Numerical Methods for Science and Engineering

Chapter Two

Solutions to Linear System

Iterative Method

Solution of Linear System by Iterative Method

Objective: Solve the system of linear equations

An iterative method converges, for any choice of the first approximation,

- if every equation satisfies the condition that the magnitude of the coefficient of solving variable
  - is greater than the sum of the absolute values of the coefficients of the other variables.
- A system satisfying this condition is called **diagonally dominant.**
- A linear system can always be reduced to diagonally dominant by **elementary operations**.

For example, in the system

$x + 2y + 10z = 10$ (E1) $\qquad |1| < |2| + |10|$

$x - 10y - z = 24$ (E2) $\qquad |-10| > |1| + |-1|$

$11x + 5y + 8z = 31$ (E3) $\qquad |8| < |11| + |5|$

is not diagonally dominant. Rearranging the system as (E3)-(E1), (E2), (E1)

$10x + 3y - 2z = 21 \qquad |10| > |3| + |-2|$

$x - 10y - z = 24 \qquad |-10| > |1| + |-1|$

$x + 2y + 10z = 10 \qquad |10| > |1| + |5|$

Output : By applying Iterative Method, system of linear equations can be solved to find roots (approximately) of the system.

$$
\begin{aligned}
10x + 3y - 2z &= 21 \\
x - 10y - z &= 24 \\
x + 2y + 10z &= 10
\end{aligned}
$$

$$x = \frac{1}{10}(21 - 3y + 2z)$$

$$y = -\frac{1}{10}(24 - x + z)$$

$$z = \frac{1}{10}(10 - x - 2y)$$

Jacobi Iterative Method:

In this method, **a fixed set of values** is used to calculate all the variables and then repeated for the next iteration with the values obtained previously. The iterative formulas of the above

$$x_{n+1} = \frac{1}{10}(21 - 3y_n + 2z_n)$$

$$y_{n+1} = -\frac{1}{10}(24 - x_n + z_n)$$

$$z_{n+1} = \frac{1}{10}(10 - x_n - 2y_n)$$

Gauss-Seidel iterative method is more efficient than **Jacobi's iterative method** and explained through an example.

Gauss-Seidel Iterative Method

In this method, the values of each variable is calculated using the most **recent approximations** to the values of the other variables. The Gauss-Seidel iterative formulas of the above system are

$$\boxed{\begin{aligned} x &= \frac{1}{10}(21 - 3y + 2z) \\ y &= -\frac{1}{10}(24 - x + z) \\ z &= \frac{1}{10}(10 - x - 2y) \end{aligned}}$$

$$x_{n+1} = \frac{1}{10}(21 - 3y_n + 2z_n)$$

$$y_{n+1} = -\frac{1}{10}(24 - x_{n+1} + z_n)$$

$$z_{n+1} = \frac{1}{10}(10 - x_{n+1} - 2y_{n+1})$$

If initial values are not supplied we can start with initial values

$$x_0 = 0, \quad y_0 = 0, \quad z_0 = 0.$$

and perform the iterations until required accuracy is achieved.

Example: Given the linear system

$$\begin{aligned} 6x + 5y + 3z &= 7 && \text{Eq(1)} \\ 8x - 3y + 2z &= 16 && \text{Eq(2)} \\ 10x - 7y - 8z &= 15 && \text{Eq(3)} \end{aligned}.$$

a. Reduce the above system to diagonally dominant form.
b. Write the corresponding Gauss-Seidel iteration formula.
c. Compute two iterations to estimate the roots to 2 d.p. with $x_0 = 1.5$, $y_0 = -1$ and $z_0 = 1$.
d. Write MATLAB code to iterate the above formula four times.

Solution: Rearranging the system as Eq(2), Eq(1)-Eq(2), Eq(2)-Eq(3)

a.

Eq(2)	$8x - 3y + 2z = 16,$	$\lvert 8\rvert > \lvert -3\rvert + \lvert 2\rvert$
Eq(1)-Eq(2)	$-2x + 8y + z = -9,$	$\lvert 8\rvert > \lvert -2\rvert + \lvert 1\rvert$
Eq(2)-Eq(3)	$-2x + 4y + 10z = 1,$	$\lvert 10\rvert > \lvert -2\rvert + \lvert 4\rvert$

$$\boxed{\begin{aligned} x &= \frac{1}{8}[16 + 3y - 2z] \\ y &= \frac{1}{8}[-9 + 2x - z] \\ z &= \frac{1}{10}[1 + 2x - 4y] \end{aligned}}$$

Gauss-Seidel formula

$$\begin{aligned} x_{n+1} &= \tfrac{1}{8}[16 + 3y_n - 2z_n] \\ y_{n+1} &= \tfrac{1}{8}[-9 + 2x_{n+1} - z_n] \\ z_{n+1} &= \tfrac{1}{10}[1 + 2x_{n+1} - 4y_{n+1}] \end{aligned}$$

c.

Starting with initial values $x_0 = 1.5, \quad y_0 = -1, \quad z_0 = 1$

$$x_{n+1} = \frac{1}{8}[16 + 3y_n - 2z_n]$$
$$y_{n+1} = \frac{1}{8}[-9 + 2x_{n+1} - z_n]$$
$$z_{n+1} = \frac{1}{10}[1 + 2x_{n+1} - 4y_{n+1}]$$

When $n = 0$, we have

$$x_1 = \frac{1}{8}[16 + 3(-1) - 2(1)] = 1.375$$

$$y_1 = \frac{1}{8}[-9 + 2(1.375) - 1] = -0.906$$

$$z_1 = \frac{1}{10}[1 + 2(1.375) - 4(-0.906)] = 0.737$$

For $n = 1$, we have

$$x_2 = \frac{1}{8}[16 + 3(-0.906) - 2(0.737)] = 1.476$$

$$y_2 = \frac{1}{8}[-9 + 2(1.476) - -0.906] = -0.848$$

$$z_2 = \frac{1}{10}[1 + 2(1.476) - 4(-0.848)] = 0.734$$

Solution to 2 d.p. is

$$x = 1.48, \quad y = -0.85, \quad z = 0.73.$$

d. MATLAB

```
>> clear
>> x(1)=1.5;      %Initial values of x, y, z
>> y(1)= -1;
>> z(1)=1;
>> iter(1)=0;
>> for n=1:4
   iter(n+1)=n;
   x(n+1)=(16+3*y(n)-2*z(n))/8;
   y(n+1)=(-9+2*x(n+1)-z(n))/8;
   z(n+1)=(1+2*x(n+1)-4*y(n+1))/10;
  end
>> Solution = [iter',x',y',z']

Solution =
         0    1.5000   -1.0000    1.0000
    1.0000    1.3750   -0.9063    0.7375
    2.0000    1.4758   -0.8482    0.7345
    3.0000    1.4983   -0.8422    0.7366
    4.0000    1.5000   -0.8421    0.7368
```

SAMPLE MCQ

1. To solve the system of equation by iterative method, the system must be in
    a) Diagonally Dominant
    b) Iterative equation
    c) Maximum value eqation
    d) None

2. In **Gauss-Seidel Iterative Method** the values of each variable is calculated using the values of the other variables
    a) Most recent
    b) old
    c) Both
    d) None

3. Which method is more convenient?
    a) Gauss-Seidal Method
    b) Gauss Jacobi method
    c) Both
    d) None

Exercise

1.

a. $x + 8y + 3z = 10,\ 3x - 5y + 7z = 4,\quad 3x - y - z = 1.$

using $x_0 = 0.85,\ y_0 = 0.8$ and $z_0 = 0.75$

b. $2x + 10y - 7z = 20,\ \ 3x - 7y - 5z = 18,\ \ 8x - 5y - 2z = 12.$

using $x_0 = 0.6,\ y_0 = -0.1$ and $z_0 = -3.$

c. $5x + 9y + 12z = 9,\qquad 8x - 4y - 11z = 14,\qquad -2x + 5y + z = 10.$

using $x_0 = 0.75,\ y_0 = 2.5$ and $z_0 = -1.5.$

d. $\mathbf{10}x + \mathbf{5}y + \mathbf{3}z = \mathbf{21},\qquad \mathbf{6x} + \mathbf{3}y - \mathbf{7}z = \mathbf{22},\qquad \mathbf{3}x + \mathbf{16}y + \mathbf{4}z = 1\mathbf{4}.$

using $x_0 = \mathbf{2},\ y_0 = \mathbf{0.8}$ and $z_0 = -1.$

e. $\mathbf{6}x + \mathbf{5}y - \mathbf{8}z = \mathbf{24},\qquad \mathbf{10x} + \mathbf{3}y + \mathbf{4}z = \mathbf{11},\qquad \mathbf{8}y + \mathbf{3}z = 1\mathbf{0}.$

using $x_0 = \mathbf{1},\ y_0 = \mathbf{1.5}$ and $z_0 = -1.$

i. Reduce the above system to diagonally dominant form.

ii. Write the corresponding Gauss-Seidel and Jacobi iteration formula.

iii. Compute two iterations to estimate the roots to 3 d.p. with the given initial values.

iv. Justify your result by direct substitution in the original equations.

v. Write MATLAB codes to solve by left division (backslash) operator.

2. Consider the linear system: $4x + 2y + z = 7,\ 4x + 5y + 3z = 4, 4x + 5y + 7z = 3$

i. Reduce the above system to diagonally dominant form.

ii. Write the corresponding Gauss-Seidel iteration formula.

iii. Compute two iterations to estimate the roots to 2 d.p with the following initial values x= 2, y=-0.75, z=-0.2.

iv. Justify your result by direct substitution in the original equations.

v. Write MATLAB codes to iterate the above formula four times.

3. Consider the linear system: $5x + 2y + z = 7, 2x - 4y + 3z = 6, 3x + 5y + 7z = 6$

i. Reduce the above system to diagonally dominant form.

ii.Write the corresponding Gauss-Seidel iteration formula.

iii. Compute two iterations to estimate the roots to 3 d.p. with the following initial values x= 1.4, y=-0.35, z=0.5.

iv. Justify your result by direct substitution in the original equations.

v. Write MATLAB codes to iterate the above formula four times.