Numerical Integration

Double Integration

Lecture-3

Specific aims

- **Discuss about double Integration Gaussian quadrature rule.**
- **Find single and multiple (definite) integration using MATLAB.**

Applications

- **Mathematics**
- **Engineering**

Advantages

- Obtain numerical results of Integral for two dimensions.
- Save time by using MATLAB to compute Integral operation and getting results.

Numerical Evaluation of Double Integrals

Multiple integrals are evaluated by expressing them as iterated integrals. For example a double integral can be expressed as follows;

$$\iint_R f(x,y)dA = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)dy \right) dx$$

Or

$$\iint_R f(x,y)dA = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)dx \right) dy$$

In computing the iterated integral of the first form, we hold *x* constant while integrating with respect to *y* and then integrate with respect to *x.*

Similarly, in computing the iterated integral of the second form, we hold *y* constant while integrating with respect to *x* and then integrate with respect to *y*.

Numerical evaluation of integrals we need to choose interval length and nodal points where the functional values to be calculated.

For fixed limits, selection of interval length and nodal points are straight forward but for variable limits interval length and nodal points should be calculated for a fixed value of other variable. For example in first form the interval at $x = x_r$ is

$$k_r = \frac{f_2(x_r) - f_1(x_r)}{n}$$

Examples

Problem 1# : Using Simpson's rule with 2-subintervals evaluate evaluate the double integral

$$\int_{0.6}^{1.0} \int_{1.0}^{1.6} \frac{2y^2}{x+y}\, dydx$$

The integral is over the rectangular region bounded by $0.6 \leq y \leq 1.0$ and $1.0 \leq x \leq 1.6$.

Solution:

Using 2-subintervals in each direction, we have

$$h = \frac{1.0 - 0.6}{2} = 0.2 \quad \text{and} \quad k = \frac{1.6 - 1.0}{2} = 0.3$$

The integrand is $f(x, y) = \dfrac{2y^2}{x+y}$.

Integration using fixed values of $x = 0.6, 0.8$ and 1.0 are as follows:

$$I(0.6) = \int_{1.0}^{1.6} f(0.6, y)dy = \frac{0.3}{3}[f(0.6,1.0) + 4f(0.6,1.3) + f(0.6,1.6)]$$

$$= \frac{0.3}{3}[1.25 + 4(1.779) + 2.327] = 1.0693$$

$$I(0.8) = \int_{1.0}^{1.6} f(0.8, y)dy = \frac{0.3}{3}[f(0.8,1.0) + 4f(0.8,1.3) + f(0.8,1.6)]$$

$$= \frac{0.3}{3}[1.111 + 4(1.610) + 2.133] = 0.9684$$

$$I(1.0) = \int_{1.0}^{1.6} f(1.0, y)dy = \frac{0.3}{3}[f(1.0,1.0) + 4f(1.0,1.3) + f(1.0,1.6)]$$

$$= \frac{0.3}{3}[1.0 + 4(1.47) + 1.969] = 0.8849$$

Finally combining the integral for *x* we have

$$\mathrm{I(Simp)} = \int_{0.6}^{1.0}\int_{1.0}^{1.6} f(x,y)dydx \approx \frac{0.2}{3}[I(0.6) + 4\,I(0.8) + I(1.0)]$$

$$= \frac{0.2}{3}[1.0693 + 4(0.9684) + 0.8849]$$

$$= 0.38852$$

Therefore, the answer is 0.38852

$$\int_{0.6}^{1.0} \int_{1.0}^{1.6} \frac{2y^2}{x+y}\, dydx$$

$$h = \frac{1.0 - 0.6}{2} = 0.2, \qquad k = \frac{1.6 - 1.0}{2} = 0.$$

x_r	y_r	$f(x,y)$	$I(x,k)$	$I(Simp)$
0.6	1	1.25		
0.6	1.3	1.779	1.0693	
0.6	1.6	2.327		
0.8	1	1.111		
0.8	1.3	1.61	0.9684	0.3852
0.8	1.6	2.133		
1	1	1		
1	1.3	1.47	0.8849	
1	1.6	1.969		

Numerical evaluation of Integrals using MATLAB

Syntax of MATLAB commands for different types of integrals are as follows:

integral1(fun, xmin, xmax)
integral2(fun, xmin, xmax, ymin, ymax)
integral3(fun, xmin, xmax, ymin, ymax,zmin,zmax)

Command for required number of digits
vpa(value, n) % to display value to n digits

Note: Define the function using "anonymous handle". Format is shown below:

fname=@(argument) <space> formula

operation / (division) and ^ (for power) must preceded with . (dot).

For example $f(x,y)=\frac{\sin(xy)}{x^2+y^2}$ may be typed as

```
ff=@(x,y) sin(x.*y)./(x.^2+y.^2))
```

[operation / (division) and ^ (for power) must preceded with . (dot).]

This will be treated as ff(x,y).

Example of single (definite) integration using MATLAB

Problem 2#: Evaluate $\int_0^1 e^x \sin x^2 dx$ numerically to 7 digits using MATLAB command.

Solution:

```
>> fun=@(x) exp(x).*sin(x.^2) % enter function as @function

fun = @(x)exp(x).*sin(x.^2)

>> int=integral(fun,0,1);        % command for integration
>> int7=vpa(int,7)               % used to increase precision

int7 =0.662701
```

Example of double (definite) integration using MATLAB

Problem 3#: Evaluate $\int_0^2 \int_{-x}^{\sqrt{x}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$ numerically using MATLAB command.

Solution:

```
>> fun2=@(x,y) 1./sqrt(x.^2+y.^2)      % define as @-function

fun2 = @(x,y)1./sqrt(x.^2+y.^2)

>> ymin=@(x) –x                         % lower limit as function of x

ymin = @(x)-x
```

Example of double (definite) integration using MATLAB

```
>> ymax=@(x) sqrt(x)              % upper limit as a function of x

ymax = @(x)sqrt(x)


>> int2=integral2(fun2, 0, 2, ymin,ymax);

>> int22=vpa(int2,7)                         % output to 7 digits

int22 =3.811758
```

Example of triple (definite) integration using MATLAB

Problem 4#: Evaluate $\iiint_R \frac{dV}{(1+x+y+z)^2}$, where R is the region bounded by the coordinate planes and the plane $x + y + z = 1$, numerically using MATLAB command.

Solution:

Given plane is $x + y + z = 1$,

Or $z = 1 - x - y$

The integral can be written as an iterated integral of the form

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(1 + x + y + z)^2} dzdydx$$

MATLAB commands and output

```
>> clear
>> fun3=@(x,y,z) 1./(x+y+z+1).^2

                        % define the integrand as a @-function

fun3 =   @(x,y,z)1./(x+y+z+1).^2


>> ymax=@(x) 1-x

ymax = @(x)1-x
```

```
>>zmax=@(x,y) 1-x-y

zmax = @(x,y)1-x-y

>> int3=integral3(fun3,0,1,0,ymax,0,zmax);
                                        % command for integration

>> int31=vpa(int3,7)                    % result using 7 digits.

int31 =0.05685282
```

Outcomes

- **Numerically solved problems by using double Integration Gaussian quadrature rule.**
- **Save time for solving problems by writing code in MATLAB software**

Multiple questions

S.No.	Questions
1	Which command is used for single integration result in MATLAB- (a) int=integral(fun,min,max), (b) int=integral(fun,max,min), (c) Both of them
2	Which command is used for double integration result in MATLAB- (a) int=integral2(fun,xmin,xmax,ymin,ymax), (b) int=integral2(fun,xmax,xmin), (c) None of them
3	Which command is used for triple integration result in MATLAB- (a) int=integral3(fun,xmin,xmax,ymin,ymax,zmin,zmax), (b) int=integral2(fun,min,max), (c) both (a) and (b)
4	Double Integration can be expressed as (a) $\iint_R f(x,y)dA = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)dy\right) dx,$ (b) $\iint_R f(x,y)dA = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)dx\right) dy,$ (c) both (a) and (b)

Try to do yourself

Exercise 1: The table below shows the values of $f(x)$ at different values of x.

x	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**
f(x)	1.831	2.592	3.515	4.643	5.926

Evaluate $\int_{1.0}^{1.8} f(x)\, dx$ using Trapezoidal rule with 1, 2 and 4 subintervals. Improve your results using Romberg integration.

Exercise 2: Write MATLAB code to evaluate the integrals given as follows:

(a) $\int_0^1 \sqrt{1+2x^3}\; dx$ (b) $\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x} dx$

Answers: (a) 1.205 , (b) 0.392699

Exercise 3: Write MATLAB code to evaluate the following integrals:

(a) $\int_{1}^{2}\int_{0}^{1}(1+8x\,)dydx$ (b) $\int_{1.4}^{2.0}\int_{1.0}^{1.5}\ln(2x+y)\,dxdy$,

(c) $\int_{0}^{1}\int_{1}^{2}2\exp(x/y)\,dydx$

Answers: (a) 13.0 , (b) 0.4295545 , (c) 2.8967

Exercise 4: Write MATLAB code to evaluate the following integrals:

(a) $\int_{0}^{2}\int_{3}^{6}\int_{-1}^{1}(e^{2x}\sin y+z\ln y)\,dxdydz$

(b) $\int_{0}^{2}\int_{0}^{3}\int_{0}^{1}xyz\;\,dzdydx$

Answers: (a) 3.6729 , (b) 4.50

Reference

[1] Applied Numerical Methods With Matlab for Engineers and Scientists (Steven C.Chapra).