Spline Interpolation: Linear Spline Interpolation (LSI)

Lecture-1

Objectives

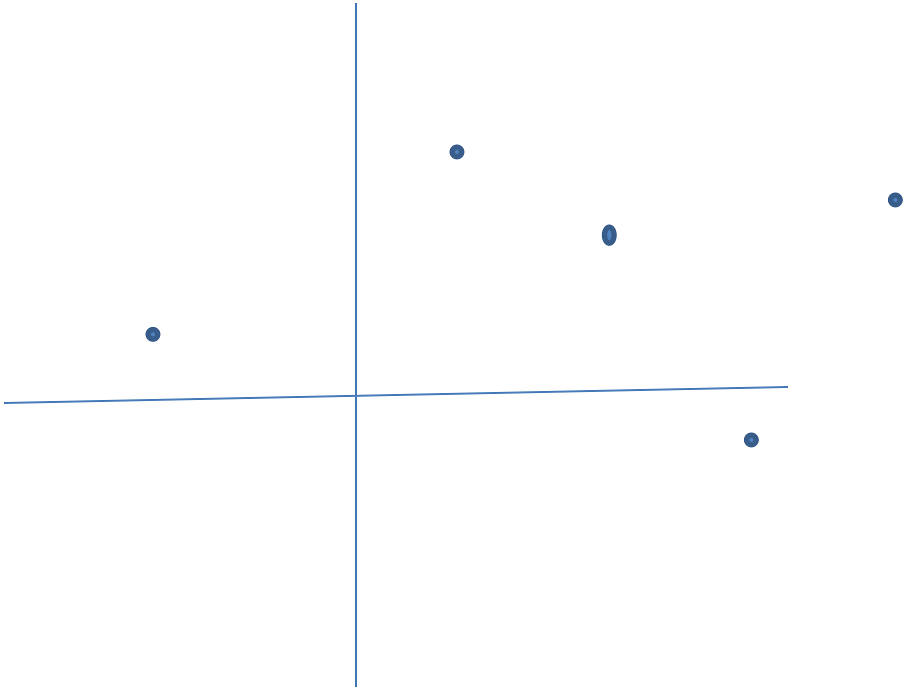
- **Plotting line or curve using providing data sets**
- **Analyze line or curve**

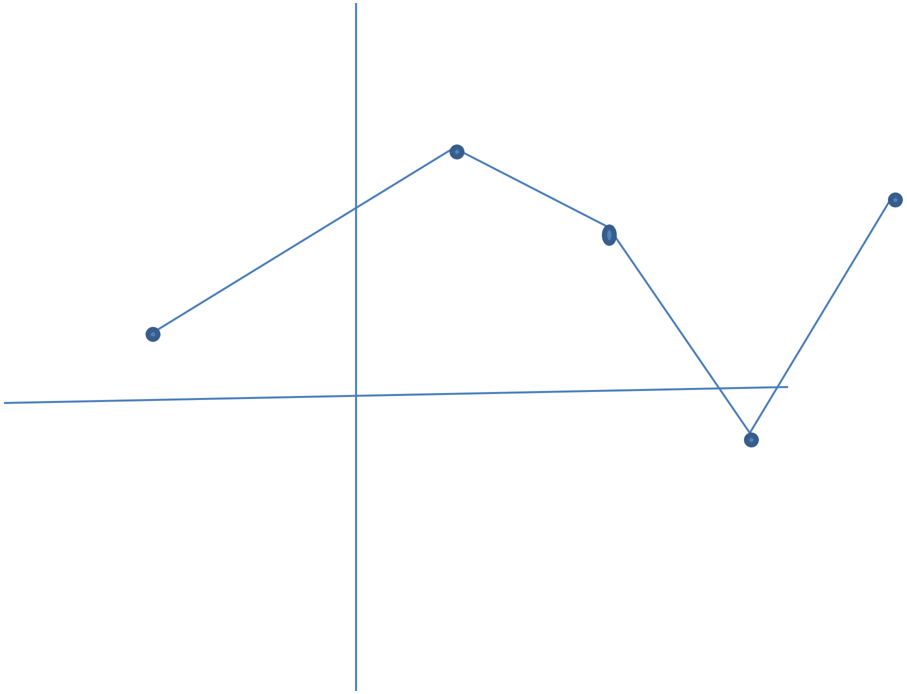
Applications

- **Data analysis numerically in Mathematics**

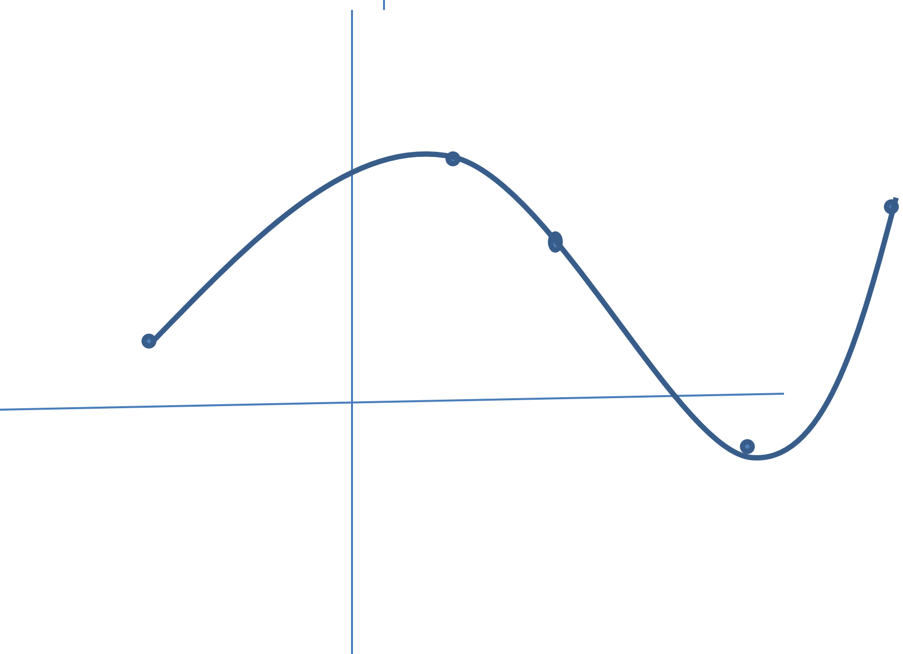
Spline Interpolation

In the mathematical field of numerical analysis, **spline interpolation** is a form of interpolation where the interpolant is a special type of piecewise polynomial called a spline. It is often preferred over polynomial interpolation because the interpolation error can be made small even when using low degree polynomials for the spline.





Linear spline



Cubic spline

x	y
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4
x_5	y_5

Definition of Spline Interpolation

Spline interpolation function is a piecewise polynomial function joined together with certain conditions satisfied by them. A function *f(x)* of the form

$$f(x) = \begin{cases} f_1(x), & x_1 \le x < x_2 \\ f_2(x), & x_2 \le x < x_3 \\ \vdots & \\ f_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases}$$

is called a **spline** of degree *m* if

(i) the domain of $f(x)$ is the interval $[x_1, x_n]$

(ii) $f(x), f'(x), f''(x) \ldots\ f^{(m-1)}$ are all continuous functions on the interval $[x_1, x_n]$

(iii) $f(x)$ is a polynomial of degree less than or equal to *m* on each subinterval $[x_k,\ \ x_{k+1}]$, $k = 1, 2,\ \cdots, n$.

Spline Interpolation

- ❑ Linear Spline Interpolation (LSI)
- ❑ Cubic Spline Interpolation(CSI)

$$f_k(x) = a_k(x - x_k) + b_k$$
$$y_k = a_k(x_k - x_k) + b_k$$
$$b_k = y_k$$
$$y_{k+1} = a_k(x_{k+1} - x_k) + y_k$$
$$a_k = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$$

- **Linear Spline Interpolation (LSI)**

For a linear spline through (x_k, y_k) we may take $f_k(x)$ is of the form

$$f_k(x) = a_k(x - x_k) + b_k, \qquad \text{for} \quad x_k \le x \le x_{k+1}$$

Since the line passes through (x_k, y_k) and (x_{k+1}, y_{k+1}) we have

$$b_k = y_k$$

and

$$a_k = \frac{y_{k+1} - y_k}{x_{k+1} - x_k} = \frac{\Delta y_k}{h_k}$$

where $\Delta y_k = y_{k+1} - y_k$ and $h_k = x_{k+1} - x_k$.

The resulting linear spline curve $f_k(x)$ in $[x_k,\ x_{k+1}]$ can be written as

$$f_k(x) = y_k + \frac{\Delta y_k}{h_k}(x - x_k), \qquad (k = 1, 2, \dots, n-1).$$

Example

Find the linear spline for the following data set

X	**–1**	**1**	**2**	**5**
Y	**2.2**	**3.5**	**5.4**	**1.5**

Hence estimate the value of $y(1.5)$.

$$f_k(x) = y_k + \frac{\Delta y_k}{h_k}(x - x_k)$$

x	y	$\frac{\Delta y_k}{h_k}$
-1	2.2	
1	3.5	0.65
2	5.4	1.9
5	1.5	-1.3

Solution:

Linear spline functions in different intervals are

$$f_1(x) = 2.2 + 0.65(x+1), \qquad -1 \le x \le 1$$

$$f_2(x) = 3.5 + 1.9(x-1), \qquad 1 \le x \le 2$$

$$f_3(x) = 5.4 - 1.3(x-2), \qquad 2 \le x \le 5$$

Linear spline function is

$$f(x) = \begin{cases} 2.2 + 0.65(x+1), & -1 \le x \le 1 \\ 3.5 + 1.9(x-1), & 1 \le x \le 2 \\ 5.4 - 1.3(x-2) & 2 \le x \le 5 \end{cases}$$

The value $x = 1.5$ is in $1 \le x \le 2$. Thus

$$y(1.5) = 3.5 + 1.9(1.5 - 1) = 4.45.$$

Linear spline is continuous at the common point. When $x = 1$

$f_1(1) = 2.2 + 0.65(2) = 3.5$ and

$f_2(1) = 3.5.$ So $f(x)$ is continuous at x=1

Outcomes

- ❑ **Numerically get value of a function at specific value belongs to the given data sets by using LSI.**

Try to do yourself

Exercise 1: In a chemical reaction the concentration level y of the product at time ***t*** (minute) was measured every half hour. The following results were found:

t	1.0	1.5	2.0	2.5
y	0.25	0.27	0.31	0.46

Construct a linear spline interpolation to estimate the concentration level at 2.2 minute.

Exercise 2: Use the portion of the given steam table for superheated H_2O at 200 MPa to find the corresponding entropy, *s*, for a specific volume, *v*, of 0.118 m^3/kg with linear spline.

V (m^3/kg)	0.2037	0.2114	0.32547	0.33213
S (kJ/kg K)	6.5147	6.6453	6.8664	6.9513

❑ Cubic Spline Interpolation(CSI)

Cubic spline interpolation is used very often. It gives smoother curves than other types. To determine the cubic spline, we need to use cubic polynomial for each subintervals.

Consider the cubic polynomial $f_k(x)$ in each subinterval $[x_k, x_{k+1}]$, $k = 1, 2, \dots, n-1$ of the form

$$f_k(x) = a_k(x - x_{k-1})^3 + b_k(x - x_{k-1})^2 + c_k(x - x_{k-1}) + d_k, \quad (a_k \neq 0).$$

where a_k, b_k, c_k and d_k are to be determined.

Since the spline passes through (x_k, y_k), and $f_k(x)$,

$$f_1(x_0) = y_0, \text{and } f_k(x_k) = y_k, k = 1, 2, 3, \cdots, n.$$

$$f_k(x_k) = f_{k+1}(x_k), \qquad k = 1, 2, 3, \cdots, n-1$$

$$f_k{}'(x_k) = f_{k+1}{}'(x_k), \qquad k = 1, 2, 3, \cdots, n-1$$

$$f_k{}''(x_k) = f_{k+1}{}''(x_k), \qquad k = 1, 2, 3, \cdots, n-1$$

We can see that there are

$$1 + n + 3(n-1) = 4n - 2$$

conditions but we need to determine 4n constants.

Boundary conditions

So we need to add two boundary conditions to get unique solution. Normally we use three types of boundary conditions:

1. Second derivatives at end points are known

$$f_1''(x_0) = M_0 \text{ and } f_n''(x_n) = M_n.$$

The special case

$$f_1''(x_0) = 0 \quad \text{and } f_n''(x_n) = 0$$

give spline called **natural cubic spline**.

2. First derivatives at end points are known

$$f_1'(x_0) = d_0 \text{ and } f_n'(x_n) = d_n.$$

give spline called **clamped cubic spline**.

3. Automatically adjusted boundary conditions known as **not-a-knot** cubic spline.

This condition assumes that $f'''(x)$ are continuous at the second and last but one points.

$$f_1'''(x_1) = f_2'''(x_1) \quad \text{and} \quad f_{n-1}'''(x_{n-1}) = f_n'''(x_{n-1}).$$

Note that minimum number of data points is four for this condition to be used.

Multiple questions:

S.No.	Questions
1	How many Spline interpolation we discussed in Numerical Analysis? (a) One, (b) Two, (c) None of them
2	What is the linear spline curve $f_k(x)$ in $[x_k,\ x_{k+1}]$ for $(k = 1, 2, \dots, n-1)$? Which rule is used for getting Modified Euler's method- Taylor series can be expresses as follows: (a) $f_k(x) = y_k + \frac{\Delta y_k}{h_k}\,(x - x_k)$, (b) $f_k(x) = y_{k-1} + \frac{\Delta y_k}{h_k}\,(x - x_k)$, (c) $f_k(x) = y_{k+1} + \frac{\Delta y_k}{h_k}\,(x - x_k)$, (d) None of them
4	Which spline is smooth? (a) Linear Spline interpolation , (b) Cubic Spline interpolation, (c) None of them, (d) Both of them

References

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[3] Numerical Analysis & Computing – W. Cheney & D. Kincaid, 6th Edition, 2007, Cengage Learning, Inc, USA.

[4] Numerical Analysis – J. Douglas Faires , Annette Burden , Richard Burden, 10th Edition, 2015, Cengage Learning, Inc, USA.