Functional Analysis

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Contents

1 Topological Vector Spaces and Distributions

In this chapter we cover the basic results of topological vector space theory and introduce Sobolev spaces and distributions.

1.1 Convex Set Separation Theorems

In this section, we study some elementary fact on the convex set theory, in particular, we will introduce the famous convex sets separation theorem.

Suppose that V is a linear space with scalar field $\mathbf{K}$. If X and Y are nonempty subsets of V, and $a \in \mathbf{K}$, we define further subsets $aX, X \pm Y$ by

$$aX = \{ax : x \in X\}, \qquad X + Y = \{x + y : x \in X, y \in Y\},$$

and

$$X - Y = X + (-1)Y.$$

When X consists of a single element x, we write $x \pm Y$ in place of $X \pm Y$. To avoid ambiguity in the use of the symbol $-$, the set theoretic difference $\{x \in A : x \notin B\}$ of two sets A and B will be denoted by $A \backslash B$. A vector of the form $a_1x_1 + \cdots + a_nx_n$, where $x_1, \cdots, x_n \in X$ and $a_1, \cdots, a_n \in \mathbf{K}$, is called a finite *linear combination* of elements of X. The zero vector is always of this form, with $\{x_1, \cdots, x_n\}$ an arbitrary finite subset of X, and $a_j = 0$ for each j. If it can be expressed as a *non-trivial* linear combination of elements of X, that is, with $x_1, \cdots, x_n$ distinct, and at least one a_j non-zero, then X is said to be *linearly dependent*; otherwise X is *linearly independent*. The set of all finite linear combinations of elements of X is a linear subspace of V, the smallest containing X; we refer to it as the *linear subspace generated* by X.

If V_0 is a linear subspace of V, we denote by V/V_0 the set of all cosets $x+V_0$ in the additive group V, where $x \in V$. Of course, V/V_0 is a group, with addition defined by $(x+V_0)+(y+V_0) = (x+y)+V_0$. If $a \in \mathbf{K}$, and $x_1+V_0 = x_2+V_0$, we have $ax_1 - ax_2 = a(x_1 - x_2) \in V_0$, so $ax_1 + V_0 = ax_2 + V_0$. From this it follows easily that V/V_0 becomes a linear space over K, the *quotient* of V by V_0, when multiplication by scalars is defined by $a(x+V_0) = ax+V_0$. If V/V_0 has finite dimension n, we say that V_0 has finite *codimension* n in V.

Suppose that V and W are linear spaces over $\mathbf{K}$. By a *linear operator*, or *linear transformation*, from V into W, we mean a mapping $T : V \to W$ such that

$$T(ax+by) = aTx + bTy$$

whenever $x, y \in V$ and $a, b \in \mathbf{K}$, where the notation $T : V \to W$ indicates that T is defined on V and takes values in W. If V_0 is a linear subspace of V, the equation $Qx = x + V_0$ defines a linear operator Q from V onto V/V_0, the *quotient mapping*. When $T : V \to W$ is a linear operator, the *null space* of T is the linear subspace $\{x \in V : Tx = 0\}$ of V, and the image $T(V) = \{Tx : x \in V\}$ is a linear subspace of W. If $T(V_0) = \{0\}$, the condition $x + V_0 = y + V_0$ entails $x - y \in V_0$, and hence $Tx - Ty = 0$; moreover, if V_0 is the null space of T, $Tx = 0$ entails $x \in V_0$. From this, the equation $T_0(x+V_0) = Tx$ defines a linear operator T_0 from V/V_0 onto $T(V)$, when $T(V_0) = \{0\}$; and T_0 is one-to-one if V_0 is the null space of T. Note that $T = T_0Q$, a fact sometimes described by saying that T *factors through* V/V_0 when $T(V_0) = \{0\}$. Given any linear operators $S, T : V \to W$ and scalars a, b, the equation $(aS+bT)x = aSx + bTx$ defines another such operator $aS + bT$, and in this way, the set of all linear operators from V into W becomes a linear space over $\mathbf{K}$.

By a *linear functional* on V we mean a linear operator $\rho : V \to \mathbf{K}$. The set of all linear functionals on V is itself a linear space over $\mathbf{K}$, the *algebraic dual space* of V. When ρ is a *non-zero* linear functional on V the image $\rho(V)$ is $\mathbf{K}$.

Proposition 1.1.1. *If ρ is a linear functional on a linear space V, then every linear functional on V that vanishes on the null space V_0 of ρ is a scalar multiple of ρ. If $\rho \neq 0$, V_0 has codimension 1 in V. Conversely each linear subspace of codimension 1 in V is the null space of a non-zero linear functional. If $\rho_1, \cdots, \rho_n$ are linear functionals on V, then every linear functional on V that vanishes on the intersection of the null space of $\rho_1, \cdots, \rho_n$ is a linear combination of $\rho_1, \cdots, \rho_n$.*

Proof. We may suppose that $\rho \neq 0$. The equation $\rho_0(x + V_0) = \rho(x)$ defines a one-to-one linear operator ρ_0 from V/V_0 onto the one-dimensional linear space K; so V/V_0 is one dimensional. Of course, ρ_0 is a non-zero linear functional on V/V_0; and in the same way, if a linear functional σ on V vanishes on V_0, there is a linear functional σ_0 on V/V_0, defined by $\sigma_0(x + V_0) = \sigma(x)$. Since V/V_0 is one dimension, $\sigma_0 = a\rho_0$ for some scalar a, and $\sigma = \sigma_0 Q = a\rho_0 Q = a\rho$, where Q is the quotient mapping from V onto V/V_0.

If V_1 is a linear subspace with codimension 1 in V, there is a non-zero linear functional τ_1 on the one-dimensional linear space V/V_1, and $\tau_1 : V/V_1 \to K$ is a one-to-one mapping. Accordingly, the equation $\tau(x) = \tau_1(x + V_1)$ defines a non-zero linear functional τ on V whose null space is V_1.

The final assertion of the proposition is proved by induction on n. We make the inductive assumption that it is valid when $n = k$, note that the initial case, in which $n = 1$, reduces to the statement in the first sentence of the proposition, and has already been proved. Now suppose that $\sigma, \rho_1, \cdots, \rho_k, \rho_{k+1}$ are linear functionals on V, and σ vanishes on the intersection of the null spaces of $\rho_1, \cdots, \rho_k, \rho_{k+1}$. Let V_{k+1} denote the null space of ρ_{k+1}, and consider the restrictions

$$\sigma|V_{k+1}, \rho_1|V_{k+1}, \cdots, \rho_k|V_{k+1}, \rho_{k+1}|V_{k+1} = 0.$$

Since $\sigma|V_{k+1}$ vanishes on the intersection of the null spaces of the k linear functionals $\rho_1|V_{k+1}, \cdots, \rho_k|V_{k+1}$, it is a linear combination $(a_1\rho_1 + \cdots + a_k\rho_k)|V_{k+1}$ of those linear functionals by our inductive assumption. Thus $\sigma - a_1\rho_1 - \cdots - a_k\rho_k$ vanishes on the null space V_{k+1} of ρ_{k+1}, and is therefore a multiple $a_{k+1}\rho_{k+1}$ by the first assertion of the proposition; and $\sigma = a_1\rho_1 + \cdots + a_k\rho_k + a_{k+1}\rho_{k+1}$. ■

Suppose that V is a linear space over $\mathbf{K}$, and $X, Y \subseteq V$. By a *finite convex combination* of elements of X, we mean a vector of the form $a_1x_1 + \cdots + a_nx_n$, where $x_1, \cdots, x_n \in X$ and $a_1, \cdots, a_n$ are real scalars satisfying $a_j > 0, j = 1, \cdots, n$ and $\sum a_j = 1$. It makes no difference in this definition if the condition $a_j > 0$ is relaxed to $a_j \geq 0$, but strict inequality is slightly more convenient for our present purposes. We say that Y is *convex* if $b_1y_1 + b_2y_2 \in Y$ whenever $y_1, y_2 \in Y$ and b_1, b_2 are positive real numbers with sum 1, that is, Y contains each convex combination of just *two* elements of Y; geometrically, this means each line segment with endpoints in Y lies wholly in Y. A simple proof, by induction on n, shows that a convex set Y contains every convex combination $a_1x_1 + \cdots + a_nx_n$ of elements $x_1, \cdots, x_n$ of Y; the inductive step up, from $n-1$

to n, depends on the observation that

$$a_1x_1 + \cdots + a_nx_n = b_1y_1 + b_2y_2,$$

where $b_1 = a_1, b_2 = a_2 + \cdots + a_n, y_1 = x_1$, and y_2 is the convex combination ${b_2}^{-1}(a_2x_2 + \cdots + a_nx_n)$ of $x_2, \cdots, x_n$.

When $X \subseteq V$, we denote by co$X$ the set of all finite convex combinations of elements of X. A straightforward calculation shows that if $y_1, \cdots, y_n \in$ co$X$, then every convex combination of $y_1, \cdots, y_n$ lies in co$X$. Thus co$X$ is a convex set, the smallest one containing X; it is called the *convex hull* of X. By an *internal point* of X we mean a vector x in X with the following property: given any y in V, there is a positive real number c such that $x + ay \in X$ whenever $0 \leq a < c$.

Our next result is concerned with *real* vector spaces. By a *hyperplane*, in a linear space V over $\mathbf{R}$, we mean a set of the form $x_0 + V_0$, where $x_0 \in V$ and V_0 is a linear subspace with codimension 1 in V. From Propostion 1.1, a subset H of V is a hyperplane if and only if it can be expressed in the form

$$H = \{x \in V : \rho(x) = k\},$$

where ρ is a non-zero linear functional on V and $k \in \mathbf{R}$; of course, ρ and k are not uniquely determined by H, but the only possible variation is to replace them by $a\rho$ and ak, respectively, where a is a non-zero real number.

In fact, let $H = \{x \in V : \rho(x) = k\} = \{x \in V : \rho_1(x) = k_1\}$. It is clear that $k_1 = 0$ if and only if $k = 0$, so in this case, the conclusion is true. If $k_1 \neq 0$, then $H = \{x \in V : \frac{\rho(x)}{k} = 1\} = \{x \in V : \frac{\rho_1(x)}{k_1} = 1\}$. So, $H \subseteq \{x \in V : \frac{\rho(x)}{k} - \frac{\rho_1(x)}{k_1} = 0\}$. Take $x_0 \in H$, then $H = x_0 + Ker(\rho)$, thus, $x_0 \notin Ker(\rho)$.

Note that $Ker(\rho)$ is codimension 1 subspace of V, so, for each $x \in V$, we have $x \in Span\{x_0, Ker(\rho)\}$. Therefore, $V = \{x \in V : \frac{\rho(x)}{k} - \frac{\rho_1(x)}{k_1} = 0\}$. Thus, $\rho = \frac{k}{k_1}\rho_1$. Moreover, $k = \frac{k}{k_1}k_1$. The conclusion is proved.

With the hyperplane H we can associate the two *closed half-spaces*, $\{x \in V : \rho(x) \geq k\}$ and $\{x \in V : \rho(x) \leq k\}$, and the two *open half-space*, which are defined similarly but with strict inequalities. We say that H separates two subsets Y and Z of V if Y is contained in one of the closed half-spaces determined by H and Z is contained in the other; *strict separation* is defined similarly in terms of the open half-spaces. If the hyperplane is described in terms of $a\rho$ and ak, the property of separation remains unchanged.

Theorem 1.1.2. *(Convex Sets Separation Theorem). If Y and Z are non-empty disjoint convex subsets of a real vector space V, at least one of which has an internal point, they are separated by a hyperplane H in V. If either Y or Z consists entirely of internal points, it is contained in one of the open half-spaces determined by H. If both Y and Z consist entirely of internal points, they are strictly separated by H.*

Lemma 1.1.3. *If ρ is a linear functional on a complex vector space V, the equation $\rho_\tau(x) = Re\rho(x)$ defines a real-linear functional ρ_τ on V, and*

$$\rho(x) = \rho_\tau(x) - i\rho_\tau(ix), x \in V.$$

Every real-linear functional on V arises, in this way, from a linear functional.

Proof. It is apparent that, given a linear functional ρ on V, the stated equation defines a real-linear functional ρ_τ. Moreover

$$Im\rho(x) = -Rei\rho(x) = -Re\rho(ix) = -\rho_\tau(ix),$$

whence $\rho(x) = Re\rho(x) + iIm\rho(x) = \rho_\tau(x) - i\rho_\tau(ix)$.

Suppose next that σ is a real-linear functional on V, and define a function $\sigma_c : V \to \mathbf{C}$ by

$$\sigma_c(x) = \sigma(x) - i\sigma(ix) \qquad (x \in V).$$

It is clear that $\sigma(x) = Re\sigma_c(x), \sigma_c(x+y) = \sigma_c(x)+\sigma_c(y)$, and $\sigma_c(ax) = a\sigma_c(x)$, whenever $x, y \in V$ and a is a *real* scalar. Since, also,

$$\begin{aligned} \sigma_c(ix) &= \sigma(ix) - i\sigma(-x) = \sigma(ix) + i\sigma(x) \\ &= i[\sigma(x) - i\sigma(ix)] = i\sigma_c(x), \end{aligned}$$

it follows that σ_c is a linear function on V. ■

We now obtain the analogue, for complex vector spaces, of Theorem 1.1.2.

Theorem 1.1.4. *If Y and Z are non-empty disjoint convex subsets of a complex vector space V, at least one of which has an internal point, there is a non-zero linear functional ρ on V, and a real number k, such that*

$$Re\rho(y) \geq k \geq Re\rho(z), y \in Y, z \in Z.$$

Moreover, $Re\rho(y) > k$, for each $y \in Y$ if Y consists entirely of internal points, and $k > Re\rho(z)$, for each $z \in Z$ if Z consists entirely of internal points.

Proof. By considering Y and Z as subsets of the real vector space V obtained from V, it follows from Theorem 1.1.2 that there is a non-zero real-linear functional σ on V and a real number k such that $\sigma(y) \geq k \geq \sigma(Z)$ whenever $y \in Y$ and $z \in Z$. Moreover, if either Y or Z consists entirely of internal points. The corresponding one of the inequalities $\geq$ can be replaced by $>$. By

Lemma 1.1.3, there is a linear functional ρ on V such that $\sigma(x) = Re\rho(x)$ for each x in V. ■

Let V be a linear space with scalar field $\mathbf{K}$. By a *sublinear functional* on V we mean a function $p : V \to \mathbf{R}$ such that

$$p(x + y) \leq p(x) + p(y), \quad p(ax) = ap(x)$$

whenever $x, y \in V$ and a is a non-negative real number. If, further,

$$p(ax) = |a|p(x), x \in V, a \in K,$$

p is described as a *semi-norm* on V. If p is a semi-norm, then

$$p(x) \geq 0, \quad |p(x) - p(y)| \leq p(x - y), x, y \in V.$$

Indeed, $2p(x) = p(x) + p(-x) \geq p(x - x) = 0$; while

$$p(x) = p((x - y) + y) \leq p(x - y) + p(y),$$

whence $p(x) - p(y) \leq p(x - y)$, and similarly

$$p(y) - p(x) \leq p(y - x) = p(x - y).$$

By a *norm* on V, we mean a semi-norm p such that $p(x) > 0$ whenever $x \in V, x \neq 0$.

A subset Y of V is said to be *balanced* if $ay \in Y$ whenever $y \in Y$, $a \in \mathbf{K}$, and $|a| \leq 1$. If p is a sublinear functional on V, it is immediately verified that the set $V_p = \{x \in V : p(x) < 1\}$ is convex, contains 0, and consists entirely of internal points; V_p is balanced if p is a semi-norm.

Proposition 1.1.5. *Suppose that W is a convex subset of a linear space V over $\mathbf{K}$, and 0 is an internal point of W. Then the equation*

$$p(x) = \inf\{c : c \in \mathbf{R}, c > 0, x \in cW\}, x \in V$$

defines a sublinear functional p on V. If W consists entirely of internal points, then $W = \{x \in V : p(x) < 1\}$. If W is balanced, p is a semi-norm.

Theorem 1.1.6. *If p is a sublinear functional on a real vector space V, while ρ_0 is a linear functional on a linear subspace V_0 of V, and*

$$\rho_0(y) \leq p(y), y \in V_0,$$

there is a linear functional ρ on V such that

$$\rho(x) \leq p(x), x \in V, \qquad \rho(y) = \rho_0(y), y \in V_0.$$

Proof. The product set $\mathbf{R} \times V$ becomes a real vector space when addition and scalar multiplication are defined by

$$(r, x) + (s, y) = (r + s, x + y), \qquad a(r, x) = (ar, ax),$$

for x, y in V and a, r, s in $\mathbf{R}$. From the defining properties of sublinear functionals, it is immediately verified that the set

$$X = \{(r, x) \in \mathbf{R} \times V : r > p(x)\}$$

is non-empty, convex, and consists entirely of internal points. The set

$$W = \{(\rho_0(y), y) : y \in V_0\}$$

is a linear subspace of $\mathbf{R} \times V$, and $X \cap W = \emptyset$. From Theorem 1.1.2, there is a linear functional σ on $\mathbf{R} \times V$ and a real number k such that

$$\sigma(v) > k \geq \sigma(w), v \in X, w \in W.$$

If $w \in W$, then $aw \in W$, and thus $a\sigma(w) = \sigma(aw) \leq k$, for every scalar a; so

$$\sigma(w) = 0, w \in W,$$

and $k \geq 0$. From this, and since $(1,0) \in X$, it follows that $\sigma((1,0)) > k \geq 0$; upon replacing σ by a suitable positive multiple of σ, we may assume that $\sigma((1,0)) = 1$.

The equation $\rho(x) = -\sigma((0,x))$ defines a linear functional ρ on V, and

$$\sigma((r,x)) = \sigma(r(1,0) + (0,x)) = r - \rho(x), r \in \mathbf{R}, x \in V.$$

Given any x in V, we have $(r,x) \in X$, and therefore

$$r - \rho(x) = \sigma((r,x)) > k \geq 0,$$

whenever $r > p(x)$; so $\rho(x) \leq p(x)$. When $y \in V_0$, $(\rho_0(y), y) \in W$, and thus

$$\rho_0(y) - \rho(y) = \sigma((\rho_0(y), y)) = 0.$$

■

Now, we show that above theorem can be used to prove the special convex set separation theorem.

Theorem 1.1.7. *If Y and Z are non-empty disjoint convex subsets of a real vector space V, at least one of which has an internal point, then there is a non-zero real linear functional ρ on V, such that*

$$\rho(y) \geq \rho(z), y \in Y, z \in Z.$$

Proof. Firstly, we consider y_0 is an internal point of Y and $Z = \{z_0\}$. It is clear that 0 is an internal point of $Y - y_0$ and $Y - y_0$ is not disjoint with $\{z_0 - y_0\}$. Let p be the sublinear functional come from $Y - y_0$, $V_0 = span\{z_0 - y_0\}$. Define $\rho_0(a(z_0 - y_0)) = a$. Then ρ_0 is a linear functional on V_0, and $\rho_0(a(z_0 - y_0)) = a \leq p(a(z_0 - y_0))$. Thus, by the functional extension theorem, we have a real linear functional ρ on V such that for each $x \in V$, we have $\rho(x) \leq p(x)$. Thus, $\rho(x - y_0) \leq p(x - y_0) \leq 1$ for each $x \in Y$. Therefore, we have $\rho(x - z_0 + z_0 - y_0) \leq 1$ for each $x \in Y$. Note that $\rho(z_0 - y_0) = 1$, thus we have $\rho(x) \leq \rho(z_0)$ for each $x \in Y$. The conclusion is proved for $Z = \{z_0\}$.

If Z is a general convex set. Then $Y - Z$ is a convex set and $0 \notin Y - Z$. Since y_0 is an internal point of Y, so $y_0 - z$ is an internal point of $Y - Z$ for each $z \in Z$. Note that $Y - Z$ and 0 satisfy the above case, thus, there is a real linear functional ρ_1 on V such that $\rho_1(u - v) \leq \rho_1(0) = 0$. That is, $\rho_1(u) \leq \rho_1(v)$ for each $u \in Y$ and $v \in Z$,

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Theorem 1.1.8. *(Hahn-Banach Functional Extension Theorem). If p is a semi-norm on a linear space V over $\mathbf{K}$, while ρ_0 is a linear functional on linear subspace V_0 of V, and*

$$|\rho_0(y)| \leq p(y), y \in V_0,$$

there is a linear functional ρ on V such that

$$|\rho(x)| \leq p(x), x \in V, \qquad \rho(y) = \rho_0(y), y \in V_0.$$

Proof. If $\mathbf{K} = \mathbf{R}$, p is a sublinear functional on V, and $\rho_0(y) \leq p(y)$ for all y in V_0. By Theorem 1.6, there is a linear functional ρ on V such that $\rho(y) = \rho_0(y)$

when $y \in V_0$ and $\rho(x) \leq p(x)$ for each x in V. Since, also,

$$-\rho(x) = \rho(-x) \leq p(-x) = p(x),$$

it follows that $|\rho(x)| \leq p(x)$ when $x \in V$.

Suppose now that $\mathbf{K} = \mathbf{C}$, and let V_τ be the real vector space obtained from V by restricting the scalar field. Then p is a sublinear functional on V_τ, the equation

$$\sigma_0(y) = Re\rho_0(y)$$

defines a linear functional σ_0 on the linear subspace V_0 of V_τ, and $\sigma_0(y) \leq p(y)$ for each y in V_0. By Theorem 1.1.6, there is a linear functional σ on V_τ such that

$$\sigma(y) = \sigma_0(y), y \in V_0, \qquad \sigma(x) \leq p(x), x \in V_\tau.$$

Thus σ is a real-linear functional on V. By Lemma 1.1.3, there is a linear functional ρ on V such that $\sigma(x) = Re\rho(x)$, and

$$\rho(y) = \sigma(y) - i\sigma(iy) = \sigma_0(y) - i\sigma_0(iy) = \rho_0(y), y \in V_0.$$

When $x \in V$, we can choose a scalar a so that $|a| = 1, |\rho(x)| = a\rho(x)$; and

$$\begin{aligned} |\rho(x)| &= \rho(ax) = Re\rho(ax) \\ &= \sigma(ax) \leq p(ax) = |a|p(x) = p(x). \end{aligned}$$

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1.2 Topological Vector Space Theory

Let X be a set and $\mathcal{T}$ be a family of subsets of X satisfy that

(1) $X, \emptyset \in \mathcal{T}$;

(2) If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$;

(3) If $\tau_1 \subseteq \mathcal{T}$, then $\bigcup_{A \in \tau_1} A \in \mathcal{T}$.

Then $\mathcal{T}$ is said to be a topology of X, $(X, \mathcal{T})$ is said to be a topology space, and each set in $\mathcal{T}$ is said to be an open subset of X. Moreover, if $U \in \mathcal{T}$, then $U^c = X \backslash U$ is said to be a closed subset of X.

Let $(X, \mathcal{T})$ be a topological space, $x \in X$. If there exists $U \in \mathcal{T}$ such that $x \in U$, then U is said to be a neighbourhood of x.

Let $(\Lambda, \geq)$ be a partially order set. If for any $\alpha_1, \alpha_2 \in \Lambda$, there is $\beta \in \Lambda$ such that $\beta \geq \alpha_1, \beta \geq \alpha_2$, then $(\Lambda, \geq)$ is said to be a directed set. Let $(\Lambda, \geq)$ be a directed set. $\{x_\alpha\}_{\alpha \in \Lambda} \subseteq X$ is said to be a net of X. Let $\{x_\alpha\}_{\alpha \in \Lambda} \subseteq X$ be a net of X and $x_0 \in X$. If for each neighbourhood U of x_0, there exists $\alpha_0 \in \Lambda$, when $\alpha \geq \alpha_0$, $x_\alpha \in U$, then we say that the net $\{x_\alpha\}_{\alpha \in \Lambda}$ converges to x_0.

Let $(X, \mathcal{T})$ be a topological space, $x_0 \in X, A \subseteq X$. If for each neighbourhood U of x_0, $U \cap A \neq \emptyset$, then x_0 is said to be an adherent point of A. All the adherent points of A is said to be the closure of A and denoted by $\overline{A}$.

It is easily to prove that if x_0 is an adherent point of A, then there exists a net $\{x_\alpha\}_{\alpha \in \Lambda} \subseteq A$ such that $x_\alpha \to x_0$.

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. Then A is a closed set iff $A = \overline{A}$.

Let $(X, \mathcal{T})$ be a topological space, $Y \subseteq X$ and $\mathcal{T}|_Y = \{U \cap Y \mid U \in \mathcal{T}\}$. Then $\mathcal{T}|_Y$ is a topology of Y, $(Y, \mathcal{T}|_Y)$ is said to be a topological subspace of $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space. If for any $x_1, x_2 \in X, x_1 \neq x_2$, there exist neighbourhood V_1 of x_1 and neighbourhood V_2 of x_2 such that $V_1 \cap V_2 = \emptyset$, then $(X, \mathcal{T})$ is said to be a Hausdorff space.

Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be two topological spaces, $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ be a map. If for each $x_0 \in X$ and each neighbourhood V of $f(x_0)$, there exists a neighbourhood U of x_0, such that $f(U) \subseteq V$, then f is said to be continuous at x_0. If f is continuous at each point of X, then f is said to be continuous on X.

Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be two topological spaces, $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ be a map. Then f is continuous at x_0 iff for each net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X, when $\{x_\alpha\}_{\alpha \in \Lambda}$ converges to x_0, $\{f(x_\alpha)\}_{\alpha \in \Lambda}$ converges to $f(x_0)$.

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{T}$. If each open set U of X there exist a subfamily $\mathcal{B}_1$ of $\mathcal{B}$ such that $U = \cup\{A : A \in \mathcal{B}_1\}$, then $\mathcal{B}$ is said to be a base of $(X, \mathcal{T})$.

The family $\mathcal{B}$ is a base of the set $X = \bigcup\{B \mid B \in \mathcal{B}\}$ with respect to some topology iff for any $U, V \in \mathcal{B}$ and $x \in U \cap V$, there exists a $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq U \cap V$.

Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ be two topological spaces, $\mathcal{T} = \{U \times V \mid U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$. If $U_1, U_2 \in \mathcal{T}_1, V_1, V_2 \in \mathcal{T}_2$, then $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, so $\mathcal{T}$ is a base of $X \times Y$ with respect to some topology, the topology is said to be the product topology of $X \times Y$, X, Y is said to be the coordinate spaces.

Let $P_X : X \times Y \longrightarrow X, (x, y) \longmapsto x$ and $P_Y : X \times Y \longrightarrow Y, (x, y) \longmapsto y$. If $U \in \mathcal{T}_1$, then $P_X^{-1}(U) = U \times Y$, so P_X is continuous, similarly, P_Y is also continuous. P_X and P_Y are said to be the projection maps of from $X \times Y$ to X and Y, respectively.

Suppose that a set V is both a linear space with scalar field $\mathbf{K}$ and also a Hausdorff topological space. If the algebraic and topological structures are so related that the mappings

$$(x, y) \to x + y : V \times V \to V,$$
$$(a, x) \to ax : \mathbf{K} \times V \to V,$$

are continuous, where $V \times V$ and $\mathbf{K} \times V$ have their product topologies, then V is said to be a *linear topological space.*

The simplest examples of linear topological spaces over $\mathbf{K}$ are the sets $\mathbf{K^n}$, $n = 1, 2, \cdots$, with their usual vector space structure and with the product topology.

It is apparent that a complex linear topological space can be viewed also as a real one, simply by restricting the scalar field. A linear subspace of a linear topological space is itself a linear topological space, with the relative topology.

The given topology, in a linear topological space V, is sometimes described as the *initial topology* in order to distinguish it from other topologies that can naturally be introduced, such as the weak topologies described later. It is usual to develop the early parts of the theory without the assumption that the initial topology is Hausdorff. However, for most purposes, an easy quotient procedure permits an immediate reduction to the Hausdorff case; and the initial topology is Hausdorff in all the cases we shall encounter later. Accordingly, we have included this condition as part of our definition of linear topological spaces.

If V_1 is a linear subspace of a linear topological space V, the closure V_0 of V_1 is also a linear subspace. Indeed, suppose that $x_0, y_0 \in V_0$ and a, b are scalars. Then (x_0, y_0) lies in the closure $V_0 \times V_0$ of $V_1 \times V_1$, and is therefore the limit of a net $\{(x_j, y_j)\}$ in $V_1 \times V_1$. Since V_1 is a subspace of V, and the mapping $(x, y) \to ax + by : V \times V \to V$ is continuous, we have

$$ax_j + by_j \in V_1, \quad ax_j + by_j \to ax_0 + by_0,$$

and $ax_0 + by_0$ lies in the closure V_0 of V_1. Similar arguments show that if a subset of V is balanced or convex, then the same is true of its closure. Note also that an open set G in V consists entirely of internal points. For this, suppose that $x \in G, y \in V$. Since the mapping $a \to x + ay : \mathbf{R} \to V$ is continuous and takes 0 into the open set G, it carries some real interval $(-c, c)$ into G; and in particular, $x + ay \in G$ whenever $0 \leq a < c$.

Suppose that V is a linear topological space with scalar field $\mathbf{K}, x_0 \in V$ and $W \subseteq V$. Since the continuous mapping $x \to x + x_0 : V \to V$ has a continuous inverse mapping $x \to x - x_0$, it follows that W is a neighborhood of 0 if and only if $x_0 + W$ is a neighborhood of x_0. Accordingly, the topology of V is determined once a base of neighborhoods of 0 has been specified. If W is a neighborhood of 0, then so is aW for each non-zero scalar a, since the one-to-one mapping $x \to ax$, from V onto V, is bicontinuous; in particular, $-W$ is a neighborhood of 0. From continuity at $(0, 0)$ of the mapping $(x, y) \to x + y$, there is a neighborhood V_0 of 0 such that $V_0 + V_0 \subseteq W$. From continuity at $(0, 0)$ of the mapping $(a, x) \to ax$, there exist a neighborhood V_1 of 0, and a positive real number ϵ, such that $ax \in W$ whenever $x \in V_1$ and $|a| \leq \epsilon$. From this, $\bigcup\{aV_1 : 0 < |a| \leq \epsilon\}$ is a balanced open subset of W; so every neighborhood of 0 contains a balanced neighborhood of 0.

Proposition 1.2.1. *Suppose that V and W are linear topological spaces with the same scalar field $\mathbf{K}$, and $T : V \to W$ is a linear operator.*

(i) If $x_0 \in V$ and T is continuous at x_0, then T is uniformly continuous on V.

(ii) If C is balanced convex subset of V and the restriction $T|C$ is continuous at 0, then $T|C$ is uniformly continuous on C.

Proof. (i). Since T is continuous at x_0, given any neighborhood W_0 of 0 in W, there is a neighborhood V_0 of 0 in V such that $Tx \in Tx_0 + W_0$ whenever $x \in x_0 + V_0$. Thus, when $y - x \in V_0$, $x_0 + y - x \in x_0 + V_0$, therefore $Tx_0 + Ty - Tx \in Tx_0 + W_0$, so $Ty - Tx \in W_0$. That is, T is uniformly continuous on V.

(ii). Since the restriction $T|C$ is continuous at 0, given any neighborhood W_0 of 0 in W, there is a balanced neighborhood V_0 of 0 in V such that $Tx \in \frac{1}{2}W_0$ whenever $x \in V_0 \cap C$.

Suppose that $x, y \in C$ and $y - x \in V_0$. Since C is convex, and both C and V_0 are balanced, we have $\frac{1}{2}y - \frac{1}{2}x \in V_0 \cap C$; hence $\frac{1}{2}Ty - \frac{1}{2}Tx \in \frac{1}{2}W_0$, thus, $Tx - Ty \in W_0$ whenever $x, y \in C$ and $x - y \in V_0$, that is, $T|C$ is uniformly continuous on C.

■

Lemma 1.2.2. *Suppose that V is a linear topological space, ρ is a linear functional on V, and p is a semi-norm on V.*

(i) If there is a non-empty open set G in V and a real number c such that $Re\rho(x) < c$ whenever $x \in G$, then ρ is uniformly continuous on V.

(ii) If p is bounded on some neighborhood of 0 in V, then p is uniformly continuous on V.

Proof. (i). If G and c have the stated properties, we can choose x_0 in G and a balanced neighborhood V_0 of 0 in V such that $x_0 + V_0 \subseteq G$. Given x in V_0, let a be a scalar such that $|a| = 1$ and $|\rho(x)| = \rho(ax)$. Then $ax \in V_0$, $x_0 + ax \in x_0 + V_0 \subseteq G$, and therefore

$$c > Re\rho(x_0 + ax) = Re\rho(x_0) + |\rho(x)|.$$

Hence $|\rho(x)| < b$ for all x in V_0, where $b = c - Re\rho(x_0)$.

Given any positive ε, $|\rho(x)| < \varepsilon$ for all x in the neighborhood $\varepsilon b^{-1}V_0$ of 0. Thus, ρ is continuous at 0, and is therefore uniformly continuous on V by Proposition 1.2.1.

(ii). Suppose that there is a neighborhood V_0 of 0 in V and a pisitive real number b such that $p(x) < b$ whenever $x \in V_0$. Given any positive ε, the set $\varepsilon b^{-1}V_0$ is a neighborhood of 0 in V, and

$$|p(y) - p(x)| \leq p(y - x) < \varepsilon$$

whenever $x, y \in V$ and $y - x \in \varepsilon b^{-1}V_0$. Thus, ρ is uniformly continuous on V.

■

Corollary 1.2.3. *A linear functional ρ on a linear topological space V is continuous if and only if its null space $\rho^{-1}(0)$ is closed in V.*

Proof. We may assume that $\rho \neq 0$; it is evident that $\rho^{-1}(0)$ is closed if ρ is continuous.

Conversely, suppose that $\rho^{-1}(0)$ is closed. We can choose x_0 in V so that $\rho(x_0) = 1$. Since $x_0 \notin \rho^{-1}(0)$, there is a balanced neighborhood V_0 of 0 in V such that $x_0 + V_0$ does not meet $\rho^{-1}(0)$.

If $x \in V_0$ and $|\rho(x)| \geq 1$, we can choose a scalar a so that $|a| \leq 1$ and $\rho(ax) = -1$. Then $ax \in V_0, x_0 + ax \in x_0 + V_0$, and $\rho(x_0 + ax) = 0$, contrary to our assumption that $x_0 + V_0$ does not meet $\rho^{-1}(0)$. From this, it follows that $|\rho(x)| < 1$ whenever $x \in V_0$, and ρ is continuous by Lemma 1.2.2 (i). ■

A *Locally convex space* is a linear topological space in which the topology has a base consisting of convex sets.

Theorem 1.2.4. *Suppose that* V *is a real or complex vector space, and* Γ *is a family of semi-norms on* V *that separates the points of* V *in the following sense: if* $x \in V$ *and* $x \neq 0$*, there is an element* p *of* Γ *for which* $p(x) \neq 0$*. Then there is a locally convex topology on* V *in which, for each* x_0 *in* V*, the family of all sets*

$$V(x_0 : p_1, \cdots, p_m; \epsilon) = \{x \in V : p_j(x - x_0) < \epsilon, j = 1, \cdots, m\},$$

where $\epsilon > 0$ *and* $p_1, \cdots, p_m \in \Gamma$*, is a base of neighborhoods of* x_0*, . With this topology, each of the semi-norms in* Γ *is continuous. Moreover, every locally convex topology on* V *arises, in this way, from a suitable family of semi-norms.*

Corollary 1.2.5. *In a locally convex space there is a base of neighborhoods of 0 consisting of balanced convex sets.*

Proof. The sets $V(0 : p_1, \cdots, p_m; \epsilon)$ are balanced and convex. ■

Proposition 1.2.6. *Suppose that* V_1 *and* V_2 *are locally convex spaces with the same scalar field* $\mathbf{K}$*; and for* $j = 1, 2$*, let* Γ_j *be a separating family of semi-norms on* V_j *that gives rise to the topology of* V_j*.*

(i) A semi-norm p on V_1 is continuous if and only if there is a positive real number C and a finite set $p_1, \cdots, p_m$ of elements of Γ_1 such that

$$p(x) \leq C \max\{p_1(x), \cdots, p_m(x)\}, x \in V_1.$$

(ii) A linear operator $T : V_1 \to V_2$ is continuous if and only if, given any q in Γ_2, there is a positive real number C and a finite set $p_1, \cdots, p_m$ of elements of Γ_1 such that

$$q(Tx) \leq C \max\{p_1(x), \cdots, p_m(x)\}, x \in V_1.$$

(iii) A linear functional ρ on V_1 is continuous if and only if there is a positive real number C and a finite set $p_1, \cdots, p_m$ of elements of Γ_1 such that

$$|\rho(x)| \leq C \max\{p_1(x), \cdots, p_m(x)\}, x \in V_1.$$

Proof. (i) If p is continuous, the set $\{x \in V_1 : p(x) < 1\}$ is a neighborhood of 0 in V_1; so it contains one of the basic neighborhoods $V(0 : p_1, \cdots, p_m; \epsilon)$, where $\epsilon > 0$, and $p_1, \cdots, p_m \in \Gamma_1$. If $x \in V_1$ and $p(x) > \epsilon^{-1} \max\{p_1(x), \cdots, p_m(x)\}$, we may assume, upon replacing x by cx, for some positive c, that $p(x) = 1$ and $p_j(x) < \epsilon, j = 1, \cdots, m$; that is, $p(x) = 1$ and $x \in V(0 : p_1, \cdots, p_m; \epsilon)$, contradicting our assumption concerning this basic neighborhood. Accordingly, the stated condition

$$p(x) \leq C \max\{p_1(x), \cdots, p_m(x)\}, x \in V_1$$

is satisfied, where $C = \epsilon^{-1}$. Conversely, if this condition is satisfied, then p is bounded on $V(0 : p_1, \cdots, p_m; 1)$; by Lemma 2.2 (ii), p is continuous on V_1.
(ii) When q is a semi-norm on V_2, the composite mapping $q \circ T$ is a semi-norm on V_1. In view of this, and taking into account part (i) of the proposition, we

have to show that T is continuous if and only if $q \circ T$ is continuous whenever $q \in \Gamma_2$. By Theorem 1.2.4, each q in Γ_2 is continuous on V_2; so continuity of T entails continuity of $q \circ T$.

Conversely, suppose that $q \circ T$ is continuous, for each q in Γ_2. Every neighborhood V of 0 in V_2 contains a basic neighborhood $V(0 : q_1, \cdots, q_m; \epsilon)$, where $\epsilon > 0$ and $q_1, \cdots, q_m \in \Gamma_2$. Since $q_j \circ T$ is continuous for each $j = 1, \cdots, m$, the set

$$W = \{x \in V_1 : q_j(Tx) < \epsilon, j = 1, \cdots, m\}$$

is a neighborhood of 0 in V_1, and it is apparent that

$$T(W) \subseteq V(0 : q_1, \cdots, q_m; \epsilon) \subseteq V.$$

Thus T is continuous at 0, and therefore throughout V_1.

(iii) The scalar field $\mathbf{K}$ is a locally convex space, its usual topology being obtained from a single norm, the modulus function; and $\rho : V_1 \to \mathbf{K}$ is a linear operator. Thus (iii) is a special case of (ii). ∎

Our next two results are Hahn-Banach separation theorems. They are formulated so as to apply to both real and complex linear topological spaces, the notation being redundant in the real case.

Theorem 1.2.7. *If Y and Z are disjoint non-empty convex subsets of a linear topological space V, and Y is open, there is a continuous linear functional ρ on V and a real number k such that*

$$Re\rho(y) > k \geq Re\rho(z), y \in Y, z \in Z.$$

If, further, Z is open, then $k > Re\rho(z)$ for each z in Z.

Proof. In view of the fact that an open set consists entirely of internal points, the assumptions of Theorem 1.1.2 are fulfilled in the real case, while those of Theorem 1.4 obtain in the complex case. From those theorems, there is a linear functional ρ on V that satisfies the stated inequalities; and by applying Lemma 1.2.2(i), with $-\rho$ and Y in place of ρ and G, it follows that ρ is continuous. ∎

Theorem 1.2.8. *If Y and Z are disjoint non-empty closed convex subsets of a locally convex space V, at least one of which is compact, there are real numbers a, b and a continuous linear functional ρ on V such that $Re\rho(y) \geq a > b \geq Re\rho(z)$, where $y \in Y, z \in Z$.*

Corollary 1.2.9. *If x is a non-zero vector in a locally convex space V, there is a continuous linear functional ρ on V such that $\rho(x) \neq 0$.*

Corollary 1.2.10. *If Z is a closed convex subset of a locally convex space V, and $y \in V \setminus Z$, there is a continuous linear functional ρ on V and a real number b such that $Re\rho(y) > b$, $Re\rho(z) \leq b, z \in Z$.*

Corollary 1.2.11. *If Z is a closed subspace of a locally convex space V, and $y \in V \setminus Z$, there is a continuous linear functional ρ on V such that*

$$\rho(y) \neq 0, \rho(z) = 0, z \in Z.$$

We now prove a Hahn-Banach extension theorem.

Corollary 1.2.12. *If ρ_0 is a continuous linear functional on a subspace V_0 of a locally convex space V, there is a continuous linear functional ρ on V such that $\rho|V_0 = \rho_0$.*

Proof. Let Γ be a family of semi-norms that gives rise, as in Theorem 1.2.4, to the topology on V. By restricting each member of Γ to V_0, we obtain a family of semi-norms that defines the relative topology on V_0. Since ρ_0 is continuous, it follows from Proposition 1.2.6(iii) that there is a positive real number C and a finite set $p_1, \cdots, p_m$ of elements of Γ such that

$$|\rho_0(y)| \leq C \max\{p_1(y), \cdots, p_m(y)\}, y \in V_0.$$

By Theorem 1.1.7, ρ_0 extends to a linear functional ρ on V, such that

$$|\rho(x)| \leq C \max\{p_1(x), \cdots, p_m(x)\}, x \in V,$$

since the equation

$$p(x) = C \max\{p_1(x), \cdots, p_m(x)\}$$

defines a semi-norm p on V. A further application of Proposition 2.6 (iii) shows that ρ is continuous. ■

Suppose that V is a linear space with scalar field $\mathbf{K}$ and F is a family of linear functionals on V, which separates the points of V in the following sense: if x is a non-zero vector in V then, for some ρ in F, $\rho(x) \neq 0$. When $\rho \in F$, the equation $p_\rho(x) = |\rho(x)|$ defines a semi-norm p_ρ on V. The separating family $\{p_\rho : \rho \in F\}$ of semi-norms gives rise, as in Theorem 2.4, to a locally convex topology on V, the *weak topology induced on V by F*. In this topology, which we denote by $\sigma(V, F)$, each point x_0 of V has a base of neighborhoods that consists of all sets of the form

$$V(x_0 : \rho_1, \cdots, \rho_m; \epsilon) = \{x \in V : |\rho_j(x) - \rho_j(x_0)| < \epsilon, j = 1, \cdots, m\},$$

where $\epsilon > 0$ and $\rho_1, \cdots, \rho_m \in F$.

Since $|\rho(x) - \rho(x_0)| < \epsilon$ when $x \in V(x_0 : \rho; \epsilon)$, each of the linear functionals ρ in F is continuous relative to the topology $\sigma(V, F)$. However, if τ is a topology on V, and each linear functional in F is $\tau-$continuous, then all the sets $V(x_0 : \rho_1, \cdots, \rho_m; \epsilon)$ are $\tau-$open; from this, $\sigma(V, F)$ is coarser than τ. Accordingly, $\sigma(V, F)$ is the coarsest topology on V relative to which each element of F is a continuous mapping from V into $\mathbf{K}$.

Theorem 1.2.13. *Suppose that V is a linear space with scalar field $\mathbf{K}$, F is a separating family of linear functionals on V, and L is the set of all finite linear combinations of elements of F. Then $\sigma(V, L)$ coincides with $\sigma(V, F)$, and L is the set of all $\sigma(V, F)-$continuous linear functionals on V.*

Proof. Since $F \subseteq L$, $\sigma(V, F)$ is coarser than $\sigma(V, L)$. Every linear functional ρ in L is a linear combination $a_1\rho_1 + \cdots + a_m\rho_m$ of elements $\rho_1, \cdots, \rho_m$ of F; moreover, each ρ_j is $\sigma(V, F)$-continuous, so the same is true of ρ. Since $\sigma(V, L)$ is the coarsest topology that makes every element of L continuous, it now follows that $\sigma(V, L)$ is coarser than $\sigma(V, F)$; so these two topologies coincide.

It remains to prove that each $\sigma(V, F)$-continuous linear functional ρ_0 lies in L. Given such ρ_0, we can apply Proposition 1.2.6(iii), bearing in mind the form of the semi-norms p_ρ(with ρ in F), which give rise to the topology $\sigma(V, F)$. It follows that there are elements $\rho_1, \cdots, \rho_m$ of F and a positive real number C such that

$$|\rho_0(x)| \leq C \max\{|\rho_1(x)|, \cdots, |\rho_m(x)|\}$$

for each x in V. In particular, $\rho_0(x) = 0$ if $\rho_j(x) = 0$ for $j = 1, \cdots, m$. From

Proposition 1.1.1, ρ_0 is a linear combination of $\rho_1, \cdots, \rho_m$, and thus $\rho_0 \in L$. ■

Proposition 1.2.14. *Suppose that V is a linear space with scalar field $\mathbf{K}(= \mathbf{R}$ or $\mathbf{C})$, F is a separating family of linear functionals on V, and φ is a mapping from a topological space S into V. Then φ is continuous, relative to the topology $\sigma(V, F)$ on V, if and only if for each $\rho \in F$, the composite mappings $\rho \circ \varphi : S \to \mathbf{K}$ is continuous .*

Proof. Since each ρ in F is $\sigma(V, F)$-continuous, it is evident that continuity of φ entails continuity of the composite mapping $\rho \circ \varphi$.

Conversely, suppose that $\rho \circ \varphi$ is continuous for each ρ in F. In order to establish the continuity of φ, it suffices to show that $\varphi^{-1}(V_0)$ is an open subset of S whenever V_0 is one of the basic $\sigma(V, F)$-open sets $V(x_0 : \rho_1, \cdots, \rho_m; \epsilon)$. Of course, $\rho_1, \cdots, \rho_m \in F$, so $\rho \circ \varphi : S \to \mathbf{K}$ is continuous for $j = 1, \cdots, m$. From this, and since

$$\begin{aligned} \varphi^{-1}(V_0) &= \{s \in S : \varphi(s) \in V_0\} \\ &= \{s \in S : |\rho_j(\varphi(s)) - \rho_j(x_0)| < \epsilon, j = 1, \cdots, m\}, \end{aligned}$$

$\varphi^{-1}(V_0)$ is open, as required. ■

Suppose that V is a locally convex space. The set of all continuous linear functionals on V is a subspace of the algebraic dual space of V; it is denoted by $V^\sharp$, and is described as the *continuous dual space* of V. By Corollary 1.2.9, it separates the points of V. The topology $\sigma(V, V^\sharp)$ is called the *weak topology on V*. It is the coarsest topology on V that makes all the linear functionals in $V^\sharp$ continuous; in particular, therefore, it is coarser than initial topology on V. When using topological concepts, such as continuity or compactness, in relation to the weak topology, we refer to weak continuity, weak compactness, etc. By Theorem 1.2.13, with $F = L = V^\sharp$, the weakly continuous linear functionals on V are precisely the members of $V^\sharp$; in other words, a linear functional on V is weakly continuous if and only if it is continuous relative to the initial topology on V.

Proposition 1.2.15. *If V and W are locally convex spaces and $T : W \to V$ is a continuous linear operator, then T is continuous also relative to the weak topologies on V and W.*

Proof. When $\rho \in V^\sharp$, the linear functional $\rho \circ T$ on W is continuous relative to the initial topology on W, and is therefore continuous also relative to the weak topology $\sigma(W, W^\sharp)$. If we now apply Proposition 1.2.14, taking for S the space W with its weak topology, it follows that T is continuous relative to the topologies $\sigma(W, W^\sharp)$ and $\sigma(V, V^\sharp)$. ∎

Proposition 1.2.16. *A convex subset Z of a locally convex space V has the same closure, relative to the initial and weak topologies on V.*

Proof. Since the weak topology on V is coarser than the initial topology τ, it suffice to prove the following assertion: if $y \in V$ and y is not in the τ-closure Z^- of Z, then y is not in the weak closure of Z. Now Z^- is convex, and it follows from Corollary 1.2.10 that there is a continuous linear functional ρ on V and a real number b such that

$$Re\rho(y) > b, \qquad Re\rho(z) \leq b, z \in Z^-.$$

The set $S = \{x \in V : Re\rho(x) \leq b\}$ is weakly closed since ρ is weakly continuous; moreover, $y \notin S$, and S contains Z and so contains the weak closure of Z. ∎

When V is a locally convex space and $x \in V$, the equation

$$\hat{x}(\rho) = \rho(x), \rho \in V^\sharp$$

defines a linear functional $\hat{x}$ on the continuous dual space $V^\sharp$. The set

$$\hat{V} = \{\hat{x} : x \in V\}$$

is a linear subspace of the algebraic dual space of $V^\sharp$. It separates the points of $V^\sharp$ since, if $\rho \in V^\sharp$ and $\hat{x}(\rho) = 0$ for all x in V, then $\rho(x) = 0$ for each $x \in V$, and thus $\rho = 0$. We often write $\sigma(V^\sharp, V)$, rather than $\sigma(V^\sharp, \hat{V})$, for the weak topology induced on $V^\sharp$ by $\hat{V}$; and we refer to $\sigma(V^\sharp, V)$ as the *weak* topology*, sometimes called the w*-topology on $V^\sharp$. Note that each ρ_0 in $V^\sharp$ has a base of neighborhoods consisting of sets of the form

$$\{\rho \in V^\sharp : |\rho(x_j) - \rho_0(x_j)| < \epsilon, j = 1, \cdots, m\},$$

where $\epsilon > 0$ and $x_1, \cdots, x_m \in V$. From Theorem 1.2.13, that is, with V replaced by $V^\sharp$, and $F = L = \hat{V}$, the weak* continuous linear functionals on $V^\sharp$ are precisely the elements of $\hat{V}$; so we have the following result.

Proposition 1.2.17. *A linear functional ω, on the continuous dual space $V^\sharp$ of a locally convex space V, is weak* continuous if and only if there is an element x of V such that $\omega(\rho) = \rho(x)$ for each ρ in $V^\sharp$.*

Suppose that V is a locally convex space. By the *closed convex hull* of a subset Y of V we mean the closure $\overline{co}Y$ of the convex hull coY; it is clear that this is the smallest closed convex set that contains Y. An element x_0 of a convex set X in V is described as an *extreme point* of X if the only way in which it can be expressed as a convex combination $x_0 = (1-a)x_1 + ax_2$, with $0 < a < 1$ and x_1, x_2 in X, is by taking $x_1 = x_2 = x_0$.

By *a face* of a convex set X in V we mean a non-empty convex subset F of X, such that the conditions

$$0 < a < 1, \qquad x_1, x_2 \in X, \qquad (1-a)x_1 + ax_2 \in F,$$

imply that $x_1, x_2 \in F$. Note that x_0 is an extreme point of X if and only if the one-point set $\{x_0\}$ is a face of X. It is an apparent that, if a family of faces of X has non-empty intersection, then this intersection is itself a face of X. Moreover, if Y is a face of X and Z is a face of Y, then Z is a face of X. In particular, therefore, an extreme point of a face of X is also an extreme point of X.

The following lemma provides a slight strengthening of the defining property of a face; and upon specializing to "one-point" face, it reduces to a similar assertion about extreme points.

Lemma 1.2.18. *If F is a face of a convex set X and $a_1x_1 + \cdots + a_nx_n \in F$, where $x_1, \cdots, x_n \in X$ and $a_1, \cdots, a_n$ are non-negative real numbers with sum 1, then $x_j \in F$ when $1 \leq j \leq n$ and $a_j > 0$.*

Proof. It suffices to show that $x_1 \in F$ if $a_1 > 0$. If $a_1 = 1$, then $a_2 = a_3 = \cdots = a_n = 0$ and $x_1 = a_1x_1 + \cdots + a_nx_n \in F$. If $0 < a_1 < 1$, let $a = 1 - a_1$ and let y be the convex combination $a^{-1}(a_2x_2 + \cdots + a_nx_n)$ of $x_2, \cdots, x_n$. Then $x_1, y \in X, 0 < a < 1$, and

$$(1-a)x_1 + ay = a_1x_1 + a_2x_2 \cdots + a_nx_n \in F.$$

Since F is a face of X, it follows that $x_1 \in F$. ■

Lemma 1.2.19. *If X is a non-empty compact convex set in a locally convex space V, ρ is a continuous linear functional on V, and*

$$c = \sup\{Re\rho(x) : x \in X\},$$

then the set $F = \{x \in X : Re\rho(x) = c\}$ is a compact face of X.

Proof. Since a continuous real-valued function on a compact set attains its supremum, F is not empty; and it is evident that F is compact and convex. If $x_1, x_2 \in X, 0 < a < 1$, and $(1-a)x_1 + ax_2 \in F$, we have $Re\rho(x_1) \leq c, Re\rho(x_2) \leq c$, and

$$(1-a)Re\rho(x_1) + aRe\rho(x_2) = Re\rho((1-a)x_1 + ax_2) = c;$$

so $Re\rho(x_1) = Re\rho(x_2) = c$, and $x_1, x_2 \in F$. Thus F is a face of X. ∎

Theorem 1.2.20. *(Klein-Milman). If X is a non-empty compact convex set in a locally convex space V, then X has an extreme point. Moreover, $X = \overline{co}E$, where E is the set of all extreme points of X.*

Proof. The family F of all compact faces of X is non-empty since $X \in F$, and is partially ordered by the inclusion relation $\subseteq$. Let F_0 be a subfamily of F that is totally ordered by inclusion. It is evident that F_0 has the finite-intersection property, so by compactness the set $G_0 = \bigcap\{G : G \in F_0\}$ is non-empty. Thus G_0 is a compact face of X, and is a lower bound of F_0 in F.

Since every totally ordered subset of F has a lower bound in F, it follows from Zorn's lemma that F has an element G that is minimal with respect to inclusion. We shall show that G consists of a single point x, and since G is a compact face of X, it then follows that x is an extreme point of X. To this end, suppose the contrary, and let x_1, x_2 be distinct elements of G. By the Hahn-Banach theorem, there is a continuous linear functional ρ on V such that $Re\rho(x_1) \neq Re\rho(x_2)$. From Lemma 1.2.19 we can choose a real number c so that the set

$$G_0 = \{x \in G : Re\rho(x) = c\}$$

is a compact face of G. Accordingly, G_0 is a compact face of X; that is, $G_0 \in F$. Since $Re\rho(x_1) \neq Re\rho(x_2)$, at least one of x_1, x_2 lies outside G_0; so G_0 is a proper subset of G, contrary to our minimality assumption. Hence G consists of a single point.

So far, we have shown that each non-empty compact convex subset of V has an extreme point. If E denotes the set of all extreme points of X, it is clear that $\overline{co}E \subseteq X$, and we have to show that equality occurs. Suppose the contrary, and let $x_0 \in X \setminus \overline{co}E$; we shall obtain a contradiction. From the Hahn-Banach theorem, we can find a continuous linear functional ρ on V and a real number a such that

$$Re\rho(x_0) > a \geq Re\rho(y), y \in \overline{co}E. \qquad (1).$$

If $c_1 = \sup\{Re\rho(x) : x \in X\}$, then $c_1 > a$, and the set

$$G_1 = \{x \in X : Re\rho(x) = c_1\}$$

is a compact face of X by Lemma 2.19. In particular, G_1 is a non-empty compact convex subset of V, and so has an extreme point x_1. Since x_1 is an extreme point of a face of X, it is an extreme point of X; that is, $x_1 \in E$. However, $Re\rho(x_1) = c_1 > a$, contradicting (1). ∎

Corollary 1.2.21. *If X is a non-empty compact convex set in a locally convex space V and ρ is a continuous linear functional on V, there is an extreme point x_0 of X such that $Re\rho(x) \leq Re\rho(x_0)$ for each x in X.*

Proof. Let
$$c = \sup\{Re\rho(x) : x \in X\}.$$
By Lemma 1.2.19, the set $\{x \in X : Re\rho(x) = c\}$ is a compact face of X. In particular, it is a non-empty compact convex set in V, and so has an extreme point x_0. Since x_0 is an extreme point of a face of X, it is an extreme point of X; and
$$Re\rho(x_0) = c \geq Re\rho(x), x \in X.$$
■

Next, we shall be concerned with the continuous dual space $X^\sharp$ of a normed space X and with the properties of the weak topology $\sigma(X, X^\sharp)$ on X and the weak* topology $\sigma(X^\sharp, X)$ on $X^\sharp$. It turns out that, in a natural way, $X^\sharp$ becomes a Banach space and X is isometrically isomorphic to a subspace of the second dual space $X^{\sharp\sharp}(= (X^\sharp)^\sharp)$. We describe a necessary and sufficient condition for this subspace to be the whole of $X^{\sharp\sharp}$.

A linear functional ρ on X is continuous if and only if it is bounded, in the sense that there is a non-negative real number C such that
$$|\rho(x)| \leq C\|x\|, x \in X.$$
When ρ is bounded, the least possible value of C is the bound $\|\rho\|$ of ρ, defined by
$$\|\rho\| = \sup\{\frac{|\rho(x)|}{\|x\|} : x \in X, x \neq 0\} = \sup\{|\rho(x)| : x \in X, \|x\| = 1\}. \qquad (2)$$

The continuous dual space $X^\sharp$ of X, defined as the linear space of all continuous linear functionals on X, coincides with $B(X, \mathbf{K})$; it is a Banach space, its norm being given by (2). We refer to $X^\sharp$, with this norm, as the *Banach dual space* of X.

For normed spaces, we have the following Hahn-Banach extension theorem.

Theorem 1.2.22. *If X_0 is a subspace of a normed space X and ρ_0 is a bounded linear functional on X_0, there is a bounded linear functional ρ on X such that $\|\rho\| = \|\rho_0\|$ and $\rho(x) = \rho_0(x)$ when $x \in X_0$.*

Proof. This follows at once from Theorem 1.7, with

$$p(x) = \|\rho_0\|\|x\|, x \in X.$$

■

Corollary 1.2.23. *If x_0 is a non-zero vector in a normed space X, there is a bounded linear functional ρ on X such that $\|\rho\| = 1$ and $\rho(x_0) = \|x_0\|$.*

Proof. The equation $\rho_0(cx_0) = c\|x_0\|$ defines a bounded linear functional ρ_0 on the one-dimensional subspace X_0 generated by x_0; moreover, $\rho_0(x_0) = \|x_0\|$, and $\|\rho_0\| = 1$. By Theorem 2.22, ρ_0 extends, still with norm 1, to a bounded linear functional ρ on X.■

Corollary 1.2.24. *If Y is a closed subspace of a normed space X and $x_0 \in X \backslash Y$, there is a bounded linear functional ρ on X such that $\|\rho\| = 1, \rho(y) = 0$ for each y in Y, and $\rho(x_0) = d$, where*

$$d = \inf\{\|x_0 + y\| : y \in Y\},$$

the distance from x_0 to Y

Proof. The quotient mapping $Q : X \to X/Y$ is a bounded linear operator, and $d = \|Qx_0\| > 0$. By Corollary 1.2.23, there is a bounded linear functional ρ_0 on X/Y such that $\|\rho_0\| = 1$ and $\rho_0(Qx_0) = d$. The equation $\rho(x) = \rho_0(Qx)$ defines a bounded linear functional ρ on X; $\rho(x_0) = d$, and $\rho(y) = 0$ for each y in Y. Since ρ has the factorization $\rho_0 Q$ through X/Y, it follows that $\|\rho\| = \|\rho_0\| = 1$. ∎

Theorem 1.2.25. *If X is a normed space and $x \in X$, the equation*

$$\hat{x}(\rho) = \rho(x), \rho \in X^\sharp$$

defines a bounded linear functional $\hat{x}$ on the Banach dual space $X^\sharp$. The mapping $x \to \hat{x}$ is an isometric isomorphism from X onto the subspace $\hat{X} = \{\hat{x} : x \in X\}$ of the second dual space $X^{\sharp\sharp}$.

Proof. It is evident that $\hat{x}$, as defined in the theorem, is a linear functional on $X^\sharp$, and that the mapping $x \to \hat{x}$ is a linear operator from X into the algebraic dual space of $X^\sharp$. When $x_0 \in X$,

$$|\hat{x_0}(\rho)| = |\rho(x_0)| \leq \|\rho\|\|x_0\|, \rho \in X^\sharp.$$

When ρ is chosen as in Corollary 1.2.23,

$$|\hat{x_0}(\rho)| = \|x_0\| = \|\rho\|\|x_0\|.$$

Thus $\hat{x_0}$ is a bounded linear functional, and $|\hat{x_0}| = \|x_0\|$; so the mapping $x \to \hat{x}$ is an isometric isomorphism from X onto a subspace $\hat{X}$ of $X^{\sharp\sharp}$. ∎

When X is a normed space, the mapping $x \to \hat{x}$ occurring in Theorem 1.2.25 is called the *natural isometric isomorphism from X into $X^{\sharp\sharp}$,* and $\hat{X}$ is described as the *natural image of* X in $X^{\sharp\sharp}$. The weak* topology $\sigma(X^\sharp, X)$ is the weak topology induced on $X^\sharp$ by $\hat{X}$.

If $\hat{X} = X^{\sharp\sharp}$, the normed space X is said to be *reflexive.* A reflexive normed space is necessarily a Banach space since it is isometrically isomorphic to the Banach dual space $X^{\sharp\sharp}$. However, many Banach spaces are not reflexive.

Theorem 1.2.26. *Suppose that* X *is a normed space and* $\hat{X}$ *is the natural image of* X *in* $X^{\sharp\sharp}$.
(i) The unit ball $(X^\sharp)_1$ *is compact in the weak* topology* $\sigma(X^\sharp, X)$ *on* $X^\sharp$.
(ii) The weak closure in* $X^{\sharp\sharp}$ *of the unit ball of* $\hat{X}$ *is the unit ball* $(X^{\sharp\sharp})_1$ *of* $X^{\sharp\sharp}$.

Theorem 1.2.27. *If* S *is a bounded weak* closed subset of the Banach dual space* $X^\sharp$ *of a normed space* X, *then* S *is weak* compact. If, in addition,* S *is convex, it is the weak* closed convex hull of its extreme points.*

Proof. For some positive r, S is a weak* closed subset of the ball $(X^\sharp)_r$, and this ball is weak* compact by Theorem 1.2.26(i); so S is weak* compact. The final assertion in the corollary now follows from the Krein-Milman theorem (1.2.20), since $X^\sharp$, with the weak* topology, is a locally convex space. ■

Theorem 1.2.28. *A normed space* X *is reflexive if and only if its unit ball* $(X)_1$ *is compact in the weak topology.*

Suppose that X and Y are normed spaces and $T : X \to Y$ is a bounded linear operator. If ρ is a continuous linear functional on Y, the composite mapping $\rho \circ T$ is a continuous linear functional on X. Accordingly, we can define a mapping $T^\sharp : Y^\sharp \to X^\sharp$ by

$$T^\sharp(\rho) = \rho \circ T, \rho \in Y^\sharp.$$

We assert that $T^\sharp$ is a bounded linear operator and that $\|T^\sharp\| = \|T\|$. The linearity of $T^\sharp$ follows from the fact that

$$(a_1\rho_1 + a_2\rho_2) \circ T = a_1(\rho \circ T) + a_2(\rho_2 \circ T),$$

when $\rho_1, \rho_2 \in Y^\sharp$ and a_1, a_2 are scalars. For each ρ in $Y^\sharp$,

$$|(T^\sharp \rho)(x)| = |\rho(Tx)| \leq \|\rho\|\|Tx\| \leq \|\rho\|\|T\|\|x\|, x \in X,$$

and thus $\|T^\sharp \rho\| \leq \|T\|\|\rho\|$; so $T^\sharp$ is bounded, and $\|T^\sharp\| \leq \|T\|$. To prove the reverse inequality, it suffices to show that $\|Tx\| \leq \|T^\sharp\|\|x\|$ for each x in X. Given such x, it follows from Corollary 1.2.23 that we can choose ρ in $Y^\sharp$ so that $\|\rho\| = 1$ and $\rho(Tx) = \|Tx\|$; and

$$\begin{aligned} \|Tx\| &= |\rho(Tx)| = |(T^\sharp \rho)(x)| \\ &\leq \|T^\sharp \rho\|\|x\| \leq \|T^\sharp\|\|\rho\|\|x\| = \|T^\sharp\|\|x\|, \end{aligned}$$

as required.

When $T_1, T_2 : X \to Y$ are bounded linear operators and $\rho \in Y^\sharp$, it results from the linearity of ρ that

$$\begin{aligned} (a_1T_1 + a_2T_2)^\sharp \rho &= \rho \circ (a_1T_1 + a_2T_2) \\ &= a_1(\rho \circ T_1) + a_2(\rho \circ T_2) = a_1T_1^\sharp \rho + a_2T_2^\sharp \rho \end{aligned}$$

for all scalars a_1, a_2. Thus

$$(a_1T_1 + a_2T_2)^\sharp = a_1T_1^\sharp + a_2T_2^\sharp,$$

and the mapping $T \to T^\sharp$ is a norm-preserving linear operator from $B(X, Y)$ into $B(Y^\sharp, X^\sharp)$.

If X, Y, Z are normed spaces, and $S \in B(Y, Z), T \in B(X, Y)$, then

$$(ST)^\sharp \rho = \rho \circ (ST) = \rho \circ (S \circ T) = (\rho \circ S) \circ T = T^\sharp (S^\sharp \rho)$$

for each ρ in $Z^\sharp$; so $(ST)^\sharp = T^\sharp S^\sharp$.

The operator $T^\sharp : Y^\sharp \to X^\sharp$ is called the *Banach adjoint* of the bounded linear operator $T : X \to Y$. When X and Y are Hilbert spaces, there is another, and, in that case, more important, adjoint operator, the *Hilbert adjoint* $T^* : Y \to X$.

Proposition 1.2.29. *If T is a bounded linear operator from a normed space X into another such space Y, then $T^\sharp$ is continuous relative to the weak* topologies on $Y^\sharp$ and $X^\sharp$.*

Proof. We use the criterion set out in Proposition 1.2.14. The weak* topology on $X^\sharp$ is $\sigma(X^\sharp, \hat{X})$, where $\hat{X}$ is the natural image of X in $X^{\sharp\sharp}$. Accordingly, it suffices to show that the linear functionals $\hat{x} \circ T^\sharp$ $(x \in X)$ on $Y^\sharp$ are weak* continuous. Suppose $x \in X$, and let $y = Tx$; for each ρ in $Y^\sharp$,

$$\begin{aligned} (\hat{x} \circ T^\sharp)(\rho) &= \hat{x}(T^\sharp \rho) = (T^\sharp \rho)(x) \\ &= \rho(Tx) = \rho(y) = \hat{y}(\rho). \end{aligned}$$

Thus $\hat{x} \circ T^\sharp = \hat{y} \in \hat{Y}$, and therefore $\hat{x} \circ T^\sharp$ is continuous in the weak* topology $\sigma(Y^\sharp, \hat{Y})$. ■

1.3 Sobolev Spaces and Distributions

Many problems arising naturally in differential equations call for a generalized definition of functions and derivatives. In this paragraph we give a brief introduction to this topic, that is, defining the concepts of weak derivative, distribution and Sobolev spaces.

Let us illustrate the basic ideas in the simplest possible context: Let $I = (a, b)$ be an open interval and $f : I \to \mathbb{R}$ a continuously differentiable function. Then for every $\varphi \in C_c^{(\infty)}(I)$ one obtains using integration by parts and taking into account that $\varphi(a) = \varphi(b) = 0$, we have

$$\int_I f'(s)\varphi(s)ds = -\int_I f(s)\varphi'(s)ds,$$

where $C_c^{(\infty)}(I)$ is the set of all infinite differentiable functions which have the compact supports in I. It is not difficult to show that considering all possible *test functions* $\varphi \in C_c^{(\infty)}(a, b)$ this equation uniquely determines f', i.e., if $g : I \to \mathbb{R}$ is continuous such that

$$\int_I g(s)\varphi(s)ds = -\int_I f(s)\varphi'(s)ds \quad \forall \varphi \in C_c^{(\infty)}(a, b), \quad (*)$$

then necessarily $g = f'$.

In fact, the equation $(*)$ might still have a solution $g \in L^1(a, b)$ even if one drops the differentiability assumption of f and considers just $f \in L^1(a, b)$. The following result shows that in this case $(*)$ uniquely determines $g \in L^1(a, b)$.

Lemma 1.3.1. *(du Bois-Reymond lemma). If $I \subset \mathbb{R}$ is open and $g : I \to \mathbb{R}$ is integrable such that*

$$\int_\Omega g\varphi d\lambda = 0, \quad \forall \varphi \in C_c^{(\infty)}(I),$$

then $g = 0$ almost everywhere, i.e. $g = 0$ considered as a functon in $L^1(I)$.

This motivates the following.

Definition 1.3.2. *If for $f \in L^1(a,b)$ there exists a $g \in L^1(a,b)$ such that* $(*)$ *holds, then we call f* weakly differentiable *and $g = f'$ the* weak derivative *of f.*

This approach enables us to differentiate many functions which are not differentiable in a classical sense. Note, however, that the weak derivative can only be defined for the whole function f and not just in a single point as the classical dervative.

Example 1.3.3. *Let $I = (-1,1)$ and consider $f : I \to \mathbb{R}$ defined by $f(s) = |s|$. Then for every $\varphi \in C_c^{(\infty)}(-1,1)$, we have*

$$\begin{aligned} -\int_{-1}^{1} f(s)\varphi'(s)ds &= \int_{-1}^{0} s\varphi'(s)ds - \int_{0}^{1} s\varphi'(s)ds \\ &= -\int_{-1}^{0} \varphi(s)ds + \int_{0}^{1} \varphi(s)ds \\ &= \int_{-1}^{1} g(s)\varphi(s)ds, \end{aligned}$$

where

$$g(s) = \begin{cases} -1 & \text{if } s \leq 0, \\ +1 & \text{if } s > 0. \end{cases}$$

This shows that $f = |\cdot|$ is weakly differentiable and $f' = g$.

Using ideas from the previous sections one can generalize things even further. We start by considering for an open set $\Omega \subseteq \mathbb{R}^n$. The set

$$\mathcal{D}(\Omega) = C_c^{(\infty)}(\Omega)$$

as the *text functions*. Obviously, $\mathcal{D}(\Omega)$ is a vector space. To do analysis, we need a notion of convergence in $\mathcal{D}(\Omega)$. Recall that for a not negative multi-index $\alpha = (\alpha_1, \dots \alpha_n)$, we defined the partial derivative

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}$$

where $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

Definition 1.3.4. *A sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ is said to converge to $\varphi \in \mathcal{D}(\Omega)$ if there exists a compact subset $K \subseteq \Omega$ such that $\varphi_n(s) = 0$ for all $s \in \Omega \backslash K$, and $D^\alpha \varphi_n \to D^\alpha \varphi$ in $(C(K), \|\cdot\|_\infty)$ for every not negative multi-index α as $n \to \infty$.*

This definition enables us to introduce the *topological dual space* $\mathcal{D}^\sharp(\Omega)$ of $\mathcal{D}(\Omega)$.

Definition 1.3.5. *The elements $J \in \mathcal{D}^\sharp(\Omega)$ are called* distributions *or* generalized functions *on Ω.*

Hence J is a distribution if it is a linear functional on $\mathcal{D}(\Omega)$ such that

$$\varphi_n \to \varphi \ \text{ in } \mathcal{D}(\Omega) \Rightarrow J(\varphi_n) \to J(\varphi) \ \text{ in } \mathbb{R}.$$

Example 1.3.6. *Consider the space of all* locally integrable functions *defined by*

$$L^1_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : f \in L^1(K) \textit{ for every compact } K \subseteq \Omega\}.$$

Then every $f \in L^1_{loc}(\Omega)$ defines a linear functional

$$J_f(\varphi) = \int_\Omega f\varphi d\lambda, \ \ \varphi \in \mathcal{D}(\Omega).$$

By Lebesgue dominated convergence theorem it follows that J_f is continuous and hence every locally integrable function f correspongs, uniquely by du Bois-Reymond, to a distribution J_f.

Not every distribution J can be represented as J_f for some $f \in L^1_{loc}(\Omega)$. For example, if $s_0 \in \Omega$ then

$$J(\varphi) = \varphi(s_0), \ \varphi \in \mathcal{D}(\Omega)$$

defines a distribution $\delta_{s_0} = J$ called the Dirac measure *in s_0. However, there is no $f \in L^1_{loc}(\Omega)$ such that $\delta_{s_0} = J_f$.*

Combining these ideas we now introduce the derivative of a distribution. Note that by the definition of convergence in $\mathcal{D}(\Omega)$ for every not negative multi-index α and every distribution $J \in \mathcal{D}^\sharp(\Omega)$, the map

$$\varphi \mapsto (-1)^{|\alpha|} J \circ D^\alpha(\varphi)$$

defines a distribution. This justifies the following

Definition 1.3.7. *For a not negative multi-index α we define the α-th derivative of $J \in \mathcal{D}^\sharp(\Omega)$ by*

$$D^\alpha J = (-1)^{|\alpha|} J \circ D^\alpha.$$

Note that a distribution admits derivatives of any order.

Example 1.3.8. *If $f \in C^{(n)}(I)$, then for every $k \in \{1, \dots, n\}$ both f and $D_k f = \frac{d^k f}{dx^k}$ are locally integrable and hence define distributions J_f and $J_{D_k f}$. Moreover, using integration by parts we conclude that for every $\varphi \in \mathcal{D}$, we have*

$$(D_k J_f)(\varphi) = (-1)^k J_f(D_k \varphi) = (-1)^k \int_I f D_k \varphi d\lambda = \int_I D_k f \varphi d\lambda = J_{D_k f}(\varphi),$$

hence

$$D_k J_f = J_{D_k f}.$$

Using the notion of distributional derivative we will now introduce an important class of function spaces. Here and in the sequel we will identify a function $f \in L^1_{loc}(\Omega)$ with the distribution J_f and denote by $D^\alpha f$ its distributional derivative $D^\alpha J_f$.

Definition 1.3.9. *For an open set $\Omega \subseteq \mathbb{R}^n$, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$ we define the* Sobolev space

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega)\ \forall\ |\alpha| \leq k\}$$

equipped with the norm

$$\|f\|_{k,p} = \begin{cases} (\sum_{|\alpha|\leq k} \|D^\alpha f\|_p^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max_{|\alpha|\leq k} \|D^\alpha f\|_\infty, & \text{if } p = \infty. \end{cases}$$

For $W^{k,2}(\Omega)$ we also use the notation $H^k(\Omega)$.

Using the completeness of $L^p(\Omega)$ one can show the following important result.

Theorem 1.3.10. *The space $W^{k,p}(\Omega)$ is a Banach space. If $p < \infty$, it is separable.*

In the study of partial differential equations, one frequently shows first by abstract arguments the existence of weak solutions in Sobolev spaces. In a second step one then wants to show that these solutions are sufficiently regular to provide classical solutions. For this reason the following result is of great importance.

Theorem 1.3.11. *(Sobolev Imbedding theorem). If $\Omega \subseteq \mathbb{R}^n$ is open and $k > \frac{n}{2} + m$, then*

$$H^k(\Omega) \subseteq C^{(m)}(\Omega)$$

with continuous imbedding, that is, for each $f \in H^k(\Omega)$, there is a $\tilde{f} \in C^{(m)}(\Omega)$ such that $\tilde{f} = f$ almost where on Ω, and $T : H^k(\Omega) \to C^{(m)}(\Omega), T(f) = \tilde{f}$ is continuous.

For example, since $1 > \frac{1}{2} + 0$ one has $H^1(I) \subseteq C(I)$ for every open interval $I \subseteq \mathbb{R}$.

If $p < \infty$, then the space of test functions $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$. This is generally not true anymore for Sobolev spaces, that is, $\mathcal{D}(\Omega)$ cannot be dense in some Sobolev spaces. In fact, all functions in $\mathcal{D}(\Omega)$ are zero on the boundary of Ω. If $\mathcal{D}(\Omega)$ is dense for each Sobolev space $H^k(\Omega)$, note that for larger k, $H^k(\Omega)$ is continuously imbedded in $C^m(\Omega)$ and so in $C(\Omega)$ by the previous theorem, this implies that all functions in the closure of $\mathcal{D}(\Omega)$ vanish on the boundary. This is impossible for all functions in $H^k(\Omega)$, unless $\partial\Omega = \emptyset$, i.e. $\Omega = \mathbb{R}^n$. This gives rise to the following:

Definition 1.3.12. *By $W_0^{k,p}(\Omega)$ we denote the closure of $\mathcal{D}(\Omega)$ in the Banach space $W^{k,p}(\Omega)$. Moreover, we set $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.*

Roughly speaking, the functions in $W_0^{k,p}(\Omega)$ are the functions in $W^{k,p}(\Omega)$ being zero on the boundary of Ω.

2 Elementary Spectral Theory

In this chapter we cover the basic results of spectral theory. The most important of these are the non-emptiness of the spectrum, Beurling's spectral radius formula, and the Gelfand representation theory for commutative Banach algebras.

2.1 Banach Algebras

We begin by setting up the basic vocabulary needed to discuss Banach algebras and by giving some examples.

An *algebra* is a vector space $\mathcal{A}$ together with a bilinear map

$$\mathcal{A}^2 \to \mathcal{A}, \quad (A, B) \mapsto AB,$$

such that

$$A(BC) = (AB)C \qquad (A, B, C \in \mathcal{A})$$

A *subalgebra* of $\mathcal{A}$ is a vector subspace $\mathcal{B}$ of $\mathcal{A}$ such that $B, B' \in \mathcal{B} \Rightarrow BB' \in \mathcal{B}$. Endowed with the multiplication got by restriction, $\mathcal{B}$ is itself an algebra.

A norm $\|.\|$ on $\mathcal{A}$ is said to be *submultiplicative* if

$$\|AB\| \leq \|A\|\|B\| \qquad (A, B \in \mathcal{A}).$$

In this case the pair $(\mathcal{A}, \|\cdot\|)$ is called a *normed algebra*. If $\mathcal{A}$ admits a unit, that is, $AI = IA = A$ for all $A \in \mathcal{A}$, and $\|I\| = 1$, we say that $\mathcal{A}$ is a *unital normed algebra*.

If $\mathcal{A}$ is a normed algebra, then it is evident from the inequality

$$\|AB - A'B'\| \leq \|A\|\|B - B'\| + \|A - A'\|\|B'\|.$$

that the multiplication operation $(A, B) \mapsto AB$ is jointly continuous.

A complete normed algebra is called a *Banach algebra*. A complete unital normed algebra is called a *unital Banach algebra*.

A subalgebra of a normed algebra is obviously itself a normed algebra with the norm got by restriction. The closure of a subalgebra in a normed algebra is a subalgebra. A closed subalgebra of a Banach algebra is a Banach algebra.

Example 2.1.1. *If $\mathcal{S}$ is a set, $l^\infty(\mathcal{S})$, the set of all bounded complex valued functions on $\mathcal{S}$, is a unital Banach algebra where the operations are defined pointwise:*

$$(f + g)(x) = f(x) + g(x),$$
$$(fg)(x) = f(x)g(x),$$
$$(\lambda f)(x) = \lambda f(x),$$

and the norm is the sup-norm

$$\|f\|_\infty = \sup_{x \in \mathcal{S}} |f(x)|.$$

Example 2.1.2. *If Ω is a topological space, the $\mathcal{C}_b(\Omega)$ of all bounded continuous complex-valued functions on Ω is a closed subalgebra of $\ell^\infty(\Omega)$. Thus, $\mathcal{C}_b(\Omega)$ is a unital Banach algebra.*

If Ω is compact, $\mathcal{C}(\Omega)$, the set of continuous functions from Ω to $\mathbb{C}$, is of course equal to $\mathcal{C}_b(\Omega)$.

Example 2.1.3. *If Ω is a locally compact Hausdorff space, we say that a continuous function f from Ω to $\mathbb{C}$* vanishes at infinity, *if for each positive number ε the set $\{\omega \in \Omega \mid |f(\omega)| \geq \varepsilon\}$ is compact. We denote the set of such functions by $\mathcal{C}_0(\Omega)$. It is a closed subalgebra of $\mathcal{C}_b(\Omega)$, and therefore, a Banach algebra. It is unital if and only if Ω is compact, and in this case $\mathcal{C}_0(\Omega) = \mathcal{C}(\Omega)$. The algebra $\mathcal{C}_0(\Omega)$ is one of the most important examples for Banach algebras, and we shall see it used constantly in C^*-algebra theory.*

Example 2.1.4. *If (Ω, μ) is a measure space, the set $\mathcal{L}^\infty(\Omega, \mu)$ of classes of essentially bounded complex-valued measurable functions on Ω is a unital Banach algebra with the usual pointwise-defined operations and the essential supremum norm $f \mapsto \|f\|_\infty$.*

Example 2.1.5. *If Ω is a measure space, let $\mathcal{B}_\infty(\Omega)$ denote the set of all bounded complex-valued measurable functions on Ω. Then $\mathcal{B}_\infty(\Omega)$ is a closed subalgebra of $\ell^\infty(\Omega)$, so it is a unital Banach algebra. This example will be used in connection with the spectral theorem in Chapter 2.*

Example 2.1.6. *The set $\mathcal{A}$ of all continuous functions on the closed unit disc $\mathbb{D}$ in the plane which are analytic on the interior of $\mathbb{D}$ is a closed subalgebra of $\mathcal{C}(\mathbb{D})$, so $\mathcal{A}$ is a unital Banach algebra, called the* disc algebra. *This is the motivating example in the theory of function algebras, where many aspects of the theory of analytic functions are extended to a Banach algebraic setting.*

All of the above examples are of course *abelian*—that is, $AB = BA$ for all elements A and B—but the following examples are not , in general.

Example 2.1.7. *If $\mathcal{X}$ is a normed vector space, denote by $\mathcal{B}(\mathcal{X})$ the set of all bounded linear maps from $\mathcal{X}$ to itself. It is routine to show that $\mathcal{B}(\mathcal{X})$ is* *normed algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(U, V) \mapsto U \circ V$, and the operator norm:*

$$\|U\| = \sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|U(x)\|.$$

If $\mathcal{X}$ is a Banach space, $\mathcal{B}(\mathcal{X})$ is complete and is therefore a Banach algebra.

Example 2.1.8. *The algebra $M_n(\mathbb{C})$ of $n \times n$-matrices with entries in $\mathbb{C}$ is identified with $\mathcal{B}(\mathbb{C}^n)$. It is therefore a unital Banach algebra. Recall that an upper triangular matrix is one of the form*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \dots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \dots & \lambda_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{nn} \end{pmatrix},$$

that is, all entries below the main diagonal are zero. These matrices form a subalgebra of $M_n(\mathbb{C})$.

We shall be seeing many more examples of Banach algebras as we proceed. Most often of these will be non-abelian, but in the first three sections of this chapter we shall be principally concerned with the abelian case.

If $(\mathcal{B}_\lambda)_{\lambda \in \Lambda}$ is a family subalgebras of an algebra $\mathcal{A}$, then $\cap_{\lambda \in \Lambda} \mathcal{B}_\lambda$ is a subalgebra, also. Hence, for any subset $\mathcal{S}$ of $\mathcal{A}$, there is a smallest subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing $\mathcal{S}$, namely, the intersection of all the subalgebras containing $\mathcal{S}$. This algebra is called the subalgebra of $\mathcal{A}$ *generated* by $\mathcal{S}$. If $\mathcal{S}$ is the singleton set $\{A\}$, then $\mathcal{B}$ is the linear span of all powers A^n of A, $n = 1, 2, \dots$. If $\mathcal{A}$ is a normed algebra, the closed subalgebra $\mathcal{C}$ of $\mathcal{A}$ *generated* by a set $\mathcal{S}$ in $\mathcal{A}$ is the smallest closed subalgebra containing $\mathcal{S}$. It is plain that $\mathcal{C} = \overline{\mathcal{B}}$, where $\mathcal{B}$ is the subalgebra generated by $\mathcal{S}$.

If $\mathcal{A} = \mathcal{C}(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, and if $z : \mathbb{T} \to \mathbb{C}$ is the inclusion function, then the closed algebra generated by z and its conjugate $\overline{z}$ is $\mathcal{C}(\mathbb{T})$ itself, this can be proved from the following Stone-Weierstrass theorem.

Theorem 2.1.9. *(Stone-Weierstrass theorem). Let X be a compact Hausdorff space, $\mathcal{A} \subseteq \mathcal{C}(\mathcal{X})$ be a closed subalgebra, separated all points of X and $\mathcal{A}$ be self-adjoint, that is, if $f \in \mathcal{A}$, we have $\overline{f} \in \mathcal{A}$. Then $\mathcal{A} = \mathcal{C}(\mathcal{X})$ or there is an $x_0 \in X$ such that $\mathcal{A}= \{f : f \in \mathcal{C}(X), f(x_0) = 0\}$.*

We omit its proof.

A *left*, respectively, *right*, ideal in an algebra $\mathcal{A}$ is a vector subspace $\mathcal{I}$ of $\mathcal{A}$ such that

$$A \in \mathcal{A} \text{ and } B \in \mathcal{I} \Rightarrow AB \in \mathcal{I}, \text{ respectively, } BA \in \mathcal{I}.$$

An *ideal* in $\mathcal{A}$ is a vector subspace that is simultaneously a left and a right ideal in $\mathcal{A}$. Obviously, 0 and $\mathcal{A}$ are ideals in $\mathcal{A}$, called the *trivial* ideals. A *maximal* ideal in $\mathcal{A}$ is a proper ideal, that is, it is not $\mathcal{A}$, that is not contained in any other proper ideal in $\mathcal{A}$. Maximal left ideals are defined similarly.

If $\mathcal{A}$ is unital, then it follows easily from Zorn's Lemma that every proper ideal is contained in a maximal ideal.

If $(\mathcal{I}_\lambda)_{\lambda\in\Lambda}$ is a family of ideals of an algebra $\mathcal{A}$, then $\cap_{\lambda\in\Lambda}\mathcal{I}_\lambda$ is an ideal of

$\mathcal{A}$. Hence, if $\mathcal{S} \subseteq \mathcal{A}$, there is a smallest ideal $\mathcal{I}$ of $\mathcal{A}$ containing $\mathcal{S}$, we call $\mathcal{I}$ the ideal *generated* by $\mathcal{S}$. If $\mathcal{A}$ is a normed algebra, then the closure of an ideal is an ideal. The closed ideal $\mathcal{J}$ *generated* by a set $\mathcal{S}$ is the smallest closed ideal containing $\mathcal{S}$. It is clear that $\mathcal{J}$ is the closure of the ideal generated by $\mathcal{S}$.

Theorem 2.1.10. *If $\mathcal{I}$ is a closed ideal in a normed algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a normed algebra when endowed with the quotient norm*

$$\|A + \mathcal{I}\| = \inf_{B \in \mathcal{I}} \|A + B\|.$$

Proof. Let $\varepsilon > 0$ and suppose that A, B belong to $\mathcal{A}$. Then $\varepsilon + \|A + \mathcal{I}\| > \|A + A'\|$ and $\varepsilon + \|B + \mathcal{I}\| > \|B + B'\|$ for some $A', B' \in \mathcal{I}$. Hence,

$$(\varepsilon + \|A + \mathcal{I}\|)(\varepsilon + \|B + \mathcal{I}\|) > \|A + A'\|\|B + B'\| \geq \|AB + C\|,$$

where $C = A'B + AB' + A'B' \in \mathcal{I}$. Thus, $(\varepsilon + \|A + \mathcal{I}\|)(\varepsilon + \|B + \mathcal{I}\|) \geq \|AB + \mathcal{I}\|$. Letting $\varepsilon \to 0$, we get $\|A + \mathcal{I}\|\|B + \mathcal{I}\| \geq \|AB + \mathcal{I}\|$; that is, the quotient norm is submultiplicative. ∎

A *homomorphism* from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathcal{A}$. Its kernel $\ker(\varphi)$ is an ideal in $\mathcal{A}$ and its image $\varphi(\mathcal{A})$ is a subalgebra of $\mathcal{B}$. We say φ is *unital* if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\varphi(I) = I$.

If $\mathcal{I}$ is an ideal in $\mathcal{A}$, the quotient map $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a homomorphism.

2.2 The Spectrum and the Spectral Radius

Let $\mathbb{C}[z]$ denote the algebra of all polynomials in an indeterminate z with complex coefficients. If A is an element of a unital algebra $\mathcal{A}$ and $p \in \mathbb{C}[z]$ is the polynomial

$$p = \lambda_0 + \lambda_1 z^1 + \ldots + \lambda_n z^n,$$

we set

$$p(A) = \lambda_0 I + \lambda_1 A^1 + \ldots + \lambda_n A^n.$$

The map

$$\mathbb{C}[z] \to \mathcal{A}, \quad p \mapsto p(A),$$

is a unital homomorphism.

We say that $A \in \mathcal{A}$ is *invertible* if there is an element B in $\mathcal{A}$ such that $AB = BA = I$. In this case B is unique and written by A^{-1}. The set

$$\mathrm{Inv}(\mathcal{A}) = \{A \in \mathcal{A} \mid A \text{ is invertible}\}$$

is a group under multiplication.

We define the spectrum of an element A in $\mathcal{A}$ to be the set

$$\sigma(A) = \sigma_{\mathcal{A}}(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \notin \mathrm{Inv}(\mathcal{A})\}.$$

We shall henceforth find it convenient to write λI simply as λ.

Example 2.2.1. *Let $\mathcal{A} = \mathcal{C}(\Omega)$, where Ω is a compact Hausdorff space. Then $\sigma(f) = f(\Omega)$ for all $f \in \mathcal{A}$.*

Example 2.2.2. *Let $\mathcal{A} = \ell^\infty(\mathcal{S})$, where $\mathcal{S}$ is a non-empty set, Then $\sigma(f) = \overline{f(\mathcal{S})}$, the closure in $\mathbb{C}$, for all $f \in \mathcal{A}$.*

Example 2.2.3. *Let $\mathcal{A}$ be the algebra of upper triangular $n \times n$-matrices. If $A \in \mathcal{A}$, say*

$$A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}.$$

it is elementary that

$$\sigma(A) = \{\lambda_{11}, \lambda_{22}, \dots, \lambda_{nn}\}.$$

Similarly, if $\mathcal{A} = M_n(\mathbb{C})$ and $A \in \mathcal{A}$, then $\sigma(A)$ is the set of eigenvalues of A.

Thus, one thinks of the spectrum as simultaneously a generalisation of the range of a function and the set of eigenvalues of a finite square matrix.

Remark 2.2.4. *If A, B are elements of a unital algebra $\mathcal{A}$, then $I - AB$ is invertible if and only if $I - BA$ is invertible. This follows from the observation that if $I - AB$ has inverse C, then $I - BA$ has inverse $I + BCA$.*

A consequence of this equivalence is that $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ for all $A, B \in \mathcal{A}$.

Theorem 2.2.5. *Let A be an element of a unital algebra $\mathcal{A}$. If $\sigma(A)$ is non-empty and $p \in \mathbb{C}[z]$, then*

$$\sigma(p(A)) = p(\sigma(A)).$$

Proof. We may suppose that p is not constant. If $\mu \in \mathbb{C}$, there are elements $\lambda_0, \dots, \lambda_n$ in $\mathbb{C}$, where $\lambda_0 \neq 0$, such that

$$p - \mu = \lambda_0(z - \lambda_1) \dots (z - \lambda_n),$$

and therefore,

$$p(A) - \mu = \lambda_0(A - \lambda_1)\dots(A - \lambda_n).$$

It is clear that $p(A) - \mu$ is invertible if and only if $A - \lambda_1, \dots, A - \lambda_n$ are. It follows that $\mu \in \sigma(p(A))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(A)$, and therefore, $\sigma(p(A)) = p(\sigma(A))$. ■

The spectral mapping property for polynomials may be generalized to continuous functions, but only for certain elements in certain algebras.

Theorem 2.2.6. *Let $\mathcal{A}$ be a unital Banach algebra and A an element of $\mathcal{A}$ such that $\|A\| < 1$. Then $I - A \in \mathrm{Inv}(\mathcal{A})$ and*

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

Proof. Since $\sum_{n=0}^{\infty} \|A^n\| \leq \sum_{n=0}^{\infty} \|A\|^n = (1 - \|A\|)^{-1} < \infty$, the series $\sum_{n=0}^{\infty} A^n$ is convergent, to B say, in $\mathcal{A}$, and since $(I - A)(I + \dots + A^n) = I - A^{n+1}$ converges to $(I - A)B = B(I - A)$ and to I as $n \to \infty$, the element B is the inverse of $I - A$. ■

The series in Theorem 2.2.6 is called the *Neumann* series for $(I - A)^{-1}$.

Theorem 2.2.7. *If $\mathcal{A}$ is a unital Banach algebra, then* $\mathrm{Inv}(\mathcal{A})$ *is open in $\mathcal{A}$, and the map*

$$\mathrm{Inv}(\mathcal{A}) \to \mathcal{A}, \quad A \mapsto A^{-1},$$

is differentiable.

Proof. Suppose that $A \in \text{Inv}(\mathcal{A})$ and $\|B-A\| < \|A^{-1}\|^{-1}$. Then $\|BA^{-1}-I\| \leq \|B-A\|\|A^{-1}\| < 1$, so $BA^{-1} \in \text{Inv}(\mathcal{A})$, and therefore, $B \in \text{Inv}(\mathcal{A})$, thus, $\text{Inv}(\mathcal{A})$ is open in $\mathcal{A}$.

If $B \in \mathcal{A}$ and $\|B\| < 1$, then $I + B \in \text{Inv}(\mathcal{A})$ and

$$\begin{aligned}\|(I+B)^{-1} - I + B\| &= \|\sum_{n=0}^{\infty}(-1)^n B^n - I + B\| = \|\sum_{n=2}^{\infty}(-1)^n B^n\| \\ &\leq \sum_{n=2}^{\infty}\|B\|^n = \|B\|^2/(1-\|B\|).\end{aligned}$$

Let $A \in \text{Inv}(\mathcal{A})$ and suppose that $\|C\| < \frac{1}{2}\|A^{-1}\|^{-1}$. Then $\|A^{-1}C\| < 1/2 < 1$, with $B = A^{-1}C$, so,

$$\|(I + A^{-1}C)^{-1} - I + A^{-1}C\| \leq \|A^{-1}C\|^2/(1 - \|A^{-1}C\|) \leq 2\|A^{-1}C\|^2,$$

since $1 - \|A^{-1}C\| > 1/2$. Now define Φ to be the linear operator on $\mathcal{A}$ given by $\Phi(B) = -A^{-1}BA^{-1}$. Then,

$$\begin{aligned}\|(A+C)^{-1} - A^{-1} - \Phi(C)\| &= \|(I + A^{-1}C)^{-1}A^{-1} - A^{-1} + A^{-1}CA^{-1}\| \\ &\leq \|(I + A^{-1}C)^{-1} - I + A^{-1}C\|\|A^{-1}\| \leq 2(\|A^{-1}\|^3\|C\|^2).\end{aligned}$$

Consequently,

$$\lim_{C\to 0}\frac{\|(A+C)^{-1} - A^{-1} - \Phi(C)\|}{\|C\|} = 0,$$

and therefore, the map $\Psi : B \mapsto B^{-1}$ is differentiable at $B = A$ with derivative $\Psi'(A) = \Phi$.

■

The algebra $\mathbb{C}[z]$ is a normed algebra where the norm is defined by setting

$$\|p\| = \sup_{|\lambda|\leq 1}|p(\lambda)|.$$

Observe that $\mathrm{Inv}(\mathbb{C}[z]) = \mathbb{C} \setminus \{0\}$, so the polynomials $p_n = 1 + z/n$ are not invertible. But $\lim_{n\to\infty} p_n = 1$, which shows that $\mathrm{Inv}(\mathbb{C}[z])$ is not open in $\mathbb{C}[z]$. Thus, the norm on $\mathbb{C}[z]$ is not complete.

Lemma 2.2.8. *Let $\mathcal{A}$ be a unital Banach algebra and let $A \in \mathcal{A}$. The spectrum $\sigma(A)$ of A is a closed subset of the disc in the plane of centre the origin and radius $\|A\|$.*

Proof. If $|\lambda| > \|A\|$, then $\| \lambda^{-1}A \| < 1$, so $I - \lambda^{-1}A$ is invertible, and therefore, so is $\lambda - A$. Hence, $\lambda \notin \sigma(A)$. Thus, $\lambda \in \sigma(A) \Rightarrow |\lambda| \leq \|A\|$. The set $\sigma(A)$ is closed, that is, $\mathbb{C} \setminus \sigma(A)$ is open, because $\mathrm{Inv}(\mathcal{A})$ is open in $\mathcal{A}$.

■

The following result can be thought of as the fundamental theorem of Banach algebras.

Theorem 2.2.9. *(Gelfand) If A is an element of a unital complex Banach algebra $\mathcal{A}$, then the spectrum $\sigma(A)$ of A is non-empty.*

Proof. Suppose that $\sigma(A) = \emptyset$ and we shall obtain a contradiction. If $|\lambda| > 2\|A\|$, then $\|\lambda^{-1}A\| < \frac{1}{2}$, and therefore, $1 - \|\lambda^{-1}A\| > \frac{1}{2}$. Hence,

$$\begin{aligned}\|(I - \lambda^{-1}A)^{-1}\| &= \|\sum_{n=0}^{\infty}(\lambda^{-1}A)^n\| \\ &\leq \frac{1}{1 - \|\lambda^{-1}A\|} \leq 2.\end{aligned}$$

Therefore,

$$\|(A - \lambda)^{-1}\| = \|\lambda^{-1}(I - \lambda^{-1}A)^{-1}\| < 2/|\lambda| < \|A\|^{-1}.$$

Moreover, since the map $\lambda \mapsto (A-\lambda)^{-1}$ is continuous, it is bounded on the compact disc $2\|A\|\mathbb{D}$. Thus, we have shown that this map is bounded on all of $\mathbb{C}$; that is, there is a positive number M such that $\|(A-\lambda)^{-1}\| \leq M$, $\lambda \in \mathbb{C}$.

If $\tau \in \mathcal{A}^*$, the function $\lambda \mapsto \tau((A-\lambda)^{-1})$ is bounded by $M\|\tau\|$. On the other hand, it is also differential by the differentiability of inversion, so the function $\lambda \to \tau((A-\lambda)^{-1})$ is entire on $\mathbb{C}$. So by Liouville's theorem in complex analysis, it is constant. In particular, $\tau(A^{-1}) = \tau((A-I)^{-1})$. Because this is true for all $\tau \in \mathcal{A}^*$, we have $A^{-1} = (A-I)^{-1}$, so $A = A - I$, which is a contradiction. ■

Theorem 2.2.10. *(Gelfand-Mazur) If $\mathcal{A}$ is a unital complex Banach algebra in which every non-zero element is invertible, then $\mathcal{A} = \mathbb{C}I$.*

Proof. This is immediate from Theorem 2.2.9. ■

If A is an element of a unital Banach algebra $\mathcal{A}$, its *spectral radius* is defined to be

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|.$$

By Remark 2.2.4, $r(AB) = r(BA)$ for all $A, B \in \mathcal{A}$.

Example 2.2.11. *If $\mathcal{A} = \mathcal{C}(\Omega)$, where Ω is a compact Hausdorff space, then $r(f) = \|f\|_\infty$ $(f \in \mathcal{A})$.*

Example 2.2.12. *Let $\mathcal{A} = M_2(\mathbb{C})$ and*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\|A\| = 1$, but $r(A) = 0$, since $A^2 = 0$.

Theorem 2.2.13. *(Beurling) If A is an element of a unital Banach algebra $\mathcal{A}$, then*

$$r(A) = \liminf_{n\geq 1} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Proof. If $\lambda \in \sigma(A)$, then $\lambda^n \in \sigma(A^n)$, so $|\lambda^n| \leq \|A^n\|$, and therefore, $r(A) \leq \liminf_{n\to\infty} \|A^n\|^{1/n}$.

Let $\triangle$ be the open disc in $\mathbb{C}$ centered at 0 and of radius $1/r(A)$, we use the usual convention that $1/0 = +\infty$. If $\lambda \in \triangle$, then $I - \lambda A \in \mathrm{Inv}(\mathcal{A})$. In fact, if $\lambda = 0$, then $I - \lambda A \in \mathrm{Inv}(\mathcal{A})$, if $\lambda \neq 0$, since $|\lambda| < \frac{1}{r(A)}$, so $\frac{1}{\lambda} > r(A)$, thus, $\frac{1}{\lambda} - A$ is invertible, and therefore $I - \lambda A$ is invertible.

If $\tau \in \mathcal{A}^*$, then the map

$$f : \triangle \to \mathbb{C}, \quad \lambda \mapsto \tau((I - \lambda A)^{-1})$$

is analytic, so there are unique complex numbers λ_n such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n, \qquad \lambda \in \triangle.$$

However, if $|\lambda| < 1/\|A\|$, then $\|\lambda A\| < 1$, so

$$(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n,$$

and therefore,

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n \tau(A^n).$$

It follows that $\lambda_n = \tau(A^n)$ for all $n \geq 0$. Hence, the sequence $(\tau(A^n)\lambda^n)$ converges to 0 for each $\lambda \in \triangle$, and therefore a fortiori, it is bounded. Since this is true for each $\tau \in \mathcal{A}^*$, it follows from the principle of uniform boundedness that $(\lambda^n A^n)$ is a bounded sequence. Hence, there is a positive number M such that $\|\lambda^n A^n\| \leq M$ for all $n \geq 0$, note that $r(A) < \infty$, we can have $\lambda \neq 0$ and $|\lambda| < 1/r(A)$, therefore, $\|A^n\|^{1/n} \leq M^{1/n}/|\lambda|$. Consequently, $\limsup_{n\to\infty} \|A^n\|^{1/n} \leq 1/|\lambda|$. We have thus shown that if $r(A) < |\lambda^{-1}|$, then $\limsup_{n\to\infty} \|A^n\|^{1/n} \leq |\lambda^{-1}|$. It follows that $\limsup_{n\to\infty} \|A^n\|^{1/n} \leq r(A)$, and since $r(A) \leq \liminf_{n\to\infty} \|A^n\|^{1/n}$, therefore $r(A) = \liminf_{n\geq 1} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n}$. ■

Example 2.2.14. *Let $\mathcal{A}$ be the set of $C^{(1)}$-functions on the interval $[0,1]$. This is an algebra when endowed with the pointwise-defined operations, and a submultiplicative norm on $\mathcal{A}$ is given by*

$$\|f\| = \|f\|_\infty + \|f'\|_\infty \qquad (f \in \mathcal{A}).$$

It is elementary that $\mathcal{A}$ is complete under this norm, and therefore, $\mathcal{A}$ is a unital Banach algebra. Let $x : [0,1] \to \mathbb{C}$ be the inclusion, so $x \in \mathcal{A}$. Clearly, $\|x^n\| = 1 + n$ for all n, so $r(x) = \lim(1+n)^{1/n} = 1 < 2 = \|x\|$.

Recall that if $\mathcal{K}$ is a non-empty compact set in $\mathbb{C}$, its complement $\mathbb{C} \setminus \mathcal{K}$ admits exactly one unbounded component, and that the bounded components of $\mathbb{C} \setminus \mathcal{K}$ are called the *holes* of $\mathcal{K}$.

Theorem 2.2.15. *Let $\mathcal{B}$ be a closed subalgebra of a unital Banach algebra $\mathcal{A}$ containing the unit of $\mathcal{A}$.*
(1) The set $\operatorname{Inv}(\mathcal{B})$ is a clopen subset in topological space $\mathcal{B} \cap \operatorname{Inv}(\mathcal{A})$.
(2) For each $B \in \mathcal{B}$,

$$\sigma_{\mathcal{A}}(B) \subseteq \sigma_{\mathcal{B}}(B) \qquad \text{and} \qquad \partial\sigma_{\mathcal{B}}(B) \subseteq \partial\sigma_{\mathcal{A}}(B).$$

(3) If $B \in \mathcal{B}$ and $\sigma_{\mathcal{A}}(B)$ has no holes, then $\sigma_{\mathcal{A}}(B) = \sigma_{\mathcal{B}}(B)$.

Proof. Clearly $\mathrm{Inv}(\mathcal{B})$ is an open set in $\mathcal{B} \cap \mathrm{Inv}(\mathcal{A})$. To see that it is also closed, Let (B_n) be a sequence in $\mathrm{Inv}(\mathcal{B})$ converging to a point $B \in \mathcal{B} \cap \mathrm{Inv}(\mathcal{A})$. Then (B_n^{-1}) converges to B^{-1} in $\mathcal{A}$, so $B^{-1} \in \mathcal{B}$, which implies that $B \in \mathrm{Inv}(\mathcal{B})$. Hence, $\mathrm{Inv}(\mathcal{B})$ is clopen in $\mathcal{B} \cap \mathrm{Inv}(\mathcal{A})$.

If $B \in \mathcal{B}$, the inclusion $\sigma_{\mathcal{A}}(B) \subseteq \sigma_{\mathcal{B}}(B)$ is immediate from the inclusion $\mathrm{Inv}(\mathcal{B}) \subseteq \mathrm{Inv}(\mathcal{A})$.

If $\lambda \in \partial\sigma_{\mathcal{B}}(B)$, then $\lambda \in \sigma_{\mathcal{B}}(B)$ and there is sequence (λ_n) in $\mathbb{C} \setminus \sigma_{\mathcal{B}}(B)$ converging to λ. Hence, $\lambda_n - B \in \mathrm{Inv}(\mathcal{B})$, and $\lambda - B \notin \mathrm{Inv}(\mathcal{B})$, so $\lambda - B \notin \mathrm{Inv}(\mathcal{A})$, by Condition (1). Thus, we have $\lambda \in \sigma_{\mathcal{A}}(B)$. This shows that for each neighbour of λ there is element belongs to $\sigma_{\mathcal{A}}(B)$. On the other hand, since $\lambda_n - B \in \mathrm{Inv}(\mathcal{A})$, so $\lambda_n \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(B)$ and note that (λ_n) converges to λ, therefore, $\lambda \in \partial\sigma_{\mathcal{A}}(B)$. This proves Condition (2).

If $B \in \mathcal{B}$ and $\sigma_{\mathcal{A}}(B)$ has no holes, then $\mathbb{C} \setminus \sigma_{\mathcal{A}}(B)$ is connected. Since $\mathbb{C} \setminus \sigma_{\mathcal{B}}(B)$ is a clopen subset of $\mathbb{C} \setminus \sigma_{\mathcal{A}}(B)$ by Condition (1) and (2), it is follows that $\mathbb{C} \setminus \sigma_{\mathcal{A}}(B) = \mathbb{C} \setminus \sigma_{\mathcal{B}}(B)$, and therefore, $\sigma_{\mathcal{A}}(B) = \sigma_{\mathcal{B}}(B)$. ■

Example 2.2.16. *Let $\mathcal{C} = \mathcal{C}(\mathbb{T})$ and let $\mathcal{A}$ be the disc algebra. If $f \in \mathcal{A}$, let $\varphi(f)$ be its restriction to $\mathbb{T}$. One easily checks that the map*

$$\varphi : \mathcal{A} \to \mathcal{C}, \quad f \mapsto \varphi(f),$$

is an isometric homomorphism onto the closed subalgebra $\mathcal{B}$ of $\mathcal{C}$ generated by the unit and the inclusion $z : \mathbb{T} \to \mathbb{C}$, the equation $\|\varphi(f)\|_\infty = \|f\|_\infty$ is given by the maximum modulus principle. Clearly, $\sigma_{\mathcal{B}}(z) = \sigma_{\mathcal{A}}(z) = \mathbb{D}$, and $\sigma_{\mathcal{C}}(z) = \mathbb{T}$.

Let A be an element of a unital Banach algebra $\mathcal{A}$. Since

$$\sum_{n=0}^{\infty} \|A^n/n!\| \leq \sum_{n=0}^{\infty} \|A\|^n/n! < \infty,$$

the series $\sum_{n=0}^{\infty} A^n/n!$ is convergent in $\mathcal{A}$. We denote its sum by e^A.

In proving the next theorem, we shall use some elementary results concerning differentiation. Suppose that f, g are differentiable maps from $\mathbb{R}$ to $\mathcal{A}$ with derivatives f', g', respectively. Then fg is differentiable and $(fg)' = fg' + f'g$. To prove this, just mimic the proof of the scalar-valued case. If $f' = 0$, then f is constant. We prove this: If $\tau \in \mathcal{A}^*$, then the function $\mathbb{R} \to \mathbb{C}, t \mapsto \tau(f(t))$, is differentiable with zero derivative, and therefore, $\tau(f(t)) = \tau(f(0))$ for all t. Since τ was arbitrary, this implies that $f(t) = f(0)$.

Theorem 2.2.17. *Let $\mathcal{A}$ be a unital Banach algebra.*
(1) If $A \in \mathcal{A}$ and $f : \mathbb{R} \to \mathcal{A}$ is differentiable, $f(0) = 1$, and $f'(t) = Af(t)$ for all $t \in \mathbb{R}$, then $f(t) = e^{tA}$ for all $t \in \mathbb{R}$.
(2) If $A \in \mathcal{A}$, then e^A is invertible with inverse e^{-A}, and if A, B are commuting elements of $\mathcal{A}$, then $e^{A+B} = e^A e^B$.

Proof. First we observe that if $f : \mathbb{R} \to \mathcal{A}$ is defined by $f(t) = e^{tA}$, then we can prove that $f'(t) = Af(t)$ and $Af(t) = f(t)A$. Now suppose g is a differentiable map from $\mathbb{R}$ to $\mathcal{A}$ such that $g'(t) = Ag(t)$ and $f(0) = g(0) = 1$. Then the map $h : \mathbb{R} \to \mathcal{A}$, $t \mapsto f(t)g(-t)$, is differentiable with zero derivative. Hence, $h(t) = 1$ for all $t \in \mathbb{R}$. Applying this to the map $t \mapsto e^{tA}$ we get $e^{tA}e^{-tA} = 1$; in particular, $e^A e^{-A} = 1$.

It follows that if $f : \mathbb{R} \to \mathcal{A}$ is differentiable, $f(0) = 1$, and $f'(t) = Af(t)$ for all t, then $f(t) = e^{tA}$ as set $g(t) = e^{tA}$ and get $g(-t)f(t) = 1$, so $f(t) = e^{tA}$.

Now suppose that A and B are commuting elements of $\mathcal{A}$ and set $f(t) = e^{tA}e^{tB}$. Then $f(0) = 1$ and $f'(t) = e^{tA}Be^{tB} + Ae^{tA}e^{tB} = (A+B)f(t)$. Hence, $f(t) = e^{t(A+B)}$ for all $t \in \mathbb{R}$, so, in particular, $e^{A+B} = f(1) = e^{A}e^{B}$. ∎

2.3 The Gelfand Representation

The idea of this section is to represent a unital abelian Banach algebra as an algebra of continuous functions on a compact Hausdorff space. This is an extremely useful way of looking at these algebras.

We begin by proving some results on ideals and multiplicative linear functions.

Theorem 2.3.1. *Let $\mathcal{I}$ be an ideal of a unital Banach algebra $\mathcal{A}$. If $\mathcal{I}$ is proper, so is its closure $\overline{\mathcal{I}}$. If $\mathcal{I}$ is maximal, then it is closed.*

Proof. If $I \in \overline{\mathcal{I}}$, there exists an element $B \in \mathcal{I}$ such that $\| I - B \| < 1$ and this implies that B is invertible in $\mathcal{A}$. Since $\mathcal{I}$ is an ideal, $I \in \mathcal{I}$ and so $\mathcal{I}$ is not proper. This contradiction shows that $\overline{\mathcal{I}}$ is proper.

If $\mathcal{I}$ is maximal, then $\mathcal{I} = \overline{\mathcal{I}}$, as $\overline{\mathcal{I}}$ is a proper ideal containing $\mathcal{I}$. ∎

Lemma 2.3.2. *If $\mathcal{I}$ is a maximal ideal of a unital abelian algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a field.*

Proof. The algebra $\mathcal{A}/\mathcal{I}$ is unital and abelian, with unit $I + \mathcal{I}$ say. If $\mathcal{J}$ is an ideal of $\mathcal{A}/\mathcal{I}$ and π is the quotient map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{I}$, then $\pi^{-1}(\mathcal{J})$ is an ideal of $\mathcal{A}$ containing $\mathcal{I}$. Hence, $\pi^{-1}(\mathcal{J}) = \mathcal{A}$ or $\mathcal{I}$, by maximality of $\mathcal{I}$. Therefore, $\mathcal{J} = \mathcal{A}/\mathcal{I}$ or 0. Thus, $\mathcal{A}/\mathcal{I}$ and 0 are the only ideals of $\mathcal{A}/\mathcal{I}$. Now suppose that $\pi(A)$ is a non-zero element of $\mathcal{A}/\mathcal{I}$. Then $\mathcal{J} = \pi(A)(\mathcal{A}/\mathcal{I})$ is a non-zero ideal of $\mathcal{A}/\mathcal{I}$, and therefore, $\mathcal{J} = \mathcal{A}/\mathcal{I}$. Hence, there is an element B of $\mathcal{A}$ such that $(A+\mathcal{I})(B+\mathcal{I}) = I+\mathcal{I}$, so $A+\mathcal{I}$ is invertible. This shows that $\mathcal{A}/\mathcal{I}$ is a field. $\blacksquare$

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a unital homomorphism between unital algebras, then $\varphi(\mathrm{Inv}(\mathcal{A})) \subseteq \mathrm{Inv}(\mathcal{B})$, so $\sigma(\varphi(A)) \subseteq \sigma(A)$, $A \in \mathcal{A}$.

A *character* on an abelian algebra $\mathcal{A}$ is a non-zero homomorphism $\tau : \mathcal{A} \to \mathbb{C}$. We denote by $\Omega(\mathcal{A})$ the set of characters on $\mathcal{A}$. Note that $\sigma(\tau(A)) \subseteq \sigma(A)$ and $\sigma(\tau(A)) = \{\tau(A)\}$, we have $\tau(A) \in \sigma(A)$.

Theorem 2.3.3. *Let $\mathcal{A}$ be a unital abelian Banach algebra.*
(1) If $\tau \in \Omega(\mathcal{A})$, then $\|\tau\| = 1$.
(2) The set $\Omega(\mathcal{A})$ is non-empty, and the map

$$\tau \mapsto \ker(\tau)$$

defines a bijection from $\Omega(\mathcal{A})$ onto the set of all maximal ideals of $\mathcal{A}$.

Proof. If $\tau \in \Omega(\mathcal{A})$ and $A \in \mathcal{A}$, then $\tau(A) \in \sigma(A)$, so $|\tau(A)| \leq r(A) \leq \|A\|$. Hence, $\|\tau\| \leq 1$. Also, $\tau(I) = 1$, since $\tau(I) = \tau(I)^2$ and $\tau(I) \neq 0$. Hence, $\|\tau\| = 1$.

Let $\mathcal{I}$ denote the closed ideal $\ker(\tau)$. This is proper, since $\tau \neq 0$, and $\mathcal{I} + \mathbb{C}I = \mathcal{A}$, since $A - \tau(A) \in \mathcal{I}$ for all $A \in \mathcal{A}$. It follows that $\mathcal{I}$ is a maximal ideal of $\mathcal{A}$.

If $\tau_1, \tau_2 \in \Omega(\mathcal{A})$ and $\ker(\tau_1) = \ker(\tau_2)$, then for each $A \in \mathcal{A}$, $\tau_2(A - \tau_2(A)) = 0$, so we have $\tau_1(A - \tau_2(A)) = 0$, so $\tau_1(A) = \tau_2(A)$. Thus, $\tau_1 = \tau_2$.

If $\mathcal{I}$ is an arbitrary maximal ideal of $\mathcal{A}$, then $\mathcal{I}$ is closed by Theorem 2.3.1 and $\mathcal{A}/\mathcal{I}$ is a unital Banach algebra in which every non-zero element is invertible, by Lemma 2.3.2. Hence, by Theorem 2.2.10 $\mathcal{A}/\mathcal{I} = \mathbb{C}(I + \mathcal{I})$. It follows that $\mathcal{A} = \mathcal{I} \oplus \mathbb{C}I$. Define $\tau : \mathcal{A} \to \mathbb{C}$ by $\tau(A + \lambda) = \lambda$, $A \in \mathcal{I}$, $\lambda \in \mathbb{C}$. Then τ is a character and $\ker(\tau) = \mathcal{I}$.

Thus, we have shown that the map $\tau \mapsto \ker(\tau)$ is a bijection from the characters onto the maximal ideals of $\mathcal{A}$.

We have seen already that $\mathcal{A}$ admits maximal ideals since it is unital. Therefore, $\Omega(\mathcal{A}) \neq \emptyset$. $\blacksquare$

Theorem 2.3.4. *Let $\mathcal{A}$ be a unital abelian Banach algebra. Then*

$$\sigma(A) = \{\tau(A) \mid \tau \in \Omega(\mathcal{A})\} \qquad (A \in \mathcal{A}).$$

Proof. If A is an element of $\mathcal{A}$ whose spectrum contains λ, then the ideal $\mathcal{I} = (A - \lambda)\mathcal{A}$ is proper, so $\mathcal{I}$ is contained in a maximal ideal $\ker(\tau)$, where $\tau \in \Omega(\mathcal{A})$. Hence, $\tau(A) = \lambda$. This shows that the inclusion $\sigma(A) \subseteq \{\tau(A) \mid \tau \in \Omega(\mathcal{A})\}$ holds, and the reverse inclusion is clear. $\blacksquare$

If $\mathcal{A}$ is a unital abelian Banach algebra, it follows from Theorem 2.3.3 that $\Omega(\mathcal{A})$ is contained in the closed unit ball of $\mathcal{A}^*$. We endow $\Omega(\mathcal{A})$ with the relative weak* topology, and call the topological space $\Omega(\mathcal{A})$ the *character* space, or *spectrum*, of $\mathcal{A}$.

Theorem 2.3.5. *If $\mathcal{A}$ is a unital abelian Banach algebra, then $\Omega(\mathcal{A})$ is a compact Hausdorff space.*

Proof. It is easily checked that $\Omega(\mathcal{A})$ is weak* closed in the closed unit ball $\mathcal{S}$ of $\mathcal{A}^*$. Since $\mathcal{S}$ is weak* compact by the Banach-Alaoglu theorem, $\Omega(\mathcal{A})$ is weak* compact. $\blacksquare$

Suppose that $\mathcal{A}$ is a unital abelian Banach algebra, thus the space $\Omega(\mathcal{A})$ is non-empty. If $A \in \mathcal{A}$, we define the function $\hat{A}$ by

$$\hat{A} : \Omega(\mathcal{A}) \to \mathbb{C}, \quad \tau \mapsto \tau(A).$$

Clearly the topology on $\Omega(\mathcal{A})$ is the smallest one making all of the functions $\hat{A}$ continuous. Hence, $\hat{A} \in \mathcal{C}(\Omega(\mathcal{A}))$.

We call $\hat{A}$ the *Gelfand transform* of A. The following result is very important:

Theorem 2.3.6. *(Gelfand Representation) Suppose that $\mathcal{A}$ is a unital abelian Banach algebra . Then the map*

$$\mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A})), \quad A \mapsto \hat{A},$$

is a norm-decreasing homomorphism,

$$r(A) = \|\hat{A}\|_\infty \qquad (A \in \mathcal{A}),$$

and $\sigma(A) = \hat{A}(\Omega(\mathcal{A}))$.

Proof. By Theorem 2.3.4 the spectrum $\sigma(A)$ is the range of $\hat{A}$. Hence, $r(A) = \|\hat{A}\|_\infty$, which implies that the map $A \mapsto \hat{A}$ is norm-decreasing. That this map is a homomorphism is easily checked. ■

The kernel of the Gelfand representation is called the *radical* of the algebra $\mathcal{A}$. It consists of the elements A such that $r(A) = 0$. It therefore contains the nilpotent elements. If the radical is zero, $\mathcal{A}$ is said to be *semisimple.*

Let A, B be two elements of an Abelian unital Banach algebra $\mathcal{A}$. Then $r(A + B) \leq r(A) + r(B)$, and $r(AB) \leq r(A)r(B)$. In fact, $r(A + B) =$

$\|(A+B)\hat{}\|_\infty \leq \|\hat{A}\|_\infty + \|\hat{B}\|_\infty = r(A) + r(B)$ by Theorem 2.3.6. Similarly, $r(AB) = \|(AB)\hat{}\|_\infty \leq \|\hat{A}\|_\infty \|\hat{B}\|_\infty = r(A)r(B)$.

The spectral radius is neither subadditive nor submultiplicative in general: Let $\mathcal{A} = M_2(\mathbb{C})$ and suppose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $r(A) = r(B) = 0$, since A and B have square zero, but $r(A+B) = r(AB) = 1$.

The interpretation of the character space as a sort of generalized spectrum is motivated by the following result.

Theorem 2.3.7. *Let $\mathcal{A}$ be a unital Banach algebra generated by I and an element A. Then $\mathcal{A}$ is abelian and the map*

$$\hat{A} : \Omega(\mathcal{A}) \to \sigma(A), \quad \tau \mapsto \tau(A),$$

is a homeomorphism.

Proof. It is clear that $\mathcal{A}$ is abelian and that $\hat{A}$ is a continuous bijection, and because $\Omega(\mathcal{A})$ and $\sigma(A)$ are compact Hausdorff spaces, $\hat{A}$ is therefore a homeomorphism. ■

2.4 Compact and Fredholm Operators

This section is concerned with the elementary spectral theory of compact operators, a class that plays an important and fundamental role in operator theory. These operators behave much like operators on finite-dimensional vector spaces, and for this reason they are relatively easy to analysis.

A linear map $U : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is *compact* if $U(\mathcal{S})$ is relatively compact in $\mathcal{Y}$, where $\mathcal{S}$ is the closed unit ball of $\mathcal{X}$. Equivalently, $U(\mathcal{S})$ is totally bounded. In this case $U(\mathcal{S})$ is bounded, and therefore, U is bounded.

Remark 2.4.1. *Note that the range of a compact operator is separable. This is immediate from the fact that a compact metric space is separable, and that the closure of the image of the ball under a compact operator is compact.*

The theory of compact operators arose out of the analysis of linear integral equations. The following example illustrates the connection.

Example 2.4.2. *Let $I = [0, 1]$ and let $\mathcal{X}$ be the Banach space $\mathcal{C}(I)$, where the norm is the supremum norm. If $k \in \mathcal{C}(I^2)$, define $U \in \mathcal{B}(\mathcal{X})$ by setting*

$$U(f)(s) = \int_0^1 k(s,t)f(t)dt \qquad (f \in \mathcal{X}, s \in I).$$

We show that $U(f) \in \mathcal{X}$. Observe first that

$$\begin{aligned}|U(f)(s) - U(f)(s')| &= |\int_0^1 (k(s,t) - k(s',t))f(t)dt| \\ &\leq \int_0^1 |k(s,t) - k(s',t)||f(t)|dt \\ &\leq \sup_{t \in I} |k(s,t) - k(s',t)| \|f\|_\infty.\end{aligned}$$

Now k is uniformly continuous because I^2 is compact, so if $\varepsilon > 0$, there exists $\delta > 0$ such that $\max\{|s - s'|, |t - t'|\} < \delta \Rightarrow |k(s,t) - k(s',t')| < \varepsilon$. Hence,

$$|s - s'| < \delta \Rightarrow |U(f)(s) - U(f)(s')| \leq \varepsilon \|f\|_\infty. \tag{1}$$

Thus, $U(f)$ is continuous, that is, $U(f) \in \mathcal{X}$, but more is true, for it is immediate from Inequality (1) that $U(\mathcal{S})$ is equicontinuous, where $\mathcal{S}$ is the closed *unit ball of $\mathcal{X}$. Also, $U(\mathcal{S})$ is bounded, since*

$$\|U(f)\| \leq \int_0^1 |k(s,t)f(t)|dt \leq \|k\|_\infty \|f\|_\infty.$$

By the Arzelà-Ascoli theorem the set $U(\mathcal{S})$ is totally bounded. Therefore, U is a compact operator on $\mathcal{X}$. The function k is called the kernel *of the operator U, and U is called an* integral *operator.*

If $\mathcal{X}, \mathcal{Y}$ are Banach spaces, we denote by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the vector space of all bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$. This is a Banach space when endowed with the operator norm. The set of all compact operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{K}(\mathcal{X}, \mathcal{Y})$.

The proof of the following is a routine exercise.

Theorem 2.4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $U \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then the following conditions are equivalent:*
(1) U is compact;
(2) For each bounded set $\mathcal{S}$ in $\mathcal{X}$, the set $U(\mathcal{S})$ is relatively compact in $\mathcal{Y}$;
(3) For each bounded sequence (x_n) in $\mathcal{X}$, the sequence $(U(x_n))$ admits a subsequence that converges in $\mathcal{Y}$.

It follows easily from (3) in Theorem 2.4.3 that $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ is a vector subspace of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. Also, if $\mathcal{X}' \xrightarrow{V} \mathcal{X} \xrightarrow{U} \mathcal{Y} \xrightarrow{W} \mathcal{Y}'$ are bounded linear maps between Banach spaces and U is compact, then WUV are compact. Hence $\mathcal{K}(\mathcal{X}) = \mathcal{K}(\mathcal{X}, \mathcal{X})$ is an ideal in $\mathcal{B}(\mathcal{X})$.

Theorem 2.4.4. *If $\mathcal{X}$ is a Banach space, then $\mathcal{K}(\mathcal{X}) = \mathcal{B}(\mathcal{X})$ if and only if $\mathcal{X}$ is finite-dimensional.*

Proof. If $\mathcal{S}$ denotes the closed unit ball of $\mathcal{X}$, then $\mathcal{K}(\mathcal{X}) = \mathcal{B}(\mathcal{X}) \Leftrightarrow \mathrm{id}_{\mathcal{X}}$ is a compact operator $\Leftrightarrow \mathcal{S}$ is compact $\Leftrightarrow \mathcal{X}$ is finite-dimensional. ∎

Theorem 2.4.5. *If $\mathcal{X}, \mathcal{Y}$ are Banach spaces, then $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ is a closed vector space of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

Proof. We want to prove that if a sequence (U_n) in $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ converges to an operator U in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, then U is compact. Let $\mathcal{S}$ denote the closed unit ball of $\mathcal{X}$ and let $\varepsilon > 0$. Choose an integer N such that $\|U_N - U\| < \varepsilon/3$. Since $U_N(\mathcal{S})$ is totally bounded, there are elements $x_1, \ldots, x_n \in \mathcal{S}$, such that for each x in $\mathcal{S}$, the inequality $\|U_N(x) - U_N(x_j)\| < \varepsilon/3$ holds for some index j. Hence,

$$\begin{aligned}\|U(x) - U(x_j)\| &\leq \|U(x) - U_N(x)\| + \|U_N(x) - U_N(x_j)\| + \|U_N(x_j) - U(x_j)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Thus, $U(\mathcal{S})$ is totally bounded, and therefore, $U \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$. ∎

Recall that a linear map $U : \mathcal{X} \to \mathcal{Y}$ is of *finite rank* if $U(\mathcal{X})$ is finite-dimensional and that $\mathrm{rank}(U) = \dim(U(\mathcal{X}))$.

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $U \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is of finite rank, then $U \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$. This is immediate from the fact that the closed unit ball of the finite-dimensional space $U(\mathcal{X})$ is compact.

It follows from this remark and Theorem 2.4.5 that norm-limits of finite-rank operators are compact, and it is natural to ask whether the converse is true. This is the case for Hilbert spaces, as we shall see in the next chapter, but it is not true for arbitrary Banach spaces. P.Enflo has given an example of a Banach space for which there are compact operators that are not norm-limits of finite-rank operators.

If $U : \mathcal{X} \to \mathcal{Y}$ is a bounded linear map between Banach spaces, we define its *transpose* $U^* \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ by $U^*(\tau) = \tau \circ U$.

Theorem 2.4.6. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $U \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$. Then $U^* \in \mathcal{K}(\mathcal{Y}^*, \mathcal{X}^*)$.*

Proof. Let $\mathcal{S}$ be the closed unit ball of $\mathcal{X}$ and let $\varepsilon > 0$. Since $U(\mathcal{S})$ is totally bounded, there exist elements $x_1, \ldots, x_n$ in $\mathcal{S}$, such that if $x \in \mathcal{S}$, then $\|U(x) - U(x_i)\| < \varepsilon/3$ for some index i. Define $V \in \mathcal{B}(\mathcal{Y}^*, \mathbb{C}^n)$ by setting $V(\tau) = (\tau \circ U(x_1), \ldots, \tau \circ U(x_n))$. Since the rank of V is finite, V is compact, and therefore $V(\mathcal{Y}_1^*)$ is totally bounded, where $\mathcal{Y}_1^*$ is the closed unit ball of $\mathcal{Y}^*$. Hence, there exist functionals $\tau_1, \ldots, \tau_m$ in $\mathcal{Y}_1^*$, such that if $\tau \in \mathcal{Y}_1^*$, then $\|V(\tau) - V(\tau_j)\| < \varepsilon/3$ for some index j. Observe that

$$\|V(\tau) - V(\tau_j)\| = [\sum_i |U^*(\tau)(x_i) - U^*(\tau_j)(x_i)|^2]^{1/2}.$$

Now suppose that $x \in \mathcal{S}$. Then $\|U(x) - U(x_i)\| < \varepsilon/3$ for some index i, and $|U^*(\tau)(x_i) - U^*(\tau_j)(x_i)| < \varepsilon/3$. Hence,

$$\begin{aligned}|U^*(\tau)(x) - U^*(\tau_j)(x)| &\leq |U^*(\tau)(x) - U^*(\tau)(x_i)| + |U^*(\tau)(x_i) - U^*(\tau_j)(x_i)| \\ &\quad + |U^*(\tau_j)(x_i) - U^*(\tau_j)(x)| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

It follows that $\|U^*(\tau) - U^*(\tau_j)\| \leq \varepsilon$, so $U^*(\mathcal{Y}_1^*)$ is totally bounded and therefore U^* is compact. ∎

A linear bounded map $U : \mathcal{X} \to \mathcal{Y}$ between Banach spaces is *bounded below* if there is a positive number δ such that $\|U(x)\| \geq \delta \|x\|$ $(x \in \mathcal{X})$. Note that in this case $U(\mathcal{X})$ is necessarily closed, for if $(U(x_n))$ is a Cauchy sequence in $U(\mathcal{X})$, then (x_n) is a Cauchy sequence in $\mathcal{X}$ and therefore converges to some element $x \in \mathcal{X}$, because $\mathcal{X}$ is complete. Hence, the sequence $(U(x_n))$ converges to $U(x)$ by continuity of U. Thus, $U(\mathcal{X})$ is complete and therefore closed in $\mathcal{Y}$.

Observe that every invertible linear bounded map is bounded below. It is easily checked that $U : \mathcal{X} \to \mathcal{Y}$ is not bounded below if and only if there is a sequence of unit vectors (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} U(x_n) = 0$. These remarks will be used in the following theorem. we first prove a lemma.

Lemma 2.4.7. *Suppose $\mathcal{X}$ be a Banach space, $\mathcal{V} \subseteq \mathcal{X}$ be a finite dimensional subspace, then there exists a closed subspace $\mathcal{M}$ such that $\mathcal{X} = \mathcal{M} \oplus \mathcal{V}$*

Proof. It is clear that $\mathcal{V}$ is closed. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathcal{V}$. For every $x \in \mathcal{V}$,

$$x = a_1(x)e_1 + \cdots + a_n(x)e_n,$$

the expression is unique. It is easy to prove that every $a_i(x)$ is a continuous linear functional. By Hahn-Banach theory, a_i can be extended to a continuous linear functional on $\mathcal{X}$, denoted by $a_1^*, \cdots, a_n^*$. Let $\mathcal{M} = \cap_{i=1}^n Ker(a_i^*)$, then $\mathcal{M}$ is a closed linear subspace.

If $x \in \mathcal{M} \cap \mathcal{V}$, for every i, $a_i^*(x) = 0$. For $x \in \mathcal{V}$, $a_i^*(x) = a_i(x) = 0$. Since $x = \sum a_i(x)e_i$, we get $x = 0$. For any $x \in \mathcal{X}$, let

$$x' = a_i^*(x)e_1 + \cdots + a_n^*(x)e_n.$$

It is clear that $x' \in \mathcal{V}$, therefore $a_i^*(x') = a_i(x')$. Since $a_i(e_i) = 1$ and $a_i(e_j) = 0$ $(i \neq j)$,

$$a_i^*(x - x') = a_i^*(x) - a_i(x') = a_i^*(x) - a_i^*(x) = 0$$

for every i, i.e., $x - x^{'} \in \mathcal{M}$. ∎

Theorem 2.4.8. *Let U be a compact operator on a Banach space $\mathcal{X}$ and suppose that $\lambda \in \mathbb{C} \setminus \{0\}$.*
(1) The space $\ker(U - \lambda)$ is finite-dimensional.
(2) The space $(U - \lambda)(\mathcal{X})$ is closed and finite-codimensional in $\mathcal{X}$. In fact, the codimension of $(U - \lambda)(\mathcal{X})$ in $\mathcal{X}$ is the dimension of $\ker(U^* - \lambda)$.

Proof. Let $\mathcal{Z} = \ker(U - \lambda)$. Then $U(\mathcal{Z}) \subseteq \mathcal{Z}$, and the restriction $U_{\mathcal{Z}}$ of U to $\mathcal{Z}$ is in $\mathcal{K}(\mathcal{Z})$. Since $U_{\mathcal{Z}} = \lambda \mathrm{id}_{\mathcal{Z}}$ and $\lambda \neq 0$, the map $\mathrm{id}_{\mathcal{Z}}$ is compact. Hence, $\mathcal{Z}$ is finite-dimensional by Theorem 2.4.4.

Because $\mathcal{Z}$ is finite-dimensional, there is a closed vector space $\mathcal{Y}$ in $\mathcal{X}$ such that $\mathcal{Z} \oplus \mathcal{Y} = \mathcal{X}$. Observe that $(U-\lambda)\mathcal{X} = (U-\lambda)\mathcal{Y}$, so to show that $(U-\lambda)\mathcal{X}$ is closed in $\mathcal{X}$ it suffices to show that the restriction $(U-\lambda)_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X}$ is bounded below. Suppose otherwise, and we shall obtain a contradiction. There is a sequence (x_n) of unit vectors in $\mathcal{Y}$ such that $\lim_{n\to\infty} \|U(x_n) - \lambda x_n\| = 0$. Using the compactness of U and going to a subsequence if necessary, we may suppose that $(U(x_n))$ is convergent. It follows from the equation $x_n = \lambda^{-1}(U(x_n) - (U - \lambda)(x_n))$ that the sequence (x_n) is convergent, to x say, and, since $\mathcal{Y}$ is closed in $\mathcal{X}$, it contains x. Obviously, $U(x) = \lambda x$, so $x \in \mathcal{Y} \cap \ker(U - \lambda)$ and therefore $x = 0$. However, x is the limit of unit vectors and is therefore itself a unit vector, a contradiction. This shows that $(U - \lambda)_{\mathcal{Y}}$ is bounded below.

Now let $\mathcal{W} = \mathcal{X}/(U-\lambda)(\mathcal{X})$. To show that $(U-\lambda)(\mathcal{X})$ is finite-codimensional in $\mathcal{X}$, we have to show $\mathcal{W}$ is finite-dimensional, and we do this by showing $\mathcal{W}^*$ is finite-dimensional. Let $\pi : \mathcal{X} \to \mathcal{W}$ be the quotient map. It is clear that the image of π^* is contained in the kernel of $U^* - \lambda$. In fact these spaces are equal. For suppose that $\sigma \in \ker(U^* - \lambda)$. Then σ annihilates $(U - \lambda)(\mathcal{X})$ and therefore induces a bounded linear functional $\tau : \mathcal{W} \to \mathbb{C}$ such that $\sigma = \tau \circ \pi = \pi^*(\tau)$. Since U^* is compact by Theorem 2.4.5, $\ker(U^* - \lambda)$ is finite-dimensional by the first part of this proof. Thus, π^* has finite-dimensional range, and clearly π^* is injective, so $\mathcal{W}^*$ is finite-dimensional, and therefore $\dim(\mathcal{W}) = \dim(\mathcal{W}^*) = \dim(\pi^*(\mathcal{W}^*)) = \dim(\ker(U^* - \lambda))$. ■

If $U : \mathcal{X} \to \mathcal{X}$ is a linear map on a vector space $\mathcal{X}$, then the sequence of spaces $(\ker(U^n))$ is clearly increasing. If $\ker(U^n) \neq \ker(U^{n+1})$ for all $n \in N$, we say that U has *infinite ascent* and set $\text{ascent}(U) = +\infty$. Otherwise we say U has *finite ascent* and we define $\text{ascent}(U)$ to be the least p such that $\ker(U^p) = \ker(U^{p+1})$. In this case, $\ker(U^p) = ker(U^n)$ for all $n \geq p$.

The sequence of spaces $(U^n(\mathcal{X}))$ is decreasing. We say that U has *infinite descent*, and we set $\text{descent}(U) = +\infty$, if $U^n(\mathcal{X}) \neq U^{n+1}(\mathcal{X})$ for all $n \in N$. Otherwise, we say that U has *finite descent* and we define $\text{descent}(U)$ to be the least $p \in N$ such that $U^p(\mathcal{X}) = U^{p+1}(\mathcal{X})$. In this case $U^p(\mathcal{X}) = U^n(\mathcal{X})$ for all $n \geq p$.

We recall now a theorem of F. Riesz from elementary functional analysis: If $\mathcal{Y}$ is a proper closed vector subspace of a normed vector space $\mathcal{X}$ and $\varepsilon > 0$, then there exists a unit vector $x \in \mathcal{X}$ such that $\|x + \mathcal{Y}\| > 1 - \varepsilon$. This simple result plays a key role in the theory of compact operators.

Theorem 2.4.9. *Let U be a compact operator on a Banach space $\mathcal{X}$ and suppose that $\lambda \in \mathbb{C} \setminus \{0\}$. Then $U - \lambda$ has finite ascent and descent.*

Proof. Suppose the ascent is infinite, and we deduce a contradiction. If $N_n = \ker(U - \lambda)^n$, then N_{n-1} is a proper subspace of N_n, and therefore, by the theorem of Riesz discussed earlier, there is a unit vector $x_n \in N_n$ such that $\|x_n + N_{n-1}\| \geq 1/2$. If $m < n$, then

$$U(x_n) - U(x_m) = \lambda x_n + (U - \lambda)(x_n) - (U - \lambda)(x_m) - \lambda x_m = \lambda x_n - z,$$

where $z \in N_{n-1}$. Hence, $\|U(x_n) - U(x_m)\| = \|\lambda x_n - z\| = |\lambda|\|x_n - \lambda^{-1}z\| \geq |\lambda|/2 > 0$. It follows that $(U(x_n))$ has no convergent subsequence, contradicting the compactness of U. Consequently, ascent$(U) < +\infty$.

The proof that $U - \lambda$ has finite descent is completely analogous and is left as an exercise. ∎

We shall have more to say about compact operators presently. One can give direct proofs of these later results, but the details are tedious and a little messy, whereas when one use the homomorphism property of the Fredholm index, which we are now going to introduce, they drop out very nicely.

The index which we shall also introduce, is the indispensible item in the operator theory. Nevertheless, many of the proofs in Fredholm theory are elementary, although often neither trivial or obvious.

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $U \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. We say U is *Fredholm* if $\ker(U)$ is finite-dimensional and $U(\mathcal{X})$ is finite-codimensional in $\mathcal{Y}$. We define the *nullity* of U to be $\dim(\ker(U))$ and denote it by $\mathrm{nul}(U)$. The *defect* of U is the codimension of $U(\mathcal{X})$ in $\mathcal{Y}$, and is denoted by $\mathrm{def}(U)$. The *index* of U

is defined to be

$$\operatorname{ind}(U) = \operatorname{nul}(U) - \operatorname{def}(U).$$

Note that because there is a finite-dimensional vector subspace $\mathcal{Z}$ of $\mathcal{Y}$, such that $U(\mathcal{X}) \oplus \mathcal{Z} = \mathcal{Y}$, it is a consequence of the following theorem that $U(\mathcal{X})$ is closed in $\mathcal{Y}$.

Theorem 2.4.10. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $U \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Suppose that there is a closed vector subspace $\mathcal{Z}$ such that $U(\mathcal{X}) \oplus \mathcal{Z} = \mathcal{Y}$. Then $U(\mathcal{X})$ is closed in $\mathcal{Y}$.*

Proof. The bounded linear map

$$\mathcal{X}/\ker(U) \to \mathcal{Y}, \quad x + \ker(U) \mapsto U(x),$$

has the same range as U and is injective, so we may suppose without loss of generality that U is injective.

The map

$$V : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}, \quad (x, z) \mapsto U(x) + z,$$

is a continuous linear isomorphism between Banach spaces, so by the open mapping theorem, V^{-1} is also continuous. . If $x \in \mathcal{X}$, then $\|x\| = \|V^{-1}U(x)\| \leq \|V^{-1}\|\|U(x)\|$, so $\|U(x)\| \geq \|V^{-1}\|^{-1}\|x\|$. Thus, U is bounded below, and therefore $U(\mathcal{X})$ is closed in $\mathcal{Y}$. ■

The following theorem is a fundamental result of Fredholm theory.

Theorem 2.4.11. *Let $\mathcal{X} \xrightarrow{U} \mathcal{Y} \xrightarrow{V} \mathcal{Z}$ be Fredholm linear maps between Banach spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$. Then VU is Fredholm and*

$$\operatorname{ind}(VU) = \operatorname{ind}(V) + \operatorname{ind}(U).$$

Proof. Set $\mathcal{Y}_2 = \ker(V) \cap U(\mathcal{X})$ and choose suitable closed vector subspaces $\mathcal{Y}_1, \mathcal{Y}_3, \mathcal{Y}_4$ of $\mathcal{Y}$, such that $U(\mathcal{X}) = \mathcal{Y}_2 \oplus \mathcal{Y}_3$, $\ker(V) = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, and $\mathcal{Y} = \mathcal{Y}_1 \oplus U(\mathcal{X}) \oplus \mathcal{Y}_4$. Note that $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_4$ are finite-dimensional, and

$$\mathrm{nul}(V) = \dim\mathcal{Y}_1 + \dim\mathcal{Y}_2,$$

$$\dim\mathcal{Y}_1 + \dim\mathcal{Y}_4 = \mathrm{def}(U),$$

so we have

$$\dim\mathcal{Y}_2 + \mathrm{def}(U) = \mathrm{nul}(V) + \dim\mathcal{Y}_4.$$

The map

$$\ker(VU) \to \mathcal{Y}_2, \quad x \mapsto U(x),$$

is surjective and it has the same kernel as U, so the kernel of VU is finite-dimensional and $\mathrm{nul}(VU) = \mathrm{nul}(U) + \dim(\mathcal{Y}_2)$.

Since $V(\mathcal{Y}) = V(\mathcal{Y}_3) \oplus V(\mathcal{Y}_4)$ and $V(\mathcal{Y}_3) = VU(\mathcal{X})$, therefore $V(\mathcal{Y}) = VU(\mathcal{X}) \oplus V(\mathcal{Y}_4)$. Choose a finite-dimensional vector subspace $\mathcal{Z}'$ of $\mathcal{Z}$ such that $V(\mathcal{Y}) \oplus \mathcal{Z}' = \mathcal{Z}$, so $\mathcal{Z} = VU(\mathcal{X}) \oplus V(\mathcal{Y}_4) \oplus \mathcal{Z}'$. Because $V(\mathcal{Y}_4) \oplus \mathcal{Z}'$ is finite-dimensional, $VU(\mathcal{X})$ is finite-codimensional in $\mathcal{Z}$. therefore, VU is a Fredholm operator.

The map

$$\mathcal{Y}_4 \to V(\mathcal{Y}_4), \quad y \mapsto V(y),$$

is a linear isomorphism, so $\dim(\mathcal{Y}_4) = \dim(V(\mathcal{Y}_4))$. Hence, $\mathrm{def}(VU) = \dim(\mathcal{Y}_4) + \dim(\mathcal{Z}') = \dim(\mathcal{Y}_4) + \mathrm{def}(V)$. Consequently, $\mathrm{nul}(VU) + \mathrm{def}(U) + \mathrm{def}(V) = \mathrm{nul}(U) + \dim(\mathcal{Y}_2) + \mathrm{def}(U) + \mathrm{def}(V) = \mathrm{nul}(U) + \dim(\mathcal{Y}_4) + \mathrm{nul}(V) + \mathrm{def}(V) =$

$\mathrm{nul}(U) + \mathrm{nul}(V) + \mathrm{def}(VU)$, and therefore, $\mathrm{ind}(VU) = \mathrm{nul}(VU) - \mathrm{def}(VU) = \mathrm{nul}(U) + \mathrm{nul}(V) - \mathrm{def}(U) - \mathrm{def}(V) = \mathrm{ind}(U) + \mathrm{ind}(V)$. ∎

We give an immediate easy application of the index:

Theorem 2.4.12. *Let U be a compact operator on a Banach space $\mathcal{X}$, and let $\lambda \in \mathbb{C} \setminus \{0\}$.*
(1) The operator $U - \lambda$ is Fredholm of index zero.
(2) If p denotes the ascent of $U - \lambda$, then

$$\mathcal{X} = \ker(U - \lambda)^p \oplus (U - \lambda)^p(\mathcal{X}).$$

Proof. That $U - \lambda$ is Fredholm follows from Theorem 2.4.8, and the ascent and descent of $U - \lambda$ are finite by Theorem 2.4.9. If we suppose that m, n are integers greater than $\max\{\mathrm{ascent}(U - \lambda), \mathrm{descent}(U - \lambda)\}$, then we have $\mathrm{nul}(U-\lambda)^m = \mathrm{nul}(U-\lambda)^n$ and $\mathrm{def}(U-\lambda)^m = \mathrm{def}(U-\lambda)^n$, so $\mathrm{ind}((U-\lambda)^m) = \mathrm{ind}((U-\lambda)^n)$, and therefore $m\,\mathrm{ind}(U - \lambda) = n\,\mathrm{ind}(U - \lambda)$ by Theorem 2.4.11. It follows that $\mathrm{ind}(U - \lambda) = 0$. Thus, Condition (1) is proved.

If $x \in \ker(U-\lambda)^p \cap (U-\lambda)^p(\mathcal{X})$, then there is an element $y \in \mathcal{X}$ such that $x = (U - \lambda)^p(y)$ and $(U - \lambda)^{2p}(y) = 0$. Since $\ker(U - \lambda)^p = \ker(U - \lambda)^{2p}$, it follows that $(U - \lambda)^p(y) = 0$; that is, $x = 0$. Moreover, since $\mathrm{nul}(U - \lambda)^p = \mathrm{def}(U - \lambda)^p$, because $\mathrm{ind}(U - \lambda)^p = 0$, it follows that $\mathcal{X} = \ker(U - \lambda)^p \oplus (U - \lambda)^p(\mathcal{X})$. ∎

Corollary 2.4.13. *(Fredholm Alternative) Let $\lambda \in \mathbb{C}\setminus\{0\}$. Then the operator $U - \lambda$ is injective if and only if it is surjective.*

Proof. Since the index of $U - \lambda$ is zero, the nullity is zero if and only if the defect is zero; that is, $U - \lambda$ is injective if and only if it is surjective. ∎

Remark 2.4.14. *If $\mathcal{Y}, \mathcal{Z}$ are complementary vector subspaces of a vector space $\mathcal{X}$, and V, U are linear maps on $\mathcal{Y}, \mathcal{Z}$, respectively, we denote by $U \oplus V$ the linear map on $\mathcal{X}$ given by*

$$(U \oplus V)(y + z) = U(y) + V(z) \qquad (y \in \mathcal{Y},\ z \in \mathcal{Z}).$$

Clearly, $U \oplus V$ is invertible if and only if U and V are invertible.

If $\mathcal{X}$ is a Banach space and $W \in \mathcal{B}(\mathcal{X})$, we write $\sigma(W)$ for $\sigma_{\mathcal{B}(\mathcal{X})}(W)$. If $\mathcal{Y}, \mathcal{Z}$ are closed complementary vector subspaces of $\mathcal{X}$, and if $U \in \mathcal{B}(\mathcal{Y})$, $V \in \mathcal{B}(\mathcal{Z})$, $W \in \mathcal{B}(\mathcal{X})$, and $W = U \oplus V$, then $\sigma(W) = \sigma(U) \cup \sigma(V)$, by the preceding observation.

Theorem 2.4.15. *Let U be a compact operator on a Banach space $\mathcal{X}$. Then $\sigma(U)$ is countable, and each non-zero point of $\sigma(U)$ is an eigenvalue of U and an isolated point of $\sigma(U)$.*

Proof. If λ is a non-zero point of $\sigma(U)$, then by the Fredholm alternative, Corollary 2.4.13, $U - \lambda$ is not injective, and therefore λ is an eigenvalue of U. The operator $U - \lambda$ has finite ascent, p say, and by Theorem 2.4.12 we can write $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$, where $\mathcal{Y} = \ker(U - \lambda)^p$ and $\mathcal{Z} = (U - \lambda)^p(\mathcal{X})$. The spaces $\mathcal{Y}, \mathcal{Z}$ are closed and invariant for U (that is, $U(\mathcal{Y}) \subseteq \mathcal{Y}$ and $U(\mathcal{Z}) \subseteq \mathcal{Z}$). Hence, $U - \lambda = (U_{\mathcal{Y}} - \lambda \mathrm{id}_{\mathcal{Y}}) \oplus (U_{\mathcal{Z}} - \lambda \mathrm{id}_{\mathcal{Z}})$, where $U_{\mathcal{Y}}, U_{\mathcal{Z}}$ are the restrictions of U to $\mathcal{Y}, \mathcal{Z}$, respectively. Since $(U_{\mathcal{Y}} - \lambda \mathrm{id}_{\mathcal{Y}})^p = 0$, the spectrum $\sigma(U_{\mathcal{Y}})$ is the singleton set $\{\lambda\}$. Also, the operator $U_{\mathcal{Z}}$ is compact and $\ker(U_{\mathcal{Z}} - \lambda \mathrm{id}_{\mathcal{Z}})^p = 0$, so $(U_{\mathcal{Z}} - \lambda \mathrm{id}_{\mathcal{Z}})^p$ is invertible (as it is injective and Fredholm of index zero), and therefore $U_{\mathcal{Z}} - \lambda \mathrm{id}_{\mathcal{Z}}$ is invertible. Hence, $\lambda \notin \sigma(U_{\mathcal{Z}})$. This implies that $\sigma(U) \setminus \{\lambda\} = \sigma(U_{\mathcal{Z}})$, so λ is an isolated point of $\sigma(U)$ because $\sigma(U_{\mathcal{Z}})$ is closed in $\sigma(U)$.

Countability of $\sigma(U)$ follows by elementary topology. ■

Example 2.4.16. *Let us interpret our results now in terms of integral equations. Let $I = [0,1]$ and suppose $k \in \mathcal{C}(I^2)$. Consider the integral equation*

$$\int_0^1 k(s,t)f(t)dt - \lambda f(s) = g(s).$$

Here λ is a non-zero scalar, $g \in \mathcal{C}(I)$ is a known function, and $f \in \mathcal{C}(I)$ is the unknown. If U is the compact integral operator corresponding to the kernel k, as in Example 2.4.2, then we can rewrite our equation as

$$(U - \lambda)(f) = g.$$

The non-zero spectrum of U is of the form $\{\lambda_n \mid 1 \leq n \leq N\}$, where N is an integer or ∞. If $0 \neq \lambda \neq \lambda_n$ for all n, then the integral equation has a unique solution: $f = (U - \lambda)^{-1}(g)$. If on the other hand $\lambda = \lambda_n$ say, then

$$\int_0^1 k(s,t)f(t)dt - \lambda f(s) = 0.$$

has a non-zero solution by the Fredholm alternative (Corollary 2.4.13), and by Theorem 2.4.8 the solution set is finite-dimensional.

Observe that if $N = \infty$, then $\lim_{n\to\infty} \lambda_n = 0$ by Theorem 2.4.15.

Example 2.4.17. *One should not be misled by Theorem 2.4.15—the spectral behaviour of compact operators is not typical of all operators. To illustrate this, let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. If (λ_n) is a* *bounded sequence of scalars, define* $U \in \mathcal{B}(\mathcal{H})$ *by setting* $U(x) = \sum_{n=1}^{\infty} \lambda_n a_n e_n$ *when* $x = \sum_{n=1}^{\infty} a_n e_n$*. We call* U *the* diagonal *operator with* diagonal (λ_n) *with respect to the basis* (e_n)*. It is readily verified that* $\|U\| = \sup_n |\lambda_n|$*, and that* U *is invertible if and only if* $\inf_n |\lambda_n| > 0$*, and in this case* U^{-1} *is the diagonal operator with respect to* (e_n) *with diagonal* (λ_n^{-1})*. These observations imply that* $\sigma(U)$ *is the closure of the set* $\{\lambda_n \mid n = 1, 2, \ldots\}$*.*

Suppose that a non-empty compact set $\mathcal{K}$ *in* $\mathbb{C}$ *is given and choose a dense sequence* λ_n *in* $\mathcal{K}$*. If* U *is the corresponding diagonal operator, then* $\sigma(U) = \mathcal{K}$*. Thus, the spectrum is an arbitrary non-empty compact set in general.*

3 Operators on the Hilbert Spaces and GNS Construction

In this chapter we study C^*-Algebras and operators on Hilbert Spaces. Hilbert Spaces are very well-behaved compared with general Banach spaces, and the same is even more true of C^*-Algebras as compared with general Banach algebras. The main results of this chapter are the theorem of Gelfand, which asserts that, up to isomorphism, all unital abelian C^*-Algebras are of the form $\mathcal{C}(\Omega)$, where Ω is a compact Hausdorff space, and two class of important operators, and the famous Gelfand-Naimark-Segal Construction theorem, it has important applications in many field of mathematics.

3.1 C*-Algebras

We begin by defining a number of concepts that make sense in any algebra with an involution.

An *involution* on an algebra $\mathcal{A}$ is a conjugate-linear map $A \mapsto A^*$ on $\mathcal{A}$, such that $A^{**} = A$ and $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$. The pair $(\mathcal{A}, *)$ is called an *involutive* algebra, or a $*$-*algebra*. If $\mathcal{S}$ is a subset of $\mathcal{A}$, we set $\mathcal{S}^* = \{A^* \mid A \in \mathcal{S}\}$, and if $\mathcal{S}^* = \mathcal{S}$ we say $\mathcal{S}$ is *self-adjoint*. A self-adjoint subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a $*$-*subalgebra* of $\mathcal{A}$ and is a $*$-algebra when endowed with the involution got by restriction. Because the intersection of a family of $*$-subalgebras of $\mathcal{A}$ is itself one, there is for every subset $\mathcal{S}$ of $\mathcal{A}$ a smallest $*$-algebra $\mathcal{B}$ of $\mathcal{A}$ containing $\mathcal{S}$, called the $*$-algebra *generated* by $\mathcal{S}$.

If $\mathcal{I}$ is self-adjoint ideal of $\mathcal{A}$, then the quotient algebra $\mathcal{A}/\mathcal{I}$ is a $*$-algebra with the involution given by $(A+\mathcal{I})^* = A^* + \mathcal{I}$ $(A \in \mathcal{A})$.

An element A in $\mathcal{A}$ is *self-adjoint* or *hermitian* if $A = A^*$. For each $A \in \mathcal{A}$ there exist unique hermitian elements $B, C \in \mathcal{A}$ such that $A = B + iC$, where $B = \frac{1}{2}(A+A^*)$ and $C = \frac{1}{2i}(A-A^*)$. The elements A^*A and AA^* are hermitian.

The set of hermitian elements of $\mathcal{A}$ is denoted by $\mathcal{A}_{sa}$.

We say A is *normal* if $A^*A = AA^*$. In this case the $*$-algebra it generates is abelian and is in fact the linear span of all $A^m A^{*n}$, where $m, n \in \mathbb{N}$ and $n + m \geq 1$.

An element P is a *projection* if $P = P^* = P^2$.

If $\mathcal{A}$ is unital, then $I^* = I$. In fact, since $I^* = (II^*)^* = I$. If $A \in \mathrm{Inv}(\mathcal{A})$, then $(A^*)^{-1} = (A^{-1})^*$. Hence, for any $A \in \mathcal{A}$,

$$\sigma(A^*) = \sigma(A)^- = \{\overline{\lambda} \in \mathbb{C} \mid \lambda \in \sigma(A)\}.$$

An element U in $\mathcal{A}$ is a *unitary* if $U^*U = UU^* = I$. If $U^*U = I$, then U is an *isometry*; and if $UU^* = I$, then U is a *co-isometry*.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ and φ preserves adjoints, that is , $\varphi(A^*) = (\varphi(A))^*$ $(A \in \mathcal{A})$, then φ is said to be a $*$-*homomorphism*. If in addition φ is a bijection, it is a $*$-*isomorphism*. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then $\ker(\varphi)$ is a self-adjoint ideal in $\mathcal{A}$ and $\varphi(\mathcal{A})$ is a $*$-subalgebra of $\mathcal{B}$.

An automorphism of a $*$-algebra $\mathcal{A}$ is a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{A}$. If $\mathcal{A}$ is unital and U is a unitary in $\mathcal{A}$, then

$$\mathrm{Ad}\, U : \mathcal{A} \to \mathcal{A}, \quad A \mapsto UAU^*,$$

is an automorphism of $\mathcal{A}$. Such automorphisms are called *inner*. We say elements A, B of $\mathcal{A}$ are *unitarily equivalent* if there exists a unitary U of $\mathcal{A}$

such that $B = UAU^*$. Since the unitaries form a group, this is an equivalence relation on $\mathcal{A}$. Note that $\sigma(A) = \sigma(B)$ if A and B are unitarily equivalent.

A *C*-algebra* is a Banach algebra and endowed with an involution $*$ such that

$$\|A^*A\| = \|A\|^2 \qquad (A \in \mathcal{A}). \tag{1}$$

Note that in a C*-algebra, it is clear that $\|A\|^2 = \|A^*A\| \leq \|A^*\|\|A\|$ imply that $\|A\| \leq \|A^*\|$ for all A. Hence, $\|A\| = \|A^*\|$. Moreover, if a non-zero C*-algebra has a unit I, then automatically $\|I\| = 1$, because $\|I\| = \|I^*I\| = \|I\|^2$. Similarly, if P is a non-zero projection, then $\|P\| = 1$.

A closed $*$-subalgebra of a C*-algebra is obviously also a C*-algebra. We shall therefore call a closed $*$-subalgebra of a C*-algebra a *C*-subalgebra*.

If U is a unitary of $\mathcal{A}$, then $\|U\| = 1$, since $\|U\|^2 = \|U^*U\| = \|I\| = 1$. Hence, $\sigma(U) \subseteq \mathbb{T}$, for if $\lambda \in \sigma(U)$, then $\lambda^{-1} \in \sigma(U^{-1}) = \sigma(U^*)$, so $|\lambda|$ and $|\lambda^{-1}| \leq 1$; that is, $|\lambda| = 1$.

Because of the existence of the involution, C*-algebra theory can be thought of as infinite-dimensional real analysis. For instance, the study of linear functionals on C*-algebras is non-commutative measure theory.

Example 3.1.1. *The scalar field $\mathbb{C}$ is a unital C*-algebra with involution given by complex conjugation $\lambda \mapsto \overline{\lambda}$.*

Example 3.1.2. *If Ω is a locally compact Hausdorff space, then $\mathcal{C}_0(\Omega)$ is a C*-algebra with involution $f \mapsto \overline{f}$. Similarly, all of the following algebras are C*-algebras with involution given by $f \mapsto \overline{f}$:*

(a) $\ell^\infty(\mathcal{S})$, where $\mathcal{S}$ is a set;

(b) $\mathcal{L}^\infty(\Omega, \mu)$, where (Ω, μ) is a measure space;

(c) $\mathcal{C}_b(\Omega)$*, where* Ω *is a topological space;*

(d) $\mathcal{B}_\infty(\Omega)$*, where* Ω *is a measurable space.*

Example 3.1.3. *If* $\mathcal{H}$ *is a Hilbert space, then* $\mathcal{B}(\mathcal{H})$ *is a C*-algebra.*

Example 3.1.4. *If* Ω *is a non-empty set and* $\mathcal{A}$ *is a C*-algebra, then* $\ell^\infty(\Omega, \mathcal{A})$ *is a C*-algebra with the pointwise-defined involution. This of course generalises Example 3.1.2 (a). If* Ω *is a locally compact Hausdorff space, we say a continuous function* $f : \Omega \to \mathcal{A}$ vanishes at infinity *if, for each* $\varepsilon > 0$*, the set* $\{\omega \in \Omega \mid \|f(\omega)\| \geq \varepsilon\}$ *is compact. Denote by* $\mathcal{C}_0(\Omega, \mathcal{A})$ *the set of all such functions. This is a C*-subalgebra of* $\ell^\infty(\Omega, \mathcal{A})$*.*

The following easy result has a surprising and important corollary:

Theorem 3.1.5. *If* A *is a self-adjoint element of a C*-algebra* $\mathcal{A}$*, then* $r(A) = \|A\|$*.*

Proof. Clearly, $\|A^2\| = \|A\|^2$, and therefore by induction $\|A^{2^n}\| = \|A\|^{2^n}$, so $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^{2^n}\|^{1/2^n} = \|A\|$. ■

Corollary 3.1.6. *There is at most one norm on a* $*$*-algebra making it a C*-algebra.*

Proof. If $\|.\|_1$ and $\|.\|_2$ are norms on a $*$-algebra $\mathcal{A}$ making it a C*-algebra, then

$$\|A\|_j^2 = \|A^*A\|_j = r(A^*A) = \sup_{\lambda \in \sigma(A^*A)} |\lambda| \qquad (j = 1, 2),$$

so $\|A\|_1 = \|A\|_2$. ■

Theorem 3.1.7. *A unital* $*$*-homomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *from a unital C*-algebra* $\mathcal{A}$ *to a unital C*-algebra* $\mathcal{B}$ *is necessarily norm-decreasing.*

Proof. If $A \in \mathcal{A}$, then $\sigma(\varphi(A)) \subseteq \sigma(A)$, so $\|\varphi(A)\|^2 = \|\varphi(A)^*\varphi(A)\| = \|\varphi(A^*A)\| = r(\varphi(A^*A)) \leq r(A^*A) \leq \|A^*A\| \leq \|A\|^2$. Hence, $\|\varphi(A)\| \leq \|A\|$.

■

Theorem 3.1.8. *If A is a hermitian element of a unital C*-algebra $\mathcal{A}$, then $\sigma(A) \subseteq \mathbb{R}$.*

Proof. Since e^{iA} is unitary, $\sigma(e^{iA}) \subseteq \mathbb{T}$. If $\lambda \in \sigma(A)$ and $B = \sum_{n=1}^{\infty} i^n(A-\lambda)^{n-1}/n!$, then $e^{iA} - e^{i\lambda} = (e^{i(A-\lambda)} - 1)e^{i\lambda} = (A-\lambda)Be^{i\lambda}$. Since B commutes with A, and since $A - \lambda$ is non-invertible, $e^{iA} - e^{i\lambda}$ is non-invertible. Hence, $e^{i\lambda} \in \mathbb{T}$, and therefore $\lambda \in \mathbb{R}$. Thus, $\sigma(A) \subseteq \mathbb{R}$. ■

Theorem 3.1.9. *If τ is a character on a unital C*-algebra $\mathcal{A}$, then it preserves adjoints.*

Proof. If $A \in \mathcal{A}$, then $A = B + iC$, where B, C are hermitian elements of $\mathcal{A}$. The numbers $\tau(B)$ and $\tau(C)$ are real because they are in $\sigma(B)$ and $\sigma(C)$ respectively, so $\tau(A^*) = \tau(B - iC) = \tau(B) - i\tau(C) = (\tau(B) + i\tau(C))^- = \tau(A)^-$.

■

We shall now completely determine the abelian C*-algebras. This result can be thought of as a preliminary form of the spectral theorem. It allows us to construct the functional calculus, a very useful tool in the analysis of non-abelian C*-algebras.

Theorem 3.1.10. *(Gelfand) If $\mathcal{A}$ is a non-zero unital abelian C*-algebra, then the Gelfand representation*

$$\varphi : \mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A})), \quad A \mapsto \hat{A},$$

is an isometric $*$-isomorphism.

Proof. That φ is a norm-decreasing homomorphism, such that $\|\varphi(A)\| = r(A)$, is given by Theorem 2.3.6. If $\tau \in \Omega(\mathcal{A})$, then $\varphi(A^*)(\tau) = \tau(A^*) = \tau(A)^- = \varphi(A)^*(\tau)$, so φ is a $*$-homomorphism. Moreover, φ is isometric, since $\|\varphi(A)\|^2 = \|\varphi(A^*)\varphi(A)\| = \|\varphi(A^*A)\| = r(A^*A) = \|A^*A\| = \|A\|^2$. Clearly, then, $\varphi(\mathcal{A})$ is a closed $*$-subalgebra of $\mathcal{C}(\Omega(\mathcal{A}))$ separating the points of $\Omega(\mathcal{A})$, and having the property that $\varphi(1)(\tau) = 1$. The Stone-Weierstrass theorem implies, therefore, that $\varphi(\mathcal{A}) = \mathcal{C}(\Omega(\mathcal{A}))$. ∎

The following result is important.

Theorem 3.1.11. *Let $\mathcal{B}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{A}$ containing the unit of $\mathcal{A}$. Then*

$$\sigma_{\mathcal{B}}(B) = \sigma_{\mathcal{A}}(B) \qquad (B \in \mathcal{B}).$$

Proof. First suppose that B is a hermitian element of $\mathcal{B}$. Since in this case $\sigma_{\mathcal{A}}(B)$ is contained in $\mathbb{R}$, it has no holes, and therefore by Theorem 2.2.15, $\sigma_{\mathcal{A}}(B) = \sigma_{\mathcal{B}}(B)$. Therefore, B is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{A}$.

Now suppose that B is an arbitrary element of $\mathcal{B}$, that is invertible in $\mathcal{A}$, so there is an element $A \in \mathcal{A}$ such that $BA = AB = I$. Then $A^*B^* = B^*A^* = I$, so $BB^*A^*A = I \Rightarrow BB^*$ is invertible in $\mathcal{A}$ and therefore in $\mathcal{B}$. Hence, there is an element $C \in \mathcal{B}$ such that $BB^*C = I$. Consequently, $B^*C = A$, so $A \in \mathcal{B}$, which implies that B is invertible in $\mathcal{B}$. Thus, for any element of $\mathcal{B}$, its invertibility in $\mathcal{A}$ is equivalent to its invertibility in $\mathcal{B}$. The theorem follows. ∎

Let $\mathcal{S}$ be a subset of a C*-algebra $\mathcal{A}$. The C*-algebra *generated* by $\mathcal{S}$ is the smallest C*-subalgebra of $\mathcal{A}$ containing $\mathcal{S}$. If $\mathcal{S} = \{A\}$, we denote by $\mathcal{C}^*(A)$ the C*-subalgebra generated by $\mathcal{S}$. If A is a normal, then $\mathcal{C}^*(A)$ is abelian. Similarly, if $\mathcal{A}$ is unital and A normal, then the C*-subalgebra generated by I and A is abelian.

Observe that $r(A) = \|A\|$ if A is a normal element of a unital C*-algebra, this can be proved by applying Theorem 3.1.10 to $\mathcal{C}^*(A, I)$.

If $\mathcal{A}$ is a unital C*-algebra and $A \in \mathcal{A}_{sa}$, then e^{iA} is a unitary, but not all unitaries are of this form. Using Theorem 3.1.10, we can give some useful conditions that ensure a unitary *does* have a logarithm.

Theorem 3.1.12. *Let U be a unitary in a unital C*-algebra $\mathcal{A}$. If $\sigma(U) \neq \mathbb{T}$, then there exists $A \in \mathcal{A}_{sa}$ such that $U = e^{iA}$. If $\|I - U\| < 2$, then $\sigma(U) \neq \mathbb{T}$.*

Proof. By replacing U by λU for some $\lambda \in \mathbb{T}$ if necessary, we may suppose that $-1 \notin \sigma(U)$. Since U is normal, we may also suppose that $\mathcal{A}$ is abelian (replacing $\mathcal{A}$ by the C*-subalgebra generated by I and U if need be). Let $\varphi : \mathcal{A} \to \mathcal{C}(\Omega)$ be the Gelfand representation, let $f = \varphi(U)$, and as usual denote by $ln : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ the principal branch of the logarithm function. Then $g = ln \circ f$ is a well-defined element of $\mathcal{C}(\Omega)$, and $e^g = f$. Since $|f(\omega)| = 1$ for all $\omega \in \Omega$, the real part of g vanishes, so $g = ih$ where $h = \overline{h} \in \mathcal{C}(\Omega)$, Let $A = \varphi^{-1}(h)$. Then $A \in \mathcal{A}_{sa}$ and $U = e^{iA}$ because $\varphi(U) = e^{ih} = e^{\varphi(iA)} = \varphi(e^{iA})$.

The parenthetical observation in the statement of the theorem follows from the equations

$$\|I - U\| = r(I - U) = \sup\{|1 - \lambda| \mid \lambda \in \sigma(U)\},$$

which imply that $-1 \notin \sigma(U)$ when $\|I - U\| < 2$. ∎

We are now going to set up the functional calculus, for which we need to make two easy observations:

If $\theta : \Omega \to \Omega'$ is a continuous map between compact Hausdorff spaces Ω and Ω', then the *transpose* map

$$\theta^t : \mathcal{C}(\Omega') \to \mathcal{C}(\Omega), \quad f \mapsto f \circ \theta,$$

is a unital $*$-homomorphism. Moreover, if θ is a homeomorphism, then θ^t is a $*$-isomorphism.

Our second observation is that a $*$-isomorphism of unital C*-algebras is necessarily isometric. This is an immediate consequence of Theorem 3.1.7.

Theorem 3.1.13. *Let A be a normal element of a unital C*-algebra $\mathcal{A}$, and suppose that z is the inclusion map of $\sigma(A)$ in $\mathbb{C}$. Then there is a unique unital $*$-isomorphism $\varphi : \mathcal{C}(\sigma(A)) \to \mathcal{C}^*(A, I) \subseteq \mathcal{A}$ such that $\varphi(z) = A$.*

Proof. Benote by $\mathcal{B}$ the abelian C*-algebra generated by I and A, and let $\psi : \mathcal{B} \to \mathcal{C}(\Omega(\mathcal{B}))$ be the Gelfand representation. Then ψ is a $*$-isomorphism by Theorem 3.1.10, and so is $\hat{A}^t : \mathcal{C}(\sigma(A)) \to \mathcal{C}(\Omega(\mathcal{B}))$, since $\hat{A} : \Omega(\mathcal{B}) \to \sigma(A)$ is a homeomorphism. Let $\varphi : \mathcal{C}(\sigma(A)) \to \mathcal{B}$ be the composition $\psi^{-1} \cdot \hat{A}^t$, so φ is a $*$-isomorphism. Then $\varphi(z) = A$, since $\varphi(z) = \psi^{-1}(\hat{A}^t(z)) = \psi^{-1}(\hat{A}) = A$, and obviously φ is unital. From the Stone-Weierstrass theorem, we know that $\mathcal{C}(\sigma(A))$ is generated by 1 and z; φ is therefore the unique unital $*$-isomorphism from $\mathcal{C}(\sigma(A))$ to $\mathcal{C}^*(A, I)$ such that $\varphi(z) = A$. ∎

As in Theorem 3.1.13, let A be a normal element of a unital C*-algebra $\mathcal{A}$, and let z be the inclusion map of $\sigma(A)$ in $\mathbb{C}$. We call the unique unital $*$-homomorphism $\varphi : \mathcal{C}(\sigma(A)) \to \mathcal{C}^*(A, I) \subseteq \mathcal{A}$ such that $\varphi(z) = A$ the *functional* *calculus* at A. If p is a polynomial, then $\varphi(p) = p(A)$, so for $f \in \mathcal{C}(\sigma(A))$ we may write $f(A)$ for $\varphi(f)$. Note that $f(A)$ is normal.

Let $\mathcal{B}$ be the image of φ, so $\mathcal{B}$ is the C*-algebra generated by I and A. If $\tau \in \Omega(\mathcal{B})$, then $f(\tau(A)) = \tau(f(A))$, since the maps $f \mapsto f(\tau(A))$ and $f \mapsto \tau(f(A))$ from $\mathcal{C}(\sigma(A))$ to $\mathbb{C}$ are $*$-homomorphisms agreeing on the generators 1 and z and hence are equal.

Theorem 3.1.14. *Let A be a normal element of a unital C*-algebra $\mathcal{A}$, and let $f \in \mathcal{C}(\sigma(A))$. Then*

$$\sigma(f(A)) = f(\sigma(A)).$$

Moreover, if $g \in \mathcal{C}(\sigma(f(A)))$, then

$$(g \circ f)(A) = g(f(A)).$$

Proof. Let $\mathcal{B}$ be the C*-subalgebra generated by I and A. Then $\sigma(f(A)) = \{\tau(f(A)) \mid \tau \in \Omega(\mathcal{B})\} = \{f(\tau(A)) \mid \tau \in \Omega(\mathcal{B})\} = f(\sigma(A))$.

If $\mathcal{C}$ denotes the C*-subalgebra generated by I and $f(A)$, then $\mathcal{C} \subseteq \mathcal{B}$ and for any $\tau \in \Omega(\mathcal{B})$ its restriction $\tau_{\mathcal{C}}$ is a character on $\mathcal{C}$. We therefore have $\tau((g \circ f)(A)) = g(f(\tau(A))) = g(\tau(f(A))) = \tau_{\mathcal{C}}(g(f(A))) = \tau(g(f(A)))$. Hence, $(g \circ f)(A) = g(f(A))$. ∎

We close this section by showing that if Ω is a compact Hausdorff spaces, then the character space of $\mathcal{C}(\Omega)$ is Ω,

Theorem 3.1.15. *Let Ω be a compact Hausdorff space, and for each $\omega \in \Omega$ let δ_ω be the character on $\mathcal{C}(\Omega)$ given by evaluation at ω; that is, $\delta_\omega(f) = f(\omega)$.*

Then the map

$$\Omega \to \Omega(\mathcal{C}(\Omega)), \quad \omega \mapsto \delta_\omega,$$

is a homeomorphism.

Proof. This map is continuous because if $(\omega_\lambda)_{\lambda\in\Lambda}$ is a net in Ω converging to a point ω, then $\lim_{\lambda\in\Lambda} f(\omega_\lambda) = f(\omega)$ for all $f \in \mathcal{C}(\Omega)$, so the net $(\delta_{\omega_\lambda})$ is weak* convergent to δ_ω. The map is also injective, because if ω, ω' are distinct points of Ω, then by Urysohn's lemma there is a function $f \in \mathcal{C}(\Omega)$ such that $f(\omega) = 0$ and $f(\omega') = 1$, and therefore $\delta_\omega \neq \delta_{\omega'}$.

Now we show surjectivity of the map. Let $\tau \in \Omega(\mathcal{C}(\Omega))$. Then $M = \ker(\tau)$ is a proper C*-algebra of $\mathcal{C}(\Omega)$. Also, M separates the points of Ω, for if ω, ω' are distinct points of Ω, then as we have just seen there is a function $f \in \mathcal{C}(\Omega)$ such that $f(\omega) \neq f(\omega')$, so $g = f - \tau(f)$ is a function in M such that $g(\omega) \neq g(\omega')$. It follows from the Stone-Weierstrass theorem that there is a point $\omega \in \Omega$ such that $f(\omega) = 0$ for all $f \in M$. Hence, $(f - \tau(f))(\omega) = 0$, so $f(\omega) = \tau(f)$, for all $f \in \mathcal{C}(\Omega)$. Therefore, $\tau = \delta_\omega$. Thus, the map is a continuous bijection between compact Hausdorff spaces and therefore is a homeomorphism. ■

3.2 Positive Elements of C*-Algebras

In this section we introduce a partial order relation on the hermitian elements of a C*-algebra. The principal results are the existence of a unique positive square root for each positive element.

Remark 3.2.1. *Let $\mathcal{A} = \mathcal{C}(\Omega)$, where Ω is a compact Hausdorff space. Then $\mathcal{A}_{sa}$ is the set of real-valued functions in $\mathcal{A}$ and there is a natural partial order* *on $\mathcal{A}_{sa}$ given by $f \leq g$ if and only if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. An element $f \in \mathcal{A}$ is positive, that is, $f \geq 0$, if and only if f is of the form $f = \overline{g}g$ for some $g \in \mathcal{A}$, and in this case f has a unique positive square root in $\mathcal{A}$, namely the function $\omega \mapsto \sqrt{f(\omega)}$. Note that if $f = \overline{f}$ we can also express the positivity condition in terms of the norm: if $t \in \mathbb{R}^+$, then f is positive if $\|f - t\| \leq t$, and in the reverse direction if $\|f\| \leq t$ and $f \geq 0$, then $\|f - t\| \leq t$. We shall presently define a partial order on an arbitrary C*-algebra that generalises that of $\mathcal{C}(\Omega)$, and we shall obtain similar, and many other, results.*

An element A of a unital C*-algebra $\mathcal{A}$ is *positive* if A is hermitian and $\sigma(A) \subseteq \mathbb{R}^+$. We write $A \geq 0$ to mean that A is positive, and denote by $\mathcal{A}^+$ the set of positive elements of $\mathcal{A}$. For any C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$, $\mathcal{B}^+ = \mathcal{B} \cap \mathcal{A}^+$.

If $\mathcal{S}$ is a non-empty set, then an element $f \in \ell^\infty(\mathcal{S})$ is positive in the C*-algebra sense if and only if $f(x) \geq 0$ for all $x \in \mathcal{S}$, because $\sigma(f)$ is the closure of the range of f. Hence, if Ω is any compact Hausdorff space, then $f \in \mathcal{C}(\Omega)$ is positive if and only if $f(\omega) \geq 0$ for all $\omega \in \Omega$.

If A is a hermitian element of a unital C*-algebra $\mathcal{A}$, observe that the C*-algebra $\mathcal{C}^*(A, I)$ is the closure of the set of polynomials in A.

Theorem 3.2.2. *Let $\mathcal{A}$ be a unital C*-algebra and $A \in \mathcal{A}^+$. Then there exists a unique element $B \in \mathcal{A}^+$ such that $B^2 = A$.*

Proof. That there exists $B \in \mathcal{C}^*(A, I)$ such that $B \geq 0$ and $B^2 = A$ follows from the Gelfand representation, since we may use it to identify $\mathcal{C}^*(A, I)$ with $\mathcal{C}(\Omega)$, where Ω is the character space of $\mathcal{C}^*(A, I)$ and then apply Remark 3.2.1.

Suppose that C is another element of $\mathcal{A}^+$ such that $C^2 = A$. As C commutes with A it must commute with B, since B is the limit of a sequence of polynomials in $\mathcal{A}$. Let $\mathcal{B}$ be the C*-subalgebra of $\mathcal{A}$ generated by B , C and I, and let $\varphi : \mathcal{B} \to \mathcal{C}(\Omega_1)$ be the Gelfand representation of $\mathcal{B}$. Then $\varphi(B)$ and $\varphi(C)$ are positive square roots of $\varphi(A)$ in $\mathcal{C}(\Omega_1)$, so by another application of Remark 3.2.1, $\varphi(B) = \varphi(C)$, and therefore $B = C$. ∎

If $\mathcal{A}$ is a unital C*-algebra and A is a positive element, we denote by $A^{1/2}$ the unique positive element B such that $B^2 = A$.

If C is a hermitian element, then C^2 is positive, and we set $|C| = (C^2)^{1/2}$, $C^+ = \frac{1}{2}(|C| + C)$, and $C^- = \frac{1}{2}(|C| - C)$. Using the Gelfand representation of $\mathcal{C}^*(C, I)$, it is easy to check that $|C|$, C^+ and C^- are positive elements of $\mathcal{A}$ such that $C = C^+ - C^-$ and $C^+C^- = 0$.

Remark 3.2.3. *If A is a hermitian element of the closed unit ball of a unital C*-algebra $\mathcal{A}$, then $I - A^2 \in \mathcal{A}^+$ and the elements*

$$U = A + i\sqrt{I - A^2} \quad \textit{and} \quad V = A - i\sqrt{I - A^2}$$

are unitaries such that $A = \frac{1}{2}(U + V)$. Therefore, the unitaries linearly span $\mathcal{A}$, a result that is frequently useful.

Lemma 3.2.4. *Suppose that $\mathcal{A}$ is a unital C*-algebra, A is a hermitian element of $\mathcal{A}$ and $t \in \mathbb{R}$. Then, $A \geq 0$ if $\|A - tI\| \leq t$. In the reverse direction, if $\|A\| \leq t$ and $A \geq 0$, then $\|A - tI\| \leq t$.*

Proof. We may suppose that $\mathcal{A}$ is the abelian C*-subalgebra generated by I and A, so by the Gelfand representation $\mathcal{A} = \mathcal{C}(\sigma(A))$. The result now follows from Remark 3.2.1. ∎

It is immediate from Lemma 3.2.4 that $\mathcal{A}^+$ is closed in $\mathcal{A}$.

Lemma 3.2.5. *The sum of two positive elements in a unital C*-algebra is a positive element.*

Proof. Let $\mathcal{A}$ be a unital C*-algebra and A, B positive elements. To show that $A + B \geq 0$. By Lemma 3.2.4, $\|A - \|A\|\,\| \leq \|A\|$ and $\|B - \|B\|\,\| \leq \|B\|$, so $\|A + B - \|A\| - \|B\|\,\| \leq \|A - \|A\|\,\| + \|B - \|B\|\,\| \leq \|A\| + \|B\|$. By Lemma 3.2.4 again, $A + B \geq 0$. ∎

Theorem 3.2.6. *If A is an arbitrary element of a unital C*-algebra $\mathcal{A}$, then A^*A is positive.*

Proof. First we show that $A = 0$ if $-A^*A \in \mathcal{A}^+$. Since $\sigma(-AA^*) \setminus \{0\} = \sigma(-A^*A) \setminus \{0\}$ by Remark 2.2.4, $-AA^* \in \mathcal{A}^+$ because $-A^*A \in \mathcal{A}^+$. Write $A = B + iC$, where $B, C \in \mathcal{A}_{sa}$. Then $A^*A + AA^* = 2B^2 + 2C^2$, so $A^*A = 2B^2 + 2C^2 - AA^* \in \mathcal{A}^+$. Hence, $\sigma(A^*A) \in \mathbb{R}^+ \cap (-\mathbb{R}^+) = \{0\}$, and therefore $\|A\|^2 = \|A^*A\| = r(A^*A) = 0$.

Now suppose A is an arbitrary element of $\mathcal{A}$, and we shall show that A^*A is positive. If $B = A^*A$, then B is hermitian, and therefore we can write $B = B^+ - B^-$. If $C = AB^-$, then $-C^*C = -B^-A^*AB^- = -B^-(B^+ - B^-)B^- = (B^-)^3 \in \mathcal{A}^+$, so $C = 0$ by the first part of this proof. Hence, $B^- = 0$, so $A^*A = B^+ \in \mathcal{A}^+$. ∎

If $\mathcal{A}$ is a C*-algebra, we make $\mathcal{A}_{sa}$ a poset by defining $A \leq B$ to mean $B - A \in \mathcal{A}^+$. The relation $\leq$ is translation-invariant; that is , $A \leq B \Rightarrow A + C \leq B + C$ for all $A, B, C \in \mathcal{A}_{sa}$. Also, $A \leq B \Rightarrow tA \leq tB$ for all $t \in \mathbb{R}^+$, and $A \leq B \Leftrightarrow -A \geq -B$.

Using Theorem 3.2.6 we can extend our definition of $|A|$: for arbitrary A set $|A| = (A^*A)^{1/2}$.

We summarise some elementary facts about $\mathcal{A}^+$ in the following result.

Theorem 3.2.7. *let $\mathcal{A}$ be a unital C*-algebra.*
(1) The set $\mathcal{A}^+$ is equal to $\{A^*A \mid A \in \mathcal{A}\}$.
(2) If $A, B \in \mathcal{A}_{sa}$ and $C \in \mathcal{A}$, then $A \leq B \Rightarrow C^*AC \leq C^*BC$.
(3) If $0 \leq A \leq B$, then $\|A\| \leq \|B\|$.
(4) if A, B are positive invertible elements, then $A \leq B \Rightarrow 0 \leq B^{-1} \leq A^{-1}$.

Proof. Conditions (1) and (2) are implied by Theorem 3.2.6 and the existence of positive square roots for positive elements. To prove Condition (3) The inequality $B \leq \|B\|I$ is given by the Gelfand representation applied to the C*-algebra generated by I and B. Hence, $A \leq \|B\|I$. Applying the Gelfand representation again, this time to the C*-algebra generated by I and A, we obtain the inequality $\|A\| \leq \|B\|$.

To prove Condition (4) we first observe that if $C \geq I$, then C is invertible and $C^{-1} \leq I$. This is given by the Gelfand representation applied to the C*-subalgebra generated by I and C. Now $A \leq B \Rightarrow I = A^{-1/2}AA^{-1/2} \leq A^{-1/2}BA^{-1/2} \Rightarrow (A^{-1/2}BA^{-1/2})^{-1} \leq I$, that is, $A^{1/2}B^{-1}A^{1/2} \leq I$. Hence, $B^{-1} \leq (A^{1/2})^{-1}(A^{1/2})^{-1} = A^{-1}$. ■

Theorem 3.2.8. *If A, B are positive elements of a unital C*-algebra $\mathcal{A}$, then the inequality $A \leq B$ implies the inequality $A^{1/2} \leq B^{1/2}$.*

Proof. We show $A^2 \leq B^2 \Rightarrow A \leq B$ and this will prove the theorem. Let $t > 0$ and let C, D be the real and imaginary hermitian parts of the element

$(t+B+A)(t+B-A)$. Then

$$\begin{aligned}C &= \frac{1}{2}((t+B+A)(t+B-A)) + (t+B-A)(t+B+A) \\ &= t^2 + 2tB + B^2 - A^2 \\ &\geq t^2.\end{aligned}$$

Consequently, C is both invertible and positive. Since $I + iC^{-1/2}DC^{-1/2} = C^{-1/2}(C+iD)C^{-1/2}$ is invertible, therefore $C+iD$ is invertible. It follows that $t+B-A$ is left invertible, and therefore invertible, because it is hermitian. Consequently, $-t \notin \sigma(B-A)$. Hence, $\sigma(B-A) \subseteq \mathbb{R}^+$, so $B-A$ is positive, that is, $A \leq B$. ■

It is not true that $0 \leq A \leq B \Rightarrow A^2 \leq B^2$ in arbitrary C*-algebras. For example, take $\mathcal{A} = M_2(\mathbb{C})$. This is a C*-algebra where the involution is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix}.$$

Let P and Q be the projections

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $P \leq P+Q$, but $P^2 = P \not\leq (P+Q)^2 = P+Q+PQ+QP$, since the matrix

$$Q + PQ + QP = \frac{1}{2}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

has a negative eigenvalue.

It can be shown that the implication $0 \leq A \leq B \Rightarrow A^2 \leq B^2$ holds only in abelian unital C*-algebras.

3.3 Operators and Sesquilinear Forms

In this section we shall interpret and apply the first two sections of this chapter in the context of operators on Hilbert spaces. We shall also prove the invaluable polar decomposition theorem. An important concern in the present section is the correspondence of operators and sesquilinear forms. This is interesting in its own right, but it also has wide applicability—for example, we shall use it in the proof of the spectral theorem.

We begin by showing that operators on Hilbert spaces have adjoints.

Theorem 3.3.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.*
(1) If $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, then there is a unique element $U^* \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that

$$\langle U(x_1), x_2\rangle = \langle x_1, U^*(x_2)\rangle \qquad (x_1 \in \mathcal{H}_1,\ x_2 \in \mathcal{H}_2).$$

(2) The map $U \mapsto U^*$ is conjugate-linear and $U^{**} = U$, also

$$\|U\| = \|U^*\| = \|U^*U\|^{1/2}.$$

Proof. If $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $x_2 \in \mathcal{H}_2$, then the function

$$\mathcal{H}_1 \to \mathbb{C}, \quad x_1 \mapsto \langle U(x_1), x_2\rangle,$$

is continuous and linear, so by the Riesz representation theorem for linear functionals on Hilbert spaces there is a unique element $U^*(x_2) \in \mathcal{H}_1$ such that $\langle U(x_1), x_2\rangle = \langle x_1, U^*(x_2)\rangle$ $(x_1 \in \mathcal{H}_1)$. Moreover,

$$\|U^*(x_2)\| = \sup_{\|x_1\|\leq 1} |\langle U(x_1), x_2\rangle| \leq \|U\|\,\|x_2\|.$$

The map $U^* : \mathcal{H}_2 \to \mathcal{H}_1$, $x_2 \mapsto U^*(x_2)$, is linear and $\|U^*\| \leq \|U\|$. Thus, U^* satisfies Condition (1), uniqueness of U^* is obvious.

If $x_1 \in \mathcal{H}_1$ and $\|x_1\| \leq 1$, then $\langle U(x_1), U(x_1)\rangle = \langle x_1, U^*U(x_1)\rangle \leq \|U^*U\|$, so

$$\|U\|^2 = \sup_{\|x_1\|\leq 1} \|U(x_1)\|^2 \leq \|U^*U\| \leq \|U\|^2.$$

Hence, $\|U\| = \|U^*U\|^{1/2}$. The other assertions in Condition (2) of the theorem have routine verifications. ■

If $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a continuous linear map between Hilbert spaces, we call the map $U^* : \mathcal{H}_2 \to \mathcal{H}_1$ the *adjoint* of U. Note that $\ker(U^*) = (\mathrm{im}(U))^\perp$, where $\mathrm{im}(U)$ is the range of U, and hence, $(\mathrm{im}(U^*))^- = \ker(U)^\perp$.

If $\mathcal{H}_1 \xrightarrow{U} \mathcal{H}_2 \xrightarrow{V} \mathcal{H}_3$ are continuous linear maps between Hilbert spaces, then $(VU)^* = U^*V^*$.

If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a C*-algebra under the involution $U \mapsto U^*$, where U^* is the adjoint of U.

It follows in particular that $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ is a C*-algebra. Observe that the involution on $M_n(\mathbb{C})$ is given by $(\lambda_{ij})^*_{ij} = (\overline{\lambda}_{ji})_{ij}$.

If $\mathcal{H}$ is a vector space, a map $\sigma : \mathcal{H}^2 \to \mathbb{C}$ is a *sesquilinear form* if it is linear in the first variable and conjugate-linear in the second. For such a form the *polarisation identity*

$$\sigma(x, y) = \frac{1}{4}\sum_{k=0}^{3} i^k \sigma(x + i^k y, x + i^k y)$$

holds. Thus, sesquilinear forms σ and σ' on $\mathcal{H}^2$ are equal if and only if $\sigma(x, x) = \sigma'(x, x)$ for all $x \in \mathcal{H}$. Sesquilinear forms are taken up in more detail later in this section.

If $\mathcal{H}$ is a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$, then $(x, y) \mapsto \langle U(x), y \rangle$ is a s-esquilinear form on $\mathcal{H}^2$. Hence, if $U, V \in \mathcal{B}(\mathcal{H})$, then $U = V$ if and only if $\langle U(x), x \rangle = \langle V(x), x \rangle$ for all $x \in \mathcal{H}$.

If $U^*U = I$ and $UU^* = I$, we say U is a *unitary operator.* This is equivalent to U being isometric and surjective. Observe that U is isometric $\Leftrightarrow U^*U = I$.

Example 3.3.2. *Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and suppose that U is an operator diagonal with respect to (e_n), with diagonal sequence (λ_n). Then U^* is also diagonal with respect to (e_n) and its diagonal sequence is $(\overline{\lambda}_n)$. This follows from the observation that $\langle U^*(e_n), e_m \rangle = \langle e_n, U(e_m) \rangle = \langle e_n, \lambda_m e_m \rangle = \overline{\lambda}_m \delta_{nm}$, where δ_{nm} is the Kronecker delta symbol, which implies that $U^*(e_n) = \overline{\lambda}_n e_n$. Since all operators diagonal with respect to the same basis commute, $UU^* = U^*U$; that is, U is normal.*

The following result on projections will be used frequently and tacitly.

Theorem 3.3.3. *Let P, Q be projections on a Hilbert space $\mathcal{H}$. Then the following conditions are equivalent:*

(1) $P \leq Q$.

(2) $PQ = P$.

(3) $QP = P$.

(4) $P(\mathcal{H}) \subseteq Q(\mathcal{H})$.

(5) $\|P(x)\| \leq \|Q(x)\| \quad (x \in \mathcal{H})$.

(6) $Q - P$ is a projection.

Proof. Equivalence of Conditions $(2), (3)$, and (4) is clear, as are the implications $(2) \Rightarrow (6) \Rightarrow (1)$. We show $(1) \Rightarrow (5) \Rightarrow (2)$, and this will prove the theorem.

If we assume Condition (1) holds, $\|Q(x)\|^2 - \|P(x)\|^2 = \langle (Q-P)(x), x\rangle = \|(Q-P)^{1/2}(x)\| \geq 0$, so Condition (5) holds.

If now we assume Condition (5) holds, $\|P(I-Q)(x)\| \leq \|(Q-Q^2)(x)\| = 0$, and therefore $P = PQ$; that is, Condition (2) holds. ∎

An operator U on a Hilbert space $\mathcal{H}$ is normal if and only if $\|U(x)\| = \|U^*(x)\|$ $(x \in \mathcal{H})$, since $\langle (UU^* - U^*U)(x), x\rangle = \|U^*(x)\|^2 - \|U(x)\|^2$. Thus, $\ker(U) = \ker(U^*)$ if U is normal.

A continuous linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is a *partial isometry* if U is isometric on $\ker(U)^\perp$, that is , $\|U(x)\| = \|x\|$ for all $x \in \ker(U)^\perp$.

Theorem 3.3.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then the following conditions are equivalent:*
(1) $U = UU^*U$.
(2) U^*U is a projection.
(3) UU^* is a projection.
(4) U is a partial isometry.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To show the converse suppose that U^*U is a projection. Note that $\|U(I-U^*U)\|^2 = \|(I-U^*U)U^*U(I-U^*U)\|$ and U^*U is a projection, so $(I-U^*U)U^*U = 0$, therefore, $U(I-U^*U) = 0$, and thus $U = UU^*U$.

To show that (2) $\Rightarrow$ (3), suppose again that U^*U is a projection. Then $(UU^*)^3 = (UU^*)^2$, so $\sigma(UU^*) \subseteq \{0,1\}$. Hence, UU^* is a projection by the functional calculus. Thus, (2) $\Rightarrow$ (3), and clearly, then, (3) $\Rightarrow$ (2) by symmetry.

To show (1) $\Rightarrow$ (4), suppose that $U = UU^*U$. Then U^*U is the projection on to $\ker(U)^\perp$, since $U^* = U^*UU^*$, and $\ker(U)^\perp = (U^*(\mathcal{H}_2))^- = U^*U(\mathcal{H}_1)$. Hence, if $x \in \ker(U)^\perp$, then $\|U(x)\|^2 = \langle U^*U(x), x\rangle = \langle x, x\rangle = \|x\|^2$. Thus, U is a partial isometry, so (1) $\Rightarrow$ (4).

Finally, we show (4) $\Rightarrow$ (2) (and this will prove the theorem). Suppose that U is a partial isometry. If P is the projection of $\mathcal{H}_1$ on $\ker(U)^\perp$ and $x \in \ker(U)^\perp$, then $\langle U^*U(x), x\rangle = \|U(x)\|^2 = \langle x, x\rangle = \langle P(x), x\rangle$. If $x \in \ker(U)$, then $\langle U^*U(x), x\rangle = 0 = \langle P(x), x\rangle$. thus, $\langle U^*U(x), x\rangle = \langle P(x), x\rangle$ for all $x \in \mathcal{H}_1$. Hence, $U^*U = P$, so (4) $\Rightarrow$ (2). $\blacksquare$

Just as we can write a complex number as the product of a unitary (= number of modulus one) times a non-negative number, the following result asserts that we can write an operator as the product of a partial isometry times a positive operator.

Theorem 3.3.5. *(Polar Decomposition) Let V be a continuous linear operator on a Hilbert space $\mathcal{H}$. Then there is a unique partial isometry $U \in \mathcal{B}(\mathcal{H})$ such that*

$$V = U|V| \quad \text{and} \quad \ker(U) = \ker(V).$$

Moreover, $U^*V = |V|$.

Proof. If $x \in \mathcal{H}$, $\|\,|V|(x)\|^2 = \langle |V|(x), |V|(x)\rangle = \langle |V|^2(x), x\rangle = \langle V^*V(x), x\rangle = \langle V(x), V(x)\rangle = \|V(x)\|^2$. Hence, the map

$$U_0 : |V|(\mathcal{H}) \to \mathcal{H}, \quad |V|(x) \mapsto V(x),$$

is well-defined and isometric. It is also linear. Therefore, it has a unique linear isometric extension (also denoted U_0) to $(|V|(\mathcal{H}))^-$. Define U in $\mathcal{B}(\mathcal{H})$

by setting

$$U = \begin{cases} U_0, & \text{on } \overline{|V|(\mathcal{H})} \\ 0, & \text{on } \overline{|V|(\mathcal{H})}^{\perp}. \end{cases}$$

Then $U|V| = V$, and U is isometric on $\ker(U)^{\perp}$, because $\ker(U) = \overline{|V|(\mathcal{H})}^{\perp}$. Thus, U is a partial isometry and $\ker(U) = \ker(|V|)$. Now $\langle U^*V(x), |V|(y)\rangle = \langle V(x), V(y)\rangle = \langle V^*V(x), y\rangle = \langle |V|(x), |V|(y)\rangle \Rightarrow \langle U^*V(x), z\rangle = \langle |V|(x), z\rangle$ for all $z \in |V|(\mathcal{H})$, and therefore for all $z \in \mathcal{H}$. Thus, $U^*V = |V|$. It follows that $\ker(|V|) = \ker(V)$, so $\ker(U) = \ker(V)$.

Now suppose that $W \in \mathcal{B}(\mathcal{H})$ is another partial isometry such that $V = W|V|$ and $\ker(W) = \ker(V)$. Then W is equal to U on $\overline{|V|(\mathcal{H})}$ and on $\overline{|V|(\mathcal{H})}^{\perp} = \ker(V) = \ker(W) = \ker(U)$. Thus, $W = U$. ∎

Before we turn on the correspondence between sesquilinear forms and operators, we present a very brief survey of the basis definitions and facts pertaining to sesquilinear forms, since these are not always covered in books on general functional analysis.

The sesquilinear form τ on a vector space $\mathcal{H}^2$ is said to be *hermitian* if $\tau(y, x) = \tau(x, y)^-$ for all $x, y \in \mathcal{H}$. It follows from the polarisation identity that a sesquilinear form τ is hermitian if and only if $\tau(x, x) \in \mathbb{R}$ $(x \in \mathcal{H})$. A sesquilinear form τ is *positive* if $\tau(x, x) \geq 0$ for all $x \in \mathcal{H}$. Thus, positive sesquilinear forms are hermitian.

The inequality

$$|\tau(x, y)| \leq \sqrt{\tau(x, x)}\sqrt{\tau(y, y)} \qquad (x, y \in \mathcal{H}),$$

which holds for any positive sesquilinear form τ, is called the *Cauchy-Schwarz inequality.* It implies that the function $p : x \mapsto \sqrt{\tau(x, x)}$ is a semi-norm on $\mathcal{H}$; that is, p satisfies the axioms of a norm except that the implication $p(x) = 0 \Rightarrow x = 0$ may not hold.

A sesquilinear form τ on a normed vector space $\mathcal{H}^2$ is bounded if there is a positive number M such that

$$|\tau(x, y)| \leq M\|x\|\,\|y\| \qquad (x, y \in \mathcal{H}).$$

The *norm* $\|\tau\|$ of τ is the infimum of all such numbers M. Obviously, $|\tau(x, y)| \leq \|\tau\|\,\|x\|\,\|y\|$. A sesquilinear form is continuous if and only if it is bounded.

The proofs of these facts are elementary and are the same as for the corresponding results on inner products.

Theorem 3.3.6. *If U is a bounded linear operator on a Hilbert space $\mathcal{H}$, then the sesquilinear form*

$$\tau_U : \mathcal{H}^2 \to \mathbb{C}, \quad (x, y) \mapsto \langle U(x), y\rangle,$$

is hermitian if and only if U is hermitian, and positive if and only if U is positive.

Proof. We show only the implication, τ_U is positive $\Rightarrow U$ is positive, since the other assertions are easy exercises.

Suppose that τ_U is positive. Then it is hermitian and therefore U is hermitian. To see that $\sigma(U) \subseteq \mathbb{R}^+$, we show that $U - \lambda$ is invertible if $\lambda < 0$. In this case if $x \in \mathcal{H}$, then

$$\begin{aligned}
\|(U - \lambda)(x)\|^2 &= \langle (U - \lambda)(x), (U - \lambda)(x)\rangle \\
&= \|U(x)\|^2 + |\lambda|^2\|x\|^2 - 2\lambda\langle U(x), x\rangle \\
&\geq |\lambda|^2\|x\|^2.
\end{aligned}$$

Thus, $\|(U-\lambda)(x)\| \geq |\lambda|\,\|x\|$, so $U-\lambda$ is bounded below, Hence, $(U-\lambda)(\mathcal{H})$ is closed in $\mathcal{H}$ and $\ker(U-\lambda) = 0$. Therefore, $(U-\lambda)(\mathcal{H}) = \ker(U^* - \overline{\lambda})^\perp = \ker(U-\lambda)^\perp = 0^\perp = \mathcal{H}$. Hence, $U-\lambda$ is invertible. ∎

By the preceding theorem, if U is a bounded linear operator on a Hilbert space $\mathcal{H}$, then U is hermitian if and only if $\langle U(x), x\rangle \in \mathbb{R}$ $(x \in \mathcal{H})$, and U is positive if and only if $\langle U(x), x\rangle \geq 0$ $(x \in \mathcal{H})$.

Theorem 3.3.7. *let τ be a bounded sesquilinear form on a Hilbert space $\mathcal{H}$. Then there is a unique bounded linear operator U on $\mathcal{H}$ such that*

$$\tau(x,y) = \langle U(x), y\rangle \qquad (x, y \in \mathcal{H}).$$

Moreover, $\|U\| = \|\tau\|$.

Proof. Uniqueness of U is obvious.

For each $y \in \mathcal{H}$, the function $\mathcal{H} \to \mathbb{C}$, $x \mapsto \tau(x,y)$, is continuous and linear, so by the Riesz representation theorem there is a unique element $V(y) \in \mathcal{H}$ such that $\tau(x,y) = \langle\, x, V(y)\rangle$ $(x \in \mathcal{H})$. Also,

$$\|V(y)\| = \sup_{\|x\|\leq 1} |\tau(x,y)| \leq \|\tau\|\,\|y\|.$$

The map $V : \mathcal{H} \to \mathcal{H}$, $y \mapsto V(y)$, is linear and $\|V\| \leq \|\tau\|$. If $U = V^*$, then $\tau(x,y) = \langle U(x), y\rangle$ $(x, y \in \mathcal{H})$, and also the inequality $|\tau(x,y)| \leq \|U\|\,\|x\|\,\|y\|$ which holds for all x, y, implies that $\|\tau\| \leq \|U\|$. Hence, $\|\tau\| = \|U\|$. ∎

3.4 The Hilbert-Schmidt Operators

In this section we analyse a class of important operators, the Hilbert-Schmidt operators. We begin by looking at general compact operators on a Hilbert space and we strengthen some of the results of Section 1.4 in this case.

We shall need to view Hilbert spaces as dual spaces. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_* = \mathcal{H}$ as an additive group, but define a new scalar multiplication on $\mathcal{H}_*$ by setting $\lambda x = \overline{\lambda} x$, and a new inner product by setting $\langle x, y\rangle_* = \langle y, x\rangle$. Then $\mathcal{H}_*$ is a Hilbert space, and obviously the norm induced by the new inner product is the same as that induced by the old one. If $x \in \mathcal{H}$, define $V(x) \in (\mathcal{H}_*)^*$ by setting $V(x)(y) = \langle y, x\rangle_* = \langle x, y\rangle$. It is a direct consequence of the Riesz representation theorem that the map

$$V : \mathcal{H} \to (\mathcal{H}_*)^*, \quad x \mapsto V(x),$$

is an isometric linear isomorphism, which we use to identify these Banach spaces. The weak* topology on $\mathcal{H}$ is called the *weak* topology. A net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point x in $\mathcal{H}$ in the weak topology if and only if $\langle x, y\rangle = \lim_\lambda \langle x_\lambda, y\rangle$ $(y \in \mathcal{H})$. Consequently, the weak topology is weaker than the norm topology, and a bounded linear map between Hilbert spaces is necessarily weakly continuous. The importance to us of the weak topology is the fact that the closed unit ball of $\mathcal{H}$ is weakly compact by the Banach-Alaoglu theorem.

Theorem 3.4.1. *Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact linear map between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the image of the closed unit ball of $\mathcal{H}_1$ under U compact.*

Proof. Let $\mathcal{S}$ be the closed unit ball of $\mathcal{H}_1$. It is weakly compact, and U is weakly continuous, so $U(\mathcal{S})$ is weakly compact and therefore weakly closed. Hence, $U(\mathcal{S})$ is norm-closed, since the weak topology is weaker than the norm topology. Since U is a compact operator, this implies that $U(\mathcal{S})$ is norm-compact. ■

Theorem 3.4.2. *let U be a compact operator on a Hilbert space $\mathcal{H}$. Then both $|U|$ and U^* are compact.*

Proof. Suppose that U has polar decomposition $U = W|U|$ say. Then $|U| = W^*U$, so $|U|$ is compact, and $U^* = |U|W^*$, so U^* is compact. ■

Corollary 3.4.3. *If $\mathcal{H}$ is any Hilbert space, then $\mathcal{K}(\mathcal{H})$ is self-adjoint.*

Thus, $\mathcal{K}(\mathcal{H})$ is a C*-algebra, since $\mathcal{K}(\mathcal{H})$ is a closed ideal in $\mathcal{B}(\mathcal{H})$.

An operator U on a Hilbert space $\mathcal{H}$ is *diagonalisable* if $\mathcal{H}$ admits an orthonormal basis consisting of eigenvectors of U. Diagonalisable operators are necessarily normal, but not all normal operators are diagonalisable.

Theorem 3.4.4. *If U is a compact normal operator on a Hilbert space $\mathcal{H}$, then it is diagonalisable.*

Proof. By Zorn's lemma there is a maximal orthonormal set $\mathcal{E}$ of eigenvectors of U. If $\mathcal{K}$ is the closed linear span of $\mathcal{E}$, then $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, and $\mathcal{K}$ reduces U. The restriction $U_{\mathcal{K}^\perp} : \mathcal{K}^\perp \to \mathcal{K}^\perp$ is compact and normal. An eigenvector of $U_{\mathcal{K}^\perp}$ is one for U also, so by maximality of $\mathcal{E}$, the operator $U_{\mathcal{K}^\perp}$ has no eigenvectors, and therefore $\sigma(U_{\mathcal{K}^\perp}) = \{0\}$ by Theorem 2.4.15. Hence, $\|U_{\mathcal{K}^\perp}\| = r(U_{\mathcal{K}^\perp}) = 0$ by normality, so $\mathcal{K}^\perp = 0$. Thus, $\mathcal{K} = \mathcal{H}$ and $\mathcal{E}$ is an orthonormal basis of eigenvectors of U, so U is diagonalisable. ■

If $\mathcal{H}$ is a Hilbert space, we denote by $\mathcal{F}(\mathcal{H})$ the set of finite-rank operators on $\mathcal{H}$. It is easy to check that $\mathcal{F}(\mathcal{H})$ is a self-adjoint ideal of $\mathcal{B}(\mathcal{H})$.

Theorem 3.4.5. *If $\mathcal{H}$ is a Hilbert space, then $\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{K}(\mathcal{H})$.*

Proof. Since $\mathcal{F}(\mathcal{H})^-$ and $\mathcal{K}(\mathcal{H})$ are both self-adjoint, it suffices to show that if U is a hermitian element of $\mathcal{K}(\mathcal{H})$, then $U \in \mathcal{F}(\mathcal{H})^-$. Let $\mathcal{E}$ be an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of U, and let $\varepsilon > 0$. By Theorem 2.4.15 the set $\mathcal{S}$ of eigenvalues λ of U such that $|\lambda| \geq \varepsilon$ is finite. From Theorem 2.4.8 it is therefore clear that the set $\mathcal{S}'$ of elements of $\mathcal{E}$ corresponding to elements of $\mathcal{S}$ is finite. Now define a finite-rank diagonal operator V on $\mathcal{H}$ by settings $V(x) = \lambda x$ if $x \in \mathcal{S}'$ and λ is the eigenvalue corresponding to x, and setting $V(x) = 0$ if $x \in \mathcal{E} \setminus \mathcal{S}'$. It is easily checked that $\|V - U\| \leq \sup_{\lambda \in \sigma(U)\setminus\mathcal{S}} |\lambda| \leq \varepsilon$. This shows that $U \in \mathcal{F}(\mathcal{H})^-$. ∎

If x, y are elements of a Hilbert space $\mathcal{H}$ we define the operator $x \otimes y$ on $\mathcal{H}$ by

$$(x \otimes y)(z) = \langle z, y\rangle x.$$

Clearly, $\|x \otimes y\| = \|x\|\,\|y\|$. The rank of $x \otimes y$ is one if x and y are non-zero. If $x, x', y, y' \in \mathcal{H}$ and $U \in \mathcal{B}(\mathcal{H})$, then the following equalities are readily verified:

$$(x \otimes x')(y \otimes y') = \langle y, x'\rangle(x \otimes y')$$
$$(x \otimes y)^* = y \otimes x$$
$$U(x \otimes y) = U(x) \otimes y$$
$$(x \otimes y)U = x \otimes U^*(y).$$

The operator $x \otimes x$ is a rank-one projection if and only if $\langle x, x\rangle = 1$, that is, x is a unit vector. Conversely, every rank-one projection is of the form $x \otimes x$ for some unit vector x. Indeed, if $e_1, \ldots, e_n$ is an orthonormal set in $\mathcal{H}$, then the operator $\sum_{j=1}^n e_j \otimes e_j$ is the orthogonal projection of $\mathcal{H}$ onto the vector subspace $\mathbb{C}e_1 + \ldots + \mathbb{C}e_n$.

If $U \in \mathcal{B}(\mathcal{H})$ is a rank-one operator and x a non-zero element of its range, then $U = x \otimes y$ for some $y \in \mathcal{H}$. For if $z \in \mathcal{H}$, the $U(z) = \tau(z)x$ for some scalar $\tau(z) \in \mathbb{C}$. It is readily verified that the map $z \mapsto \tau(z)$ is a bounded linear functional on $\mathcal{H}$, and therefore, by the Riesz representation theorem, there exists $y \in \mathcal{H}$ such that $\tau(z) = \langle z, y\rangle$ for all $z \in \mathcal{H}$. Therefore, $U = x \otimes y$.

Theorem 3.4.6. *If $\mathcal{H}$ is a Hilbert space, then $\mathcal{F}(\mathcal{H})$ is linearly spanned by the rank-one projections.*

Proof. Let $U \in \mathcal{F}(\mathcal{H})$ and we shall show it is a linear combination of rank-one projections. The real and imaginary parts of U are in $\mathcal{F}(\mathcal{H})$, since $\mathcal{F}(\mathcal{H})$ is self-adjoint, so we may suppose that U is hermitian. Now $U = U^+ - U^-$, and by the polar decomposition $|U| \in \mathcal{F}(\mathcal{H})$, so U^+ and U^- belong to $\mathcal{F}(\mathcal{H})$. Hence, we may assume that $U \geq 0$. The range $U(\mathcal{H})$ is finite-dimensional, and therefore it is a Hilbert space with an orthonormal basis, $e_1, \ldots, e_n$ say. Let $P = \sum_{j=1}^n e_j \otimes e_j$, so P is projection of $\mathcal{H}$ onto $U(\mathcal{H})$. Then $U = PU = U^{1/2}PU^{1/2} \Rightarrow U = \sum_{j=1}^n x_j \otimes x_j$, where $x_j = U^{1/2}(e_j)$. Now $x_j = \lambda_j f_j$ for some unit vector f_j and scalar λ_j, so $U = \sum_{j=1}^n |\lambda_j|^2 f_j \otimes f_j$, and since the operators $f_j \otimes f_j$ are rank-one projections we are done. ∎

Theorem 3.4.7. *If $\mathcal{H}$ is a Hilbert space and $\mathcal{I}$ a non-zero ideal in $\mathcal{B}(\mathcal{H})$, then $\mathcal{I}$ contains $\mathcal{F}(\mathcal{H})$.*

Proof. Let U be a non-zero operator in $\mathcal{I}$. Then for some $x \in \mathcal{H}$ we have $U(x) \neq 0$. If P is a rank-one projection, then $P = y \otimes y$ for some unit vector $y \in \mathcal{H}$, and clearly there exists $V \in \mathcal{B}(\mathcal{H})$ such that $VU(x) = y$, in fact, take

$V = (y \otimes U(x))/\|U(x)\|^2$, for instance. Hence, $P = VU(x \otimes x)U^*V^*$, so $P \in \mathcal{I}$ as $U \in \mathcal{I}$. Thus, $\mathcal{I}$ contains all the rank-one projections and therefore by Theorem 3.4.6 it contains $\mathcal{F}(\mathcal{H})$. ■

Before we introduce the Hilbert Schmidt operators, it is convenient to make a few observations about summable families. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a family of elements of a Banach space $\mathcal{X}$. Let Λ' denote the set of all non-empty finite subsets of Λ, and for each $\mathcal{F} \in \Lambda'$, set $x_{\mathcal{F}} = \sum_{\lambda \in \mathcal{F}} x_\lambda$. Then $(x_{\mathcal{F}})_{\mathcal{F} \in \Lambda'}$ is a net where $\mathcal{F} \leq \mathcal{G}$ in Λ' if $\mathcal{F} \subseteq \mathcal{G}$. We say $(x_\lambda)_{\lambda \in \Lambda}$ is *summable* to an element $x \in \mathcal{X}$ if the net $(x_{\mathcal{F}})_{\mathcal{F} \in \Lambda'}$ converges to x, and in this case we write $x = \sum_{\lambda \in \Lambda} x_\lambda$.

If all x_λ are in $\mathbb{R}^+$, then the family $(x_\lambda)_{\lambda \in \Lambda}$ is summable if and only if $\sup_{\mathcal{F}} \sum_{\lambda \in \mathcal{F}} x_\lambda < \infty$, and in this case

$$\sum_{\lambda \in \Lambda} x_\lambda = \sup_{\mathcal{F} \in \Lambda'} \sum_{\lambda \in \mathcal{F}} x_\lambda.$$

We thus can use the right-hand side of this expression to define $\sum_{\lambda \in \Lambda} x_\lambda$ whether $(x_\lambda)_{\lambda \in \Lambda}$ is summable or not, provided all x_λ are in $\mathbb{R}^+$.

Let U be an operator on a Hilbert space $\mathcal{H}$, and suppose that $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$. We define the *Hilbert-Schmidt norm* of U to be

$$\|U\|_2 = (\sum_{x \in \mathcal{E}} \|U(x)\|^2)^{1/2}.$$

This definition is independent of the choice of basis. To see this let $\mathcal{E}'$ be another orthonormal basis for $\mathcal{H}$. Then for each finite non-empty set $\mathcal{F}$ of $\mathcal{E}$,

$$\begin{aligned}\sum_{x\in\mathcal{F}} \|U(x)\|^2 &= \sum_{x\in\mathcal{F}}\sum_{y\in\mathcal{E}'} |\langle U(x), y\rangle|^2 \\ &= \sum_{y\in\mathcal{E}'}\sum_{x\in\mathcal{F}} |\langle U(x), y\rangle|^2 \\ &\leq \sum_{y\in\mathcal{E}'} \|U^*(y)\|^2,\end{aligned}$$

so

$$\sum_{x\in\mathcal{E}} \|U(x)\|^2 \leq \sum_{y\in\mathcal{E}'} \|U^*(y)\|^2.$$

By symmetry, therefore,

$$\sum_{x\in\mathcal{E}} \|U(x)\|^2 = \sum_{x\in\mathcal{E}} \|U^*(x)\|^2 = \sum_{y\in\mathcal{E}'} \|U(y)\|^2.$$

This shows not only that the expression for $\|U\|_2$ is independent of the choice of basis, but also that $\|U^*\|_2 = \|U\|_2$.

An operator U is a Hilbert-Schmidt operator if $\|U\|_2 < \infty$. We denote the class of an Hilbert-Schmidt operators on $\mathcal{H}$ by $\mathcal{L}^2(\mathcal{H})$.

Example 3.4.8. *Let $(e_n)_{n=1}^\infty$ be an orthonormal basis for a Hilbert space $\mathcal{H}$ and let U be an operator on $\mathcal{H}$ diagonal with respect to (e_n), with diagonal sequence (λ_n). Then U is a Hilbert-Schmidt operator if and only if $\sum_{n=1}^\infty |\lambda_n|^2 < \infty$, since $\|U\|_2 = \sqrt{\sum_{n=1}^\infty |\lambda_n|^2}$.*

More generally, if U is an arbitrary operator in $\mathcal{B}(\mathcal{H})$ and $(\alpha_{n,m})$ is its matrix with respect to the basis (e_n), so that $\alpha_{n,m} = \langle U(e_m), e_n\rangle$, then from the definition

$$\|U\|_2 = \sqrt{\sum_{m=1}^\infty\sum_{n=1}^\infty |\alpha_{n,m}|^2},$$

and, therefore, U is Hilbert-Schmidt if and only if $\sum_m \sum_n |\alpha_{n,m}|^2 < \infty$.

Theorem 3.4.9. *Let U, V be operators on Hilbert space $\mathcal{H}$, and $\lambda \in \mathbb{C}$. Then*
(1) $\|U+V\|_2 \leq \|U\|_2 + \|V\|_2$ and $\|\lambda U\|_2 = |\lambda|\,\|U\|_2$;
(2) $\|U\| \leq \|U\|_2 = \|U^*\|_2$;
(3) $\|UV\|_2 \leq \|U\|\,\|V\|_2$ and $\|UV\|_2 \leq \|U\|_2\|V\|$.

Proof. If $\mathcal{F}$ is any finite set of orthonormal vectors of $\mathcal{H}$, then

$$\sqrt{\sum_{x\in\mathcal{F}} \|U(x)+V(x)\|^2} \leq \sqrt{\sum_{x\in\mathcal{F}} (\|U(x)\| + \|V(x)\|)^2}$$
$$\leq \sqrt{\sum_{x\in\mathcal{F}} \|U(x)\|^2} + \sqrt{\sum_{x\in\mathcal{F}} \|V(x)\|^2}.$$

It follows that $\|U+V\|_2 \leq \|U\|_2 + \|V\|_2$. The equality $\|\lambda U\|_2 = |\lambda|\,\|U\|_2$ is trivial.

If x is a unit vector of $\mathcal{H}$, there is an orthonormal basis $\mathcal{E}$ containing x. Hence, $\|U(x)\|^2 \leq \sum_{y\in\mathcal{E}} \|U(y)\|^2 = \|U\|_2^2$, so $\|U\| \leq \|U\|_2$.

If $\mathcal{E}$ is an arbitrary orthonormal basis of $\mathcal{H}$, then

$$\|UV\|_2^2 = \sum_{x\in\mathcal{E}} \|UV(x)\|^2 \leq \|U\|^2 \sum_{x\in\mathcal{E}} \|V(x)\|^2 = \|U\|^2\|V\|_2^2.$$

Hence, $\|UV\|_2 \leq \|U\|\,\|V\|_2$. Therefore, $\|UV\|_2 = \|V^*U^*\|_2 \leq \|V^*\|\,\|U^*\|_2 = \|U\|_2\|V\|$. ∎

Corollary 3.4.10. *The set $\mathcal{L}^2(\mathcal{H})$ is a self-adjoint ideal of $\mathcal{B}(\mathcal{H})$, and a normed $*$-algebra, that is, a normed algebra with an isometric involution, where the norm is given by $U \mapsto \|U\|_2$.*

Note that if $x, y \in \mathcal{H}$, then $\|x \otimes y\|_2 = \|x\|\|y\|$, so $x \otimes y \in \mathcal{L}^2(\mathcal{H})$. Hence, $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{L}^2(\mathcal{H})$.

3.5 The Trace-Class Operators

The trace-class operators have important applications in mathematical physics, in particular, in quantum physics.

Lemma 3.5.1. *Let U_1, U_2 be Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$. If $\mathcal{E}$ is an orthonormal basis of $\mathcal{H}$ and $V = U_1^* U_2$, then the family $(\langle V(x), x\rangle)_{x\in\mathcal{E}}$ is absolutely summable, that is, $\sum_{x\in\mathcal{E}} |\langle V(x), x\rangle| < +\infty$, and*

$$\sum_{x\in\mathcal{E}} \langle V(x), x\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|U_2 + i^k U_1\|_2^2$$

Proof. If $\mathcal{F}$ is a finite non-empty subset of $\mathcal{E}$, then

$$\begin{aligned}\sum_{x\in\mathcal{F}} |\langle V(x), x\rangle| &= \sum_{x\in\mathcal{F}} |\langle U_2(x), U_1(x)\rangle| \\ &\leq \sum_{x\in\mathcal{F}} \|U_2(x)\|\|U_1(x)\| \\ &\leq \sqrt{\sum_{x\in\mathcal{F}} \|U_2(x)\|^2}\sqrt{\sum_{x\in\mathcal{F}} \|U_1(x)\|^2}\end{aligned}$$

Hence, $(\langle V(x), x\rangle)_{x\in\mathcal{E}}$ is absolutely summable. Also,

$$\langle V(x), x\rangle = \langle U_2(x), U_1(x)\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|U_2(x) + i^k U_1(x)\|^2,$$

by the polarisation identity, so

$$\sum_{x\in\mathcal{E}} \langle V(x), x\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \sum_{x\in\mathcal{E}} \|(U_2 + i^k U_1)(x)\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \|U_2 + i^k U_1\|_2^2,$$

which is the required result. ∎

If U is an operator on a Hilbert space $\mathcal{H}$, we define its trace-class norm to be $\|U\|_1 = \||U|^{1/2}\|_2^2$. If $\mathcal{E}$ is an orthonormal basis of $\mathcal{H}$, then

$$\|U\|_1 = \sum_{x \in \mathcal{E}} \langle |U|(x), x \rangle.$$

If $\|U\|_1 < +\infty$, we call U a trace-class operator. The connection between trace-class operators and Hilbert-Schmidt operators is given in the following result.

Theorem 3.5.2. *Let V be an operator on a Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

(1) V is trace-class.

(2) $|V|$ is trace-class.

(3) $|V|^{1/2}$ is a Hilbert-Schmidt operator.

(4) There exist Hilbert-Schmidt operators U_1, U_2 on $\mathcal{H}$ such that $V = U_1 U_2$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are easy (for (3) $\Rightarrow$ (4) use the polar decomposition of V), so we prove (4) $\Rightarrow$ (1) only.

Assume that $V = U_1 U_2$, where $U_1, U_2 \in \mathcal{L}^2(\mathcal{H})$. If $V = W|V|$ is the polar decomposition of V, then $|V| = W^* V = (W^* U_1) U_2$. If $\mathcal{E}$ is any orthonormal basis of $\mathcal{H}$, then by the polarisation identity of the preceding lemma, $\sum_{x \in \mathcal{E}} \langle |V|(x), x \rangle < +\infty$, so $\|V\|_1 < +\infty$. ∎

It is clear from Theorem 3.5.2 that if V is a trace-class operator and U is an arbitrary operator on $\mathcal{H}$, then UV and VU are also trace-class operators. We define the trace of a trace-class operator V to be

$$tr(V) = \sum_{x \in \mathcal{E}} \langle V(x), x \rangle,$$

where $\mathcal{E}$ is any orthonormal basis of $\mathcal{H}$. By Lemma 3.5.1 the definition of tr is independent of the choice of orthonormal basis.

Theorem 3.5.3. *Let U and V be operators on a Hilbert space $\mathcal{H}$. Then*

$$tr(UV) = tr(VU)$$

if either
(1) U and V are both Hilbert-Schmidt operators,
or
(2) V is trace-class.

Proof. In Case (1),

$$\begin{aligned} tr(UV) &= \frac{1}{4}\sum_{k=0}^{3} i^k \|V + i^k U^*\|_2^2 \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \|(V + i^k U^*)^*\|_2^2 \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \|U + i^k V^*\|_2^2 \\ &= tr(VU). \end{aligned}$$

In Case (2), $V = U_1U_2$ for some $U_1, U_2 \in \mathcal{L}^2(\mathcal{H})$, so $tr(UV) = tr((UU_1)U_2) = tr(U_2(UU_1)) = tr(U_1(U_2U)) = tr(VU)$ by Case (1). ∎

There are similar results for the trace-class norm as for the Hilbert-Schmidt norm, but the proofs require more work:

Theorem 3.5.4. *Let U, V be operators on a Hilbert space $\mathcal{H}$ and $\lambda \in \mathbb{C}$.*
(1) $\|U + V\|_1 \leq \|U\|_1 + \|V\|_1$ and $\|\lambda U\|_1 = |\lambda|\|U\|_1$.

(2) $\|U\| \leq \|U\|_1 = \|U^*\|_1$.

(3) $\|UV\|_1 \leq \|U\|\|V\|_1$ *and* $\|UV\|_1 \leq \|U\|_1\|V\|$

Proof. Beginning with Condition (2) we have $\|U\|_1 = \||U|^{1/2}\|_2^2 \geq \||U|^{1/2}\|^2 = \||U|\| = \|U\|$. If $U = W|U|$ is the polar decomposition of U, then $UU^* = W|U|^2W^*$, so $|U^*|^2 = (W|U|W^*)^2$, and therefore $|U^*| = W|U|W^*$. Hence, $\|U^*\|_1 = tr(|U^*|) = tr(W|U|W^*) = tr(W^*U) = tr(|U|) = \|U\|_1$. This proves Condition (2).

Next we show that Condition (3) holds. Let $VU = W'|VU|$ be the polar decomposition of VU and $W'' = W'^*VW$.Then $|VU| = W'^*VU = W''|U|$. Hence, $|VU|^2 = |U|W''^*W''|U| \leq |U|^2\|W''\|^2 \leq |U|^2\|V\|^2$, so $|VU| \leq |U|\|V\|$ by Theorem 3.2.8. Consequently, if $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$,

$$\begin{aligned}
\|VU\|_1 &= \sum_{x\in\mathcal{E}} \langle |VU|(x), x\rangle \\
&\leq \sum_{x\in\mathcal{E}} \langle |U|(x), x\rangle \|V\| \\
&= \|U\|_1\|V\|.
\end{aligned}$$

Also, $\|UV\|_1 = \|V^*U^*\|_1 \leq \|V\|\|U\|_1$. This proves Condition (3).

Finally, we show Condition (1). The equality $\|\lambda U\|_1 = |\lambda|\|U\|_1$ follows from the corresponding statement for the norm $\|.\|_2$. Suppose that U and V are trace-class operators, and let $U = W|U|, V = W'|V|$, and $U + V = W''|U + V|$ be the respective polar decompositions. Then

$$|U + V| = W''^*(U + V) = W''^*W|U| + W''^*W'|V|.$$

If $\mathcal{E}$ is an orthonormal basis of $\mathcal{H}$,

$$\begin{aligned}
\|U+V\|_1 &= \sum_{x\in\mathcal{E}}\langle |U+V|(x),x\rangle \\
&= |\sum_{x\in\mathcal{E}}\langle W''^*W|U|(x),x\rangle + \sum_{x\in\mathcal{E}}\langle W''^*W'|V|(x),x\rangle| \\
&\leq \sum_{x\in\mathcal{E}}|\langle |U|^{1/2}(x),|U|^{1/2}W^*W''(x)\rangle| + \sum_{x\in\mathcal{E}}|\langle |V|^{1/2}(x),|V|^{1/2}W'^*W''(x)\rangle| \\
&\leq (\sum_{x\in\mathcal{E}}\||U|^{1/2}(x)\|^2)^{1/2}(\sum_{x\in\mathcal{E}}\||U|^{1/2}W^*W''(x)\|^2)^{1/2} \\
&+ (\sum_{x\in\mathcal{E}}\||V|^{1/2}(x)\|^2)^{1/2}(\sum_{x\in\mathcal{E}}\||V|^{1/2}W'^*W''(x)\|^2)^{1/2} \\
&= \|U\|_1^{1/2}\||U|^{1/2}W^*W''\|_2 + \|V\|_1^{1/2}\||V|^{1/2}W'^*W''\|_2 \\
&\leq \|U\|_1^{1/2}\|U\|_1^{1/2} + \|V\|_1^{1/2}\|V\|_1^{1/2} \\
&= \|U\|_1 + \|V\|_1,
\end{aligned}$$

so $\|U+V\|_1 \leq \|U\|_1 + \|V\|_1$. ∎

If $\mathcal{H}$ is a Hilbert space, we denote the set of trace-class operators on $\mathcal{H}$ by $\mathcal{L}^1(\mathcal{H})$. From the preceding theorem it is clear that $\mathcal{L}^1(\mathcal{H})$ is a self-adjoint ideal of $\mathcal{B}(\mathcal{H})$, and the function $U \longmapsto \|U\|_1$ is a norm on $\mathcal{L}^1(\mathcal{H})$ making it a normed *-algebra.

Theorem 3.5.5. *Let $\mathcal{H}$ be a Hilbert space. The function*

$$tr : \mathcal{L}^1(\mathcal{H}) \longrightarrow \mathbb{C}, U \longmapsto tr(U),$$

is linear, and

$$|tr(VU)| \leq \|V\|\|U\|_1, \quad V \in \mathcal{B}(\mathcal{H}), U \in \mathcal{L}^1(\mathcal{H}).$$

Proof. Linearity of the trace is clear. To show inequality let $U = W|U|$ be the polar decomposition of U and let $\mathcal{E}$ be an orthonormal basis of $\mathcal{H}$. Then

$$\begin{aligned}
|tr(VU)| &= |\sum_{x\in\mathcal{E}}\langle VU(x), x\rangle| \\
&= |\sum_{x\in\mathcal{E}}\langle |U|^{1/2}(x), |U|^{1/2}W^*V^*(x)\rangle| \\
&\leq \sum_{x\in\mathcal{E}}\||U|^{1/2}(x)\|\||U|^{1/2}W^*V^*(x)\| \\
&\leq (\sum_{x\in\mathcal{E}}\||U|^{1/2}(x)\|^2)^{1/2}(\sum_{x\in\mathcal{E}}\||U|^{1/2}W^*V^*(x)\|^2)^{1/2} \\
&= \|U\|_1^{1/2}\||U|^{1/2}W^*V^*\|_2 \\
&\leq \|U\|_1^{1/2}\||U|^{1/2}\|_2\|V\| \\
&= \|U\|_1\|V\|,
\end{aligned}$$

so $|tr(VU)| \leq \|U\|_1\|V\|$. ■

If $x, y \in \mathcal{H}$, then $\|x\otimes y\|_1 = \|x\|\|y\|$, and $tr(x\otimes y) = \langle x, y\rangle$. The inclusions $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^2(\mathcal{H})$ hold.

3.6 Gelfand-Naimark-Segal Construction

In this section, we will study the representations of C^*-algebras, that is, the famous Gelfand-Naimark-Segal construction.

Definition 3.6.1. *A representation of a C^*-algebra $\mathscr{A}$ is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathscr{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism. If $\mathscr{A}$ has an identity I, it is assumed that $\pi(I) = I$. Often $\mathcal{H}$ is deleted and we say that π is a representation.*

Example 3.6.2. *If $(X, \mathcal{F}, \mu)$ is a σ-finite measure space and $\mathcal{H} = L^2(\mu)$, then $\pi : L^\infty(\mu) \to \mathcal{B}(\mathcal{H})$ defined by $\pi(\phi) = M_\phi$ is a representation.*

Example 3.6.3. *If X is a compact space and μ is a positive regular Borel measure on X, then $\pi : \mathcal{C}(X) \to \mathcal{B}(L^2(\mu))$ defined by $\pi(f) = M_f$ is a representation.*

Definition 3.6.4. *A representation π of a C^*-algebra $\mathscr{A}$ is said to be cyclic if there is a vector e in $\mathcal{H}$ such that $\overline{\pi(\mathscr{A})e} = \mathcal{H}$; e is said to be a cyclic vector for the representation π.*

Note that the representations in the above two examples are cyclic. Also, the identity representation $i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is cyclic and every nonzero vector is a cyclic vector for this representation.

There is another way to obtain representations.

Definition 3.6.5. *If $\{(\pi_i, \mathcal{H}_i) : i \in I\}$ is a family of representations of $\mathscr{A}$, then the direct sum of this family is the representation $(\pi, \mathcal{H})$, where $\mathcal{H} = \oplus_i \mathcal{H}_i$ and $\pi(a) = \{\pi_i(a)\}$ for every a in $\mathscr{A}$.*

In fact, since $\|\pi_i(a)\| \leq \|a\|$ for every i, $\pi(a)$ is a bounded operator on $\mathcal{H}$. It is easy to check that π is a representation.

Definition 3.6.6. *Two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of a C^*-algebra $\mathscr{A}$ is said to be equivalent if there is an unitary transform $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(a)U^{-1} = \pi_2(a)$ for every a in $\mathscr{A}$.*

Theorem 3.6.7. *If π is a representation of the C^*-algebra $\mathscr{A}$ with identity I, then there is a family of cyclic representations $\{\pi_i\}$ of $\mathscr{A}$ such that π and $\oplus_i \pi_i$ are equivalent.*

Proof. Let $\mathcal{E}$ be the collection of all subsets E of nonzero vectors in $\mathcal{H}$ such that $\pi(\mathscr{A})e \perp \pi(\mathscr{A})f$ for e, f in E with $e \neq f$. Order $\mathcal{E}$ by inclusion. An application of Zorn Lemma implies that $\mathcal{E}$ has a maximal element E_0. Let $\mathcal{H}_0 = \overline{span\{\pi(\mathscr{A})e : e \in E_0\}}$. If $h \in \mathcal{H}_0^{\perp}$, then $0 = \langle \pi(a)e, h\rangle$ for every a in $\mathscr{A}$ and e in E_0. So if $a, b \in \mathscr{A}$ and $e \in E_0$, $0 = \langle \pi(b^*a)e, h\rangle = \langle \pi(b)^*\pi(a)e, h\rangle = \langle \pi(a)e, \pi(b)h\rangle$. That is, $\pi(\mathscr{A})e \perp \pi(\mathscr{A})h$ for all e in E_0. Since $I \in \mathscr{A}$, so for any $e \in E_0$, $h \neq e$. Hence $E_0 \cup \{h\} \in \mathcal{E}$; by the maximality of E_0 it must be that $h = 0$. Therefore $\mathcal{H} = \mathcal{H}_0$.

For e in E_0 let $\mathcal{H}_e = \overline{\pi(\mathscr{A})e}$. If $a \in \mathscr{A}$, clearly $\pi(a)\mathcal{H}_e \subseteq \mathcal{H}_e$. Since $a^* \in \mathscr{A}$ and $\pi(a)^* = \pi(a^*)$, $\mathcal{H}_e$ reduces $\pi(a)$. So if $\pi_e : \mathscr{A} \to \mathcal{B}(\mathcal{H}_e)$ is defined by $\pi_e(a) = \pi(a)|\mathcal{H}_e$, π_e is a representation of $\mathscr{A}$. Clearly $\pi = \oplus\{\pi_e : e \in E_0\}$. ■

In light of the preceding theorem, it becomes important to understand cyclic representations. To do this, let $\pi : \mathscr{A}$ be a cyclic representation with cyclic vector e. Define $f : \mathscr{A} \to \mathbb{C}$ by $f(a) = \langle \pi(a)e, e\rangle$. Note that f is a bounded linear functional on $\mathscr{A}$ with $\|f\| \leq \|e\|^2$. Since $f(I) = \|e\|^2$, $\|f\| = \|e\|^2$. Moreover, $f(a^*a) = \langle \pi(a^*a)e, e\rangle = \langle \pi(a)^*\pi(a)e, e\rangle = \|\pi(a)e\|^2 \geq 0$.

Definition 3.6.8. *If $\mathscr{A}$ is a C^*- algebra, a linear functional $f : \mathscr{A} \to \mathbb{C}$ is positive if $f(a) \geq 0$ whenever $a \in \mathscr{A}_+$. A state on $\mathscr{A}$ is a positive linear functional on $\mathscr{A}$ of norm 1*

Proposition 3.6.9. *If f is a positive linear functional on a C^*-algebra $\mathscr{A}$, then $\mid f(y^*x) \mid^2 \leq f(y^*y)f(x^*x)$ for every x, y in $\mathscr{A}$.*

Proof. If $[x, y] = f(y^*x)$ for x, y in $\mathscr{A}$, then $[\cdot, \cdot]$ is a semi-inner product on $\mathscr{A}$, the proposition now follows by the Cauchy-Schwarz inequality. ■

Corollary 3.6.10. *If f is a non-zero positive linear functional on the C^*- algebra $\mathscr{A}$ with identity I, then f is bounded and $\| f \|= f(I)$.*

Proof. Note that if $a \geq 0$, then $a \leq ||a||I$, so $f(a) \leq ||a||f(I)$. Thus, $|f(x)|^2 \leq f(I)f(I)||x^*x||$. It showed that $||f|| \leq f(I)$ and so $||f|| = f(I)$. ■

Example 3.6.11. *If X is a compact space, then the positive linear functionals on C(X) correspond to the positive regular Borel measures on X. The sates correspond to the probability measures on X.*

As was shown above, each cyclic representation gives rise to a positive linear functional. It turns out that each positive linear functional gives rise to a cyclic representation.

Theorem 3.6.12 (Gelfand-Naimark-Segal Construction)**.** *Let $\mathscr{A}$ be a C^*-algebra with identity.*

(1) If f is a positive linear functional on $\mathscr{A}$, then there is a cyclic representation $(\pi_f, \mathcal{H}_f)$ of $\mathscr{A}$ with cyclic vector e_f such that $f(a) = \langle \pi_f(a)e_f, e_f\rangle$ for all a in $\mathscr{A}$.

(2) If $(\pi, \mathcal{H})$ is a cyclic representation of $\mathscr{A}$ with cyclic vector e and $f(a) \equiv \langle \pi(a)e, e\rangle$ and if $(\pi_f, \mathcal{H}_f)$ is constructed as in (1), then π and π_f are equivalent.

Before beginning the proof, it will be helpful if the theorem is examined when $\mathscr{A}$ is abelian. So let $\mathscr{A} = C(X)$ where X is compact. If f is a positive linear functional on $\mathscr{A}$, then there is a positive regular Borel measure μ on X such that $f(\phi) = \int \phi d\mu$ for all ϕ in $\mathscr{A}$. The representation $(\pi_f, \mathcal{H}_f)$ is the one obtained by letting $\mathcal{H}_f = L^2(\mu)$ and $\pi_f(\phi) = M_\phi$, but let us look a little closer. One way to obtain $L^2(\mu)$ from $C(X)$ and μ is to let $\Phi = \{\phi \in C(X) :$

$f(\phi^*\phi) = \int |\phi|^2 d\mu = 0\}$. Note that Φ is an ideal in $C(X)$. Define an inner product on $C(X)/\Phi$ by $\langle \phi + \Phi, \psi + \Phi\rangle = \int \phi\overline{\psi} d\mu$. The completion of $C(X)/\Phi$ with respect to this inner product is $L^2(\mu)$.

To see part (2) in the abelian case, let $\pi : C(X) \to \mathcal{B}(\mathcal{H})$ be a cyclic representation with cyclic vector e. Let μ be the positive regular Borel measure on X such that $\int \phi d\mu = \langle \pi(\phi)e, e\rangle = f(\phi)$. Now define $U_1 : C(X) \to \mathcal{H}$ by $U_1(\phi) = \pi(\phi)e$. Note that U_1 is linear and has dense range. If Φ is as in the preceding paragraph and $\phi \in \Phi$, then $\|U_1(\phi)\|^2 = \langle \pi(\phi)e, \pi(\phi)e\rangle = \langle \pi(\phi^*\phi)e, e\rangle = \int |\phi|^2 d\mu = 0$. So $U_1\Phi = 0$. Thus U_1 induces a linear map $U : C(X)/\Phi \to \mathcal{H}$ where $U(\phi + \Phi) = \pi(\phi)e$. Since $\langle \phi + \Phi, \psi + \Phi\rangle = \int \phi\overline{\psi} d\mu$, so $\langle U(\phi + \Phi), U(\psi + \Phi)\rangle = \langle \pi(\phi)e, \pi(\psi)e\rangle = \langle \pi(\phi\psi^*)e, e\rangle = \int \phi\overline{\psi} d\mu = \langle \phi + \Phi, \psi + \Phi\rangle$. Thus U extends to an unitary transform U from the completion $L^2(\mu)$ of $C(X)/\Phi$ onto $\mathcal{H}$. So $U : L^2(\mu) \to \mathcal{H}$ and if $\phi \in C(X)$ and we think of $C(X)$ as a subset of $L^2(\mu)$, $U\phi = \pi(\phi)e$. If $\phi, \psi \in C(X)$, then $UM_\phi\psi = U(\phi\psi) = \pi(\phi\psi)e = \pi(\phi)\pi(\psi)e = \pi(\phi)U(\psi)$; that is, $UM_\phi = \pi(\phi)U$ on a dense subset of $L^2(\mu)$ and, hence, $UM_\phi = \pi(\phi)U$ for every ϕ in $C(X)$. In other words, π is equivalent to the representation $\phi \mapsto M_\phi$.

Proof. Let f be a positive linear functional on $\mathscr{A}$ and put $\Phi = \{x \in \mathscr{A} : f(x^*x) = 0\}$. It is easy to see that Φ is closed in $\mathscr{A}$. Also if $a \in \mathscr{A}$ and $x \in \Phi$, then Proposition 3.5.9 implies that

$$\begin{aligned} f((ax)^*(ax))^2 &= f(x^*(a^*ax))^2 \\ &\leq f(x^*x)f(x^*a^*aa^*ax) \\ &= 0. \end{aligned}$$

That is, Φ is a closed left ideal in $\mathscr{A}$. Now consider $\mathscr{A}/\Phi$ as a vector space. For x, y in $\mathscr{A}$, define

$$\langle x + \Phi, y + \Phi \rangle = f(y^* x).$$

It is left as an exercise for the reader to show that $\langle \cdot, \cdot \rangle$ is a well-defined inner product on $\mathscr{A}/\Phi$. Let $\mathcal{H}_f$ be the completion of $\mathscr{A}/\Phi$ with respect to the norm defined on $\mathscr{A}/\Phi$ by this inner product.

Because Φ is a left ideal of $\mathscr{A}$, $x + \Phi \to ax + \Phi$ is a well-defined linear transformation on $\mathscr{A}/\Phi$. Also, $\|ax + \Phi\|^2 = \langle ax + \Phi, ax + \Phi \rangle = f(x^* a^* ax)$. Note that $\|a^* a\| I - a^* a \geq 0$, hence

$$0 \leq x^* (\|a^* a\| - a^* a) x = \|a\|^2 x^* x - x^* a^* ax,$$

that is, $x^* a^* ax \leq \|a\|^2 x^* x$. Therefore $\|ax + \Phi\|^2 \leq \|a\|^2 f(x^* x) = \|a\|^2 \|x + \Phi\|^2$. Thus if $\pi_f(a) : \mathscr{A}/\Phi \to \mathscr{A}/\Phi$ is defined by

$$\pi_f(a)(x + \Phi) = ax + \Phi,$$

then $\pi_f(a)$ is a bounded linear operator with $||\pi_f(a)|| \leq \|a\|$. Hence $\pi_f(a)$ extends to an element of $\mathcal{B}(\mathcal{H}_f)$. It is left to the reader to verify that π_f is a representation.

Put $e_f = I + \Phi$ in $\mathcal{H}_f$. Then $\pi_f(\mathscr{A}) e_f = \{a + \Phi : a \in \mathscr{A}\} = \mathscr{A}/\Phi$ which, by definition, is dense in $\mathcal{H}_f$. Thus e_f is a cyclic vector for $\mathcal{H}_f$. Also note that $\langle \pi_f(a) e_f, e_f \rangle = f(a)$. This proves (1).

Now let $(\pi, \mathcal{H})$, e, and f be as in (2) and let $(\pi_f, \mathcal{H}_f)$ be the representation constructed. Let e_f be the cyclic vector for π_f so that $f(a) = \langle \pi_f(a) e_f, e_f \rangle$ for all a in $\mathscr{A}$. Hence $\langle \pi_f(a) e_f, e_f \rangle = \langle \pi(a) e, e \rangle$ for all a in $\mathscr{A}$. Define U on the dense subspace $\pi_f(\mathscr{A}) e_f$ of $\mathcal{H}_f$ by $U \pi_f(a) e_f = \pi(a) e$. Note that $\|\pi(a) e\|^2 =$

$\langle \pi(a)e, \pi(a)e \rangle = \langle \pi(a^*a)e, e \rangle = \langle \pi_f(a^*a)e_f, e_f \rangle = \|\pi_f(a)e_f\|^2$. This implies that U is well defined and an isometry. Thus U extends to an unitary transform of $\mathcal{H}_f$ onto $\mathcal{H}$. if $x, a \in \mathscr{A}$, then $U\pi_f(a)\pi_f(x)e_f = U\pi_f(ax)e_f = \pi(a)\pi(x)e = \pi(a)U\pi_f(x)e_f$. Thus $\pi(a)U = U\pi_f(a)$ so that π and π_f are equivalent. ∎

The Gelfand-Naimark-Segal construction is often called the GNS construction. It is not difficult to show that if f is a positive linear functional on $\mathscr{A}$ and $\alpha > 0$, then the representations π_f and $\pi_{\alpha f}$ are equivalent. So it is appropriate to only consider the cyclic representations corresponding to states. If $\mathscr{A}$ is a C^*-algebra, let $S_{\mathscr{A}}$ be the collection of all states on $\mathscr{A}$. Note that $S_{\mathscr{A}} \subseteq$ ball of $\mathscr{A}^*$. $S_{\mathscr{A}}$ is called the sate space of $\mathscr{A}$.

Theorem 3.6.13. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras with $\mathscr{B} \subseteq \mathscr{A}$ and identity I. If f is a state on $\mathscr{B}$, then there is a state $\hat{f}$ on $\mathscr{A}$ such that $\hat{f}|\mathscr{B} = f$.*

Proof. Consider the real linear spaces $L =$ self-adjoint elements of $\mathscr{A}$ and $L_1 =$ self-adjoint elements of $\mathscr{B}$. Let $P = \{x \in L : x \geq 0\}$ and $L_2 = L_1 + P - P$. It is easy to show that L_2 is a real linear subspace. If there is a positive linear functional g on L_2 that extends f, let f_0 be any linear functional on L that extends g by using the Hamel basis. If $x \geq 0$, then $x \in P$ so $x \in L_2$ so $f_0(x) = g(x) \geq 0$. Hence f_0 is positive. Thus, we may assume that $L = L_1 + P - P$.

Claim (I). $L = L_1 + P = L_1 - P$.

In fact, let $x \in L$, so $x = y + p_1 - p_2$, $y \in L_1, p_1, p_2 \in P$. Since $I \in L_1$, so there is a $y_1 \in L_1$ such that $y_1 \geq p_1$. Hence $p_1 = y_1 - (y_1 - p_1) \in L_1 - P$. Thus, $x = y - p_2 + p_1 \in (L_1 - P) + (L_1 - P) \subseteq L_1 - P$. So $L = L_1 - P$. Also, $L = -L = -L_1 + P = L_1 + P$.

Claim (II). If $x \in L$, there are $y_1, y_2 \in L_1$ such that $y_2 \leq x \leq y_1$.

In fact, Claim (I) stated that we can write $x = y_1 - p_1 = y_2 + p_2, p_1, p_2 \in P$ and $y_1, y_2 \in L_1$. Thus, $y_2 \leq x \leq y_1$.

By Claim (II), it is possible to define for each $x \in L$,

$$q(x) = \inf\{f(y) : y \in L_1,\ y \geq x\}.$$

It is easy to prove that:

Claim (III). The function q is a sublinear functional on L.

Now, we prove the theorem.

For $y \in L_1$, let $y_1 \in L_1$ such that $y_1 \geq y$. Because f is positive, $f(y) \leq f(y_1)$. Hence, $f(y) \leq q(y)$ for all $y \in L_1$. The Hahn-Banach Theorem implies that there is a linear functional $\hat{f}$ on L such that $\hat{f}|L_1 = f$ and $\hat{f}(x) \leq q(x)$ for $x \in L$. If $x \in P$, then $-x \leq 0$. Hence $q(-x) \leq f(0)$. Thus, $-\hat{f}(x) = \hat{f}(-x) \leq q(-x) \leq 0$, or $\hat{f}(x) \geq 0$. Therefore, $\hat{f}$ is positive. Since f is a state on L_1, so $\hat{f}$ is also a state on L.

Now let $\hat{f}(a) = \hat{f}(\frac{a+a^*}{2}) + i\hat{f}(\frac{a-a^*}{2i})$ for $a \in \mathscr{A}$. Thus, $\hat{f}$ satisfies the condition of theorem. ■

Theorem 3.6.14. *Let $\mathscr{A}$ be a C^*-algebra with identity. Then $S_{\mathscr{A}}$ is a weak* compact convex subset of $\mathscr{A}^*$. Moreover, if $a \in \mathscr{A}$ is a positive element, then $||a|| = \sup\{f(a) : f \in S_{\mathscr{A}}\}$ and this supremum is attained.*

Proof. If $\mathscr{A} = C(X)$ with X compact and $g \in C(X)_+$, then there is a point $x \in X$ such that $g(x) = ||g||$. Note that δ_x is a positive regular Borel measure on X, so, it is a state. Thus, $||g|| = \int_X g\delta_x = \sup\{f(g) : f \in S_{\mathscr{A}}\}$. If $\mathscr{A}$ is arbitrary and $a \in \mathscr{A}_+$, then $||a|| \geq \sup\{f(a) : f \in S_{\mathscr{A}}\}$. Also, from the argument in the abelian case, there is a state f_1 on the C^*-algebra $C(\sigma(a))$

such that $f_1(a) = ||a||$. By Theorem 3.6.13 we have the conclusion of this theorem. ∎

Theorem 3.6.15. *Let $\mathscr{A}$ be a C^*-algebra with identity. Then there is a representation $(\pi, \mathscr{H})$ of $\mathscr{A}$ such that π is an isometry. If $\mathscr{A}$ is separable, then $\mathscr{H}$ can be chosen separable.*

Proof. Let F be a $\sigma(\mathscr{A}^*, \mathscr{A})$ dense subset of $S_{\mathscr{A}}$ and let $\pi = \oplus\{\pi_f : f \in F\}$, $\mathscr{H} = \oplus\{H_f : f \in F\}$. Thus, $||a||^2 \geq ||\pi(a)||^2 = \sup_f ||\pi_f(a)||^2$. If e_f is the cyclic vector for π_f, then $||e_f||^2 = \langle e_f, e_f \rangle = \langle \pi_f(I)e_f, e_f \rangle = f(I) = 1$. Hence, $||\pi_f(a)||^2 \geq ||\pi_f(a)e_f||^2 = \langle \pi_f(a^*a)e_f, e_f \rangle = f(a^*a)$, and $||a||^2 \geq ||\pi(a)||^2 \geq \sup\{f(a^*a) : f \in F\}$. Since F is a $\sigma(\mathscr{A}^*, \mathscr{A})$ dense subset of $S_{\mathscr{A}}$, thus, by Theorem 3.6.14 that $\sup\{f(a^*a) : f \in F\} = ||a^*a|| = ||a||^2$. Hence, π is an isometry.

If $\mathscr{A}$ is separable, then $(\{f \in \mathscr{A}^* : ||f|| \leq 1\}, \sigma(\mathscr{A}^*, \mathscr{A}))$ is a compact metric space. Hence $S_{\mathscr{A}}$ is $\sigma(\mathscr{A}^*, \mathscr{A})$ separable so that the set F of the preceding paragraph can be chosen to be countable. Now, if $f \in F, \pi(\mathscr{A})e_f$ is a separable dense linear subspace in $\mathscr{H}_f$ since $\mathscr{A}$ is separable. Thus, $\mathscr{H}_f$ is separable. It follows that $\mathscr{H}$ is separable. ∎

4 Spectral Theorems of Bounded Normal Operators

The normal operators form one of the best understood and most tractable of classes of operators. The principal reason for this is the spectral theorem, which enables us to synthesize a normal operator from linear combinations of projections where the coefficients lie in the spectrum. It is a very powerful result that answers many questions about these operators. In this section we actually prove a more general result than the spectral theorem for normal operators, and we get this extra useful generality without any increase in difficulty of the proofs. Indeed, the more general situation illustrates nicely the connection between spectral measures and representations of abelian C*-algebras. Finally, we introduce the famous Fuglede-Putnam Theorem.

4.1 Spectral Measures and Spectral Integrals

Let X be a set and $\mathcal{F}$ be a σ-algebra of subsets of X, $\mathcal{H}$ be a Hilbert space. A spectral measure E relative to $(X, \mathcal{F}, \mathcal{H})$ is a map from the σ-algebra $\mathcal{F}$ to $\mathcal{B}(\mathcal{H})$ such that

(1) for every $S \in \mathcal{F}$, $E(S)$ is an orthogonal projection operator;

(2) $E(\emptyset) = 0, E(X) = I$;

(3) if $\{S_n\}$ are pairwise disjoint sets in $\mathcal{F}$, then

$$E(\bigcup_{n=1}^{\infty} S_n) = \sum_{n=1}^{\infty} E(S_n).$$

Now, we show that for all sets $S_1, S_2 \in \mathcal{F}$, we have

$$E(S_1 \cap S_2) = E(S_1)E(S_2).$$

In fact, note that

$$\begin{aligned} E(S_1\cap S_2)+E(S_1\cup S_2) &= E(S_1\cap S_2)+E(S_1\backslash S_2)+E(S_1\cap S_2)+E(S_2\backslash S_1) \\ &= E(S_1)+E(S_2), \end{aligned}$$

use $E(S_1)$ multiply the equality, we have $E(S_1\cap S_2)+E(S_1) = E(S_1)+E(S_1)E(S_2)$, and so $E(S_1\cap S_2)=E(S_1)E(S_2)$.

Let Ω be a compact Hausdorff space and $\mathcal{F}$ be the set of all Borel subsets of Ω, $\mathcal{H}$ be a Hilbert space. A spectral measure E relative to $(\Omega,\mathcal{F},\mathcal{H})$ is called a spectral measure relative to $(\Omega,\mathcal{H})$.

Denote by $M(\Omega)$ the set of all finite total variation Borel complex measures on Ω, and by $\mathcal{B}_\infty(\Omega)$ the C*-algebra of all bounded Borel-measurable complex-valued functions on Ω.

If E is a spectral measure relative to $(\Omega,\mathcal{H})$, then for all $x,y\in\mathcal{H}$, the function $E_{x,y}: S\longmapsto\langle E(S)x,y\rangle$ is a finite total variation Borel complex measure.

In fact, it is clear that $E_{x,y}$ is a complex measure on $\mathcal{F}$. If $S_1,S_2,\cdots,S_n$ are pairwise disjoint Borel sets, let $a_1,a_2,\cdots,a_n$ be complex numbers such that $|a_i|=1$ and $|\langle E(S_i)x,y\rangle| = a_i\langle E(S_i)x,y\rangle$. So

$$\sum_i |\langle E(S_i)x,y\rangle| = \sum_i\langle E(S_i)a_ix,y\rangle \le ||\sum_i E(S_i)a_ix||||y||.$$

Now $\{E(S_i)a_ix\}_{i=1}^n$ is a finite sequence of pairwise orthogonal vectors so that $||\sum_{i=1}^n E(S_i)a_ix||^2 = \sum_{i=1}^n ||E(S_i)a_ix||^2 = ||E(\cup_{i=1}^n S_i)x||^2 \le ||x||^2$, hence $\sum_{i=1}^n |E_{x,y}(S_i)| \le ||x||||y||$, that is, $E_{x,y}$ is a finite total variation Borel complex measure.

Lemma 4.1.1. *Let Ω be a compact Hausdorff space, let $\mathcal{H}$ be a Hilbert space, and suppose that $\mu_{x,y} \in M(\Omega)$ for all $x, y \in \mathcal{H}$. Suppose also that for each Borel set S of Ω the function*

$$\tau_S : \mathcal{H}^2 \longrightarrow \mathbb{C}, (x, y) \longmapsto \mu_{x,y}(S),$$

is a sesquilinear form. Then for each $f \in \mathcal{B}_\infty(\Omega)$ the function

$$\tau_f : \mathcal{H}^2 \longrightarrow \mathbb{C}, (x, y) \longmapsto \int f d\mu_{x,y},$$

is a sesquilinear form.

Proof. Suppose first that f is simple, so we can write $f = \sum_{j=1}^{n} \lambda_j \chi_{S_j}$, where $S_1, \ldots, S_n$ are pairwise disjoint Borel sets of Ω, and $\lambda_1, \ldots, \lambda_n$ are complex numbers. Then

$$\int f d\mu_{x,y} = \sum_{j=1}^{n} \lambda_j \int \chi_{S_j} d\mu_{x,y} = \sum_{j=1}^{n} \lambda_j \mu_{x,y}(S_j).$$

The set of sesquilinear forms on $\mathcal{H}^2$ is a vector space with the pointwise-defined operations, and we have just shown that τ_f is a linear combination of the τ_{S_j}, so τ_f is a sesquilinear form.

Now suppose that f is an arbitrary element of $\mathcal{B}_\infty(\Omega)$. Then f is the uniform limit of a sequence (f_n), where each f_n is a simple function in $\mathcal{B}_\infty(\Omega)$. Hence, $\int |f_n - f| d|\mu_{x,y}| \leq \|f_n - f\|_\infty |\mu_{x,y}|(\Omega)$, so $\int f d\mu_{x,y} = \lim_{n\to\infty} \int f_n d\mu_{x,y}$ for each $x, y \in E$. It follows immediately that τ_f is a sesquilinear form on $\mathcal{H}^2$.

■

Theorem 4.1.2. *Let Ω be a compact Hausdorff space, $\mathcal{H}$ a Hilbert space, and E a spectral measure relative to $(\Omega, \mathcal{H})$. Then for each $f \in \mathcal{B}_\infty(\Omega)$ the function*

$$\tau_f : \mathcal{H}^2 \longrightarrow \mathbb{C}, (x, y) \longmapsto \int f dE_{x,y},$$

is a bounded sesquilinear from on $\mathcal{H}$, and $\|\tau_f\| \leq \|f\|_\infty$.

Proof. That τ_f is a sesquilinear form follows from the preceding lemma, so we need only show $\|\tau_f\| \leq \|f\|_\infty$. This follows from $|\int f dE_{x,y}| \leq \|f\|_\infty |E_{x,y}|(\Omega) \leq ||f||_\infty ||x|| ||y||$, so $\|\tau_f\| \leq \|f\|_\infty$. ∎

Theorem 4.1.3. *Let Ω be a compact Hausdorff space, $\mathcal{H}$ a Hilbert space, and E a spectral measure relative to $(\Omega, \mathcal{H})$. Then for each $f \in \mathcal{B}_\infty(\Omega)$ there is a unique bounded operator U on $\mathcal{H}$ such that*

$$\langle U(x), y \rangle = \int f dE_{x,y} \quad (x, y \in \mathcal{H}).$$

Proof. Immediate from the preceding theorem and Theorem 3.4.7.∎

We write $\int f dE$ for U and call it the integral of f with respect to E. Note that $\int \chi_S dE = E(S)$ for each Borel set S.

Theorem 4.1.4. *With the same assumptions on $\Omega, \mathcal{H}$, and E as in the preceding theorem, the map*

$$\Phi : \mathcal{B}_\infty(\Omega) \longrightarrow \mathcal{B}(\mathcal{H}), f \longmapsto \int f dE$$

*is a unital *-homomorphism.*

Proof. Linearity is routine and boundedness follows from Theorems 4.1.2 and 3.3.7. To show that $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = (\Phi(f))^*$, we need only show these results when f, g are simple, because the simple elements of $\mathcal{B}_\infty(\Omega)$ are dense. Hence, we may reduce further and suppose that $f = \chi_S$ and $g = \chi_{S'}$

by linearity of Φ and the fact that all simple elements of $\mathcal{B}_\infty(\Omega)$ are linear combinations of such characteristic functions. Then $\Phi(fg) = \int \chi_S \chi_{S'} dE = E(S \cap S') = E(S)E(S') = \int \chi_S dE \int \chi_{S'} dE = \Phi(f)\Phi(g)$. Also, $\Phi(\overline{f}) = \Phi(f) = E(S) = (\Phi(f))^*$. ■

Theorem 4.1.5. *Let Ω be a compact Hausdorff space and $\mathcal{H}$ a Hilbert space, and suppose that $\varphi : \mathcal{C}(\Omega) \longrightarrow \mathcal{B}(\mathcal{H})$ is a unital *-homomorphism. Then there is a unique spectral measure E relative to $(\Omega, \mathcal{H})$ such that*

$$\varphi(f) = \int f dE \quad (f \in \mathcal{C}(\Omega)).$$

Moreover, φ can be extended to the map Φ such that

$$\Phi : \mathcal{B}_\infty(\Omega) \longrightarrow \mathcal{B}(\mathcal{H}), f \longmapsto \int f dE$$

*is a unital *-homomorphism.*

If $U \in \mathcal{B}(\mathcal{H})$, then U commutes with $\varphi(f)$ for all $f \in \mathcal{C}(\Omega)$ if and only if U commutes with $E(S)$ for all Borel sets S of Ω.

Proof. If $x, y \in \mathcal{H}$, then the function

$$\tau_{x,y} : \mathcal{C}(\Omega) \longrightarrow \mathbb{C}, f \longmapsto \langle \varphi(f)(x), y \rangle,$$

is linear and $\|\tau_{x,y}\| \leq \|x\|\|y\|$. By the Riesz-Kakutani theorem, there is a unique measure $\mu_{x,y}$ in $M(\Omega)$ such that $\tau_{x,y}(f) = \int f d\mu_{x,y}$ for all $f \in \mathcal{C}(\Omega)$. Also, $|\mu_{x,y}|(\Omega) = \|\tau_{x,y}\|$. Since the function

$$\mathcal{H}^2 \longrightarrow \mathbb{C}, (x, y) \longmapsto \langle \varphi(f)(x), y \rangle$$

is sesquilinear, the maps from $\mathcal{H}$ to $M(\Omega)$ given by

$$x \longmapsto \mu_{x,y} \quad \textit{and} \quad y \longmapsto \mu_{x,y}$$

are, respectively, linear and conjugate-linear. Hence, for each $f \in \mathcal{B}_\infty(\Omega)$ the function

$$\mathcal{H}^2 \longrightarrow \mathbb{C}, (x, y) \longmapsto \int f d\mu_{x,y},$$

is a sessquilinear form, by Lemma 4.1.1. Also,

$$|\int f d\mu_{x,y}| \leq \|f\|_\infty |\mu_{x,y}|(\Omega) \leq \|f\|_\infty \|x\| \|y\|,$$

so this sesquilinear form is bounded and its norm is not great than $\|f\|_\infty$. By Theorem 3.3.7, there is a unique operator, $\Phi(f)$ say, in $\mathcal{B}(\mathcal{H})$ such that

$$\langle \Phi(f)(x), y\rangle = \int f d\mu_{x,y} \quad (x, y \in \mathcal{H}).$$

Moreover, $\|\Phi(f)\| \leq \|f\|_\infty$.

Now suppose that $f \in \mathcal{C}(\Omega)$. Then

$$\langle \Phi(f)(x), y\rangle = \int f d\mu_{x,y} = \tau_{x,y}(f) = \langle \varphi(f)(x), y\rangle \quad (x, y \in \mathcal{H}),$$

so $\Phi(f) = \varphi(f)$.

It is straightforward to check that the map

$$\Phi : \mathcal{B}_\infty(\Omega) \longrightarrow \mathcal{B}(\mathcal{H}), f \longmapsto \Phi(f),$$

is linear and we already know it is norm-decreasing. We show now that Φ is a *-homomorphism.

If $f \in \mathcal{C}(\Omega)$ and $\overline{f} = f$, then $\varphi(f)$ is hermitian, so $\int f d\mu_{x,x} = \langle \varphi(f)(x), x\rangle$ is a real number. Thus, the measure $\mu_{x,x}$ is real, that is, $\overline{\mu}_{x,x} = \mu_{x,x}$, and therefore if f is an arbitrary function in $\mathcal{B}_\infty(\Omega)$ such that $\overline{f} = f$, then $\langle \Phi(f)(x), x\rangle =$

$\int f d\mu_{x,x}$ is real. Since x is arbitrary, this shows that $\Phi(f)$ is hermitian. Therefore, Φ preserves the involutions.

Let $f \in \mathcal{B}_\infty(\Omega)$ and $x \in \mathcal{H}$.

Assertion: If the equation

$$\langle \Phi(fg)x, x\rangle = \langle \Phi(f)\Phi(g)(x), x\rangle \tag{1}$$

holds for all $g \in \mathcal{C}(\Omega)$, then it also holds for all $g \in \mathcal{B}_\infty(\Omega)$.

Observe that Eq. (1) is equivalent to

$$\int gf d\mu_{x,x} = \int g d\mu_{x,\Phi(\overline{f})(x)}. \tag{2}$$

To prove the assertion, note that if Eq. (1) holds for all $g \in \mathcal{C}(\Omega)$, then the measures $f d\mu_{x,x}$ and $\mu_{x,\Phi(\overline{f})(x)}$ are equal because Eq. (2) holds for all $g \in \mathcal{C}(\Omega)$. Hence, Eq.(2) holds for all $g \in \mathcal{B}_\infty(\Omega)$; that is, Eq. (1) holds for all such g, as claimed.

Since φ is a *-homomorphism, Eq.(1) holds for all $f, g \in \mathcal{C}(\Omega)$. Hence, by the assertion, Eq.(1) holds if $f \in \mathcal{C}(\Omega)$ and $g \in \mathcal{B}_\infty(\Omega)$. Replacing f, g with their conjugates, we get $\langle \Phi(\overline{f}\overline{g})(x), x\rangle = \langle \Phi(\overline{f})\Phi(\overline{g})(x), x\rangle$. Taking conjugates of both sides of this equation and using the fact Φ preserves the involutions, we get

$$\langle \Phi(gf)(x), x\rangle = \langle \Phi(g)\Phi(f)(x), x\rangle, \tag{3}$$

for all $g \in \mathcal{B}_\infty(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Using the assertion again (with the roles of f and g interchanged), we get Eq.(3) holds for all $f, g \in \mathcal{B}_\infty(\Omega)$. Since x was an arbitrary element of $\mathcal{H}$, this implies that $\Phi(gf) = \Phi(g)\Phi(f)$, so Φ is a homomorphism.

If S is a Borel set of Ω, we put $E(S) = \Phi(\chi_S)$. Obviously, $E(S)$ is a projection on $\mathcal{H}$, and it is easily verified that the map $E : S \longmapsto E(S)$ from the σ-algebra of Borel sets of Ω to $\mathcal{B}(\mathcal{H})$ is a spectral measure relative to $(\Omega, \mathcal{H})$, we have $E_{x,y} = \mu_{x,y} \in M(\Omega)$, since $E_{x,y}(S) = \langle E(S)(x), y\rangle = \langle \Phi(\chi_S)(x), y\rangle = \int \chi_S d\mu_{x,y} = \mu_{x,y}(S)$.

If $f \in \mathcal{B}_\infty(\Omega)$, then

$$\langle (\int f dE)(x), y\rangle = \int f dE_{x,y} = \int f d\mu_{x,y} = \langle \Phi(f)(x), y\rangle,$$

so $\Phi(f) = \int f dE$. In particular, $\varphi(f) = \int f dE$ for all $f \in \mathcal{C}(\Omega)$.

To see uniqueness of E, suppose that E' is another spectral measure relative to $(\Omega, \mathcal{H})$ such that $\varphi(f) = \int f dE'$ for all $f \in \mathcal{C}(\Omega)$. Then $\int f dE'_{x,y} = \langle \varphi(f)(x), y\rangle = \int f dE_{x,y}$. Hence, $E'_{x,y} = E_{x,y}$, and therefore $\langle E'(S)(x), y\rangle = \langle E(S)(x), y\rangle$, so $E = E'$.

Now suppose U is an operator on $\mathcal{H}$ commuting with all of the elements of the range of φ. Then if $f \in \mathcal{C}(\Omega)$, $\int f d\mu_{U(x),y} = \langle \varphi(f)U(x), y\rangle = \langle U\varphi(f)(x), y\rangle = \langle \varphi(f)(x), U^*(y)\rangle = \int f d\mu_{x,U^*(y)}$, hence $E_{U(x),y} = E_{x,U^*(y)}$. So $E(S)U = UE(S)$ for all Borel sets S.

Conversely, suppose now that U commutes with all the projections $E(S)$. Then

$$\langle E(S)U(x), y\rangle = \langle UE(S)(x), y\rangle = \langle E(S)(x), U^*(y)\rangle,$$

so $E_{U(x),y} = E_{x,U^*(y)}$. Hence, for every $f \in \mathcal{C}(\Omega)$,

$$\int f dE_{U(x),y} = \int f dE_{x,U^*(y)},$$

that is, $\langle \varphi(f)U(x), y\rangle = \langle \varphi(f)(x), U^*(y)\rangle$, so $\varphi(f)U = U\varphi(f)$. ■

4.2 Spectral Theorem and Applications

The next result, which is special case of Theorem 4.1.5, is one of the most important in single operator theory, and is called the spectral theorem.

Theorem 4.2.1. *Let U be a bounded normal operator on a Hilbert space $\mathcal{H}$. Then*

(1) there is a unique spectral measure E relative to $(\sigma(U), \mathcal{H})$ such that $U = \int z dE$, where z is the inclusion map of $\sigma(U)$ in $\mathbb{C}$;

(2) if G is a nonempty relative open subset of $\sigma(U)$, then $E(G) \neq 0$;

(3) if $A \in B(\mathcal{H})$, then $AU = UA$ and $AU^* = U^*A$ if and only if $AE(\Delta) = E(\Delta)A$ for every Borel subset Δ of $\sigma(U)$.

Proof. (1) By the Theorem 3.1.14, let $\varphi : \mathcal{C}(\sigma(U)) \longrightarrow \mathcal{B}(\mathcal{H})$ be the functional calculus at U, then φ is a unital *-homomorphism. It follows from the preceding theorem, there exists a unique spectral measure E relative to $(\sigma(U), \mathcal{H})$ such that $\varphi(f) = \int f dE$ for all $f \in \mathcal{C}(\sigma(U))$. In particular, $U = \varphi(z) = \int z dE$. If E' is another spectral measure such that $U = \int z dE'$, then $\int f dE' = \int f dE = \varphi(f)$ for all $f \in \mathcal{C}(\sigma(U))$, since 1 and z generate $\mathcal{C}(\sigma(U))$. Therefore, $E = E'$.

(2) If G is a nonempty relative open subset of $\sigma(U)$, then by the Uryson theorem, there exists a nonzero $f_0 \in \mathcal{C}(\sigma(U))$ such that $0 < f_0 \leq \chi_G$. Thus, it follows from Theorem 4.1.6 that $E(G) = \Phi(\chi_G) \geq \varphi(f_0) > 0$. So, $E(G) \neq 0$.

(3) If $A \in B(\mathcal{H})$ such that $AU = UA$ and $AU^* = U^*A$, by the Stone-Weierstrass Theorem, it is easily to show that for each $f \in \mathcal{C}(\sigma(U))$, $Af(U) = f(U)A$, so by Theorem 4.1.6, $AE(\Delta) = E(\Delta)A$ for every Borel subset Δ of $\sigma(U)$. The converse is clear. ■

The spectral measure E in Theorem 4.1.7 is called the resolution of the identity for U.

Since $f(U) = \int f dE$ for all $f \in \mathcal{C}(\sigma(U))$. By Theorem 4.1.6, we can define $f(U) = \int f dE$ for all $f \in \mathcal{B}_\infty(\sigma(U))$. The unital *-homomorphism

$$\mathcal{B}_\infty(\sigma(U)) \longrightarrow \mathcal{B}(\mathcal{H}), f \longmapsto f(U),$$

is called the Borel functional calculus at U.

Now, we give out several applications of Spectral Theorem and the famous Fuglede-Putnam Theorem.

Proposition 4.2.2. *If N is a bounded normal operator and $N = \int z dE(z)$, then N is compact if and only if for every $\varepsilon > 0$, $E(\{z : \mid z \mid > \varepsilon\})$ has finite rank.*

Proof. If $\varepsilon > 0$, let $\Delta_\varepsilon = \{z : \mid z \mid > \varepsilon\}$ and $E_\varepsilon = E(\Delta_\varepsilon)$. Then

$$N - NE_\varepsilon = \int z dE(z) - \int z \chi_{\Delta_\varepsilon}(z) dE(z)$$

$$= \int z \chi_{\mathbb{C} \setminus \Delta_\varepsilon}(z) dE(z) = \phi(N).$$

where $\phi(z) = z\chi_{\mathbb{C}\setminus\Delta_\varepsilon}(z)$. Thus $\| N - NE_\varepsilon \| \leq \sup\{\mid z \mid : z \in \mathbb{C} \setminus \Delta_\varepsilon\} \leq \varepsilon$. If E_ε has finite rank for every $\varepsilon > 0$, then so does NE_ε. Thus N is compact.

Converse, if N is compact and let $\varepsilon > 0$. Put $\phi(z) = z^{-1}\chi_{\Delta_\varepsilon}(z)$, then ϕ is a bounded Borel measurable function. Since N is compact, so is $N\phi(N)$. But $N\phi(N) = \int z z^{-1} \chi_{\Delta_\varepsilon}(z) dE(z) = E_\varepsilon$. Since E_ε is a compact projection, it must have finite rank. ∎

Theorem 4.2.3. *If $\mathcal{H}$ is separable and $\mathcal{I}$ is an ideal of $\mathcal{B}(\mathcal{H})$ that contains a noncompact operator, then $\mathcal{I} = \mathcal{B}(\mathcal{H})$.*

Proof. If $A \in \mathcal{I}$ and A is not compact, consider A^*A; let $A^*A = \int tdE(t)$. By the preceding proposition, there is an $\varepsilon > 0$ such that $P = E(\varepsilon, \infty)$ has infinite rank. But $P = (\int t^{-1}\chi_{(\varepsilon,\infty)}(t)dE(t))A^*A \in \mathcal{I}$. Since $\mathcal{H}$ is separable, dim $P\mathcal{H} = dim\mathcal{H} = \aleph_0$. Let $U : \mathcal{H} \to P\mathcal{H}$ be a surjective isometry. It is easy to check that $I = U^*PU$. But $P \in \mathcal{I}$, so $I \in \mathcal{I}$. Hence $\mathcal{I} = \mathcal{B}(\mathcal{H})$. ■

It follows from Theorem 3.4.7 and above theorem that

Corollary 4.2.4. *If $\mathcal{H}$ is separable, then the only nontrivial closed ideal of $\mathcal{B}(\mathcal{H})$ is the ideal of all compact operators.*

Proposition 4.2.5. *If $\mathscr{A}$ is an S.O.T closed C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathscr{A}$ is the norm closed linear span of the projections in $\mathscr{A}$.*

Proof. If $A \in \mathscr{A}$, $A + A^*$ and $A - A^* \in \mathscr{A}$; hence $\mathscr{A}$ is the linear span of $\mathscr{A}_{sa}$. Suppose $A \in \mathscr{A}_{sa}$ and $A = \int tdE(t)$. If $[a, b] \subseteq \mathbb{R}$, then there is a sequence of functions $\{u_n\}$ defined on $\mathbb{R}$ such that $0 \le u_n \le 1$, $u_n(t) = 1$ for $a \le t \le b - n^{-1}$, $u_n(t) = 0$ for $t \le a - n^{-1}$ or $t \ge b$. Hence $u_n(t) \to \chi_{[a,b)}(t)$ as $n \to \infty$. If $h \in \mathcal{H}$, then

$$\| u_n(A)h - E[a, b)h \|^2 = \int | u_n(t) - \chi_{[a,b)}(t) |^2 \, dE_{h,h}(t) \to 0$$

by the Lebesgue Dominated Convergence Theorem. That is, $u_n(A) \to E[a, b)$ with respect to $S.O.T$. Since $\mathscr{A}$ is $S.O.T$-closed, $E[a, b) \in \mathscr{A}$. Now let (α, β) be an open interval containing $\sigma(A)$. If $\varepsilon > 0$, then there is a partition $\alpha =$

$t_0 < \cdots < t_n = \beta$ such that $| t - \sum_{k=1}^{n} t_k \chi_{[t_{k-1},t_k)}(t) | < \varepsilon$ for t in $\sigma(A)$; hence

$$\| A - \sum_{k=1}^{n} t_k E[t_{k-1}, t_k) \| < \varepsilon.$$

Thus every self-adjoint operator in $\mathscr{A}$ belongs to the closed linear span of the projections in $\mathscr{A}$. ■

Theorem 4.2.6 (Fuglede-Putnam). *If N and M are bounded normal operators on $\mathcal{H}$ and $\mathcal{K}$, and $B : \mathcal{K} \to \mathcal{H}$ is a bounded linear operator such that $NB = BM$, then $N^*B = BM^*$.*

Proof. Note that it follows from the hypothesis that $N^kB = BM^k$ for all $k \geq 1$. So if $p(z)$ is a polynomial, then $p(N)B = Bp(M)$. Since for a fixed z in $\mathbb{C}$, $\exp(i\overline{z}N)$ and $\exp(i\overline{z}M)$ are limits of polynomials in N and M, respectively, it follows that $\exp(i\overline{z}N)B = B\exp(i\overline{z}M)$ for all z in $\mathbb{C}$. Equivalently, $B = e^{-i\overline{z}N}Be^{i\overline{z}M}$. Because $\exp(X+Y) = (\exp X)(\exp Y)$ when X and Y commute, the fact that N and M are bounded normal implies that

$$\begin{aligned} f(z) &\equiv e^{-izN^*}Be^{izM^*} \\ &= e^{-izN^*}e^{-i\overline{z}N}Be^{i\overline{z}M}e^{izM^*} \\ &= e^{-i(zN^*+\overline{z}N)}Be^{i(\overline{z}M+zM^*)}. \end{aligned}$$

But for every z in $\mathbb{C}$, $zN^*+\overline{z}N$ and $zM^*+\overline{z}M$ are hermitian operators. Hence $\exp[-i(zN^*+\overline{z}N)]$ and $\exp[i(zM^*+\overline{z}M)]$ are unitary. Therefore $\|f(z)\| \leq \|B\|$. But $f : \mathbb{C} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an entire function. By Liouville Theorem, f is constant. Thus, $0 = f'(z) = -iN^*e^{-izN^*}Be^{izM^*} + ie^{-izN^*}BM^*e^{izM^*}$. Putting $z = 0$ gives $0 = -iN^*B + iBM^*$, whence the theorem. ■

5 The Unbounded Operator Theory

5.1 Basic Properties of Unbounded Operators

It is unfortunate for the world we live in that all of the operators that arise naturally are not bounded. But that is indeed the case. Thus it is important to study such operators.

The idea here is not to study an arbitrary linear transformation on a Hilbert space. In fact, such a study is the province of linear algebra rather than analysis. The operators that are to be studied do possess certain properties that connect them to the underlying Hilbert space. The properties that will be isolated are inspired by natural examples.

Now, all Hilbert spaces are assumed to be separable.

The first relaxation in the concept of operator is not to assume that the operators are defined everywhere on the Hilbert space.

Definition 5.1.1. *If $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces, a linear operator $A : \mathcal{H} \to \mathcal{K}$ whose domain of definition is a linear subspace $dom(A)$ of $\mathcal{H}$ and such that $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for x, y in $dom(A)$ and α, β in $\mathbb{C}$. A is bounded if there is a constant $c > 0$ such that $\|Ax\| \leq c\|x\|$ for all x in $dom(A)$.*

Note that if A is bounded, then A can be extended to a bounded linear operator on $\overline{dom(A)}$ and then extended to $\mathcal{H}$ by letting be 0 on $(dom(A))^{\perp}$. So unless it is specified to the contrary, a bounded operator will always be assumed to be defined on all of $\mathcal{H}$.

If A is a linear operator from $\mathcal{H}$ into $\mathcal{K}$, then A is also a linear operator from $\overline{dom(A)}$ into $\mathcal{K}$. So we will often only consider those A such that $dom(A)$

is dense in $\mathcal{H}$; such an operator A is said to be densely defined.

If A, B are linear operators from $\mathcal{H}$ into $\mathcal{K}$, then $A+B$ is defined with $dom(A+B) = domA \cap domB$. If $B : \mathcal{H} \to \mathcal{K}$ and $A : \mathcal{K} \to \mathcal{L}$, then AB is a linear operator from $\mathcal{H}$ into $\mathcal{L}$ with $dom(AB) = B^{-1}(dom(A))$.

Definition 5.1.2. *If A, B are linear operators from $\mathcal{H}$ into $\mathcal{K}$, then A is an extension of B if $domB \subseteq domA$ and $Ah = Bh$ whenever $h \in domB$. In symbols this is denoted by $B \subseteq A$.*

Note that if $A \in B(\mathcal{H})$, then the only extension of A is itself. So this concept is only of value for unbounded operators.

If $A : \mathcal{H} \to \mathcal{K}$, the graph of A is the set

$$gra(A) = \{h \oplus Ah \in \mathcal{H} \oplus \mathcal{K} : h \in dom(A)\}.$$

It is easy to see that $B \subseteq A$ if and only if $gra(B) \subseteq gra(A)$.

Definition 5.1.3. *An operator $A : \mathcal{H} \to \mathcal{K}$ is closed if its graph is closed in $\mathcal{H} \oplus \mathcal{K}$. An operator is closable if it has a closed extension.*

Denote $C(\mathcal{H}, \mathcal{K})$ be the set of all closed densely defined operators from $\mathcal{H}$ into $\mathcal{K}$.

Proposition 5.1.4. *An operator $A : \mathcal{H} \to \mathcal{K}$ is closable if and only if $\overline{gra(A)}$ is a graph, that is, there is an operator $B : \mathcal{H} \to \mathcal{K}$ such that $gra(B) = \overline{gra(A)}$.*

Proof. Let $\overline{gra(A)}$ be a graph, then there is an operator $B : \mathcal{H} \to \mathcal{K}$ such that $gra(B) = \overline{gra(A)}$. Clearly $gra(A) \subseteq gra(B)$, so A is closable.

Now assume that A is closable; that is, there is a closed operator $B : \mathcal{H} \to \mathcal{K}$ with $A \subseteq B$. If $0 \oplus k \in \overline{gra(A)}, 0 \oplus k \in gra(B)$ and hence $k = 0$. Then it not hard to see that $\overline{gra(A)}$ is a graph. ∎

Definition 5.1.5. *Let operator $A : \mathcal{H} \to \mathcal{K}$ be densely defined. Denote*

$$dom(A^*) = \{k : k \in \mathcal{K}, h \mapsto \langle Ah, k\rangle \text{ is a bounded linear functional on } dom(A)\}.$$

Because $dom(A)$ is dense in $\mathcal{H}$, if $k \in dom(A^*)$, then there is a unique vector f in $\mathcal{H}$ such that $\langle Ah, k\rangle = \langle h, f\rangle$ for all h in $dom(A)$. Denote this unique vector f by $f = A^*k$. Thus

$$\langle Ah, k\rangle = \langle h, A^*k\rangle$$

for all h in $dom(A)$ and k in $dom(A^*)$.

Lemma 5.1.6. *If $A : \mathcal{H} \to \mathcal{K}$ is densely defined and $J : \mathcal{H} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ is defined by $J(h \oplus k) = (-k) \oplus h$, then J is an isomorphism and*

$$gra(A^*) = [J(gra(A))]^{\perp}.$$

Proof. It is clear that J is an isomorphism. To prove the formula for $gra(A^*)$, note that $gra(A^*) = \{k \oplus A^*k \in \mathcal{K} \oplus \mathcal{H} : k \in dom(A^*)\}$. So if $k \in dom(A^*)$ and $h \in dom(A)$,

$$\begin{aligned} \langle k \oplus A^*k, J(h \oplus Ah)\rangle &= \langle k \oplus A^*k, -Ah \oplus h\rangle \\ &= -\langle k, Ah\rangle + \langle A^*k, h\rangle = 0. \end{aligned}$$

Thus $gra(A^*) \subseteq [J(gra(A))]^{\perp}$. Conversely, if $k \oplus f \in [J(gra(A))]^{\perp}$, then for every h in $dom(A)$, $0 = \langle k \oplus f, -Ah \oplus h\rangle = -\langle k, Ah\rangle + \langle f, h\rangle$, so $\langle k, Ah\rangle = \langle f, h\rangle$. By definition $k \in dom(A^*)$ and $A^*k = f$. ∎

Proposition 5.1.7. *If $A : \mathcal{H} \to \mathcal{K}$ is densely defined, then*

(1) A^* is a closed operator;

(2) A^* is densely defined if and only if A is closable;

(3) if A is closable, then its closure is $(A^*)^* = A^{**}$.

Proof. The proof of (1) is clear from Lemma 5.1.6. For the remainder of the proof notice that because the map J defined above is an isomorphism, $J^* = J^{-1}$ and so $J^*(k \oplus h) = h \oplus (-k)$.

(2) Assume A is closable and let $k_0 \in [dom(A^*)]^{\perp}$. We want to show that $k_0 = 0$. Thus $k_0 \oplus 0 \in [gra(A^*)]^{\perp} = [J(gra(A))]^{\perp\perp} = \overline{J(gra(A))} = J(\overline{gra(A)})$. So $0 \oplus (-k_0) = J^*(k_0 \oplus 0) \in J^*(J(\overline{gra(A)})) = \overline{gra(A)}$. But because A is closable, $\overline{gra(A)}$ is a graph; hence $k_0 = 0$. For the converse, assume $dom(A^*)$ is dense in $\mathcal{K}$. Thus $A^{**} = (A^*)^*$ is defined. By (1), A^{**} is a closed operator. It is easy to see that $A \subseteq A^{**}$, so A has a closed extension.

(3) Note that by Lemma 5.1.6,

$$gra(A^{**}) = [J^*((gra(A^*))]^{\perp} = [J^*([J(gra(A))]^{\perp})]^{\perp}.$$

But for any linear subspace $\mathcal{M}$ and isomorphism J, $[J(\mathcal{M})]^{\perp} = J(\mathcal{M}^{\perp})$. Hence $J^*([J(\mathcal{M})]^{\perp}) = \mathcal{M}^{\perp}$ and, thus, $[J^*([J(\mathcal{M})]^{\perp})]^{\perp} = \mathcal{M}^{\perp\perp} = \overline{\mathcal{M}}$. Putting $\mathcal{M} = gra(A)$ gives that $gra(A^{**}) = \overline{gra(A)}$. ■

Corollary 5.1.8. *If $A \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then $A^* \in \mathcal{C}(\mathcal{K}, \mathcal{H})$ and $A^{**} = A$.*

Example 5.1.9. *Let $(X, \mathcal{F}, \mu)$ be a $\sigma-$finite measure space and let $\phi : X \to \mathbb{C}$ be an $\mathcal{F}-$measurable function. Let $\mathcal{D} = \{f \in \mathcal{L}^2(\mu) : \phi f \in \mathcal{L}^2(\mu)\}$ and define $Af = \phi f$ for all f in $\mathcal{D}$. Then $A \in \mathcal{C}(\mathcal{L}^2(\mu))$, $dom(A^*) = \mathcal{D}$, and $A^* f = \overline{\phi} f$ for all f in $\mathcal{D}$.*

Proposition 5.1.10. *If $A : \mathcal{H} \to \mathcal{K}$ is densely defined, then*

$$(ran\ (A))^{\perp} = ker(A^*).$$

If A is also closed, then

$$(ran\ (A^*))^{\perp} = ker(A).$$

Proof. If $h \perp ran(A)$, then for every f in $dom(A)$, $0 = \langle Af, h\rangle$. Hence $h \in dom(A^*)$ and $A^*h = 0$. The other inclusion is clear. By Corollary 5.1.8, if $A \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, $A^{**} = A$. So the second equality follows from the first. ■

Definition 5.1.11. *If $A : \mathcal{H} \to \mathcal{K}$ is a linear operator, A is boundedly invertible if there is a bounded linear operator $B : \mathcal{K} \to \mathcal{H}$ such that $AB = I_{\mathcal{K}}$ and $BA \subseteq I_{\mathcal{H}}$. We call B to be the inverse of A.*

Proposition 5.1.12. *If $A : \mathcal{H} \to \mathcal{K}$ is a linear operator.*

(1) A is boundedly invertible if and only if $Ker(A) = 0, ran(A) = \mathcal{K}$, and the graph of A is closed.

(2) If A is boundedly invertible, its inverse is unique and denoted by A^{-1}

Proof. (1) Let B be a bounded inverse of A. So $dom(B) = \mathcal{K}$. Since $BA \subseteq I_{\mathcal{H}}$, $Ker(A) = (0)$; since $AB = I_{\mathcal{K}}$, $ran(A) = \mathcal{K}$. Also, $gra(A) = \{h \oplus Ah : h \in dom(A)\} = \{Bk \oplus k : k \in \mathcal{K}\}$. Since B is bounded, $gra(A)$ is closed. Conversely, if A has the stated properties, $Bk = A^{-1}k$ for k in $\mathcal{K}$ is a well-defined operator on $\mathcal{K}$. Because $gra(A)$ is closed, $gra(B)$ is closed. By the Closed Graph Theorem, $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

(2) This is an exercise. ■

Definition 5.1.13. *If $A : \mathcal{H} \to \mathcal{H}$ is a linear operator, denote $\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A$ is boundedly invertible $\}$. The spectrum of A is the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$.*

It is easy to see that if $A : \mathcal{H} \to \mathcal{K}$, is a linear operator and $\lambda \in \mathbb{C}$, $gra(A)$ is closed if and only if $gra(A - \lambda)$ is closed. So if A does not have closed graph, $\sigma(A) = \mathbb{C}$, so $\rho(A) = \emptyset$. Even if A has closed graph, it is possible that $\rho(A)$ is empty. The spectrum of an unbounded operator, however, does enjoy some *of the properties possessed by the spectrum of an element of a Banach algebra. The proofs of the next two results are left to the reader.*

Proposition 5.1.14. *If $A : \mathcal{H} \to \mathcal{H}$ is a linear operator, then $\sigma(A)$ is closed and $z \mapsto (z - A)^{-1}$ is an analytic function on $\rho(A)$.*

Proposition 5.1.15. *Let $A \in \mathcal{C}(\mathcal{H})$.*

(1) $\lambda \in \rho(A)$ if and only if ker$(A - \lambda) = (0)$ and ran$(A - \lambda) = \mathcal{H}$.

(2) $\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$ and for λ in $\rho(A)$, $(A - \lambda)^{*-1} = [(A - \lambda)^{-1}]^*$.

5.2 Symmetric and Self-Adjoint Operators

In this section, we point out the distinction between symmetric and self-adjoint operators that is necessary to make in the theory of unbounded operators.

Definition 5.2.1. *An operator $A : \mathcal{H} \to \mathcal{H}$ is symmetric if A is densely defined and $\langle Af, g\rangle = \langle f, Ag\rangle$ for all f, g in dom(A).*

Proposition 5.2.2. *If A is densely defined, the following statements are equivalent.*

(1) A is symmetric.

(2) $\langle Af, f\rangle \in \mathbb{R}$ for all f in dom(A).

(3) $A \subseteq A^*$.

If A is symmetric, then $A \subseteq A^*$ implies *dom*(A^*) is dense. Hence A is closable.

Definition 5.2.3. *A densely defined operator $A : \mathcal{H} \to \mathcal{H}$ is self-adjoint if $A = A^*$.*

Proposition 5.2.4. *Suppose A is a symmetric operator on $\mathcal{H}$.*

(1) If $ran(A)$ is dense, then A is injective.

(2) If $A = A^*$ and A is injective, then $ran(A)$ is dense and A^{-1} is self-adjoint.

(3) If $dom(A) = \mathcal{H}$, then $A = A^*$ and A is bounded.

(4) If $ran(A) = \mathcal{H}$, then $A = A^*$ and $A^{-1} \in \mathcal{B}(\mathcal{H})$.

Proof. The proof of (1) is trivial and (2) is an easy consequence of Proposition 5.1.10 and some manipulation.

(3) we have $A \subseteq A^*$. If $dom(A) = \mathcal{H}$, then $A = A^*$ and so A is closed. By the Closed Graph Theorem $A \in \mathcal{B}(\mathcal{H})$.

(4) If $ran(A) = \mathcal{H}$, then A is injective by (1). Let $B = A^{-1}$ with $dom(B) = ran(A) = \mathcal{H}$. If $f = Ag$ and $h = Ak$, with g, h in $dom(A)$, then $\langle Bf, h\rangle = \langle g, Ak\rangle = \langle Ag, k\rangle = \langle f, k\rangle = \langle f, Bh\rangle$. Hence B is symmetric. By (3), $B = B^* \in \mathcal{B}(\mathcal{H})$. By (2), $A = B^{-1}$ is self-adjoint. ∎

We now will turn our attention to the spectral properties of symmetric and self-adjoint operators.

Proposition 5.2.5. *Let A be a symmetric operator and let $\lambda = \alpha + i\beta, \alpha$ and β real numbers.*

(1) For each f in $dom(A)$, $\|(A-\lambda)f\|^2 = \|(A-\alpha)f\|^2 + \beta^2\|f\|^2$.

(2) If $\beta \neq 0$, $ker(A-\lambda) = (0)$.

(3) If A is closed and $\beta \neq 0$, then $ran(A-\lambda)$ is closed.

Proof. Note that

$$\begin{aligned}
\|(A-\lambda)f\|^2 &= \|(A-\alpha)f - i\beta f\|^2 \\
&= \|(A-\alpha)f\|^2 + 2Rei\langle (A-\alpha)f, \beta f\rangle + \beta^2\|f\|^2.
\end{aligned}$$

But

$$\langle (A-\alpha)f, \beta f\rangle = \beta\langle Af, f\rangle - \alpha\beta\|f\|^2 \in \mathbb{R},$$

so (1) follows. Part (2) is immediate from (1). To prove (3), note that $\|(A-\lambda)f\|^2 \geq \beta^2\|f\|^2$. Let $(f_n) \subseteq dom(A)$ such that $(A-\lambda)f_n \to g$. The preceding inequality implies that (f_n) is a Cauchy sequence in $\mathcal{H}$; let $f = \lim f_n$. But $f_n \oplus (A-\lambda)f_n \in gra(A-\lambda)$ and $f_n \oplus (A-\lambda)f_n \to f \oplus g$. Hence $f \oplus g \in gra(A-\lambda)$ and so $g = (A-\lambda)f \in ran(A-\lambda)$. This prove (3).■

Lemma 5.2.6. *If $\mathcal{M}, \mathcal{N}$ are closed subspaces of $\mathcal{H}$ and $\mathcal{M} \cap \mathcal{N}^\perp = (0)$, then $\dim \mathcal{M} \leq \dim \mathcal{N}$.*

Proof. Let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}$ and define $T : \mathcal{M} \to \mathcal{N}$ by $Tf = Pf$ for f in $\mathcal{M}$. Since $\mathcal{M} \cap \mathcal{N}^\perp = 0$, T is injective. If $\mathcal{L}$ is a finite dimensional subspace of $\mathcal{M}$, $\dim \mathcal{L} = \dim T\mathcal{L} \leq \dim \mathcal{N}$. Since $\mathcal{L}$ was arbitrary, $\dim \mathcal{M} \leq \dim \mathcal{N}$. ■

Theorem 5.2.7. *If A is a closed symmetric operator, then $\dim ker(A^* - \lambda)$ is constant for $Im\lambda > 0$ and constant for $Im\lambda < 0$.*

Proof. Let $\lambda = \alpha + i\beta$, α and β real numbers and $\beta \neq 0$.

> **Claim.** $If\ |\lambda - \mu| < |\beta|, \quad ker(A^* - \mu) \cap [\ker(A^* - \lambda)]^\perp = (0).$

Suppose this is not so. Then there is an f in $ker(A^* - \mu) \cap [ker(A^* - \lambda)]^\perp$ with $\|f\| = 1$. By Proposition 5.2.5, $ran(A - \overline{\lambda})$ is closed. Hence $f \in [ker(A^* - \lambda)]^\perp = ran(A - \overline{\lambda})$. Let $g \in dom(A)$ such that $f = (A - \overline{\lambda})g$. Since $f \in$

$ker(A^* - \mu)$,

$$\begin{aligned} 0 &= \langle (A^* - \mu)f, g\rangle = \langle f, (A - \overline{\mu})g\rangle \\ &= \langle f, (A - \overline{\lambda} + \overline{\lambda} - \overline{\mu})g\rangle \\ &= \|f\| + (\lambda - \mu)\langle f, g\rangle. \end{aligned}$$

Hence $1 = \|f\| = |\lambda - \mu||\langle f, g\rangle| \leq |\lambda - \mu|\|g\|$. But Proposition 5.2.5 implies that $1 = \|f\| = \|(A - \overline{\lambda})g\| \geq |\beta|\|g\|$; so $\|g\| \leq |\beta|^{-1}$. Hence $1 \leq |\lambda - \mu|\|g\| \leq |\lambda - \mu||\beta|^{-1} < 1$ if $|\lambda - \mu| < |\beta|$. This contradiction establishes the claim.

Combining the claim with Lemma 5.2.6 gives that $\dim ker(A^* - \mu) \leq \dim ker(A^* - \lambda)$ if $|\lambda - \mu| < |\beta| = |Im\lambda|$. Note that if $|\lambda - \mu| < \frac{1}{2}|\beta|$, then $|\lambda - \mu| < |Im\mu|$, so that the other inequality also holds. This shows that the function $\lambda \mapsto \dim ker(A^* - \lambda)$ is locally constant on $\mathbb{C} \setminus \mathbb{R}$. A simple topological argument demonstrates the theorem. ■

Theorem 5.2.8. *If A is a closed symmetric operator, then one and only one of the following possibilities occurs:*

(1) $\sigma(A) = \mathbb{C}$;

(2) $\sigma(A) = \{\lambda \in \mathbb{C} : Im\lambda \geq 0\}$;

(3) $\sigma(A) = \{\lambda \in \mathbb{C} : Im\lambda \leq 0\}$;

(4) $\sigma(A) \subseteq \mathbb{R}$.

Proof. Let $H_\pm = \{\lambda \in \mathbb{C} : \pm Im\lambda > 0\}$. By Proposition 5.2.5 for λ in $H_\pm$, $A - \lambda$ is injective and has closed range. So if $A - \lambda$ is surjective, $\lambda \in \rho(A)$. But $[ran(A - \lambda)]^\perp = ker(A^* - \overline{\lambda})$. So the preceding theorem implies that either $H_\pm \subseteq \sigma(A)$ or $H_\pm \cap \sigma(A) = \emptyset$. Since $\sigma(A)$ is closed, if $H_\pm \subseteq \sigma(A)$, then either $\sigma(A) = \mathbb{C}$ or $\sigma(A) = \overline{H_\pm}$. If $H_\pm \cap \sigma(A) = \emptyset$, $\sigma(A) \subseteq \mathbb{R}$. ■

Corollary 5.2.9. *If A is a closed symmetric operator, the following statements are equivalent:*

(1) A is self-adjoint.

(2) $\sigma(A) \subseteq \mathbb{R}$.

(3) $ker(A^* - i) = ker(A^* + i) = (0)$.

Proof. If A is symmetric, every eigenvalue of A is real. So if $A = A^*$ and $Im\lambda \neq 0$, $ker(A^* - \lambda) = ker(A^* + \lambda) = (0)$. Thus $A - \lambda$ is injective and has dense range. By Propositin 5.2.5, $A - \lambda$ has a bounded inverse whenever $Im\lambda \neq 0$. That is, $\sigma(A) \subseteq \mathbb{R}$ and so (1) implies (2).

If $\sigma(A) \subseteq \mathbb{R}$, $ker(A^* \pm i) = [ran(A \mp i)]^\perp = \mathcal{H}^\perp$ =0. Hence (2) implies (3).

If (3) holds, then this, combined with Propositions 5.2.5 and 5.1.10, implies $A + i$ is surjective. Let $h \in dom(A^*)$. Then there is an f in $dom(A)$ such that $(A + i)f = (A^* + i)h$. But $(A + i) \subseteq (A^* + i)$, so $(A^* + i)f = (A^* + i)h$. But $(A^* + i)$ is injective, so $h = f \in dom(A)$. Thus $A = A^*$. ■

Corollary 5.2.10. *If A is a closed symmetric operator and $\sigma(A)$ does not contain $\mathbb{R}$, then $A = A^*$.*

This can follow from Theorem 3.2.9 immediately.

5.3 The Cayley Transform

In this section, we introduce an important transform, the Cayley transform.

Definition 5.3.1. *Let A be a closed symmetric operator. The deficiency subspaces of A are the spaces*

$$\Psi_+ = ker(A^* - i) = [ran(A + i)]^\perp,$$

$$\Psi_- = ker(A^* + i) = [ran(A - i)]^\perp.$$

The deficiency indices of A are the numbers $n_\pm = \dim \Psi_\pm$.

Theorem 5.3.2. *(1) If A is a closed symmetric operator with deficiency subspace $\Psi_\pm$, and if $U : \mathcal{H} \to \mathcal{H}$ is defined by letting $U = 0$ on Ψ_+ and*

$$U = (A - i)(A + i)^{-1}, \tag{3.1}$$

on $\Psi_+^\perp$, then U is a partial isometry with initial space $\Psi_+^\perp$, final space $\Psi_-^\perp$, and such that $(I - U)(\Psi_+^\perp)$ is densely in $\mathcal{H}$.

(2) If U is a partial isometry with initial space and final spaces $\mathcal{M}$ and $\mathcal{N}$, respectively, and such that $(I - U)\mathcal{M}$ is dense in $\mathcal{H}$, then

$$A = i(I + U)(I - U)^{-1} \tag{3.2}$$

is a closed symmetric operator with deficiency subspaces $\Psi_+ = \mathcal{M}^\perp$ and $\Psi_- = \mathcal{N}^\perp$

(3) If A is given as in (1) and U is defined by (3.1), then A and U satisfy (3.2). If U is given as in (2) and A is defined by (3.2), then A and U satisfy (3.1).

Proof. By Proposition 5.2.5, $ran(A \pm i)$ is closed and so $\Psi_\pm^\perp = ran(A \pm i)$. By Proposition 5.2.5 again, $ker(A + i) = 0$, so $(A + i)^{-1}$ is well defined on $\Psi_+^\perp$. Moreover, $(A + i)^{-1}\Psi_+^\perp \subseteq dom(A)$, so that U defined by (3.1) makes sense and gives a well-defined operator. If $h \in \Psi_+^\perp$, then $h = (A + i)f$ for a unique f in $dom(A)$. Hence $\|Uh\|^2 = \|(A - i)f\|^2 = \|Af\|^2 + \|f\|^2 = \|(A + i)f\|^2 = \|h\|^2$. Hence U is a partial isometry, $[ker(U)]^\perp = \Psi_+^\perp$, and $ran(U) = \Psi_-^\perp$. Once again, if $f \in dom(A)$ and $h = (A + i)f$, then $(I - U)h = h - (A - i)f = (A + i)f - (A - i)f = 2if$. So $(I - U)\Psi_+^\perp = domA$ and is dense in $\mathcal{H}$.

(2) Now assume that U is a partial isometry as in (2). It follows that $ker(I-U)=(0)$. In fact, if $f\in ker(I-U)$, then $Uf=f$; so $\|f\|=\|Uf\|$ and hence f in the initial space of U. Since U^*U is the projection onto initial space of U, $f=U^*Uf=U^*f$; so $f\in ker(I-U^*)=[ran(I-U)]^{\perp}\subseteq[(I-U)\mathcal{M}]^{\perp}=(0)$ by hypothesis. Thus $f=0$ and $I-U$ is injective.

Let $\mathcal{D}=(I-U)\mathcal{M}$ and define $(I-U)^{-1}$ on $\mathcal{D}$. Because $I-U$ is bounded, $gra(I-U)^{-1}$ is closed. If A is defined as in (3.2), it follows that A is a closed densely defined operator. If $f,g\in\mathcal{D}$, let $f=(I-U)h$ and $g=(I-U)k$, $h,k\in\mathcal{M}$. Hence

$$\begin{aligned}\langle Af,g\rangle &= i\langle (I+U)h,(I-U)k\rangle\\ &= i[\langle h,k\rangle+\langle Uh,k\rangle-\langle h,Uk\rangle-\langle Uh,Uk\rangle].\end{aligned}$$

Since $h,k\in\mathcal{M}$, $\langle Uf,Uk\rangle=\langle h,k\rangle$; hence $\langle Af,g\rangle=i[\langle Uh,k\rangle-\langle h,Uk\rangle]$. Similarly, $\langle f,Ag\rangle=-i\langle (I-U)h,(I+U)k\rangle=-i[\langle h,Uk\rangle-\langle Uh,k\rangle]=\langle Af,g\rangle$. Hence A is symmetric.

Finally, if $h\in\mathcal{M}$ and $f=(I-U)h$, then $(A+i)f=Af+if=i(I+U)h+i(I-U)h=2ih$. Thus $ran(A+i)=\mathcal{M}$. Similarly, $(A-i)f=i(I+U)h-i(I-U)h=2Uh$, so that $ran(A-i)=ran(U)=\mathcal{N}$.

(3) Suppose A is as in (1) and U is defined as in (3.1). If $g\in(I-U)\Psi_+^{\perp}$, put $g=(I-U)h$, where $h\in\Psi_+^{\perp}=ran(A+i)$. Hence $h=(A+i)f$ for some f in $dom(A)$. Thus $g=h-Uh=(A+i)f-(A-i)f=2if$; so $f=-\frac{1}{2}ig$. Also,

$$\begin{aligned}i(I+U)(I-U)^{-1}g &= i(I+U)h=i(h+Uh)\\ &= i[(A+i)f+(A-i)f]=2iAf=Ag.\end{aligned}$$

Therefore (3.2) holds.

The proof of the remainder of (3) is left to the reader. ■

Usually, U defined by (3.1) is called the Cayley transform of A.

Corollary 5.3.3. *If A is a self-adjoint operator and U is its Cayley transform, then U is a unitary operator with $ker(I-U) = 0$. Conversely, if U is a unitary operator and 1 is not its eigenvalue, then the operator A defined by (3.2) is self-adjoint.*

Proof. Since A is a symmetric operator, then A is self-adjoint if and only if $\Psi_{\pm} = (0)$ by Corollary 3.2.9. A partial isometry is a unitary operator if and only if its intitial and final spaces are all of $\mathcal{H}$. The corollary is now seen to follow from Theorem 3.3.2. ■

Ones use of the Cayley transform is to study self-adjoint operators by using the theory of unitary operators . Indeed, the preceding results say that there is a bijective correspondence between self-adjoint operators and the set of unitary operators does not with 1 as an eigenvalue.

5.4 Unbounded Normal Operator Spectral Theory

In this section, we study the spectral theorem of unbounded normal operators.

Definition 5.4.1. *A linear operator N on $\mathcal{H}$ is said to be normal if N is closed, densely defined, and $N^*N = NN^*$.*

Proposition 5.4.2. *If $A \in \mathcal{C}(\mathcal{H})$, then*

(1) $I + A^*A$ has a bounded inverse defined on all of $\mathcal{H}$.

(2) If $B = (I + A^*A)^{-1}$, then $\|B\| \leq 1$ and $B \geq 0$.

(3) The operator $C = A(I + A^*A)^{-1}$ is a contraction.

(4) A^*A is self-adjoint.

(5) $\{h \oplus Ah : h \in dom(A^*A)\}$ is dense in $gra(A)$.

Proof. Define $J : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ by $J(h \oplus k) = (-k) \oplus h$. By Lemma 5.1.6, $gra(A^*) = [J(gra(A))]^{\perp}$. So if $h \in \mathcal{H}$, there are $f \in dom(A)$ and $g \in dom(A^*)$ such that $0 \oplus h = J(f \oplus Af) + g \oplus A^*g = (-Af) \oplus f + g \oplus A^*g$. Hence $0 = -Af + g$, or $g = Af$; also, $h = f + A^*g = f + A^*Af = (I + A^*A)f$. Thus $ran(I + A^*A) = \mathcal{H}$.

Also, for f in $dom(A^*A)$, $Af \in dom(A^*)$ and $\|f + A^*Af\|^2 = \|f\|^2 + 2\|Af\|^2 + \|A^*Af\|^2 \geq \|f\|^2$. Hence $ker(I + A^*A) = (0)$. Thus $(I + A^*A)^{-1}$ exists and is defined on all of $\mathcal{H}$. In the next paragraph it will be shown that $(I + A^*A)^{-1}$ is a contraction, completing the proof of (1).

It was shown that $\|(I + A^*A)f\| \geq \|f\|$ whenever $f \in dom(A^*A)$. If $h = (1 + A^*A)f$ and $B = (1 + A^*A)^{-1}$, then this implies that $\|Bh\| \leq \|h\|$. Hence $\|B\| \leq 1$. In addition, $\langle Bh, h\rangle = \langle f, (I + A^*A)f\rangle = \|f\|^2 + \|Af\|^2 \geq 0$, so (2) holds.

Put $C = A(I + A^*A)^{-1} = AB$; if $f \in dom(A^*A)$, and $(I + A^*A)f = h$, then $\|Ch\|^2 = \|Af\|^2 \leq \|(I + A^*A)f\|^2 = \|h\|^2$ by the argument used to prove (1). Hence $\|C\|^2 \leq 1$, so (3) is proved.

Now to prove (5). Since A is closed, it suffices to show that no nonzero vector in $gra(A)$ is orthogonal to $\{h \oplus Ah : h \in dom(A^*A)\}$. So let $g \in dom(A)$ and suppose that for every h in $dom(A^*A)$,

$$\begin{aligned} 0 &= \langle g \oplus Ag, h \oplus Ah \rangle \\ &= \langle g, h \rangle + \langle Ag, Ah \rangle \\ &= \langle g, h \rangle + \langle g, A^*Ah \rangle \\ &= \langle g, (I + A^*A)h \rangle. \end{aligned}$$

So $g \perp ran(I + A^*A) = \mathcal{H}$; hence $g = 0$.

To prove (4), note that (5) implies that $dom(A^*A)$ is dense. Now let $f, g \in dom(A^*A)$; so $f, g \in dom(A)$ and $Af, Ag \in dom(A^*)$. Hence $\langle A^*Af, g \rangle = \langle Af, Ag \rangle = \langle f, A^*Ag \rangle$. Thus A^*A is symmetric. Also $1 + A^*A$ has a bounded inverse. This implies two things. First, $I + A^*A$ is closed, and so A^*A is closed. Also, $-1 \notin \sigma(A^*A)$ so that by Corollary 5.2.10, A^*A is self-adjoint. ∎

Proposition 5.4.3. *If N is a normal operator, then $dom(N) = dom(N^*)$ and $\|Nf\| = \|N^*f\|$ for every f in $dom(N)$.*

Proof. First observe that if $h \in dom(N^*N) = dom(NN^*)$, then $Nh \in dom(N^*)$ and $N^*h \in dom(N)$. Hence $\|Nh\|^2 = \langle N^*Nh, h \rangle = \langle NN^*h, h \rangle = \|N^*h\|$. Now if $f \in dom(N)$, Proposition 5.4.2 implies that there is a sequence (h_n) in $dom(N^*N)$ such that $h_n \oplus Nh_n \to f \oplus Nf$; so $\|Nh_n - Nf\| \to 0$. But from the first part of this proof, $\|N^*h_n - N^*h_m\| = \|Nh_n - Nh_m\|$. So there is an g in $\mathcal{H}$ such that $N^*h_n \to g$. Thus $h_n \oplus N^*h_n \to f \oplus g$. But N^* is closed; thus $f \in dom(N^*)$ and $g = N^*f$. So $dom(N) \subseteq dom(N^*)$ and $\|Nf\| = \lim \|Nh_n\| = \lim \|N^*h_n\| = \|N^*f\|$.

On the other hand, N^* is normal, and so $dom(N^*) \subseteq dom(N^{**}) = dom(N)$.

∎

Proposition 5.4.4. *Let $\mathcal{H}_1,\mathcal{H}_2,\cdots$ be Hilbert spaces and let $A_n \in \mathcal{B}(\mathcal{H}_n)$ for all $n \geq 1$. If $\mathcal{D} = \{(h_n) \in \oplus_n \mathcal{H}_n : \sum_{n=1}^{\infty} \|A_n h_n\|^2 < \infty\}$ and A is defined on $\mathcal{H} = \oplus_n \mathcal{H}_n$ by $A(h_n) = (A_n h_n)$ whenever $(h_n) \in \mathcal{D}$, then $A \in \mathcal{C}(\mathcal{H})$. A is a normal operator if and only if each A_n is normal.*

Proof. Since $\mathcal{H}_n \subseteq \mathcal{D}$ for each n, $\mathcal{D}$ is dense in $\mathcal{H}$. Clearly A is linear. If $(h^{(j)}) \subseteq dom(A)$ and $h^{(j)} \oplus Ah^{(j)} \to h \oplus g$ in $\mathcal{H} \oplus \mathcal{H}$, then for each n, $h_n^{(j)} \oplus A_n h_n^{(j)} \to h_n \oplus g_n$. Since A_n is bounded, $A_n h_n = g_n$. Hence $\sum_n \|A_n h_n\|^2 = \sum_n \|g_n\|^2 = \|g\|^2 < \infty$, so $h \in dom(A)$. Clearly $Ah = g$, so $A \in \mathcal{C}(\mathcal{H})$.

It is left to the reader to show that $dom(A^*) = \{(h_n) \in \mathcal{H} : \sum_{n=1}^{\infty} \|A_n^* h_n\|^2 < \infty\}$ and $A^*(h_n) = (A_n^* h_n)$ when $(h_n) \in dom(A^*)$. From this the rest of the proposition easily follows. ∎

Let $(X, \mathcal{F})$ be a measurable space and $\mathcal{H}$ be a Hilbert space, E be a spectral measure relative to $(X, \mathcal{F}, \mathcal{H})$, $h, k \in \mathcal{H}$, $E_{h,k}$ be the complex-valued measure given by $E_{h,k}(S) = \langle E(S)h, k\rangle$ for all sets $S \in \mathcal{F}$. If $\phi : X \to \mathbb{C}$ is an $\mathcal{F}-$measurable function and for each n let $\Delta_n = \{x \in X : n-1 \leq |\phi(x)| < n\}$, then $\chi_{\Delta_n}\phi$ is a bounded $\mathcal{F}-$measurable function. Put $\mathcal{H}_n = E(\Delta_n)\mathcal{H}$. Since $\cup_{n=1}^{\infty}\Delta_n = X$ and the set $\{\Delta_n\}$ are pairwise disjoint, $\oplus_{n=1}^{\infty}\mathcal{H}_n = \mathcal{H}$. If $E_n(S) = E(S \cap \Delta_n)$, then E_n is a spectral measure for $(X, \mathcal{F}, \mathcal{H}_n)$. Also, $\int \phi dE_n$ is a normal operator on $\mathcal{H}_n$. Define

$$\mathcal{D}_\phi = \{h \in \mathcal{H} : \sum_{n=1}^{\infty} \|(\int \phi dE_n)E(\Delta_n)h\|^2 < \infty\}.$$

By Proposition 4.4.4, $N_\phi : \mathcal{H} \to \mathcal{H}$ given by

$$N_\phi h = \sum_{n=1}^{\infty}(\int \phi dE_n)E(\Delta_n)h$$

for h in $\mathcal{D}_\phi$ is a normal operator. The operator N_ϕ is also denoted by

$$N_\phi = \int \phi dE.$$

Note the fact that $dom(N_\phi) = \mathcal{D}_\phi$.

In order to prove the following theorem, we first need the following fact:

Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space and μ be a complex valued measure on $\mathcal{F}$ and $f \in L(X, \nu)$ and for each $A \in \mathcal{F}$, $\mu(A) = \int_A f d\nu$, then we have

$$|\mu|(A) = \int_A |f| d\nu,$$

where $|\mu|$ is the variation measure of μ.

In fact, let $\{A_i\}$ be a partition of A, then $\sum_i |\mu(A_i)| \leq \sum_i \int_{A_i} |f| d\nu$. So $|\mu|(A) \leq \int_A |f| d\nu$. For the reverse inequality, let $g(x) = \frac{|f(x)|}{f(x)}$ if $x \in A$ and $f(x) \neq 0$; and $g(x) = 0$ if $f(x) = 0$ or x off A. Let $\{g_n\}$ be a sequence of $\mathcal{F}$-measureable simple functions such that $g_n = 0$ off A, $|g_n| \leq |g| \leq 1$, and $g_n \to g$ everywhere. Thus, $f g_n \to |f| \chi_A$. By the Lebesgue Dominated Convergent Theorem, $\int f g_n d\nu \to \int_A |f| d\nu$. If $g_n = \sum_i a_i \chi_{A_i}$, where $|a_i| \leq 1$ and $\{A_i\}$ is a partition of A, then $|\int f g_n d\nu| = |\int g_n d\mu| \leq |\mu|(A)$.

Moreover, we have also the fact:

Let μ be a σ-finite complex valued measure on $\mathcal{F}$. Then the Radon-Nikodym derivative of μ with respect to $|\mu|$, satisfies $|\frac{d\mu}{d|\mu|}| = 1$ almost everywhere with respect to $|\mu|$.

In fact, applied the above conclusion in the case where $f = \frac{d\mu}{d|\mu|}$ and $\nu = |\mu|$, then

$$|\mu|(A) = \int_A |f| d|\mu|$$

holds for each $A \in \mathcal{F}$. Thus, $|\frac{d\mu}{d|\mu|}|$ is a Radon-Nikodym derivative of $|\mu|$ with respect to $|\mu|$. Since the constant 1 is another such Radon-Nikodym derivative, it follows that $|\frac{d\mu}{d|\mu|}| = 1$ almost everywhere with respect to $|\mu|$.

Theorem 5.4.5. *Let $(X, \mathcal{F})$ be a measurable space and $\mathcal{H}$ be a Hilbert space, E be a spectral measure relative to $(X, \mathcal{F}, \mathcal{H})$, $\phi : X \to \mathbb{C}$ be a complex $\mathcal{F}-$measurable function, and $\mathcal{D}_\phi$ be defined as above. Then:*

(1) $\mathcal{D}_\phi = \{h \in \mathcal{H} : \int |\phi|^2 dE_{h,h} < \infty\}$;

(2) for h in $\mathcal{D}_\phi$ and f in $\mathcal{H}$, $\phi \in L^1(|E_{h,f}|)$ with

$$\int |\phi| d|E_{h,f}| \leq \|f\| (\int |\phi|^2 dE_{h,h})^{1/2}, \tag{3.3}$$

$$\langle (\int \phi dE) h, f \rangle = \int \phi dE_{h,f}, \tag{3.4}$$

and

$$\|(\int \phi dE) h\|^2 = \int |\phi|^2 dE_{h,h}.$$

Proof. Note that

$$\begin{aligned} \|(\int \phi dE_n) E(\Delta_n) h\|^2 &= \langle (\int \chi_{\Delta_n} \phi dE)^* (\int \chi_{\Delta_n} \phi dE) h, h \rangle \\ &= \int_{\Delta_n} |\phi|^2 dE_{h,h}. \end{aligned}$$

From here, let $n \to \infty$, (1) is immediate.

Now let $h \in \mathcal{D}_\phi$, $f \in \mathcal{H}$. By the Radon-Nikodym Theorem, there is an $\mathcal{F}-$measurable function u such that $|u| \equiv 1$ and $dE_{h,f} = ud|E_{h,f}|$, that is, for each $A \in \mathcal{F}$, $E_{h,f}(A) = \int_A ud|E_{h,f}|$, where $|E_{h,f}|$ is the variation for $E_{h,f}$. Let $\phi_n = \sum_{k=1}^n \chi_{\Delta_k} \phi$; so ϕ_n is bounded. Thus

$$\begin{aligned}\int |\phi_n| d|E_{h,f}| &= \int |\phi_n|\overline{u}u d|E_{h,f}| \\ &= \int |\phi_n|\overline{u} dE_{h,f} \\ &= \langle (\int |\phi_n|\overline{u} dE)h, f\rangle \\ &\leq \|f\| \|(\int |\phi_n|\overline{u} dE)h\|.\end{aligned}$$

But

$$\begin{aligned}\|(\int |\phi_n|\overline{u} dE)h\|^2 &= \langle (\int |\phi_n|\overline{u} dE)h, (\int |\phi_n|\overline{u} dE)h\rangle. \\ &= \langle (\int |\phi_n|^2 dE)h, h\rangle. \\ &= \int |\phi_n|^2 dE_{h,h} \\ &\leq \int |\phi|^2 dE_{h,h}.\end{aligned}$$

Combining this with the preceding inequality gives that

$$\int |\phi_n| d|E_{h,f}| \leq \|f\| (\int |\phi|^2 dE_{h,h})^{1/2}$$

for all n. Letting $n \to \infty$ give (3.3). Since ϕ_n is bounded, $\langle (\int \phi_n dE)h, f\rangle = \int \phi_n dE_{h,f}$. If $h \in \mathcal{D}_\phi$ and $f \in \mathcal{H}$, then (3.3) and the Lebesgue Dominated Convergence Theorem imply that $\int \phi_n dE_{h,f} \to \int \phi dE_{h,f}$ as $n \to \infty$. But

$$\begin{aligned}(\int \phi_n dE)h &= (\int \phi dE) E(\cup_{j=1}^n \Delta_j)h \\ &= E(\cup_{j=1}^n \Delta_j)(\int \phi dE)h.\end{aligned}$$

Since $E(\cup_{j=1}^{n}\Delta_j) \to E(X) = I$ in the strong operator topology as $n \to \infty$, $\langle (\int \phi_n dE)h, f\rangle \to \langle (\int \phi dE)h, f\rangle$ as $n \to \infty$. This prove (3.4). ∎

Note that as a consequence of Theorem 5.4.5, $dom(N_\phi)$ and the definition of N_ϕ do not depend on the choice of the sets $\{\Delta_n\}$.

Theorem 5.4.6 (Functional Calculus). *Let $(X, \mathcal{F})$ be a measurable space and $\mathcal{H}$ be a Hilbert space, E be a spectral measure relative to $(X, \mathcal{F}, \mathcal{H})$, $\Phi(X, \mathcal{F})$ be the algebra of all complex $\mathcal{F}-$measurable functions, we define*

$$F : \Phi(X, \mathcal{F}) \to \mathcal{C}(\mathcal{H}) \quad by \quad F(\phi) = N_\phi = \int \phi dE,$$

F is said to be the Functional Calculus. For any ϕ, ψ in $\Phi(X, \mathcal{F})$, we have

(1) $\mathcal{D}_\phi \cap \mathcal{D}_\psi \subseteq \mathcal{D}_{\phi+\psi}$, $F(\phi) + F(\psi) \subseteq F(\phi + \psi)$, and if one of ϕ and ψ is bounded, then $F(\phi) + F(\psi) = F(\phi + \psi)$;

(2) $dom(F(\phi)F(\psi)) = \mathcal{D}_\psi \cap \mathcal{D}_{\phi\psi}$ and $F(\phi\psi) \supseteq F(\phi)F(\psi)$;

(3) If ψ is bounded, then $F(\phi)F(\psi) = F(\psi)F(\phi) = F(\phi\psi)$;

(4) $F(\phi)^* = F(\overline{\phi})$, $F(\phi)^*F(\phi) = F(|\phi|^2)$.

Proof. We only prove (2) and (4). First suppose ϕ is bounded, then $\mathcal{D}_{\phi\psi} \supseteq \mathcal{D}_\psi$. Let $x \in \mathcal{D}_\psi$, $z \in \mathcal{H}$, $v = F(\overline{\phi})z$, it follows from Theorem 3.4.5 that

$$\begin{aligned}
\langle F(\phi)F(\psi)x, z\rangle &= \langle F(\psi)x, F(\phi)^* z\rangle \\
&= \langle F(\psi)x, F(\overline{\phi})z\rangle \\
&= \langle F(\psi)x, v\rangle = \int \psi(\lambda) dE_{x,v} \\
&= \int \psi(\lambda)\phi(\lambda) dE_{x,z} \\
&= \langle F(\phi\psi)x, z\rangle.
\end{aligned}$$

Thus, for any $x \in \mathcal{D}_\psi$, we have

$$F(\phi)F(\psi)x = F(\phi\psi)x.$$

Now let $y = F(\psi)x$, then by Theorem 3.4.5,

$$\begin{aligned}
\int |\phi(\lambda)|^2 dE_{y,y}(\lambda) &= \|F(\phi)y\|^2 \\
&= \|F(\phi)F(\psi)x\|^2 \\
&= \|F(\phi\psi)x\|^2 \\
&= \int |\phi(\lambda)\psi(\lambda)|^2 dE_{x,x}(\lambda).
\end{aligned}$$

Next suppose ϕ is a general measurable function, let $\phi_n = \sum_{k=1}^n \chi_{\Delta_k}\phi$, where $\Delta_k = \{x \in X : k-1 \leq |\phi(x)| < k\}$. Obviously, ϕ_n is bounded, and $\phi_n \to \phi$ as $n \to \infty$. So

$$\int |\phi_n(\lambda)|^2 dE_{y,y}(\lambda) = \int |\phi_n(\lambda)\psi(\lambda)|^2 dE_{x,x}(\lambda), n = 1, 2, 3, \cdots$$

Note that $|\phi_n|^2 \nearrow |\phi|^2$, $|\phi_n\psi|^2 = |\phi_n|^2|\psi|^2 \nearrow |\phi|^2|\psi|^2 = |\phi\psi|^2$. By Levi Theorem,

$$\int |\phi(\lambda)|^2 dE_{y,y}(\lambda) = \int |\phi(\lambda)\psi(\lambda)|^2 dE_{x,x}(\lambda). \tag{3.5}$$

Since

$$\begin{aligned}
dom(F(\phi)F(\psi)) &= \{x \in dom(F(\psi)) : F(\psi)x \in dom(F(\phi))\} \\
&= \{x \in \mathcal{D}_\psi : F(\psi)x \in \mathcal{D}_\phi\}.
\end{aligned}$$

So it follows from (3.5) that $y \in \mathcal{D}_\phi \iff x \in \mathcal{D}_{\phi\psi}$. Thus, we have

$$dom(F(\phi)F(\psi)) = \mathcal{D}_\psi \cap \mathcal{D}_{\phi\psi}.$$

If $x \in \mathcal{D}_\psi \cap \mathcal{D}_{\phi\psi}$, since $\phi_n \to \phi$, $\phi_n\psi \to \phi\psi$ in $L^2(E_{x,x})$, and $F(\phi_n)F(\psi)x = F(\phi_n\psi)x, n = 1, 2, \cdots$, we have

$$F(\phi)F(\psi)x = \lim_n F(\phi_n)F(\psi)x = \lim_n F(\phi_n\psi)x = F(\phi\psi)x.$$

Thus, (2) is proved.

Now, we prove (4). Note that for given measurable function ϕ, $\mathcal{D}_\phi = \mathcal{D}_{\overline{\phi}}$. Let $\{\phi_n\}$ defined as above, $x \in \mathcal{D}_\phi$ and $y \in \mathcal{D}_{\overline{\phi}}$, we have

$$\langle F(\phi_n)x, y\rangle = \langle x, F(\phi_n)^* y\rangle = \langle x, F(\overline{\phi_n})y\rangle = \overline{\langle F(\overline{\phi_n})y, x\rangle},$$

so

$$\langle F(\phi)x, y\rangle = \overline{\langle F(\overline{\phi})y, x\rangle} = \langle x, F(\overline{\phi})y\rangle.$$

This implies $y \in dom(F(\phi)^*)$, and $F(\phi)^* y = F(\overline{\phi})y$, thus $F(\overline{\phi}) \subseteq F(\phi)^*$.

For the converse, It is enough to show that if $z \in dom(F(\phi)^*)$, then $z \in dom(F(\overline{\phi})) = \mathcal{D}_{\overline{\phi}} = \mathcal{D}_\phi$, that is

$$\int |\phi(\lambda)|^2 dE_{z,z}(\lambda) < \infty.$$

Consider $\{f_n\}$, where f_n is defined by

$$f_n = \begin{cases} 1, & (|\phi(\lambda)| \leq n), \\ 0, & (|\phi(\lambda|) > n). \end{cases}$$

Obviously, $\phi_n = f_n\phi, n = 1, 2, \cdots$. It follows from (3) that

$$F(\phi_n) = F(f_n)F(\phi), \ n = 1, 2, \cdots ,$$

where $\{F(f_n)\}$ is a sequence of bounded self-adjoint operators. Therefore, it is easy to show that

$$F(\overline{\phi_n}) = F(\phi_n)^* = [F(\phi)F(f_n)]^* \supseteq F(f_n)^* F(\phi)^* = F(f_n)F(\phi)^*.$$

Set $v = F(\phi)^* z$. Then for each $n = 1, 2, \cdots$,

$$F(f_n)v = F(f_n)F(\phi)^* z = F(\overline{\phi_n})z,$$

so

$$\int |\phi_n(\lambda)|^2 dE_{z,z}(\lambda) = \|F(\overline{\phi_n})z\|^2 = \|F(f_n)v\|^2 \leq \|F(f_n)\|^2\|v\|^2 \leq \|v\|^2.$$

Let $n \to \infty$, we have

$$\int_{\mathbb{C}} |\phi(\lambda)|^2 dE_{z,z}(\lambda) \leq \|v\|^2 < \infty.$$

This showed that $z \in \mathcal{D}_\phi = \mathcal{D}_{\overline{\phi}} = dom(F(\overline{\phi}))$. Thus $F(\overline{\phi}) = F(\phi)^*$.

Suppose $x \in \mathcal{D}_{|\phi|^2} = dom(F(|\phi|^2))$, then $\int |\phi(\lambda)|^4 dE_{x,x}(\lambda) < \infty$. So

$$\int |\phi(\lambda)|^2 dE_{x,x}(\lambda) \leq [\int |\phi(\lambda)|^4 dE_{x,x}(\lambda)]^{1/2}[\int dE_{x,x}(\lambda)]^{1/2} < \infty,$$

that is, $x \in \mathcal{D}_\phi$. Thus $\mathcal{D}_{|\phi|^2} \subseteq \mathcal{D}_\phi$, so

$$F(\phi)^* F(\phi) = F(\overline{\phi})F(\phi) = F(\overline{\phi}\phi) = F(|\phi|^2) = F(\phi)F(\phi)^*.$$

■

In order to prove the Spectral Theorem, the following lemmas are needed.

Lemma 5.4.7. *If N is a normal operator on $\mathcal{H}$, $B = (I + N^*N)^{-1}$, and $C = N(I + N^*N)^{-1}$, then $BC = CB$ and $(I + N^*N)^{-1}N \subseteq C$.*

Proof. From Proposition 5.4.2, B and C are contractions and $B \geq 0$. It will first be shown that $(I + N^*N)^{-1}N \subseteq C$; that is, $BN \subseteq NB$. If $f \in dom(BN)$, then $f \in dom(N)$. Let $g \in dom(N^*N)$ such that $f = (I + N^*N)g$. Then $N^*Ng \in dom(N)$; hence $Ng \in dom(NN^*) = dom(N^*N)$. Thus $Nf = Ng +$

$NN^*Ng = (I + N^*N)Ng$. Therefore $BNf = B(1 + N^*N)Ng = Ng$. But $NBf = Ng$, so $BN = NB$ on $dom(N)$. Thus $BN \subseteq NB$.

If $h \in \mathcal{H}$, let $f \in dom(N^*N)$ such that $h = (I + N^*N)f$. So $BCh = BNBh = BNf = NBf = NBBh = CBh$. Hence $CB = BC$. ■

Lemma 5.4.8. *With the same notation as in Lemma 5.4.7, if $B = \int t dP(t)$, is its spectral representation, $1 > \delta > 0$, and Δ is a Borel subset of $[\delta, 1]$, then $\mathcal{H}_\Delta = P(\Delta)\mathcal{H} \subseteq dom(N)$, $\mathcal{H}_\Delta$ is invariant for both N and N^*, and $N|\mathcal{H}_\Delta$ is a bounded normal operator with $\|N|\mathcal{H}_\Delta\| \leq [(1-\delta)/\delta]^{1/2}$.*

Proof. If $h \in \mathcal{H}_\Delta$, then because $P(\Delta) = \chi_\Delta(B)$, $\|Bh\|^2 = \langle B^2P(\Delta)h, h\rangle = \int_\Delta t^2 dP_{h,h} \geq \delta^2\|h\|^2$. So $B|\mathcal{H}_\Delta$ is invertible. Since $BP(\Delta) = P(\Delta)B$, so $BP(\Delta)\mathcal{H} = P(\Delta)B\mathcal{H}$, thus, $B\mathcal{H}_\Delta = P(\Delta)B\mathcal{H} \subseteq \mathcal{H}_\Delta$. Note that $B\mathcal{H} = Dom(N^*N)$ and N^*N is self-adjoint, thus, $B\mathcal{H}$ is dense in $\mathcal{H}$, so $B\mathcal{H}_\Delta = P(\Delta)B\mathcal{H}$ is dense in $\mathcal{H}_\Delta$. Let $h_0 \in \mathcal{H}_\Delta$. Then there exists $h_n \in \mathcal{H}_\Delta$ such that $Bh_n \to h_0$, so $||h_n - h_m|| \leq \frac{1}{\delta}||Bh_n - Bh_m||$, so there is a $g_0 \in \mathcal{H}_\Delta$ such that $h_n \to g_0$, thus, we have $Bg_0 = h_0$. So $B|\mathcal{H}_\Delta$ is an one to one map. Thus, there is an g in $\mathcal{H}_\Delta$ such that $h = Bg$. But $ranB = dom(1 + N^*N) \subseteq dom(N)$. Hence $h \in dom(N)$; that is, $\mathcal{H}_\Delta \subseteq dom(N)$.

Let $h \in \mathcal{H}_\Delta$ and again let $g \in \mathcal{H}_\Delta$ such that $h = Bg$. Hence $Nh = NBg = Cg$. By Lemma 5.4.7, $BC = CB$; then by Theorem 2.5.6 that $P(\Delta)C = CP(\Delta)$. Since $g \in \mathcal{H}_\Delta$, $Nh = Cg \in \mathcal{H}_\Delta$. Note that if $M = N^*$ and $B_1 = (1 + M^*M)^{-1}$, then $B_1 = B$. From the preceding argument $N^*\mathcal{H}_\Delta = M\mathcal{H}_\Delta \subseteq \mathcal{H}_\Delta$. It easily follows that $N|\mathcal{H}_\Delta$ is normal.

Finally, if $h \in \mathcal{H}_\Delta$, then it follows from Theorem 3.4.6(3) that

$$
\begin{aligned}
\|Nh\|^2 &= \langle N^*NP(\Delta)h, h\rangle \\
&= \langle((N^*N+I)-I)P(\Delta)h, h\rangle \\
&= \int_\Delta (t^{-1}-1)dP_{h,h}(t) \\
&\leq \int_\delta^1 (t^{-1}-1)dP_{h,h}(t) \leq \|h\|^2(1-\delta)/\delta.
\end{aligned}
$$

Hence $\|N|\mathcal{H}_\Delta\| \leq [(1-\delta)/\delta]^{1/2}$.■

Theorem 5.4.9 (Spectral Theorem for Unbounded Normal Operator)**.** *If N is a normal operator on $\mathcal{H}$, then there is a unique spectral measure E defined on the Borel subsets of $\mathbb{C}$ such that:*

(1) $N = \int zdE(z)$;

(2) $E(\Delta) = 0$ if $\Delta \cap \sigma(N) = \emptyset$;

(3) if O is an open subset of $\mathbb{C}$ and $O \cap \sigma(N) \neq \emptyset$, then $E(O) \neq 0$;

(4) if $A \in B(\mathcal{H})$ such that $AN \subseteq NA$ and $AN^* \subseteq N^*A$, then $A(\int \phi dE) \subseteq (\int \phi dE)A$ for every Borel function ϕ on $\mathbb{C}$

Proof. Let $B = (I+N^*N)^{-1}$, and $C = N(I+N^*N)^{-1}$, $B = \int tdP(t)$ be the spectral decomposition of B and put $P_n = P(1/(n+1), 1/n]$ for all $n \geq 1$. Since $Ker(B) = 0 = P(\{0\})\mathcal{H}$, $\sum_{n=1}^\infty P_n = I$. Let $\mathcal{H}_n = P_n\mathcal{H}$. By Lemma 5.4.8, $\mathcal{H}_n \subseteq dom(N)$, $\mathcal{H}_n$ reduces N, and $N_n \equiv N|\mathcal{H}_n$ is bounded normal operator with $\|N_n\| \leq n^{1/2}$. Also, if $h \in \mathcal{H}_n, (I+N_n^*N_n)Bh = B(I+N_n^*N_n)h = h$; that is,

$$B|\mathcal{H}_n = (I+N_n^*N_n)^{-1}.$$

Thus if $\lambda \in \sigma(N_n)$, $(1+|\lambda|^2)^{-1} \in \sigma(B|\mathcal{H}_n) \subseteq [1/(n+1), 1/n]$. Thus $\sigma(N_n) \subseteq \{z \in \mathbb{C} : (n-1)^{1/2} \leq |z| \leq n^{1/2}\} \equiv \Delta_n$. Let $N_n = \int zdE_n(z)$ be the spectral decomposition of N_n. For every Borel subset Δ of $\mathbb{C}$, let $E(\Delta)$ be defined by

$$E(\Delta) = \sum_{n=1}^{\infty} E_n(\Delta \cap \Delta_n).$$

Note that $E_n(\Delta \cap \Delta_n)$ is a projection with range in $\mathcal{H}_n$. Since $\mathcal{H}_n \perp \mathcal{H}_m$ for $m \neq n$, $E(\Delta)$ is a projection. Next we show E is a spectral measure. Clearly, $E(\mathbb{C}) = I$ and $E(\emptyset) = 0$. If $h \in \mathcal{H}$, then $\langle E(\Delta)h, h\rangle = \sum_{n=1}^{\infty} \langle E_n(\Delta \cap \Delta_n)h, h\rangle$. So if $\{\Lambda_j\}_{j=1}^{\infty}$ are pairwise disjoint Borel sets, then

$$\begin{aligned}\langle E(\cup_{j=1}^{\infty}\Lambda_j)h, h\rangle &= \sum_{n=1}^{\infty} \langle E_n((\cup_{j=1}^{\infty}\Lambda_j) \cap \Delta_n)h, h\rangle \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \langle E_n(\Lambda_j \cap \Delta_n)h, h\rangle.\end{aligned}$$

Since each term in this double summation is non-negative, the order of summation can be reversed. Thus

$$\begin{aligned}\langle E(\cup_{j=1}^{\infty}\Lambda_j)h, h\rangle &= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \langle E_n(\Lambda_j \cap \Delta_n)h, h\rangle \\ &= \sum_{j=1}^{\infty} \langle E(\Lambda_j)h, h\rangle.\end{aligned}$$

So $E(\cup_{j=1}^{\infty}\Lambda_j) = \sum_{j=1}^{\infty} E(\Lambda_j)$; therefore E is a spectral measure.

Let $M = \int z dE(z)$ be defined as in Theorem 5.4.5. Thus $\mathcal{H}_n \subseteq dom(M)$ and by the spectral Theorem for bounded operators, $Mh = N_n h = Nh$ if $h \in \mathcal{H}_n$. If h is any vector in $dom(M)$, $h = \sum_1^{\infty} h_n$, $h_n \in \mathcal{H}_n$, and $\sum_1^{\infty} \|Nh_n\|^2 < \infty$. Because N is closed, $h \in dom(N)$ and $Nh = Mh$. Thus $M \subseteq N$. To prove the other inclusion, note that M is a closed operator by Proposition 3.4.4.

Thus by Proposition 3.4.2(5) that it is suffices to show that $\{h \oplus Nh : h \in dom(N^*N)\} \subseteq gra(M)$. If $h \in dom(N^*N)$, there is a vector g such that $h = Bg$. Then $P_nNh = P_nNBg = P_nCg = CP_ng = NP_nh$. If $h_n = P_nh$, then $\sum \|Nh_n\|^2 = \sum \|P_nNh\|^2 = \|Nh\|^2 < \infty$. Therefor $h \in dom(M)$ and so, by the above argument, $Nh = Mh$. That is, $h \oplus Nh \in gra(M)$. This prove (1).

Next we show that

$$\sigma(N) = \overline{\cup_{n=1}^{\infty}\sigma(N_n)}. \tag{3.5}$$

It is not hard to prove $\cup_{n=1}^{\infty}\sigma(N_n) \subseteq \sigma(N)$, so $\overline{\cup_{n=1}^{\infty}\sigma(N_n)} \subseteq \sigma(N)$. If $\lambda \notin \overline{\cup_{n=1}^{\infty}\sigma(N_n)}$, then there is an $\delta > 0$ such that $|\lambda - z| \geq \delta$ for all z in $\cup_{n=1}^{\infty}\sigma(N_n)$. Thus $(N_n - \lambda I)^{-1}$ exists and by Theorem 2.1.15 that $\|(N_n - \lambda I)^{-1}\| \leq \delta^{-1}$ for all n. Thus $A = \oplus_{n=1}^{\infty}(N_n - \lambda I)^{-1}$ is a bounded operator. It follows that $A = (N - \lambda)^{-1}$, so $\lambda \notin \sigma(N)$.

By (3.5) if $\Delta \cap \sigma(N) = \emptyset, \quad \Delta \cap \sigma(N_n) = \emptyset$ for all n. Thus $E_n(\Delta) = 0$ for all n. Hence $E(\Delta) = 0$ and (2) holds.

If O is open and $O \cap \sigma(N) \neq \emptyset$, then (3.5) implies $O \cap \sigma(N_n) \neq \emptyset$ for some n. Since $E_n(O) \neq 0$, so $E(O) \neq 0$ and (3) is true.

Now let $A \in \mathcal{B}(\mathcal{H})$ such that $AN \subseteq NA$ and $AN^* \subseteq N^*A$. Thus $A(I + N^*N) \subseteq (I + N^*N)A$. It follows that $AB = BA$. By the Spectral Theorem for bounded operator, A commutes with the spectral projections of B. In particular, each $\mathcal{H}_n$ reduces A and if $A_n \equiv A|\mathcal{H}_n$, then $A_nN_n = N_nA_n$. Hence $A_nE_n(\Delta) = E_n(\Delta)A_n$ for all Borel set Δ contained in Δ_n. It follows that $AE(\Delta) = E(\Delta)A$ for all Borel set Δ. The remaining details of the proof of (4) are left to the reader. ∎

Theorem 5.4.10. *Let Ω and Ω' be two Borel subsets of $\mathbb{C}$, $\mathcal{B}$ and $\mathcal{B}'$ be the all Borel subsets of Ω and Ω', respectively. Suppose E is a spectral measure defined on Ω such that $E(\Omega) = I$, and that $\phi : \Omega \to \Omega'$ satisfies*

$$\phi^{-1}(S') \in \mathcal{B}, \ S' \in \mathcal{B}'.$$

Denote $E'(S') = E(\phi^{-1}(S'))$, then E' is a spectral measure on $\mathcal{B}'$ satisfying $E'(\Omega') = I$. Moreover, for any given Borel function $f : \Omega' \to \mathbb{C}$, if the following two integrals exist, then we have

$$\int_{\Omega'} f(\lambda) dE'_{(x,y)}(\lambda) = \int_{\Omega} (f \circ \phi)(\lambda) dE_{(x,y)}(\lambda).$$

Proof. Firstly, we claim that E' is strong countable addition. To see this, let $S' = \cup_{k=1}^{\infty} S'_k$ and $S'_k \cap S'_j = \emptyset$ for $k \neq j$. if we put $S = \phi^{-1}(S'), S_k = \phi^{-1}(S'_k), k = 1, 2, \cdots$, then

$$S, S_k \in \mathcal{B}, S = \cup_{k=1}^{\infty} S_k, S_k \cap S_j = \emptyset \ \text{ for } \quad k \neq j.$$

And so

$$E'(S')x = E(S)x = \sum_{k=1}^{\infty} E(S_k)x = \sum_{k=1}^{\infty} E'(S'_k)x$$

for any $x \in H$, this proved the claim.

Secondly, from the assumption we know that $\Omega = \phi^{-1}(\Omega'), E'(\Omega') = E(\Omega) = I$, and that $E'(\emptyset) = E(\phi^{-1}(\emptyset)) = E(\emptyset) = 0$. For $S' \in \mathcal{B}'$, denote $S = \phi^{-1}(S')$. Then we have $\chi_{S'} \circ \phi = \chi_S$, and so

$$\begin{aligned}
\int_{\Omega'} \chi_{S'}(\lambda) dE'_{(x,y)}(\lambda) &= \langle E'(S')x, y\rangle = \langle E(\phi^{-1}(S'))x, y\rangle \\
&= \langle E(S)x, y\rangle = \int_{\Omega} \chi_S(\lambda) dE_{(x,y)}(\lambda) \\
&= \int_{\Omega} \chi_{S'} \circ \phi(\lambda) dE_{(x,y)}(\lambda).
\end{aligned}$$

Noting that any Borel function is the limit of some simple functions, thus, it is easily to prove the theorem. ■

By Theorems 4.4.6 and 4.4.9 and 4.4.10, we have the following

Corollary 5.4.11. *If A is a normal operator and f and g are Borel functions on $\mathbb{C}$, then*

$$f \circ g(A) = f(g(A)).$$

5.5 Stone Theorem

If A is a self-adjoint operator on $\mathcal{H}$, then $\exp(iA)$ is a unitary operator. Hence $U(t) = \exp(itA)$ is unitary for all t in $\mathbb{R}$. The purpose of this section is not to investigate the individual operators $\exp(itA)$, but rather the entire collection of operators $\{\exp(itA) : t \in \mathbb{R}\}$. In fact, as the first theorem shows, $U : \mathbb{R} \to$ unitaries on $\mathcal{H}$, is a group homomorphism with certain properties. Stone Theorem provides a converse to this; every such homomorphism arises in this way.

Theorem 5.5.1. *If A is self-adjoint and $U(t) = \exp(itA)$ for t in $\mathbb{R}$, then*

(1) $U(t)$ is unitary;

(2) $U(s+t) = U(s)U(t)$ for all s in $\mathbb{R}$;

(3) if $h \in \mathcal{H}$, then $\lim_{s\to t} U(s)h = U(t)h$;

(4) if $h \in dom(A)$, then

$$\lim_{t\to 0} 1/t[U(t)h - h] = iAh;$$

(5) if $h \in \mathcal{H}$ and $\lim_{t\to 0} t^{-1}[U(t)h - h]$ exists, then $h \in dom(A)$. Consequently, $dom(A)$ is invariant under each $U(t)$.

Proof. As was mentioned, part (1) is an exercise. Since $\exp(itx)\exp(isx) = \exp(i(s+t)x)$ for all s in $\mathbb{R}$, (2) is a consequence of the functional calculus for normal operators. Also note that $U(0)U(t) = U(t)$, so that $U(0) = I$.

(3) If $h \in \mathcal{H}$, then $\|U(t)h - U(s)h\| = \|U(t-s+s)h - U(s)h\| = \|U(s)[U(t-s)h - h]\| = \|U(t-s)h - h\|$ since $U(s)$ is unitary. Thus (3) will be shown if it is proved that $\|U(t)h - h\| \to 0$ as $t \to 0$. If $A = \int_{-\infty}^{+\infty} x dE(x)$ is the spectral decomposition of A, then

$$\|U(t)h - h\|^2 = \int_{-\infty}^{+\infty} |e^{itx} - 1|^2 dE_{h,h}(x).$$

Now $E_{h,h}$ is a finite measure on $\mathbb{R}$; for each x in $\mathbb{R}$, $|e^{itx} - 1|^2 \to 0$ as $t \to 0$; $|e^{itx} - 1|^2 \leq 4$. So the Lebesgue Dominated Convergence Theorem implies that $U(t)h \to h$ as $t \to 0$.

(4) Note that $t^{-1}[U(t) - I] - iA = f_t(A)$, where $f_t(x) = t^{-1}[\exp(itx) - 1] - ix$. So if $h \in dom(A)$,

$$\begin{aligned} \|1/t[U(t)h - h] - iAh\|^2 &= \|f_t(A)h\|^2 \\ &= \int_{-\infty}^{+\infty} |(e^{itx} - 1)/t - ix|^2 dE_{h,h}(x). \end{aligned}$$

As $t \to 0$, $t^{-1}[e^{itx} - 1] - ix \to 0$ for all x in $\mathbb{R}$. Also, $|e^{is} - 1| \leq |s|$ for all real numbers s, hence, $|f_t(x)| \leq |t|^{-1}|e^{itx} - 1| + |x| \leq 2|x|$. But $|x| \in L^2(E_{h,h})$ by Theorem 5.4.5. So again the Lebesgue Dominated Convergence Theorem implies that (4) is true.

(5) Let $\mathcal{D} = \{h \in \mathcal{H} : \lim_{t\to 0} t^{-1}[U(t)h - h] \text{ exists in } \mathcal{H}\}$. For $h \in \mathcal{D}$, let Bh be defined by

$$Bh = -i \lim_{t \to 0} \frac{U(t)h - h}{t}.$$

It is easy to see that $\mathcal{D}$ is a linear subspace in $\mathcal{H}$ and B is linear on $\mathcal{D}$. Also, by (4), $B \supseteq A$ so that B is densely defined. Moreover, if $h, g \in \mathcal{D}$, then

$$\langle Bh, g\rangle = -i \lim_{t\to 0}\langle \frac{U(t)h - h}{t}, g\rangle.$$

By (2) and the fact that each $U(t)$ is unitary, it follows that $U(t)^* = U(t)^{-1} = U(-t)$. Hence

$$\begin{aligned}\langle Bh, g\rangle &= -i \lim_{t\to 0}\langle h, \frac{U(-t)g - g}{t}\rangle \\ &= \lim_{t\to 0}\langle h, -i[\frac{U(-t)g - g}{-t}]\rangle = \langle h, Bg\rangle.\end{aligned}$$

Hence B is a symmetric extension of A. Since self-adjoint operators are maximal symmetric operators, $B = A$ and $\mathcal{D} = dom(A)$. ■

Definition 5.5.2. *A strongly continuous one-parameter unitary group is a function $U : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ such that for all s and t in $\mathbb{R}$:*

(1) $U(t)$ is a unitary operator;

(2) $U(s+t) = U(s)U(t)$;

(3) if $h \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, then $U(t)h \to U(t_0)h$ as $t \to t_0$.

Note that by Theorem 5.5.1, if A is self-adjoint, then $U(t) = \exp(itA)$ defines a strongly continuous one-parameter unitary group. Also, $U(0) = I$ and $U(-t) = U(t)^{-1}$, so that $[U(t) : t \in \mathbb{R}\}$ is indeed a group. Property (3) also implies that $U : \mathbb{R} \to (\mathcal{B}(\mathcal{H}), S.O.T)$ is continuous. Moreover, it can be proved that if U is only assumed to be $W.O.T$-continuous, then U is $S.O.T$-continuous. However, this condition can be relaxed even further as the following result of von Neumann showed in 1932.

Theorem 5.5.3. *If $\mathcal{H}$ is separable, $U : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ satisfies conditions (1) and (2) of Definition 3.5.2, and if for all h, g in $\mathcal{H}$ the function $t \mapsto \langle U(t)h, g\rangle$ is Lebesgue measurable, then U is a strongly continuous one-parameter unitary group.*

Proof. If $0 < a < \infty$ and $h, g \in \mathcal{H}$, then $t \mapsto \langle U(t)h, g\rangle$ is a bounded measurable function on $[0, a]$ and

$$\int_0^a |\langle U(t)h, g\rangle| dt \leq a\|h\|\|g\|.$$

Thus

$$h \mapsto \int_0^a \langle U(t)h, g\rangle dt$$

is a bounded linear function on $\mathcal{H}$. Therefore there is an g_a in $\mathcal{H}$ such that

$$\langle h, g_a\rangle = \int_0^a \langle U(t)h, g\rangle dt$$

and $\|g_a\| \leq a\|g\|$.

$$\textbf{Claim.} \quad \{g_a : g \in \mathcal{H}, a > 0\} \quad \text{is total in} \quad \mathcal{H}. \tag{3.5}$$

In fact, suppose $h \in \mathcal{H}$ and $h \perp \{g_a : g \in \mathcal{H}, a > 0\}$. Then for every $a > 0$ and every g in $\mathcal{H}$,

$$0 = \int_0^a \langle U(t)h, g\rangle dt.$$

Thus for every g in $\mathcal{H}$, $\langle U(t)h, g\rangle = 0$ a.e. on $\mathbb{R}$. Because $\mathcal{H}$ is separable there is a subset Δ of $\mathbb{R}$ having measure zero such that if $t \notin \Delta$, $\langle U(t)h, g\rangle = 0$ whenever g belongs to a preselected countable dense subset of $\mathcal{H}$. Thus $U(t)h = 0$ if $t \notin \Delta$. But $\|h\| = \|U(t)h\|$, so $h = 0$ and the claim is established.

Now if $s \in \mathbb{R}$,

$$\begin{aligned}\langle h, U(s)g_a\rangle &= \langle U(-s)h, g_a\rangle \\ &= \int_0^a \langle U(t-s)h, g\rangle dt \\ &= \int_{-s}^{a-s} \langle U(t)h, g\rangle dt.\end{aligned}$$

Thus $\langle h, U(s)g_a\rangle \to \langle h, g_a\rangle$ as $s \to 0$. By the claim and the fact that the group is uniformly bounded, $U : \mathbb{R} \to (\mathcal{B}(\mathcal{H}), W.T.O)$ is continuous at 0. By the group property, $U : \mathbb{R} \to (\mathcal{B}(\mathcal{H}), W.T.O)$ is continuous. Moreover, it can be proved that U is also $S.O.T$-continuous. ■

We now turn our attention to the principal result of this section, Stone Theorem, which states that the converse of Theorem 5.5.1 is valid. Note that if $U(t) = \exp(itA)$ for a self-adjoint operator A, then part (4) of Theorem 5.5.1 instructs us how to recapture A. This is the route followed in the proof of Stone Theorem, proved in 1932.

Theorem 5.5.4 (Stone)**.** *If U is a strongly continuous one-parameter unitary group, then there is a self-adjoint operator A such that $U(t) = \exp(itA)$.*

Proof. Begin by defining $\mathcal{D}$ to be the set of all vectors h in $\mathcal{H}$ such that $\lim_{t\to 0} t^{-1}[U(t)h - h]$ exists; since $0 \in \mathcal{D}$, $\mathcal{D} \neq \emptyset$. Clearly $\mathcal{D}$ is a linear subspace of $\mathcal{H}$.

Claim 1. $\mathcal{D}$ is dense in $\mathcal{H}$.

Let Φ be the set of all continuous functions ϕ on $\mathbb{R}$ such that $\phi \in L^1(0, \infty)$. Hence for any h in $\mathcal{H}, t \mapsto \phi(t)U(t)h$ is a continuous function of $\mathbb{R}$ into $\mathcal{H}$. Because $\|U(t)h\| = \|h\|$ for all t, a Riemann integral, $\int_0^\infty \phi(t)U(t)hdt$, can be defined and is a vector in $\mathcal{H}$. Put

$$T_\phi h = \int_0^\infty \phi(t)U(t)hdt.$$

It is easy to see that $T_\phi : \mathcal{H} \to \mathcal{H}$ is linear and bounded with $\|T_\phi\| \leq \int_0^\infty |\phi(t)|dt$. Similarly, for each ϕ in Φ,

$$S_\phi h = \int_0^\infty \phi(t)U(-t)hdt$$

defines a bounded operator on $\mathcal{H}$. For any ϕ in Φ and t in $\mathbb{R}$,

$$\begin{aligned}
U(t)T_\phi h &= U(t)\int_0^\infty \phi(s)U(s)hds \\
&= \int_0^\infty \phi(s)U(t+s)hds \\
&= \int_t^\infty \phi(s-t)U(s)hds.
\end{aligned}$$

Similarly,

$$U(t)S_\phi h = \int_{-t}^\infty \phi(s+t)U(-s)hds.$$

Now let $\Phi^{(1)}$ be the set of all ϕ in Φ that are continuously differentiable with ϕ' in Φ. For ϕ in $\Phi^{(1)}$,

$$\begin{aligned}
-\frac{i}{t}[U(t)-I]T_\phi h &= -\frac{i}{t}\int_t^\infty \phi(s-t)]U(s)hds + \frac{i}{t}\int_0^\infty \phi(s)U(s)hds \\
&= -i\int_t^\infty [\frac{\phi(s-t)-\phi(s)}{t}]U(s)hds + \frac{i}{t}\int_0^t \phi(s)U(s)hds.
\end{aligned}$$

Now

$$\| \int_0^t [\frac{\phi(s-t)-\phi(s)}{t}]U(s)hds\| \leq \|h\| \sup\{|\phi(s-t)-\phi(s)| : 0 \leq s \leq 1\} \to 0$$

as $t \to 0$. Hence

$$\lim_{t\to 0} \int_t^\infty [\frac{\phi(s-t)-\phi(s)}{t}]U(s)hds = -\int_0^\infty \phi'(s)U(s)hds = -T_{\phi'}h.$$

Since $s \mapsto \phi(s)U(s)h$ is continuous and $U(0) = I$, the Fundamental Theorem of Calculus implies that

$$\lim_{t\to 0} \frac{1}{t}\int_0^t \phi(s)U(s)hds = \phi(0)h.$$

Hence for ϕ in $\Phi^{(1)}$ and h in $\mathcal{H}$,

$$\lim_{t\to 0} -\frac{i}{t}[U(t)-I]T_\phi h = iT_{\phi'}h + i\phi(0)h. \tag{3.6}$$

Similarly, for ϕ in $\Phi^{(1)}$ and h in $\mathcal{H}$,

$$\lim_{t\to 0} -\frac{i}{t}[U(t)-I]S_\phi h = -iS_{\phi'}h - i\phi(0)h. \tag{3.7}$$

So we have

$$\mathcal{D} \supseteq \{T_\phi h : \phi \in \Phi^{(1)} \quad \text{and} \quad h \in \mathcal{H}\}.$$

But for every positive integer n there is an ϕ_n in $\Phi^{(1)}$ such that $\phi_n \geq 0, \;\; \phi_n(t) = 0$ for $t \geq 1/n$, and $\int_o^\infty \phi_n(t)dt = 1$. Hence

$$T_{\phi_n}h - h = \int_0^{1/n} \phi_n(t)[U(t)-I]hdt$$

and so $\|T_{\phi_n}h - h\| \leq \sup\{\|U(t)h - h\| : 0 \leq t \leq 1/n\}$. Therefore $\|T_{\phi_n}h - h\| \to 0$ as $n \to 0$ since U is strongly continuous. This says that $\mathcal{D}$ is dense.

For h in $\mathcal{D}$, define

$$Ah = -i \lim_{t\to 0} \frac{1}{t}[U(t)-I]h. \tag{3.8}$$

Claim 2. A is symmetric.

The proof of this is left to the reader.

By Proposition 3.2.2, A is closable; also denote the closure of A by A. According to Corollary 3.2.9, to prove that A is self-adjoint it suffices to prove that $ker(A^* \pm i) = (0)$. Equivalently, it suffices to show that $ran(A \pm i)$ is dense. It will be shown that there are operators $B_\pm$ such that $(A \pm i)B_\pm = I$, so that $A \pm i$ is surjective. Notice that according to (3.6),

$$(A+i)T_\phi = AT_\phi + iT_\phi = i(T_{\phi'} + T_\phi) + i\phi(0).$$

So taking $\phi(t) = -ie^{-t}$, $(A+i)T_\phi = I$. According to (3.7),

$$(A-i)S_\psi = AS_\psi - iS_\psi = -i(S_{\psi'} + S_\psi) - i\psi(0).$$

Taking $\psi(t) = ie^{-t}$, $(A-i)S_\psi = I$. Hence A is self-adjoint.

Put $V(t) = \exp(iAt)$. It remains to show that $V = U$. Let $h \in \mathcal{D}$. By Theorem 5.5.1,

$$s^{-1}[V(t+s) - V(t)]h = s^{-1}[V(s) - I]V(t)h \to iAV(t)h;$$

that is, $V'(t)h = iAV(t)h$. Similarly,

$$s^{-1}[U(t+s) - U(t)]h = s^{-1}[U(s) - I]U(t)h \to iAU(t)h.$$

So if $h(t) = U(t)h - V(t)h$, then $h : \mathbb{R} \to \mathcal{H}$ is differentiable and

$$h'(t) = iAU(t)h - iAV(t)h = iAh(t).$$

But

$$\begin{aligned}(d/dt)\|h(t)\|^2 &= \langle h'(t), h(t)\rangle + \langle h(t), h'(t)\rangle \\ &= \langle iAh(t), h(t)\rangle + \langle h(t), iAh(t)\rangle.\end{aligned}$$

Thus $(d/dt)\|h(t)\|^2 = 0$ and so $\|h\| : \mathbb{R} \to \mathbb{R}$ is a constant function. But $h(0) = 0$, so $h(t) = 0$. This says that $U(t)h = V(t)h$ for all h in $\mathcal{D}$ and all t in $\mathbb{R}$. Since $\mathcal{D}$ is dense, $U = V$.■

Definition 5.5.5. *If U is a strongly continuous one-parameter unitary group, then the self-adjoint operator A such that $U(t) = \exp(itA)$ is called the infinitesimal generator of U.*

By virtue of Stone Theorem and Theorem 5.5.1, there is a one-to-one correspondence between self-adjoint operators and strongly continuous one-parameter unitary groups. Thus, it should be able to characterize certain properties of a group in terms of its infinitesimal generator and vice versa. For example, suppose the infinitesimal generator is bounded; what can be said about the group ?

Proposition 5.5.6. *If U is a strongly continuous one-parameter unitary group with infinitesimal generator A, then A is bounded if and only if $lim_{t\to 0}\|U(t) - I\| = 0$.*

Proof. First assume that A is bounded. Hence $\|U(t) - I\| = \|\exp(itA) - I\| = \sup\{|e^{itx} - 1| : x \in \sigma(A)\} \to 0$ as $t \to 0$ since $\sigma(A)$ is compact.

Now assume that $\|U(t)-I\| \to 0$ as $t \to 0$. Let $0 < \epsilon < \pi/4$. Then there is a $t_0 > 0$ such that $\|U(t)-I\| < \varepsilon$ for $|t| < t_0$. Since $U(t)-I = \int_{\sigma(A)}(e^{ixt}-1)dE(t)$, $\sup\{|e^{ixt} - 1| : x \in \sigma(A)\} = \|U(t) - I\| < \varepsilon$ for $|t| < t_0$. Thus for a small δ, $tx \in \cup_{n=-\infty}^{\infty}(2\pi n - \delta, 2\pi n + \delta) \equiv G$ whenever $x \in \sigma(A)$ and $|t| < t_0$. In fact, if ε is chosen sufficiently small, then δ is small enough that the intervals $\{(2\pi n - \delta, 2\pi n + \delta)\}$ are the components of G. If $x \in \sigma(A)$, $\{tx : 0 \leq t < t_0\}$ is the interval from 0 to $t_0 x$ and is contained in G. Hence $tx \in (-\delta, \delta)$ for x in

$\sigma(A)$ and $|t| < t_0$. In particular, $t_0\sigma(A) \subseteq [-\delta, \delta]$ so $\sigma(A)$ is compact and A is bounded. ■

Let μ be a positive measure on $\mathbb{R}$ and let $A_x f = xf$ for f in $\mathcal{D}_x = \{f \in L^2(\mu) : xf \in L^2(\mu)\}$. We have already seen that A is self-adjoint. Clearly $\exp(itA_x) = M_{e_t}$ on $L^2(\mathbb{R})$, where e_t is the function $e_t(x) = \exp(itx)$. This can be generalized a bit.

Proposition 5.5.7. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let ϕ be a real-valued $\mathcal{F}$-measurable function on X. If $A = M_\phi$ on $L^2(\mu)$ and $U(t) = \exp(itA)$, then $U(t) = M_{e_t}$, where $e_t(x) = \exp(it\phi(x))$.*

6 Tensor Product Theory

There are several ways of defining the Hilbert tensor product $\mathcal{H}$ of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, each method having advantages in particular circumstances. Our approach, set out below, emphasizes the universal property of the tensor product. The Hilbert space $\mathcal{H}$ is characterized by the existence of a bilinear mapping p, from the Cartesian product $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{H}$, with the following property: each suitable bilinear mapping $\mathcal{L}$ from $\mathcal{H}_1 \times \mathcal{H}_2$ into a Hilbert space $\mathcal{K}$ has a unique factorization $L = Tp$, with T a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}$.

6.1 Tensor Product of Hilbert Spaces

Proposition 6.1.1. *Suppose that $\mathcal{H}_1, \cdots, \mathcal{H}_n$ are Hilbert spaces and φ is a bounded multilinear functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}n$.*

(1) The sum

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi(y_1, \cdots, y_n)|^2, \tag{1}$$

has the same value for all orthonormal bases Y_1 of $\mathcal{H}_1, \cdots, Y_n$ of $\mathcal{H}_n$.

(2) If $\mathcal{K}_1, \cdots, \mathcal{H}_n$ are Hilbert spaces, $A_m \in \mathcal{B}(\mathcal{H}_m, \mathcal{K}_m), m = 1, \cdots, n$, ψ is a bounded multilinear functional on $\mathcal{K}_1 \times \cdots \times \mathcal{K}_n$, and

$$\varphi(x_1, \cdots, x_n) = \psi(A_1 x_1, \cdots, A_n x_n), (x_1 \in \mathcal{H}_1, \cdots, x_n \in \mathcal{H}_n$$

then

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi(y_1, \cdots, y_n)|^2 \leq \|A_1\|^2 \cdots \|A_n\|^2 \sum_{z_1 \in Z_1} \cdots \sum_{z_n \in Z_n} |\psi(z_1, \cdots, z_n)|^2$$

when Y_m and Z_m are orthonormal bases of $\mathcal{H}_m$ and $\mathcal{K}_m$, respectively, $m = 1, \cdots, n$.

Proof. In order to prove (i), it is sufficient to show that

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi(y_1, \cdots, y_n)|^2 \leq \sum_{z_1 \in Z_1} \cdots \sum_{z_n \in Z_n} |\psi(z_1, \cdots, z_n)|^2$$

whenever Y_m, Z_m, are orthonormal bases of $\mathcal{H}_m, m = 1, \cdots, n$, since equality of the two sums then follows by exchanging the roles of Y_m and Z_m. The required inequality is a special case of part (2) of the proposition, with $\psi = \varphi$, $\mathcal{K}_m = \mathcal{H}_m$, and $A_m = I, m = 1, \cdots, n$.

It now suffices to prove (2). For this, suppose that $1 \leq m \leq n$, and choose and fix vectors y_1 in $Y_1, \cdots, y_{m-1}$ in Y_{m-1}, z_{m+1} in $Z_{m+1}, \cdots, z_n$ in Z_n. The mapping

$$z \rightarrow \psi(A_1 y_1, \cdots, A_{m-1} y_{m-1}, z, z_{m+1}, \cdots, z_n) : \mathcal{K}_m \rightarrow \mathbf{C}$$

is a bounded linear functional on $\mathcal{K}_m$, so there is a vector w in $\mathcal{K}_m$ such that

$$\psi(A_1 y_1, \cdots, A_{m-1} y_{m-1}, z, z_{m+1}, \cdots, z_n) = \langle z, w \rangle, z \in \mathcal{K}_m.$$

From Parseval's equation

$$\begin{aligned}
&\sum_{y_m \in Y_m} |\psi(A_1 y_1, \cdots, A_{m-1} y_{m-1}, A_m y_m, z_{m+1}, \cdots, z_n)|^2 \\
&\quad = \sum_{y_m \in Y_m} |\langle A_m y_m, w \rangle|^2 = \sum_{y_m \in Y_m} |\langle y_m, A_m^* w \rangle|^2 \\
&\quad = \|A_m^* w\|^2 \leq \|A_m\|^2 \|w\|^2 = \|A_m\|^2 \sum_{z_m \in Z_m} |\langle z_m, w \rangle|^2 \\
&\quad = \|A_m\|^2 \sum_{z_m \in Z_m} |\psi(A_1 y_1, \cdots, A_{m-1} y_{m-1}, z_m, z_{m+1}, \cdots, z_n)|^2.
\end{aligned}$$

A further summation now yields

$$\sum_{y_1\in Y_1}\cdots\sum_{y_m\in Y_m}\sum_{z_{m+1}\in Z_{m+1}}\cdots\sum_{z_n\in Z_n}|\psi(A_1y_1,\cdots,A_my_m,z_{m+1},\cdots,z_n)|^2$$

$$\leq \|A_m\|^2\sum_{y_1\in Y_1}\cdots\sum_{y_{m-1}\in Y_{m-1}}\sum_{z_m\in Z_m}\cdots\sum_{z_n\in Z_n}|\psi(A_1y_1,\cdots,A_{m-1}y_{m-1},z_m,\cdots,z_n)|^2.$$

Thus

$$\sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n}|\varphi(y_1,\cdots,y_n)|^2=\sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n}|\psi(A_1y_1,\cdots,A_ny_n)|^2$$

$$\begin{aligned}
&\leq \|A_n\|^2\sum_{y_1\in Y_1}\cdots\sum_{y_{n-1}\in Y_{n-1}}\sum_{z_n\in Z_n}|\psi(A_1y_1,\cdots,A_{n-1}y_{n-1},z_n)|^2\\
&\leq \|A_{n-1}\|^2\|A_n\|^2\\
&\quad\times\sum_{y_1\in Y_1}\cdots\sum_{y_{n-2}\in Y_{n-2}}\sum_{z_{n-1}\in Z_{n-1}}\sum_{z_n\in Z_n}|\psi(A_1y_1,\cdots,A_{n-2}y_{n-2},z_{n-1},z_n)|^2\\
&\leq \cdots\leq \|A_1\|^2\cdots\|A_n\|^2\sum_{z_1\in Z_1}\cdots\sum_{z_n\in Z_n}|\psi(z_1,\cdots,z_n)|^2.
\end{aligned}$$

■

With $\mathcal{H}_1,\cdots,\mathcal{H}_n$ Hilbert spaces, a mapping $\varphi:\mathcal{H}_1\times\cdots\times\mathcal{H}_n\to\mathbf{C}$ is described as a *Hilbert-Schmidt functional* on $\mathcal{H}_1\times\cdots\times\mathcal{H}_n$ if it is a bounded multilinear functional, and the sum (1) is finite for one choice of the orthonormal bases Y_1 in $\mathcal{H}_1,\cdots,Y_n$ in $\mathcal{H}_n$.

Proposition 6.1.2. *If $\mathcal{H}_1,\cdots,\mathcal{H}_n$ are Hilbert spaces, the set $\mathcal{HSF}$ of all Hilbert-Schmidt functionals on $\mathcal{H}_1\times\cdots\times\mathcal{H}_n$ is itself a Hilbert space when the linear structure, inner product, and norm are defined by*

$$(a\varphi+b\psi)(x_1,\cdots,x_n)=a\varphi(x_1,\cdots,x_n)+b\psi(x_1,\cdots,x_n),$$

$$\langle \varphi, \psi \rangle = \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} \varphi(y_1, \cdots, y_n) \overline{\psi(y_1, \cdots, y_n)}, \tag{2}$$

$$\|\varphi\|^2 = \left[\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi(y_1, \cdots, y_n)|^2 \right]^{\frac{1}{2}}, \tag{3}$$

where Y_m is an orthonormal basis in $\mathcal{H}_m, m = 1, \cdots, n$. The sum in (2) is absolutely convergent, and the inner product and norm do not depend on the choice of the orthonormal bases $Y_1, \cdots, Y_n$.

For each $v(1)$ in $\mathcal{H}_1, \cdots, v_n$ in $\mathcal{H}_n$, the equation

$$\varphi_{v(1), \cdots, v(n)}(x_1, \cdots, x_n) = \langle x_1, v(1) \rangle \cdots \langle x_n, v(n) \rangle, \; x_1 \in \mathcal{H}_1, \cdots, x_n \in \mathcal{H}_n.$$

defines an element $\varphi_{v(1), \cdots, v(n)}$ of $\mathcal{HSF}$, and

$$\begin{aligned} \langle \varphi_{v(1), \cdots, v(n)}, \varphi_{w(1), \cdots, w(n)} \rangle &= \langle w(1), v(1) \rangle \cdots \langle w(n), v(n) \rangle, \\ \|\varphi_{v(1), \cdots, v(n)}\|_2 &= \|v(1)\| \cdots \|v(n)\|. \end{aligned}$$

The set $\{\varphi_{y(1), \cdots, y(n)} : y(1) \in Y_1, \cdots, y(n) \in Y_n\}$ is an orthonormal basis of $\mathcal{HSF}$. There is a unitary transformation U from $\mathcal{HSF}$ onto $l_2(Y_1 \times \cdots \times Y_n)$, such that $U\varphi$ is the restriction $\varphi | Y_1 \times \cdots \times Y_n$ when $\varphi \in \mathcal{HSF}$.

Proof. Having chosen an orthonormal basis Y_m in $\mathcal{H}_m, m = 1, \cdots, n$, we can associate with *each* bounded multilinear functional φ on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ the complex-valued function $U\varphi$ obtained by restricting φ to $Y_1 \times \cdots \times Y_n$. Note that φ is a Hilbert-Schmidt functional if and only if

$$U\varphi \in l_2(Y_1 \times \cdots \times Y_n).$$

If $U\varphi = 0$, then

$$\varphi(y_1, \cdots, y_n) = 0, y_1 \in Y_1, \cdots, y_n \in Y_n.$$

Since Y_m has closed linear span $\mathcal{H}_m,\ m = 1, \cdots, n$, it follows from the multilinearity and continuity of φ that φ vanishes throughout $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$.

If φ and ψ are Hilbert-Schmidt functionals on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, the same is true of $a\varphi + b\psi$ for all scalars a, b; for $a\varphi + b\psi$ is a bounded multilinear functional, $U\varphi, U\psi \in l_2(Y_1 \times \cdots \times Y_n)$, and therefore

$$U(a\varphi + b\psi) = aU\varphi + bU\psi \in l_2(Y_1 \times \cdots \times Y_n).$$

The summation occurring in (2) can be written in the form

$$\sum_{y \in Y_1 \times \cdots \times Y_n} (U\varphi)(y)\overline{(U\psi)(y)},$$

and is absolutely convergent with sum $\langle U\varphi, U\psi \rangle$, the inner product in $l_2(Y_1 \times \cdots \times Y_n)$ of $U\varphi$ and $U\psi$.

From the preceding argument, the set $\mathcal{HSF}$ of all Hilbert-Schmidt functionals on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ is a complex vector space, (2) defines an inner product on $\mathcal{HSF}$, the restriction $U|\mathcal{HSF}$ is a one-to-one linear mapping from $\mathcal{HSF}$ into $l_2(Y_1 \times \cdots \times Y_n)$, and $\langle U\varphi, U\psi \rangle = \langle \varphi, \psi \rangle$ when $\varphi, \psi \in \mathcal{HSF}$. Since the inner product on $l_2(Y_1 \times \cdots \times Y_n)$ is definite, so is that on $\mathcal{HSF}$; for if $\varphi \in \mathcal{HSF}$ and $\langle \varphi, \varphi \rangle = 0$, we have $\langle U\varphi, U\varphi \rangle = 0$, whence $U\varphi = 0$ and so $\varphi = 0$. From this, $\mathcal{HSF}$ is an inner product space and it is apparent from (2) that the norm $\| \ \|_2$ in $\mathcal{HSF}$ is given by (3). From Proposition 5.1.1, this norm is independent of the choice of the orthonormal bases $Y_1, \cdots, Y_n$; by polarization, the same is true of the inner product on $\mathcal{HSF}$.

We prove next that U carries $\mathcal{HSF}$ onto the whole of the l_2 space. With f in $l_2(Y_1 \times \cdots \times Y_n)$ and x_m in $H_m, m = 1, \cdots, n$, the Cauchy-Schwarz inequality and Parseval equation give

$$
\begin{aligned}
& \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |f(y_1, \cdots, y_n)\langle x_1, y_1\rangle \cdots \langle x_n, y_n\rangle| \\
\leq & \left[\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |f(y_1, \cdots, y_n)|^2 \right]^{\frac{1}{2}} \\
& \times \left[\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\langle x_1, y_1\rangle|^2 \cdots |\langle x_n, y_n|^2 \right]^{\frac{1}{2}} \\
= & \|f\| \left(\sum_{y_1 \in Y_1} |\langle x_1, y_1\rangle|^2 \right)^{\frac{1}{2}} \cdots \left(\sum_{y_n \in Y_n} |\langle x_n, y_n\rangle|^2 \right)^{\frac{1}{2}} = \|f\| \|x_1\| \cdots \|x_n\|.
\end{aligned}
$$

From this, the equation

$$
\varphi(x_1, \cdots, x_n) = \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} f(y_1, \cdots, y_n)\langle x_1, y_1\rangle \cdots \langle x_n, y_n\rangle
$$

defines a bounded multilinear functional φ on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, with $\|\varphi\| \leq \|f\|$. From orthonormality of the sets $Y_1, \cdots, Y_n$,

$$
(U\varphi)(y_1, \cdots, y_n) = \varphi(y_1, \cdots, y_n) = f(y_1, \cdots, y_n), y_1 \in Y_1, \cdots, y_n \in Y_n,
$$

so $U\varphi = f$. Moreover, $\varphi \in \mathcal{HSF}$ since $U\varphi \in l_2(Y_1 \times \cdots \times Y_n)$, whence U carries $\mathcal{HSF}$ onto the l_2 space.

Since U is a norm-preserving linear mapping from $\mathcal{HSF}$ onto $l_2(Y_1 \times \cdots \times Y_n)$, completeness of the space l_2 entails completeness of $\mathcal{HSF}$; so $\mathcal{HSF}$ is a Hilbert space, and U is a unitary operator.

When $v(1) \in \mathcal{H}_1, \cdots, v(n) \in \mathcal{H}_n, \varphi_{v_{(1)}, \cdots, v_{(n)}}$, as defined in Proposition 5.1.2, is a multilinear functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, and is bounded since

$$
|\varphi_{v(1), \cdots, v(n)}(x_1, \cdots, x_n)| \leq \|v(1)\| \cdots \|v(n)\| \|x_1\| \cdots \|x_n\|
$$

by the Cauchy-Schwarz inequality. Moreover, Parseval's equation gives

$$\begin{aligned}
&\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi_{v(1),\cdots,v(n)}(y_1, \cdots, y_n)|^2 \\
= &\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\langle y_1, v(1)\rangle|^2 \cdots |\langle y_n, v(n)\rangle|^2 \\
= &\left(\sum_{y_1 \in Y_1} |\langle y_1, v(1)\rangle|^2\right) \cdots \left(\sum_{y_n \in Y_n} |\langle y_n, v(n)\rangle|^2\right). \\
= &\|v(1)\|^2 \cdots \|v(n)\|^2.
\end{aligned}$$

Hence $\varphi_{v(1),\cdots,v(n)} \in \mathcal{HSF}$ and $\|\varphi_{v(1),\cdots,v(n)}\|_2 = \|v(1)\| \cdots \|v(n)\|$. Again, by Parseval's equation and absolute convergence, .

$$\begin{aligned}
&\langle \varphi_{v(1),\cdots,v(n)}, \varphi_{w(1),\cdots,w(n)} \rangle \\
= &\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} \varphi_{v(1),\cdots,v(n)}(y_1, \cdots, y_n)\overline{\varphi_{w(1),\cdots,w(n)}(y_1, \cdots, y_n)} \\
= &\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} \langle y_1, v(1)\rangle \cdots \langle y_n, v(n)\rangle \langle w(1), y_1\rangle \cdots \langle w(n), y_n\rangle \\
= &\left(\sum_{y_1 \in Y_1} \langle w(1), y_1\rangle \langle y_1, v(1)\rangle\right) \cdots \left(\sum_{y_n \in Y_n} \langle w(n), y_n\rangle \langle y_n, v(n)\rangle\right) \\
= &\langle w(1), v(1)\rangle \cdots \langle w(n), v(n)\rangle.
\end{aligned}$$

When $y(1) \in Y_1, \cdots, y(n) \in Y_n$, the orthonormality of $Y_1, \cdots, Y_n$ implies that $U\varphi_{y(1),\cdots,y(n)}$ is the function that takes the value 1 at $(y(1), \cdots, y(n))$ and 0 elsewhere on $Y_1 \times \cdots \times Y_n$. Thus

$$\{U\varphi_{y(1),\cdots,y(n)} : y(1) \in Y_1, \cdots, y(n) \in Y_n\}$$

is an orthonormal basis of $l_2(Y_1 \times \cdots \times Y_n)$, and therefore

$$\{\varphi_{y(1),\cdots,y(n)} : y(1) \in Y_1, \cdots, y(n) \in Y_n\}$$

is such a basis of $\mathcal{HSF}$. ■

Definition 6.1.3. *Suppose that $\mathcal{H}_1, \cdots, \mathcal{H}_n$ and $\mathcal{K}$ are Hilbert spaces and L is a mapping from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into $\mathcal{K}$. We describe L as a bounded multilinear mapping if it is linear in each of its variables, and there is a real number c such that*

$$\|L(x_1, \cdots, x_n)\| \leq c\|x_1\| \cdots \|x_n\|, x_1 \in \mathcal{H}_1, \cdots, x_n \in \mathcal{H}_n.$$

In these circumstances, the least such constant c is denoted by $\|L\|$.

By a weak Hilbert-Schmidt mapping from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into $\mathcal{K}$, we mean a bounded multilinear mapping L with the following properties:

(1) For each u in $\mathcal{K}$, the mapping L_u defined by

$$L_u(x_1, \cdots, x_n) = \langle L(x_1, \cdots, x_n), u \rangle$$

is a Hilbert-Schmidt functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$;

(2) There is a real number d such that $\|L_u\|_2 \leq d\|u\|$ for each u in $\mathcal{K}$.

When these conditions are satisfied, the least possible value of the constant d in (2) is denoted by $\|L\|_2$.

Theorem 6.1.4. *Suppose that $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ are Hilbert spaces.*

(1) There is a Hilbert space $\mathcal{H}$ and a weak Hilbert-Schmidt mapping $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H}$ with the following property: given any weak Hilbert-Schmidt mapping L from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into a Hilbert space $\mathcal{K}$, there is a unique bounded linear mapping T from $\mathcal{H}$ into $\mathcal{K}$, such that $L = Tp$; moreover, $\|T\| = \|L\|_2$.

(2) If $\mathcal{H}'$ and p' have the properties attributed in (1) to $\mathcal{H}$ and p, there is a unitary transformation U from $\mathcal{H}$ onto $\mathcal{H}'$ such that $p' = Up$.

(3) If $v_m, w_m \in H_m$ and Y_m is an orthonormal basis of $\mathcal{H}_m, m = 1, \cdots, n$, then

$$\langle p(v_1, \cdots, v_n), p(w_1, \cdots, w_n) \rangle = \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle,$$

the set $\{p(y_1, \cdots, y_n) : y_1 \in Y_1, \cdots, y_n \in Y_n\}$ *is an orthonormal basis of* $\mathcal{H}$*, and* $\|p\|_2 = 1$*.*

Proof. With ${}_*\mathcal{H}_m$ the dual Hilbert space of $\mathcal{H}_m$, let $\mathcal{H}$ be the set of all Hilbert-Schmidt functionals on ${}_*\mathcal{H}_1 \times \cdots \times_* \mathcal{H}_n$ with the Hilbert space structure described in Proposition 5.1.2. When $v(1) \in \mathcal{H}_1, \cdots, v(n) \in \mathcal{H}_n$, let $p(v(1), \cdots, v(n))$ be the Hilbert-Schmidt functional $\varphi_{v(1),\cdots,v(n)}$ defined on

$$ {}_*\mathcal{H}_1 \times \cdots \times_* \mathcal{H}_n $$

by

$$ \begin{aligned} \varphi_{v(1),\cdots,v(n)}(x_1, \cdots, x_n) &= \langle x_1, v(1)\rangle_* \cdots \langle x_n, v(n)\rangle_* \\ &= \langle v(1), x_1\rangle \cdots \langle v(n), x_n\rangle. \end{aligned} $$

Since Y_j is an orthonormal basis of $H_j, j = 1, \cdots, n$, it follows from Proposition 5.1.2 that the set $\{p(y_1, \cdots, y_n) : y_1 \in Y_1, \cdots, y_n \in Y_n\}$ is an orthonormal basis of H, and that

$$ \begin{aligned} \langle p(v_1, \cdots, v_n), p(w_1, \cdots, w_n)\rangle &= \langle w_1, v_1\rangle_* \cdots \langle w_n, v_n\rangle_* \\ &= \langle v_1, w_1\rangle \cdots \langle v_n, w_n\rangle, \\ \|p(v_1, \cdots, v_n\|_2 &= \|v_1\| \cdots \|v_n\|. \end{aligned} $$

From the preceding paragraph, $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H}$ is a bounded multilinear mapping, we prove next that it is a weak Hilbert-Schmidt mapping. For this, suppose that $\varphi \in \mathcal{H}$, and consider the bounded multilinear functional $p_\varphi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbf{C}$ defined by

$$ p_\varphi(x_1, \cdots, x_n) = \langle p(x_1, \cdots, x_n), \varphi\rangle. $$

With $y(1)$ in $Y_1, \cdots, y(n)$ in Y_n, orthonormality of the bases implies that $\varphi_{y(1),\cdots,y(n)}$ takes the value 1 at $(y(1), \cdots, y(n))$ and 0 elsewhere on $Y_1 \times \cdots \times Y_n$. Thus

$$\begin{aligned} p_\varphi(y(1), \cdots, y(n)) &= \langle p(y(1), \cdots, y(n)), \varphi \rangle = \langle \varphi_{y(1),\cdots,y(n)}, \varphi \rangle \\ &= \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} \varphi_{y(1),\cdots,y(n)}(y_1, \cdots, y_n) \overline{\varphi(y_1, \cdots, y_n)} \\ &= \overline{\varphi(y(1), \cdots, y(n))}, \end{aligned}$$

$$\sum_{y(1) \in Y_1} \cdots \sum_{y(n) \in Y_n} |p_\varphi(y(1), \cdots, y(n))|^2 = \|\varphi\|_2^2.$$

From this, p_φ is a Hilbert-Schmidt functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ and $\|p_\varphi\|_2 = \|\varphi\|_2$; so $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H}$ is a weak Hilbert-Schmidt mapping with $\|p\|_2 = 1$.

Suppose next that L is a weak Hilbert-Schmidt mapping from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into another Hilbert space $\mathcal{K}$. If $u \in \mathcal{K}$ and L_u is the Hilbert-Schmidt functional occurring in Definition 5.1.3, while $\varphi \in \mathcal{H}$ and $\mathbf{F}$ is a finite subset of $Y_1 \times \cdots \times Y_n$, we have

$$
\begin{aligned}
&|\langle \sum_{(y_1,\cdots,y_n)\in\mathbf{F}} \varphi(y_1,\cdots,y_n)L(y_1,\cdots,y_n),u\rangle| \\
\leq & \sum_{(y_1,\cdots,y_n)\in\mathbf{F}} |\varphi(y_1,\cdots,y_n)||L_u(y_1,\cdots,y_n)| \\
\leq & \left[\sum_{(y_1,\cdots,y_n)\in\mathbf{F}} |\varphi(y_1,\cdots,y_n)|^2\right]^{\frac{1}{2}} \left[\sum_{(y_1,\cdots,y_n)\in\mathbf{F}} |L_u(y_1,\cdots,y_n)|^2\right]^{\frac{1}{2}} \\
\leq & \|L_u\|_2 \left[\sum_{(y_1,\cdots,y_n)\in\mathbf{F}} |\varphi(y_1,\cdots,y_n)|^2\right]^{\frac{1}{2}} \\
\leq & \|u\|\|L\|_2 \left[\sum_{(y_1,\cdots,y_n)\in\mathbf{F}} |\varphi(y_1,\cdots,y_n)|^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\| \sum_{(y_1,\cdots,y_n)\in\mathbf{F}} \varphi(y_1,\cdots,y_n)L(y_1,\cdots,y_n)\| \\
&\leq \|L\|_2 \left[\sum_{(y_1,\cdots,y_n)\in\mathbf{F}} |\varphi(y_1,\cdots,y_n)|^2\right]^{\frac{1}{2}}. \qquad (4)
\end{aligned}
$$

Since

$$\sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n} |\varphi(y_1,\cdots,y_n)|^2 = \|\varphi\|_2^2 < \infty,$$

it follows from (4) and the Cauchy criterion that the sum

$$\sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n} \varphi(y_1,\cdots,y_n)L(y_1,\cdots,y_n)$$

converges to an element $T\varphi$ of K, and $\|T\varphi\| \leq \|L\|_2\|\varphi\|_2$. Thus T is a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}$, and $\|T\| \leq \|L\|_2$. When $y(1) \in Y_1, \cdots, y(n) \in$

Y_n, we have

$$
\begin{aligned}
Tp((y(1), \cdots, y(n)) &= T\varphi_{y(1),\cdots,y(n)} \\
&= \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} \varphi_{y(1),\cdots,y(n)}(y_1, \cdots, y_n) L(y_1, \cdots, y_n) \\
&= L(y(1), \cdots, y(n)).
\end{aligned}
$$

Since L and Tp are both bounded and multilinear and Y_m has closed linear span H_m $(m = 1, \cdots, n)$, it follows that $L = Tp$.

The condition $Tp = L$ uniquely determines the bounded linear operator T, because the range of contains the orthonormal basis $p(Y_1 \times \cdots \times Y_n)$ of $\mathcal{H}$. For each u in $\mathcal{K}$, Parseval's equation gives

$$
\begin{aligned}
\|L_u\|_2^2 &= \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\langle L(y_1, \cdots, y_n), u\rangle|^2 \\
&= \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\langle Tp(y_1, \cdots, y_n), u\rangle|^2 \\
&= \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\langle p(y_1, \cdots, y_n), T^*u\rangle|^2 \\
&= \|T^*u\|^2 \leq \|T\|^2 \|u\|^2;
\end{aligned}
$$

so $\|L\|_2 \leq \|T\|$, and thus $\|L\|_2 = \|T\|$.

It remains to prove part (2) of the theorem. For this, suppose that $\mathcal{H}'$ and $p' : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H}'$ have the properties set out in (1). When $\mathcal{K}$ is $\mathcal{H}'$ and L is p', the equation $L = Tp'$ is satisfied when T is the identity operator on $\mathcal{H}'$, and also when T is the projection from $\mathcal{H}'$ onto the closed subspace $[p'(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)]$ generated by the range $p'(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)$ of p'. From the uniqueness of T,

$$[p'(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)] = \mathcal{H}',$$

moreover,

$$\|p'\|_2 = \|L\|_2 = \|T\| = \|I\| = 1.$$

With the same choice, $\mathcal{K} = \mathcal{H}'$ and $L = p'$, it follows, from the properties of $\mathcal{H}$ and p set out in (1), that there is a bounded linear operator U from $\mathcal{H}$ into $\mathcal{H}'$ such that $p' = Up$ and

$$\|U\| = \|L\|_2 = \|p'\|_2 = 1.$$

The roles of $\mathcal{H}, p$, and $\mathcal{H}', p'$, can be reversed in this argument, so there is a bounded linear operator U' from $\mathcal{H}'$ into $\mathcal{H}$ such that $p = U'p'$ and $\|U'\| = 1$. Since

$$U'Up(x_1, \cdots, x_n) = U'p'(x_1, \cdots, x_n) = p(x_1, \cdots, x_n),$$

for all x_1 in $\mathcal{H}_1, \cdots, x_n$ in $\mathcal{H}_n$, while

$$[p(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)] = \mathcal{H},$$

it follows that $U'U$ is the identity operator on $\mathcal{H}$; and similarly, UU' is the identity operator on $\mathcal{H}'$. Finally,

$$\|x\| = \|U'Ux\| \leq \|Ux\| \leq \|x\|, x \in \mathcal{H};$$

so $\|Ux\| = \|x\|$, and U is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$. ■

By part (2) of Theorem 5.1.4, the Hilbert space $\mathcal{H}$ appearing in that theorem, together with the multilinear mapping $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H}$, is uniquely determined by the universal property set out in (1). We describe $\mathcal{H}$ as the *Hilbert tensor product* of $\mathcal{H}_1, \cdots, \mathcal{H}_n$, denoted by $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, and refer to p as the canonical mapping from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. The vector $p(x_1, \cdots, x_n)$ in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is usually denoted by $x_1 \otimes \cdots \otimes x_n$. Finite linear combinations of these simple tensors form an everywhere-dense subspace of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$; indeed, if Y_m is an orthonormal basis of $\mathcal{H}_m, m = 1, \cdots, n$, then

$$\{y_1 \otimes \cdots \otimes y_n : y_1 \in Y_1, \cdots, y_n \in Y_n\}$$

is an orthonormal basis of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Thus

$$\dim(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) = \dim \mathcal{H}_1 \dim \mathcal{H}_2 \cdots \dim \mathcal{H}_n.$$

As the notation suggests, the vector $x_1 \otimes \cdots \otimes x_n$ behaves in some respects like a formal product of $x_1, \cdots, x_n$; for example, it results from the multilinearity of p, and from Theorem 5.1.4, that

$$\begin{aligned} & x_1 \otimes \cdots \otimes x_{m-1} \otimes (ax_m^{'} + bx_m^{''}) \otimes x_{m+1} \otimes \cdots \otimes x_n \\ & = a(x_1 \otimes \cdots \otimes x_{m-1} \otimes x_m^{'} \otimes x_{m+1} \otimes \cdots \otimes x_n) \\ & \quad + b(x_1 \otimes \cdots \otimes x_{m-1} \otimes x_m^{''} \otimes x_{m+1} \otimes \cdots \otimes x_n), \end{aligned} \tag{5}$$

$$\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \rangle = \langle x_1, y_1 \rangle \cdots \langle x_n, y_n \rangle, \tag{6}$$

$$\|x_1 \otimes \cdots \otimes x_n\| = \|x_1\| \cdots \|x_n\|. \tag{7}$$

In studying tensor products of Hilbert spaces, the properties just listed are usually more important than the detailed constructions employed in the proof of Theorem 5.1.4. Many of the arguments involve two stages; the first stage deals with the linear span $\mathcal{H}_0$ of the simple tensors, and is based on the identities (5)-(7), while the second employs extension by continuity from $\mathcal{H}_0$ to its closure $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Since

$$a(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (ax_1) \otimes x_2 \otimes \cdots \otimes x_n,$$

$\mathcal{H}_0$ consists of all finite *sums* of simple tensors. In dealing with $\mathcal{H}_0$, it is important to bear in mind that the simple tensors are not linearly independent. Relation (5) can be viewed as the assertion that a certain linear combination of three simple tensors is zero, and repeated application of (5) yields more complicated identities of this type. We shall look at this question in more detail in Proposition 5.1.5. In the meantime, we establish the associativity of the tensor product.

Proposition 6.1.5. *If $\mathcal{H}_1, \cdots, \mathcal{H}_{m+n}$ are Hilbert spaces, there is a unique unitary transformation U from $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{m+n}$ onto*

$$(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m) \otimes (\mathcal{H}_{m+1} \otimes \cdots \otimes \mathcal{H}_{m+n})$$

such that

$$U(x_1 \otimes \cdots \otimes x_{m+n}) = (x_1 \otimes \cdots \otimes x_m) \otimes (x_{m+1} \otimes \cdots \otimes x_{m+n}) \tag{8}$$

whenever $x_j \in \mathcal{H}_j, j = 1, \cdots, m+n$.

Proof. Since the set of all simple tensors in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{m+n} = \mathcal{K}$ has linear span everywhere dense in $\mathcal{K}$, there is at most one unitary operator U with the stated property; so it suffices to prove the existence of such an isomorphism. For this, let

$$\mathcal{K}' = (\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m) \otimes (\mathcal{H}_{m+1} \otimes \cdots \otimes \mathcal{H}_{m+n}),$$

and when $x_j \in \mathcal{H}_j$ $(j = 1, \cdots, m+n)$, define

$$\begin{aligned} p(x_1, \cdots, x_{m+n}) &= x_1 \otimes \cdots \otimes x_{m+n} \\ p'(x_1, \cdots, x_{m+n}) &= (x_1 \otimes \cdots \otimes x_m) \otimes (x_{m+1} \otimes \cdots \otimes x_{m+n}). \end{aligned}$$

The ranges of p and $p^{'}$ contain orthonormal bases of $\mathcal{K}$ and $\mathcal{K}^{'}$, and so generate everywhere-dense subspaces X and $X^{'}$. If $x_j, y_j \in \mathcal{H}_j, j = 1, \cdots, m+n$, we have

$$\begin{aligned} & \langle p(x_1, \cdots, x_{m+n}), p(y_1, \cdots, y_{m+n})\rangle \\ = & \langle x_1, y_1\rangle \cdots \langle x_m, y_m\rangle\langle x_{m+1}, y_{m+1}\rangle \cdots \langle x_{m+n}, y_{m+n}\rangle \\ = & \langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m\rangle\langle x_{m+1} \otimes \cdots \otimes x_{m+n}, y_{m+1} \otimes \cdots \otimes y_{m+n}\rangle \\ = & \langle p^{'}(x_1, \cdots, x_{m+n}), p^{'}(y_1, \cdots, y_{m+n})\rangle. \end{aligned}$$

From this,

$$\begin{aligned} \|\sum_{k=1}^{q} a_k p(x_1^{(k)}, \cdots, x_{m+n}^{(k)})\| &= \sum_{k=1}^{q}\sum_{l=1}^{q} a_k \overline{a_l}\langle p(x_1^{(k)}, \cdots, x_{m+n}^{(k)}), p(x_1^{(l)}, \cdots, x_{m+n}^{(l)})\rangle \\ &= \sum_{k=1}^{q}\sum_{l=1}^{q} a_k \overline{a_l}\langle p^{'}(x_1^{(k)}, \cdots, x_{m+n}^{(k)}), p^{'}(x_1^{(l)}, \cdots, x_{m+n}^{(l)})\rangle \\ &= \|\sum_{k=1}^{q} a_k p^{'}(x_1^{(k)}, \cdots, x_{m+n}^{(k)})\|^2 \end{aligned}$$

whenever $a_k \in \mathbf{C}$ and $x_j^{(k)} \in \mathcal{H}_j, j = 1, \cdots, m+n; k = 1, \cdots, q$.

The remainder of the argument is of frequently recurring type. The equation

$$U_0(\sum_{k=1}^{q} a_k p(x_1^{(k)}, \cdots, x_{m+n}^{(k)})) = \sum_{k=1}^{q} a_k p^{'}(x_1^{(k)}, \cdots, x_{m+n}^{(k)})$$

defines a norm-preserving linear mapping U_0 from X onto $X^{'}$. The definition is unambiguous since, given two expressions $\sum a_k p(x_1^{(k)}, \cdots, x_{m+n}^{(k)})$ and $\sum b_l p(y_1^{(l)}, \cdots, y_{m+n}^{(l)})$ for a vector x in X, it follows, upon replacing $\sum a_k p(x_1^{(k)}, \cdots, x_{m+n}^{(k)})$ by $\sum a_k p(x_1^{(k)}, \cdots, x_{m+n}^{(k)}) - \sum b_l p(y_1^{(l)}, \cdots, y_{m+n}^{(l)})$ in the last chain of equations, that the two corresponding expressions

$$\sum a_k p^{'}(x_1^{(k)}, \cdots, x_{m+n}^{(k)}),\ \ \sum b_l p^{'}(y_1^{(l)}, \cdots, y_{m+n}^{(l)})$$

for $U_0 x$ are equal. By continuity, U_0 extends to an isomorphism U from $\mathcal{K}$ onto $\mathcal{K}^{'}$, and

$$\begin{aligned} U(x_1 \otimes \cdots \otimes x_{m+n}) &= U(p(x_1, \cdots, x_{m+n})) \\ &= p^{'}(x_1, \cdots, x_{m+n}) \\ &= (x_1 \otimes \cdots \otimes x_m) \otimes (x_{m+1} \otimes \cdots \otimes x_{m+n}). \end{aligned}$$

■

By use of the associativity established in the preceding proposition, questions concerning the n-fold tensor product of Hilbert spaces can usually be reduced to the particular case $n = 2$. Finally, we consider the question of linear dependence of simple tensors.

Proposition 6.1.6. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and $\mathcal{H}_0$ is the everywhere-dense subspace of $\mathcal{H}$ generated by the simple tensors.*

(1) If $x_1, \cdots, x_n \in \mathcal{H}_1$, $y_1, \cdots, y_n \in \mathcal{H}_2$, then $\sum_{j=1}^n x_j \otimes y_j = 0$ if and only if there is an $n \times n$ complex matrix $[c_{jk}]$ such that

$$\sum_{j=1}^n c_{jk} x_j = 0,\ k = 1, \cdots, n,$$

$$\sum_{k=1}^n c_{jk} y_k = y_j,\ j = 1, \cdots, n.$$

(2) If L is a bilinear mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ into a complex vector space $\mathcal{K}$, there is a (unique) linear mapping T from $\mathcal{H}_0$ into $\mathcal{K}$ such that $L(x, y) = T(x \otimes y)$ for each x in $\mathcal{H}_1$ and y in $\mathcal{H}_2$.

Proof. (1). If there is a matrix $[c_{jk}]$ with the stated properties, bilinearity of the mapping $(x, y) \to x \otimes y$ implies that

$$\begin{aligned} \sum_{j=1}^{n} x_j \otimes y_j &= \sum_{j=1}^{n} x_j \otimes (\sum_{k=1}^{n} c_{jk} y_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j \otimes y_k \\ &= \sum_{k=1}^{n} (\sum_{j=1}^{n} c_{jk} x_j) \otimes y_k = 0. \end{aligned}$$

Conversely, suppose that $\sum_{j=1}^{n} x_j \otimes y_j = 0$. If $v_1, \cdots, v_r$ is an orthonormal basis of the linear subspace of $\mathcal{H}_2$ generated by $y_1, \cdots, y_n$, we can choose an $n \times r$ matrix $A = [a_{jk}]$ and an $r \times n$ matrix $B = [b_{jk}]$ such that

$$y_j = \sum_{l=1}^{r} a_{jl} v_l, j = 1, \cdots, n,$$

$$v_l = \sum_{k=1}^{n} b_{lk} y_k, l = 1, \cdots, r.$$

With $[c_{jk}]$ the $n \times n$ matrix AB, we have

$$y_j = \sum_{l=1}^{r} a_{jl} (\sum_{k=1}^{n} b_{lk} y_k) = \sum_{k=1}^{n} c_{jk} y_k, j = 1, \cdots, n,$$

and

$$0 = \sum_{j=1}^{n} x_j \otimes y_j = \sum_{j=1}^{n} x_j \otimes (\sum_{l=1}^{r} a_{jl} v_l) = \sum_{l=1}^{r} u_l \otimes v_l,$$

where

$$u_l = \sum_{j=1}^{n} a_{jl} x_j, l = 1, \cdots, r.$$

For each $m = 1, \cdots, r$,

$$0 = \sum_{l=1}^{r} \langle u_l \otimes v_l, u_m \otimes v_m \rangle = \sum_{l=1}^{r} \langle u_l, u_m \rangle \langle v_l, v_m \rangle = \|u_m\|^2.$$

Thus $u_1 = u_2 = \cdots = u_r = 0$, and

$$\sum_{j=1}^{n} c_{jk} x_j = \sum_{j=1}^{n} \sum_{l=1}^{r} a_{jl} b_{lk} x_j = \sum_{l=1}^{r} b_{lk} u_l = 0, k = 1, \cdots, n.$$

(2). Suppose that L is a bilinear mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ into $\mathcal{K}$. If $x_1, \cdots, x_n \in \mathcal{H}_1$, $y_1, \cdots, y_n \in \mathcal{H}_2$ and $\sum_{j=1}^{n} x_j \otimes y_j = 0$, we can choose a matrix $[c_{jk}]$ as in (1). The bilinearity of L then entails

$$\begin{aligned} \sum_{j=1}^{n} L(x_j, y_j) &= \sum_{j=1}^{n} L\left(x_j, \sum_{k=1}^{n} c_{jk} y_k\right) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} c_{jk} L(x_j, y_k) = \sum_{k=1}^{n} L\left(\sum_{j=1}^{n} c_{jk} x_j, y_k\right) = 0. \end{aligned}$$

Suppose next that $x_1, \cdots, x_n, u_1, \cdots, u_m \in \mathcal{H}_1$, $y_1, \cdots, y_n, v_1, \cdots, v_m \in \mathcal{H}_2$ and $\sum_{j=1}^{n} x_j \otimes y_j = \sum_{j=1}^{m} u_j \otimes v_j$. Then

$$\sum_{j=1}^{n} x_j \otimes y_j + \sum_{j=1}^{m} (-u_j) \otimes v_j = 0;$$

the preceding paragraph shows that

$$\sum_{j=1}^{n} L(x_j, y_j) + \sum_{j=1}^{m} L(-u_j, v_j) = 0,$$

and therefore

$$\sum_{j=1}^{n} L(x_j, y_j) = \sum_{j=1}^{m} L(u_j, v_j)$$

From this, it follows that the equation

$$T(\sum_{j=1}^{n} x_j \otimes y_j) = \sum_{j=1}^{n} L(x_j, y_j).$$

defines a linear operator T from $\mathcal{H}_0$ into $\mathcal{K}$; and $T(x \otimes y) = L(x, y)$, $x \in \mathcal{H}_1, y \in \mathcal{H}_2$. ■

6.2 Tensor Product of Bounded Linear Operators

Now, we study the tensor product of bounded linear operators.

Proposition 6.2.1. *If $\mathcal{H}_1, \cdots, \mathcal{H}_n, \mathcal{K}_1, \cdots, \mathcal{K}_n$ are Hilbert spaces, and let $A_m \in B(\mathcal{H}_m, \mathcal{K}_m)$, $m = 1, \cdots, n$, then there is a bounded linear operator A from $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ into $\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_n$ such that*

$$A(x_1 \otimes \cdots \otimes x_n) = A_1 x_1 \otimes \cdots \otimes A_n x_n, x_1 \in \mathcal{H}_1, \cdots, x_n \in \mathcal{H}_n.$$

Proof. The canonical mapping $p : \mathcal{K}_1 \times \cdots \times \mathcal{K}_n \to \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_n = K$ is a weak Hilbert-Schmidt mapping, with $\|p\|_2 = 1$. With u in $\mathcal{K}$, and p_u defined by

$$p_u(z_1, \cdots, z_n) = \langle p(z_1, \cdots, z_n), u\rangle,$$

p_u is a Hilbert-Schmidt functional on $\mathcal{K}_1 \times \cdots \times \mathcal{K}_n$, and $\|p_u\|_2 \leq \|u\|$. The equation

$$\varphi(x_1, \cdots, x_n) = p(A_1 x_1, \cdots, A_n x_n)$$

defines a bounded multilinear mapping $\varphi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{K}$, and

$$\begin{aligned} \varphi_u(x_1, \cdots, x_n) &= \langle \varphi(x_1, \cdots, x_n), u\rangle \\ &= \langle p(A_1 x_1, \cdots, A_n x_n), u\rangle \\ &= p_u(A_1 x_1, \cdots, A_n x_n). \end{aligned}$$

It now follows from Proposition 5.1.1 that φ_u is a Hilbert-Schmidt functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, with

$$\|\varphi_u\|_2 \leq \|A_1\| \cdots \|A_n\| \|p_u\|_2 \leq \|A_1\| \cdots \|A_n\| \|u\|.$$

Accordingly, $\varphi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{K}$ is a weak Hilbert-Schmidt mapping, with $\|\varphi\|_2 \le \|A_1\| \cdots \|A_n\|$. By the universal property of the tensor product, there is a unique bounded linear operator A, from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into $\mathcal{K}$, such that $\varphi = Ap^{'}$, where $p^{'}$ is the canonical mapping from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Moreover,

$$\|A\| = \|\varphi\|_2 \le \|A_1\| \cdots \|A_n\|.$$

Also,

$$\begin{aligned} A(x_1 \otimes \cdots \otimes x_n) &= Ap^{'}(x_1, \cdots, x_n) = \varphi(x_1, \cdots, x_n) \\ &= p(A_1x_1, \cdots, A_nx_n) = A_1x_1 \otimes \cdots \otimes A_nx_n, \end{aligned}$$

when $x_1 \in \mathcal{H}_1, \cdots, x_1 \in \mathcal{H}_n$. ■

The operator A described in Proposition 5.2.1 is called the tensor product of $A_1, \cdots, A_n$ and denoted by $A_1 \otimes \cdots \otimes A_n$. It is apparent that $A_1 \otimes \cdots \otimes A_n$ depends linearly on each A_m and that

$$(A_1 \otimes \cdots \otimes A_n)(B_1 \otimes \cdots \otimes B_n) = A_1B_1 \otimes \cdots \otimes A_nB_n.$$

Since

$$\begin{aligned} & \langle (A_1 \otimes \cdots \otimes A_n)(x_1 \otimes \cdots \otimes x_n), y_1 \otimes \cdots \otimes y_n \rangle \\ = \ & \langle A_1x_1 \otimes \cdots \otimes A_nx_n, y_1 \otimes \cdots \otimes y_n \rangle \\ = \ & \langle A_1x_1, y_1 \rangle \cdots \langle A_nx_n, y_n \rangle \\ = \ & \langle x_1, A_1^*y_1 \rangle \cdots \langle x_n, A_n^*y_n \rangle \\ = \ & \langle x_1 \otimes \cdots \otimes x_n, A_1^*y_1 \otimes \cdots \otimes A_n^*y_n \rangle \\ = \ & \langle x_1 \otimes \cdots \otimes x_n, (A_1^* \otimes \cdots \otimes A_n^*)(y_1 \otimes \cdots \otimes y_n) \rangle, \end{aligned}$$

it follows by linearity and continuity that

$$\langle (A_1 \otimes \cdots \otimes A_n)u, v\rangle = \langle u, (A_1^* \otimes \cdots \otimes A_n^*)v\rangle$$

for all vectors u and v in the appropriate tensor product spaces. Thus

$$(A_1 \otimes \cdots \otimes A_n)^* = A_1^* \otimes \cdots \otimes A_n^*.$$

We assert also that

$$\|A_1 \otimes \cdots \otimes A_n\| = \|A_1\| \cdots \|A_n\|.$$

Indeed, given by any unit vectors x_1 in $\mathcal{H}_1, \cdots, x_n$ in $\mathcal{H}_n$, we have

$$\begin{aligned} \|A_1 \otimes \cdots \otimes A_n\| &= \|A_1 \otimes \cdots \otimes A_n\| \|x_1 \otimes \cdots \otimes x_n\| \\ &\geq \|(A_1 \otimes \cdots \otimes A_n)(x_1 \otimes \cdots \otimes x_n)\| \\ &= \|A_1 x_1 \otimes \cdots \otimes A_n x_n\| = \|A_1 x_1\| \cdots \|A_n x_n\|. \end{aligned}$$

Upon taking the supremum of the right-hand side, as the unit vectors $x_1, \cdots, x_n$ vary, we obtain

$$\|A_1 \otimes \cdots \otimes A_n\| \geq \|A_1\| \cdots \|A_n\|;$$

the reverse inequality is clear.

Suppose next that $\mathcal{H}_1, \cdots, \mathcal{H}_{m+n}, \mathcal{K}_1, \cdots, \mathcal{K}_{m+n}$ are Hilbert spaces and that $A_j \in B(\mathcal{H}_j, \mathcal{K}_j)$ $(j = 1, \cdots, m+n)$. We can construct unitary transforms

$$U : \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{m+n} \to (\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m) \otimes (\mathcal{H}_{m+1} \otimes \cdots \otimes \mathcal{H}_{m+n}),$$

$$V : \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_{m+n} \to (\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_m) \otimes (\mathcal{K}_{m+1} \otimes \cdots \otimes \mathcal{K}_{m+n}),$$

as in Proposition 5.1.5, and it is at once verified that

$$V(A_1 \otimes \cdots \otimes A_{m+n})U^{-1} = (A_1 \otimes \cdots \otimes A_m) \otimes (A_{m+1} \otimes \cdots \otimes A_{m+n}).$$

This proves the associativity of the tensor product of bounded linear operators on Hilbert spaces.