PDE Homework #6

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Problem 1. *Let Ω be an open bounded domain in $\mathbb{R}^n$, c and q nonnegative smooth functions. Assume $\exists\delta > 0$ s.t. $\delta < c(\mathbf{x}) < \delta^{-1}$ for any $\mathbf{x} \in \Omega$. Consider the real-valued variable-coefficient wave equation*

$$u_{tt} - \nabla \cdot (c^2(\mathbf{x})\nabla u) + q(\mathbf{x})u = 0, u(0, \mathbf{x}) = f(\mathbf{x}), u_t(0, \mathbf{x}) = g(\mathbf{x}), \mathbf{x} \in \Omega, \tag{1}$$

with homogoneous Neumann boundary condition

$$\mathbf{n}(\mathbf{x}) \cdot \nabla u(t, \mathbf{x}) = 0, \mathbf{x} \in \partial\Omega,$$

where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal. Multiply the equation by u_t and apply the integration-by-parts formula to find the energy corresponding to this problem and prove that the energy so defined is conserved, assuming $u \in \mathcal{C}^2$ with compact support for any time t.

Proof. Multiplying the wave equation by u_t and integrating over Ω yields

$$\int_\Omega u_t u_{tt} \mathrm{d}\mathbf{x} - \int_\Omega u_t \nabla \cdot (c^2 \nabla u) \mathrm{d}\mathbf{x} + \int_\Omega u_t q u = 0.$$

Use the fact that $\frac{1}{2}(u_t^2)_t = u_t u_{tt}$, $\frac{1}{2}(u^2)_t = u u_t$ and integrate by part, and we have

$$\begin{aligned} 0 &= \int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} u_t^2 \mathrm{d}\mathbf{x} + \int_\Omega \frac{q}{2}\frac{\mathrm{d}}{\mathrm{d}t} u^2 \mathrm{d}\mathbf{x} + \int_\Omega c^2 \nabla u \cdot (\nabla u)_t \mathrm{d}\mathbf{x} - \int_{\partial\Omega} u_t c^2 \nabla u \cdot \mathbf{n} \mathrm{d}S \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}\mathbf{x}\right), \end{aligned}$$

where the second equality follows by applying the boundary condition. If we define the energy $E(t)$ by

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}\mathbf{x},$$

the above argument shows that $E'(t) \equiv 0$, and therefore the energy $E(t)$ is conserved. □

Problem 2. *Prove the following finite speed of propagation property for problem (1), with homogeneous Dirichlet or Neumann boundary conditions. If $f = g = 0$ in $B_1(\mathbf{0}) \subset \Omega$, then there exists a region in $\mathbb{R}_+ \times \Omega$ where u vanishes identically.*

Proof. First we introduce some notations for convenience. Denote the ball B_t in $\mathbb{R}_+ \times \Omega$ by

$$B_t = \{(\mathbf{x}, t) | \|\mathbf{x}\| \le 1 - t\},$$

and the cone $K(\mathbf{0}, 1)$ in $\mathbb{R}_+ \times \Omega$ by

$$K(\mathbf{0}, 1) = \cup_{t\in[0,1]} B_t = \{(\mathbf{x}, t) | 0 \le t \le 1, \|\mathbf{x}\| \le 1 - t\}.$$

Define the energy $e(t)$ by

$$e(t) = \frac{1}{2}\int_{B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}\mathbf{x},$$

compute that

$$\begin{aligned}
e'(t) &= \int_{B_t} u_t u_{tt} + quu_t + c^2\nabla u\cdot(\nabla u)_t \mathrm{d}\mathbf{x} - \frac{1}{2}\int_{\partial B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}S\\
&= \int_{B_t} u_t\nabla\cdot(c^2\nabla u) + c^2\nabla u\cdot(\nabla u)_t \mathrm{d}\mathbf{x} - \frac{1}{2}\int_{\partial B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}S\\
&= \int_{\partial B_t} u_t(c^2\nabla u)\cdot\mathbf{n}\mathrm{d}S - \frac{1}{2}\int_{\partial B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}S\\
&\le \int_{\partial B_t} \frac{1}{2}u_t^2 + \frac{1}{2}\left(c^2\nabla u\cdot\mathbf{n}\right)^2 \mathrm{d}S - \frac{1}{2}\int_{\partial B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}S\\
&\le \int_{\partial B_t} \frac{1}{2}u_t^2 + \frac{1}{2}\|c^2\nabla u\|^2\|\mathbf{n}\|^2 \mathrm{d}S - \frac{1}{2}\int_{\partial B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}S\\
&= -\frac{1}{2}\int_{\partial B_t} qu^2 \mathrm{d}S - \frac{1}{2}\int_{\partial B_t} c^2(1-c^2)\|\nabla u\|^2 \mathrm{d}S\\
&\le 0
\end{aligned}$$

where the second step follows from (1), the third from the integration-by-parts formula, the fourth from Cauchy's inequality, the fifth from Cauchy-Schwarz inequality and the last from the assumption that $p(\mathbf{x})$ is nonnegative and $0 < c(\mathbf{x}) < 1, \forall \mathbf{x}\in\Omega$. We have

$$e(0) = \frac{1}{2}\int_{B_1(\mathbf{0})} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}\mathbf{x} = \frac{1}{2}\int_{B_1(\mathbf{0})} g^2 + qf^2 + c^2\|\nabla f\|^2 \mathrm{d}\mathbf{x} = 0,$$

and

$$e(t) = \frac{1}{2}\int_{B_t} u_t^2 + qu^2 + c^2\|\nabla u\|^2 \mathrm{d}\mathbf{x} \ge 0$$

$e(t)$ is non-increasing in $[0,\infty)$ since we have shown that $e'(t) < 0$ for $t \ge 0$, and therefore

$$e(t) = 0, \quad \forall t \ge 0 \Rightarrow u \equiv 0 \text{ in } B_t.$$

Consequently,

$$u \equiv 0 \text{ in } K(\mathbf{0}, 1).$$

□

Problem 3. *Consider the (uniformly) elliptic operator with (continuous) variable-coefficient*

$$L = \sum_{i,j} a^{ij}(\mathbf{x})\partial_i\partial_j + \sum_i b^i(\mathbf{x})\partial_i.$$

Assume that $\exists \mu > 0$ s.t. the matrix $A - \mu I$ is positive semidefinite, i.e.,

$$\exists \mu > 0 \text{ s.t. } \forall \mathbf{x}\in\Omega, \forall \xi\in\mathbb{R}^n, \sum_{i,j=1}^n a^{ij}(\mathbf{x})\xi_i\xi_j \ge \mu\|\xi\|^2.$$

Try to formulate and prove the corresponding weak maximum principle in Ω. (Hint: You may want to construct $\phi = e^{\lambda x_1}$ with $1 \ll \lambda$.)

Solution. The following is a reiteration of the classical PDE textbook by Evans.

Theorem (Weak maximum principle). *Let Ω be a connected open bounded set. Assume $u\in\mathcal{C}^2(\Omega)\cap\mathcal{C}(\overline{\Omega})$, if*

$$Lu \ge 0 \text{ in } \Omega,$$

then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. 1. Let us first suppose we have the strict inequality

$$Lu > 0 \text{ in } \Omega, \tag{2}$$

and yet there exists a point $\mathbf{x}_0\in\Omega$ with

$$u(\mathbf{x}_0) = \max_{\overline{\Omega}} u. \tag{3}$$

Now at this maximum point $\mathbf{x}_0$, we have

$$Du(\mathbf{x}_0) = 0 \tag{4}$$

and

$$D^2u(\mathbf{x}_0) \leq 0. \tag{5}$$

2. Since the matrix $A = ((a^{jk}(\mathbf{x}_0)))$ is symmetric and positive definite, there exists an orthogonal matrix $O = ((o_{ij}))$ so that

$$OAO^T = \text{diag}(d_1, \dots, d_n), \quad OO^T = I. \tag{6}$$

with $d_k > 0 (k = 1, \dots, n)$. Write $\mathbf{y} = \mathbf{x}_0 + O(\mathbf{x} - \mathbf{x}_0)$. Then $\mathbf{x} - \mathbf{x}_0 = O^T(\mathbf{y} - \mathbf{x}_0)$, and so

$$u_{x_i} = \sum_{k=1}^n u_{y_k} o_{ki}, \quad u_{x_i x_j} = \sum_{k,l=1}^n u_{y_k y_l} o_{ki} o_{lj} \quad (i, j = 1, \dots, n).$$

Hence at the point $\mathbf{x}_0$,

$$\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = \sum_{k,l=1}^n \sum_{i,j=1}^n a^{ij} u_{y_k y_l} o_{ki} o_{lj} = \sum_{k=1}^n d_k u_{y_k y_k} \text{ (by (6))} \leq 0, \tag{7}$$

since $d_k > 0$ and $u_{y_k y_k}(\mathbf{x}_0) \leq 0 (k = 1, \dots, n)$, according to (5).

3. Thus at $\mathbf{x}_0$

$$Lu = \sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1} b^i u_{x_i} \leq 0,$$

in light of (4) and (7). So (2) and (3) are incompatible, and we have a contradiction.

4. In the general case, write

$$u^\epsilon(\mathbf{x}) := u(\mathbf{x}) + \epsilon e^{\lambda x_1} \quad (\mathbf{x} \in \Omega),$$

where $\lambda > 0$ will be selected below and $\epsilon > 0$. Recall that the uniform ellipticity condition implies $a^{ii}(\mathbf{x}) \geq \mu (i = 1, \dots, n, \mathbf{x} \in \Omega)$. Therefore

$$Lu^\epsilon = Lu + \epsilon L(e^{\lambda x_1}) \geq \epsilon e^{\lambda x_1}(\lambda^2 a^{11} + \lambda b^1) \geq \epsilon e^{\lambda x_1}(\lambda^2 \mu - \|\mathbf{b}\|_{L^\infty} \lambda) > 0 \text{ in } \Omega,$$

provided we choose $\lambda > 0$ sufficiently large. Then according to steps 1 and 2 above $\max_{\overline{\Omega}} u^\epsilon = \max_{\partial\Omega} u^\epsilon$. Let $\epsilon \to 0$ to find $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$. This completes the proof.

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Problem 4. *Consider the (uniformly) elliptic operator with (continuous) variable-coefficient*

$$L = \sum_{i,j} a^{ij}(\mathbf{x}) \partial_i \partial_j + \sum_i b^i(\mathbf{x}) \partial_i + c(\mathbf{x})$$

Assume that $\exists \mu > 0$ s.t. the matrix $A - \mu I$ is positive semidefinite, i.e.,

$$\exists \mu > 0 \ s.t. \ \forall \mathbf{x} \in \Omega, \forall \xi \in \mathbb{R}^n, \sum_{i,j=1}^n a^{ij}(\mathbf{x}) \xi_i \xi_j \geq \mu \|\xi\|^2.$$

Prove the following theorem.

Theorem (Weak maximum principle(version 2))**.** *If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that*

$$Lu \geq 0 \geq c(\mathbf{x}) \ \text{in} \ \Omega,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where $u^+(\mathbf{x}) = \max(u(\mathbf{x}), 0)$.

Proof. Set $V := \{\mathbf{x} \in \Omega | u(\mathbf{x}) > 0\}$. Then

$$Ku := Lu - cu \geq -cu \geq 0 \text{ in } V.$$

The operator K has no zeroth-order term and consequently the theorem in Problem 3 implies

$$\max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial U} u^+.$$

This gives the desired result in the case that $V \neq \emptyset$. Otherwise $u \leq 0$ everywhere in U, and the desired result likewise follows. $\square$

Problem 5. *Let Ω be an open bounded smooth domain in $\mathbb{R}^n$, $\delta < c(\mathbf{x}) < \delta^{-1}$ for any $\mathbf{x} \in \Omega$. Consider*

$$u_t - \nabla \cdot (c^2(\mathbf{x})\nabla u) = F \tag{8}$$

with Neumann boundary conditions $\partial_\nu u = f$, prove uniqueness for the solutions.

Proof. First, we introduce some notations.

$$\Omega_T := \Omega \times (0, T], \quad \Gamma_T = \overline{\Omega}_T - \Omega_T.$$

Hence Ω_T denotes the parabolic cylinder and Γ_T the parabolic boundary of Ω_T.
If $\tilde{u}$ is another solution, $w := u - \tilde{u}$ solves

$$\begin{cases} w_t - \nabla \cdot (c^2(\mathbf{x})\nabla w) = 0 \text{ in } \Omega_T, \\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_T. \end{cases} \tag{9}$$

Set

$$e(t) := \frac{1}{2}\int_\Omega w^2(\mathbf{x}, t)\mathrm{d}\mathbf{x} \quad (0 \leq t \leq T).$$

Then

$$\begin{aligned} e'(t) &= \int_\Omega w w_t \mathrm{d}\mathbf{x} = \int_\Omega w \nabla \cdot (c^2 \nabla w)\mathrm{d}\mathbf{x} = -\int_\Omega c^2 \|\nabla w\|^2 \mathrm{d}\mathbf{x} + \int_{\partial\Omega} c^2 w \frac{\partial w}{\partial \nu}\mathrm{d}\mathbf{x} \\ &= -\int_\Omega c^2 \|\nabla w\|^2 \mathrm{d}\mathbf{x} \leq 0, \end{aligned}$$

and so

$$e(t) \leq e(0) = 0 \quad (0 \leq t \leq T).$$

Consequently $w = u - \tilde{u} \equiv 0$ in Ω_T, which shows the uniqueness of the solutions. $\square$