Numerical Integration

Romberg Integration

Lecture-2

Specific aims

- ❑ **Discuss about Composite trapezoidal, Simpson's rules and Romberg integration methods.**
- ❑ **Use this numerical integration methods to obtain approximate values for definite integrals that cannot be solved analytically.**

Applications

- ➢ **Mathematics**
- ➢ **Engineering**

Advantages

- To avoid the use of higher order methods and still obtain accurate results, we use the composite integration methods.
- The interval [*a*, *b*] is divided into a number of subintervals and evaluate the integral in each required number of subintervals by a particular method.

Composite Trapezoidal Rules

We divide the interval [*a*, *b*] into N subintervals $[x_{r-1}, x_r]$ each of length h=(b-a)/N, $x_0 = a, x_N = b$, and $x_r = x_0 + rh$, r=1,2,3,….,N. We can write,

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)dx$$

Evaluating each integral on the right hand side by the trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_{21}) + \cdots + \frac{h}{2}(f_{N-1} + f_N)$$

$$\approx \frac{h}{2}[f_0 + f_N + 2(f_1 + f_2 + f_3 + \cdots + f_{N-1})]$$

The error in the formula is

$$E_T = -\frac{b-a}{12} h^2 f''(e) \qquad a \le e \le b$$

Composite Simpson's Rules

We divide the interval [a, b] into N, an even number of subintervals $[x_{r-1}, x_r]$, each of length h=(b-a)/N, $x_0 = a, x_N = b$, and $x_r = x_0 + rh$, r=1,2,3,….,N. We can write,

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x2}^{x_4} f(x)dx + \cdots + \int_{x_{N-2}}^{x_N} f(x)dx$$

Evaluating each integral on the right hand side by the Simpson's rule

$$\int_a^b f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2) + \frac{h}{3}(f_2 + 4f_3 + f_4) + \cdots + \frac{h}{3}(f_{N-2} + 4f_{N-1} + f_N)$$

$$\approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{N-2} + 4f_{N-1} + f_N)$$

The error in the formula is

$$E_S = -\frac{h^4}{180}(b-a)f^{(4)}(e) \qquad a \le e \le b$$

Romberg Integration

Romberg integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integration by repeated applications of the Richardson's Extrapolation formula. It is known that the Trapezoidal rule approximation $I_T(h)$ to an integral I has error behavior

$$\mathrm{I} = \mathrm{I}_T(h) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots$$

Suppose that $I^{(0)}(n, h)$ is a Trapezoidal estimation of the integral, I with n subintervals and step size h. By doubling the step size the corresponding estimate is $I^{(0)}(2n, h/2)$.

The first Richardson extrapolated value $I^{(1)}\left(2n, \frac{h}{2}\right)$ *is*

$$\mathrm{I}^{(1)}(2n, h/2) = \mathrm{I}^{(0)}(2n, h/2) + \frac{\mathrm{I}^{(0)}(2n. h/2) - \mathrm{I}^{(0)}(n. h)}{2^2 - 1}$$

has error of order $O(h^4)$.

The Simpson's rule has an error of order $O(h^4)$. In fact, $I^{(1)}\left(2n, \frac{h}{2}\right)$ is exactly the Simpson's rule estimate. The second improved estimate $I^{(2)}\left(2n, \frac{h}{2}\right)$ is

$$\mathrm{I}^{(2)}(4n, h/4) = \mathrm{I}^{(1)}(4n, h/4) + \frac{\mathrm{I}^{(1)}(4n. h/4) - \mathrm{I}^{(1)}(2n. h/2)}{2^4 - 1}$$

has error of order $O(h^6)$ and so on.

Example

Problem 1: The table below shows the values of $f(x)$ at different Values of x.

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	1.000	1.780	1.954	2.000	1.976	1.909	1.814

Evaluate $I = \int_0^{1.2} f(x)dx$ by using extrapolation and (i) trapezoidal rule
(ii) Simpson's rule

Solution: (i) Given table is

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	1.000	1.780	1.954	2.000	1.976	1.909	1.814
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

$$\text{Here, } h = 0.2 - 0 = 0.2,$$
$$= 0.4 - 0.2 = 0.2,$$
$$= 0.6 - 0.4 = 0.2,$$

$$h = 0.8 - 0.6 = 0.2,$$
$$= 1.0 - 0.8 = 0.2$$

Trapezoidal rule:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] \qquad O(h^2), \qquad h = x_1 - x_0$$

$$\int_0^{1.2} f(x)\ dx = [\int_0^{0.2} + \int_{0.2}^{0.4} + \int_{0.4}^{0.6} + \int_{0.6}^{0.8} + \int_{0.8}^{1.0} + \int_{1.0}^{1.2}]\ f(x)\,dx$$

$$I_T(0.2) = \int_0^{1.2} f(x)\,dx$$

$$= \frac{0.2}{2}\left[f(0) + 2\big(\boldsymbol{f(0.4)} + \boldsymbol{f(0.4)} + \boldsymbol{f(0.6)} + \boldsymbol{f(0.8)} + \boldsymbol{f(1.0)}\big) + f(1.2)\right]$$

$$= 0.1 \times 22.078 = 2.208$$

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	1.000	1.780	1.954	2.000	1.976	1.909	1.814

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Let us consider $h = 0.4$

$$\int_0^{1.2} f(x)\ dx = \left[\int_0^{0.4} + \int_{0.4}^{0.8} + \int_{0.8}^{1.2}\right] f(x)dx$$

$$I_T(0.4) = \frac{0.4}{2}\left[f(0) + 2\big(f(0.4) + f(0.8)\big) + f(1.2)\right]$$

$$= 0.2 \times 10.674 = 2.135$$

Richardson Extrapolation

$$I_R = I(h_1) + \frac{I(h_1) - I(h_2)}{r^n - 1}$$

Where, $r = \frac{h_2}{h_1}$, $\quad h_2 > h_1 \ \ and\ n = 2\ for\ Trapezoidal\ rule$

Using Richardson Extrapolation,

$$I_R = I_T(0.2) + \frac{I_T(0.2) - I_T(0.4)}{2^2 - 1}$$

$$= 2.208 + \frac{2.208 - 2.135}{3} = 2.232$$

Simpson's Rule

$$\int_{x_0}^{x_2} f(x)\ dx = \frac{h}{3}[f_0 + 4f_1 + f(2)], \qquad O(h^4), \qquad h = \frac{x_2 - x_0}{2}$$

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	1.000	1.780	1.954	2.000	1.976	1.909	1.814

$$\int_0^{1.2} f(x)dx = \left[\int_0^{0.4} \quad + \int_{0.4}^{0.8} \quad + \int_{0.8}^{1.2} \quad \right] f(x)dx$$

Now, $h = \frac{0.4-0}{2} = 0.2,$

$$I_S(0.2) = \int_0^{0.4} f(x)dx + \int_{0.4}^{0.8} f(x)\ dx + \int_{0.8}^{1.2} f(x)dx$$

$$I_S(0.2) = \frac{0.2}{3}\Big[\big(\boldsymbol{f(0) + 4f(0.2) + f(0.4)}\big) + \big(\boldsymbol{f(0.4) + 4f(0.6) + f(0.8)}\big) + \big(\boldsymbol{f(0.8) + 4f(1.0) + f(1.2)}\big)\Big]$$

$I_S(0.2)$ =2.229

With step size , h=0.4 , the number of subintervals, 1.2/0.4=3 (odd number intervals)and Simpson's rule cannot be used

Taking h = 0.6, now the number of subintervals is 2 that is even. From given table we get

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	1.000	1.780	1.954	2.000	1.976	1.909	1.814
	f_0			f_1			f_2

So the integral with composite Simpson's rule is

$$I_S = \int_a^b f(x)dx \approx \frac{h}{2}(f_0 + 4f_1 + f_2)$$

$$I_S(0.6) = \frac{0.6}{3}\left[1 + 4\times 2 + 1.814\right] = 2.163$$

Using Richardson extrapolation

$$I_R = I_S(0.2) + \frac{I_S(0.2) - I_S(0.6)}{3^4 - 1} = 2.229 + \frac{2.229 - 2.163}{80} = 2.223$$

Example

Problem 2: Evaluate $\int_0^1 x \exp\left(\frac{x^2}{2}\right) dx$

using Trapezoidal rule with 1, 2 and 4 subintervals. Improve the results using **Romberg** integration.

Solution:

Here, $f(x) = x\, e^{x^2/2}$

$$n = 1, h = (1-0) = 1; \mathrm{I}^{(0)}(1,1) = \frac{1}{2}[f(0) + f(1)] = 0.8244$$

$$n = 2, h = \tfrac{1}{2} = 0.5; \mathrm{I}^{(0)}(2, 0.5) = \frac{0.5}{2}[f(0) + f(1) + 2f(0.5)]$$

$$= 0.6955$$

$$n = 4, h = \tfrac{1}{4} = 0.25; \mathrm{I}^{(0)}(4, 0.25) = \frac{1}{2}[f(0) + f(1) + 2(f(0.25) + f(0.5) + f(0.75))]$$

$$= \quad 0.6606$$

First order extrapolated values are

$$I^{(1)}(2, 0.5) = I^{(0)}(2, 0.5) + \frac{I^{(0)}(2, 0.5) - I^{(0)}(1, 1)}{2^2 - 1}$$

$$= 0.6955 + \frac{0.6955 - 0.6244}{3} = 0.6525$$

And $$I^{(1)}(4, 0.25) = I^{(0)}(4, 0.25) + \frac{I^{(0)}(4, 0.25) - I^{(0)}(2, 0.5)}{2^2 - 1}$$

$$= 0.6606 + \frac{0.6606 - 0.6955}{3} = 0.6490$$

Second extrapolated value is

$$I^{(2)}(4, 0.25) = I^{(1)}(4, 0.25) + \frac{I^{(1)}(4, 0.25) - I^{(1)}(2, 0.5)}{2^4 - 1}$$

$$= 0.6490 + \frac{0.6490 - 0.6525}{15} = 0.6488$$

Results are summarized below in a Table:

N	H	$O(h^2)$	$O(h^4)$	$O(h^6)$
1	1	0.8244		
2	0.5	0.6955	0.6528	
4	0.25	0.6606	0.6490	0.6488

Trapezoidal rule: $\int_{x_0}^{x_1} f(x)\ dx \approx \frac{h}{2}(f_0 + f_1)$ $\quad O(h^2),\ \ h = x_1 - x_0$

$$\int_0^1 x \exp\left(\frac{x^2}{2}\right) dx \qquad f(x) = x \exp\left(\frac{x^2}{2}\right)$$

0 0.25 0.5 0.75 1

$$n = 1, h = 1, \qquad I^0(h = 1) = \frac{1}{2}[f(0) + f(1)] =$$

$$n = 2, h = 0.5, \qquad I^0(h = 0.5) = \frac{0.5}{2}[f(0) + 2f(0.5) + f(1)] =$$

$$n = 4, h = 0.25, I^0(h = 0.25) = \frac{0.25}{2}\left[f(0) + 2\big(f(0.25) + f(0.5) + f(0.75)\big) + f(1)\right]$$

$$I^{11}(h = 0.5) = 0.6955 + \frac{0.6955 - 0.8244}{2^2 - 1}$$

$$I = 0.6490 + \frac{0.6490 - 0.6528}{2^4 - 1}$$

$$I^{12}(h = 0.25) = 0.6606 + \frac{0.6606 - 0.6955}{2^2 - 1}$$

Multiple questions

S.No.	Questions
1	Which one can be used for numerical integration? (a) Composite Simpson's rule, (b) Composite Trapezoidal rule, (c) Romberg Integration, (d) all of them
2	What type of solution could be by applying above rules? (a) Analytical solution, (b) Numerical solution
5	Which formula can be used for Composite Trapezoidal rule? (a) $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f_0 + f_1]$, (b) $\int_a^b f(x)dx \approx \frac{h}{2}[f_0 + f_N + 2(f_1 + f_2 + .. + f_{N-1})]$
6	Which formula can be used for Composite Simpson's rule? (a) $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f_0 + f_1]$, (b) $\int_a^b f(x)dx \approx \frac{h}{3}[f_0 + f_N + 4(f_1 + f_3 + .. + f_{N-1}) + 2(f_2 + f_4 + .. + f_{N-2})]$ (c) Both of them

Try to do yourself

Exercise 1: A river is 50 meters wide. The depth 'd' in meters at distance *x* meters from one back is given by the following table. Calculate the area of cross-section of the river using Simpson's rule.

x	0	10	20	30	40	50
d	0	4	7	9	12	15

Exercise 2: The table below shows the values of $f(x)$ at different values of x.

x	1.0	1.2	1.4	1.6	1.8
f(x)	1.831	2.592	3.515	4.643	5.926

Evaluate $\int_{1.0}^{1.8} f(x)\,dx$ using Trapezoidal rule with 1, 2 and 4 subintervals.

Improve your results using Romberg integration.

x	1.0	1.2	1.4	1.6	1.8
f(x)	1.831	2.592	3.515	4.643	5.926

R
O
M
B
E
R
G

n	h	$O(h^2)$	$O(h^4)$	$O(h^6)$
1	0.8	3.1028		
2	0.4	2.9574	2.908933	
4	0.2	2.9257	2.915133	2.915547

$$n = 1, h = 0.8, \qquad I(h = 0.8) = \frac{0.8}{2}\big(f(1.0) + f(1.8)\big) =$$

$$n = 2, h = 0.4, \qquad I(h = 0.4) = \frac{0.4}{2}\big(f(1.0) + 2f(1.4) + f(1.8)\big) =$$

$$n = 4, h = 0.2, \qquad I(h = 0.2) = \frac{0.2}{2}\Big(f(1.0) + 2\big(f(1.2) + f(1.4) + f(1.6)\big) + f(1.8)\Big) =$$

$$I_{R_{11}} = I(h = 0.4) + \frac{I(h = 0.4) - I(h = 0.8)}{2^2 - 1}$$

$$I_{R_2} = I(h = 0.2) + \frac{I(h = 0.2) - I(h = 0.4)}{2^4 - 1}$$

$$I_{R_{12}} = I(h = 0.2) + \frac{I(h = 0.2) - I(h = 0.4)}{2^2 - 1}$$