Nonlinear Equations in One Variable

(Fixed Point Iteration Method)

Lecture-3

Objective:

To find the root of a nonlinear equation in on variable with the help of Fixed point iteration methods.

Fixed Point Iteration Method

A **fixed point** of a function $g(x)$ is a real number α such that $\alpha = g(\alpha)$

This means α is a root of the equation $x = g(x)$

Algorithm:

Step 1: Consider nonlinear equation *f(x)=0*

***Step* 2:** Rewrite the given equation as follows:

$$\boldsymbol{x = g(x)\ (NOT\ UNIQUE)}$$

***Step* 3:** Find $|g'(x_0)|$ & $\boldsymbol{check\ if}$ $|g'(x_0)| < 1$

Step 4: Initial guess $\boldsymbol{x_0}$ compute $\boldsymbol{x_{n+1} = g(x_n)}$, $where\ n = 0,1,2,\ldots,n$

***Step* 5:** Then calculate x_1 by using initial guess value and continue this process till $|x_{n+1} - x_n| \le \varepsilon$ where ε is the specified accuracy.

Problems and Solutions

Example: Given that $f(x) = 2\cos 2x + 2 - x$

An iterative formula $x_{n+1} = \frac{1}{4}(2 + 3x_n + 2\cos 2x_n)$ can be used to estimate the root of $f(x) = 0$

i. State with reason whether the iterative formula will converge to the root near $x_0 = 3.5$.

ii. If the iterative formula converges to the root, then perform the iteration **two** times to estimate the root to 3 decimal places

iii. Write down MATLAB commands to execute the iterations five times

Solution:

i. Consider $f(x) = 2\cos 2x + 2 - x = 0$ Eq.(1)

Given $x_{n+1} = \frac{1}{4}(2 + 3x_n + 2\cos 2x_n)$ $\therefore \; g(x_n) = \frac{1}{4}(2 + 3x_n + 2\cos 2x_n)$

$\boxed{|g'(x_0)| < 1}$

Now, $g'(x_n) = \frac{1}{4}(3 - 4\sin 2x_n)$

$g'(x_0) = \frac{1}{4}(3 - 4\sin 2x_0) = \frac{1}{4}(3 - 4\sin 2(3.5)) =$ 0.0930133

The sequence will converge because $|g'(3.5)| < 1$.

ii. 1st iteration: We have, x_0=3.5, so consider n = 0 then Eq. (2) become

$$x_1 = \frac{1}{4}(2 + 3x_0 + 2\cos 2\,x_0)$$

[Substitute the value of x_0] $x_{n+1} = \frac{1}{4}(2 + 3x_n + 2\cos 2x_n)$

$$= \frac{1}{4}(2 + 3(3.5) + 2\cos 2\,(3.5))$$

$$= \frac{1}{4}(12.5 + 2(0.753902))$$

$$x_1 = 3.501951$$

2nd iteration: We have now, x_1=3.501951, so consider n=1 then Eq. (2)

$$x_2 = \frac{1}{4}(2 + 3x_1 + 2\cos 2\,x_1)$$

$$= \frac{1}{4}(2 + 3(3.501951) + 2\cos 2\,(3.501951)) = \frac{1}{4}(12.505853 + 2(0.751332))$$

$$x_2 = 3.502129 = 3.5021$$

The root correct to 3 decimal places ≈ 3.502

MATLAB code for the solution for $f(x) = 2\cos 2x + 2 - x = 0$ using Fixed Point iteration. (using five iteration)

$$g(x_n) = \frac{1}{4}(2 + 3x_n + 2\cos 2\, x_n)$$

$$\boldsymbol{x_{n+1} = g(x_n)}$$

iii. MATLAB CODE

% Function using handle f=@(x) *f(x)*

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f=@(x) (2+3*x+2*cos(2*x))/4;        % Function using handle
```

% initial value $x_0 = 3.5$

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x(1)=3.5;
   for n=1:5
    x(n+1)=f(x(n));
    end
```

Output: Solution = x

```
     3.5000    3.5020    3.5021    3.5021    3.5021    3.5021
```

Exercise

Given the equation $5x + \text{sinhx} - e^{x} - 6 = 0.$

The following iterative formulae are suggested to estimate the root of the above equation.

a. $x_{n+1} = \frac{1}{5}(e^{x} - \text{sinhx} + 6)$ b. $x_{n+1} = \frac{1}{6}(e^{x} - \text{sinhx} + 6)\ )$

- State with reason which iterative formula will converge faster to the root near $x = 1$.

- Use the suitable iterative formula from the above two (a) and (b) to find the root correct to 2 decimal places.

- Write MATLAB codes to execute the above iterative formula in five times.

Advantages and Drawbacks: Fixed Point Iteration Method

Advantages:

- ❑ Fast
- ❑ Fewer calculations than bracketing methods
- ❑ Requires one guess only
- ❑ Easier to program

Drawbacks:

- ❑ Convergence is not guaranteed

Outcome

By applying Fixed point iteration method, nonlinear equations in one variable can be solved to find roots (approximately) of the equation, although it has few drawbacks.

Multiple Choice Questions

1. The fixed point iteration method defined as $\boldsymbol{x_{n+1} = g(x_n)}$ **converges if**

a) $|g'(x_1)| < 1$ **, b)** $|g'(x_1)| > 1$ **, c)** $|g'(x_1)| = 1$ d) None

2. Fixed Point Iteration method can be used to find roots of the following types of equations:

(a) Linear equations (b) Non-linear equations (c) both (a) and (b) (d) None

3. A point, say, α is called a fixed point of a function $g(x)$

(a) If it satisfies the equation α = g(α).

(b) If it does not satisfy the equation α = g(α).

(c) both (a) and (b)

d) None

4. To find a root of the equation $f(x) = 0$ by an iterative method, we have to first rearrange the equation into a form $x = g(x)$. What is the name of $g(x)$ here?

a) iteration function

b) non-iteration function

c) both a) and b)

d) None

5. If the equation $x^3 + 2x - 5 = 0$ has a root near $x = 1.4$. Which of the following can be an iteration formulae ?

a) $x_{n+1} = \frac{1}{2}(5 - x_n^{\ 3})$

b) $x_{n+1} = (5 - 2x_n)^{1/3}$

c) both a) and b)

d) Neither