BOOLEAN ALGEBRA OF THREE-DIMENSIONAL CONTINUA WITH ARBITRARILY COMPLEX TOPOLOGY

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1. Introduction.

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2. Analysis. We start from a few definitions.

Definition 2.1 (generalized radii). *A* generalized radius *of a point $p \in \mathbb{R}^3$ in an open ball $\mathcal{N}(p) \subset \mathbb{R}^3$ centered at p is a simple curve from p to some point in $\partial\mathcal{N}(p)$.*

Definition 2.2 (generalized disks). *Denote by $\mathbb{D}$ the closed unit disk in $\mathbb{R}^2$ and $\overline{\mathcal{N}(p)}$ the closure of an open ball centered at $p \in \mathbb{R}^3$. A* generalized disk *D centered at a point p is the homeomorphic image of $\mathbb{D}$ under a continuous map $f : \mathbb{D} \to \overline{\mathcal{N}(p)}$ such that $p = f(\mathbf{0})$.*

Definition 2.3 (generalized sectors). *Let D be a generalized disk centered at p and suppose r_1, r_2 are two distinct generalized radii in D. Then r_1 and r_2 cut D into two connected components whose boundary relative to D are both $r_1 \cup r_2$. We call the closure of these two connected components* generalized sectors *and r_1, r_2 the generalized radii of these sectors. If r is a radius of a generalized sector F, F is called a generalized sector* on *r.*

For each generalized radius r, the number of the sectors on r is clearly even.

Lemma 2.4. *Let $p \in \partial Y$ be a boundary point of a Yin set $Y \subset \mathbb{R}^3$. For any sufficiently small $\mathcal{N}(p)$, we have*

(a) $\partial Y \cap \mathcal{N}(p)$ is the union of finitely many generalized disks;

(b) Pairwise intersection of the generalized disks in (a) is either p itself or the union of finitely many generalized radii pairwise intersecting only at p;

(c) $\mathcal{N}(p) - \partial Y$ consists of disjoint regular open sets; for two such sets sharing a common sector as part of their boundary, one is a subset of Y while the other that of $Y^\perp$.

Proof. Since Y is semianalytic, by definition $\partial Y \cap \mathcal{N}(p)$ is defined by a finite number of analytic functions $g_i : \mathbb{R}^3 \to \mathbb{R}$. The fact of $Y \cap \mathcal{N}(p)$ being regular open implies that $\partial Y \cap \mathcal{N}(p)$ contains neither 0-dimensional features such as isolated points nor 1-dimensional features such as a curve segment. By the implicit function theorem, each $g_i(x, y, z) = 0$ defines a surface, which is the graph of some height function $\chi = H_i(\xi, \eta)$. Hence $\partial Y \cap \mathcal{N}(p)$ can only contain the heomorphic images of the open disk, which, by definition, are generalized disks. Therefore (a) holds.

Because each g_i is analytic, so is H_i. Then H_i can be replaced by its Taylor series in terms of the local coordinates ξ and η. For sufficiently small r, H_i can be approximated to arbitrary degree of accuracy by a bivariate polynomial. Within the local neighborhood of p, the graphs of two such bivariate polynomials intersect either solely at p or at a curve segment. Therefore (b) holds.

By (a) and (b), we get a finite number of disjoint regular open sets after deleting from $\mathcal{N}(p)$ a finite number of surfaces. For two such regular open sets sharing a common sector, they cannot simultaneouly be contained in Y because this would contradict Y being regular open. Similarly, they cannot simultaneouly be contained in $Y^\perp$ either. Hence (c) holds. □

Definition 2.5. *A* good neighborhood *at $p \in \partial Y$ is a neighborhood $\mathcal{N}(p)$ satisfying the conclusion of Lemma 2.4.*

Whenever there is a generalized sector F with one radius r, i.e. F is itself a generalized disk, we pick another generalized radius r' in F intersecting r only at p. Then r and r' divide F into two generalized sectors. When a good neighborhood $\mathcal{N}(p)$ is given, we will use "disks" to mean generalized disks in $\partial Y \cap \mathcal{N}(p)$ given by (1) of the above lemma; "radius" will be used to mean generalized radius along which disks in $\mathcal{N}(p)$ intersect or those we put in to divide generalized sectors. The word "sector" will be used to mean generalized sector which is a connected component of $\partial Y \cap \mathcal{N}(p)$ minus all the radii in $\mathcal{N}(p)$.

For a good neighborhood at $p \in \partial Y$, we denote by $\mathcal{R}(p)$, $\mathcal{D}(p)$, and $\mathcal{S}(p)$ the collections of all radii, disks, and sectors in $\mathcal{N}(p)$, respectively.

DEFINITION 2.6. *A point $p \in \partial Y$ is* non-singular *if $|\mathcal{D}(p)| = 1$, i.e., the number of disks at p is 1; otherwise p is called* singular. *The boundary point p is* isolated singular *if it is singular and $\mathcal{R}(p) = \emptyset$; it is* non-isolated singular *if it is singular and $\mathcal{R}(p) \neq \emptyset$.*

Note that whether p is singular or non-singular does not depend on the choice of the good neighborhood $\mathcal{N}(p)$.

COROLLARY 2.7. *When p is non-isolated singular any point in $\partial Y \cap \mathcal{N}(p) - \bigcup_{r \in \mathcal{R}(p)} r$ is non-singular and any point in $\bigcup_{r \in \mathcal{R}(p)} r$ is non-isolated singular.*

Proof. This follows directly from Lemma 2.4 and Definition 2.6. □

LEMMA 2.8. *A Yin set $Y \subset \mathbb{R}^3$ satisfies*

(a) ∂Y contains finitely many isolated singular points;

(b) the non-isolated singular points in ∂Y form a compact one-dimensional CW-complex.

Proof. Suppose ∂Y contains infinitely many isolated singular points. Since ∂Y is compact, there exists a point $y \in \partial Y$ such that any neighborhood of y contains infinitely many isolated singular point. Hence none of the neighborhood of y is a good neighborhood, and this contradicts Lemma 2.4. Therefore (a) holds.

Let l be a connected component of the set of all non-isolated singular points and let $p \in l$. Let $\mathcal{N}(p)$ be a good neighborhood of p and $\mathcal{R}(p)$ be the corresponding set of radii. Then by Corollary 2.7 a neighborhood of p in l is homeomorphic to $\bigcup_{r \in \mathcal{R}(p)} r$. So l is locally homeomorphic to a star in a graph. Therefore l is homeomorphic to a graph and hence can be given a one-dimensional CW-complex structure.

To show that the set of non-isolated singular points is closed in ∂Y, we show that any convergent sequence of non-isolated singular points converges to a non-isolated singular point. Suppose the point q to which the sequence converges is not non-isolated singular. Then q is either non-singular or isolated singular. In both cases there would exist a sufficiently small neighborhood of q that contains no points in the sequence; this contradicts the convergence of the sequence. Therefore the set of non-isolated singular points must be compact. □

DEFINITION 2.9. *Let S be a topological space and l_1 and l_2 be subsets of S. Let $f : l_1 \to l_2$ be a map. The* gluing of S along $f(l_1)$, *denoted by S_f, is the quotient of S by identifying a with $f(a)$ for all $a \in l_1$.*

DEFINITION 2.10. *A* folded disk *is the gluing of a generalized disk D along a map $f : r_1 \cup \cdots \cup r_k \to l_1 \cup \cdots \cup l_t$ such that $r_1, \ldots, r_k$ and $l_1, \ldots, l_t$ are generalized radii in D and f maps each r_i homeomorphically to l_j for some j.*

DEFINITION 2.11. *A* good pairing *of generalized sectors on r is a decomposition of these sectors into pairs such that no two pairs (F, F') and (G, G') intersect properly.*

LEMMA 2.12. *Let $\mathcal{N}(p)$ be a good neighborhood of any point $p \in \partial Y$. Then $\partial Y \cap \mathcal{N}(p)$ can be realized as the union of a finite collection of disks and folded disks that admits a good pairing of them.*

Proof. Let $r \in \mathcal{R}(p)$. By the definition of good neighborhood, there are an even number of sectors on r. Note that a good pairing always exists: two sectors form a pair only if they are next to each other. Suppose for each $r \in \mathcal{R}(p)$ we have a good pairing of sectors on r. Then the sectors are naturally arranged into groups that form disks or folded disks: given any sector F_0, the pairing gives a unique sequence of sectors $F_0, F_1, \ldots, F_k, F_{k+1} = F_0$, where $F_i \neq F_j$ unless $\{i, j\} = \{0, k+1\}$ and F_i and F_{i+1} form a pair in the given good pairing of $r = F_i \cap F_{i+1}$. As one goes from

F_0 to F_k, if there exists one radius which is passed twice, then $\bigcup_{0\le i\le k} F_i$ is a folded disk. Otherwise $\bigcup_{0\le i\le k} F_i$ is a disk. By construction, these disks and folded disks intersect only at radius. As a result, by the definition of good pairing, none of them intersect properly. □

Definition 2.13. *A* good disk decomposition *of $\partial Y \cap \mathcal{N}(y)$ is a decomposition of $\partial Y \cap \mathcal{N}(y)$ into disks and folded disks as in Lemma 2.12.*

Lemma 2.14. *Let p and q be non-isolated singular points in ∂Y and let $\mathcal{N}(p)$ and $\mathcal{N}(q)$ be good neighborhoods of p and q, respectively. Suppose a radius r_1 in $\mathcal{N}(p)$ has non-empty intersection with a radius r_2 in $\mathcal{N}(q)$. Then there is a one-one correspondence between the sectors of $\mathcal{N}(p)$ on r_1 and sectors of $\mathcal{N}(q)$ on r_2, where the corresponding sectors intersect in a two-dimensional subset.*

Proof. Let $x \in \mathcal{N}(p)$ be a point in $r_1 \cap r_2$. Consider a good disk decomposition of a good neighborhood $\mathcal{N}(x)$. By making $\mathcal{N}(x)$ sufficiently small, we can assume that $\mathcal{N}(x)$ is contained in both $\mathcal{N}(p)$ and $\mathcal{N}(q)$. Note that $\mathcal{N}(x)$ has only two radii, both of which are subset of $r_1 \cap r_2$ as long as we make $\mathcal{N}(x)$ small enough. Each sector in $\mathcal{N}(x)$ is contained in a sector of $\mathcal{N}(p)$ on r_1 and each sector of $\mathcal{N}(p)$ on r_1 contains exactly one sector of $\mathcal{N}(x)$. Similarly, the same is true for sectors in $\mathcal{N}(x)$ and sectors in $\mathcal{N}(q)$. Therefore, there is a natural one-one correspondence two sectors in $\mathcal{N}(p)$ and $\mathcal{N}(q)$ with corresponding pair being those that contains the same sector of $\mathcal{N}(x)$. □

Lemma 2.15. *Let l be a connected component of the set of non-isolated singular points of ∂Y. Let $\{\mathcal{N}(y_1), \ldots, \mathcal{N}(y_k)\}$ be a finite collection of good neighborhoods that cover l. Then a good disk decomposition of each $\mathcal{N}(y_i)$ can be chosen such that: for any y and z in l, if radius $r_1 \in \mathcal{N}(y_i)$ and radius $r_2 \in \mathcal{N}(y_j)$ intersect, then any sectors F_1 and F_2 on r_1 form a good pair if and only if the corresponding F_1' and F_2' on r_2 form a good pair.*

Proof. For each radius of these good neighborhood, choose any good pairing of the sectors on it. If a radius r_1 in $\mathcal{N}(y_1)$ intersects any radius r_2 in one of these good neighborhood $\mathcal{N}(y_2), \ldots, \mathcal{N}(y_k)$, then the good pairing of sectors on r_1 induces a good pairing of sectors on r_2 via the one-one correspondence between these two sets of sectors. Then we use this induced good pairing for sectors on r_2 instead of the previously randomly chosen one. After repeating this process for all the radii in these good neighborhoods, we have good pairing for sectors on each radius satisfying the conclusion of the lemma. Now by Lemma 2.12, these good pairing of sectors induce good disk decomposition of each of the good neighborhoods satisfying the conclusion of the lemma. □

Theorem 2.16. *∂Y is homeomorphic to the gluing of a collection of closed surfaces alone subsets homeomorphic to one dimensional CW complex.*

Proof. We begin by defining two operations that we can perform on good neighborhoods with good disk decompositions. Let $y \in \partial Y$ and $\mathcal{N}(y)$ be a good neighborhood with a good disk decomposition for $\partial Y \cap \mathcal{N}(y) = \bigcup_i D_i$, where D_i is either a disk or a folded disk.

For each radius r in $\mathcal{N}(y)$, there are $2n$ sectors on r for some positive integer n. There is a good pairing on these set of sectors. The unfolding operation is defined to do the following: Turn r into n radius $r_1, \ldots, r_n$ at y. The $2n$ sectors on r are now attached to these new radii, each of which has a good pair of sectors on it. So the result of an unfolding operation is n "big" sectors (the union of a good pair of sectors) having y as the only common point. Note that this operation can be done in an arbitrarily small neighborhood of r. By "unfolding" one good pair of sectors (pick the outer most one) at a time, the unfolding operation can be done "continuously" and without introducing any new intersections along the process. To be more precise, let $\mathcal{S}$ be the union of the sectors on r, i.e. the subspace of $\partial Y \cap \mathcal{N}(y)$ consisting of all sectors on r and let $\mathcal{S}'$ be the space of n "big" sectors attached at y. Then there exists a continuous map $f : \mathcal{S}' \times I \to \mathcal{N}(y)$ such that $f\mid_{\mathcal{S}'\times\{t\}}$ is a homeomorphism between $\mathcal{S}'$ and its image $f(\mathcal{S} \times \{t\})$ for any $t \in (0, 1]$ and $f(\mathcal{S}' \times \{0\}) = \mathcal{S}$. Also $f\mid_{\mathcal{S}'\times\{0\}}$ maps each good pair of sectors homeomorphically to its image in $\mathcal{S}$. In particular, it maps each r_j homeomorphically to r.

After applying the unfolding operation to each of the radii in $\mathcal{N}(y)$, the resulting space is a finite collection of k disks intersecting only at y for some positive integer k. Note that all the folded disks have been "unfolded" into disks after the unfolding operation. Now we apply the detaching operation, which, by definition, turns the above space into k disjoint disks in $\mathcal{N}(y)$. Similarly as above, the detaching operation can also be done "continuously" and without introducing any new intersections along the process. Note that the detaching operation can be done in an arbitrarily small neighborhood of y. As a result, if we let R be the union of all radii in $\mathcal{N}(y)$, then both of the unfolding operation and the detaching operation can be done within an arbitrarily small neighborhood of R.

Note that after applying the unfolding and detaching operations, there are no singular points in resulting space in $\mathcal{N}(y)$. This is the reason for performing these operations.

Let l be a connected component of the set of singular points of ∂Y. Let $\mathcal{C}$ be a finite collection of points in l such that $\{\mathcal{N}(y) \mid y \in C\}$ covers l, where $\mathcal{N}(y)$ is a good neighborhood of y. By making these good neighborhood small, we can assume that each $y \in C$ is in only one of these neighborhoods. Choose good disk decomposition for these good neighborhoods satisfying the conclusion of Lemma **??**. Now we perform the unfolding and detaching operations on all these good neighborhoods. By Lemma **??**, the good pairings at intersecting radii from different good neighborhoods are compatible. Hence, since any unfolding operation is done with respect to some good pairing of sectors on each radius, the unfolding operations on each of the good neighborhood are compatible. On the other hand, each of the detaching operations is performed in a neighborhood of the center. So these operations do not interfere with each other. By performing all these compatible operations, we "get rid of " the connected component l.

Apply the above process to all connected components of singular points of ∂Y. Let $\tilde{\partial} Y$ be the resulting space. Then each point $y \in \tilde{\partial} Y$ has a neighborhood homeomorphic to an open disk in $\mathbb{R}^2$. Since ∂Y is compact, $\tilde{\partial} Y$ is also compact. Therefore $\tilde{\partial} Y$ is a compact surface without boundary. Note that $\tilde{\partial} Y$ might not be connected. Hence it is homeomorphic to a finite collection of compact closed surface with finite genus.

The unfolding and detaching operation can be undone by gluing operation. The unfolding operation can be undone by gluing along a radius. The detaching operation can be undone by gluing at a point. □