Solution of System of Nonlinear Equations: Newton-Raphson Method

Lecture-1

Objective:

Solve the system of nonlinear equations to find roots of the system.

Methodologies:

Two methods can be used to find roots of system of nonlinear equations. They are

1. **Newton-Raphson Method**
2. **Fixed Point Iteration Method**

Newton-Raphson Method

To find the root of nonlinear equation in one variable *f(x)=0,* Newton-Raphson formula can be written as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For a system $f_i(X)$ **Newton-Raphson formula in a matrix form that can be written by introducing inverse of Jacobian matrix as follows:**

$$X_{n+1} = X_n - J_n^{-1} F(X_n) \qquad Eq.(1)$$

Where

$X \Rightarrow$ **Variables matrix**

$F(X) \Rightarrow$ **Function matrix**

$J^{-1}(X) \Rightarrow$ **Inverse of Jacobian matrix**

Variables matrix can be written for n variables as follows:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ : \\ x_n \end{pmatrix} \qquad Eq.(2)$$

Function matrix for n equations can be written:

$$F = \begin{pmatrix} f_1(x_1 \quad x_2 \quad \dots \quad x_n) \\ f_2(x_1 \quad x_2 \quad \dots \quad x_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ f_n(x_1 \quad x_2 \quad \dots \quad x_n) \end{pmatrix} \qquad Eq.(3)$$

While non-linear system of equations are →

$$\begin{aligned} f_1(x_1 \quad x_2 \quad \dots \quad x_n) &= 0 \\ f_2(x_1 \quad x_2 \quad \dots \quad x_n) &= 0 \\ \dots\dots\dots\dots\dots&\dots\dots\dots \\ f_n(x_1 \quad x_2 \quad \dots \quad x_n) &= 0 \end{aligned} \qquad \text{Eq.(4)}$$

Inverse of Jacobian matrix: $J^{-1}(X) = \dfrac{Adj(J)}{Det(J)}; Det(J) \neq 0 \qquad Eq.(5)$

Jacobian matrix

Lets calculate

$$J(x_1, x_2, .., x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \qquad Eq.(6)$$

$$Det(J) = |J| = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \qquad Eq.(7)$$

Adjoint matrix of J(X),

$$Adj(J) = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T \; Eq.(8)$$

Where A_{ij} (i=1,2,…,n; j=1,2,3,…,n) are cofactors of the Matrix J

Perform 1st iteration of Newton-Raphson method, substitute guess value initially in formula given in Eq.(1) , then we get x_1

Further iterations can be done by updating initialized value, i.e., updating x_0 by x_1.

Advantages and Drawbacks: Newton-Raphson Method

Advantages of Newton-Raphson Method

- Converges fast if it converges
- Requires one set of guess values only

Drawbacks of Newton-Raphson Method

- Division by Zero
- Root Jumping
- Oscillations near Local Maxima or Minima
- Inflection Point Issues

Problems and Solutions

Example # : Consider the systems

$$x^2 + xy - 7 = 0$$

$$y^3 - 2x + 4 = 0$$

1. Find Jacobian matrix for the above system
2. Evaluate the inverse of the Jacobian matrix at $(1.5, -1)$.
3. Write down the iterative formula for the above system based on the Newton-Raphson method.
4. Estimate the root to 3 decimal point using the using the above iterative formula once starting with $x_0 = 1.5$ and $y_0 = -1$.
5. Write down MATLAB commands to execute the iteration seven times.
6. Use MATLAB function "**fsolve(fun,x0)**" to find the above root.

Example

1. Find Jacobian matrix for the following system

$$x^2 + xy - 7 = 0$$
$$y^3 - 2x + 4 = 0$$

$$J(x,y) = \begin{pmatrix} \frac{\partial}{\partial x}(x^2 + xy - 7) & \frac{\partial}{\partial y}(x^2 + xy - 7) \\ \frac{\partial}{\partial x}(y^3 - 2x + 4) & \frac{\partial}{\partial y}(y^3 - 2x + 4) \end{pmatrix}$$

Functions matrix $F(X) = \begin{pmatrix} x^2 + xy - 7 \\ y^3 - 2x + 4 \end{pmatrix}$

Therefore, Jacobian $J(x,y) = \begin{pmatrix} 2x + y & x \\ -2 & 3y^2 \end{pmatrix}$

2. Evaluate the inverse of the Jacobian matrix at $(1.5, -1)$.

At point (1.5,-1), Jacobian, $J(x,y) = \begin{pmatrix} 2x + y & x \\ -2 & 3y^2 \end{pmatrix} = \begin{pmatrix} 2 & 1.5 \\ -2 & 3 \end{pmatrix}$

$\det(J) = 6 + 3 = 9 \neq 0$

$$adj(J) = \begin{pmatrix} 3 & 2 \\ -1.5 & 2 \end{pmatrix}^T = \begin{pmatrix} 3 & -1.5 \\ 2 & 2 \end{pmatrix}$$

We know inverse of a matrix A can be calculated as $A^{-1} = \frac{1}{|A|} adj(A)$

So,

$$J^{-1} = \frac{1}{9}\begin{pmatrix} 3 & -1.5 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3333 & -0.1666 \\ 0.2222 & 0.2222 \end{pmatrix}$$

3. Write down the iterative formula for the above system based on the Newton-Raphson method.

The iterative formula for the system of nonlinear equations is

$$X_{n+1} = X_n - J_n^{-1} F(X_n) \qquad \text{[Using Eq. (1)]}$$

Where $X = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ [Using Eq. (2)]

4. Estimate the root to 3 d.p. using the using the above iterative formula once starting with $x_0 = 1.5$ and $y_0 = -1$.

We have, $x_0 = 1.5$ and $y_0 = -1$. Using n = 0 in Eq.(1), we obtain

$$X_1 = X_0 - J_0^{-1} F(X_0) \qquad \text{Eq.(9)}$$

$$F(x_0, y_0) = \begin{pmatrix} {x_0}^2 + x_0 y_0 - 7 \\ {y_0}^3 - 2x_0 + 4 \end{pmatrix} \rightarrow F(1.5, -1) \quad = \begin{pmatrix} (1.5)^2 + (1.5 \times (-1)) - 7 \\ (-1)^3 - 2(1.5) + 4 \end{pmatrix}$$
$$= \begin{pmatrix} -6.25 \\ 0.00 \end{pmatrix}$$

Eq. (9) become

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} 0.3333 & -0.1666 \\ 0.2222 & 0.2222 \end{pmatrix} \begin{pmatrix} -6.25 \\ 0.00 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix} - \begin{pmatrix} 0.3333 & -0.1666 \\ 0.2222 & 0.2222 \end{pmatrix} \begin{pmatrix} -6.25 \\ 0.00 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3.583125 \\ 0.38875 \end{pmatrix} = \begin{pmatrix} 3.583 \\ 0.388 \end{pmatrix}$$

5. Define given functions as follows: $F(x,y) = \begin{pmatrix} x^2 + xy - 7 \\ y^3 - 2x + 4 \end{pmatrix}$

Jacobian, $J(x,y) = \begin{pmatrix} 2x+y & x \\ -2 & 3y^2 \end{pmatrix}$

Program code:

```
clear all
close all
clc;

F=@(x,y) [x.^2+x.*y-7; y.^3-2.*x+4];
J=@(x,y)[2.*x+y,  x;  -2,  3.*y.^2];
x(1)=1.5;
y(1)=-1;
for n=1:7
    Xn=[x(n); y(n)];
    Fn=F(x(n), y(n));
    Jn=J(x(n), y(n));
    Xn1=Xn-Jn\Fn;
  x(n+1)=Xn1(1);
 y(n+1)=Xn1(2);
end

Roots=[x',y']
```

Output:

```
Roots =
      x          y
   1.5000   -1.0000
   3.5833    0.3889
   2.2224    1.2397
   2.2239    0.9236
   2.2655    0.8232
   2.2698    0.8142
   2.2698    0.8141
   2.2698    0.8141
```

6. Find above root by using MATLAB function using fsolve

Program code:

```
clear all
close all
clc;
% Left_hand side of equations as vector
F=@(x) [x(1)^2+x(1)*x(2)-7; x(2)^3-2*x(1)+4];
% Guess solution
x0=[1.5; -1];
% Solve the system
x=fsolve(F, x0)
```

$$F(x,y) = \begin{pmatrix} x^2 + xy - 7 \\ y^3 - 2x + 4 \end{pmatrix}$$

$$x \to x(1)$$

$$y \to x(2)$$

Output:

x = 2.2698

Y = 0.8141

Outcome

By applying Newton-Raphson method, system of nonlinear equations can be solved to find roots (approximately) of the system, although it has few drawbacks.

Multiple questions:

S.No.	Questions
1	Newton-Raphson Method is- (a) Closed method, (b) Open method, (c) Bracketing method
2	What type of solution could be by applying Newton-Raphson method? (a) Analytical solution, (b) Numerical solution
3	Newton-Raphson method can be used to find roots of the following system of equations: (a) Linear equations , (b) Non-linear equations , (c) both (a) and (b)
4	Jacobian matrix should be (a) m$\times m$ order, (b) $\mathrm{m} \times n$ order
5	Which formula can be used to find a root of the system of nonlinear equations (a) $X_{n+1} = X_n - j_n^{-1} F(X_n)$, (b) $X_{n+1} = X_n - F(X_n)/F'(X_n)$
6	Inverse matrix is in (a) m$\times m$ order, (b) $\mathrm{m} \times n$ order
7	How many guess value requires for applying Newton-Raphson method to find roots of equations (a) one, (b) many (c) two

S.No.	Questions
8	In Newton-Raphson formula, convergence fast if a) It converges, (b) it doesn't converges
9	Inverse matrix exists if (a) Determinant of matrix is zero, (b) Determinant of matrix is non-zero
10	Find Jacobian of the following system of nonlinear equations: $$f = x^3 + xy^2 + x - 1$$ $$g = x^3 + y^3 + xy$$ (a) $J = \begin{pmatrix} 3x^2 + y^2 + 1 & 2xy \\ 3x^2 + y & x + 3y^2 \end{pmatrix}$, (b) $J = \begin{pmatrix} 3x^2 - y^2 + 1 & 2xy \\ 3x^2 - y & x + 3y^2 \end{pmatrix}$, (c) both (a) and (b)
11	At the point x=1, y=2, what will be the value of Jacobian in question (10)? (a) $J = \begin{pmatrix} 13 & 4 \\ 5 & 8 \end{pmatrix}$, (b) $J = \begin{pmatrix} 8 & 4 \\ 5 & 13 \end{pmatrix}$, (c) both (a) and (b)

Exercise

 # Consider the following systems

$$x^2 + x^2y^2 - 7 = 0$$

$$y^3 - 2xy + 4 = 0$$

1. Find Jacobian matrix for the above system
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3. Write down the iterative formula for the above system based on the Newton-Raphson method.
4. Estimate the root to 3 decimal point using the using the above iterative formula once starting with $x_0 = 2.5$ and $y_0 = -2$.
5. Write down MATLAB commands to execute the iteration eight times.
6. Use MATLAB function "**fsolve(fun,x0)**" to find the above root.

References

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