Functions of several Variables

- **Partial Derivatives:** If $\mathrm{z} = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

- **Rule:** To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x, and
  - To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

- **Examples:**
  - Introduction: Page – 914 # 1, 2 ,3, 4.
  - Higher Derivatives: Page – 918 # 7, 8.
  - Partial Differential Equations: Page – 920 # 9, 10.
  - **Exercises:** Page – 924 # 15, 17, 18, 19, 26, 41, 42. Page – 925 # 59, 60, 63, 64, 65, 76(a, e, f), 78(a, d).

<u>Example1</u> Given $f(x,y) = x^4 + 5xy^3$. Find $f_{xx}, f_{xy}, f_{yy}, f_{yx}, f_{xxx}$.

$$f_x(x,y) = \frac{\partial}{\partial x}(x^4 + 5xy^3) = 4x^3 + 5y^3, \quad f_y(x,y) = \frac{\partial}{\partial y}(x^4 + 5xy^3) = 5x \cdot 3y^2 = 15xy^2$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(4x^3 + 5y^3) = 12x^2, \qquad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(4x^3 + 5y^3) = 15y^2,$$

$$f_{yy} = \left(f_y\right)_y = \frac{\partial}{\partial y}(15xy^2) = 30xy, \qquad f_{yx} = \left(f_y\right)_x = \frac{\partial}{\partial x}(15xy^2) = 15y^2, \quad f_{xxx} = (f_{xx})_x = \frac{\partial}{\partial x}(12x^2) = 24x.$$

<u>Example2</u> $u = x^2 - y^2$, verify the Laplace equation: $u_{xx} + u_{yy} = 0.$

<u>Solution</u> $u_x = 2x, \implies u_{xx} = 2$ and $u_y = -2y, \implies u_{yy} = -2 \cdot \quad \therefore u_{xx} + u_{yy} = 0.$ (verified)

<u>Exercise</u> (P-925)#78(c) Given $u = (x - at)^6 + (x + at)^6.$

Show that the function $u(x,t)$ is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.

<u>Solution</u> $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = (u_x)_x = 30(x - at)^4 + 30(x + at)^4$

$$u_t = -6a(x - at)^5 + 6a(x + at)^5, \quad u_{tt} = (u_t)_t = 30a^2(x - at)^4 + 30a^2(x + at)^4$$

$u_{tt} = a^2[30(x - at)^4 + 30(x + at)^4] = a^2 u_{xx}$ (shown)

• Maximum and Minimum Values

- **Definition:**

A function of two variables $f(x, y)$ has a

- **local maximum** at (a, b), if $f(x, y) \leq f(a, b)$
- **local minimum** at (a, b), if $f(x, y) \geq f(a, b)$

when (x, y) is near (a, b).

Then $f(a, b)$ is a **local maximum value** or **local minimum value.**

- **Stationary point or critical point :**

A point (a, b) is called a Critical point (or stationary point) of $f(x, y)$ if

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

• Maximum and Minimum Values

- **Second Derivatives Test**

Let (a, b) be a critical point of $f(x, y)$ and

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - \left[\, f_{xy}(a, b)\right]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then $f(a, b)$ is not a local maximum or minimum, f has a saddle point at (a, b).
- If $D = 0$, then no conclusion can be drawn.

  - Examples: Page - 961 # 3.
  - Exercises for practice Page – 968 # 3, 5, 6, 9, 11.

Find the local maximum and minimum values and saddle point(s) of the functions (Page – 968)

Exercise -#11. $f(x,y) = x^3 - 3x + 3xy^2$ ·..........(11.1)

Solution: Here, $f_x = 3x^2 - 3 + 3y^2$ and $f_y = 6xy$

For stationary points, $\boldsymbol{f_x = 0} \rightarrow 3x^2 - 3 + 3y^2 = 0 \rightarrow x^2 + y^2 = 1 \quad (11.2)$

$$\boldsymbol{f_y = 0} \rightarrow 6xy = 0 \rightarrow x = 0 \text{ or } y = 0 \quad (11.3)$$

Now putting $x = 0$ in (11.2), $y = \pm 1$ and $y = 0$ in (11.2), $x = \pm 1$

$\therefore$ Stationary points are $(0, 1)$, $(0, -1), (1, 0)$and $(-1, 0)$ ·

$$f_{xx} = 6x, \qquad f_{yy} = 6x, \qquad f_{xy} = 6y$$

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \{f_{xy}(x,y)\}^2$$
$$= 6x \cdot 6x - (6y)^2 = 36x^2 - 36y^2.$$

Stationary points (a,b)	$\boldsymbol{D}(a,b)$	$f_{xx}(\boldsymbol{a},b)$	Results
$(0,1)$	-36		**Saddle point**
$(0,-1)$	-36		**Saddle point**
$(1,0)$	36	6	**local minimum**
$(-1,0)$	36	-6	**local maximum**

MCQ

1. If $f(s,t) = s^4t^3$, then find $f_{st}\,(-1,1)$

(a) (b) (c) (d)

2. If $u = \sin(xy)$, then find $u_x\left(1, \frac{\pi}{2}\right)$

(a) (b) (c) (d)

3. If (a, b) is a critical point of any given function $f(x, y)$ and D(a, b) <0, then (a, b) is a

(a) (b) (c) (d)

4. If (a, b) is a critical point and D(a, b)>0 and $f_{xx}(a, b) > 0$, then at (a, b) there is a

(a) (b) (c) (d)

5. If $f(x, y) = 4 + x^3 + y^3 - 3xy$, then f_x =?

(a) (b) (c) (d)

6. If $f(x, y) = 4 + x^3 + y^3 - 3xy$, then f_y =?

(a) (b) (c) (d)

7. If $u(r, t) = r^2 - t^2$, then $u_{rr} + u_{tt}$ is equal to

(a) (b) (c) (d)