PDE Homework #3

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Problem 1. *Solve the following heat equation:*

$$\begin{cases} \partial_t u - \Delta u = F(t, \mathbf{x}) \in L^1([0,\infty) \times \mathbb{R}^n), \\ u(0, \mathbf{x}) = f(\mathbf{x}) \in L^1(\mathbb{R}). \end{cases} \tag{1}$$

Solution. From the lecture notes, we know that the solution of the following Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = 0 \text{ in } (0,\infty) \times \mathbb{R}^n, \\ u = f \text{ on } \{t = 0\} \times \mathbb{R}^n. \end{cases} \tag{2}$$

is given by

$$u(t, \mathbf{x}) = (K_t * f)(\mathbf{x}) = \int_{\mathbb{R}^n} K_t(\mathbf{x} - y) f(\mathbf{y}) \mathrm{d}\mathbf{y}, \tag{3}$$

where K_t is the heat kernel defined as

$$K_t(\mathbf{x}) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|\mathbf{x}\|^2}{4t}}.$$

Duhamel's principle asserts that we can build a solution of the following nonhomogeneous problem

$$\begin{cases} \partial_t u - \Delta u = F(t, \mathbf{x}) \text{ in } (0,\infty) \times \mathbb{R}^n, \\ u = 0 \text{ on } \{t = 0\} \times \mathbb{R}^n. \end{cases} \tag{4}$$

out of the solutions of the following homogeneous problem:

$$\begin{cases} \partial_t u(s, \cdot) - \Delta u(s, \cdot) = 0 \text{ in } (s,\infty) \times \mathbb{R}^n, \\ u(s, \cdot) = F(s, \cdot) \text{ on } \{t = s\} \times \mathbb{R}^n, \end{cases} \tag{5}$$

by integrating with respect to s. The idea is to consider

$$u(t, \mathbf{x}) = \int_0^t u(t, \mathbf{x}; s) \mathrm{d}s,$$

where $u(t, \mathbf{x}; s)$ is the solution to (5),

$$u(t, \mathbf{x}; s) = \int_{\mathbb{R}^n} K_{t-s}(\mathbf{x} - \mathbf{y}) F(s, \mathbf{y}) \mathrm{d}\mathbf{y}.$$

Rewriting, we have

$$u(t, \mathbf{x}) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(\mathbf{x} - \mathbf{y}) F(s, \mathbf{y}) \mathrm{d}\mathbf{y} \mathrm{d}s, \tag{6}$$

for $t > 0, \mathbf{x} \in \mathbb{R}^n$. □

Since (1) is a linear PDE, we conclude that adding the solution of (2) and (4) gives the solution to (1). Hence

$$u(t, \mathbf{x}) = \int_{\mathbb{R}^n} K_t(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \mathrm{d}\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(\mathbf{x} - \mathbf{y}) F(s, \mathbf{y}) \mathrm{d}\mathbf{y} \mathrm{d}s,$$

for $t > 0, \mathbf{x} \in \mathbb{R}^n$.

Problem 2. *Find a solution to the following Dirichlet problem for the Laplace equation, by using the Fourier transform:*

$$\begin{cases} (\partial_x^2 + \partial_y^2) u = 0, (x, y) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x, 0) = f(x) \in \mathcal{S}(\mathbb{R}). \end{cases}$$

Solution. Take the Fourier transform, and we have

$$\begin{cases} \partial_y^2 \hat{u}(\xi, y) + \xi^2 \hat{u}(\xi, y) = 0, \\ \hat{u}(\xi, 0) = \hat{f}(\xi). \end{cases}$$

The general solution of this ordinary differential equation in y (with ξ fixed) takes the form

$$\widehat{u}(\xi, y) = A(\xi) e^{-|\xi| y} + B(\xi) e^{|\xi| y}.$$

If we disregard the second term because of its rapid increase we find, after setting $y = 0$, that

$$\widehat{u}(\xi, y) = \widehat{f}(\xi) e^{-|\xi| y}.$$

Therefore u is given in terms of the convolution of f with a kernel whose Fourier transform is $e^{-|\xi| y}$.

Lemma. *Define the Poisson kernel $\mathcal{P}_y(x)$ for the upper half-plane*

$$\mathcal{P}_y(x) = \frac{2y}{x^2 + y^2} \quad \text{where } x \in \mathbb{R} \text{ and } y > 0.$$

Then the following two identities hold:

$$\int_{-\infty}^{\infty} e^{-|\xi| y} e^{i \xi x} \mathrm{d}\xi = \mathcal{P}_y(x),$$

$$\int_{-\infty}^{\infty} \mathcal{P}_y(x) e^{i x \xi} \mathrm{d}x = e^{-|\xi| y}.$$

Proof of Lemma. The first formula is fairly straightforward since we can split the integral from $-\infty$ to 0 and 0 to ∞. Then, since $y > 0$ we have

$$\int_0^{\infty} e^{-\xi y} e^{i \xi x} \mathrm{d}\xi = \int_0^{\infty} e^{i(x+iy)\xi} \mathrm{d}\xi = \left[\frac{e^{i(x+iy)\xi}}{i(x+iy)} \right]_0^{\infty} = -\frac{1}{i(x+iy)},$$

and similarly,

$$\int_{-\infty}^{0} e^{\xi y} e^{i \xi x} \mathrm{d}\xi = \frac{1}{i(x - iy)}.$$

Therefore

$$\int_{-\infty}^{\infty} e^{-|\xi| y} e^{i \xi x} \mathrm{d}\xi = \frac{1}{i(x-iy)} - \frac{1}{i(x+iy)} = \frac{2y}{x^2 + y^2}.$$

The second formula is now a consequence of the Fourier inversion theorem applied in the case when f and $\hat{f}$ are of moderate decrease. □

Therefore

$$u(x, y) = (\mathcal{P}_y * f)(x) = \int_{\mathbb{R}} \mathcal{P}_y(x - z) f(z) \mathrm{d}z = \int_{\mathbb{R}} \frac{2y}{(x-z)^2 + y^2} f(z) \mathrm{d}z.$$

□

Problem 3. *Check that any polynomial $p(\mathbf{x}) \in \mathcal{S}'(\mathbb{R}^n)$, however, $f(x) = e^{x^2} \notin \mathcal{S}'(\mathbb{R})$, $g(x) = e^x \notin \mathcal{S}'(\mathbb{R})$. (Hint: you may want to use test functions like $e^{-\sqrt{1+x^2}}$.)*

Solution.

- For a polynomial $p(\mathbf{x})$, define

$$L(\phi) = \int_{\mathbb{R}^n} p(\mathbf{x}) \phi(\mathbf{x}) \mathrm{d}\mathbf{x}, \quad \phi \in \mathcal{S}.$$

Now we show that $L(\phi)$ is a continuous linear functional on the Schwartz space $\mathcal{S}$. Choose $N \in \mathbb{N}^+$ large enough so that

$$\int_{\mathbb{R}^n} \left(1 + \|\mathbf{x}\|^2\right)^{-N} |p(\mathbf{x})| \mathrm{d}\mathbf{x} = C < \infty.$$

Then

$$\begin{aligned} |L(\phi)| &= \left| \int_{\mathbb{R}^n} p(\mathbf{x}) \phi(\mathbf{x}) \mathrm{d}\mathbf{x} \right| = \left| \int_{\mathbb{R}^n} \left(1 + \|\mathbf{x}\|^2\right)^{-N} p(\mathbf{x}) (1 + \|\mathbf{x}\|^2)^N \phi(\mathbf{x}) \mathrm{d}\mathbf{x} \right| \\ &\leq \int_{\mathbb{R}^n} \left(1 + \|\mathbf{x}\|^2\right)^{-N} |p(\mathbf{x})| \mathrm{d}\mathbf{x} \sup_{\mathbf{x} \in \mathbb{R}^n} \left\| (1 + \|\mathbf{x}\|^2)^N \phi(\mathbf{x}) \right\| \\ &= C \sup_{\mathbf{x} \in \mathbb{R}^n} \left\| (1 + \|\mathbf{x}\|^2)^N \phi(\mathbf{x}) \right\|, \end{aligned}$$

which shows the continuity of L, L is easily seen to be a linear functional. Therefore, the polynomial $p(\mathbf{x}) \in \mathcal{S}'(\mathbb{R}^n)$.

- Choose a function $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)\mathrm{d}x = 1$, let

$$\phi_j(x) = \frac{\psi(x-j)}{f(x)} = e^{-x^2}\psi(x-j).$$

It is easily verified that $\phi_j \to 0$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$, but

$$\int_{\mathbb{R}} f(x)\phi_j(x)\mathrm{d}x = \int_{\mathbb{R}} \psi(x)\mathrm{d}x = 1$$

for all j. Therefore $f(x) = e^{x^2} \notin \mathcal{S}'(\mathbb{R})$.

- Similarly, we can show that $g(x) = e^x \notin \mathcal{S}'(\mathbb{R})$.

□

Problem 4. *Let $u \in \mathcal{S}'$, calculate $\mathcal{F}(\partial_j u) \in \mathcal{S}'(\mathbb{R}^n)$ by definition.*

Solution. $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \mathcal{F}(\partial_j u), \phi \rangle &= \langle \partial_j u, \mathcal{F}(\phi) \rangle = -\langle u, \partial_j \mathcal{F}(\phi) \rangle \\
&= -\langle u, -i\mathcal{F}(\xi_j \phi) \rangle = \langle \mathcal{F}(u), i\xi_j \phi \rangle \\
&= \langle i\xi_j \mathcal{F}(u), \phi \rangle,
\end{aligned}$$

where the first and third step follow from the definition of the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, the second step from the definition of derivatives of tempered distribution and the third step follows from the property of Fourier transforms. Therefore

$$\mathcal{F}(\partial_j u) = i\xi_j \mathcal{F}(u).$$

□

Lemma (Gaussian function)**.** *If $G_\lambda = e^{-\lambda\|\mathbf{x}\|^2}$, where $\mathscr{R}\lambda > 0$, then*

$$\widehat{G}_\lambda(\xi) = \left(\frac{\pi}{\lambda}\right)^{n/2} e^{-\frac{\|\xi\|^2}{4\lambda}} = \left(\frac{\pi}{\lambda}\right)^{n/2} G_{1/(4\lambda)}. \tag{7}$$

Problem 5. *Based on* (7) *with $\lambda = \epsilon - it$, $\epsilon > 0$, $t \in \mathbb{R}\backslash\{0\}$. By considering the limit in $\mathcal{S}'(\mathbb{R})$ as $\epsilon \to 0^+$, deduce that*

$$\mathcal{F}_{\mathbf{x}} e^{it\|\mathbf{x}\|^2}(\xi) = \left(\frac{\pi}{|t|}\right)^{n/2} e^{i\frac{n\pi}{4}\operatorname{sgn}t - \frac{i\|\xi\|^2}{4t}}. \tag{8}$$

Proof. Based on (7) with $\lambda = \epsilon - it$, we obtain

$$\mathcal{F}e^{-(\epsilon - it)\|\mathbf{x}\|^2}(\xi) = \left(\frac{\pi}{\epsilon - it}\right)^{n/2} e^{-\frac{\|\xi\|^2}{4(\epsilon - it)}},$$

considering the limit in $\mathcal{S}'(\mathbb{R})$ as $\epsilon \to 0^+$, we have

$$\begin{aligned}
\mathcal{F}_{\mathbf{x}} e^{it\|\mathbf{x}\|^2}(\xi) &= \left(\frac{\pi}{-it}\right)^{n/2} e^{-\frac{i\|\xi\|^2}{4t}} \\
&= \begin{cases} \left(\frac{\pi}{t}\right)^{n/2} i^{n/2} e^{-\frac{i\|\xi\|^2}{4t}} = \left(\frac{\pi}{|t|}\right)^{n/2} e^{i\frac{n\pi}{4}\operatorname{sgn}t - \frac{i\|\xi\|^2}{4t}} & \text{if } t > 0, \\ \left(\frac{\pi}{|t|}\right)^{n/2} (-i)^{n/2} e^{-\frac{i\|\xi\|^2}{4t}} = \left(\frac{\pi}{|t|}\right)^{n/2} e^{i\frac{n\pi}{4}\operatorname{sgn}t - \frac{i\|\xi\|^2}{4t}} & \text{if } t < 0, \end{cases}
\end{aligned}$$

where we have used Euler's identity $e^{ix} = \cos x + i\sin x$, in particular, $i = e^{i\frac{\pi}{2}}$ and $-i = e^{-i\frac{\pi}{2}}$. Therefore, we have the desired result. □