Interpolation

Lecture-1

Objective:

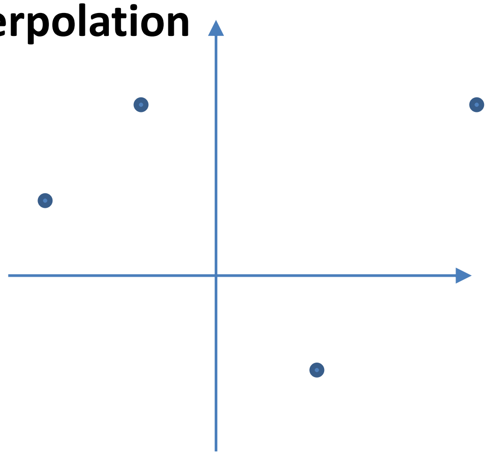
To study the behavior of the function through those points a technique known as **interpolation** is introduced.

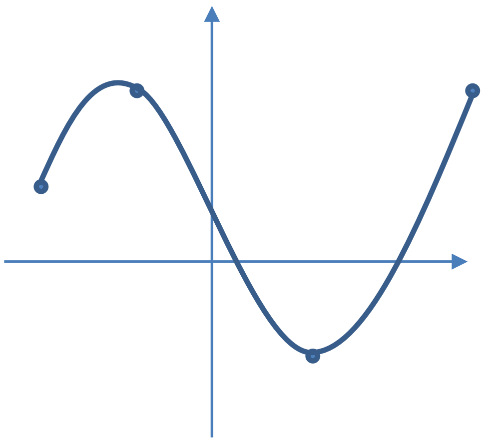
Methodologies:

Two methods can be used for interpolation. They are

1. **Newton's divided difference interpolation**
2. **Lagrange Interpolating Polynomial**

Interpolation





Suppose some sets of values are given $(x_1, y_1), (x_2, y_2), (x_3, y_3) and\ (x_4, y_4)$.

A polynomial can be approximated such that it passes through all the points.

$$f(x) \approx p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Since the polynomial is passing through four points, we will get four equations with four unknowns.

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3$$

$$y_3 = a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3$$

$$y_4 = a_0 + a_1 x_4 + a_2 x_4^2 + a_3 x_4^3$$

To find the unknowns a_0, a_1, a_2, a_3 it will be difficult to find all the constants.

There is another type of polynomial called **Newtonian polynomial.**

$$f(x) \cong p(x) = a_0 + a_1(x - x_1) + a_2(x - x_1)(x - x_2) + a_3(x - x_1)(x - x_2)(x - x_3)$$

Putting $x = x_1, y = y_1,$ we get

$$y_1 = a_0$$

x	y
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

Putting , $x = x_2, y = y_2,$

$$y_2 = a_0 + a_1(x_2 - x_1)$$

$$a_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

And so on.

In this way we can find out all the constants.

If we have n number of points, then the polynomial is of degree n-1.

Newton's Divided difference Interpolation formula

Let us consider the divided difference Table

x	y	1DD	2DD	3DD
x_1	y_1			
x_2	y_2	$f[x_1, x_2]$		
x_3	y_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	
x_4	y_4	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$

$$f[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}, \qquad f[x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}, \qquad f[x_3, x_4] = \frac{y_4 - y_3}{x_4 - x_3}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}, f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$$

$$f(x) \approx y_1 + f[x_1, x_2](x - x_1) + f[x_1, x_2, x_3](x - x_1)(x - x_2) + f[x_1, x_2, x_3, x_4](x - x_1)(x - x_2)(x - x_3)$$

Example

The table below gives the values of x and $f(x)$:

x :	-1	1	2	3	4
$f(x)$:	-7	-1	8	29	68

(i) Construct a divided-difference table for the above data.

(ii) Find the polynomial of least degree that incorporates the values in the table and find $f(5)$

(iii) Find by linear interpolation a real root of $f(x) = 0$

(iv) Find the polynomial $g(x)$ that takes the values of the above table and $g(5) = 203$

(i) **Divided difference Table**

x	Y=f(x)	1DD	2DD	3DD	4DD
-1	-7				
1	-1	3			
2	8	9	2		
3	29	21	6	1	
4	68	39	9	1	0

(ii) The no. of given points = 5

The degree of the polynomial is 4

The polynomial is

$$f(x) \approx -7 + 3(x+1) + 2(x+1)(x-1) + 1(x+1)(x-1)(x-2)$$

Put x=5 , $f(5) =$ 131

(iii) Find the root of f(x)=0 by linear interpolation

x	Y=f(x)	1DD	2DD	3DD	4DD
-1	-7				
1	-1	3			
2	8	9	2		
3	29	21	6	1	
4	68	39	9	1	0

Let us consider

x	Y=f(x)	1DD
1	-1	
2	8	9

$$y = -1 + 9(x - 1)$$

To get the root of f(x)=0, you put y=0 and get the value of x x=1.111

(iv) Find the polynomial $g(x)$ that takes the values of the above table and $g(5) = 203$

Let $g(x) = f(x) + b(x+1)(x-1)(x-2)(x-3)(x-4)$

x
-1
1
2
3
4

Put $x = 5$

$$g(5) = f(5) + b(6)(4)(3)(2)(1)$$

$$203 = 131 + 144b$$

$$b = \frac{1}{2}$$

Now, $g(x) = f(x) + \frac{1}{2}(x+1)(x-1)(x-2)(x-3)(x-4)$

Advantages and Drawbacks:
Newton divided difference interpolation

Advantages of Newton divided difference interpolation:

- Higher-order polynomials can exactly fit larger datasets (by construction).
- They are simpler to evaluate than non-polynomial approximations.

Drawbacks of Newton divided difference interpolation

- Because of their rigidity (due to smoothness), they tend to over-fit the data.
- This over-fitting is a serious issue, which is why it is often much better to use a spline, i.e., a collection of polynomials stitched together

Lagrange Interpolating Polynomial

x	x_0	x_1
y	y_0	y_1

Lagrange polynomial of degree one passing through two points (x_0, y_0) and (x_1, y_1) is written as

$$L_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

Lagrange polynomial of degree two passing through three points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) is written as

$$L_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

x	x_0	x_1	x_2
y	y_0	y_1	y_2

Lagrange polynomial of degree three passing through four points (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) is written as

$$L_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

In general, the Lagrange polynomial of degree *n* passing through $(n+1)$ points (x_0, y_0), (x_1, y_1), $\cdots$, (x_n, y_n) is written as

$$L_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+\cdots+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Example

The following table gives the values of an empirical function

x	0	1	2	3
$f(x)$	–4	–1	8	29

(i) Use the Lagrange interpolation formula to estimate $f(2.5)$

(ii) the root of the equation $f(x) = 0$ to 3 decimal places

Solution:

(i) Applying Lagrange's formula, we have

$$f(x) = -4\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} - 1\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$
$$+8\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} + 29\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

$$f(2.5) = -4\frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)} - 1\frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)}$$

$$+8\frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} + 29\frac{(2.5)(1.5)(0.5)}{(3)(2)(1)}$$

$$= -0.25 + 0.3125 + 7.5 + 9.0625$$

$$= 16.625$$

(ii) Let $y = f(x)$. Then the root of $f(x) = 0$ corresponds to $y = 0$. To find the root let us use the Lagrange formula in reverse order i.e. consider the polynomial in terms of *y*.

y	–4	–1	8	29
x	0	1	2	3

$$x = 0 + \frac{(y+4)(y-8)(y-29)}{(-1+4)(-1-8)(-1-29)}1 + \frac{(y+4)(y+1)(y-29)}{(8+4)(8+1)(8-29)}2$$

$$+\frac{(y+4)(y+1)(y-8)}{(29+4)(29+1)(29-8)}3$$

Put $y = 0,\ x =$1.2434

Advantages and Drawbacks: Lagrange Interpolation

Advantages of Lagrange Interpolation

- The answers for Higher-order polynomials will be more accurate.
- For Higher-order polynomials the approximate result convergesmto the exact solution very quickly.

Drawbacks of Lagrange interpolation

- It becomes a tedious job to do when the polynomial order increases the number of point increases and we need to evaluate approximate solutions for each point.

MATLAB CODE

Write down MATLAB codes using "**polyfit(x, y, n)**" and "**polyval(p, x)**" for the following.

t (s)	10	15	20	22.5	30
$v(t)$ (m/s)	227	363	517	603	903

a. Find the polynomial of least degree that incorporates all the values in the table. and estimate the velocities corresponding to $.t = 17$, 25 and 35 seconds.

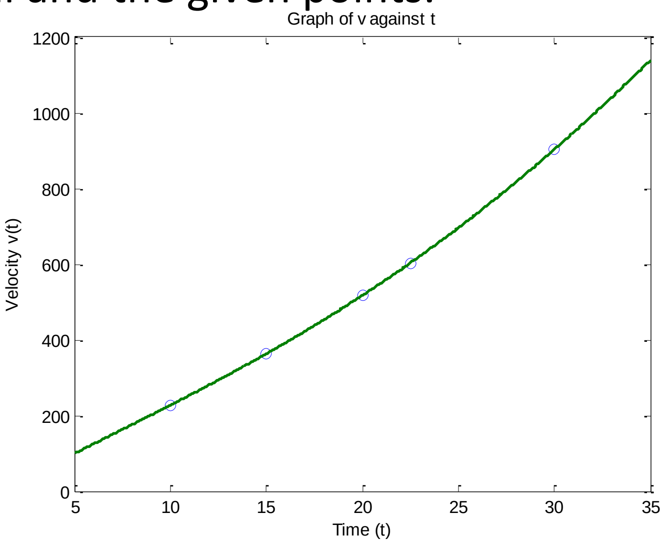
b. Draw the figure showing fitted polynomial and the given points.

a)
```
>> t=[10  15  20  22.5  30];
>> v=[227  363  517  603  903];
>> pt=polyfit(t,v,4)

pt =     -0.0002     0.0240    -0.4267    28.2000   -34.2000

>> t1=[17  25  30];
>> v1=polyval(pt,t1);
>> % Output value of v for t
>> t_v =[t1',v1']
t_v =    17.0000     421.9875
         25.0000     695.8000
         30.0000     903.0000
```

b)



SAMPLE MCQ

1. To study the behavior of the function through only a few discrete sets of values points a technique known as

(a) Curve fitting, (b) Interpolation, (c)Trapezoidal Rule, (d) None

2. By inverse Lagrange method find the real root for $f(x) = 0$ from the sets of data:

x	-1	0
$f(x)$	-2.5	3

(a) 0.45,

(b) -0.54,

(c) -1,

(d) 2.

3. What is the MATLAB command for finding polynomial from some discrete sets of values.

(a) polyal(x,y,x0) , (b) polyfit(x,y,n) , (c) Both, (d) None.

4. By Newtons Divided Difference method find the polynomial from the data:

x	1	2
$f(x)$	-1	8

(a) $9x - 1$ (b)$10x - 9$, (c) $9x - 10$, (d) $x - 9$,

5. From the polynomial from question 4, Find $\boldsymbol{f(5)}$.

(a) 40, (b) 45, (c) 35, (d) 30.

6. With the help of question 4 and 5, Find the polynomial $\boldsymbol{g(x)}$ which takes the value $g(5) = 59$

(a)$x^2 + 6x - 9$, (b) $2x^2 + 3x - 6$, (c) $x^2 + 3x - 5$, (d) $3x^2 + 2x - 6$.

Exercises

1. The table below gives the velocity *v* at time *t*

t(s)	1	3	4	7
v(m/s)	3	5	21	201

i. Construct a divided-difference table for the above data.
ii. Find the polynomial of least degree that incorporates the values in the table.
iii. Find the acceleration at time $t = 6s$.
iv. Find the distance function when $S(0) = 2$.

2. The table below gives the values of x and $f(x)$

x	-2	0	3	6	7
f(*x*)	2	-4	-58	842	1802

(i) Construct a divided-difference table for the above data.
(ii) Find the polynomial which passes through all the points of the table and find $f(5)$

(iii) Find the polynomial $f(5)$ that takes the values of the above table and $g(5) = 549$

3. The table below gives the values of x and $f(x)$

x	4	5	7	9	11
$f(x)$	62	95	185	307	461

i. Construct a divided-difference table for the above data.
ii. Find the polynomial which passes through all the points of the table and find f(12).
iii.Find the polynomial g(x) that takes the values of the above table and g (12)=1280.
iv. Use Lagrange interpolating polynomial to estimate
  a. The value of f(8)using two points.
  b. The value of x for f(x)=380 using three points.
v. Write down MATLAB codes using "**polyfit(x, y, n)**" and "**polyval(p, x)**" for the following.
vi. Find the polynomial of least degree that incorporates all the values in the table and estimate the values corresponding to x=1, 3 and 5

4. The table below gives the values of *x* and *f*(*x*):

x	-2	-1	0	3
$f(x)$	12	14	10	22

i.Construct a divided-difference table for the above data.
ii.Find the polynomial of least degree that incorporates the values in the table and find *f(8)*.
iii.Given *g(8)=1202*, find the polynomial g(x) that also takes the values of the above table.
iv. Use Lagrange interpolation formula to find
  a. a real root of $f(x) = 0$ using linear approximation.
  b. a real root of *fx=0* using all the points.
v. Write down MATLAB codes using "**polyfit(x, y, n)**" and "**polyval(p, x)**" to plot the figure showing fitted polynomial and the given points.