Nonlinear Equations in One Variable
Lecture-1

No. of real roots, Location, Bisection Method

Objective:

To find the root of a nonlinear equation in one variable with the help of different methods .

Methodologies:

Five methods can be used to find roots of nonlinear equation in one variable. They are-

1. **Graphical Method (to find the nature of the roots)**
2. **Bisection Method**
3. **The Secant Method**
4. **Newton-Raphson Method**
5. **Fixed Point Iteration Method**

What is nonlinear equation?

An equation in which one or more terms have a variable of degree 2 or higher is called a nonlinear equation.

Example of nonlinear equations

1. $x^3+4x-3=0$
2. $x^2+x+2=25$

Number of Real Roots by Graphical Method

- **A polynomial equation** $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ **of degree *n* has exactly *n* roots.**

- **Some of them are real and others are complex.**

- **Geometrically, if the graph of** $y = f(x)$ **crosses the *x*-axis at x = a , then x = a is a real root of** $f(x) = 0$

- **Now we shall consider graphically to find the number of real roots and its location.**

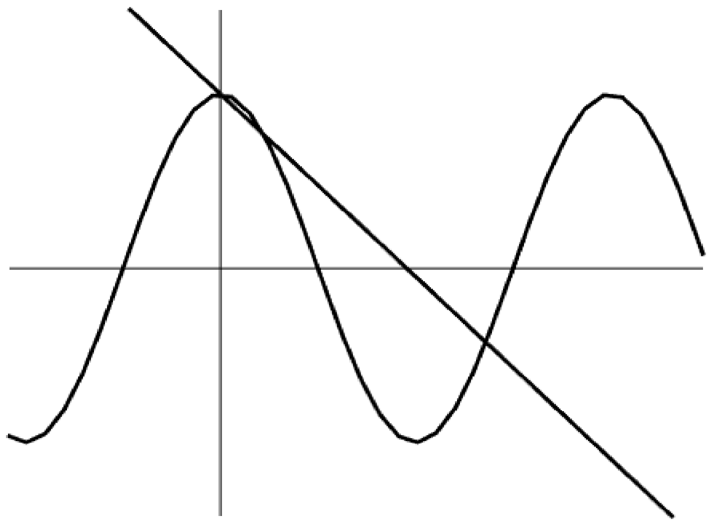
Rewrite the equation $f(x)=0$ as $f_1(x)=f_2(x)$

The point of intersection $x=x_1$ (say) of the graphs $y=f_1(x)$ and $y=f_2(x)$ is a root of the equation.

Thus the number of intersections of the two graphs will be the number of real roots (See the following figure)

How many points of interaction are there?

How many real roots are there?



Problems and Solutions

Example: **(a) Find the number of real roots of $x^3 + 4x - 3 = 0$ by graphical method**

(b) Find the nmber of complex roots, if any.

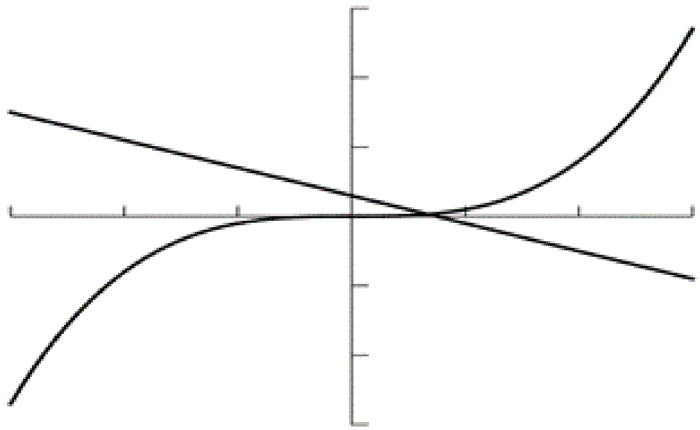
(c) Use MATLAB function “roots” to find all the roots including complex roots.

(a) $x^3 + 4x - 3 = 0$ rewrite the equation as $f_1(x) = f_2(x)$

$\Longrightarrow x^3 = 3 - 4x$

$\Longrightarrow f_1(x) = f_2(x)$

Now, $f_1(x) = x^3$ and $f_2(x) = 3 - 4x$



Number of point of intersection = 1

The no. of real roots =1

(b) It is a polynomial equation of degree **three** . [$x^3+4x-3=0$]

So the **total number of roots is three**.

So, number of complex roots = Total number of roots - Number of real roots

=3-1=2.

MATLAB program

(c) >> p=[1 0 4 -3] % coefficients of the polynomial $x^3+4x-3=0$

```
p =    1     0     4    -3
```

```
>> Roots = roots(p)
Roots =
-0.3368 + 2.0833i
-0.3368 - 2.0833i
 0.6736 + 0.0000i
```

Exercise

Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$,

b. $x^3 + x^2 - 2x - 5 = 0$;

c. $x^4 - 2x - 5 = 0$;

d. $x^4 + x^2 - 80 = 0$;

- Find the number of real roots of the equation by graphical method. Find also the number of complex roots, if any.

- Write MATLAB commands "**roots**" to find all the roots including complex roots.

(a) Solve the nonlinear equation

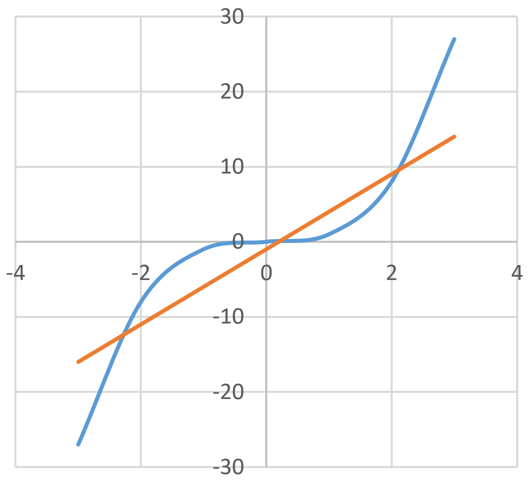
$$x^3 - 5x + 1 = 0$$
$$x^3 = 5x - 1$$
$$f_1(x) = f_2(x) \quad i.e. \quad f_1(x) = x^3 \;\; and \;\; f_2(x) = 5x - 1$$

The degree of the polynomial = ?

No. of point of intersection= ?

The No. of real roots =?

The No. of complex roots =?



(b)

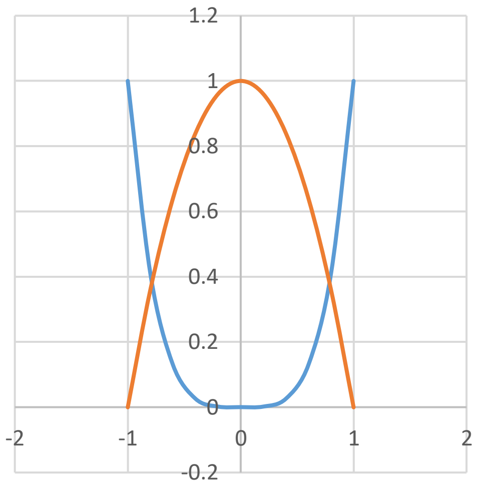
$$x^4 + x^2 - 1 = 0$$
$$x^4 = 1 - x^2$$
$$f_1(x) = f_2(x) \quad i.e. \quad f_1(x) = x^4 \;\; and \;\; f_2(x) = 1 - x^2$$

The degree of the polynomial=?

No. of point of intersection= ?

The No. of real roots =?

The No. of complex roots =?



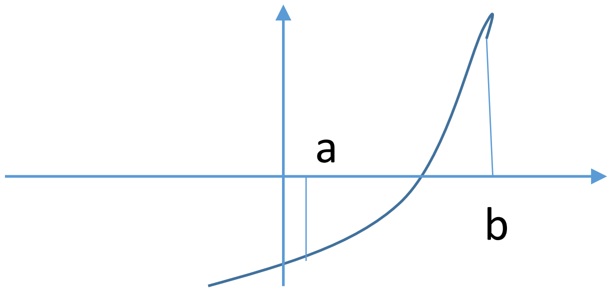
Location of Roots

- To locate the roots of $f(x) = 0$ first study the graph of $y = f(x)$ as shown below (Fig-4.1)

- If we can find two values of x, say a and b such that $f(a)$ is positive, and $f(b)$ is negative, then the curve must have crossed the x-axis where $f(x) = 0$

- In general, if $f(x) = 0$ is continuous in $[a, b]$ and $f(a)$ and $f(b)$ are opposite in signs i.e., $f(a)f(b) < 0$, then there exists odd number of real roots (at least one root) of $f(x) = 0$ in (a, b)

- But the only exception where it does not work is when curve touches the x-axis. For this case, the existence of a root can be determined by the sign of $f'(x)$ in the interval (a, b) containing the root and it will satisfy the condition $f'(a)f'(b) < 0$.



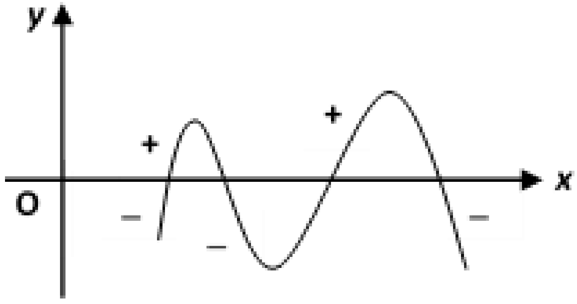


Fig 4.1

Example: The equation $(x-1)e^x - x = 0$ has two real roots. For each root, find an interval where it lies.

Solution: Consider the values of function $f(x) = (x-1)e^x - x$ for different values of x:

x	$f(x)$
-2	1.59
-1	0.26
0	-1
1	-1
2	5.39

From the above table, we see that

Since $f(-1)f(0) = -0.26 < 0$, a root lies in $(-1,\ 0)$,

Another interval

Since $f(1)f(2) = -5.39 < 0$, a root lies in $(1,2)$

.

Techniques to find real roots

Method of Bisection

Algorithm:

Step-1: If $f(x)$ is continuous in $[a, b]$, Choose two approximations a and b such that $f(a)f(b) < 0$.

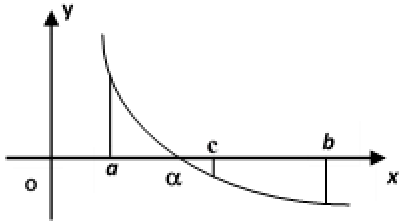


Fig 4.2Bisection Method

Step-2: Evaluate the mid-point c of $[a, b]$ such that $c = \frac{a+b}{2}$

Step-3: If $f(c) = 0$ we conclude that c is a root of $f(x) = 0$.

If $f(c) \neq 0$ and

i. If $f(a)f(c) < 0$ the root lies in $[a, c]$

ii. If $f(b)f(c) < 0$ the root lies in [c, b]

Step-4: By designating the new interval $[a_1, b_1]$ we can calculate the next root (approximation) x_1 by the formula $x_{n+1} = \frac{a_n + b_n}{2}$, $n = 1, 2, 3, \ldots$

Step-5: Repeat the process until $|x_{n+1} - x_n| \leq \varepsilon$, where ε is the specified accuracy.

Problems and Solutions

Example: Find the root of $f(x) = x^2 - 3$ using bisection method with accuracy $\varepsilon = 0.01$ in the interval [1, 2].

Solution:

a	C=(a+b)/2	b	f(a)	f (c)	f(b)	Condition	Next interval (a,c) or (c,b)
1.0	1.5	2.0	-2.0	-0.75	1.0	$f(c)(f(b) < 0$	(c,b)
1.5	1.75	2.0	-0.75	0.062	1.0	$f(a)(f(c) < 0$	(a,c)
1.5	1.625	1.75	-0.75	-0.359	0.062	$f(c)(f(b) < 0$	(c,b)
1.625	1.6875	1.75	-0.359	-0.1523	0.062	$f(c)(f(b) < 0$	(c,b)

$f(x) = x^2 - 3$ Initially the root lies in [1, 2]

Let us write the values to 3 decimal places.

a	C=(a+b)/2	b	f(a)	f (c)	f(b)	Condition (i) f(a)f(c)<0 (ii) f(c)f(b)<0	Next interval (a,c) or (c,b)
1.0	1.5	2.0	-2.0	-0.75	1.0	$f(c)f(b) < 0$	(c, b)
1.5	1.75	2.0	-0.75	0.062	1.0	$f(a)f(c) < 0$	(a, c)
1.5	1.625	1.75	-0.75	-0.359	0.062	$f(c)f(b) < 0$	(c, b)

Completing the operations three times write down the interval of shorter length, where the root lies.

The interval is (1.625, 1.75)

The root of the equation to 2 decimal places ≈ 1.63

Exercise

$$x_{n+1} = \frac{a_n + b_n}{2}$$

Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$; $[2, 3]$,

b. $x^3 + x^2 - 2x - 5 = 0$; $[1,2]$,

c. $x^4 - 2x - 5 = 0$; $[0, 2]$,

d. $x^4 + x^2 - 80 = 0$; $[2.90,2.92]$.

Apply bisection method two times in the given interval to find the new smaller interval of this root.

Solution(a): find $f(2) =$ -1

Find $f(3) =$ 13

The root lies in (2, 3)

Since $f(2)f(3) = -13 < 0$

The first approximation of the root = 2.5

(c) Find $f(0) =$ -5 $f(2) =$ 7

The root lies in [0, 2]

Since $f(0)f(2) = -35 < 0$

The first approximation of the root = 1

Advantages and Drawbacks: Bisection Method

Advantages:

- ❑ **Always Convergent.**

Drawbacks:

- ❑ **Slow convergence.**
- ❑ **If one of the initial guess is closer to the root , the convergence is slower.**

Outcome

By applying Bisection method, nonlinear equations in one variable can be solved to find roots (approximately), although it has few drawbacks.

Multiple Choice Questions

1. Find the degree of the polynomial $x^4 + x^2 - 80 = 0.$

a) 3 , b) 2 , c) 4 , d) 5

2. Find the number of roots of the polynomial $x^3 + x^2 - 2x - 5 = 0.$

a) 3 , b) 1 , c) 4 , d) 5

3. Find the number of complex roots of the polynomial $x^4 + x^2 - 80 = 0$

a) 3 , b) 2 , c) 4 , d) 5

4. If $f(x) = x^3 - 5x + 1 = 0$ **is continuous on the interval** $[\mathbf{2}, \mathbf{3}]$, **What is the value of** $\boldsymbol{f}(\mathbf{2})$?

a) -3 , b) -1 , c) 4 , d) 0

5. If $f(x)$ is continuous in $[a, b]$ then what is the condition that there exists at least one root of $f(x)$=0 ?

a) $f(a)f(b) < 0$ **,** **b)** $f(a)f(b) > 0$ **, c)** $f(a) < 0$ **,d) Neither**