Lecture 0. Preparation

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Let R^m and R^n denote two Euclidean spaces of m and n dimensions, respectively. Let O and O' be open subsets, $O \subset R^m, O' \subset R^n$ and suppose φ is a mapping of O into O'. Let $(x^1, ..., x^m)$ and $(y^1, ..., y^n)$ be coordinates of R^m and R^n, respectively. Then

$$\varphi(x^1, ..., x^m) = (y^1(x^1, ..., x^m), ..., y^n(x^1, ..., x^m))$$

or simply

$$\varphi(x) = (y^1(x), ..., y^n(x)).$$

The mapping φ is called *differentiable* if the coordinates $y^j\left(\varphi\left(p\right)\right) = y^j(x)$ of $\varphi\left(p\right)$ are differentiable (that is, indefinitely differentiable) functions of the coordinates $x^i\left(p\right), p \in O$. The mapping φ is called *analytic* if for each point $p \in O$ there exists a neighborhood U of p and n convergent power series P^j $(1 \leq j \leq n)$ in m variables such that $y^j\left(\varphi\left(q\right)\right) = P^j\left(x^1\left(q\right) - x^1\left(p\right), ..., x^m\left(q\right) - x^m\left(p\right)\right)$ $(1 \leq j \leq n)$ for $q \in U$.

A differentiable mapping $\varphi : O \to O'$ is called a diffeomorphism of O onto O' if $\varphi\left(O\right) = O'$, φ is one-to-one, and the inverse mapping φ^{-1} is differentiable. In the case when $n = 1$ it is customary to replace the term "mapping" by the term "function".

An analytic function on R^m which vanishes on an open set is identically 0. For differentiable functions the situation is completely different. In fact if A and B are disjoint subsets of R^m, A compact and B closed, then there exists a differentiable function φ which is identically 1 on A and identically 0 on B. The standard procedure for constructing such a function φ is as follows:

First step: Let $0 < a < b$ and consider the function f on R defined by

$$f\left(x\right) = \begin{cases} \exp\left(\frac{1}{x-b} - \frac{1}{x-a}\right) & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is differentiable and the same holds for the function

$$F(x) = \int_x^b f(t)\,dt / \int_a^b f(t)\,dt,$$

which has value 1 for $x \leq a$ and 0 for $x \geq b$. The function ψ on R^m given by

$$\psi(x_1, ..., x_m) = F\left(x_1^2 + ... + x_m^2\right)$$

is differentiable and has values 1 for $x_1^2 + ... + x_m^2 \leq a$ and 0 for $x_1^2 + ... + x_m^2 \geq b$.

Second step: Let S and S' be two concentric spheres in R^m, S' lying inside S. Starting from ψ we can by means of a linear transformation of R^m construct a differentiable function on R^m with value 1 in the interior of S' and value 0 outside S.

Third step: Turning to the sets A and B, we can, owing to the compactness of A, find finitely many spheres S_i $(1 \leq i \leq n)$, such that the corresponding open balls B_i $(1 \leq i \leq n)$, form a covering of A (that is, $A \subset \cup_{i=1}^n B_i$) and such that the closed balls $\overline{B}_i$ $(1 \leq i \leq n)$ do not intersect B. Each sphere S_i can be shrunk to a concentric sphere S_i' such that the corresponding open balls B_i' still form a covering of A. Let ψ_i be a differentiable function on R^m which is identically 1 on B_i' and identically 0 in the complement of B_i. Then the function

$$\varphi = 1 - (1 - \psi_1)(1 - \psi_2) ... (1 - \psi_n)$$

is a differentiable function on R^m which is identically 1 on A and identically 0 on B.

Let M be a topological space. We assume that M satisfies the Hausdorff separation axiom which states that any two different points in M can be separated by disjoint open sets. An open chart on M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of R^m.