PDE Homework #5

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Problem 1. *Consider the operator $L = \partial_x + i\partial_y$ in $\mathbb{R}^2$, is it hypoelliptic? Explain in details.*

Solution. The Cauchy-Riemann operator $L = \partial_x + i\partial_y$ is hypoelliptic.
Consider L on $\Omega \subset \mathbb{R}^2$, we know from complex analysis that solutions of $Lu = 0$ are homomorphic functions of $z = x + iy$ on Ω. Therefore L is hypoelliptic by definition.

□

Problem 2. *Consider the heat operator $L = \partial_t - \Delta$, is it hypoelliptic? Explain in details.*

Solution. The heat operator $L = \partial_t - \Delta$ is hypoelliptic.
Recall the regularity of solutions of the heat equation:

Theorem (Smoothness)**.** *Suppose $u \in C_1^2(U_T)$ solves the heat equation in U_T. Then*

$$u \in C^\infty(U_T).$$

This regularity assertion is valid even if u attains nonsmooth boundary values on Γ_T.

Hence the heat operator L is hypoelliptic by definition. □

Problem 3. *Prove (2), i.e.,*

$$\partial_\xi^\alpha \hat{K}_1 = \mathcal{O}(\langle \xi \rangle^{-m-|\alpha|}).$$

Proof. Recall that

$$\hat{K}_1 = \frac{1-\chi(\xi)}{P(i\xi)} \in L^\infty \cap C^\infty \subset \mathcal{S}',$$

where

$$P(i\xi) = \sum_{|\alpha| \le m} a_\alpha (i\xi)^\alpha.$$

Lemma. *A derivative of order k of $1/P(\xi)$ is of the form $Q(\xi)/P(\xi)^{k+1}$ with Q of degree no greater than $(m-1)k$.*

Proof of Lemma. We proceed by induction on k.
For $k = 0$, the conclusion clearly holds.
Assume the conclusion to be true for some k, then for some $k+1$, by the induction hypothesis,

$$\exists Q(\xi) \text{ s.t. } \deg(Q(\xi)) \le (m-1)k \text{ and } (1/P(\xi))^{(k)} = Q(\xi)/P(\xi)^{k+1},$$

hence

$$(1/P(\xi))^{(k+1)} = \frac{Q'(\xi)P(\xi)^{k+1} - (k+1)Q(\xi)P(\xi)^k P'(\xi)}{P(\xi)^{2k+2}} = \frac{Q'(\xi)P(\xi) - (k+1)Q(\xi)P'(\xi)}{P(\xi)^{k+2}},$$

and

$$\det(Q'(\xi)P(\xi) - (k+1)Q(\xi)P'(\xi)) \le \max\{(m-1)k - 1 + m, (m-1)k + m - 1\} = (m-1)(k+1),$$

which completes the inductive proof. □

Now the result follows immediately from the above Lemma and the expression of $\hat{K}_1$. □

Problem 4. *For 3D wave equation, perform a rigorous derivation of E_3 for $t > 0$. Hint: As for Laplace operator, given $t > 0$, we have*

$$\hat{K}_\delta = \hat{E}_3 e^{-\delta|\xi|^2} \to \hat{E}_3 \text{ in } \mathcal{S}'(\mathbb{R}^3),$$

when $\delta > 0$ tends to zero.

Solution.

Lemma. *Define $f_\nu(x) = \frac{1}{\pi}\frac{\sin \nu x}{x}$, then $f_\nu \to \delta$ as $\nu \to \infty$.*

Proof of Lemma. In fact, for $\forall \varphi \in C_c^\infty$, we have from Riemann-Lebesgue lemma

$$\langle f_\nu, \varphi \rangle = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \nu x}{x}\varphi(x)\mathrm{d}x \to \varphi(0) = \langle \delta, \varphi \rangle.$$

□

From the lecture notes, we know that

$$\hat{E}_3(t,\xi) = \frac{\sin t|\xi|}{|\xi|}.$$

We need to apply the inverse Fourier transformation to get E_3:

$$\begin{aligned} E_3(t,x_1,x_2,x_3) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\sin t|\xi|}{|\xi|} e^{i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\ &= \frac{1}{(2\pi)^3}\int_0^\infty \int_{\mathbb{S}^2} \frac{\sin t\rho}{\rho} e^{i\xi\cdot\mathbf{x}}\rho^2 \mathrm{d}\omega \mathrm{d}\rho, \end{aligned}$$

establish spherical coordinates (θ, φ) on the sphere with x-direction as the north direction, then we have

$$\xi \cdot \mathbf{x} = \rho r \cos\theta, \quad r = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \mathrm{d}\omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi.$$

Thus

$$\begin{aligned} E(t,x_1,x_2,x_3) &= \frac{1}{(2\pi)^3}\int_0^\infty \int_0^{2\pi}\int_0^\pi \rho \sin\rho t e^{i\rho r\cos\theta}\sin\theta \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}\rho \\ &= \frac{1}{(2\pi)^2}\int_0^\infty \sin\rho t \left(\int_0^\pi e^{i\rho r\cos\theta}\rho\sin\theta \mathrm{d}\theta\right)\mathrm{d}\rho \\ &= \frac{1}{4\pi^2 r}\int_0^\infty 2\sin\rho t\cdot \sin\rho r \mathrm{d}\rho \\ &= \frac{1}{4\pi^2 r}\lim_{A\to\infty}\int_0^A [\cos\rho(r-t) - \cos\rho(r+t)] dif\rho \\ &= \frac{1}{4\pi^2 r}\lim_{A\to\infty}\left(\frac{\sin A(r-t)}{r-t} - \frac{\sin A(r+t)}{r+t}\right). \end{aligned}$$

From the above lemma, we know that $\frac{1}{\pi}\frac{\sin\nu x}{x}$ converges to $\delta(x)$ as $\nu \to \infty$. Therefore

$$E(t,x_1,x_2,x_3) = \frac{1}{4\pi r}[\delta(r-t) - \delta(r+t)].$$

Since $r + t > 0$, we have $\delta(r+t) \equiv 0$. Hence

$$E(t,x_1,x_2,x_3) = \frac{1}{4\pi r}\delta(r-t).$$

□

Problem 5. *For 2D wave, viewing $u(t,\mathbf{x}) = w(t,\mathbf{x},0)$ with $(\mathbf{x},0) \in \mathbb{R}^3$, using (9) to write down the solution representation formula for $n = 2$.*

Solution. We use the method of descent.
Let us write

$$\bar{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t).$$

Then

$$\begin{cases} \bar{u}_{tt} - \Delta\bar{u} = 0 \text{ in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u} = \bar{g}, \bar{u}_t = \bar{h} \text{ on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

for

$$\bar{g}(x_1,x_2,x_3) := g(x_1,x_2), \quad \bar{h}(x_1,x_2,x_3) := h(x_1,x_2).$$

If we write $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\bar{\mathbf{x}} = (x_1, x_2, 0) \in \mathbb{R}^3$, then Kirchhoff's formula implies

$$u(\mathbf{x}, t) = \bar{u}(\bar{\mathbf{x}}, t) = \frac{\partial}{\partial t}\left(t \fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} \mathrm{d}\bar{S}\right) + t \fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{h} \mathrm{d}\bar{S},$$

where $\bar{B}(\bar{\mathbf{x}}, t)$ denotes the ball in $\mathbb{R}^3$ with center $\bar{\mathbf{x}}$, radius $t > 0$ and where $\mathrm{d}\bar{S}$ denotes two-dimensional surface measure on $\partial \bar{B}(\bar{\mathbf{x}}, t)$. We simplify the above formula by observing

$$\fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} \mathrm{d}\bar{S} = \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} \mathrm{d}\bar{S} = \frac{2}{4\pi t^2} \int_{B(\mathbf{x}, t)} g(\mathbf{y})(1 + \|D\gamma(\mathbf{y})\|^2)^{1/2} \mathrm{d}\mathbf{y},$$

where $\gamma(\mathbf{y}) = (t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}$ for $\mathbf{y} \in B(\mathbf{x}, t)$. The factor "2" enters since $\partial \bar{B}(\bar{\mathbf{x}}, t)$ consists of two hemispheres. Observe that $(1 + \|D\gamma\|^2)^{1/2} = t(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{-1/2}$. Therefore

$$\fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} \mathrm{d}\bar{S} = \frac{1}{2\pi t} \int_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y} = \frac{t}{2} \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y}.$$

Consequently

$$u(\mathbf{x}, t) = \frac{1}{2} \frac{\partial}{\partial t}\left(t^2 \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y}\right) + \frac{t^2}{2} \fint_{B(\mathbf{x}, t)} \frac{h(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y}.$$

But

$$t^2 \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y} = t \fint_{B(\mathbf{0}, 1)} \frac{g(\mathbf{x} + t\mathbf{z})}{(1 - \|\mathbf{z}^2\|)^{1/2}} \mathrm{d}\mathbf{z},$$

and so

$$\begin{aligned} \frac{\partial}{\partial t}\left(t^2 \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y}\right) &= \fint_{B(\mathbf{0}, 1)} \frac{g(\mathbf{x} + t\mathbf{z})}{(1 - \|\mathbf{z}\|^2)^{1/2}} \mathrm{d}\mathbf{z} + t \fint_{B(\mathbf{0}, 1)} \frac{Dg(\mathbf{x} + t\mathbf{z}) \cdot \mathbf{z}}{(1 - \|\mathbf{z}\|^2)^{1/2}} \mathrm{d}\mathbf{z} \\ &= t \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y} + t \fint_{B(\mathbf{x}, t)} \frac{Dg(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y}. \end{aligned}$$

Hence we obtain

$$u(\mathbf{x}, t) = \frac{1}{2} \fint_{B(\mathbf{x}, t)} \frac{tg(\mathbf{y}) + t^2 h(\mathbf{y}) + tDg(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x})}{(t^2 - \|\mathbf{y} - \mathbf{x}\|^2)^{1/2}} \mathrm{d}\mathbf{y} \text{ for } \mathbf{x} \in \mathbb{R}^2, t > 0.$$

□