Probability Distribution

Binomial Distribution

- The binomial distribution is a probability distribution.
- It is used when an action is repeated and we are interested in the number of successes.
- It gives the probability for each number of successes.
- There is a formula for calculating these probabilities.

- The probability of success must be the same in each trial.
- The trials must be independent.

What is the probability of getting x successes in n trials?

What is the probability of winning 2 games out of 5?

Probability of winning a game $= \frac{1}{4}$

$$P(\text{2 wins out of 5}) = \begin{array}{l}\text{Number of ways}\\ \text{of winning 2 games}\\ \text{out of 5}\end{array} \times P(\text{SSFFF})$$

$$P(\text{SSFFF}) = \left(\frac{1}{4}\right)^2 \times \left(\frac{3}{4}\right)^3$$

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$$\begin{array}{l}\text{Number of ways}\\ \text{of winning 2 games}\\ \text{out of 5}\end{array} = 10$$

$$P(\text{2 wins out of 5}) = 10 \times \left(\tfrac{1}{4}\right)^2 \times \left(\tfrac{3}{4}\right)^3 = \frac{135}{512}$$

What is the probability of getting x successes in n trials?

Probability of success $= p$

Probability of failure $= 1 - p$

$$\text{Probability of } x \text{ successes in a row followed by } n - x \text{ failures in a row} = p^x \times (1-p)^{n-x}$$

$$\text{Number of ways of getting } x \text{ successes in } n \text{ trials} = \binom{n}{x} = {}^nC_x$$

$$\mathrm{P}(x \text{ successes out of } n) = \binom{n}{x} \times p^x \times (1-p)^{n-x}$$

n = number of trials

p = probability of success in one trial

X = number of successes

$$X \sim \mathrm{B}(n, p)$$

$$\mathrm{P}(X = x) = \frac{n!}{x!(n-x)!} \times p^{x} \times (1-p)^{(n-x)}$$

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If $X \sim B(6, \frac{2}{5})$, what is $P(X = 4)$?

$$P(X = 4) = {}^{6}C_{4} \times \left(\frac{2}{5}\right)^{4} \times \left(\frac{3}{5}\right)^{2}$$

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Five dice are thrown. What is the probability of getting two multiples of 3?

$$X \sim \mathrm{B}(12, 0.35)$$

$$\mathrm{P}(X \leq 3)$$

If $X \sim \mathrm{B}(15, 0.3)$, what is $\mathrm{P}(X \geq 5)$?

$$\boxed{0 \quad 1 \quad 2 \quad 3 \quad 4} \; \boxed{5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15}$$

$$\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4)$$

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We need to show

$$\binom{n}{x} p^x (1-p)^{n-x} \longrightarrow \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \frac{n!}{(n-x)!} \frac{1}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\frac{n!}{(n-x)!} \frac{1}{n^x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)(n-x)!}{(n-x)!n^x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{n^x}$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-x+1}{n}$$

$$= 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x+1}{n}\right)$$

$$e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$$

$$\binom{n}{x}p^x(1-p)^{n-x}$$

$$= \frac{\lambda^x}{x!}(1-\tfrac{1}{n})(1-\tfrac{2}{n})\ldots(1-\tfrac{x+1}{n})(1-\tfrac{\lambda}{n})^{n-x}$$

$$\frac{\lambda^x}{x!}(1-\tfrac{1}{n})(1-\tfrac{2}{n})\ldots(1-\tfrac{x+1}{n})(1-\tfrac{\lambda}{n})^{n}(1-\tfrac{\lambda}{n})^{-x}$$

$$\begin{aligned}
&\lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{n-x}\\
&\quad=\frac{\lambda^x}{x!}\\
&\qquad\times\lim_{n\to\infty}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{x+1}{n})\\
&\qquad\times\lim_{n\to\infty}(1-\frac{\lambda}{n})^{n}\\
&\qquad\times\lim_{n\to\infty}(1-\frac{\lambda}{n})^{-x}
\end{aligned}$$

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$$