Solution of System of Nonlinear Equations: Fixed Point Iteration Method (FPIM)

Lecture-2

Objective:

Solve the system of nonlinear equations to find roots of the system using FPIM

Fixed Point Iteration Method (one variable)

The algorithm and convergence criteria (CC) of FPIM on in one variable are as follows:

Algorithm and CC:

Step 1: Consider nonlinear equation in **one variable**,

$f(x)=0$ Eq.(1)

***Step* 2:** Rewrite the given equation as follows:

$$x = g(x)\,(NOT\ UNIQUE) \qquad \text{Eq.(2)}$$

Step 3: Initial guess x_0, compute

$$x_{n+1} = g(x_n) \qquad \text{Eq.(3)}$$

$$\text{where } n = 0, 1, 2, \ldots, n$$

Step 4: Use iteration formula given in Eq. (3),

Then calculate x_1 by using initial guess value and continue this process till hold **stop criteria**.

Stop criteria:

$$|g'(x_0)| \quad \& \ check \ \ if \ \ |g'(x_0)| < 1.$$

Example

Example 1: Comment on the results of the given equation $x^3+2x-5=0$ has a root near x=1.4.

Solution: Consider

$$f(x)=x^3+2x-5=0 \qquad \text{Eq.(1)}$$

Arrange Eq.(1) and we get three iterative functions follows as:

$$(a)x_{n+1}=\frac{1}{2}\left(5-x_n^3\right),\ (b)x_{n+1}=\frac{\left(2x_n^3+5\right)}{\left(3x_n^2+2\right)},\quad (c)x_{n+1}=\left(5-2x_n\right)^{1/3}$$

Choose iterative function (a), and we can write $g_a(x_n)=\frac{1}{2}\left(5-x_n^3\right)$

Now, calculate $g_a'(x_n) = -\frac{3}{2}x_n^2$

We have, x_0=1.4, so consider n=0 and we get

$$g_a'(x_0) = -\frac{3}{2}x_0^2 = -\frac{3}{2}(1.4)^2 = -2.94$$

[Substitute the value of x_0]

The sequence will not converge because $\left|g_a'(1.4)\right| > 1.$

Similarly, choose iterative function (b), and we can write

$$g_b(x_n) = \frac{\left(2x_n^3 + 5\right)}{\left(3x_n^2 + 2\right)}$$

Now, calculate $g_b'(x_n) = \frac{6x_n(x_n{}^3 + 2x_n - 5)}{(3x_n{}^2 + 2)^2}$

We have, x_0=1.4, so consider n=0 and we get

$$g_b'(x_0) = \frac{6x_0(x_0{}^3 + 2x_0 - 5)}{(3x_0{}^2 + 2)^2} = \frac{6(1.4)((1.4)^3 + 2(1.4) - 5)}{(3(1.4)^2 + 2)^2} = 0.0735$$

[Substitute the value of x_0]

The sequence will converge because $\left| g_b'(1.4) \right| < 1.$

Similarly, choose iterative function (c), and we can write

$$g_c(x_n) = \left(5 - 2x_n\right)^{1/3}$$

Now, calculate $g_c'(x_n) = -\dfrac{2}{3}\dfrac{1}{(5-2x_n)^{2/3}}$

We have, x_0=1.4, so consider n=0 and we get

$$g_c'(x_0) = -\frac{2}{3}\frac{1}{(5-2x_0)^{2/3}} = -\frac{2}{3}\frac{1}{(5-2(1.4))^{2/3}} = -0.394$$

[Substitute the value of x_0]

The sequence will converge because $\left|g_c'(1.4)\right| < 1.$

Example 2: Given that $f(x) = 2\cos 2x + 2 - x$. If the iterative formula converge to the root near the point x_0=3.5, do the iteration two times to estimate the root to 4 decimal places. Also, Write down MATLAB commands to execute the iterations five times.

Solution: Consider

$$f(x) = 2\cos 2x + 2 - x = 0 \qquad \text{Eq.(1)}$$

Rewrite the given equation as follows:

$$x_{n+1} = \frac{1}{4}\left(2 + 3x_n + 2\cos 2x_n\right) = g(x_n) \qquad \text{Eq.(2)}$$

$$or,\, g(x_n) = \frac{1}{4}\left(2 + 3x_n + 2\cos 2x_n\right)$$

Calculate $g'(x_n) = \frac{1}{4}\left(3 - 4\sin 2x_n\right)$

We have, x_0=3.5, so consider n=0 and we get

$$g'(x_0) = \frac{1}{4}\left(3 - 4\sin 2x_0\right) = \frac{1}{4}\left(3 - 4\sin 2(3.5)\right) = 0.093013$$

[Substitute the value of x_0]

The sequence will converge because $\left|g'(3.5)\right| < 1.$

1st iteration: We have, x_0=3.5, so consider n=0 then Eq. (2) become

$$x_1 = \frac{1}{4}\left(2 + 3x_0 + 2\cos 2x_0\right)$$

$$= \frac{1}{4}\left(2 + 3(3.5) + 2\cos 2(3.5)\right)$$ [Substitute the value of x_0]

$$= \frac{1}{4}\left(12.5 + 2(0.753902)\right)$$

$$x_1 = 3.501951$$

2nd iteration: We have now, x_1=3.501951, so consider n=1 then Eq. (2) become

$$x_2 = \frac{1}{4}\left(2+3x_1+2\cos 2x_1\right)$$

$$= \frac{1}{4}\left(2+3(3.501951)+2\cos 2(3.501951)\right)$$ [Substitute the value of x_1]

$$= \frac{1}{4}\left(12.505853+2(0.751332)\right)$$

$$x_2 = 3.502129 = 3.5021$$

3rd iteration: We have now, x_2=3.502129, so consider n=2 then Eq. (2) become

$$x_3 = \frac{1}{4}\left(2+3x_2+2\cos 2x_2\right)$$

$$= \frac{1}{4}\left(2+3(3.502129)+2\cos 2(3.502129)\right)$$

[Substitute the value of x_2]

$$= \frac{1}{4}\left(2+3(3.502129)+2(0.751097)\right)$$

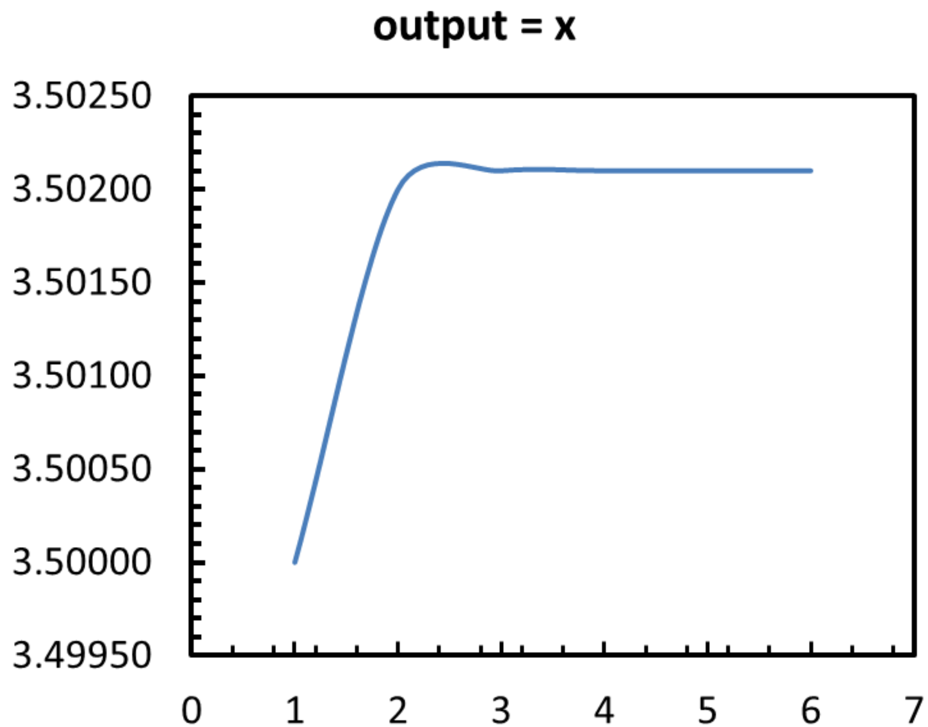
$$x_3 = 3.502145 = 3.5021$$

Program code in MATLAB:

Program code:

```
clear all
close all
clc

f=@(x)(2+3.*x+2.*cos(2.*x))./4;
x(1)=3.5;
for n=1:5
   x(n+1)=f(x(n));
end
output=x'
```



Output:

```
   x
3.5000
3.5020
3.5021
3.5021
3.5021
3.5021
```

FPIM for system of NLE in two variables

The Algorithm and stop criteria of iteration to find the root of nonlinear equation in two variables by applying Fixed point iteration are as follows:

Algorithm:

Step 1: Consider nonlinear equation in **two variables**,

$$f_1(x, y) = 0 \text{ and } f_2(x, y) = 0 \qquad Eq.(1)$$

***Step* 2:** Rewrite the given equation as follows:

$$x = g_1(x, y) \text{ and } y = g_2(x, y) \qquad Eq.(2)$$

Step 3: If there is a point (p,q) such that, $p = g_1(p, q)$ and $q = g_2(p, q)$,

then (p,q) is a fixed point of the system and this (p,q) is a root of the system.

Step 4: Arrange Eq.(2) and assume an iteration formula such as

$$x_{n+1} = g_1(x_n, y_n) \text{ and } y_{n+1} = g_2(x_n, y_n) \text{Eq.(3)}$$
where $n = 0,1,2,\ldots,$

Then calculate x_1, y_1 by using initial guess value and continue this process till hold **stop criteria**.

Stop criteria:

1. Iterative formula will converge close to fixed point (p,q)

Convergence and divergence criteria:

(x_0, y_0) is close to the fixed point (p,q).

(i) if Eq.(4a) and Eq.(4b) holds.

$$\left|\frac{\partial}{\partial x} g_1(x_0, y_0)\right| + \left|\frac{\partial}{\partial y} g_1(x_0, y_0)\right| < 1 \qquad \text{Eq.(4a)}$$

$$\left|\frac{\partial}{\partial x} g_2(x_0, y_0)\right| + \left|\frac{\partial}{\partial y} g_2(x_0, y_0)\right| < 1 \qquad \text{Eq.(4b)}$$

Then the iterative formula given in Eq. (3) will converge to the fixed point (p,q).

(ii) If the conditions given in Eq.(4a) and Eq.(4b) are not satisfied, the iterative process might diverse. The above method can be extended for more than two variables.

Example

Example 3: Consider the system $y = x^2 - 5x + 3\,;\;\; x^2 + 4y^2 = 4$

A fixed point iteration formula is suggested to estimate root at $(x_0,y_0)=(0.9,1)$ given as follows:

$$x_{n+1} = \frac{1}{5}\left(x_n^2 - y_n + 3\right)$$

$$y_{n+1} = \frac{1}{10}\left(4 - x_n^2 - 4y_n^2 + 10y_n\right)$$

(a) Verify whether the above iterative formula will converge to the root near $(x_0, y_0) = (0.9, -1)$. If converges, perform one iteration otherwise suggest another fixed point iterative formula which converge to the root.

(b) Write MATLAB commands to execute the above iterative formula five times.

Solution: From the iterative formula, let us define

$$g_1(x, y) = \frac{1}{5}(x^2 - y + 3)$$
$$g_2(x, y) = \frac{1}{10}(4 - x^2 - 4y^2 + 10y)$$

$$\frac{\partial}{\partial x} g_1(x, y) = \frac{2x}{5}, \qquad \frac{\partial}{\partial y} g_1(x, y) = -\frac{1}{5}$$

$$\frac{\partial}{\partial x} g_2(x, y) = -\frac{2x}{10}, \qquad \frac{\partial}{\partial y} g_2(x, y) = \frac{1}{10}(-8y + 10)$$

Near point (0.9, -1.0) we have, $\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0.36| + |-0.2| = 0.56 < 1$

$g_1(x, y)$ Satisfy convergence criteria

But it can be seen, that $\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |-0.18| + \left|\frac{-8(-1)+10}{10}\right| = 1.98 > 1$

Test fails and the convergence is not guaranteed. Rearranging the second equation by

$$x^2 + 4y^2 - 4 + 10y = 10y$$

Now, we have $y = \frac{1}{10}(x^2 + 4y^2 - 4 + 10y)$

So, suggested iterative formula can be

$$g_3(x, y) = \frac{1}{10}(x^2 + 4y^2 - 4 + 10y)$$

Let us calculate

$$\frac{\partial}{\partial x} g_3(x, y) = \frac{2x}{10}, \qquad \frac{\partial}{\partial y} g_3(x, y) = \frac{1}{10}(8y + 10)$$

At near point $(x_0,y_0)=(0.9,-1)$

$$\left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| = \left|\frac{1.8}{10}\right| + \left|\frac{8(-1)+10}{10}\right| = 0.38 < 1$$

So, $g_3(x,y)$ satisfy convergence criteria.

Thus a fixed point iterative formula which converge to root near point $(0.9,-1)$ is

$$x_{n+1} = \frac{1}{5}(x_n^2 - y_n + 3)$$
$$y_{n+1} = \frac{1}{10}(x_n^2 + 4y_n^2 - 4 + 10y_n)$$

Program code in MATLAB:

Program code:

```
clear all
close all
clc
x(1)=0.9;
y(1)=-1;
for n=1:5
     x(n+1)=(x(n)^2-y(n)+3)/5;
     y(n+1)=(x(n)^2+4*y(n)^2-4+10*y(n))/10;
end
Iterative_Roots = [x',y'];
```

Output:

```
Iterative_Roots :
      x          y
   0.9000   -1.0000
   0.9620   -0.9190
   0.9689   -0.8886
   0.9655   -0.8789
   0.9622   -0.8767
   0.9605   -0.8767
```

Advantages and Drawbacks: Fixed Point Iteration Method

Advantages:

- Fast
- Fewer calculations than bracketing methods
- Requires one guess only
- Easier to program

Drawbacks:

- Convergence is not guaranteed

Outcome

System of nonlinear equations can be solved by applying Fixed Point Iteration Method (FPIM), to find roots (approximately) of the system, although it has few drawbacks.

Also, behavior of function can be predicted by analyzing roots.

Multiple questions:

S.No.	Questions
1	Fixed Point Iteration Method is- (a) Closed method, (b) Open method, (c) Bracketing method
2	What type of solution could be by applying above method? (a) Analytical solution, (b) Numerical solution
3	Fixed Point Iteration method can be used to find roots of the following system of equations: (a) Linear equations , (b) Non-linear equations , (c) both (a) and (b)
5	What is the stop criteria of Fixed Point Iteration Method in one variable? (a) $\lvert g'(x_0)\rvert$ and check $\lvert g'(x_0)\rvert < 1$, (b) $\lvert g'(x_0)\rvert$ and check $\lvert g'(x_0)\rvert > 1$, (c) None of them (d) Both of them
6	What is the stop criteria of Fixed Point Iteration Method in two variables? (a) $\left\lvert\frac{\partial}{\partial x} g_1(x_0, y_0)\right\rvert + \left\lvert\frac{\partial}{\partial y} g_1(x_0, y_0)\right\rvert$<1, (b) $\left\lvert\frac{\partial}{\partial x} g_2(x_0, y_0)\right\rvert + \left\lvert\frac{\partial}{\partial y} g_2(x_0, y_0)\right\rvert$<1, (c) Both of them

S.No.	Questions
7	How many guess value requires for applying Fixed Point Iteration Method to find roots of equations (a) one, (b) many, (c) Both of them (d) None of them
8	What is the drawback of Fixed point iteration method? a) It converges, b) it doesn't converges, c) Convergence is not guaranteed

Try to do yourself

#1. The following system has a root near $(x_0, y_0) = (0.2,\ 0.3)$.

$$2x^2 + y^2 - 15x + 2 = 0,$$
$$xy^2 + x - 10y + 5 = 0.$$

Estimate the root correct to four decimal places using fixed point iterative method.

#2. Determine the roots of the following simultaneous non linear equations using

$$x = y + x^2 - 2.5,$$
$$y = x^2 - 5xy$$

Employ initial guesses of $x_0 = y_0 = 1.0$ and discuss the results.

References

[1] Applied Numerical Methods With Matlab for Engineers and Scientists (Steven C.Chapra).

[2] Applied Numerical Analysis – C.F.Gerald & P.O.Wheatley, 7th Edition, 2003, Pearson Education Limited, USA.

[3] Numerical Analysis & Computing – W. Cheney & D. Kincaid, 6th Edition, 2007, Cengage Learning, Inc, USA.

[4] Numerical Analysis – J. Douglas Faires , Annette Burden , Richard Burden, 10th Edition, 2015, Cengage Learning, Inc, USA.