime. Thus, there are no solutions. □

> *Remark 3.2.* Notice how factorizations were **secondary** to solving the problem. This is something that happens a lot – in NT problems, it is more common to use factorizations to prove a certain number is not prime.

Sometimes Sophie Germain is applied more obviously.

**Example 3.3 (AIME 1987/14)**
Compute

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

*Solution.* Using Sophie Germain, we get

$$\frac{[(10(10-6)+18)(10(10+6)+18)][(22(22-6)+18)(22(22+6)+18)]\cdots[(58(58-6)+18)(58(58+6)+18)]}{[(4(4-6)+18)(4(4+6)+18)][(16(16-6)+18)(16(16+6)+18)]\cdots[(52(52-6)+18)(52(52+6)+18)]}$$

$$= \frac{(10(4)+18)(10(16)+18)(22(16)+18)(22(28)+18)\cdots(58(52)+18)(58(64)+18)}{(4(-2)+18)(4(10)+18)(16(10)+18)(16(22)+18)\cdots(52(46)+18)(52(58)+18)}.$$

Many of these terms cancel out, giving us

$$\frac{58(64)+18}{4(-2)+18} = \frac{3730}{10} = \boxed{373}.$$

□

**Example 3.4**
Find the remainder of $x^{2020} + 1$ when divided by $x^2 + x + 1$.

Most have seen how when we are dividing by a linear function, we can plug in the $x$-intercept to get the constant remainder. This is known as ***remainder theorem***. Now we apply roots of unity to this (there actually isn't a large difference, we still plug roots in, but understanding the fundamental property of $\omega^n = 1$ is crucial).

*Solution.* This is equivalent to finding $r$, where

$$x^{2020}+1=(x^2+x+1)q(x)+r(x).$$

Note that $r$ is at most linear (since the degree of $r$ is less than the degree of $x^2+x+1$ always). Let $\omega$ and $\omega^2$ be the two third of roots of unity not equal to 1. Specifically, these two numbers are the roots of $x^2+x+1$. Furthermore, $x^3=1$ where $x=\omega,\omega^2$. Plugging in $\omega$, we get

$$\begin{aligned}\omega^{2020}+1&=r(\omega),\\(\omega^3)^{673}\cdot\omega+1&=r(\omega),\\\omega+1&=r(\omega).\end{aligned}$$

Plugging $\omega^2$, we get

$$\begin{aligned}\omega^{4040}+1&=r(\omega^2),\\(\omega^3)^{1346}\cdot\omega^2+1&=r(\omega^2),\\\omega^2+1&=r(\omega^2).\end{aligned}$$

Thus, if $r(x)=ax+b$, we have

$$r(\omega)=a\omega+b=\omega+1,$$

$$r(\omega^2)=a\omega^2+b=\omega^2+1,$$

which gives us very easily that $r(x)=\boxed{x+1}$. □

# §4 Problem Solving Strategies

## §4.1 Obvious Factorizations

This type of problem involves some polynomial we have to factor or some remainder we have to find.

1. Use remainder theorem, or more generally, the fact all polynomials are in the form $f(x)=g(x)\cdot q(x)+r(x)$, where $q$ is the quotient and $r$ is the remainder after dividing $f$ by $g$.

2. Roots of unity can help with somewhat nice polynomials (usually of the form $x^n\pm 1$ or $x^n+x^{n-1}+\ldots+1$).

3. Algebraic manipulations in the AIME often end in Vieta's formulas.

I should mention the opposite of factorization: distribution. A trick for these is to multiply. For example, if we have $a+\frac{1}{b}=7$ and $b+\frac{1}{a}=5$, multiplying them can get us $ab+\frac{1}{ab}=33$.

## §4.2 Hidden Factorizations

These appear more in number theoretic problems when trying to prove certain numbers must be factored in a certain form (e.g. prime). These also appear in "fake NT", where the problem is more about arithmetic than NT.

1. **Generalize**: specifically, if we have to factorize some weird arithmetic expression, it is useful to replace terms with variables. Note this often requires replacing consecutive numbers as well (for example, if we replace 2020 with $x$, it is good to replace 2019 with $x-1$ as well).

2. **Engineer's Induction**: after we generalize, we can try smaller cases to see if we can find a pattern. This is used a lot when the problem involves the current year, i.e. it was just a arbitrary number that could really be any number.

3. **Break the Rules**: this seems counterintuitive, but some conditions are meant for breaking. For example, say we have $\frac{x^3}{(x-1)(x+1)}+\frac{5x^3+1}{(x+1)(x+2)}+\frac{2x^3+x}{(x+2)(x-1)}=3$, it is an implied rule to "not plug in something that will make the denominator 0". We can bypass this by multiplying both sides by the common denominator.

4. **Use Symmetry**: this should be pretty obvious. When there's an arithmetic sequence, use the median, when there's a geometric sequence, set the middle number as $a$ and the ones surrounding it as $\frac{a}{r}$ and $ar$ (and so on), etc. If the terms are unequally weighted (e.g. $2a+3b+4c$), it may be a good idea to substitute variables to see if this was actually symmetry in disguise (e.g. $4a^2+9b^2+16c^2+12ab+24bc+16ca=(x+y+z)^2$ where $x=2a, y=3b, z=4c$).

It is often good to chain generalizations. Basically, after we generalize one variable and get a factorization, convert back and see if we can make something else a variable. This works because one number can have so many properties that are lost when we specify its form (by turning it into a variable). For example, if we are trying to calculate $\frac{2019+2020+2021}{505}$, we can first set the variable as $x=2020$ to get $\frac{x-1+x+x+1}{505}=\frac{3x}{505}$, then plug in 2020 again to get the desired answer of 12. This is an extremely weak example, because converting back was necessary to end the problem. However, with a problem with more intermediate steps, this will make more sense.

# §5 Miscellaneous Factorizations

Just for fun.

**24** $\left(\sum_{i=1}^{n-1} x^{n+i}\right)+1=(x^n-x+1)\left(\sum_{i=0}^{n-1} x^i\right)$.

**25** $x^3+y^3+3xy-1=((x+y)-1)((x+y)^2-(x+y)-3xy+1)$.

**26** $2(a^2+b^2)(b^2+c^2)(c^2+a^2)=(a^2b+b^2c+c^2a-abc)^2+(b^2a+c^2b+a^2c-abc)^2$.

**27 (Problems from the Book)** $x^2+y^2+z^2=xyz+4 \implies x=\frac{a+b}{c}, y=\frac{a+c}{b}, z=\frac{b+c}{a}$.

**28** $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)=\frac{1}{4}\cdot(x^2+y^2+(x+y)^2)(x^2+y^2+(x-y)^2)$.

**29** $xyz = 1 \iff x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$

**30 (Ravi Substitution)** If $a, b, c$ are sides of a non-degenerate triangle, then we can find positive $x, y, z$ such that $a = x + y, b = y + z, c = z + x$.

**31** $a^4 + b^4 + (a+b)^4 = 2(a^2 + ab + b^2)^2$.

**32** $xy + yz + zx + xyz = 4 \implies x = \frac{2a}{b+c}, y = \frac{2b}{c+a}, z = \frac{2c}{a+b}$.

**33** The following factorizations are of similar forms:

1. $(a+b)^3 - a^3 - b^3 = 3ab(a+b)$
2. $(a+b)^5 - a^5 - b^5 = 5ab(a+b)(a^2+ab+b^2)$
3. $(a+b)^7 - a^7 - b^7 = 7ab(a+b)(a^2+ab+b^2)^2$

**34** $\frac{(a^2+bc)(b^2+ac)}{(a+c)(b+c)} + \frac{(a^2+bc)(c^2+ab)}{(a+b)(b+c)} + \frac{(b^2+ac)(c^2+ab)}{(a+b)(a+c)} = a^2 + b^2 + c^2$.

**35** The following factorizations are of similar forms:

1. $a + b + c = abc \implies \arctan a + \arctan b + \arctan c = 180^\circ$
2. $ab + bc + ca = 1 \implies \arctan a + \arctan b + \arctan c = 90^\circ$

This is known as trigonometric substitution, which could be a seperate topic entirely.

**36** $\left(xy^2 + yz^2 + x^2z - xyz\right)^2 + \left(x^2y + y^2z + z^2x - xyz\right)^2 = \left(x^2+y^2\right)\left(y^2+z^2\right)\left(x^2+z^2\right)$.

> *Remark 5.1.* The point of these manipulations are not actually to memorize them – rather, they serve to (a) illustrate there are too many to keep track of and (b) offer ideas on how to come up with them during contest.

## §6 Problems

**Problem 1 (AIME 1987/5).** Find $3x^2y^2$ if $x$ and $y$ are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

**Problem 2 (AMC 12B 2007/23).** How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

**Problem 3 (AIME 1998/14).** An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where $m, n$, and $p$ are integers, and $m \le n \le p$. What is the largest possible value of $p$?

**Problem 4 (AMC 12 2000/11).** Two non-zero real numbers, $a$ and $b$, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

**Problem 5 (Mathcounts 2020).** What is the value of $\sqrt{111,111,111 \cdot 1,000,000,011 + 4}$?

**Problem 6 (AHSME 1969/34).** The remainder $R$ obtained by dividing $x^{100}$ by $x^2 - 3x + 2$ is a polynomial of degree less than 2. Find $R$.

**Problem 7 (AMC 10A 2003/18).** What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

**Problem 8 (AMC 10B 2020/22).** What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

**Problem 9 (AIME 1991/1).** Find $x^2 + y^2$ if $x$ and $y$ are positive integers such that

$$xy + x + y = 71$$
$$x^2y + xy^2 = 880.$$

**Problem 10 (AMC 10A 2019/24).** Let $p$, $q$, and $r$ be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers $A$, $B$, and $C$ such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

**Problem 11 (AIME I 2015/3).** There is a prime number $p$ such that $16p + 1$ is the cube of a positive integer. Find $p$.

**Problem 12 (Titu Andreescu).** Find all pairs $(x, y)$ of integers such that

$$xy + \frac{x^3 + y^3}{3} = 2007$$

**Problem 13 (BMO 2013/1).** Calculate the value of

$$\frac{2014^4 + 4 \times 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \times 2013^4}{2013^2 + 4025^2}.$$

**Problem 14 (ARML Individual 2016/10).** Find the largest prime factor of $13^4 + 16^5 - 172^2$, given that it is the product of three distinct primes.

**Problem 15 (AIME 1988/13).** Find $a$ if $a$ and $b$ are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

**Problem 16 (AIME I 2000/9).** The system of equations

$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0 \end{aligned}$$

has two solutions $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. Find $y_1 + y_2$.