Application of Linear Algebra in Quantum Computing

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(E18CSE187)

Contents

Quantum Computing

- Quantum Computing is an area of computing focused on developing computer technologies governed by the laws of **Quantum Mechanics*** to store data and perform computations.

- A Quantum Computer employs 'Qubits'.

- A Quantum Computation consists of a sequence of operations performed on Quantum States.

- The operations and states can be represented by matrices and vectors, respectively, and are required by the underlying physics (**Quantum Mechanics**) to follow the rules of **Linear Algebra**.

*It is a branch of physics dealing with the behaviour of matter and light at a subatomic level that are subject to the uncertainty principle.

Uses of Quantum Computing

- Healthcare
  - Research & Drug Development
  - Diagnostics
  - Treatment
- Financial Modelling
  - Automated, High Frequency Trading
- Cybersecurity & Cryptography
- Computational Chemistry
- Machine Learning & Deep Learning
- Marketing
  - Big Data Analytics
- Meteorology & Weather Forecasting
- Logistics Optimization

- Quantum computers are great for solving optimization problems from figuring out the best way to schedule flights at an airport to determining the best delivery routes for the FedEx truck.

- Quantum systems are exponentially powerful. A system of 500 particles has 2^{500} "Computing power".

- Quantum Computers provide a neat shortcut for solving a range of mathematical tasks known as NP-complete problems.

- For example, factorization is an exponential time task for classical computers.

- But Shor's quantum algorithm for factorization is a polynomial time algorithm. It has successfully broken RSA cryptosystem.

Motivations for Quantum Computation

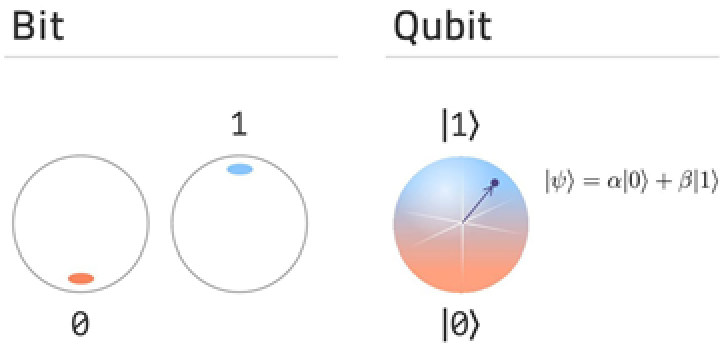
- Faster than light (?) communication.
- Highly parallel and efficient Quantum Algorithms.
- Quantum Cryptography and many more..

How do Quantum Computers process a lot of information at a faster rate?

- The next few sections of this PPT will answer the question!

Superposition

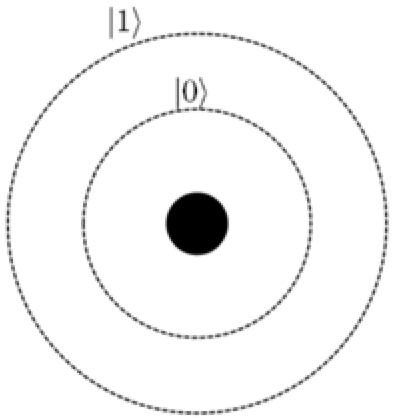
- In classical computation, bits are either represented with 0 or 1.
- In Quantum Computation, these bits are replaced by a superposition of both 0 and 1.
- If a quantum system can be in one of k states, it can also be in any linear superposition of those k states.
- Two level systems are called Qubits (k=2)
- $|0\rangle,\ |1\rangle,\ \dots,\ |k-1\rangle$ are called the basis states, the superposition is denoted as a linear combination of these basis.



$$\alpha_0\,|0\rangle + \alpha_1\,|1\rangle + \dots + \alpha_{k-1}\,|k-1\rangle$$

where, $\alpha_i \in \mathbf{C}$ and $\sum_i |\alpha_i|^2 = 1$

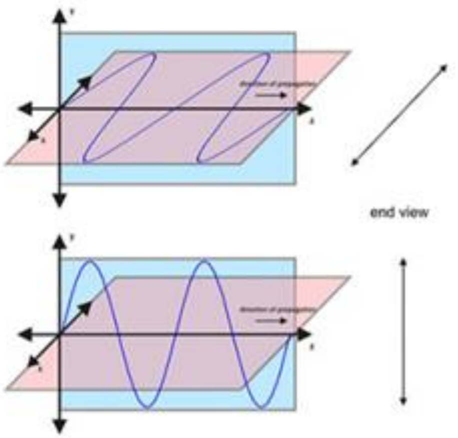
Qubits



Hydrogen Atom

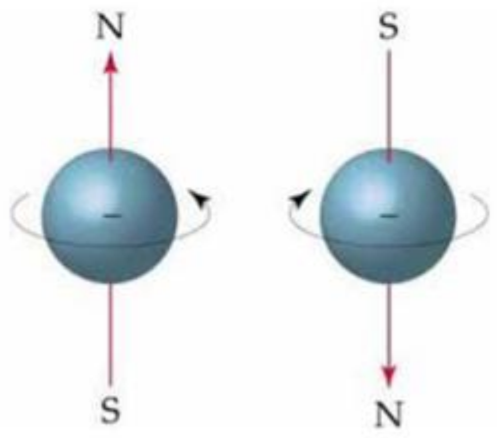
- In Quantum Computing, a Quantum Bit or a 'Qubit' is a building block or a basic unit of Quantum Information.
- We have various physical interpretations of a Qubit in Quantum Mechanics.
- Qubits are represented as $|0\rangle$ and $|1\rangle$ or their linear combination.
- Qubits have been created in the laboratory using photons, ions and certain sorts of atomic nuclei.
- A Hydrogen atom could be interpreted as a Qubit. An electron in its ground state, could be represented as $|0\rangle$ and in its first energy state as $|1\rangle$.
- The electron dwells in some linear superposition of these two energy levels. But during measurement, it can only be found in one of these energy states.

Physical Interpretation of Qubits



Photon Polarization: The orientation of electrical field oscillation is either horizontal or vertical.

Electron Spin: The electron spin is either up or down.

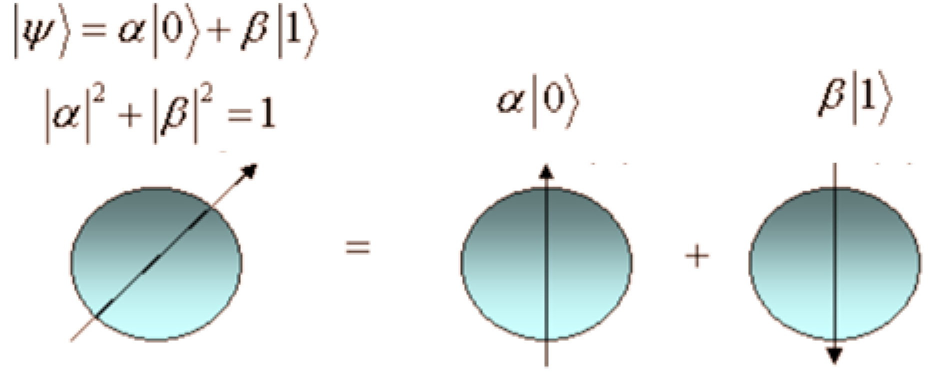


Representation of Qubits

Matrix Representation of $|0\rangle$ and $|1\rangle$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A Qubit is Mathematically represented as a Quantum State of the form $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$



Non-Determinism

Attempt at 3rd Postulate of Quantum Mechanics

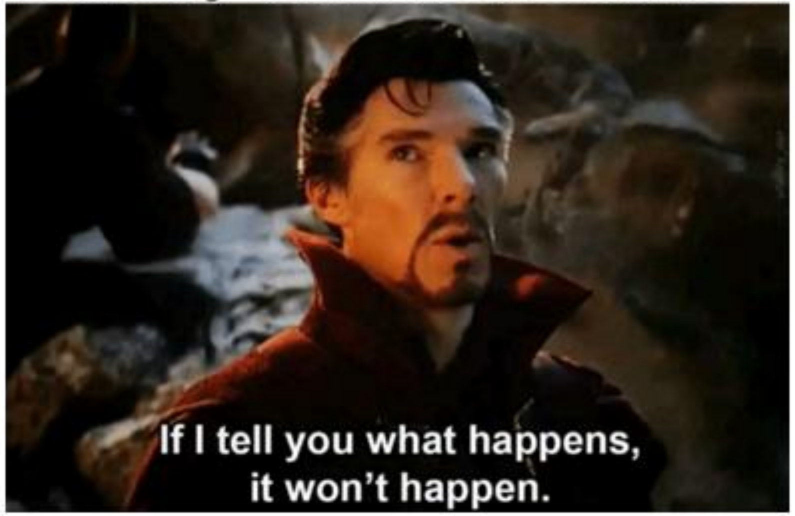
- Unlike classical physics, measurement in quantum mechanics is not deterministic. Even if we have the complete knowledge of a system, we can at most predict the probability of a certain outcome from a set of possible outcomes.

- If we have a quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then the probability of getting outcome $|0\rangle$ is $|\alpha|^2$ and that of $|1\rangle$ is $|\beta|^2$.

- After measurement, the state of the system collapses to either $|0\rangle$ or $|1\rangle$ with the said probability.

- However, after measurement if the new state of the system is $|0\rangle$ (say), then further measurements in the same basis gives outcome $|0\rangle$ with probability 1.

Basically Quantum mechanics:



Postulate 3

of Quantum Mechanics

- The postulate states that a quantum system stays in a superposition when it is not observed.
- When a measurement is done, it immediately collapses to one of its eigenstates*
- Hence, we can never observe what the original superposition of the system was. We merely observe the state after it collapses.
- This inherent ambiguity provides an excellent security in Quantum Cryptography.

*Eigenstate (physics) is a dynamic quantum mechanical state whose wave function is an eigenvector that corresponds to a physical quantity.

Imagine to have 500 qubits, then 2^{500} complex coefficients describe their state.

- How to store this state?

- 2^{500} is larger than the number of atoms in the universe.
- It is impossible in classical bits.
- This is also why it is hard to simulate quantum systems on classical computers.

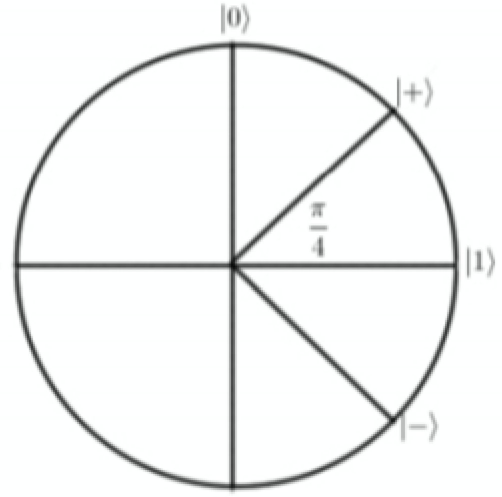
- A quantum computer would be much more efficient than a classical computer at simulating quantum systems.

- Superposition and Entanglement are what give quantum computers the ability to process so much more information so much faster.

Sign Basis

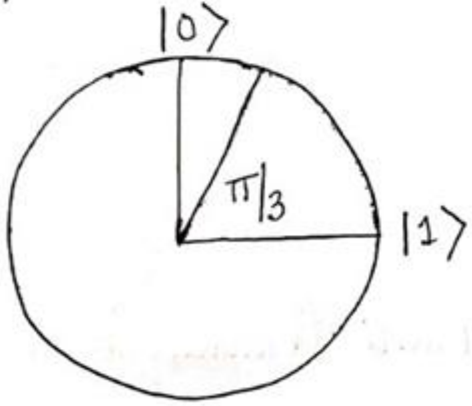
- $|0\rangle$ and $|1\rangle$ are called the bit basis. However, they aren't the only possible basis.
- We can have infinitely many orthonormal basis for a given space.
- Another basis, called the sign basis is denoted as $|+\rangle$ and $|-\rangle$.

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



Geometrical Representation of Basis

Ques. Check if the Quantum State represented by the given Geometrical Representation exists.



$$|\psi\rangle = \sqrt{3}/_2\,|0\rangle + \,^1/_2\,|1\rangle$$

→ Linear Combination of the bit basis $|0\rangle, |1\rangle$

→ $\left(\sqrt{3}/_2\right)^2 + \left(^1/_2\right)^2 = 1$

$\left(\sqrt{3}/_2\right)^2$ → probability of $|0\rangle$

$\left(^1/_2\right)^2$ → probability of $|1\rangle$

The sum of probabilities = 1.

Hence, the Quantum State exists.

Vector Space in Quantum Computing

A Vector Space consists of Quantum States (Vectors) – $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ together with a set of Scalars – a, b, c (belonging to Complex Numbers). It is closed under two operations:

- Vector Addition
- Scalar Multiplication

Vector Addition

- Closure Property

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

- Commutative Property

$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$

- Associative Property

$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$$

- Additive Identity

$$|\alpha\rangle + |0\rangle = |\alpha\rangle\,, \qquad \forall\,|\alpha\rangle$$

- Additive Inverse

$$|\alpha\rangle + |-\alpha\rangle = |0\rangle$$

Scalar Multiplication

- Closure Property

$$a|\alpha\rangle = |\gamma\rangle$$

- Distributive w.r.t Vector Addition

$$a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle$$

- Distributive w.r.t Scalar Addition

$$(a+b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

- Associative w.r.t Ordinary Scalar Multiplication

$$a(b|\alpha\rangle) = (ab)|\alpha\rangle$$

- Multiplication with Scalars 0 and 1

$$0|\alpha\rangle = |0\rangle; \quad 1|\alpha\rangle = |\alpha\rangle$$

Basis

- Linear Combination of Quantum States $|\alpha\rangle,\ |\beta\rangle,\ |\gamma\rangle$ is in the form of $|\alpha\rangle + |\beta\rangle + |\gamma\rangle + \ldots$

- A Quantum State $|\lambda\rangle$ is said to be linearly independent of the set of Quantum States $|\alpha\rangle,\ |\beta\rangle,\ |\gamma\rangle$, if it cannot be written as a Linear Combination of them.

- If every Quantum State can be written as a Linear Combination of this set, then the collection of Quantum States said to Span the space.

- A set of Linearly Independent Quantum States that span the space is called a Basis.

Inner Product

- An inner product is a function which takes two Quantum States as input and gives a Complex Number as output.

- The Dual (or Complex Conjugate) of any Quantum State $|\alpha\rangle$ (ket notation – column vector) is $\langle\alpha|$ (bra notation – row vector)

$$|\alpha\rangle^* = \langle\alpha|$$

- The inner product of two Quantum States $(|\alpha\rangle, |\beta\rangle)$ is written as $\langle\alpha|\beta\rangle$

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$$

$$\langle\alpha|\alpha\rangle \geqslant 0, \quad and\ \langle\alpha|\alpha\rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle$$

$$\langle\alpha|\left(b|\beta\rangle + c|\gamma\rangle\right) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$$

Inner Product Space

- A Vector Space with an Inner Product is called Inner Product Space.
- $\mathbf{C}^n$ has an inner product defined by

$$\langle \alpha | \beta \rangle \equiv \sum_i a_i^* b_i = [a_1^* \ldots a_n^*] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

where, $|\alpha\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $\quad |\beta\rangle = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Outer Product

The inner product of $|x\rangle$ and $|y\rangle$ is $\langle x|y\rangle \in \mathbb{C}$.

The outer product of $|x\rangle$ and $|y\rangle$ is $|x\rangle\langle y|$

The outer product $|x\rangle\langle y|$, is the tensor product $|x\rangle \otimes \langle y|$.

$$
\begin{aligned}
|0\rangle\langle 0| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
|0\rangle\langle 1| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
|1\rangle\langle 0| &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
|1\rangle\langle 0| &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Outer Product

- 1-Bit Transformation (linear combination of the outer products of vectors)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(|0\rangle\langle 0|) + b(|0\rangle\langle 1|) + c(|1\rangle\langle 0|) + d(|1\rangle\langle 1|).$$

- Property of Outer Product

$$\begin{aligned} (|x\rangle\langle x|)\,|y\rangle &= |x\rangle\langle x|y\rangle\,, \\ &= \langle x|y\rangle\,|x\rangle\,. \end{aligned}$$

- Optimization of Quantum State Expressions plays a crucial role in Quantum Computation and Quantum Information Processing.

Ques. Simplify the expression of the quantum state –

$$(|0\rangle\langle 0|)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left((|0\rangle\langle 0|)|0\rangle + (|0\rangle\langle 0|)|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0|0\rangle + |0\rangle\langle 0|1\rangle\right)$$

$$\langle 0|0\rangle \rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1} = [1]_{1\times 1}$$

$$\langle 0|1\rangle \rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1} = [0]_{1\times 1}$$

as, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}[1]_{1\times 1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1} = |0\rangle \quad \Bigg| \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}[0]_{1\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}$$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}(|0\rangle)$$

Orthonormal Set

- Inner product of any Quantum State with itself gives a non-negative number – its square root of is real which is called norm. (Also termed as Vector Length)

$$\| \alpha \| = \sqrt{\langle \alpha | \alpha \rangle}$$

- A Quantum State with Norm 1 (Unit Vector) is said to be Normalized.

- Two Quantum States with Inner Product zero are said to be Orthogonal.

$$\langle \alpha | \alpha \rangle = 0$$

- A Mutual Collection of Orthogonal Normalized Quantum States is called an Orthonormal Set.

$$\langle \alpha_i | \alpha_j \rangle = \delta_{ij}, \quad where \quad \delta_{ij} \begin{cases} = 0, & \mathrm{i} = \mathrm{j} \\ \neq 0, & \mathrm{i} \neq \mathrm{j} \end{cases}$$

Pauli Matrices

- Pauli Matrices are named after the physicist Wolfgang Pauli.
- These are a set of 2 x 2 complex matrices which are Hermitian and Unitary.

$$\sigma_0 \equiv \mathbb{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues of Y

We have

$$
\begin{aligned}
\det\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \lambda I\right) &= 0 \\
\Rightarrow \det(\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix}) &= 0 \\
\Rightarrow \lambda^2 - 1 &= 0 \\
\Rightarrow \lambda &= \pm 1.
\end{aligned}
$$

In fact all Pauli matrices have $\lambda = \pm 1$.

Eigenvectors of Y

So we have

$$\begin{aligned} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \pm \begin{bmatrix} a \\ b \end{bmatrix} \\ \Rightarrow \begin{bmatrix} -ib \\ ia \end{bmatrix} &= \pm \begin{bmatrix} a \\ b \end{bmatrix} \\ \Rightarrow \ -ib = \pm a \quad \text{and} & \quad ia = \pm b \\ \Rightarrow a^2 &= -b^2 \\ \Rightarrow a &= \pm ib. \end{aligned}$$

We also have $|a|^2 + |b|^2 = 1$. So the eigenvectors of Y are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \pm i \frac{1}{\sqrt{2}} \end{bmatrix}$

- Similarly, Eigenvectors can be computed for the remaining Pauli Matrices – X, Z

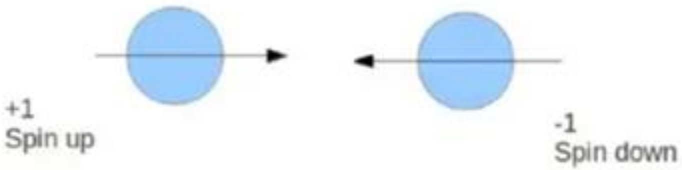
Properties of Pauli Matrices

- $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$
- $det(\sigma_i) = -1$
- $Tr(\sigma_i) = 0$
- Each Pauli Matrix has eigenvalues +1 and -1
- Normalised Eigenvectors are

$$\psi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \psi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\psi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \psi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$
$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli Transformations

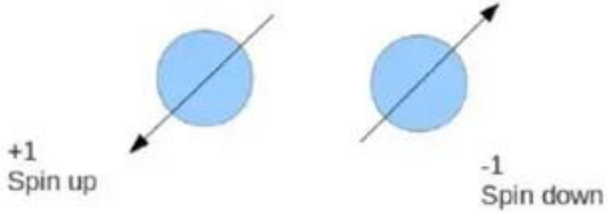
- The Pauli-X gate is the quantum equivalent of **NOT** gate. It maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.



$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \beta|0\rangle + \alpha|1\rangle$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

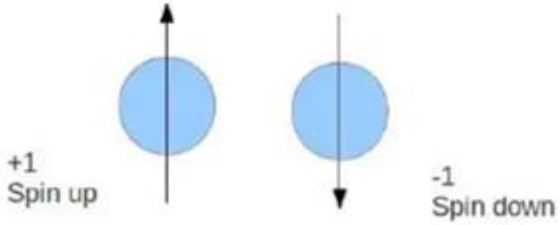
- The Pauli-Y gate maps $|0\rangle$ to $i\,|1\rangle$ and $|1\rangle$ to $-i\,|0\rangle$.



$$\alpha|0\rangle + \beta|1\rangle \longrightarrow -i\beta|0\rangle + i\alpha|1\rangle$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}$$

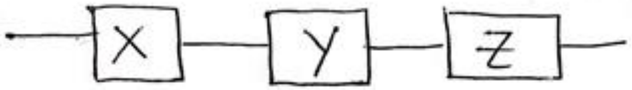
- The Pauli-Z gate leaves the basis state $|0\rangle$ unchanged and maps $|1\rangle$ to -$|1\rangle$.



$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

Ques. A Particle can be in state up and down, with the amount in up = $3i/5$ and the amount in down = $4/5$.
This particle passes through the given gate:



Calculate the amount in up and down of the resulting particle.

$$|\psi\rangle = \frac{3}{5}i|\uparrow\rangle + \frac{4}{5}|\downarrow\rangle$$

$$\rightarrow \quad |\psi\rangle = \frac{3}{5}i|0\rangle + \frac{4}{5}|1\rangle$$

X Gate $\rightarrow$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3i/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3i/5 \end{bmatrix}$$

Y Gate $\rightarrow$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 4/5 \\ 3i/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4i/5 \end{bmatrix}$$

Z Gate $\rightarrow$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4i/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4i/5 \end{bmatrix}$$

$$|\psi'\rangle = \frac{3}{5}|0\rangle - \frac{4i}{5}|1\rangle$$

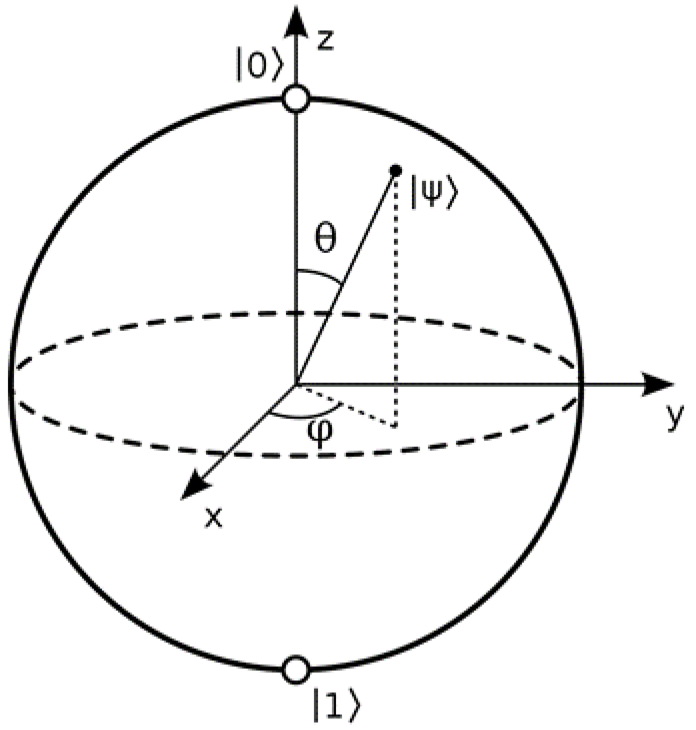
$$\rightarrow |\psi'\rangle = \frac{3}{5}|\uparrow\rangle - \frac{4}{5}i|\downarrow\rangle$$

Amount in up = $3/5$

Amount in down = $-4i/5$

Bloch Sphere

- In quantum mechanics and computing, the Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit), named after the physicist Felix Bloch.



Rotation Operators on Bloch Sphere

- When the Pauli X, Y and Z matrices are exponentiated, they give rise to the rotation operators, which rotate the Bloch vector of the Quantum State about the ˆx, ˆy and ˆz axes, by a given angle θ:

$$
\begin{aligned}
R_x(\theta) &\equiv e^{-i\frac{\theta}{2}X} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\
R_y(\theta) &\equiv e^{-i\frac{\theta}{2}Y} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\
R_z(\theta) &\equiv e^{-i\frac{\theta}{2}Z} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}
\end{aligned}
$$

Ques Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, it is rotated by $\pi/3$ about the x axis and $\pi/2$ about the y axis. Find the resultant Quantum State

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$\rightarrow$ rotated along x axis by $\pi/3$.

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = R_x(\pi/3).A \text{, where } A = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\pi/6 & -i\sin\pi/6 \\ -i\sin\pi/6 & \cos\pi/6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \cdot \alpha - i/2 \cdot \beta \\ -i/2 \cdot \alpha + \sqrt{3}/2\,\beta. \end{bmatrix}$$

After rotation about x axis,

$$|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle.$$

$$(\sqrt{3}/2 \cdot \alpha - i/2 \cdot \beta)|0\rangle + (-i\alpha/2 + \sqrt{3}/2\,\beta)|1\rangle$$

$$|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$$

$\longrightarrow$ rotated along y axis by $\pi/2$

$$\begin{bmatrix} \alpha'' \\ \beta'' \end{bmatrix} = R_y(\pi/2).A' \text{, where } A' = \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}$$

$$= \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} \begin{bmatrix} \sqrt{3}\alpha/2 - i\beta/2 \\ -i\alpha/2 + \sqrt{3}\beta/2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\sqrt{3}\alpha - \sqrt{3}\beta + i(\alpha - \beta)}{2\sqrt{2}} \\ \dfrac{\sqrt{3}\alpha + \sqrt{3}\beta - i(\alpha + \beta)}{2\sqrt{2}.} \end{bmatrix} = \begin{bmatrix} \alpha'' \\ \beta'' \end{bmatrix}$$

Resultant Quantum State $\rightarrow$ $|\psi''\rangle = \alpha''|0\rangle + \beta''|1\rangle$

$$\alpha'' = (\sqrt{3}\alpha - \sqrt{3}\beta + i(\alpha - \beta))/2\sqrt{2}$$

$$\beta'' = (\sqrt{3}\alpha + \sqrt{3}\beta - i(\alpha + \beta))/2\sqrt{2}$$

Quantum Entanglement

- The term ‘Entanglement’ means that the two members of a pair exist in a single quantum state.

- Changing the state of one of the qubits will instantaneously change the state of the other one in a predictable fashion.

- This property is used as a resource in Quantum Computing such as
    - Super dense coding
    - Quantum Teleportation
    - Error Correction

We write product states $|a\rangle \otimes |b\rangle$ in the abbreviated form $|ab\rangle$. The computational (standard) basis of the two qubit system is formed by the four states

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle.$$

Ques. Represent $|\psi\rangle = 1/\sqrt{2}\left(|00\rangle + |11\rangle\right)$ as a product state.

$$|\psi\rangle = 1/\sqrt{2}\left(|00\rangle + |11\rangle\right)$$

Let's try to write it in the form of a Product State.

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \; ; \; |\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$|\psi_1\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

$$|\psi\rangle = |\psi_1\psi_2\rangle \Rightarrow \alpha_1\alpha_2 = 1/\sqrt{2} \text{ and } \beta_1\beta_2 = 1/\sqrt{2}.$$

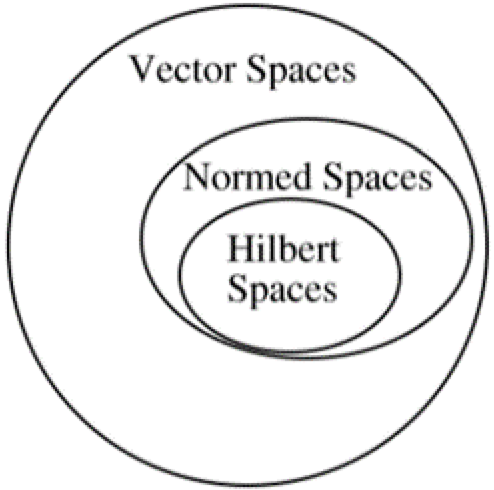
but $\Rightarrow \alpha_1\beta_2 \neq 0$ and $\alpha_2\beta_1 \neq 0$!

Hence, this is not possible, as this quantum state is entangled!

Hilbert Space

- ‘Wave functions live in the Hilbert Space.’
- Hilbert space which contains the "wavefunctions" that stand for the possible states of the system.
- Absolute value of each wave function is interpreted as being a probability distribution function.

Hilbert Space



- The Mathematical Concept of Hilbert Space has been named after David Hilbert.
- Hilbert Space is an abstract vector space with inner product defined in it, that allow the length and angle to be measured.
- Hilbert Space must be complete.
- It is an Infinite Dimensional Inner Product Space
- A Hilbert Space H of $d = 3$ is referred to as a Qutrit, $d = 4$ is sometimes called a Ququart, and the generic term for any $d > 2$ is Qudit.

Unitary Transformation

- A unitary transformation $U : H \to H$ is an isomorphism, where H an inner product space (Hilbert space).
- So U is a bijection that preserves inner product.
- In our notation $U : |x\rangle \mapsto |y\rangle$ and if $|x\rangle, |x'\rangle \in H$, then $< |x\rangle, |x'\rangle > = < |Ux\rangle, |Ux'\rangle >$.

Observables

- A system 'Observable' is a measurable operator, where the property of the system state can be determined by some sequence of physical operations.

- In Quantum Mechanics, each 'Dimensional Variable' – Position, Translational Momentum, Orbital Angular Momentum, Energy, Spin, etc. is associated with a Hermitian Operator that acts on the state of the quantum system, whose Eigenvalues correspond to the possible values of the dynamical variable.

- Let $|\alpha\rangle$ be an Eigenvector of the Observable 'A' with Eigenvalue a, and exists in a D-Dimensional Hilbert Space, then

$$\mathbf{A}\,|\alpha\rangle = a\,|\alpha\rangle$$

- This equation states that if a measurement of the Observable A is made while the system is in $|\alpha\rangle$ Quantum State, then the observed value of that measurement must return the eigenvalue a with certainty.

Hermitian Operator

- Self-Adjoint Operator
- Hermitian Matrix is a square matrix with complex entries that is equal to its own conjugate transpose.

$$\mathbf{A} = (\mathbf{A}^T)^* \text{ or } \mathbf{A} = \mathbf{A}^\dagger$$

- Eigenvalues of Hermitian operators are real.
- This is important because the eigenvalues correspond to physical properties of a system, which cannot be imaginary or complex.

1st Postulate: State Space

of Quantum Mechanics

- Associated to any isolated system is a Hilbert Space called the state space. The system is completely defined by the state vector, which is a unit vector in the state space.

Let us consider a quantum state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Since a state is a unit vector, the norm of the vector must be unity, or in mathematical notation,

$$\langle\psi|\psi\rangle = 1$$

Hence, the condition that $|\psi\rangle$ is a unit vector is equivalent to

$$|\alpha|^2 + |\beta|^2 = 1$$

This condition is called the *normalization condition* of the state vector.

2nd Postulate: Evolution

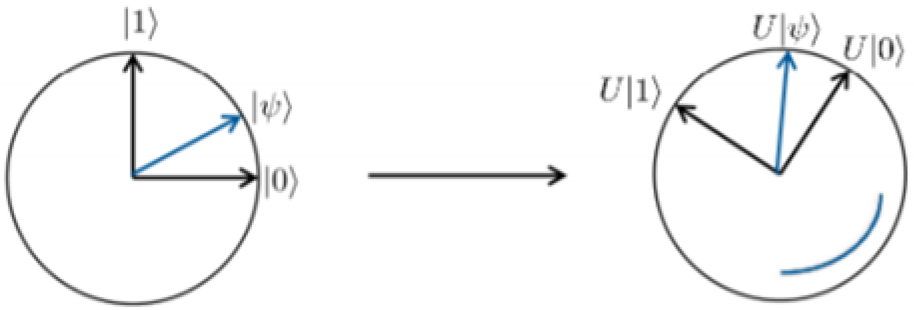
of Quantum Mechanics

- The evolution of a closed Quantum System is described by a Unitary Transformation.

That is, if $|\psi_1\rangle$ is the state of the system at time t_1 and $|\psi_2\rangle$ at time t_2, then:

$$|\psi_2\rangle = U(t_1, t_2)\,|\psi_1\rangle$$

where $U(t_1, t_2)$ is a unitary operator.



Quantum systems evolve by the rotation of the Hilbert Space

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Thank You!

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