Numerical Solution of Ordinary Differential Equations (ODE): Initial Value Problem (IVP)

Lecture-1

Specific aims

- **Applications**
- **Introduction of IVP**
- **Discuss about the solution of IVP using Taylor series, Euler's and modified Euler's method**
- **Examples**
- **Multiple questions**
- **Exercises**

Applications

- Science and Engineering

Introduction of IVP

Problems in which all the initial conditions are specified at one point only are called ***initial value problem*** **(IVP).**

To describe various numerical methods for the solution of ordinary differential equations, we consider the general first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.

Taylor Series Method

Consider the differential equation with the initial condition

$$\frac{dy}{dx} = f(x, y); y(x_0) = y_0 \qquad \text{..........................(1)}$$

The solution of the equation is a function of *x*. The Taylor series expansion of $y(x)$ about x_0 is

$$y(x_0 + h) = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{3!} {y_0}^{(iv)} + \cdots$$

..........................(2)

Or $$y(x) = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{3!} {y_0}^{(iv)} + \cdots$$

..........................(3) [around x_0=0]

Where ${y_0}^{(n)}$ is the value of $\frac{d^n y}{dx^n}$ at $x = x_0$.

The values of the derivatives can be found by differentiating repeatedly and substituting $x = x_0$ and $y = y_0$.

The method can also be used for higher order differential equations.

Euler's Method

Consider the first order and first degree ordinary differential equation with the initial condition

$$\frac{dy}{dx} = f(x, y);\ \mathrm{y}(\mathrm{x}_0)=\mathrm{y}_0 \qquad \text{..........................(4)}$$

From Taylor series expansion of $y(x_0 + h)$ given in Eq. (2) and neglecting terms containing h^2 and higher powers of h, we have

$$y(x_0 + h) \approx y(x_0) + hy'(x_0)$$

or equivalently $y_1 = y_0 + hf(x_0, y_0)$

This is Euler's formula and the order of the error is $O(h^2)$.

Alternatively, integrating Eq.(4) from x_0 to x_1 with respect to x, we have

$$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y)dx$$

$$\boxed{\frac{dy}{dx} = f(x, y)}$$

or $$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx$$

Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \leq x \leq x_1$, we have

$$y_1 = y_0 + hf(x_0, y_0) \text{ where } h = x_1 - x_0$$

The process can be continued for the next interval by taking x_1 and y_1 as the starting values. In general, iteration formula be

$$y_{n+1} = y_n + hf(x_n, y_n); n = 0, 1, 2, \ldots$$

..........................(5)

Modified Euler's Method

Consider the first order and first degree ordinary differential equation with the initial condition

$$\frac{dy}{dx} = f(x,y);\; y(x_0)=y_0 \quad \text{..........................(6)}$$

Integrating the ODE (3) from x_0 to x_1, we have

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)dx$$

Using trapezoidal rule to the integration we obtain the modified Euler's formula

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$$

An iterative formula to the above equation can be taken as

$$y_1^{(n+1)} = y_n + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

The iteration can be started by choosing $y_1^{(0)}$ from the Euler's formula

$$y_1^{(0)} \approx y_1 = y_0 + hf(x_0, y_0)$$

This procedure is known as one-step ***predictor-corrector*** method.

General form of Modified Euler's method:

$$y_r^{(n+1)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n)})\right] \quad \text{..........................(7)}$$

Examples

Question 1#: From the Taylor's series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies $y' = x - y^2$, where $y = 1$ at $x = 0$.

Solution:

Here, $x_0 = 0$ and $y_0 = 1$. Taylor's series for $y(x)$ is given in Eq. (3) as follows:

$$y(x) = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{3!} y_0^{(iv)} + \cdot \cdot \cdot$$

Or $$y(x) = 1 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{3!} y_0^{(iv)} + \cdot \cdot \cdot$$

[Since $y_0 = 1$]

The derivatives $y_0', y_0'', y_0''', \ldots\ldots$ are obtained thus:

$$y'(x) = x - y^2 \qquad y_0' = 0 - (1)^2 = -1$$

$$y''(x) = 1 - 2yy' \qquad y_0'' = 1 - 2(1)(-1) = 3$$

$$y'''(x) = -2yy'' - 2(y')^2 \qquad y_0''' = -2(1)(3) - 2(-1)^2 = -8$$

$$y'^{v}(x) = -2yy''' - 2y'y'' - 4y'y'' = -2yy''' - 6y'y'' \qquad y_0''' = -2(1)(-8) - 6(-1)(3) = 34$$

$$y^{v}(x) = -2yy'^{v} - 2y'y''' - 6y'y''' - 6(y'')^2 = -2yy'^{v} - 8y'y''' - 6(y'')^2$$

$$y_0^{v} = -2(1)(34) - 8(-1)(-8) - 6(3)^2 = -186$$

Using these values, the Taylor series becomes

$$y(x) = 1 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{3!}y_0^{(iv)} + \cdot\cdot\cdot$$

$$= 1 + x(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \frac{x^4}{4!}(34) + \frac{x^5}{5!}(-186) + \ldots.$$

$$y(x) = 1 - x + \frac{3}{2!}x^2 - \frac{8}{3!}x^3 + \frac{34}{4!}x^4 - \frac{186}{5!}x^5 + \ldots.$$

To obtain the value of $y(0.1)$ correct to four decimal places, it is found that the terms up to x^4 should be considered, we have

$$y(0.1) = 1 - (0.1) + \frac{3}{2!}(0.1)^2 - \frac{8}{3!}(0.1)^3 + \frac{34}{4!}(0.1)^4$$

$$= 0.9135$$

Answer: 0.9135

Question 2#: Given that $y' = 2xy^2 - y$, where $y = 1$ at $x = 0$.

Estimate the values of $y(0.2)$ using Euler's method with step size $h = 0.1$.

Solution:

Here $f(x, y) = 2xy^2 - y$

and $x_0 = 0$, $y_0 = 1$, h=0.1.

Euler's method: $y_{n+1} = y_n + hf(x_n, y_n); n = 0, 1, 2, ...$

[Using Eq. (5)]

For n=0, $x_0 = 0$, $y_0 = 1, h = 0.1$,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + h[2x_0(y_0)^2 - y_0]$$

$$y_1 = 1 + (0.1)[2(0)(1)^2 - 1]$$

$$y_1 = 0.9$$

For n=1, $x_1 = 0 + 0.1 = 0.1,\ y_1 = 0.9,$ **[Since $\boldsymbol{x_1 = x_0 + h}$]**

General form of Euler Method become

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = y_1 + h[2x_1(y_1)^2 - y_1]$$

$$y_2 = 0.9 + (0.1)[2(0.1)(0.9)^2 - 0.9]$$

$$y_2 = 0.9 - 0.0738$$

$$y_2 = 0.8262$$

Answer: 0.8262

Question 3#: Solve by Modified Euler's method the following differential equation for $\mathrm{x} = 0.1$ with step size $\mathrm{h}=0.05$
$y' = y + x^2$, where $y = 1$ at $x = 0$.

Solution: Here $f(x, y) = y + x^2$

and $x_0 = 0$, $y_0 = 1$, h=0.05, $x_0 = 0.05$.

Modified Euler's method:

$$y_r^{(n+1)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n)}) \right]$$ **[Using Eq. (7)]**

$$y_1^{(0)} \approx y_1 = y_0 + hf(x_0, y_0)$$ **[r=1]**

For n=0, r=1, $x_0 = 0$, $y_0 = 1, h = 0.05, x_1 = 0.05,$

$$y_1^{(0)} \approx y_1 = y_0 + hf(x_0, y_0) = y_0 + h[x_0^2 + y_0] = 1 + 0.05[0+1] = 1.05$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)}) \right]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$ [For n=0, r=1]

$$= y_0 + \frac{h}{2}\left[x_0^2 + y_0 + x_1^2 + y_1^{(0)} \right]$$

$$= 1 + \frac{0.05}{2}\left[0 + 1 + (0.05)^2 + 1.05 \right]$$

$$= 1.05131$$

Again, $y_1^{(n+1)} = y_n + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$ [For n=1]

$$= y_0 + \frac{h}{2}\left[x_0^2 + y_0 + x_1^2 + y_1^{(1)}\right]$$

$$= 1 + \frac{0.05}{2}\left[0 + 1 + (0.05)^2 + 1.05131\right]$$

$$\boxed{= 1.05135}$$

And $y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$ [For n=2]

$$= y_0 + \frac{h}{2}\left[x_0^2 + y_0 + x_1^2 + y_1^{(2)}\right]$$

$$= 1 + \frac{0.05}{2}\left[0 + 1 + (0.05)^2 + 1.05135\right]$$

$$\boxed{= 1.05135}$$

It is clear that $y_1^{(2)} = y_1^{(3)} = 1.05135$. $\therefore y(0.05) = 1.05135$

Now we can take $x_1 = 0.05$, $y_1 = 1.05135, h = 0.05, x_2 = 0.1$.

$$y_2^{(0)} \approx y_2 = y_1 + hf(x_1, y_1) = y_1 + h[x_1^2 + y_1]$$ [For r=2]

$$y_2^{(0)} \approx y_2 = 1.05135 + 0.05[(0.05)^2 + 1.05135] = 1.10404$$

$$y_2^{(n+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(n)}) \right]$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)}) \right]$$ [For n=0]

$$= y_1 + \frac{h}{2}\left[x_1^2 + y_1 + x_2^2 + y_2^{(0)} \right]$$

$$= 1.05135 + \frac{0.05}{2}\left[(0.05)^2 + 1.05135 + (0.1)^2 + 1.10404 \right]$$

$$= 1.10555$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)}) \right]$$

$$= y_1 + \frac{h}{2}\left[x_1^2 + y_1 + x_2^2 + y_2^{(1)} \right]$$

[For n=1]

$$= 1.05135 + \frac{0.05}{2}\left[(0.05)^2 + 1.05135 + (0.1)^2 + 1.10555 \right]$$

$$= 1.10558$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)}) \right]$$

[For n=2]

$$= y_1 + \frac{h}{2}\left[x_1^2 + y_1 + x_2^2 + y_2^{(2)} \right]$$

$$= 1.05135 + \frac{0.05}{2}\left[(0.05)^2 + 1.05135 + (0.1)^2 + 1.10558 \right]$$

$$\boxed{= 1.10559}$$

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)}) \right]$$ [For n=3]

$$= y_1 + \frac{h}{2}\left[x_1^2 + y_1 + x_2^2 + y_2^{(3)} \right]$$

$$= 1.05135 + \frac{0.05}{2}\left[(0.05)^2 + 1.05135 + (0.1)^2 + 1.10559 \right]$$

$$\boxed{= 1.10559}$$

It is clear that $y_2^{(4)} = y_2^{(3)} = 1.10559 \;\therefore\; y(0.1) = 1.10559$

Answer: 1.10559

Outcomes

- **Solved problems numerically to obtain approximate solutions of ODE by using Taylor series, Euler's and Modified Euler's Methods.**

Multiple questions:

S.No.	Questions
1	Which rule is used for getting Modified Euler's method- (a) Newton-Raphson method, (b) Trapezoidal rule, (c) Fixed point method
2	Taylor series can be expresses as follows: (a) $y(x) = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \cdots$ (b) $y(x) = y_0 + x'y_0' + \frac{x^2}{2!}y_0'' + \cdots$
3	Euler method can be written by- (a) $yy_{n+1} = y_n + h\, f(x_n, y_n); n = 0,1,2,..$ (b) $y_{n+1} = y_n + h\, f(x_n, y_n); n = 0,1,2,..$ (c) $y_{n+1} = y_n + h\, f(x_n, y_n); n = 1,2,..$
4	Which one could be Modified Euler method? (a) $y_r{}^{(n+1)} = y_{r-1} + \frac{h}{2}[f(x_{r-1}, y_{r-1}) + f(x_r, y_r{}^{(n)})]$, (b) None of them (c) $y_r{}^{(n+1)} = y_r + \frac{h}{2}[f(x_{r-1}, y_{r-1}) + f(x_r, y_r{}^{(n)})]$,

Try to do yourself

Exercise 1: Given the initial value problem $y' = y^2 + 1$ with $y(0) = 1$. Estimate the values of $y(0.1)$ using Modified Euler's method with step size $h = 0.05$.

Exercise 3: Given the initial value problem $y' = 2x^2 - y + 3y^2$ with $y(2) = 0.5$. Estimate the values of $y(2.2)$ using Euler's method with step size $h = 0.2$.

Reference

[1] Applied Numerical Methods With Matlab for Engineers and Scientists (Steven C.Chapra).