Islamic University of Madinah

Faculty of Science

Mathematics Department

Algebra and Geometry - Math 3140[1]

Course References:

- **R. N. Aufmann, V.C. Barker, and Nation, R.D., *College Algebra and Trigonometry*, 7th Edition, Brooks, 2011.**
- **J. Stewart, L.Redlin, S.Watson, *College Algebra*, 5th Edition, Cengage Learning (2008).**
- **Saudi Digital Library:** https://sdl.edu.sa/SDLPortal/ar/Publishers.aspx

Lecture Notes 1: Chapter 1: Equations and Inequalities

[1] **Course Materials will be covered from:** Course Textbooks and The Lecture notes

Section 1.1: Linear and Absolute Value Equations

This section covers the topics:
- Linear Equations
- Contradictions, Conditional
- Equations, and Identities
- Absolute Value Equations

Definition of a Linear Equation

A **linear equation,** or first-degree equation, in the single variable x is an equation that can be written in the form

$$ax + b = 0$$

where a and b are real numbers, with $a \neq 0$.

Linear equations are solved by applying the properties of real numbers and the properties of equality.

EXAMPLE 1 Solve a Linear Equation in One Variable

Solve: $3x - 5 = 7x - 11$

Solution

When an equation contains parentheses, use the distributive property to remove the parentheses.

EXAMPLE 2 Solve a Linear Equation in One Variable

Solve: $8 - 5(2x - 7) = 3(16 - 5x) + 5$

Solution

If an equation involves fractions, it is helpful to multiply each side of the equation by the least common denominator (LCD) of all denominators to produce an equivalent equation that does not contain fractions.

EXAMPLE 3 Solve by Clearing Fractions

Solve: $\dfrac{2}{3}x + 10 - \dfrac{x}{5} = \dfrac{36}{5}$

Solution

Contradictions, Conditional Equations, and Identities

An equation that has no solutions is called a **contradiction.** The equation $x = x + 1$ is a contradiction. No number is equal to itself increased by 1.

An equation that is true for some values of the variable but not true for other values of the variable is called a **conditional equation.** For example, $x + 2 = 8$ is a conditional equation because it is true for $x = 6$ and false for any number not equal to 6.

An **identity** is an equation that is true for all values of the variable for which all terms of the equation are defined. Examples of identities include the equations $x + x = 2x$ and $4(x + 3) - 1 = 4x + 11$.

EXAMPLE 4 Classify Equations

Classify each equation as a contradiction, a conditional equation, or an identity.

a. $x + 1 = x + 4$

b. $4x + 3 = x - 9$

c. $5(3x - 2) - 7(x - 4) = 8x + 18$

Solution

Absolute Value Equations

A Property of Absolute Value Equations

For any variable expression E and any nonnegative real number k,

$$|E| = k \quad \text{if and only if} \quad E = k \quad \text{or} \quad E = -k$$

EXAMPLE

If $|x| = 5$, then $x = 5$ or $x = -5$.

If $|x| = \frac{3}{2}$, then $x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

If $|x| = 0$, then $x = 0$.

EXAMPLE 5 Solve an Absolute Value Equation

Solve: $|2x - 5| = 21$

Solution

Exer: Solve the equation: $\frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$.

Exer: solve each absolute value equation $|2x + 14| = 60$.

Homework Exercises: Sec. 1.1

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Q1: Solve the equation: $\frac{12+x}{-4} = \frac{5x-7}{3} + 2.$

Q2: Classify each equation as a contradiction, a conditional equation, or an identity:

- $2x + \frac{1}{3} = \frac{6x+1}{3}$

- $\frac{4x+8}{4} = x + 8$

- $2x - 8 = -x + 9$

Q3: Solve each absolute value equation for x:

- $|x - 5| = 2$

- $|2x - 3| = 21$

- $\left|\frac{x+3}{4}\right| = 6$

- $\left|\frac{x-4}{2}\right| = 8$

Formulas

A **formula** is an equation that expresses known relationships between two or more variables. Table 1.2 lists several formulas from geometry that are used in this text. The variable P represents perimeter, C represents circumference of a circle, A represents area, S represents surface area of an enclosed solid, and V represents volume.

It is often necessary to solve a formula for a specified variable. Begin the process by isolating all terms that contain the specified variable on one side of the equation and all terms that do not contain the specified variable on the other side.

EXAMPLE 1 Solve a Formula for a Specified Variable

a. Solve $2l + 2w = P$ for l.

b. Solve $S = 2(wh + lw + hl)$ for h.

Solution

Solve the formula for the specified variable.

- $V = \frac{1}{3}\pi r^2 h, \qquad h.$

- $A = \frac{1}{2}h(b_1 + b_2), \qquad b_1.$

- $\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}, \qquad V_2.$

Sec.1.3: Quadratic Equations

This section covers the topics:

- Solving Quadratic Equations by Factoring
- Solving Quadratic Equations by Taking Square Roots
- Solving Quadratic Equations by Completing the Square
- Solving Quadratic Equations by Using the Quadratic Formula
- The Discriminant of a Quadratic Equation

Solving Quadratic Equations by Factoring

In Section 1.1 you solved linear equations. In this section you will learn to solve a type of equation that is referred to as a *quadratic equation*.

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the **standard quadratic form**

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

Several methods can be used to solve a quadratic equation. For instance, if you can factor $ax^2 + bx + c$ into linear factors, then $ax^2 + bx + c = 0$ can be solved by applying the following property.

The Zero Product Principle

If A and B are algebraic expressions such that $AB = 0$, then $A = 0$ or $B = 0$.

The zero product principle states that if the product of two factors is zero, then at least one of the factors must be zero. In Example 1, the zero product principle is used to solve a quadratic equation.

EXAMPLE 1 Solve by Factoring

Solve each quadratic equation by factoring.

a. $x^2 + 2x - 15 = 0$ **b.** $2x^2 - 5x = 12$

Solution

Solving Quadratic Equations by Taking Square Roots

Recall that $\sqrt{x^2} = |x|$. This principle can be used to solve some quadratic equations by taking the square root of each side of the equation.

We will refer to the preceding method of solving a quadratic equation as the **square root procedure.**

The Square Root Procedure

If $x^2 = c$, then $x = \sqrt{c}$ or $x = -\sqrt{c}$, which can also be written as $x = \pm\sqrt{c}$.

EXAMPLE

If $x^2 = 9$, then $x = \sqrt{9} = 3$ or $x = -\sqrt{9} = -3$. This can be written as $x = \pm 3$.

If $x^2 = 7$, then $x = \sqrt{7}$ or $x = -\sqrt{7}$. This can be written as $x = \pm\sqrt{7}$.

If $x^2 = -4$, then $x = \sqrt{-4} = 2i$ or $x = -\sqrt{-4} = -2i$. This can be written as $x = \pm 2i$.

EXAMPLE 2 Solve by Using the Square Root Procedure

Use the square root procedure to solve each equation.

a. $3x^2 + 12 = 0$ **b.** $(x + 1)^2 = 48$

Solution

Solving Quadratic Equations by Completing the Square

EXAMPLE 3 Solve by Completing the Square

Solve $x^2 = 2x + 6$ by completing the square.

Solution

EXAMPLE 4 Solve by Completing the Square

Solve $2x^2 + 8x - 1 = 0$ by completing the square.

Solution

Solving Quadratic Equations by Using the Quadratic Formula

Completing the square for $ax^2 + bx + c = 0$ $(a \neq 0)$ produces a formula for x in terms of the coefficients a, b, and c. The formula is known as the *quadratic formula*, and it can be used to solve *any* quadratic equation.

The Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 5 Solve by Using the Quadratic Formula

Use the quadratic formula to solve each of the following.

a. $x^2 = 3x + 5$ **b.** $4x^2 - 4x + 3 = 0$

Solution

The Discriminant of a Quadratic Equation

The solutions of $ax^2 + bx + c = 0$, $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Discriminant and the Solutions of a Quadratic Equation

The equation $ax^2 + bx + c = 0$, with real coefficients and $a \neq 0$, has as its discriminant $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has *two distinct real solutions.*
- If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has *one real solution*. The solution is a double solution.
- If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has *two distinct nonreal complex solutions.* The solutions are conjugates of each other.

EXAMPLE 6 Use the Discriminant to Determine the Number of Real Solutions

For each equation, determine the discriminant and state the number of real solutions.

a. $2x^2 - 5x + 1 = 0$

b. $3x^2 + 6x + 7 = 0$

c. $x^2 + 6x + 9 = 0$

Solution

Exer 1: solve by factoring and applying the zero product principle

$$2x^2 + 5x = 3.$$

Exer 2: use the square root procedure to solve the equation

$$(x - 5)^2 = 36.$$

Exer 3: solve by completing the square

$$x^2 - 2x - 15 = 0.$$

Exer 4: solve each equation by he quadratic formula

$$x^2 - 2x = 15.$$

Exer 5: determine the discriminant of the quadratic equation and then state the number of real solutions of the equation

$$2x^2 - 5x - 7 = 0\ .$$

Homework Exercises: Sec. 1.3

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Exer 1: solve by factoring and applying the zero product principle

$$(x-5)^2 - 9 = 0.$$

Exer 2: use the square root procedure to solve the equation

$$(z+1)^2 + 64 = 0.$$

Exer 3: solve by completing the square

$$x^2 - 2x - 15 = 0.$$

Exer 4: solve each equation by the quadratic formula

$$x^2 - 2x = 15.$$

Exer 5: determine the discriminant of the quadratic equation and then state the number of real solutions of the equation

$$2x^2 - 5x - 7 = 0\,.$$

Polynomial Equations

Some polynomial equations that are neither linear nor quadratic can be solved by the various techniques presented in this section. For instance, the **third-degree equation,** or **cubic equation,** in Example 1 can be solved by factoring the polynomial and using the zero product principle.

EXAMPLE 1 Solve a Polynomial Equation

Solve: $x^3 + 3x^2 - 4x - 12 = 0$

Rational Equations

A **rational equation** is one that involves rational expressions. The following two equations are rational equations.

$$\frac{2x}{x+3} - 5 = \frac{x+4}{x-1} \qquad \frac{x+1}{x^2-1} + \frac{x}{2x-3} = \frac{4}{x-1}$$

EXAMPLE 2 Solve a Rational Equation

Solve.

a. $\dfrac{2x + 1}{x + 4} + 3 = \dfrac{-2}{x + 4}$ **b.** $3x + \dfrac{4}{x - 2} = \dfrac{-4x + 12}{x - 2}$

c. $\dfrac{2x}{x - 3} + \dfrac{x + 1}{x + 4} = \dfrac{x - 1}{x - 3}$

Solution

Radical Equations

Some equations that involve radical expressions can be solved by using the following principle.

The Power Principle

If P and Q are algebraic expressions and n is a positive integer, then every solution of $P = Q$ is a solution of $P^n = Q^n$.

EXAMPLE 3 Solve a Radical Equation

Use the power principle to solve $\sqrt{x + 4} = 3$.

Solution

Definition of an Extraneous Solution

Any solution of $P^n = Q^n$ that is not a solution of $P = Q$ is called an **extraneous solution**. Extraneous solutions *may* be introduced whenever each side of an equation is raised to an *even* power.

EXAMPLE 4 Solve Radical Equations

Solve.

a. $2\sqrt{2x - 1} - x = 1$ **b.** $\sqrt{x + 1} - \sqrt{2x - 5} = 1$

Solution

Rational Exponent Equations

Recall that $\sqrt[n]{b^n} = |b|$ when n is a positive even integer and $\sqrt[n]{b^n} = b$ (the absolute value sign is not necessary) when n is a positive odd integer. These results can be restated using rational exponents.

$$(b^n)^{1/n} = |b|, n \text{ is a positive even integer}$$
$$(b^n)^{1/n} = b, n \text{ is a positive odd integer}$$

EXAMPLE 5 **Solve an Equation That Involves a Variable with a Rational Exponent**

Solve.

a. $2x^{4/5} - 47 = 115$ **b.** $5x^{3/4} + 4 = 44$

Solution

Equations That Are Quadratic in Form

The equation $4x^4 - 25x^2 + 36 = 0$ is said to be **quadratic in form,** which means that it can be written in the form

$$au^2 + bu + c = 0, \quad a \neq 0$$

where u is an algebraic expression involving x. For example, if we make the substitution $u = x^2$ (which implies that $u^2 = x^4$), then our original equation can be written as

$$4u^2 - 25u + 36 = 0$$

This quadratic equation can be solved for u, and then, using the relationship $u = x^2$, we can find the solutions of the original equation.

EXAMPLE 6 Solve an Equation That Is Quadratic in Form

Solve: $4x^4 - 25x^2 + 36 = 0$

Solution

EXAMPLE 7 Solve an Equation That Is Quadratic in Form

Solve: $3x^{2/3} - 5x^{1/3} - 2 = 0$

Solution

Exer 1: solve by factoring and using the principle of zero products.

$$x^3 - 2x^2 - x + 2 = 0\,.$$

Exer 2: solve the rational equation

$$x - \frac{2x+3}{x+3} \quad = \frac{2x+9}{x+3}\,.$$

Exer 3: solve the radical equation

$$\sqrt{x-4} - 6 = 0\,.$$

Exer 4: solve each equation containing a rational exponent on the variable.

$$x^{\frac{3}{2}} = 27.$$

Exer 5: find all real solutions by first rewriting the equation as a quadratic equation

$$x^4 + 8x^2 - 9 = 0.$$

Homework Exercises: Sec. 1.4

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Exer 1: solve by factoring and using the principle of zero products.

$$x^4 - 2x^3 + 27x - 54 = 0\ .$$

Exer 2: solve the rational equation

$$\frac{x+4}{x-2} + 3 = \frac{-2}{x-2}\,.$$

Exer 3: solve the radical equation

$$\sqrt{9x - 20} = x\ .$$

Exer 4: Solve each equation containing a rational exponent on the variable.

$$x^{\frac{3}{4}} = 125.$$

Exer 5: Find all real solutions of each equation by first rewriting each equation as a quadratic equation

$$x^6 + x^3 - 6 = 0.$$

Sec. 1.5: Inequalities

This section covers the topics:

- Properties of Inequalities
- Compound Inequalities
- Absolute Value Inequalities
- Polynomial Inequalities
- Rational Inequalities

Properties of Inequalities

Let a, b, and c be real numbers.

1. ***Addition–Subtraction Property*** If the same real number is added to or subtracted from each side of an inequality, the resulting inequality is equivalent to the original inequality.

$$a < b \text{ and } a + c < b + c \text{ are equivalent inequalities.}$$

2. ***Multiplication–Division Property***

   a. Multiplying or dividing each side of an inequality by the same *positive* real number produces an equivalent inequality.

$$\text{If } c > 0, \text{ then } a < b \text{ and } ac < bc \text{ are equivalent inequalities.}$$

   b. Multiplying or dividing each side of an inequality by the same *negative* real number produces an equivalent inequality provided the direction of the inequality symbol is *reversed*.

$$\text{If } c < 0, \text{ then } a < b \text{ and } ac > bc \text{ are equivalent inequalities.}$$

EXAMPLE 1 Solve Linear Inequalities

Solve each of the following inequalities.

a. $2x + 1 < 7$ **b.** $-3x - 2 \leq 10$

Solution

Compound Inequalities

A **compound inequality** is formed by joining two inequalities with the connective word *and* or *or*. The inequalities shown below are compound inequalities.

$$x + 1 > 3 \qquad \text{and} \qquad 2x - 11 < 7$$
$$x + 3 > 5 \qquad \text{or} \qquad x - 1 < 9$$

The solution set of a compound inequality with the connective word *or* is the *union* of the solution sets of the two inequalities. The solution set of a compound inequality with the connective word *and* is the *intersection* of the solution sets of the two inequalities.

EXAMPLE 2 Solve Compound Inequalities

Solve each compound inequality. Write each solution in set-builder notation.

a. $2x < 10$ or $x + 1 > 9$ **b.** $x + 3 > 4$ and $2x + 1 > 15$

Solution

Absolute Value Inequalities

Properties of Absolute Value Inequalities

For any variable expression E and any nonnegative real number k,

$$|E| \leq k \quad \text{if and only if} \quad -k \leq E \leq k$$
$$|E| \geq k \quad \text{if and only if} \quad E \leq -k \quad \text{or} \quad E \geq k$$

These properties also hold true when the $<$ symbol is substituted for the $\leq$ symbol and when the $>$ symbol is substituted for the $\geq$ symbol.

EXAMPLE

If $|x| < 5$, then $-5 < x < 5$.

If $|x| > 7$, then $x < -7$ or $x > 7$.

EXAMPLE 3 Solve Absolute Value Inequalities

Solve each of the following inequalities.

a. $|2 - 3x| < 7$ **b.** $|4x - 3| \geq 5$

Solution

Polynomial Inequalities

Any value of x that causes a polynomial in x to equal zero is called a **zero of the polynomial.** For example, -4 and 1 are both zeros of the polynomial $x^2 + 3x - 4$ because $(-4)^2 + 3(-4) - 4 = 0$ and $1^2 + 3 \cdot 1 - 4 = 0$.

Sign Property of Polynomials

Polynomials in x have the following property: for all values of x between two consecutive real zeros, all values of the polynomial are positive or all values of the polynomial are negative.

EXAMPLE 4 Solve a Polynomial Inequality

Find the solution set of $x^3 + 3x^2 - 4x - 12 \geq 0$. Write the answer in interval notation.

Solution

Solving a Polynomial Inequality by the Critical Value Method

1. Write the inequality so that one side of the inequality is a nonzero polynomial and the other side is 0.
2. Find the real zeros of the polynomial. They are the critical values of the original inequality.
3. Use test values to determine which of the consecutive intervals formed by the critical values are to be included in the solution set.

Rational Inequalities

A rational expression is the quotient of two polynomials. **Rational inequalities** involve rational expressions, and they can be solved by an extension of the critical value method.

Definition of a Critical Value of a Rational Expression

A **critical value of a rational expression** is a number that causes the numerator of the rational expression to equal zero or the denominator of the rational expression to equal zero.

Solving a Rational Inequality Using the Critical Value Method

1. Write the inequality so that one side of the inequality is a rational expression and the other side is zero.
2. Find the real zeros of the numerator of the rational expression and the real zeros of its denominator. They are the critical values of the inequality.
3. Use test values to determine which of the consecutive intervals formed by the critical values are to be included in the solution set.

EXAMPLE 5 Solve a Rational Inequality

Solve: $\dfrac{3x + 4}{x + 1} \leq 2$

Solution

Exer 1: solve the inquality $|4-5x| \geq 24$.

Exer 2: solve the inquality $x^2+7x+10<0$.

Exer 3: solve the rational inquality $\frac{x-4}{x+6} < 1$.

Homework Exercises: Sec. 1.4

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Exer 1: solve the compound inequality: $4x + 1 > -2 \ and \ 4x + 1 \leq 17$.

Exer 2: solve the compound inequality: $x + 1 > 4 \ \ or \ \ x + 2 \leq 3$.

Exer 3: solve the inquality $|3 - 2x| \leq 5$.

Exer 4: solve the inquality $x^2 - 3x \geq 28$.

Exer 5: solve the rational inquality $\frac{x-5}{x+8} \geq 3$.