

## Propositional Logic

**Definition:** A **proposition** or **statement** is a sentence which is either true or false.

**Definition:** If a proposition is true, then we say its **truth value** is true, and if a proposition is false, we say its truth value is false.

**Are these propositions?**

1. The sun is shining.
2. The sum of two prime numbers is even.
3.  $3+4=7$
4. It rained in Austin, TX, on October 30, 1999.
5.  $x+y > 10$
6. Is it raining?
7. Come to class!
8.  $n$  is a prime number.
9. The moon is made of green cheese.

**Definition:** A **propositional variable** represents an arbitrary proposition. We represent propositional variables with uppercase letters.

## What is an Argument?

**Definition:** An **argument** consists of a sequence of statements called **premises** and a statement called a **conclusion**. An argument is **valid** if the conclusion is true whenever the premises are all true.

### Example:

My program won't compile or it produces a division by 0 error.

My program does not produce a division by 0 error.

Therefore my program will not compile.

**Now:** Rewrite this argument in its general form by defining appropriate propositional variables.

This is one example of an argument form that is called **disjunctive syllogism**.

**Example:** Another disjunctive syllogism example

Ladybugs are purple or green.

Ladybugs are not green.

Therefore ladybugs are purple.

Let P be: Ladybugs are purple.

Let Q be: Ladybugs are green.

**Rewritten argument:**

P or Q

not Q

Therefore P

This argument is valid, but it isn't very meaningful since P and Q are not true.

## LOGICAL CONNECTIVES

- Use logical connectives to build complex propositions from simpler ones.

### The First Three Logical Connectives

- $\neg$  denotes not.  $\neg P$  is the **negation** of P.
- $\vee$  denotes or.  $P \vee Q$  is the **disjunction** of P and Q.
- $\wedge$  denotes and.  $P \wedge Q$  is the **conjunction** of P and Q.

## Order of Operations

- $\neg$  first
- $\wedge/\vee$  second
- implication and biconditionals last (more on these later)
- parentheses can be used to change the order

## Translating English Into Logic

**Note:** The word “but” in English is often translated as  $\wedge$ .

**Example:** Today is hot but it is not sunny.

Because the second part of the sentence is a surprise, “but” is used instead of “and”.

**Example:** Write each sentence in symbols, assigning propositional variables to statements as follows:

P: It is hot.

Q: It is sunny.

1. It is not hot but it is sunny.
2. It is neither hot nor sunny.

Answers: (in class)

## Truth Tables

**Truth tables** represent the relationship between the truth values of propositions and compound propositions formed from those propositions.

**Examples:** Give truth tables for the logical connectives not, and, or.

**Example:** P: Today is Friday. What is  $\neg P$ ?

**Example:** P: At least two inches of rain fell in Austin today.  
 $\neg P$ ?

**Example:**

P: Today it is raining.

Q: Today is it snowing.

$P \wedge Q$ ?

$P \vee Q$ ?

## Inclusive vs. Exclusive Or

**Note:** The  $\vee$  operation represents “inclusive” or.  
 $P \vee Q$  is true if either P and Q are true, or if both P and Q are true.

“Or” has two meanings in English:

1. Inclusive Or: I will take cs 305j or cs 313k this fall. (Possibly I will take both).
2. Exclusive Or: I will order a steak or enchiladas for dinner tonight. (But hopefully **not** both!).

Make sure you are interpreting “or” correctly when translating English statements into logic!!

**Exercise:** Write a truth table for exclusive or operation, sometimes denoted  $\oplus$  or  $\underline{\vee}$ .

## Implications

For propositions P and Q, the **implication** or **conditional statement**  $P \rightarrow Q$  is false when P is true and Q is false, and is true otherwise. P is called the **premise** or **hypothesis**, and Q is called the **conclusion**.

**Ways to read**  $P \rightarrow Q$ :

If P then Q.

P is sufficient for Q.

Q is necessary for P.

P implies Q.

Q follows from P.

P only if Q.

Q if P.

## Implication Example

**Example:** If you earn 90% of the possible points in cs 313k, then you will get an A.

Rewritten in logical notation:  $P \rightarrow Q$ , where

P is: You earn 90% of the possible points in cs 313k.

Q is: You will get an A in cs 313k.

**Interpretation of  $P \rightarrow Q$ :**

Think of  $P \rightarrow Q$  being true as long as I do what I promised:

If P is false, you should not expect an A (though I could give you an A), so I have honored the commitment ( $P \rightarrow Q$  is true).

If P is true,  $P \rightarrow Q$  is true only if Q is also true, ie if I give you an A.

**Example:** Give the truth table for implication  $P \rightarrow Q$ .

## More on the Meaning of Implications

**Note:** In English, a sentence of the form “if A then B” can have different meanings.

1. Typically there is a causal relationship between A and B, which is not required in logic.
2. We are often implying more than simply that if A holds, then B holds as well.

### Example:

If I finish my work early, then I will pick you up before lunch.

The common-sense interpretation of this sentence is that the **inverse** statement is also true: If I do *not* finish my work early, then I will *not* pick you up before lunch. But this is not implied by the logical implication  $P \rightarrow Q$ , where P is “I finish my work early” and Q is “I will pick you up before lunch”. English is not as precise as logic.

### Example:

If 2 is prime, then grape juice will pour from the sky today.

Logical implication  $P \rightarrow Q$  can be formed using propositions P and Q that are not related in any way. We would never say this in English.

## **Examples: Are these implications true or false?**

1. If it rains today, then 2 is a prime number.
2. If  $2+2=5$ , then all dogs are black.
3. If  $4*4 = 16$ , then  $(-1)*(-1) = 1$ .
4. If there are 100 students in class today, then we will order pizza.

## Biconditionals

The **biconditional** of statements  $P$  and  $Q$ , denoted  $P \leftrightarrow Q$ , is read “ $P$  if and only if  $Q$ ” (or “ $P$  is necessary and sufficient for  $Q$ ”), and is true if  $P$  and  $Q$  have the same truth values, and false otherwise.

**Example:** Write down the truth table of  $P \leftrightarrow Q$ .

**Example: True or false?**  $2+2 = 5$  if and only if 2 is prime.

**Example: true or false?**  $2+2 = 5$  if and only if 10 is prime.

**Note:** If  $P \leftrightarrow Q$  is true, then  $P \rightarrow Q$  and  $Q \rightarrow P$  are true. Conversely if both  $P \rightarrow Q$  and  $Q \rightarrow P$  are true, then  $P \leftrightarrow Q$  is true. Prove this claim using a truth table.

## Translating English into Logic

### **Example:**

You can use the microlab only if you are a cs major or not a freshman.

P: You can use the microlab.

Q: You are a cs major.

R: You are a freshman.

Rewrite the statement using logical connectives.

### **Example:**

If it snows or rains today, I will not go for a walk.

Rewrite this proposition using logical connectives and propositional variables.

## Rearrangements of an Implication

**Definition:** For conditional statement  $P \rightarrow Q$ , the **converse** statement is  $Q \rightarrow P$ , the **contrapositive** statement is  $\neg Q \rightarrow \neg P$ , and the **inverse** statement is  $\neg P \rightarrow \neg Q$ .

**Note:** As we saw in a previous example, we can use a truth table to determine if two compound propositions are **logically equivalent**, ie if they always have the same truth values. If two propositions  $R$  and  $S$  are logically equivalent, we write  $R \equiv S$ .

**Example:** Use a truth table to determine whether or not the converse, contrapositive and inverse statements are logically equivalent to  $P \rightarrow Q$ .

**Definition:** If a compound proposition is always true, it is a **tautology**. Note that if  $R \leftrightarrow S$  is a tautology, then  $R \equiv S$ .

**Note:**  $\equiv$  is **not** a logical connective and  $R \equiv S$  is not a compound proposition.  $R \equiv S$  just means that  $R \rightarrow S$  is a tautology. Alternate notation:  $R \Leftrightarrow S$ .

## LOGICAL IDENTITIES

$P \equiv P \wedge P$	idempotence of $\wedge$
$P \equiv P \vee P$	idempotence of $\vee$
$P \vee Q \equiv Q \vee P$	commutativity of $\vee$
$P \wedge Q \equiv Q \wedge P$	commutativity of $\wedge$
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	associativity of $\vee$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	associativity of $\wedge$
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	DeMorgan's Laws
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	distributivity of $\wedge$ over $\vee$
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	distributivity of $\vee$ over $\wedge$
$P \vee T \equiv T$	domination laws
$P \wedge F \equiv F$	
$P \wedge T \equiv P$	identity laws
$P \vee F \equiv P$	
$P \vee \neg P \equiv T$	negation laws
$P \wedge \neg P \equiv F$	
$\neg(\neg P) \equiv P$	double negation law
$P \vee (P \wedge Q) \equiv P$	absorption laws
$P \wedge (P \vee Q) \equiv P$	
$P \rightarrow Q \equiv \neg P \vee Q$	implication
$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	contrapositive
$P \leftrightarrow Q \equiv [(P \rightarrow Q) \wedge (Q \rightarrow P)]$	equivalence
$[(P \wedge Q) \rightarrow R] \equiv [P \rightarrow (Q \rightarrow R)]$	exportation

## More on Propositional Equivalence

**Recall:** A compound proposition that is always true is a **tautology**. Example:  $P \vee \neg P$

**Definition:** A compound proposition that is always false is a **contradiction**. Example:  $P \wedge \neg P$ .

**Definition:** A compound proposition that is not a tautology or a contradiction is a **contingency**.

**Homework:** Look at the table of logical equivalences and become familiar with the laws and their names.

**Example:**

$P$ : Ann will go to the movie.

$Q$ : Scott will go to the movie.

Using DeMorgan's law,  $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ , write down  $\neg(P \vee Q)$ .

## Equivalence Proofs using the Logical Identities

### Example - our first step by step equivalence proof

Show  $P \wedge \neg Q \equiv \neg(P \rightarrow Q)$ . Hint: You will need to use the fact that  $P \rightarrow Q \equiv \neg P \vee Q$  (we just proved this).

**Example:** Show that  $(P \wedge Q) \rightarrow P$  is a tautology.

**Note:** We can also use a truth table to prove a compound proposition is a tautology, but this is only practical if the number of propositional variables is small.

**Exercise:** Show  $\neg P \rightarrow (P \rightarrow Q)$  is a tautology, first using a truth table and then with an equivalence proof.

## Grammar/Syntax

It is very important to use logical connectives correctly. When writing out logical statements and formulas, consider the rules of syntax to be as strict as in a programming language. If your statements do not use correct grammar/syntax, then others will not know what you mean.

**Note:** We will often mix logical notation and English, but even when we do this, logical symbols must obey the same strict rules. Remember that every logical symbol can only be used according to strict syntax rules. Do NOT simply plug in a symbol anywhere that the corresponding English word would fit. For example, the  $\wedge$  connective cannot always be used to replace the word “and” in a sentence.

**Exercise:** Which statements are correctly formed propositions? Which are nonsense?

1.  $C$  implies  $D$
2. The proposition  $A \vee B$  is false
3. Either  $A \vee B$  is false
4.  $A$  iff  $B$
5.  $A \wedge B$
6.  $B \neg C$
7. Proposition  $A \rightarrow$  that  $B$  is true.