#Calculation of VaR for a Portfolio

Submitted By:

170123051 TANVI OHRI

170123053 TEJASVEE PANWAR

170123057 KARTIK SETHI

In [47]:

```
import pandas as pd
import numpy as np
from pandas_datareader import data, wb
import datetime
import scipy.optimize as sco
from scipy import stats
import matplotlib.pyplot as plt
%matplotlib inline
```

What are Quantiles?

An outcome at a given probability can be expressed using quantiles. Let (Ω, F, P) be a probability space and let $X : \Omega \to R$ be a random variable. The cumulative distribution function $F_X : \mathbb{R} \to [0, 1]$, defined by $F_X(x) = P(X \leq x)$ is right-continuous and non-decreasing. Then,

For $\alpha \in (0, 1)$, the number

> $q^{\alpha}(X) = inf\ \{x : \alpha < F_X(\text{x})\}$

is called the upper α-quantile of X. The number

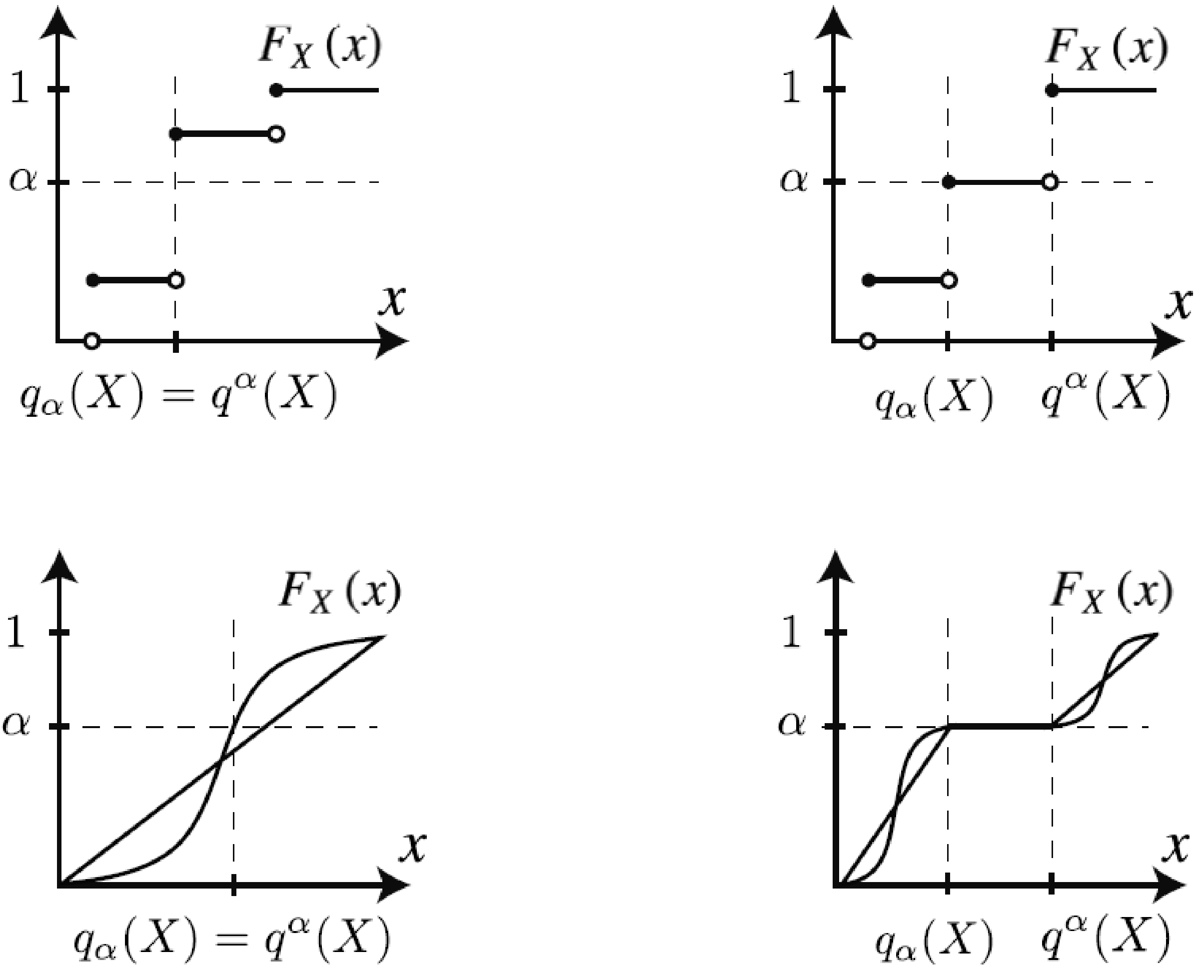
> $q_{\alpha}(X) = inf\ \{x : \alpha \leq F_X(\text{x})\}$

is called the lower α-quantile of X. Any

> $q \in [q_{\alpha}(X), q^{\alpha}(X)]$

is called an α-quantile of X.

The image below shows the upper and lower quantiles for various distribution functions:

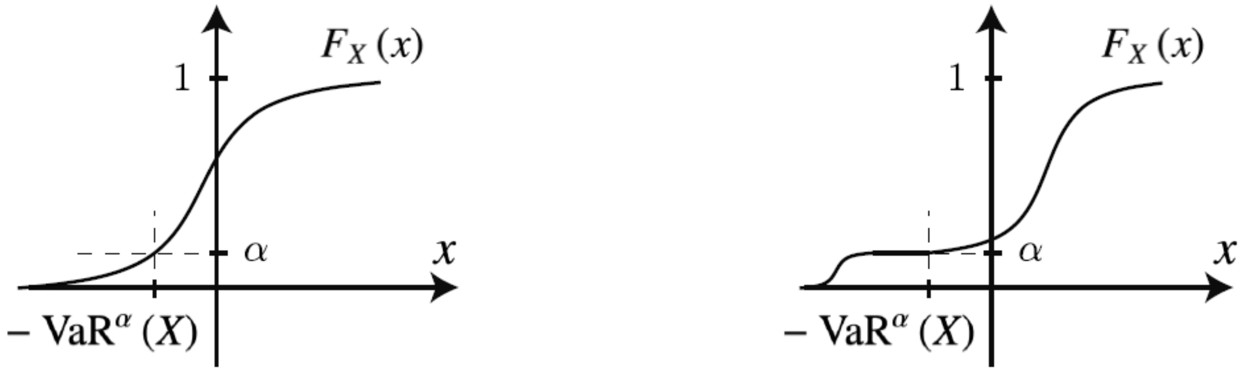


What is VaR?

Value at risk (VaR) is a statistic that measures and quantifies the level of financial risk within a firm, portfolio or position over a specific time frame.

For α in (0, 1), we define the Value at Risk (VaR) of X, at confidence level $1 - \alpha$, as

$$VaR^\alpha(X) = -q^\alpha(X) = -inf\,\{x : \alpha < F_X(x)\}$$



In [48]:

```
tickers = ['FB','AAPL', 'AMZN', 'NFLX', 'GOOG', 'MSFT']
start = datetime.datetime(2014, 1, 1)
end = datetime.datetime(2019, 12, 31)
df = pd.DataFrame([data.DataReader(ticker, 'yahoo', start, end)['Adj Close'] for ti
"""
   DataReader returns a Panel object, which can be thought of as a 3D matrix. The f
   consists of the various fields Yahoo Finance returns for a given instrument, nam
   Adj Close (adjusted closing price) prices for each date. The second dimension co
"""
df.columns = tickers
```

In [49]:

```
df.describe()
```

Out[49]:

	FB	AAPL	AMZN	NFLX	GOOG	MSFT
count	1511.000000	1511.000000	1511.000000	1511.000000	1511.000000	1511.000000
mean	130.621496	34.048545	984.070947	177.324537	854.370829	70.748682
std	44.592705	12.014982	566.591955	112.094569	248.999708	33.566073
min	53.529999	15.795355	286.950012	44.887142	491.201416	30.246181
25%	86.364998	24.756407	438.975006	89.359287	599.400818	42.178766
50%	129.070007	29.458752	817.880005	128.350006	797.070007	58.729496
75%	172.570000	42.311350	1602.989990	291.539993	1083.875000	97.865253
max	217.500000	72.192863	2039.510010	418.970001	1361.170044	157.293686

VaR of individual stocks according to historical stock data

The historical method simply calculates the Quantiles/VaR for the actual historical returns. It then assumes that history will repeat itself, from a risk perspective.

In [50]:

```
pct_returns = df.pct_change()
"""Computes the percentage change from the immediately previous row. """
pct_returns.describe()
print(pct_returns.quantile(0.05))
```

```
FB     -0.027015
AAPL   -0.023623
AMZN   -0.027120
NFLX   -0.036799
GOOG   -0.022540
MSFT   -0.020494
Name: 0.05, dtype: float64
```

VaR Using Variance Covariance Method

The variance-covariance method is an analytical way to calculate VaR. To use it we need different information than the other methods because of the assumptions it makes.

- The variance-covariance method assumes that a stock investment's returns will be normally distributed around the mean of a normal or bell-shaped probability distribution.
- Since returns are distributed in a normal or bell curve format, we need the standard deviation of the returns.
- A complicating factor of this method is that stocks can have a tendency to move up and down together, usually caused by some external factor. That means we need the covariance of returns for all of the stocks in a portfolio against all of the other stocks.

In [13]:

```
# Assuming stock returns are normally distributed
def portfolio_stats(weights, mean_returns, cov, alpha, days):
    portfolio_return = np.sum(mean_returns * weights) * days
    portfolio_std = np.sqrt(np.dot(weights.T, np.dot(cov, weights))) * np.sqrt(days
    portfolio_VaR = abs(portfolio_return - (portfolio_std * stats.norm.ppf(1 - alph
    return portfolio_return, portfolio_std, portfolio_VaR
```

In [14]:

```
def simulate_random_portfolios(num_portfolios, mean_returns, cov, alpha, days):
    results_matrix = np.zeros((len(mean_returns)+3, num_portfolios))
    for i in range(num_portfolios):
        weights = np.random.random(len(mean_returns))
        weights /= np.sum(weights)
        portfolio_return, portfolio_std, portfolio_VaR = portfolio_stats(weights, m
        results_matrix[0:3,i]=[portfolio_return, portfolio_std, portfolio_VaR]
        results_matrix[3:3+len(weights),i]=weights
    results_df = pd.DataFrame(results_matrix.T,columns=['ret','stdev','VaR'] + [tic
    return results_df
```

In [15]:

```
mean_returns = df.pct_change().mean()
cov = df.pct_change().cov()
num_portfolios = 100000
rf = 0.0
days = 252
alpha = 0.05
results_frame = simulate_random_portfolios(num_portfolios, mean_returns, cov, alpha
```

In [16]:

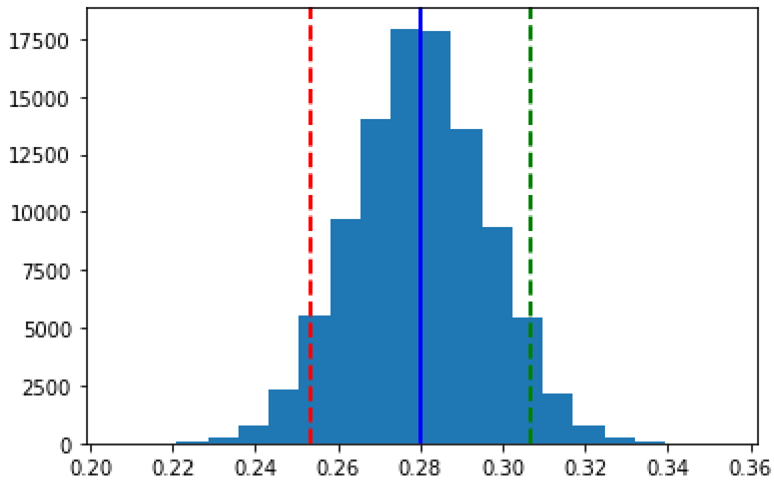
```
#locate positon of portfolio with minimum VaR
min_VaR_port = results_frame.iloc[results_frame['VaR'].idxmin()]
min_VaR_port.to_frame().T
```

Out[16]:

	ret	stdev	VaR	FB	AAPL	AMZN	NFLX	GOOG	MSI
38730	0.295547	0.207181	0.045235	0.036745	0.288984	0.058906	0.143883	0.001522	0.4699!

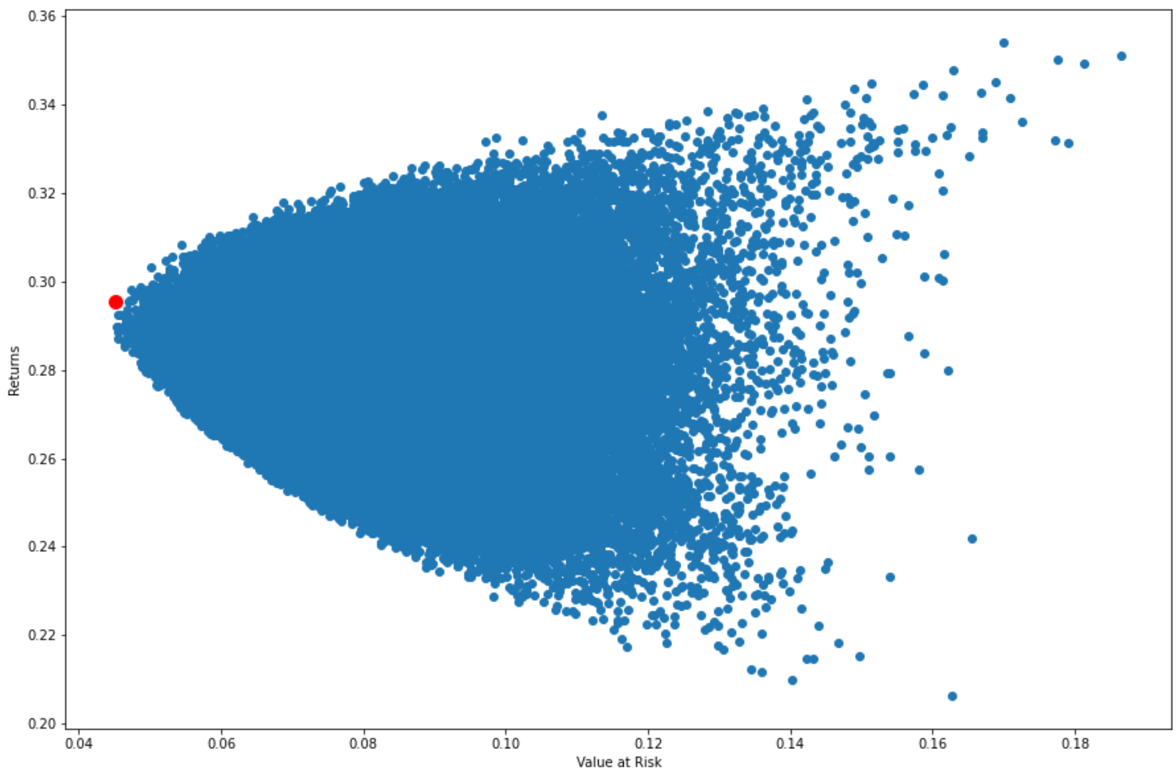
In [17]:

```
sample_returns=results_frame['ret']
plt.hist(sample_returns,bins=20)
plt.axvline(np.percentile(sample_returns,5), color='r', linestyle='dashed', linewid
plt.axvline(np.percentile(sample_returns,95), color='g', linestyle='dashed', linewi
plt.axvline(np.mean(sample_returns), color='b', linestyle='solid', linewidth=2)
plt.show()
```



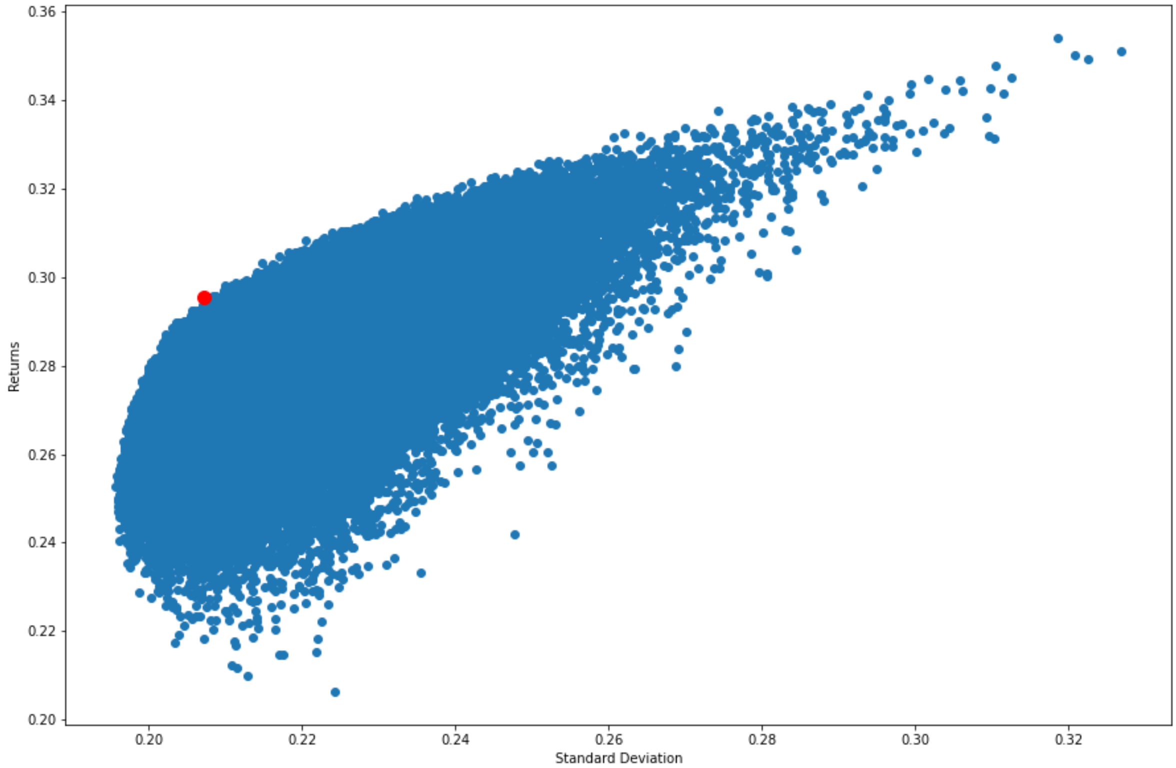
In [18]:

```
#create scatter plot coloured by VaR
plt.subplots(figsize=(15,10))
plt.scatter(results_frame.VaR,results_frame.ret)
plt.xlabel('Value at Risk')
plt.ylabel('Returns')
#plot red star to highlight position of minimum VaR portfolio
plt.scatter(min_VaR_port[2],min_VaR_port[0],color='r',s=100)
plt.show()
```



In [19]:

```
#create scatter plot coloured by Returns
plt.subplots(figsize=(15,10))
plt.scatter(results_frame.stdev,results_frame.ret)
plt.xlabel('Standard Deviation')
plt.ylabel('Returns')
#plot red star to highlight position of minimum VaR portfolio
plt.scatter(min_VaR_port[1],min_VaR_port[0],color='r',s=100)
plt.show()
```



In [20]:

```
constraints = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1})
def calc_portfolio_VaR(weights, mean_returns, cov, alpha, days):
    portfolio_return = np.sum(mean_returns * weights) * days
    portfolio_std = np.sqrt(np.dot(weights.T, np.dot(cov, weights))) * np.sqrt(days
    portfolio_VaR = abs(portfolio_return - (portfolio_std * stats.norm.ppf(1 - alph
    return portfolio_VaR
```

In [21]:

```
# Numerical Solver for Minimum VaR portfolio
def min_VaR(mean_returns, cov, alpha, days):
    num_assets = len(mean_returns)
    args = (mean_returns, cov, alpha, days)
    constraints = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1})
    bound = (0.0,1.0)
    bounds = tuple(bound for asset in range(num_assets))
    result = sco.minimize(calc_portfolio_VaR, num_assets*[1./num_assets,], args=arg
                          method='SLSQP', bounds=bounds, constraints=constraints)
    return result
```

In [22]:

```
min_port_VaR = min_VaR(mean_returns, cov, alpha, days)
```

In [23]:

```
pd.DataFrame([round(x,2) for x in min_port_VaR['x']],index=tickers).T
```

Out[23]:

	FB	AAPL	AMZN	NFLX	GOOG	MSFT
0	0.07	0.28	0.08	0.11	0.0	0.47

VaR using Monte Carlo Simulation

Properties of $W(t)$

1. $W(0) = 0$
2. W has independent increments: for every $t > 0$, the future increments $W_{t+u} - W_t, u \geq 0$ are independent of the past values W_s, $s \leq t$.
3. W has Gaussian increments: $W_{t+u} - W_t \sim \mathcal{N}(0, u)$.
4. W has continuous paths: W_t is continuous in t.

###The Geometric Brownian Motion

We have assumed that the stock price follows a Geometric Brownian motion (GBM).

A stochastic process St is said to follow a GBM if it satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Brownian motion, and μ ('the percentage drift') and σ ('the percentage volatility') are constants.

Solving this SDE

Consider an arbitrary initial value S_0. This derivation requires the use of Itô calculus. Applying Itô's formula leads to

$$d(\ln S_t) = (\ln S_t)' dS_t + \tfrac{1}{2}(\ln S_t)'' dS_t dS_t$$
$$= \tfrac{dS_t}{S_t} - \tfrac{1}{2}\tfrac{1}{S_t^2} dS_t dS_t$$

where $dS_t dS_t$ is the quadratic variation of the SDE.

Now,

$$dS_t dS_t = \sigma^2 S_t^2 dW_t{}^2 + 2\sigma S_t^2 \mu dW_t dt + \mu^2 S_t^2 dt^2$$

When $dt \to 0$, dt converges to 0 faster than dW_t, since $dW_t{}^2 = O(dt)$. So the above infinitesimal can be simplified by

$$dS_t dS_t = \sigma^2 S_t^2 dt$$

Plugging the value of $dS_t dS_t$ in the above equation and simplifying we obtain

$$\ln \frac{S_t}{S_0} = (\mu - \frac{\sigma^2}{2})t + \sigma W_t.$$

Taking the exponential and multiplying both sides by S_0 we get

$$S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t)$$

In [24]:

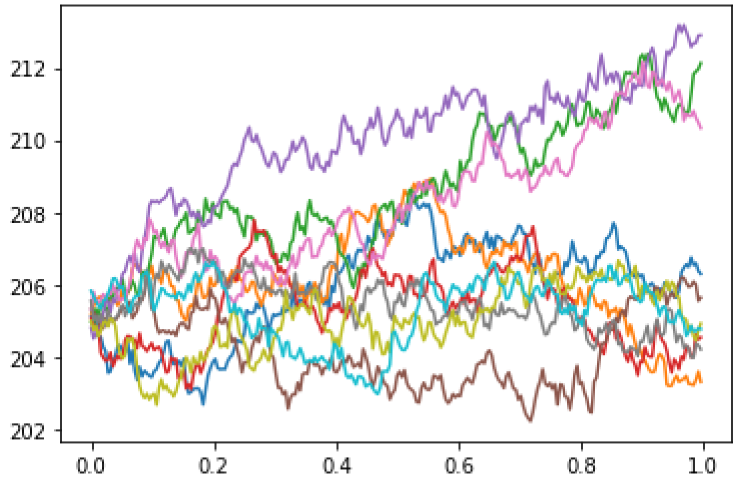
```
def Brownian(N):
    '''
    N: Number of increments
    '''
    dt = 1./N
    b = np.random.normal(0., 1., int(N))*np.sqrt(dt)
    W = np.cumsum(b)
    return W
```

In [25]:

```
def GBM(S0, mu, sigma, W, N):
    '''
    S0:Initial stock price
    mu:returns (drift coefficient)
    sigma:Volatility (diffusion coefficient)
    W:brownian motion
    N:number of increments
    '''
    t = np.linspace(0.,1.,N+1)
    results_matrix = np.zeros((2,N))
    for i in range(1,int(N+1)):
        drift = (mu - 0.5 * sigma**2) * t[i]
        diffusion = sigma * W[i-1]
        S_temp = S0*np.exp(drift + diffusion)
        results_matrix[0:2,i-1]=[S_temp,t[i-1]]
    results_df = pd.DataFrame(results_matrix.T,columns=['S','t'])
    return results_df
```

In [26]:

```
# Checking everything works
mu = df['FB'].pct_change().mean()
sigma = df['FB'].pct_change().std()
S0=df['FB'].loc[datetime.datetime(2019,12,31)]
for i in range(10):
    W=Brownian(252)
    df1=GBM(S0, mu, sigma,W, 252)
    plt.plot(df1['t'],df1['S'])
```



Monte Carlo Simulations

Lemma : Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables, $X_i\ : \Omega \to \mathbb{R}$, with the same distribution as X. Let $x \in \mathbb{R}$ be fixed. If we take a sequence of random variables $F_N(x) : \Omega \to \mathbb{R}$ defined as

$$F_N(x) = \frac{1}{N}\sum_{i=1}^{N} 1_{\{X_i \leq x\}}$$

Then $\quad F_N(x) \xrightarrow{p} F_X(x)$

Proof: $Y_i\ = 1_{\{X_i \leq x\}}$ and $\quad Y\ = 1_{\{X_i \leq x\}}$ By the weak law of large numbers $\frac{1}{N}\sum_{i=1}^{N} Y_i \xrightarrow{p} \mathbb{E}[Y]$ hence

$$F_N(x) = \frac{1}{N}\sum_{i=1}^{N} Y_i \xrightarrow{p} \mathbb{E}[Y] = \mathbb{E}[1_{\{X_i \leq x\}}] = P(X \leq x) = F_X(x)$$

Hence proved

Using this result we can see that if we have $\hat{X}_1, \hat{X}_2 \cdots$ as results of a simulation such that they follow the same distribution as X and let

$$\hat{F}_N(x) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{\hat{X}_i \leq x\}}$$

by the above lemma we know that

$$F_X(x) = \lim_{N \to \infty} \hat{F}_N(x)$$

Let Y_N denote a discrete r.v distribution given by

$$P(Y_N = \hat{X}_i) = \frac{1}{N}, \text{for } i = 1, 2 \cdots N$$

the distribution function $F_{Y_N}(x) = F_N(x)$

Taking sufficiently large N we can approximate $Var_\alpha(X)$

$$Var_\alpha(X) \approx Var_\alpha(Y_N)$$

So we can approximate VaR of simulations using the r.v. Y_N and we know how to get VaR of discrete r.v. using quantiles.

This motivates our algorithm for VaR using Monte Carlo simulation

This method involves developing a model for future stock price returns and running multiple hypothetical trials through the model. A Monte Carlo simulation refers to any method that randomly generates trials, but by itself does not tell us anything about the underlying methodology.

Steps in the algorithm

1. Given a fixed weight vector and stocks
2. Get the μ and σ for the GBM model of every stock in our portfolio
3. Final_Portfolio_value = []
4. For 1000 trials repeat:
5. $V = 0$
6. for i^{th} Stock in portfolio repeat:
7. $T = 252$ get $r_i = \frac{S_i(T) - S_i(0)}{S_i(0)}$ or the stock value at final time point using GBM equation and from that get returns
8. $V = V + w_i r_i$ i.e. $V = \sum_{i=1}^{n} w_i r_i$
9. Append V to array of Final_portfolio_value
10. Get the $(1 - \alpha)^{th}$ percentile of Final_Portfolio_values

In [27]:

```
def GBM_returns(tickers,df,trials = 1000):
    mean_returns = df.pct_change().mean()
    returns_std = df.pct_change().std()
    returns=np.zeros((len(mean_returns),trials))
    for j in range(trials):
        for i in range(len(mean_returns)):
            mu = mean_returns[tickers[i]]
            sigma = returns_std[tickers[i]]
            S0=df[tickers[i]].loc[datetime.datetime(2019,12,31)]
            W=Brownian(252)
            df_GBM=GBM(S0,mu,sigma,W,252)
            returns[i,j]=(df_GBM['S'].iloc[-1] - df_GBM['S'].iloc[0])/df_GBM['S'].i
    returns_df=pd.DataFrame(returns.T,columns=[ticker for ticker in tickers])
    return returns_df
```

In [28]:

```
def get_VaR(weights,returns_df):
    return_arr=returns_df.to_numpy()
    portfolio_returns=return_arr*weights
    return -np.quantile(portfolio_returns,0.05)
```

In [29]:

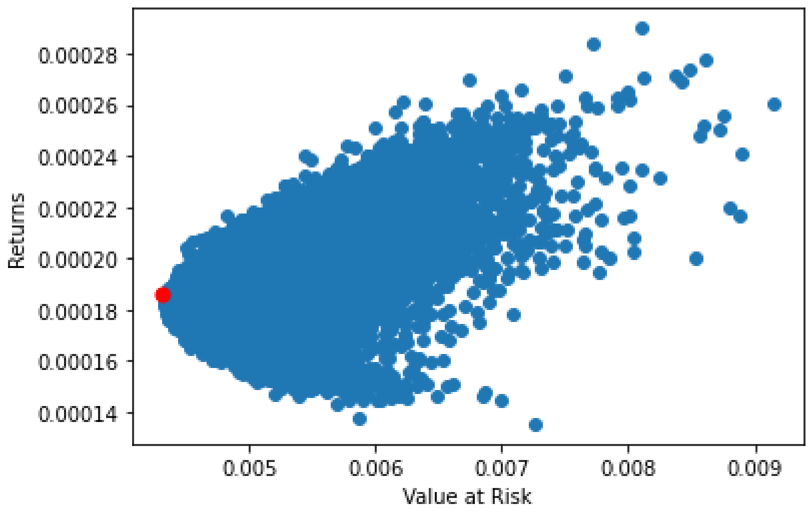
```
def simulate_portfolios_VaR(returns_df,num_of_portfolios=1000):
    n=returns_df.shape[1]
    results_matrix = np.zeros((2+n, num_of_portfolios))
    for i in range(num_of_portfolios):
        weights = np.random.random(n)
        weights /= np.sum(weights)
        port_var=get_VaR(weights,returns_df)
        port_mean_return=np.mean(returns_df.to_numpy()*weights)
        results_matrix[0:2,i]=[port_mean_return, port_var]
        results_matrix[2:2+n,i]=weights
    results_df = pd.DataFrame(results_matrix.T,columns=['ret','VaR'] + [ticker for
    return results_df
```

In [30]:

```
returns_df=GBM_returns(tickers,df,trials=1000)
results_frame=simulate_portfolios_VaR(returns_df,10000)
min_VaR_port = results_frame.iloc[results_frame['VaR'].idxmin()]
plt.scatter(results_frame.VaR,results_frame.ret)
plt.scatter(min_VaR_port[1],min_VaR_port[0],color='r',s=50)
plt.xlabel('Value at Risk')
plt.ylabel('Returns')
```

Out[30]:

```
Text(0, 0.5, 'Returns')
```



In [31]:

```
# Numerical Solver for Minimum VaR portfolio
def min_VaR(returns_df):
    num_assets = returns_df.shape[1]
    args = (returns_df)
    constraints = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1})
    bound = (0.0,1.0)
    bounds = tuple(bound for asset in range(num_assets))
    result = sco.minimize(get_VaR, num_assets*[1./num_assets,], args=args,
                          method='SLSQP', bounds=bounds, constraints=constraints)
    return result
```

In [32]:

```
returns_df=GBM_returns(tickers,df,trials=1000)
min_var = min_VaR(returns_df)
weights=pd.DataFrame([round(x,3) for x in min_var['x']],index=tickers).T
print(get_VaR(weights.to_numpy(),returns_df))
weights
```

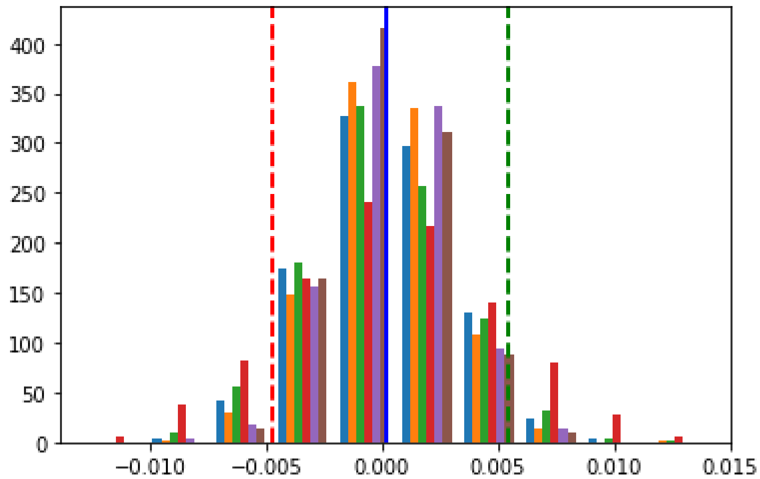
0.004759401540890839

Out[32]:

	FB	AAPL	AMZN	NFLX	GOOG	MSFT
0	0.154	0.169	0.169	0.169	0.169	0.169

In [33]:

```
sample_returns=returns_df.to_numpy()*weights.to_numpy()
plt.hist(sample_returns,bins=10)
plt.axvline(np.percentile(sample_returns,5), color='r', linestyle='dashed', linewid
plt.axvline(np.percentile(sample_returns,95), color='g', linestyle='dashed', linewi
plt.axvline(np.mean(sample_returns), color='b', linestyle='solid', linewidth=2)
plt.show()
```



In [ ]: