Islamic University of Madinah
Faculty of Science
Mathematics Department

Algebra and Geometry - Math 3140[1]

Course References:

- **R. N. Aufmann, V.C. Barker, and Nation, R.D., *College Algebra and Trigonometry*, 7th Edition, Brooks, 2011.**
- **J. Stewart, L.Redlin, S.Watson, *College Algebra*, 5th Edition, Cengage Learning (2008).**
- **Saudi Digital Library**: https://sdl.edu.sa/SDLPortal/ar/Publishers.aspx

Lecture Notes 1: Chapter P: Preliminary concepts

Section P.5: Complex numbers

- Introduction to Complex Numbers
- Addition and Subtraction of Complex Numbers
- Multiplication of Complex Numbers
- Division of Complex Numbers
- Powers of i

[1] **Course Materials will be covered from:** Course Textbook and These Lectures

Section P.5: Complex numbers

Recall that $\sqrt{9} = 3$ because $3^2 = 9$. Now consider the expression $\sqrt{-9}$. To find $\sqrt{-9}$, we need to find a number c such that $c^2 = -9$. However, the square of any real number c (except zero) is a *positive* number. Consequently, we must expand our concept of number to include numbers whose squares are negative numbers.

Around the seventeenth century, a new number, called an *imaginary number,* was defined so that a negative number would have a square root. The letter i was chosen to represent the number whose square is -1.

Definition of *i*

The **imaginary unit,** designated by the letter i, is the number such that $i^2 = -1$.

The principal square root of a negative number is defined in terms of i.

Definition of an Imaginary Number

If a is a positive real number, then $\sqrt{-a} = i\sqrt{a}$. The number $i\sqrt{a}$ is called an **imaginary number.**

EXAMPLE

$\sqrt{-36} = i\sqrt{36} = 6i$ $\qquad$ $\sqrt{-18} = i\sqrt{18} = 3i\sqrt{2}$

$\sqrt{-23} = i\sqrt{23}$ $\qquad$ $\sqrt{-1} = i\sqrt{1} = i$

Definition of a Complex Number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. The number a is the **real part** of $a + bi$, and b is the **imaginary part.**

EXAMPLE

$-3 + 5i$	• Real part: -3; imaginary part: 5
$2 - 6i$	• Real part: 2; imaginary part: -6
5	• Real part: 5; imaginary part: 0
$7i$	• Real part: 0; imaginary part: 7

EXAMPLE 1 Write a Complex Number in Standard Form

Write $7 + \sqrt{-45}$ in the form $a + bi$.

Addition and Subtraction of Complex Numbers

All the standard arithmetic operations that are applied to real numbers can be applied to complex numbers.

Definition of Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are complex numbers, then

Addition $\quad (a + bi) + (c + di) = (a + c) + (b + d)i$

Subtraction $\quad (a + bi) - (c + di) = (a - c) + (b - d)i$

EXAMPLE 2 Add or Subtract Complex Numbers

Simplify. **a.** $(7 - 2i) + (-2 + 4i)$ **b.** $(-9 + 4i) - (2 - 6i)$

Solution

Multiplication of Complex Numbers

When multiplying complex numbers, the term i^2 is frequently a part of the product. Recall that $i^2 = -1$. Therefore,

$$3i(5i) = 15i^2 = 15(-1) = -15$$
$$-2i(6i) = -12i^2 = -12(-1) = 12$$
$$4i(3 - 2i) = 12i - 8i^2 = 12i - 8(-1) = 8 + 12i$$

When multiplying square roots of negative numbers, first rewrite the radical expressions using i. For instance,

$$\begin{aligned} \sqrt{-6} \cdot \sqrt{-24} &= i\sqrt{6} \cdot i\sqrt{24} && \bullet\ \sqrt{-6} = i\sqrt{6},\ \sqrt{-24} = i\sqrt{24} \\ &= i^2\sqrt{144} = -1 \cdot 12 \\ &= -12 \end{aligned}$$

Note from this example that it would have been incorrect to multiply the radicands of the two radical expressions. To illustrate:

$$\sqrt{-6} \cdot \sqrt{-24} \neq \sqrt{(-6)(-24)}$$

Definition of Multiplication of Complex Numbers

If $a + bi$ and $c + di$ are complex numbers, then

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Because every complex number can be written as a sum of two terms, it is natural to perform multiplication on complex numbers in a manner consistent with the operation defined on binomials and the definition $i^2 = -1$. By using this analogy, you can multiply complex numbers without memorizing the definition.

EXAMPLE 3 Multiply Complex Numbers

Multiply.

a. $3i(2 - 5i)$ **b.** $(3 - 4i)(2 + 5i)$

Solution

Division of Complex Numbers

Recall that the number $\frac{3}{\sqrt{2}}$ is not in simplest form because there is a radical expression in the denominator. Similarly, $\frac{3}{i}$ is not in simplest form because $i = \sqrt{-1}$. To write this expression in simplest form, multiply the numerator and denominator by i.

$$\frac{3}{i} \cdot \frac{i}{i} = \frac{3i}{i^2} = \frac{3i}{-1} = -3i$$

Here is another example.

$$\frac{3-6i}{2i} = \frac{3-6i}{2i} \cdot \frac{i}{i} = \frac{3i-6i^2}{2i^2} = \frac{3i-6(-1)}{2(-1)} = \frac{3i+6}{-2} = -3 - \frac{3}{2}i$$

Recall that to simplify the quotient $\frac{2+\sqrt{3}}{5+2\sqrt{3}}$, we multiply the numerator and denominator by the conjugate of $5 + 2\sqrt{3}$, which is $5 - 2\sqrt{3}$. In a similar manner, to find the quotient of two complex numbers, we multiply the numerator and denominator by the conjugate of the denominator.

The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates** or **conjugates** of each other. The conjugate of the complex number z is denoted by $\bar{z}$. For instance,

$$\overline{2+5i} = 2 - 5i \qquad \text{and} \qquad \overline{3-4i} = 3 + 4i$$

Consider the product of a complex number and its conjugate. For instance,

$$\begin{aligned}(2+5i)(2-5i) &= 4 - 10i + 10i - 25i^2 \\ &= 4 - 25(-1) = 4 + 25 \\ &= 29\end{aligned}$$

Note that the product is a *real* number. This is always true.

Product of Complex Conjugates

The product of a complex number and its conjugate is a real number. That is, $(a+bi)(a-bi) = a^2 + b^2$.

EXAMPLE

$(5+3i)(5-3i) = 5^2 + 3^2 = 25 + 9 = 34$

The next example shows how the quotient of two complex numbers is determined by using conjugates.

EXAMPLE 4 Divide Complex Numbers

Simplify: $\dfrac{16 - 11i}{5 + 2i}$

Solution

Powers of i

The following powers of i illustrate a pattern:

$$\begin{aligned}
i^1 &= i & i^5 &= i^4 \cdot i = 1 \cdot i = i \\
i^2 &= -1 & i^6 &= i^4 \cdot i^2 = 1(-1) = -1 \\
i^3 &= i^2 \cdot i = (-1)i = -i & i^7 &= i^4 \cdot i^3 = 1(-i) = -i \\
i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1 & i^8 &= (i^4)^2 = 1^2 = 1
\end{aligned}$$

Because $i^4 = 1$, $(i^4)^n = 1^n = 1$ for any integer n. Thus it is possible to evaluate powers of i by factoring out powers of i^4, as shown in the following.

$$i^{27} = (i^4)^6 \cdot i^3 = 1^6 \cdot i^3 = 1 \cdot (-i) = -i$$

The following theorem can also be used to evaluate powers of i.

Powers of i

If n is a positive integer, then $i^n = i^r$, where r is the remainder of the division of n by 4.

EXAMPLE 5 Evaluate a Power of i

Evaluate: i^{153}

Solution

Homework Exercises: Sec. P6

Name: .... ID: .... Section: ....

Write each expression as a complex number in standard form:

1. $\frac{5-i}{4+5i}$

2. $\frac{4+i}{3+5i}$

3. $(3-5i)^2$

Evaluate the power of i:

4. i^{15}

5. i^{66}

6. $\frac{1}{i^{25}}$

7. i^{-52}