Comprehensive Study of Pendulum Systems and ODE Solvers

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Abstract

This report examines two types of pendulum systems: a simple pendulum and a double pendulum. We compare different numerical methods for solving the equations that describe pendulum motion and discuss their applications in physics simulations.

1 Introduction

We study two different types of pendulum systems in this report:

1. The Simple Pendulum: A weight suspended from a pivot point, moving in a circular arc.
2. The Double Pendulum: A pendulum with another pendulum attached to its end, capable of exhibiting chaotic motion.

Both systems are described by ordinary differential equations (ODEs), which we solve using various numerical methods. Our goal is to compare these methods in terms of accuracy, stability, and energy conservation.

2 Theoretical Background

2.1 Simple Pendulum

The motion of a simple pendulum is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

where θ is the angle from the vertical, g is the gravitational acceleration, and L is the length of the pendulum. Here, $\sin\theta = \theta$. This is called small angle approximation

2.2 Numerical Methods

2.2.1 Euler Method

The Euler method is the simplest numerical method for solving ODEs:

$$\theta_{n+1} = \theta_n + \omega_n \Delta t \tag{2}$$

$$\omega_{n+1} = \omega_n - \frac{g}{L} \sin\theta_n \Delta t \tag{3}$$

2.2.2 Fourth-order Runge-Kutta Method (RK4)

A more accurate method using a weighted average of four increments:

$$k_1 = f(t_n, y_n) \tag{4}$$

$$k_2 = f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1) \tag{5}$$

$$k_3 = f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_2) \tag{6}$$

$$k_4 = f(t_n + \Delta t, y_n + \Delta t k_3) \tag{7}$$

$$y_{n+1} = y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{8}$$

2.2.3 odeint

The odeint function from SciPy, which uses adaptive step size control for accurate and efficient ODE solving.

2.3 Results and Discussion

2.3.1 Angle vs. Time

2.3.2 Total Energy vs. Time

2.3.3 Error Analysis

The Euler method, while simple to implement, shows significant errors in both the angle and energy calculations. The error in the Euler method increases at every time step due to its first-order approximation. This accumulation of error leads to the diverging behavior seen in Figure 3. The method fails to conserve energy, as evident from the increasing total energy over time in Figure 2. The RK4 shows increasing amount of accuracy as compared to the Euler method due to a higher order approximation.

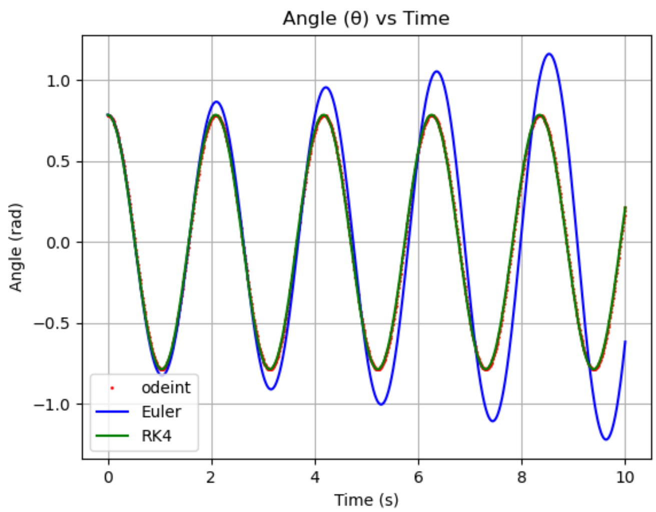

Figure 1: Graph showing the variation of angle θ as a function of time

3 Double Pendulum Study

3.1 System Description

The double pendulum consists of one pendulum attached to another. This system is known for its chaotic behavior, making it an excellent test case for numerical methods in more complex scenarios.

3.2 Equations of Motion

The double pendulum system is described by the following set of coupled, non-linear differential equations:

$$(m_1 + m_2)l_1\ddot{\theta_1} + m_2l_2\ddot{\theta_2}\cos(\theta_1 - \theta_2) + m_2l_2\dot{\theta_2}^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)g\sin\theta_1 = 0 \tag{9}$$

$$m_2l_2\ddot{\theta_2} + m_2l_1\ddot{\theta_1}\cos(\theta_1 - \theta_2) - m_2l_1\dot{\theta_1}^2\sin(\theta_1 - \theta_2) + m_2g\sin\theta_2 = 0 \tag{10}$$

where m_1, m_2 are the masses, l_1, l_2 are the lengths, and θ_1, θ_2 are the angles of the first and second pendulum respectively.

3.3 Numerical Method: Symplectic Euler

For the double pendulum simulation, we use the Symplectic Euler method. The animation can be found here.

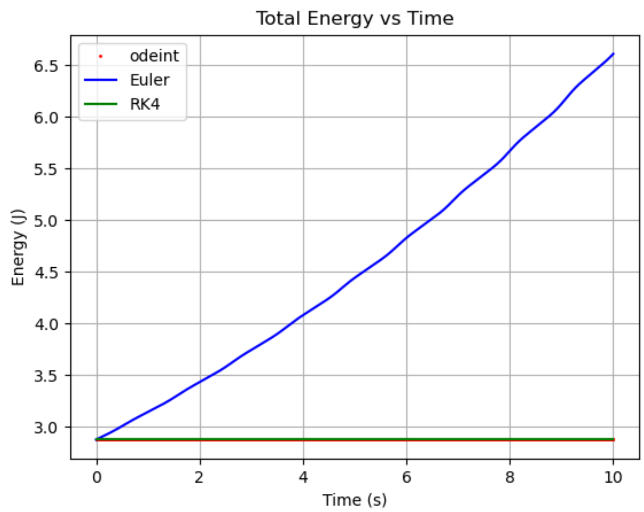

Figure 2: Graph showing the variation of total energy as a function of time

4 Comparison and Discussion

4.1 Simple vs. Double Pendulum

The double pendulum is a generalization of the simple pendulum. When the mass and length of the second pendulum approach zero, the system behaves like a simple pendulum. The animation demonstrating this transition can be found here.

5 Conclusion

This comprehensive study of pendulum systems has demonstrated the application of various numerical methods to solve ODEs in physics.

The exploration of the double pendulum system highlighted the challenges involved in simulating more complex, chaotic systems.

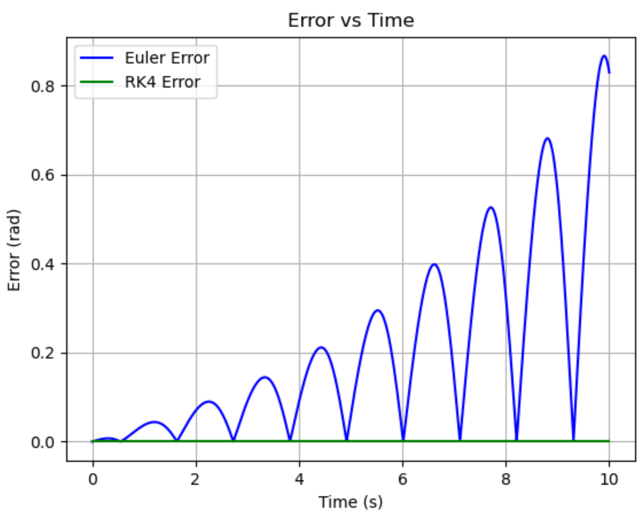

Figure 3: Error comparison between Euler and RK4 methods