

Propositional Logic (PL) & First Order Logic (FOL)

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Proposition/Statement [1]

A **Proposition/Statement** is a declarative sentence which is either *true* or *false* but not both.

Example: This robot is active.

Sofia is a robot.

Don't talk in the class.

Come to the class.

Propositional Variable

- A **Propositional Variable** represents an arbitrary proposition.
- Propositional variables are represented with uppercase letters. (A,B,...,Z)

Example:

P = This robot is active

Q = Sofia is a robot

Logical Connectives [2]

These are the **Words or Symbols** used to combine two or more sentences to form a **Compound Statement**.

logic	Name	rank
\sim	Negation	1
\wedge	Conjunction	2
\vee	Disjunction	3
\Rightarrow	Conditional	4
\Leftrightarrow	Biconditional	5

Basic Logical Operations

A	B	\wedge	\vee	$\sim A$	$=>$	\Leftrightarrow	NOR	NAND	XOR	EX-NOR
T	T	T	T	F	T	T	F	F	F	T
T	F	F	T	F	F	F	F	T	T	F
F	T	F	T	T	T	F	F	T	T	F
F	F	F	F	T	T	T	T	T	F	T

Rearrangement of Implication [2]

For conditional statement $P \rightarrow Q$, the **Converse Statement** is $Q \rightarrow P$, the **Contrapositive Statement** is $\neg Q \rightarrow \neg P$, and the **Inverse Statement** is $\neg P \rightarrow \neg Q$.

H.W. Use truth table to show whether converse, contrapositive and inverse statements of $P \rightarrow Q$ are equivalent to it or not.

Propositional Logic (PL) [1]

- Also Called Propositional Logic/ Propositional Calculus/ Statement Logic/Sentential Calculus/Sentential Logic/Zeroth Order Logic
- The branch of Logic concerned with the study of Propositions that are formed by other propositions with the use of Logical Connectives

PL-Example

- $P = \text{"It is hot"}, Q = \text{"It is humid"}, R = \text{"It is raining"}$
 - $(P \wedge Q) \rightarrow R$
“If it is hot and humid, then it is raining”
 - $Q \rightarrow P$
“If it is humid, then it is hot”
- A better way:
 - $H_o = \text{"It is hot"}$
 - $H_u = \text{"It is humid"}$
 - $R = \text{"It is raining"}$

Tautology

- A tautology is a proposition which is True for all truth values of its sub-propositions or components.
- A tautology is also called **logically valid** or **logically true**.
- All entries in the column of tautology are **true**.

Tautology...

P	q	$p \wedge q$	q	$p \wedge q \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

Contradiction

- Contradiction is a proposition which is always **false** for all truth values of its sub-propositions or components.
- A contradiction is also called **logically invalid** or logically false.
- All entries in the column of contradiction are false.

Contradiction...

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$(P \vee Q) \wedge (\sim P) \wedge (\sim Q)$
T	T	T	F	F	F
T	F	T	F	T	F
F	T	T	T	F	F
F	F	F	T	T	F

Contingency

- It is a proposition which is either true or false depending on the truth value of its components or proposition.

Contingency...

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Wff [3]

- A sentence (**Well Formed Formula-wff**) is defined as follows:
 - A symbol is a sentence
 - If S is a sentence, then $\neg S$ is a sentence
 - If S is a sentence, then (S) is a sentence
 - If S and T are sentences, then $(S \vee T)$, $(S \wedge T)$, $(S \rightarrow T)$, and $(S \leftrightarrow T)$ are sentences
 - A sentence results from a finite number of applications of the above rules

Logical Equivalence

Two statements are called Logically Equivalent if the truth values of both the statements are **always identical..**

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim q \Rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Identities

$P \equiv P \wedge P$	idempotence of \wedge
$P \equiv P \vee P$	idempotence of \vee
$P \vee Q \equiv Q \vee P$	commutativity of \vee
$P \wedge Q \equiv Q \wedge P$	commutativity of \wedge
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	associativity of \vee
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	associativity of \wedge
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	DeMorgan's Laws
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	distributivity of \wedge over \vee
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	distributivity of \vee over \wedge
$P \vee T \equiv T$	domination laws
$P \wedge F \equiv F$	
$P \wedge T \equiv P$	identity laws
$P \vee F \equiv P$	
$P \vee \neg P \equiv T$	negation laws
$P \wedge \neg P \equiv F$	
$\neg(\neg P) \equiv P$	double negation law
$P \vee (P \wedge Q) \equiv P$	absorption laws
$P \wedge (P \vee Q) \equiv P$	
$P \rightarrow Q \equiv \neg P \vee Q$	implication
$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	contrapositive
$P \leftrightarrow Q \equiv [(P \rightarrow Q) \wedge (Q \rightarrow P)]$	equivalence
$[(P \wedge Q) \rightarrow R] \equiv [P \rightarrow (Q \rightarrow R)]$	exportation

Inference Rules

- **Inference Rule** or **Transformation Rule** is a logical form consisting of a function which takes premises, analyzes their syntax, and returns a conclusion (or conclusions).
- Popular Rules of Inference in Propositional Logic include **Modus Ponens**, **Modus Tollens**, and **Contraposition**.

Rules of Inference [1]

Rule of inference	Tautology	Name
$p \rightarrow q$ \underline{p} $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollen
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
\underline{p} $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
\underline{p} \underline{q} $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\underline{p \vee q}$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Problems With PL [4]

- Consider the problem of representing the following information:
 - Every person is mortal.
 - Confucius is a person.
 - Confucius is mortal.
- How can these sentences be represented so that we can infer the *third sentence* from the *first two*?

Problems With PL...

- In PL we have to create propositional symbols to stand for all or part of each sentence. For example, we might have:
 $P = \text{"person"}; Q = \text{"mortal"}; R = \text{"Confucius"}$
- So the above 3 sentences are represented as:
 $P \rightarrow Q; R \rightarrow P; R \rightarrow Q$
We can't infer the third proposition using the above rules.
- To represent **other individuals** we must introduce separate symbols for each one, with some way to represent the fact that all individuals who are “people” are also “mortal”.

PL is a Weak Language

- It applies only to atomic propositions. There is no way to talk about properties that apply to categories of objects, or about relationships between those properties.
- Propositional Logic doesn't allow you to access the components of an assertion:

It rained on Tuesday

But if we write

weather (Tuesday, rain)

or weather(X, rain)

First Order Logic (FOL) [4]

- First Order Logic/ First Order Predicate Calculus/Predicate Logic is expressive enough to concisely represent this kind of information

FOL adds Relations, Variables,
and Quantifiers

- “*Every elephant is gray*”: $\forall x (\text{elephant}(x) \rightarrow \text{gray}(x))$
- “*There is a white alligator*”: $\exists x (\text{alligator}(X) \wedge \text{white}(X))$

FOL...

- ✓ An **Argument** can be either a constant or a variable.
- ✓ It is conventional to write variables with an Initial Upper-case Letter.
- ✓ Constants can be numbers, names that begin with a lower-case letter, or quoted strings.
- ✓ A **Relation** is one way to represent a predicate.
 - ✓ **P(X,3)**: True iff X is greater than 3
 - ✓ **Greater(X,3)**: True iff X is greater than 3

FOL...

✓ More Examples:

- Domain of ASCII characters - $\text{IsAlpha}(x)$: TRUE iff x is an alphabetical character.
- Domain of floating point numbers - $\text{IsInt}(x)$: TRUE iff x is an integer.
- Domain of integers: $\text{Prime}(x)$ - TRUE if x is prime.

FOL...

- A propositional function may associate with **two or more variables**.
- ✓ Examples:
 - $P(x,y)$: “ $x = y+3$ ”, and $P(1,2)$ is false.
 - $Q(x,y,z)$: “ $x+y=z$ ”, then $Q(1,2,3)$ is true.

FOL...

- **Predicate logic** has two additional operators not found in propositional logic (called **Quantifiers**) to express truth values about predicates with variable arguments.
- **Quantifiers:** Quantifiers specifies the quantity of samples in the domain of discourse.
 - ✓ Universal Quantifiers \forall
 - ✓ Existential Quantifiers \exists

FOL...

- **Universal:** $\forall x P(x)$ – “ $P(x)$ for all x in the domain”
- **Existential:** $\exists x P(x)$ – “ $P(x)$ for some x in the domain”
or “There exists x such that $p(x)$ is TRUE”.
- **Examples**
(Domain: Real numbers)
 - ✓ $\forall x (x^2 \geq 0)$
 - ✓ $\exists x (x > 1)$
 - ✓ $(\forall x > 1) (x^2 > x)$ – Quantifier with restricted domain

FOL...

- (Domain: Integers)

✓ **Using implications:** The cube of all negative integers is negative.

$$\forall x (x < 0) \rightarrow (x^3 < 0)$$

✓ **Expressing sums :** $\forall x (r = \sum_{i=1}^x \frac{x(x+1)}{2})$

FOL...

- **Translation Examples**

✓ Every student in this class has studied calculus.

Sol-1: *Domain of x is “the students in this class”*

C(x) represents “x has studied calculus”

$\forall x C(x)$

Sol-2: *Domain of x is “all people”*

S(x) stands for “the person x is in this class”

$\forall x (S(x) \rightarrow C(x))$ (**Caution:** Not $\forall x (S(x) \wedge C(x))$)

Sol-3: *Domain of x is*

“the students in this class” or “all people”

Q(x,y) represents “x has studied subject y”

$\forall x Q(x, \text{calculus})$ or $\forall x (S(x) \rightarrow Q(x, \text{calculus}))$

FOL...

- **Translation Examples**

✓ Some students in this class has visited Mexico.

Sol: Let $M(x)$ = “ x has visited Mexico”

$S(x)$ = “ x is a student in this class”

$\exists x M(x)$ if x ’s domain is “the students in this class”

$\exists x (S(x) \wedge M(x))$ if x ’s domain is defined as “all people” (**Caution:** Not $\exists X (S(x) \rightarrow M(x))$ Why?)

FOL...

- **Translation Examples**

✓ “Every student in this class has visited either Canada or Mexico”

Sol: $M(x) = \text{“}x \text{ has visited Mexico”}$

$C(x) = \text{“}x \text{ has visited Canada”}$

$S(x) = \text{“}x \text{ is a student in this class”}$

$V(x,y)$ denotes “ x has visited y ”

Process-1: $\forall x (S(x) \rightarrow (C(x) \vee M(x)))$

Process-2: $\forall x (S(x) \rightarrow (V(x,\text{Mexico}) \vee V(x,\text{Canada})))$
if x ’s domain is defined as “all people”

References

- [1] https://en.wikipedia.org/wiki/Propositional_calculus
- [2] <https://www.cs.utexas.edu/~schrum2/cs301k/lec/topic01-propLogic.pdf>
- [3] https://en.wikipedia.org/wiki/Well-formed_formula
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THANKS