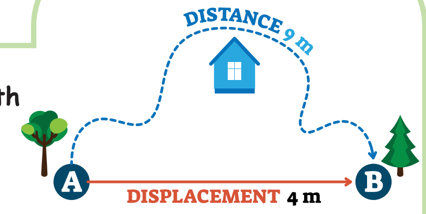
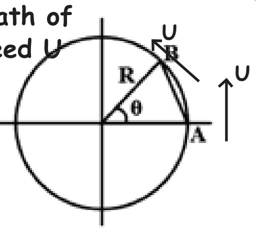
MOTION ALONG A STRAIGHT LINE

- Distance = Length of actual path
- Displacement = Length of shortest path
- Distance $\geq$ |displacement|



A particle moves from A to B in a circular path of radius R covering an angle θ with uniform speed U



- Distance $= \overset{\frown}{AB} = R\theta$
- Displacement $= AB = 2R\sin\left(\frac{\theta}{2}\right)$
- Ratio of Displacement to Distance $= \dfrac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}}$
- Time $t = \dfrac{R\theta}{U}$
- Average Velocity $= \dfrac{2U\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}}$
- Average Acceleration $= \dfrac{U^2\sin\left(\frac{\theta}{2}\right)}{R\frac{\theta}{2}}$

- **For uniform motion**

Displacement = velocity × time

Average speed = |average velocity| = |instantaneous velocity|

- **Time average speed**

$$v_{av} = \frac{\text{Total distance covered}}{\text{Totaltime elapsed}} = \frac{S_1 + S_2 + S_3 + \ldots + S_n}{t_1 + t_2 + t_3 + \ldots + t_n} = \frac{v_1 t_1 + v_2 t_2 + v_3 t_3 + \ldots}{t_1 + t_2 + t_3 + \ldots}$$

If $t_1 = t_2 = t_3 = \ldots = t_n$ then

$$v_{av} = \frac{v_1 + v_2 + v_3 + \ldots + v_n}{n}$$

for v_1 & v_2, $V_{avg} = \dfrac{v_1 + v_2}{2}$ (Arithmetic mean of speeds)

- **Distance average speed**

$$v_{av} = \frac{\text{Total distance covered}}{\text{Totaltime elapsed}} = \frac{S_1 + S_2 + S_3 + \ldots + S_n}{t_1 + t_2 + t_3 + \ldots + t_n} = \frac{S_1 + S_2 + S_3 + \ldots + S_n}{\frac{S_1}{v_1} + \frac{S_2}{v_2} + \frac{S_3}{v_3} + \ldots + \frac{S_n}{v_n}}$$

If $S_1 = S_2 = S_3 = \ldots \ldots S_n$ then

$$v_{av} = \frac{n}{\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots + \frac{1}{v_n}}$$

for v_1 & v_2, $V_{avg} = \dfrac{2V_1V_2}{V_1 + V_2}$ (Harmonic mean of speeds)

Instantaneous Velocity $v = \dfrac{dx}{dt}$ $\quad \Delta x = \int v\,dt$

Instantaneous Acceleration $a = \dfrac{dv}{dt}$ $\quad \Delta v = \int a\,dt$

Case 1	Case 2	Case 3
$v = f(t)$ or $x = f(t)$	$V = f(x)$	$t = f(x)$ then
$a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$	$a = V\dfrac{dV}{dx}$	$a = -$(double diff. of t w.r.t. x) $\times V^3$

$$\text{Displacement} \underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftharpoons}} \text{Velocity} \underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftharpoons}} \text{Acceleration}$$

Motion with constant acceleration: Equations of motion

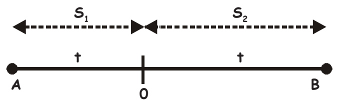
(i) $\boxed{v = u + at}$

(ii) $\boxed{S = ut + \frac{1}{2}at^2}$

- A Person travels from A to B covers unequal distances in equal interval of time with constant acceleration a then

initial velocity $U = \dfrac{3S_1 - S_2}{2t}$

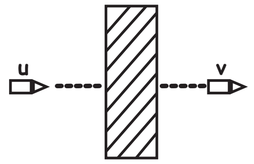
Acceleration $a = \dfrac{S_2 - S_1}{t^2}$



(iii) $\boxed{v^2 = u^2 + 2a.s}$

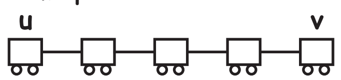
- The number of planks required to stop the bullet

$N = \dfrac{u^2}{u^2 - v^2}$

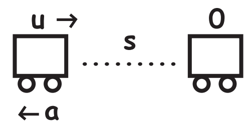


- The two ends of a train moving with constant acceleration pass a certain point with velocities u and v. The velocity with which the middle point of the train passes the same point is

$V_{Mid} = \sqrt{\dfrac{u^2 + v^2}{2}}$



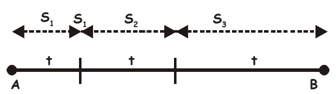
- Calculation of stopping distance $S = \dfrac{u^2}{2a}$



(iv) $\boxed{S_n = u + \frac{a}{2}(2n - 1)}$

- Ratio of distance travelled in equal interval of time in a uniformly accelerated motion from rest

$S_1 : S_2 : S_3 = 1 : 3 : 5$



- for uniform accelerated motion $V_{avg} = \dfrac{u + v}{2}$

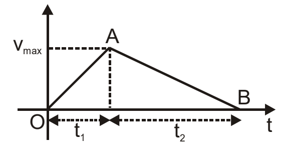
Different Cases	v-t graph	s-t graph
1. Uniform motion	v = constant	s = vt
2. Uniformly accelerated motion with u = 0 at t = 0	v = at	$s = \frac{1}{2}at^2$
3. Uniformly accelerated with u ≠ 0 at t = 0 & s = 0 at t = 0	v = u + at	$s = ut + \frac{1}{2}at^2$
4. Uniformly accelerated motion with u ≠ 0 and s = s_0 at t = 0	v = u + at	$s = s_0 + ut + \frac{1}{2}at^2$
5. Uniformly retarded motion till velocity becomes zero	v = u − at	$s = ut - \frac{1}{2}at^2$
6. Uniformly retarded then accelerated in opposite direction	v = u − at	$s = ut - \frac{1}{2}at^2$

Important points about graphical analysis of motion

- Instantaneous velocity is the slope of position-time curve $\left[v = \dfrac{dx}{dt}\right]$
- Area of v-t curve gives displacement. $\left[\Delta x = \int v\,dt\right]$
- Slope of velocity-time curve = instantaneous acceleration $\left[a = \dfrac{dv}{dt}\right]$
- Area of a-t curve gives change in velocity. $\left[\Delta v = \int a\,dt\right]$

A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is t, then

$V_{max} = \dfrac{\alpha\beta}{\alpha + \beta}t$ $\qquad$ Total Distance $= \dfrac{1}{2}\left(\dfrac{\alpha\beta}{\alpha + \beta}\right)t^2$



MOTION UNDER GRAVITY

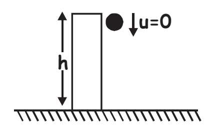
Sign Convention

(i) initial velocity
- +ve = upward motion
- −ve = downward motion

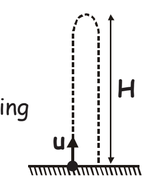
(ii) Acceleration
- Always −ve

(iii) Displacement
- +ve = final position is above initial position
- −ve = final position is below initial position
- Zero = final position & initial position are at same level

- Object is dropped from top of a tower
  - (i) Ratio of displacement in equal interval of time $S_1 : S_2 : S_3 \ldots = 1 : 3 : 5 \ldots$
  - (ii) Ratio of time of covering equal distance $t_1 : (t_2 - t_1) : (t_3 - t_2) : \ldots : (t_n - t_{n-1}) = 1 : (\sqrt{2} - \sqrt{1}) : (\sqrt{3} - \sqrt{2}) : \ldots : (\sqrt{n} - \sqrt{n-1})$
  - (iii) Ratio of total distance covered at the end of time $t : 2t : 3t : \ldots = 1^2 : 2^2 : 3^2 \ldots$



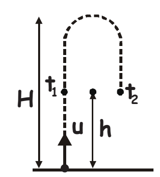
- If a body is thrown vertically up with a velocity u in the uniform gravitational field (neglecting air resistance) then
  - (i) Maximum height attained $H = \dfrac{u^2}{2g}$
  - (ii) Time of ascent = time of descent $\dfrac{u}{g}$
  - (iii) Total time of flight $= \dfrac{2u}{g}$
  - (iv) Velocity of fall at the point of projection = u (downwards)



- At any point on its path the body will have same speed for upward journey and downward journey. If a body thrown upwards crosses a point in time t_1 & t_2 respectively then

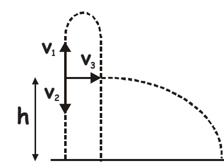
height of point $h = \frac{1}{2}gt_1t_2$ $\qquad$ Maximum height $H = \dfrac{1}{8}g(t_1 + t_2)^2$

Time of flight $= t_1 + t_2 = \dfrac{2u}{g}$



- A body is thrown upward, downward & horizontally with same speed takes time t_1, t_2 & t_3 respectively to reach the ground then

$t_3 = \sqrt{t_1t_2}$ & height from where the particle was throw is $h = \frac{1}{2}gt_1t_2$



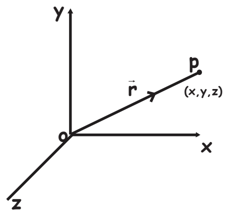
MOTION IN A PLANE

VECTORS

Position Vector

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

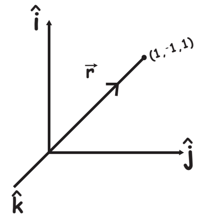
$|\vec{r}| = \sqrt{x^2+y^2+z^2}$



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}|$ or $r = \sqrt{x^2+y^2+z^2}$

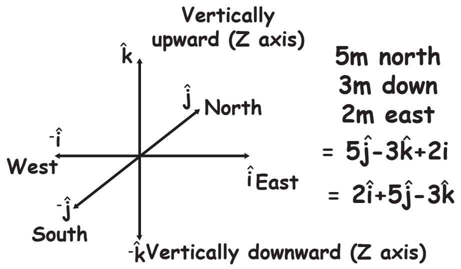
unit vector $= \hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$



$\vec{r} = \hat{i} - \hat{j} + \hat{k}$

$|\vec{r}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

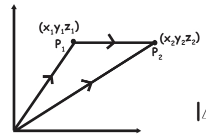
$\hat{r} = \dfrac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$



5m north
3m down
2m east
$= 5\hat{j} - 3\hat{k} + 2\hat{i}$
$= 2\hat{i} + 5\hat{j} - 3\hat{k}$

Displacement Vector

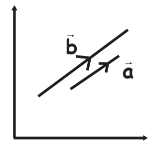
Particle displaces from position P_1 to position P_2



$\Delta\vec{r} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$

$|\Delta\vec{r}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Parallel Vectors



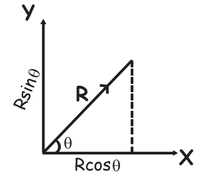
$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} = m\vec{b}$

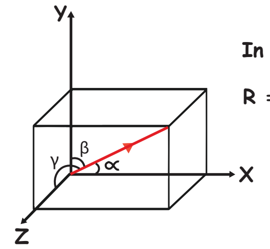
$\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3} = m$

Components of Vector



In two dimentions

$\vec{R} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}$



In three dimentions

$R = R\cos\alpha\,\hat{i} + R\cos\beta\,\hat{j} + R\cos\gamma\,\hat{k}$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

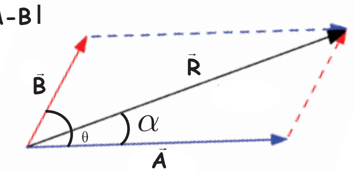
$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Addition Of Vectors

$R = \sqrt{A^2+B^2+2AB\cos\theta}$

$R_{max} = A+B$ $\quad R_{min} = |A-B|$

$\tan\alpha = \dfrac{B\sin\theta}{A+B\cos\theta}$



Vector product

$\vec{C} = \vec{A}\times\vec{B}$ $\quad |\vec{A}\times\vec{B}| = AB\sin\theta$

$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}[A_yB_z - B_yA_z] - \hat{j}[A_xB_z - A_zB_x] + \hat{k}[A_xB_y - A_yB_x]$

Dot product

$x = \vec{A}.\vec{B} = AB\cos\theta$

PROJECTILE MOTION

Projectile motion

Horizontal component $= U\cos\theta$

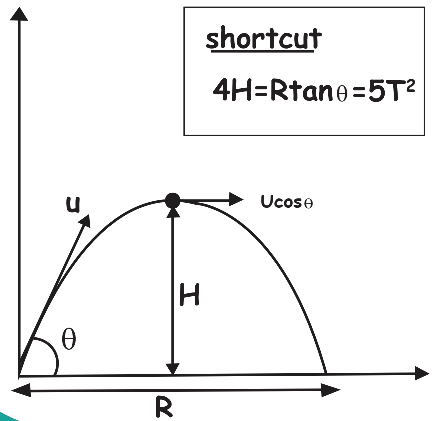
Vertical component $= U\sin\theta$

$H = \dfrac{U^2\sin^2\theta}{2g} = \dfrac{(U\sin\theta)^2}{2g} = \dfrac{U_y^{\,2}}{2g}$

$R = \dfrac{U^2\sin 2\theta}{g} = \dfrac{2U\sin\theta\,U\cos\theta}{g} = \dfrac{2U_xU_y}{g}$

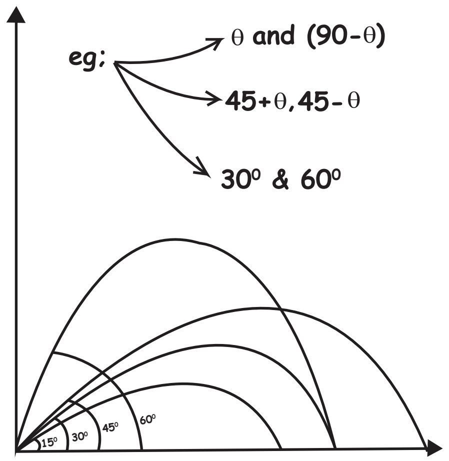
shortcut

$4H = R\tan\theta = 5T^2$

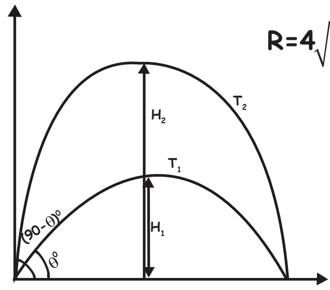


Same range for θ and (90−θ)

eg; θ and (90−θ)
45+θ, 45−θ
30° & 60°



$R = 4\sqrt{H_1.H_2}$



- $\dfrac{T_1}{T_2} = \tan\theta$
- $\dfrac{H_1}{H_2} = \tan^2\theta$
- $T_1 \times T_2 = \dfrac{2R}{g}$
- $H_1 \times H_2 = \dfrac{R^2}{16}$

$T_1^{\,2} + T_2^{\,2} = \dfrac{4R_{max}}{g}$ $\qquad H_1 + H_2 = \dfrac{R_{max}}{2}$

Maximum range

For $\theta = 45°$

$R_{max} = \dfrac{U^2}{g}$

From the relation,

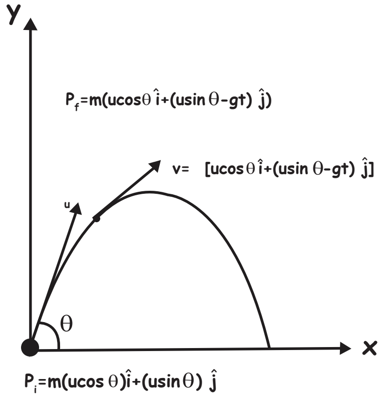
$4H = R\tan\theta = 5T^2$

$4H = R_{max}\tan 45 \Rightarrow H = \dfrac{R_{max}}{4}$

$4H = R_{max} \quad \Rightarrow H = \dfrac{u^2}{4g}$

KE at maximum height $= K\cos^2\theta$

Momentum



$\Delta p = -mgt\,\hat{j}$

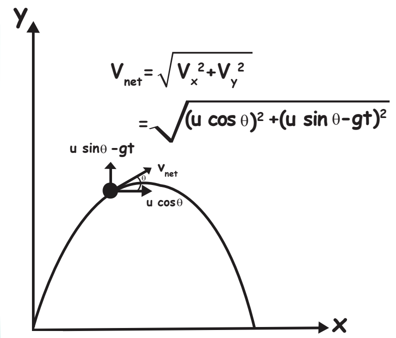
$= -mg \times \dfrac{u\sin\theta}{g}\,\hat{j}$

$= -mu\sin\theta\,\hat{j}$

Equation of Velocity

$V_{net} = \sqrt{V_x^{\,2} + V_y^{\,2}}$

$= \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$



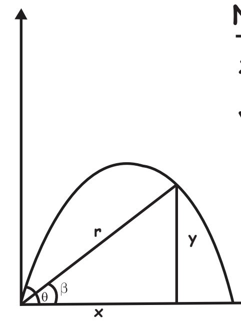
Net displacement

$x = U\cos\theta\, t$

$y = U\sin\theta\, t - \dfrac{1}{2}gt^2$

$r = \sqrt{x^2+y^2}$

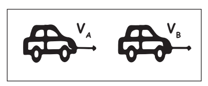
$\beta = \tan^{-1}(y/x)$



RELATIVE MOTION

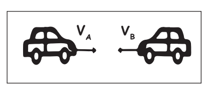
Relative Motion

1) Velocity of A with respect to B $V_{AB} = V_A - V_B$



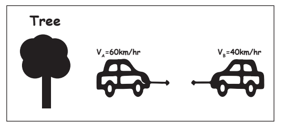
2) $V_{A/B} = V_A - V_B$

$= V_A - (-V_B) = V_A + V_B$



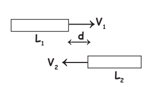
3) $V_{A/Tree} = V_A - V_{Tree} = 60 - 0 = 60$

$V_{B/Tree} = V_B - V_{Tree} = {}^{-}40$

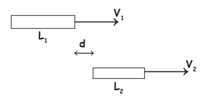


Relative Motion in one dimension (overtaking & chasing)

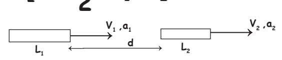
1) $t = \dfrac{d+L_1+L_2}{V_1+V_2}$



2) $t = \dfrac{d+L_1+L_2}{V_1-V_2}$



3) $d + L_1 + L_2 = (u_1-u_2)t + \dfrac{1}{2}(a_1-a_2)t^2$



Minimum separation to avoid collision

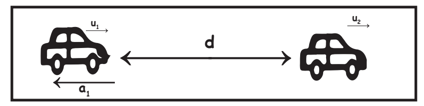
$u_{rel} = u_1 - u_2$ $\qquad a_{rel} = -a_1$

$0 = u^2_{rel} + 2a_{rel}s_{rel}$

$0 = (u_1-u_2)^2 - 2a_1 s$ $\qquad S = \dfrac{(u_1-u_2)^2}{2a_1}$

to avoid collision,
$s \leq d$

$\Rightarrow d \geq \dfrac{(u_1-u_2)^2}{2a_1}$



a_1 = retardation of car 1

RELATIVE MOTION

Man-river problem

1) $\vec{V}_{MR}$ or $\vec{V}_{M/Still\ water}$ = velocity due to effort of man, OR velocity of man in still water
2) $\vec{V}_R$ = velocity of River
3) $\vec{V}_m$ = Resultant velocity of man with respect to ground

1) Upstream

$V_{M/R}$, V_R

$V_M = V_{M/R} - V_R$

2) Down stream

$V_{M/R}$, V_R

$V_M = V_{M/R} + V_R$

Swimming across the river

$V_{MR}\sin\theta$, V_{MR}, V_m, θ, $V_{MR}\cos\theta$, V_R

1) $V_\perp = V_{MR}\sin\theta$
2) $V_\parallel = V_{MR}\cos\theta + V_R$

$t_{cross} = \frac{d}{V_{MR}\cos\theta} = \frac{d}{V_\perp}$

$X_{drift} = (V_{MR}\cos\theta + V_R) \times t$

$= V_{MR}\cos\theta \times t \quad + \quad V_R t$

Due to effort of man — Additional due to push of river

Shortest time	Shortest Path (No drift)
V_{MR}, V_m, V_R	V_{MR}, V_m, $V_{MR}\cos\theta$, θ, $V_{MR}\sin\theta$, V_R
$\theta = 90$, $\sin\theta = 1$	Horizontal component of V_m becomes 0
• $t = \frac{d}{V_{MR}\sin\theta}$	$\Rightarrow V_{MR}\sin\theta = V_R$ $\quad V_{MR}\cos\theta = V_m$
• $t_{min} = \frac{d}{V_{MR}}$	Condition for no drifting
• $X_{drift} = V_R \times t$	$\Rightarrow \sin\theta = \frac{V_R}{V_{MR}}$
• $V_m = \sqrt{(V_{MR})^2 + (V_R)^2}$	$\Rightarrow t_{cross} = \frac{d}{V_m}$
	$\Rightarrow V_m = \sqrt{V_{MR}^2 - V_R^2}$
	$\Rightarrow$ Drift = 0

Escalator

$t_3 = \frac{d}{V_E + V_{M/E}} = \frac{d}{\frac{d}{t_2} + \frac{d}{t_1}} = \frac{t_1 t_2}{t_1 + t_2}$

t_1 = Time taken by a man to move distance d on a stationary escalator

t_2 = Time taken by a stationary man to move distance d along with moving escalator

t_3 = Time taken by a man to move distance d while walking along a moving escalator

V_E = Velocity of escalator

$V_{M/E}$ = Velocity of man w.r.t escalator

MAN RAIN PROBLEM

Man-rain problem

V_A, V_B, A, B

In order to find the relative velocity of B with respect to A we have to reverse the direction of vector A and add it with vector B

w.r.t. A

V_A, V_B

$V_{B/A} = V_B - V_A$, $\quad V_B$ w.r.t A

$V_{A/B} = V_B - V_A$, $\quad V_A$ w.r.t B

20 m/s, 10 m/s, A, B

$V_{B/A}$, 20 m/s, 10 m/s

Terms

$V_R \Rightarrow$ Velocity of rain w.r.t stationary man

$V_m \Rightarrow$ Velocity of man

$V_{R/m} \Rightarrow$ Velocity of Rain w.r.t man

Method

V_m, $V_R\sin\theta$, θ, V_R, $V_R\cos\theta$, V_m

$V_{R/m} = \sqrt{(V_R\sin\theta - V_m)^2 + (V_R\cos\theta)^2}$

Case 1

$V_R\sin\theta > V_m$

$V_R\sin\theta - V_m$, $\propto$, $V_{R/m}$, $V_R\cos\theta$

$V_{R/m} = \sqrt{(V_R\cos\theta)^2 + (V_R\sin\theta - V_m)^2}$

$\tan\propto = \frac{V_R\sin\theta - V_m}{V_R\cos\theta}$

$\propto = \tan^{-1}\left(\frac{V_R\sin\theta - V_m}{V_R\cos\theta}\right)$

Case 2

$V_m > V_R\sin\theta$

$V_m - V_R\sin\theta$, $\propto$, $V_{R/m}$, $V_R\cos\theta$

$V_{R/m} = \sqrt{(V_R\cos\theta)^2 + (V_m - V_R\sin\theta)^2}$

$\propto = \tan^{-1}\left(\frac{V_m - V_R\sin\theta}{V_R\cos\theta}\right)$

Case 3

$V_m = V_R\sin\theta$

V_m, $V_R\sin\theta$, $V_{R/m}$, V_R, $V_R\cos\theta$

$V_m = V_R\sin\theta$

$V_{R/m} = V_R\sin\theta$

$\propto = 0$

minimum distance

A, v, u, d, B, A, $V_{A/B}$, d_{min}, In frame of B, B

$d_{min} = \frac{d \times V}{\sqrt{V^2 + U^2}}$

Time taken to reach minimum distance $= \frac{d \times U}{V^2 + U^2}$

CIRCULAR MOTION

$1° = \frac{\pi}{180}$ rad

Angular velocity

$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T}$ (in uniform circular motion), $V = \omega r$

$\vec{V} = \vec{\omega} \times \vec{r}$

Angular acceleration

$\alpha = \frac{d\omega}{dt}$ $\quad a_t = r\alpha$ $\quad \vec{a}_t = \vec{\alpha} \times \vec{r}$

Equation of angular motion

1) Constant angular velocity : ω = constant
2) Constant angular acceleration

$\Rightarrow \omega = \omega_o + \alpha t$

$\Rightarrow \Delta\theta = \omega_o t + 1/2\,\alpha t^2$

$\Rightarrow \omega^2 = \omega_o^2 + 2\alpha(\Delta\theta)$

Centripetal acceleration

Directed towards centre

Not a constant vector

v, a_c, v, v, v

· a_c is due to change in the direction of velocity

· $a_c = \frac{v^2}{R} = a_c = r\omega^2$

· $\vec{a}_c \perp \vec{v}$

· $\vec{F}_c \perp \vec{s}$

Tangential acceleration

In direction of tangent

$a_t = \frac{dv}{dt} = r\alpha \rightarrow \alpha = \frac{d\omega}{dt}$

Tangential Force

$F_t = ma_t$

a_t is due to change in magnitude of velocity

Resultant acceleration

$a_r = \sqrt{a_c^2 + a_t^2}$

Velocity is tangent to the circle

Circular Motion

Uniform Circular Motion
- Speed Constant
- Direction of velocity changes

- ω = Constant
- $a_c = \frac{v^2}{r} = r\omega^2$
- $a_t = 0$

Non-uniform Circular Motion

- Speed not Constant
- Velocity changes in direction and magnitude
- a_c = Centripetal acceleration
- a_t = tangential acceleration $= \frac{dv}{dt}$
- $\alpha = \frac{d\omega}{dt}$
- ω = Changes $\longrightarrow \alpha$ angular acceleration

Horizontal circular motion

$T = \frac{mv^2}{r} = mr\omega^2 = ml\omega^2$

ω, T_1, m_1, T_2, m_2, T_3, m_3, l_1, l_2, l_3

$T_1 = m_1 l_1\omega^2 + m_2(l_1 + l_2)\omega^2 + m_3(l_1 + l_2 + l_3)\omega^2$

$T_2 = m_2(l_1 + l_2)\omega^2 + m_3(l_1 + l_2 + l_3)\omega^2$

$T_3 = m_3(l_1 + l_2 + l_3)\omega^2$

Flat circular track

R

$V_{max} = \sqrt{\mu_s Rg}$

Banking of Road

θ

$V_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}}$

$V_{opt} = \sqrt{rg\tan\theta}$ (for smooth road)

$V_{min} = \sqrt{\frac{rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$

$\sqrt{gR}$, $\sqrt{3gR}$, $\sqrt{5gR}$

At Bottom

a) $T_{max} = \frac{mv^2}{r} + mg$

b) min velocity at bottom to complete circle $= \sqrt{5gR}$

At Top

a) $T_{min} = \frac{mv^2}{r} - mg$

b) min velocity at top to complete the circle $= \sqrt{gR}$

MOTION IN A PLANE - 02