Probability

Uncertainty in the World

- An person can often be uncertain about the state of the world/domain since there is often ambiguity and uncertainty
- Plausible/**probabilistic inference**
  - I’ve got this evidence; what’s the chance that this conclusion is true?
    - I’ve got a sore neck; how likely am I to have meningitis?
    - A mammogram test is positive; what’s the probability that the patient has breast cancer?

Uncertainty

- Say we have a rule:

  if *toothache* ***then*** *problem is cavity*

- But not all patients have toothaches due to cavities, so we could set up rules like:

  if *toothache and ¬gum-disease and ¬filling and ...*
  then *problem = cavity*

- This gets complicated; better method:

  if *toothache* ***then*** *problem is cavity with 0.8 probability*

  or $P(cavity \mid toothache) = 0.8$

  the probability of cavity is 0.8 given toothache is observed

Example of Uncertainty

- Assume a camera and vision system is used to estimate the curvature of the road ahead
- There's uncertainty about which way it curves
  - limited pixel resolution, noise in image
  - algorithm for "road detection" is not perfect
- This uncertainty can be represented with a simple probability model:

$$P(\textit{road curves to left} \mid E) = 0.6$$
$$P(\textit{road goes straight} \mid E) = 0.3$$
$$P(\textit{road curves to right} \mid E) = 0.1$$

  - where the probability of an event is a measure of observer's belief in the event given the evidence E

Uncertainty in the World and our Models

- True uncertainty: rules *are* probabilistic in nature
  - quantum mechanics
  - rolling dice, flipping a coin
- Laziness: too hard to determine exception-less rules
  - takes too much work to determine *all* of the relevant factors
  - too hard to use the enormous rules that result
- Theoretical ignorance: don't know all the rules
  - problem domain has no complete, consistent theory (e.g., medical diagnosis)
- Practical ignorance: do know all the rules BUT
  - haven't collected all relevant information for a particular case

Logics

Logics are characterized by what they commit to as "primitives"

Logic	**What Exists in World**	**Knowledge States**
Propositional	facts	true/false/unknown
First-Order	facts, objects, relations	true/false/unknown
Temporal	facts, objects, relations, times	true/false/unknown
Probability Theory	facts	degree of belief 0..1
Fuzzy	degree of truth	degree of belief 0..1

Probability Theory

- Probability theory serves as a formal means for
  - Representing and reasoning with uncertain knowledge
  - Modeling **degrees of belief** in a proposition (event, conclusion, diagnosis, etc.)
- *Probability is the "language" of uncertainty*
  - A key modeling method in modern AI

Source of Probabilities

- Frequentists
  - probabilities come from experiments
  - if 10 of 100 people tested have a cavity, $P(cavity) = 0.1$
  - probability means the fraction that would be observed in the limit of infinitely many samples
- Objectivists
  - probabilities are real aspects of the world
  - objects have a propensity to behave in certain ways
  - coin has propensity to come up heads with probability 0.5
- Subjectivists
  - probabilities characterize an agent's belief
  - have no external physical significance

Sample Space/Outcome Space

- S is a outcome space: collection of all possible outcome
- Let, A be a part of the collection of outcomes in S; that is, $A \subset S$. Then A is called an event.
- Events can be binary, multi-valued, or continuous

Outcome and Event

- Outcome and event are not synonymous.
- **Outcome** is the result of a random experiment. Example: rolling a die has **six** possible outcomes.
- **Event** is a set of outcomes to which a probability is assigned. Example: One possible event is "rolling a number less than 3".

Mutually Exclusive Event

- **Mutually exclusive** events are events that **cannot occur together (simultaneously).**
- $A_1, A_2, \dots., A_k$ are mutually exclusive events means that $A_i \cap A_j = \emptyset, i \neq j$; that is, $A_1, A_2, \dots., A_k$ are disjoint sets.
- **Example:**
  - A = queen of diamonds; B = queen of clubs
  - Events A and B are mutually exclusive if only one card is selected

Mutually Exhaustive Event

- $A_1, A_2, \ldots., A_k$ are mutually exhaustive events means that $A_i \cup A_j \cup \cdots \cup A_k = S$

Example:

Consider the experiment of throwing a die.

Sample space S = {1, 2, 3, 4, 5, 6}

Assume that A, B and C are the events associated with this experiment.
Define: A be the event of getting a number greater than 3

B be the event of getting a number greater than 2 but less than 5

C be the event of getting a number less than 3

We can write these events as:

A = {4, 5, 6}

B = {3, 4}

and C = {1, 2}

We observe that

A ⋃ B ⋃ C = {4, 5, 6} ⋃ {3, 4} ⋃ {1, 2} = {1, 2, 3, 4, 5, 6} = S

The Axioms of Probability

1. $0 \leq P(A) \leq 1$
2. $P(\text{true}) = 1$, $P(\text{false}) = 0$

3. For any two disjoint events A and B, we have

$$P(A \cup B) = P(A) + P(B)$$

4. For any infinite sequence of mutually disjoint events $A_1, A_2, A_3, \ldots$, we have

$$\begin{aligned} &P(A_1 \cup A_2 \cup A_3 \cup \cdots) \\ &= P(A_1) + P(A_2) + P(A_3) + \cdots \end{aligned}$$

Empirical Probablity

- Refers to a probability that is based on historical data.

$$P(A) = \frac{\text{\# of times event A occurs}}{\text{total \# of observed occurences}}$$

Empirical Probablity

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

$$\text{Probability of Male Taking Stats} = \frac{\text{number of males taking stats}}{\text{total number of people}} = \frac{84}{439} = 0.191$$

Equiprobable Probability Space

- All outcomes equally likely (fair coin, fair die...)
- Laplace's definition of probability (only in finite equiprobable space)

$$P(A) = \frac{|A|}{|S|}$$

Theoritical Probablity

- Theoretical probability is finding the probability of events that come from an equiprobable sample space.

$$P(A) = \frac{\#\ of\ outcomes\ in\ A}{number\ of\ outcomes\ in\ S} = \frac{|A|}{|S|}$$

Theoritical Probablity

Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.

$$P(Face\ Card) = \frac{|A|}{|S|} = \frac{12}{52} = \frac{3}{13}$$

Simple vs Joint Probability

- Simple (Marginal) Probability refers to the probability of a simple event.
  - Example: P(King)

- Joint Probability refers to the probability of an occurrence of two or more events.
  - Example: P(King and Spade)

Simple vs Joint Probability

Computing Joint and Marginal Probabilities:

- The probability of a **joint** event, A and B:

$$P(A\ and\ B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a **marginal (or simple)** probability:

$$P(A) = P(A\ and\ B_1) + P(A\ and\ B_2) + \cdots + P(A\ and\ B_k)$$

Where $B_1, B_2, \dots, B_k$ are k mutually exclusive and collectively exhaustive events

Example of Joint Probability

	Ace	Not Ace	Total
Black	2	24	26
Red	2	24	26
Total	4	48	52

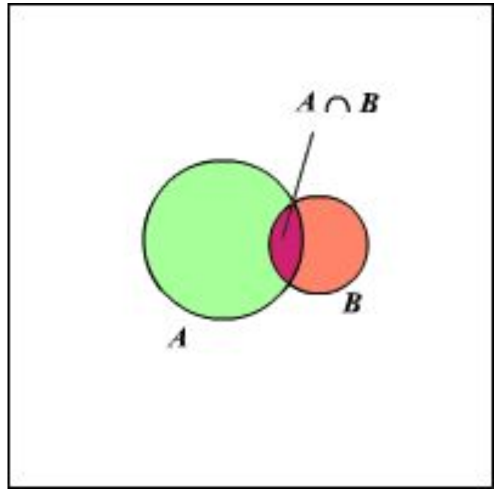
$$P(\text{Red and Ace}) = \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Example of Marginal Probability

	Ace	Not Ace	Total
Black	2	24	26
Red	2	24	26
Total	4	48	52

$$P(\text{Ace}) = P(\text{Ace and Red}) + P(\text{Ace and Black}) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52}$$

Laws of Probability: Additive Rule



- If A and B are two events in a probability experiment, then the probability that either one of the events will occur is

$$P(A\, or\, B) = P(A) + P(B) - P(A\, and\, B)$$

Or

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

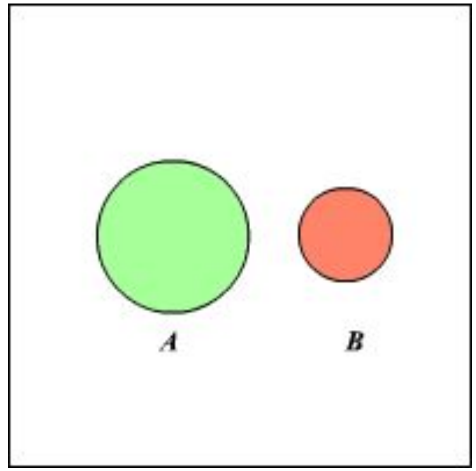
Laws of Probability: Additive Rule (Example)

- Example: If I roll a number cube and flip a coin, What is the probability I will get a tails or a 3?

Answer:

$$P(\text{tails or a 3}) = \frac{1}{2} + \frac{1}{6} = \frac{8}{12} = \frac{2}{3}$$

Laws of Probability: Additive Rule



- If A and B are two mutually exclusive events then $P(A \cap B) = 0$.

$$P(A\ or\ B) = P(A) + P(B)$$

Or

$$P(A \cup B) = P(A) + P(B)$$

Laws of Probability: Additive Rule (Example)

- If you take out a single card from a regular pack of cards, what is probability that the card is either an ace or spade?

Answer

Let *X* be the event of picking an ace and *Y* be the event of picking a spade.

$$P(X) = \frac{4}{52}$$

$$P(Y) = \frac{13}{52}$$

The two events are not mutually exclusive, as there is one favorable outcome in which the card can be both an ace and spade.

$$P(X \cap Y) = \frac{1}{52}$$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$$

Complement Rule

- For any event A, we have

$$P(A^c) = 1 - P(A)$$

Complement Rule

- Suppose that we flip eight fair coins. What is the probability that we have at least one head showing?

Answer:

The complement of the event "we flip at least one head" is the event "there are no heads."

$$P(\text{At least one head}) = 1 - P(\text{No head})$$
$$= 1 - \frac{1}{256} = 0.99609375$$

Complement Rule

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Random Variable

- A variable, *X*, whose domain is a sample space, and whose value is (somewhat) uncertain
- Examples:

  X = coin flip outcome

  X = first word in tomorrow's NYT newspaper

  X = tomorrow's high temperature

Random Variable

- **Random Variables** (RV):
  - are capitalized (usually) e.g., *Sky*, *Weather*, *Temperature*
  - refer to attributes of the world whose "status" is *unknown*
  - have one and only one value at a time
  - have a domain of **values** that are possible states of the world:
    - Boolean: domain = $<true,\ false>$

      $Cavity = true$ (often abbreviated as $cavity$) $Cavity = false$ (often abbreviated as $\leftarrow cavity$)
    - Discrete: domain is countable (includes Boolean)

      values are ***mutually exclusive and exhaustive***

      e.g. Sky domain = $<clear, partly\_cloudy, overcast>$

      $Sky = clear$ abbreviated as $clear$

      $Sky \neq clear$ also abbreviated as $\neg clear$
    - Continuous: domain is real numbers

Conditional Probability

- Conditional probabilities
  - formalizes the process of accumulating evidence and updating probabilities based on new evidence
  - specifies the belief in a proposition (event, conclusion, diagnosis, etc.) that is *conditioned on* a proposition (evidence, feature, symptom, etc.) being true
- $P(a \mid e)$: conditional probability of $A=a$ *given* $E=e$ evidence is *all that is known true*
  - $P(a \mid e) = P(a \wedge e) / P(e) = P(a, e) / P(e)$
  - conditional probability can viewed as the joint probability $P(a, e)$ normalized by the prior probability, $P(e)$

Conditional Probability

Conditional probabilities behave exactly like standard probabilities; for example:

$$0 \leq P(a \mid e) \leq 1$$

conditional probabilities are between 0 and 1 inclusive

$$P(a_1 \mid e) + P(a_2 \mid e) + \ldots + P(a_k \mid e) = 1$$

conditional probabilities sum to 1 where a_1, …, a_k are all values in the domain of random variable A

$$P(\neg a \mid e) = 1 - P(a \mid e)$$

negation for conditional probabilities

Conditional Probability

$P(conjunction\ of\ events \mid e)$

$P(a \wedge b \wedge c \mid e)$ or as $P(a, b, c \mid e)$

is the agent's belief in the sentence $a \wedge b \wedge c$

conditioned on e being true

$P(a \mid conjunction\ of\ evidence)$

$P(a \mid e \wedge f \wedge g)$ or as $P(a \mid e, f, g)$

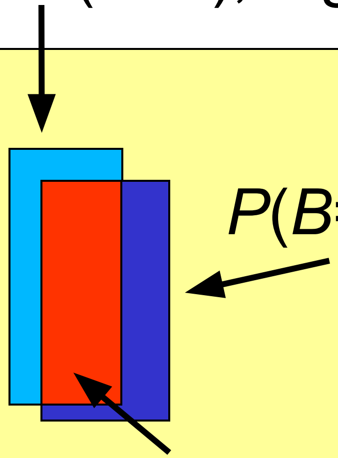
is the agent's belief in the sentence a

conditioned on $e \wedge f \wedge g$ being true

Conditional Probability

The conditional probability $P(A\text{=a} \mid B\text{=b})$ is the fraction of time A=a, within the region where B=b

$P(A\text{=a})$, e.g. $P(1^{st}$ word on a random page = “San”) = 0.001



$P(B\text{=b})$, e.g. $P(2^{nd}$ word = “Francisco”) = 0.0008

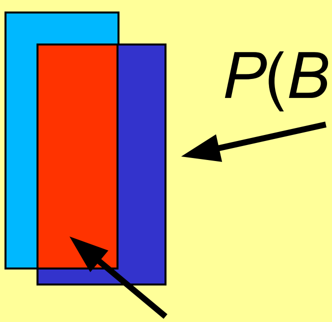
$P(A\text{=a} \mid B\text{=b})$, e.g. $P(1^{st}$=“San” | 2^{nd} =“Francisco”) = **?**

(possibly: San, Don, Pablo …)

Conditional Probability

- $P(\text{san} \mid \text{francisco})$
  $= \#(1^{st}{=}s \text{ and } 2^{nd}{=}f) / \#(2^{nd}{=}f)$
  $= P(\text{san} \wedge \text{francisco}) / P(\text{francisco})$
  $= 0.0007 / 0.0008$
  $= 0.875$

$P(s)=0.001$
$P(f)=0.0008$
$P(s,f)=0.0007$



$P(B{=}b)$, e.g. $P(2^{nd}$ word = "Francisco"$) = 0.0008$

$P(A{=}a \mid B{=}b)$, e.g. $P(1^{st}{=}$"San" $\mid 2^{nd}{=}$"Francisco"$) =$ **0.875**

(possibly: San, Don, Pablo …)

Although "San" is rare and "Francisco" is rare,
given "Francisco" then "San" is quite likely!

Conditional Probability

- In general, the conditional probability is

$$P(A=a \mid B)=\frac{P(A=a,B)}{P(B)}=\frac{P(A=a,B)}{\sum_{\text{all } a_i} P(A=a_i,B)}$$

- We can have everything *conditioned* on some other event(s), *C*, to get a conditionalized version of conditional probability:

$$P(A \mid B,C)=\frac{P(A,B \mid C)}{P(B \mid C)}$$

'|' has low precedence.
This should read: P(*A* | (*B*,*C*))

The Chain Rule

- From the definition of conditional probability we have

$$P(A, B) = P(B) * P(A \mid B) = \mathrm{P}(A \mid B) * \mathrm{P}(B)$$

- It also works the other way around:

$$P(A, B) = P(A) * P(B \mid A) = \mathrm{P}(B \mid A)\, \mathrm{P}(A)$$

- It works with more than 2 events too:

$$P(A_1, A_2, \ldots, A_n) =$$

$$P(A_1) * P(A_2 \mid A_1) * P(A_3 \mid A_1, A_2) * \ldots$$

$$* P(A_n \mid A_1, A_2, \ldots, A_{n-1})$$

Called "**Product Rule**"

Called "**Chain Rule**"

Probabilistic Reasoning

How do we use probabilities in AI?

- You wake up with a headache
- Do you have the flu?
- H = headache, F = flu

Logical Inference: if H then F
(but the world is usually not this simple)

Statistical Inference: compute the probability of a query/diagnosis/decision given (i.e., conditioned on) evidence/symptom/observation, i.e., $P(F \mid H)$

[Example from Andrew Moore]

Example

Statistical Inference: Compute the probability of a diagnosis, F, given symptom, H, where H = "has a headache" and F = "has flu"

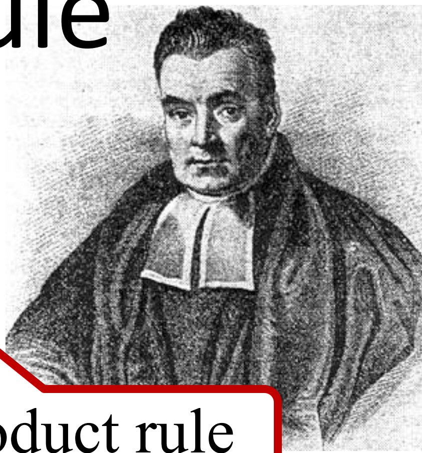
That is, compute $P(F \mid H)$

You know that

- $P(H) = 0.1$ "one in ten people has a headache"
- $P(F) = 0.01$ "one in 100 people has flu"
- $P(H \mid F) = 0.9$ "90% of people who have flu have a headache"

[Example from Andrew Moore]

Inference with Bayes's Rule



Thomas Bayes, "Essay Towards Solving a Problem in the Doctrine of Chances," 1764

$$P(F \mid H) = \frac{P(F,H)}{P(H)} = \frac{P(H \mid F)P(F)}{P(H)}$$

Def of cond. prob.

Product rule

- $P(H) = 0.1$ "one in ten people has a headache"
- $P(F) = 0.01$ "one in 100 people has flu"
- $P(H|F) = 0.9$ "90% of people who have flu have a headache"

- $P(F|H) = 0.9 * 0.01 / 0.1 = 0.09$
- So, there's a 9% chance you have flu – much less than 90%
- But it's higher than $P(F) = 1\%$, since you have a headache

Bayes’s Rule

- Bayes’s Rule is the basis for probabilistic reasoning given a prior model of the world, P(Q), and a new piece of evidence, E, Bayes’s rule says how this piece of evidence decreases our ignorance about the world
- Initially, know P(Q)   (“prior”)
- Update after knowing E (“posterior”):

$$P(Q|E) = P(Q)\frac{P(E|Q)}{P(E)}$$

Inference with Bayes's Rule

$\boldsymbol{P(A|B) = P(B \mid A)P(A) \,/\, P(B)}$ **Bayes's rule**

- Why do we make things this complicated?
  - Often $P(B|A)$, $P(A)$, $P(B)$ are easier to get
  - Some names:
    - **Prior** $\boldsymbol{P(A)}$: probability of A *before* any evidence
    - **Likelihood** $\boldsymbol{P(B|A)}$: assuming A, how likely is the evidence
    - **Posterior** $\boldsymbol{P(A|B)}$: probability of A after knowing evidence B
    - **(Deductive) Inference**: deriving an unknown probability from known ones
- If we have the full joint probability table, we can simply compute $P(A|B) = P(A, B) \,/\, P(B)$

Bayes's Rule in Practice

Summary of Important Rules

- **Conditional Probability**: $P(A|B) = P(A,B)/P(B)$
- **Product rule**: $P(A,B) = P(A|B)P(B)$
- **Chain rule**: $P(A,B,C,D) = P(A|B,C,D)P(B|C,D)P(C|D)P(D)$
- **Conditionalized version of Chain rule**:

  $P(A,B|C) = P(A|B,C)P(B|C)$
- **Bayes's rule**: $P(A|B) = P(B|A)P(A)/P(B)$
- **Conditionalized version of Bayes's rule**:

  $P(A|B,C) = P(B|A,C)P(A|C)/P(B|C)$
- **Addition / Conditioning rule**: $P(A) = P(A,B) + P(A,\neg B)$

  $P(A) = P(A|B)P(B) + P(A|\neg B)P(\neg B)$

Common Mistake

- $P(A) = 0.3$ so $P(\neg A) = 1 - P(A) = 0.7$
- $P(A|B) = 0.4$ so $P(\neg A|B) = 1 - P(A|B) = 0.6$

  because $P(A|B) + P(\neg A|B) = 1$

but $P(A|\neg B) \neq 0.6$ (in general)

because $P(A|B) + P(A|\neg B) \neq 1$ in general

Quiz

- A doctor performs a test that has 99% reliability, i.e., 99% of people who are sick test positive, and 99% of people who are healthy test negative. The doctor estimates that 1% of the population is sick.
- Question: A patient tests positive. What is the chance that the patient is sick?
- 0-25%, 25-75%, 75-95%, or 95-100%?

Quiz

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- Question: A patient tests positive. What is the chance that the patient is sick?
- 0-25%, 25-75%, 75-95%, or 95-100%?
- Common answer: 99%; Correct answer: 50%

Given:

$P(TP \mid S) = 0.99$

$P(\neg TP \mid \neg S) = 0.99$

$P(S) = 0.01$

TP = "tests positive"
S = "is sick"

Query:

$P(S \mid TP) = ?$

$P(TP \mid S) = 0.99$

$P(\neg TP \mid \neg S) = 0.99$

$P(S) = 0.01$

$P(S \mid TP) =$

$P(TP \mid S)\, P(S) \,/\, P(TP)$

$= (0.99)(0.01) \,/\, P(TP) = 0.0099/P(TP)$

$P(\neg S \mid TP) = P(TP \mid \neg S)P(\neg S) \,/\, P(TP)$

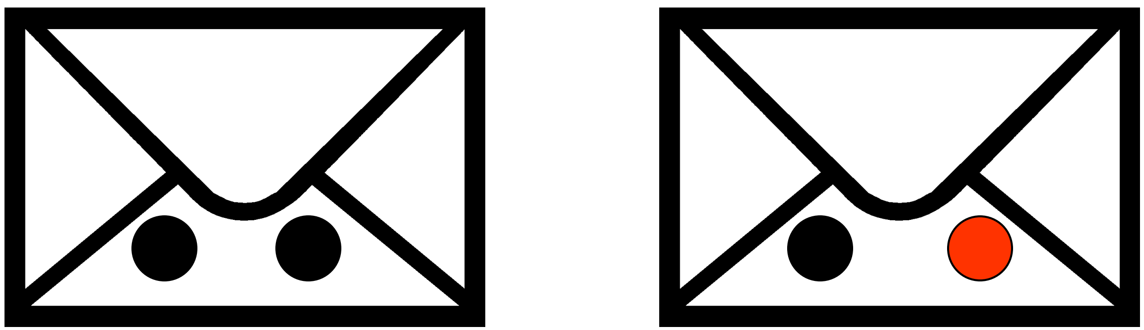
$= (1 - 0.99)(1 - 0.01) \,/\, P(TP) = 0.0099/P(TP)$

$0.0099/P(TP) + 0.0099/P(TP) = 1$, so $P(TP) = 0.0198$

So, $P(S \mid TP) = 0.0099 \,/\, 0.0198 = 0.5$

Inference with Bayes's Rule

- In a bag there are two envelopes
  - one has a red ball (worth $100) and a black ball
  - one has two black balls. Black balls are worth nothing



- You randomly grab an envelope, and randomly take out one ball – it's **black**
- At this point you're given the option to switch envelopes. Should you switch or not?

Similar to the "Monty Hall Problem"