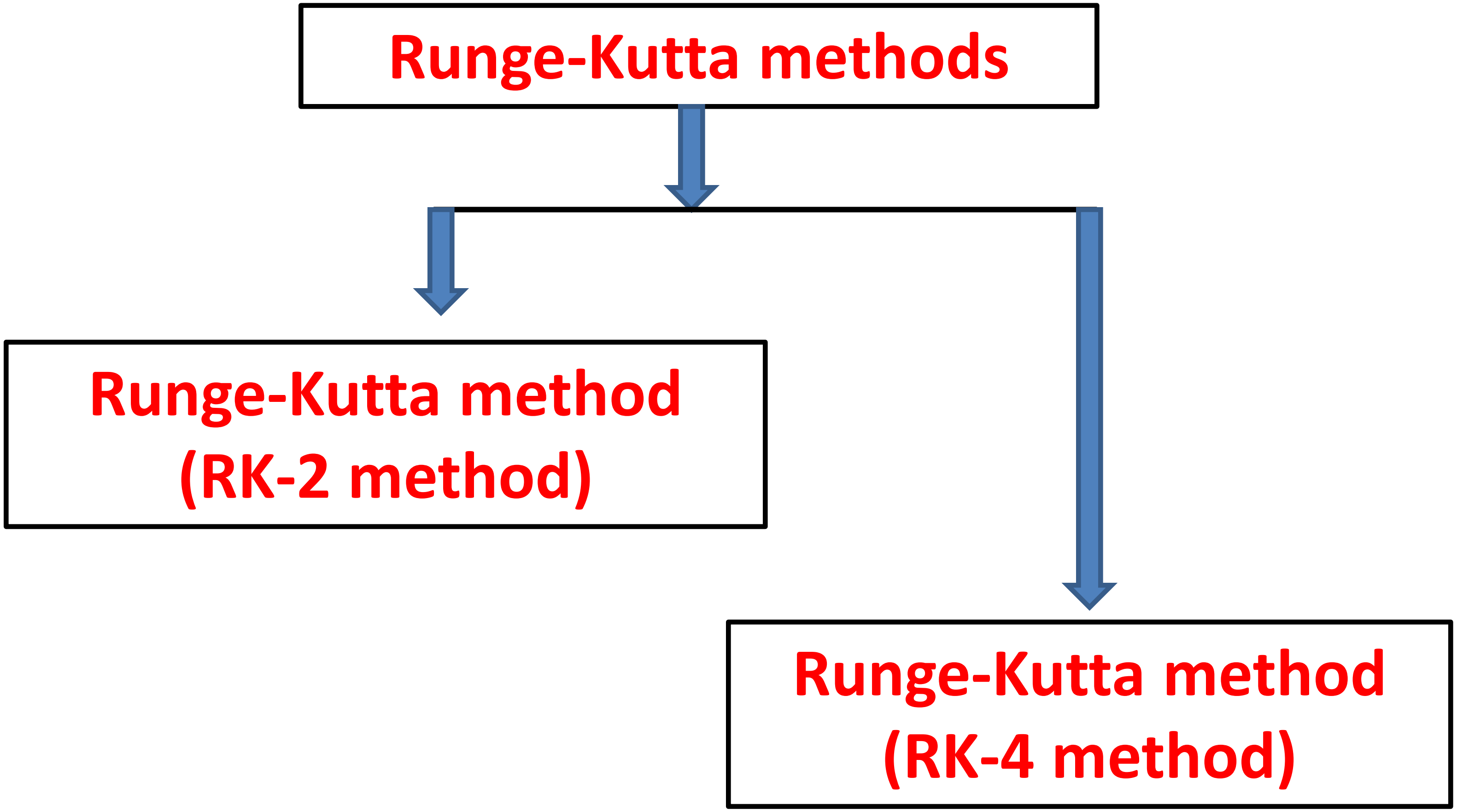
Numerical Solution of Ordinary Differential Equations (ODE): Initial Value Problem (IVP)

Lecture-2

Specific aims

- **Discuss about the Runge-Kutta methods (RK-2 and RK-4 methods)**
- **Examples**
- **Multiple questions**
- **Exercises**


Runge-Kutta methods
Runge-Kutta method
(RK-2 method)
Runge-Kutta method
(RK-4 method)

Runge-Kutta method (RK-2 method)

If we use $y_1 = y_0 + h\,f(x_0, y_0)$ on the right side of the modified Euler's formula, we have

$$y_1 = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f\big(x_1, y_0 + h\,f(x_0, y_0)\big)]$$

Setting

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = h\,f(x_0 + h, y_0 + k_1)$$

we have

$$y_1 = y_0 + \frac{1}{2}\,[k_1 + k_2]$$

This is known as the second order Runge-Kutta formula. It can be shown that RK-2 method is equivalent to Taylor series of order two and the order of the error is $O(h^3)$.

Runge-Kutta method (RK-4 method)

The fourth order Runge-Kutta formula, the most commonly used one in practice, is stated without proof.

$$y_1 = y_0 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

Note that the **process is not unique**, and many other variations are possible. In fact the fourth order process is very accurate and most frequently used. This formula is equivalent to Taylor series of order four and the order of the error is $O(h^5)$

Examples

Question 1#: Given that $y' = 2xy^2 - y$, where $y = 1$ at $x = 0$.

Estimate the values of $y(0.2)$ using (a) the Runge-Kutta method of order two, (b) the Runge-Kutta method of order four; with step size $h = 0.1$.

Solution:

Here $f(x, y) = 2xy^2 - y$

and $x_0 = 0$, $y_0 = 1$, h=0.1.

(a) we know, the formula of second order Runge-Kutta method

$\Downarrow$

$$y_1 = y_0 + \frac{1}{2}\,[k_1 + k_2]$$

where

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f(x_0 + h, y_0 + k_1)$$

Now

$$k_1 = h\,f(x_0, y_0)$$

$$= h\,[2x_0{y_0}^2 - y_0]$$

$$= 0.1\,[2(0)(1)^2 - 1] = -0.1$$

$$f(x,y) = 2xy^2 - y$$
$$x_0 = 0, \qquad y_0 = 1$$
$$h = 0.1$$

$$k_2 = h\,f(x_0 + h, y_0 + k_1)$$

$$k_2 = h\,f(0 + 0.1, 1 - 0.1)$$

$$= h\,f(0.1, 0.9)$$

$$= 0.1\,[2(0.1)(0.9)^2 - 0.9] = -0.0738$$

Therefore,

$$y_1 = y_0 + \frac{1}{2}\,[k_1 + k_2]$$

$$y_1 = y(0.1) = 1 + \frac{1}{2}\,[-0.1 - 0.0738] = 0.9131$$

For y(0.2): $x_1 = 0.1$, $y_1 = 0.9131$, h=0.1.

$$f(x, y) = 2xy^2 - y$$

Now

$$k_1 = h\, f(x_1, y_1)$$

$$= h\, [2x_1 {y_1}^2 - y_1]$$

$$= 0.1\, [2(0.1)(0.9131)^2 - 0.9131] = -0.0746$$

$$k_2 = h\, f(x_1 + h, y_1 + k_1)$$

$$k_2 = h\, f(0.1 + 0.1, 0.9131 - 0.0746)$$

$$= h\, f(0.2, 0.84)$$

$$= 0.1\, [2(0.2)(0.84)^2 - 0.84] = -0.0558$$

Therefore,

$$y_2 = y_1 + \frac{1}{2}\, [k_1 + k_2]$$

$$y_2 = y(0.2) = 0.9131 + \frac{1}{2}\, [-0.0746 - 0.0558] = 0.848$$

(b) we know, the formula of second order Runge-Kutta method

$$y_1 = y_0 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

Now

$$k_1 = h\,f(x_0, y_0)$$

$$= h\,[2x_0 {y_0}^2 - y_0]$$

$$= 0.1\,[2(0)(1)^2 - 1] = -0.1$$

$$k_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$= h\, f(0 + \frac{0.1}{2}, 1 + \frac{-0.1}{2})$$ [Since $x_0 = 0$, $y_0 = 1$, k_1=-0.1, h=0.1]

$$= 0.1\, f(0.05, 0.95)$$

$$= 0.1\,[2(0.05)(0.95)^2 - 0.95]$$ [Since $f(x,y) = 2xy^2 - y$]

$$= -0.08598$$

$$k_3 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$= h\, f(0 + \frac{0.1}{2}, 1 + \frac{-0.08598}{2})$$ [Since $x_0 = 0$, $y_0 = 1$, k_2= -0.08598, h=0.1]

$$= 0.1\, f(0.05, 0.957)$$

$$= 0.1\,[2(0.05)(0.957)^2 - 0.957]$$ [Since $f(x,y) = 2xy^2 - y$]

$$= -0.08654$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

$$= h\,f(0 + 0.1, 1 - 0.08654)$$

[Since $x_0 = 0$, $y_0 = 1$, k_3= - 0.08654, h=0.1]

$$= h\,f(0.1, 0.913)$$

$$= 0.1\,[2(0.1)(0.913)^2 - 0.913]$$

[Since $f(x, y) = 2xy^2 - y$]

$$= -0.0745$$

Therefore,

$$y_1 = y_0 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

Substituting the values of y_0, k_1, k_2, k_3 and k_4 , we get

$$y_1 = y_0 + \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}\ [-0.1 + 2(-0.08598) + 2(-0.08654) - 0.0745]$$

$$= 1 + \frac{1}{6}\ [-0.1 - 0.17196 - 0.17308 - 0.0745]$$

$$= 0.91341$$

For y(0.2): $x_1 = 0.1,\ y_1 = 0.91341$, h=0.1.

$$k_1 = h\, f(x_1, y_1)$$

$$= h\ [2x_1{y_1}^2 - y_1]$$ [Since $f(x, y) = 2xy^2 - y$]

$$= 0.1\ [2(0.1)(0.91341)^2 - 0.91341] = -0.0747$$

$$k_2 = h\,f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$= h\,f(0.1 + \frac{0.1}{2}, 0.91341 + \frac{-0.0747}{2})$$

[Since $x_1 = 0.1,\; y_1 = 0.91341,$
k_1= - 0.0747, h=0.1]

$$= 0.1\,f(0.15, 0.876)$$

$$= 0.1\,[2(0.15)(0.876)^2 - 0.876]$$ [Since $f(x, y) = 2xy^2 - y$]

$$= -0.065$$

$$k_3 = h\,f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$$

$$= h\,f(0.1 + \frac{0.1}{2}, 0.91341 + \frac{-0.065}{2})$$

[Since $x_1 = 0.1,\; y_1 = 0.91341,$
k_2= - 0.065, h=0.1]

$$= 0.1\,f(0.15, 0.881)$$

$$= 0.1\,[2(0.15)(0.881)^2 - 0.881]$$ [Since $f(x, y) = 2xy^2 - y$]

$$= -0.0648$$

$$k_4 = h\, f(x_1 + h, y_1 + k_3)$$

$$= h\, f(0.1 + 0.1, 0.91341 - 0.0648)$$

[Since $x_1 = 0.1,$
$y_1 = 0.91341,$
k_3= - 0.0648, h=0.1]

$$= h\, f(0.2, 0.849)$$

$$= 0.1\, [2(0.2)(0.849)^2 - 0.849]$$

[Since $f(x, y) = 2xy^2 - y$]

$$= -0.056$$

Therefore,

$$y_2 = y_1 + \frac{1}{6}\, [k_1 + 2k_2 + 2k_3 + k_4]$$

Substituting the values of y_1, k_1, k_2, k_3 and k_4 , we get

$$y_2 = y_1 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0.91341 + \frac{1}{6}\,[-0.0747 + 2(-0.065\,) + 2(-0.0648) - 0.056]$$

$$= 0.91341 + \frac{1}{6}\,[-0.0747 - 0.13 - 0.129 - 0.056]$$

$$= 0.8483$$

Answer: (a) 0.848, (b) 0.8483

Outcomes

- **Numerically solved problems of ODE by using Runge-Kutta method in order second and four.**

Multiple questions:

S.No.	Questions
1	Which formula refers to Runge-Kutta method of second order? (a) $y_1 = y_0 + \frac{1}{2}\ [k_1 + k_2]$, (b) $y_1 = y_0 + \frac{1}{2}\ [k_0 + k_2]$ (c) Both of them, (d) None of them
2	What is the expression of k_2 for the Runge-Kutta method of second order? (a) $k_2 = h\, f(x_0 + h, y_0 + k_1)$ (b) $k_2 = h\, f(x_0 + \frac{h}{2}, y_0 + k_1)$
3	Which formula refers to Runge-Kutta method of order four? (a) $y_1 = y_0 + \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]$, (b) $y_1 = y_0 + \frac{1}{2}\ [k_0 + k_2 + 2k_3 + k_4]$ (c) Both of them
4	What is the expression of k_4 for the Runge-Kutta method of four order? (a) $k_4 = h\, f(x_0 + h, y_0 + k_1)$, (b) $k_4 = h\, f(x_0 + h, y_0 + k_3)$ (c) None of them

Try to do yourself

Exercise 1: Given that $y' = 2x^2 - y + 3y^2$, where $y = 0.5$ at $x = 2$.

(a) Estimate the values of $y(2.2)$ using the Runge-Kutta method of order two,

(b) Estimate the values of $y(2.4)$ using the Runge-Kutta method of order four.

Where consider step size $h = 0.2$.

Exercise 2: Given that $y' = 2x + x^2y$, where $y(0) = -1$ with step size h=0.1.

(a) Estimate the values of $y(0.2)$ using the Runge-Kutta method of order two,

(b) Estimate the values of $y(0.2)$ using the Runge-Kutta method of order four.