Islamic University of Madinah
Faculty of Science
Mathematics Department

Calculus I (MATH 3111)

Course References:

◇ J. Stewart, Calculus: Early Transcendentals; Publisher, Cengage Learning; 8th edition (February 4, 2015); Language, English; Hardcover, 1368 pages; ISBN-10, 1285741552.

◇ Saudi Digital Library: https://sdl.edu.sa/SDLPortal/ar/Publishers.aspx

CHAPTER 1: Functions and Models

Section 1.1: Functions Representation

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 1.1 (Functions Representation)

Definition 1: A relation is a set of ordered pairs
Example: (Grade , Letter) (Amount of Gas , Amount of money)

Definition 2: A function is a relation in which, for each distinct value of the first component of the ordered pair, there is exactly one value of the second component.

Definition 3: A function f is a rule that assigns to each element x in in a set D exactly **ONE** element, called $f(x)$, in a set E.
i.e. A function takes an element from a set and maps it to a **UNIQUE** element in another set
Note: We can visualize a function as a machine that accepts elements of the set D as inputs and produces corresponding elements of E as outputs.

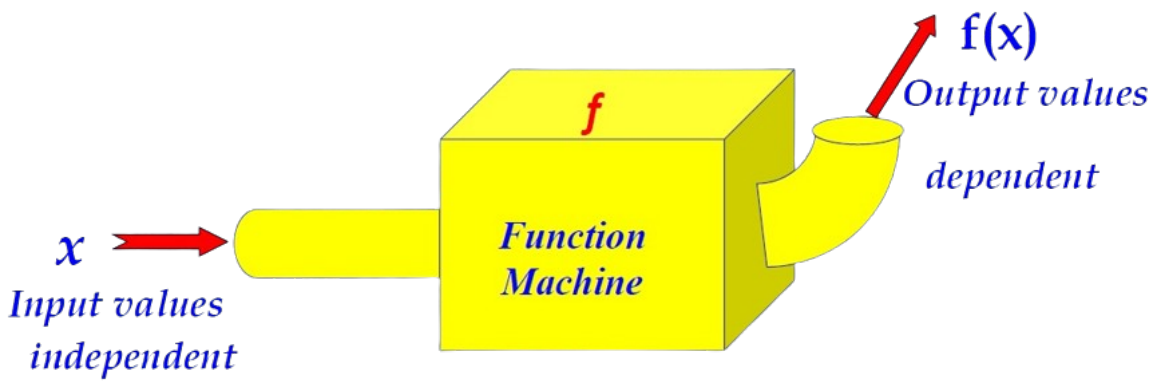


Fig.1 Function as a machine

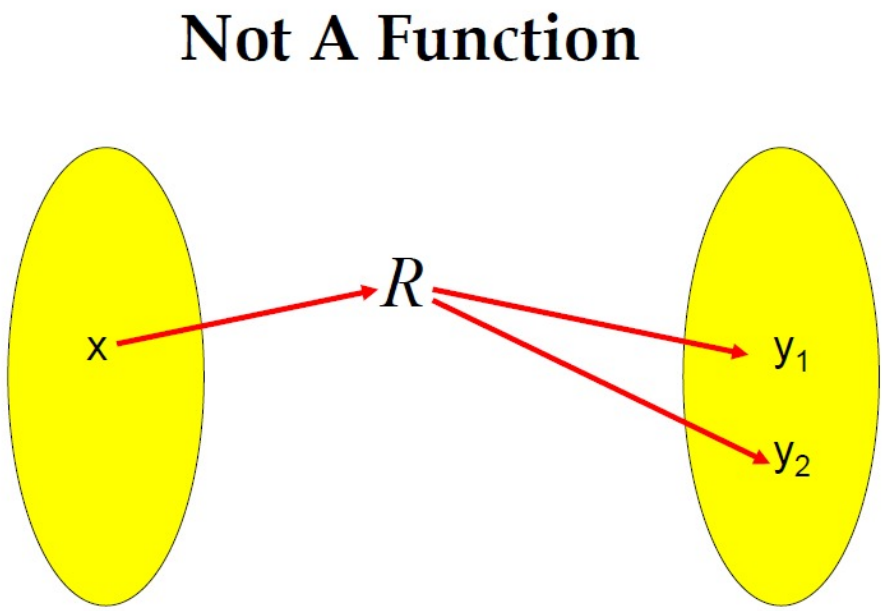


Fig.2 Not a Function

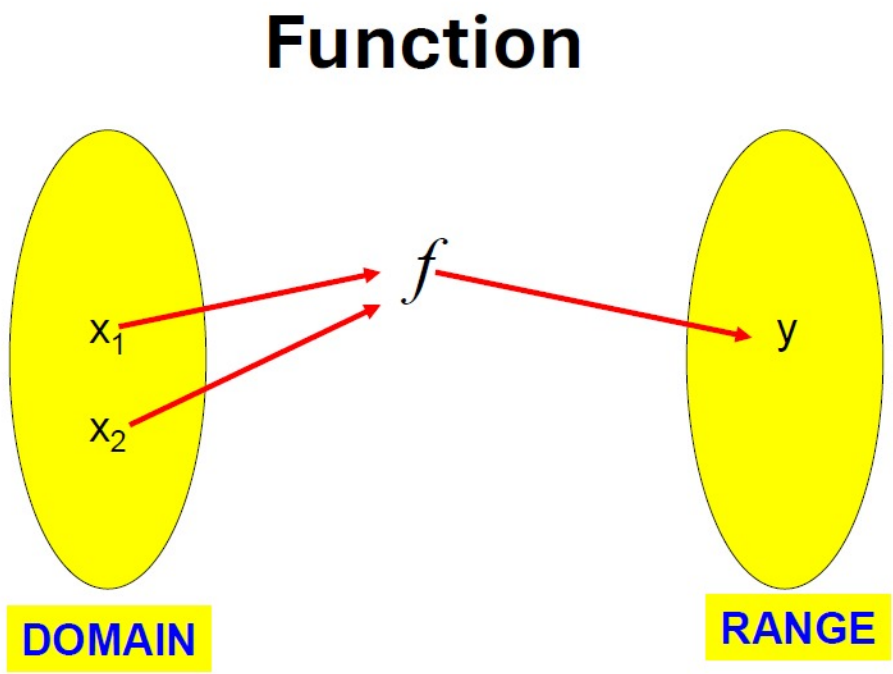


Fig.3 Function

Ex. 1: Let $A = \{-1, 1, 2, 3\}$ and let $B = \{1, 2, 4, 5, 9, 10\}$ and let $f : A \to B$ be a relation such that $f(x) = x^2 + 1$ for each $x \in A$.
Is f represents a function? If so, find the domain and the range of f.

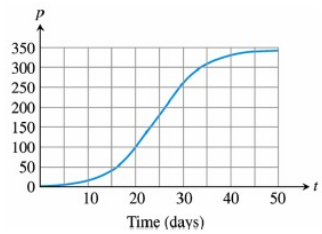
Ex. 2: Let $A = \{-1, 1, 2, 3, 4\}$ and let $B = \{1, 2, 4, 5, 9, 10\}$ and let $f : A \to B$ be a relation such that $f(x) = x^2 + 1$ for each $x \in A$.
Is f represents a function? If so, find the domain and the range of f.

Representing a Function

An algebraic expression

$$y = \sqrt{x+9}$$

A graph in the xy-plane



A table of selected pairs (x,y)

Grade	Letter
97	A
72	C
40	F

Determining the domain and the range of a function represented graphically.

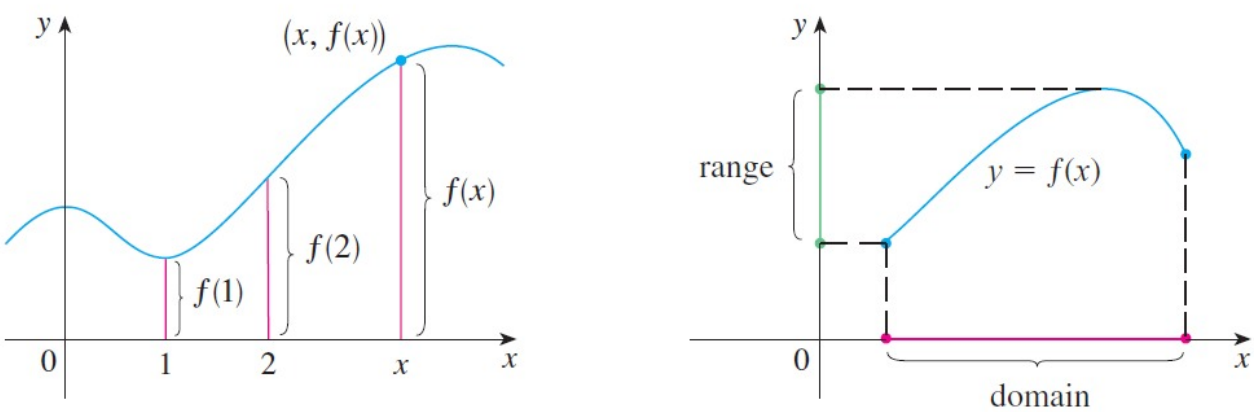
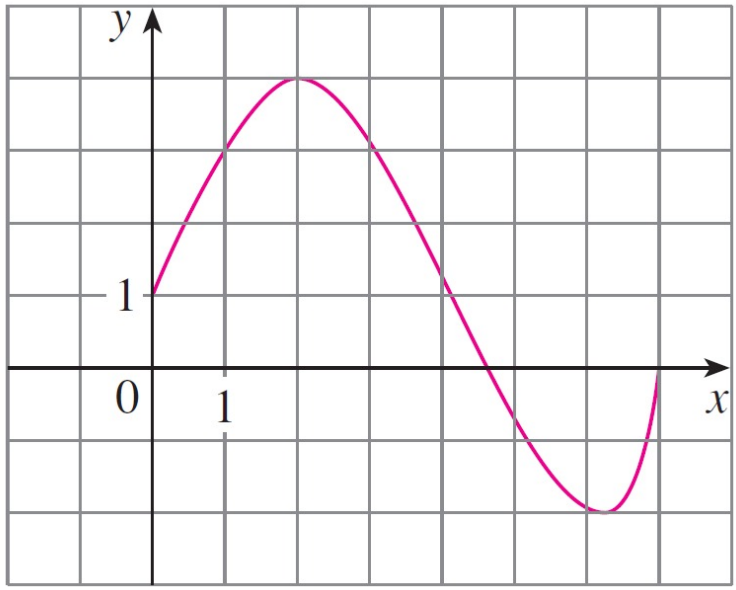


Fig.4 The domain and the range of a function represented graphically

Ex.3 (Example 1, P. 11): The graph of a function f is shown in the following figure:



(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f ?

Ex.4 (Example 2, P. 12): Sketch the graph and find the domain and range of each function:
(a) $f(x) = 2x - 1$.

(b) $g(x) = x^2$.

Ex.5 (Example 3, P. 12): If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

Ex.6 (Example 6, P. 14): Find the domain of each function:

(a) $f(x) = \sqrt{x+2}$.

(b) $g(x) = \dfrac{1}{x^2 - x}$.

Ex.7: Find the domain of each function:

(a) $f(x) = 4x - 8$.

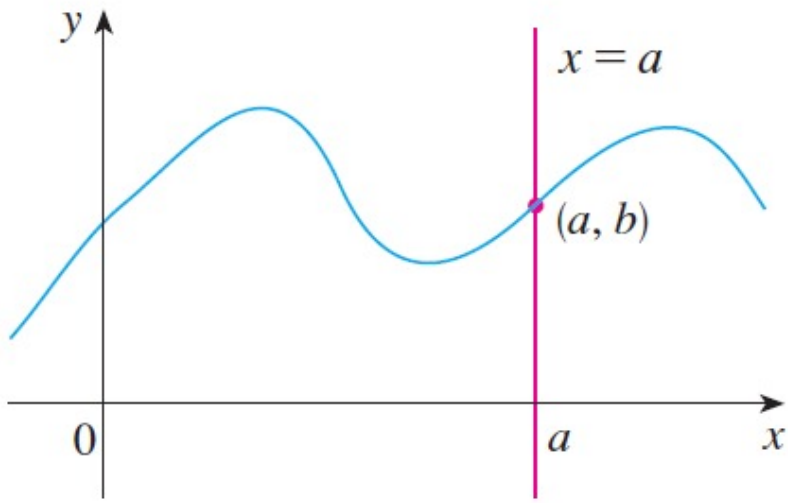
(b) $h(x) = \sqrt{x - 5}$.

Ex.8: Find the domain of each function:

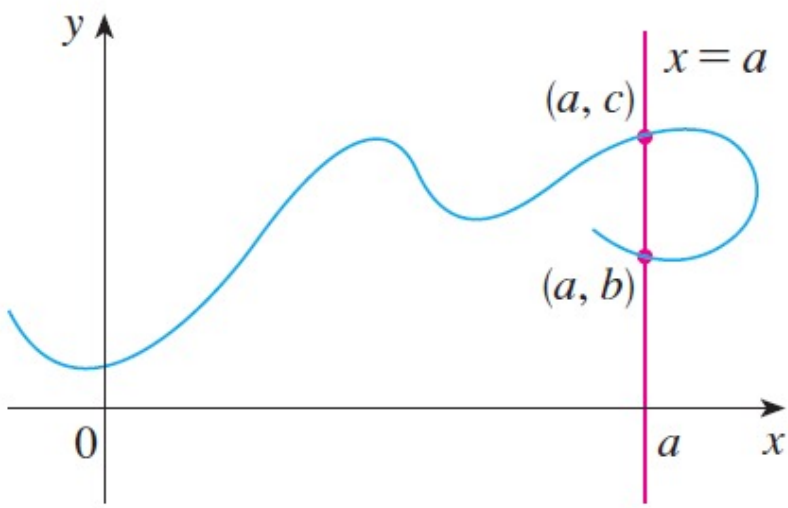
(a) $f(x) = \dfrac{1}{\sqrt{1-x}}$.

(b) $f(x) = \sqrt{3-\sqrt{x}}$.

The Vertical Line Test: A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.



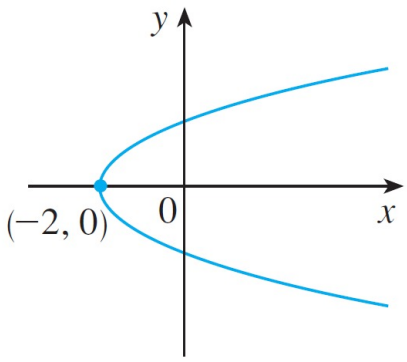
(a) This curve represents a function.



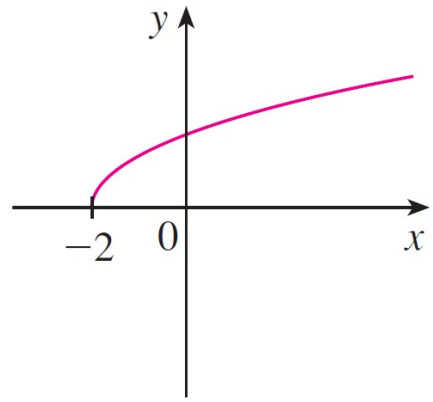
(b) This curve doesn't represent a function.

Fig.5 The Vertical Line Test

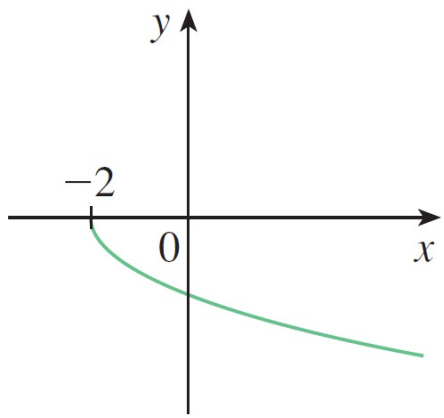
Ex.9 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.



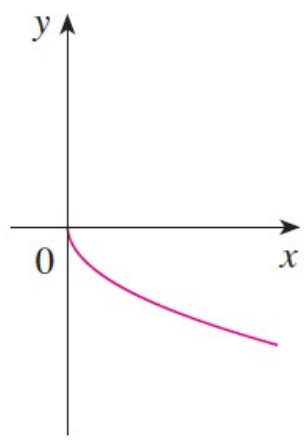
(a) $x = y^2 - 2$



(b) $y = \sqrt{x+2}$



(c) $y = -\sqrt{x+2}$



(d) $y = -\sqrt{x}$

Even/Odd Functions: (The domain should be symmetric around 0)

$\diamondsuit$ If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even** function. That is, the curve is symmetric about y**-axis**.

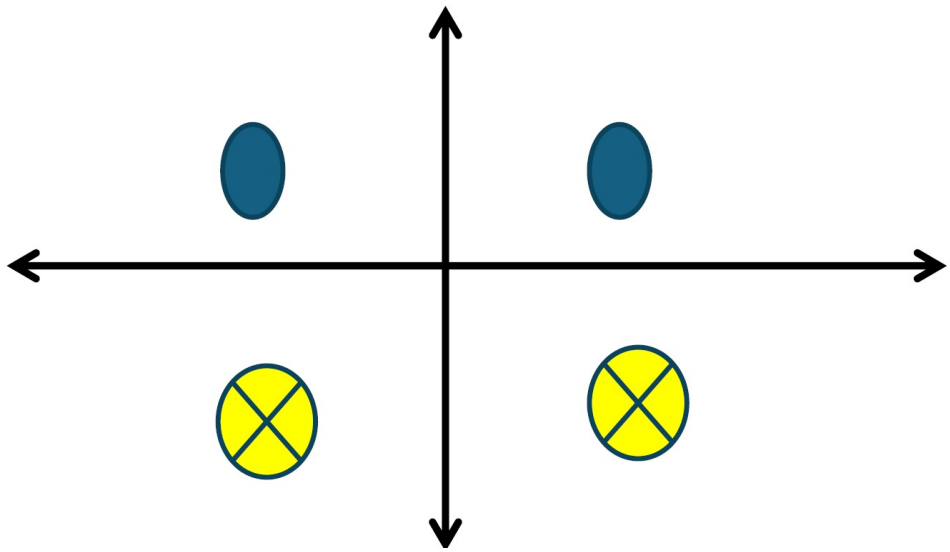


Fig.6 Even Function

$\diamondsuit$ If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd** function. That is, the curve is symmetric about **the origin**.

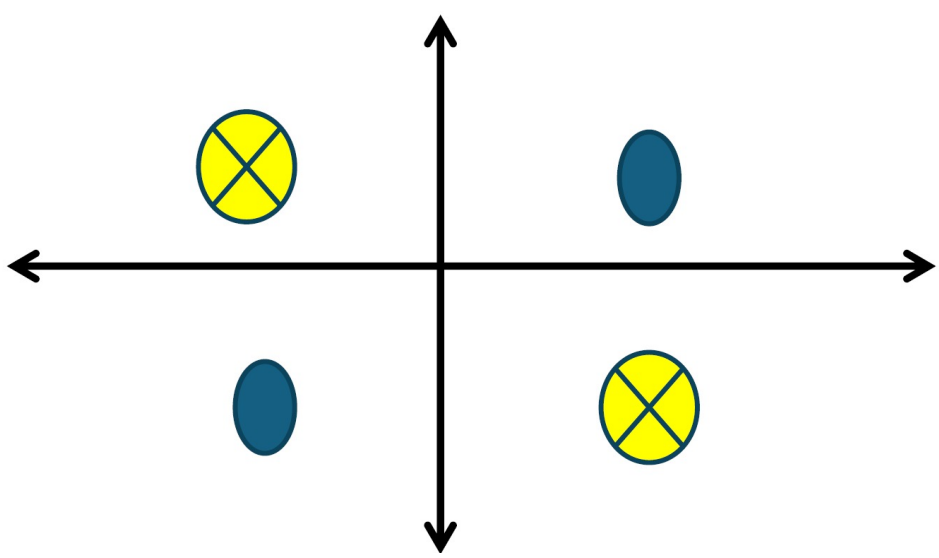
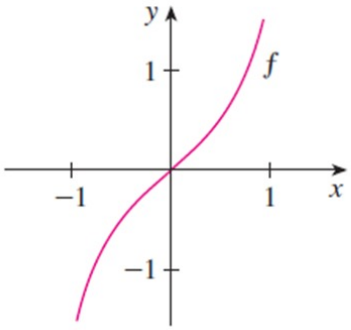
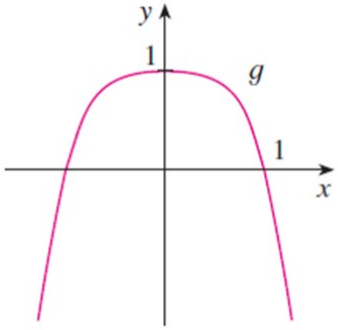
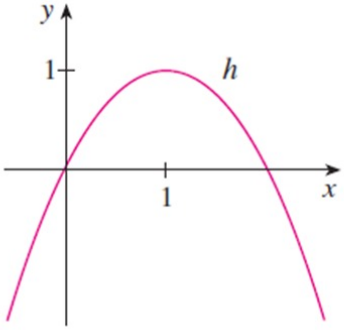


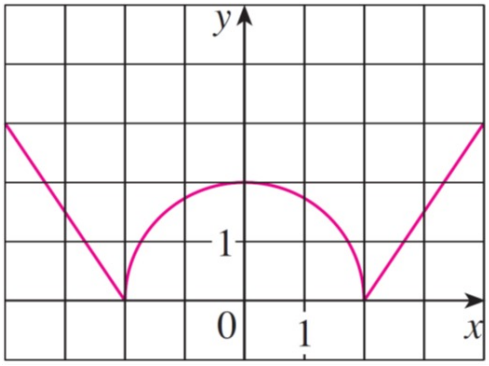
Fig.7 Odd Function

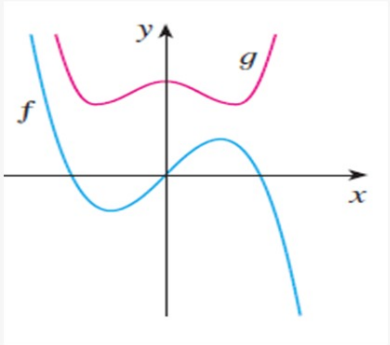
Ex.10: Determine whether each of the following functions is even, odd, or neither even nor odd:

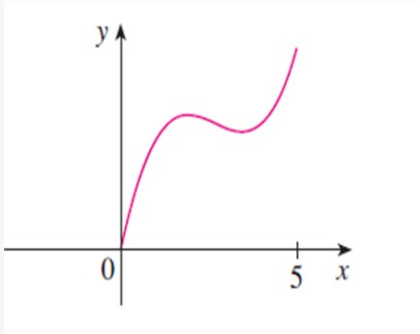












Ex.11 (Example 11, P. 18): Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$.

(b) $g(x) = 1 - x^4$.

(c) $h(x) = 2x - x^2$.

Ex.12: Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^4 - 4x^2$.

(b) $g(x) = \dfrac{1}{x^3 - x}$.

(c) $h(x) = x^3 + 1$.

Absolute Value Function (Definition):

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note that $\sqrt{x^2} = |x|$.

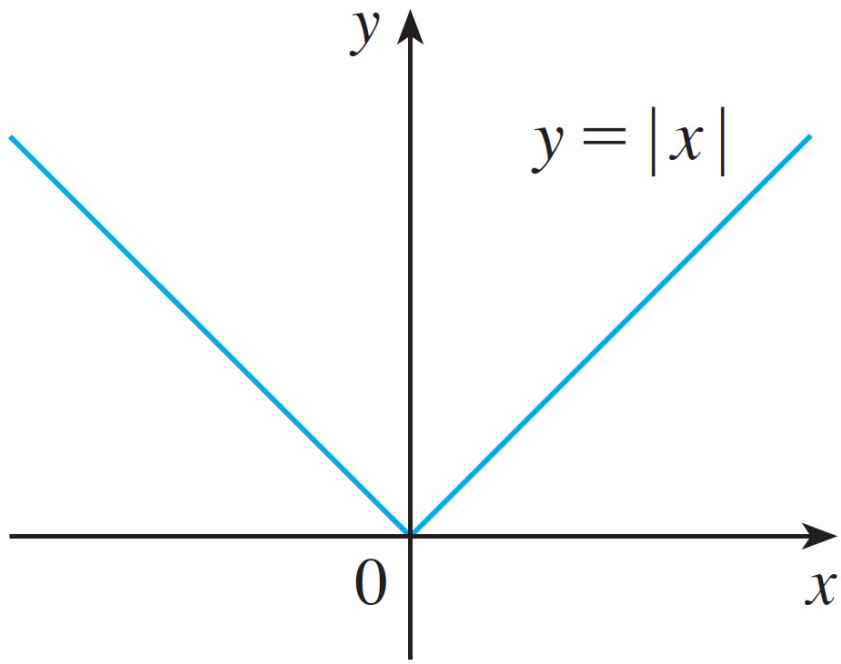


Fig.8 Absolute Value Function

Some useful rules:

Suppose $a > 0$. Then

$|x| = a$ if and only if $x = \pm a$

$|x| < a$ if and only if $-a < x < a$

$|x| > a$ if and only if $x > a$ or $x < -a$

$|x| = |b|$ if and only if $x = b$ or $x = -b$

Ex.14 (Example 7, P. 16): A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } \leq -1, \\ x^2 & \text{if } x > -1. \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$, then sketch the graph of f.

Ex.13: Rewrite the expression without using the absolute value symbol:

(a) $f(x) = |x - 3|$.

(b) $g(x) = |2x - 1|$.

Sections 1.1 Exercises: Page 19: 3,4,7,8,9,10,31,32,33,37,41,43,73,74,75,76,77.

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CHAPTER 1: Functions and Models

Section 1.2: Mathematical Models

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 1.2 (Mathematical Models)

OBJECTIVES:

- Define the polynomial and the rational functions.
- Graph some basic power functions.
- Classify functions according to their types.
- Trigonometric functions.

Type 1: LINEAR function: A function ? is a linear function if it is of the form $f(x) = ax+b$, where m is the slope and b is the y-intercept.

Ex. 1: Let $f(x) = -2x + 6$. Sketch the function $f(x)$, then find: D_f, R_f, the slope, and the y-intercept.

Ex. 2: Determine whether a function is linear or not. If so, find the slope and the y-intercept:

(a) $f(x) = 3x + \frac{1}{5}$.

(b) $f(x) = -\frac{1}{5}x + 3$.

(c) $f(x) = 3\sqrt{x} + 1$.

(d) $f(x) = x^2 - 1$.

Special cases: Equations of Vertical and Horizontal lines:
An equation of the vertical line through the point (a, b) is $x = a$.
An equation of the horizontal line through the point (a, b) is $y = b$.

Ex. 3: Sketch the straight lines $x = 5$ and $y = 5$. Are these equations represent functions? If so, find the domain and the range.

Type 2: POLYNOMIAL function (POLYNOMIAL): A function $P(x)$ is called an nth degree polynomial if it is of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0.$$

Where n is a non-negative integer and the numbers $a_n, a_{n-1}, ..., a_0$ are real constants called coefficients of the polynomial.
The domain of any polynomial is the set of all real numbers $\mathbb{R} = (-\infty, \infty)$.
The highest power of x appearing in the polynomial is said to be the degree of the polynomial.

Ex. 4: State whether each function is a polynomial or not. Find its degree if it is a polynomial:

(a) $f(x) = 3x^3 - x + \frac{1}{5}$.

(b) $f(x) = 4 - \sqrt{5}x^4 - \frac{1}{5}x$.

(c) $f(x) = 3\sqrt{x} + 1$.

(d) $f(x) = x^2 - 1 + \frac{1}{x}$.

Ex.5 : Sketch each of the following polynomial, then determine D_f, R_f, its degree, odd or even or neither, and where is this function increasing and decreasing:

(a) $f(x) = x$.

(b) $f(x) = x^2$.

(c) $f(x) = x^3$.

(d) $f(x) = x^4$.

(e) $f(x) = x^5$.

Type 3: POWER function: The power functions $f(x) = x^n$, n is a constant.

Ex.6 : Sketch each of the following power functions, then determine D_f, R_f, its degree, odd or even or neither, and where is this function increasing and decreasing:

(a) $f(x) = x^{\frac{1}{2}}$.

(b) $f(x) = x^{\frac{1}{3}}$.

Remark:

1. The behavior of all power functions with $n = \dfrac{1}{4}$, $n = \dfrac{1}{6}$, $n = \dfrac{1}{8}$, ... is similar to that of $f(x) = x^{\frac{1}{2}}$.
2. The behavior of all power functions with $n = \dfrac{1}{5}$, $n = \dfrac{1}{7}$, $n = \dfrac{1}{9}$, ...is similar to that of $f(x) = x^{\frac{1}{3}}$

Ex.7: Sketch each of the following power functions, then determine D_f, R_f, its degree, odd or even or neither, and where is this function increasing and decreasing:

(a) $f(x) = \dfrac{1}{x} = x^{-1}$.

(b) $f(x) = \dfrac{1}{x^2} = x^{-2}$.

Remark:

1. The behavior of all power functions with $n = -3$, $n = -5$, $n = -7$, ... is similar to that of $f(x) = x^{-1}$.
2. The behavior of all power functions with $n = -4$, $n = -6$, $n = -8$, ... is similar to that of $f(x) = x^{-2}$.

Type 4: RATIONAL function: This is a quotient of two polynomials $f(x) = \dfrac{p(x)}{q(x)}$.

For example: $f(x) = \dfrac{x^2 - 3x + 4}{x^2 + 1}$ and $f(x) = \dfrac{1}{x + 2}$ are rational functions.

Type 5: ALGEBRAIC function: These are built from x and real numbers using algebraic operations (addition, subtraction, multiplication, division, and taking roots).

For example: $f(x) = \dfrac{x^{\frac{2}{3}} - 3x + 4}{x^2 + 1}$ and $f(x) = \sqrt{x + 2}$ are algebraic functions.

Type 6: TRANSCENDENTAL function: These are functions that are not algebraic

For example: $f(x) = \sin x$, $f(x) = \log_3 x$, and 4^x are Transcendental functions.

Ex.8: Classify the following functions.:

(a) $f(x) = \dfrac{1}{\sqrt{1-x}}$.

(b) $f(x) = \sqrt{3-\sqrt{x}}$.

(c) $f(x) = x^5 + x$.

(d) $g(x) = 1 - x^4$.

(c) $h(x) = tanx$.

(e) $g(x) = \dfrac{1-\sqrt{x}}{x^3 - x}$.

(c) $h(x) = x^3$.

Ex.9 (Ex.6, P. 33): Classify the following functions.:

(a) $f(x) = 5^x$.

(b) $g(x) = x^5$.

(c) $h(x) = \dfrac{1+x}{1-\sqrt{x}}$.

(d) $u(t) = 1 - t + 5t^4$.

Trigonometric Functions:

1. $f(x) = \sin(x)$:

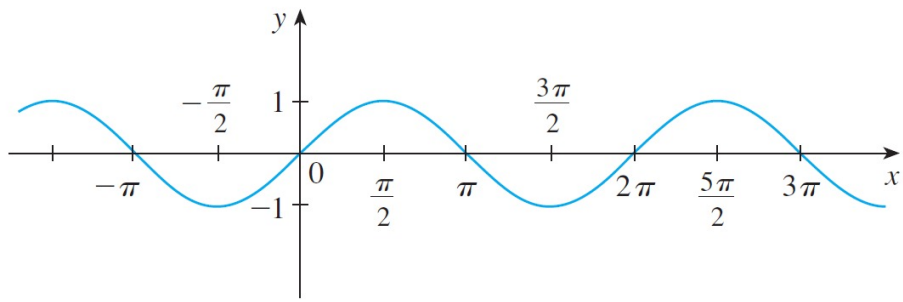

Fig.1: Graph of $f(x) = \sin(x)$.

You can note that the function $f(x) = \sin(x)$ has the following properties:
Odd Functions ($\sin(-x) = -\sin x$).
Domain is $\mathbb{R}$.
Range $[-1, 1]$.
Period is 2π ($\sin(x + 2\pi) = \sin x$).

2. $f(x) = \cos(x)$:

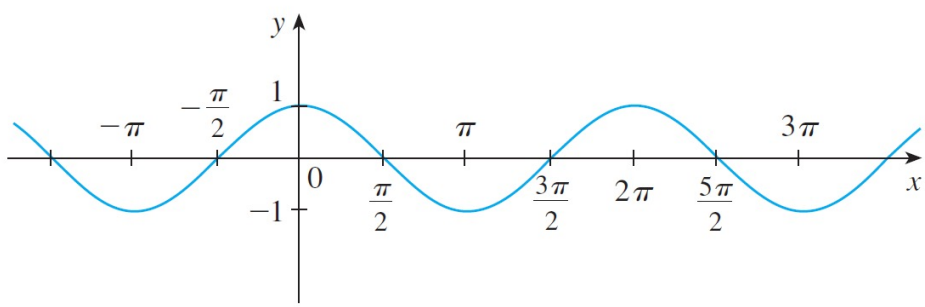

Fig.2: Graph of $f(x) = \cos(x)$.

You can note that the function $f(x) = \cos(x)$ has the following properties:
Even Functions ($\cos(-x) = \cos x$).
Domain is $\mathbb{R}$.
Range $[-1, 1]$.
Period is 2π ($\cos(x + 2\pi) = \cos x$).

3. $f(x) = \tan(x)$:

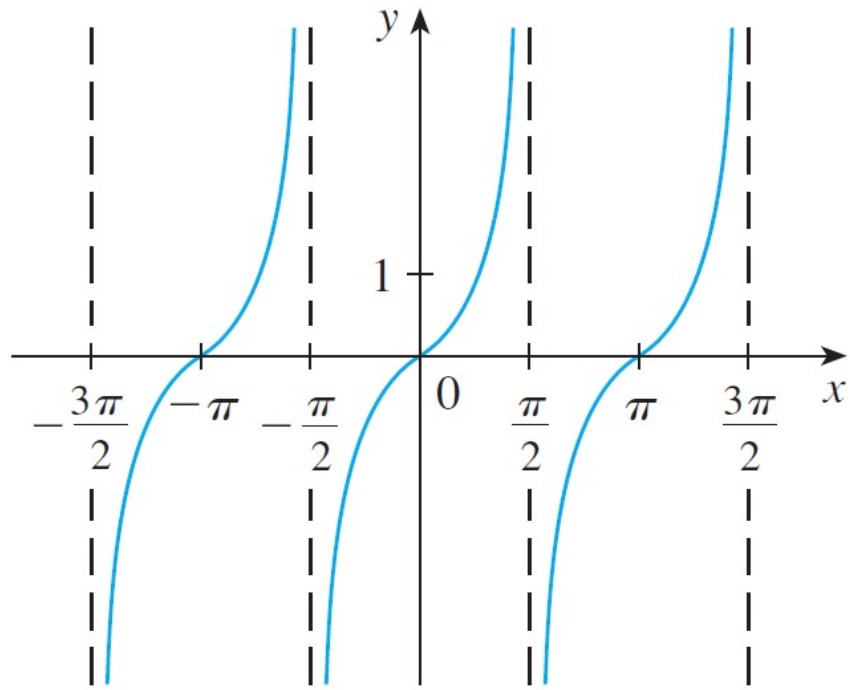

Fig.3: Graph of $f(x) = \tan(x)$.

You can note that the function $f(x) = \tan(x)$ has the following properties:
Odd Functions ($\tan(-x) = -\tan x$).
Domain is $\mathbb{R} - \{-\frac{\pi}{2}, \frac{\pi}{2}, -\frac{3\pi}{2}, \frac{3\pi}{2}, -\frac{5\pi}{2}, \frac{5\pi}{2}, ...\}$.
Range $\mathbb{R}$.
Period is π ($\tan(x + \pi) = \tan x$).

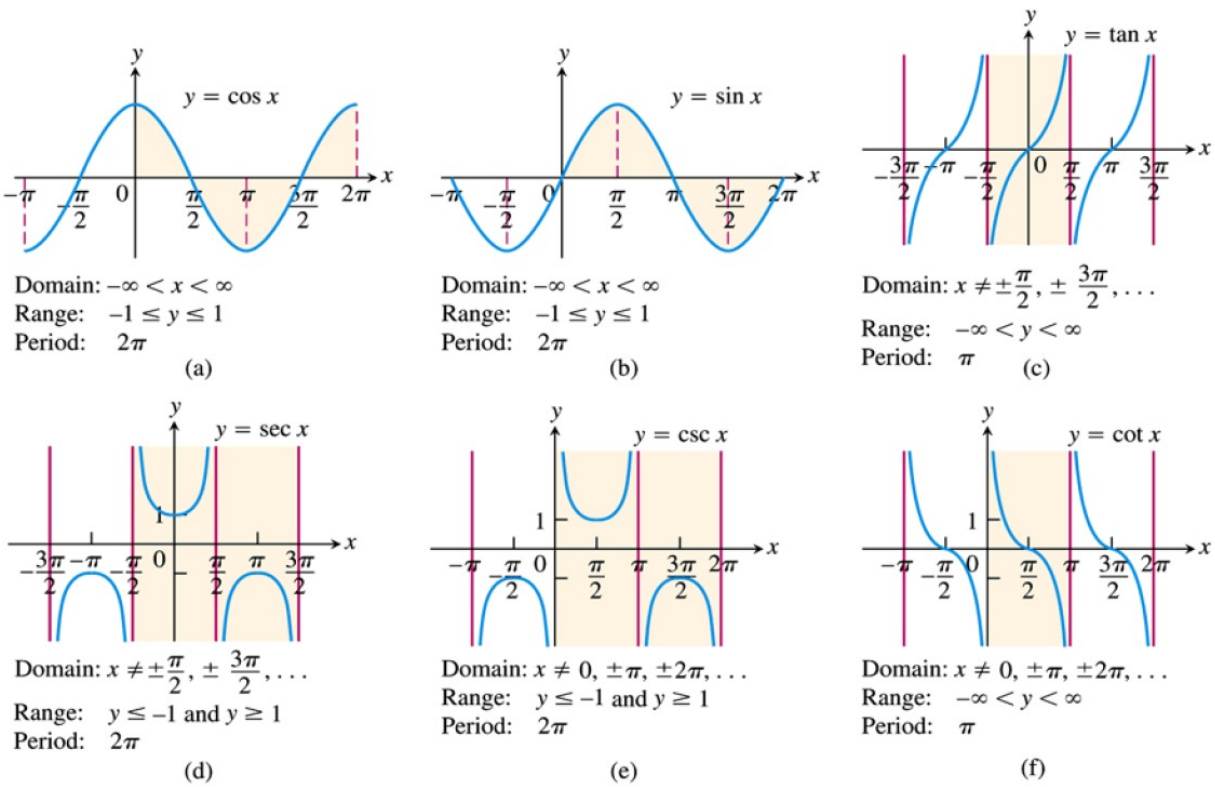

Fig.4: Graph of ALL trigonometric functions.

Remark: The trigonometric identities in Appendix D, Pages A28, A29, and A30 are important:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{\cos x}{\sin x}$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x\cos x$$

$$\cos(2x) = 1 - 2\sin^2 x$$

$$\cos(2x) = 2\cos^2 x - 1$$

$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$\sin x\cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\sin x\sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\cos x\cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$$

$$\cos x\sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$$

Sections 1.1 Exercises: P. 33: 1,2.

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CHAPTER 1: Functions and Models

Section 1.3: New Functions From Old Functions

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 1.3 (New Functions From Old Functions)

OBJECTIVES:

• Find algebraic expressions and sketch functions that are shifted up/down/right/left.

• Find algebraic expressions and domains of functions that are combined using addition, subtraction, multiplication, division, and composition.

Transformations of Functions: By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Vertical and Horizontal Shifts: Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward.

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward.

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left.

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right.

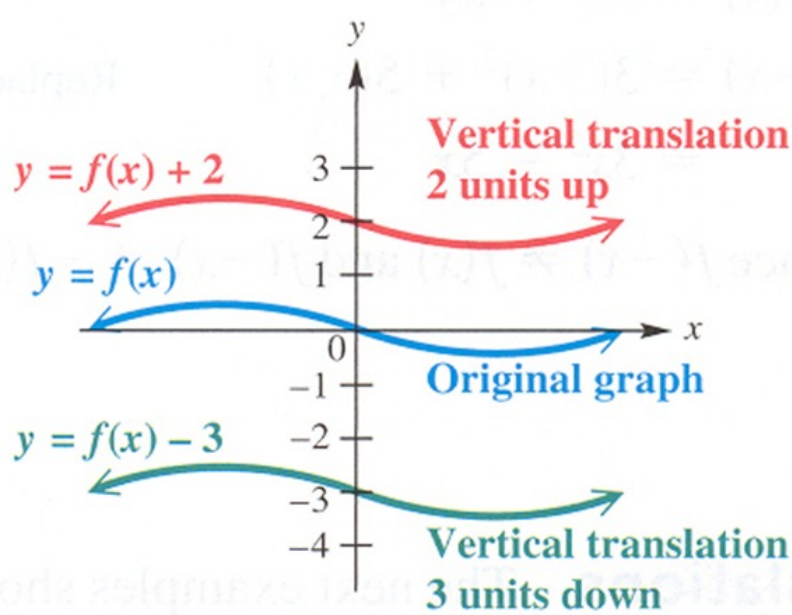

Fig.1: Vertical Shifting.

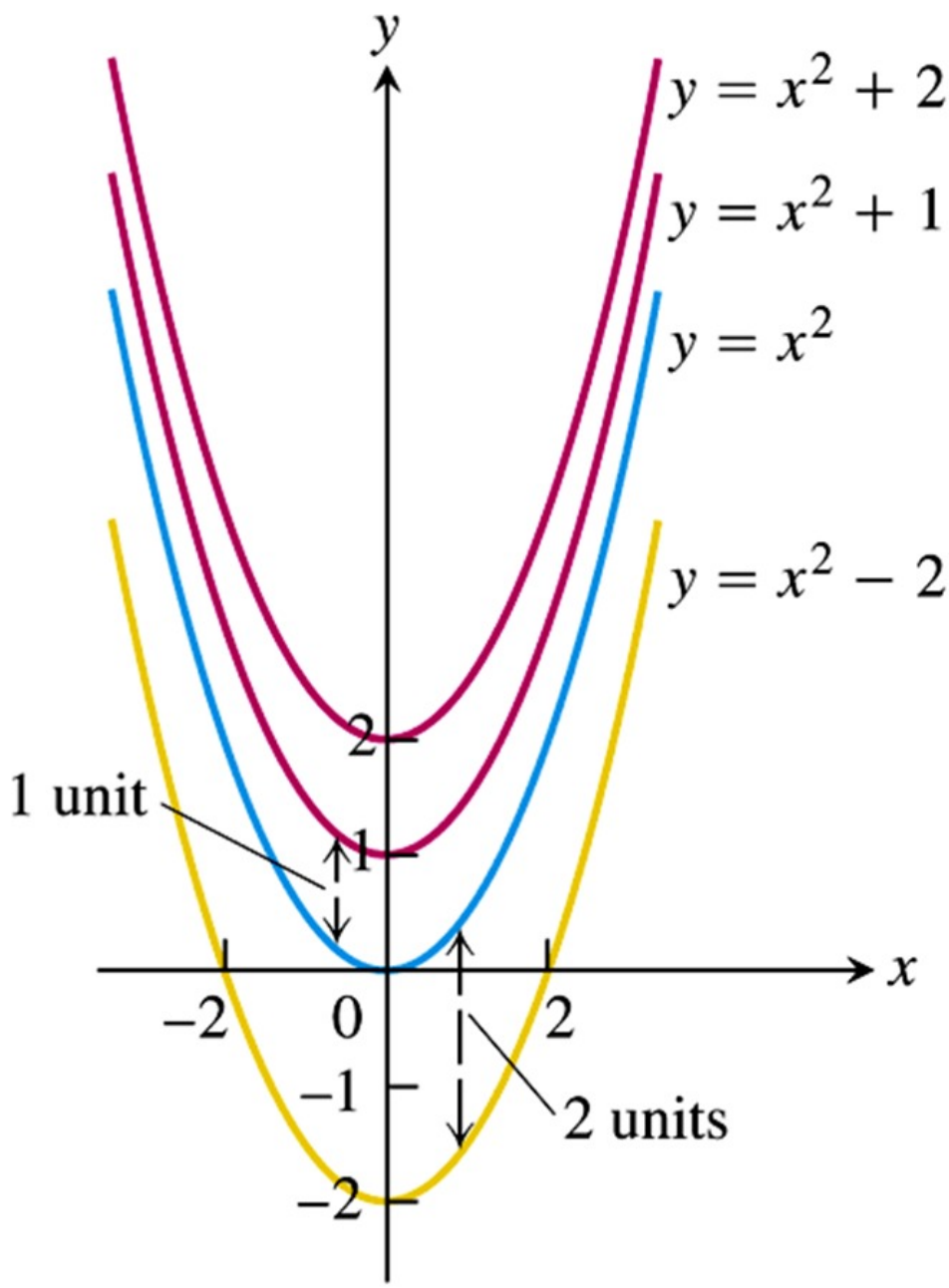

Fig.2: Examples for Vertical Shifting.

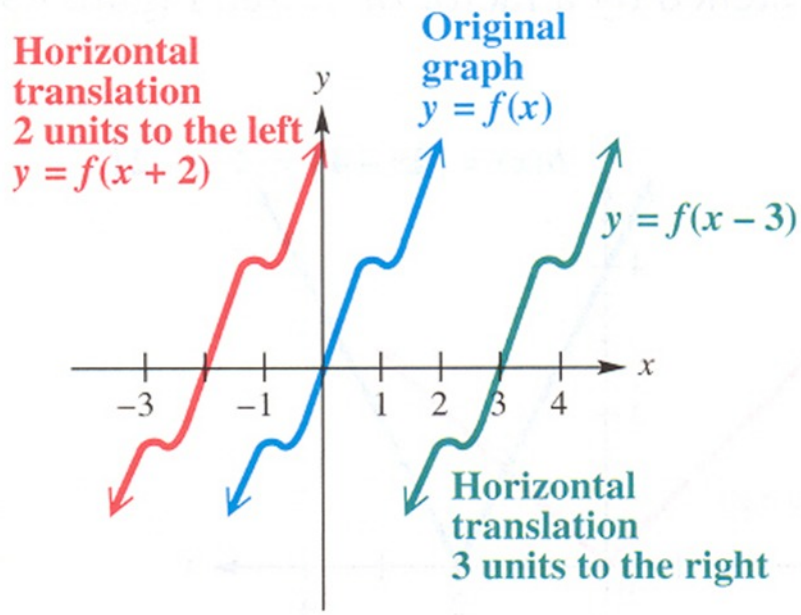

Fig.3: Horizontal Shifting.

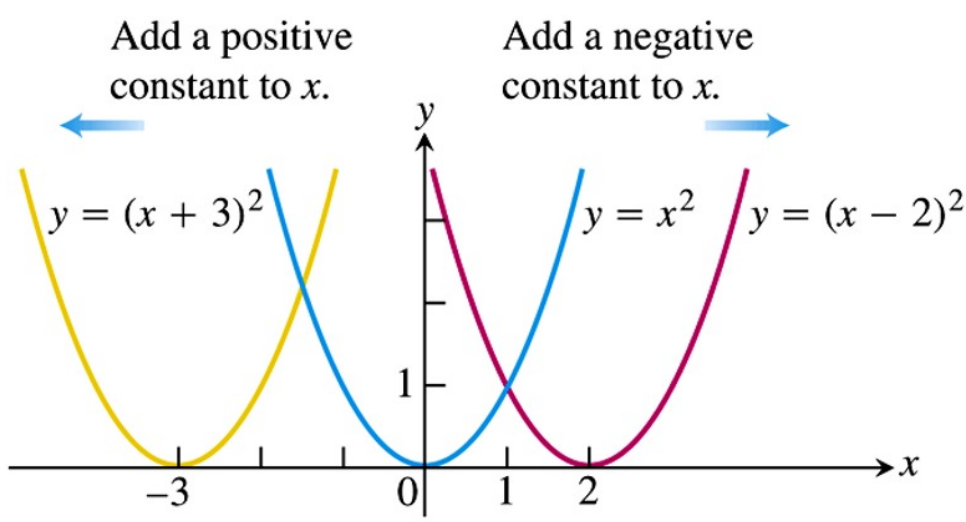

Fig.4: Examples for Horizontal Shifting.

Ex. 1 (Ex.1, P. 38): Start by the function $f(x) = \sqrt{x}$, then sketch the following functions:

(a) $\sqrt{x} + 2$. (b) $\sqrt{x} - 2$. (c) $\sqrt{x - 2}$. (d) $\sqrt{x + 2}$. (e) $-\sqrt{x}$.

Ex. 2 (Ex.2, P. 38): Graph the function: $f(x) = x^2 + 6x + 10$.

Ex. 3 (Ex.3, P. 39): Graph the function: $f(x) = 1 - \sin x$.

Ex. 4: Graph the function: $f(x) = |x - 2| - 1$.

Ex. 5: Graph the function: $f(x) = |x| + 4$.

Ex. 6: Graph the function: $f(x) = -|x + 3| + 1$.

Combining Functions: We can combine two functions $f(x)$ and $g(x)$ to get new functions using:

- Addition: $(f+g)(x) = f(x) + g(x)$.
- Subtraction: $(f-g)(x) = f(x) - g(x)$.
- Multiplication: $(f.g)(x) = f(x).g(x)$, and
- Division: $(f/g)(x) = f(x)/g(x)$.

Note: The domain of the combined function is the intersection (common part) of the two domains of f and g, with Division having the condition of $g(x) \neq 0$.

Ex. 7: Let $f(x) = 8x - 9$ and $g(x) = \sqrt{2x-1}$. Find $f+g$, $f-g$, $f.g$,$\dfrac{f}{g}$, and the domains.

Ex. 8: Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Find $f+g$, $f-g$, $f.g$, $\dfrac{f}{g}$, $\dfrac{g}{f}$, and the domains.

Composition of Functions : In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$.

- If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$.
- The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f.
- It is called the composition (or composite) of f and g.
- It is denoted by fog (f circle g).

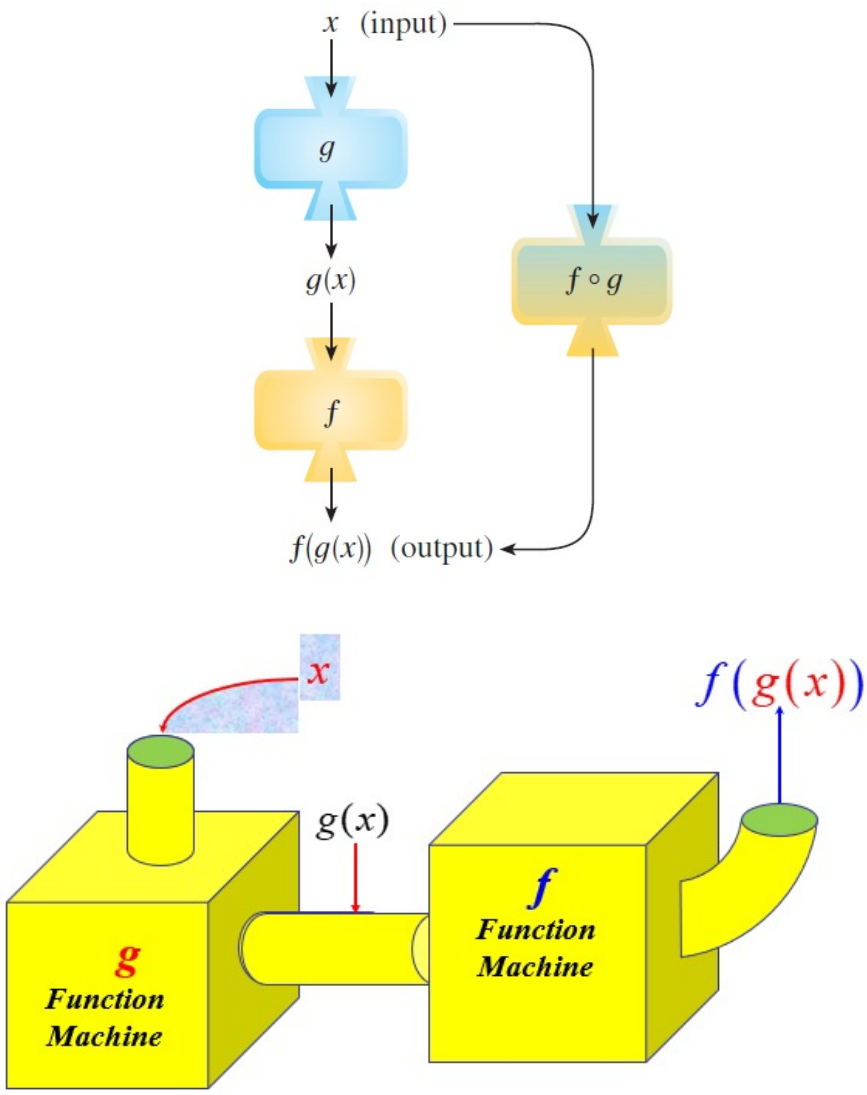

Fig.5: Composition of Functions.

Ex. 9 (Ex.6, P. 41): $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions fog and gof.

Ex. 10 (Ex.7, P. 41): $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the composite functions fog, gof, fof, and gog.

You can see from the previous Example that, in general, $fog \neq gof$.

Ex. 11 (Ex.8, P. 42): Find $fogoh$ if $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}$, and $h(x) = x + 3$.

Ex. 12 (Ex.9, P. 42): Let $F(x) = \cos^2(x + 9)$. Find the functions $f(x)$, $g(x)$, and $h(x)$ such that $F(x) = fogoh$.

Sections 1.3 Exercises: P.42: 9,10,11,14,15,17, 20,21,31,33,35,37,39,43,44,46,47,48.

Islamic University of Madinah
Faculty of Science
Mathematics Department

Calculus I (MATH 3111)

Course References:

◇ J. Stewart, Calculus: Early Transcendentals; Publisher, Cengage Learning; 8th edition (February 4, 2015); Language, English; Hardcover, 1368 pages; ISBN-10, 1285741552.

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CHAPTER 1: Functions and Models

Sections 1.4 and 1.5: Inverse Functions & Exponential and Logarithmic Functions

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 1.4 and 1.5 (Inverse Functions & Exponential and Logarithmic Functions)

OBJECTIVES:

- Find algebraic expressions and sketch the inverse functions.
- Introduce the exponential and logarithmic functions.
- Introduce the inverse trigonometric functions.

One-to-One Function: A function f is called a **one-to-one** function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

Horizontal Line Test: function is **one-to-one** if and only if no horizontal line intersects its graph more than once.

Ex. 1 (Ex.1, P. 56): Is the function $f(x) = x^3$ one-to-one?

Ex. 2 (Ex.2, P. 56): Is the function $f(x) = x^2$ one-to-one?

Inverse Function: Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. The following arrow diagnam indicates that f^{-1} reverses the effect of f.

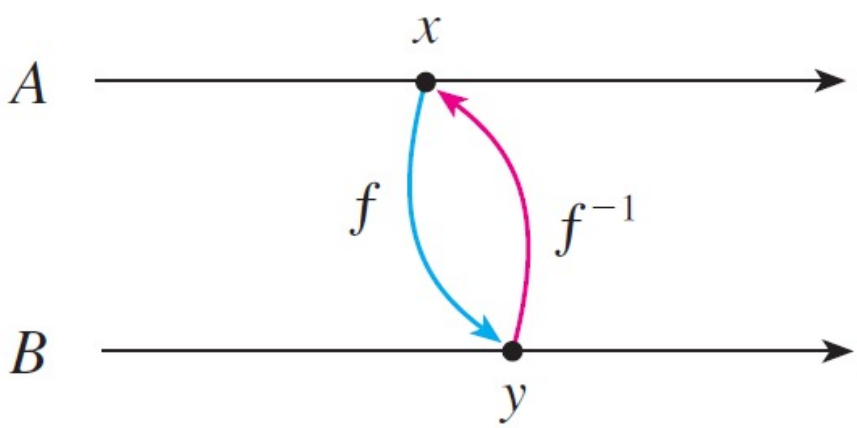


Fig.1: Arrow diagnam indicates that f^{-1} reverses the effect of f.

Note that Domain of f^{-1} = Range of f, and Range of f^{-1} = Domain of f.
i.e.

$$D_f = R_{f^{-1}}, \text{ and } R_f = D_{f^{-1}}$$

CAUTIOM Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in the above definition of the inverse function and write

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

Then, we get the following **cancelation** equations: $f^{-1}(f(x)) = x \quad$ for every x in D_f
$f\left(f^{-1}(x)\right) = x \quad$ for every x in $D_{f^{-1}}$

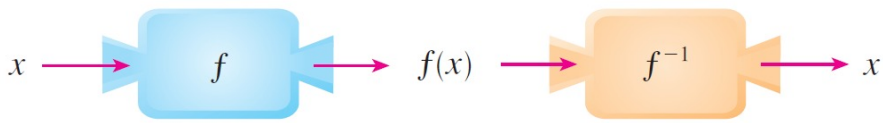


Fig.2: Diagram shows that f^{-1} undoes what f does (Cancelation Equations).

Ex. 3 (Ex.3, P. 57): If $f(1) = 5, f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7), f^{-1}(5)$, and $f^{-1}(-10)$.

How to Find the Inverse Function of a One-to-One Function f?

1. Write $y = f(x)$. 2. Solve this equation for x in terms of y (if possible). 3. To express f as a function of x, interchange x and y. The resulting equation $y = f^{-1}(x)$.

Ex. 4 (Ex.3, P. 58): Find the inverse function of $f(x) = x^3 + 2$.

Ex. 5 (Exer.21, P. 67): Find the inverse function of $f(x) = 1 + \sqrt{2 + 3x}$.

How to Sketch the Graph of the Inverse Function f?

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

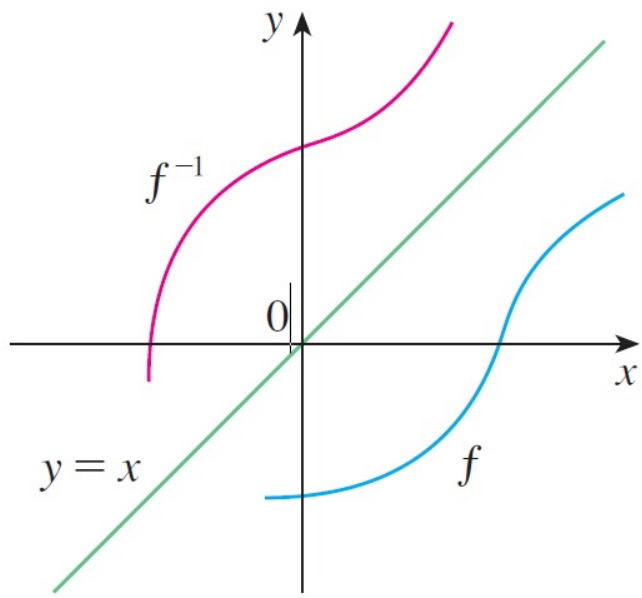


Fig.3: The graph of f^{-1} from f.

Ex. 6: Sketch the graphs of $f(x) = \sqrt{x-2}$ and its inverse function using the same coordinate axes.

Exponential Functions

Exponential Functions: An exponential function is a function of the form

$$f(x) = b^x,$$

where b is a positive constant.

The graphs of members of the family of functions $y = b^x$ are shown in the following Figure for various values of the base b. Notice that all of these graphs pass through the same point $(0, 1)$ because $b^0 = 1$ for $b \neq 0$.

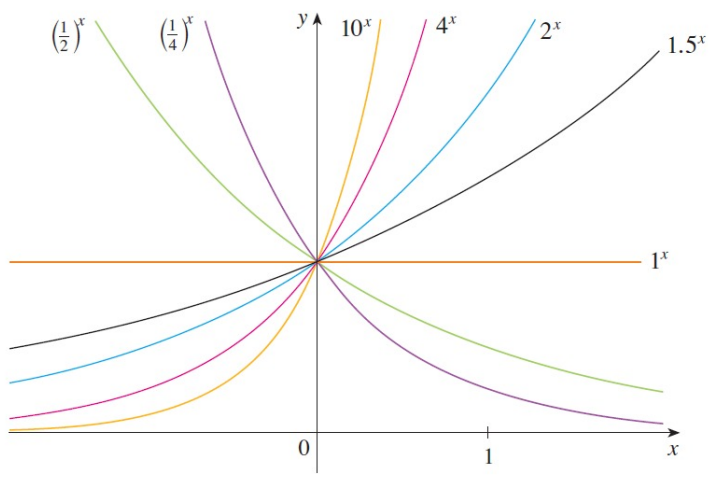


Fig.4: The graphs of members of the family of functions $y = b^x$.

You can see from the above Figure that there are basically three kinds of exponential functions $y = b^x$. If $0 < b < 1$, the exponential function decreases; if $b = 1$, it is a constant; and if $b > 1$, it increases. These three cases are illustrated in the following Figure. Observe that if $b \neq 1$, then the exponential function $y = b^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

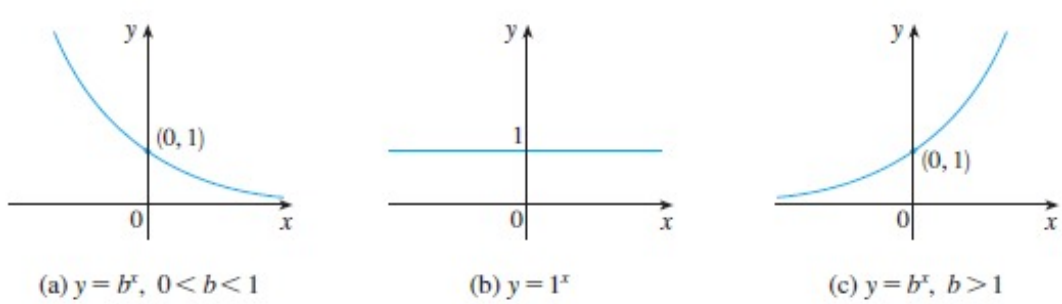


Fig.5: The graphs of the there are basically three kinds of exponential functions $y = b^x$.

Laws of Exponents: If a and b are positive numbers and x and y are any real numbers, then

1. $b^{x+y} = b^x b^y$, 2. $b^{x-y} = \frac{b^x}{b^y}$,

3. $(b^x)^y = b^{xy}$, 4. $(ab)^x = a^x b^x$.

Ex. 7 (Ex.1, P. 47): Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

Natural Exponential Functions

The Number e: Is the number satisfying the following limit:

$$e = \lim n \to \infty (1 + \frac{1}{n})^n.$$

Its approximate value: A very special irrational number denoted by $e = 2.71828....$

Exponential Functions: We call the function $y = e^x$ the natural exponential function.

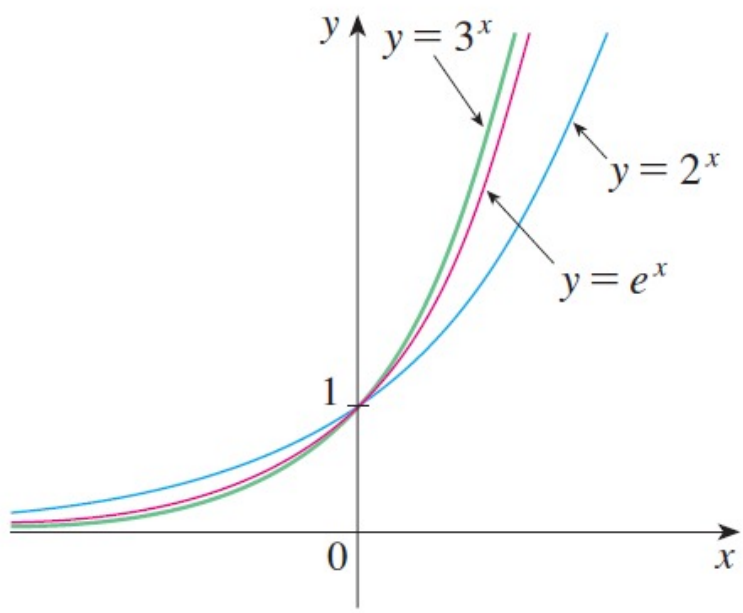


Fig.6: The graphs of the natural exponential function $y = e^x$.

Ex. 8: Sketch the graph of the function $y = -2 - e^x$ and determine its domain

Logarithmic Functions

Logarithmic Functions: If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function** with base b and is denoted by $\log_b$. If we use the formulation of an inverse function, then we have

$$\log_b x = y \quad \Longleftrightarrow \quad b^y = x$$

Thus, if $x > 0$, then $\log_b x$ is the exponent to which the base b must be raised to give x. For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations, when applied to the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, become

$$\log_b (b^x) = x \text{ for every } x \in \mathbb{R}$$
$$b^{\log_b x} = x \text{ for every } x > 0$$

The logarithmic function $\log_b$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$.

The following Figure shows the case where $b > 1$. The fact that $y = b^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.

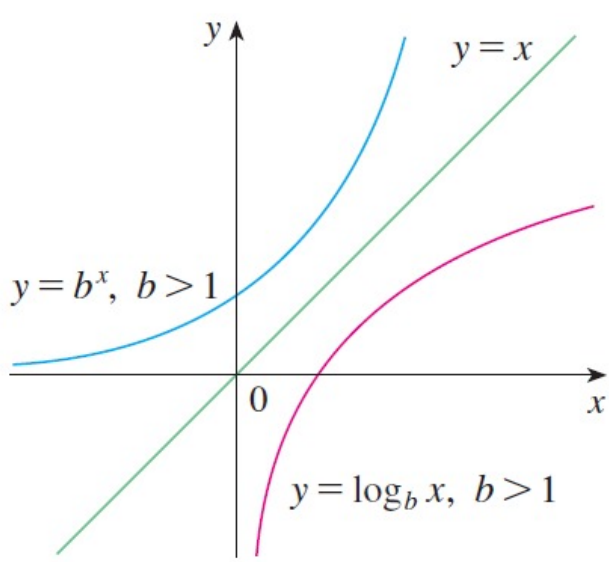


Fig.7: The graphs of the logarithmic function $y = \log_b x$ where $b > 1$.

The following Figure shows the graphs of $y = \log_b x$ with various values of the base $b > 1$. Since $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

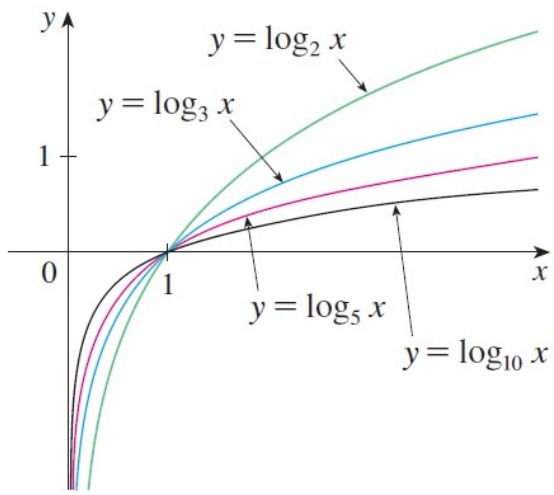


Fig.8: The graphs of $y = \log_b x$ with various values of the base $b > 1$.

Laws of Logarithms: If x and y are positive numbers, then 1. $\log_b(xy) = \log_b x + \log_b y$,

2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$,

3. $\log_b(x^r) = r\log_b x$ (where r is any real number).

Ex. 9 (Ex.6, P. 60): Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Ex. 10 (Exer.20, P. 53): Find the domain of each function:

$$f(x) = \sqrt{10^x - 100}.$$

Natural Logarithmic Functions

Natural Logarithmic Functions: It is the logarithmic function when its base is the number e and is usually denoted by:

$$\log_e x = \ln x.$$

As $\ln x$ is a logarithmic function, we have

$$\ln x = y \quad \Longleftrightarrow \quad e^y = x,$$

and the cancellation equations:

$$\ln(e^x) = x \quad x \in \mathbb{R}$$
$$e^{\ln x} = x \quad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

It is clear that the natural logarithmic function $\ln x$ is the inverse function of the natural exponential function e^x, then their graphs can be shown as in the following Figure

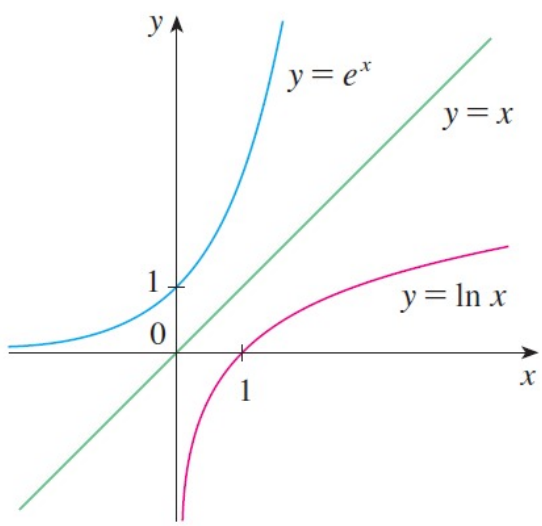


Fig.9: The graphs of the natural exponential function e^x and its inverse function, the natural logarithmic function $\ln x$.

Change of Base Formula: For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}.$$

Ex. 11 (Ex.7, P. 61): Find x if $\ln x = 5$.

Ex. 12 (Ex.8, P. 61): Solve the equation $e^{5-3x} = 10$.

Ex. 13 (Exer.51, P. 67): Solve each equation for x:

(a) $e^{7-4x} = 6$.

(b) $\ln(3x - 10) = 2$.

Ex. 14 (Exer.19, P. 53): Find the domain of each function:

$$f(x) = \frac{1 - e^{x^2}}{1 - e^{x^2-1}}.$$

Ex. 15 (Ex.9, P. 61): Express $\ln a + \frac{1}{2} \ln b$ as a single logarithm.

Ex. 16 (Ex.10, P. 62): Evaluate $\log_8 5$ correct to six decimal places.

Ex. 17 (Ex.11, P. 62): Sketch the function $f(x) = \ln(x-2) - 1$.

Ex. 18 (Exer.23, P. 67): Find a formula for the inverse of the function:

$$f(x) = e^{2x-1}.$$

Ex. 19 (Exer.25, P. 67): Find a formula for the inverse of the function:

$$f(x) = \ln(x+2).$$

Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from the following Figure that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test).

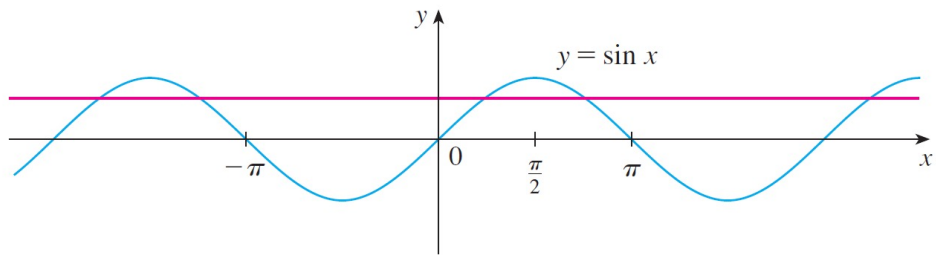


Fig.10: The graphs of the sine function $y = \sin x$, which is not one-to-one.

But the new function

$$\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$$

is one-to-one (check the following figure of this restricted f).

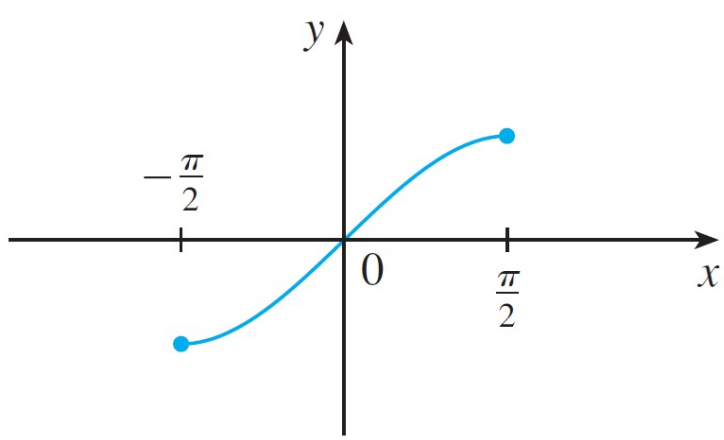


Fig.11: The graphs of the restricted sine function $y = \sin x, \;\; x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, which is one-to-one.

The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1} x$ or arcsin. It is called the inverse sine function or the arcsine function:

$$\sin^{-1} x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

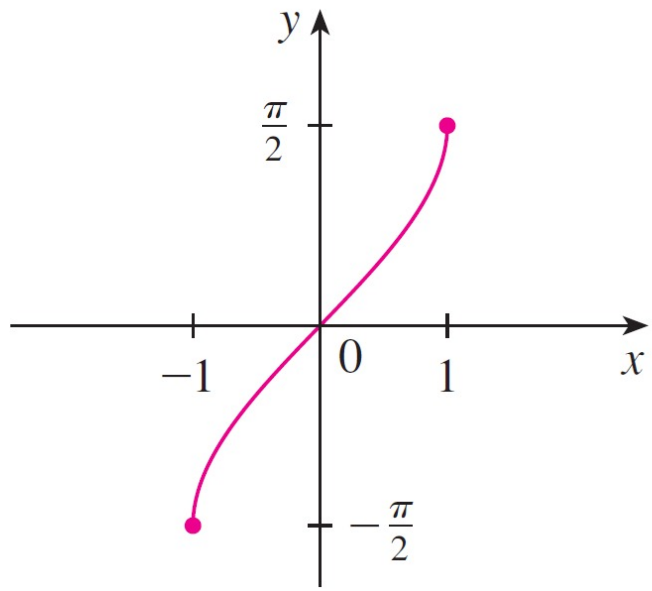


Fig.12: The graphs of the inverse sine function $\sin^{-1} x$.

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$

The cancellation equations for inverse functions become, in this case,

$$\begin{aligned} \sin^{-1}(\sin x) &= x \quad \text{for } -\tfrac{\pi}{2} \leqslant x \leqslant \tfrac{\pi}{2} \\ \sin\left(\sin^{-1} x\right) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

The **inverse cosine** function is handled similarly (check that the regular cosine function $y = \cos x,\ x \in \mathbb{R}$ is not one-to-one). The restricted cosine function $f(x) = \cos x, 0 \leqslant x \leqslant \pi$ is shown in the following Figure

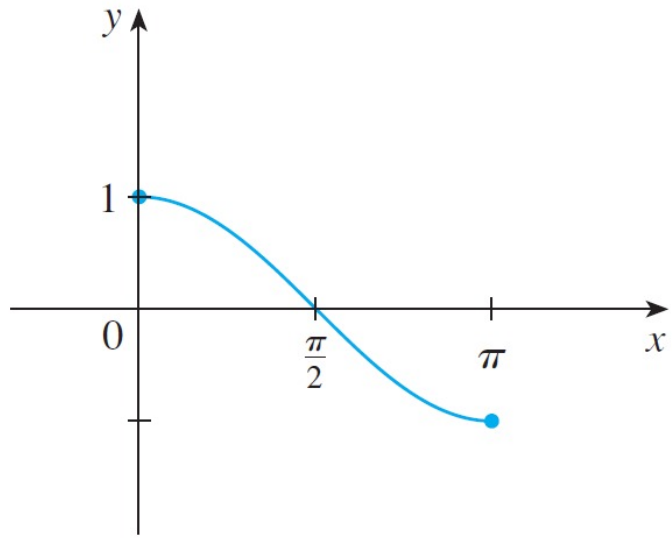


Fig.13: The graphs of the restricted sine function $y = \cos x,\ \ x \in [0, \pi]$, which is one-to-one.

is one-to-one and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

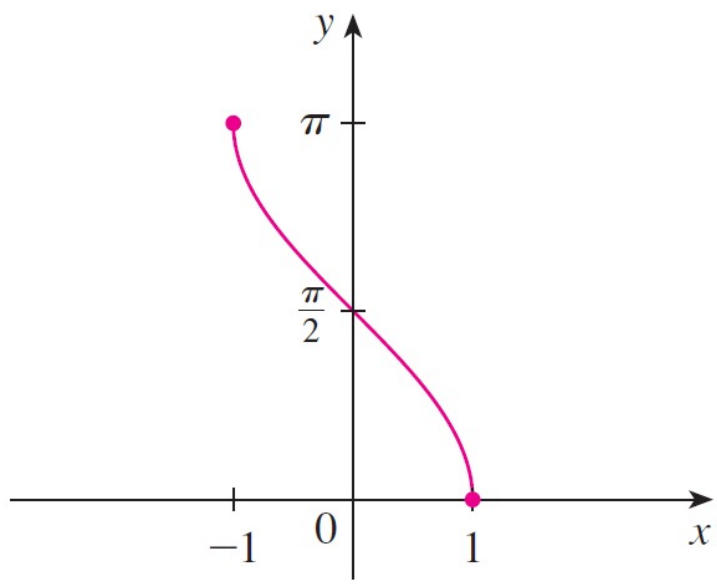


Fig.14: The graphs of the inverse cosine function $\cos^{-1} x$.

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$

The cancellation equations are

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$
$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

he **inverse tangent** function is handled similarly (check that the regular tangent function $y = \tan x$ is not one-to-one). The restricted cosine function $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ is shown in the following Figure

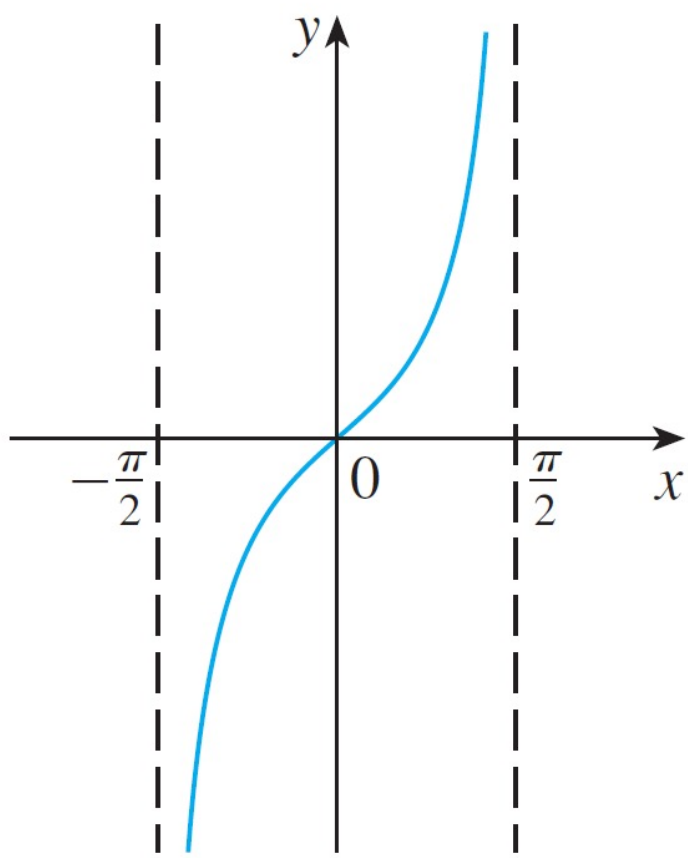


Fig.15: The graphs of the restricted sine function $y = \tan x, \ x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which is one-to-one.

is one-to-one and so it has an inverse function denoted by $\tan^{-1}$ or arctan.

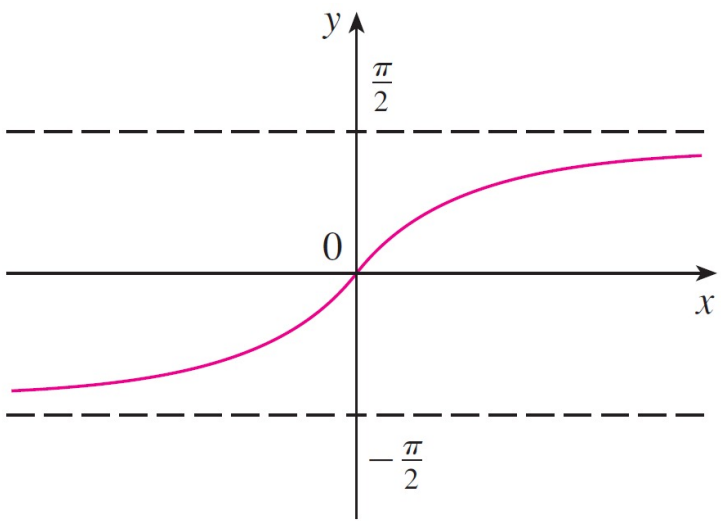


Fig.16: The graphs of the inverse tangent function $\tan^{-1} x$.

The inverse tangent function, $\tan^{-1}$, has domain $\mathbb{R}$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The cancellation equations for inverse functions become, in this case,

$$\begin{aligned} \tan^{-1}(\tan x) &= x \quad \text{for } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2} \\ \tan(\tan^{-1} x) &= x \quad \text{for } x \in \mathbb{R} \end{aligned}$$

Sections 1.4 Exercises: P.53: 1,4,7,8,11,15,16,19,20.

Sections 1.5 Exercises: P.66:

6,7,8,9,10,11,12,17,18,2123,25,35,36,37,51,52,53,69,70,71.

Islamic University of Madinah
Faculty of Science
Mathematics Department

Calculus I (MATH 3111)

Course References:

◇ J. Stewart, Calculus: Early Transcendentals; Publisher, Cengage Learning; 8th edition (February 4, 2015); Language, English; Hardcover, 1368 pages; ISBN-10, 1285741552.

◇ Saudi Digital Library: https://sdl.edu.sa/SDLPortal/ar/Publishers.aspx

CHAPTER 2: LIMITS and DERIVATIVES

Section 2.2: Limit of a Function

Section 2.3: Calculating Limits Using the Limits Laws

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 2.2: Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in geneml and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2 . The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.06	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985005	2.006	4.015005
1.999	3.997001	2.001	4.003001

Table.1 The values of $f(x)$ for values of x close to 2 but not equal to 2.

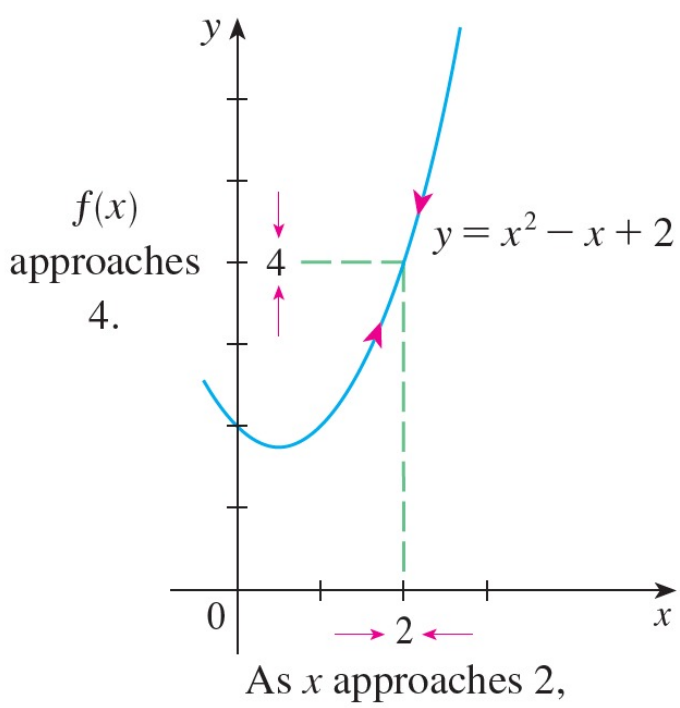


Fig.1 The meaning of the limit graphically.

From the table and the graph of f (a parabola) shown in Figure 1 we see that the closer x is to 2 (on either side of 2), the closer $f(x)$ is to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2 . We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} \left(x^2 - x + 2\right) = 4$$

In general, we use the following notation.

Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L a x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined near a.

Vertical Asymptotes: Will be covered in Section 2.6 with the Horizontal Asymptotes.

Sections 2.3: Calculating Limits Using the Limits Laws

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the Limit Laws, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Ex.1 (Ex.2, P. 97): Evaluate the following limits:

(a) $\lim_{x\to 2}(2x^2 - 3x + 4)$.

(b) $\lim_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$.

Ex.2 (Ex.3, P. 98): Evaluate the following limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Ex.3 (Ex.5, P. 99): Evaluate the following limit:

$$\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h}.$$

Ex.4 (Ex.6, P. 99): Evaluate the following limit:

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}.$$

Ex.5 (Exer.12, P. 102): Evaluate the following limit:

$$\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12}.$$

Ex.6 (Exer.19, P. 103): Evaluate the following limit:

$$\lim_{x \to -2} \frac{x+2}{x^3+8}.$$

Ex.7 (Ex.20, P. 103): Evaluate the following limit:

$$\lim_{t \to 1} \frac{t^4-1}{t^3-1}.$$

Ex.8 (Ex.22, P. 103): Evaluate the following limit:

$$\lim_{u \to 2} \frac{\sqrt{4u+1} - 3}{u - 2}.$$

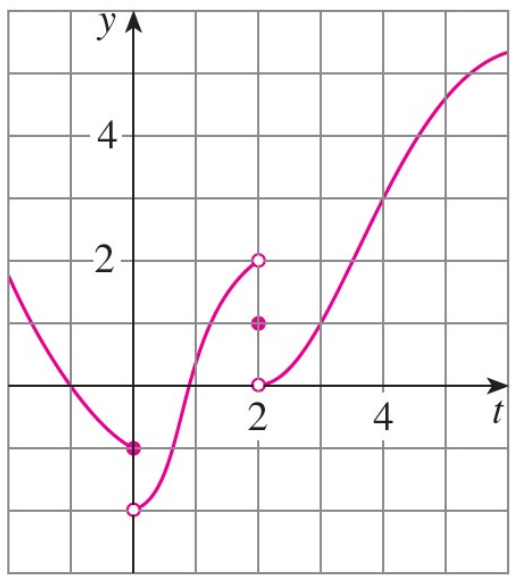
Ex.9 (Ex.29, P. 103): Evaluate the following limit:

$$\lim_{t \to 0} \left(\frac{1}{t\sqrt{t+1}} - \frac{1}{t} \right).$$

TWO sided limits: Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

Theorem: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

Ex.10 (Exer.7, P. 92): Let $g(x)$ be the function that given in the following graph:



State the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 0^+} g(x)$, $\lim_{x\to 0^-} g(x)$, $\lim_{x\to 0} g(x)$

(b) $\lim_{x\to 2^+} g(x)$, $\lim_{x\to 2^-} g(x)$, $\lim_{x\to 2} g(x)$

(c) $g(2)$, $\lim_{x\to 4} g(x)$

Ex.11 (Ex.9, P. 100): If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4, \\ 8 - 2x & \text{if } x < 4. \end{cases}$$

Determine whether $\lim_{x \to 4} f(x)$ exists.

Ex.12 (Exer.12, P. 93): If

$$f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0, \\ \cos x & \text{if } 0 \leq x \leq \pi, \\ \sin x & \text{if } x > \pi. \end{cases}$$

Determine whether $\lim_{x \to 0} f(x)$ and $\lim_{x \to \pi} f(x)$ exist.

Ex.13: If

$$f(x) = \begin{cases} \frac{x^3-27}{x^2-9} & \text{if } x > 3, \\ 1 + \sin(\pi x) & \text{if } x \leq 3. \end{cases}$$

Determine whether $\lim_{x \to 3} f(x)$ exists.

Ex.14 (Ex.7, P. 100): Find $\lim_{x \to 0} |x|$ if exists.

Ex.15 (Ex.8, P. 100): Find $\lim_{x \to 0} \frac{|x|}{x}$ if exists.

The Squeeze Theorem (Sandwich Theorem). If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then

$$\lim_{x \to a} g(x) = L.$$

Ex.16 (Ex.11, P. 101): Find $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$ if exists.

Ex.17 (Exer.38, P. 103): If

$$2x \leq g(x) \leq x^4 - x^2 + 2$$

Find $\lim_{x \to 1} g(x)$ if exists.

Ex.18 (Exer.39, P. 103): Find $\lim_{x\to 0} x^4 \cos(\frac{2}{x})$ if exists.

Sections 2.2 Exercises: Page 92: 4,5,6,7,8,9,11,12.

Sections 2.3 Exercises: Page 102: 1,2, odd No. from 11 to 31+ 36,37,38,39,40,49,50,51,52.

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CHAPTER 2: LIMITS and DERIVATIVES

Section 2.5: Continuity

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 2.5: Continuity

Definition 2.5.1: A function f is said to be a **continuous** function at a point a if:

1. $f(a)$ is defined (that is, a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $f(a) = \lim_{x \to a} f(x)$.

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous** at a (or f has a discontinuity at a) if f is not continuous at a.

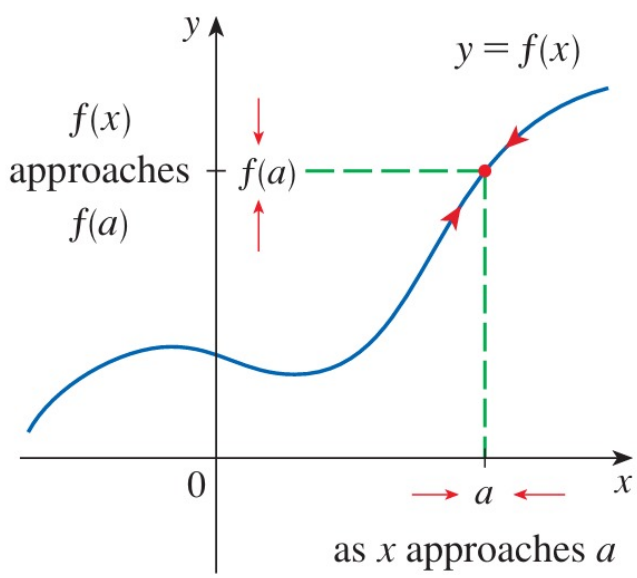

Fig.1 If f is continuous, then the points $(x, f(x)$ on the graph of f approach the point $(a, f(a)$ on the graph. So there is no gap in the curve.

Ex.1 (Ex.2(b), P. 116): Study the continuity of the following function at the point $x = 2$:

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Ex.2 (Ex.2(c), P. 116): Study the continuity of the following function at the point $x = 0$:

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Ex.3: Study the continuity of the following function at the point $x = 0$:

$$f(x) = \begin{cases} \frac{x^3 - x}{x} & \text{if } x < 0, \\ -2 + \cos(\pi x) & \text{if } x \geq 0. \end{cases}$$

Ex.4 (Exer.45, P. 125): Study the continuity of the following function at the points $x = 0$ and $x = 1$:

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0, \\ e^x & \text{if } 0 \leq x \leq 1, \\ 2 - x & \text{if } x > 1. \end{cases}$$

Types of discontinuities: There are three types as follows:

$\Downarrow\Downarrow\Downarrow$

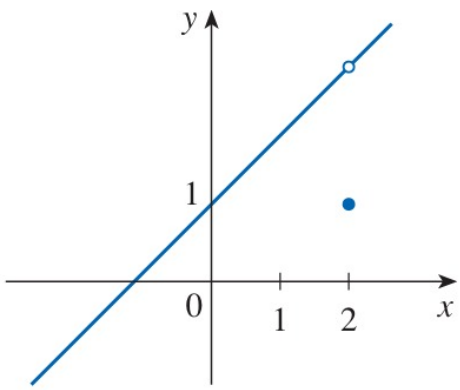

Fig.2 A **removable** discontinuity as in Ex.2(b).

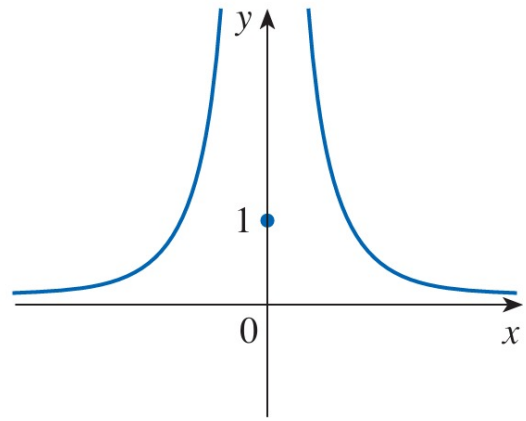

Fig.3 An **infinite** discontinuity as in Ex.2(c).

The third type of discontinuities are called jump discontinuities because the function jumps from one value to another as in the following graph:

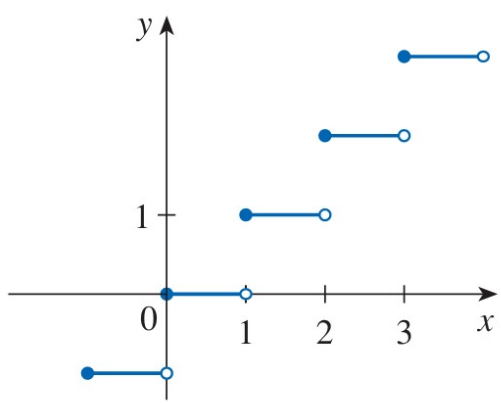

Fig.4 **Jump** discontinuity.

Definition 2.5.2: A function f is continuous from the right at a point a if

$$f(a) = \lim_{x \to a^+} f(x),$$

and f is continuous from the left at a if

$$f(a) = \lim_{x \to a^-} f(x).$$

Definition 2.5.3: A function f is continuous on an interval if it is continuous at every number in the interval.

Theorem 1. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$f+g, \quad f-g, \quad cf, \quad fg, \quad \frac{f}{g}, \text{ if } g(a) \neq 0.$$

Theorem 2. The following types of functions are continuous at every number in their domains:

Polynomials - Rational functions - Trigonometric functions - Inverse trigonometric functions - Exponential functions - Logarithmic functions.

Theorem 3. If g is continuous at a and f is continuous at $g(a)$, then the composite function fog is continuous at a.

Ex.5 (Exer.41, P. 125): Study the continuity of the following function on $\mathbb{R}$:

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1, \\ \ln x & \text{if } x > 1. \end{cases}$$

Ex.6, P. 120: Where is the function $\dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Ex.7: Where is the function $\dfrac{\sqrt{x-2} + \cos x}{x - 7}$ continuous?

Ex.8 (Exer.42, P. 125): Study the continuity of the following function on $\mathbb{R}$:

$$f(x) = \begin{cases} \sin x & \text{if } x < \frac{\pi}{4}, \\ \cos x & \text{if } x \geq \frac{\pi}{4}. \end{cases}$$

Ex.9 (Exer.47, P. 125): For what value of the constant c is the function f continuous at $x = 2$:

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2, \\ x^3 - cx & \text{if } x \geq 2. \end{cases}$$

H.W. Try to solve exercise 48.

The Intermediate Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and assume that $f(a)f(b) < 0$. Then there exists a number c in (a, b) such that $f(c) = 0$.

Ex.10 (Exer.55, P. 126): Use the Intermediate Value Theorem to find a solution of the following equation:

$$-x^3 + 4x + 1 = 0, \qquad x \in [-1, 0]$$

Ex.11 (Exer.57, P. 126): Use the Intermediate Value Theorem to find a solution of the following equation:

$$e^x = 3 - 2x.$$

Sections 2.5 Exercises: Page 124:

18,19,20,21,22,25,27,29,31,39,40,41,42,43,45,46,53,54,55, 56, 57, 58

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CHAPTER 2: LIMITS and DERIVATIVES

Section 2.6: Limits at Infinity; Horizontal Asymptotes

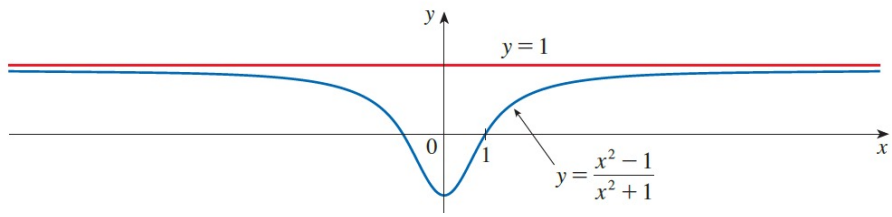
•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 2.6: Limits at Infinity; Horizontal and Vertical Asymptotes

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in the following Figure:



x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

You can see that as x grows larger and larger, the values of $f(x)$ get closer and closer to 1. (The graph of f approaches the horizontal line $y = 1$ as we look to the right.) In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \longrightarrow \infty} \frac{x^2 - 1}{x^2 + 1} = 1.$$

In general, we use the notation $\lim\limits_{x \longrightarrow \infty} = L$ to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

Ex.1, P. 129: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} \tan^{-1} x \quad \text{and} \quad \lim_{x \longrightarrow -\infty} \tan^{-1} x.$$

Ex.2, P. 129: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \longrightarrow -\infty} \frac{1}{x}.$$

Ex.3: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} \frac{1}{x^2} \quad \text{and} \quad \lim_{x \longrightarrow -\infty} \frac{1}{x^2}.$$

Theorem. If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \longrightarrow \infty} \frac{1}{x^r} = 0 \quad \text{and} \quad \lim_{x \longrightarrow -\infty} \frac{1}{x^r} = 0.$$

Ex.4 (Ex.3, P. 130): Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Ex.5, P. 131: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} \left(\sqrt{x^2 + 1} - x \right).$$

Ex.6 (Exer.19, P. 138): Evaluate the following limits:

$$\lim_{r \longrightarrow \infty} \frac{r - r^3}{2 - r^2 + 3r^3}.$$

Ex.7 (Exer.29, P. 138): Evaluate the following limits:

$$\lim_{t \longrightarrow \infty} \left(\sqrt{25t^2 + 2} - 5t \right).$$

Ex.8, P. 132: Evaluate the following limits:

$$\lim_{x \longrightarrow 0^-} e^{\frac{1}{x}}.$$

Ex.9, P. 133: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} x^3 \quad \text{and} \quad \lim_{x \longrightarrow -\infty} x^3.$$

Ex.10, P. 133: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} (x^3 - x).$$

Ex.11, P. 133: Evaluate the following limits:

$$\lim_{x \longrightarrow \infty} \frac{x^2 + x}{3 - x}.$$

Horizontal and Vertical Asymptote:

Definition 2.6.1 (From section 2.2): The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty$$

Ex.12 (Ex.7, P. 91): Find the vertical asymptotes of the graph of the function

$$f(x) = \frac{2x}{x-3}.$$

Ex.13 (Ex.8, P. 91): Find the vertical asymptotes of the graph of the function

$$f(x) = \tan x.$$

Ex.14: Find the vertical asymptotes of the graph of the function

$$f(x) = \frac{x+5}{x^2-4}.$$

Definition 2.6.2: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$if either

$$\lim_{x \longrightarrow \infty} = L \qquad \text{or} \qquad \lim_{x \longrightarrow -\infty} = L.$$

Ex.15: Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

Ex.16: Find the horizontal asymptotes of the graph of the function

$$f(x) = \tan^{-1} x.$$

Ex.17: Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{3 - x}{2 + x}.$$

Ex.18 (Exer.49, P. 138): Find the vertical and horizontal asymptotes of the graph of the function

$$f(x) = \frac{2x^2 + x - 1}{x^2 + x - 2}.$$

Ex.19 (Exer.52, P. 138): Find the vertical and horizontal asymptotes of the graph of the function

$$f(x) = \frac{2e^x}{e^x - 5}.$$

Sections 2.5 Exercises: Page 137: Odd No. from 15 to 39 + 47,48,49,52.

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CHAPTER 2: LIMITS and DERIVATIVES

Section 2.7: Derivatives and Rates of Change
Section 2.8: The Derivative as a Function

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

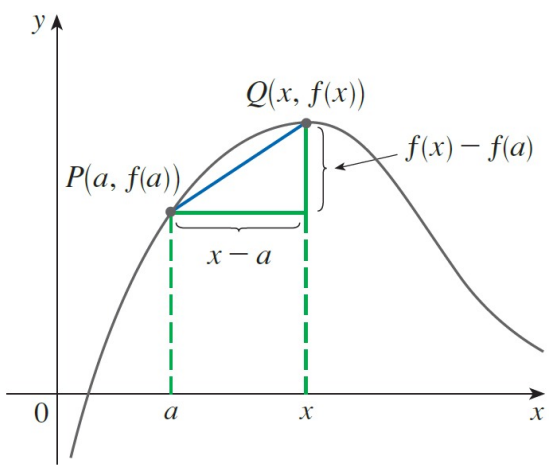
Sections 2.7: Derivatives and Rates of Change

Sections 2.8: The Derivative as a Function

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the tangent line ℓ to be the line through P with slope m.



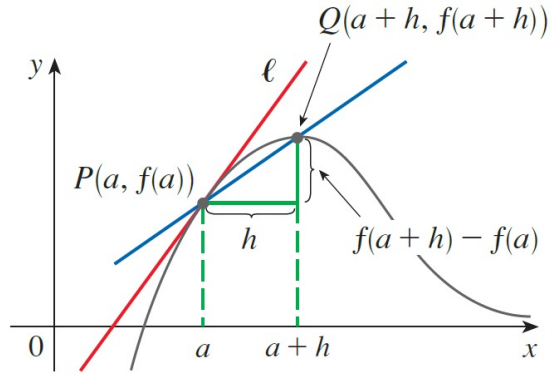
Definition 2.7.1 (Tangent) The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

provided that this limit exists.

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$



where the case $h > 0$ is illustrated and Q is located to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.
Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 2.7.1 becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{2}$$

Note: You can use any of equations (1) and (2) to obtain the slope of the tangent line to the curve at the given point.

Definition 2.7.2 (Derivative) The **derivative** of a function f at a number a, denoted by $f^{'}(a)$, is

$$f^{'}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{3}$$

provided that this limit exists.

and similarly, we can get a different expression for the derivative, which is

$$f^{'}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{4}$$

Note: You can use any of equations (3) and (4) to obtain the slope of the derivative of the function at the given point.

In equation (3), we considered the derivative of a function f at a fixed number a. Here we change our point of view and let the number a vary. If we replace a in Equation (3) by a variable x, we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{5}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the derivative of f and defined by Equation (5). We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.
The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation (5). The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

Ex.1, P. 141: Use the definition to find an equation of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.

Ex.2, P. 142: Use the definition to find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Ex.3, (Exer.3, P. 149): Use the definition to find an equation of the tangent line to the hyperbola $y = x^2 + 3x - 9$ at the point $(-2, -11)$.

Ex.4, P. 144: Use the definition to find the derivative of the function $f(x) = x^2 - 8x + 9$.

Ex.5, P. 145: Use the definition to find the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$.

Ex.6, P. 146: Use the definition to find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Ex.7, (Ex.2, P. 154): Use the definition to find the derivative of the function $f(x) = x^3 - x$.

Ex.8, (Ex.3, P. 155): Use the definition to find the derivative of the function $f(x) = \sqrt{x}$. Then find the domain of $f^{'}$.

Ex.9, (Ex.4, P. 156): Use the definition to find the derivative of the function $f(x) = \dfrac{1-x}{2+x}$.

Ex.10, (Ex.5, P. 157): Use the definition to find the derivative of the function $f(x) = |x|$.

Theorem If f is differentiable at a, then f is continuous at a.

Is the inverse true ? Discuss the inverse or show by example.

Section 2.7 Exercises: Page 148: 5,6,7,31,32,33.

Section 2.8 Exercises: Page 160: 21,23,25,27,29,31.

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CHAPTER 3: DIFFERENTIATION RULES

Section 3.1: Derivatives of Polynomials and Exponential Functions
Section 3.2: The Product and Quotient Rules
Section 3.3: Derivatives of Trigonometric Functions

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 3.1: Derivatives of Polynomials and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

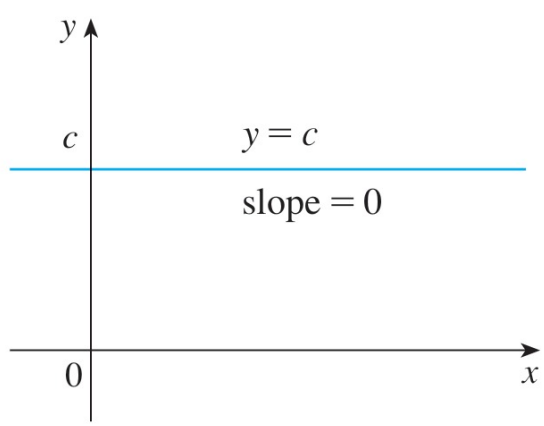
You can prove each of the following rules by limits as was covered in Section 2.8 and some of them can be shown using the slope of tangent line as was covered in Section 2.7, for example:

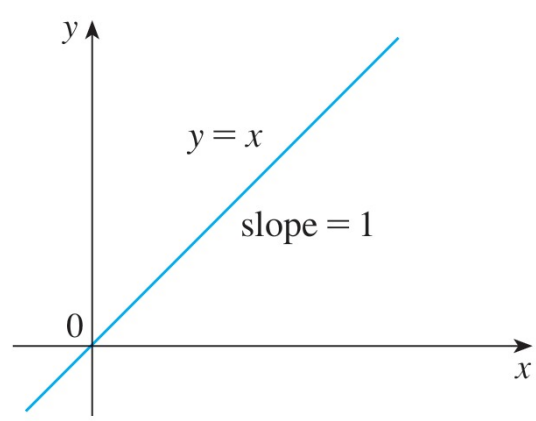
(1) The constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$ (See the following Figure), which has slope 0, so we must have $f'(x) = 0$.
A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0}(0) = 0.$$

(2) The linear function $f(x) = x$. The graph of this function is the line $y = x$ (See the following Figure), which has slope 1, so we must have $f'(x) = 1$.
A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0}(1) = 1.$$





Similarly, we can get the following **differentiation rules:**

(1) Constant Rule: $f(x) = c$ then $f'(x) = 0$

(2) Constant Multiple Rule: $g(x) = c \cdot f(x)$ then $g'(x) = c \cdot f'(x)$

(3) Power Rule: $f(x) = x^n$ then $f'(x) = nx^{n-1}$

(4) Exponential Derivative $f(x) = e^x$ then $f'(x) = e^x$

(5) Sum and Diff Rules: $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$

Ex.1, P. 174: find the derivative of the following functions:

(a) $f(x) = x^6$.

(b) $y = x^{1000}$.

(c) $y = t^4$.

(d) $y = r^3$.

Ex.2, P. 174: find the derivative of the following functions:

(a) $f(x) = \dfrac{1}{x^2}$.

(b) $y = \sqrt[3]{x^2}$.

Ex.3, P. 175: Find an equation of the tangent line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Ex.4, P. 176: Find the derivative of the following functions:

(a) $f(x) = 3x^4$.

(b) $y = -x$.

Ex.5 (Ex.8, P. 179): Let $f(x) = e^x - x$. Find f' and f''.

Ex.6 (Exer.19, P. 180): Find the derivative of the function:

$$f(x) = 3e^x + \frac{4}{\sqrt[3]{x}}.$$

Ex.7 (Exer.23, P. 180): Find the derivative of the function:

$$f(x) = \frac{x^2 + 4x + 3}{\sqrt{x}}.$$

Ex.8 (Exer.33, P. 180): Find an equation of the tangent line to the curve $y = 2x^3 - x^2 + 2$ at the point $(1, 3)$.

Section 3.1 Exercises: Page P. 180: All from 3 to 37 + 47,48.

Sections 3.2: The Product and Quotient Rules

(6) **Product Rule:** $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$

(7) **Quotient or Division Rule:** $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Ex.1, P. 184: Let $f(x) = xe^x$

(a) Find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$. Then find $f^{(1000)}(0)$.

Ex.2, P. 185: Find the derivative of the function:

$$f(t) = \sqrt{t}(a + bt).$$

Ex.3, P. 185: If $f(x) = \sqrt{x}\, g(x)$, $g(4) = 2$ and $g^{'}(4) = 3$. Find $f^{'}(4)$.

Ex.4 (Exer.4, P. 188): Find the derivative of the function:

$$g(x) = (x + 2\sqrt{x})e^{x}.$$

Ex.5 (Exer.10, P. 188): Find the derivative of the function:

$$J(v) = (v^3 - 2v)(v^{-4} + v^{-2}).$$

Ex.6 (Ex.4, P. 186): Find the derivative of the function:

$$y = \frac{x^2 + x - 2}{x^3 + 6}.$$

Ex.7 (Exer.23, P. 188): Find the derivative of the function:

$$f(x) = \frac{x^2\, e^x}{x^2 + e^x}.$$

Ex.8 (Exer.26, P. 186): Find the derivative of the function:

$$f(x) = \frac{ax + b}{cx + d}.$$

Ex.9 (Ex.5, P. 187): Find an equation of the tangent line to the curve $y = \dfrac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

Ex.10 (Exer.31, P. 188): Find an equation of the tangent line to the curve $y = \dfrac{x^2-1}{x^2+x+1}$ at the point $(1, 0)$.

Sections 3.2 Exercises: Page P. 188: All from 3 to 34.

Sections 3.3: Derivatives of Trigonometric Functions

As we mentioned in the introduction of Section 3.1, all the differentiation rules can be obtained using the definition that we covered in Section 2.8, so we have the following **trigonometric derivatives:**

(8) $f(x) = \sin(x)$ then $f'(x) = \cos(x)$

(9) $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$

(10) $f(x) = \tan(x)$ then $f'(x) = \sec^2(x)$

(11) $f(x) = \sec(x)$ then $f'(x) = \sec(x)\tan(x)$

(12) $f(x) = \cot(x)$ then $f'(x) = -\csc^2(x)$

(13) $f(x) = \csc(x)$ then $f'(x) = -\csc(x)\cot(x)$

Important Limits Rules: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$

Ex.1, P. 193: Find the derivative of the function:

$$f(x) = x^2 \sin x.$$

Ex.2, P. 194: Find the derivative of the function:

$$f(x) = \frac{\sec x}{1 + \tan x}.$$

Ex.3, (Exer.13, P. 196): Find the derivative of the function:

$$f(t) = \frac{t \sin t}{1+t}.$$

Ex.4, (Exer.22, P. 196): Find an equation of the tangent line to the curve $y = e^x \cos x$ at the point $(0, 1)$.

Ex.5, P. 195: Find

$$\lim_{x\to 0} \frac{\sin(7x)}{4x}.$$

Ex.6, P. 195: Find

$$\lim_{x\to 0} x \cot x.$$

Ex.7 (Exer.16, P. 196): Find the derivative of the function:

$$f(t) = te^t \cos t.$$

Sections 3.3 Exercises: Page P. 196: All from 1 to 19 + 21,22,23,24,39,41,43,44,45.

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CHAPTER 3: DIFFERENTIATION RULES

Section 3.4: Chain Rule

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 3.4: Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = fog$ defined by $F(x) = f\big(g(x)\big)$ is differentiable at x and F' is given by the product

$$F'(x) = f'\big(g(x)\big)g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Using the chain rule, we can get the following **differentiation rules:**

(1) $\dfrac{d}{dx}(u(x))^n = (u(x))^{n-1} \cdot u'(x)$,

(2) $\dfrac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)} \cdot u'(x)$

(3) $\dfrac{d}{dx}\sin(u(x)) = \cos(u(x)) \cdot u'(x)$,

(4) $\dfrac{d}{dx}\cos(u(x)) = -\sin(u(x)) \cdot u'(x)$

(5) $\dfrac{d}{dx}\tan(u(x)) = \sec^2(u(x)) \cdot u'(x)$,

(6) $\dfrac{d}{dx}\cot(u(x)) = -\csc^2(u(x)) \cdot u'(x)$

(7) $\dfrac{d}{dx}\sec(u(x)) = \sec(u(x))\tan(u(x)) \cdot u'(x)$

(8) $\dfrac{d}{dx}\csc(u(x)) = -\csc(u(x))\cot(u(x)) \cdot u'(x)$

Ex.1, P. 199: Find the derivative of the following functions:

$$f(x) = \sqrt{x^2 + 1}.$$

Ex.2, P. 200: Find the derivative of the following functions:

(a) $f(x) = \sin(x^2)$.

(b) $y = \sin^2(x)$.

Ex.3, P. 201 Find the derivative of the following functions:

$$f(x) = (x^3 - 1)^{100}.$$

Ex.4, P. 201: Find the derivative of the following functions:

$$f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}.$$

Ex.5, P. 201 Find the derivative of the following functions:

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9.$$

Ex.6, P. 201: Find the derivative of the following functions:

$$y = (2x+1)^5 \left(x^3 - x + 1\right)^4.$$

Ex.7, P. 202 Find the derivative of the following functions:

$$y = e^{\sin x}.$$

Ex.8, P. 202 Find the derivative of the following functions:

$$y = \sin[\cos(\tan x)].$$

Ex.9, P. 203 Find the derivative of the following functions:

$$y = e^{\sec(3\theta)}.$$

Ex.10 (Exer.35, P. 204): Find the derivative of the following functions:

$$y = \cos\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right).$$

Ex.11 (Exer.37, P. 204): Find the derivative of the following functions:

$$y = \cot^2(\sin\theta).$$

Ex.12 (Exer.52, P. 204): Find an equation of the tangent line to the curve $y = \sqrt{1 + x^3}$ at the point $(2, 3)$.

Ex.13 (Exer.54, P. 204): Find an equation of the tangent line to the curve $y = xe^{-x^2}$ at the point $(0, 0)$.

Section 3.4 Exercises: Page P. 204: All from 7 to 54.

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CHAPTER 3: DIFFERENTIATION RULES

Section 3.5: Iimplicit differentiation

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 3.5: Iimplicit differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable as $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 25 \quad \text{or} \quad x^3 + y^3 = 6xy.$$

In some cases it is possible to solve such an equation for y as an explicit function of x and in other cases, it's not easy to solve for y explicitly as a function of x by hand.

Fortunately, it is unnecessary to express y as a function of x to get the derivative of y. We can employ the technique of implicit differentiation instead. This involves differentiating both sides of the equation with regard to x and subsequently solving the resulting equation for y'. This section's examples and exercises consistently assume that the provided equation implicitly defines y as a differentiable function of x, hence allowing the use of implicit differentiation.

Ex.1, P. 209: Let $x^2 + y^2 = 25$:

(a) Find y'.

(b) Find an equation of the tangent line to the above curve at the point $(3, 4)$.

Ex.2, P. 210: Let $x^3 + y^3 = 6xy$:

(a) Find y'.

(b) Find an equation of the tangent line to the above curve at the point $(3, 3)$.

Ex.3, P. 211 Find $y^{'}$ if

$$\sin(x+y) = y^2 \cos x.$$

Ex.4, P. 212: Find y'' if

$$x^4 + y^4 = 16.$$

Ex.5 (Exer.15, P. 215) Find $y^{'}$ if

$$e^{\frac{x}{y}} = x - y.$$

Ex.6 (Exer.19, P. 215) Find $y^{'}$ if

$$\sin(xy) = \cos(x + y).$$

Ex.7 (Exer.25, P. 215) Find an equation of the tangent line to the curve $y\ \sin(2x) = x\ \cos(2y)$ at the point $(\frac{\pi}{2}, \frac{\pi}{4})$.

Derivatives of Inverse trigonometric Functions

Here we use implicit differentiation to find the derivatives of the inverse trigonometric functions, assuming that these functions are differentiable. We can prove that:

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

Ex.8 (Exers.49-60, P. 215) Find y' if:

(a) $y = \left(\tan^{-1} x\right)^2$

(b) $y = \tan^{-1}\left(x^2\right)$

(c) $y = \sin^{-1}(2x+1)$

(d) $g(x) = \arccos\sqrt{x}$

(e) $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$

(f) $R(t) = \arcsin(1/t)$

(g) $y = x\sin^{-1}x + \sqrt{1-x^2}$

(h) $y = \arctan\sqrt{\dfrac{1-x}{1+x}}$

Section 3.5: Page P. 215: All from 5 to 22 + All from 25 to 28 + All from 49 to 58.

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CHAPTER 3: DIFFERENTIATION RULES

Section 3.6: Derivatives of Logarithmic Functions

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 3.6: Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic and the natural logarithmic functions. We have the following differentiation rules:

$$\frac{d}{dx}(\log_b x) = \frac{1}{x}\frac{1}{\ln b}, \quad \text{and} \quad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Also, we have the following differentiation rules for the exponential and natural exponential functions:

$$\frac{d}{dx}(e^x) = e^x, \quad \text{and} \quad \frac{d}{dx}(a^x) = a^x \ln a.$$

Using the chain rule, we have the following rules:

(1) $\dfrac{d}{dx}[\log_b(f(x))] = \dfrac{1}{f(x)}\dfrac{1}{\ln b} f'(x).$

(2) $\dfrac{d}{dx}[\ln(f(x))] = \dfrac{1}{f(x)} f'(x).$

(3) $\dfrac{d}{dx}[e^{f(x)}] = e^{f(x)} f'(x).$

(4) $\dfrac{d}{dx}[a^{f(x)}] = a^{f(x)} f'(x) \ln a.$

Ex.1, P. 218: Find the derivative of $y = \ln(x^3 + 1)$.

Ex.2, P. 219: Find the derivative of $y = \ln(x \sin x)$.

Ex.3, P. 219: Find the derivative of $f(x) = \sqrt{\ln x}$.

Ex.4, P. 219: Find the derivative of $y = \log_{10}(2 + \sin x)$.

Ex.5, P. 219: Find the derivative of $y = \ln(\frac{x+1}{\sqrt{x-2}})$.

Ex.6, P. 220: Find the derivative of $y = \ln|x|$.

Logarithmic Differentiation: The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called logarithmic differentiation. We need to remember that:

$$y = \ln(a.b) = \ln a + \ln b, \quad \ln(\frac{a}{b}) = \ln a - \ln b, \quad \ln(x^n) = n \ln x.$$

Ex.7, P. 220: Find y' if $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Ex.8, P. 222: Find y' if $y = x^{\sqrt{x}}$.

Ex.9 (Exers.12,17,21,22,27,42, P. 223) Find y' if:

(1) $y = \ln(x + \sqrt{x^2 - 1})$.

(2) $y = 2^z \log_2 z$.

(3) $y = \tan[\ln(ax + b)]$.

(4) $y = \log_2(x \log_5 x)$.

(5) $y = \dfrac{x}{1 - \ln(x - 1)}$.

(6) $y = \sqrt{x}\, e^{x^2 - x}\, (x + 1)^{\frac{3}{2}}$.

Ex.10 (Exer.40, P. 223) Find y' if:

$$y = \frac{e^{-x}\cos^2 x}{x^2 + x + 1}.$$

Ex.10 (Exer.50, P. 223) Find y' if:

$$y = (\ln x)^{\cos x}.$$

Ex.11 (Exer.33, P. 223) Find equations of the tangent lines to the curve $y = \ln(x^2 - 3x + 1)$ at the points $(3, 0)$.

Section 3.6: Page P. 223: All from 2 to 15 + 23,24,26,31,32,33,34 + All from 39 to 50.

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CHAPTER 3: DIFFERENTIATION RULES

Section 3.11: Hyperbolic Functions

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Sections 3.11: Hyperbolic Functions

Particular even and odd combinations of the exponential functions e^x and e^{-x} occur so frequently in mathematics and its applications that they warrant distinct nomenclature. They are, in numerous respects, akin to the trigonometric functions, sharing a similar connection to the hyperbola as the trigonometric functions do to the circle. Consequently, we collectively refer to them as hyperbolic functions and individually designate them as hyperbolic sine, hyperbolic cosine, and so on.

Definition of the Hyperbolic Functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x},$$

$$\operatorname{csch} \mathrm{x} = \frac{1}{\sinh x}, \qquad \operatorname{sech} \mathrm{x} = \frac{1}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x}.$$

You can note that

$$D_{\sinh x} = D_{\cosh x} = D_{\tanh x} = D_{\operatorname{sech} \mathrm{x}} = \mathbb{R},$$

where

$$D_{\operatorname{csch} \mathrm{x}} = D_{\coth x} = \mathbb{R} - \{0\}.$$

The graphs of $\sinh x$, $\cosh x$ and $\tanh x$ are shown in the following Figures 1, 2 and 3.

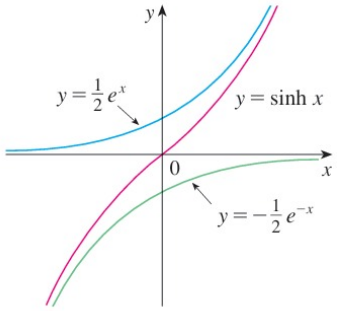


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

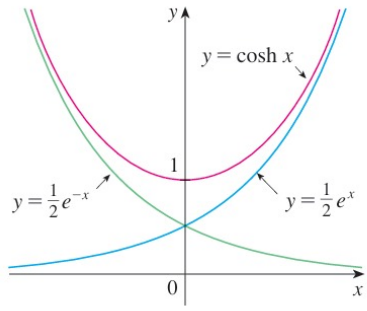


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

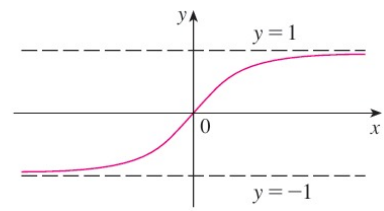


FIGURE 3
$y = \tanh x$

Note that $\sinh x$ range $\mathbb{R}$, while $\cosh x$ has range $[1, \infty)$ and $\tanh x$ has the horizontal asymptotes $y = 1, -1$.
It is easy to show that:

$$\sinh(-x) = -\sinh x, \quad \cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x.$$

This shows that sinh is an odd function, cosh is an even function, and tanh is an odd function.

The hyperbolic functions adhere to several identities analogous to established trigonometric identities. We enumerate many of them here and defer the majority of the proofs to the exercises.

Hyperbolic Identities:

1. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
2. $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$
3. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

4. $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $\cosh^2 x + \sinh^2 x = \cosh(2x)$
7. $1 - \tanh^2 x = \operatorname{sech}^2 x$
8. $\coth^2 x - 1 = \operatorname{csch}^2 x$
9. $\cosh x + \sinh x = e^x$
10. $\cosh x - \sinh x = e^{-x}$
11. $\sinh 2x = 2 \sinh x \cosh x$
12. $\dfrac{1+\tanh x}{1-\tanh x} = e^{2x}$
13. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

Ex.0: Prove any identity from (1) to (4).

Ex.1, P. 261: Prove:

(a) $\cosh^2 x - \sinh^2 x = 1$.

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Ex.2 (Exer.17, P. 264): Prove that$\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$.

Ex.3 (Exer.19, P. 264): Prove that

$$(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx), \quad n \in \mathbb{R}.$$

Derivatives of Hyperbolic Functions: We can prove the following derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh x) = \cosh x, \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x, \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x, \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Note that: Any of these differentiation rules can be combined with the Chain Rule.

Ex.4 (Ex.2, P. 261): Find the derivative of $y = \cosh\sqrt{x}$.

Ex.5 (Exers.30,31,36,37,38 P. 264): Find the derivative of the following functions:

(a) $f(x) = e^x \cosh x$.

(b) $f(x) = \tanh\sqrt{x}$.

(c) $y = (1 + \ln(\operatorname{sech} \mathrm{x}))\ \operatorname{sech} \mathrm{x}$.

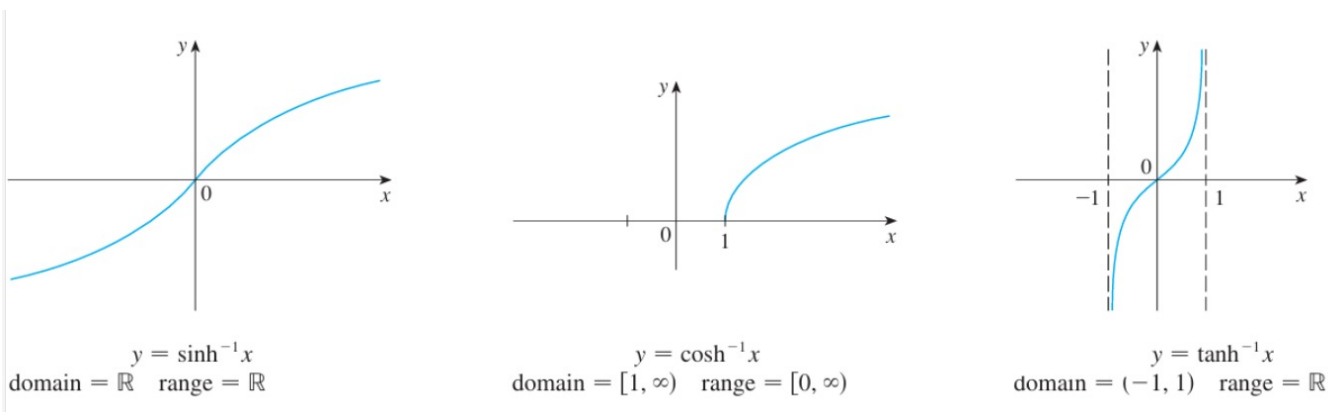
(d) $y = e^{\cosh(e3x)}$.

(e) $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$.

Inverse Hyperbolic Functions: You can see from Figures 1 and 3 that sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. Figure 2 shows that cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

$$y = \sinh^{-1} x \quad \Longleftrightarrow \sinh y = x,$$
$$y = \cosh^{-1} x \Leftrightarrow \cosh y = x \text{ and } y \geqslant 0,$$
$$y = \tanh^{-1} x \Leftrightarrow \tanh y = x.$$

We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in the following Figures by using Figures 1, 2, and 3.



$y = \sinh^{-1} x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

$y = \cosh^{-1} x$
domain $= [1, \infty)$ range $= [0, \infty)$

$y = \tanh^{-1} x$
domain $= (-1, 1)$ range $= \mathbb{R}$

Derivatives of Inverse Hyperbolic Functions: We can prove the following derivatives of inverse hyperbolic functions::

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}, \qquad \frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}},$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}.$$

Ex.6 (Ex.5, P. 264): Find the derivative of $y = \tanh^{-1}(\sinh x)$.

Ex.7 (Exers.41,42,43, P. 265): Find the derivative of the following functions:

(a) $y = \cosh^{-1}\sqrt{x}$.

(b) $y = x\tanh^{-1} x + \ln(\sqrt{1 -^2 x})$.

(c) $y = x\sinh^{-1}(\frac{x}{3}) - \sqrt{9+x^2}$.

Section 3.11: Page P. 264: All from 1 to 19 + All from 30 to 43.

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Mathematics Department

Calculus I (MATH 3111)

Course References:

◇ J. Stewart, Calculus: Early Transcendentals; Publisher, Cengage Learning; 8th edition (February 4, 2015); Language, English; Hardcover, 1368 pages; ISBN-10, 1285741552.

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CHAPTER 4: APPLICATIONS of DIFFERENTIATION

Section 4.1: Maximum and Minimum Values

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Section 4.1: Maximum and Minimum Values

Definition 1.

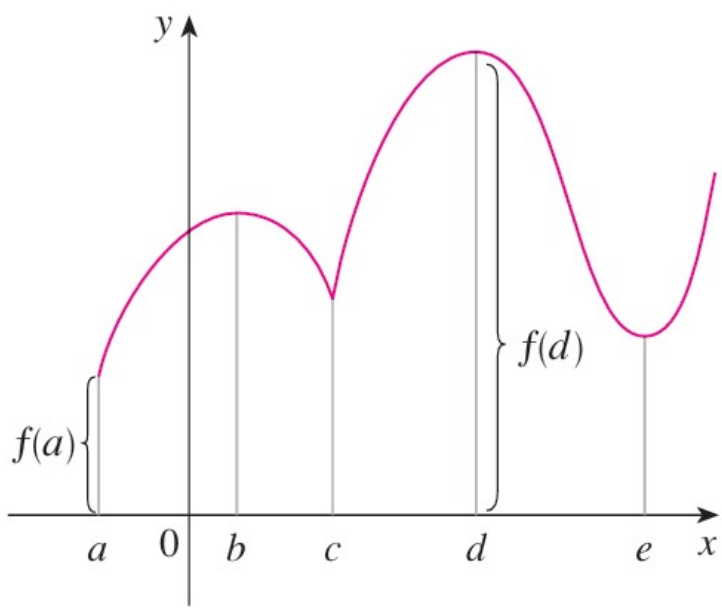
Let c be a number in the domain D of a function f. Then $f(c)$ is the

- *Absolute maximum* value of f on D if $f(c) \geq f(x)$ for all x in D.
- *Absolute minimum* value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition 2.

Let c be a number in the domain D of a function f. Then $f(c)$ is the

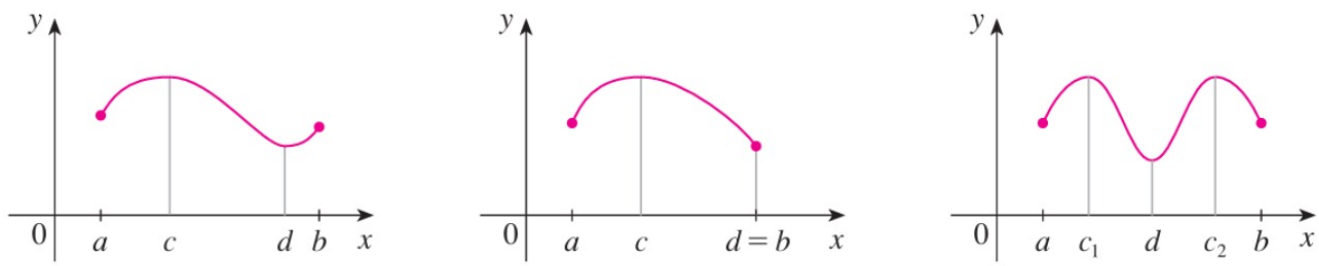
- *Local maximum* value of f on D if $f(c) \geq f(x)$ when x is near c.
- *Local minimum* value of f on D if $f(c) \leq f(x)$ when x is near c.



Abs min $f(a)$, abs max $f(d)$,
loc min $f(c), f(e)$, loc max $f(b), f(d)$

In Definition 2 (and elsewhere), if we say that something is true near c, we mean that it is true on some open interval containing c.

Theorem 1. (The Extreme Value Theorem) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



Note that an extreme value can be taken on more than once.

Theorem 2. (Fermat's Theorem) If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Is the inverse true?

Definition 3. A *critical point* of a function f is a point c in the domain of f such that either $f^{'}(c) = 0$ or $f^{'}(c)$ does not exist.

Ex.1 (Exer.30, P. 283): Find the critical points of the function:

$$f(x) = x^3 + 6x^2 - 15x.$$

Ex.2 (Ex.7, P. 280): Find the critical points of the function:

$$f(x) = x^{\frac{3}{5}}(4 - x).$$

Theorem 3. If f has a local maximum or minimum at c, then c is a critical number of f.

Is the inverse true?

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval. Thus the following three-step procedure always works:

The Closed interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the critical points of f.
2. Find the values of f at the critical points of f in $[a, b]$.
3. Find the values of f at the endpoints of the interval.
4. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex.3 (Ex.8, P. 281): Find the absolute maximum and minimum values of the function:

$$f(x) = x^3 - 3x^2 + 1, \qquad x \in [-\frac{1}{2}, 4].$$

Ex.4 (Exer.48, P. 284): Find the absolute maximum and minimum values of the function:

$$f(x) = 5 + 54x - 2x^3, \qquad x \in [0, 4].$$

Section 4.1: Page P. 283: From 29 to 33 + All from 47 to 51.

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CHAPTER 4: APPLICATIONS of DIFFERENTIATION

Section 4.3: Curve Sketching

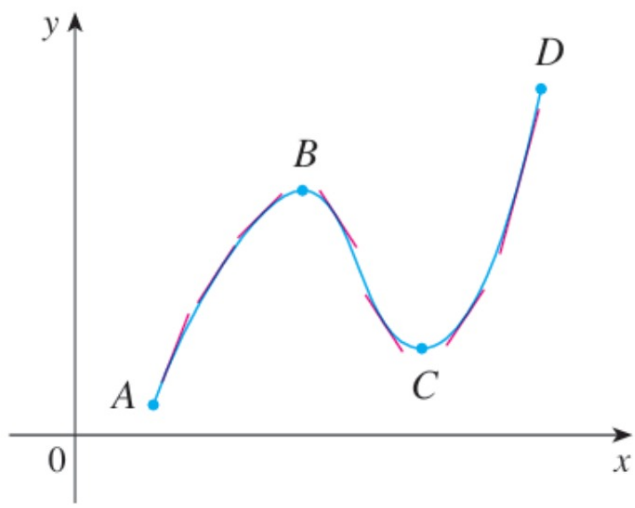
•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Section 4.3: Sketching the Graph of a Function

Many of the applications of calculus depend on our ability to deduce facts about a function from information concerning its derivatives. Because the derivative represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about $f^{'}(x)$ will provide us with information about $f(x)$.

What Does $f^{'}$ Say about f ?

To see how the derivative of f can tell us where a function is increasing or decreasing, look at the following Figure.

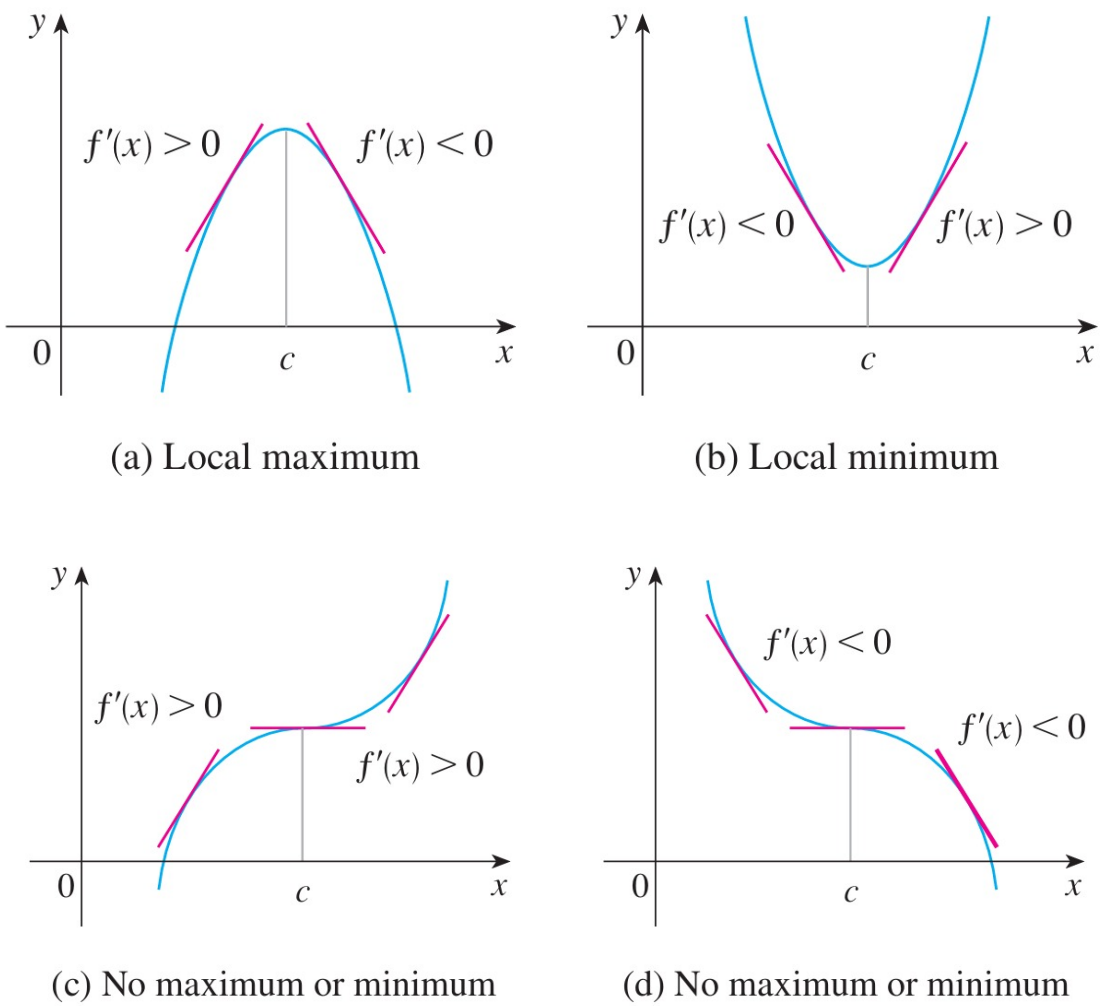


From this figure, we can note that between (A and B) and between (C and D), the tangent lines have positive slope and so $f^{'}(x) > 0$. Between (B and C), the tangent lines have negative slope and so $f^{'}(x) < 0$. Thus it appears that f increases when $f^{'}(x)$ is positive and decreases when $f^{'}(x)$ is negative.

The First Derivative Test:

- If $f^{'}(x) > 0$ on an interval, then f is increasing on that interval.
- If $f^{'}(x) < 0$ on an interval, then f is decreasing on that interval.
- If $f^{'}(x)$ changes from +ve to -ve at c, then f has a *local maximum* at c.
- If $f^{'}(x)$ changes from -ve to +ve at c, then f has a *local minimum* at c.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in the following Figure:



(a) Local maximum (b) Local minimum

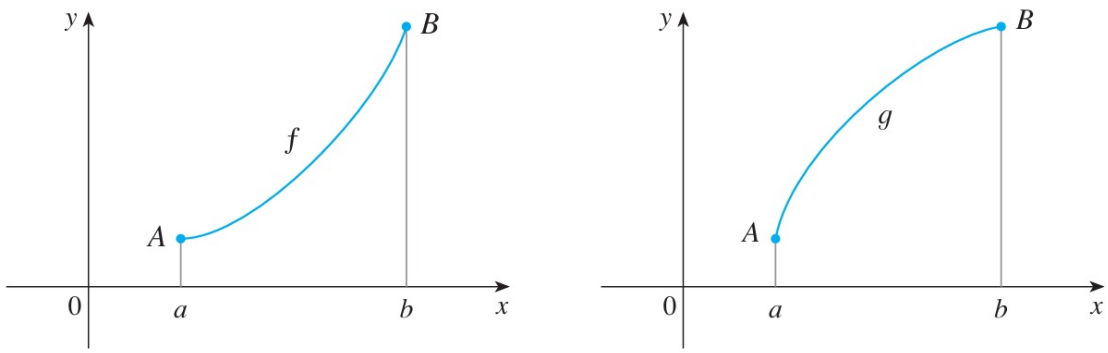
(c) No maximum or minimum (d) No maximum or minimum

Ex.1 (Exs. 1 and 2, P. 293): Find the intervals on which f is increasing or decreasing. Then Find the local maximum and minimum values of f:

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5.$$

What Does f'' Say about f ?

The following Figure shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behavior?



(a) (b)

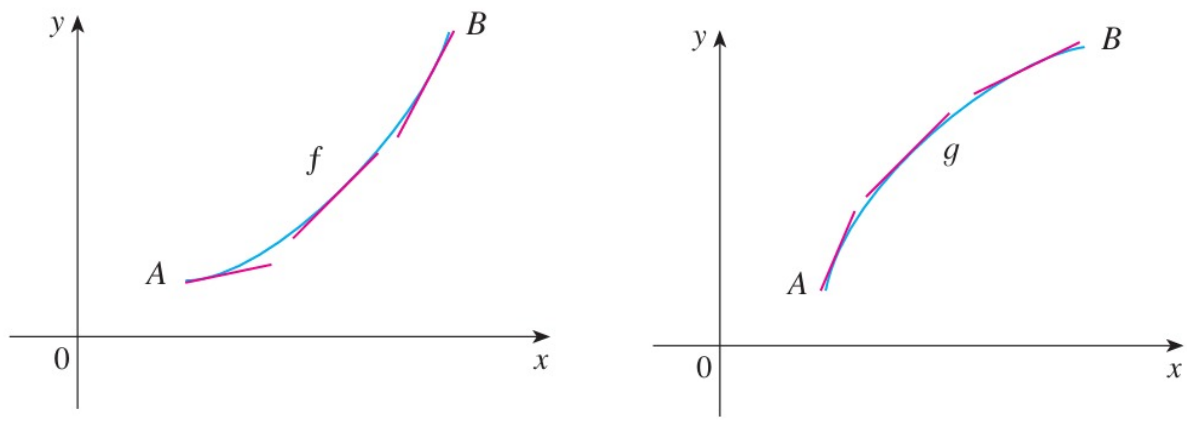
In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and t is called *concave downward* on (a, b).

Concavity Test:

• If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

• If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition. A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous there and the curve changes from concave upward to concave downward **or** from concave downward to concave upward at P.



(a) Concave upward (b) Concave downward

Ex.2: Find the intervals of concavity and the inflection points of the following function:

$$f(x) = \frac{1}{3}x^3 - 2x^2 - 4x + 1.$$

Ex.3 (Exer.11, P. 301): Let

$$f(x) = x^4 - 2x^2 + 3$$

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity and the inflection points.

(d) Sketch the graph of the function f.

Ex.4 (Ex.6, P. 298): Sketch the graph of the following function:

$$f(x) = x^4 - 4x^3.$$

Section 4.3: Page P. 300: 9,10,11,19,37,38, 39,40,42.

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CHAPTER 4: APPLICATIONS of DIFFERENTIATION

Section 4.4: Indeterminate Forms and L'Hospital's Rule

•• **Course Materials will be covered from:** Course Text Book and These Lecture Notes

Section 4.4: Indeterminate Forms and L'Hospital's Rule

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Ex.0, P. 305: Find the following limits if exist:

$$(a) \lim_{x \to 0} \frac{x^2 - x}{x - 1} \qquad\qquad (b) \lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1}$$

Ex.1, P. 306: Find the following limit if exists:

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Ex.2, P. 306: Find the following limit if exists:

$$\lim_{x \to \infty} \frac{e^x}{x^2}.$$

Ex.3, P. 307: Find the following limit if exists:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}.$$

Ex.4, P. 307: Find the following limit if exists:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}.$$

Ex.5, P. 308: Find the following limit if exists:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}.$$

Ex.6, P. 308: Find the following limit if exists:

$$\lim_{x \to 0^+} x \ln x.$$

Ex.7, P. 309: Find the following limit if exists:

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right).$$

Ex.8, P. 309: Find the following limit if exists:

$$\lim_{x\to\infty}(e^x-x).$$

Ex.9: Find the following limit if exists:

$$\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x).$$

Section 4.4: Page P. 311: Odd No. from 9 to 35 + Odd No. from 57 to 67.