Applications of Symmetric Sums

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As visible in the title, this handout is focused primarily on applications of elementary symmetric sums in AMC and AIME level computational problems. I'd like to give a special thanks to David Altizio for suggesting a lot of the exercises.

§1 Defining Elementary Symmetric Sums

It seems natural to begin with a discussion of what exactly the elementary symmetric sums are.

Definition 1.1 (Elementary Symmetric Sums)**.** Assume that we are dealing with the variables $x_1, x_2, \ldots, x_k$. For positive integer n such that $n \leq k$, let s_n denote the nth elementary symmetric sum of the set of k variables. Then, we have:

$$s_n = \sum_{1 \leq a_1 \leq \ldots \leq a_n \leq k} x_{a_1} \ldots x_{a_n}$$

This is a pretty formal definition of elementary symmetric sums. Let's use some examples to try and understand this better.

Example 1.2

What are the elementary symmetric sums of the three variables x, y, z?

From **Definition 1.1**, s_1 is the sum of all combinations of 1 distinct variable. So, it is easy to see that $s_1 = x + y + z$. Similarly, s_2 is the sum of all combinations of 2 distinct variables. So, $s_2 = xy + yz + zx$. And lastly, s_3 is the sum of all combinations of 3 distinct variables so s_3 is simply equal to xyz.

Something important to notice from the above is that s_1 is always the sum of all variables and s_k where k is the number of variables is always equal to the product of all of the variables.

I'm going to give one more example of the elementary symmetric sums just to ensure that the point is driven home.

Example 1.3

What are the elementary symmetric sums of the four variables a, b, c, d?

Note that we have:

$$s_1 = a + b + c + d$$

$$s_2 = ab + ac + ad + bc + bd + cd$$

$$s_3 = abc + abd + acd + bcd$$

$$s_4 = abcd$$

Now that we've defined the elementary symmetric sums, let's look into a few popular applications of them in math competitions and beyond.

§2 Vieta's Formulas

Elementary symmetric sums and Vieta's Formulas are two concepts that go hand-in-hand. It's nearly impossible to think about one without thinking of the other.

Let's start off small by looking at Vieta's Formulas when applied to quadratics.

Theorem 2.1 (Vieta's Formulas on Quadratics)
Given any quadratic equation written as $P(x) = a_2x^2 + a_1x + a_0$, it is always true that $s_1 = -\frac{a_1}{a_2}$ and $s_2 = \frac{a_0}{a_2}$ where s_n is the nth symmetric sum.

This might look complex to those of you who have never seen it before, but it's actually just a formulation of a very simple idea:

Proof. Note that if r_1 and r_2 are the roots of the quadratic $P(x)$, we have:

$$P(x) = a_2x^2 + a_1x + a_0 = a_2(x - r_1)(x - r_2)$$

Expanding this out yields:

$$P(x) = a_2(x - r_1)(x - r_2) = a_2x^2 - a_2(r_1 + r_2)x + a_2r_1r_2$$

Since $a_1 = -a_2(r_1 + r_2)$ and $a_2r_1r_2 = a_0$, it follows easily that $s_1 = -\frac{a_1}{a_2}$ and $s_2 = \frac{a_0}{a_2}$. $\square$

Basically, the entire idea behind Vieta's formulas is that expanding out the polynomial $P(x) = a_n(x - r_1)\dots(x - r_n)$ leads to the coefficient of each term being $(-1)^k(s_k)a_n$ where a_n is the coefficient of the leading term. Let's expanding this to all polynomials, not just quadratics:

Theorem 2.2 (Vieta's Formulas on All Polynomials)
Given any polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, we have:

$$s_k = (-1)^k \frac{a_{n-k}}{a_n}$$

Just like the proof for quadratics, the proof for Vieta's Formulas on all polynomials is pretty intuitive:

Proof. We have $P(x) = a_n(x - r_1)\dots(x - r_n) = a_nx^n + \dots + a_0$. In order to create a term with degree d, we must choose x from d out of the n $(x - r_k)$ terms and (-1) times the root from the other $n - d$ terms. So, it follows quite simply that the coefficient of the x^d term is the sum of all combinations of negative roots with size $n - d$. This means that the coefficient of x^d would end up being $(-1)^{n-d}s_{n-d}a_n$. This is equivalent to the above formulation of Vieta's Formulas, so we are done. □

There is one thing about Vieta's Formulas that typically confuses people. What happens if I'm looking at a polynomial and one of the terms is missing? Let's take a look at this:

Example 2.3

Find the elementary symmetric sums of the roots of $P(x) = 2x^3 + 6$.

So, there aren't any x^2 or x terms. However, the roots of the polynomial must still have first and second elementary symmetric sums. Notice that:

$$P(x) = 2x^3 + 6 = 2x^3 + 0x^2 + 0x + 6$$

If the term is missing, this is equivalent to the coefficient being a 0. Now that we have this, we can proceed using Vieta's formulas to find:

$$s_1 = -\frac{0}{2} = 0, s_2 = \frac{0}{2} = 0, s_3 = -\frac{6}{2} = -3$$

The idea behind Vieta's Formulas might seem super simple, but this simple technique plays a large role in solving many difficult problems as we'll see later during the sections for Examples and Exercises.

§3 Newton Sums

Newton Sums are a technique that is also very helpful for math competitions. It allows us to express power sums like $r_1^n + r_2^n + \dots + r_n^n$ in terms of the elementary symmetric sum. Like Vieta's Formulas, we'll start by looking at smaller degree polynomials before moving on to a general statement:

Theorem 3.1 (Newton Sums on Quadratics)

Let s_k denote the kth elementary symmetric sum and let p_k denote the kth power sum (the sum of each term to the kth power). Then:

$$p_k = p_{k-1}s_1 - p_{k-2}s_2$$

for each $k \geq 3$. For p_1 and p_2, we have:

$$p_1 = s_1, p_2 = p_1s_1 - 2s_2$$

Note that in this case we are dealing with the roots of quadratics, meaning that we only have 2 variables to deal with.

It should be intuitive why this works. However, let's just test out the formula for p_2 and p_3 just to see if it works.

Example 3.2

Check the Newton Sums for p_2 and p_3.

From the above, we have:

$$p_2 = p_1 s_1 - 2s_2$$

$$p_3 = p_2 s_1 - p_1 s_2$$

Since we are dealing with quadratics, let $s_1 = x + y$ and $s_2 = xy$. Let's start with p_2. We have:

$$p_2 = (x+y)^2 - 2xy = x^2 + 2xy + y^2 - 2xy = x^2 + y^2$$

So, the definition of p_2 is correct. Let's try p_3:

$$p_3 = (x^2+y^2)(x+y) - (x+y)(xy) = x^3 + x^2y + xy^2 + y^3 - x^2y - xy^2 = x^3 + y^3$$

Thus, the Newton Sum for p_3 works as well. Now, let's move on to cubics:

Theorem 3.3 (Newton Sums for Cubics)

Assume the same definitions for s_k and p_k as the previous version. For $k \leq 3$ we have:

$$p_1 = s_1, p_2 = s_1 p_1 - 2s_2, p_3 = s_1 p_2 - p_1 s_2 + 3s_3$$

For larger k, we have:

$$p_k = s_1 p_{k-1} - s_2 p_{k-2} + s_3 p_{k-3}$$

Note that in this case we are dealing with cubics, so we only have 3 variables.

Just like the case for quadratics, this should be fairly intuitive and easy to check. The main thing to note is that for more variables, we have more elementary symmetric sums, and the maximum number of distinct terms in a Newton Sum is the number of variables being used.

Without further ado, let's look at the generalized version of Newton Sums.

Theorem 3.4 (General Form of Newton Sums)

Let k be the number of variables we are dealing with and define s_k and p_k as we did previously. When $i < k$, we have:

$$p_1 = 1 \cdot s_1$$

$$p_2 = p_1 s_1 - 2s_2$$

$$p_3 = p_2 s_1 - p_1 s_2 + 3s_3$$

$$\vdots$$

$$p_{k-1} = p_{k-2} s_1 - p_{k-3} s_2 + \ldots + (-1)^{k-2} s_{k-1}(k-1)$$

For $i > k$, we have:

$$p_i = p_{i-1} s_1 - p_{i-2} s_2 + \ldots + (-1)^{k-1} p_{i-k} s_k$$

The above works for any number of variables k. Note that the forms for quadratics and cubics that we mentioned above are just specific cases of the above. Also, note that because we can write any power sum in terms of the elementary symmetric sums, it follows that any symmetric polynomial can be written in terms of the elementary symmetric sums.

Now that we've developed the two primary tools (Vieta's Formulas, Newton Sums) used when applying symmetric sums, let's dive into some examples of how they can be used in contest problems.

§4 Examples

Let's start with a standard application of Vieta's Formulas.

> **Example 4.1** (2014 AIME II 5)
>
> Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

Let's go through how one would approach this problem. Please do these steps and follow along with the walkthrough.

1. Compare the symmetric sums of the roots of $p(x)$ and $q(x)$. Use this to figure out the value of the third roots of both polynomials.
2. Rewrite both polynomials in terms of their roots and expand.
3. Use the differences between the symmetric sums of both polynomials to set up a system of equations by equating coefficients of $p(x)$ and $q(x)$ in their new forms. For example, if you know the difference between s_k of $p(x)$ and s_k of $q(x)$, you can write it as $a = b + c$ where a, b are s_k of $p(x), q(x)$ respectively and c is the different $a - b$ between them.
4. Use the system of equations and solve to determine the possible values of pairs (r, s).
5. Figure out what the problem is asking for and use the pairs (r, s) to find the answer.

Note that the most important part of this practice problem is seen in part 3. The idea of equating coefficients when differences between them are known is a very standard technique in Vieta's Formulas problems.

Here is one more clever application of Vieta's Formulas on a very recent AIME.

> **Example 4.2** (2020 AIME II 11)
>
> Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Once again, work through the walkthrough using the steps provided.

1. Write $Q(x)$ and $R(x)$ in the form $x^2 - bx + c$. Use the given information in the problem statement to minimize the number of variables used.
2. Add up $P(x)$ and the forms of $Q(x)$ and $R(x)$ from part 1 to find $P + Q, P + R$ and $Q + R$.
3. Let r, s, t be the three distinct common roots mentioned in the problem statement. Use Vieta's Formulas to write r, s, t in terms of the coefficients of $Q(x)$ and $R(x)$.
4. Apply Vieta's Formulas on $P + Q, P + R, Q + R$ and combine this with your values of r, s, t from part 2 to figure out the constant values of r, s, t.
5. Since we know r, s, t in terms of the coefficients of Q and R, we can use the values of r, s, t to find what we are looking for, which is the constant term of $R(x)$.

Now, let's move on from Vieta's Formulas for a bit. The third and final walkthrough will be about Newton Sums. Admittedly, these do not show up in math competitions as often as Vieta's Formulas but they can be a powerful tool used in dealing with these problems when they show up.

Example 4.3 (1973 USAMO 4)
Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Don't be intimidated by the fact that this problem is from the USAMO. Since this is a rather old problem, it is actually easier than a lot of problems from the modern-day AIME.

1. Start by defining a polynomial $P(t) = t^3 + at^2 + bt + c$ where x, y, z are the roots of $P(t)$. Since one of the elementary symmetric sums, s_1, is given, use the constant value rather than assigning that coefficient a variable name.
2. Since you know s_1 and p_2, use the Newton Sum for p_2 to set up an equation that allows you to solve for s_2. Once you find s_2, use this value to replace a variable in the expansion of $P(x)$.
3. By now, you should know s_1, s_2, p_2, p_3. Use the Newton Sum for p_3 to set up an equation and solve for s_3. The value of s_3 should allow you to replace the last variable in the expansion of $P(x)$.
4. Surprisingly, $P(x)$ comes out to something with obvious roots. So, all distinct triples of these roots make up your answer.

Now that I've gone over how to do a few of these problems, here is a section with practice problems for you to go over:

§5 Exercises

Note that each of these problems will have a source given so that you can find the solutions once you are done using the AoPS Forums. These problems are ordered arbitrarily. Some of these problems are quite hard so it's okay if you are not able to solve all of the problems. An additional disclaimer is that although some of these problems may have alternate solutions without using Vieta's Formulas or Newton Sums, there is a way to solve each of them using the techniques learned in this handout.

Exercise 5.1 (2003 AIME I 3)**.** Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.

Exercise 5.2 (1996 AIME 5)**.** Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c$, and $c + a$. Find t.

Exercise 5.3 (USMCA 2020 Challenger 12)**.** Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^2}{d^3+2}$$

Exercise 5.4 (David's Problem Stash 2)**.** Let a, b, and c be nonzero real numbers such that $a + b + c = 0$ and

$$28(a^4 + b^4 + c^4) = a^7 + b^7 + c^7.$$

Find $a^3 + b^3 + c^3$.

Exercise 5.5 (2019 AIME I 10)**.** For distinct complex numbers $z_1, z_2, \dots, z_{673}$, the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 5.6 (CMIMC 2017 Team 10)**.** The polynomial $P(x) = x^3 - 6x - 2$ has three real roots, α, β, and γ. Depending on the assignment of the roots, there exist two different quadratics Q such that the graph of $y = Q(x)$ pass through the points (α, β), (β, γ), and (γ, α). What is the larger of the two values of $Q(1)$?

Exercise 5.7 (CMIMC 2016 Algebra 8)**.** Let r_1, r_2, …, r_{20} be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$

can be written in the form $\frac{m}{n}$ where m and n are positive coprime integers, find $m + n$.

Exercise 5.8 (CMIMC 2019 Algebra/Number Theory 6)**.** Let a, b and c be the distinct solutions to the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find the value of

$$\frac{1}{a(b^2+c^2-a^2)} + \frac{1}{b(c^2+a^2-b^2)} + \frac{1}{c(a^2+b^2-c^2)}.$$

Exercise 5.9 (HMMT 2007 Algebra 9)**.** The complex numbers α_1, α_2, α_3, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Exercise 5.10 (2003 AIME II 9)**.** Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Exercise 5.11 (2008 AIME II 7)**.** Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

Exercise 5.12 (2019 AMC12A 17)**.** Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\,s_k + b\,s_{k-1} + c\,s_{k-2}$ for $k = 2$, 3, ... What is $a + b + c$?

Exercise 5.13 (2011-2012 Winter OMO 25)**.** Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.

Exercise 5.14 (2011-2012 Winter OMO 38)**.** Let S denote the sum of the 2011th powers of the roots of the polynomial $(x - 2^0)(x - 2^1) \cdots (x - 2^{2010}) - 1$. How many ones are in the binary expansion of S?

Exercise 5.15 (2010 Stanford Math Tournament Algebra 10)**.** Find the sum of all solutions of the equation

$$\frac{1}{x^2 - 1} + \frac{2}{x^2 - 2} + \frac{3}{x^2 - 3} + \frac{4}{x^2 - 4} = 2010x - 4$$

Exercise 5.16 (2015 AIME II 14)**.** Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.