3.2.1

1. $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right)$

$= \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$= \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{15}{4}\Gamma\left(\frac{1}{2}+1\right)$

$= \frac{15}{4}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{15}{8}\Gamma\left(\frac{1}{2}\right)$

$= \frac{15}{8}\sqrt{\pi}$ (Ans)

2. $\dfrac{\Gamma\left(\frac{9}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

$= \dfrac{\frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

$= \dfrac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

(P.t.o.)

$$= \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{105}{16} \quad \text{(Ans)}$$

3. $$\frac{2\Gamma\left(\frac{10}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)} = \frac{2\Gamma\left(\frac{7}{3}+1\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\Gamma\left(\frac{7}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\Gamma\left(\frac{4}{3}+1\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\cdot\frac{4}{3}\Gamma\left(\frac{4}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\cdot\frac{4}{3}\cdot\Gamma\left(\frac{1}{3}+1\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= 2\cdot\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}$$

$$= \frac{56}{81} \quad \text{(Ans)}$$

4. $\Gamma(10) = \Gamma(9+1) = 9! = 362880$.

5. $\Gamma\left(-\frac{7}{2}\right) = \dfrac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}}$

$= -\frac{2}{7} \cdot \Gamma\left(-\frac{5}{2}\right)$

$= -\frac{2}{7} \cdot \dfrac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}}$

$= -\frac{2}{7} \cdot \left(-\frac{2}{5}\right) \Gamma\left(-\frac{3}{2}\right)$

$= \frac{4}{35} \cdot \dfrac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}$

$= \frac{4}{35} \cdot \left(-\frac{2}{3}\right) \Gamma\left(-\frac{1}{2}\right)$

$= -\frac{8}{105} \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$

$= -\frac{8}{105} \times \left(-\frac{2}{1}\right) \Gamma\left(\frac{1}{2}\right)$

$= \frac{16}{105} \cdot \sqrt{\pi}$

(Ans)

3.2.2

a) $\int_0^\infty x^6 e^{-x}\, dx$

$= \int_0^\infty x^{7-1} e^{-x}\, dx$

$= \Gamma(7)$

$= 6!$

$= 720$ (Ans)

b) $\int_0^\infty \sqrt{x}\, e^{-3x}\, dx$

Let,

$u = 3x$

$\therefore x = \frac{u}{3}$

$\therefore dx = \frac{du}{3}$

Changing limit

x	0	∞
u	0	∞

So, the integral becomes,

$\int_0^\infty \left(\frac{u}{3}\right)^{1/2} e^{-u} \frac{du}{3}$

$= \frac{1}{3} \cdot \left(\frac{1}{3}\right)^{1/2} \int_0^u u^{1/2} e^{-u}\, du$

$= \frac{-1}{3\sqrt{3}} \int_0^\infty e^{-u} u^{3/2-1}\, du$

$= \frac{1}{3\sqrt{3}} \cdot \Gamma\left(\frac{3}{2}\right)$

$$= \frac{1}{3\sqrt{3}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{6\sqrt{3}} \sqrt{\pi} \quad \text{(Ans)}$$

c) $\int_0^\infty x^4 e^{-x^2} dx$

$$= \frac{1}{2} \times 2 \int_0^\infty x^{\left(2 \cdot \frac{5}{2} - 1\right)} e^{-x^2} dx$$

$$= \frac{1}{2} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{8} \sqrt{\pi} \quad \text{(Ans)}$$

d) Given, $\int_0^\infty x^5 \cdot e^{-2x^2} dx$

Let, $u = 2x^2$

$\Rightarrow x^2 = \frac{u}{2}$

& $du = 4x\, dx$

$\therefore dx = \frac{du}{4x}$

Limit changing

x	0	∞
u	0	∞

So, $\int_0^\infty x^5 e^{-x} \cdot \frac{du}{4x}$

$= \frac{1}{4}\int_0^\infty x^4 \cdot e^{-\cancel{2x}} \cdot du$

$= \frac{1}{4}\int_0^\infty \left(\frac{u}{2}\right)^2 e^{-u} du$

$= \frac{1}{4}\cdot\frac{1}{2^2}\int_0^\infty u^2 e^{-u} du$

$= \frac{1}{4}\cdot\frac{1}{4}\int_0^\infty u^{3-1} e^{-u} du$

$= \frac{1}{6}\sqrt{3}$

$= \frac{1}{16}\cdot 2!$

$= \frac{1}{8}$ (Ans)

e) $\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy$

Let $u = y^2$

$\therefore du = 2y\, dy$

$\therefore dy = \frac{du}{2y}$

y	0	∞
u	0	∞

So, $\int_0^\infty y^{1/2} \cdot e^{-u} \cdot \frac{du}{\cancel{dy}\, 2y}$

$= \frac{1}{2} \int_0^\infty y^{-1/2}\, e^{-u}\, du$

$= \frac{1}{2} \int_0^\infty (y^2)^{-1/4} \cdot e^{-u}\, du$

$= \frac{1}{2} \int_0^\infty u^{-1/4} \cdot e^{-u}\, du$

$= \frac{1}{2} \int_0^\infty u^{\left(\frac{3}{4}-1\right)} e^{-u}\, du$

$= \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$

$= \frac{1}{2}\left(-\frac{1}{4}\right) \Gamma\left(-\frac{1}{4}\right)$

$= -\frac{1}{8} \Gamma\left(-\frac{1}{4}\right)$ (Ans)

3.2.3

a) $\int_0^1 x^4(1-x)^3 dx$

$= \int_0^1 x^{5-1} (\cancel{x}1-x)^{4-1}$

$= \dfrac{\Gamma 5 \, \Gamma 4}{\Gamma 5+4}$

$= \dfrac{4! \times 3!}{8!}$

$= \dfrac{1}{280}$ (Au)

b) $I = \int_0^4 \dfrac{x^2}{\sqrt{4-x}} dx$

$= \int_0^4 \dfrac{x^2}{\sqrt{4\left(1-\frac{x}{4}\right)}} dx$

$\therefore I = \dfrac{1}{\sqrt{4}} \int_0^4 x^2 \left(1-x/_4\right)^{-\frac{1}{2}} dx$

putting $\frac{x}{4} = u$

$\therefore x = 4u$

$\therefore dx = 4\,du$

x	0	4
u	0	1

So, $I = \frac{1}{\sqrt{4}} \int_0^1 (4u)^2 (1-u)^{-1/2} \cdot 4\,du$

$= \frac{1}{\sqrt{4}} \cdot 4^2 \cdot 4 \int_0^1 u^2 (1-u)^{-1/2} du$

$= 16\sqrt{4} \cdot B\left(3, \frac{1}{2}\right)$

$= 16\sqrt{4} \cdot \frac{\Gamma 3 \cdot \Gamma 1/2}{\Gamma\left(3+\frac{1}{2}\right)}$

$= 16\sqrt{4} \times \frac{2!\sqrt{\pi}}{\Gamma 3/2}$

$= 16\sqrt{4} \times \frac{2\sqrt{\pi}}{1/2 \cdot \Gamma 1/2}$

$= 16\sqrt{4} \times \frac{2\sqrt{\pi}}{1/2\sqrt{\pi}}$

$= 32\sqrt{4} \times \frac{2}{1} = 64\sqrt{4} = 128$ (Ans)

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c) $\int_0^1 y^4 \sqrt{1-y^2}\, dy$

Let

$y^2 = u$

$\therefore du = 2y\,dy$

x	0	1
u	0	1

$= \int_0^1 y^4 \sqrt{1-u} \cdot \frac{du}{2y}$

$= \frac{1}{2} \int_0^1 y^3 \sqrt{1-u} \cdot du$

$= \frac{1}{2} \int_0^1 (y^2)^{3/2} (1-u)^{1/2} du$

$= \frac{1}{2} \cdot \int_0^1 u^{3/2} (1-u)^{1/2} du$

$= \frac{1}{2} \int_0^1 u^{5/2-1} (1-u)^{3/2-1} du$

$= \frac{1}{2} B\left(\frac{5}{2}, \frac{3}{2}\right)$

$= \frac{1}{2} \frac{\Gamma 5/2 \cdot \Gamma 3/2}{\Gamma\left(5/2 + \frac{3}{2}\right)}$

$= \frac{1}{2} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{\Gamma 4}$

$= \frac{1}{2} \cdot \frac{\frac{3}{8} \cdot \sqrt{\pi}}{3!}$

$= \frac{1}{2} \cdot \frac{3}{8} \sqrt{\pi} \cdot \times \frac{1}{3 \times 2}$

$= \frac{\sqrt{\pi}}{32}$ (Ans)

d) $\int_0^{\pi/2} \sin^6\theta \, d\theta$

$= \int_0^{\pi/2} \sin^6\theta \cos^0\theta \, d\theta$

$= \frac{1}{2} B\left(\frac{6+1}{2}, \frac{0+1}{2}\right)$

$= \frac{1}{2} B\left(\frac{7}{2}, \frac{1}{2}\right)$

$= \dfrac{\Gamma\frac{7/2+1}{2} \, \Gamma\frac{1/2+1}{2}}{\Gamma\frac{7/2+1/2+2}{2}}$

$= \dfrac{\Gamma 9/4 \, \Gamma 3/4}{\Gamma 3}$

$= \dfrac{5/4 \, \Gamma 5/4 \, \Gamma 3/4}{2!}$

$= \dfrac{5/4 \cdot \frac{1}{4} \Gamma 1/4 \quad \Gamma 3/4}{2!}$

$= \dfrac{\frac{5}{16} \cdot \sqrt{2}\pi}{2}$

$= \frac{5}{16} \sqrt{2} \cdot \pi \cdot \times \frac{1}{2}$

$= \frac{5}{16\sqrt{2}} \pi$ (Ans)

e). $\int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta$

Let $f(x) = \cos^3 x$

$\therefore f(-x) = \cos^3(-x)$

$= \cos^3 x$

$= f(x)$

So, $f(x)$ is an even function.

So, $\int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta$

$= 2\int_{0}^{\pi/2} \sin^0\theta \cos^3\theta \, d\theta$

$= 2 \cdot \frac{1}{2} B\left(\frac{0+1}{2}, \frac{3+1}{2}\right)$

$= B\left(\frac{1}{2}, 2\right)$

$= \frac{\Gamma 1/2 \; \Gamma 2}{\Gamma 1/2 + 2}$

$= \frac{\sqrt{\pi} \times 1!}{\Gamma 5/2}$

$= \frac{\sqrt{\pi}}{3/2 \cdot \frac{1}{2} \cdot \sqrt{\pi}} = 4/3$ (Ans)

f) Given $\int_0^{\pi/2} \sin^4\theta \cos^5\theta \, d\theta$

$$= \frac{1}{2} B\left(\frac{4+1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{5}{2}, 3\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma 5/2 \cdot \Gamma 3}{\Gamma 5/2+3}$$

$$= \frac{1}{2} \cdot \frac{3/2 \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot 2!}{\Gamma 11/2}$$

$$= \frac{1}{2} \cdot \frac{\frac{3}{2} \cdot \sqrt{\pi}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$= \frac{8}{315} \quad \text{(Ans)}$$

g) Given, $B\left(\frac{7}{2}, 1\right)$

$$= \frac{\Gamma 7/2 \; \Gamma 1}{\Gamma 9/2}$$

$$= \frac{\Gamma 7/2 \cdot \times 1}{\frac{7}{2} \cdot \Gamma 7/2}$$

$$= \frac{2}{7} \quad \text{(Ans)}$$

h) Given $B(10, 11)$

$$= \frac{\Gamma 10 \, \Gamma 11}{\Gamma 10+11}$$

$$= \frac{9!\,10!}{\Gamma 21}$$

$$= \frac{9!\,10!}{20!}$$

$$= 5.41 \times 10^{-7} \text{ (Ans)}$$