Numerical Integration

Lecture-1

Objectives

- **Develop the basic methods of numerical integration**
- **Use numerical integration to obtain approximate values for definite integrals that cannot be solved analytically**

Applications

- **Mathematics**
- **Engineering**

Background

- **Numerical Integration**

It is an essential tool used by scientists and engineers to obtain approximate values for definite integrals that cannot be solved analytically. For example, the integral

$$\int_0^{0.5} e^{-x^2}\,dx$$

has no closed form solution. The function e^{-x^2} is a continuous bounded function over the interval [0, 0.5] and hence the integral exists. But it is not possible to evaluate it analytically.

Newton-Cotes Quadrature Rule

If the nodes x_r's are uniformly distributed in $x_0 = a, x_n = b$ and the spacing h=(b-a)/n, the method is known as Newton-Cotes integration method and has the order n. When both the end points of the interval are included as nodes, the methods are called closed type methods, otherwise they are called open type methods.

Closed Newton Cotes Quadrature Rule

Assume that $x_r = x_0 + rh$ are equally spaced nodes and $f_r = f(x_r)$ the first few Newton-Cotes quadrature formulas are listed below:

- **Trapezoidal Rule**
- **Simpson's Rule (Simpson 1/3 Rule)**
- **Simpson's 3/8-Rule**

Trapezoidal Rule

For $n = 1$, we obtain the Trapezoidal rule. In this case, the quadrature rule is of the form

$$\int_{x_0}^{x_1} f(x)dx \approx af(x_0) + bf(x_1)$$

Where $h = x_1 - x_0$

To make the simplification short and simple, the axis is translated to make x_0 as the origin. Thus the formula we are looking for is of the form

$$\int_0^h f(x)dx \approx af(0) + bf(h)$$

For two unknown parameters we may assume that the method is exact for $f(x) = 1$ and x.

$$\int_0^h f(x)dx \approx af(0) + bf(h)$$

Let $f(x) = 1$,

LHS= $\int_0^h dx = [x]_0^h = h$

RHS=$a + b$

Then, $a + b = h \qquad \dots.(1)$

Let $f(x) = x$,

LHS= $\int_0^h xdx = \left[\frac{x^2}{2}\right]_0^h = \frac{h^2}{2}$

RHS= bh

Then, $bh = \frac{h^2}{2} \quad \rightarrow \quad b = \frac{h}{2} \qquad \dots..(2)$

From (1) $a = h - b = h - \frac{h}{2} = \frac{h}{2}$

Now the Trapezoidal rule becomes

$$\int_0^h f(x)\ dx \approx \frac{h}{2}[f(0) + f(h)]$$

Precision and an estimate of Error

Let us consider the rule as

$$\int_0^h f(x)dx = \frac{h}{2}[f(0) + f(h)] + E$$

Let $f(x) = kx^2$

$$E = \int_0^h kx^2 dx - \frac{h}{2}kh^2 = \frac{kh^3}{3} - \frac{kh^3}{2} = -\frac{1}{6}kh^3$$

Here $E \neq 0$, Hence the degree of precision is 1

Note that $f''(x) = 2k$

Assuming that $2k = f''(e)$

For $0 \le e \le h$ we can write the error term as

$$E = -\frac{1}{12}h^3 f''(e)$$

The Trapezoidal rule for arbitrary points x_0, x_1 with step size h can be obtained by identifying the points 0 by x_0 and h by $x_1 = x_0$+h. In this case $f(0) = f(x_0)$ and $f(h) = f(x_1)$. Thus

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f(x_0) + f(x_1)]$$

$$\approx \frac{h}{2}[f_0 + f_1]$$

where the notation $f(x_r) = f_r$ is used.

The Trapezoidal rule is

$$\int_{x_0}^{x_1} f(x)\ dx \approx \frac{h}{2}[f(x_0) + f(x_1)] \qquad O(h^2)$$

Solve problems

Example 1#: Evaluate $\int_{0.4}^{1.0} f(x)dx$ numerically using the values given below.

X	0.4	0.5	0.7	1.0
f(x)	1.083	1.133	1.287	1.649

Solutions: Here subinterval sizes are unequal. Using the Trapezoidal rule in each subinterval separately, we have

$$\int_{0.4}^{1.0} f(x)dx = \frac{0.5-0.4}{2}(1.133+1.083) + \frac{0.7-0.5}{2}(1.287+1.133) + \frac{1.0-0.7}{2}(1.649+1.287)$$

$$= \frac{0.1}{2}(2.216) + \frac{0.2}{2}(2.420) + \frac{0.3}{2}(2.936)$$

$$= 0.7932$$

Simpson's Rule (Simpson 1/3 Rule)

For $n = 2$, we obtain the Simpson's rule (Simpson's 1/3 rule)

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$$

Where $h = (x_2 - x_0)/2$

Simpson's rule can easily be proved by considering the integral

$$\int_0^{2h} f(x)dx \approx af(0) + bf(h) + cf(2h)$$

or

$$\int_{-h}^{h} f(x)dx \approx af(-h) + bf(0) + cf(h)$$

and then by translating the axis. This is left as an exercise for the reader. The Simpson's rule has degree of precision three. The error in the formula is

$$E = -\frac{(x_2 - x_0)}{90} h^4 f^{(4)}(e) \qquad x_0 \le e \le x_2$$

Solve problems

Example 2#: The table below shows the values of $f(x)$ at different Values of x.

x	0.4	0.5	0.6	0.8	1.0
f(x)	1.083	1.133	1.197	1.377	1.649

Evaluate $\int_{0.4}^{1.0} f(x)\, dx$ using Simpson's rule.

Solutions: Simpson's rule is applied to two consecutive subintervals of equal length. Thus for the given data we may divide the subintervals as follows:

$$\int_{0.4}^{1.0} f(x)dx = \int_{0.4}^{0.6} f(x)dx + \int_{0.6}^{1.0} f(x)dx$$

$$= \frac{0.1}{3}[f(0.4) + 4 * f(0.5) + f(0.6)]$$

$$+ \frac{0.2}{3}[f(0.6) + 4 * f(0.8) + f(1.0)$$

$$= \frac{0.1}{3}[1.083 + 4(1.133) + 1.197]$$

$$+ \frac{0.2}{3}[1.197 + 4(1.377) + 1.649]$$

$$= 0.2271 + 0.5569 = 0.784$$

$$\int_{0.4}^{1.0} f(x)dx \ = 0.784$$

Simpson's 3/8 - Rule

For $n = 3$, we obtain the Simpson's 3/8 rule

$$\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3 \right)$$

Where $h = (x_3 - x_0)/3$

The precision of the rule is 3. The error in the formula is

$$\mathrm{E} = -\frac{(x_3 - x_0)}{80} h^4 f^{(4)}(e) \qquad x_0 \le e \le x_3$$

Higher order formula can be derived in a similar way.

Outcomes

- **Numerically solved problems by using Trapezoidal Rule and Simpson's Rule.**

Multiple questions

S.No.	Questions
1	Which is not closed Newton Cotes Quadrature Rule? (a) Simpsons 1/3 rule, (b) Trapezoidal rule, (c) Finite difference method
2	What type of solution could be by applying Trapezoidal rule? (a) Analytical solution, (b) Numerical solution
5	Which formula can be used for Trapezoidal rule? (a) $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f_0 + f_1]$, (b) $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f_0 + f_1 + 2f_2]$
6	Which formula can be used for Simpson's 1/3 rule? (a) $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f_0 + f_1]$, (b) $\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}[f_0 + 4f_1 + f_2]$
7	Which formula can be used for Simpson's 3/8 rule? (a) $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f_0 + f_1]$, (b) $\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3]$

Try to do yourself

Exercise 1: The table below shows the values of $f(x)$ at different values of x.

x	1.2	1.4	1.6	1.8	2.0
f(x)	3.728	4.124	4.525	5.123	5.626

Use Trapezoidal rule to estimate $\int_{1.2}^{2.0} f(x)\, dx$

Exercise 2: The table below shows the values of $f(x)$ at different values of x.

x	1.2	1.35	1.5	1.65	1.8
f(x)	3.32	3.86	4.48	5.21	6.05

Use Simpson's rule $\int_{1.2}^{1.8} f(x)\, dx$

Reference

[1] Applied Numerical Methods With Matlab for Engineers and Scientists (Steven C.Chapra).