Proof of Slides 93-96

Properties for $\mathbb{E}[X]$

1. $\mathbb{E}[A-B]=\mathbb{E}[A]-\mathbb{E}[B]$
2. $\mathbb{E}[X]=\Pr(B)\cdot\mathbb{E}[X\mid B]+\Pr(\text{not }B)\,\mathbb{E}[X\mid\text{not }B]$

Proof of Property (2):

- First, we know $\Pr(A)=\Pr(B)\Pr(A\mid B)+\Pr(\text{not }B)\Pr(A\mid\text{not }B)$, because:

$$\begin{aligned}\Pr(A)&=\Pr(A\cap B)+\Pr(A\cap\text{not }B)\\&=\Pr(B)\cdot\frac{\Pr(A\cap B)}{\Pr(B)}+\Pr(\text{not }B)\frac{\Pr(A\cap\text{not }B)}{\Pr(\text{not }B)}\\&=\Pr(B)\cdot PR(A\mid B)+\Pr(\text{not }B)\Pr(A\mid\text{not}B)\end{aligned}$$

- Applying this to the definition of $\mathbb{E}[X]$:

$$\begin{aligned}\mathbb{E}[X]&=\sum x\Pr(X=x)\\&=\sum x\{\Pr(B)\Pr(X=x\mid B)+\Pr(\text{not }B)\Pr(X=x\mid\text{not }B)\}\\&=\sum x\Pr(B)\Pr(X=x\mid B)+\sum x\Pr(\text{not }B)\Pr(X=x\mid\text{not }B)\\&=\Pr(B)\left\{\sum x\Pr(X=x\mid B)\right\}+\Pr(\text{not }B)\left\{\sum x\Pr(X=x\mid\text{not }B)\right\}\\&=\Pr(B)\,\mathbb{E}[X\mid B]+\Pr(\text{not }B)\,\mathbb{E}[X\mid\text{not }B]\end{aligned}$$

- Q.E.D.

Now, we can expand the expression for ATE using Properties 1 and 2:

$$\begin{aligned}\text{ATE}&=\mathbb{E}[Y_i(1)-Y_i(0)]\\&=\mathbb{E}[Y_i(1)]-\mathbb{E}[Y_i(0)]\\&=\Pr(D_i=1)\,\mathbb{E}[Y_i(1)\mid D_i=1]+\Pr(D_i=0)\,\mathbb{E}[Y_i(1)\mid D_i=0]\\&\quad-\{\Pr(D_i=1)\,\mathbb{E}[Y_i(0)\mid D_i=1]+\Pr(D_i=0)\,\mathbb{E}[Y_i(0)\mid D_i=0]\}\end{aligned}$$

For our convenience, let's define $\Pr(D_i=1)=p$, such that $\Pr(D_i=0)=(1-p)$, we arrive at the expression on slide 94:

$$\begin{aligned}\text{ATE}=&p\,\color{red}{\mathbb{E}[Y_i(1)\mid D_i=1]}+(1-p)\,\mathbb{E}[Y_i(1)\mid D_i=0]\\&-p\,\mathbb{E}[Y(0)\mid D_i=1]-(1-p)\,\color{red}{\mathbb{E}[Y_i(0)\mid D_i=0]}\end{aligned}$$

Recall that out of the four terms in the previous equation, only the first and fourth terms (in red) are observable (and are indeed the constituent terms of SDO). Our goal here is to try to express the SDO as a combination of ATE and the other two unobservable terms

For simplicity, I will reexpress the conditional probabilties as

$$\begin{cases}\mathbb{E}[Y_i(1)\mid D_i=1]&=e_{11}\\\mathbb{E}[Y_i(1)\mid D_i=0]&=e_{00}\\\mathbb{E}[Y_i(0)\mid D_i=1]&=e_{01}\\\mathbb{E}[Y_i(0)\mid D_i=0]&=e_{00}\end{cases}$$

Such that

$$\begin{cases}\text{SDO}=e_{11}-e_{00}\\\text{ATE}=pe_{11}+(1-p)e_{10}-pe_{01}-(1-p)e_{00}\end{cases}$$

Now, we can try rearranging the terms:

$$
\begin{aligned}
\mathrm{ATE} &= pe_{11} + (1-p)e_{10} - pe_{01} - (1-p)e_{00} \\
e_{11} - e_{00} &= (1+p)e_{11} + (1-p)e_{10} - pe_{01} - pe_{00} - \mathrm{ATE}
\end{aligned}
$$