Propositional Logic (PL) & First Order Logic (FOL)

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Proposition/Statement [1]

A **Proposition/Statement** is a declarative sentence which is either ***true*** or ***false*** but not both.

Example: This robot is active.

Sofia is a robot.

Don’t talk in the class.

Come to the class.

Propositional Variable

- A **Propositional Variable** represents an arbitrary proposition.
- Propositional variables are represented with uppercase letters. (A,B,…,Z)

Example:

P = This robot is active

Q = Sofia is a robot

Logical Connectives [2]

These are the **Words or Symbols** used to combine two or more sentences to form a **Compound Statement.**

logic	Name	rank
~	Negation	1
˄	Conjunction	2
V	Disjunction	3
=>	Conditional	4
⇔	Biconditional	5

Basic Logical Operations

A	B	^	V	~A	=>	⇔	NOR	NAND	XOR	EX-NOR
T	T	T	T	F	T	T	F	F	F	T
T	F	F	T	F	F	F	F	T	T	F
F	T	F	T	T	T	F	F	T	T	F
F	F	F	F	T	T	T	T	T	F	T

Rearrangement of Implication [2]

For conditional statement $\mathbf{P \rightarrow Q}$, the **Converse Statement** is $\mathbf{Q \rightarrow P}$, the **Contrapositive Statement** is $\mathbf{\neg Q \rightarrow \neg P}$, and the **Inverse Statement** is $\mathbf{\neg P \rightarrow \neg Q}$.

H.W. Use truth table to show whether converse, contrapositive and inverse statements of $P \rightarrow Q$ are equivalent to it or not.

Propositional Logic (PL) [1]

- Also Called Propositional Logic/ Propositional Calculus/ Statement Logic/Sentential Calculus/Sentential Logic/ Zeroth Order Logic

- The branch of **Logic** concerned with the study of Propositions that are formed by other propositions with the use of **Logical Connectives**

PL-Example

- P = “It is hot”, Q = “It is humid”, R = “It is raining”
  - $(P \wedge Q) \rightarrow R$

    “If it is hot and humid, then it is raining”
  - $Q \rightarrow P$

    “If it is humid, then it is hot”

- A better way:

  Ho = “It is hot”

  Hu = “It is humid”

  R = “It is raining”

Tautology

- A tautology is a proposition which is **True for all truth values** of its sub-propositions or components.
- A tautology is also called **logically valid or logically true.**
- All entries in the column of tautology are **true**.

Tautology…

P	q	p^q	q	p^q=> q
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

Contradiction

- Contradiction is a proposition which is always **false for all truth values** of its sub-propositions or components.
- A contradiction is also called **logically invalid** or logically false.
- All entries in the column of contradiction are false.

Contradiction…

P	Q	P V Q	~P	~Q	(P v Q)^(~P)^(~Q)
T	T	T	F	F	F
T	F	T	F	T	F
F	T	T	T	F	F
F	F	F	T	T	F

Contingency

- It is a proposition which is **either true or false** depending on the truth value of its components or proposition.

Contingency…

p	q	~p	~q	~p ^ ~q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Wff [3]

- A sentence (**Well Formed Formula-wff**) is defined as follows:
  - A symbol is a sentence
  - If S is a sentence, then $\neg$S is a sentence
  - If S is a sentence, then (S) is a sentence
  - If S and T are sentences, then $(S \vee T)$, $(S \wedge T)$, $(S \rightarrow T)$, and $(S \leftrightarrow T)$ are sentences
  - A sentence results from a finite number of applications of the above rules

Logical Equivalence

Two statements are called Logically Equivalent if the truth values of both the statements are always identical..

p	q	~p	~q	p=>q	~q=>~p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Identities

$P \equiv P \wedge P$ $P \equiv P \vee P$	idempotence of $\wedge$ idempotence of $\vee$
$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	commutativity of $\vee$ commutativity of $\wedge$
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	associativity of $\vee$ associativity of $\wedge$
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	DeMorgan's Laws
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	distributivity of $\wedge$ over $\vee$ distributivity of $\vee$ over $\wedge$
$P \vee T \equiv T$ $P \wedge F \equiv F$	domination laws
$P \wedge T \equiv P$ $P \vee F \equiv P$	identity laws
$P \vee \neg P \equiv T$ $P \wedge \neg P \equiv F$	negation laws
$\neg(\neg P) \equiv P$	double negation law
$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$	absorption laws
$P \rightarrow Q \equiv \neg P \vee Q$	implication
$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	contrapositive
$P \leftrightarrow Q \equiv [(P \rightarrow Q) \wedge (Q \rightarrow P)]$	equivalence
$[(P \wedge Q) \rightarrow R] \equiv [P \rightarrow (Q \rightarrow R)]$	exportation

Inference Rules

- **Inference Rule** or **Transformation Rule** is a logical form consisting of a function which takes premises, analyzes their syntax, and returns a conclusion (or conclusions).

- Popular **Rules** of **Inference** in Propositional Logic include **Modus Ponens, Modus Tollens**, and **Contraposition**.

Rules of Inference [1]

Rule of inference	Tautology	Name
$p \to q$ $\underline{p \qquad}$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p \quad}$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\underline{p \qquad}$ $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q \qquad}$ $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Problems With PL [4]

- Consider the problem of representing the following information:
  - Every person is mortal.
  - Confucius is a person.
  - Confucius is mortal.
- How can these sentences be represented so that we can infer the *third sentence* from the *first two*?

Problems With PL...

- In PL we have to create propositional symbols to stand for all or part of each sentence. For example, we might have:

  P = "person"; Q = "mortal"; R = "Confucius"

- So the above 3 sentences are represented as:

  $P \rightarrow Q$; $R \rightarrow P$; $R \rightarrow Q$

  We can't infer the third proposition using the above rules.

- To represent **other individuals** we must introduce separate symbols for each one, with some way to represent the fact that all individuals who are "people" are also "mortal".

PL is a Weak Language

- It applies only to **atomic propositions**. There is no way to talk about properties that apply to categories of objects, or about relationships between those properties.

- Propositional Logic doesn't allow you to access the components of an assertion:

  It rained on Tuesday

  But if we write

  weather (Tuesday, rain)

  or weather(X, rain)

First Order Logic (FOL) [4]

➢**First Order Logic/ First Order Predicate Calculus/Predicate Logic** is expressive enough to concisely represent this kind of information

FOL adds Relations, Variables, and Quantifiers

- *"Every elephant is gray"*: $\forall$x (elephant(x) → gray(x))
- *"There is a white alligator"*: $\exists$x (alligator(X) ∧ white(X))

FOL…

- ✓ An **Argument** can be either a constant or a variable.
- ✓ It is conventional to write variables with an Initial Upper-case Letter.
- ✓ **Constants** can be numbers, names that begin with a lower-case letter, or quoted strings.
- ✓ A **Relation** is one way to represent a predicate.
  - ✓ **P(X,3):** True iff X is greater than 3
  - ✓ **Greater(X,3):** True iff X is greater than 3

FOL...

- More Examples:
  - Domain of ASCII characters - IsAlpha(x): TRUE iff x is an alphabetical character.
  - Domain of floating point numbers - IsInt(x): TRUE iff x is an integer.
  - Domain of integers: Prime(x) - TRUE if x is prime.

FOL…

- A propositional function may associate with **two or more** variables.

  ✓ Examples:

    - P(x,y): "x = y+3", and P(1,2) is false.
    - Q(x,y,z): "x+y=z", then Q(1,2,3) is true.

FOL…

- **Predicate logic** has two additional operators not found in propositional logic (called **Quantifiers**) to express truth values about predicates with variable arguments.
- **Quantifiers:** **Quantifiers** specifies the quantity of samples in the **domain of discourse.**

  ✓ **Universal Quantifiers** $\forall$

  ✓ **Existential Quantifiers** $\exists$

FOL…

- **Universal:** $\forall$x P(x) – "P(x) for all x in the domain"
- **Existential:** $\exists$x P(x) – "P(x) for some x in the domain"

  or "There exists x such that p(x) is TRUE".
- **Examples**

  (Domain: Real numbers)

  ✓ $\forall$x (x^2 >= 0)

  ✓ $\exists$x (x > 1)

  ✓ ($\forall$x > 1) (x^2 > x) – Quantifier with restricted domain

FOL…

- (Domain: Integers)
  - ✓ **Using implications:** The cube of all negative integers is negative.

    $$\forall x\ (x < 0) \rightarrow (x^3 < 0)$$

  - ✓ **Expressing sums :** $\forall \boldsymbol{x}(\boldsymbol{r} = \sum_{\boldsymbol{i}=1}^{\boldsymbol{x}} \frac{\boldsymbol{x}(\boldsymbol{x}+1)}{2})$

FOL...

- **Translation Examples**
  - ✓ Every student in this class has studied calculus.

    Sol-1: *Domain of x is* "the students in this class"

    $C(x)$ represents "x has studied calculus"

    $\forall x\, C(x)$

    Sol-2: *Domain of x is* "all people"

    $S(x)$ stands for "the person x is in this class"

    $\forall x\, (S(x) \rightarrow C(x))$ (Caution: Not $\forall x\, (S(x) \land C(x))$)

    Sol-3: *Domain of x is*

    "the students in this class" or "all people"

    $Q(x,y)$ represents "x has studied subject y"

    $\forall x\, Q(x,\text{calculus})$ or $\forall x\, (S(x) \rightarrow Q(x,\text{calculus}))$

FOL…

- **Translation Examples**
  - ✓ Some students in this class has visited Mexico.

Sol: Let $M(x)$ = "x has visited Mexico"
$S(x)$ = "x is a student in this class"

$\exists x\, M(x)$ if x's domain is "the students in this class"

$\exists x\, (S(x) \wedge M(x))$ if x's domain is defined as "all people" (Caution: Not $\exists X\, (S(x) \rightarrow M(x))$ Why?)

FOL…

- **Translation Examples**
  - ✓ "Every student in this class has visited either Canada or Mexico"

Sol: $M(x)$ = "x has visited Mexico"
$C(x)$ = "x has visited Canada"
$S(x)$ = "x is a student in this class"
$V(x,y)$ denotes "x has visited y"

Process-1: $\forall x\,(S(x) \rightarrow (C(x) \vee M(x))$

Process-2: $\forall x\,(S(x) \rightarrow (V(x,Mexico) \vee V(x,Canada)))$
if x's domain is defined as "all people"

References

[1] https://en.wikipedia.org/wiki/Propositional_calculus
[2] https://www.cs.utexas.edu/~schrum2/cs301k/lec/topic01-propLogic.pdf
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THANKS