Vectors

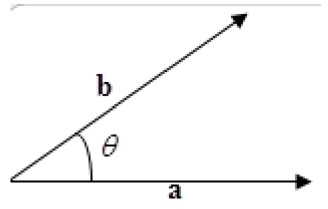
Vector: A vector is a quantity having both magnitude and direction.

A vector is represented by a directed line segment. When denoted by a single letter we use bold face letter $\boldsymbol{a}$ or in manuscript by $\bar{a}$. The magnitude of $\boldsymbol{a}$ is denoted by $|\bar{a}|$ or a.

- Magnitude (P-801) : If $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, then $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.
- Unit vector A vector with magnitude 1 is called a unit vector. Unit vector along $\bar{a}$ is denoted by $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$
- Dot product :

The **dot** or **scalar** product of two vectors **a** and **b**, denoted by $\bar{a} \cdot \bar{b}$, is defined as

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta,$$ where θ is the angle between **a** and **b.**



If $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Example:(P-807) # 1, Exercise: Dot product (P-812) # 4,8

- **Angle between two vectors**(P-809) # 3, Ex: (P-812) Angle # 17-20.

Ex# 17: Find the angle between the two vectors $\bar{a} = \langle 1, -4, 1\rangle$ and $\bar{b} = \langle 0, 2, -2\rangle$.

Solution: We have, $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta$

$$\bar{a} \cdot \bar{b} = \langle 1, -4, 1\rangle . \langle 0, 2, -2\rangle = 1(0) + (-4)2 + 1(-2) = 0 - 8 - 2 = -10$$

$$|\bar{a}| = \sqrt{1^2 + (-4)^2 + 1^2} = \sqrt{18}, \ |\bar{b}| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{8}$$

$$\therefore \cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{-10}{\sqrt{18}\sqrt{8}} \qquad \Longrightarrow \theta = cos^{-1}\left(\frac{-10}{\sqrt{18}\sqrt{8}}\right) = cos^{-1}(-0.833) = 146^0$$

- **Perpendicular/ orthogonal vectors** (P-809) # 4 Exercise: (P-813) #23(b, c, d), 24

Ex: Vectors orthogonal, parallel # 23(b, c, d), 24.

If $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

- the vectors are parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$
- the vectors are orthogonal or perpendicular if $\bar{a} \cdot \bar{b} = 0$.

Ex# 23 (c): Determine whether the vectors $\bar{a} = -8\hat{\imath} + 12\hat{\jmath} + 4\hat{k}$ and $\bar{b} = 6\hat{\imath} - 9\hat{\jmath} - 3\hat{k}$ are orthogonal, parallel, or neither.

Soln: As $\frac{-8}{6} = \frac{12}{-9} = \frac{4}{-3} \Longrightarrow \frac{-4}{3} = \frac{4}{-3} = \frac{4}{-3} \rightarrow -\frac{4}{3} = -\frac{4}{3} = -\frac{4}{3}$ The vectors are parallel.

- **<u>Direction angle, Direction Cosine</u>** (P-810) # 5. Ex: Direction cosine, Direction angle (P-813) #33-37.

The **direction angles** of a nonzero vector $\bar{a}$ are the angles α, β and γ (in the interval $[0, \pi]$) that $\bar{a}$ makes with the positive x- , y- and z-axes, respectively.

The cosines of these direction angles $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of the vector $\bar{a}$.

$$\therefore \cos\alpha = \frac{\bar{a}.\hat{\imath}}{|\bar{a}||\hat{\imath}|} = \frac{a_1}{|\bar{a}|},\ \cos\beta = \frac{\bar{a}.\hat{\jmath}}{|\bar{a}||\hat{\jmath}|} = \frac{a_2}{|\bar{a}|} \text{ and } \cos\gamma = \frac{\bar{a}.\hat{k}}{|\bar{a}||\hat{k}|} = \frac{a_3}{|\bar{a}|}.$$

Ex# 35: Find the direction cosines and direction angles of the vector $\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$.

Soln: Let $\bar{a} = \hat{\imath} - 2\hat{\jmath} - 3\hat{k}$ $\quad\therefore |\bar{a}| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}$

For direction cosines: $\cos\alpha = \frac{a_1}{|\bar{a}|} = \frac{1}{\sqrt{14}}$, $\cos\beta = \frac{a_2}{|\bar{a}|} = \frac{-2}{\sqrt{14}}$ and $\cos\gamma = \frac{a_3}{|\bar{a}|} = \frac{-3}{\sqrt{14}}$.

Direction angles are $\alpha = cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$, $\beta = cos^{-1}\left(\frac{-2}{\sqrt{14}}\right)$ and $\gamma = cos^{-1}\left(\frac{-3}{\sqrt{14}}\right)$.

- **<u>Projection</u>** (P-811) # 6; **<u>Work done</u>** (P-812) # 7, 8. Ex: Scalar projection (P-813) #39, 41-44; Work done # 49.

Ex# 44: Find the projection of $\bar{b} = 5\hat{\imath} - \hat{k}$ onto $\bar{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Soln: $|\bar{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ $\quad \bar{a} \cdot \bar{b} = 1(5) + 2(0) + 3(-1) = 5 - 3 = 2$

The projection of $\bar{b}$ onto $\bar{a}$ $= \bar{b} \cdot \hat{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|} = \frac{2}{\sqrt{14}}$.

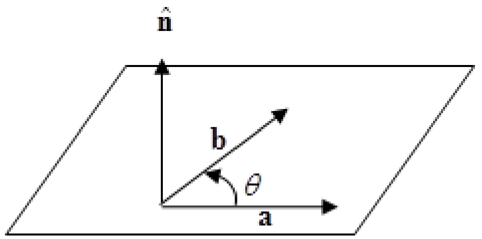
Ex# 49: Find the work done by a force $\bar{F} = 8\hat{\imath} - 6\hat{\jmath} + 9\hat{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

Solution: The displacement vector $\bar{D} = \overline{PQ} = \langle 6-0,\ \ 12-10,\ \ 20-8\rangle = \langle 6, 2, 12\rangle$.

The work done is $W = \bar{F} \cdot \bar{D} = \langle 8, -6, 9\rangle \cdot \langle 6, 2, 12\rangle = 48 - 12 + 108 = 144\, J$.

- **Cross product** (P-815) :

The cross or vector product of two vectors $\bar{a}$ and $\bar{b}$, denoted by $\bar{a} \times \bar{b}$, is a vector which is Perpendicular to both $\bar{a}$ and $\bar{b}$.



If $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{\imath} - (a_1b_3 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\,\hat{k}$$

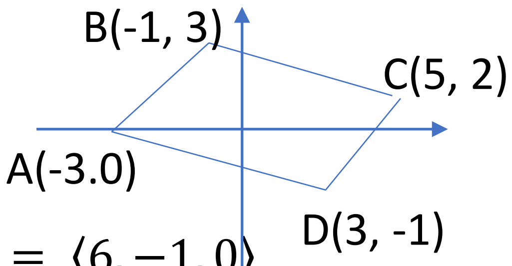
Theorem If θ is the angle between $\bar{a}$ and $\bar{b}$, then

$$|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin\theta$$

Application:

The area of a parallelogram with sides as two vectors $\bar{a}$ and $\bar{b}$, is the magnitude of the cross product $\bar{a} \times \bar{b}$.

$$\text{Area}= |\bar{a} \times \bar{b}|$$



Exercise (P-821)#27

Find the area of the parallelogram with vertices A(-3, 0), B(-1, 3), C(5, 2) and D(3, -1)

Solution $\overline{AB} = \bar{a} = \langle -1+3, 3-0, 0\rangle = \langle 2,3,0\rangle$, and $\overline{AD} = \bar{b} = \langle 3+3, -1-0, 0\rangle = \langle 6,-1,0\rangle$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 0 \\ 6 & -1 & 0 \end{vmatrix} = (0+0)\hat{\imath} - (0+0)\hat{\jmath} + (-2-18)\hat{k} = -20\hat{k}$$

Area $=|\bar{a} \times \bar{b}| = 20$ (unit)

Exercise (P-821)# 20: Find the unit vector perpendicular/ orthogonal to both $\hat{\jmath} - \hat{k}$ and $\hat{\imath} + \hat{\jmath}$.

Solution: $\hat{n} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$ $\quad \bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = (0+1)\hat{\imath} - (0+1)\hat{\jmath} + (0-1)\hat{k}$

$$\bar{a} \times \bar{b} = \hat{\imath} - \hat{\jmath} - \hat{k} \quad \rightarrow \quad |\bar{a} \times \bar{b}| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3} \qquad \therefore \ \hat{n} = \frac{1}{\sqrt{3}}(\hat{\imath} - \hat{\jmath} - \hat{k}).$$

Example: Area (P-818 # 4), **Exercise**: (P-821) Unit vector #19, 20; Area # 27,28.

- **Triple product** (P-819): The product $\bar{a} \cdot (\bar{b} \times \bar{c})$ is called the **scalar triple product** of the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$.

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- **Application:**

  The volume of the parallelepiped determined by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is the absolute value of the scalar triple product: $V = |\bar{a} \cdot (\bar{b} \times \bar{c})|$.

- Ex# 34 (P-822): Find the volume of the parallelepiped determined by the vectors $\bar{a} = \hat{\imath} + \hat{\jmath}$, $\bar{b} = \hat{\jmath} + \hat{k}$ and $\bar{c} = \hat{\imath} + \hat{\jmath} + \hat{k}$.

  Solution: The volume of the parallelepiped determined by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is the absolute value of the scalar triple product.

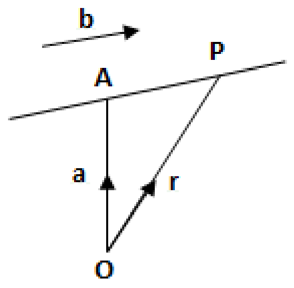
$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (1-1)1 - (0-1)1 + (0-1)0 = 1 \qquad \therefore\ V = |\bar{a} \cdot (\bar{b} \times \bar{c})| = 1.$$

> If the volume of the parallelepiped determined by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is 0 (zero) then the vectors must lie in the same plane; that is they are coplanar.

Example: Volume (P-820), Coplanar(# 5).

Exercise: (P-822) Volume # 33, 34; Coplanar #37, 38.

Straight lines and planes in 3-space



- **Equation of a straight line:** A particular line is uniquely located in space if
  - ✓ it has a known direction and passes through a point,
  - ✓ it passes through two known points.
- **Vector form**

Consider the vector equation of a straight line passing through a point *A* with position vector $\bar{a}$ and parallel to the vector $\bar{b}$. Since $\overline{AP}$ is parallel to $\bar{b}$,

we get $\overline{AP} = t\bar{b}$, t is a scalar.

If *P* is any point on the line with position vector $\bar{r}$, then $\bar{r} = \bar{a} + t\bar{b}$, t is a scalar parameter.

Scalar form

If $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$\bar{r} = \bar{a} + t\bar{b}$ can be expressed as

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} + t\left(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}\right)$$

Equating corresponding components, we have the scalar form as follows:

(i) **Parametric form:** $x = a_1 + b_1 t,\ y = a_2 + b_2 t, z = a_3 + b_3 t,$

(ii) **Symmetric form:** $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} = t,$ which is the Cartesian equation of the line.

➢ **Note that** $b_1 : b_2 : b_3$ are the **direction ratios(D.R.'s)** of the line.

- Examples : Equations of lines (P-824) # 1,2,3. Exercise: P-831 Lines- #2, 4, 7, 10,11, 19-22.

Ex# 2 (P-831): Find the vector equation and parametric equations for the line which passes through the point $(6, -5, 2)$ and parallel to the vector $\left\langle 1, 3, -\frac{2}{3} \right\rangle$.

Solution: Let $\bar{r} = \langle x, y, z \rangle$ be the position vector of the point P.

The position vector of the given point $(6, -5, 2) = \langle 6, -5, 2 \rangle$.

Then, the vector equation of the line is $\bar{r} = \bar{a} + t\bar{b}$

$$\Rightarrow \langle x, y, z \rangle = \langle 6, -5, 2 \rangle + t \left\langle 1, 3, -\frac{2}{3} \right\rangle$$

The parametric equations of the line is $x = 6 + t,\ y = -5 + 3t, z = 2 - \frac{2}{3}t$

Symmetric form: $\frac{x-6}{1} = \frac{y+5}{3} = \frac{z-2}{\frac{-2}{3}} (= t).$

➢ The vector equation of the line that passes through two points:

$$\bar{r} = \bar{a} + t(\bar{b} - \bar{a})$$

➢ The vector equation of the line that passes through a point and orthogonal to two vectors:

$$\bar{r} = \bar{a} + t(\bar{b} \times \bar{c})$$

Ex# 9 (P-831): Find the parametric equations and symmetric equations for the line through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

Solution: Let $\bar{r} = \langle x, y, z\rangle$ be the position vector of any point P on the straight line.

The position vector of the given points :

$$(-8, 1, 4) \to \langle -8, 1, 4\rangle \quad \text{and} \quad (3, -2, \ 4) \to \langle 3, -2, \ 4\rangle.$$

Then, the vector equation of the line

$$\bar{r} = \bar{a} + t(\bar{b} - \bar{a}) \quad \Rightarrow \quad \langle x, y, z\rangle = \langle -8, 1, 4\rangle + t\langle 3 + 8, -2 - 1, 4 - 4\rangle = \langle -8, 1, 4\rangle + t\langle 11, -3, 0\rangle$$

The parametric equations:

$$x = -8 + 11t, \quad y = 1 - 3t, \quad z = 4$$

The symmetric equations:

$$\frac{x+8}{11} = \frac{y-1}{-3}, \quad z = 4.$$

Ex# 10 (P-831): Find the parametric equations and symmetric equations for the line through the point $(2, 1, 0)$ and perpendicular to both $\hat{\imath} + \hat{\jmath}$ and $\hat{\jmath} + \hat{k}$.

Solution: Let $\bar{r} = \langle x, y, z\rangle$ be the position vector of any point P on the straight line.

The position vector of the given point

$(2, 1, 0) \rightarrow \langle 2, 1, 0\rangle$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1-0)\hat{\imath} - (1-0)\hat{\jmath} + (1-0)\hat{k} = \hat{\imath} - \hat{\jmath} + \hat{k}$$

The vector equation of the line $\quad \bar{r} = \bar{a} + t(\bar{b} \times \bar{c})$

$$\Rightarrow \qquad \langle x, y, z\rangle = \langle 2,1,0\rangle + t\langle 1, -1, 1\rangle$$

The parametric equations:

$$x = 2 + t,\ \ y = 1 - t, z = t$$

The symmetric equations:

$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z}{1}.$$

Ex# 21 (P-831): Determine whether the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect find the point of intersection.

$$L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}, \qquad L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

Soln: Let us write the equations as $L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} = \lambda, \qquad L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7} = \mu$

The D.R. of the first line is $1 : -2 : -3,$ the D.R. of the second line is $1 : 3 : -7,$

and they are not proportional so the lines are not parallel.

Coordinates of any point on the lines are

$L_1 : x = \lambda + 2, y = -2\lambda + 3, z = -3\lambda + 1$ and $L_2 : x = \mu + 3, y = 3\mu - 4, z = -7\mu + 2$

Equating x and z $\quad \lambda + 2 = \mu + 3, \qquad -3\lambda + 1 = -7\mu + 2$

Solving the above equations, we have $\lambda = 2$ and $\mu = 1$

The values of y are:

from first line $y = -4 + 3 = -1$ and from second line $y = -4 + 3 = -1$

The values are equal and the lines intersect.

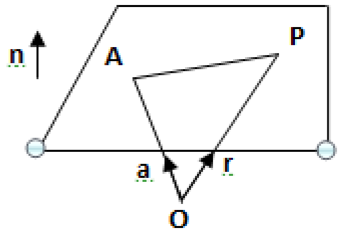
Now $x = 2 + 2 = 4, \qquad y = -2(2) + 3 = -1, \qquad z = -3(2) + 1 = -5$

The point of intersection is $(4, -1, -5)$.

❑ Equation of a plane:

A plane in 3D coordinate space is determined

- by a point and a vector perpendicular to the plane. The perpendicular vector is called normal vector
- by three points on the plane.



- **Scalar product form**

The vector equation of a plane passing through the point with position vector $\bar{r}_0$ and perpendicular to the vector $\bar{n}$ (i.e. $\bar{n}$ is a normal to the plane) is

$$(\bar{r} - \bar{r}_0) \cdot \bar{n} = 0 \qquad \text{(since } (\bar{r} - \bar{r}_0) \text{ and } \bar{n} \text{ are perpendicular)}$$

$$\bar{r} \cdot \bar{n} = \bar{r}_0 \cdot \bar{n} = d$$

- **Cartesian form**

If $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\bar{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, then $\bar{r} \cdot \bar{n} = \bar{r}_0 \cdot \bar{n} = d$ becomes

$$ax + by + cz = d$$

This is the Cartesian equation of the plane.

Here $[a : b : c]$ is the D.R.s of the normal to the plane.

- Example: Equation of planes (P-827) # 4, 5.
- Ex: Planes- (P-831) # 23, 24, 26, 27, 33, 34; Points of intersection # 6, Ex: Angle # 51, 53, 55;
- Angle between planes # 7.

Ex# 24 (P-831): Find an equation of the plane through the point $(5, 3, 5)$ and with normal vector $2\hat{\imath} + \hat{\jmath} - \hat{k}$.

Solution:

$\bar{r_0} = \langle 5,3,5 \rangle$ be the position vector of the given point $P_0 = (5,3,5)$,

$\bar{n} = \langle 2, 1, \quad -1 \rangle$ normal to the plane and

$\bar{r} = \langle x, y, z \rangle$ position vector of any point P on the plane.

Then, the equation of the plane $\bar{r}.\bar{n} = \bar{r_0}.\bar{n} \quad \rightarrow \quad \langle x, y, z \rangle \cdot \langle 2,1,-1 \rangle = \langle 5,3,5 \rangle \cdot \langle 2,1,-1 \rangle$

$$\Longrightarrow \quad 2x + y - z = 10 + 3 - 5 = 8 \qquad \therefore 2x + y - z = 8.$$

Ex#26 (P-831): Find an equation of the plane through the point $(2,0,1)$ and perpendicular to the line $x = 3\mathbf{t}, y = 2 - t, z = 3 + 4t.$

Solution:

The direction vector of the given line $\bar{v} = \langle 3, -1, 4 \rangle$

the normal vector to the plane $\bar{n} = \langle 3, -1, 4 \rangle$.

Then $\bar{r_0} = \langle 2, 0, 1 \rangle$ is the position vector of the given point $P_0 = (2, 0, 1)$.

The equation of the plane $\bar{r}.\bar{n} = \bar{r_0}.\bar{n} \quad \rightarrow \quad \langle x, y, z \rangle \cdot \langle 3, -1, 4 \rangle = \langle 2, 0, 1 \rangle \cdot \langle 3, -1, 4 \rangle$

$$\Longrightarrow 3x - y + 4z = 6 - 0 + 4 = 10 \qquad \therefore 3x - y + 4z = 10.$$

Ex# 27 (P-831): Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$**.**

Solution:

$\bar{r_0} = \langle 1, -1, -1 \rangle$ is the position vector of the given point $P_0 = (1, -1, -1)$.

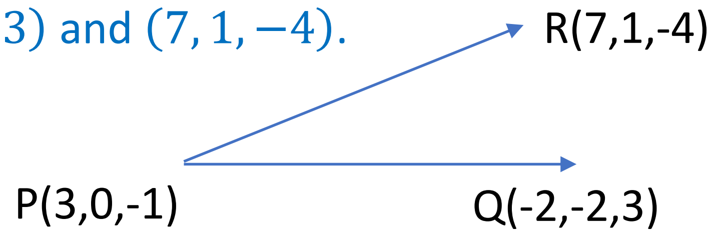
The normal vector to the plane $= \langle 5, -1, -1 \rangle$.

Since the plane is parallel to the given plane, $\bar{n} = \langle 5, -1, -1 \rangle$

The equation of the plane is $\bar{r}.\bar{n} = \bar{r_0}.\bar{n} \quad \rightarrow \quad \langle x, y, z \rangle \cdot \langle 5, -1, -1 \rangle = \langle 1, -1, -1 \rangle \cdot \langle 5, -1, -1 \rangle$

$\Longrightarrow 5x - y - z = 5 + 1 + 1 = 7 \qquad \therefore 5x - y - z = 7.$

Ex# 33: Find an equation of the plane through the points $(3, 0, -1)$, $(-2, -2, 3)$ and $(7, 1, -4)$.



Solution: The vectors $\bar{a}$ and $\bar{b}$ corresponding to $\overline{PQ}$ and $\overline{PR}$ are

$$\bar{a} = \langle -2-3,\ \ -2-0,\ \ 3+1\rangle = \langle -5,\ \ -2,\ \ 4\rangle$$

$$\bar{b} = \langle 7-3, 1-0, -4+1\rangle = \langle 4, 1, -3\rangle$$

Since both $\bar{a}$ and $\bar{b}$ lie in the plane, their cross product $\bar{a} \times \bar{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus,

$$\bar{n} = \bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -5 & -2 & 4 \\ 4 & 1 & -3 \end{vmatrix}$$

$$= (6-4)\hat{\imath} - (15-16)\hat{\jmath} + (-5+8)\hat{k} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$$

With the point $P(3, 0, -1)$ and the normal vector $\bar{n}$, the equation of the plane $\bar{r}.\bar{n} = \bar{r_0}.\bar{n}.$

$$\langle x, y, z\rangle \cdot \langle 2, 1, 3\rangle = \langle 3, 0, -1\rangle \cdot \langle 2, 1, 3\rangle$$

$$\Longrightarrow 2x + y + 3z = 6 + 0 - 3 = 3 \qquad \therefore 2x + y + 3z = 3.$$

Ex# 53 (P-832): Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. $x + 2y - z = 2$ and $2x - 2y + z = 1$.

Solution: The normal vectors of these planes are $\bar{n}_1 = \langle 1, 2, -1\rangle$ and $\bar{n}_2 = \langle 2, -2, 1\rangle$.

$$\frac{1}{2} \neq \frac{2}{-2} = \frac{-1}{1}$$

Since the D.R's are not proportional, so they are not parallel.

Now, $\bar{n}_1 \cdot \bar{n}_2 = 2 - 4 - 1 = -3$

Since $\overline{n_1} \cdot \overline{n_2} \neq 0$, the two planes are not orthogonal.

If θ is the angle between the planes,

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}$$

$$= \frac{\langle 1, 2, -1\rangle \cdot \langle 2, -2, 1\rangle}{\sqrt{1+4+1}\sqrt{4+4+1}}$$

$$= \frac{2-4-1}{3\sqrt{6}}$$

$$= \frac{-3}{3\sqrt{6}} = \frac{-1}{\sqrt{6}}$$

$$\therefore \theta = cos^{-1}\left(\frac{-1}{\sqrt{6}}\right) = 114^0$$

MCQ

1. The magnitude of the vector $\vec{a} = 4\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

   (a) (b) (c) (d)

2. Given the vectors $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = -\hat{\imath} + \hat{\jmath} + 3\hat{k}$. Find $\vec{a} \cdot \vec{b}$.

   (a) (b) (c) (d)

3. The vectors $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = -\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ are

   (a) Parallel (b) Orthogonal (c) None

4. Given the vectors $\vec{a} = 2\hat{\imath} + 4\hat{k}$ and $\vec{b} = 2\hat{\jmath} - 3\hat{k}$. Find $\vec{a} \times \vec{b}$.

   (a) (b) (c) (d)

5. For the two vectors $\vec{a} = 2\hat{\imath} + 4\hat{k}$ and $\vec{b} = 2\hat{\jmath} - 3\hat{k}$, it is given $\vec{a} \cdot \vec{b} = -12$ and $\vec{a} \times \vec{b} = -8\hat{\imath} + 6\hat{\jmath} + 4\hat{k}$.

The area of the parallelogram with sides $\vec{a}$ and $\vec{b}$ is

   (a) (b) (c) (d)

6. For the two given points P(0, 3, -4) and Q(-2, 5, 2) find $\overrightarrow{PQ}$.

   (a) (b) (c) (d)

7. The equation of the line $\frac{x}{3} = \frac{y-2}{-1} = \frac{z-3}{4}$ is in

   (a) Vector form, (b) Symmetric form, (c) parametric form.

8. For the equation of a line $\frac{x}{3} = \frac{y-2}{-1} = \frac{z-3}{4}$, find the point on the line and component of the direction vector.

   (a) (b) (c) (d)

9. The normal vector of the equation of the plane $2(x-5) + (y-3) - (z-5) = 0$

   (a) (b) (c) (d)