1. Group of order $pq$ (distinct primes)

Let $p<q$, By sylow's theorem, the number of sylow $q$-subgroup divides $p$ and is $\equiv (1 \bmod q)$. So it must be 1, hence the sylow $q$-subgroup is normal. Thus $G$ is semidirect.

If $p \nmid (q-1)$, the action is trivial

If $p \mid (q-1)$, a nonabelian case may exist.

So groups of order $pq$ are abelian in many cases, but not always.

2. cyclic group $C_{p^2}$: has exactly 1 subgroup of order $p$.

Thus, having $p+1$ subgroup of order $p$ characterizes the second case.

3. Union of conjugates of proper subgroup

suppose $G = \cup_{x \in G} xHx^{-1}$ with $H < G$.
Consider the action of $G$ on cosets $G/H$. Every element in a conjugate fixes at least one coset, so the assumption implies all elements of $G$ fix some point.

4. If $N$ is cyclic and central, and $G/N$ is cyclic then $G$ must be abelian (since every element looks like $g^k n$, and they commute).

5. Element of finite order.

In abelian groups, the sets of all elements of finite order (torsion subgroup) is a subgroup, because is the lcm of finite orders is finite.

6. Subgroup of index smallest prime p.

Let $H \leq G$ with index p.

Action of G on consists $G/H$ gives a homomorphism $\varphi: G \to S_p$.

If order 1, then H = kernal normal

If order p, then the kernal also has index p, so kernal = H.