CALC 3D OVERVIEW

I. Vectors

dot product : $\vec{A} \cdot \vec{B} = AB\cos\theta$

cross product : $\vec{A} \times \vec{B}$ → gives area in space
→ gives vector $\perp$ to $\vec{A}$ and $\vec{B}$

Equation of a plane: $ax + by + cz = d$, where $\langle a, b, c \rangle$ is the normal vector

Equation of a line: parametrization...

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

, where (x_0, y_0, z_0) is a point on the line and $\langle a, b, c \rangle$ is a vector parallel to the line.

velocity $\vec{v} = \frac{d\vec{r}}{dt}$ → always tangent to the trajectory

speed $= |\vec{v}|$, acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

II. Matrices, Determinants, Linear Systems

Matrix dimensions: rows x columns.

$\underset{(3\times3)}{A} \xrightarrow{\text{minors}}$ [entries are 2x2 determinants formed by deleting 1 row & 1 column] (3×3)

$\xrightarrow{\text{cofactors}}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ $\xrightarrow{\text{transpose}}$ flip rows and columns

Note: $A^{-1} = \dfrac{\text{transpose}}{\det(A)}$ as long as $\det(A) \neq 0$

Ⓐ if $AX = B$, then $X = A^{-1}B$

Overview Continued

II. Matrices, Determinants, Linear Systems (continued)

- You can think of a 3x3 matrix A being the equations of 3 planes (each row describes one plane).

  Thus, det(A) describes the point of intersection of all 3 planes.

  If det(A) = 0, this means that either all 3 planes intersect at a single line (∞ points) or there is no common point of intersection between the 3 planes.

III. Functions of Several Variables & Partial Derivatives

- viewing functions → 3D graph
  → contour plot (aka level curves)
- partial derivatives → $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$

  ↳ linear approximation: $\Delta f = f_x \Delta x + f_y \Delta y$
  (this is just the tangent line at f)

- differentials: $df = f_x\, dx + f_y\, dy$
- chain rule: if $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ then $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$
- gradient $\nabla f = \langle f_x, f_y \rangle$ (normal vector to curve)
  - always $\perp$ to level curves & points towards higher ground.
- directional derivative: $\left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}$
- different types of partial derivatives
  - $\frac{\partial f}{\partial x} \Rightarrow$ x varies, y and z are held constant
  - $\left(\frac{\partial f}{\partial x}\right)_y \Rightarrow$ x varies, y is held constant, z is dependent on x and y

Overview Continued

IV. Min/max problems

- find critical pts (where $\nabla f = 0$)
- then do 2nd derivative test (min vs max vs saddle pt)
- check values of f at boundary / at ∞

To find max/min of f with no independent variables, use ...

- lagrange multipliers
  - ↳ solve $\nabla f = \lambda \nabla g$ where constraint $g = c$ applies
  - ⊛ Second Derivative Test <u>DOES NOT</u> apply here

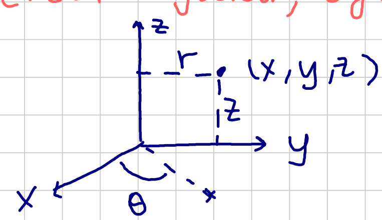
V. Double Integrals (rectangular, polar, u-v plane)

- draw a picture of region R to find integral bounds.
- $dA = dx\,dy = r\,dr\,d\theta$
- Rectangular → Polar : $r^2 = x^2 + y^2$, $\tan\theta = y/x$
- Polar → Rectangular . $x = r\cos\theta$, $y = r\sin\theta$
- Changing to/from u-v coordinates : $du\,dv = \underbrace{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}_{\text{Jacobian}} dx\,dy$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \leftarrow \text{Jacobian}$$

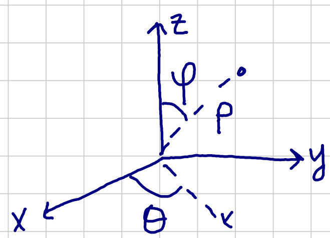
VI. Triple Integrals (rectangular, cylindrical, spherical)

cylindrical :
($dV = r\,dz\,dr\,d\theta$)



$r^2 = x^2 + y^2$ $\quad$ $x = r\cos\theta$
$\tan\theta = y/x$ $\quad$ $y = r\sin\theta$
$z = z$ $\quad$ $z = z$

spherical :
($dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$)



$\rho^2 = x^2 + y^2 + z^2$ $\quad$ $x = \rho\cos\theta\sin\varphi$
$r = \rho\sin\varphi$ $\quad$ $y = \rho\sin\theta\sin\varphi$
$z = \rho\cos\varphi$ $\quad$ $z = \rho\cos\varphi$
$\theta = \theta$

Overview Continued

VII. Applications of Double and Triple Integrals

- area $= \iint dA$, volume $= \iiint dV$, mass $= \iint \delta \, dA = \iiint \delta \, dV$
- avg value of function:

$$\bar{f} = \frac{1}{\text{volume}} \iiint f \, dV$$

$$\bar{f}_{\text{weighted}} = \frac{1}{\text{mass}} \iiint f \, dV$$

- center of mass $\vec{r}_{cm} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$
- moment of Inertia about z-axis:

$$I_z = \iiint (x^2+y^2)\, \delta \, dV$$

- gravitational attraction

$$F = Gm \iiint \frac{\delta \cos\varphi}{\rho^2} \, dV$$

VIII. Work and Line Integrals

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy, \text{ where } \vec{F} = \langle M, N \rangle$$

<u>Hint</u>: express x + y in terms of t & use t bounds on integral.

- Gradient fields:
  - curl $\vec{F} = 0$ $\Longrightarrow$ $\begin{cases} \text{in 2D: } N_x - M_y = 0 \\ \text{in 3D: } \langle (R_y - Q_z), (P_z - R_x), (Q_x - P_y) \rangle = \langle 0,0,0 \rangle \end{cases}$
  - $\vec{F}$ is defined in a simply connected region

Thus, $\vec{F} = \nabla f$. To find f . . .

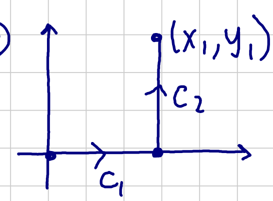
① start with $f_x = M$.

$\int M dx = f =$ ________ $+ g(y)$

compare $\frac{\partial f}{\partial y}$ to f_y to find $g(y)$

<u>OR</u>

②

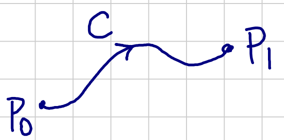


$\int_{C_1+C_2} \vec{F} \cdot d\vec{r}$ gives $f(x_1, y_1)$

Overview Continued

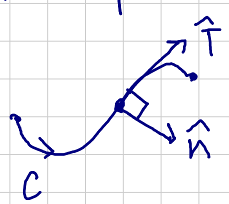
VIII. Work and Line Integrals (continued)

- Fundamental theorem of line integrals



$\Longrightarrow \int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$
(path independence)

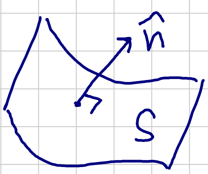
- Flux
  - In a plane:



$\hat{n}\, ds = \langle dy, -dx \rangle$, so...

$\int_C \vec{F} \cdot \hat{n}\, ds = \int_C -Q\,dx + P\,dy$

  - in space:



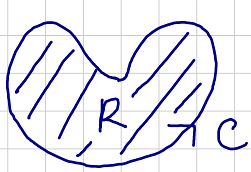
- express $\hat{n}\, dS$ geometrically if possible.

If $z = f(x,y)$, $\hat{n}\, dS = \langle -f_x, -f_y, 1 \rangle\, dx\, dy$

then compute $\iint_S \vec{F} \cdot \hat{n}\, dS$

IX. Important Theorems

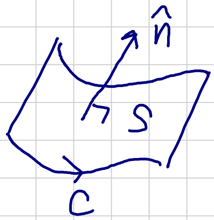
- Green's Theorem (Work in 2D)



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F}\, dA$$

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

- Stoke's Theorem (Work in 3D)



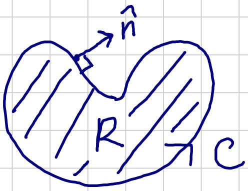
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS$$

$$= \iint_S \langle (R_y - Q_z), (P_z - R_x), (Q_x - P_y) \rangle \cdot \hat{n}\, dS$$

Overview Continued

IX. Important Theorems (continued)

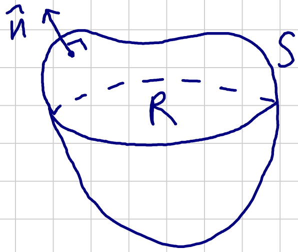
· Green's Theorem (Flux in 2D)



$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$$

$$= \iint_R (M_x + N_y)\,dA$$

· Divergence Theorem (Flux in 3D)

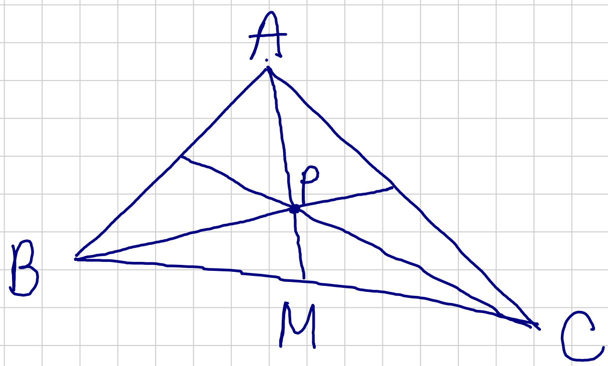


$$\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_R \operatorname{div}\vec{F}\,dV$$

$$= \iiint_R (P_x + Q_y + R_z)\,dV$$

PROBLEM SET

① Show that the three medians of a triangle intersect at a point 2/3 of way from each vertex.



Let the origin be labeled as "point O".

Knowing that the diagonals of a parallelogram always bisect each other, we can say that . .

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC})$$

Finding $\overrightarrow{OP}$,

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{3}(\overrightarrow{AM}) = \overrightarrow{OA} + \frac{2}{3}(\overrightarrow{OM} - \overrightarrow{OA})$$

$$= \overrightarrow{OA} + \frac{2}{3}\left(\frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) - \overrightarrow{OA}\right)$$

$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} + \frac{1}{3}\overrightarrow{OC} - \frac{2}{3}\overrightarrow{OA}$$

$$= \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$$

By symmetry, we would get the same answer if we used a different midpoint and the opposite corner to solve for $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

② Find the angle between the following vectors:

$\hat{i} + \hat{j} + 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$

By the definition of dot product

$$\vec{a} \cdot \vec{b} = ab\cos\theta$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab}$$

$$= \frac{(1 \cdot 2) + (1 \cdot -1) + (2 \cdot 1)}{\left(\sqrt{1^2+1^2+2^2}\right)\left(\sqrt{2^2+(-1)^2+1^2}\right)}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$\therefore \boxed{\theta = \pi/3}$$

(2b) Let $P = (a, 1, -1)$, $Q = (0, 1, 1)$, and $R = (a, -1, 3)$. For what values of a is $\angle PQR$ a right angle?

$\angle PQR$ is between vectors $\vec{QP}$ and $\vec{QR}$

$$\vec{QP} = (a, 1, -1) - (0, 1, 1) = \langle a, 0, -2 \rangle$$

$$\vec{QR} = (a, -1, 3) - (0, 1, 1) = \langle a, -2, 2 \rangle$$

Dotting $\vec{QP}$ and $\vec{QR}$,

$$\cos\theta = \frac{\vec{QP} \cdot \vec{QR}}{\|\vec{QP}\| \ \|\vec{QR}\|} = \frac{(a \cdot a) + (0 \cdot -2) + (-2 \cdot 2)}{\left(\sqrt{a^2+0^2+(-2)^2}\right)\left(\sqrt{a^2+(-2)^2+2^2}\right)}$$

next page →

(2b) Continued

If $\theta = 90°$

$$\cos 90° = \frac{a^2 - 4}{\sqrt{(a^2+4)(a^2+8)}}$$

$$0 = \frac{a^2-4}{\sqrt{(a^2+4)(a^2+8)}}$$

$$0 = a^2 - 4$$

$$\therefore \boxed{a = \pm 2}$$

(3) Compute the component of $2\hat{i} - 2\hat{j} + \hat{k}$ in the direction of $\hat{i} + \hat{j} + \hat{k}$

To find $\vec{a}_{\vec{b}}$,

$|\vec{a}_{\vec{b}}| = (a\cos\theta)$, where θ is the angle b/w $\vec{a}$ & $\vec{b}$.

From the definition of dot product, we know...

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$$

$$|\vec{a}_{\vec{b}}| = \left(\frac{\vec{a}\cdot\vec{b}}{b}\right)$$

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③ continued

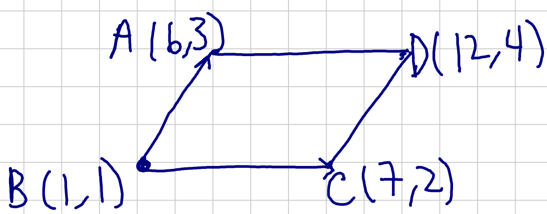
Here, $\vec{a} = 2\hat{i} - 2\hat{j} + \vec{k}$ and $\vec{b} = \hat{i} + \hat{j} + \vec{k}$, so . . .

$$\left|\vec{a}_{\vec{b}}\right| = \left(\frac{(2\cdot 1) + (-2\cdot 1) + (1\cdot 1)}{\sqrt{1^2+1^2+1^2}}\right)$$

$$= \boxed{\frac{1}{\sqrt{3}}}$$

∴ The component of $2\hat{i} - 2\hat{j} + \hat{k}$ in the direction of $\hat{i} + \hat{j} + \vec{k}$ is $1/\sqrt{3}$.

④ Compute the area of :



Area of parallelogram ABCD $= \|\vec{BA} \times \vec{BC}\|$

$\vec{BA} = (6,3) - (1,1) = \langle 5,2 \rangle$

$\vec{BC} = (7,2) - (1,1) = \langle 6,1 \rangle$

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & 0 \\ 6 & 1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - 7\hat{k}$$

$$\|\vec{BA} \times \vec{BC}\| = \boxed{7}$$

⑤ Compute the following determinants:

$$\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix} = (3 \cdot -2) - (-4 \cdot -1) = -6 - 4 = \boxed{-10}$$

$$\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & 1 \end{vmatrix} = -1\left[(2 \cdot 1) - (2 \cdot -2)\right] - 0\left[(1 \cdot 1) - (2 \cdot 3)\right] + 4\left[(1 \cdot -2) - (2 \cdot 3)\right]$$

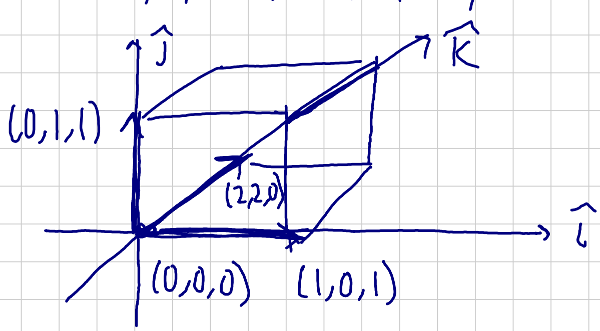
$$= -(6) - 0 + 4(-8)$$

$$= -6 - 32 = \boxed{-38}$$

Ⓐ This technique of computing the determinant of a 3x3 matrix is called <u>LaPlace Expansion</u>.

⑥ A parallelopiped has one vertex at the origin. The edges coming from the origin are the following vectors:

$\langle 2,2,0 \rangle$, $\langle 1,0,1 \rangle$, and $\langle 0,1,1 \rangle$.



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⑥ Continued

To find the volume of the parallelopiped,

Volume $= (\vec{a} \times \vec{b}) \cdot \vec{c}$ where $\vec{a}, \vec{b}$, and $\vec{c}$ are the edges of the parallelopiped coming from the origin

Let...

$\vec{a} = \langle 2,2,0 \rangle$

$\vec{b} = \langle 1,0,1 \rangle$

$\vec{c} = \langle 0,1,1 \rangle$

Thus,

$$\text{Volume} = (\langle 2,2,0\rangle \times \langle 1,0,1\rangle) \cdot \langle 0,1,1\rangle$$

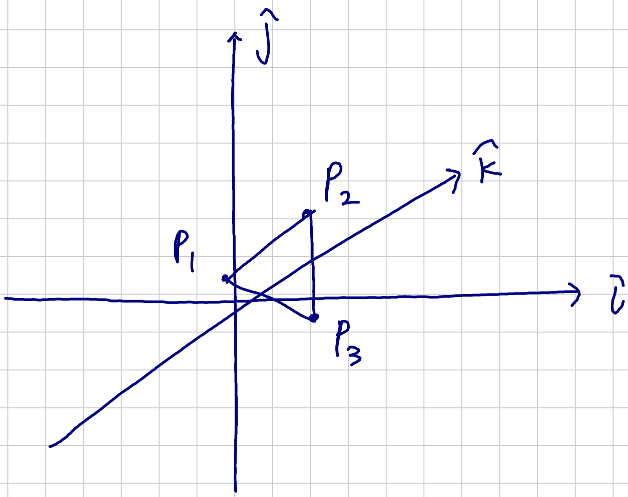
$$= \pm \begin{vmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \pm \left\{ 2\left[(0\cdot1)-(1\cdot1)\right] - 2\left[(1\cdot1)-(1\cdot0)\right] + 0\left[(1\cdot1)-(0\cdot0)\right] \right\}$$

$$= |2(-1) - 2(1)| = |-4| = \boxed{4}$$

⑦ Find the area of the triangle with vertices:

$P_1 = (-1,0,1)$, $P_2 = (0,2,2)$, $P_3 = (0,-1,2)$



$$\text{Area of } \Delta P_1P_2P_3 = \frac{1}{2}\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \right\|$$

$$\overrightarrow{P_1P_2} = (0,2,2) - (-1,0,1) = \langle 1,2,1 \rangle$$

$$\overrightarrow{P_1P_3} = (0,-1,2) - (-1,0,1) = \langle 1,-1,1 \rangle$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}\left[(2\cdot 1) - (1\cdot -1)\right] - \hat{j}\left[(1\cdot 1) - (1\cdot 1)\right] + \hat{k}\left[(1\cdot -1) - (1\cdot 2)\right]$$

$$= 3\hat{i} - 3\hat{k}$$

$$\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \right\| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\text{Area} = \frac{1}{2} \cdot 3\sqrt{2} = \boxed{\frac{3\sqrt{2}}{2}}$$

⑧ Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$

2x2 2x3 3x2

a) compute AB

AB will be a 2x3 matrix

$$\begin{bmatrix} (6\cdot 2)+(5\cdot 1) & (6\cdot -1)+(5\cdot 0) & (6\cdot 3)+(5\cdot 4) \\ (1\cdot 2)+(2\cdot 1) & (1\cdot -1)+(2\cdot 0) & (1\cdot 3)+(2\cdot 4) \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} 17 & -6 & 38 \\ 4 & -1 & 11 \end{bmatrix}}$$

b) Compute BA → $\boxed{\text{Not defined}}$

c) Compute BC

$$BC = \begin{bmatrix} (2\cdot 1)+(-1\cdot 2)+(3\cdot -1) & (2\cdot -1)+(-1\cdot 3)+(3\cdot 2) \\ (1\cdot 1)+(0\cdot 2)+(4\cdot -1) & (1\cdot -1)+(0\cdot 3)+(4\ 2) \end{bmatrix}$$

$$\boxed{BC = \begin{bmatrix} -3 & 1 \\ -3 & 7 \end{bmatrix}}$$

d) Compute AC → $\boxed{\text{Not defined}}$

(9) Let $A = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$ (3x3)

Solve $A\vec{x} = \vec{b}$ where

a) $b = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ (3x1)

So x must be a 3x1 matrix, such that

$x = A^{-1} b$

$$A^{-1} = \frac{1}{\det A} \left(\text{transpose of cofactor of } [A] \right)$$

$$\det A = 3[(2\cdot -1) - (0 \cdot -1)] - 1[(-1 \cdot -1) - (0 \cdot -1)] - 1[(-1 \cdot -1) + (-1 \cdot 2)]$$

$$= 3(-2) - 1(1) - 1(-3) = -10$$

matrix of minors: $\begin{bmatrix} -2 & 1 & -3 \\ -2 & -4 & -2 \\ 2 & -1 & 7 \end{bmatrix}$

cofactor matrix: $\begin{bmatrix} -2 & -1 & -3 \\ 2 & -4 & 2 \\ 2 & 1 & 7 \end{bmatrix}$

$$A^{-1} = \frac{1}{-10} \begin{bmatrix} -2 & 2 & 2 \\ -1 & -4 & 1 \\ -3 & 2 & 7 \end{bmatrix}$$

$$A^{-1} b = \frac{-1}{10} \begin{bmatrix} [(-2 \cdot 1) + (2 \cdot 2) + (2 \cdot -3)] \\ [(-1 \cdot 1) + (-4 \cdot 2) + (1 \cdot -3)] \\ [(-3 \cdot 1) + (2 \cdot 2) + (7 \cdot -3)] \end{bmatrix} = \frac{-1}{10} \begin{bmatrix} -4 \\ -12 \\ -20 \end{bmatrix} = \boxed{\begin{bmatrix} 2/5 \\ 6/5 \\ 2 \end{bmatrix}}$$

9 Continued

b) Let $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x = A^{-1} b = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

10 Calculate the equations of the planes defined as follows:

a) With normal $\vec{N} = \langle 1, 2, 3 \rangle$ through $(1, 0, -1)$

$\vec{N} \perp (\vec{r} - \vec{r_0})$ for any $\vec{r}$, so $\vec{N} \cdot (\vec{r} - \vec{r_0}) = 0$. Thus...

vector equation of a plane: $0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$,

where $\vec{N} = \langle a, b, c \rangle$ and (x_0, y_0, z_0) is a pt on the plane

$$\boxed{0 = 1(x-1) + 2(y-0) + 3(z+1)}$$

b) Through the origin and parallel to $\langle 1, 0, -1 \rangle$ and $\langle -1, 2, 0 \rangle$.

To find the normal vector $\vec{N}$, cross the two known vectors that are parallel to the plane.

$$\langle 1, 0, -1 \rangle \times \langle -1, 2, 0 \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{vmatrix} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

plane: $\boxed{0 = 2(x-0) + 1(y-0) + 2(z-0)}$

(10) Continued

c) Through the 3 points $(1,2,0), (3,1,1), (2,0,0)$

From the 3 points, we can solve for 2 vectors that are parallel to the plane

$(3,1,1)-(1,2,0) = \langle 2,-1,1\rangle$

$(3,1,1)-(2,0,0) = \langle 1,1,1\rangle$

Crossing these 2 vectors to find $\vec{N}$...

$$\langle 2,-1,1\rangle \times \langle 1,1,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2\hat{i} - \hat{j} + 3\vec{k}$$

Using $\vec{N} = \langle -2,-1,3\rangle$ and pt $(1,2,0)$...

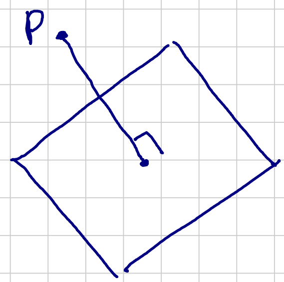
plane: $\boxed{0 = -2(x-1) - 1(y-2) + 3(z-0)}$

d) Parallel to plane in (a) passing through $(1,2,3)$

If the plane is parallel to the plane in part (a), the 2 planes must have the same $\vec{N}$. Thus,

plane: $\boxed{0 = 1(x-1) + 2(y-2) + 3(z-3)}$

(11) Compute the distance from $P=(0,0,0)$ to the plane with equation $2x+y-2z=4$



Pick a point X on the plane. $X=(2,0,0)$

Find $\vec{PX}$ from point P to the plane...

$$\vec{PX} = (2,0,0)-(0,0,0) = \langle 2,0,0\rangle$$

To find the distance between pt P and the plane, we solve for the component of $\vec{PX}$ in the direction of $\vec{N}$.

$$|\vec{PX}_{\vec{N}}| = \frac{\vec{PX}\cdot\vec{N}}{N} = \frac{\langle 2,0,0\rangle \cdot \langle 2,1,-2\rangle}{\|\langle 2,1,-2\rangle\|}$$

$$= \frac{(2\cdot 2)+(0\cdot 1)+(0\cdot -2)}{\sqrt{2^2+1^2+(-2)^2}} = \boxed{\frac{4}{3}}$$

(12) Consider the system. Find c for which the system has a unique solution.

$$\begin{aligned} 2x + 0y + cz &= 4 \\ x - 1y + 2z &= \pi \\ x - 2y + 2z &= -12 \end{aligned}$$

$$\begin{bmatrix} 2 & 0 & c \\ 1 & -1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \pi \\ -12 \end{bmatrix}$$

⑫ Continued

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{1}{\det[A]}\right)\left(\begin{array}{l}\text{transpose of the} \\ \text{cofactor of } [A]\end{array}\right)\begin{bmatrix} 4 \\ \pi \\ -12 \end{bmatrix}, \text{ where}$$

$[A]$ is the coefficient matrix.

When $\det[A] = 0$, the system either has zero or infinitely many solutions (x, y, z). Thus, when $\det[A] \neq 0$, the system has a <u>single</u> unique solution.

$$\det[A] = 2\left[(-1\cdot 2)-(-2\cdot 2)\right] - 0\left[(1\cdot 2)-(2\cdot 1)\right] + c\left[(1\cdot -2)-(-1\cdot 1)\right]$$

$$0 = 2(2) + c(-1)$$

$$c = 4$$

This means that when $c = 4$, the system either has zero or infinitely many solutions, so...

When $\boxed{c \neq 4}$, the system has a unique solution.

⑬ Do the following lines intersect? If so, where?

L_1: $x = 2 - t$
$y = 1 + t$

L_2: $x = 2 + t \rightarrow x = 2 + u$
$y = 4 + 2t \rightarrow y = 4 + 2u$

$$\begin{array}{r} 2 - t = 2 + u \\ + \quad 1 + t = 4 + 2u \\ \hline 3 = 6 + 3u \end{array}$$

$-3 = 3u$

$u = -1$

$t = 1$

$x = 2 - t = 2 - 1 = 1$
$y = 1 + t = 1 + 1 = 2$ — $(1,2)$ on L_1

$x = 2 + u = 2 - 1 = 1$
$y = 4 + 2u = 4 + 2(-1) = 2$ — $(1,2)$ on L_2

$\therefore$ L_1 and L_2 intersect at $\boxed{(1,2)}$

(14) Where does the line through $(0,-1,1)$ and $(2,3,3)$ meet the plane with equation $2x+y-z=1$?

$\langle 0,-1,1\rangle + t\langle 2,4,2\rangle$

$x=2t$

$y=4t-1$

$z=2t+1$

Substituting into $2x+y-z=1$,

$2(2t)+(4t-1)-(2t+1)=1$

$4t+4t-1-2t-1=1$

$6t=3$

$t=\frac{1}{2}$

If $t=\frac{1}{2}$,

$x=2t=2(\frac{1}{2})=1$

$y=4t-1=4(\frac{1}{2})-1=1$

$z=2t+1=2(\frac{1}{2})+1=2$

Therefore, the line meets the plane at $\boxed{(1,1,2)}$

(15) Consider a position vector $\vec{r}(t) = (x(t), y(t), 0)$. Suppose $\vec{r}(t)$ has constant length and $\vec{a}(t) = c\vec{r}(t)$ where $c \neq 0$ is a constant. Use vector differentiation to show $\vec{r}\cdot\vec{v} = 0$ and $\vec{r}\times\vec{v}$ is constant. Give an example of such an $\vec{r}(t)$.

Knowing that $\vec{r}\cdot\vec{r}$ is a constant,

$$\vec{r}\cdot\vec{r} = c_1$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d}{dt}(\vec{r})\cdot\vec{r} + \vec{r}\cdot\frac{d}{dt}(\vec{r})$$

$$0 = \vec{v}\cdot\vec{r} + \vec{r}\cdot\vec{v}$$

$\leftarrow$ $\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 0$ because $\vec{r}\cdot\vec{r}$ = constant

$$= \vec{r}\cdot\vec{v} + \vec{r}\cdot\vec{v}$$

$$= 2(\vec{r}\cdot\vec{v})$$

$$\boxed{0 = \vec{r}\cdot\vec{v}}$$

To show that $\vec{r}\times\vec{v}$ is a constant, we should calculate the derivative of $\vec{r}\times\vec{v}$ to be zero.

$$\frac{d}{dt}(\vec{r}\times\vec{v}) = \frac{d}{dt}(\vec{r})\times\vec{v} + \frac{d}{dt}(\vec{v})\times\vec{r}$$

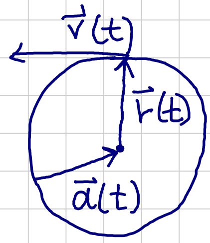
$$= \vec{v}\times\vec{v} + \vec{a}\times\vec{r}$$

$$= 0 + c\vec{r}\times\vec{r}$$

$\frac{d}{dt}(\vec{r}\times\vec{v}) = 0$ $\therefore$ $\vec{r}\times\vec{v}$ is a constant

Example of $\vec{r}(t)$:

$$\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$$



(16) Let $\vec{r}(t) = (1-2t^2)\hat{\imath} + t^2\hat{\jmath} + (-2+2t^2)\hat{k}$

a) Compute velocity, speed, and acceleration, and find the unit tangent vector.

$$\vec{v}(t) = \langle -4t, 2t, 4t \rangle$$

$$\|\vec{v}(t)\| = \sqrt{(-4t^2) + (2t)^2 + (4t)^2}$$

$$= \sqrt{36t^2} = 6t$$

$$\vec{a}(t) = \langle -4, 2, 4 \rangle$$

$$\text{unit tangent vector} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$$

$$= \frac{1}{6t}\langle -4t, 2t, 4t \rangle$$

$$= \left\langle \frac{-2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

b) Compute the arc length of the trajectory from $t=0$ to $t=2$

$$\text{arc length} = \int_0^2 (\text{speed})\,dt = \int_0^2 6t\,dt$$

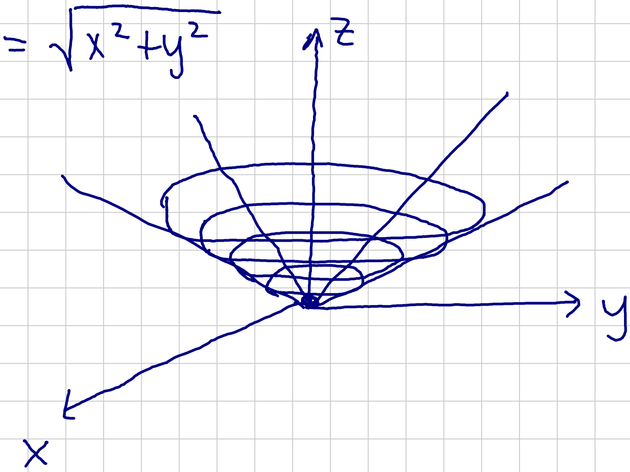
$$= \frac{6t^2}{2}\Big|_0^2$$

$$= 3t^2\Big|_0^2$$

$$= \boxed{12}$$

(17) Sketch a graph of each function

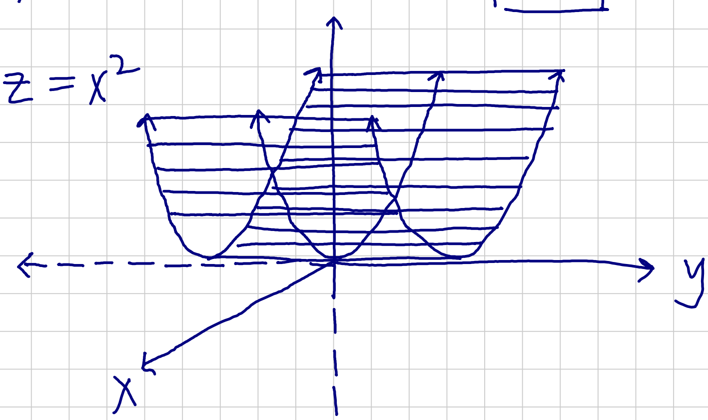
a) $z = \sqrt{x^2+y^2}$



if $x=0$, $z = \sqrt{y^2} = \pm y$ (2 lines)
if $y=0$, $z = \sqrt{x^2} = \pm x$ (2 Lines)
if $z=2$, $2 = \sqrt{x^2+y^2} \rightarrow x^2+y^2=4$ (circles)

$\therefore$ $z = \sqrt{x^2+y^2}$ is a $\boxed{\text{cone}}$

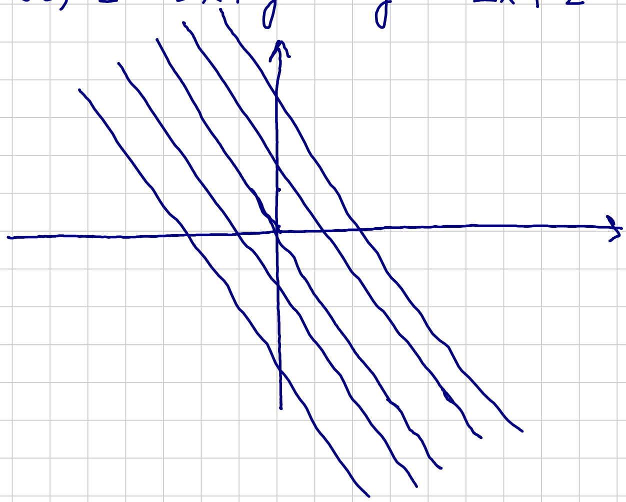
b) $z = x^2$



Since the formula has no dependence on y, the parabola $z=x^2$ is extended to every value of y, creating a parabola-shaped solid.

18 Draw the level curves for:

a) $z = 2x + y \rightarrow y = -2x + z$



Substituting different values for z,

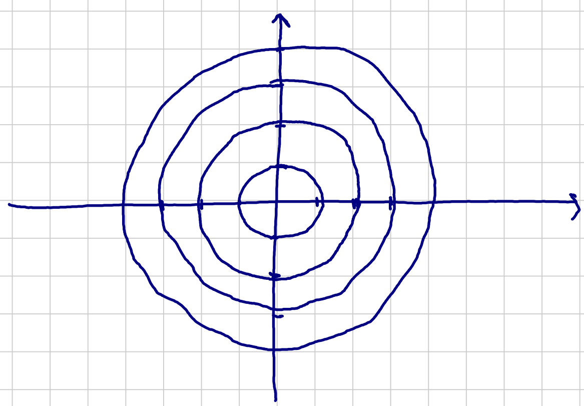
$y = -2x + 1$

$y = -2x + 2$

$y = -2x + 3$

$\vdots$

b) $z = x^2 + y^2$



Substituting different values for z

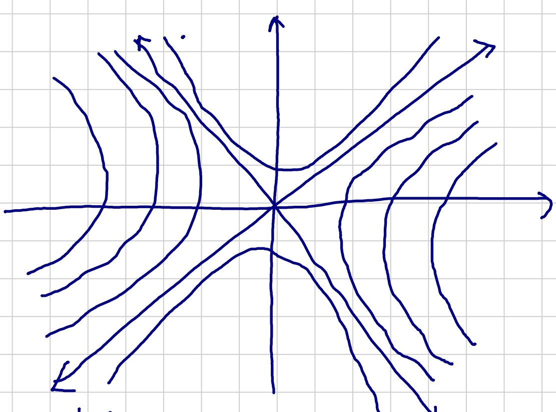
$1 = x^2 + y^2$

$2 = x^2 + y^2$

$3 = x^2 + y^2$

$\vdots$

c) $z = x^2 - y^2 \rightarrow z = (x+y)(x-y)$



asymptote: $0 = x - y$

asymptote: $0 = x + y$

Substituting different values for z

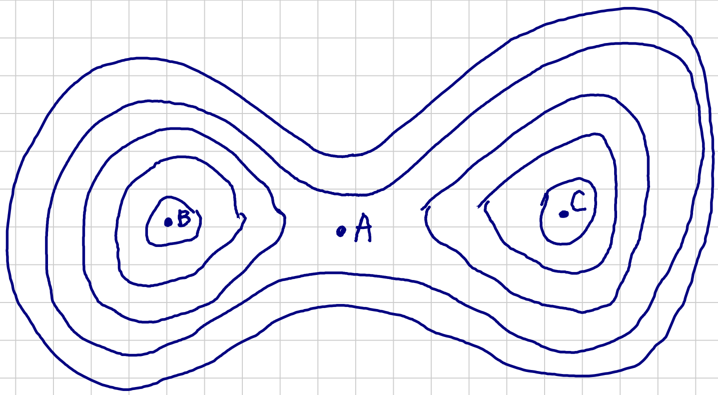
$1 = x^2 - y^2$

$2 = x^2 - y^2$

$3 = x^2 - y$

$\vdots$

(19) The contour plot below has a saddle point A and two min/max points B and C. Label them.



min/max points exist when concentric level curves converge to single points.

saddle points typically exist between 2 min/max points where the slopes in orthogonal directions are equal to zero, but it is important to remember that saddle points are <u>not</u> local extremum.

(20) $f(x,y) = xy^2 + x^2y$

a) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial x}(1,2)$.

$$\frac{\partial f}{\partial x} = \boxed{y^2 + 2xy}, \quad \frac{\partial f}{\partial y} = \boxed{2xy + x^2}, \quad \frac{\partial f}{\partial x}(1,2) = 2^2 + 2(1)(2) = \boxed{8}$$

b) Compute second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(y^2 + 2xy\right) = \boxed{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2xy + x^2\right) = \boxed{2y + 2x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}\left(y^2 + 2xy\right) = \boxed{2y + 2x}$$

(21) A rectangle has sides x and y. Approximate the area for $x = 2.1$ and $y = 2.8$
Near $x=2, y=3$, which has a greater effect — a change in x or an equal change in y?

$$\text{Area}(x,y) = x \cdot y$$

$$A(x,y) \approx A\Big|_{(2,3)} + A_x\Big|_{(2,3)}(x-2) + A_y\Big|_{(2,3)}(y-3)$$

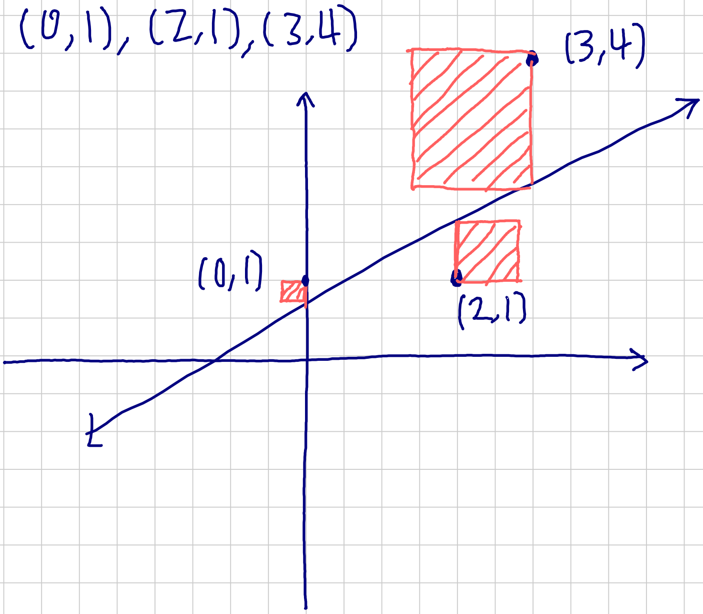
$$\approx xy\Big|_{(2,3)} + y(x-2)\Big|_{(2,3)} + x(y-3)\Big|_{(2,3)}$$

$$\approx 2(3) + 3(x-2) + 2(y-3) \quad \text{where } x = 2.1,\ y = 2.8$$

$$\approx 6 + 3(2.1-2) + 2(2.8-3) = \boxed{5.9}$$

A change in x would have a greater effect on the approximation than an equal change in y. This can be seen in simplified area approximation formula, where the coefficient of x is 3 and the coefficient of y is only 2.

(22) Use least squares to fit a line to the following data:
(0,1), (2,1), (3,4)



line of best fit:
$y = ax + b$

* The goal of least-squares linear regression is to determine a line of best fit for the data provided, for which the sum of the squares of the deviances is the least possible value.

(22) Continued

If $D(a,b)$ is a function of the sum of the squares,

$$D(a,b) = \Sigma\left[y_i - (ax_i+b)\right]^2$$

$$= \Sigma\left[y_i^2 - 2y_i(ax_i+b) + (ax_i+b)^2\right]$$

$$= \Sigma\left[y_i^2 - 2y_i a x_i - 2y_i b + a^2x_i^2 + 2ax_i b + b^2\right]$$

To determine the values of a and b that minimize $D(a,b)$, $\frac{\partial D}{\partial a}$ and $\frac{\partial D}{\partial b}$ must be zero.

$$\frac{\partial D}{\partial a} = \Sigma\left[-2y_i x_i + 2ax_i^2 + 2x_i b\right] = 0$$

$$2a\Sigma x_i^2 + 2b\Sigma x_i = 2\Sigma x_i y_i$$

$$a\Sigma(x_i^2) + b\Sigma(x_i) = \Sigma(x_i y_i) \longrightarrow \text{Eq } 1$$

$$\frac{\partial D}{\partial b} = \Sigma\left[-2y_i + 2ax_i + 2b\right] = 0$$

$$2a\Sigma x_i + 2bn = 2\Sigma y_i$$

$$a\Sigma(x_i) + bn = \Sigma(y_i) \longrightarrow \text{Eq } 2$$

Knowing that $\Sigma(x_i) = 5$, $\Sigma(y_i) = 6$, $\Sigma(x_i^2) = 13$, $\Sigma(x_i y_i) = 14$, we get the following system of equations from 1 and 2 ...

$$13a + 5b = 14$$
$$5a + 3b = 6$$

$$\left.\begin{aligned} a &= 6/7 \\ b &= 4/7 \end{aligned}\right\}$$

Thus the equation for the line of best fit for the given data is

$$\boxed{y = 6/7\, x_i + 4/7}$$

(23) Find and classify the critical points of the function
$w = x^3 - 3xy + y^3$

A critical point occurs when the derivative of the function is equal to zero.

$$w_x = \frac{\partial}{\partial x}(w) = 3x^2 - 3y = 0 \rightarrow y = x^2$$

$$w_y = \frac{\partial}{\partial y}(w) = -3x + 3y^2 = 0 \rightarrow x = y^2$$

$$x = (x^2)^2 = x^4$$

so $\underline{x=0}$ or $\underline{x=1}$

Thus the function has 2 critical points: $\boxed{(0,0) \text{ and } (1,1)}$

To classify these critical points as local max/min or saddle points, we do the second derivative test...!

$$w_{xx} = \frac{\partial}{\partial x}(w_x) = 6x = A$$

$$w_{yy} = \frac{\partial}{\partial y}(w_y) = 6y = C$$

$$w_{xy} = \frac{\partial}{\partial x}(w_y) = -3 = B$$

Evaluating the discriminant:

(1) critical point (0,0).

$AC - B^2$

$= (6x)(6y) - (-3)^2$

$= 36xy - 9$

$= 36(0)(0) - 9 = -9 < 0$, so...

$\boxed{(0,0) \text{ is a saddle point}}$

(2) critical point (1,1):

$AC - B^2$

$= (6x)(6y) - (-3)^2$

$= 36xy - 9$

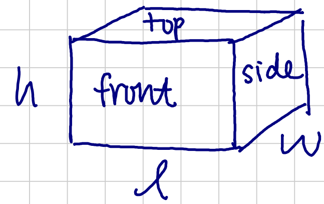
$= 36(1)(1) - 9 = 27 > 0$

$A = 6(1) = 6 > 0$, so...

$\boxed{(1,1) \text{ is a local min}}$

(24) A cardboard box of volume 3 has front and ack of equal thickness, sides of double thickness, bottom of triple thickness, and no top. What dimensions use the least amount of cardboard?

$$V = lwh = 3 \rightarrow h = \frac{3}{lw}$$



$$C(l,w,h) = 1(3lw) + 2(lh) + 2(2hw)$$

$$= 3lw + 2lh + 4hw$$

C(l,w,h) tells you how much cardboard is needed to make the box given l, w, h dimensions.

since $h = 3/lw$...

$$= 3lw + \frac{2l \cdot 3}{lw} + \frac{4w \cdot 3}{lw}$$

$$= 3lw + 6/w + 12/l$$

At the minimum value of C, $C_l = C_w = 0$, so...

$$C_l = \frac{\partial}{\partial l}(c) = 3w - 12/l^2 = 0$$

$$3w = 12/l^2$$

$$w = 4/l^2 \quad \text{— Eq (1)}$$

$$C_w = \frac{\partial}{\partial w}(c) = 3l - 6/w^2 = 0$$

$$3l = 6/w^2$$

$$l = 2/w^2 \quad \text{— Eq (2)}$$

$$l = 2/(4/l^2)^2$$

$$= \frac{2}{(16/l^4)} = \frac{2l^4}{16} = \frac{l^4}{8}$$

$$8l = l^4$$

$$\underline{l = 0} \text{ or } \underline{l = 2}$$

(24) Continued

Practically, l cannot equal 0, so we consider $\underline{l=2}$

If $l=2$, then...

$$w = 4/l^2 \quad \text{— Eq } ①$$

$$= 4/(2)^2$$

$$\underline{w = 1}$$

Since $h = \frac{3}{lw}$

$$= \frac{3}{(2)(1)}$$

$$h = 3/2$$

✱ Checking the boundaries for l and w, we find that as either l and w both approach ∞ or 0, C approaches ∞, so the minimum value of C does not exist at a boundary point, but rather at a critical point. Thus the one critical point we got is a local min (max would $= \infty$).

Thus the box should have dimensions $\boxed{2 \times 1 \times 3/2}$ to have a total volume of 3 and minimize the amount of cardboard used.

(25) Suppose $z = x^2 + y^2$, $x = u^2 - v^2$, and $y = uv$

a) Write the differential dz in terms of dx and dy

$$dz = 2x\,dx + 2y\,dy$$

b) Compute $\frac{\partial z}{\partial u}$ in two ways — using ① chain rule and ② differentials.

① $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = (2x)(2u) + (2y)(v)$$
$$= \boxed{4ux + 2vy}$$

② $$dz = 2x\,dx + 2y\,dy$$

$$dx = 2u\,du - 2v\,dv$$

$$dy = v\,du + u\,dv$$

$$dz = 2x(2u\,du - 2v\,dv) + 2y(v\,du + u\,dv)$$

$$= (4ux + 2vy)\,du + (2uy - 4vx)\,dv; \quad \frac{\partial z}{\partial u} = \boxed{4ux + 2vy}$$

(26) Use gradients to find

a) the tangent plane to the surface $z = x^3 + 3xy^2$ at $(1,2,13)$

To find the equation of a plane, we must know $\vec{N}$.

$$0 = x^3 + 3xy^2 - z$$

$$\vec{N} = \langle 3x^2 + 3y^2, 6xy, -1 \rangle$$

at pt $(1,2,13)$, $\vec{N} = \langle 3(1)^2 + 3(2)^2, 6(1)(2), -1 \rangle$

$$\vec{N} = \langle 15, 12, -1 \rangle$$

Thus, the equation of the plane is...

$$\boxed{0 = 15(x-1) + 12(y-2) - 1(z-13)}$$

b) the tangent line to the curve $x^3 + 2xy + y^2 = 9$ at $(1,2)$

$$\vec{N} = \langle 3x^2 + 2y, 2x + 2y \rangle$$

at pt $(1,2)$, $\vec{N} = \langle 3(1)^2 + 2(2), 2(1) + 2(2) \rangle$

$$\vec{N} = \langle 7, 6 \rangle$$

Thus, the equation of the line is...

$$\boxed{0 = 7(x-1) + 6(y-2)}$$

(27) For each of the following functions, compute the gradient, evaluate it at point P, and compute the directional derivative at P in direction $\vec{v}$.

a) $f(x,y) = x^2y + xy^2$, $P = (-1,2)$, $\vec{v} = \langle 3,4 \rangle$

$$\nabla f(-1,2) = \langle 2xy + y^2, x^2 + 2xy \rangle \Big|_{(-1,2)} = \boxed{\langle 0,-3 \rangle}$$

$$D_{\vec{v}} = \nabla f(x,y) \cdot \hat{v}$$

$$\hat{v} = \frac{\langle 3,4 \rangle}{\| \langle 3,4 \rangle \|} = \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$

$$D_{\vec{v}} = \langle 8,5 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle = \left(\frac{0 \cdot 3}{5} + \frac{-3 \cdot 4}{5} \right) = \boxed{\frac{-12}{5}}$$

b) $g(x,y,z) = \sqrt{x^2+y^2+z^2}$, $P = (2,6,-3)$, $\vec{v} = \langle 1,1,1 \rangle$

$$\nabla g(x,y,z) = \left\langle \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x), \tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2y), \tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2z) \right\rangle$$

$$\nabla g(2,6,-3) = \left\langle \tfrac{1}{2}(\tfrac{1}{7})(2 \cdot 2), \tfrac{1}{2}(\tfrac{1}{7})(2 \cdot 6), \tfrac{1}{2}(\tfrac{1}{7})(2 \cdot -3) \right\rangle = \boxed{\langle \tfrac{2}{7}, \tfrac{6}{7}, \tfrac{-3}{7} \rangle}$$

$$\hat{v} = \frac{\langle 1,1,1 \rangle}{\| \langle 1,1,1 \rangle \|} = \langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \rangle$$

$$D_{\vec{v}} = \langle \tfrac{2}{7}, \tfrac{6}{7}, \tfrac{-3}{7} \rangle \cdot \langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \rangle = \tfrac{1}{\sqrt{3}}\left(\tfrac{2}{7} + \tfrac{6}{7} - \tfrac{3}{7}\right) = \frac{5}{7\sqrt{3}} = \boxed{\frac{5\sqrt{3}}{21}}$$

c) $h(w,x,y,z) = wx + wy + wz + xy + xz + yz$, $P = (2,0,-1,-1)$, $\vec{v} = \langle 1,-1,1,-1 \rangle$

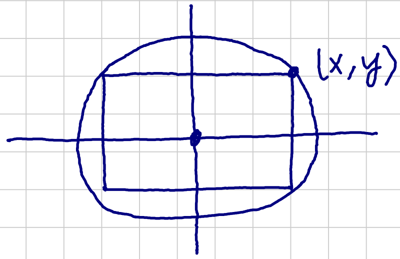
$$\nabla h(w,x,y,z) = \langle x+y+z, w+y+z, w+x+z, w+x+y \rangle$$

$$\nabla h(2,0,-1,-1) = \langle 0-1-1, 2-1-1, 2+0-1, 2+0-1 \rangle = \boxed{\langle -2,0,1,1 \rangle}$$

$$\hat{v} = \frac{\langle 1,-1,1,-1 \rangle}{\| \langle 1,-1,1,-1 \rangle \|} = \langle \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \rangle$$

$$D_{\vec{v}} = \langle -2,0,1,1 \rangle \cdot \langle \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \rangle = \boxed{\langle -1, 0, \tfrac{1}{2}, -\tfrac{1}{2} \rangle}$$

(28) Find the rectangle of maximal perimeter that can be inscribed in the ellipse with equation $x^2+4y^2=4$



Based on the point labeled (x,y) above, the perimeter of the rectangle should be...

$$P(x,y) = 2x+2y$$

* here we are only considering the top half of the rectangle.

Given the restriction of the ellipse,

$$x^2+4y^2=4$$
$$x^2 = 4-4y^2$$
$$x = \sqrt{4-4y^2}, \text{ so...}$$

* here we are only considering the top half of the ellipse

$$P(y) = 2\left(\sqrt{4-4y^2}\right)+2y = 4\sqrt{1-y^2}+2y$$

Finding the critical point(s) of P(x,y)...

$$\frac{d}{dy}(P) = 4\left(\tfrac{1}{2}\right)(1-y^2)^{-1/2}(-2y)+2$$
$$0 = 2 - \frac{4y}{\sqrt{1-y^2}}$$
$$2 = \frac{4y}{\sqrt{1-y^2}}$$
$$4y = 2\sqrt{1-y^2} \rightarrow 16y^2 = 4(1-y^2)$$
$$16y^2 = 4-4y^2$$
$$20y^2 = 4$$
$$y = \frac{1}{\sqrt{5}}, \quad x = \frac{4}{\sqrt{5}}$$

Therefore the dimensions of the rectangle are $\boxed{2/\sqrt{5} \times 8/\sqrt{5}}$

28 Continued.

Alternatively, this problem can be solved using LaGrange multipliers.

Let $P(x,y) = 4x + 4y$

$g(x,y) = x^2 + 4y^2 - 4$

Using LaGrange Multipliers...

$P_x = \lambda g_x$

$4 = \lambda(2x)$ — Eq 1

$P_y = \lambda g_y$

$4 = \lambda(8y)$ — Eq 2

From equations 1 and 2...

$\lambda(2x) = 4 = \lambda 8y$

$2x = 8y$

$x = 4y$

Substituting into the equation of the ellipse

$4 = x^2 + 4y^2$

$= (4y)^2 + 4y^2$

$= 16y^2 + 4y^2 = 20y^2$

$y^2 = 4/20$

$y = 1/\sqrt{5}$, $x = 4/\sqrt{5}$

$\therefore$ A rectangle with dimensions $\boxed{2/\sqrt{5} \times 8/\sqrt{5}}$ would still be inscribed in the ellipse and have a maximal perimeter.

(29) Find the maximum and minimum values of the function $f(x,y,z) = x^2 + x + 2y^2 + 3z^2$ as (x,y,z) varies on the unit sphere $x^2+y^2+z^2=1$

Using LaGrange Multipliers...

① $f_x = \lambda g_x$
$2x+1 = \lambda(2x)$

② $f_y = \lambda g_y$
$4y = \lambda(2y)$

③ $f_z = \lambda g_z$
$6z = \lambda(2z)$

Equation ② is satisfied if $y=0$ or $\lambda = 2$

Equation ③ is satisfied if $z=0$ or $\lambda=3$

a) If $y=0$ and $z=0$, then $x = \pm 1$
$(1,0,0)$, $(-1,0,0)$

b) If $y=0$ and $\lambda=3$, then $x=\frac{1}{4}$, $z = \pm\frac{\sqrt{15}}{4}$
$\left(\frac{1}{4}, 0, \frac{\sqrt{15}}{4}\right)$, $\left(\frac{1}{4}, 0, \frac{-\sqrt{15}}{4}\right)$

c) If $\lambda = 2$, $z=0$, then $x=\frac{1}{2}$, $y=\pm\frac{\sqrt{3}}{2}$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0\right)$

$f(1,0,0) = 2$, $f(-1,0,0) = 0$

$f\left(\frac{1}{4},0,\frac{\sqrt{15}}{4}\right) = \frac{25}{8}$, $f\left(\frac{1}{4},0,\frac{-\sqrt{15}}{4}\right) = \frac{25}{8}$

$f\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right) = \frac{9}{4}$, $f\left(\frac{1}{2},\frac{-\sqrt{3}}{2},0\right) = \frac{9}{4}$

So the max value of f is $25/8$, which occurs at $\boxed{\left(\frac{1}{4}, 0, \frac{\pm\sqrt{15}}{4}\right)}$

(30) Suppose $w = x^3y - z^2t$, $xy = zt$. Find $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ using...

a) total differentials.

$$dw = 3x^2y\,dx + x^3dy - 2zt\,dz - z^2dt$$

Since x and y are held constant...

$$dw = -2zt\,dz - z^2dt \quad \text{——— Eq (1)}$$

We know that $xy = zt$, so $t = \frac{xy}{z}$

$$dt = \frac{y}{z}dx + \frac{x}{z}dy - \frac{xy}{z^2}dz$$

x and y are constants, so $dx = dy = 0$...

$$dt = -\frac{xy}{z^2}dz$$

Substituting into Equation (1)...

$$dw = -2zt\,dz - z^2\left(-\frac{xy}{z^2}\right)dz$$

$$dw = -2zt\,dz + xy\,dz$$

$$\frac{\partial w}{\partial z} = xy - 2zt = zt - 2zt = \boxed{-zt}$$

b) implicit differentiation and chain rule

$$\frac{\partial w}{\partial z} = -2zt - z^2\left(\frac{\partial t}{\partial z}\right) \quad \text{——— Eq (2)}$$

Using the constraint equation

$$\frac{\partial}{\partial z}(xy = zt)$$

$$0 = t + z\left(\frac{\partial t}{\partial z}\right) \rightarrow \frac{\partial t}{\partial z} = \frac{-t}{z}$$

Substituting into Equation (2)

$$\frac{\partial w}{\partial z} = -2zt - z^2\left(\frac{-t}{z}\right) = -2zt + zt = \boxed{-zt}$$

(31) Let $P = (1,-1,1)$, $z = x^2 + y + 1$. Suppose $f(x,y,z)$ is differentiable with $\nabla f = 2\hat{i} + \hat{j} - 3\hat{k}$ at P. Let $g(x,z) = f(x, y(x,z), z)$. Find ∇g at $(x,z) = (1,1)$.

$$\nabla g(x,z) = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial z} \right\rangle$$

$$= \left\langle \left(\frac{\partial f}{\partial x}\right)_z, \left(\frac{\partial f}{\partial z}\right)_x \right\rangle$$

Knowing that $\nabla f = \langle 2, 1, -3 \rangle$...

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

$$df = 2dx + dy - 3dz \qquad \text{Eq ①}$$

From $z = x^2 + y + 1$ ...

$$dz = 2x\,dx + dy$$

$$dy = dz - 2x\,dx$$

Substituting into Equation ①

$$df = 2dx + (dz - 2x\,dx) - 3dz$$

$$= (2 - 2x)dx - 2dz$$

At $P(1,-1,1)$,

$$df = (2 - 2(1))dx - 2dz = -2dz$$

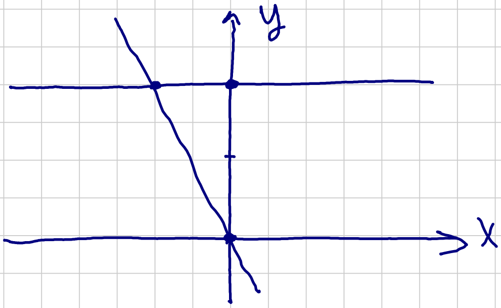
Thus, at $P(1,-1,1)$,

$$\left(\frac{\partial f}{\partial x}\right)_z = 0 \quad \text{and} \quad \left(\frac{\partial f}{\partial z}\right)_x = -2, \text{ so ...}$$

$$\nabla g(1,1) = \boxed{\langle 0, -2 \rangle}$$

(32) Write the double integrals $\iint_R dx\,dy$ and $\iint_R dy\,dx$ as iterated integrals, where...

a) R is the triangle with vertices $(0,0), (0,2), (-1,2)$



<u>boundaries</u>

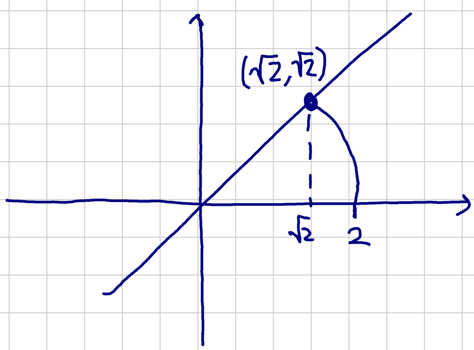
$x=0$

$y=2$

$y=-2x$

$$\boxed{\int_{x=-1}^{x=0} \int_{y=-2x}^{y=2} dy\,dx}$$

b) R is the sector of the circle of radius 2 centered at the origin above the x-axis and below $y=x$



<u>boundaries</u>

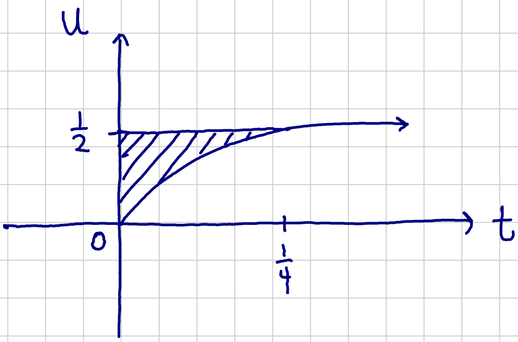
$y=x$

$x^2+y^2=4$

$$\boxed{\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=x} dy\,dx + \int_{x=\sqrt{2}}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} dy\,dx}$$

33 Compute the integral:

$$\int_{t=0}^{1/4} \int_{u=\sqrt{t}}^{1/2} \frac{e^u}{u} \, du \, dt$$



Reversing the order of integration...

$$= \int_{u=0}^{1/2} \int_{t=0}^{u^2} \frac{e^u}{u} \, dt \, du$$

$$= \int_{u=0}^{1/2} \frac{e^u}{u} t \Big|_0^{u^2} du$$

$$= \int_{u=0}^{1/2} u e^u \, du$$

By integration by parts...

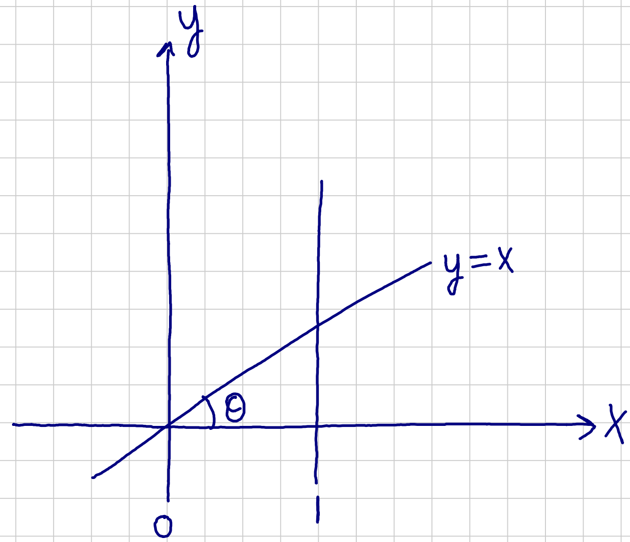
$$= u e^u - e^u \Big|_{u=0}^{1/2}$$

$$= \left(\tfrac{1}{2}\sqrt{e} - \sqrt{e}\right) - (0 - 1)$$

$$= \boxed{1 - \frac{\sqrt{e}}{2}}$$

(34) Evaluate in polar coordinates

a) $\displaystyle\int_{x=1}^{2}\int_{y=0}^{x} \frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx$



Knowing that $r=\sqrt{x^2+y^2}$,

$$=\int_{x=1}^{2}\int_{y=0}^{x}\frac{1}{r^3}\,r\,dr\,d\theta$$

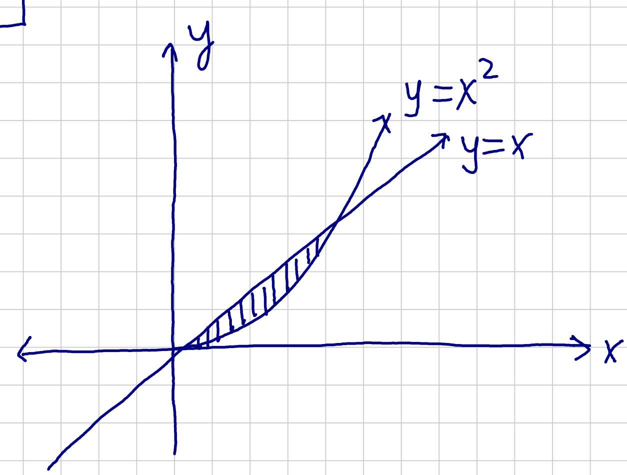
$$=\int_{\theta=0}^{\pi/4}\int_{r=\sec\theta}^{2\sec\theta}\frac{1}{r^2}\,dr\,d\theta$$

$$=\int_{\theta=0}^{\pi/4}\left[-\frac{1}{r}\right]\Bigg|_{\sec\theta}^{2\sec\theta} d\theta = \int_{\theta=0}^{\pi/4}\left(\cos\theta-\tfrac{1}{2}\cos\theta\right)d\theta$$

$$=\frac{1}{2}\sin\theta\Big|_0^{\pi/4} = \frac{1}{2}\sin\frac{\pi}{4} = \boxed{\frac{\sqrt{2}}{4}}$$

b) $\displaystyle\int_{x=0}^{1}\int_{y=x^2}^{x} f\,dy\,dx$

$$\boxed{=\int_{\theta=0}^{\pi/4}\int_{r=0}^{\tan\theta\sec\theta} f\,r\,dr\,d\theta}$$



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<u>boundaries</u>

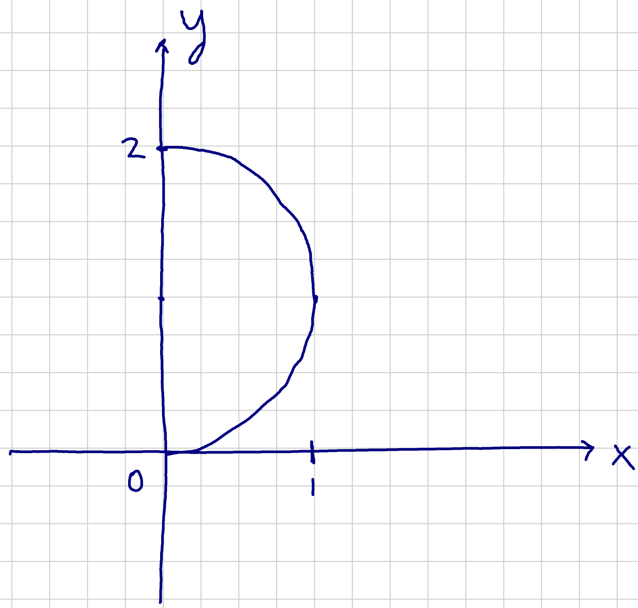
$y=x^2$

$r\sin\theta = r^2\cos^2\theta$

$r = \tan\theta\sec\theta$

(34) Continued

c) $\displaystyle\int_{y=0}^{2}\int_{x=0}^{\sqrt{2y-y^2}} f\,dx\,dy$



From the diagram on the left, we decide that $0 \leq \theta \leq \pi/2$.

Boundaries

$$x = \sqrt{2y-y^2}$$
$$x^2 = 2y - y^2$$
$$0 = x^2 + y^2 - 2y$$

Completing the square...

$$1 = x^2 + (y^2 - 2y + 1)$$
$$1 = x^2 + (y-1)^2$$

To find r bounds...

$$0 = x^2 + y^2 - 2y$$
$$= r^2 - 2y$$
$$= r^2 - 2r\sin\theta$$

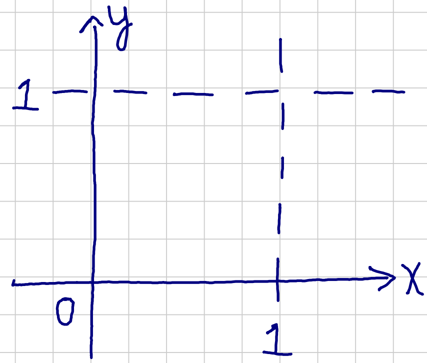
$$r^2 = 2r\sin\theta$$
$$r = 2\sin\theta$$

Thus $0 \leq r \leq 2\sin\theta$

$$\boxed{= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\sin\theta} f\,r\,dr\,d\theta}$$

(35) Let R be the unit square

Assume its density is
$\delta = xy$



a) Find its mass m

$$dm = \delta \, dA$$

$$m = \iint_R \delta \, dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} xy \, dy \, dx$$

$$= \int_{x=0}^{1} \frac{1}{2} x y^2 \Big|_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} \frac{x}{2} \, dx$$

$$= \frac{x^2}{4} \Big|_{x=0}^{1} = \boxed{\frac{1}{4}}$$

(35) Continued

b) Find its center of mass $\vec{r}_{cm} = \langle x_{cm}, y_{cm} \rangle$

$$x_{cm} = \frac{\iint x\,dm}{\iint dm}$$

$$= \frac{\int_{x=0}^{1}\int_{y=0}^{1} x(xy)\,dy\,dx}{(1/4)}$$

$$= 4\int_{x=0}^{1} \frac{1}{2}x^2 y^2 \Big|_{y=0}^{1} dx$$

$$= 4\int_{x=0}^{1} \frac{1}{2}x^2\,dx$$

$$= \frac{4}{2}\left(\frac{1}{3}\right) x^3 \Big|_{x=0}^{1}$$

$$x_{cm} = \frac{2}{3}$$

By symmetry, $y_{cm} = \frac{2}{3}$

$$\therefore \boxed{\vec{r}_{cm} = \left\langle \frac{2}{3}, \frac{2}{3} \right\rangle}$$

35 Continued

c) Find its moment of inertia I

$$dI = r^2 dm$$

$$I = \iint_R r^2 dm$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} (x^2+y^2)(xy)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} x^3y + xy^3 \, dy\, dx$$

$$= \int_{x=0}^{1} \frac{1}{2}x^3y^2 + \frac{1}{4}xy^4 \Big|_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} \frac{1}{2}x^3 + \frac{1}{4}x \, dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\right)x^4 + \frac{1}{4}\left(\frac{1}{2}\right)x^2 \Big|_{x=0}^{1}$$

$$= \frac{1}{8} + \frac{1}{8}$$

$$= \boxed{\frac{1}{4}}$$

(35) Continued

d) Find its moment of inertia about the x-axis

$$dI = y^2 \delta \, dm$$

$$= \iint_R y^2 (xy) \, dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} x y^3 \, dy \, dx$$

$$= \int_{x=0}^{1} \frac{1}{4} x y^4 \Big|_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} \frac{1}{4} x \, dx$$

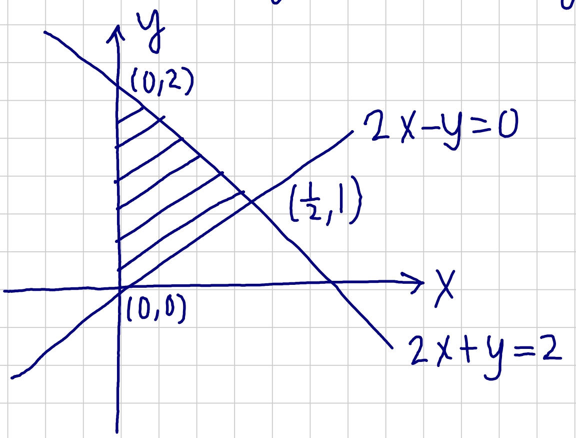
$$= \frac{1}{4}\left(\frac{1}{2}\right) x^2 \Big|_{x=0}^{1}$$

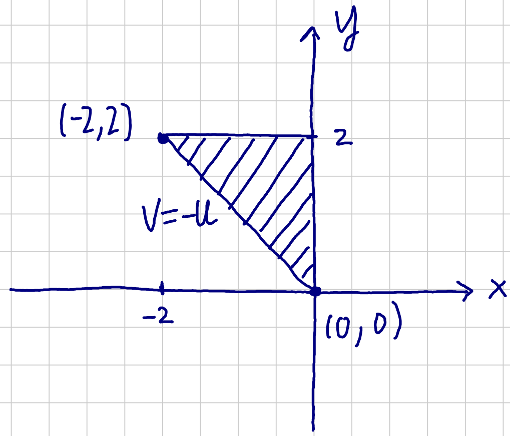
$$= \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \boxed{\frac{1}{8}}$$

(36) Given the region R shown below, compute $\iint_R (4x^2-y^2)^4\,dx\,dy$ by changing variables

$u = 2x - y \quad , \quad v = 2x + y$





Evaluating the Jacobian...

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$$

Thus,

$$du\,dv = 4\,dx\,dy$$
$$\frac{1}{4}\,du\,dv = dx\,dy$$

Substituting into the original integral...

$$\iint_R (uv)^4 \left(\tfrac{1}{4}\right) du\,dv$$

$$= \int_{u=-2}^{0} \int_{v=-u}^{2} \frac{u^4 v^4}{4}\,dv\,du = \frac{1}{4}\int_{u=-2}^{0} \left(\tfrac{1}{5}\right) u^4 v^5 \Big|_{v=-u}^{2}\,du$$

$$= \frac{1}{4}\left(\frac{1}{5}\right)\int_{u=-2}^{0} u^4\left[2^5 - (-u^5)\right]du = \frac{1}{20}\int_{u=-2}^{0} \left(32u^4 + u^9\right) du$$

$$= \frac{1}{20}\left[\frac{32}{5}u^5 + \frac{1}{10}u^{10}\right]\Bigg|_{u=-2}^{0} = \frac{1}{20}\left[\frac{32}{5}\left(0-(-2)^5\right) + \frac{1}{10}\left(0-(-2)^{10}\right)\right] = \boxed{\frac{128}{25}}$$

37 Evaluate ...

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Let $\int_{-\infty}^{\infty} e^{-x^2}dx = I$, then

$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}dy\right]$$

$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx$$

Turning this into polar coordinates ...

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

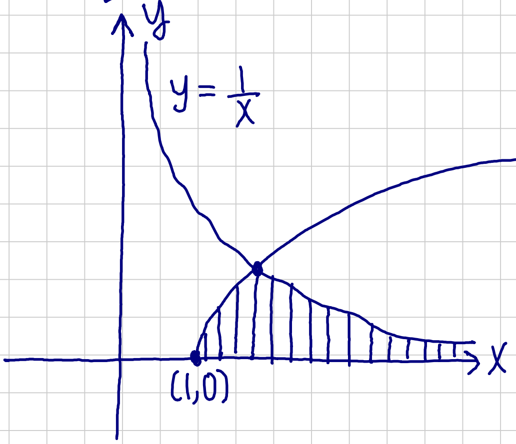
$$= 2\pi \int_{r=0}^{\infty} e^{-r^2} r\,dr$$

$$= 2\pi \left.\frac{(-e^{-r^2})}{2}\right|_{r=0}^{\infty}$$

$$= 0 - 2\pi\left(-\frac{1}{2}\right)$$

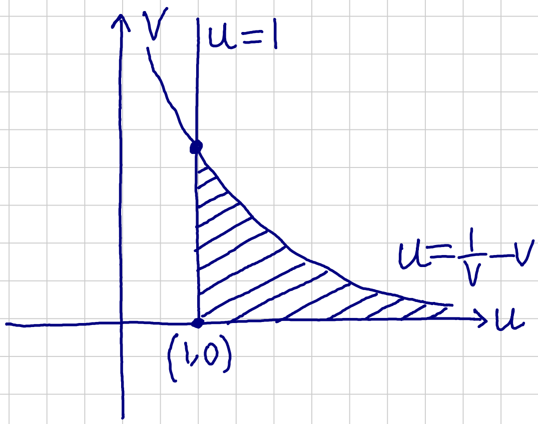
$$= \boxed{\pi}$$

(38) Using the change of variables $u = x^2 - y^2$, $v = y/x$, supply the limits and integrand for $\iint_R \frac{dx\,dy}{x^2}$ where R is the infinite region in the first quadrant under $y = 1/x$ and to the right of $x^2 - y^2 = 1$.



Determining the point at which the two curves intersect...

$$x^2 - \left(\frac{1}{x}\right)^2 = 1$$
$$0 = x^4 - x^2 - 1$$
$$x^2 = \frac{1 \pm \sqrt{5}}{2}$$



Determining u-v boundaries...

at $(x,y) = (1,0)$, $(u,v) = (1,0)$

at $y=0$, $v=0$

at $x^2-y^2=1$, $u=1$

at $y = \frac{1}{x}$, $v = y/x = 1/x^2$

and $u = x^2 - y^2 = x^2 - \frac{1}{x^2} = \frac{1}{v} - v$

To determine where the two curves intersect in u-v space ...

$$v = \frac{1}{x^2} = \frac{2}{1+\sqrt{5}}, \quad u = 1$$

Evaluating the Jacobian..

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2 - 2\left(\frac{y^2}{x^2}\right) = 2 - 2v^2$$

Determining the integrand in terms of u and v..

$$1 - v^2 = \frac{u}{x^2} \rightarrow \frac{1}{x^2} = \frac{1-v^2}{u}$$

next page →

(38) Continued

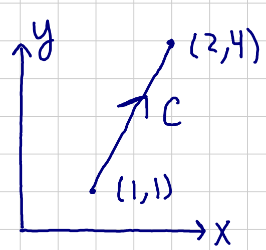
Putting all the parts together...

$$\iint_R \frac{1}{x^2}\,dx\,dy$$

$$= \int_{v=0}^{\frac{2}{1+\sqrt{5}}} \int_{u=1}^{\frac{1}{v}-v} \frac{1-v^2}{u}\left(\frac{1}{2-2v^2}\right) dv\,du$$

$$= \boxed{\int_{v=0}^{\frac{2}{1+\sqrt{5}}} \int_{u=1}^{\frac{1}{v}-v} \frac{1}{2u}\,dv\,du}$$

(39) Let $\vec{F} = \langle xy, x^2+y^2 \rangle$. Compute $\int_C \vec{F}\cdot d\vec{r}$ where...

a) 

$C: \langle 1,1 \rangle + t\langle (2-1),(4-1) \rangle$

$= \langle 1,1 \rangle + t\langle 1,3 \rangle$, where $0 \le t \le 1$

Thus...

$$\left.\begin{aligned} x &= 1+t \\ y &= 1+3t \end{aligned}\right\rangle \text{parametrization}$$

Substituting the evaluated parametrization into $\vec{F}$...

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \langle (1+t)(1+3t), (1+t)^2 + (1+3t)^2 \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

$$= \int_{t=0}^{1} \langle 1+4t+3t^2, 2+8t+10t^2 \rangle \cdot \langle 1,3 \rangle\, dt$$

next page ⟶

(39) Continued

$$= \int_{t=0}^{1} \left[1(1+4t+3t^2)+3(2+8t+10t^2)\right] dt$$

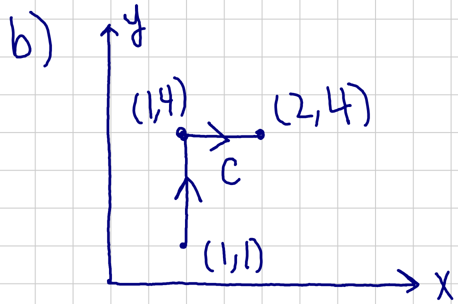
$$= \int_{t=0}^{1} (7+28t+33t^2)\, dt$$

$$= \left[7t+14t^2+11t^3\right]\Big|_{t=0}^{1}$$

$$= 7+14+11$$

$$= \boxed{32}$$

b)



$$C = C_1 + C_2$$

$$C_1 = \langle 1,1\rangle + t\langle (1-1),(4-1)\rangle$$
$$= \langle 1,1\rangle + t\langle 0,3\rangle$$

$$C_2 = \langle 1,4\rangle + t\langle (2-1),(4-4)\rangle$$
$$= \langle 1,4\rangle + t\langle 1,0\rangle$$

Thus,

C1) $x=1$, $y=1+3t$ C2) $x=1+t$, $y=4$ where $0\le t\le 1$

Substituting the evaluated parametrizations into $\vec{F}$...

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C1} \vec{F}\cdot d\vec{r} + \int_{C2} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \langle 1(1+3t),\ 1^2+(1+3t)^2\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt$$

$$+ \int_0^1 \langle 4(1+t),\ (1+t)^2+4^2\rangle \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt$$

next page ⟶

(39) Continued

$$= \int_0^1 \langle 1+3t, 2+6t+9t^2 \rangle \cdot \langle 0,3 \rangle \, dt$$

$$+ \int_0^1 \langle 4+4t, 17+2t+t^2 \rangle \cdot \langle 1,0 \rangle \, dt$$

$$= \int_0^1 [0(1+3t)+3(2+6t+9t^2)] \, dt + \int_0^1 [1(4+4t)+0(17+2t+t^2)] \, dt$$

$$= \int_0^1 (6+18t+27t^2) \, dt + \int_0^1 (4+4t) \, dt$$

$$= 6t+9t^2+9t^3 \Big|_0^1 + 4t+2t^2 \Big|_0^1$$

$$= 6+9+9+4+2$$

$$= \boxed{30}$$

(40) Let $\vec{F} = \langle xy, x^2+y^2 \rangle$ and C be the arc of $y=x^2$ connecting $(1,1)$ and $(2,4)$. Compute $\int_C \vec{F} \cdot d\vec{r}$ twice — using $x=t, y=t^2$, then $x=e^t$, $y=e^{2t}$

If $x=t, y=t^2 \ldots$

$$\vec{F} = \langle t(t^2), t^2+(t^2)^2 \rangle = \langle t^3, t^2+t^4 \rangle$$

$$d\vec{r} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt = \langle 1, 2t \rangle \, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 \langle t^3, t^2+t^4 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_1^2 [1(t^3) + 2t(t^2+t^4)] \, dt$$

next page

(40) Continued

$$= \int_1^2 (3t^3 + 2t^5)\,dt$$

$$= \frac{3}{4}t^4 + \frac{1}{3}t^6 \Big|_1^2$$

$$= \frac{3}{4}(2^4 - 1) + \frac{1}{3}(2^6 - 1)$$

$$= \frac{45}{4} + \frac{63}{3} = \boxed{\frac{129}{4}}$$

If $x = e^t$, $y = e^{2t}$

$$\vec{F} = \langle e^t \cdot e^{2t}, (e^t)^2 + (e^{2t})^2 \rangle = \langle e^{3t}, e^{2t} + e^{4t} \rangle$$

$$d\vec{r} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt = \langle e^t, 2e^{2t} \rangle\, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\ln(2)} \langle e^{3t}, e^{2t} + e^{4t} \rangle \cdot \langle e^t, 2e^{2t} \rangle\, dt$$

$$= \int_0^{\ln(2)} \left[e^t(e^{3t}) + 2e^{2t}(e^{2t} + e^{4t}) \right] dt$$

$$= \int_0^{\ln(2)} \left[e^{4t} + 2e^{4t} + 2e^{6t} \right] dt$$

$$= \frac{1}{4}e^{4t} + \frac{1}{2}e^{4t} + \frac{1}{3}e^{6t} \Big|_0^{\ln(2)}$$

$$= \frac{3}{4}e^{4\ln(2)} + \frac{1}{3}e^{6\ln(2)} - \frac{3}{4}e^{4\cdot 0} - \frac{1}{3}e^{6\cdot 0}$$

$$= \frac{3}{4}(2^4) + \frac{1}{3}(2^6) - \frac{3}{4} - \frac{1}{3}$$

$$= \boxed{\frac{129}{4}}$$

41. Let C be the circle of radius b centered at the origin, oriented counter-clockwise. Compute $\oint_C \vec{F}\cdot d\vec{r}$ where...

a) $\vec{F} = x\hat{i} + y\hat{j}$

$$\left.\begin{aligned} x &= b\cos t \\ y &= b\sin t \end{aligned}\right\} \text{ parametrization, where } 0 \le t \le 2\pi$$

$\vec{F} = \langle b\cos t, b\sin t \rangle$

$d\vec{r} = \langle -b\sin t, b\cos t \rangle dt$

✻ You can also think of this as the force vector always being perpendicular to the displacement vector, so the dot product will equal zero.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle b\cos t, b\sin t \rangle \cdot \langle -b\sin t, b\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \langle -b^2\sin t\cos t + b^2\sin t\cos t \rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt \quad = \boxed{0}$$

b) $\vec{F} = g(x,y)(x\hat{i} + y\hat{j})$

This $\vec{F}$ is just a multiple of the $\vec{F}$ in part (a), thus...

$$\oint_C \vec{F}\cdot d\vec{r} = \boxed{0}$$

c) $\vec{F} = -y\hat{i} + x\hat{j}$

Let $x = b\cos t$, $y = b\sin t$, where $0 \le t \le 2\pi$

$\vec{F} = \langle -b\sin t, b\cos t \rangle$

$d\vec{r} = \langle -b\sin t, b\cos t \rangle dt$

✻ You can also think of this as the force vector being parallel to the displacement vector, so you can just multiply force & displacement instead of computing an integral.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -b\sin t, b\cos t \rangle \cdot \langle -b\sin t, b\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \langle b^2\sin^2 t + b^2\cos^2 t \rangle\, dt = b^2 t \Big|_0^{2\pi} = \boxed{2\pi b^2}$$

(42) Let $f(x,y) = x^5 + 3xy^3$, and C = upper semi-circle, from $(1,0)$ to $(-1,0)$.

a) Compute $\vec{F} = \nabla f$

$$\boxed{\vec{F} = \langle 5x^4 + 3y^3, 9xy^2 \rangle}$$

b) Compute $\int_C \vec{F} \cdot d\vec{r}$ directly

Let $x = \cos t$, $y = \sin t$, where $0 \le t \le \pi$

$$\vec{F} = \langle 5\cos^4 t + 3\sin^3 t, 9\cos t \sin^2 t \rangle$$

$$d\vec{r} = \langle -\sin t, \cos t \rangle \, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \langle 5\cos^4 t + 3\sin^3 t, 9\cos t \sin^2 t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$\boxed{= \int_0^{\pi} \left[-5\sin t \cos^4 t - 3\sin^4 t + 9\cos^2 t \sin^2 t \right] dt}$$

c) Compute $\int_C \vec{F} \cdot d\vec{r}$ using path independence

Line segment : $\langle 1,0 \rangle + t \langle (-1-1), (0-0) \rangle = \langle 1,0 \rangle + t \langle -2,0 \rangle$

parametrization. $x = 1-2t$, $y = 0$, where $0 \le t \le 1$

$$\vec{F} = \langle 5(1-2t)^4 + 3(0)^3, 9(1-2t)(0)^2 \rangle = \langle 5(1 - 8t + 24t^2 - 32t^3 + 16t^4), 0 \rangle$$

$$d\vec{r} = \langle -2, 0 \rangle \, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = 5\int_0^1 \langle 1 - 8t + 24t^2 - 32t^3 + 16t^4, 0 \rangle \cdot \langle -2, 0 \rangle \, dt$$

$$= 5\int_0^1 (-2 + 16t - 48t^2 + 64t^3 - 32t^4) \, dt$$

$$= 5\left(-2t + 8t^2 - 16t^3 + 16t^4 - \frac{32}{5}t^5\right)\Big|_0^1 = \boxed{-2}$$

(42) Continued

d) Compute $\int_C \vec{F}\cdot d\vec{r}$ using the fundamental theorem of line integrals.

Since $\vec{F} = \nabla f$,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\, dr = f(P_1) - f(P_0)$$

Knowing that $f(x,y) = x^5 + 3xy^3$ ...

$$f(P_1) - f(P_0) = f(-1,0) - f(1,0)$$

$$= [(-1)^5 + 3(1)(0)^3] - [(1)^5 + 3(-1)(0)^3]$$

$$= (-1) - 1 = \boxed{-2}$$

(43) Show that $\dfrac{-y\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$ is not conservative.

$$f = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(-1)(x^2+y^2) - (2y)(-y)}{(x^2+y)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(1)(x^2+y^2) - (2x)(x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

while $N_x = M_y$, the vector field is not defined at every point (it is undefined at the origin), so we cannot prove that the function is conservative.

next page →

(43) Continued

Thus, we must try a different method...

$\oint_C \vec{F} \cdot d\vec{r}$ $\longleftarrow$ If the function $\vec{F}$ is conservative, $\oint \vec{F} \cdot d\vec{r} = 0$.

Let $x = \cos t$, $y = \sin t$, where $0 \le t \le 2\pi$

$$= \int_0^{2\pi} \left\langle \frac{-\sin t}{\sin^2 t + \cos^2 t}, \frac{\cos t}{\sin^2 t + \cos^2 t} \right\rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

$$= t \Big|_0^{2\pi}$$

$$= \boxed{2\pi} \neq 0$$

$\therefore$ The function is $\boxed{\text{NOT conservative}}$

(44) What values of b will make $\vec{F} = e^{x+y}[(x+b)\hat{i} + x\hat{j}]$ a gradient field? For this b, find a potential function f using both methods from lecture.

The field is a gradient field when $N_x - M_y = 0$

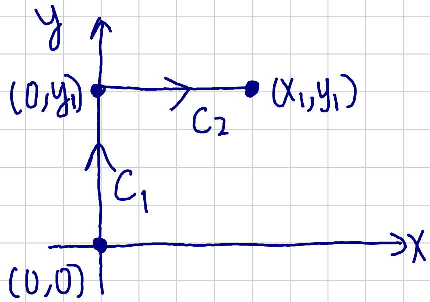
$$N_x = \frac{\partial}{\partial x}(xe^{x+y}) = e^{x+y} + xe^{x+y}$$

$$M_y = \frac{\partial}{\partial y}(xe^{x+y} + be^{x+y}) = xe^{x+y} + be^{x+y}$$

$$N_x - M_y = (e^{x+y} + xe^{x+y}) - (xe^{x+y} + be^{x+y}) = 0$$

$$\boxed{b = 1}$$

Finding a potential function using line integrals...



Let $C = C_1 + C_2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C1} M\,dx + N\,dy + \int_{C2} M\,dx + N\,dy$$

C_1: $x = 0$, $dx = 0$, $0 \le y \le y$

C_2: $y = y$, $dy = 0$, $0 \le x \le x$

$$\int_C \vec{F} \cdot dr = \int_{C1} M(0) + (0 \cdot e^{0+y})\,dy + \int_{C2} (e^{x+y_1} \cdot (x+1))\,dx + N(0)$$

$$= \int_{C2} xe^{x+y_1} + e^{x+y_1}\,dx = \int_0^{x_1} xe^{x+y_1} + e^{x+y_1}\,dx$$

$$= xe^{x+y_1}\Big|_0^{x_1} = \boxed{x_1e^{x_1+y_1} + C = f(x_1,y_1)}$$

next page $\longrightarrow$

(44) Continued

Finding a potential function using antiderivatives...

$$f_x = M = e^{x+y}(x+1) \rightarrow f = xe^{x+y} + g(y)$$

$$f_y = \frac{\partial}{\partial y}\left(xe^{x+y} + g(y)\right) = xe^{x+y} + g'(y)$$

$\hookrightarrow$ This should be equal to N.

Solving for $g(y)$...

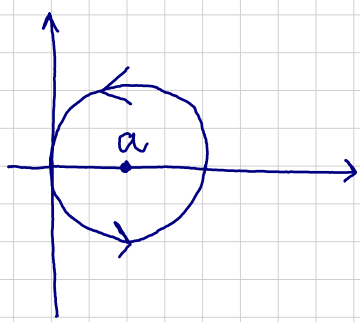
$$xe^{x+y} + g'(y) = f_y = N = xe^{x+y}$$

$$g'(y) = 0$$

$$g(y) = C$$

$$\boxed{\therefore\ f(x,y) = xe^{x+y} + C}$$

(45) Use Green's theorem to compute $\oint_C 3x^2y^2dx + 2x^2(1+xy)\,dy$, where C is the circle shown below



$$\oint_C M dx + N dy = \iint_R (N_x - M_y)\,dy\,dx$$

$$= \iint_R \left((4x + 6x^2y) - 6x^2y\right) dA = \iint_R 4x\, dA$$

$$\bar{x} = \frac{1}{\text{volume}} \iint_R x\, dA$$

$$\iint_R x\,dA = \bar{x}\,(\text{volume}) = a(\pi a^2) = \pi a^3$$

$$\therefore \iint_R 4x\,dA = \boxed{4\pi a^3}$$

(45) Continued

Alternatively, you can solve $\iint_R 4x\,dA$.

Switching to polar coordinates...

$$\iint_R 4x\,dA = \int_{\theta=0}^{2\pi}\int_{r=a}^{2a} 4(r+r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{2a} 4r^2+4r^2\sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} 4\left(\tfrac{1}{3}\right)r^3 + 4\left(\tfrac{1}{3}\right)r^3\sin\theta \Big|_{r=a}^{2a}\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \tfrac{4}{3}(8a^3-a^3) + \tfrac{4}{3}(8a-a^3)\sin\theta\,d\theta$$

$$= \tfrac{4}{3}(7a^3)\theta - \tfrac{4}{3}(7a^3)\cos\theta \Big|_{\theta=0}^{2\pi}$$

$$= \frac{28a^3}{3}(2\pi) - \frac{28a^3}{3}(\cos 2\pi) - \frac{28a^3}{3}(0) + \frac{28a^3}{3}(\cos 0)$$

$$= \frac{56\pi a^3}{3}$$

Green's Theorem

If C is a closed curve enclosing a region R, counter-clockwise, and vector field $\vec{F}$ is defined and differentiable everywhere in R

then $\oint \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \, dA$ ——— Eq ①

$\Rightarrow \oint M dx + N dy = \iint_R (N_x - M_y) \, dA$ ——— Eq ②

Thus, by Green's Theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \text{ given } \text{curl } \vec{F} = 0$$

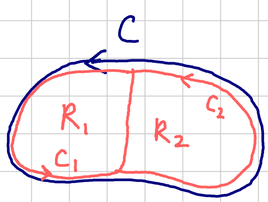
⊛ This proves that if curl $\vec{F} = 0$ everywhere in R, then $\oint_C \vec{F} \cdot d\vec{r} = 0$ and $\vec{F}$ is conservative

↳ NOTE: Curl $\vec{F}$ must be defined everywhere in R to apply Green's Theorem.

Proof for Green's Theorem

Prove that $\oint_C M dx + N dy = \iint_R (N_x - M_y) \, dA$

Instead, we'll prove that $\oint_C M dx = \iint_R -M_y \, dA$ (special case where $N=0$)



Dividing R into simpler regions $R_1 + R_2$

If we prove $\oint_{C1} M dx = \iint_{R_1} -M_y \, dA$

AND that $\oint_{C2} M dx = \iint_{R_2} -M_y \, dA$

then we can prove that $\oint_C M dx = \iint_R -M_y \, dA$,

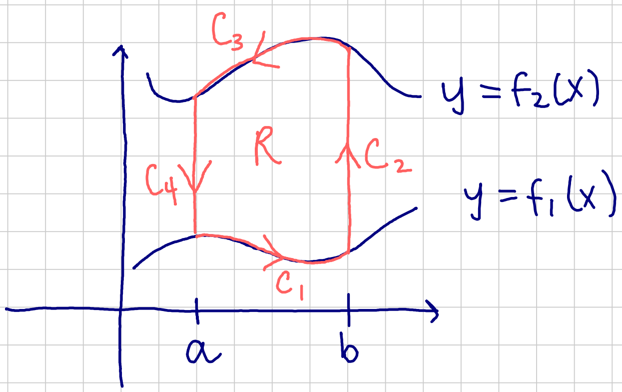
since $\oint_C M dx = \oint_{C1} M dx + \oint_{C2} M dx$

$= \iint_{R1} -M_y \, dA + \iint_{R2} -M_y \, dA = \iint_R -M_y \, dA$.

next page ⟶

Green's Theorem Continued

Proving that $\oint_C M\,dx = \iint_R -M_y\,dA$ if R is a vertically simple region and C is the boundary of R in the counter-clockwise direction...



$$\int_{C_2} M\,dx = \int_a^b M(x,y)\,dx \text{ where } x=b,\ dx=0,$$
thus $\int_{C_2} M\,dx = 0$

$\int_{C_4} M\,dx = 0$ (for the same reason that $\int_{C_2} M\,dx = 0$)

$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx$ where $y=f_1(x)$ and $a \le x \le b$.

$$\int_{C_3} M\,dx = \int_b^a M(x, f_2(x))\,dx \text{ where } y=f_2(x) \text{ and } b \le x \le a$$
$$= -\int_a^b M(x, f_2(x))$$

Thus, $\oint_C M\,dx = \left[\int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx\right]$

$$\iint_R -M_y\,dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx = -\int_a^b M\Big|_{f_1(x)}^{f_2(x)}\,dx$$
$$= -\int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\,dx$$
$$= \left[\int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx\right]$$

$\therefore\ \oint_C M\,dx = \iint_R -M_y\,dA \implies \boxed{\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA}$

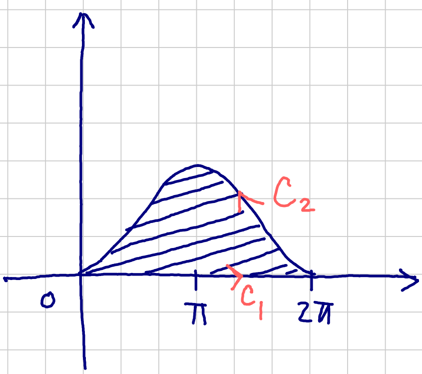
Q.E.D

(4b) Use Green's Theorem to find the area between on arch of the cycloid given by

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$
$$y = 0 \quad (x\text{-axis})$$

Green's Theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}\, dr$

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$



R: shaded region

Shaded area $= \iint_R dA$

$$\boxed{\iint_R dA = \oint_C x\,dy}$$

⊛explanation on next page

Let $C = C_1 + C_2$, so...

$$\iint_R dA = \int_{C1} x\,dy + \int_{C2} x\,dy$$

Over C_1: $y = 0$, $dy = 0$

Over C_2: $x = a(\theta - \sin\theta)$, $y = a - a\cos\theta$, $dy = a\sin\theta\, d\theta$, $0 \le \theta \le 2\pi$

$$\iint_R dA = \int_{C1} x(0) + \int_{\theta=0}^{2\pi} [a(\theta - \sin\theta)][a\sin\theta]\, d\theta$$

$$= 0 + \int_{\theta=0}^{2\pi} a^2(\theta\sin\theta - \sin^2\theta)\, d\theta$$

$$= a^2\left[(\sin\theta - \theta\cos\theta) - \left(\tfrac{1}{2}\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\right)\right]_{\theta=2\pi}^{0}$$

$$= a^2\left[0 - \left(-2\pi - \left(\tfrac{1}{2}(2\pi)\right)\right)\right]$$

$$= \boxed{3\pi a^2}$$

Ⓐ Explanation of Identity on Previous page

Starting with Green's Theorem,

$$\iint_R (N_x - M_y)\, dA = \oint_C M\,dx + N\,dy$$

To evaluate $\iint_R dA$, $N_x - M_y$ must $= 1$.

$N_x - M_y = 1$ if..

① $N_x = 1$, $M_y = 0$

② $N_x = 0$, $M_y = -1$

Case ①

$$\iint_R (N_x - M_y)\, dA = \iint_R (N_x)\, dA = \oint_C N\,dy$$

$$= \oint_C (x + g(y))\,dy = \oint_C x\,dy + \oint_C g(y)\,dy$$

$$= \boxed{\oint_C x\,dy}$$

Case ②

$$\iint_R (N_x - M_y)\,dA = \iint_R -M_y\, dA = \oint_C -M\,dx$$

$$= \oint_C -(y + g(x))\,dx = \oint_C -y\,dx - \oint_C g(x)\,dx$$

$$= \boxed{\oint_C -y\,dx}$$

Summary: $\iint_R dA = \oint_C x\,dy = \oint -y\,dx$

(47) For what simple closed curve C (oriented positively around the region it encloses) does

$$\oint_C -(x^2y+3x-2y)\,dx + (4y^2x-2x)\,dy$$ achieve its minimum value?

By Green's Theorem...

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

$M = -(x^2y + 3x - 2y)$ and $N = 4y^2x - 2x$, so

$$\oint_C M\,dx + N\,dy = \iint_R (x^2+4y^2-4)\,dA$$

When the given integral achieves its minimum value, $N_x - M_y$ will be equal to 0..

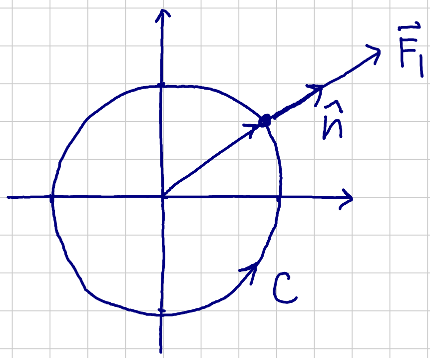
$$0 = N_x - M_y = x^2 + 4y^2 - 4$$

$$4 = x^2 + 4y^2$$

$$\boxed{C: \ 1 = \frac{x^2}{4} + y^2 \quad \text{(ellipse)}}$$

(48) Use geometric methods to compute the flux of $\vec{F}$ across the curve C.

a) $\vec{F}_1 = g(r)\langle x, y\rangle$, C = unit circle



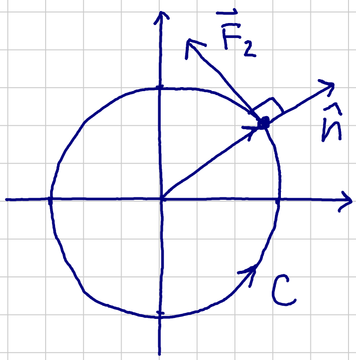
$$\int_C \vec{F}_1 \cdot \hat{n}\, ds$$

$$= \int_C |\vec{F}_1||\hat{n}| \cos\theta\, ds$$

$$= (g(1))(1)\cos(0) \int_C ds$$

$$= g(1)\int_C ds = g(1)(2\pi r) = \boxed{2\pi g(1)}$$

b) $\vec{F}_2 = g(r)\langle -y, x\rangle$, C is a unit circle

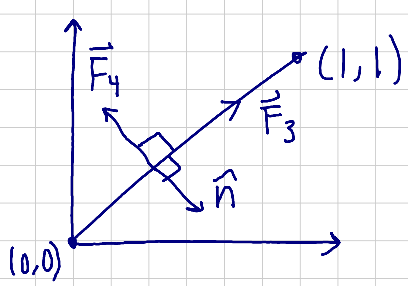


$$\int_C \vec{F}_2 \cdot \hat{n}\, ds$$

$$= \int_C |\vec{F}_2||\hat{n}| \cos\theta\, ds$$

$$= \cos(90^\circ)\int_C |\vec{F}_2||\hat{n}|\, ds = \boxed{0}$$

c) $\vec{F}_3 = 3\langle 1, 1\rangle$, C is segment from $(0,0)$ to $(1,1)$



$$\int_C \vec{F}_3 \cdot \hat{n}\, ds$$

$$= \int_C |\vec{F}_3||\hat{n}| \cos\theta\, ds$$

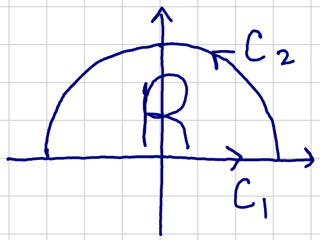
$$= \cos(90^\circ)\int_C |\vec{F}_2||\hat{n}|\, ds = \boxed{0}$$

d) $\vec{F}_4 = 3\langle -1, 1\rangle$, C is same as in part (c)

$$\int_C \vec{F}_4 \cdot \hat{n}\, ds = |\vec{F}_4||\hat{n}|\cos\theta \int_C ds = \left(\sqrt{(-3)^2+3^2}\right)(1)(\cos\pi)\left(\sqrt{1^2+1^2}\right)$$

$$= -(\sqrt{18})(\sqrt{2}) = \boxed{-6}$$

(49) Verify Green's Theorem in normal form for the field $\vec{F} = x\hat{i} + y\hat{j}$ and the curve C that consists of the upper half of the unit circle and the x-axis over the domain $[-1,1]$.



Green's Theorem in normal form:

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$$

$$\oint_C \vec{F}\cdot\hat{n}\,ds$$

$$= \oint_C M\,dy - N\,dx$$

$$= \oint_C x\,dy - y\,dx$$

Let $C = C_1 + C_2 \ldots$

$$= \oint_{C_1}\left[x(0) - (0)dx\right] + \oint_{C_2} x\,dy - y\,dx$$

Let $x = \cos t$, $y = \sin t$,
$dx = -\sin t\,dt$, $dy = \cos t\,dt \ldots$

$$= 0 + \int_{t=0}^{\pi} (\cos t)^2 dt + (\sin t)^2 dt$$

$$= \int_0^{\pi} dt = \boxed{\pi}$$

$$\iint_R \operatorname{div}\vec{F}\,dA$$

$$= \iint_R (M_x + N_y)\,dA$$

$$= \iint_R 2\,dA = 2\iint_R dA$$

$$= 2\left(\frac{\pi}{2}\right) = \boxed{\pi}$$

$\pi = \pi \Rightarrow$ Green's Theorem holds

Extended Form of Green's Theorem

To Review:

① Original Green's Theorem:

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_R \operatorname{curl} \vec{F}\, dA$$

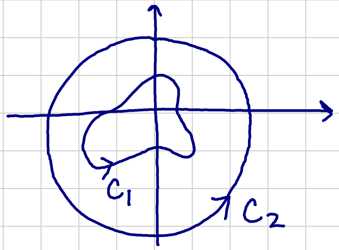
② Normal Form of Green's Theorem:

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA$$

Only works if $\vec{F}$ is defined & differentiable everywhere in R.

If $\vec{F}$ is not defined & differentiable everywhere in R, we cannot use ① or ②. In such cases we must use the extended form of Green's Theorem.

If a closed curve C_1 exists such that $\vec{F}$ is not defined and differentiable at every point with the region R enclosed by the curve C_1..



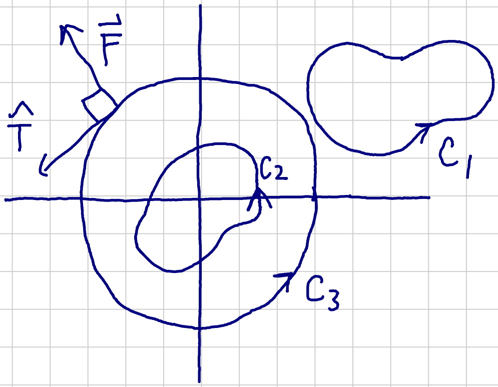
$$\oint_{C1} \vec{F} \cdot d\vec{r} = \oint_{C2} \vec{F} \cdot d\vec{r}$$

as long as $\vec{F}$ is defined and differentiable at every point in the region between C_1 and C_2.
(and C_1 and C_2 are oriented the same way)

Definition: a connected region R in the plane is simply connected if the interior of any closed curve in R is also contained in R.

Ⓐ If the domain where $\vec{F}$ is defined and differentiable is simply connected, then you can always apply Green's Thm.

(50) Let $\vec{F} = r^n(x\vec{i} + y\vec{j})$. Use extended Green's Theorem to show that $\vec{F}$ is conservative for all integers n. Then find a potential function.



If the curve does not contain the origin such as the case in C_1, regular Green's Theorem can be applied...

$$\boxed{\oint_{C_1} \vec{F} \cdot d\vec{r} = 0}$$

Next we consider a curve containing the origin such as C_2...

$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \oint_{C_3} \vec{F} \cdot dr ,$$

where C_3 is a closed curve that surrounds C_2 and $\vec{F}$ is defined and differentiable at every point in the region between C_2 and C_3.

By Extended Green's Thm. $\longrightarrow$

Let C_3 be a circle surrounding C_2.

At any point along C_3 $\vec{F} \perp d\vec{r}$,

$$\oint_{C_3} \vec{F} \cdot dr = \boxed{\oint_{C_2} \vec{F} \cdot dr = 0}$$

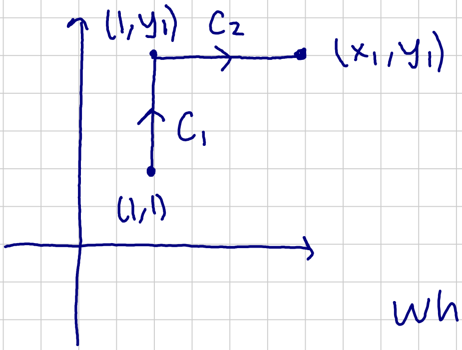
Since every curve can be modeled by either C_1 or C_2, the work above shows that $\vec{F}$ is conservative for all integers n.

50 Continued

To find the potential function...

$$f(x_1, y_1) = \int_C \vec{F} \cdot d\vec{r}$$

Since $\vec{F}$ is not defined and differentiable at $(0,0)$, we begin our integration at $(1,1)$



$$f(x_1, y_1) = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C P\,dx + Q\,dy$$

where $\langle P, Q \rangle = \langle r^n x, r^n y \rangle$

along C_1 : $x = 1$, $dx = 0$, $1 \leq y \leq y_1$

along C_2 : $y = y_1$, $dy = 0$, $1 \leq x \leq x_1$

Let $C = C_1 + C_2$ ...

$$f(x_1, y_1) = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$= \int_{y=1}^{y_1} (r^n x)dx + (r^n y)dy + \int_{x=1}^{x_1} (r^n x)dx + (r^n y)dy$$

$$= \int_{y=1}^{y_1} r^n y\,dy + \int_{x=1}^{x_1} r^n x\,dx$$

$$= \int_{y=1}^{y_1} (1^2 + y^2)^{n/2} y\,dy + \int_{x=1}^{x_1} (x^2 + y_1^2)^{n/2} x\,dx$$

Case ①:
If $n \neq -2 \rightarrow$

$$= \left.\frac{(1+y^2)^{\frac{n+2}{2}}}{n+2}\right|_{y=1}^{y_1} \quad \left.\frac{(x^2+y_1^2)^{\frac{n+2}{2}}}{n+2}\right|_{x=1}^{x_1} = \frac{(x_1^2+y_1^2)^{\frac{n+2}{2}}}{n+2} - \frac{2^{\frac{n+2}{2}}}{n+2}$$

$$\boxed{f(x_1, y_1) = \frac{r^{n+2}}{n+2} + C}$$

(50) Continued

Case (2): If $n=-2$ ...

$$f(x_1,y_1) = \frac{\ln(1+y^2)}{2}\Bigg|_{y=1}^{y_1} + \frac{\ln(x^2+y_1^2)}{2}\Bigg|_{x=1}^{x_1}$$

$$= \frac{\ln(x_1^2+y_1^2)}{2} - \frac{\ln(2)}{2}$$

$$\boxed{f(x_1,y_1) = \ln(r) + C}$$

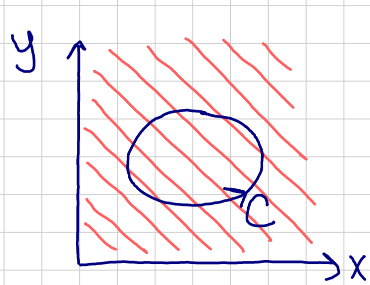
(51) For which of the following vector fields is the domain where it is defined and continuously differentiable a simply connected region.

a) $\sqrt{x}\,\vec{i} + \sqrt{y}\,\vec{j}$

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$
defined when $x \geq 0$
diff. when $x > 0$
both: $x > 0$

$g(y) = \sqrt{y}$, $g'(y) = \frac{1}{2\sqrt{y}}$
defined when $y \geq 0$
diff. when $y > 0$
both: $y > 0$

$\therefore \sqrt{x}\,\vec{i} + \sqrt{y}\,\vec{j}$ is defined and differentiable where



$x > 0$ and $y > 0$

According to the definition on page 61, the region above is $\boxed{\text{simply connected}}$

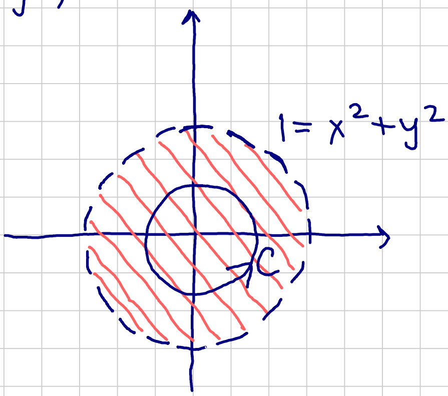
(51) Continued

b) $\dfrac{\vec{i}+\vec{j}}{\sqrt{1-x^2-y^2}}$

$f(x,y)$ is defined when $1-x^2-y^2>0$
$1>x^2+y^2$

$f'(x,y) = \dfrac{(\vec{i}+\vec{j})(x\,dx+y\,dy)}{(1-x^2-y^2)^{3/2}}$ is defined when $1-x^2-y^2>0$
$1>x^2+y^2$

both: $1>x^2+y^2$

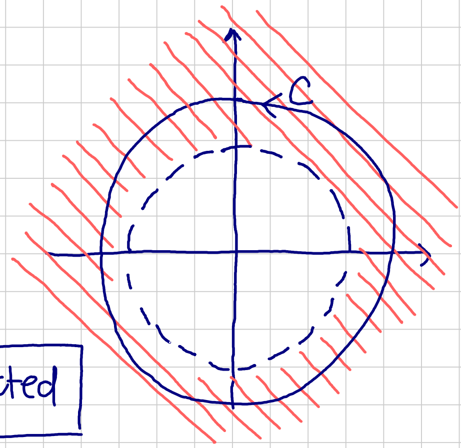


This region is simply connected

c) $\dfrac{\vec{i}+\vec{j}}{\sqrt{x^2+y^2-1}}$ $f(x,y)$ is defined when $x^2+y^2-1>0$
$x^2+y^2>1$

$f'(x,y) = \dfrac{(\vec{i}+\vec{j})(x\,dx+y\,dy)}{(x^2+y^2-1)^{3/2}}$ is defined when
$x^2+y^2-1>0 \rightarrow x^2+y^2>1$

both: $x^2+y^2>1$



C contains some non-region, so the region is

NOT simply connected

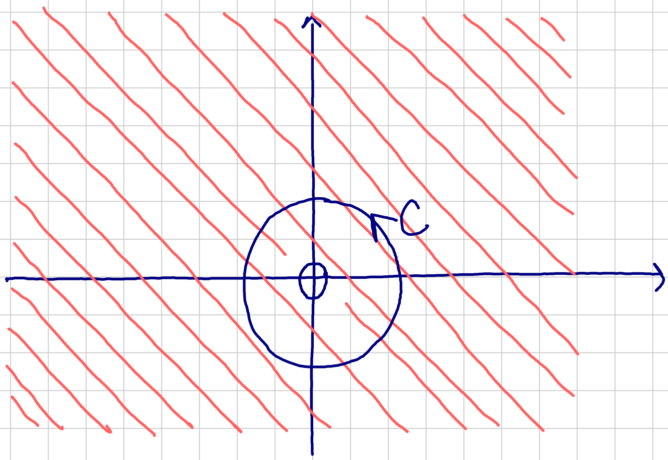
51 Continued

d) $(\vec{i}+\vec{j})(\ln(x^2+y^2))$

$f(x,y)$ is defined when $x^2+y^2>0$

$f'(x,y) = \frac{2x\,dx}{x^2+y^2} + \frac{2y\,dy}{x^2+y^2}$ is defined when $x^2+y^2 \neq 0$

both: $x^2+y^2>0$

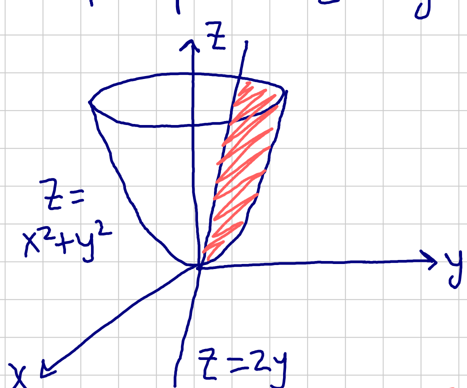


← region is defined and differentiable everywhere but the origin.

C contains some non-region, so the region is

$\boxed{\text{NOT simply connected}}$

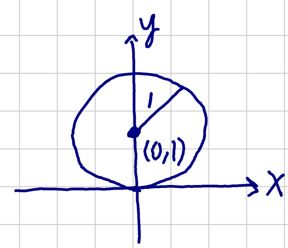
52 Find the volume between the paraboloid $z=x^2+y^2$ and the plane $z=2y$.



$x^2+y^2 = z = 2y$

$x^2+y^2-2y = 0$

$x^2+(y-1)^2 = 1 \Longrightarrow$



$x^2+y^2 = 2y$

$r^2 = 2r\sin\theta$

$r = 2\sin\theta$

$0 \le r \le 2\sin\theta$

$0 \le \theta \le \pi$

$r^2 \le z \le 2r\sin\theta$

$$V = \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=r^2}^{2r\sin\theta} r\,dz\,dr\,d\theta$$

52 Continued

Evaluating the integral...

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} rz \Big|_{z=r^2}^{2r\sin\theta} \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} 2r^2\sin\theta - r^3 \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{2}{3}r^3\sin\theta - \frac{1}{4}r^4 \Big|_{r=0}^{2\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{2}{3}\cdot 8\sin^4\theta - \frac{1}{4}\cdot 16\sin^4\theta\right) d\theta$$

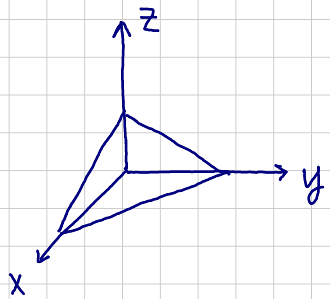
$$= \int_{\theta=0}^{\pi} \frac{4}{3}\sin^4\theta \, d\theta$$

$$= \frac{4}{3}\left[\frac{3}{8}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right]\Big|_{\theta=0}^{\pi}$$

$$= \frac{4}{3}\left[\frac{3\pi}{8}\right]$$

$$= \boxed{\frac{\pi}{2}}$$

(53) Find the average distance of a point in the tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,1) to the x-y plane.



$$\text{avg}(f) = \frac{1}{V}\iiint_V f(x,y,z)\,dV$$

$$= \frac{1}{V}\iiint_V z\,dV$$

$$= \frac{\iiint_V z\,dV}{\frac{1}{3}(A_{base})(\text{height})}$$

$$= \frac{\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z\,dz\,dy\,dx}{\frac{1}{3}\left(\frac{1}{2}\cdot 1\ 1\right)(1)}$$

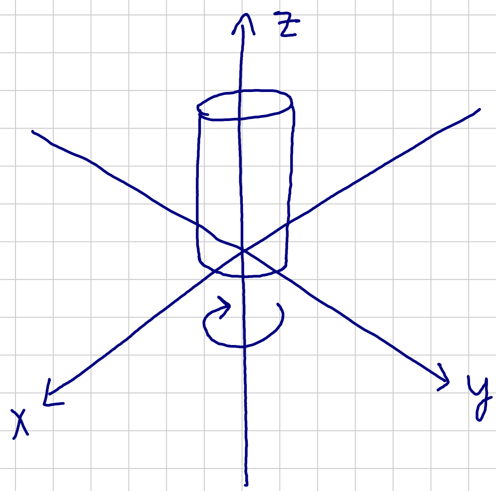
$$= \frac{\int_{x=0}^{1}\int_{y=0}^{1-x} \frac{1}{2}\left[(1-x-y)^2 - 0^2\right]dy\,dx}{1/6}$$

$$= \frac{\int_{x=0}^{1} -\frac{1}{6}(1-x-y)^3 \Big|_{y=0}^{1-x} dx}{1/6}$$

$$= \frac{\int_{x=0}^{1} \frac{1}{6}(1-x)^3\,dx}{1/6}$$

$$= \frac{1/24}{1/6} = \boxed{\frac{1}{4}}$$

(54) Find the moment of inertia of a cylinder of height h, radius b & constant density $\delta = 1$ around its central axis.



$$\text{moment of inertia} = \iiint_V \left(\begin{matrix}\text{distance} \\ \text{to axis}\end{matrix}\right)^2 \delta \, dV$$

$$I = \iiint_V (\text{distance to axis})^2 \, \delta \, dV$$

$$= \iiint_V (x^2 + y^2) \, \delta \, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{b} \int_{z=0}^{h} r^2 (1) \, r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{b} r^3 h \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{4} h b^4 \, d\theta$$

$$= \frac{1}{4} h b^4 (2\pi)$$

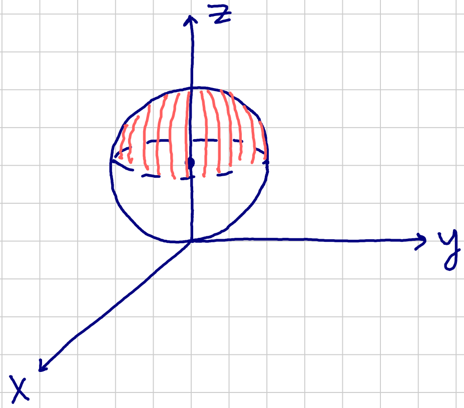
$$= \frac{\pi h b^4}{2}$$

Since mass $=$ volume $\cdot (\delta) = \pi b^2 h (1)$,

$$\boxed{I = \frac{\pi h b^4}{2} = \frac{1}{2} (\text{mass}) \, b^2}$$

(55) Let D be the portion of the solid sphere of radius 1 centered at (0, 0, 1) which lies above the plane $z=1$.

a) Supply the limits for D in spherical coordinates.



Evaluating ρ limits...

sphere: $x^2+y^2+(z-1)^2=1$

$$r^2+(z-1)^2=1$$

$$\rho^2\sin^2\varphi+(\rho\cos\varphi-1)^2=1$$

$$\rho^2\sin^2\varphi+\rho^2\cos^2\varphi-2\rho\cos\varphi+1=1$$

$$\rho^2-2\rho\cos\varphi=0$$

$$\rho^2=2\rho\cos\varphi$$

upper limit $\longrightarrow$ $\underline{\rho=2\cos\varphi}$

$z=\rho\cos\varphi$

at the plane $z=1$,

$1=\rho\cos\varphi \rightarrow \underline{\rho=\sec\varphi} \leftarrow$ lower limit

∴ The limits of D are...

$$\boxed{\begin{gathered}\sec\varphi<\rho\le 2\cos\varphi\\ 0\le\theta\le 2\pi\\ 0\le\varphi\le \pi/4\end{gathered}}$$

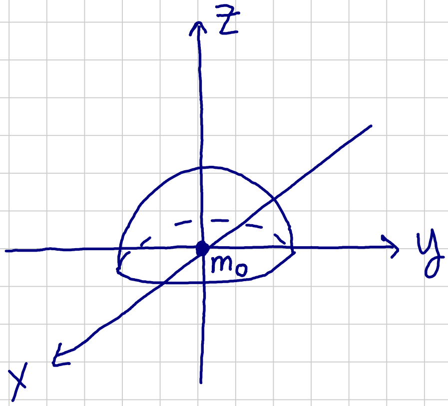
b) Set up the integral for the average distance of a point in D from the origin.

$$\text{avg}(f)=\frac{1}{V}\iiint_V \rho\,dV$$

$$=\frac{1}{\frac{1}{2}\cdot\frac{4}{3}\pi(1)^3}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/4}\int_{\rho=\sec\varphi}^{2\cos\varphi}\rho\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\boxed{\frac{3}{2\pi}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/4}\int_{\rho=\sec\varphi}^{2\cos\varphi}\rho^3\sin\varphi\,d\rho\,d\varphi\,d\theta}$$

(56) Find the gravitational attraction of an upper solid half sphere of radius a & center O on a mass m_0 at O. Assume the density $\delta = \sqrt{x^2+y^2}$.



$$|\vec{F}_g| = \frac{GMm}{\rho^2} \; ; \; \text{dir}\,\vec{F} = \frac{\langle x,y,z\rangle}{\rho}$$

$$\text{so...} \quad \vec{F} = \frac{GMm\langle x,y,z\rangle}{\rho^3}$$

$$M = \iiint_V \Delta M \, dV$$

$$\Delta M = \delta \, dV$$

Thus...

$$\vec{F} = \iiint_V \frac{Gm_0\langle x,y,z\rangle}{\rho^3} \cdot \delta \, dV$$

$$\langle F_x, F_y, F_z\rangle = Gm_0 \iiint_V \frac{\langle x,y,z\rangle}{\rho^3} \cdot \left(\sqrt{x^2+y^2}\right) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$F_z = Gm_0 \iiint_V \frac{z}{\rho^3} (r) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= Gm_0 \iiint_V \frac{\rho\cos\varphi}{\rho^3} (\rho\sin\varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

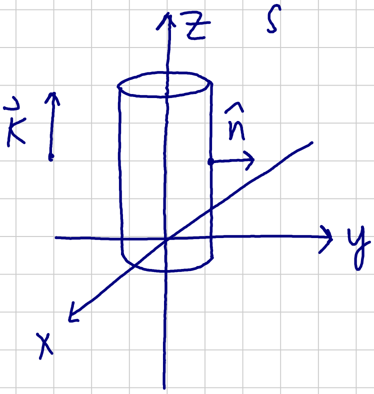
$$= Gm_0 \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{\rho=0}^{a} \rho \sin^2\varphi \cos\varphi \, d\rho \, d\varphi \, d\theta$$

$$= Gm_0 \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \frac{1}{2} a^2 \sin^2\varphi \cos\varphi \, d\varphi \, d\theta = \frac{Gm_0 a^2}{2} \int_{\theta=0}^{2\pi} \frac{1}{3} \sin^3\left(\frac{\pi}{2}\right) d\theta$$

$$= \frac{Gm_0 a^2}{6} \cdot 2\pi = \frac{\pi G m_0 a^2}{3} \Rightarrow \boxed{\vec{F} = \left\langle 0, 0, \frac{\pi G m_0 a^2}{3} \right\rangle}$$

57 a) Find the flux of $\vec{K}$ through $x^2+y^2=1$

$$\text{flux} = \iint_S \vec{K}\cdot\hat{n}\,ds$$

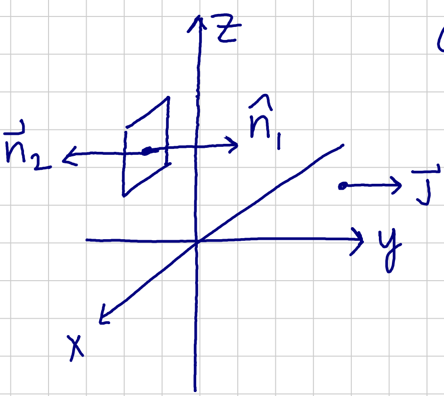


$\vec{K} \perp \hat{n}$, so...

$$\vec{K}\cdot\hat{n} = 0$$

$$\text{flux} = \iint_S 0\,ds = \boxed{0}$$

b) Find the flux of $\vec{J}$ through one square (with side length one) in the x-z plane.



Case 1) If $\hat{n}$ points in the same direction as $\vec{J}$...

$$\text{flux} = \iint_S \vec{J}\cdot\hat{n}_1\,ds$$

$$= |\vec{J}|\,|\hat{n}_1| \cos\theta \iint_S ds$$

$$= (1)(1)\cos(0)\left[1^2\right]$$

$$= \boxed{1}$$

Case 2) If $\hat{n}$ points in opposite direction of $\vec{J}$...

$$\text{flux} = \iint_S \vec{J}\cdot\vec{n}_2\,ds$$

$$= |\vec{J}|\,|\vec{n}_2| \cos\theta \iint_S ds$$

$$= (1)(1)\cos(\pi)\left[1^2\right]$$

$$= \boxed{-1}$$

(58) Let $z = x^2 + y$. Let S be the graph of z above the unit square in the x-y plane. Let $\vec{F} = z\vec{i} + x\vec{k}$. Find the upward flux of $\vec{F}$ through S.

$$f(x,y) = z = x^2 + y$$

$$\text{flux} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iint_S \vec{F} \cdot \langle -f_x, -f_y, 1 \rangle \, dx\, dy$$

$$= \iint_S \langle z, 0, x \rangle \cdot \langle -2x, -1, 1 \rangle \, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} (x - 2xz) \, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} (x - 2x(x^2 + y)) \, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} (x - 2x^3 - 2xy) \, dx\, dy$$

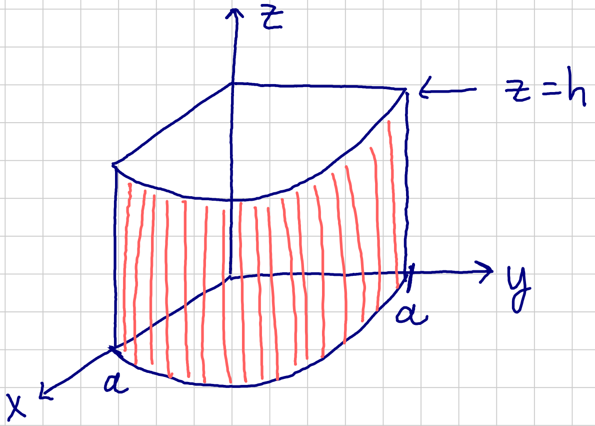
$$= \int_{y=0}^{1} \frac{1}{2}x^2 - \frac{1}{2}x^4 - x^2 y \Big|_{x=0}^{1} dy$$

$$= \int_{y=0}^{1} \frac{1}{2} - \frac{1}{2} - y \, dy$$

$$= -\frac{1}{2} y^2 \Big|_{y=0}^{1}$$

$$= \boxed{-\frac{1}{2}}$$

(59) a) Find the outward flux of $\vec{F} = \langle z, x, y \rangle$ through the piece of the cylinder shown:



$$\text{flux} = \iint_S \vec{F} \cdot \hat{n} \, dS$$

$$\hat{n} = \frac{\langle x, y, 0 \rangle}{a}$$

$$\vec{F} \cdot \hat{n} = \langle z, x, y \rangle \cdot \frac{\langle x, y, 0 \rangle}{a}$$

$$= \frac{xz + xy}{a}$$

$$\text{flux} = \int_{z=0}^{h} \int_{\theta=0}^{\pi/2} \frac{z(a\cos\theta) + (a\cos\theta)(a\sin\theta)}{a} \cdot a \, d\theta \, dz$$

$$= \int_{z=0}^{h} \int_{\theta=0}^{\pi/2} az\cos\theta + a^2 \sin\theta \cos\theta \, d\theta \, dz$$

$$= \int_{z=0}^{h} az\sin\theta + \frac{a^2 \sin^2\theta}{2} \Big|_{\theta=0}^{\pi/2} dz$$

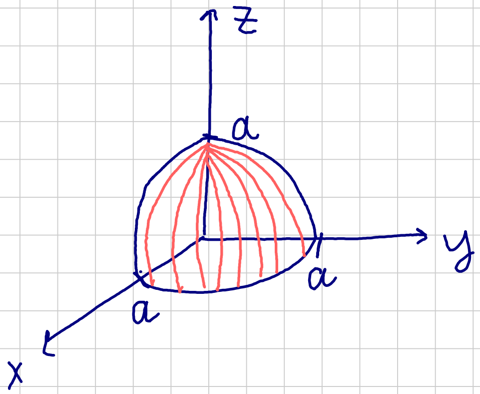
$$= \int_{z=0}^{h} az + \frac{1}{2}a^2 \, dz$$

$$= \frac{1}{2}az^2 + \frac{1}{2}a^2 z \Big|_{z=0}^{h}$$

$$= \boxed{\frac{1}{2}ah^2 + \frac{1}{2}a^2 h}$$

59 Continued

b) Find the flux (outward) of $\vec{F} = \langle xz, yz, z^2 \rangle$ through the piece of the sphere of radius a in the first octant.



$$\text{flux} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

$$\hat{n} = \frac{\langle x, y, z \rangle}{a}$$

$$\vec{F} \cdot \hat{n} = \langle xz, yz, z^2 \rangle \cdot \frac{\langle x, y, z \rangle}{a}$$

$$= \frac{x^2 z + y^2 z + z^3}{a}$$

$$= \frac{z(x^2 + y^2 z^2)}{a} = \frac{z a^2}{a} = az$$

$$= a \cdot a\cos\varphi = a^2 \cos\varphi$$

$$\text{flux} = \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} a^2 \cos\varphi \cdot a^2 \sin\varphi \, d\theta \, d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} a^4 \sin\varphi \cos\varphi \, d\theta \, d\varphi$$

$$= a^4 (\pi/2) \int_{\varphi=0}^{\pi/2} \sin\varphi \cos\varphi \, d\varphi$$

$$= \frac{\pi a^4}{2} \left(\frac{\sin^2\varphi}{2} \right) \Bigg|_{\varphi=0}^{\pi/2}$$

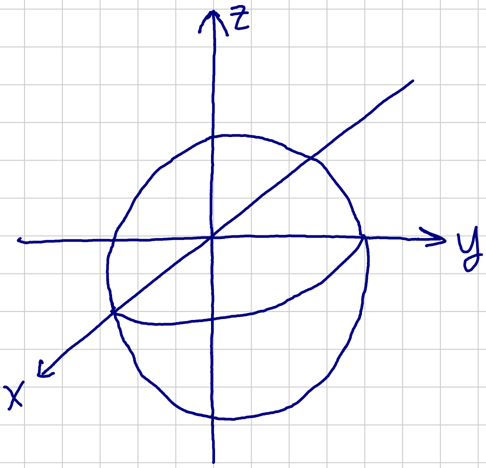
$$= \frac{\pi a^4}{2} \cdot \frac{1}{2} \cdot (1)^2$$

$$= \boxed{\frac{\pi a^4}{4}}$$

(60) Use the divergence theorem to find the flux of $\vec{F} = y\hat{j}$ through the right half of the sphere of radius R centered at the origin.

S: right half of sphere

S_2: the circular face on the x-z plane



Divergence Theorem: $\displaystyle\oiint_{S+S_2} \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\, dV$

$$\iiint_D \operatorname{div}\vec{F}\, dV = \iiint_D (1)\, dV$$

$$= \int_{\varphi=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\rho=0}^{R} (1)\,\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\pi}\int_{\theta=0}^{\pi} \tfrac{1}{3}R^3\sin\varphi\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\pi} \tfrac{1}{3}R^3(\sin\varphi)(\pi)\, d\varphi$$

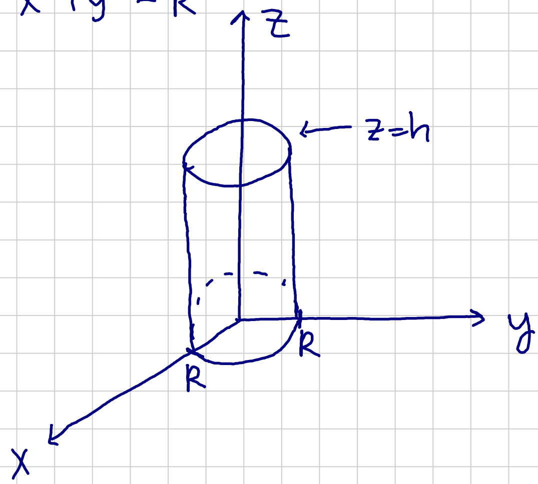
$$= \frac{\pi}{3}R^3(-\cos\varphi)\Big|_{\varphi=0}^{\pi}$$

$$= -\frac{\pi}{3}R^3((-1)-1) = \frac{2\pi R^3}{3}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\, dV - \iint_{S_2}\vec{F}\cdot d\vec{S}$$; $y=0$ on S_2 and $\vec{F} = y\hat{j} = (0)\hat{j} = 0$, so...

$$= \frac{2\pi R^3}{3} - 0 = \boxed{\frac{2\pi R^3}{3}}$$

(61) Compute the flux of $\vec{F} = \langle x^4y, -2x^3y^2, z^2 \rangle$ through the surface of the solid bounded by $z=0$, $z=h$, and $x^2+y^2=R^2$



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\, dV$$

$$= \iiint_D 4x^3-4x^3y+2z\, dV$$

$$\iiint_D \operatorname{div}\vec{F}\, dV = \iiint_D 4x^3y-4x^3y+2z\, dV$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R}\int_{z=0}^{h} (4x^3y-4x^3y+2z)\, r\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R}\int_{z=0}^{h} 2zr\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} h^2 r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{2}h^2R^2\, d\theta$$

$$= \frac{1}{2}h^2R^2(2\pi)$$

$$= \boxed{\pi h^2 R^2}$$

(62) Let $\vec{F} = \frac{1}{\rho^3}\langle x,y,z\rangle$ and let S be the surface of the box with vertices $(\pm 2, \pm 2, \pm 2)$.

a) Show $\operatorname{div}\vec{F}=0$ wherever $\vec{F}$ is defined.

$$\vec{F} = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\operatorname{div}\vec{F} = \frac{3(x^2+y^2+z^2)^{3/2} - (x^2+y^2+z^2)^{1/2}[3x^2+3y^2+3z]}{(x^2+y^2+z^2)^3}$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$$

$$= \boxed{0} \text{ as long as } x^2+y^2+z^2 \neq 0$$

b) Can we conclude flux S is zero?

By the divergence theorem, $\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\,dV$, given that $\vec{F}$ is defined and differentiable everywhere in D. At $(x,y,z)=(0,0,0)$, $\vec{F}$ is undefined, so $\boxed{\text{NO}}$ we cannot conclude that $\iint_S \vec{F}\cdot d\vec{S}=0$.

c) Use extended Gauss' Theorem to compute flux through S.

Extended Gauss' Theorem: $\iiint_D \operatorname{div}\vec{F} = \iint_{S_2} \vec{F}\cdot\hat{n}\,ds - \iint_S \vec{F}\cdot\hat{n}\,ds$

where S is the cube and S_2 is a sphere surrounding it with radius R.

Since $\iiint_D \operatorname{div}\vec{F}\,dV = 0$ (as calculated in part (a)).

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = |\vec{F}||\hat{n}|\cos\theta \iint_S ds$$

$$= \frac{1}{R^2}(1)\cos(0)\cdot 4\pi R^2 = \boxed{4\pi}$$

(63) Let $\vec{F}$ be a vector field and u be a scalar function. Show the following "product rule" for the del operator:

$$\nabla\cdot(u\vec{F}) = \nabla u\cdot\vec{F} + u\nabla\cdot\vec{F}$$

$$\nabla\cdot(u\vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle\cdot\langle uP, uQ, uR\rangle$$

$$= \frac{\partial}{\partial x}(uP) + \frac{\partial}{\partial y}(uQ) + \frac{\partial}{\partial z}(uR)$$

$$= \left(P\cdot\frac{\partial u}{\partial x} + u\frac{\partial P}{\partial x}\right) + \left(Q\frac{\partial u}{\partial y} + u\frac{\partial Q}{\partial y}\right) + \left(R\frac{\partial u}{\partial z} + u\frac{\partial R}{\partial z}\right)$$

$$= \left(P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z}\right) + \left(u\frac{\partial P}{\partial x} + u\frac{\partial Q}{\partial y} + u\frac{\partial R}{\partial z}\right)$$

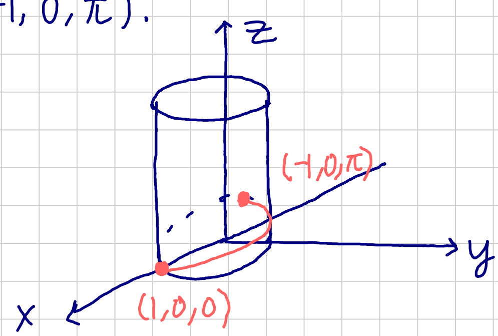
$$= \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\rangle\cdot\langle P,Q,R\rangle + u\left\langle \frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}\right\rangle$$

$$\boxed{\nabla\cdot(u\vec{F}) = \nabla u\cdot\vec{F} + u\nabla\cdot\vec{F}}$$

(64) Let $\vec{F} = zx\vec{i} + zy\vec{j} + x\vec{k}$. Let C be the helix $(\cos t, \sin t, t)$ going from $(1,0,0)$ to $(-1,0,\pi)$.

a) Sketch the curve

Curve shown in red →



b) Compute $\int_C \vec{F}\cdot d\vec{r}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[\langle zx, zy, x\rangle\cdot\langle dx, dy, dz\rangle\right] \text{ where } \begin{cases} x=\cos t \\ y=\sin t \\ z=t \end{cases},\ 0\le t\le\pi$$

$$= \int_{t=0}^{\pi}(zx\,dx + zy\,dy + x\,dz)$$

$$= \int_{t=0}^{\pi}\left(-t\sin t\cos t\,dt + t\sin t\cos t\,dt + \cos t\,dt\right) = \int_{t=0}^{\pi}\cos t\,dt$$

$$= \sin(\pi) - \sin(0) = \boxed{0}$$

(65) a) For what value(s) of b is $\vec{F} = y\vec{i} + (x + byz)\vec{j} + (y^2+1)\vec{k}$ conservative?

$\vec{F}$ is conservative when curl $\vec{F} = 0$

$$\text{curl } \vec{F} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$$

$$\vec{0} = (2y - by)\vec{i} + (0-0)\vec{j} + (1-1)\vec{k}$$

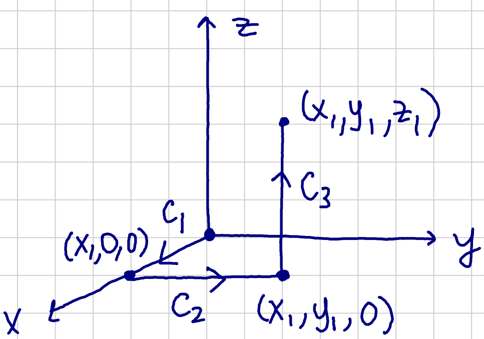
$$= (2y - by)\vec{i}$$

$$= 2 - b$$

$$\boxed{b = 2}$$

b) For each b value determined by part (a), find a potential function.

$$f(x_1, y_1, z_1) = \int_C \vec{F} \cdot d\vec{r}$$



On C_1: $y = z = dy = dz = 0$, $0 \leq x \leq x_1$

On C_2: $z = dz = dx = 0$, $x = x_1$, $0 \leq y \leq y_1$

On C_3. $dx = dy = 0$, $x = x_1$, $y = y_1$, $0 \leq z \leq z_1$

Let $C = C_1 + C_2 + C_3$ . . .

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C1} P\,dx + \int_{C2} Q\,dy + \int_{C3} R\,dz \text{, where } \vec{F} = \langle P, Q, R \rangle$$

$$= \int_{x=0}^{x_1} y\,dx + \int_{y=0}^{y_1} (x + byz)\,dy + \int_{z=0}^{z_1} (y^2+1)\,dz$$

$$= (x_1 y) + (x y_1 + \tfrac{1}{2} b y_1 z) + (y^2 z_1 + z_1)$$

$$= (x_1(0)) + (x_1 y_1 + \tfrac{1}{2}(2) y_1 (0)) + (y_1^2 z_1 + z_1)$$

$$\boxed{f(x_1, y_1, z_1) = x_1 y_1 + y_1^2 z_1 + z_1 + C}$$

(65) Continued

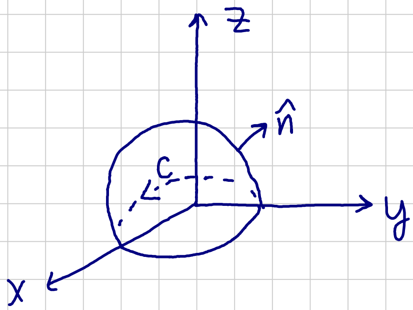
c) Explain why $\vec{F}\cdot d\vec{r}$ is exact, for b value(s) determined in part (a)

$\vec{F}\cdot d\vec{r}$ is exact when $b=2$ because

1) $\vec{F}$ is defined and differentiable everywhere on C.

2) $\vec{F}$ is a gradient field. (curl $\vec{F}=0$ when $b=2$)

(66) Verify Stoke's Theorem for $\vec{F}=\langle 2z, x, y\rangle$ where S is the top half of the unit sphere.



Stoke's Theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

On C: $x=\cos t$, $y=\sin t$, $z=0$, $0\le t\le 2\pi$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_C 2z\,dx + x\,dy + y\,dz$$

$$= \int_C 2(0)\,dx + x\,dy + y(0)$$

$$= \int_C x\,dy$$

$$= \int_{t=0}^{2\pi} \cos^2 t\,dt$$

$$= \frac{t}{2} + \frac{\sin(2t)}{4}\Big|_{t=0}^{2\pi}$$

$$= \frac{2\pi}{2} = \boxed{\pi}$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

$$= \iint_S \langle (R_y-Q_z),(P_z-R_x),(Q_x-P_y)\rangle\cdot\hat{n}\,dS$$

$$= \iint_S \langle 1,2,1\rangle\cdot\hat{n}\,dS$$

$$= \int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{2\pi} \langle 1,2,1\rangle\cdot\langle x,y,z\rangle \rho^2\sin\varphi\,d\varphi\,d\theta$$

$$= \int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{2\pi} (x+2y+z)\rho^2\sin\varphi\,d\varphi\,d\theta$$

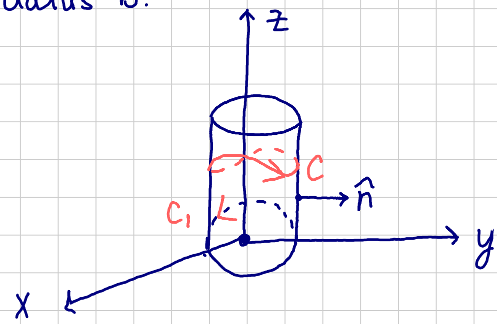
$$= \rho^2\int_{\varphi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi}\left(\rho\sin\varphi\cos\theta + 2\rho\sin\varphi\sin\theta + \rho\cos\varphi\right)\sin\varphi\,d\varphi\,d\theta$$

$$= \rho^3\int_{\varphi=0}^{\pi/2}\left(0+0+2\pi\cos\varphi\sin\varphi\right)d\varphi$$

$$= 2\pi\rho^3\,\frac{\sin^2(\pi/2)}{2} = \boxed{\pi}$$

* $\pi=\pi \Rightarrow$ Stoke's Theorem *

(67) Let $\vec{F} = \langle 2xz - 2y, 2yz + 2x, x^2 + y^2 + z^2 \rangle$. Use Stoke's Theorem to compute $\oint_C \vec{F} \cdot d\vec{r}$ where C is the curve shown on the cylinder of radius b.



Stoke's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

Extended Stoke's Theorem: $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r}$

$$\nabla \times \vec{F} = \langle (R_y - Q_z), (P_z - R_x), (Q_x - P_y) \rangle$$
$$= \langle (2y - 2y), (2x - 2x), (2 + 2) \rangle = \langle 0, 0, 4 \rangle$$

For any point (x, y) on the cylinder, $\hat{n} = \langle x, y, 0 \rangle$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iint_S \langle 0,0,4 \rangle \cdot \langle x, y, 0 \rangle\, dS = \iint_S 0\, dS = 0$$

On C_1: $x = b\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$, $z = dz = 0$

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_1} (2xz - 2y)\,dx + (2yz + 2x)\,dy + 2x(dz)$$

$$= \oint_{C_1} (0 - 2y)\,dx + (0 + 2x)\,dy + 0$$

$$= \oint_{t=0}^{2\pi} (-2b\sin t)(-b\sin t\, dt) + (2b\cos t)(b\cos t\, dt)$$

$$= 2b^2 \int_{t=0}^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2b^2\, t \Big|_{t=0}^{2\pi} = 4\pi b^2$$

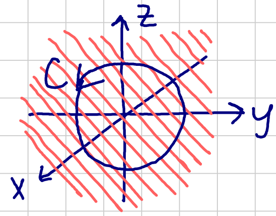
Thus,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$
$$= 4\pi b^2 - 0$$
$$= \boxed{4\pi b^2}$$

(68) For each of the following, say whether or not the region is simply connected.

a) $\mathbb{R}^3$

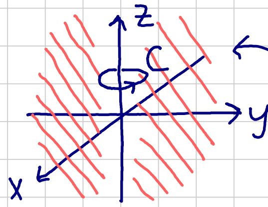
yes, it is $\boxed{\text{simply connected}}$



C does not contain non-region when collapsed

b) $\mathbb{R}^3$ minus the z-axis

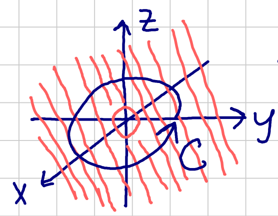
this is $\boxed{\text{not simply connected}}$



C contains non-region when collapsed

c) $\mathbb{R}^3 - \{0\}$

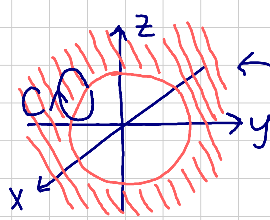
yes, it is $\boxed{\text{simply connected}}$



C does not contain non-region when collapsed

d) $\mathbb{R}^3$ minus a circle

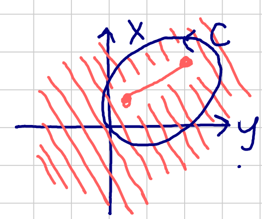
this is $\boxed{\text{not simply connected}}$



C contains non-region when collapsed

e) $\mathbb{R}^2$ minus a line segment

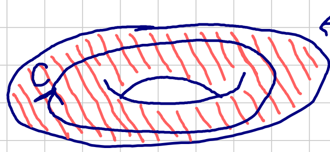
this is $\boxed{\text{not simply connected}}$



C contains non-region when collapsed

f) A solid taurus.

this is $\boxed{\text{not simply connected}}$



C contains non-region when collapsed

(69) Show that $\vec{F} = \rho^n (x\vec{i} + y\vec{j} + z\vec{k})$ is a gradient field

$\vec{F}$ is a gradient field if $(\nabla \times \vec{F}) = \vec{0}$.

$$\vec{F} = \langle x(x^2+y^2+z^2)^{n/2} + y(x^2+y^2+z^2)^{n/2} + z(x^2+y^2+z^2)^{n/2} \rangle$$

$$\nabla \times \vec{F} = \langle (R_y - Q_z), (P_z - R_x), (Q_x - P_y) \rangle$$

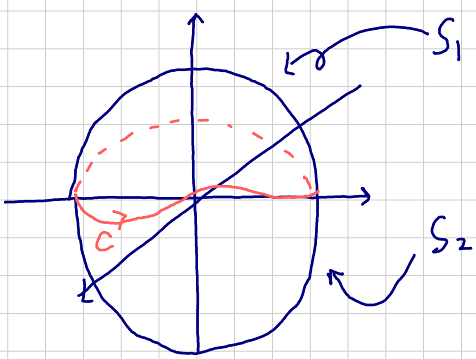
$$= \langle yzn(\rho^{n-2}) - yzn(\rho^{n-2}), xzn(\rho^{n-2}) - xzn(\rho^{n-2}), xyn(\rho^{n-2}) - xy(\rho^{n-2}) \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

$\nabla \times \vec{F} = \vec{0}$, so $\boxed{\vec{F} \text{ is a gradient field}}$

(70) Let $\vec{F} = \langle x, y, z \rangle$. Show there is no $\vec{G}$ such that $\vec{F} = \vec{\nabla} \times \vec{G}$ as follows: let S be a sphere of radius b centered at the origin & let C be a curve on S. Assume $\vec{F} = \vec{\nabla} \times \vec{G}$ & use Stoke's Theorem to interpret $\oint_C \vec{G}\, d\vec{r}$ & get $\Rightarrow\!\Leftarrow$ (a contradiction)

Note: we are going to solve this using proof by contradiction.



By stoke's theorem...

$$\oint_C \vec{G} \cdot d\vec{r} = \iint_{S_1} (\vec{\nabla} \times \vec{G}) \cdot \hat{n}\, dS$$

<u>Assume</u>: $\vec{F} = \vec{\nabla} \times \vec{G}$

$$\oint_C \vec{G} \cdot d\vec{r} = \iint_{S1} (\vec{F}) \cdot \hat{n}\, ds$$

$$= |\vec{F}||\hat{n}| \cos\theta \iint_{S_1} ds$$

$$= (b)(1)(\cos 0) \cdot \text{area}(S_1)$$

$$= \boxed{b \cdot \text{area}(S_1) > 0}$$

next page →

70 Continued.

Now taking a look at S_2 ...

By stoke's theorem...

$$-\oint_C \vec{G}\cdot d\vec{r} = \iint_{S_2} (\vec{\nabla}\times\vec{G})\cdot\hat{n}\,ds$$

(the negative sign is necessary to maintain the outward normal vector required for applying Stoke's Theorem.)

Assume: $\vec{F} = \vec{\nabla}\times\vec{G}$

$$-\oint_C \vec{G}\cdot d\vec{r} = \iint_{S_2} (\vec{F})\cdot\hat{n}\,ds$$

$$= |\vec{F}||\hat{n}|\cos\theta \iint_{S_2} ds$$

$$= (b)(1)(\cos 0)\cdot \text{area}(S_2)$$

$$= \boxed{b\cdot \text{area}(S_2) > 0}$$

Assuming that $\vec{F} = \vec{\nabla}\times\vec{G}$ led to the result that both $\oint_C \vec{G}\cdot d\vec{r}$ and $-\oint_C \vec{G}\cdot d\vec{r}$ are positive values, which cannot logically be the case.

From this logical contradiction, we have proved by contradiction that...

$$\boxed{\text{If } \vec{F} = \langle x,y,z\rangle, \text{ there is no such } \vec{G} \text{ such that } \vec{F} = \vec{\nabla}\times\vec{G}.}$$