Partial Differential Equations: an introduction

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CHAPTER 1

Introduction

Background; Basic concepts; Real-valued first order PDE and method of characteristics; Power series method and Cauchy-Kowalevskaya theorem; Well-posed problem

Webpage: http://www.mathweb.zju.edu.cn:8080/wang/21PDE/

Our main references will be Folland [**9**] and 陈恕行[**35**] (in Chinese).

Some other references can also be useful:

- Undergraduate PDE books: Strauss [**26**], 谷超豪等[**34**] (in Chinese), 齐民友等[**36**] (in Chinese), 姜礼尚等[**33**] (in Chinese), and Folland [**8**].
- Some other books on PDE: Evans [**7**], John [**16**], Courant-Hilbert [**5**], Jost [**17**], Rauch [**19**], Klainerman [**12**, IV.12].
- Some online resource: Klainerman [**18**].
- History: Brezis-Browder [**3**], Dieudonné [**6**].

1.1. Background

Where do Partial Differential Equations (PDE) come from? c.f. Brezis-Browder [**3**] Section 1& 2, Strauss [**26**] Chapter 1& 13.

The study of partial differential equations started in the 18th century in the works of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continuum and more generally, as the principal model of analytical study of models in the physical science. The analysis of physical models has remained to the present day one of the fundamental concerns of the development of PDE's. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDE's also became an essential tool in other branches of mathematics.

1.1.1. Early history.

(1) wave equations

$$u_{tt} - c^2 \Delta u = 0, \text{ where } \Delta u = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u$$

The one dimensional wave equation ($n = 1$) was introduced and analyzed by d'Alembert in 1752 as a model of a vibrating string. His work was extended by Euler (1759) and later by D. Bernoulli (1762) to 2 and 3 dimensional wave equations in the study of acoustic waves. See See Subsection 1.1.5 for a derivation in the case of vibrating string.

(2) the Laplace equation

$$\Delta u = 0$$

was first studied by Laplace in his work on gravitational fields, in around 1780 (outside the support of the mass distribution). See subsection 1.1.6.

(3) the Poisson equation for gravity:

$$\Delta V = 4\pi G\rho \ . \tag{1.1}$$

Poisson 1813: gravitational potential V generated by mass distribution ρ. See subsection 1.1.6.

(4) heat equation $u_t - k\Delta u = 0$ (also known as the diffusion equation). See Subsection 1.1.7 for a derivation. Fourier 1810-22: Fourier series, Fourier transform.

1.1.2. (Mathematical) Physics.

(1) The Euler equation of fluid flows, 1755.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0, \frac{\partial v_j}{\partial t} + v\cdot\nabla v_j = F_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j}, p = f(\rho)$$

equation of continuity (conservation of mass), motion (conservation of momentum) and state. ρ density, v velocity, p pressure, ν viscosity.

(2) The Navier-Stokes equations for fluid flows, with viscosity, in 1822 1827 by Navier, followed by Poisson (1831) and Stokes (1845).

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0, \frac{\partial v_j}{\partial t} + v\cdot\nabla v_j - \nu\Delta v_j = F_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j}, p = f(\rho)$$

(3) Maxwell's equation in electromagnetic theory in 1864. For the electric field E and the magnetic field B (which are both maps $\Omega\times\mathbb{R}\to\mathbb{R}^3$) corresponding to a charge density ρ and current density J (which takes value in $\mathbb{R}$ and $\mathbb{R}^3$ respectively), it states that

$$\nabla\cdot E = \frac{\rho}{\epsilon_0}, \nabla\times E = -\partial_t B, \nabla\times B = \mu_0(J + \epsilon_0\partial_t E), \nabla\cdot B = 0\ , \tag{1.2}$$

where ϵ_0 and μ_0 are (positive) physical constants, reflecting the choice of physical units, called the permittivity and permeability of free space, respectively. The equations (1.2) imply the continuity equation $\partial_t\rho + \nabla\cdot J = 0$.

Special cases of the Maxwell equations are well known by themselves. The first equation is called Coulomb's law, while the second is called Faraday's law. In case E does not depend on time, the third equation reduces to $\nabla\times B = \mu_0 J$, which is Ampère's law. The trouble with Ampère's law is that it requires $\nabla\cdot J = 0$. So if $\nabla\cdot J \neq 0$, Maxwell proposed adding the term $\partial_t E$ for the sake of mathematical consistency, thereby coming up with his complete set of equations (1.2).

One special case of the Maxwell system is electrostatics, i.e., when there is no current and the fields are independent of time t. Then we have $\nabla\times E = 0$, which implies that E is the gradient of a function (in a simply connected region Ω). Conventionally, we write $E = -\nabla V$ and calls V the electrostatic potential. Then

$$\nabla\cdot E = -\Delta V = \frac{\rho}{\epsilon_0},$$

which is the Poisson equation.

On the other hand, in the vacuum region (where there are no charges and currents, $\rho = 0$ and $J = 0$), we have

$$\epsilon_0\mu_0\partial_t E = \nabla \times B, \partial_t B = -\nabla \times E, \nabla \cdot E = \nabla \cdot B = 0 \ .$$

Let $c = (\epsilon_0\mu_0)^{-1/2}$. Recall that

$$\nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \Delta u \ ,$$

one gets

$$\Delta E = -\nabla \times (\nabla \times E) + \nabla(\nabla \cdot E) = \nabla \times B_t = c^{-2}\partial_t^2 E \ ,$$

i.e., E satisfies the wave equation with speed of waves c. Similarly, B satisfies the same wave equation

$$\partial_t^2 B = -\nabla \times (\partial_t E) = -c^2\nabla \times (\nabla \times B) = c^2\Delta B - c^2\nabla(\nabla \cdot B) = c^2\Delta B \ .$$

These waves are the electromagnetic waves, i.e. light.

(4) The Korteweg-de Vries equation (1896) as a model for solitary water waves

$$u_t + uu_x + u_{xxx} = 0 \ .$$

(5) General relativity (Einstein 1915): the Einstein field equations for Lorentzian $1+3$ space-time manifolds (M, g),

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}, 0 \le \alpha, \beta \le 3 \ ,$$

which could be viewed as the geometrization of the gravity.

Hilbert figured that this could be viewed as Lagrangian, and proposed

$$S_H = \int \sqrt{-|g|}RdV$$

known as the **Hilbert action** (or sometimes the Einstein-Hilbert action)[1].

In the Newtonian limit (weak, static, gravitational fields with speed much less than the speed of limit), it recovers the Poisson equation.

In the vacuum, we have $T = 0$ and so we obtain the Einstein's vacuum equation:

$$R_{\alpha\beta} = 0 \ .$$

If we introduce the harmonic gauge/wave gauge, $\Box_g x^\alpha = 0$, it turns out that is could be rewritten in the following form

$$\Box_g g_{\alpha\beta} = Q_{\alpha\beta}(g, \partial g) \ ,$$

where Q is quadratic in ∂g. This equation could be viewed as a nonlinear version of the wave equations.

(6) Quantum mechanics (wave mechanics, Schrödinger 1926). In classical mechanics, single particle of mass m moving in fields with potential energy V, satisfies

$$E = \frac{p^2}{2m} + V$$

where p is the momentum and E is the energy. Through quantization,

$$E \to i\hbar\partial_t, p \to -i\hbar\nabla \ ,$$

[1]See, e.g., Sean M. Carroll, Spacetime and Geometry: An Introduction to General Relativity, 2003, p159–165.

where $h = 2\pi\hbar$ is the Planck's constant, we obtain the Schrödinger equation, for the wave functions ψ of the particle

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \ .$$

(7) Quantum field theory: Klein-Gordon equation $u_{tt} - c^2\Delta u + m^2 u = 0$; Dirac's equation for electron ("square root" of the Klein-Gordon equation)

$$c^{-1}\gamma^0 u_t + \gamma^1 u_x + \gamma^2 u_y + \gamma^3 u_z + imu = 0$$

γ^j certain 4×4 matrices; nonlinear Klein-Gordon equation $u_{tt} - c^2\Delta u + m^2 u + gu^3 = 0$, a model for mesons (介子); Yang-Mills equations; etc.

1.1.3. Various areas of mathematics. It is even more surprising that equations, originally introduced to describe specific physical phenomena, also play a fundamental role in areas of mathematics, which are considered pure, such as Complex Analysis, Differential Geometry, Topology and Algebraic Geometry. Other branches of math, for example

(1) Fourier analysis: originated in the study of the heat equation
(2) Complex analysis: Cauchy 1827-Riemann 1851 equation for analytic $U = u + iv$ of $z = x + iy$, if

$$u_x = v_y, u_y = -v_x, \Leftrightarrow \partial_{\bar z} U = 0 \ . \tag{1.3}$$

(3) The minimal surface equation by Lagrange in 1760 (the first major application of the Euler-Lagrange principle in PDE's)

$$\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \ . \tag{1.4}$$

(4) Uniformization theorem for compact Riemann surface (Poincaré, Koebe 1907)

Theorem 1.1. *Let S be a 2-dimensional, compact, Riemann surface with metric g, Gauss curvature $K = K(g)$ and Euler characteristic $\chi(S)$. There exists a conformal transformation of the metric g, i.e. $\tilde g = \Omega^2 g$, for some smooth non-vanishing function Ω, such that the Gauss curvature $\tilde K$ of the new metric g is identical equal to $1, 0$ or -1 according to whether $\chi(S) > 0$, 0 or $\chi(S) < 0$.*

Let $\Omega = e^u$, reduced to $\Delta_S u + e^{2u} = K$.

(5) Geometry/Topology, Thurston's geometrization conjecture, including the well known Poincaré conjecture. Poincaré conjecture: a compact simply connected smooth n-dimensional manifold must be homeomorphic to the n-sphere S^n. Introduction of Ricci flow (R. Hamilton 1982) $\frac{\partial g_{jk}}{\partial t} = -2R_{jk}$
(6) Hodge theory, which plays a fundamental role in topology and algebraic geometry, is based on studying the space of solutions to a class of linear systems of partial differential equations on manifolds which generalize the Cauchy-Riemann equations

1.1.4. applied science. Finally we need to note that PDE arise not only in Physics and Geometry but also in many fields of applied science.

(1) Control Theory.
(2) stochastic differential equations
(3) mathematical finance. A particularly interesting example of a PDE, which is derived from a stochastic process, related to the price of stock options in finance, is the well known Black-Scholes equation.

1.1.5. Derivation of the d'Alembert equation of the vibrating string: variational approach. To find the Lagrangian so that the d'Alembert equation of the vibrating string is the Euler-Lagrange equation.

(1) Object: flexible, elastic string of length L, which undergoes relatively small transverse vibrations. (guitar string, plucked violin string)
(2) Introduce mathematical variables:
    - coordinates: position x, time t, displacement from equilibrium position $u(x,t)$
    - physical consideration: density $\rho(x)$, tension $T = T(x)$, external force per unit mass.
(3) Physical principle: stationary with respect to the Lagrangian density

$$L = K - V$$

where K is the kinetic energy and V is the potential energy.
    - The kinetic energy is given by

$$K = \frac{1}{2}\int_a^b \rho(x)u_t^2 dx$$

    - Tension: tangent to $y = u(x,t)$. Let $T(x)$ be the magnitude of this tension vector, with direction $\theta(x,t) \in (-\pi/2, \pi/2)$

$$\tan\theta(x,t) = \frac{\partial u(x,t)}{\partial x}$$

    - Fix any piece $[a,b]$, the transverse components of the tension force is given by

$$T(b)\sin\theta(b,t) - T(a)\sin\theta(a,t) = \int_a^b \frac{d}{dx}(T(x)\frac{u_x}{\sqrt{1+u_x^2}})dx$$

    - To move the string from the equilibrium position to $u(x)$. Suppose this move is defined by the "curve" $u(x,\alpha)$, $\alpha \in [0,1]$, so that $u(x,0) = 0$, $u(x,1) = u(x)$. For the interval $(x, x+\delta x)$, the force is roughly $F = \frac{d}{dx}(T(x)\frac{u_x}{\sqrt{1+u_x^2}})\delta x$, and the work need to move from α to $\alpha + \delta\alpha$, (moving from $u(x,\alpha)$ to $u(x,\alpha+\delta\alpha)$, $\delta h \sim \partial_\alpha u \delta\alpha$) is roughly

$$-F\delta h = -\frac{d}{dx}(T(x)\frac{u_x}{\sqrt{1+u_x^2}})\partial_\alpha u \delta x \delta\alpha$$

    Integrating with respect to x and α, we see that the total work need to be performed over the segment (a,b) is, which is exactly the potential energy,

$$V = -\int_0^1 \int_a^b \frac{d}{dx}(T(x)\frac{u_x}{\sqrt{1+u_x^2}})\partial_\alpha u dx d\alpha$$

Let's fix the endpoint a, b: $u(a) = u(b) = 0$, then, after integration by parts, we have

$$\begin{aligned} V &= -\int_0^1 \int_a^b \frac{d}{dx}(T(x)\frac{u_x}{\sqrt{1+u_x^2}})\partial_\alpha u dx d\alpha \\ &= -\int_0^1 (T(x)\frac{u_x}{\sqrt{1+u_x^2}})\partial_\alpha u d\alpha|_{x=a}^b + \int_0^1 \int_a^b (T(x)\frac{u_x}{\sqrt{1+u_x^2}})\partial_\alpha \partial_x u dx d\alpha \\ &= \int_0^1 \int_a^b T(x)\partial_\alpha(\sqrt{1+u_x^2}) dx d\alpha \\ &= \int_a^b T(x)(\sqrt{1+u_x^2}-1)dx \end{aligned}$$

(4) the Lagrangian

$$L[u, u_t] = K - V = \frac{1}{2}\int_a^b \rho(x)u_t^2 dx - \int_a^b T(x)(\sqrt{1+u_x^2}-1)dx$$

(5) Calculus of variation: view $\tilde{L} = \int L dt$ as a functional on $u \in C^1([a,b] \times [t_0, t_1])$ with $u(a,t) = u(b,t) = 0$. It's Fréchet derivative

$$\tilde{L}(w) - \tilde{L}(u) = \langle \tilde{L}'(u), w - u\rangle + o(\|w-u\|) \ .$$

Let $w = u + \epsilon\phi$ with $\phi \in C^1$ with $\phi(a,t) = \phi(b,t) = 0$, we see that

$$\langle \tilde{L}'(u), \phi\rangle = \int_{t_0}^{t_1} \int_a^b (\rho(x)u_t\phi_t - T(x)\frac{u_x}{\sqrt{1+u_x^2}}\phi_x dx dt$$

and so

$$\langle \tilde{L}'(u), \phi\rangle = \int_{t_0}^{t_1} \int_a^b [-\rho(x)u_{tt} + \partial_x(T(x)\frac{u_x}{\sqrt{1+u_x^2}})]\phi dx dt$$

As ϕ is arbitrary, we get

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{u_x}{\sqrt{1+u_x^2}}\right) = \frac{T_x u_x}{\sqrt{1+u_x^2}} + T\frac{u_{xx}}{[1+u_x^2]^{3/2}} \ . \tag{1.5}$$

(6) For further simplification, if small transverse vibrations, with $|u_x| \ll 1$, $1 + u_x^2 \simeq 1$, and (as a linear approximation)

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}(T(x)u_x)$$

(7) If homogeneous $\rho(x) = \rho > 0$. Ideally, $T(x) = T > 0$, we obtain the d'Alembert's equation of vibrating string, wave equation

$$u_{tt} = \frac{T}{\rho}u_{xx}$$

Dimension analysis:

$$T \sim m\frac{d^2x}{dt^2} = kg * m * s^{-2}, \rho \sim kg * m^{-1}, \frac{T}{\rho} \sim m^2 * s^{-2} \sim v^2$$

with dimension of speed square, denoted by c^2. Here, in view of the dimension analysis, c may be related with certain speed. Actually, as we shall see, it is the wave speed

$$u_{tt} = c^2 u_{xx} \ . \tag{1.6}$$

1.1.6. Derivation of the Laplace/Poisson equation (1.1)**.** Newton's law of gravitation: for point mass M at $Y \in \mathbb{R}^3$, its (attractive) gravitational force acted on mass m at $X \in \mathbb{R}^3 \backslash \{Y\}$ is given by

$$-GMm\frac{X-Y}{|X-Y|^3} \ ,$$

i.e., it generate a *gravitational field*

$$F(X) = -GM\frac{X-Y}{|X-Y|^3} \ .$$

Thus the *gravitational potential* (V), generated by M, is the gravitational potential energy per unit mass:

$$F = -\nabla V \ .$$

Assuming $V = 0$ at infinity, we see that the field has a potential

$$V(X) = -\frac{GM}{|X-Y|} \ ,$$

which is known as the **Newton potential**.

It is easy to check that

$$\Delta V = 0, \forall X \neq Y \ . \tag{1.7}$$

In conclusion, we see that in vacuum region, the gravitational fields satisfy the Laplace equation.

More generally, by the principle of superposition, the field generated by a massive body with mass density $\rho(Y)$ is given by

$$F(X) = -G\int_{\mathbb{R}^3} \rho(Y)\frac{X-Y}{|X-Y|^3} dY \ ,$$

with *gravitational potential*

$$u(X) = -G\int_{\mathbb{R}^3} \frac{\rho(Y)}{|X-Y|} dY = -G\int_{\mathbb{R}^3} \frac{\rho(X+Y)}{|Y|} dY \ . \tag{1.8}$$

From (1.8), it follows that u satisfies the Poisson equation (1.1).

1.1.6.1. *Proof of* (1.1). Assuming for simplicity that $\rho \in C_c^2(\mathbb{R}^3)$, by (1.8), we have

$$\Delta u(X) = -G \int_{\mathbb{R}^3} \frac{\Delta_X \rho(X+Y)}{|Y|} dY = -G \lim_{\epsilon\to 0+} \int_{|Y|\ge\epsilon} \frac{\Delta_Y \rho(X+Y)}{|Y|} dY \ .$$

Recall (1.7), we see that

$$\begin{aligned} \Delta u(X) &= -G \lim_{\epsilon\to 0+} \int_{|Y|\ge\epsilon} [\frac{1}{|Y|}\Delta_Y \rho(X+Y) - \rho(X+Y)\Delta_Y \frac{1}{|Y|}] dY \\ &= G \lim_{\epsilon\to 0+} \int_{|Y|=\epsilon} \frac{Y}{|Y|} [\frac{1}{|Y|}\nabla_Y \rho(X+Y) - \rho(X+Y)\nabla_Y \frac{1}{|Y|}] dS_Y \\ &= 4\pi G \rho(X) \ , \end{aligned}$$

which is the Poisson equation (1.1).

1.1.7. Derivation of the heat equation. Let $x \in \Omega \subset \mathbb{R}^3$, $u(x,t)$ be the temperature and $H(t)$ be the amount of heat contained in a region $D \subset \Omega$. Then

$$H(t) = \int_D c\rho u dx \ .$$

Heuristically, we know that heat flows from hot to cold regions, which might be related with the gradient fo temperature. **Fourier's law** claims that heat flows proportionately to the temperature gradient $\nabla_x u$ (with direction from cold to hot).

By conservation of the heat energy in D, the change of heat energy in D equals the heat flux across the boundary,

$$H(t+dt) - H(t) = [\int_{\partial D} k(x,t)(n \cdot \nabla_x u) dS] dt$$

where n is the outward unit normal vector. Thus, by the divergence theorem, we get the heat equation

$$c\rho u_t = \nabla \cdot (k \nabla u) \ .$$

1.2. Basic concepts

Differential equations, in it's most general form, is given by

$$F(x, u, \{\nabla^\alpha u\}_{1\le|\alpha|\le k}) = 0, x \in \Omega \subset \mathbb{R}^n$$

where u is the unknown functions(s), F is given function(s), $x = (x^1, x^2, \cdot, x^n) \in \Omega \subset \mathbb{R}^n$, $\partial_j = \frac{\partial}{\partial_{x^j}}$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $\nabla^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} u$, $|\alpha| = \sum \alpha_j$.

order k; solution $u \in C^k(\Omega)$:

$$F(x, u(x), \{\nabla^\alpha u(x)\}_{1\le|\alpha|\le k}) = 0, \ \forall x \in \Omega \ .$$

$n = 1$ ODE, $n > 1$ PDE

linear PDE:

$$Lu := \sum_{|\alpha|\le k} a_\alpha(x) \nabla^\alpha u = f(x),$$

semilinear PDE::

$$\sum_{|\alpha|=k} a_\alpha(x) \nabla^\alpha u = f(x, \{\nabla^\alpha u\}_{|\alpha|\le k-1}),$$

quasilinear PDE:

$$\sum_{|\alpha|=k} a_\alpha(x, (\nabla^\alpha u\}_{|\alpha|\le k-1}))\nabla^\alpha u = f(x, \{\nabla^\alpha u\}_{|\alpha|\le k-1}))$$

Otherwise, fully nonlinear PDE.

General questions: existence, uniqueness, stability, behavior of solutions

Auxiliary conditions: initial conditions (IC), boundary conditions (BC), etc.

IVP=PDE+initial conditions (also known as the Cauchy problem); BVP=PDE+boundary conditions; IBVP=PDE+IC+BC.

1.3. Real-valued first order PDE and method of characteristics

Simplest PDE, method of characteristics; Real-valued first order linear PDE and notion of characteristics; Some higher order PDE; Quasilinear PDE; elliptic operator

1.3.1. Example: Some simplest PDE, reduction to ODE.

Example 1.1. A simplest PDE is

$$\frac{\partial u(x,y)}{\partial x} = 0$$

for $u = u(x, y)$. In practice, it is just ODE in x, if we viewed u as function of x, for fixed y. It is easy to see

$$u(x, y) = g(y)$$

where $g(y)$ is any function depend only on y.

Example 1.2. Let $c \in \mathbb{R}$, consider the general transport equation (传输方程)

$$u_x + cu_y = 0 \tag{1.9}$$

Inspired by **Example 1.1.**, we want to reduce the problem to ODE.

Geometric method: the equation is equivalent to $(1, c) \cdot (u_x, u_y) = 0$, if we denote $V = (1, c)$, then

$$\frac{du}{dV}(x,y) = \lim_{\epsilon\to 0} \frac{u((x,y)+\epsilon V) - u(x,y)}{\epsilon} = V_1 u_x + V_2 u_y = u_x + cu_y = 0$$

which means that $u(x, y)$ must be constant in the direction of V. The vector $(c, -1)$ is orthogonal to V, and the equation of the line parallel to V is $cx - y = constant$ (which are called the characteristic lines [2], see also Section 1.4). The solution is constant on each such line. Therefore, $u(x, y)$ depends on cx–y only. Thus the general solution is given by $u = g(cx - y)$.

Coordinate method Let $x' = x + cy$, $y' = cx - y$,

$$u_x + cu_y = (1 + c^2)u_{x'}$$

and so $u = g(y')$. Here, we see that in the new coordinate system, derivative wrt y' disappear and y' corresponds exactly to the (characteristic) vector $(c, -1)$.

[2] In general, for linear operators $L = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha$ on $\Omega \subset \mathbb{R}^n$, we introduce its characteristic form (also known as principal symbol) at $x \in \Omega$: the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by

$$\chi_L(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

A nonzero vector ξ is called characteristic for L at x if $\chi_L(x, \xi) = 0$, and the set of all such ξ is called the characteristic variety of L at x and is denoted by $\mathrm{char}_x(L) := \{\xi \ne 0 : \chi_L(x, \xi) = 0\}$.

This means that along the (characteristic) direction $(c, -1)$, the PDE fail to be "genuinely 1st order". See also Section 1.4.

IVP: If $u(0, y) = y$, then we see that $g(cx - y)|_{x=0} = g(-y) = y$, $g(x) = -x$, and so $u(x, y) = g(cx - y) = y - cx$.

REMARK 1.2. *We observe that, in general, we could not impose initial condition on the characteristic line.*

1.3.2. Real-valued first order linear PDE and the method of characteristics.

Consider more general linear first order PDE with **real-coefficients** ($a^j \in C(\mathbb{R}^n), b, f \in C(\mathbb{R}^n)$, real-valued, $n \geq 2$)

$$Lu := \sum_{j=1}^{n} a^j(x)\partial_j u + b(x)u := a^j(x)\partial_j u + b(x)u = f(x) \ . \tag{1.10}$$

We see that if we can find certain curve $x = x(t)$, such that the derivative of $u = u(x(t))$ along this curve

$$\frac{du(x(t))}{dt} = \sum (x^j)'(t)(\partial_j u)(x(t)) \approx \sum_{j=1}^{n} a^j(x)\partial_j u \ ,$$

then the solution along the curve satisfies an ODE. Such curves are called characteristic curves. Furthermore, if the characteristic curves can cover certain domain, the equation can be solved in such domain.

Thus, the parametrized curves $x(t)$, characteristic curves, satisfy the following system of ordinary differential equations

$$\frac{dx}{dt} = A(x), \quad \text{i.e.} \quad \frac{dx^j}{dt} = a^j(x), 1 \leq j \leq n \ . \tag{1.11}$$

Along one of those curves a solution u of (1.10) must satisfy

$$\frac{du(x(t))}{dt} = \sum \frac{dx^j}{dt}\frac{du}{dx^j} = \sum a^j u_j = f - bu \ . \tag{1.12}$$

By the fundamental existence and uniqueness theorem for ordinary differential equations (see, e.g., [**4**], [**29**]), through each point $x_0 \in S$, there passes a unique integral curve $x(t)$ of A, namely the solution of (1.11) with $x(0) = x_0$. Along this curve the solution u of (1.10) with data $u(x_0) = y_0$ must be the solution of the ordinary differential equation (1.12), i.e., $g' + b(x(t))g = f(x(t))$, where $g(t) = u(x(t))$ with $g(0) = u(x_0) = y_0$.

Suppose we wish to find a solution u of (1.10) with given initial values $u = \phi$, on the hypersurface S. To avoid possible over-determined situation, i.e., to make the initial value problem well-behaved in general, it is natural to ask that $x(t) \notin S$ for $t \neq 0$, at least for $|t|$ small, and the curves $x(t)$ fill out a neighborhood of S. In effect, we could ask $\dot{x}(0) \notin T_x S$, $\Leftrightarrow A(x) \notin T_x S$,

$$\Leftrightarrow A(x) \cdot \gamma(x) \neq 0 \ , \ \forall x \in S \ ,$$

where $\gamma(x)$ is the unit normal vector of S.

$$\Leftrightarrow \gamma(x) \notin \mathrm{char}_x(L) := \{\xi \neq 0 : A(x) \cdot \xi = 0\} \ , \ \forall x \in S \ ,$$

In other words, by definition, we require that the hypersurface S is *non-characteristic* for any $x \in S$.

Here, it is appropriate the introduce the notion of characteristic vector/surface, for the linear operator in (1.10). A nonzero vector ξ is called characteristic for L at x if $\chi_L(x,\xi) := A(x)\cdot\xi = 0$, and the set of all such ξ is called the characteristic variety of L at x and is denoted by $\mathrm{char}_x(L) := \{\xi \neq 0 : \chi_L(x,\xi) = 0\}$. A hypersurface S is called characteristic for L at $x \in S$ if the normal vector ν_x to S at x is in $\mathrm{char}_x(L)$, and S is called non-characteristic if it is not characteristic at any point. We remark that $\nu(x)$ can be defined in a coordinate-free way as a cotangent vector at x, $T^*_x\mathbb{R}^n$, i.e., it transforms under coordinate changes by the same rule as $\mathrm{char}_x(L)$. (In fact, ν_x is one of the two unit cotangent vectors at x that annihilate the tangent space to S at x) Hence the condition "S is non-characteristic" is independent of the choice of coordinates.

Now with the notion just introduced, we see that it is natural (or necessary) to assume that S is non-characteristic, and thus the initial value problem for (1.10) with $u(x) = \phi(x)$ on S, is reduced to the initial value problem for the ordinary differential equations (1.11) and (1.12), and we have:

THEOREM 1.3. *Assume that S is a hypersurface of class C^1 which is non-characteristic for (1.10), and that the functions a^j, b, f, and ϕ are C^1 and real-valued. Then for sufficiently small neighborhood Ω of S in $\mathbb{R}^n$, there is a unique solution $u \in C^1$ of (1.10) on Ω that satisfies $u = \phi$ on S.*

Example 1. In $\mathbb{R}^3$, solve $x^1\partial_1 u + 2x^2\partial_2 u + \partial_3 u = 3u$ with $u = \phi(x^1, x^2)$ on the plane $x^3 = 0$.

It is non-characteristic. The solution is then given by

$$u = \phi(x^1 e^{-x^3}, x^2 e^{-2x^3}) e^{3x^3}$$

1.3.3. Some Quasilinear PDE*. It turns out that the method of characteristics could also be applied to quasilinear PDEs:

$$\sum_{j=1}^{n} a^j(x,u)\partial_j u = b(x,u) \ . \tag{1.13}$$

If u is a function of x, the tangent space $T_{(x,u(x))}$ is the span of the vectors $(e_j, \partial_j u)$, and the normal to the graph of u in $\mathbb{R}^{n+1}$ is proportional to $(\partial_1 u, \cdots, \partial_n u, -1)$. So (1.13) just says that the vector field

$$A(x,y) = (a^1(x,y), \cdots, a^n(x,y), b(x,y))$$

is tangent to the graph $y = u(x)$ at any point. This suggests that we look at the integral curves of the vector field A in $\mathbb{R}^{n+1}$ given by solving the ordinary differential equations

$$\frac{dx^j}{dt} = a^j(x,y),\ 1 \le j \le n\ ,\ \frac{dy}{dt} = b(x,y) \ . \tag{1.14}$$

It is clear that any graph $y = u(x)$ in $\mathbb{R}^{n+1}$ which is the union of an $(n-1)$-parameter family of these integral curves will define a solution of (1.13).

Conversely, suppose u is a solution of (1.13). If we solve the equations

$$\frac{dx^j}{dt} = a^j(x, u(x)),\ x^j(0) = x^j_0,\ 1 \le j \le n$$

to obtain a curve $x(t)$ passing through x_0, and then set $y(t) = u(x(t))$, we have

$$\frac{dy}{dt} = u_j(x(t))(x^j)'(t) = u_j a^j = b(x, u(x)) = b(x,y) \ .$$

Thus if the graph $y = u(x)$ intersects an integral curve of A in one point $(x_0, u(x_0))$, it contains the whole curve.

Suppose we are given initial data $u = \phi$ on a hypersurface S. If we form the submanifold

$$S^* = \{(x, \phi(x)), x \in S\}$$

of $\mathbb{R}^{n+1}$, the graph of the solution should be the hypersurface generated by the integral curves of A passing through S^*. Again, we need to assume that S is non-characteristic in some sense. This is more complicated than in the linear case because the coefficients a^j depend on u as well as x, but the geometric interpretation is exactly the same: for $x \in S$, the vector

$$(a^1(x, \phi(x)), \cdots, a^n(x, \phi(x))) \tag{1.15}$$

should not be tangent to S at x. (Note that this condition involves ϕ as well as S.) If S is represented parametrically by a mapping $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$ and we take coordinates $s^1, \cdots, s^{n-1}$ on $\mathbb{R}^{n-1}$, this condition is just

$$\det \begin{pmatrix} \partial_{s_1} g^1 & \dots & \partial_{s^{n-1}} g^1 & a^1(g(s), \phi(g(s))) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \partial_{s_1} g^n & \dots & \partial_{s^{n-1}} g^n & a^n(g(s), \phi(g(s))) \end{pmatrix} \neq 0 \; . \tag{1.16}$$

THEOREM 1.4. *Suppose S is a hypersurface of class C^1 in $\mathbb{R}^n$, and a^j, b, f, and ϕ are C^1 real-valued functions. Suppose also that the vector (1.15) is not tangent to S at any $x \in S$. Then for any sufficiently small neighborhood Ω of S in $\mathbb{R}^n$ there is a unique solution $u \in C^1$ of (1.13) on Ω that satisfies $u = \phi$ on S.*

See, e.g., Folland [9, 1.B].

Example 1. In $\mathbb{R}^2$, solve $u\partial_x u + \partial_y u = 1$ with $u = s/2$ on the segment $x = y = s$, $0 < s < 1$.

First, (1.16) is satisfied, for

$$\det \begin{pmatrix} \partial_s x & u \\ \partial_s y & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & s/2 \\ 1 & 1 \end{pmatrix} = 1 - s/2 \neq 0 \; .$$

With parameter $s \in (0, 1)$, we obtain $y = s + t$, $x = s + (st + t^2)/2$, $u = s/2 + t$. Notice that $u = y - s/2$, to write $u = u(x, y)$, it suffices to find $s = s(x, y)$. As $t = y - s$, we see that

$$2x = 2s + st + t^2 = 2s + s(y - s) + (y - s)^2 = y^2 - ys + 2s, s = \frac{2x - y^2}{2 - y} \; .$$

Plugging it into the equation, we obtain

$$u = \frac{4y - 2x - y^2}{2(2 - y)} \; ,$$

with the maximal domain of definition given by

$$\{(x, y) : y < 2, y^2 < 2x < y^2 - y + 2\} \; .$$

1.4. Linear differential operator: characteristic

1.4.1. 1-D Wave equation. Some higher order PDE could be reduced to first order PDE and solved by the method of characteristics.

As a classical example: we could show that

$$u_{tt} - c^2 u_{xx} = 0 \Rightarrow u = F(x - ct) + G(x + ct)$$

Notice that, as differential operators, we have $\partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$.

i) Let $w = u_t - cu_x$, then

$$(\partial_t + c\partial_x)w = 0 \Rightarrow w = f(x - ct)$$

Let $v = u_t - cu_x$, then

$$(\partial_t - c\partial_x)v = 0 \Rightarrow v = g(x + ct)$$

and so

$$2u_t = f(x - ct) + g(x + ct), u = F(x - ct) + G(x + ct)$$

ii) Let $w = u_t - cu_x$, then

$$(\partial_t + c\partial_x)w = 0 \Rightarrow w = f(x - ct)$$

Since $u_t - cu_x = f(x - ct)$, then

$$\frac{d}{dt}u(t, x-ct) = f(x-2ct), u(t, x-ct) = G(x)+F(x-2ct), u(t, x) = F(x-ct)+G(x+ct)$$

Notice that, we have been convinced that c in the wave equation (1.6) has the meaning of wave speed.

Concerning the Cauchy's problem, we obtain the D'Alembert's formula : if $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, then

$$u = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \ . \tag{1.17}$$

REMARK 1.5. *We see the well-known phenomenon for the waves: finite speed of propagation, which is the characteristic property enjoyed for a larger class of PDEs, the hyperbolic equations.*

1.4.2. Characteristics. In general, for linear PDE operators

$$L = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha$$

on $\Omega \subset \mathbb{R}^n$, we introduce its characteristic form (also known as the principal symbol) at $x \in \Omega$: the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by

$$\chi_L(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \ .$$

A nonzero vector ξ is called characteristic for L at x if $\chi_L(x, \xi) = 0$, and the set of all such ξ is called the characteristic variety of L at x and is denoted by

$$\mathrm{char}_x(L) := \{\xi \ne 0 : \chi_L(x, \xi) = 0\}$$

Now, note that if $\xi \ne 0$ is a vector in the x^j-direction (e.g., $\xi = e_j$), then $\xi \in \mathrm{char}_x(L)$ if and only if the coefficient of ∂_j^k in L vanishes at x. Moreover, given any $\xi \ne 0$, by a rotation of coordinates we can arrange for ξ to lie in a coordinate direction. Thus the condition $\xi \in \mathrm{char}_x(L)$ means that, in some sense, L fails to be "genuinely k-th order" in the ξ direction at x.

L is said to be **elliptic** at x if $\mathrm{char}_x(L) = \emptyset$, and elliptic in Ω if it is elliptic at every $x \in \Omega$. At least on the formal level, an elliptic operator of order k exerts control on all derivatives of order k. We shall see later that this formal statement is also valid analytically.

1.4.2.1. *characteristics and cotangent vectors.* To see more clearly the meaning of the characteristic variety, let us consider the effect of a change of coordinates. Let F be a smooth one-to-one mapping of Ω onto some open set $\Omega' = F(\Omega)$, and set $y = F(x)$. Assume that the Jacobian matrix $J_x = [J_{jk}] = [(\partial y^j/\partial x^k)(x)]$ is nonsingular for $x \in \Omega$, so that $y_1, \cdots, y_n$ is a coordinate system on Ω. We have

$$\frac{\partial}{\partial x^j} = \sum_k \frac{\partial y^k}{\partial x^j}\frac{\partial}{\partial y^k}, \partial_x = J_x^t \partial_y \ ,$$

and thus L is transformed into the operator

$$L' = \sum_{|\alpha|\le k} a_\alpha(F^{-1}(y))(J^t_{F^{-1}(y)}\partial_y)^\alpha$$

on Ω'. Then

$$\chi_{L'}(y,\xi) = \sum_{|\alpha|=k} a_\alpha(F^{-1}(y))(J^t_{F^{-1}(y)}\xi)^\alpha \ .$$

Therefore

$$\begin{aligned}
\mathrm{char}_x(L) &= \{\xi \neq 0 : \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha = 0\} \\
&= \{J^t_{F^{-1}(y)}\eta \neq 0 : \sum_{|\alpha|=k} a_\alpha(F^{-1}(y))(J^t_{F^{-1}(y)}\eta)^\alpha = 0\} \\
&= J^t_{F^{-1}(y)}\mathrm{char}_y(L') \ .
\end{aligned}$$

Recall that if we associate to $\xi \in \mathbb{R}^n$ the differential form $d\xi = \sum \xi_j dy^j$ and use the chain rule $dy^j = \sum(\partial y^j/\partial x^k)dx^k$, we have

$$d\xi = \sum \xi_j dy^j = \sum \xi_j(\partial y^j/\partial x^k)dx^k = \sum(\sum \xi_j \partial y^j/\partial x^k)dx^k$$

Thus in the x-coordinates, $d\xi$ corresponds to the vector $J_x^t\xi$. In the terminology of differential geometry, $\mathrm{char}_x(L)$ is intrinsically defined as a subset of the cotangent space at x, $T_x^*(\mathbb{R}^n)\backslash\{0\}$ (in effect, subset of the cotangent sphere bundle, $S^*(\mathbb{R}^n)$).

1.5. Power series method and Cauchy-Kowalevskaya theorem

1.5.1. Complex-valued. What happens if we allow the quantities in (1.10) to be complex-valued?

If the coefficients, a^j's and b, are real, this system is uncoupled and can be solved by Theorem 1.3.

Moreover, as we could observe from the method, we need only to assume that the coefficients for derivatives, i.e., a^j, are real, which is sufficient to construct characteristics, along which the equation becomes linear ODE.

However, as we shall see, in general, if the coefficients are complex, this system may possess no solutions.

1.5.2. Example, Power series method. What happens if $c \in \mathbb{C}\backslash\mathbb{R}$ in (1.9)? Let $c = i$ in (1.9), then with $z = x + iy$, $f(z) = u(x, y)$, we see that

$$(\partial_x + i\partial_y)u = 2\partial_{\bar{z}} f$$

which tells us that this is exactly the well-known **Cauchy-Riemann** equation.

LEMMA 1.6. *Let $u(x, y) \in C^1(\Omega)$ be a solution to $u_x + iu_y = 0$ ($c = i$ in (1.9)), then $f(x+iy) := u(x, y)$ is a analytic function in $\Omega' = \{z = x+iy \in \mathbb{C} : (x, y) \in \Omega\}$. Thus $g(y) = u(0, y) \in C^\omega \Rightarrow g \in C^\infty$*

Here, for functions defined on $\Omega \in \mathbb{R}^n$, we say $g \in C^\omega$ (**real analytic**) near x_0, if $\exists\delta_1 > 0$, so that it agrees with the Taylor series,

$$g(y) = \sum_{k\in\mathbb{Z}^n_+} a_k(y - y_0)^k \ , \ \forall|y - y_0| < \delta_1 \ ,$$

where $a_k = \frac{g^k(y_0)}{k!}$.

In particular, we see that there is no C^1 solution for $u_x + iu_y = 0$, $u(0, y) = g(y) \in C^1\backslash C^\omega$, in contrast to the result for $c \in \mathbb{R}$, for which we have $u \in C^1$ for any $g \in C^1$, Theorem 1.3.

On the other hand, suppose $g \in C^\omega$ near y_0, By Lemma 1.6, if u is a solution, u must be analytic in $x + iy$, that is,

$$u(x, y) = \sum_{n=0}^{+\infty} c_n(x + i(y - y_0))^n, \forall|x| + |y - y_0| < \delta_2,$$

Due to the data, $c_n = i^{-n}g^{(n)}(y_0)/n!$, then

THEOREM 1.7. *Let $g \in C^\omega$ near y_0, then there exists a local unique solution $u \in C^\omega$ for IVP $u_x + iu_y = 0$, $u(0, y) = g(y)$.*

Here, we remark that real analytic C^ω functions are not necessarily real-valued, because a_n could also be complex numbers. It is called real analytic, simply because it is defined in the real domain, instead of complex domain.

1.5.3. Cauchy-Kowalevskaya theorem. In general, the power series method could be applied in the context of C^ω functions, as long as the initial hypersurface is non-characteristic. More precisely, we have the following Cauchy-Kowalevskaya theorem, where for simplicity we have only stated the result for flat hyperplane.

THEOREM 1.8 (Cauchy-Kowalevskaya theorem). *Consider the initial data problem*

$$\begin{cases} \partial_t^k u = F(t, x, \partial_t^j \partial_x^\alpha u, j < k, j + \alpha \leq k) \\ \partial_t^j u(0, x) = u_0^j(x), j \leq k - 1 \end{cases} \tag{1.18}$$

If F, u_0^j are (real) analytic near the origin, then there exists a unique analytic solution near the origin.

See, e.g., Folland [9, 1.D].

1.5.4. Non-characteristic condition is necessary. Finally, let us give an example showing that the non-characteristic condition is in general necessary for the Cauchy-Kowalevskaya theorem, by using power series method.

Consider the initial value problem for the heat equation:

$$u_t = u_{xx},\ u(0,x) = g(x) = \frac{1}{1-ix} = \sum_{n\geq 0} i^n x^n \ ,$$

where it is clear that $g(x) \in C^\omega((-1,1))$. Let us presume that we could apply power series method, that is, for $|t|+|x| \ll 1$, we have

$$u(t,x) = \sum \frac{c_{jk}}{j!k!} t^j x^k \ ,$$

where

$$c_{jk} = \partial_t^j \partial_x^k u(0,0) = \partial_x^{2j+k} u(0,0) = \partial_x^{2j+k} g(0) = i^{2j+k}(2j+k)! \ .$$

Then $u(t,x) = \sum \frac{i^{2j+k}(2j+k)!}{j!k!} t^j x^k = \sum c_k(t) x^k$. In particular, we have $u(t,0) = c_0(t) \in C^\omega(-\delta,\delta)$.

However, as we can see, for any $t \neq 0$, the coefficients

$$\lim_{j\to\infty} \frac{(2j)!}{j!} |t|^j \geq (j+1)^j |t|^j = +\infty \ ,$$

the series $\sum_{j\geq 0} \frac{i^{2j}(2j)!}{j!} t^j$ is not convergent, which is the desired contradiction.

1.6. Well-posed problem (in the sense of Hadamard)

Definition 1.1. *A given problem for a PDE is said to be well posed if both existence and uniqueness of solutions can be established for arbitrary data which belong to a specified large space of functions, which includes certain class of smooth functions. Moreover the solutions must depend continuously on the data.*

As we have seen, IVP (also known as the Cauchy problem) for non-characteristic data is well posed for first order scalar equations (real-valued), by method of characteristics.

How about the Cauchy-Kowalevskaya theorem?

1.6.1. Uniqueness: Holmgren uniqueness theorem. [16, Section 3.5]

In the special case of linear equations, an important companion theorem, due to Holmgren, asserts that the analytic solution given by the Cauchy-Kowalevskaya theorem is unique in the class of all smooth solutions and smooth non-characteristic hypersurfaces H.

Theorem 1.9 (Holmgren uniqueness theorem). *Consider the initial value problem for a linear equations of the type*

$$\sum_{|\alpha|\leq k} A_\alpha(x) \partial^\alpha u(x) = F(x)$$

with analytic coefficients. If the hypersurface H is also analytic and non-characteristic at $x_0 \in H$, then the corresponding Cauchy problem is unique in the class of C^k solutions, in a small neighborhood of x_0.

Basically, the proof follows from the duality argument (existence of the dual problem yields the uniqueness of the original problem). For example, let $\Omega \subset \mathbb{R}^n$ be sufficiently small with $\partial\Omega = Z \cup S$ and $L = \sum a^j(x)\partial_j + b(x)$ with $a^j, b \in C^\omega(\Omega)$.

Assume $u \in C^1$ with $Lu = 0$ in Ω and $u = 0$ on Z, introduce the adjoint equation:

$$\tilde{L}v = bv - \sum \partial_j(a^j v)$$

(if $u, v \in C_0^1(\Omega)$, then $\int vLudx = \int u\tilde{L}vdx$).

For any $w \in C^\omega(S)$, considering

$$\tilde{L}v = bv - \sum \partial_j(a^j v) = 0, x \in \Omega;\ \ v = w, x \in S\ ,$$

we know from the Cauchy-Kowalevskaya theorem that we have $v \in C^\omega(\Omega)$. Notice that

$$vLu - u\tilde{v} = \sum \partial_j(a^j vu)\ ,$$

divergence theorem gives us (as $u|_Z = 0$)

$$0 = \int_\Omega vLu - u\tilde{L}vdx = \int_{Z\cup S} \nu_j a^j vud\sigma(x) = \int_S \nu_j a^j wud\sigma(x)\ .$$

We are assuming non-characteristic condition on S, $\nu_j a^j \neq 0$. Also, $C^\omega(S)$ is dense in $C(S)$ [e.g., the Weierstrass theorem], then we see that $\int_S \nu_j a^j wud\sigma(x) = 0$ for all $w \in C(S)$, and so $u \equiv 0$ on S.

REMARK 1.10. *The remarkable thing about Holmgren's theorem is that it proves uniqueness even in cases where existence of solutions cannot be guaranteed. For example, the Cauchy problem for the (three-dimensional) wave equation with data on the hyperplane $x_1 = 0$ does not, in general, have solutions, yet Holmgren's theorem asserts that if a solution exists it must be unique.*

REMARK 1.11. *In the linear case, there is a more general version of the Cauchy-Kowalevskaya theorem that requires analyticity only in x, not in t; see Treves* [**30**].

1.6.2. Stability. A major drawback of the Cauchy-Kowalevskaya theorem is that it gives little control over the dependence of the solution on the Cauchy data. Consider the following example in $\mathbb{R}^2$, due to **Hadamard**:

Consider Laplace equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

with initial data

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = ke^{-\sqrt{k}}\sin(kx + \pi/2),$$

where k is a positive integer.

Since the problem is elliptic, it is non-characteristic, and one easily checks (or solve by power series method) that the solution is

$$u(x,t) = e^{-\sqrt{k}}\sin(kx + \pi/2)\sinh kt\ .$$

By Holmgren's uniqueness theorem, this analytic solution is unique in C^2. But, this solution is not stable in the usual C-topology. In fact, as $k \to \infty$, the Cauchy data and their derivatives of all orders tend uniformly to zero, since $e^{-\sqrt{k}}$ decays faster than any polynomials.

But if $t \neq 0$,

$$\lim_{k\to\infty} u(0,t) = \lim_{k\to\infty} e^{-\sqrt{k}} \sinh kt = \infty \ ,$$

so as $k \to +\infty$ the solution oscillates more and more rapidly with greater and greater amplitude, and in the end it blows up altogether. The solution for the limiting case $k = \infty$ is of course $u \equiv 0$. This example shows that the solution of the Cauchy problem may not depend continuously on the Cauchy data in most of the usual topologies on functions.

1.6.3. Importance of stability. The continuous dependence on the data is very important. Indeed solutions to the IVP for a PDE would be of little use if very small changes of the initial conditions will result, instantaneously, in very large changes in the corresponding solutions. It is only in the class of smooth solutions that the theory of PDE becomes really interesting, relevant and challenging. It means that we have to give up hope for a all encompassing result and look instead for special classes of equations which have common features, or really just on special important equations. It is in that sense that the generality of the Cauchy-Kowalevskaya theorem is really an illusion. The true study of partial differential equations only begins when we give up on analyticity.

1.6.4. Another remark on the Cauchy-Kowalevskaya theorem. At first glance it may seem that the Cauchy-Kowalevskaya theorem is a perfect analogue of the fundamental theorem for ODE's. It turns out, however, that the analyticity conditions required by the Cauchy-Kowalevskaya theorem are much too restrictive and thus the apparent generality of the result is misleading.

As example, let us consider the wave equation

$$\Box u \equiv u_{tt} - c^2 u_{xx} = 0 \ , u(x,0) = f(x), u_t(x,0) = g(x) \ ,$$

for which we know the D'Alembert's formula (1.17), that is,

$$u = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \ .$$

From this representation formula, we see that it is well-posed in C^2 for $C^2 \times C^1$ data. Moreover, we see the fundamental feature of finite speed of propagation.

However, as we know, in the analytic category, compactly supported functions must be identically zero. Thus, the limitation becomes immediately obvious when we consider the wave equation $\Box u = 0$, whose fundamental feature of finite speed of propagation is impossible to make sense in the class of real analytic solutions.

1.7. Homeworks

Section 1.1

(1) (Newton potential) Check

$$\Delta(1/r) = 0, \forall x \in \mathbb{R}^3 \backslash \{0\} \ .$$

(2) Find general solutions for spherically symmetric functions $V = V(|x|)$ such that $\Delta V = 0$, $x \in \mathbb{R}^n \backslash \{0\}$ with $n \geq 2$.

(3) (minimal surface) Let $u : \Omega \to \mathbb{R}$ with $\Omega \in \mathbb{R}^2$. For fixed function f on the boundary of Ω, suppose u is a function such that it minimizes the area

$$A[u] = \int_\Omega \sqrt{1 + |\nabla u|^2} dx \ ,$$

among all surfaces with $u = f$ on the boundary of Ω. Try to derive a PDE for u.

(4) (Cauchy-Riemann equation) By (1.3), check that both u and v satisfy the Laplace equation. Assume u is known in $\mathbb{R}^2$, try to find v such that $v(0,0) = 0$.

Section 1.3

(1) Solve $u_t + xu_x = x$ with data $u(0,x) = x^2$
(2) Solve $u_t + uu_x = 0$, $u(0,x) = x$
(3) Solve $u_t + uu_x = 0, u(0,x) = -x$
(4) Solve $u_t + uu_x = 2u$ with data $u(0,x) = x$

Section 1.4-5

(1) Solve $u_{tt} - c^2 u_{xx} = F(t,x)$ for $t > 0$ and $x \in \mathbb{R}$, with data $u(0,x) = 0$, $u_t(0,x) = 0$ for $x \in \mathbb{R}$.
(2) Find all elliptic operators among $L_1 = \partial_x + \partial_y$, $L_2 = \partial_x^2 + \partial_y^2$, $L_3 = \partial_x + i\partial_y$, $L_4 = \partial_x + \partial_y^2$.
(3) Let $c \in \mathbb{C}\backslash\mathbb{R}$, try to find new coordinates to reduce the problem $u_x + cu_y = 0$ to the case $c = i$.
(4) Solve $u_t + xu_x = x$ with data $u(0,x) = 2x$, by applying the power series method.
(5) Solve $u_{xx} + u_{yy} = 0$ with data $u(0,y) = e^y$, $u_x(0,y) = 0$, by applying the power series method.
(6) Find a solution to $u_{tt} - u_{xx} = xt$ for $t > 0$, $x \in \mathbb{R}$.

CHAPTER 2

Method of separation of variables, Fourier series and Fourier transform

Last chapter: Basic concepts; Real-valued first order PDE and method of characteristics; Power series method and Cauchy-Kowalevskaya theorem; Well-posed problem

This chapter: Method of separation of variables, Fourier series, Fourier transform, Schwartz class $\mathcal{S}$

2.1. Method of separation of variables: IBVP and Fourier series

We consider the initial boundary value problem of wave equation, with (homogeneous) Dirichlet's boundary condition:

$$\begin{cases} \rho(x)\partial_t^2 u(x,t) = \partial_x\left(p(x)u_x\right), \quad x\in[0,L],\ t>0. \\ u(0,t)=u(L,t)=0, \\ u(x,0)=f(x), u_t(x,0)=g(x)\ . \end{cases} \tag{2.1}$$

Here, $\rho, p>0$, $\rho\in C^2[0,L]$, $p\in C^1[0,L]$.

2.1.1. Reduction, simplification. At first, we try to reduce the problem to the case $\rho\equiv 1$, which could be achieved by change of variable.

Let $u=zw$, with $z=z(x)>0$ to be determined, we see that the PDE is equivalent to

$$\rho(x)z(x)\partial_t^2 w(x,t) = pz\partial_x^2 w + [(pz)_x + pz']\partial_x w + (pz')'w\ ,$$

$$\partial_t^2 w(x,t) = \frac{p}{\rho}\partial_x^2 w + \frac{(pz)_x+pz'}{\rho z}\partial_x w + \frac{(pz')'}{\rho z}w\ .$$

Choosing z such that the first two terms behave like the right of (2.1), that is,

$$\partial_x\left(\frac{p}{\rho}\right) = \frac{(pz)_x+pz'}{\rho z} \Leftrightarrow \rho' = -2\rho\frac{z'}{z} \Leftarrow z=\rho^{-1/2}\ ,$$

then we arrive at the new PDE for w,

$$\partial_t^2 u(x,t) = \partial_x\left(P(x)w_x\right) - Q(x)w := -\mathcal{L}w\ , \tag{2.2}$$

where $P(x)=p(x)/\rho(x)\in C^1[0,L]$ and $Q(x)=-\frac{(pz')'}{\rho z}\in C[0,L]$.

2.1.2. Compatibility condition (necessary conditions). If $u\in C^2([0,L]\times[0,\infty))$, we must have $f\in C^2$, $g\in C^1$,

$$f(0)=f(L)=0; g(0)=g(L)=0; f''(0)=f''(L)=0 \tag{2.3}$$

2.1.3. Method of separation of variables: general solutions for PDE+BC. Based on the **principle of superposition**[1], we try to find sufficiently many simple solutions, which are many enough to generate "all" solutions.

Separation of variables Seeking all of the possible nontrivial solutions of the form $u(t,x) = X(x)T(t)$ (**standing wave**) satisfying PDE+BC, then

$$T_{tt}(t)X(x) = -T(t)\mathcal{L}X \Rightarrow -\frac{T_{tt}}{T} = \frac{\mathcal{L}X}{X}, \forall (x,t) \text{ s.t. } \quad X(x) \neq 0 \neq T(t) \ .$$

Thus, provided that the set $A = \{(x,t) : u(x,t) = 0\}$ is relatively small (e.g., measure zero), we obtain that

$$-\frac{T_{tt}}{T} = \frac{\mathcal{L}X}{X} = -\lambda, \forall (x,t) \in A^c$$

where λ is independent of t, x. Based on BC, as T is non-trivial, we see that $X(0) = X(L) = 0$.

The boundary value problem

$$\mathcal{L}X = -\partial_x (P(x)X_x) + Q(x)X = \lambda X, \quad X(0) = X(L) = 0 \ , \tag{2.4}$$

is the well-known **Sturm-Liouville** eigenvalue problem.

2.1.4. Sturm-Liouville eigenvalue problem: sample case $P \equiv 1$ and $Q \equiv 0$. We first consider the boundary value problem

$$\varphi_{xx} = -\lambda\varphi, \quad \varphi(0) = \varphi(L) = 0. \tag{2.5}$$

We claim that if φ is nontrivial, then $\lambda \in \mathbb{R}$ and $\lambda > 0$. Actually, if $\lambda = 0$, $\varphi = C_1 + C_2 x$, BC could not be satisfied. Else, if $\lambda \in \mathbb{C}\backslash\{0\}$, set $\gamma = \sqrt{-\lambda} \neq 0$, then the general solutions for ODE are

$$\varphi = C_1 e^{\gamma x} + C_2 e^{-\gamma x}$$

BC tells us that (nontrivial solution (C_1, C_2), or by direct calculation)

$$\left| \begin{pmatrix} 1 & e^{\gamma L} \\ 1 & e^{-\gamma L} \end{pmatrix} \right| = 0, e^{2\gamma L} = 1, 2\gamma L = i2\pi n, n \in \mathbb{Z}\backslash\{0\}; \gamma = \frac{i\pi n}{L}, \lambda = -\gamma^2 = (\frac{n\pi}{L})^2 \ .$$

2.1.4.1. *Positiveness of the eigenvalue and the Rayleigh quotient.* Alternatively, if there is nontrivial solution φ, we calculate

$$-2\lambda\bar{\varphi}\varphi = \bar{\varphi}\varphi_{xx} + \bar{\varphi}_{xx}\varphi = \partial_x(\bar{\varphi}\varphi_x + \bar{\varphi}_x\varphi) - 2\bar{\varphi}_x\varphi_x$$

then

$$\int_0^L |\varphi_x|^2 - \lambda|\varphi|^2 dx = 0,$$

$$\lambda = \frac{\int_0^L |\varphi_x|^2 dx}{\int_0^L |\varphi|^2 dx} \ , \tag{2.6}$$

which is known as the Rayleigh quotient. Hence we see that $\lambda \in \mathbb{R}$ and $\lambda \geq 0$.

Moreover, $\lambda = 0$ could be excluded, since if $\lambda = 0$, then $\varphi'(x) = 0$ and $\varphi(0) = 0$, and so $\varphi(x) \equiv 0$, which is a contradiction.

Actually, in view of the functional analysis (more specifically, spectrum theorem), let $\mathcal{L} = -\partial_x^2$ and $H = L^2([0,L])$, we know that $\mathcal{L}$ is non-negative (unbounded) self-adjoint operator on H, which yields that the spectrum λ must be positive real number.

[1] If u and v are solutions to PDE+BC, then $c_1 u + c_2 v$ are also solutions, for any $c_1, c_2 \in \mathbb{C}$.

2.1.4.2. *Eigen-system.* **Conclusion**: $\phi_n(x) = \sin\frac{n\pi x}{L}$, $n \geq 1$, $\lambda_n = \frac{n^2\pi^2}{L^2}$.

2.1.5. General Sturm-Liouville eigenvalue problem. In general, similar proof tells us that, for nontrivial solution X to (2.4), we get the **Rayleigh quotient**.

$$\lambda = \frac{\int_0^L P|X_x|^2 + Q|X|^2 dx}{\int_0^L |X|^2 dx}, \tag{2.7}$$

which gives that $\lambda \in \mathbb{R}$, and $\lambda > \min Q$. Moreover, we have the following **Sturm–Liouville theory**.

THEOREM 2.1 (Sturm-Liouville).

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, Ky(a) + Ly'(a) = 0, My(b) + Ny'(b) = 0, \tag{2.8}$$

where λ is a parameter,

$$K^2 + L^2 > 0, M^2 + N^2 > 0,$$

$p(x) \in C^1([a,b])$, $q(x), r(x) \in C([a,b])$ *and* $p(x), r(x) > 0$.

The value of λ is not specified in the equation, finding the values of λ for which there exists a non-trivial solution of (2.8) *satisfying the boundary conditions is part of the (regular) Sturm-Liouville (S-L) problem. Any such λ is called eigenvalue and the corresponding non-trivial solution is called to be the corresponding eigenfunction.*

Under the above assumption, the Sturm-Liouville theory states that:

- *The eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the (regular) Sturm-Liouville problem are real and can be ordered such that*
$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_j < \lambda_{j+1} < \cdots$$
and $\lim_{j\to\infty} \lambda_j = \infty$.
- *Corresponding to each eigenvalue λ_n, there is a unique (up to a normalization constant) eigenfunction $y_n(x)$ which has exactly $n-1$ zeros in (a,b). The eigenfunction $y_n(x)$ is called the n-th fundamental solution satisfying the regular Sturm-Liouville problem.*
- *The normalized eigenfunctions form a complete orthonormal basis*
$$\int_a^b y_n(x) y_m(x) r(x)\, \mathrm{d}x = \delta_{mn},$$
in the Hilbert space $L^2([a,b], r(x)dx)$.

It could be proved by using the Oscillation theorem of C. Sturm (1836). See, e.g., [**22**, Chapter 3], [**29**, Chapter 5].

2.1.6. Separation of variables: all of the possible nontrivial solutions. Then by $T_{tt} = -\lambda_n T$, we obtain $\psi_n(t) = A_n\cos(\sqrt{\lambda_n}t) + B_n\sin(\sqrt{\lambda_n}t)$. Thus we find the solution

$$u_n(t,x) = (A_n\cos(\sqrt{\lambda_n}t) + B_n\sin(\sqrt{\lambda_n}t))X_n(x)$$

Based on that, we propose the "general solutions":

$$u(t,x) = \sum_n u_n(t,x). \tag{2.9}$$

But how to define A_n, B_n and justify it is really a solution?

Our problem is thus reduced to the study of the expansion/convergence of functions by generalized Fourier series.

In particular, when $P \equiv 1$ and $Q \equiv 0$, we have: $\psi_{tt} = -\lambda_n \psi = -\frac{n^2\pi^2}{L^2}\psi$, we obtain $\psi_n(t) = A_n \cos(\frac{n\pi}{L}t) + B_n \sin(\frac{n\pi}{L}t)$. Thus we find the solution

$$u_n(t,x) = (A_n \cos(\frac{n\pi}{L}t) + B_n \sin(\frac{n\pi}{L}t)) \sin \frac{n\pi x}{L} \ .$$

2.1.7. Fourier sine series.

Let $\varphi_n(x) = \sin \frac{n\pi x}{L}$, we know that they are orthogonal

$$\int_0^L \varphi_n(x)\varphi_m(x) = \frac{L}{2}\delta_{nm} \ ,$$

and complete basis of $L^2([0,L])$. Thus we have for any $f \in L^2([0,L])$,

$$f(x) = \sum_{n\geq 1} A_n \varphi_n(x), A_n = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx \ ,$$

and, in view of Parseval's identity,

$$\int_0^L |f|^2(x)dx = \frac{L}{2}\sum_{n\geq 1}|A_n|^2,$$

which, particularly gives us Bessel's inequality

$$\sum |A_n|^2 \leq \frac{2}{L}\int_0^L |f|^2(x)dx, \forall f \in L^2 \ .$$

Moreover, if $f \in C^1$ with $f(0) = f(L) = 0$, we have $A_n = \mathcal{O}(1/n)$

$$A_n = \frac{L}{n\pi}a_n, \ a_n = \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}dx. \tag{2.10}$$

By the Cauchy-Schwarz and Bessel inequality, we obtain

$$|\sum A_n\varphi_n(x)| \leq \sum |A_n| \leq \|a_n\|_{l^2}\|\frac{L}{n\pi}\|_{l^2} \leq C\|f'\|_{L^2} \ , \tag{2.11}$$

which tells us that the Fourier sine series is absolutely convergent to f, and so is the uniform convergence.

2.1.8. Return to the wave equation.

THEOREM 2.2. *Let $f \in C^3$, $g \in C^2$ satisfying the compatibility condition (2.3). then there exists a unique solution $u \in C^2([0,L]\times\mathbb{R})$, given by (2.9), for (2.1) with $\rho = p = 1$.*

To ensure that (2.9) gives the desired solution, it is clear that a necessary condition is that

$$f(x) = u(0,x) = \sum_n A_n \sin\frac{n\pi x}{L}, \ g(x) = u_t(0,x) = \sum_n B_n \frac{n\pi}{L}\sin\frac{n\pi x}{L} \ ,$$

which gives us (by orthogonality)

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, B_n = \frac{2}{n\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx \ . \tag{2.12}$$

Notice that the (uniform) convergence to f, g is ensured by $f, g \in C^1$.

With (2.12), we need to check

$$\Box\sum_n u_n=\sum_n \Box u_n,\ \lim_{x\to 0,L}\sum_n u_n=\sum_n \lim_{x\to 0,L} u_n,\ \lim_{t\to 0}\partial_t^m\sum_n u_n=\sum_n \lim_{t\to 0}\partial_t^m u_n, m=0,1\ ,$$

for which purpose, it suffices to prove the convergence of the series

$$\sum \partial_t^j\partial_x^k u_n\ , \tag{2.13}$$

in $[0,L]\times[0,\infty)$ for $j+k\le 2$.

Actually, by similar approach of (2.10)-(2.11), we know that the series are uniformly convergent.

For uniqueness, it could be proved by the completeness of the eigen-system $\{X_n\}$, method of characteristics, or (odd) extension, or even idea of Holmgren's uniqueness theorem, which we leave as exercise.

2.1.9. Fourier series. Assume f is a periodic function with period $2L$, we want to represent f as

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}},\quad |x|\le L. \tag{2.14}$$

If we assume, for example, that the series converges uniformly, then by multiplying by $e^{-\frac{im\pi x}{L}}$ and integrating term by term on $[-L,L]$, we get

$$C_m\equiv \hat f(m)=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{im\pi x}{L}}dx \tag{2.15}$$

because of the orthogonality relationship

$$\int_{-L}^{L} e^{\frac{in\pi x}{L}}e^{-\frac{im\pi x}{L}}dx=\begin{cases}0, n\neq m\\ 2L, n=m\end{cases} \tag{2.16}$$

It is well-known that we have (2.14) in L^2 sense, or in the classical sense for $f\in C^{0+}$, or absolutely and uniformly if $f\in C^{1/2+}$.

Let $f\in C^1$ be a periodic function with period $2L$, it is easy to prove that

$$\hat{f'}(m)=i\frac{m\pi}{L}\hat f(m),$$

and its Fourier series $\sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}}$ converges absolutely and uniformly.

We also have the Parseval's identity (which implies Bessel's inequality)

$$\sum |C_n|^2=\frac{1}{2L}\int_{-L}^{L}|f|^2(x)dx, \forall f\in L^2\ .$$

2.1.10. Other models. The method of separation of variables applies in many different contexts and various PDEs.

It includes wave equations or heat equations with Dirichlet's boundary condition, Neumann's boundary condition or periodic boundary conditions. The following is an example of the heat equation with Neumann's boundary condition:

$$\begin{cases}u_t=u_{xx}, \quad x\in[0,L],\ t>0.\\ u_x(0,t)=u_x(L,t)=0,\\ u(x,0)=f(x).\end{cases} \tag{2.17}$$

It could also be applied to higher dimensional equations, provided that the domain has certain symmetric property.

2.2. Sturm-Liouville theory and asymptotic behavior*

As we have seen, when $P \equiv 1$ and $Q \equiv 0$, we have

$$\lambda_n = (n\pi/L)^2, X_n = \sin(n\pi x/L) \ .$$

It is natural to ask the asymptotic behavior for the general case.

For simplicity, we consider the sample case of (2.4) with $P \equiv 1$ (which is also known as the Schrödinger operator):

$$\mathcal{L}X = -X'' + Q(x)X = \lambda X, \quad X(0) = X(L) = 0 \ , \tag{2.18}$$

THEOREM 2.3 (Sturm Comparison/Oscillation Theorem). *Let $Q_1 \geq Q_2$, X, Y be real solutions to*

$$X'' + Q_1(x)X = 0, Y'' + Q_2(x)Y = 0, Y(a) = Y(b) = 0$$

and Y is nontrivial, then either there exists $x_0 \in (a, b)$ such that $X(x_0) = 0$, or $Q_1 \equiv Q_2$ and $X = cY$ for some $c \in \mathbb{R}$.

2.2.1. Eigenfunctions. Without loss of generality ($\lambda - Q = (\lambda - c) - (Q - c)$), we could assume $Q \geq 1$, which, in view of (2.7), tells us that the eigenvalues have lower bound: $\lambda_n \geq 1$.

Let $X(x, s)$ be the solution to

$$(\mathcal{L} - s)X(x, s) = -X'' + QX - sX = 0, X(0, s) = 0, X'(0, s) = 1 \ .$$

For fixed $s \in \mathbb{R}^+$, $X(x, s)$ is an eigenfunction iff $X(L, s) = 0$. To record the variance with respect to s, we denote the number of interior zeros by

$$N(s) = |\{x \in (0, L) : X(x, s) = 0\}| \ .$$

As $\lambda \geq 1$, we know that $X(\cdot, s)$ are not eigenfunctions for $s < 1$. Moreover, we claim that we have $N(s) = 0$ for $s < 1$. Actually, suppose, by contradiction, that $N(s_0) \geq 1$ for some $s_0 < 1$, then there exists $x_0 \in (0, L)$ so that $X(x_0, s_0) = 0$. Let $\phi(x) = X(x, s_0)$, as $s_0 - Q < -\delta^2 < 0$ and

$$\phi'' + (s_0 - Q)\phi = 0, \ ,$$

we know from Theorem 2.3 that there exists at least one zeros in $(0, x_0)$, for any real solutions to

$$y'' - \delta^2 y = 0 \ ,$$

which is clearly a contradiction (say $y = e^{\delta x}$).

Similar argument tells us that $N(s)$ is an increasing function of s.

In addition, $\lim_{s\to\infty} N(s) = \infty$. Actually, there exists M such that $Q < M$ and $s - Q > s - M$. Comparing

$$X'' + (s - Q)X = 0, X(0, s) = 0, X'(0, s) = 1; Y'' + (s - M)Y = 0, Y(0) = 0, Y'(0) = 1 \ ,$$

we know that

$$N(s) \geq \tilde{N}(s) \to \infty \ ,$$

where $\tilde{N}(s) = \frac{L}{\pi}\sqrt{s - M} + O(1)$ is the corresponding function for $\sin\sqrt{s - M}x$. As a byproduct, we see that

$$\frac{L}{\pi}\sqrt{s} + O(1) \geq N(s) \geq \frac{L}{\pi}\sqrt{s - M} + O(1) \ . \tag{2.19}$$

Moreover, x_j are decreasing and continuous functions, where $x_j(s)$ is the j-th interior zeros for $X(x, s)$, with the convention that $x_0(s) = 0$. Recall that

$X(x_j(s), s) = 0$, we must have $X_x(x_j(s), s) \neq 0$ as $X(\cdot, s)$ is a nontrivial solution. By the implicit function theorem, the function $x_j \in C^1$ and

$$X_s + X_x x_j' = 0, x_j' = -X_s/X_x \ .$$

The eigenvalues correspond to the values s for which a new zero appears at L. Since the number of these zeros is finite for any s, this implies that the eigenvalues form a discrete sequence

$$0 < \lambda_1 < \lambda_2 < \cdots$$

with $x_j(\lambda_j) = L$. The n-th eigenfunction $X_n(x)$ is related with the eigenvalue λ_n, and has exactly $n-1$ zeros on the open interval $(0, L)$.

This verifies part of Theorem 2.1.

2.2.2. Eigenvalues: asymptotic behavior. The behavior of eigenvalues is easily derived with the help of oscillation theorem, Theorem 2.3.

By the similar argument which yields (2.19), we conclude that

$$N(s) \geq \tilde{N}(s),$$

Let $s = (n\pi/L)^2 + M$, we have $N(s) \geq \tilde{N}(s) = n - 1$ and so $\lambda_n \leq (n\pi/L)^2 + M$. Similarly, $\lambda_n \geq (n\pi/L)^2$. Thus,

$$\lambda_n = (\frac{n\pi}{L})^2 + O(1) = (\frac{n\pi}{L})^2(1 + O(\frac{1}{n^2})), \tag{2.20}$$

$$k_n \equiv \sqrt{\lambda_n} = \frac{n\pi}{L}(1 + O(\frac{1}{n^2})) = \frac{n\pi}{L} + O(\frac{1}{n}) \ . \tag{2.21}$$

2.2.3. Eigenfunctions: asymptotic behavior. With the help of (2.21), we could try to derive the asymptotic behavior of eigenfunctions for $n \gg 1$.

For (2.18), with $\lambda = k^2$, we write it in the form

$$X'' + k^2 X = Q(x)X \ , X(0) = 0, X'(0) = 1$$

which satisfied

$$X(x) = \frac{\sin kx}{k} + \frac{1}{k}\int_0^x \sin k(x-t)X(t)dt := \frac{\sin kx}{k} + A_k(X)(x) \ .$$

It is an integral equation of Volterra type, where $A_k : C[0, L] \to C[0, L]$ with

$$\|A_k\| \leq \frac{L}{k} < \frac{1}{2}, \forall k > 2L \ .$$

The equation is of the form

$$(I - A_k)X = \frac{\sin kx}{k},$$

and so

$$X = \frac{1}{k}\sum_{n\geq 0} A_k^n \sin kx, \ \|kX - \sin kx\| \leq \sum_{n\geq 1}\|A_k\|^n = O(k^{-1}) \ .$$

Based on this estimate, let $k = k_n$ and so $X(x) = X_n(x)$, then we obtain

$$k_n X_n(x) = \sin[k_n x] + O(\frac{1}{k_n}) = \sin[(\frac{n\pi}{L} + O(\frac{1}{n}))x] + O(\frac{1}{k_n}) = \sin(\frac{n\pi x}{L}) + O(\frac{1}{n}) \ ,$$

which gives us the high-frequency asymptotics or short-wave asymptotics.

2.2.4. Proof of Theorem 2.3. At first, we make an observation. If $Y(x_0) = 0$, then, in view of the structure theorem for linear homogeneous ODE, as Y is nontrivial, we must have $Y'(x_0) \neq 0$. In particular, the set of zeros of Y is a discrete set (and finite).

Without loss of generality, we suppose that a, b are neighboring zero points, and $Y(x) > 0$ for $x \in (a, b)$, which tells us that $Y'(a) > 0 > Y'(b)$. If there exists $x_0 \in (a, b)$ such that $X(x_0) = 0$, we are done. Suppose next that $X > 0$ in (a, b).

Introducing the *Wronskian*

$$W(x) = X'Y - Y'X, x \in [a, b] ,$$

we get from the equation that

$$W' = X''Y - Y''X = (Q_2 - Q_1)XY \leq 0, \ W(b) \leq W(a) ,$$

where $W(b) = W(a)$ iff $Q_2 \equiv Q_1$. Noticing that

$$W(b) - W(a) = -Y'(b)X(b) + Y'(a)X(a) \geq 0 ,$$

we conclude that $W(b) = W(a)$, $Q_2 \equiv Q_1$, and $X(a) = X(b) = 0$. As X, Y satisfy the same ODE

$$y'' + Q_1(x)y = 0, y(a) = 0,$$

there exists $c = X'(a)/Y'(a)$ so that $X = cY$, which completes the proof.

2.2.5. The Green function and completeness of the system of eigenfunctions. To show the completeness of the system of eigenfunctions, we consider the operator $\mathcal{L}X = -X'' + Q(x)X$ in the framework of the functional analysis.

Let $E = C[0, L]$ be the Banach space of continuous functions. Consider the (unbounded) operator $\mathcal{L}$ with domain

$$D_{\mathcal{L}} = \{X \in C^2[0, L] : X(0) = X(L) = 0\} .$$

We claim that, for $Q \geq 0$, the operator is invertible, and the inverse operator $\mathcal{L}^{-1}$ can be expressed as an integral operator,

$$\mathcal{L}^{-1}f(x) = \int_0^L G(x, y)f(y)dy ,$$

where $G \in C[0, L]^2$. The function $G(x, y)$ is called the Green function of $\mathcal{L}$.

To find $\mathcal{L}^{-1}f$, we need to solve

$$\mathcal{L}X = f, X(0) = X(L) = 0 . \tag{2.22}$$

For that purpose, we introduce two homogeneous solutions to

$$\mathcal{L}X_1 = 0, X_1(0) = 0, X_1'(0) = 1; \ \mathcal{L}X_2 = 0, X_2(L) = 0, X_1'(L) = -1 .$$

As 0 is not an eigenvalue of $\mathcal{L}$ (due to $Q \geq 0$), it is clear that $X_1(L) \neq 0$, and X_1, X_2 are linear independent.

By the variation of parameters, we try to find a solution to (2.22), by

$$X(x) = C_1(x)X_1(x) + C_2(x)X_2(x) ,$$

with $C_1(L) = 0$ and $C_2(0) = 0$ (so that the boundary condition is satisfied). Then

$$C_1'X_1 + C_2'X_2 = 0, C_1'X_1' + C_2'X_2' = -f$$

and so

$$C_1' = fX_2/W, C_2' = -fX_1/W$$

where

$$W = X_1 X_2' - X_2 X_1' = const \neq 0 \ .$$

Thus we arrive at

$$X(x) = \int_0^x C_2'(t)dt X_2(x) - \int_x^L C_1'(t)dt X_1(x) = \int_0^L G(x,t)f(t)dt \ ,$$

where

$$G(x,t) = -\frac{X_1(t)H(x-t)X_2(x) + H(t-x)X_2(t)X_1(x)}{W}$$

and $H(t) = \chi_{[0,\infty)}(t)$ is the Heaviside function. It is clear that $G(x,t)$ is continuous and symmetric.

Consider now the operator T in $H = L^2[0,L]$ with the kernel G

$$Tf(x) = \int_0^L G(x,t)f(t)dt \ .$$

Since the kernel is continuous and symmetric, this is a symmetric compact operator. By the Hilbert-Schmidt theorem, it has a complete orthogonal system of eigenfunctions $\phi_n(x)$ with real eigenvalues μ_n, where $n = 1, 2, \cdots$, and $\mu_n \to 0$ as $n \to \infty$.

We claim that T is injective, and thus $\mu_n \neq 0$ for all n. Actually, if $u \in H$ with $Tu = 0$, then

$$\langle u, Tw \rangle = \langle Tu, w \rangle = 0, \forall w \in H \ ,$$

that is, $u \perp TH$. On the other hand, as we have seen, for any $w \in D_{\mathcal{L}}$, $w = T\mathcal{L}w$ and so $D_{\mathcal{L}} \subset TH$. Since $D_{\mathcal{L}}$ is dense in H, we have

$$D_{\mathcal{L}} \subset TH \Rightarrow (TH)^{\perp} \subset (D_{\mathcal{L}})^{\perp} = H^{\perp} = \{0\} \ ,$$

and so $u = 0$.

As $T\phi_n = \mu_n \phi_n$, and $\mu_n \neq 0$, by Cauchy-Schwarz inequality, we have

$$\phi_n = \frac{1}{\mu_n} T\phi_n \in C[0,L] \ .$$

It is possible to prove that $\phi_n \in C^2[0,L]$[2], so that we could apply $\mathcal{L}$, to get

$$\mathcal{L}\phi_n = \frac{1}{\mu_n}\mathcal{L}T\phi_n = \frac{1}{\mu_n}\phi_n \ .$$

Thus, we see that $\phi_n = c_n X_n$ and $\lambda_n = 1/\mu_n$.

In conclusion, the set of eigenfunctions of T coincides exactly with the set of eigenfunctions of $\mathcal{L}$. In particular, we have proved the completeness of the system of eigenfunctions of $\mathcal{L}$ in $L^2([0,L])$.

[2]For the regularity issue, see, e.g., Brezis [2, Section 8.6]. Actually, $\mathcal{L}u = f$ has a unique weak solution $u = Tf \in H^2 \cap H_0^1$ for any $f \in L^2$. Thus $\phi_n \in L^2$ gives us $\phi_n \in H^2 \cap H_0^1 \subset C^1$. Then

$$\phi_n'' = q\phi_n - \lambda_n \phi_n \in C$$

and so $\phi_n \in C^2$. Concerning spectral theorem for self-adjoint compact operators, see, e.g., [2, Section 6.4].

2.3. Fourier transform: basics

2.3.1. Heuristic derivation of Fourier transform from Fourier series. As occurred in (2.15), set

$$\hat{f}(\xi) = \int_{-L}^{L} f(y)e^{-iy\xi}dy \ .$$

If $f \in C_c^2$, then for any $L > L_0$ with supp $f \subset [-L_0, L_0]$, we have

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(y)e^{-iy\xi}dy = O((1+|\xi|^2)^{-1}) \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}) \ ,$$

and for any $|x| \le L_0$,

(2.23)

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\hat{f}(\frac{n\pi}{L})e^{\frac{in\pi x}{L}} = \sum_{n=-\infty}^{\infty} \frac{\Delta\xi_n}{2\pi}\hat{f}(\xi_n)e^{i\xi_n x} \to \frac{1}{2\pi}\int \hat{f}(\xi)e^{ix\xi}d\xi \quad ,$$

by (2.14) and (2.15).

In general, with spatial dimension $n \ge 1$, we have the **Fourier inversion formula**

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)d\xi \tag{2.24}$$

for $f \in C_c^\infty(\mathbb{R}^n)$, where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx \ .$$

2.3.2. Fourier transform. Definition. If $f \in L^1(\mathbb{R}^n)$, then the Fourier transform is defined as

$$\mathcal{F}(f) = \hat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}dx$$

Lemma 2.4. *Suppose $f, g \in C_c^\infty(\mathbb{R}^n)$.*

(1) *Let $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, then*

$$\widehat{D_j f} = \xi_j \hat{f}, \qquad \widehat{x_j f} = -D_j \hat{f}.$$

In particular, $\mathcal{F} : C_c^\infty \to C^\infty$.

(2) (Riemann-Lebesgue lemma) $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$; $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$.

(3) (Parseval's identity) $\int \hat{f}g = \int f\hat{g}$.

(4) (Convolution) $(f*g)^\wedge(\xi) = \hat{f}\cdot\hat{g}$, *where* $(f*g)(x) = \int f(x-y)g(y)dy$.

(5) (Scaling, translation) *Let* $S_\lambda f(x) = f(\lambda x)$, $\tau_h f(x) = f(x-h)$, *then*

$$\mathcal{F}(S_\lambda f) = \lambda^{-n}\hat{f}(\xi/\lambda),\ (\mathcal{F}\tau_h f)(\xi) = e^{-ih\cdot\xi}\hat{f}(\xi).$$

(6) (Gaussian function) *If* $G_\lambda(x) = e^{-\lambda|x|^2}$, *where* $\Re\lambda > 0$, *then*

$$\hat{G_\lambda}(\xi) = \left(\frac{\pi}{\lambda}\right)^{n/2} e^{-\frac{|\xi|^2}{4\lambda}} = \left(\frac{\pi}{\lambda}\right)^{n/2} G_{1/(4\lambda)} \ . \tag{2.25}$$

2.3.3. Sample application to PDE. Similar to Fourier series, the Fourier transform is useful to solve PDEs, in view of Lemma 2.4 (1), which could be used to transform PDE to algebraic equations.

For example, we could try to find a solution to the Cauchy problem for the heat equation, by using the Fourier transform:

$$\partial_t u - \Delta u = F(t,x) \in L^1([0,\infty)\times\mathbb{R}^n), u(0,x) = f(x) \in L^1(\mathbb{R}^n) \tag{2.26}$$

If we can do the Fourier transform for u, with respect to x, then

$$\partial_t \hat{u} + |\xi|^2\hat{u} = \hat{F}(t,\xi), \hat{u}(0,\xi) = \hat{f}(\xi)$$

For simplicity, take $F = 0$, we have

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{f}(\xi)$$

In view of Lemma 2.4 (4), we need to find some function $K_t(x)$ such that $\mathcal{F}(K_t)(\xi) = e^{-t|\xi|^2}$ so that one solution is given by

$$u(t,x) = K_t * f, \ \forall t > 0 \ . \tag{2.27}$$

To calculate K_t, we use Lemma 2.4 (6), with $\lambda = 1/(4t)$, to conclude that

$$K_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \ ,$$

which is known as the heat kernel.

2.3.4. Schwartz space $\mathcal{S}$ (and its topology). In view of Lemma 2.4 (1), we find the dual relation between differential operator D_j and multiplication operator x_j.

It turns out that $\mathcal{F}(C_c^\infty)\cap C_c^\infty = \{0\}$[3], but $\mathcal{F}(C_c^\infty) \subset C^\infty$. Then it is natural to seek a minimal function space, which are invariant under both D_j, x_j, the Fourier transform, and include C_c^∞.

Notice that if $f \in C_c^\infty$, then

$$D^\alpha x^\beta f \in C_c^\infty \subset L^1, \ \forall \alpha,\beta \in \mathbb{N}^n \ ,$$

which is equivalent to

$$x^\alpha D^\beta \hat{f} \in L^\infty, \ \forall \alpha,\beta \in \mathbb{N}^n \ ,$$

which is also equivalent to

$$x^\alpha D^\beta \hat{f} \in L^1, \ \forall \alpha,\beta \in \mathbb{N}^n \ .$$

Thus, a natural candidate of the (invariant) space is simply the collection of smooth functions with

$$x^\alpha D^\beta f \in L^1 \ ,$$

which is the Schwartz space $\mathcal{S}(\mathbb{R}^n)$[4].

[3]When $f \in C_c^\infty$, we could replace $\xi \in \mathbb{R}^n$ by $\xi \in \mathbb{C}^n$ in the Fourier transform, which defines an analytic functions. If $\hat{f} \in C_c^\infty$, then it is clear that $\hat{f} \equiv 0$ (by analyticity). By inversion, we see that $f \equiv 0$.

[4]The Schwartz space was named in honour of Laurent Schwartz by Alexander Grothendieck. A function in the Schwartz space is sometimes called a Schwartz function.

Definition 2.1. *A function $\phi \in C^\infty(\mathbb{R}^n)$ is said to be in the Schwartz class, if for all multi indices α, β we have*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty.$$

The topology in $\mathcal{S}$ is given by the (counted) semi-norms

$$p_{\alpha,\beta}(\phi) = \|x^\alpha \partial^\beta \phi(x)\|_{L^\infty} \ ,$$

which makes the space a Fréchet space[5].

In other words, a sequence of functions ϕ_m converges to ϕ in this topology if, for all $k \in \mathbb{N}$

$$P_k(\phi_m - \phi) \to 0, \quad as\ m \to \infty,$$

where $P_k(\phi) = \sum_{|\beta|\le k} \|(1+|x|^2)^{\frac{k}{2}} \partial^\beta \phi\|_{L^\infty}$.

Equivalently,

$$\lim_{m\to\infty} \|x^\alpha \partial^\beta (\phi_m - \phi)\|_{L^1} = 0, , \ \forall \alpha, \beta \in \mathbb{N}^n \ ,$$

or

$$\lim_{m\to\infty} \|x^\alpha \partial^\beta (\phi_m - \phi)\|_{L^\infty} = 0, , \ \forall \alpha, \beta \in \mathbb{N}^n \ ,$$

or

$$\lim_{m\to\infty} \|(1+|x|^2)^k \partial^\beta (\phi_m - \phi)\|_{L^\infty} = 0, , \ \forall k \in \mathbb{N}, \forall \beta \in \mathbb{N}^n \ .$$

It is obviously invariant under the Fourier transform, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, in view of Lemma 2.4 (1).

2.3.5. Fourier inversion. Here we prove the heuristically verified Fourier inversion formula (2.24), in the framework of $\mathcal{S}(\mathbb{R}^n)$.

Theorem 2.5 (The Fourier inversion formula). *Suppose $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi \ . \tag{2.28}$$

Proof. Inspired by the solutions to the heat equation, (2.27), with the heuristics that it converges to the data as $t \to 0+$, we try to use heat operator to prove it. Notice that, for any $t > 0$,

$$\int_{\mathbb{R}^n} K_t dx = \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} dx = \int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx = (2\pi)^{-\frac{n}{2}} (\int_{\mathbb{R}^2} e^{-\frac{|x|^2}{2}} dx)^{\frac{n}{2}} = 1 \ ,$$

and $K_t > 0$. This ensures that, for any $f \in \mathcal{S}$, we have

$$f(x) = \lim_{t\to 0+} f * K_t(x) \ .$$

[5]By definition, Fréchet space (X, ρ) is a locally convex, topological vector space, which topology is induced from a complete, invariant metric. In practice, the metric ρ could be defined through countably many semi-norms. Here, the metric for $f, g \in \mathcal{S}$ could be

$$\rho(f, g) = \|f - g\| = \sum_{k\ge 0} 2^{-k} \frac{P_k(f-g)}{1 + P_k(f-g)} \ ,$$

which is easy to check that it satisfies the following axioms for the quasi-norm:

(1) (Positive definite) $\|f\| \ge 0$, $=$ iff $f = 0$;
(2) (Triangle inequality) $\|f + g\| \le \|f\| + \|g\|$;
(3) (Symmetry) $\|-f\| = \|f\|$;
(4) (Continuity) $\lim_{\alpha_n\to 0} \|\alpha_n f\| = 0$, $\lim_{\|f_n\|\to 0} \|\alpha f_n\| = 0$, $\forall \alpha, \alpha_n \in \mathbb{C}$.

Actually, here, as $\nabla f \in L^\infty$, $f(x)-f(x-y)=\mathcal{O}(|y|)$, we have

$$f(x)-f*K_t(x)=\int_{\mathbb{R}^n} K_t(y)(f(x)-f(x-y))dy\ ,$$

$$|f(x)-f*K_t(x)|\le \int_{\mathbb{R}^n} K_t(y)|f(x)-f(x-y)|dy \le C\int_{\mathbb{R}^n} K_t(y)|y|dy = Ct^{1/2}\int_{\mathbb{R}^n} K_1(y)|y|dy\ .$$

However, recall that

$$(2\pi)^{-n}\mathcal{F}_\xi(e^{-t|\xi|^2})(x)=K_t(x)\ ,$$

we see that

$$\begin{aligned} f*K_t(x) &= \int K_t(y)f(x-y)dy \\ &= (2\pi)^{-n}\int \mathcal{F}(e^{-t|\xi|^2})(y)f(x-y)dy \\ &= (2\pi)^{-n}\int e^{-t|\xi|^2}\mathcal{F}_y(f(x-y))(\xi)d\xi \\ &= (2\pi)^{-n}\int e^{-t|\xi|^2}\hat f(-\xi)e^{-ix\cdot\xi}d\xi \\ &= (2\pi)^{-n}\int e^{-t|\xi|^2}\hat f(\xi)e^{ix\cdot\xi}d\xi\ , \end{aligned}$$

where we have used Lemma 2.4 (3), (5).

Now, as $\hat f \in \mathcal{S}\subset L^1$, we get by dominated convergence theorem that

$$f(x)=\lim_{t\to 0+} f*K_t(x)=\lim_{t\to 0+}(2\pi)^{-n}\int e^{-t|\xi|^2}\hat f(\xi)e^{ix\cdot\xi}d\xi=(2\pi)^{-n}\int \hat f(\xi)e^{ix\cdot\xi}d\xi\ ,$$

which completes the proof. ■

With the help of Theorem 2.5, we have the following

Proposition 2.6. *The Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.*

In view of Theorem 2.5 and Lemma 2.4 (1), we see that $\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is a bijection. To prove the continuity of $\mathcal{F}$ and $\mathcal{F}^{-1}$, we need only to check that $\hat f_j \to \hat f$ in $\mathcal{S}$, whenever $f_j\to f$ in $\mathcal{S}$. The details are left as exercise.

2.3.6. Plancherel, Hausdorff-Young and Heisenberg.

Lemma 2.7 (Plancherel theorem, Parseval's identity). *Let $f,g\in\mathcal{S}$, then*

$$\int f\hat h dx=\int \hat f h dx\ ,\ \int f\bar g dx=(2\pi)^{-n}\int \hat f\bar{\hat g}dx\ . \tag{2.29}$$

The Fourier transform on $\mathcal{S}$ extends uniquely to a isomorphism of L^2 onto itself.

Let $\hat h=\bar g$, then $h=(2\pi)^{-n}\bar{\hat g}(x)$, which verifies (2.29). If $f\in L^2$, we could find $f_j\in\mathcal{S}$ with $f_j\to f$ in L^2, then $\mathcal{F}f_j$ is Cauchy in L^2 and we could define $\mathcal{F}f=\lim\mathcal{F}f_j$. This is independent of the choice of f_j. By density of $\mathcal{S}$ in L^2, we also know that (2.29) applies for $f,g\in L^2$. Notices that, the operator $T=(2\pi)^{-n/2}\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R}^n)$.

LEMMA 2.8 (Hausdorff-Young inequality). *Let $f \in L^p$ with $p \in [1,2]$, then $\hat{f}$ is well-defined and we have*

$$\|\hat{f}\|_{p'} \le (2\pi)^{n/p'} \|f\|_p$$

The case $p = 1$ is trivial. For $p = 2$, it follows directly from the Plancherel theorem. Then the general case for $p \in (1,2)$ follows from the the Riesz-Thorin interpolation theorem.

LEMMA 2.9 (Heisenberg's uncertainty principle). *Let $f \in \mathcal{S}$, then*

$$\|(x - x_0)f\|_2 \|(\xi - \xi_0)\hat{f}\|_2 \ge C_n \|f\|_2^2$$

where $C_n = n(2\pi)^{n/2}/2$.

WLOG, $x_0 = \xi_0 = 0$.

$$\begin{aligned} \int |f|^2 dx &= \frac{1}{n}\int (\nabla \cdot x)|f|^2 dx = -\frac{1}{n}\int x \cdot \nabla |f|^2 dx \\ &= -\frac{1}{n}\int x \cdot (\nabla f \bar{f} + f \nabla \bar{f}) dx \\ &\le \frac{2}{n}\|xf\|\|\nabla f\| = \frac{2}{n}(2\pi)^{-n/2}\|xf\|\|\xi \hat{f}\| \ . \end{aligned}$$

2.3.7. Paley-Wiener theory. Let $f \in C_c^\infty$, then it is clear that $\hat{f}$ is well-defined for $\xi \in \mathbb{C}^n$, and becomes an entire function. Moreover, if supp $(f) \subset B_R$, then

$$|\hat{f}(\xi)| = |\int f(x) e^{-ix\cdot\xi} dx| \le \int_{|x|\le R} |f(x)| e^{R|\Im \xi|} dx \le C e^{R|\Im \xi|} \ .$$

As $\mathcal{F}\nabla f(\xi) = i\xi \hat{f}(\xi)$, we have

$$|\hat{f}(\xi)| \le C_N (1 + |\xi|)^{-N} e^{R|\Im\xi|} \ , \forall N \ge 0, \xi \in \mathbb{C}^n. \tag{2.30}$$

It turns out that this is an equivalent condition for the Fourier transform of $f \in C_c^\infty(B_R)$.

THEOREM 2.10 (Paley-Wiener theory). *Let g be an entire function satisfies*

$$|g(\xi)| \le C_N (1 + |\xi|)^{-N} e^{R|\Im\xi|} \ , \forall N \ge 0, \xi \in \mathbb{C}^n. \tag{2.31}$$

Then there exists $f \in C_c^\infty(B_R)$ so that $g = \hat{f}$.

Proof. At first, as $g \in L^1$, we could define

$$f(x) = \mathcal{F}^{-1}(g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} g(\xi) e^{ix\cdot\xi} d\xi \ .$$

It is clear that (2.31) ensures that $f \in C^\infty$, for example,

$$\lim_{\epsilon\to 0} \frac{f(x + \epsilon e_1) - f(x)}{\epsilon} = \lim (2\pi)^{-n} \int_{\mathbb{R}^n} g(\xi) e^{ix\cdot\xi} \frac{e^{i\epsilon\xi_1} - 1}{\epsilon} d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} g(\xi) e^{ix\cdot\xi} i\xi_1 d\xi = \partial_1 f \ .$$

It remains to prove the support information. As g is holomorphic, we could consider deformations of the region of integration off of the real axis in the complex space $\mathbb{C}^n$:

$$\begin{aligned}(2\pi)^n f(x) &= \lim_{T\to\infty}\int_{\mathbb{R}^{n-1}}\int_{-T}^{T} g(\xi)e^{i(x_1\xi_1+x'\cdot\xi')}d\xi_1 d\xi' \\ &= \lim_{T\to\infty}\int_{\mathbb{R}^{n-1}}\int_{-T}^{T} g(\xi_1+i\eta_1,\xi')e^{i(x_1(\xi_1+i\eta_1)+x'\cdot\xi')}d\xi \\ &\quad +O(\int_{\mathbb{R}^{n-1}}\int_{0}^{\eta_1} g(\pm T+it,\xi')e^{i(x_1(\pm T+it)+x'\cdot\xi')}dtd\xi') \\ &= \int_{\mathbb{R}^n} g(\xi_1+i\eta_1,\xi')e^{i(x_1(\xi_1+i\eta_1)+x'\cdot\xi')}d\xi\ ,\end{aligned}$$

where in the last equality, we used the assumption (2.31) so that

$$|g(\pm T+it,\xi')e^{i(x_1(\pm T+it)+x'\cdot\xi')}|\lesssim T^{-N}(1+|\xi'|)^{-N}e^{(R+|x_1|)t}\ ,$$

and no contributions come from the boundaries $\xi_1=\pm T$ in the limit. Repeating this argument in all variables, we show that

$$f(x)=(2\pi)^{-n}\int_{\mathbb{R}^n} g(\xi+i\eta)e^{ix\cdot(\xi+i\eta)}d\xi\ ,\forall\eta\in\mathbb{R}^n\ .$$

Then, in view of (2.31), for any x with $|x|>R$, we choose $\eta=tx$ with $t>0$, and obtain

$$|f(x)|\lesssim\int_{\mathbb{R}^n}(1+|\xi|)^{-N}e^{Rt|x|-t|x|^2}d\xi\lesssim e^{(R-|x|)t|x|}\ ,\forall t>0\ .$$

As $t\to\infty$, we conclude $f=0$, which completes the proof. ∎

2.4. Homeworks

Section 2.1

(1) Prove the uniqueness in Theorem 2.2, for $(x,t)\in[0,L]^2$. Hint: it could be proved by method of characteristics, or (odd) extension, or idea of Holmgren's uniqueness theorem.

(2) When $g=0$, $f=\sum A_n\sin nx\in C^3$, with the compatibility condition (2.3), prove the series

$$\sum\partial_t\partial_x u_n\ ,u_n(t,x)=A_n\cos nt\sin nx \tag{2.32}$$

is uniformly convergent, in $[0,\pi]\times[0,\infty)$.

(3) Prove the strengthened version of Theorem 2.2, for $(x,t)\in[0,L]^2$, with $f\in C^2$, $g\in C^1$ instead. [Hint: use the general solutions for wave equations]

(4) (Bernstein's theorem) We have similar result of uniform convergence as that of (2.11), when $f\in C^\alpha$ with $1/2<\alpha\le 1$. Here f is Hölder continuous with exponent α ($\in C^\alpha(\Omega)$), if

$$|f(x)-f(y)|\le C|x-y|^\alpha,\qquad\forall x,y\in\Omega,$$

for some constant C.

- Let f be the periodic extension of $f\in C^\alpha([-L,L])$. Set $g(x)=f(x+h)-f(x-h)$, prove that

$$\frac{1}{8L}\int_{-L}^{L}|g(x)|^2dx=\sum_{n\in\mathbb{Z}}\sin^2(\frac{n\pi h}{L})|\hat f(n)|^2$$

- Let $h = 2^{-j}L$, show that

$$\sum_{2^{j-2}<|n|\le 2^{j-1}} |\hat{f}(n)|^2 \le C_{\alpha,L} 2^{-2j\alpha}$$

- Estimate $\sum_{2^{j-2}<|n|\le 2^{j-1}} |\hat{f}(n)|$ and conclude that the Fourier series of f converges absolutely and hence uniformly.

(5) (Poisson's formula) Solve the boundary value problem of the Laplace equation in disc:

$$\begin{cases} \Delta u = u_{xx} + u_{yy}, \quad x^2 + y^2 < R^2, \\ u(R\cos\theta, R\sin\theta) = f(\theta) \end{cases} \tag{2.33}$$

by the method of separation of variables (in polar coordinates). You should finally obtain the celebrated Poisson's formula

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi = \frac{R^2 - |X|^2}{2\pi R} \int_{|Y|=R} \frac{u(Y)}{|X-Y|^2} dS(Y) \,. \tag{2.34}$$

(6) Specify conditions on f, and solve the initial boundary value problem of heat equations

$$\begin{cases} u_t = u_{xx}, \quad x \in [0,1], \ t > 0. \\ u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = f(x) \end{cases} \tag{2.35}$$

by the method of separation of variables.

(7) Solve the boundary value problem of the Laplace equation in annulus

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, \quad x^2 + y^2 \in (a^2, b^2), \\ u(a\cos\theta, a\sin\theta) = f(\theta), u(b\cos\theta, b\sin\theta) = g(\theta) \,. \end{cases} \tag{2.36}$$

Section 2.3

(1) Prove the Riemann-Lebesgue lemma: $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$.
(2) Find a solution for (2.26):

$$\partial_t u - \Delta u = F(t,x) \in L^1([0,\infty) \times \mathbb{R}^n), u(0,x) = f(x) \in L^1(\mathbb{R}^n) \,.$$

(3) Find a solution to the following Dirichlet problem for the Laplace equation, by using the Fourier transform:

$$(\partial_x^2 + \partial_y^2)u = 0, (x,y) \in \mathbb{R} \times \mathbb{R}_+, u(x,0) = f(x) \in \mathcal{S}(\mathbb{R})$$

(4) Check that the Fourier transform is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

2.5. Notes and references

In Section 2.2, we mainly follow [22, Chapter 3].

CHAPTER 3

Elements of distribution theory

Last chapter: Method of separation of variables, Fourier series, Fourier transform, Schwartz class $\mathcal{S}$.

3.1. Tempered distribution $\mathcal{S}'$ (and its topology)

QUESTION 3.1. *Could we define the Fourier transform for larger class of functions?*

We know that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extend to L^2 by Plancherel. Well-defined for $f \in L^p$ with $p \in (2, \infty]$?

Alternatively, we can define by duality: L^2 functions are exactly bounded linear functionals of L^2! Recall Parseval's identity: $\int \hat{f} g = \int f \hat{g}$. If $f \in L^2$, let $f_j \in \mathcal{S}$ convergent to f in L^2, then

$$\int \hat{f}_j g = \int f_j \hat{g} \to \int f \hat{g}, \ \forall g \in \mathcal{S} \ ,$$

and we define the functional $T : g \to \int f \hat{g}$ which can be represented by $\int \hat{f} g$, $T = \hat{f}$.

To define $\mathcal{F}$ for bigger class, we need to find smaller space of "test functions" A which is reasonably big (include all "good" functions, say C_c^∞). For $f \in A'$, which is a linear functional of A, define $\hat{f}(g) := f(\hat{g})$, which naturally require $\mathcal{F} : A \to A$. We will choose $A = \mathcal{S}$.

DEFINITION 3.2 (Tempered distribution). *The dual of $\mathcal{S}$, denoted $\mathcal{S}'$, is the space of tempered distributions. In other words, any $u \in \mathcal{S}'$ is a continuous functional on $\mathcal{S}$, we write $\langle u, \phi \rangle$ for u applied to $\phi \in \mathcal{S}$. The space is equipped with the weak-$*$ topology. Thus, $u_m \to u$ in $\mathcal{S}'$ if and only if*

$$\langle u_m, \phi \rangle \to \langle u, \phi \rangle \quad \textit{as } m \to \infty, \quad \forall \phi \in \mathcal{S}.$$

PROPOSITION 3.1 (Characterization of tempered distribution). *A linear form u on $\mathcal{S}$ is in $\mathcal{S}'$ if and only if there exist $N > 0$ and C_N such that $|u(\phi)| \le C_N P_N(\phi)$ for every $\phi \in \mathcal{S}$.*

Proof. $\Leftarrow$ If $\phi_j \in \mathcal{S}$ is convergent to 0, then $P_k(\phi_j) \to 0, \forall k$. By assumption, $\exists N$, $|u(\phi_j)| \le C_N P_N(\phi_j) \to 0$. This proves $u \in \mathcal{S}'$.

$\Rightarrow$ We argue by contradiction. If not, $\forall k > 0$, $\exists \phi_k \ne 0 \in \mathcal{S}$, such that

$$u(\phi_k) \ge k^2 P_k(\phi_k) \ .$$

Let $\Phi_k = \frac{\phi_k}{k P_k(\phi_k)} \in \mathcal{S}$, we have for any $N \le k$, $P_N(\Phi_k) \le k^{-1} \to 0$, i.e., $\Phi_k \to 0$ in $\mathcal{S}$. However, by the construction, we know that $u(\Phi_k) \ge k$, which gives us the desired contradiction (to the continuity). ■

3.1.1. Examples.

EXAMPLE 3.1. *If* $f \in L^1_{loc}(\mathbb{R}^n)$*, we define* $f(g) = \int fg dx$*. It is easy to check that any polynomial* $P(x) \in \mathcal{S}'(\mathbb{R}^n)$*,* $L^1(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$*, however,* $f(x) = e^{x^2} \notin \mathcal{S}'(\mathbb{R})$*,* $g(x) = e^x \notin \mathcal{S}'(\mathbb{R})$*.*

EXAMPLE 3.2. *the function* $H(x) = 1$ *for* $x > 0$*, and* 0 *for* $x \le 0$ *is known as the Heaviside function. It is clear that* $H \in \mathcal{S}'(\mathbb{R})$*.*

3.1.2. Operators on $\mathcal{S}'$. Let T be a continuous linear operator on $\mathcal{S}$, suppose there is another continuous linear operator T' (dual or transpose or adjoint of T) such that

$$\int (T\phi)\psi = \int \phi T'\psi,\ \forall \phi, \psi \in \mathcal{S}\ ,$$

then we can extend T to $\mathcal{S}'$, by

$$(Tu)(\psi) = u(T'\psi),\ \forall \psi \in \mathcal{S}\ \ .$$

3.1.2.1. *Rigid transform.* translation, rotation, inversion, scaling, change of variables, etc.

3.1.2.2. *Derivative.* Let $T = \partial_j$, i.e., $T\phi := \frac{\partial \phi}{\partial x_j}$, then $T' = -\partial_j$, and so we can define $\partial_j u \in \mathcal{S}'$ by $(\partial_j u)(\phi) = -u(\partial_j \phi)$ for any $u \in \mathcal{S}'$.

EXAMPLE 3.3. *Let* $\delta = \delta_0 = H' \in \mathcal{S}'(\mathbb{R})$*, which is known as the Dirac* δ*-function:* $\delta(\phi) = -H(\phi') = -\int_0^\infty \phi'(x)dx = \phi(0)$*. In general, we define* $\delta_{x_0}(\phi) = \phi(x_0)$*.*

3.1.2.3. *Multiplication by* $\mathcal{S}$*.* $f \in \mathcal{S}$, $g \in \mathcal{S}'$, define $fg \in \mathcal{S}'$ by

$$\langle fg, h\rangle = \langle g, fh\rangle$$

Actually: multiplication by $f \in C^\infty$, with $\partial^\alpha f = \mathcal{O}(\langle x\rangle^{N_\alpha})$, could also be well-defined. We will denote such functions to be $f \in \mathcal{O}_M(\mathbb{R}^n)$. As, for any given N and any $|\alpha| \le N$, we have $\partial^\alpha f = \mathcal{O}(\langle x\rangle^{\tilde N})$ where $\tilde N = \max_{|\alpha|\le N} N_\alpha$. As $h \in \mathcal{S}$, it is clear that $\partial^\alpha h = \mathcal{O}(\langle x\rangle^{-N-\tilde N})$, and so $\partial^\alpha(fh) = \mathcal{O}(\langle x\rangle^{-N})$, for any $|\alpha| \le N$. This tells us that $fh \in \mathcal{S}$ and $T : h \to fh$ is continuous on $\mathcal{S}$.

3.1.2.4. *Convolution by* $\mathcal{S}$*.* Let $f \in \mathcal{S}'$, $g \in \mathcal{S}$, we define

$$\langle f * g, \phi\rangle := \langle f(x), \langle g(y), \phi(x+y)\rangle\rangle, \forall \phi \in \mathcal{S}\ ,$$

as $\langle g(y), \phi(x+y)\rangle \in \mathcal{S}(\mathbb{R}^n_x)$.

3.1.3. Fourier transform. By Parseval's identity, $\mathcal{F}$ is self-adjoint operator on $\mathcal{S}$ and we define $\mathcal{F}$ for $u \in \mathcal{S}'$:

$$(\mathcal{F}(u))(\phi) := u(\mathcal{F}\phi) = \int u(x)\hat\phi(x)dx, \forall \phi \in \mathcal{S}\ . \tag{3.1}$$

PROPOSITION 3.2. *The Fourier transform is an isomorphism of* $\mathcal{S}'(\mathbb{R}^n)$ *onto* $\mathcal{S}'(\mathbb{R}^n)$*. Both* $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *are sequentially continuous.*

Proof. By definition and (2.28), we have

$$\langle \hat{\hat u}, \phi\rangle = \langle \hat u, \hat\phi\rangle = \langle u, \hat{\hat\phi}\rangle = (2\pi)^n \langle u, \tilde\phi\rangle\ ,$$

for any $\phi \in \mathcal{S}$, where $\tilde\phi(x) = \phi(-x)$. Thus,

$$\langle u, \phi\rangle = (2\pi)^{-n}\langle \hat u, \hat{\tilde\phi}\rangle\ , \forall \phi \in \mathcal{S}\ ,$$

which gives us the inversion theorem for the Fourier transform on $\mathcal{S}'$. So one has $u = 0$ if $\hat{u} = 0$, which is the injection. Moreover, as $\hat{\hat{\phi}} = \tilde{\tilde{\phi}}$, we have

$$\langle u, \phi\rangle = (2\pi)^{-n}\langle \hat{u}, \tilde{\hat{\phi}}\rangle = (2\pi)^{-n}\langle \tilde{\hat{u}}, \hat{\phi}\rangle$$

which tells us that, for any $u \in \mathcal{S}'$, it is the Fourier transform of $(2\pi)^{-n}\tilde{\hat{u}} = \mathcal{F}^{-1}(u)$. This proves the surjection. Concerning the sequentially continuity of $\mathcal{F}$, it follows directly from the definition (3.1). This completes the proof. ∎

As in Lemma 2.4, we have the following

COROLLARY 3.3. *Suppose $f \in \mathcal{S}'(\mathbb{R}^n)$.*

(1) *Let $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, then*

$$\mathcal{F}(D^\alpha f) = \xi^\alpha \hat{f}, \qquad \mathcal{F}(x^\alpha f) = (-1)^{|\alpha|} D^\alpha \hat{f}.$$

(2) *Let $\tau_h f(x) = f(x-h)$, then*

$$\mathcal{F}(\tau_h f)(\xi) = e^{-ih\cdot\xi}\hat{f}(\xi),\ \mathcal{F}(e^{ih\cdot x} f)(\xi) = \tau_h \hat{f}\ .$$

(3) *Let $g \in \mathcal{S}$, we have $\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g$.*

3.2. Compactly supported smooth functions $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$ and Distribution $\mathcal{D}'(\Omega)$

Seen: linear partial differential operator $L = \sum_{|\alpha|\le N} a_\alpha(x)\partial^\alpha$ is well-defined on $\mathcal{S}'$, for any $a_\alpha \in C^\infty$, with $\partial^\beta a_\alpha = \mathcal{O}(\langle x\rangle^{N_{\alpha,\beta}})$. However, the standard operator like $L = \partial_x + e^x$ is not well-defined for $\mathcal{S}'$. On the other hand, $\mathcal{S}'$ is defined only on $\mathbb{R}^n$, but not for a local domain, which is typically the region for PDE.

To find a better setting for PDE, we introduce more general definition of distribution, by using smaller space of test functions.

3.2.1. Definition.

Let $\Omega \subset \mathbb{R}^n$ be an open set.

DEFINITION 3.3 (Support). *If $u \in C(\Omega)$ then the support of u, denoted as* supp u, *is the closure in Ω of the set*

$$\{x \in \Omega; u(x) \ne 0\}.$$

We denote by $C^k(\Omega)$ the set of complex valued functions on Ω which are k times continuously differentiable and by $C_c^k(\Omega)$ the subset of those which are also compactly supported.

DEFINITION 3.4 ($C_c^\infty(\Omega)$). *We denote by $C^\infty(\Omega) = \cap_{k\in\mathbb{N}} C^k(\Omega)$ the space of infinitely differentiable functions, and by $C_c^\infty(\Omega)$ the subset of those which also have compact support.*

The topology of space $C_c^\infty(\Omega)$ is rather delicate, which is defined as the (strong) inductive limit of $\mathcal{D}_K$ of compact subset $K \subset \Omega$, see, e.g., Yosida [32, P28] and Rudin [20, Section 6.2-3].

Here, we only give the meaning of convergence. We say that $\{\phi_j\} \subset C_c^\infty(\Omega)$ is convergent to ϕ in $C_c^\infty(\Omega)$ if

(1) there is a compact set $K \subset \Omega$, supp $\phi_j, \phi \subset K$, for all $j \ge 1$,
(2) for any k,

$$p_{K,k}(\phi_j - \phi) := \sup_{|\alpha|\le k}\sup_{x\in K} |\partial^\alpha(\phi_j - \phi)(x)| \to 0.$$

We also sometimes denote $C_c^\infty(\Omega)$ as $\mathcal{D}(\Omega)$.

DEFINITION 3.5 (Distribution). *A distribution $u \in \mathcal{D}'(\Omega)$ is a continuous linear functional $u : C_c^\infty(\Omega) \to \mathbb{C}$. A sequence of distributions $u_j \in \mathcal{D}'(\Omega)$ is said to be convergent to a distribution $u \in \mathcal{D}'(\Omega)$ if $u_j(\phi) \to u(\phi)$, for all $\phi \in \mathcal{D}(\Omega)$*

REMARK 3.4. *The action of a distribution u on a test function ϕ is commonly denoted by $u(\phi) = \langle u, \phi\rangle$, or even $\int u\phi dx$, when there is no possibility of confusion.*

PROPOSITION 3.5. *A linear form u on $\mathcal{D}(\Omega)$ is continuous ($u(\phi_j) \to 0$ for every sequence $\phi_j \in \mathcal{D}(\Omega)$ converging to 0) iff it verifies the following property: for any compact set $K \subset \Omega$ there exists an integer k and a constant $C = C_{K,k}$ such that*

$$|\langle u, \phi\rangle| \le C p_{K,k}(\phi),\ \forall \phi \in C_c^\infty(K).$$

If the integer k can be used in the above definition for every K, then the smallest k is called the order of the distribution. In that case, we denote $u \in \mathcal{D}'^k(\Omega)$.

The proof is similar to that of Proposition 3.1.

3.3. Examples and basic operation

EXAMPLE 3.4. *Let $g(x) = e^{-(1-|x|^2)^{-1}}$ for $|x| < 1$ and 0 for $|x| \ge 1$, it is easy to check that $g \in C_c^\infty(\mathbb{R}^n)$.*

EXAMPLE 3.5. *$L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$, here $f \in L^1_{loc}(\Omega)$ is understood as the following linear form: $f(\phi) = \int_\Omega f\phi dx$. It is of order 0, $L^1_{loc}(\Omega) \subset \mathcal{D}'^0(\Omega)$.*

3.3.1. Multiplication by smooth functions. $f \in C^\infty(\Omega)$, $g \in \mathcal{D}'(\Omega)$, define $fg \in \mathcal{D}'$ by $\langle fg, \phi\rangle = \langle g, f\phi\rangle$.

3.3.2. Differentiation of distributions. Let $T = \partial^\alpha$, then $T' = (-1)^{|\alpha|}\partial^\alpha$, and so we can define $\partial^\alpha u \in \mathcal{D}'$ by $(\partial^\alpha u)(\phi) = u((-1)^{|\alpha|}\partial^\alpha\phi)$ for any $u \in \mathcal{D}'$.

LEMMA 3.6. *If $u \in \mathcal{D}'((a,b))$ and $u' = 0$ in $\mathcal{D}'((a,b))$, then u is a constant.*

Proof. By definition, $u' = 0$ in $\mathcal{D}'((a,b))$ tells us that

$$u(\phi') = 0, \forall \phi \in \mathcal{D}((a,b))\ .$$

Fix $\phi_0 \in \mathcal{D}((a,b))$ such that $\int_a^b \phi_0 dx = 1$. Then for any $\phi \in \mathcal{D}((a,b))$, we set

$$\psi(x) = \phi(x) - \phi_0(x)\int_a^b \phi(t)dt \in \mathcal{D}((a,b))\ ,$$

with $\int_a^b \psi(x)dx = 0$, which ensures that $\Phi(x) = \int_a^x \psi(t)dt \in \mathcal{D}((a,b))$ and $\psi = \Phi'$. Thus

$$0 = \langle u, \psi\rangle = \langle u, \phi\rangle - \langle u, \phi_0\rangle\langle 1, \phi\rangle\ ,$$

which tells us that $u = C = \langle u, \phi_0\rangle$ in $\mathcal{D}'$. ∎

COROLLARY 3.7. *If $u \in \mathcal{D}'((a,b))$ and $u' + pu = f$ in $\mathcal{D}'((a,b))$, with $f \in C((a,b))$, $p \in C^\infty((a,b))$, then $u \in C^1((a,b))$ and $u' + pu = f$ in the classical sense.*

Proof. This follows essentially from the method of integration factors. Let $w = e^{\int_{x_0}^x p(t)dt} u$ with fixed $x_0 \in (a, b)$. We see that

$$w' = e^{\int_{x_0}^x p(t)dt}(u' + pw) = e^{\int_{x_0}^x p(t)dt} f(x) := F(x) \ .$$

As $F \in C$, we know that $G(x) = \int_{x_0}^x F(t)dt \in C^1$ is one of the classical solutions. Then $(w - G)' = F - F = 0$ and thus $w = C + G$, which in turn gives us $u \in C^1$. ■

3.3.3. Linear partial differential operators. We can now define the action of a general linear partial differential operator on distributions. Given $a_\alpha \in C^\infty(\Omega)$, let

$$L(x, \partial) = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega) \ .$$

3.3.4. Dirac delta.

LEMMA 3.8. *Let $g \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} g dx = 1$, then $g_\epsilon(x) = \epsilon^{-n} g(\epsilon^{-1}x)$ converges to δ as $\epsilon \to 0+$, in $\mathcal{D}'(\mathbb{R}^n)$.*

LEMMA 3.9. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ with* supp $f = \{0\}$, *then there exists k such that*

$$f = \sum_{|\alpha| \le k} C_\alpha \nabla^\alpha \delta \ .$$

See, e.g., Hörmander [**14**, Theorem 2.3.4].

LEMMA 3.10. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ with $x^j f = 0$ for all $1 \le j \le n$, then $f = C\delta$ for some C.*

Proof. Let $\psi(x) \in \mathcal{D}$ with $\psi = 1$ in B_1, then for any $\phi \in \mathcal{D}$, we see that $|x|^{-2}(1 - \psi)\phi \in \mathcal{D}$, and so

$$\langle f, (1 - \psi)\phi \rangle = \langle f, |x|^2 |x|^{-2}(1 - \psi)\phi \rangle = \sum_j \langle x^j f, x^j |x|^{-2}(1 - \psi)\phi \rangle = 0 \ .$$

In general, for $\phi \in \mathcal{D}$, we could use Taylor expansion to conclude that

$$\phi(x) - \phi(0) = \int_0^1 \frac{d}{dt}\phi(tx)dt = \sum x^j \int_0^1 (\partial_j \phi)(tx)dt = \sum x^j \phi_j(x) \ ,$$

where $\phi_j(x) = \int_0^1 (\partial_j \phi)(tx)dt \in C^\infty(\mathbb{R}^n)$. With the help of the previous observations, we get

$$\langle f, \phi \rangle = \langle f, \psi\phi \rangle = \phi(0)\langle f, \psi \rangle + \sum \langle f, x^j \psi \phi_j \rangle = \phi(0)\langle f, \psi \rangle, \forall \phi \in \mathcal{D} \ .$$

This tells us that $f = C\delta$ with $C = \langle f, \psi \rangle$. ■

3.3.5. Dirichlet kernel. Let $f_\nu(x) = \frac{\sin \nu x}{\pi x}$ be the Dirichlet kernel, then

$$f_\nu(x) = \frac{\sin \nu x}{\pi x} \to \delta \ . \tag{3.2}$$

Actually, $f_\nu = \mathcal{F}^{-1}(\chi_{[-\nu,\nu]})$ is the Dirichlet kernel for the Fourier transform, which comes from the approximation of Fourier inversion, for $u \in \mathcal{S}$ or C_c^∞:

$$u(x) = \frac{1}{2\pi} \int e^{ix\xi} \hat{u}(\xi) d\xi = \lim \frac{1}{2\pi} \int_{-\nu}^{\nu} e^{ix\xi} \hat{u}(\xi) d\xi = \lim \mathcal{F}^{-1}(\chi_{[-\nu,\nu]} \hat{u}) = \lim f_\nu * u \ .$$

In particular, as f_ν is even, the equality with $x=0$ gives us

$$u(0)=\lim_{\nu\to+\infty}\int f_\nu(y)u(y)dy\ ,\forall u\in C_c^\infty(\mathbb{R})\ ,$$

which is $\lim_{\nu\to+\infty} f_\nu=\delta$.

3.3.6. Principle value distribution. Let $f=\ln|x|\in L^1_{loc}(\mathbb{R})\subset\mathcal{D}'(\mathbb{R})$, classically we know that $f'=1/x$ for all $x\neq 0$. In the sense of distribution, we denote $f'=\mathrm{pv}\frac{1}{\mathrm{x}}$, from which we know that (why?)

$$\langle \mathrm{pv}\frac{1}{\mathrm{x}},\phi\rangle=\lim_{\epsilon\to 0+}\int_{|\mathrm{x}|\geq\epsilon}\frac{\phi(\mathrm{x})}{\mathrm{x}}\mathrm{dx}\ . \tag{3.3}$$

Note that $\mathrm{pv}(\frac{1}{\mathrm{x}})$ is an odd distribution (it is orthogonal to even test functions), and is of order 1 even though it is of order 0 away from the origin. In fact, decomposing $\phi=\phi_{ev}+\phi_{odd}$ into even and odd parts, we have

$$\langle \mathrm{pv}\frac{1}{\mathrm{x}},\phi\rangle=\lim_{\epsilon\to 0+}\int_{|\mathrm{x}|\geq\epsilon}\frac{\phi_{\mathrm{odd}}(\mathrm{x})}{\mathrm{x}}\mathrm{dx}=\int\frac{\phi_{\mathrm{odd}}(\mathrm{x})}{\mathrm{x}}\mathrm{dx}=\int\frac{\phi(\mathrm{x})-\phi(-\mathrm{x})}{2\mathrm{x}}\mathrm{dx}=\frac{1}{2}\int\int_{-1}^{1}\phi'(\mathrm{tx})\mathrm{dtdx}\ .$$

EXAMPLE 3.6 (Principle value distribution). *By* (3.3)*, we know that* $\mathrm{pv}\frac{1}{\mathrm{x}}$ *is also in* $\mathcal{S}'(\mathbb{R})$*. Let* $g(\xi)=\mathcal{F}(\mathrm{pv}\frac{1}{\mathrm{x}})(\xi)$*. We know that* $D_\xi g=\mathcal{F}((-x)\mathrm{pv}\frac{1}{\mathrm{x}})(\xi)$*. By definition,*

$$\langle(-x)\mathrm{pv}\frac{1}{\mathrm{x}},\phi\rangle=\langle \mathrm{pv}\frac{1}{\mathrm{x}},-\mathrm{x}\phi\rangle=\lim_{\epsilon\to 0+}\int_{|\mathrm{x}|>\epsilon}\frac{1}{\mathrm{x}}(-\mathrm{x}\phi)\mathrm{dx}=-\int\phi\mathrm{dx}=\langle -1,\phi\rangle\ ,$$

we see that $D_\xi g=-2\pi\delta$*. Then* $D_\xi(g+2\pi iH)=0$*, which, by Lemma 3.6, tells us that* $g=c-2\pi iH$*, for some* c *to be determined.*

Notice that $\mathrm{pv}\frac{1}{\mathrm{x}}$ *is odd, and so is* g*, which determines* $c=\pi i$*:*

$$\mathcal{F}(\mathrm{pv}\frac{1}{\mathrm{x}})(\xi)=\pi\mathrm{i}-2\pi\mathrm{iH}=-\pi\mathrm{isgn}\xi\ .$$

By Fourier inversion, we see that $\mathcal{F}H(\xi)=\pi\delta-\mathrm{ipv}\frac{1}{\xi}$.

3.3.7. $(x\pm i0)^{-1}$**.** * For $\lambda=-1$, we define

$$(x\pm i0)^{-1}=\lim_{\epsilon\to 0+}(x\pm i\epsilon)^{-1}\ .$$

Recall that classically, we have

$$(x\pm i\epsilon)^{-1}=\frac{d}{dx}\ln(x\pm i\epsilon),\epsilon>0\ .$$

As both $(x\pm i\epsilon)^{-1}$ and $\ln(x\pm i\epsilon)$ are in L^1_{loc}, this is also true in the sense of distribution. Thus, as d/dx is continuous map on $\mathcal{D}'$, we have

$$(x\pm i0)^{-1}=\lim_{\epsilon\to 0+}(x\pm i\epsilon)^{-1}=\frac{d}{dx}\lim_{\epsilon\to 0+}\ln(x\pm i\epsilon),\epsilon>0\ .$$

Since, with the typical convention of $\arg z=0$ for positive z, we have for any $\epsilon>0$,

$$\ln(x\pm i\epsilon)=\ln\sqrt{x^2+\epsilon^2}+i\arg(x\pm i\epsilon)\to\ln|x|\pm\pi i(1-H(x))\ ,$$

then we obtain

$$(x\pm i0)^{-1}=\frac{d}{dx}\lim_{\epsilon\to 0+}\ln(x\pm i\epsilon)=\frac{d}{dx}(\ln|x|\pm\pi i(1-H(x)))=\mathrm{pv}\frac{1}{\mathrm{x}}\mp\pi\mathrm{i}\delta\ . \tag{3.4}$$

Observe then that

$$\delta = \frac{i}{2\pi}((x+i0)^{-1} - (x-i0)^{-1}),\ \mathrm{pv}\frac{1}{\mathrm{x}} = \frac{(\mathrm{x}+\mathrm{i}0)^{-1} + (\mathrm{x}-\mathrm{i}0)^{-1}}{2}\ . \tag{3.5}$$

3.3.8. x_+^λ**.** * For $\lambda \in \mathbb{C}$ with $\Re\lambda > -1$, we set

$$x_+^\lambda = \begin{cases} x^\lambda & x > 0 \\ 0 & x < 0 \end{cases} \in L^1_{loc}, (\ln x)_+ = \begin{cases} \ln x & x > 0 \\ 0 & x < 0 \end{cases} \in L^1_{loc}.$$

We know that

$$\frac{d}{dx} x_+^\lambda = \lambda x_+^{\lambda-1}, \Re\lambda > 0\ , \tag{3.6}$$

by which, similar to the Gamma function, we can extend the definition to $\lambda \in \mathbb{C}\backslash\{-1,-2,\cdots\}$.

When λ is a negative integer, it is reasonable to define

$$x_+^{-k}(\phi) = -\frac{\int_0^\infty (\ln x)\phi^{(k)}(x)dx}{(k-1)!} + \phi^{(k-1)}(0)\frac{\sum_{j=1}^{k-1} 1/j}{(k-1)!}$$

i.e.,

$$x_+^{-1} = \frac{d}{dx}(\ln x)_+, x_+^{-k} = \frac{(-1)^{k-1}}{(k-1)!}[(\frac{d}{dx})^{(k)}(\ln x)_+ + \delta^{(k-1)}\sum_{j=1}^{k-1} 1/j] \tag{3.7}$$

Then

$$x x_+^\lambda = x_+^{\lambda+1}, \forall \lambda \in \mathbb{C}$$

$$\frac{d}{dx} x_+^{-k} = -k x_+^{-k-1} + \frac{(-1)^k}{k!}\delta^{(k)}.$$

See, e.g., [**14**, P68-69] for more details. Similarly, we define

$$\langle x_-^\alpha, \phi\rangle = \langle x_+^\alpha, \tilde{\phi}\rangle, \tilde{\phi}(x) = \phi(-x)\ ,$$

In particular,

$$\langle x_-^{-1}, \phi\rangle = \langle x_+^{-1}, \tilde{\phi}\rangle = \langle \frac{d}{dx}(\ln x)_+, \phi(-x)\rangle = \int_0^\infty (\ln x)\phi'(-x)dx = \int_{-\infty}^0 (\ln(-x))\phi'(x)dx,$$

i.e., $x_-^{-1} = -\frac{d}{dx}(\ln|x|)_-$. Observe then that

$$\mathrm{pv}\frac{1}{\mathrm{x}} = \mathrm{x}_+^{-1} - \mathrm{x}_-^{-1}\ . \tag{3.8}$$

3.4. Smooth functions $C^\infty(\Omega) = \mathcal{E}(\Omega)$ and Distribution with compact support $\mathcal{E}'(\Omega)$

$C^\infty(\Omega) = \mathcal{E}(\Omega)$ equipped with the topology generated by seminorms $p_{K,k}$: $f_j \to f$ if

$$\forall K \subset\subset \Omega, \forall k \geq 0, p_{K,k}(f_j - f) \to 0\ ,$$

which makes it a Fréchet space.

$\mathcal{E}'(\Omega)$ is the set of continuous linear functional $u : C^\infty(\Omega) \to \mathbb{C}$.

Proposition 3.11. *The following conditions are equivalent for a linear form u on $\mathcal{E}(\Omega)$*

(1) *it is continuous ($u(\phi_j) \to 0$ for every sequence $\phi_j \in \mathcal{E}(\Omega)$ converging to 0)*

(2) *there exist compact set* $K \subset \Omega$*, an integer* k *and a constant* $C = C_{K,k}$ *such that*

$$|\langle u, \phi \rangle| \le C \sup_{|\alpha| \le k} |\partial^\alpha \phi| = C \ p_{K,k}(\phi), \ \forall \phi \in \mathcal{E}(\Omega).$$

(3) $u \in \mathcal{D}'$ *with* supp $u \subset\subset \Omega$.

Sketch of the proof. (1)→ 2: contradiction. If not, then for (K_k, k, C_k), where $K_k := \{x \in \Omega, d(x, \partial\Omega) \ge 1/k, |x| \le k\}$, and $C = k^2$, there exists $\phi_k \in \mathcal{E}$ such that

$$|u(\phi_k)| > k^2 p_{K_k,k}(\phi_k) \ .$$

Let

$$\Phi_k = \frac{\phi_k}{k p_{K_k,k}(\phi_k)} \ ,$$

then, as $\cup_{k \ge 1} K_k = \Omega$, we have $\Phi_k \to 0$ in $\mathcal{E}(\Omega)$, while $u(\Phi_k) > k$.

(2)→ 3: Let $O = K^c$, then for all $\phi \in \mathcal{D}(O)$, we have $p_{K,k}(\phi) = 0$ and so $u(\phi) = 0$. This tells us that supp $u \subset K$.

(3)→ 1: $d(K, \partial\Omega) \ge 3\epsilon$, then $u = \chi u$ ($\chi = 1$ on $K_\epsilon = B_\epsilon(K)$ and 0 on $K_{2\epsilon}^c$). Notice that $\chi\phi_j \to 0$ in $\mathcal{D}(\Omega)$ if $\phi_j \to 0$ in $\mathcal{E}(\Omega)$.

Proposition 3.12. *We have the following continuous embedding*

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E} \ , \ \mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}' \ .$$

3.5. Convolution

3.5.1. Convolution with smooth functions. Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{D}(\mathbb{R}^n)$, (or $f \in \mathcal{S}'$, $g \in \mathcal{S}$, or $f \in \mathcal{E}'(\mathbb{R}^n)$, $g \in \mathcal{E}(\mathbb{R}^n)$,) we define the convolution by the following smooth functions

$$(f * g)(x) := \langle f(\cdot), g(x - \cdot) \rangle \ .$$

Actually, recall that, as functions of y, $g(x_j - y)$ is convergent to $g(x - y)$, in the space of test functions, as $x_j \to x$. Then by definition that $f(g(x_j - \cdot))$ is convergent to $f(g(x - \cdot))$, which means that $(f * g)(x) := f(g(x - \cdot)) \in C(\mathbb{R}^n)$. Similarly, we can show that $\partial_j(f * g) = f * (\partial_j g)$, by classical definition of derivative. Thus, $f * g \in C^\infty(\mathbb{R}^n)$ by induction.

Lemma 3.13. *Let* $f \in X = \mathcal{D}(\mathbb{R}^n), \mathcal{S}$*, or* $\mathcal{E}(\mathbb{R}^n)$. $f \in X'$, $g \in X$*, then*

(1) $f * g \in \mathcal{E}(\mathbb{R}^n)$,
(2) $\delta * g = g$,
(3) supp $(f * g) \subset$ supp $f +$ supp g,
(4) $\partial^\alpha (f * g) = (\partial^{\alpha_1} f) * (\partial^{\alpha_2} g)$*, for all* $\alpha = \alpha_1 + \alpha_2$.

Lemma 3.14 (Associativity). *Let* $f \in \mathcal{D}'(\mathbb{R}^n)$*, and* $g, h \in \mathcal{D}(\mathbb{R}^n)$*, then*

$$(f * g) * h = f * (g * h) \ .$$

Lemma 3.15. *Let* $0 \le \psi \in \mathcal{D}(\mathbb{R}^n)$ *with* $\int \psi dx = 1$. *Then* $f * \psi_\epsilon \to f$ *in* $\mathcal{D}'$ *for any* $f \in \mathcal{D}'(\mathbb{R}^n)$*, as* $\epsilon \to 0+$*, where* $\psi_\epsilon(x) = \epsilon^{-n} \psi(x/\epsilon)$.

Although we have seen the generality and complexity of $\mathcal{D}'(\Omega)$, it turns out it could still be approximated by $\mathcal{D}(\Omega)$.

Proposition 3.16. $\mathcal{D}(\Omega)$ *is dense in* $\mathcal{D}'(\Omega)$.

See, e.g., Hörmander [**14**, Theorem 4.1.5]. It is also based on convolution with well-chosen test functions.

3.5.2. Convolution of distributions. * Notice that if $f \in \mathcal{D}'(\mathbb{R}^n)$, and $g, h \in \mathcal{D}(\mathbb{R}^n)$, then

$$\langle f * g, h\rangle = (f * g) * \tilde{h}(0) = f * (g * \tilde{h})(0) = \langle f, \widetilde{g * \tilde{h}}\rangle \ ,$$

where

$$\widetilde{g * \tilde{h}}(x) = \int g(y)\tilde{h}(-x-y)dy = \int g(y)h(x+y)dy = \langle g, h(x+\cdot)\rangle \ .$$

Thus we obtain an alternative definition of convolution

$$\langle f * g, h\rangle = \langle f, \widetilde{g * \tilde{h}}\rangle = \langle f(x), \langle g(y), h(x+y)\rangle\rangle \ . \tag{3.9}$$

Notice that $g * \tilde{h} \in \mathcal{E}(\mathbb{R}^n)$ is well-defined even if $g \in \mathcal{D}'(\mathbb{R}^n)$, we could see that $\langle f * g, h\rangle$ is well-defined if $g \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{E}'(\mathbb{R}^n)$. Similarly, if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in \mathcal{E}'(\mathbb{R}^n)$, we have $g * \tilde{h} \in \mathcal{D}(\mathbb{R}^n)$ and $\langle f * g, h\rangle$ is well-defined. It turns out that $f * g \in \mathcal{D}'$.

LEMMA 3.17. *Let $f, g \in \mathcal{D}'(\mathbb{R}^n)$ and one of which has compact support, then $f * g \in \mathcal{D}'(\mathbb{R}^n)$ is well-defined and commutative, with*

$$\text{supp } (f * g) \subset \text{supp } f + \text{supp } g \ ,$$

$$\text{singsupp} f * g \subset \text{singsupp} f + \text{singsupp} g \ ,$$

$$f * \delta = f \ .$$

DEFINITION 3.6 (singular support). *Let $f \in \mathcal{D}'(\Omega)$, the singular support of f, denoted by* singsupp$f$*, is defined to be the the complement of the maximal open set, on which the function f is smooth.*

For example, $\text{singsupp}\delta = \text{supp}\delta = \{0\}$, $\text{singsupp}H = \{0\}$.

Differentiation could be interpreted as convolution. Actually, for any $u \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in \mathcal{D}(\mathbb{R}^n)$, we have

$$(\partial^\alpha u) * g = u * (\partial^\alpha g) = u * (\delta * \partial^\alpha g) = u * ((\partial^\alpha \delta) * g) = (u * \partial^\alpha \delta) * g$$

which gives us

$$\partial^\alpha u = (\partial^\alpha \delta) * u \ .$$

More generally, for linear PDE operator with constant coefficients, $L = \sum a_\alpha \partial^\alpha$, we have

$$L(u_1 * u_2) = (Lu_1) * u_2 = u_1 * (Lu_2), \forall u_1 \in \mathcal{D}'(\mathbb{R}^n), u_2 \in \mathcal{E}'(\mathbb{R}^n) \ . \tag{3.10}$$

3.5.3. Fourier transform.

LEMMA 3.18. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$, then $\mathcal{F}f \in \mathcal{O}_M(\mathbb{R}^n)$ could be defined as*

$$\mathcal{F}f(\xi) = \langle f(x), e^{-ix\cdot\xi}\rangle \ .$$

Proof. Let ψ_ϵ be the functions as that in Lemma 3.15. As $f \in \mathcal{E}'(\mathbb{R}^n)$, we have $f * \psi_\epsilon \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{E}(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ and $f * \psi_\epsilon \to f \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'$. Then $\mathcal{F}(f * \psi_\epsilon) \to \hat{f} \in \mathcal{S}'$. Notice that

$$\begin{aligned} \mathcal{F}(f * \psi_\epsilon)(\xi) &= \int \langle f(y), \psi_\epsilon(x-y)\rangle e^{-ix\xi} dx \\ &= \langle f(y), \int \psi_\epsilon(x-y) e^{-ix\xi} dx\rangle \ . \end{aligned}$$

Moreover, for $\xi \in B_R$ with given $R > 0$, $\int \psi_\epsilon(x-y)e^{-ix\xi}dx \to e^{-iy\xi}$ uniformly in $\mathcal{E}$. This gives us that $\hat{f}(\xi) = \langle f(y), e^{-iy\xi}\rangle \in \mathcal{E}$. The property $\hat{f} \in \mathcal{O}_M(\mathbb{R}^n)$ follows then from the fact that $\partial^\alpha \hat{f}(\xi) = \langle f(y), (-iy)^\alpha e^{-iy\xi}\rangle$ and $|\langle f, g\rangle| \le Cp_{K,k}(g)$. ∎

THEOREM 3.19. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$, $g \in \mathcal{S}'(\mathbb{R}^n)$, then $f*g \in \mathcal{S}'(\mathbb{R}^n)$, and $\mathcal{F}(f*g) = \mathcal{F}(f)\cdot\mathcal{F}(g)$.*

Proof. As $\hat{f} \in \mathcal{O}_M$, we have $\hat{f}\hat{g} \in \mathcal{S}'$, and then there exists $w \in \mathcal{S}'$ so that $\hat{w} = \hat{f}\hat{g}$. Then, for any $h \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f}\check{h} \in \mathcal{S}(\mathbb{R}^n)$ and

$$\langle w, h\rangle = \langle \hat{f}\hat{g}, \check{h}\rangle = \langle \hat{g}, \hat{f}\check{h}\rangle = \langle g, \mathcal{F}(\hat{f}\check{h})\rangle = (2\pi)^n \langle \tilde{g}, \mathcal{F}^{-1}(\hat{f}\check{h})\rangle = \langle \tilde{g}, f * \tilde{h}\rangle \ .$$

In view of (3.9), it shows that $w = f * g$. ∎

THEOREM 3.20 (Structure theorem for $\mathcal{E}'(\mathbb{R}^n)$). *Let $f \in \mathcal{E}'(\mathbb{R}^n)$. Then there exist N and $g \in C_0(\mathbb{R}^n)$, such that*

$$f = (1-\Delta)^N g \ .$$

Proof. By Lemma 3.18, we have for some m, C,

$$|\hat{f}| \le C(1+|\xi|^2)^m \ ,$$

and so $(1+|\xi|^2)^{-m-n}\hat{f}(\xi) \in L^1$. Let $g = \mathcal{F}^{-1}((1+|\xi|^2)^{-m-n}\hat{f}(\xi)) \in C_0$, we have

$$\mathcal{F}((1-\Delta)^{n+m}g) = (1+|\xi|^2)^{m+n}\hat{g} = \hat{f}(\xi) \ ,$$

which verifies $(1-\Delta)^{n+m}g = f$. ∎

THEOREM 3.21 (Structure theorem for $\mathcal{S}'(\mathbb{R}^n)$). *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then there exist $N > 0$ and $g \in C(\mathbb{R}^n)$, such that*

$$g = \mathcal{O}(\langle x\rangle^{2N}),\ f = (\partial_1\partial_2\cdots\partial_n)^N g \ .$$

Proof. For simplicity, let us consider only the case $n = 1$.

Without loss of generality, we assume supp $f \subset \mathbb{R}_+$. Then there exists some $g \in C^\infty(\mathbb{R})$ so that $g = 1$ in supp f with supp $g \subset \mathbb{R}_+$, and then $f = gf$. As $f \in \mathcal{S}'(\mathbb{R})$, there exists m such that

$$|\langle f, \phi\rangle| = |\langle f, g\phi\rangle| \le C_m \sum_{j+k\le m} \sup |x^j \partial_x^k (g\phi)|,\ \forall \phi \in \mathcal{S}(\mathbb{R}) \ ,$$

that is, we have

$$|\langle f, \phi\rangle| \le C \sum_{j+l\le m} \sup_{x>0} x^j |\partial_x^l \phi|,\ \forall \phi \in \mathcal{S}(\mathbb{R}) \ .$$

Recall that $(\partial_x)^{N+1}x_+^N = N!\delta$, we have

$$N!f = N!f * \delta = f * (\partial_x)^{N+1}x_+^N \ .$$

Let $\phi_{N,k}(x) = x^N h_{[1/k,k]}(x) \to x_+^N$ in $\mathcal{D}'$, we have

$$N!f = \lim \partial_x^{N+1}[f * \phi_{N,k}] = \partial_x^{N+1} \lim f * \phi_{N,k} \ .$$

Observing that, when $N - 1 > m$, there exists $C > 0$, which is independent of k, so that

$$|f * \phi_{N,k}(x)| = |\langle f(t), \phi_{N,k}(x-t)\rangle| \le C(1+|x|)^{N+m} \ ,$$
$$|\partial_x f * \phi_{N,k}(x)| = |\langle f(t), \partial_x\phi_{N,k}(x-t)\rangle| \le C(1+|x|)^{N+m-1} \ .$$

This tells us that, for any given $L > 0$, the series of smooth functions $f * \phi_{N,k}(x)$, restricted in $[-L, L]$, are uniformly bounded and equicontinuous, which tells us that (Arzela-Ascoli theorem), up to subseries,

$$f * \phi_{N,k}(x) \to u \in C[-L, L] \ ,$$

with $u = 0$ for $x < 0$. In conclusion, we know that, for some $u \in C(\mathbb{R})$ with

$$|u(x)| \le C(1 + |x|)^{N+m}, N!f = \partial_x^{N+1} u \ .$$

The proof is complete if we set $N = m + 2$. ∎

In addition, we also have the following structure theorem, for $\mathcal{D}'$, see, e.g., Hörmander [14, Theorem 4.4.7].

THEOREM 3.22. *Let $\Omega \subset \mathbb{R}^n$ and $f \in \mathcal{D}'(\Omega)$. Then there exists $g_\alpha \in C(\Omega)$, such that*

$$\langle f, \phi \rangle = \sum \int g_\alpha \partial^\alpha \phi dx, \forall \phi \in \mathcal{D}(\Omega) \ ,$$

and the sets supp g_α *are locally finite. If $f \in \mathcal{D}'^k(\Omega)$, the sum could be taken finite.*

3.6. Homeworks

Section 3.1

(1) Check that any polynomial $P(x) \in \mathcal{S}'(\mathbb{R}^n)$, however, $f(x) = e^{x^2} \notin \mathcal{S}'(\mathbb{R})$, $g(x) = e^x \notin \mathcal{S}'(\mathbb{R})$. Hint: you may want to use test functions like $e^{-\sqrt{1+x^2}}$.
(2) Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\langle g(y), \phi(x + y) \rangle \in \mathcal{S}(\mathbb{R}^n_x)$.
(3) Let $u \in \mathcal{S}'$, prove that $\mathcal{F}(u) \in \mathcal{S}'$.
(4) Let $u \in \mathcal{S}'$, calculate $\mathcal{F}(\partial_j u) \in \mathcal{S}'(\mathbb{R}^n)$ by definition.
(5) Let $a \in \mathbb{R}$, calculate $\mathcal{F}(\delta_a) \in \mathcal{S}'(\mathbb{R})$, by definition.
(6) Calculate $\mathcal{F}(1) \in \mathcal{S}'(\mathbb{R})$, by definition.
(7) Based on (2.25) with $\lambda = \epsilon - it$, $\epsilon > 0$, $t \in \mathbb{R}\backslash\{0\}$. By considering limit in $\mathcal{S}'(\mathbb{R})$ as $\epsilon \to 0+$, deduce that

$$\mathcal{F}_x e^{it|x|^2}(\xi) = \left(\frac{\pi}{|t|}\right)^{n/2} e^{i\frac{n\pi}{4}\mathrm{sgnt} - \frac{\mathrm{i}|\xi|^2}{4\mathrm{t}}} \ . \tag{3.11}$$

Section 3.2-3.3

(1) Prove Proposition 3.5, similar to Proposition 3.1.
(2) Prove Lemma 3.8.
(3) Let $f = \ln|x| \in \mathcal{D}'(\mathbb{R})$, check that $f' \in \mathcal{D}'(\mathbb{R})$ is given by (3.3). In addition, prove that it is not distribution of order 0.

CHAPTER 4

Fundamental solutions and local solvability

Last time: FT, $\mathcal{S}'$, $\mathcal{D}'$

4.1. Local solvability and fundamental solutions

Let $a_\alpha \in C^\infty(\Omega)$, consider Linear PDE operator with smooth coefficients :

$$L = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha \ . \tag{4.1}$$

Definition 4.1 (local solvability). *The linear PDE operator with smooth coefficients (4.1) is said to be locally solvable in Ω, if for any $x_0 \in \Omega$, there exists a neighborhood $\Omega_{x_0} \subset \Omega$ of x_0, so that for any $f \in \mathcal{D}(\Omega)$, there is $u \in \mathcal{D}'(\Omega)$ such that $Lu = f$ in Ω_{x_0}.*

Extreme case: support of f shrink to a single point, say, x_0? tend to δ_{x_0}. If, in addition, the solution map is continuous, then there exists $K_{x_0} \in \mathcal{D}'(\Omega)$ so that $LK_{x_0} = \delta_{x_0}$. Such $K_{x_0} \in \mathcal{D}'(\Omega)$ is called to be a fundamental solution of L at x_0.

If, for any $x_0 \in \Omega$, there exists $K_{x_0} \in \mathcal{D}'(\Omega)$ such that

$$LK_{x_0} = \delta_{x_0}(x) = \delta(x - x_0) \ ,$$

then for $f \in \mathcal{D}(\Omega)$, by setting $u(x) = \int K_{x_0}(x) f(x_0) dx_0$, we have (informally)

$$Lu(x) = \int LK_{x_0}(x) f(x_0) dx_0 = f(x) \ .$$

In particular, for linear PDE operator with constant coefficients:

$$L = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$$

we have translation invariance and can set $K_{x_0}(x) = K_0(x - x_0)$.

Definition 4.2 (fundamental solution). *For linear PDE operator with constant coefficients L, $K \in \mathcal{D}'(\mathbb{R}^n)$ is called to be a fundamental solution of L, if $LK = \delta$.*

As we have seen, with the help of fundamental solution, it is easy to find a solution for the equation $Lu = f \in \mathcal{D}(\mathbb{R}^n)$, that is,

$$u(x) = K * f(x) = K(f(x - \cdot)) \in C^\infty(\mathbb{R}^n) \ .$$

4.2. Some examples

4.2.1. d/dx**.** In view of Lemma 3.6, we see that $d/dxK = \delta$ if and only if $K = H(x) + C$.

4.2.2. Cauchy-Riemann operator. For the Cauchy-Riemann operator $L = \partial_x + i\partial_y$, to find a $K \in \mathcal{S}'$ with $LK = \delta$, we use FT on y, to obtain

$$(\partial_x - \eta)\hat{K}(x,\eta) = \delta(x) \ .$$

Multiplying the integrating factor $e^{-x\eta}$, we obtain

$$\partial_x(e^{-x\eta}\hat{K}(x,\eta)) = \delta(x)e^{-x\eta} = \delta(x) \ .$$

View as ODE in x, we get, for any fixed η,

$$e^{-x\eta}\hat{K}(x,\eta) = H(x) + C \ .$$

But it is PDE, we have in general

$$e^{-x\eta}\hat{K}(x,\eta) = H(x) + C(\eta) \ , \ \hat{K}(x,\eta) = (H(x) + C(\eta))e^{x\eta} \ .$$

As we have seen, it is not in $\mathcal{S}'$ in general, because of the factor $e^{x\eta}$ in the case of $x\eta > 0$.

To avoid the bad behavior, for $x > 0$, we want to set for $\eta > 0$, $1 + C(\eta) = 0$; while for $x < 0$, we want to set for $\eta < 0$, $0 + C(\eta) = 0$. In conclusion, we choose to set $C(\eta) = -H(\eta)$ for $\eta \neq 0$. In general, we could set $C(\eta) = -H(\eta) + c\delta$. Here for simplicity, let us set $C(\eta) = -H(\eta)$.

With this choice, we see that $\hat{K} \in L^1$ for fixed $x \neq 0$, and we can find $2\pi K$ by (classical) Fourier inversion:

$$\int (H(x)-H(\eta))e^{x\eta}e^{iy\eta}d\eta = \int_{-\infty}^{0} H(x)e^{x\eta}e^{iy\eta}d\eta + \int_0^\infty (H(x)-1)e^{x\eta}e^{iy\eta}d\eta = \frac{1}{x+iy} \ ,$$

$K = \mathcal{O}(1/r) \in L^1_{loc} \cap \mathcal{S}'$.

Let us check that $LK = \delta$. Actually, in polar coordinates $(x, y) = r(\cos\theta, \sin\theta)$, we have

$$r\partial_r = x\partial_x + y\partial_y, \partial_\theta = -y\partial_x + x\partial_y \ ,$$

$$\partial_x = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta, \partial_y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta \ .$$

$$\partial_x + i\partial_y = e^{i\theta}(\partial_r - \frac{i}{r}\partial_\theta) \ ,$$

$$\langle LK, \phi\rangle = -\langle K, L\phi\rangle = -\int_0^{2\pi}\int_0^\infty (2\pi re^{i\theta})^{-1}e^{i\theta}[(\partial_r - \frac{i}{r}\partial_\theta)\phi]rdrd\theta = \phi(0) \ ,$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$.

Notice that, however, in general K does not necessarily belong to $\mathcal{S}'$. For example, $K = e^{x+iy} + \frac{1}{2\pi(x+iy)}$.

4.2.3. Heat operator. We want to find $K \in \mathcal{D}'(\mathbb{R}^{1+n})$, such that

$$\partial_t K - \Delta K = \delta(t, x) = \delta(t)\delta(x) \tag{4.2}$$

Suppose there exists $K \in \mathcal{S}'$[1]. If we apply the Fourier transform with respect to the space variables, (4.2) becomes an ordinary differential equation:

$$\partial_t\hat{K} + |\xi|^2\hat{K} = \delta(t),$$

[1]We know that K does not necessarily belong to $\mathcal{S}'$. For example, if $K_1 \in \mathcal{S}'$, then $K = K_1 + e^{t+x_1} \notin \mathcal{S}'$.

$(e^{t|\xi|^2}\hat{K})_t = \delta(t)$, $e^{t|\xi|^2}\hat{K} = H(t) + C$[2]. It gives us

$$\hat{K} = (H(t) + C)e^{-t|\xi|^2} \in \mathcal{S}'$$

which forces us to set $C = 0$. Thus $\hat{K}(t,\xi) = H(t)e^{-t|\xi|^2}$, and so

$$K(t,x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} H(t) \in \ C^\infty(\mathbb{R}^{1+n}\backslash 0) \ . \tag{4.3}$$

Henceforth, we can find a solution of equation

$$\partial_t u - \Delta u = f(t,x),$$

which is given by $u = K *_{t,x} f$.

4.3. Laplace operator

In the ODE case $n = 1$, it is clear that $K = C_1 + C_2 x + xH(x)$. Next, we consider PDE case $n \geq 2$.

Let us begin with the important case $n = 3$. Actually, in this case, we have seen from subsection 1.1.6 that

$$K = -(4\pi|x|)^{-1}$$

is a fundamental solution.

4.3.1. Dimension $n = 3$: Fourier approach.

If there is $K \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\Delta K = \delta \ ,$$

then $K(x) = -\frac{1}{4\pi|x|}$ modulo harmonic functions (for example, any harmonic polynomials, e.g. $a + b \cdot x + c(x_1^2 - x_2^2)$).

Actually, by Fourier transform, $-|\xi|^2\hat{K} = 1$, and so for $\xi \neq 0$, $\hat{K} = -|\xi|^{-2} \in L^1_{loc} \cap \mathcal{S}'$. For our purpose, we need only to construct $K \in \mathcal{S}'$ such that $\hat{K} = -|\xi|^{-2}$.

To calculate K, we use L^1 functions (with $\delta > 0$) to approximate $-\hat{K}$,

$$\hat{K}_\delta = |\xi|^{-2} e^{-\delta|\xi|^2} \to |\xi|^{-2}, \text{ in } \mathcal{S}'(\mathbb{R}^3) \ ,$$

and then $K_\delta \in C_0(\mathbb{R}^3)$,

$$-K = \lim_{\delta\to 0+} K_\delta = \lim_{\delta\to 0+} \mathcal{F}^{-1}\hat{K}_\delta \ .$$

A direct calculation yields that

$$\mathcal{F}^{-1}\hat{K}_\delta = \frac{1}{8\pi^{3/2} r} \int_{|\lambda|\leq r/\sqrt{\delta}} e^{-\frac{\lambda^2}{4}} d\lambda \to \frac{(4\pi)^{1/2}}{8\pi^{3/2} r} = \frac{1}{4\pi r} \ . \tag{4.4}$$

[2]Strictly speaking, we missed some terms in the general solution. For example, $\delta(\xi)$ or $\delta'(\xi)$. However, the point here is just to find one fundamental solution.

4.3.2. General dimension ($n \geq 3$). If $K \in \mathcal{S}'(\mathbb{R}^n)$,

$$\Delta K = \delta$$

then we want to determine K, modulo harmonic functions (for example, any harmonic polynomials, e.g. $a + b \cdot x + c(x_1^2 - x_2^2)$).

Actually, by FT, $-|\xi|^2 \hat{K} = 1$, and so for $\xi \neq 0$, $\hat{K} = -|\xi|^{-2} \in L^1_{loc} \cap \mathcal{S}'$ (if $n \geq 3$). WLOG, we consider the case $\hat{K} = -|\xi|^{-2}$. Recall that for any γ with $\Re\gamma \in (0, n)$, in relation to the Riesz potential[3],

$$\mathcal{F}(|x|^{-\gamma})(\xi) = \frac{2^{n-\gamma}\pi^{n/2}\Gamma((n-\gamma)/2)}{\Gamma(\gamma/2)}|\xi|^{\gamma-n} \ . \tag{4.5}$$

Based on (4.5) with $\gamma = n - 2 > 0$, we conclude that

$$K = -\mathcal{F}^{-1}|\xi|^{-2} = -\frac{\Gamma(n/2-1)}{2^2\pi^{n/2}\Gamma(1)}|x|^{2-n} = -\frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}\Gamma(1)}|x|^{2-n} = -\frac{1}{(n-2)w_n}|x|^{2-n} \ .$$

4.3.3. remaining case $n = 2$. **Question**: For the remaining case $n = 2$, can we use FT to find K?

In this case, as $|\xi|^{-2}$ is not a well-defined distribution. You might think the situation is hopeless, but it isn't. We have already seen examples of regularization of non-L^1 functions. While $1/x$ is not in $L^1(\mathbb{R})$, we defined a distribution $PV(1/x)$ with $PV(1/x)(f) = \int x^{-1} f(x) dx$ if $f \in \mathcal{S}(\mathbb{R})$ with $f(0) = 0$.

By (4.5), with $\gamma = \epsilon \in (0, 2)$ and $n = 2$, we see that $|\xi|^{-(2-\epsilon)}$ has a Fourier inverse transform $g_\epsilon = d_\epsilon \Gamma(\epsilon/2)|x|^{-\epsilon}$, where $d_\epsilon = \frac{1}{2^{2-\epsilon}\pi\Gamma((2-\epsilon)/2)}$, $\lim_{\epsilon\to 0+} d_\epsilon = \frac{1}{4\pi}$. Since $x\Gamma(x) = \Gamma(x+1)$, we see that

$$\lim_{\epsilon\to 0+} \epsilon\Gamma(\epsilon/2) = 2 \lim_{\epsilon\to 0+} \epsilon/2\Gamma(\epsilon/2) = 2 \lim_{\epsilon\to 0+} \Gamma(1+\epsilon/2) = 2$$

This confirms that $g_\epsilon(x)$ diverges as $\epsilon \to 0+$ as $(2\pi\epsilon)^{-1} \sim d_\epsilon\Gamma(\epsilon/2) = c_\epsilon \epsilon^{-1}$ (since for $x \neq 0$, $|x|^{-\epsilon} \to 1$), with $c_\epsilon = d_\epsilon \Gamma(\epsilon/2)\epsilon \to 1/(2\pi)$.

It suggests we subtract the constant of order ϵ^{-1} and note that

$$(|x|^{-\epsilon} - 1)\epsilon^{-1} = [\exp(\epsilon \ln |x|^{-1}) - 1]\epsilon^{-1} \to \ln(|x|^{-1}) \ , \tag{4.6}$$

as $\epsilon \to 0+$, for any $x \neq 0$. We have thus shown formally that

$$h_\epsilon = g_\epsilon(x) - d_\epsilon\Gamma(\epsilon/2) = c_\epsilon(|x|^{-\epsilon} - 1)\epsilon^{-1} \to (2\pi)^{-1}\ln(|x|^{-1}) \ . \tag{4.7}$$

It is not hard to show that (4.7) holds in the sense of tempered distributions. Actually, as $\ln(|x|^{-1}) \in L^1_{loc}(\mathbb{R}^2) \subset \mathcal{S}'$, $c_\epsilon \to 1/(2\pi)$, it reduces to prove (4.6) in $\mathcal{S}'$, which we leave as an exercise.

Now, we see that

$$\mathcal{F}(\Delta h_\epsilon) = \mathcal{F}(\Delta g_\epsilon) = -|\xi|^2|\xi|^{-(2-\epsilon)} = -|\xi|^\epsilon \to -1, \text{in } \mathcal{S}'$$

which shows that $\Delta h_\epsilon \to -\delta$ and so

$$\Delta \frac{\ln(|x|)}{2\pi} = \delta, n = 2 \ . \tag{4.8}$$

[3]In mathematics, the Riesz potential is a potential named after its discoverer, the Hungarian mathematician Marcel Riesz. In a sense, the Riesz potential defines an inverse for a power of the Laplace operator on Euclidean space. They generalize to several variables the Riemann–Liouville integrals of one variable.

4.3.4. Proof of (4.5). Notice that

$$\Gamma(\gamma/2) = \int_0^\infty e^{-s} s^{\gamma/2-1} ds \ ,$$

for fixed $x \neq 0$, write $s = \lambda |x|^2$, we get

$$\Gamma(\gamma/2) = \int_0^\infty e^{-\lambda|x|^2} \lambda^{\gamma/2-1} d\lambda |x|^\gamma \ , \ |x|^{-\gamma} = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-\lambda|x|^2} \lambda^{\gamma/2-1} d\lambda \ . \tag{4.9}$$

Let $G_\lambda(x) = e^{-\lambda|x|^2}$, we know from (2.25) that

$$\mathcal{F} G_\lambda = (\frac{\pi}{\lambda})^{n/2} G_{1/(4\lambda)} \ .$$

Then we see that

$$\mathcal{F}|x|^{-\gamma} = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty \lambda^{\gamma/2-1} \mathcal{F} G_\lambda d\lambda = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty \lambda^{\gamma/2-1} (\frac{\pi}{\lambda})^{n/2} G_{1/(4\lambda)}(\xi) d\lambda \ .$$

Let $t = 1/(4\lambda)$, then

$$\mathcal{F}|x|^{-\gamma} = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty (4t)^{1-\gamma/2} (4\pi t)^{n/2} G_t(\xi) \frac{1}{4t^2} dt = \frac{2^{n-\gamma}\pi^{n/2}}{\Gamma(\gamma/2)} \int_0^\infty t^{(n-\gamma)/2-1} G_t(\xi) dt$$

which equals to $\frac{2^{n-\gamma}\pi^{n/2}\Gamma((n-\gamma)/2)}{\Gamma(\gamma/2)} |\xi|^{\gamma-n}$ by applying (4.9) again.

4.3.5. Alternative method (Spherical average). Observation: commute with rotation. If there is one fundamental solution E, then the spherical average of E is still a fundamental solution, but radial. Thus we could just try to find a radial fundamental solution.

What we want to solve is to find a radial solution to

$$\Delta K = \delta \ .$$

Notice that for radial functions K, we have

$$\nabla K = \frac{x}{r} \partial_r K, \Delta K = \nabla \cdot \frac{x}{r} \partial_r K = (\nabla \cdot \frac{x}{r}) \partial_r K + (\frac{x}{r} \cdot \nabla) \partial_r K = (\partial_r^2 + \frac{n-1}{r} \partial_r) K \ .$$

Then, away from 0, it is just an ODE of Euler type

$$(\partial_r^2 + \frac{n-1}{r} \partial_r) K = 0,$$

with general solutions

$$K = \begin{cases} C_1 r^{2-n} + C_2 & n \geq 3 \ , \\ C_1 \ln r + C_2 & n = 2 \ , \end{cases} \tag{4.10}$$

as we have the characteristic equation $\lambda(\lambda-1) + (n-1)\lambda = 0 \Rightarrow \lambda = 0, 2-n$.

Next, let us calculate the distributional ΔK (notice that $K, K' \in L^1_{loc}$): for any $\Phi \in \mathcal{D}$, we denote $\phi(r\omega) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \Phi(r\omega) d\omega$ be the spherical average of Φ, and get

$$\begin{aligned} \langle \Delta K, \Phi \rangle &= \langle \Delta K, \phi \rangle \\ &= \langle K, \Delta \phi \rangle \\ &= \langle K, (\partial_r^2 + \frac{n-1}{r} \partial_r) \phi \rangle \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty K \partial_r r^{n-1} \partial_r \phi dr d\omega = -\int_{\mathbb{S}^{n-1}} \int_0^\infty (\partial_r K) r^{n-1} \partial_r \phi dr d\omega \end{aligned}$$

which equals

$$\begin{cases} (n-2)C_1 \int_{\mathbb{S}^{n-1}} \int_0^\infty \partial_r \phi dr d\omega = (2-n)C_1 |\mathbb{S}^{n-1}| \phi(0) & n \geq 3 \ , \\ -C_1 \int_{\mathbb{S}^1} \int_0^\infty \partial_r \phi dr d\omega = 2\pi C_1 \phi(0) & n = 2 \ . \end{cases}$$

That is, for the purpose of $\Delta K = \delta$, we need only to choose (all possible radial fundamental solutions)

$$K = C + N(x) = C + \begin{cases} \frac{r^{2-n}}{(2-n)|\mathbb{S}^{n-1}|} & n \geq 3 \\ -\frac{1}{4\pi r} & n = 3 \\ \frac{\ln r}{2\pi} & n = 2 \end{cases} \tag{4.11}$$

where N is the Newtonian potential and

$$|\mathbb{S}^{n-1}| = w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \ . \tag{4.12}$$

4.4. Wave equation: fundamental solution

In this section, we investigate the fundamental solution of wave equation

$$\Box E = (\partial_t^2 - \Delta) E = \delta(t, x) \ .$$

FT wrt x,

$$\partial_t^2 \hat{E} + |\xi|^2 \hat{E} = \delta(t) \ .$$

Let $\partial_t \hat{E} + i|\xi|\hat{E} = F$, we have

$$\partial_t F - i|\xi| F = \delta(t)$$

$(e^{-it|\xi|}F)' = \delta(t)$, $e^{-it|\xi|}F = H(t) + C$,

$$\partial_t \hat{E} + i|\xi|\hat{E} = F = (H(t) + C)e^{it|\xi|}$$

$(e^{it|\xi|}\hat{E})' = (H(t) + C)e^{2it|\xi|}$,

$$\hat{E} = c_1 e^{it|\xi|} + c_2 e^{-it|\xi|} + e^{-it|\xi|} H(t) \int_0^t H(s) e^{2is|\xi|} ds = c_1 e^{it|\xi|} + c_2 e^{-it|\xi|} + H(t) \frac{\sin t|\xi|}{|\xi|}$$

WLOG, let's set $c_1 = c_2 = 0$, and try to calculate E, with $\hat{E} = \frac{e^{it|\xi|} - e^{-it|\xi|}}{2i|\xi|} H(t) = H(t) \frac{\sin t|\xi|}{|\xi|}$ for which we notice that $E(t) = 0$ for $t < 0$.

We claim that

$$\mathbf{E_n(t, x) = 2^{-1}} \pi^{\frac{1-n}{2}} \chi_+^{\frac{1-n}{2}} \mathbf{(t^2 - |x|^2)} H(t) \ , \tag{4.13}$$

which is the unique fundamental solution with support in the forward light cone $\{|x| \leq t\}$. When n is odd, the support is in the boundary of the cone if $n \neq 1$. See, e.g., [**14**, Theorem 6.2.3].

Here, for $\Re(\lambda) > -1$, we have

$$\chi_+^\lambda = \begin{cases} \frac{x^\lambda}{\Gamma(\lambda+1)} & x > 0 \\ 0 & x < 0 \ . \end{cases}$$

It is analytic in λ, which admits analytic extension for all $\lambda \in \mathbb{C}$ and

$$\frac{d}{dx} \chi_+^\lambda = \chi_+^{\lambda - 1} \ .$$

Noticing that $\chi_+^0 = H$, we have

$$\chi_+^{-k} = \delta^{(k-1)}, k \geq 1 \ .$$

4.4.1. Fundamental solution: 1D. For $n = 1$, we observe that it is even function, and so

$$\hat{E} = H(t)\frac{\sin t|\xi|}{|\xi|} = H(t)\frac{\sin t\xi}{\xi}$$

$$\frac{d}{dt}\hat{E} = H(t)\cos t\xi = \frac{e^{it\xi} + e^{-it\xi}}{2}H(t)$$

Recall that if we let $(\tau_t f)(x) = f(x-t)$, then $(\tau_t f)\check{}(\xi) = e^{-it\xi}\hat{f}(\xi)$. As $\hat{\delta} = 1$, we see that

$$\frac{d}{dt}E = \frac{\delta(x+t) + \delta(x-t)}{2}H(t)$$

Thus, for $t > 0$, we have

$$E(t) = \frac{1}{2}(H(t+x) - H(x-t)) = \frac{1}{2}H(t-|x|).$$

Notice that it is $L^1_{loc}(\mathbb{R}^{1+n})$, we obtain that

$$E(t,x) = \frac{1}{2}H(t)H(t^2 - |x|^2)$$

4.4.2. Fundamental solution: 3D. Note that

$$\hat{E}_3(t,\xi) = H(t)\frac{\sin t|\xi|}{|\xi|}$$

is radial (tempered) distribution. As h is radial, $\check{h}$ is also radial. WLOG, let $x = (0,0,r)$, then (heuristically) for $t > 0$,

$$\begin{aligned}
E_3(t,0,0,r) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ir\xi_3}\frac{\sin(t|\xi|)}{|\xi|}d\xi \\
= \frac{1}{(2\pi)^2}\int_0^\pi\int_0^\infty e^{ir\lambda\cos\theta}\sin(t\lambda)\lambda\sin\theta d\lambda d\theta &= \frac{1}{(2\pi)^2}\int_{-1}^1\int_0^\infty e^{ir\lambda k}\sin(t\lambda)\lambda d\lambda dk \\
= \frac{2}{(2\pi)^2}\int_0^1\int_0^\infty \cos(r\lambda k)\sin(t\lambda)\lambda d\lambda dk &= \frac{2}{(2\pi)^2 r}\int_0^\infty \sin(r\lambda)\sin(t\lambda)d\lambda \\
= \frac{1}{(2\pi)^2 r}\int_0^\infty \cos((t-r)\lambda) - \cos((r+t)\lambda)d\lambda &= \frac{1}{2(2\pi)^2 r}\int_{\mathbb{R}} e^{i(t-r)\lambda} - e^{i(r+t)\lambda}d\lambda \\
= \frac{1}{4\pi r}(\check{1}(t-r) - \check{1}(r+t)) &= \frac{1}{4\pi r}\delta(t-r) = \frac{1}{2\pi}\delta(t^2 - r^2)
\end{aligned}$$

Notice that, letting $E_+(t) = \frac{1}{2\pi}\delta(t^2 - r^2)$ for $t > 0$, we have

$$E_+ \in C(\mathbb{R}, \mathcal{E}'(\mathbb{R}^n)) \subset \mathcal{D}'(\mathbb{R}^{1+n}), E'_+(0+) = \lim_{t\to 0+}\frac{d}{dt}E_+(t) = \delta$$

Check*. For $n = 3$, recall that

$$\widehat{d\sigma}(\xi) = \mathfrak{F}(\delta(1-|x|))(\xi) = \int_{S^2} e^{-iw\cdot\xi}d\sigma(\omega) = \int_0^\pi\int_{-\pi}^\pi e^{-i|\xi|\cos\theta}\sin\theta d\phi d\theta$$

which equals to

$$2\pi\int_0^\pi e^{-i|\xi|\cos\theta}\sin\theta d\theta = 2\pi\int_{-1}^1 e^{-i|\xi|y}dy = 4\pi\frac{\sin|\xi|}{|\xi|}.$$

As in the case $n = 1$, we then need only to scale the equality with $t > 0$, then we get

$$\mathfrak{F}(\delta(t-|x|))(\xi) = \int_{S^2} e^{-itw\cdot\xi} t^2 d\sigma(\omega) = t^2 \widehat{d\sigma}(t\xi) = 4\pi t \frac{\sin t|\xi|}{|\xi|}$$

i.e.

$$E_3(t,x) = \frac{1}{4\pi t}\delta(t-|x|) = \frac{1}{2\pi}\delta(t^2 - |x|^2).$$

4.4.3. Dimension two: method of descent. When $n = 2$, for $x \in \mathbb{R}^2$, we could just view it as $n = 3$ by adding an artificial variable z, so that $X = (x, z) \in \mathbb{R}^3$, then we could construct

$$E_2(t,x) = \int E_3(t,x,z)dz \ .$$

4.4.4. Another physical method, inspired by Fourier method*. In the previous section, we see that

$$E_1 = \frac{H(t^2 - r^2)}{2}, E_3 = \frac{\delta(t^2 - r^2)}{2\pi}$$

which are connected with the derivative wrt r^2 or t^2. Inspired by this phenomenon, we may infer to get high dimensional fundamental solution by differentiate low dimensional one by r^2 or t^2.

4.4.4.1. *Low to high dimension.* Of course, $2\partial_{t^2} = \frac{1}{t}\partial_t$ will not yield high dimensional solution. Let us try $2\partial_{r^2} = \frac{1}{r}\partial_r$ instead.

PROPOSITION 4.1. *If u is radial solution to the wave equation with dimension n, then $r^{-1}\partial_r u = 2\partial_{r^2} u$ is a radial solution to the wave equation with dimension $n + 2$.*

Proof. Recall that we have

$$\Delta = \Delta_n = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_\omega = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\sum_{i<j}\Omega_{ij}^2 \ .$$

We need only to check that

$$(\partial_r^2 + \frac{n+1}{r}\partial_r)r^{-1}\partial_r = r^{-1}\partial_r(\partial_r^2 + \frac{n-1}{r}\partial_r) \ . \tag{4.14}$$

Direct calculation tells us that

$$LHS = r^{-1}\partial_r^3 + 2(-r^{-2})\partial_r^2 + 2r^{-3}\partial_r + \frac{n+1}{r}(r^{-1}\partial_r^2 - r^{-2}\partial_r) \ ,$$

$$LHS = r^{-1}\partial_r^3 + \frac{n-1}{r^2}\partial_r^2 - \frac{n-1}{r^3}\partial_r = r^{-1}\partial_r^3 + \frac{n-1}{r}\partial_r r^{-1}\partial_r = RHS \ ,$$

which completes the proof. ∎

As a byproduct, we see that

$$(\partial_r^2 + \frac{n}{r}\partial_r)(r^{-1}\partial_r)^k = r^{-1}\partial_r(\partial_r^2 + \frac{n-2}{r}\partial_r)(r^{-1}\partial_r)^{k-1} = (r^{-1}\partial_r)^k(\partial_r^2 + \frac{n-2k}{r}\partial_r) \ . \tag{4.15}$$

Then, with $L = r^{-1}\partial_r$,

$$\partial_r^2 L^k = L^k\partial_r^2 - 2kL^{k+1} \ . \tag{4.16}$$

4.4.4.2. *High dimensional fundamental solutions.* On the basis of this result, and

$$E_1(t,x) = \frac{1}{2}H(t-|x|) = \frac{1}{2}H(t^2-|x|^2),$$

we see that for $n = 2k+1$,

$$\Xi_+^k = \delta^{(k-1)}(t^2-|x|^2), t > 0$$

is a solution to n-D wave equation. We claim that modulo a constant multiplier, it is the desired fundamental solution.

4.5. Homeworks

Section 4.3

(1) Let $\Omega_{jk} = x^j\partial_k - x^k\partial_j$ be the generators of rotation, check $[\Delta, \Omega_{jk}] = 0$.
(2) Check (4.4).
(3) Prove that (4.6) holds in the sense of tempered distributions $\mathcal{S}'(\mathbb{R}^2)$.

$$(|x|^{-\epsilon}-1)\epsilon^{-1} \to \ln(|x|^{-1}) \ .$$

Section 4.4

(1) For 1D wave, we have

$$E(t,x) = \frac{1}{2}H(t)H(t^2-|x|^2) \ ,$$

such that $(\partial_t^2 - \partial_x^2)E = \delta$. Let $F \in \mathcal{D}(\mathbb{R}^{1+1})$. Use E to construct a solution to $\Box u = F$.
(2) For 3D wave, do a rigorous derivation of E_3 for $t > 0$. Hint: Similar to the proof for the Laplace operator, given $t > 0$, we have

$$\hat{K}_\delta = \hat{E}_3 e^{-\delta|\xi|^2} \to \hat{E}_3, \text{in } \mathcal{S}'(\mathbb{R}^3) \ ,$$

when $\delta > 0$ tends to zero.
(3) For 2D wave, calculate $E_2(t,x) = \int E_3(t,x,z)dz$ for $t > 0$, and compare it with (4.13). Are they the same fundamental solution?

CHAPTER 5

Implication of fundamental solutions

5.1. Malgrange-Ehrenpreis theorem

So far, for some explicit examples, we constructed fundamental solution and so is the local solvability. A natural question is if it apply for general operators.

Malgrange (1955-56) and Ehrenpreis (1954) independently proved that every linear differential operator with constant coefficients has a fundamental solution. An immediate corollary is that every constant-coefficient operator is locally solvable, and one can deduce regularity properties of the solutions by examination of the fundamental solution.

THEOREM 5.1 (Malgrange-Ehrenpreis theorem). *For $\forall c_\alpha \in \mathbb{C}$, the linear PDE operator with constant coefficients:*

$$L = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha$$

has a fundamental solution $K \in \mathcal{D}'(\mathbb{R}^n)$.

The result has many proofs, see e.g. Folland [**9**, chapter 1, F], Simon [**24**, Section 6.9] for two different proofs.

COROLLARY 5.2 (Local solvability). *The problem $Lu = f$ is locally solvable. More precisely, $\forall f \in C_c^\infty(\mathbb{R}^n), R > 0$, there exists $u \in C^\infty(B_R)$, such that*

$$Lu = f \in \mathcal{D}'(B_R)\ . \tag{5.1}$$

With the help of the fundamental solution K, we could just set $u = K * f \in C^\infty(\mathbb{R}^n)$.

Proof. We will present a proof, avoiding fundamental solution K, following Taylor [**28**, Chapter 3, Section 10].

Let $L(D) = \sum a_\alpha D^\alpha$ with $D = -i\nabla$. The starting point is the fundamental observation that

$$D(e^{ix\cdot\xi} = \xi e^{ix\cdot\xi},\ D(e^{ix\cdot\xi}w) = e^{ix\cdot\xi}(D+\xi)w,\ L(D)(e^{ix\cdot\xi}w) = e^{ix\cdot\xi}L(D+\xi)w\ .$$

With the help of this, we find that, for given $f \in \mathcal{D}(\mathbb{R}^n)$ and $R > 0$, solving (5.1), which is equivalent to

$$L(D+\xi)(e^{-ix\cdot\xi}u) = fe^{-ix\cdot\xi} := g \in \mathcal{D}'(B_R)\ ,$$

could be reduced to solving

$$L(D+\xi)w = \Phi g := h \tag{5.2}$$

in $\mathcal{D}'(B_{2R})$, where $\Phi \in \mathcal{D}(B_{2R})$ equals one in B_R. In this case, we could just take $u = e^{ix\cdot\xi}w \in \mathcal{D}'(B_{2R})$, and noticing that the parameter ξ could be arbitrary.

As $B_{2R} \subset [-2R, 2R]^n$, we could further reduce it to the solvability in $\mathcal{D}'(\mathbb{R}^n/(4R\mathbb{Z}^n))$. By scaling, we could assume without loss of generality that $R = \pi/2$ so that $\mathbb{R}^n/(4R\mathbb{Z}^n) = \mathbb{T}^n$. As $h \in \mathcal{D}(B_{2R})$, it follows directly from the following **claim**: for almost all $\xi \in A := [0,1)^n$, there exist $c = c(\xi), N = N(\xi)$ so that we have

$$|L(k+\xi)| \geq c\langle k\rangle^{-N}, \forall k \in \mathbb{Z}^n \ . \tag{5.3}$$

where $\langle k\rangle = \sqrt{1+|k|^2}$. Actually, for any such ξ, by Fourier series, we know that solving (5.2) is equivalent to solving

$$L(k+\xi)\hat{w}(k) = \hat{h}(k) \Leftrightarrow \hat{w}(k) = L(k+\xi)^{-1}\hat{h}(k)$$

As

$$h \in \mathcal{E}(\mathbb{T}^n) \Leftrightarrow \hat{h}(k) = \mathcal{O}(\langle k\rangle^{-M}), \forall M \ , \tag{5.4}$$

we see that (5.3) ensures that $w \in \mathcal{E}(\mathbb{T}^n)$ for any $h \in \mathcal{E}(\mathbb{T}^n)$.

Proof of (5.3): Step 1. **Claim 2**: for any nontrivial polynomial L, of degree m, and any $\delta \in (0, 1/m)$, we have

$$L^{-\delta}(\xi) \in L^1_{loc}(\mathbb{R}^n) \ . \tag{5.5}$$

Then, for any sufficiently small $\delta_0 > 0$ and large $M > 0$ ($\delta_0 < 1/(2m)$ and $M = 2n$ will be sufficient),

$$\int |L(\xi)|^{-\delta_0}\langle\xi\rangle^{-M} d\xi < \infty \ . \tag{5.6}$$

Actually, the local part is trivial in view of the claim (5.5). For the part $|\xi| > 1$, we set $\eta = \xi/|\xi|^2$ to see that

$$d\xi = r^{n-1}drd\omega = \lambda^{-1-n}d\lambda d\omega = \lambda^{-2n}\lambda^{n-1}d\lambda d\omega = |\eta|^{-2n}d\eta \ ,$$

$$\int_{|\xi|\geq 1} |L(\xi)|^{-\delta_0}|\xi|^{-M}d\xi = \int_{|\eta|\leq 1} |L(|\eta|^{-2}\eta)|^{-\delta_0}|\eta|^{M-2n}d\eta \ .$$

As $L(|\eta|^{-2}\eta) = \sum_{|\alpha|\leq m} a_\alpha |\eta|^{-2|\alpha|}\eta^\alpha = |\eta|^{-2m}Q_{2m}(\eta)$, we see that,

$$|L(|\eta|^{-2}\eta)|^{-\delta_0}|\eta|^{M-2n} \leq |\eta|^{2m\delta_0+M-2n}|Q_{2m}(\eta)|^{-\delta_0} \leq |Q_{2m}(\eta)|^{-\delta_0} \in L^1_{loc} \ ,$$

provided that $\delta_0 < 1/(2m)$ and $2m\delta_0 + M - 2n \geq 0$.

Step 2. By (5.6), we see that

$$\int_A \sum_k |L(k+\xi)|^{-\delta_0}\langle k+\xi\rangle^{-M}d\xi = \int_{\mathbb{R}^n} |L(\xi)|^{-\delta_0}\langle\xi\rangle^{-M}d\xi < \infty \ . \tag{5.7}$$

Then it is clear that, for almost all $\xi \in A$, we have

$$\sum_k |L(k+\xi)|^{-\delta_0}\langle k+\xi\rangle^{-M} = C(\xi) < \infty \ ,$$

which gives us

$$|L(k+\xi)|^{-\delta_0} \leq C(\xi)\langle k+\xi\rangle^M \lesssim_\xi \langle k\rangle^M$$

and so is (5.3).

Step 3. It remains proving (5.5). Without loss of generality, we assume

$$L(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha = \Pi_{j=1}^m(\xi_1 - \lambda_j(\xi')) \ ,$$

and so for any fixed $\xi' \in B_R$, we have some $\eta_j(\xi') \in \mathbb{R}$ (which are uniformly bounded) and $k = k(\xi') \le m$ so that

$$|L(\xi_1, \xi')| \ge \Pi_{j=1}^k |\xi_1 - \eta_j(\xi')|^{m_j} \ ,$$

with $\sum m_j = m$. Then it is clear that, provided that $\delta_0 \in (0, 1/m)$, we have $|L(\xi_1, \xi')|^{-\delta_0} \in L^1(-R, R)$ uniformly.

■

5.2. Lewy example

We know that every constant coefficient PDE has solutions. It is natural to ask if it is the case for nonconstant coefficients cases.

It was long believed that any "reasonable" partial differential equation (with no boundary conditions imposed) should have many solutions. In particular, consider the linear PDE operator with smooth coefficients :

$$Lu = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha u = f$$

with $f, a_\alpha \in C^\infty(\Omega)$. Given $x_0 \in \mathbb{R}^n$, can we find a solution u of this equation in some neighborhood of x_0?

Notice that, thanks to the Cauchy-Kowalevskaya theorem Theorem 1.8, if f and a_α are real analytic, the answer is YES.

One might well expect that the assumption of analyticity can be omitted. But, surprising enough, in 1957, Hans Lewy destroyed all hopes for such a theorem with the following simple counterexample.

The example is, in spirit, similar to the Cauchy-Riemann operator for IVP, where local solution is possible if and only if the data is real analytic.

Consider the differential operator L defined on $\mathbb{R}^3$ with coordinates (x, y, t) by

$$L = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} - 2i(x + iy) \frac{\partial}{\partial t}.$$

THEOREM 5.3 (Lewy example). *Let f be a continuous real-valued function depending only on t. If there is a C^1 function u of (x, y, t) satisfying $Lu = f$ in some neighborhood of the origin, then f is analytic at $t = 0$.*

Proof. See e.g. [**9**, §1E].

Actually, let $z = x + iy$, then $2\partial_{\bar{z}} = \partial_x + i\partial_y$ and

$$\tilde{L} = L/2 = \partial_{\bar{z}} - iz\partial_t \ .$$

Suppose $u \in C^1$ near the origin ($|z| \le R$, $|t| \le R$) is a solution to $\tilde{L}u = f(t) \in C$, we introduce

$$V(r, t) = \int_{|z|=r} u dz = \int_0^{2\pi} ure^{i\theta} i d\theta = i \int_0^{2\pi} uz d\theta = i \int_0^{2\pi} (\cos\theta, \sin\theta) \cdot (u, iu) r d\theta = i \int_{|z|<r} (\partial_x, \partial_y) \cdot (u, iu) dx dy$$

where in the last inequality, we have used the divergence theorem (Green's theorem). Then, with $ds = r d\theta$,

$$\partial_r V(r, t) = i \int_{|z|=r} (\partial_x, \partial_y) \cdot (u, iu) r ds = 2i \int_0^{2\pi} \partial_{\bar{z}} u r d\theta = 2i \int_0^{2\pi} (iz\partial_t u + f(t)) r d\theta = 2ir\partial_t V + 4\pi i r f(t) \ .$$

That is, $\partial_r V(r, t) = 2ir\partial_t V + 4\pi i r f(t)$, $\frac{1}{2r}\partial_r V(r, t) = i\partial_t V + 2\pi i f(t) = i\partial_t(V + 2\pi F)$, with $F' = f$.

Making change of variable: $s = r^2$, $W(t, s) = V(t, r^2) + 2\pi F(t)$, we see that

$$\partial_s W = \partial_s V(s, t) = i\partial_t V + 2\pi i f(t) = i\partial_t W, \ (\partial_t + i\partial_s)W = 0 \ ,$$

which is Cauchy-Riemann equation for $t+is$, with $0 < s < R^2$, $|t| < R$. In addition, as u is continuous, W is continuous up to the line $s = 0$.

By assumption, $u \in C^1$ and so $|u| \leq M$, $|V(r,t)| \leq 2\pi M r$, $V(0,t) = 0$. By setting $F(0) = 0$, $F(t)$ is real valued and

$$W(0, t) = 2\pi F(t)$$

Set $W(s, t) = \bar{W}(-s, t)$ for $s < 0$, we get that, by the Schwarz reflection principle, W is a holomorphic continuation of W to a full neighborhood of the origin, $|s| < R^2$, $|t| < R$. In particular, $F(t) \in C^\omega((-R, R))$ and $f = F'(t) \in C^\omega((-R, R))$. ■

Mizohata[1] proved the same for the even simpler equation on $\mathbb{R}^2$,

$$\partial_x u + ix\partial_y u = F(x, y) \ .$$

A couple of years after Lewy proved Theorem 5.3, Hörmander embedded it into a more general result that initiated the theory of local solvability of differential operators. We make a formal definition: a linear differential operator L with C^∞ coefficients is said to be locally solvable at x_0 if there is a neighborhood Ω of x_0 such that for every $f \in \mathcal{D}(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ with $Lu = f$. Hörmander's theorem is then as follows[2].

THEOREM 5.4. *Let L be a linear differential operator with C^∞ coefficients on Ω, let $P(x, \xi) := \chi_L(x, \xi)$ be the characteristic form of L, and let*

$$Q(x, \xi) := \frac{\partial P}{\partial \xi} \cdot \frac{\partial \bar{P}}{\partial x} - \frac{\partial \bar{P}}{\partial \xi} \cdot \frac{\partial P}{\partial x}$$

(1) *If $x_0 \in \Omega$ and there is a $\xi \in \mathbb{R}^n$ such that $P(x_0, \xi) = 0$ but $Q(x_0, \xi) \neq 0$, then L is not locally solvable at x_0.*
(2) *If for each $x \in \Omega$ there is a $\xi \in \mathbb{R}^n$ such that $P(x, \xi) = 0 \neq Q(x, \xi)$, then there is an $f \in C^\infty(\Omega)$ such that the equation $Lu = f$ has no distribution solution on any open subset of Ω.*

See Folland [**9**] 1E for more discussion.

5.3. Elliptic operator with constant coefficients, parametrix

In this section, for elliptic operators with constant coefficients, we will construct a **parametrix** (approximation to the fundamental solution) to prove a property enjoyed by the elliptic operators. See, e.g., Taylor [**28**, Chapter 3, Section 9]; 齐民友等[**36**, Chapter 6, Section 3].

For given elliptic operators (with constant coefficients) $P(\partial) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$, with symbol $P(i\xi) = \sum_{|\alpha| \leq m} a_\alpha (i\xi)^\alpha$ and principle symbol $P_m(i\xi) = \sum_{|\alpha| = m} a_\alpha (i\xi)^\alpha$.

Heuristically, we want to use FT to find the fundamental solution

$$\mathcal{F}(PK) = P(i\xi)\hat{K} = \hat{\delta} = 1$$

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[1] S. Mizohata, Solutions nulles et solutions non analytiques, J. Math. Kyoto Univ. 2 (1962) 271–302.

[2] For the proof, see L. Hormander, Linear Partial Differential Operators, Springer-Verlag, Berlin, 1963.

which gives us $\hat{K} = 1/P(i\xi)$ provided that there no zeros for $P(i\xi)$ in $\xi \in \mathbb{R}^n$. However, in general, there may have zeros for some $\xi \in \mathbb{R}^n$.

By definition, $P_m(i\xi) \neq 0$ for any $\xi \neq 0$. Thus we see that

$$\inf_{|\xi|=1} |P_m| = c > 0 \ ,$$

and so

$$|P| \geq |P_m| - |P - P_m| \geq c|\xi|^m - C(1+|\xi|)^{m-1} \geq c|\xi|^m$$

provided that $|\xi| \geq R \gg 1$.

With the help of this observation, we could invert P at least in the region $|\xi| \geq R$. To achieve this, we simply introduce a cutoff function $\chi \in C_0^\infty$, which equals one on B_R, and define $K_1 \in \mathcal{S}'$ by

$$\hat{K}_1 = \frac{1-\chi(\xi)}{P(i\xi)} \in L^\infty \cap C^\infty \subset \mathcal{S}' \ .$$

Then we see that

$$\mathcal{F}PK_1 = P(i\xi)\frac{1-\chi(\xi)}{P(i\xi)} = 1-\chi(\xi), PK_1 = \delta - \mathcal{F}^{-1}(\chi)$$

where $\mathcal{F}^{-1}(\chi) \in \mathcal{S} \cap C^\omega$. Thus, up to a nice factor, we find the fundamental solution. In this sense, we call K_1 be a parametrix.

LEMMA 5.5. *The parametrix* $K_1 \in \mathcal{S}'$ *has the property* $\mathrm{singsupp} K = \{0\}$.

Proof. Noticing that $\hat{K}_1 \in C^\infty$ and

$$\hat{K}_1 = \mathcal{O}(\langle\xi\rangle^{-m})$$

By induction, it is easy to conclude that (exercise)

$$\partial_\xi^\alpha \hat{K}_1 = \mathcal{O}(\langle\xi\rangle^{-m-|\alpha|}), \forall \alpha \in \mathbb{N}^n \ , \tag{5.8}$$

which gives us that

$$\xi^\beta \partial_\xi^\alpha \hat{K}_1 = \mathcal{O}(\langle\xi\rangle^{-m-|\alpha|+|\beta|}), \partial_\xi^\alpha \xi^\beta \hat{K}_1 = \mathcal{O}(\langle\xi\rangle^{-m-|\alpha|+|\beta|}) \ .$$

If $-m-|\alpha|+|\beta| < -n$, then $\partial_\xi^\alpha \xi^\beta \hat{K}_1 \in L^1$ and so $x^\alpha \partial_x^\beta K_1 \in C_0$. Thus, for any fixed j, we see that for any $|\beta| = j$ and $|\alpha| > n+j-m$ we have

$$x^\alpha \partial_x^\beta K_1 \in C_0, |x|^{2+2|n+j-m|} \partial_x^\beta K_1 \in C_0, \partial_x^\beta K_1 \in C_0(\mathbb{R}^n \backslash \{0\}), K_1 \in C^j(\mathbb{R}^n \backslash \{0\}) \ .$$

Then $K_1 \in C^\infty(\mathbb{R}^n \backslash \{0\})$. ∎

THEOREM 5.6. *Any elliptic operators P (with constant coefficients) are hypoelliptic, that is, for any $u \in \mathcal{D}'(\mathbb{R}^n)$ with $Pu \in \mathcal{E}(\Omega)$, then $u \in \mathcal{E}(\Omega)$.*

Suppose $Pu = f \in C^\infty(\Omega')$, we need to show $u \in C^\infty(\Omega')$. So it is sufficient to show $\forall x_0 \in \Omega'$, and $\delta_0 > 0$, with $B_{2\delta_0}(x_0) \subset \Omega'$, we have $u \in C^\infty(B_{\delta_0}(x_0))$.

Let $\phi \in C_c^\infty(B_{2\delta_0}(x_0))$, and $\phi = 1$ on $B_{\delta_0}(x_0)$. Then $P[\phi u] = \phi Pu + v = \phi f + v$, where $v = 0$ in $B_{\delta_0}(x_0)$ and outside $B_{2\delta_0}(x_0)$, i.e.,

$$\mathrm{supp}\ v \subset B_{2\delta_0}(x_0) \backslash B_{\delta_0}(x_0) \ .$$

$$K_1 * P[\phi u] = PK_1 * (\phi u) = (\delta - \mathcal{F}^{-1}(\chi)) * (\phi u) = \phi u - \mathcal{F}^{-1}(\chi) * (\phi u),$$

As $\mathcal{F}^{-1}(\chi) \in \mathcal{S} \subset \mathcal{E} = C^\infty$ and $\phi u \in \mathcal{E}'$, we have $\mathcal{F}^{-1}(\chi) * (\phi u) \in C^\infty$. Similarly, as $K_1 \in \mathcal{S}' \subset \mathcal{D}'$, $\phi f \in C_0^\infty = \mathcal{D}$, we get $K_1 * (\phi f) \in C^\infty$. Thus, as $K_1 * P[\phi u] = K_1 * (\phi f) + K_1 * v$, we have

$$\phi u = K_1 * (\phi f) + K_1 * v + \mathcal{F}^{-1}(\chi) * (\phi u) = K_1 * v + C^\infty$$

By Lemma 5.5 and 5.7[3], we know that $K_1 * v \in C^\infty(B_{\delta_0}(x_0))$. Thus on $B_{\delta_0}(x_0))$, we have $u = \phi u \in C^\infty(B_{\delta_0}(x_0))$ and this completes the proof.

LEMMA 5.7. *Suppose $f \in C^\infty(\mathbb{R}^n \backslash 0) \cap \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{E}'$, then $f * g$ is C^∞ on $\mathbb{R}^n \backslash (\text{supp } g)$.*

Proof. $\forall x \notin \text{supp } g$, $\exists \delta > 0$, $B_{4\delta}(x) \cap \text{supp } g = \emptyset$. Let $\phi \in C_0^\infty(B_{2\delta})$ with $\phi(x) = 1$ on B_δ, then

$$f * g = (\phi f) * g + ((1 - \phi) f) * g$$

As $(1 - \phi) f \in C^\infty$, we have $((1 - \phi) f) * g \in C^\infty$. For ϕf, it has support in $B_{2\delta}$ and so

$$\text{supp } (\phi f) * g \subset \text{supp } \phi f + \text{supp } g \subset B_{2\delta}(\text{supp } g)$$

which means that $(\phi f) * g = 0$ in $B_{2\delta}(x)$ and so $f * g \in C^\infty(B_{2\delta}(x))$. ∎

5.4. Hypoelliptic operators

Consider general partial differential operator

$$L = \sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha.$$

DEFINITION 5.1. *Let $a_\alpha \in C^\infty(\Omega)$. Then the operator L is said to be hypoelliptic if any distribution u such that Lu is $C^\infty(\Omega')$ on $\Omega' \subset \Omega$ must itself be $C^\infty(\Omega')$.*

THEOREM 5.8. *If $a_\alpha \in \mathbb{C}$, then the following are equivalent:*
(1) L is hypoelliptic.
(2) $\forall K \in \mathcal{D}'$, $LK = \delta \Rightarrow K \in C^\infty(\mathbb{R}^n \backslash 0)$.
(3) $\exists K \in \mathcal{D}'$, $LK = \delta \Rightarrow K \in C^\infty(\mathbb{R}^n \backslash 0)$.

Proof. (1) $\Rightarrow$ (2): Since L is hypoelliptic, and $LK = \delta \in C^\infty(\mathbb{R}^n \backslash 0)$, hence $K \in C^\infty(\mathbb{R}^n \backslash 0)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Suppose $Lu = f \in C^\infty(\Omega')$, we need to show $u \in C^\infty(\Omega')$. So it is sufficient to show that for any $x_0 \in \Omega'$, there exists $\delta_0 > 0$, such that $B_{\delta_0}(x_0) \subset \Omega'$, and $u \in C^\infty(B_{\delta_0}(x_0))$.

It turns out that we can just choose $\delta_0 > 0$ such that $B_{2\delta_0}(x_0) \subset \Omega'$. Pick $\phi \in C_c^\infty(B_{2\delta_0}(x_0))$, and $\phi = 1$ on $B_{\delta_0}(x_0)$. Then $L[\phi u] = \phi L u + v = \phi f + v$. Notice that where $v = 0$ in $B_{\delta_0}(x_0)$ and outside $B_{2\delta_0}(x_0)$, i.e.,

$$\text{supp } v \subset B_{2\delta_0}(x_0) \backslash B_{\delta_0}(x_0) \ ,$$

we have $v \in \mathcal{E}'$, and we could define $K * v$. By Lemma 5.7, which is a special case of Lemma 3.17, and recall that

$$\text{singsupp} K \subset \{0\} \ ,$$

we know that

$$\text{singsupp} K * v \subset \text{singsupp} K + \text{singsupp} v \subset B_{2\delta_0}(x_0) \backslash B_{\delta_0}(x_0) \ ,$$

[3] It is a special case of Lemma 3.17.

and thus $K * v \in C^\infty(B_{\delta_0}(x_0))$.

By assumption, $\phi f \in \mathcal{D}$, we have $K * (\phi f) \in C^\infty$, and in view of (3.10), we get

$$\phi u = \delta * (\phi u) = LK * (\phi u) = K * L[\phi u] = K * (\phi f) + K * v \in C^\infty(B_{\delta_0}(x_0)) \ .$$

This tells us that, on $B_{\delta_0}(x_0)$, $u = \phi u$ is C^∞ and this completes the proof. ■

Here we remark that singsupp$K = \{0\}$ for any K in Theorem 5.8 (2) [4].

5.5. Laplace operator: regularity properties

5.5.1. Hypoelliptic property. $C^\infty(\Omega)$,
$\Delta u = f \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega)$
Question: $f \in C^k \Rightarrow u \in C^{k+2}$?

5.5.2. $C^0(\Omega)$: counterexample. See, e.g., [**10**, Problem 4.9]. See also Subsection 8.6.3 for another example.

If $n = 1$, $u'' = f \in C$, then $u \in C^2$ for sure.

On the other hand, if $n \geq 2$, it is not true in general. For example, with $X = (x, y, \cdots)$ and $p = xy$, we have

$$\Delta p = 0, \partial_x \partial_y p = 1 \ .$$

Let $\eta \in C_0^\infty(B_2)$ which equals 1 on B_1, $t_k = 2^k$, $c_k = 1/k$, we define

$$f(X) = \sum_{k \geq 1} c_k [\Delta(\eta p)](2^k X)$$

and claim that $f \in C(\mathbb{R}^n)$, and there are no Ω including 0, such that there exists $u \in C^2(\Omega)$ such that $\Delta u = f$.

At first, notice that $\Delta(\eta p) \in C_0^\infty(B_2)$ vanishes also on B_1, then $[\Delta(\eta p)](2^k X) \neq 0$ only if $2^k |X| \in (1, 2)$. Based on this fact, we have $f(0) = 0$, and for any $X \neq 0$, there exist $\delta > 0$ and at most two successive k_0 such that $[\Delta(\eta p)](2^{k_0} Y) \neq 0$ for any $Y \in B_\delta(X)$, and thus $f \in C(B_\delta(X))$, which confirms that $f \in C(\mathbb{R}^n)$.

Then let $w(X) = \sum_{k \geq 1} 2^{-2k} c_k (\eta p)(2^k X)$ be the natural candidate of solution, which can be easily checked to be in $C^1 \backslash C^2$. Actually, for any open set $0 \in U$, we have $w \notin C^2(U)$. But still $\partial_j^2 w \in C(\mathbb{R}^n)$ and $\Delta w = f$, for all $X \in \mathbb{R}^n$.

Now if there exists a $C^2(\Omega)$ solution, u, with $0 \in \Omega$, then $\Delta u = f$ in Ω, and so $\Delta(u - w) = 0$ in Ω. But as Δ is hypoelliptic, we obtain $u - w \in C^\infty(\Omega)$, and so $w = u - (u - w) \in C^2(\Omega)$, which is a contradiction.

5.5.3. Hölder space $C^\alpha(\Omega)$ and Hölder theorem.

DEFINITION 5.2. *$\Omega \subset \mathbb{R}^n$ is open and $0 < \alpha \leq 1$. If u satisfying*

$$|u(x) - u(y)| \leq C|x - y|^\alpha, \qquad (x, y \in \Omega),$$

for some constant C. Such a function is said to be Hölder continuous with exponent α.

If k is a positive integer, $C^{k+\alpha}$ denote the set of all $u \in C^k(\Omega)$ such that $\partial^\beta u \in C^\alpha(\Omega)$ for all muli-indices β with $|\beta| \leq k$.

THEOREM 5.9 (Hölder's theorem). *Suppose $k \geq 0$, $0 < \alpha < 1$. $\Omega \subset \mathbb{R}^n$ is open. If $f \in C^{k+\alpha}(\Omega)$, u is a distribution solution of $\Delta u = f$ on Ω, then $u \in C^{k+2+\alpha}(\Omega)$.*

[4]Notice that K can not be smooth near 0, if so, LK is smooth near 0, which is a contradiction to the fact that $LK = \delta$. Thus singsupp$K \neq \emptyset$. But, Theorem 5.8 (2) tells us that singsupp$K \subseteq \{0\}$.

See Folland [9, 2C, Theorem 2.28, page 78-82] for a proof.

Here, we present a glimpse of the proof for the simpler case $k = 0$ and $u \in C^2$. At first, a cutoff argument tells us that, to show $u \in C^2(B_\epsilon(x_0))$, it suffices to prove for the case $f \in C_c^\alpha(B_{2\epsilon}(x_0))$.

Due to the fact that Δ is elliptic (and so hypoelliptic), we need only to prove $K * f \in C^2$. As $K, \nabla K \in C^\infty(\mathbb{R}^n \backslash \{0\}) \cap L^1_{loc}$, it is clear that $K * f \in C^1$ and what remains is to prove $\nabla^2(K * f) \in C$, here ∇^2 is the derivative in the sense of distribution.

To calculate $\nabla^2(K * f)$, in the sense of distribution, we use approximation (for $n \geq 3$, and the simple adaption for $n = 2$ is trivial)

$$K_\epsilon = c_n(\epsilon^2 + |x|^2)^{(2-n)/2} \to K, \text{ in } \mathcal{D}' \ .$$

As $K_\epsilon \in C^\infty$, we have $K_\epsilon * f \in C^\infty$ and

$$\nabla K_\epsilon = \tilde{c}_n(\epsilon^2 + |x|^2)^{-n/2} x,$$

$$\nabla^2 K_\epsilon = \bar{c}_n(\epsilon^2 + |x|^2)^{-n/2}(\delta_{jk} - n(\epsilon^2 + |x|^2)^{-1} x_j x_k) = (K^1_\epsilon + K^2_\epsilon)_{jk},$$

where

$$(K^1_\epsilon)_{jk} = \bar{c}_n(\epsilon^2 + |x|^2)^{-n/2}\epsilon^2 \delta_{jk}, \ (K^2_\epsilon)_{jk} = \bar{c}_n(\epsilon^2 + |x|^2)^{-1-n/2}(\delta_{jk}|x|^2 - n x_j x_k) \ .$$

Concerning K^1_ϵ, we observe that

$$\sum_j (K^1_\epsilon)_{jj} = \Delta K_\epsilon - \sum_j (K^2_\epsilon)_{jj} = \Delta K_\epsilon \to \delta \ ,$$

and so, in view of symmetry,

$$\lim_{\epsilon \to 0+} (K^1_\epsilon)_{jj} * f = \frac{1}{n} f \ . \tag{5.9}$$

Turning to K^2_ϵ, which is of order $|x|^{-n} \notin L^1_{loc}$, we need to exploit the fact that $f \in C_c^\alpha$. For this purpose, we observe that

$$\int_{a \leq r \leq b} (K^2_\epsilon)_{jk} dx = 0, \ \forall 0 \leq a < b < \infty \ , \tag{5.10}$$

and thus, for any fixed $R > 0$, for any $x \in B_R$, we have

$$(K^2_\epsilon)_{jk} * f(x) = \int_{B_b} (K^2_\epsilon)_{jk}(y) f(x-y) dy = \int_{B_b} (K^2_\epsilon)_{jk}(y)[f(x-y) - f(x)] dy \tag{5.11}$$

for some b ($b = 2R$ will be sufficient if supp $f \subset B_R$).

Using the assumption that $f \in C_c^\alpha$, there exists a uniform constant C such that we have

$$|(K^2_\epsilon)_{jk}(y)[f(x-y) - f(x)]| \leq C|y|^{\alpha - n} \in L^1(B_b) \ ,$$

and thus an application of the dominated convergence theorem gives us that the limit of $(K^2_\epsilon)_{jk} * f(x)$ is uniform in x, which ensures that $\lim (K^2_\epsilon)_{jk} * f(x) \in C(B_R)$. As R is arbitrary, $\lim (K^2_\epsilon)_{jk} * f(x) \in C(\mathbb{R}^n)$.

Recalling (5.9), this completes the proof of $K * f \in C^2$.

Remark 5.10. *It is in relation with the Calderon-Zygmund singular integral operators*

$$T : C^\alpha \to C^\alpha, L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), 1 < p < \infty$$

but not bounded operator $L^\infty \to L^\infty$ *or* $C \to C$ *in general. For example* $T = (-\Delta)^{-1}\partial_j \partial_k$.

5.6. Fundamental solutions for the Cauchy problem: heat equation

Considering $Lu = 0$ with certain initial data, we also have the notion of fundamental solutions.

In this section, let us consider the Cauchy problem for the heat equation,

$$\partial_t u - \Delta u = 0, u(0,x) = f(x) \in L^1(\mathbb{R}^n) \ . \tag{5.12}$$

It could be solved by using FT: for $t > 0$,

$$u(t,x) = K_H(t) * f = (4\pi t)^{-n/2} \int e^{-\frac{|x-y|^2}{4t}} f(y) dy \tag{5.13}$$

where $K_H(t,x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ is known as the heat kernel.

Equivalently, we see that

$$\partial_t K_t - \Delta K_t = 0, t > 0, \lim_{t \to 0+} K_t(x) = \delta(x)$$

which can be viewed as the (fundamental) solution with data δ.

With the help of this fundamental solution, we could write down a solution for

$$\partial_t u - \Delta u = F, u(0,x) = f(x) \ ,$$

that is,

$$u = K(t, \cdot) *_x f + \int_0^t K(t-\tau, \cdot) *_x F(\tau, \cdot) d\tau \ ,$$

whenever the convolution makes sense. Here, we see that a solution formula for the homogeneous problem yields a formula for the inhomogeneous problem, which is a general phenomena, known as the Duhamel's principle.

5.6.1. Existence. Our first applications rest on the fact that for $t > 0$,

$$K_H(t) \in \mathcal{S}(\mathbb{R}^n), K_H(t) > 0, \forall x \in \mathbb{R}^n$$

$$\int K_H(t) dx = \mathcal{F}(K_H(t))(0) = 1 \ .$$

Young's inequality for convolutions implies that for any $p \in [1, \infty]$, we have

$$\|u(t)\|_{L^p} = \|K_H(t) * f\|_{L^p} \le \|K_H(t)\|_{L^1} \|f\|_{L^p} = \|f\|_{L^p} \tag{5.14}$$

Let $u(t) = S_H(t) f := K_H(t) * f$, we see that it solve the IVP for $t > 0$.

Actually, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, as $K_H(t) \in C^\infty((0,\infty), \mathcal{S}(\mathbb{R}^n))$, we have

$$u(t,x) = K_H(t) * f \in C^\infty((0,\infty) \times \mathbb{R}^n)$$

and $(\partial_t - \Delta) u = 0$ for $t > 0$.

As $K_H(t) \to \delta$ in $\mathcal{S}'$, we have $S_H(t) f \to f$ uniformly, if $f \in \mathcal{S}$. Moreover, approximation of identity results give us that, $S_H(t) f \to f$ in $L^p(\mathbb{R}^n)$, provided that $f \in L^p$ with $p \in [1, \infty)$.

In view of Plancherel's theorem and DCT, we also have

$$\|S_H(t) f - f\|_{H^s} \le c \|(e^{-t|\xi|^2} - 1)(1 + |\xi|^2)^{s/2} \hat{f}\|_{L^2} \to 0 \ ,$$

if we have $f \in H^s = (1 - \Delta)^{-s/2} L^2$ for some $s \in \mathbb{R}$.

Anyway, we obtained

THEOREM 5.11. *If $f \in H^s$ for some $s \in \mathbb{R}$, or $f \in L^p$ for some $p \in [1, \infty)$, we have a solution to IVP (5.12), given by $u = S_H(t) f$.*

PROPOSITION 5.12. *Suppose $f \in H^s$ for some s or $f \in L^p$ with $p \in [1, \infty]$. If $f \geq 0$, then for all $t \geq 0$, we have $S_H(t)f \geq 0$*

COROLLARY 5.13 (Comparison theorem). *If $X = H^s$ for some s or $X = L^p$ with $p \in [1, \infty]$. If $f, g \in X$ with $f \geq g$, then for all $t \geq 0$, we have $S_H(t)f \geq S_H(t)g$.*

For $f \in L^\infty$ we may take g to be the constant function $\operatorname{essinf}(f)$ for which $S_H(t)g$ is independent of t, x. Similarly, $\operatorname{esssup}(f)$ is an upper bound for u.

COROLLARY 5.14. *If $f \in L^\infty$, then for all $t > 0$, we have* $\operatorname{essinf}(f) \leq S_H(t)f \leq \operatorname{esssup}(f)$.

Thus the temperature is always between its initial extremes, consistent with the intuition that heat flows from hot to cold.

5.6.2. large time and asymptotical behavior. Next we examine the large time behavior of $u = K_H(t) * f$ when $f \in L^1$. Recall that the L^1 norm of nonnegative solutions is the physical energy.

$$\|u(t)\|_{L^\infty} \leq \|K_H(t)\|_{L^\infty} \|f\|_{L^1} \leq (4\pi t)^{-n/2} \|f\|_{L^1}$$

More generally, we have a sharp rate of decay for the L^p norm for any $1 \leq p \leq \infty$, by the Riesz-Thorin Theorem.

PROPOSITION 5.15 ($L^1 - L^p$ decay estimates). *Suppose $f \in L^1$, $p \in [1, \infty]$. Then for all $t > 0$, we have*

$$\|u(t)\|_{L^p} \leq (4\pi t)^{-\frac{n}{2}(1-1/p)} \|f\|_{L^1} \tag{5.15}$$

The behavior as $t \to \infty$ can be described even more precisely if f decays sufficiently rapidly as $|x| \to \infty$. As we see from the solution on the Fourier side, we have

$$\hat{u}(t) = e^{-t|\xi|^2} \hat{f} \ ,$$

which decays exponentially fast as $t \to \infty$, except $\xi = 0$. This suggests replacing $\hat{f}$ by a Taylor expansion at $\xi = 0$, with the main term $e^{-t|\xi|^2} \hat{f}(0)$, which has Fourier inversion

$$K_H(t)(x) \int f(y) dy \ .$$

To estimate the error term note that,

$$|\hat{f}(\xi) - \hat{f}(0)| \leq |\xi| \|\nabla \hat{f}(\xi)\|_{L^\infty} \leq |\xi| \|xf\|_{L^1} \ .$$

Then

$$\|u(t) - K_H(t)(x) \int f(y) dy\|_{L^\infty_x} \leq \|\hat{u}(t,x) - e^{-t|\xi|^2} \hat{f}(0)\|_{L^1_\xi} \leq \|e^{-t|\xi|^2} \xi\|_{L^1_\xi} \|xf\|_{L^1} = ct^{-(n+1)/2} \|xf\|_{L^1} \ .$$

THEOREM 5.16 (asymptotical behavior). *If $\langle x \rangle f \in L^1$, and $u = K_H(t) * f$, then there is a $c = c(n)$ such that for $t > 0$,*

$$\|u(t) - K_H(t)(x) \int f(y) dy\|_{L^\infty_x} \leq ct^{-(n+1)/2} \|xf\|_{L^1} \ . \tag{5.16}$$

Note that the error on the right decays faster, by a factor $t^{-1/2}$, than the solution itself. We known that the energy, $\int u(t,x)dx$, is independent of time. The above theorem shows that for t large, u is close to the Gaussian with the same energy. For large time, solutions with the same initial energy are essentially indistinguishable. Thus for t large, there is only one observable, the energy. This is a rather striking degradation of information. For example, one could code the Encyclopaedia Britannica as a sequence of 0's and 1's and encode that as a step function. Then asymptotically, all one could measure is the number of bits of information rather than the information itself. This is a strongly irreversible process. Time's arrow is clearly visible.

5.6.3. Backward uniqueness *. It is clear that the problem is not well-posed, for $t \to -\infty$ (in the sense that continuous dependence may fail quickly). Still we could ask if we have backward uniqueness, for $u \in C_t H^2 \cap C_t^1 H_x^1$.

Somewhat remarkably, it turns out that the energy argument could also be exploited yield backward uniqueness. Let $E(t) = \|u(t)\|_{L^2}^2$, we have seen that

$$E' = 2\int u_t u dx = -2\|\nabla u\|_{L^2}^2 \le 0 \ ,$$

$$E'' = -4\int \nabla u \cdot \nabla u_t dx = 4\int \Delta u u_t dx = 4\int u_t^2 dx \ge 0 \ .$$

Thus we get

$$|E'| \le 2\|u_t\|\|u\| \le E^{1/2}(E'')^{1/2} \ .$$

For any $\epsilon > 0$, we introduce $f = \ln(E+\epsilon)$, then $f' = E'/(E+\epsilon)$,

$$f'' = \frac{(E+\epsilon)E'' - (E')^2}{E^2} \ge 0$$

from which we see it is convex. Thus, for any $t \in [0,T]$ with $t = \theta T$, we get $f(t) \le \theta f(0) + (1-\theta)f(T)$, i.e.,

$$E(t) + \epsilon \le (E(0)+\epsilon)^\theta (E(T)+\epsilon)^{1-\theta} \ .$$

Let $\epsilon \to 0+$, we obtain

$$E(t) \le E(0)^\theta E(T)^{1-\theta} \ , \tag{5.17}$$

for all $t \in [0,T]$.

Based on this, we claim that, provided that $E(T) = 0$ and $E(0) < \infty$, we have $E(t) = 0$ for any $t \in [0,T]$, which gives us the backward uniqueness, for $u \in C_t H^2 \cap C_t^1 H_x^1$.

5.6.4. Non-uniqueness. It turns out that $u = S_H(t)f$ is only one out of infinitely many solutions of the IVP. For example, there exists solution $u \in C^\infty(\mathbb{R}^{1+n})$ to $u_t = \Delta u$, with $u = 0$ for $t < 0$ but $u \ne 0$ for $t > 0$.

Following Tychonoff, we construct such u for $n = 1$ (and so for $n \ge 1$). Formally, $x = 0$ is non-characteristic and we prescribe data on $x = 0$ to be

$$u = g(t), u_x = 0.$$

$u = \sum_{j\ge 0} g_j(t)x^j$

$$u_{xx} = \sum_{j\ge 2} j(j-1)g_j(t)x^{j-2} = \sum_{j\ge 0}(j+2)(j+1)g_{j+2}(t)x^j = \sum_{j\ge 0} g_j'(t)x^j = u_t$$

$$g_0 = g, g_1 = 0, (j+2)(j+1)g_{j+2}(t) = g_j'(t)$$

$$g_{2k} = g_0^{(k)}(t)/(2k)!$$

which gives us the formal solution

$$u = \sum_{k\geq 0} \frac{g_0^{(k)}(t)}{(2k)!} x^{2k} \ .$$

To find g so that it really gives a solution, we set for $\alpha > 1$,

$$g(t) = e^{-t^{-\alpha}}, t > 0;\ g = 0, t \leq 0$$

See John [**16**, Theorem, p211-212].

5.6.5. Non-negative solutions.

THEOREM 5.17 (D. V. Widder). *Let $u \in C([0,T) \times \mathbb{R})$ and $u_t, \partial_j u, \partial_j\partial_k u \in C((0,T) \times \mathbb{R})$, with*

$$u_t - \Delta u = 0, (t,x) \in (0,T) \times \mathbb{R}$$
$$u(x,0) = f(x), u(x,t) \geq 0, (t,x) \in (0,T) \times \mathbb{R}$$

then $u(x,t)$ is determined uniquely for $(t,x) \in (0,T) \times \mathbb{R}$, is real analytic and given by $K_H(t) * f$.

See John [**16**, Theorem, p222].

5.7. The free Schrödinger equation

Quantum mechanics: Schrödinger 1926: single particle of mass m moving in fields with potential energy V ($h = 2\pi\hbar$ Planck's constant):

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$$

For simplicity, consider the case with $V = 0$, the free Schrödinger equation.

$$\frac{\partial\psi}{\partial t} = i\frac{\hbar}{2m}\Delta\psi, \psi(0,x) = f(x) \tag{5.18}$$

On the Fourier side, we see that

$$\hat{\psi}(t,\xi) = e^{-i\frac{\hbar}{2m}t|\xi|^2}\hat{f}(\xi) = \hat{E}\hat{f}(\xi) \ .$$

Although similar to the heat equation, $\hat{E} = e^{-i\frac{\hbar}{2m}t|\xi|^2}$ is not in $L^1(\mathbb{R}^n)$. Anyway, $\hat{E}(t) \in \mathcal{S}'(\mathbb{R}^n)$. Notice that $\hat{E}(t) \in \mathcal{S}(\mathbb{R}^n)$ if $\Im t < 0$, and, as $\epsilon \to 0+$, we have

$$e^{-i\frac{\hbar}{2m}(t-i\epsilon)|\xi|^2} \to \hat{E}(t) \text{ , in } \mathcal{S}'$$

Recall that for any $s \in \mathbb{C}$ with $\Re s > 0$, we have

$$\hat{K}_s(x) = e^{-s|\xi|^2} \ ,$$

equivalently, with $s = i\frac{\hbar}{2m}t$ for $t \in \mathbb{C}$ with $\Im t < 0$, we have

$$\hat{K}_s(x) = e^{-s|\xi|^2} \ .$$

We obtain, for the limit case with $s = i\frac{\hbar}{2m}t$ with $\Im t = 0$, $t \neq 0$,

$$E(t,x) = K_s(x) = (4\pi s)^{-n/2}e^{-\frac{|x|^2}{4s}} = (4\pi i\frac{\hbar}{2m}t)^{-n/2}e^{-\frac{|x|^2}{4i\frac{\hbar}{2m}t}} = (\frac{2\pi\hbar}{m})^{-n/2}(it)^{-n/2}e^{i\frac{m|x|^2}{2\hbar t}} \ ,$$

where $(it)^{-n/2} = \lim_{\epsilon\to 0+}(it+\epsilon)^{-n/2}$, which is exactly $|t|^{-n/2}e^{-\mathrm{sgn(t)n\pi i}/4}$.

That is,

$$E(t,x) = (\frac{2\pi\hbar|t|}{m})^{-n/2} e^{-\mathrm{sgn(t)n\pi i/4}} e^{i\frac{m|x|^2}{2\hbar t}} \ .$$

Notice that, we have seen this in HW7 of Section 3.1.

THEOREM 5.18. *For $t \neq 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$u(x,t) = \int_{\mathbb{R}^n} (\frac{2\pi\hbar|t|}{m})^{-n/2} e^{-sgn(t)n\pi i/4} e^{i\frac{m|x-y|^2}{2\hbar t}} f(y)dy \tag{5.19}$$

defines a function in $\mathcal{S}(\mathbb{R}^n)$ for each t, which is C^∞ in $(x,t) \in \mathbb{R}^{n+1}$, and which solves (5.18).

Proof. As we have calculated, $u = E(t) * f$, so

$$g(t,\xi) = \hat{u}(t,\xi) = e^{-i\frac{\hbar}{2m}t|\xi|^2} \hat{f}(\xi) \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)) \ . \tag{5.20}$$

Similarly, we could check $g \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$. Fourier inversion gives us $u \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)) \subset C^\infty$. Clearly $g(t)$, as an $\mathcal{S}$-valued function, obeys $i\hbar\partial_t g = \frac{\hbar^2}{2m}|\xi|^2 g$, $g(0) = \hat{f}$, so taking Fourier transforms, u obeys (5.18). ■

Multiplication by $e^{-ic|\xi|^2 t}$ clearly preserves L^2 norms so by the Plancherel theorem

$$U(t)f = u(t,x) = \mathcal{F}^{-1}(e^{-i\frac{\hbar}{2m}t|\xi|^2}\hat{f}(\xi))$$

defines a unitary map of L^2 to L^2 that agrees with (5.19) if $f \in \mathcal{S}$. As with the Fourier transform if $f \in L^1 \cap L^2$, (5.19) holds pointwise and, in general, (5.19) holds in a limit in mean sense.

In the following, for simplicity we set $\hbar = 2m$.

5.7.1. dispersive estimates. For any $p \in [2, \infty]$,

$$\|e^{it\Delta} f\|_{L^p} \le C_p t^{-n(1/2-1/p)} \|f\|_{L^{p'}}$$

Based on duality, Hardy-Littlewood(-Sobolev) inequality of fractional integration, we obtain the Strichartz estimates (1977, Ginibre-Velo 1985 [5]):

$$\|e^{it\Delta} f\|_{L^q_t L^p_x} \le C_{p,q} \|f\|_{L^2}$$

provided that

$$\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{p}), q > 2 \ .$$

The endpoints $q = 2$ and $n \ge 3$ are also admissible and more delicate, see Keel-Tao 1998.

[5] The global Cauchy problem for the nonlinear Schrodinger equation revisited. Ann. Inst. H. Poincar, Anal. Non Lineaire 2: 309–327, 1985.

5.7.2. asymptotical behavior. A better idea of the behavior as $t \to +\infty$ comes from expanding the exponent in (5.19):

$$e^{i\frac{|x-y|^2}{4t}} = e^{i\frac{|x|^2}{4t}} e^{-ix\cdot\frac{y}{2t}} e^{i\frac{|y|^2}{4t}} \ .$$

For f of compact support, we expect $e^{i\frac{|y|^2}{4t}} \to 1$ on supp f, and

$$\int e^{-ix\cdot\frac{y}{2t}} f(y)dy = \hat{f}(x/2t) \ .$$

This suggests that

$$V(t)f = (4\pi t)^{-n/2} e^{-i\frac{n\pi}{4}} e^{i|x|^2/(4t)} \hat{f}(x/2t) \tag{5.21}$$

is an approximation to $u(t,x) = U(t)f$.

THEOREM 5.19. *Let $t > 0$ and (5.21) Then, for any $f \in L^2$,*

$$\lim_{t\to\infty} \|U(t)f - V(t)f\|_{L^2} = 0 \ .$$

Proof. Notice that

$$V(t)f(x) = (4\pi t)^{-n/2} e^{-i\frac{n\pi}{4}} \int e^{i|x|^2/(4t)} f(y) e^{-iyx/2t} dy = (4\pi t)^{-n/2} e^{-i\frac{n\pi}{4}} \int e^{i\frac{|x|^2-2x\cdot y}{4t}} f(y)dy$$

Then for $t > 0$, we have

$$V(t)f(x) = (4\pi t)^{-n/2} e^{-i\frac{n\pi}{4}} \int e^{i\frac{|x-y|^2}{4t}} e^{-i\frac{|y|^2}{4t}} f(y)dy = U(t)C(t)f$$

where $C(t)f(x) = e^{-i\frac{|x|^2}{4t}} f(x)$. Based on this observation, we see that $V(t)^*V(t) = I$, unitary. Thus

$$\|U(t)f - V(t)f\|_{L^2} = \|f - C(t)f\|_{L^2} = \|(1 - e^{-i\frac{|x|^2}{4t}})f\|_{L^2} \to 0 \ ,$$

by the DCT (dominated convergence theorem). ■

The asymptotics give a physical intuition into the decay rates for the L^p norms. If the momentum density decays rapidly at infinity, then the solution u is concentrated over a regions which dilates linearly with time, so has volume growing like $|t|^n$. The amplitude decays like $|t|^{-n/2}$, which is to be expected given conservation of the L^2 norm and is verified in formulas. Amplitude $|t|^{-n/2}$ over a region of size t^n yields the rates of decay. Exactly such spread is present in the explicit Gaussian solutions.

COROLLARY 5.20 (Dollard). *If $f \in L^2$, $\Gamma \subset \mathbb{R}^n$ is a measurable cone, and $u(t) = U(t)f$, then*

$$\lim \int_\Gamma |u(t,x)|^2 dx = (2\pi)^{-n} \int_\Gamma |\hat{f}(\xi)|^2 d\xi \ .$$

Actually, for any $t > 0$,

$$\int_\Gamma |V(t)f|^2 dx = (4\pi t)^{-n} \int_\Gamma |\hat{f}(x/2t)|^2 dx = (2\pi)^{-n} \int_\Gamma |\hat{f}(x)|^2 dx \ .$$

Then since

$$\int_\Gamma |U(t)f - V(t)f|^2 \to 0$$

as $t \to \infty$, the proof is complete.

This corollary shows that for large positive time the probability that a particle lies in r converges to the probability that its momentum lies in Γ. Similarly, the probability that a particle lies in $-\Gamma$ for large negative time converges to the probability that the momentum lies in Γ. The fact that these two probabilities are equal shows that there is no change in direction of motion in the scattering of particles by the Schrodinger equation.

Dispersive phenomena like those for Schrodinger's equation can also be studied directly from the Fourier representation. From that point of view, one is lead to oscillatory integrals which are estimated using integration by parts in what is called the method of (non)stationary phase.

Holmgren's Theorem shows that if $u = 0$ on a neighborhood of $(t_1, t_2) \times \{x\}$, then u vanishes on $(t_1, t_2) \times \mathbb{R}^n$. This result leads to the quantum mechanical way to catch a lion: If there exists a lion, then putting a cage anywhere, there is a strictly positive probability that the lion is in the cage. It also shows that it is not reasonable to look for solutions compactly supported in x.

5.8. Wave equations

Consider the wave equations

$$\Box u = F, t > 0; u(0) = f(x), u_t(0) = g(x); x \in \mathbb{R}^n.$$

Applying the Fourier transform, we obtain

$$(\partial_t^2 + |\xi|^2)\hat{u}(t,\xi) = \hat{F}(t,\xi); \hat{u}(0) = \hat{f}, \partial_t \hat{u}(0) = \hat{g}$$

which has general solutions, for $t > 0$

$$u(t,\xi) = \cos(t|\xi|)\hat{f} + \frac{\sin(t|\xi|)}{|\xi|}\hat{g} + \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\hat{F}(\tau)d\tau \ .$$

Let E be the distributional solution for

$$\Box E_n = (\partial_t^2 - \Delta)E_n = 0, E_n(0) = 0, E_n'(0) = \delta$$

then we obtain for $t > 0$ that

$$E_n(t,x) = \mathcal{F}^{-1}\frac{\sin(t|\xi|)}{|\xi|} \ ,$$

and then we could construct a general solution for wave equations by

$$u = \partial_t E_n(t,\cdot) *_x f + E_n(t,\cdot) *_x g + \int_0^t E_n(t-\tau,\cdot) *_x F(\tau,\cdot)d\tau \ ,$$

whenever the convolution makes sense.

Similar to Section 4.4, we claim that

$$\mathbf{E_n(t,x) = 2^{-1}}\pi^{\frac{\mathbf{1-n}}{\mathbf{2}}}\chi_+^{\frac{\mathbf{1-n}}{\mathbf{2}}}(\mathbf{t^2 - |x|^2}), E_3(t,x) = \frac{1}{2\pi}\delta(t^2 - |x|^2) = \frac{1}{4\pi t}\delta(t - |x|).$$

Here, for $\Re(\lambda) > -1$, we have

$$\chi_+^\lambda = \begin{cases} \frac{x^\lambda}{\Gamma(\lambda+1)} & x > 0 \\ 0 & x < 0 \ . \end{cases}$$

It is analytic in λ, which admits analytic extension for all $\lambda \in \mathbb{C}$ and

$$\frac{d}{dx}\chi_+^\lambda = \chi_+^{\lambda-1} \ .$$

Noticing that $\chi_+^0 = H$, we have

$$\chi_+^{-k} = \delta^{(k-1)}, k \geq 1 \ .$$

Notice that

$$\mathrm{singsupp}\ \mathrm{E_n(t, \cdot)} \subset \{\mathrm{x} \in \mathbb{R}^\mathrm{n}, |\mathrm{x}| = \mathrm{t}\} \ .$$

which is the light cone originated from the origin.

In particular, for $n = 1$, with the help of E, we recover the D'Alembert's formula:

$$u = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{1}{2}\int_0^t \int_{x-s}^{x+s} F(t-s, y)dyds \ .$$

When $n = 3$ and $F = 0$, we find Kirchoff's formula[6]

$$u(t, x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} [f(x+t\omega) + t\omega \cdot \nabla f(x+t\omega) + tg(x+t\omega)]d\omega \ , \tag{5.22}$$

if we recall that

$$E_3(t) * g(x) = \frac{1}{4\pi t}\int g(x-y)\delta(t-|y|)dy = \frac{t}{4\pi}\int_{\mathbb{S}^2} g(x+t\omega)d\omega \ .$$

In the following, we consider the homogeneous wave equation:

$$\Box u = 0, u(0) = f, u_t(0) = g \ . \tag{5.23}$$

5.8.1. Huygens principle and propagation of singularities. As we have

$$\mathrm{supp}\ E(t, \cdot) \subset \{x \in \mathbb{R}^n, |x| \leq t\} \ ,$$

and

$$\mathrm{supp}\ E(t, \cdot) \subset \{x \in \mathbb{R}^n, |x| = t\} \ , n = 2k+1, k \geq 1 \ ,$$

we get immediately that

THEOREM 5.21 (Huygens principle). *Suppose* supp $(f, g) \subset B_R = \{x; |x| \leq R\}$, *then* supp $u \subset B_{R+t}$ *for* $t > 0$. *If* $n \geq 3$ *odd, then* supp $u \subset B_{R+t} \backslash B_{t-R}$ *for* $t > 0$.

Moreover,

$$\mathrm{singsuppE_n(t, \cdot)} \subset \{\mathrm{x} \in \mathbb{R}^\mathrm{n}, |\mathrm{x}| = \mathrm{t}\} \ .$$

which is the light cone originated from the origin, we have by Lemma 3.17 that

$$\mathrm{singsuppu(t)} \subset \mathrm{singsuppE(t, \cdot)} + (\mathrm{singsuppg} \cup \mathrm{singsuppf}) \ .$$

We see that the singular support of u is at most affected by the singular support of f and g, spreading along the light cones. This is an example of the propagation of

[6]This famous formula is due to Poisson but is known as Kirchoff's formula, aka the Kirchoff-Poisson formula.

Remarkably, Poission is not only responsible for the Green's kernel for Lapalacians, but also the heat kernel and the solution of the wave equation in two and three dimensions. These appear in his 1818–19 papers.

Kirchoff's formula, appeared first in the 1818 paper of Poisson and written 65 years before Kirchoff's paper. What Kirchoff had was a formula involving integration over other surfaces inside the cone but (5.22) is a special case of his results.

Poisson also had the wave equation Riemann function for $n = 2$ kernel which he got from what we call the method of descent; he referred to an earlier paper of Parseval (1800) for the idea of this method.

The Riemann kernel for general n is due to Tédone in 1898. It was rediscovered by Sobolev 35 years later and is often known by his name in the Russian literature.

singularities, which occurs even for variable coefficient wave equations, and is one of the most fundamental facts about wave propagation.

5.8.2. Huygens principle: Fourier approach. It is natural (actually also very important) to ask if we could derive important properties of the solutions, without consulting the fundamental solutions.

It turns out that the non-sharp Huygens principle could also be obtained, with help of the Paley-Wiener theory.

Let us suppose that the initial data are smooth functions with compact support: supp $(f,g) \subset B_R$. Then by Fourier transform, we know that

$$\hat{u}(\xi,t) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin t|\xi|}{|\xi|}\hat{g}(\xi) \ .$$

As $f, g \in \mathcal{D}(\mathbb{R}^n)$, we know that $\hat{f}, \hat{g}$ are entire functions for $\xi \in \mathbb{C}^n$ and we try to extend $\hat{u}(\xi,t)$ as entire function either. To achieve this purpose, recall that

$$\cos t|\xi| = \sum_{n\ge 0}(-1)^n \frac{t^{2n}|\xi|^{2n}}{(2n)!} = \sum_{n\ge 0}(-1)^n \frac{t^{2n}(\xi\cdot\xi)^n}{(2n)!} := C(t,\xi),$$

$$\frac{\sin t|\xi|}{|\xi|} = \sum_{n\ge 0}(-1)^n \frac{t^{2n+1}|\xi|^{2n+1}}{(2n+1)!|\xi|} = \sum_{n\ge 0}(-1)^n \frac{t^{2n+1}(\xi\cdot\xi)^n}{(2n+1)!} := S(t,\xi) \ ,$$

it is clear that they are entire functions of $\xi \in \mathbb{C}^n$.

Moreover, we claim that, there exists some constant $C > 0$, so that

$$|C(t,\xi)| \le Ce^{|\Im(\xi)||t|}, \ |S(t,\xi)| \le C(1+|t|)e^{|\Im(\xi)||t|} \ , \ \forall \xi \in \mathbb{C}^n. \tag{5.24}$$

By the Paley-Wiener theory, Theorem 2.10, we see that

$$|\hat{u}(\xi,t)| \le C(1+|t|)e^{|\Im(\xi)|(|t|+R)} \ , \ \forall \xi \in \mathbb{C}^n,$$

which tells us that for any $t \in \mathbb{R}$,

$$\text{supp } u(x,t) \subset B_{R+|t|} \ .$$

This is a proof that is valid for all spatial dimensions $n \ge 1$, which is independent of the parity of the dimension. Because of this independence, techniques based on the Fourier transform are, however, not so well adapted for a proof of the sharp Huygens principle that holds in odd space dimensions $n \ge 3$, as stated in Theorem 5.21.

5.8.3. Decay estimates and asymptotic behavior.

LEMMA 5.22 (Decay estimates). *For* (5.23) *with nice enough data, we have*

$$\|u(t,\cdot)\|_{L^\infty} = \mathcal{O}(t^{-(n-1)/2}) \text{ as } t \to \infty.$$

Proof. Here we present a sample proof for the case $n = 3$. If $f = 0$, then

$$u = \frac{t}{4\pi}\int_{\mathbb{S}^2} g(x+t\omega)d\omega = -\frac{t}{4\pi}\int_{\mathbb{S}^2}\int_t^\infty \frac{d}{ds}g(x+s\omega)dsd\omega = -\frac{t}{4\pi}\int_{\mathbb{S}^2}\int_t^\infty (\omega\cdot\nabla_x g)(x+s\omega)dsd\omega$$

and thus

$$|u(x,t)| \le \frac{C}{t}\|\nabla g\|_{L^1} \ .$$

Similarly, if $g = 0$, as $u(t,x) = \frac{1}{4\pi}\int_{\mathbb{S}^2}[f(x+t\omega) + t\omega\cdot\nabla f(x+t\omega)]d\omega$, we have

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} f(x+t\omega)d\omega = -\frac{1}{4\pi}\int_{\mathbb{S}^2}\int_t^\infty \frac{d}{ds}f(x+s\omega)dsd\omega = \frac{1}{4\pi}\int_{\mathbb{S}^2}\int_t^\infty (s-1)\frac{d^2}{ds^2}f(x+s\omega)dsd\omega$$

and so

$$|u(x,t)| \le \frac{C}{t}(\|\nabla^2 f\|_{L^1} + \|\nabla g\|_{L^1}) \ . \tag{5.25}$$

∎

Question What is the asymptotic behavior?

THEOREM 5.23 (asymptotic behavior). *Let $n = 3$, $\Box u = 0$, $u(0) = 0$, $u_t(0) = g \in C_c^\infty(B_M)$, then for $t \ge 2M$, there exists $F \in C^\infty(\mathbb{R} \times \mathbb{S}^2 \times [0, 1/(2M)])$ with $F = 0$ for $\rho > M$ such that*

$$u(x,t) = \frac{1}{r} F(r - t, \omega, \frac{1}{r}) \ .$$

Here, it can be shown that

$$F_0(\rho, \omega) = F(\rho, \omega, 0) = \frac{1}{4\pi} \int_{z\cdot\omega=\rho} g(z)dz$$

which is known to be the Friedlander's radiation fields (Friedlander 1962, 1964), see Alinhac [**1**, Section 5.1.5], Hörmander [**13**, Section 6.2] for more information and proof. It is also interesting to observe that

$$\int_{z\cdot\omega=\rho} g(z)dz = Rg(\rho, \omega)$$

is the well-known Radon transform, see Folland [**9**, Section 5.F], Hörmander [**13**, Section 6.2].

5.8.4. Strichartz estimates. Based on the decay estimates, recall also (if $f = 0$)

$$\|u(\cdot, t)\|_{L^2} \le \|D^{-1} g\|_{L^2}$$

heuristically, we have by interpolation that (which could be proved by Littlewood-Paley theory)

$$\|u(\cdot, t)\|_{L^4} \le C t^{-1/2} \|g\|_{L^{4/3}} \tag{5.26}$$

and so for inhomogeneous solutions ($\Box u = F$ with vanishing data, by using the Hardy-Littlewood inequality of fractional integration)

$$\|u(\cdot, t)\|_{L^4_{t>0} L^4} \le \tilde{C} \|F\|_{L^{4/3}_{t,x}} \ . \tag{5.27}$$

This is part of the well-known result of Strichartz (1977).

THEOREM 5.24 (Strichartz's estimate). *Let $n \ge 2$ and $q = 2(n+1)/(n-1)$, we have*

$$\|u\|_{L^q_{t,x}} \lesssim \|u(0)\|_{\dot{H}^{1/2}} + \|\partial_t u(0)\|_{\dot{H}^{-1/2}} + \|\Box u\|_{L^{q'}_{t,x}} \ .$$

We will omit the proof for the homogeneous estimate. Instead, we present a boundedness result for $n = 3$. In fact, if $(f, g) \in C_c^\infty$, we know that

$$u = \mathcal{O}(\langle t \rangle^{-1}), \text{supp } u \subset \{||x| - t| \le R\} \tag{5.28}$$

from which, we see that for $t > 2R$,

$$\|u\|^p_{L^p_x} \lesssim \langle t \rangle^{-p} [(t+R)^3 - (t-R)^3] \lesssim \langle t \rangle^{2-p} \in L^1_t$$

provided that $p > 3$. That is, for any $q > 3$, we have

$$\|u\|_{L^q_{t,x}} \le C(f, g) \ . \tag{5.29}$$

Notice that we have a bigger range of admissible powers.

5.9. Homeworks

Section 5.1-5.2

(1) Prove (5.4).
(2) Apply Theorem 5.4 to Lewy and Mizohata's operators.

Section 5.3-5.4

(1) Prove (5.8), at least for $|\alpha| \le 2$.
(2) Consider the operator $L = \partial_x + x$ in $\mathbb{R}$, is it hypoelliptic? Explain in details.
(3) Consider the operator $L = \partial_x + i\partial_y$ in $\mathbb{R}^2$, is it hypoelliptic? Explain in details.
(4) Consider the operator $L = \partial_t^2 - \partial_x^2$ in $\mathbb{R}^2$, is it hypoelliptic? Explain in details.
(5) Consider the heat operator $L = \partial_t - \Delta$, is it hypoelliptic? Explain in details.

Section 5.7-5.8

(1) For g given in (5.20), check that $g \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$.
(2) Calculate the solution to

$$\frac{\partial \psi}{\partial t} = i\Delta\psi, \psi(0,x) = f(x) \tag{5.30}$$

with $f = e^{-a|x|^2/2}$, $a > 0$.
(3) Prove the claim (5.24).

5.10. Homeworks

Section 6.1

(1) Similar to (5.22), use the fundamental solutions to write down the solution representation formula for $n = 2$.
(2) Prove the energy conservation (6.3) for solutions to (5.23) with $f, g \in \mathcal{S}(\mathbb{R}^n)$ by Fourier transform.
(3) Prove (5.24), that is,

$$|C(t,\xi)| \le Ce^{|\Im(\xi)||t|},\ |S(t,\xi)| \le C(1+|t|)e^{|\Im(\xi)||t|}\ ,\ \forall \xi \in \mathbb{C}^n. \tag{5.31}$$

(4) Let Ω be an open bounded smooth domain in $\mathbb{R}^n$, c, q be nonnegative smooth functions. In addition, we assume that $\delta < c(x) < \delta^{-1}$ for any $x \in \Omega$, for some $\delta > 0$. Considering the variable coefficient (real-valued, i.e., u is real) wave equation

$$u_{tt} - \nabla \cdot (c^2(x)\nabla u) + q(x)u = 0, u(0,x) = f(x), u_t(0,x) = g(x), x \in \Omega\ .$$

Imposing the homogeneous Dirichlet boundary condition, that is,

$$u(t,x) = 0, x \in \partial\Omega\ .$$

Use the multiplier argument (multiply by u_t and integration by part) to find the energy of the problem and prove energy conservation, assuming $u \in C^2$.

(5) Let Ω be an open bounded smooth domain in $\mathbb{R}^n$, c, q be nonnegative smooth functions. In addition, we assume that $\delta < c(x) < \delta^{-1}$ for any $x \in \Omega$, for some $\delta > 0$. Considering the variable coefficient (real-valued, i.e., u is real) wave equation

$$u_{tt} - \nabla \cdot (c^2(x)\nabla u) + q(x)u = 0, u(0,x) = f(x), u_t(0,x) = g(x), x \in \Omega \; . \tag{5.32}$$

Imposing the homogeneous Neumann boundary condition, that is,

$$n(x) \cdot \nabla u(t,x) = 0, x \in \partial\Omega \; ,$$

where $n(x)$ denotes the outward unit normal vector. Use the multiplier argument (multiply by u_t and integration by part) to find the energy of the problem and prove energy conservation, assuming $u \in C^2$.

(6) Prove the following finite speed of propagation property for problem (5.32). If $f = g = 0$ in $B_1(0) \subset \Omega$, then there exists a region in $\mathbb{R}_+ \times \Omega$ such that u vanishes there.

CHAPTER 6

Properties of solutions of second order PDE: energy estimates and the maximum principle

It turns out that many important properties for solutions to PDE could also be understood, without consulting to the fundamental solutions, which is of fundamental importance in applications (considering that all of our model PDEs are approximations to the real problems).

6.1. The energy estimate

While the finite speed of propagation (i.e., Huygens' principle) and the propagation of singularities are very general phenomena, i.e., they hold for variable coefficient equations as well, so far we could only justify them for our special constant coefficient equation. Next, we consider the conservation of energy, which is more robust to analysis even in general, without having to develop further tools.

Inspired by the variational approach for the d'Alembert equation of the vibrating string, in subsection 1.1.5, the total energy

$$E(t) = K + V = \int_{\mathbb{R}^n} \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 dx \ ,$$

should be conserved along with the evolution. Actually, provided that $u \in C_t^2 C_c^2$,

$$\frac{d}{dt}E = \int \partial_t u u_{tt} + \nabla u \cdot \nabla u_t dx = \int \partial_t u(u_{tt} - \Delta u)dx = 0 \ .$$

6.1.1. Multiplier approach. Inspired by the previous calculation, or using the translation invariance in time, which has generator u_t, we use the multiplier u_t to the equation[1]. We try to write $\partial_t u \Box u$ as a space-time divergence

$$\partial_t u \Box u = \frac{1}{2}\partial_t |u_t|^2 + \frac{1}{2}\partial_t |\nabla u|^2 - \sum_{i=1}^n \partial_i(\partial_t u \partial_i u) = \partial_t e - \nabla \cdot p, \tag{6.1}$$

where

$$e = \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2, p_j = u_t u_j \ .$$

Assuming $u \in C^2(\mathbb{R}^{1+n})$ and compactly supported for any t, integration with respect to x yields that,

$$\frac{d}{dt}E = \int \partial_t u \Box u dx \le \|\partial u\|_{L^2} \|\Box u\|_{L^2} \ .$$

[1]There is a general result concerning the invariance and conservation laws, which is known as the Nöther's principle.

where $E(t) = \int e(t)dx$. For $t > 0$, we get $\partial_t \|\partial u\|_{L^2} \le \|\Box u\|_{L^2}$ and then

$$\|\partial u(t)\|_{L^2} \le \|\partial u(0)\|_{L^2} + \int_0^t \|\Box u(\tau)\|_{L^2} d\tau, \tag{6.2}$$

which is called the energy estimate. In particular, when $\Box u = 0$, we have the energy conservation

$$E(t) = E(0) \ . \tag{6.3}$$

6.1.2. Uniqueness. It is easy to conclude uniqueness (actually also stability) by energy estimates.

6.1.3. Decay. In view of propagation of singularity, it is also convincing to have decay properties $\|u(t,\cdot)\|_{L^\infty} \le t^{-(n-1)/2}$. A rigorous version is provided by the Klainerman-Sobolev inequality.

6.1.4. Huygens principle. When deriving the energy estimate, we assumed that the solution are compactly supported with respect to x. For data with compactly support, it turns out that this is a technical condition which is actually unnecessary.

Actually, we have the following (weak) Huygens principle:

THEOREM 6.1 (Finite speed of propagation). *Suppose $u \in C^2$ is the solution of (5.23). If $f = g = 0$ in $B_{t_0}(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < t_0\}$, then $u \equiv 0$ in $K(x_0, t_0)$.*

$$K(x_0, t_0) = \{(t, x) | 0 \le t \le t_0, |x - x_0| \le t_0 - t\}.$$

Proof. Let $B_t = \{x : |x - x_0| \le t_0 - t\}$ and define the energy carried in B_t by

$$E_0(t) = \int_{B_t} e(t)dx = \frac{1}{2}\int_{B_t} |\partial u(t,x)|^2 dx \ .$$

Under differentiation, we get

$$E_0'(t) = \int_{B_t} u_t u_{tt} + \nabla u \cdot \nabla u_t dx - \frac{1}{2}\int_{\partial B_t} |\partial u|^2 d\sigma(x).$$

Since

$$e' = \nabla u_t \cdot \nabla u + u_t u_{tt} = \nabla u_t \cdot \nabla u + u_t \Delta u + u_t \Box u = \mathrm{div}(u_t \nabla u) + u_t \Box u \ ,$$

it follows from the divergence theorem that

$$\begin{aligned} E_0'(t) &= \int_{B_t} \mathrm{div}(u_t \nabla u) dx - \frac{1}{2}\int_{\partial B_t} |\partial u|^2 d\sigma(x) \\ &= \int_{\partial B_t} u_t \nabla u \cdot \mathbf{n} d\sigma(x) - \frac{1}{2}\int_{\partial B_t} |\partial u|^2 d\sigma(x) \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal of ∂B_t. But

$$|u_t \nabla u \cdot \mathbf{n}| \le |u_t||\nabla u| \le \frac{1}{2}(|u_t|^2 + |\nabla u|^2),$$

then we see $E_0'(t) \le 0$ for $0 \le t \le t_0$, $E_0(t) \le E_0(0) = 0$, but certainly $E_0(t) \ge 0$. It follows that $\partial u = 0$ in $K(x_0, t_0)$, and hence $u = 0$ in $K(x_0, t_0)$. ∎

REMARK 6.2. *We could actually read off more information from the argument. Noticing that*

$$u_t \nabla u \cdot \mathbf{n} - \frac{1}{2}|\partial u|^2 = -\frac{1}{2}(|u_t - \partial_n u|^2 + |\not\nabla_n u|^2) ,$$

where $\partial_n u = \nabla u \cdot \mathbf{n}$ *and* $\not\nabla_n u = \nabla u - n\partial_n u$ *are the radial and angular component of the gradient. In effect, we obtain*

$$E_0'(t) + \frac{1}{2}\int_{\partial B_t} (|u_t - \partial_n u|^2 + |\not\nabla_n u|^2) d\sigma(x) = \int_{B_t} u_t \Box u dx .$$

6.2. The maximum principle for Laplace/Poisson's equation and the heat equation

We now investigate analogous issues for Laplace's equation and the heat equation. In this case the simplest method is the maximum principle.

6.2.1. Laplace's equation. We start with Laplace's equation, where we restate a related version of Theorem 8.17 here.

THEOREM 6.3 (Weak maximum principle). *Suppose* Ω *is a connected open bounded set. If* $u \in C^2(\Omega) \cap C(\overline{\Omega})$*,* u *is real-valued and* $\Delta u \geq 0$*, then* u *attains its maximum on* $\partial\Omega$*:*

$$\sup_{x\in\Omega} u(x) = \max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) .$$

Proof. Since $\overline{\Omega}$ is closed and bounded, the continuous function u attains its maximum on $\overline{\Omega}$ and we need to prove that this happens on $\partial\Omega$. To see why this should be the case, we notice first that if x_0 is a local maximum for u in Ω, then $\partial_j u(x_0) = 0$, $\partial_j^2 u(x_0) \leq 0$ and so $\Delta u(x_0) \leq 0$.

If the inequality were strict, $\Delta u(x_0) < 0$, we would have obtained a contradiction, to the condition $\Delta u \geq 0$. Alternatively, we are done if we know that $\Delta u > 0$.

To get around this problem, we introduce auxiliary functions $w = u + \phi$, then

$$\Delta w = \Delta u + \Delta\phi \geq \Delta\phi .$$

If we could choose ϕ such that $\Delta\phi > 0$ everywhere, then we could conclude maximum principle for w, so that, if $\phi \geq 0$,

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\Omega} w(x) = \sup_{x\in\partial\Omega} w(x) \leq \sup_{x\in\partial\Omega} u(x) + \sup_{x\in\partial\Omega} \phi(x) .$$

Here, it is easy to construct such ϕ. For example, $\phi = |x|^2 \geq 0$, which enjoys

$$\Delta\phi = 2n > 0 .$$

With the help of these ϕ, there exists $C = \|\phi\|_{L^\infty(\Omega)}$ so that

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) + C .$$

To remove the constant C, we simply revise the argument by adding small parameters $\epsilon > 0$: $u_\epsilon = u + \epsilon\phi$:

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) + C\epsilon .$$

Let $\epsilon \to 0$, we obtain the desired result. ∎

We can also apply the maximum principle to $-u$ (or run a similar argument directly for $\inf u$) to deduce that

$$\inf_{x\in\Omega} u(x) = \min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x) ,$$

provided that $\Delta u \le 0$.

Remark 6.4. *It may be natural to ask whether we have weak maximum principle for general elliptic operators? It turns out that it is not always valid. For example, $u'' + u = 0$, $u(0) = u(2\pi) = 0$, $u = \sin x$.*

As a consequence, we get the uniqueness for solutions of the Dirichlet problem for Poisson's equation: Corollary 8.18. Moreover, we have the following enhanced version of the stability result of Corollary 8.19:

Corollary 6.5 (Stability). *Suppose $\overline{\Omega}$ is compact. If $f_1 \ge f_2$ and $u_1, u_2 \in C^2(\Omega)\cap C(\overline{\Omega})$ solve*

$$\Delta u_j = f_j, x\in\Omega,\ u_j = \phi_j, x\in\partial\Omega ,$$

then

$$\max_{x\in\Omega} u_1(x) - u_2(x) \le \max_{y\in\partial\Omega} \phi_1(y) - \phi_2(y) ,$$

which particularly implies, if $f_1 = f_2$,

$$\max_{\overline{\Omega}} |u_1 - u_2| \le \max_{y\in\partial\Omega} |\phi_1(y) - \phi_2(y)| .$$

6.2.2. Heat equation. There is a complete analogue of this for the heat equation, $u_t = \Delta u$.

Theorem 6.6 (Weak maximum principle). *Suppose Ω is a connected open bounded set and $T > 0$. If $u \in C^2(\Omega\times(0,T])\cap C(\overline{\Omega}\times[0,T])$, u is real-valued and $u_t - \Delta u \le 0$, then u attains its maximum on the parabolic boundary*

$$(\partial\Omega\times[0,T])\cup(\Omega\times\{0\}) .$$

Proof. We observe as above that if u had a local maximum at $y\in\Omega\times(0,T)$, then $\Delta u(y) - u_t(y) \le 0$ there. If u has a local maximum at $y \in \Omega\times\{T\}$, then we still have $\partial_{x^j} u(y) = 0$ and $\partial^2_{x^j} u(y) \le 0$, and hence $\Delta u(y) \le 0$ as above. For $\partial_t u$, instead, we conclude that $\partial_t u(y) \ge 0$ there. Thus, we conclude that

$$\Delta u(y) - u_t(y) \le 0 .$$

This does not contradict the heat equation, but we can modify the argument as before by replacing u by

$$u_\epsilon = u + \epsilon |x|^2 .$$

For any $\epsilon > 0$, we conclude that

$$\sup_{(x,t)\in\Omega\times[0,T]} u_\epsilon(x,t) = \max_{(x,t)\in(\partial\Omega\times[0,T])\cup(\Omega\times\{0\})} u_\epsilon(x,t) .$$

A limiting argument as before now gives the desired maximum principle. ∎

Similar to the Poisson equation, we have stability result for heat equations (and also uniqueness)

COROLLARY 6.7 (Stability). *Suppose $\overline{\Omega}$ is compact. If $u_1, u_2 \in C^2(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$ solve*

$$\partial_t u_j - \Delta u_j = f, x \in \Omega, t \in (0,T],\ u_j(x,0) = \phi_j, x \in \Omega\ ,\ u_j(x,t) = h_j, x \in \partial\Omega \times [0,T]\ ,$$

then

$$\sup_{(x,t)\in\Omega\times[0,T]} |u(x,t)| \le \max\Big(\max_{\partial\Omega\times[0,T]} |h_1 - h_2|, \max_{\Omega\times\{0\}} |\phi_1 - \phi_2|\Big)\ .$$

COROLLARY 6.8 (Uniqueness). *Suppose $\overline{\Omega}$ is compact. There exists at most one solution $u \in C^2(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$ for*

(6.4)

$$\partial_t u - \Delta u = f, x \in \Omega, t \in (0,T],\ u(x,0) = \phi, x \in \Omega\ ,\ u(x,t) = h, x \in \partial\Omega \times [0,T]\ .$$

6.3. Energy for Poisson's equation and the heat equation

In physical applications of the heat equation, u is temperature or concentration, and the corresponding conserved quantity is the total heat or mass,

$$M(t) = \int_\Omega u(x,t)dx\ ,$$

at least if there is no heat flux/flow through $\partial\Omega$, i.e., $\frac{\partial u}{\partial \nu} = 0$ (the Neumann boundary condition). This is easy to check:

$$M'(t) = \int_\Omega u_t(x,t)dx = \int_\Omega \Delta u dx = \int_{\partial\Omega} n \cdot \nabla u dS = 0\ ,$$

where the last step used the boundary condition.

If $\Omega = \mathbb{R}^n$ and there is sufficient decay of u at infinity, then there is no need for the boundary condition (one can think of the decay as the boundary condition), and one again concludes that M is conserved. The main issue here is that just because this integral vanishes, we cannot conclude that u vanishes, unlike in the case of the energy for the wave equation.

6.3.1. Energy for the heat equation.

There is, however, an unphysical "energy" akin to the energy for the wave equation, namely

$$E(t) = \frac{1}{2}\int_\Omega u(x,t)^2 dx\ .$$

Then

$$\frac{d}{dt}E = \int_\Omega u u_t dx \int_\Omega u\Delta u dx = \int_\Omega \nabla\cdot(u\nabla u) - |\nabla u|^2 dx = -\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} n\cdot(u\nabla u)dS\ .$$

As $\int_\Omega |\nabla u|^2 dx \ge 0$, we deduce that for either homogeneous Dirichlet or Neumann boundary conditions,

$$E' \le 0\ .$$

Thus, this "energy" is not conserved. However it is decreasing, so in particular if $E(0) = 0$, then $E(t) = 0$ for any $t > 0$.

6.3.2. Uniqueness. This gives us another proof of uniqueness for the heat equation, i.e. that there is at most one solution of (6.4), Corollary 6.8. And, moreover, this also works for the Neumann boundary condition:

$$\partial_t u - \Delta u = f, x \in \Omega, t \in (0,T],\ u(x,0) = \phi, x \in \Omega\ ,\ \partial_\nu u(x,t) = h, x \in \partial\Omega \times [0,T]\ . \tag{6.5}$$

Further, we have a new stability result: any solutions u_1 and u_2 of either

$$\partial_t u_j - \Delta_j u = f, x \in \Omega, t \in (0,T],\ u_j(x,0) = \phi_j, x \in \Omega\ ,\ u_j(x,t) = h, x \in \partial\Omega \times [0,T]\ , \tag{6.6}$$

or

$$\partial_t u_j - \Delta_j u = f, x \in \Omega, t \in (0,T],\ u_j(x,0) = \phi_j, x \in \Omega\ ,\ \partial_\nu u_j(x,t) = h, x \in \partial\Omega \times [0,T]\ , \tag{6.7}$$

we have

$$\int_\Omega |u_1(t) - u_2(t)|^2 dx \leq \int_\Omega |\phi_1(t) - \phi_2(t)|^2 dx\ .$$

Here we had matching boundary conditions so that $u_1 - u_2$ has vanishing boundary conditions, thus satisfying the criteria in our energy calculation.

6.3.3. Energy estimate for Laplace's equation. We finally work out an energy estimate for a generalized version of Laplace's equation:

$$\nabla \cdot (c^2(x)\nabla u) - q(x)u = F \tag{6.8}$$

with homogeneous Dirichlet/Neumann conditions.

Let

$$E = \int_\Omega c^2 |\nabla u(x)|^2 + qu^2 dx\ .$$

We may also write $E = E(u)$, which is sometimes called the Dirichlet form. Now note that

$$\begin{aligned} E &= \int_\Omega c^2 |\nabla u(x)|^2 + qu^2 dx \\ &= \int_\Omega \nabla \cdot (c^2 u \nabla u) - u \nabla \cdot (c^2 \nabla u) + qu^2 dx \\ &= \int_{\partial\Omega} \nu \cdot (c^2 u \nabla u) - \int_\Omega uF dx = -\int_\Omega uF dx\ . \end{aligned}$$

Let us assume that $q \geq q_0 > 0$ and $c \geq c_0 > 0$. Then we can use the Cauchy-Schwarz inequality,

$$c_0^2 \|\nabla u\|_{L^2}^2 + q_0 \|u\|_{L^2}^2 \leq E \leq \|F\|_{L^2} \|u\|_{L^2} \leq \frac{q_0}{2} \|u\|_{L^2}^2 + \frac{1}{2q_0} \|F\|_{L^2}^2\ ,$$

which gives us

$$2c_0^2 \|\nabla u\|_{L^2}^2 + q_0 \|u\|_{L^2}^2 \leq \frac{1}{q_0} \|F\|_{L^2}^2\ . \tag{6.9}$$

In particular, if we have two solutions u_j, $j = 1,2$, of (6.8) with either the same inhomogeneous Dirichlet or Neumann boundary conditions, then $u = u_1 - u_2$ solves

$$\nabla \cdot (c^2(x)\nabla u) - q(x)u = 0 \tag{6.10}$$

with homogeneous Dirichlet/Neumann conditions. So an application of (6.9) gives us the uniqueness of the solutions of the PDE. Indeed, we get stability also, for the PDE

$$\nabla \cdot (c^2(x)\nabla u_j) - q(x)u_j = F_j \tag{6.11}$$

with either the same inhomogeneous Dirichlet or Neumann boundary conditions, since we have, for $w = u_1 - u_2$

$$2c_0^2\|\nabla w\|_{L^2}^2 + q_0\|w\|_{L^2}^2 \le \frac{1}{q_0}\|F_1 - F_2\|_{L^2}^2 \ . \tag{6.12}$$

6.3.4. Case $q \ge 0$. The only somewhat serious assumption here was that $q \ge q_0 > 0$ since we might want to solve Laplace's equation itself. It turns out that we can merely assume $q \ge 0$, at least if we impose the Dirichlet boundary condition. This is due to the Poincaré inequality:

PROPOSITION 6.9 (Poincaré inequality). *Suppose Ω is bounded C^1 domain, and $u \in C^1(\bar{\Omega})$ with $u = 0$ on $\partial\Omega$, then*

$$\|u\|_{L^2(\Omega)} \le \operatorname{diam}(\Omega)\|\nabla u\|_{L^2(\Omega)} \ ,$$

where $\operatorname{diam}(\Omega) = \sup_{x,y\in\Omega}|x - y|$.

Proof. We just give the proof for the case $\Omega = [a, b]^n$ and $u \in C_c^1(\Omega)$. We have $u(a, x') = 0$ and then

$$|u(x_1, x')| = |\int_a^{x_1} \partial_1 u(s, x')ds| \le \int_a^b |\partial_1 u(s, x')|ds \le |b - a|^{\frac{1}{2}}\|\partial_1 u(\cdot, x')\|_{L^2([a,b])}$$

$$\int |u(x_1, x')|^2 dx' \le |b - a| \int \|\partial_1 u(\cdot, x')\|_{L^2([a,b])}^2 dx' \le |b - a|\|\partial_1 u\|_{L^2(\Omega)}^2$$

So we have $\|u\|_{L^2} \le (b - a)\|\nabla u\|_{L^2}$. ∎

CHAPTER 7

Dirichlet problem, Sobolev space, Nonlinear phenomenon

Suppose $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is C^1. Let $c \in C^1(\bar{\Omega})$, $c > 0$ and

$$\Delta_c = \nabla \cdot c^2(x)\nabla \ .$$

Ill-posed for IVP, possible WP for BVP (Dirichlet problem):

$$\Delta u = f, x \in \Omega, u = g, x \in \partial\Omega \tag{7.1}$$

Essentially, it reduced to the "equivalent" two problems

$$\Delta u = 0, x \in \Omega, u = g, x \in \partial\Omega \tag{7.2}$$

$$\Delta u = f, x \in \Omega, u = 0, x \in \partial\Omega \tag{7.3}$$

For (7.2), by WMP $\Rightarrow$ uniqueness, stability
For (7.1), by energy $\Rightarrow$ uniqueness, stability
Question: existence for BVP?
Many ideas available: Green's function; layer potential; Perron's method (MP); L^2 method/energy method; variational method; etc.

7.1. Existence via duality

Let Ω be a bounded domain, $c \in C^1(\bar{\Omega})$, $c > 0$ and

$$\Delta_c = \nabla \cdot c^2(x)\nabla \ .$$

We have seen in Section 6.3, we have an energy estimate for the generalized version of Laplace's equation

$$\nabla \cdot (c^2(x)\nabla u) = f, u \in C^2(\bar{\Omega}), u|_{\partial\Omega} = 0 \Rightarrow \|u\|^2_{H^1(\Omega)} \le C\|f\|^2_{L^2(\Omega)} \tag{7.4}$$

where

$$\|u\|^2_{H^1} = \int_\Omega c^2(x)|\nabla u(x)|^2 + |u|^2(x)dx \ . \tag{7.5}$$

We may note that different choices of c give different, but equivalent, norms. When $c \equiv 1$, it is the standard H^1 norm.

As discussed in Section 6.3, this gives uniqueness and stability with

$$S = \{u \in C^2(\bar{\Omega}) : u|_{\partial\Omega} = 0\}, N = C(\bar{\Omega})$$

equipped with the norms

$$\|u\|_S = \|u\|_{H^1}, \|f\|_N = \|f\|_{L^2} \ ,$$

where the norms is important to prove stability.

However, as we see in Section 5.5.2, existence is false, even if $c = 1$: there exists $f \in N$ such that there is no $u \in S$ with $\Delta u = f$. Thus, we would like to enlarge the solution space S, while keeping the uniqueness and stability.

The simplest way of getting existence is dualizing as above. Note that $\Delta_c^* = \Delta_c$, so the energy estimate gives that

$$\|\phi\|_S \le C\|\Delta_c^*\phi\|_N$$

so Δ_c^* is injective on $\mathcal{D}(\Omega)$. If $\phi_j, \phi \in \mathcal{D}(\Omega)$, $\Delta_c^*\phi_j \to \Delta_c^*\phi$ in the L^2-norm, then $\phi_j \to \phi$ in the H^1-norm.

Correspondingly, for given $f \in L^2$, if we define a linear functional

$$T : \Delta_c S \to \mathbb{C},\ T(\Delta_c\phi) = \int_\Omega f\phi dx$$

for any $\phi \in S$, then

$$|T(\Delta_c\phi)| \le \|f\|_{L^2}\|\phi\|_{L^2} \le \|f\|_{L^2}\|\phi\|_{H^1} \le C\|f\|_{L^2}\|\Delta_c\phi\|_{L^2}\ .$$

By the Hahn-Banach theorem, T could be extended to $N^*(\Omega)$ so that $T(\Delta_c\phi) = \int_\Omega f\phi dx$ for any $\phi \in S$. By the Riesz representation theorem, we find $u \in L^2(\Omega)$ so that

$$\int_\Omega u\Delta_c\phi dx = \int_\Omega f\phi dx\ , \forall\phi \in S\ ,$$

in particular, it gives us that $\Delta_c u = f$ in $\mathcal{D}'(\Omega)$.

In summary, we have

THEOREM 7.1. *For any $f \in L^2(\Omega)$, there exists $u \in L^2(\Omega) \subset \mathcal{D}'(\Omega)$ such that $\Delta_c u = f$.*

While this solves the PDE, it is not clear whether we have solved the (original) problem, since the boundary conditions may not make any sense!

7.2. Sobolev spaces

At this point, it is natural to introduce Sobolev spaces.

DEFINITION 7.1 (Sobolev spaces). *Let $\Omega \subset \mathbb{R}^n$ is bounded, open and connected, with smooth boundary. The Sobolev space $H_0^1(\Omega)$ is defined as completion of $\mathcal{D}(\Omega)$ wrt H^1 norm. Similarly, $H^1(\Omega)$ is defined as completion of $C^1(\bar\Omega)$ wrt H^1 norm.*

Fact (see e.g. Folland [9] 6E) :

$$H^1(\Omega) = \{u \in \mathcal{D}'(\Omega) : u, \nabla u \in L^2(\Omega)\}.$$

PROPOSITION 7.2 (Trace lemma). *Let Ω be bounded and $\partial\Omega$ is C^1, then there exists $C > 0$, such that for any $u \in H^1(\Omega)$*

$$\|u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)}. \tag{7.6}$$

Proof. We only need to prove (7.6) for $u \in C^1(\bar\Omega)$, as $C^1(\bar\Omega)$ is dense in $H^1(\Omega)$. For any $x_0 \in \partial\Omega$, we can introduce a $C^1(\bar\Omega)$ extension of the outward unit normal ν from $B_\delta(x_0)$, with support in $B_{2\delta}(x_0)$ and $\nu \cdot v \ge 0$, for sufficiently small $\delta > 0$. Then

$$\begin{aligned}\int_{\partial\Omega\cap B_\delta(x_0)} |u|^2 dx &\le \int_{\partial\Omega\cap B_{2\delta}(x_0)} |u|^2 \nu\cdot v(x) dx = \int_\Omega \nabla\cdot(v(x)u^2)dx \\ &\le \int_\Omega |\nabla v||u|^2 + 2\int_\Omega |v||u||\nabla u|dx \le C\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\end{aligned}$$

Gluing together and partition of unit give us the desired result.

The proof will be even more clear, if we have $\Omega = \{(x, y) : x > 0, y \in \mathbb{R}^{n-1}\}$, then

$$|u(0,y)|^2 = |u(0,y)|^2 - |u(\infty,y)|^2 = -\int_0^\infty \partial_x |u(x,y)|^2 dx \le \|u(x,y)\|_{L_x^2}^2 + \|\partial_x u(x,y)\|_{L_x^2}^2$$

and so

$$\|u(0,y)\|_{L_y^2}^2 \le \|u(x,y)\|_{L^2}^2 + \|\partial_x u(x,y)\|_{L^2}^2 \ .$$

■

With the help of this fact, we can define trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ and we see that H_0^1 are H^1 functions with vanishing trace on $\partial\Omega$. Actually, it can be shown that any H^1 functions with vanishing trace on $\partial\Omega$ lie in H_0^1.

The following has been essentially proved in Theorem 6.9.

PROPOSITION 7.3 (Poincaré inequality). *Suppose Ω is bounded, and $u \in H_0^1(\Omega)$, then*

$$\|u\|_{L^2(\Omega)} \le C \|\nabla u\|_{L^2(\Omega)}$$

Proof. Let us denote $x = (x_1, x') = (x_1, x_2, \cdots, x_n)$, suppose also $\Omega \subset [a, b]^n$.

i) For any $u \in C_c^1(\Omega)$, see Theorem 6.9.

ii) In general, $u \in H_0^1$, find $u_j \in C_c^1(\Omega)$ convergent to u in H^1, we get $u_j \to u$, $\nabla u_j \to \nabla u$ in L^2, and so is the estimate for u. ■

PROPOSITION 7.4 (Sobolev embedding). *Suppose $u \in H_0^s(\Omega)$, if $s > \frac{n}{2}$, then $u \in C(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \le C \|u\|_{H_0^s(\Omega)}.$$

Moreover if $u \in H_0^{s+k}(\Omega)$, then $u \in C^k$, for any $k \in \mathbb{N}$.

For it's proof, by density (why? how? can you write down a complete proof?), we need only to prove the inequality for $u \in C_0^\infty(\Omega)$, which is reduced to the case of $\Omega = \mathbb{R}^n$. Actually, we need only to show $\hat{u} \in L^1(\mathbb{R}^n)$, which follows from Cauchy-Schwartz:

$$\|u\|_{C^0(\mathbb{R}^n)} \le (2\pi)^{-n} \|\hat{u}\|_{L^1} = (2\pi)^{-n} \|(1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}\hat{u}\|_{L^1} \le (2\pi)^{-n} \|(1+|\xi|^2)^{-s/2}\|_{L^2} \|(1+|\xi|^2)^{s/2}\hat{u}\|_{L^2}$$

which, by Plancherel theorem 2.7, equals to

$$(2\pi)^{-n/2} \|(1+|\xi|^2)^{-s/2}\|_{L^2} \|(1-\Delta)^{s/2} u\|_{L^2} \le C_s \|u\|_{H^s} \ ,$$

provided that $s > n/2$.

For $u \in H^s(\Omega)$, with C^1 domain Ω, we have similar results, see, e.g., Evans [7, 5.6.3].

PROPOSITION 7.5 (Sobolev inequality). *Let $0 < k < n/2$. Suppose $u \in H_0^k(\Omega)$, $\frac{n}{p*} = -k + \frac{n}{2}$, where $p*$ is Sobolev conjugate of p and $\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}$, then*

$$\|u\|_{L^p} \le C \|u\|_{H^k},$$

for any $2 \le p \le p*$.

Similar proof as in Proposition 7.4 could be employed to give a proof for $p < p*$. For example, $n = 3$, $p = 4$ and $k = 1$, we could prove as follows,

$$\|u\|_{L^4} \lesssim \|\hat{u}\|_{L^{4/3}} \lesssim \|(1+|\xi|)\hat{u}\|_{L^2} \|(1+|\xi|)^{-1}\|_{L^4} \lesssim \|(1+|\xi|)\hat{u}\|_{L^2} \lesssim \|u\|_{H^1} \ ,$$

where we have used the Hausdorff-Young inequality in Lemma 2.8.

For the critical $p*$, we could use the following

THEOREM 7.6 (Hardy-Littlewood-Sobolev Inequality). *Fix $s \in (0, n)$ and exponents $1 < p < q < \infty$ satisfying $n/q - n/p = s$. Then, with $I_s = D^{-s} = C_s J_{n-s}$, where $J_a f = \int_{\mathbb{R}^n} f(x-y)|y|^{-a}dy$, we have*

$$\|I_s f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \ .$$

For example, $n = 3$ and $p* = 6$, we have

$$\|D^{-1}f\|_{L^6} \lesssim \|f\|_{L^2}$$

Then for any $g \in C_0^\infty(\mathbb{R}^n)$, let $f \in L^2 \cap C_0 \subset \mathcal{S}'$ be such that $\hat{f} = |\xi|\hat{g} \in L^1 \cap L^\infty \subset L^2$, thus $\mathcal{F}(D^{-1}f)(\xi) = |\xi|^{-1}\hat{f}(\xi) = \hat{g}$ in L^1. We see that $D^{-1}f = g$, and so

$$\|g\|_{L^6} = \|D^{-1}f\|_{L^6} \lesssim \|f\|_{L^2} \simeq \|\hat{f}\|_{L^2} = \||\xi|\hat{g}\|_{L^2} \simeq \|\nabla g\|_{L^2} \ .$$

Alternatively, we could exploit the Littlewood-Paley theory and Bernstein inequality to conclude the Sobolev inequality.

7.3. Returning to the Dirichlet problem (7.4): well-posedness

Now we can make a well posedness statement:

THEOREM 7.7. *The PDE $\Delta_c u = f$ is well-posed for $u \in H_0^1$, $f \in L^2(\Omega)$.*

At first, in view of the Poincaré inequality, we know that H_0^1 is also the completion of C_c^∞ wrt

$$\|u\|_{\dot{H}^1} = \left(\int_\Omega c(x)^2|\nabla u(x)|^2 dx\right)^{1/2} ,$$

which is a Hilbert space with (real) inner product

$$\langle u, v\rangle_{H_0^1} = \int_\Omega c(x)^2 \nabla u(x) \cdot \nabla v(x) dx \ .$$

Recall that this arose from the divergence theorem: for $u \in C^2(\bar{\Omega})$, and $\phi \in \mathcal{D}(\Omega)$

$$\langle \Delta_c u, \phi\rangle_{L^2} = \int_{\partial\Omega} c^2 \nu \cdot \nabla u \phi dS - \langle c^2 \nabla u, \nabla\phi\rangle_{L^2} = -\langle u, \phi\rangle_{H_0^1} \ .$$

Thus to solve $\Delta_c u = f \in L^2(\Omega)$ in H_0^1, we need to find u such that

$$\langle u, \phi\rangle_{H_0^1} = -\langle f, \phi\rangle_{L^2}, \forall \phi \in \mathcal{D}(\Omega) \ .$$

Let $\ell(\phi) = -\langle f, \phi\rangle_{L^2}$ for $\phi \in \mathcal{D}(\Omega)$. By the Cauchy-Schwarz equality and the Poincaré inequality, we know that

$$|\ell(\phi)| \le \int |f\phi| dx \le \|f\|_{L^2}\|\phi\|_{L^2} \le C\|f\|_{L^2}\|\phi\|_{H_0^1} \ .$$

Thus, $\ell : H_0^1 \to \mathbb{R}$ is a continuous linear map.

By the Riesz representation theorem for Hilbert spaces, there exists a unique $u \in H_0^1$ such that

$$\ell(\phi) = \langle u, \phi\rangle_{H_0^1} \ ,$$

which tells us that for $\phi \in \mathcal{D}(\Omega)$,

$$\langle \Delta_c u, \phi\rangle_{\mathcal{D}', \mathcal{D}} = -\langle u, \phi\rangle_{H_0^1} = \langle f, \phi\rangle$$

i.e., it is a solution of (7.4) with vanishing trace.

Notice that for the weak solutions, we also have the energy estimates and so is the uniqueness and stability. Actually,

$$\|u\|_{H_0^1} = \sup_{\|\phi\|_{H_0^1}\le 1} \langle u, \phi\rangle_{H_0^1} = \sup_{\|\phi\|_{H_0^1}\le 1} -\langle f, \phi\rangle \le C\|f\|_{L^2} .$$

7.3.1. general Dirichlet problem. Consider the general Dirichlet problem:

$$\nabla \cdot (c^2(x)\nabla u) - q(x)u = F$$

with $u = \phi$ on $\partial\Omega$ (inhomogeneous Dirichlet conditions).

If we have an extension of ϕ in $H^1(\Omega)$, then we could formulate the weak solutions as follows

$$u - \phi \in H_0^1(\Omega), \nabla \cdot (c^2(x)\nabla u) - q(x)u = F, \text{in } \mathcal{D}'(\Omega) .$$

7.3.2. General case. Returning to the general problem (6.8):

$$\nabla \cdot (c^2(x)\nabla u) - q(x)u = F \tag{7.7}$$

with homogeneous Dirichlet/Neumann conditions.

Consider first the case of homogeneous Dirichlet conditions and we are seeking $H_0^1(\Omega)$ solutions.

THEOREM 7.8 (Dirichlet's problem). *Let $0 < c \in C^1(\bar\Omega)$ and $0 \le q \in C(\bar\Omega)$. The PDE $\Delta_c u - qu = f$ (with homogeneous Dirichlet boundary condition) is well-posed for $u \in H_0^1$, $f \in L^2(\Omega)$.*

In the special case of q and c constant, we have already shown hypoellipticity, so if $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$. With some additional work, for $q, c \in C^\infty(\bar\Omega)$, one can show that if $f \in C^\infty(\bar\Omega)$ then $u \in C^\infty(\bar\Omega)$, so this is in fact a classical solution of the PDE. Thus, even to obtain a classical solution we had to go through the machinery of weak solutions.

In fact, it is achieved by elliptic regularity (See, e.g., Evans [**7**, 6.3.2]) and Sobolev embedding (Theorem 7.4).

THEOREM 7.9 (Elliptic regularity). *Let $m \ge 0$, $c, q \in C^{m+1}(\bar\Omega)$, $f \in H^m(\Omega)$, $\partial\Omega \in C^{m+2}$, then the weak solution $u \in H_0^1$ to $\Delta_c u - qu = f$ is in $H^{m+2}(\Omega)$, and*

$$\|u\|_{H^{m+2}(\Omega)} \le C(m, \Omega, c, q)(\|u\|_{L^2(\Omega)} + \|f\|_{H^m(\Omega)})$$

We finish with a statement that for Neumann boundary conditions. Notice however that we need $q > 0$ since we do not have the Poincaré inequality.

THEOREM 7.10 (Neumann's problem). *Let $0 < c \in C^1(\bar\Omega)$ and $0 < q \in C(\bar\Omega)$. The PDE $\Delta_c u - qu = f$ (with homogeneous Neumann boundary condition) is well-posed for $u \in H^1$, $f \in L^2(\Omega)$. Here, the PDE holds in the sense that*

$$\langle \nabla u, c^2\nabla\phi\rangle_{L^2} + \langle u, q\phi\rangle_{L^2} = -\langle f, \phi\rangle_{L^2}$$

for any $\phi \in$ $\mathbf{C^\infty(\bar\Omega)}$.

Note that the boundary condition simply disappeared here, for H^1 functions it does not make sense to talk about their normal derivative at the boundary. However, we allow all $\phi \in C^\infty(\bar\Omega)$. So, if $u \in C^2(\bar\Omega)$, which holds if for instance $f \in C^\infty(\bar\Omega)$, the integration by parts gives us that

$$\langle f, \phi\rangle_{L^2} = -\int_\Omega c^2\nabla u \cdot \nabla\phi + qu\phi dx = -\int_{\partial\Omega} c^2\nu \cdot \nabla u\phi dS + \int_\Omega [\nabla \cdot (c^2\nabla u) + qu]\phi dx$$

for any $\phi \in C^\infty(\bar{\Omega})$. This shows that $\nu \cdot \nabla u$ vanishes at $\partial\Omega$. Thus, this should be considered a weak formulation of the Neumann boundary condition.

We could also state results for $\mathbb{R}^n$

THEOREM 7.11. *Let $c \in C^1(\mathbb{R}^n)$ with $c \in [c_1, c_2] \subset (0, \infty)$ and $q \in L^\infty(\mathbb{R}^n)$ with $q \in [q_1, q_2] \subset (0, \infty)$ The PDE $\Delta_c u - qu = f$ is well-posed for $u \in H^1$, $f \in L^2$.*

While we did not get to a discussion of other problems, such as the wave equation, similar methods apply.

7.4. Sample non-linear wave equations

7.4.1. Basic local existence theory for non-linear wave equations. Consider the Cauchy problem for the non-linear wave equation

$$\Box u = u^p \quad in (0, T) \times \mathbb{R}^3, \quad 1 < p \in \mathbb{N}. \tag{7.8}$$

$$u(0, x) = f, \quad u_t(0, x) = g. \tag{7.9}$$

THEOREM 7.12. *Suppose $f, g \in C_0^\infty(\mathbb{R}^3)$. If $1 < p \le 3$, then there exist $T > 0$, such that (7.8) (7.9) has a unique solution in $C([0, T]; H^1(\mathbb{R}^3)) \cap C^1([0, T]; L^2(\mathbb{R}^3))$.*

Proof. We define a closed subset of Banach space

$$E_T = \{u \in C([0, T]; H^1(\mathbb{R}^3)) \cap C^1([0, T]; L^2(\mathbb{R}^3)), u(0) = f, u_t(0) = g, \|u\|_{E_T} \le 2M\},$$

where $\|u\|_{E_T} = \|u\|_{L^\infty([0,T];H^1)} + \|u_t\|_{L^\infty([0,T];L^2)}$, $M = 2(\|f\|_{H^1} + \|g\|_{L^2})$.

Let $T \in (0, 1]$ to be determined, we define a map $\Gamma : v \to u$, satisfying

$$\Box u = v^p, \quad u(0, x) = f(x), \quad u_t(0, x) = g(x). \tag{7.10}$$

Hence, to show existence, we are reduced to show Γ is contraction map from E_T to E_T.

We first show $\Gamma : E_T \to E_T$.

For any $v \in E_T$. Applying energy estimate (6.2) to equation (7.10), we have

$$\|\partial \Gamma v\|_{L_t^\infty L^2} \le \|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|v^p\|_{L^1([0,T];L^2)}$$

$$\begin{aligned}\|\Gamma v\|_{E_T} = \|u\|_{E_T} &\le (1 + T)(\|f\|_{H^1} + \|g\|_{L^2} + \|v^p\|_{L^1([0,T];L^2)}) \\ &\le M + 2T\|v\|^p_{L^\infty([0,T];L_x^{2p})}\end{aligned}$$

by Sobolev embedding $\|v\|_{L^{2p}} \le C(\|v\|_{L^2} + \|v\|_{L^6}) \le C\|v\|_{H^1}$, as $1 < p \le 3$. Thus, $\|\Gamma v\|_{E_T} \le 2M$ if we choose $T \le \frac{1}{2(2C)^p M^{p-1}}$.

$\Gamma : E_T \to E_T$ is a contraction.

For any $v_1, v_2 \in E_T$, with

$$\Box u_1 = v_1^p, \quad \Box u_2 = v_2^p, \quad u_1(0) = u_2(0) = f, \quad \partial_t u_1(0) = \partial_t u_2(0) = g.$$

we have

$$\Box(u_1 - u_2) = v_1^p - v_2^p, \quad (u_1 - u_2)(0) = 0, \quad \partial_t(u_1 - u_2)(0) = 0. \tag{7.11}$$

Applying energy estimate (6.2) to equation (7.11), we have

$$\|\Gamma(v_1 - v_2)\|_{E_T} = \|u_1 - u_2\|_{E_T} \le 2\|v_1^p - v_2^p\|_{L^1([0,T];L^2)},$$

by Hölder's equality and Sobolev embedding

$$\begin{aligned}\|v_1^p - v_2^p\|_{L^2} &\le C_0\|v_1 - v_2\|_{L^{2p}}\|(v_1, v_2)\|^{p-1}_{L^{2p}} \\ &\le C_1\|v_1 - v_2\|_{H^1}(\|v_1\|^{p-1}_{H^1} + \|v_2\|^{p-1}_{H^1})\end{aligned}$$

If we choose $T \le \frac{1}{8C_1(2M)^{p-1}}$, then

$$\|\Gamma(v_1 - v_2)\|_{E_T} \le \frac{1}{2}\|v_1 - v_2\|_{E_T}$$

Hence we show Γ is a contraction map from E_T to E_T if $T \le \min\{1, \frac{1}{2(2C)^p M^{p-1}}, \frac{1}{8C_1(2M)^{p-1}}\}$. Thus, (7.8) (7.9) has a solution $u \in C([0,T]; H^1(\mathbb{R}^3)) \cap C^1([0,T]; L^2(\mathbb{R}^3))$.

The uniqueness and continuous dependence follows similarly, as that for the ODE. ∎

Actually, by Sobolev embedding, Proposition 7.4, we could prove LWP in H^s for $s > n/2$ for

$$\Box u = F(u)$$

with sufficiently regular F. See, e.g., Evans [**7**, Section 12.2.2, Section 12.3.2].

7.4.2. global existence. $\Box u = -u^3$, $x \in \mathbb{R}^3$ in H^1

Hamiltonian+LWP

Actually, for all $p \in (1,5)$ and $n = 3$, we have global well-posedness and regularity for $\Box u = -|u|^{p-1}u$. See, e.g., Evans [**7**, Section 12.3.3]. The critical case $p = 5$ is also admissible, known from the end of 1980's to 1990's (Struwe, Grillakis, Shatah-Struwe, et al), see Evans [**7**, Section 12.4] or Sogge [**25**].

7.4.3. blow up for large data and global existence with small data. $\Box u = u^3$, $x \in \mathbb{R}^3$ in H^1

ODE argument+finite seed of propagation; LWP+Hamiltonian implies global existence with small data, as $H^1 \subset L^6$.

7.5. Long time existence for solutions with small data

Let $p > 1$, consider

$$\Box u = |u|^{p-1}u \quad in(0,T) \times \mathbb{R}^3, u(0,x) = f, \quad u_t(0,x) = g. \tag{7.12}$$

When the data are sufficiently regular, small, with compactly support, we would like to determine the maximal range of $p > 1$ so that we have global existence with small data.

As we have seen, we have global existence for $p = 3$. It is natural to expect similar result for $p > 3$. However, the similar argument is not sufficient to conclude this seemingly trivial result.

7.5.1. Strichartz estimates and non-linear wave equations. Alternatively, if we use Theorem 5.24 L^4_{tx} Strichartz estimates: local well-posedness and global existence with small data.

Based on this, together with the Sobolev embedding, Theorem 7.4, it is not so difficult to conclude global existence, for $p \ge 3$ and small $H^2 \times H^1$ data.

Moreover, if we use the fact (5.29), as well as the Harmse-Oberlin inequality

$$\|u\|_{L^q} \lesssim \|F\|_{L^\lambda}, q, \lambda' > 3, 1/\lambda - 1/q = 1/2,$$

we have similar result for $p > 2.5$, with $q = p\lambda$, $q = 3+$ (such that $\lambda = 6/5-$).

By using further delicate space-time estimates, it is possible to improve it further to $p > 1 + \sqrt{2}$. The space-time estimates could be the pointwise estimates

$$\langle t+r\rangle\langle t-r\rangle^{p-2}|u| \lesssim C(f,g) + \|\langle t+r\rangle^p\langle t-r\rangle^{p(p-2)}\Box u\|_{L^\infty},\ p \in (1+\sqrt{2}, 3)\ ,$$

or the (radius) weighted Strichartz estimates

$$\|r^{-\alpha}u\|_{L^p_{t,r}L^2_\omega}\lesssim C(f,g)+\|r^{-\beta}\Box u\|_{L^1_{t,r}L^2_\omega}\ ,$$

or the $(t^2-|x|^2)$ weighted Strichartz estimates

$$\|(t^2-r^2)^{-\alpha}u\|_{L^p_{t,r}L^2_\omega}\lesssim C(f,g)+\|(t^2-r^2)^{-\beta}\Box u\|_{L^1_{t,r}L^2_\omega}\ .$$

See [**31**] for more discussion, history and references.

On the other hand, it is known that there are no global solutions, in general, if $1<p<1+\sqrt{2}$, see John [**15**].

For the critical case, Schaeffer [**21**] showed that there is no global solution in general.

7.5.2. blow up. To reduce it to ODE, we want to remove the term Δu.

To achieve this purpose, we exploit the finite speed of propagation.

Let $(f,g)\in C_0^\infty(\mathbb{R}^n)$, nonnegative with support in $B_R(0)$, then for any $t>0$, $u(t)\geq 0$ (so that $|u|^{p-1}u=|u|^p$),

$$\text{supp } u(x,t)\subset\{|x|\leq t+R\}\ , \tag{7.13}$$

and

$$\int_{\mathbb{R}^n}\Delta u(t,x)dx=0\ .$$

Thus if we set

$$F(t)=\int_{\mathbb{R}^n}u(x,t)dx\ ,$$

$$F''=\int|u|^{p-1}udx=\int_{|x|\leq t+R}|u|^pdx\geq 0\ . \tag{7.14}$$

Choosing data such that $F(0)\geq\varepsilon$, $F'(0)\geq\varepsilon$, we see that for all $t>0$, $F'(t)\geq\varepsilon$,

$$F(t)\geq\varepsilon(1+t)\ . \tag{7.15}$$

To connect (7.14) with $F(t)$, we use Hölder's inequality to get

$$F=\int u\chi_{r\leq t+R}dx\leq\|\chi_{r\leq t+R}\|_{p'}\|u\|_p\leq C(t+R)^{n(p-1)/p}\|u\|_p\ .$$

$$F''\geq cF^p(t+R)^{-n(p-1)}\ . \tag{7.16}$$

By(7.15) and (7.16), we see that there are no global solutions if $(n-1)(p-1)<2$.

Actually, if $(n-1)(p-1)<2$, we have $p-n(p-1)>-1$, plugging (7.15) into (7.16), we find that we could improve (7.15) to

$$F''\geq c_1(t+R)^{p-n(p-1)}, F'\geq c_2(t+R)^{1+p-n(p-1)}, F\geq c_3(t+R)^{2+p-n(p-1)}\ . \tag{7.17}$$

By iteration, we have

$$F\geq c_4(t+R)^{2n}\ .$$

By (7.16),

$$F''\geq cF^p(t+R)^{-n(p-1)}\geq cF^{(p+1)/2}F^{(p-1)/2}(t+R)^{-n(p-1)}\geq cc_4^{(p-1)/2}F^{(p+1)/2}\ . \tag{7.18}$$

As $(p+1)/2>1$, it is clear that there is no global solutions for this ordinary differential inequality.[1].

[1]This is essentially the Kato type lemma. See, e.g., Sideris [**23**], Takamura [**27**].

7.5.3. Further result. As we have $n = 3$, we have seen the result for $p < 2$. To improve it further, we exploit more properties of the wave equations.

Let w be the solution to $\Box w = 0$ with same data, then

$$u \geq w, \ \mathrm{supp} w \subset \{t - R \leq |x| \leq t + R\} \equiv A \ .$$

Based on this, we see that $\frac{d^2}{dt^2} \int w(t, x) dx = 0$,

$$\varepsilon(t + R) \lesssim F(0) + F'(0)t = \int w dx = \int \chi_A w dx \leq \int \chi_A u dx \lesssim \|u\|_p (t + R)^{2(p-1)/p} \ .$$

Thus

$$F'' = \int |u|^p \geq c\varepsilon^p (t + R)^{p-2(p-1)} = c\varepsilon^p (t + R)^{2-p},$$

$$F \geq c\varepsilon^p (t + R)^{4-p} \ . \tag{7.19}$$

We notice that it improved (7.15) only if $p < 3$. Based on this, by Kato type lemma, we see blow up results, provided that [2]

$$(4 - p)p - 3(p - 1) + 2 > 4 - p \Leftrightarrow (p - 1)^2 < 2 \Leftrightarrow p < 1 + \sqrt{2}$$

7.6. Quadratic wave equations: existence vs blow up

almost global existence with small data $\Box u = Q(u_t, \nabla u)$: $T_\varepsilon \geq \exp(c\varepsilon^{-1})$.

7.6.1. Quadratic wave equations: blow up with small data. The lower bound is sharp in general. Actually, considering

$$\Box u = u_t^2$$

with nontrivial nonnegative data of size ε, we know that

$$T_\varepsilon \leq \exp(C\varepsilon^{-1})$$

for some $C > 0$.

7.6.2. Example of global existence: nonlinear structure. When certain nonlinear structure is satisfied by the equation, it is still possible to have global existence. For example:

$$\Box u = u_t^2 - |\nabla u|^2 \ .$$

[2] We essentially follow Glassey [11], see also Evans [7, 12.5.2].

7.7. Homeworks

Section 6.2

(1) Considering (uniformly) elliptic operator with (continuous) variable coefficients $L = \sum_{j,k} a^{jk}(x)\partial_j\partial_k + \sum_j b^j(x)\partial_j$. Here, the conditions on the coefficients ensures that there exists $\mu > 0$, such that $(a^{jk} - \mu\delta^{jk}) \geq 0$, and $|a^{jk}| + |b^j| \leq \mu^{-1}$. Try to formulate and prove the corresponding weak maximum principle in Ω. (Hint: you may want to try to construct $\phi = e^{\lambda x_1}$ with $1 \ll \lambda$)

(2) Considering (uniformly) elliptic operator with (continuous) variable coefficients $L = \sum_{j,k} a^{jk}(x)\partial_j\partial_k + \sum_j b^j(x)\partial_j + c(x)$. Here, the conditions on the coefficients ensures that there exists $\mu > 0$, such that $(a^{jk} - \mu\delta^{jk}) \geq 0$, and $|a^{jk}| + |b^j| + |c| \leq \mu^{-1}$. As we see from the example, $u'' + u = 0$, we need to add some condition on the zeroth order term c so that we have the maximum principle. Actually, we could prove

THEOREM 7.13 (Weak maximum principle, v2). *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $Lu \geq 0 \geq c(x)$ in Ω, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$, where $u^+(x) = \max(u(x), 0)$.*

Try to prove this weak maximum principle.

(3) Let Ω be an open bounded smooth domain in $\mathbb{R}^n$, c be nonnegative smooth functions. In addition, we assume that $\delta < c(x) < \delta^{-1}$ for any $x \in \Omega$, for some $\delta > 0$. Consider

$$u_t - \nabla \cdot (c^2(x)\nabla u) = 0 \tag{7.20}$$

Try to formulate and prove the corresponding weak maximum principle.

Section 6.3

(1) Let Ω be an open bounded smooth domain in $\mathbb{R}^n$, $\delta < c(x) < \delta^{-1}$ for any $x \in \Omega$. Consider

$$u_t - \nabla \cdot (c^2(x)\nabla u) = F \tag{7.21}$$

with Neumann boundary conditions $\partial_\nu u = f$, prove uniqueness for the solutions.

(2) Prove uniqueness for (6.8) with $q \geq 0$ and $c \geq 1$.

Section 7.1-7.3

(1) Prove Proposition 7.5 for $2 \leq p < p*$.
(2) Prove Theorem 7.10 in detail.
(3) Consider the PDE $\sum_{1\leq j,k\leq n} \partial_j(a^{jk}(x)\partial_k u) = qu + f$, on a bounded region Ω where $q \geq 0$ and $A(x) = (a^{jk}(x))$ is symmetric, positive definite. Assume Dirichlet boundary conditions. Show that for $f \in L^2(\Omega)$ the PDE is well-posed for $u \in H^1(\Omega)$.

Section 7.4

(1) Prove the uniqueness result for Theorem 7.12.
(2) Prove the existence result (in $CH^2 \cap C^1H^1$) for $\Box u = u^5$, similar to Theorem 7.12.

CHAPTER 8

Additional material, Nonlinear phenomenon

8.1. Dirichlet's principle (variational approach, calculus of variation)

Two situations, $g = 0$ and $f = 0$ are basically equivalent. WLOG, $f = 0$.

$$\Delta u = 0, x \in \Omega, u = g, x \in \partial\Omega \tag{8.1}$$

We define the energy functional

$$I[w] := \int_\Omega |\nabla w|^2 dx,$$

where w belonging to the admissible set

$$\mathcal{A} := \{w \in C^2(\bar{\Omega}), w = g \textit{ on } \partial\Omega\}.$$

Theorem 8.1 (Dirichlet's principle). *Assume $u \in C^2(\bar{\Omega})$ solves (8.1). Then*

$$I[u] = \min_{w \in \mathcal{A}} I[w] \tag{8.2}$$

Conversely, if $u \in \mathcal{A}$ satisfies (8.2), then u solves the boundary-value problem (8.1).

Proof. 1. For any $w \in \mathcal{A}$. Then (8.1) implies

$$0 = \int_\Omega \Delta u(u - w) dx.$$

An integration by parts yields

$$0 = -\int_\Omega \nabla u \cdot \nabla(u - w) dx,$$

and there is no boundary term since $u - w = g - g \equiv 0$ on $\partial\Omega$. Hence

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u \cdot \nabla w dx \le \int_\Omega \frac{1}{2}|\nabla u|^2 dx + \int_\Omega \frac{1}{2}|\nabla w|^2 dx$$

we conclude

$$I[u] \le I[w] \quad (w \in \mathcal{A}) \tag{8.3}$$

Since $u \in \mathcal{A}$, (8.2) follows from (8.3).

2. Now, conversely, suppose (8.2) holds. Fix any $v \in C_c^\infty(\Omega)$ and write

$$i(\tau) := I[u + \tau v] \quad \tau \in \mathbb{R}.$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\tau)$ has a minimum at zero, and thus

$$i'(0) = 0,$$

provided this derivative exists. But

$$i(\tau) = \int_\Omega |\nabla u + \tau \nabla v|^2 dx = \int_\Omega |\nabla u|^2 + 2\tau \nabla u \cdot \nabla v + \tau^2 |\nabla v|^2 dx \in C_\tau^\infty .$$

Consequently

$$0 = i'(0) = 2\int_\Omega \nabla u \cdot \nabla v dx = -2\langle \Delta u, v\rangle.$$

This identity is valid for each function $v \in C_c^\infty(\Omega)$ and so $\Delta u = 0$ in $\mathcal{D}'(\Omega)$. ■

Inspired by Dirichlet's principle, try to construct a minimization sequences, reduction to convergence issue (limit exist? in $\mathcal{A}$?) and regularity issue, provided $K < \infty$.

Natural: $K = \inf_{w\in\mathcal{A}} I[w]$, and find $u_j \in \mathcal{A}$ with $I[u_j] \to K$.

As $v \to |v|^2$ is convex, we have

LEMMA 8.2 (Convexity). *We have*

$$I[tu + (1-t)v] \le tI[u] + (1-t)I[v], \forall t \in [0,1]$$

Thus $K \le I[(u_j + u_k)/2] \le \frac{1}{2}(I[u_j] + I[u_k]) \to K$,

$$I[u_j - u_k] = 2(I[u_j] + I[u_k]) - I[u_j + u_k] \to 2K + 2K - 4K = 0$$

that is ∇u_j is Cauchy in L^2, which could be viewed as one of the motivations to introduce Sobolev spaces.

8.1.1. Crisis for the Dirichlet's principle: examples.

REMARK 8.3. *Weierstrass argued that even if the functional E is bounded from below, it is possible that the infimum is never attained by an admissible function, in which case there would be no admissible function that minimizes the energy. He backed his reasoning by an explicit example of an energy that has no minimizer. Let $I = (-1,1)$ with $E(u) = \int_I x^2(u')^2 dx$ for all $u \in C(I)$ with piecewise continuous derivatives in I, satisfying the boundary conditions $u(-1) = 0$ and $u(1) = 1$.*

In 1871, Prym constructed a striking example of a continuous function g on the boundary of a disk, such that there is not a single function u with finite energy that equals g on the boundary. This makes it impossible even to talk about a minimizer since all functions with the correct boundary condition would have infinite energy. Here is a variation of Friedrich Prym's example from 1871.

EXAMPLE 8.1 (Hadamard 1906). *Let $D = B_1 \subset \mathbb{R}^2$, and consider the harmonic function*

$$u(x) = \sum_{k\ge 1} k^{-2} r^{k!} \sin(k!\theta) \in C(\bar{D}) \ .$$

As $\partial_r u(x) = \sum_{k\ge1} k^{-2} k! r^{k!-1}\sin(k!\theta)$, we have

$$E(u) = \int_D |\nabla u|^2 dx \ge \int_D |\partial_r u|^2 dx = \sum_{k\ge 1}\int_0^1 (k^{-2}k! r^{k!-1})^2 \pi r dr \ge \sum_{k=1}^m \frac{k!\pi}{2k^4}\rho^{2k!}$$

for any $\rho < 1$ and $m \in \mathbb{Z}_+$. This implies that $E(u) = \infty$. To conclude, there exists a Dirichlet datum $g \in C(\partial D)$ for which the Dirichlet problem is perfectly solvable, but the solution cannot be obtained by minimizing the Dirichlet energy. There is no full equivalence between the Dirichlet problem and the minimization problem.

8.1.2. Weak solution: definition. Weak solution for (8.1) in $H^1(\Omega)$, with given $g \in H^1(\Omega)$: if

$$0 = \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall\, v \in C_c^\infty(\Omega),$$

and $u - g \in H_0^1(\Omega)$.

8.1.3. Weak solutions: existence. Inspired by the Dirichlet's principle, we let

$$\mathcal{A}' := \{w \in H^1(\Omega) | w = g \textit{ on } \partial\Omega \textit{ in the trace sence}\}$$

to denote this class of admissible functions w.

THEOREM 8.4. *Suppose $u \in \mathcal{A}'$ satisfies*

$$I[u] = \min_{w \in \mathcal{A}'} I[w].$$

Then u is a weak solution of (7.1).

Proof. Fix any $v \in C_c^\infty(\Omega)$ and set

$$i(\tau) := I[u + \tau v] \quad \tau \ \in \ \mathbb{R}$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\tau)$ has a minimum at zero. But

$$i(\tau) = \frac{1}{2}\int_\Omega |\nabla u + \tau \nabla v|^2 dx$$

Consequently

$$0 = i'(0) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall\, v \in C_c^\infty(\Omega),$$

which gives us that $\Delta u = 0$ in $\mathcal{D}'(\Omega)$. ■

THEOREM 8.5 (Existence of minimizer). *There exists $u \in \mathcal{A}'$ solving*

$$I[u] = \min_{w \in \mathcal{A}'} I[w].$$

Proof. As we have seen, for a minimizing sequence $\{u_k\}_{k=1}^\infty$, we have ∇u_k Cauchy in L^2.

As we have $u_k - g \in H_0^1(\Omega)$, by Poincare inequality

$$\|u_k\|_{L^2(\Omega)} \le \|u_k - g + g\|_{L^2(\Omega)} \le \|u_k - g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \le \|\nabla u_k - \nabla g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}.$$

Then $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$.

Consequently there exist a subsequence $\{u_{k_j}\}_{k=1}^\infty \subset \{u_k\}_{k=1}^\infty$ and a function $u \in H^1(\Omega)$ such that

$$u_{k_j} \rightharpoonup u \quad \textit{weakly in } H^1(\Omega).$$

To show this, we need only to see that $u_{k_j} \rightharpoonup u$, $\partial_l u_{k_j} \rightharpoonup w_l$ in L^2, and then $w_l = \partial_l u$ in $\mathcal{D}'(\Omega)$.

It remains to show that $u \in \mathcal{A}'$. To see this, note that $g \in \mathcal{A}'$ as above, $u_k - g \in H_0^1(\Omega)$. Now $H_0^1(\Omega)$ is a closed, linear subspace of $H^1(\Omega)$ and so is weakly closed. Hence $u - g \in H_0^1(\Omega)$, the trace of u on $\partial\Omega$ is g. ■

8.2. general Dirichlet problem: existence

8.2.1. Weak solution: definition. Weak solution for (7.1) in $H^1(\Omega)$, with given $g \in H^1(\Omega)$, $f \in L^2(\Omega)$.

Definition. Suppose $u \in H^1(\Omega)$ and $u = g$ on $\partial\Omega$ (i.e., $u - g \in H_0^1(\Omega)$) , such that

$$\int_\Omega \nabla u \cdot \nabla v + fvdx = 0, \quad \forall\ v\ \in H_0^1(\Omega). \tag{8.4}$$

Then u is the weak solution of (7.1).

8.2.2. Weak solutions: existence.

We define the energy functional for equation (7.1).

$$I[u] = \int_\Omega \frac{|\nabla u|^2}{2} + fudx$$

and we write

$$\mathcal{A} := \{w \in H^1(\Omega)|w = g \textit{ on } \partial\Omega \textit{ in the trace sence}\}$$

to denote this class od admissible functions w and assume $f \in L^2(\Omega)$.

THEOREM 8.6. *Suppose $u \in \mathcal{A}$ satisfies*

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Then u is a weak solution of (7.1).

Proof. Fix any $v \in C_c^\infty(\Omega)$ and set

$$i(\tau) := I[u + \tau v] \quad \tau\ \in\ \mathbb{R}$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\tau)$ has a minimum at zero, and thus

$$i'(0) = 0,$$

provided this derivative exists. But

$$i(\tau) = \frac{1}{2}\int_\Omega |\nabla u + \tau \nabla v|^2 + fu + f\tau vdx$$

Consequently

$$0 = i'(0) = \int_\Omega \nabla u \cdot \nabla v + fvdx, \quad \forall\ v \in C_c^\infty(\Omega).$$

Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, u is a weak solution of (7.1) by definition. ■

THEOREM 8.7. *(Existence of minimizer) There exists $u \in \mathcal{A}$ solving*

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Proof. Set $m = \inf_{w \in \mathcal{A}} I[w]$. As $g \in H^1$, $g \in \mathcal{A}$ and m is finite. Select a minimizing sequence $\{u_k\}_{k=1}^\infty$. Then

$$I[u_k] \to m.$$

Then we claim that

$$\sup_k \|\nabla u_k\|_{L^2(\Omega)} < \infty.$$

Actually, fix any $w \in \mathcal{A}$, e.g., $w = g$, we have $u_k - w \in H_0^1(\Omega)$, by Poincare inequality

$$\begin{aligned}\|u_k\|_{L^2(\Omega)} \le \|u_k - w + w\|_{L^2(\Omega)} &\le \|u_k - w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \\ &\le \|\nabla u_k - \nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}.\end{aligned}$$

Then

$$2I[u] = \int_\Omega |\nabla(u-w+w)|^2 + 2fudx \ge \int_\Omega |\nabla(u-w)|^2 + 2\nabla(u-w)\nabla w dx - 2\|f\|\|u\|$$

$$\ge \|\nabla(u-w)\|^2 - 2\|\nabla(u-w)\|\|\nabla w\| - 2\|f\|\|\nabla(u-w)\| - 2\|f\|\|w\| \ge (\|\nabla(u-w)\| - \|f\| - \|\nabla w\|)^2 - C$$

Then $\sup \|\nabla(u_k - w)\| < \infty$ and so $\sup_k \|u_k\|_{L^2(\Omega)} < \infty$. Hence $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$.

Consequently there exist a subsequence $\{u_{k_j}\}_{k=1}^\infty \subset \{u_k\}_{k=1}^\infty$ and a function $u \in H^1(\Omega)$ such that

$$u_{k_j} \rightharpoonup u \quad \textit{weakly in } H^1(\Omega).$$

To show this, we need only to see that $u_{k_j} \rightharpoonup u$, $\partial_l u_{k_j} \rightharpoonup w_l$ in L^2, and then $w_l = \partial_l u$ in $\mathcal{D}'(\Omega)$.

We assert next that $u \in \mathcal{A}$. To see this, note that $g \in \mathcal{A}$ as above, $u_k - g \in H_0^1(\Omega)$. Now $H_0^1(\Omega)$ is a closed, linear subspace of $H^1(\Omega)$ and so is weakly closed. Hence $u - g \in H_0^1(\Omega)$, the trace of u on $\partial\Omega$ is g.

Finally, since $u \in \mathcal{A}$ and

$$I[u] \le \liminf_{j\to\infty} I[u_{k_j}] = m,$$

it follows that $I[u] = m = \min_{w\in\mathcal{A}} I[w]$. ■

8.2.3. Weak solutions: regularity. Sobolev embedding and regularity of weak solutions

8.3. Green's function

Green's identities:

$$\int_{\partial\Omega} u\partial_\nu v dS = \int_{\partial\Omega} \nu\cdot(u\nabla v)dS = \int_\Omega \nabla\cdot(u\nabla v)dx = \int_\Omega \nabla u\cdot\nabla v + u\Delta v dx$$

$$\int_\Omega u\Delta v - v\Delta u dx = \int_{\partial\Omega} u\partial_\nu v - v\partial_\nu u dS$$

Let $v = K_\xi(x) = N(x-\xi)$ be fundamental solution at $\xi \in \Omega$, we obtain

$$u(\xi) = \int_\Omega K_\xi \Delta u dx + \int_{\partial\Omega} u\partial_\nu K_\xi - K_\xi \partial_\nu u dS \tag{8.5}$$

which seems to be a representation formula for the IVP for Poisson's equation

$$\Delta u = f, x \in \Omega, \;\; u = g, \partial_\nu u = h, x \in \partial\Omega \tag{8.6}$$

However, as we know, the IVP (8.6) is not well-posed in general. For example, WMP gives uniqueness for (7.1). If well-posed, then u is determined by f, g, and so $h = \partial_\nu u$ is determined by f, g and can not be freely chosen.

Thus, to make (8.5) useful, we need to avoid the occurrence of the term involving $\partial_\nu u$.

Definition 8.1. *If there is a function $G(x,y) : \Omega \times \bar{\Omega} \to \mathbb{R}$, such that*

$$\Delta_y G(x,y) = \delta_x(y), \quad \forall x, y \in \Omega, \; G(x,y) = 0, \quad \forall x \in \Omega, \quad y \in \partial\Omega.$$

Then $G(x,y)$ is the Green's function for the region Ω. (which is basically a fundamental solution, at x, for the Dirichlet problem)

If exist, must be unique. Actually, for any fundamental solution $K \in \mathcal{D}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash 0)$, it is equivalent to find solution to $w = G(x,y) - K(x-y)$

$$\Delta_y w = 0, w = -K(x-y), y \in \partial\Omega$$

which solution is unique (by, e.g., WMP).

Question: existence?

We claim that

Theorem 8.8. *For any bounded domain Ω with smooth boundary $\partial\Omega$, there exists a Green's function. Moreover, for each $x \in \Omega$, $G(x,\cdot) \in C^\infty(\bar{\Omega} \backslash \{x\})$.*

See, e.g., Folland [**9**, §7H].

Lemma 8.9 (Symmetry of Green's function). *The Green's function satisfies $G(x,y) = G(y,x)$, for any $x, y \in \Omega$.*

Proof. Formally, for fixed $x \neq y$, let $u(z) = G(x,z) \in C^\infty(\bar{\Omega} \backslash \{x\})$, $w(z) = G(y,z) \in C^\infty(\bar{\Omega} \backslash \{y\})$, and so both $\langle \delta_y, u \rangle$, $\langle \delta_x, w \rangle$ are well-defined. Then we have

$$u(y) - w(x) = \langle \delta_y, u \rangle - \langle \delta_x, w \rangle = \langle \Delta w, u \rangle - \langle \Delta u, w \rangle = \int_{\partial\Omega} (u \partial_\nu w - w \partial_\nu u) dS = 0$$

in view of the Green's identity. ∎

Because of this symmetry, G may be extended naturally to $\bar{\Omega} \times \bar{\Omega}$ by setting $G(x,y) = 0$ for $x \in \partial\Omega$. Also, $G(\cdot,y) - N(\cdot,y)$ is a harmonic function on Ω for each y.

In view of (8.5) with the Green's function G instead of the fundamental solution $K = N$, we could write down the solution formula, for (7.1).

For the Poisson equation (7.3), we have

$$u(x) = \int_\Omega G(x,y) f(y) dy \tag{8.7}$$

which satisfies $v(x) = 0$ on $\partial\Omega$.

For the Laplace equation (7.2), we have

$$u(x) = \int_{y \in \partial\Omega} g(y) \partial_{\nu_y} G(x,y) dS \; . \tag{8.8}$$

We claim that when $g \in C(\partial\Omega)$, the function u given in (8.8) is harmonic and extend continuously to $\bar{\Omega}$ with $u = g$ on $\partial\Omega$.

Definition 8.2 (Poisson integral formula). *The function $\partial_{\nu_y} G(x,y)$ on $\Omega \times \partial\Omega$ is called the Poisson kernel for Ω, and (8.8) is called the Poisson integral formula for the solution of the Dirichlet problem.*

By symmetric consideration, for example, we could obtain the Green's function for half-spaces $\mathbb{R}^n_+$ and balls.

8.3.1. Half-space $\mathbb{R}^n_+$, $n \geq 2$. Recall the Newtonian potential $N(X)$ is given in (4.11):

$$N(X) = \begin{cases} \frac{|X|^{2-n}}{(2-n)|\mathbb{S}^{n-1}|} & n \geq 3 \\ -\frac{1}{4\pi r} & n = 3 \\ \frac{\ln |X|}{2\pi} & n = 2 \end{cases}$$

with $|\mathbb{S}^{n-1}| = w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, which satisfies $\Delta N(X) = \delta(X)$ and so $\Delta_Y N(Y-X) = \delta_X(Y)$.

For $X = (x,t), Y = (y,s) \in \mathbb{R}^n_+$, we would like to construct the Green's function

$$G(X,Y) = N(Y-X) + M(X,Y),$$

which is equivalent to find a solution of

$$\Delta_Y M(X,Y) = 0, Y \in \mathbb{R}^n_+, M(X,Y) = -N(Y-X), Y = (y,0) \in \partial\mathbb{R}^n_+ \ .$$

Here, by geometric consideration, we see M could be

$$M(X,Y) = N((y,s),(x,-t)) \ .$$

In conclusion, we find a Green's function:

$$G((x,t),(y,s)) = N((y,s)-(x,t)) - N((y,s)-(x,-t)).$$

When $n \geq 3$, we have

$$-\partial_s |((y,s)-(x,t))|^{2-n} = (n-2)|((y,s)-(x,t))|^{-n}(s-t),$$

$$-\partial_s G((x,t),(y,s)) = \frac{(n-2)}{(2-n)|\mathbb{S}^{n-1}|}(|((y,s)-(x,t))|^{-n}(s-t) - |((y,s)-(x,-t))|^{-n}(s+t))$$

and so, with $s = 0$,

$$\partial_{\nu_Y} G(X,Y)|_{Y\in\partial\Omega} = \frac{2t}{|\mathbb{S}^{n-1}||((y,0)-(x,t))|^n} = \frac{2t}{|\mathbb{S}^{n-1}||(x-y,t)|^n} := P_t(x-y) \ .$$

It is easy to check that this formula applies also for $n = 2$. Thus we obtain a harmonic function on $\mathbb{R}^n_+$ with prescribed boundary data g

$$u(x,t) = \frac{2t}{|\mathbb{S}^{n-1}|}\int_{\mathbb{R}^{n-1}} \frac{g(y)}{|(x-y,t)|^n} dy = (g * P_t)(x) \tag{8.9}$$

where

$$P_t(x) = \frac{2t}{|\mathbb{S}^{n-1}|}(|x|^2+t^2)^{-n/2} = \frac{\Gamma(n/2)t}{\pi^{n/2}(|x|^2+t^2)^{n/2}} = t^{-(n-1)}P(t^{-1}x)$$

with $x \in \mathbb{R}^{n-1}$. With the help of the beta integral, we see that

$$\int_{\mathbb{R}^{n-1}} P_1(x)dx = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\int_0^\infty \frac{2r^{n-2}}{(r^2+1)^{n/2}}dr = \frac{2\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)}\int_0^\infty \frac{r^{n-2}}{(r^2+1)^{n/2}}dr$$

with $s = r^2/(r^2+1) = 1 - 1/(r^2+1)$, $ds = 2(1+r^2)^{-2}rdr$, we see that

$$\int_{\mathbb{R}^{n-1}} P_1(x)dx = \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)}\int_0^1 s^{(n-1)/2-1}(1-s)^{1/2-1}ds = 1 \ .$$

THEOREM 8.10. *Let $g \in L^p(\mathbb{R}^{n-1})$ with $1 \leq p \leq \infty$. Then $u(x,t) = g * P_t$ is well-defined and harmonic on $\mathbb{R}^n_+$. When $g \in C \cap L^\infty$, u is continuous on $\overline{\mathbb{R}^n_+}$ and $u(x,0) = g(x)$. When $p < \infty$, we have $u(x,t) \to g(x)$ in L^p as $t \to 0+$.*

8.3.2. Ball. Considering the unit ball $B_1 \subset \mathbb{R}^n$, we use similar strategy to find, for fixed $x \in B_1 \backslash \{0\}$,

$$G(x,y) = N(x,y) - CN(x^*,y)$$

with $x^* = |x|^{-2}x$. Then for $y \in \partial B_1 = \mathbb{S}^{n-1}$, we get

$$G(x,y) = \frac{1}{(2-n)\omega_n}(|x-y|^{2-n} - C|x^*-y|^{2-n}) \ .$$

By geometry, we see that $\Delta(Oxy) \sim \Delta(Oyx^*)$, and so

$$|x-y| = \frac{|x-y|}{|y|} = \frac{|x^*-y|}{|x^*|} = |x||x^*-y| = ||x|^{-1}x - |x|y| \ , \tag{8.10}$$

to make G vanish for $y \in \mathbb{S}^{n-1}$ we naturally take $C = |x|^{2-n}$.

In conclusion, the Green's functions for $B_1 \subset \mathbb{R}^n$ is given by

(8.11)

$$G(x,y) = \frac{1}{(2-n)\omega_n}(|x-y|^{2-n} - |x|^{2-n}|x^*-y|^{2-n}) = \frac{1}{(2-n)\omega_n}(|x-y|^{2-n} - ||x|^{-1}x - |x|y|^{2-n})$$

for $x \in B_1\backslash\{0\}$ and $y \in \bar{B}_1$. It also makes clear how to define G for the case $x = 0$:

$$G(0,y) = \frac{1}{(2-n)\omega_n}(|y|^{2-n} - 1) \ .$$

When $n = 2$, the analogous formula is

$$G(x,y) = \frac{1}{2\pi}\begin{cases} (\ln|x-y| - \ln||x|^{-1}x - |x|y|) & x \neq 0 \\ \ln|y| & x = 0 \end{cases}$$

Now that we know the Green's function, we can compute the Poisson kernel

$$P(x,y) = \partial_{\nu_y}G(x,y) = y \cdot \nabla_y G(x,y), y \in \mathbb{S}^{n-1} \ .$$

For all $n \geq 2$, we have

$$y \cdot \nabla_y G(x,y) = \frac{1}{\omega_n}(|x-y|^{-n} y \cdot (y-x) - |x|^{2-n}|x^*-y|^{-n} y \cdot (y - x^*))$$

and so with $y \in \mathbb{S}^{n-1}$, we get

$$P(x,y) = \frac{1}{\omega_n |x-y|^n}(1 - y\cdot x - |x|^2(1 - y \cdot x^*)) = \frac{1-|x|^2}{\omega_n|x-y|^n} \ . \tag{8.12}$$

Similarly, we have for all $x \in B_1$,

$$\int_{\mathbb{S}^{n-1}} P(x,y)\omega(y) = 1,$$

and

THEOREM 8.11. *Let $g \in L^p(\mathbb{S}^{n-1})$ with $1 \leq p \leq \infty$. Then $u(x) = \int_{\mathbb{S}^{n-1}} g(y)P(x,y)d\omega(y)$ is well-defined and harmonic on B_1. When $g \in C$, u is continuous on $\overline{B_1}$ and $u(x) = g(x)$ on $\mathbb{S}$. When $p < \infty$, we have $u(r\omega) \to g(\omega)$ in L^p as $r \to 1-$.*

8.4. Nonlinear elliptic problem

8.4.1. (Defocusing) nonlinear elliptic problem*.

$$\Delta u = |u|^{p-1}u, x \in \Omega, u = g, x \in \partial\Omega \tag{8.13}$$

$$I[u] = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} dx$$

HW Let $q \in 2\mathbb{N}$, for the nonlinear elliptic problem

$$\Delta u = |u|^q u, x \in \Omega, u = g, x \in \partial\Omega \tag{8.14}$$

formulate and prove the corresponding Dirichlet's principle (in $C^2(\bar\Omega)$, for $g \in C^2(\bar\Omega)$) (Hint: $I[u] = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{|u|^{q+2}}{q+2} dx$) [1]

Similar arguments: existence $u \in H^1 \cap L^{p+1}$ for $g \in H^1 \cap L^{p+1}$

8.4.2. (Focusing) nonlinear elliptic problem*.

$$\Delta u = -|u|^{p-1}u, x \in \Omega, u = g, x \in \partial\Omega \tag{8.15}$$

New difficulty: non-definite sign for action functional

$$I[u] = \int_\Omega \frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1} dx$$

For simplicity, concentrate on the **problem:** any other solutions if $g = 0$?

8.4.3. Derrick-Pohozaev identity and the super-critical powers.

THEOREM 8.12. *Let $n \geq 3$. If $p > 1 + 4/(n-2)$ then $u \equiv 0$, provided Ω is star-shaped.*

[7, Section 9.4.2]

Scaling: if u solution for $\Omega = \mathbb{R}^n$, then

$$u_\lambda(x) = \lambda^k u(x/\lambda), \lambda > 0$$

are family of solutions provided that $k - 2 = kp$,

$$k = -2/(p-1) \ .$$

---

[1] Let

$$A = \{w \in C^2(\bar\Omega), w = g, x \in \partial\Omega\} \ .$$

For any $w \in A$, $u - w = 0$ on $\partial\Omega$ and $u - w \in C^2(\bar\Omega)$. If $u \in C^2(\bar\Omega)$ is a solution, then

$$\int |u|^q u(u-w)dx = \int (u-w)\Delta u dx \ .$$

An integration by parts yields

$$\int |u|^q u(u-w)dx = -\int \nabla(u-w)\nabla u dx \ ,$$

$$\int |\nabla u|^2 dx + \int |u|^{q+2} dx = \int \nabla w \nabla u dx + \int |u|^q u w dx \ .$$

Recall that

$$\nabla w \nabla u \leq \frac{|\nabla w|^2 + |\nabla u|^2}{2}, |u|^q u w \leq \frac{(q+1)|u|^{q+2}}{q+2} + \frac{|w|^{q+2}}{q+2}$$

we get $I[u] \leq I[w]$.

On the other hand, if $u \in A$ with $I[u] \leq I[w]$, then it is a weak solution.

$I[u + th]$

$$\frac{d}{dt} I[u+th]|_{t=0} = \int \nabla u \cdot \nabla h dx + \int |u|^q u h dx = 0$$

and so $-\Delta u + |u|^q u = 0$ in $\mathcal{D}'(\Omega)$.

$$I[u_\lambda] = \lambda^{n-2+2k} I[u] = \lambda^{n-2-4/(p-1)} I[u]$$

which is invariant iff $p = 1 + 4/(n-2)$. This suggests that this power may be certain critical power for the problem.

On the other hand, the generator of u_λ:

$$\frac{d}{d\lambda}|_{\lambda=1} u_\lambda(x) = ku(x) - x\cdot\nabla u(x) = -(x\cdot\nabla + \frac{2}{p-1})u$$

This suggests that the operator $L = x\cdot\nabla + \frac{2}{p-1}$ may be relavant for the problem.

We call Ω is star-shaped, with respect to the origin 0, if

$$\lambda x \in \bar{\Omega}, \forall x \in \bar{\Omega}, \lambda \in [0,1].$$

By which we see that $\nu(x)\cdot x \geq 0$.

$$\begin{aligned} |u|^{p-1}uLu &= x\cdot\nabla\frac{|u|^{p+1}}{p+1} + \frac{2}{p-1}|u|^{p+1} \\ &= \nabla\cdot(x\frac{|u|^{p+1}}{p+1}) + (\frac{2}{p-1} - \frac{n}{p+1})|u|^{p+1} \end{aligned}$$

As $-\Delta uu = -\nabla\cdot(u\nabla u) + |\nabla u|^2$,

$$-\Delta u(x\cdot\nabla u + \frac{2}{p-1}u) = (\frac{2-n}{2} + \frac{2}{p-1})|\nabla u|^2 - \partial^j(Lu\partial_j u) + \partial_j(x^j\frac{|\nabla u|^2}{2})$$

$$\int_{\partial\Omega}\nu\cdot x\frac{|u|^{p+1}}{p+1} + \int_\Omega(\frac{2}{p-1} - \frac{n}{p+1})|u|^{p+1} = \int(\frac{2-n}{2} + \frac{2}{p-1})|\nabla u|^2 - \int_{\partial\Omega}\nu^j(Lu\partial_j u) + \int_{\partial\Omega}\nu_j(x^j\frac{|\nabla u|^2}{2})$$

Notice that Dirichlet BC gives us that, on $\partial\Omega$,

$$u = 0,\ \nabla u \propto \nu$$

and so $\nabla u = \pm|\nabla u|\nu$. Then

(8.16)
$$\int_\Omega(\frac{2}{p-1} - \frac{n}{p+1})|u|^{p+1} = \int(\frac{2-n}{2} + \frac{2}{p-1})|\nabla u|^2 - \int_{\partial\Omega}\nu_j(x^j\frac{|\nabla u|^2}{2}) \leq \int(\frac{2-n}{2} + \frac{2}{p-1})|\nabla u|^2 .$$

However, by definition of weak solution, we have

$$\langle -\Delta u, u\rangle = \langle |u|^{p-1}u, u\rangle, \int_\Omega|\nabla u|^2 = \int_\Omega|u|^{p+1} . \tag{8.17}$$

Thus

$$\int_\Omega(\frac{n-2}{2} - \frac{n}{p+1})|u|^{p+1} \leq 0$$

which tells us that either $\int_\Omega|u|^{p+1} = 0$ or

$$\frac{n-2}{2} - \frac{n}{p+1} \leq 0$$

However, as we have assumed, $p > 1 + 2/(n-2)$,

$$\frac{n-2}{2} - \frac{n}{p+1} > 0$$

. Then $\int_\Omega|u|^{p+1} = 0$ and so $u \equiv 0$.

Alternatively, we could just use $x \cdot \nabla$ [2]

8.4.4. sub-critical powers. If $1 < p < 1+4/(n-2)$, two distinct approaches: existence of solution $u \neq 0$:

(1) critical point, mountain pass theorem
(2) minimal value with constraint

[**7**, Section 8.5.2]

8.5. Sample nonlinear heat problems

8.5.1. Nonexistence of global solutions.

THEOREM 8.13 (Fujita 1966). *Consider*

$$u_t - \Delta u = u^p \tag{8.18}$$

on $(0,T)\times\mathbb{R}^n$, with data $u = g$. If $1 < p < 1+2/n$ then for any nonnegative non-trivial data g, there cannot exist non-negative integrable and smooth global solutions for all $T > 0$.

See Evans [**7**, Section 9.4.1]

Let

$$\Phi(x,s) = K_H(s,x) = (4\pi s)^{-n/2}e^{-\frac{|x|^2}{4s}} \ ,$$

we know that

$$\Delta\Phi = \partial_s\Phi = -\frac{n}{2s}\Phi + \frac{|x|^2}{4s^2}\Phi \geq -\frac{n}{2s}\Phi \ .$$

Introduce

$$\eta(t) = \int_{\mathbb{R}^n} u(x,t)\Phi(x,s)dx$$

---

[2]

$$\begin{aligned}|u|^{p-1}u(x\cdot\nabla u) &= x\cdot\nabla\frac{|u|^{p+1}}{p+1}\\ &= \nabla\cdot(x\frac{|u|^{p+1}}{p+1}) - n\frac{|u|^{p+1}}{p+1}\end{aligned}$$

$$-\Delta u(x\cdot\nabla u) = -\partial^j(x^i\partial_i u\partial_j u) + \frac{2-n}{2}|\nabla u|^2 + \partial_j(x^j\frac{|\nabla u|^2}{2})$$

$\int_\Omega$ gives us

$$\int_{\partial\Omega}\nu\cdot(x\frac{|u|^{p+1}}{p+1}) - \int_\Omega n\frac{|u|^{p+1}}{p+1} = \int_{\partial\Omega}[-(\nu\cdot\nabla u)(x\cdot\nabla u) + (x\cdot\nu)\frac{|\nabla u|^2}{2}] + \frac{2-n}{2}\int_\Omega|\nabla u|^2$$

Notice that Dirichlet BC gives us that

$$u = 0, \ \nabla u \propto \nu$$

and so $\nabla u = \pm|\nabla u|\nu$.

$$-\int_\Omega n\frac{|u|^{p+1}}{p+1} = -\int_{\partial\Omega}(x\cdot\nu)\frac{|\nabla u|^2}{2} + \frac{2-n}{2}\int_\Omega|\nabla u|^2 \leq \frac{2-n}{2}\int_\Omega|\nabla u|^2$$

where the first equality is known as the Derrick-Pohozaev identity. Thus, for $p > 1+2/(n-2)$,

$$\frac{n-2}{2}\int_\Omega|\nabla u|^2 \leq \int_\Omega\frac{n}{p+1}|u|^{p+1} < \frac{n-2}{2}\int_\Omega|u|^{p+1} \ .$$

However, by definition of weak solution, we have

$$\langle-\Delta u, u\rangle = \langle|u|^{p-1}u, u\rangle, \int_\Omega|\nabla u|^2 = \int_\Omega|u|^{p+1} \ ,$$

which tells us $\int_\Omega|u|^{p+1} = 0$ and so $u \equiv 0$.

then

$$\eta' = \int_{\mathbb{R}^n} u_t(x,t)\Phi(x,s)dx = \int_{\mathbb{R}^n} (\Delta u + u^p)\Phi(x,s)dx = \int_{\mathbb{R}^n} (\Delta u + u^p)\Phi(x,s)dx = \int_{\mathbb{R}^n} u\Delta\Phi + u^p\Phi(x,s)dx$$

$$\eta' \geq -\frac{n}{2s}\eta + \int_{\mathbb{R}^n} u^p\Phi(x,s)dx$$

To relate $\int u^p\Phi(x,s)$ with η, we use Hölder's inequality to conclude that

$$\eta \leq \|u\|_{L^p(\Phi dx)}\|1\|_{L^{p'}(\Phi dx)} \leq \|u\|_{L^p(\Phi dx)}$$

where we used the fact that $\int \Phi(x,s)dx = 1$. Plugging into the previous inequality, we get

$$\eta' \geq -\frac{n}{2s}\eta + \eta^p$$

Let $\xi(t) = e^{\lambda t}\eta(t)$, with $\lambda = n/(2s)$, we see that

$$\xi' = e^{-\lambda(p-1)t}\xi^p$$

$$\frac{\xi(p)^{1-p}}{p-1} - \frac{\xi(T)^{1-p}}{p-1}\frac{\xi(t)^{1-p}}{1-p}|_{t=0}^T \geq \int_0^T e^{-\lambda(p-1)t} = \frac{1-e^{-\lambda(p-1)T}}{\lambda(p-1)}$$

Blows up in finite time if

$$\frac{\xi(0)^{1-p}}{p-1} < \frac{1}{\lambda(p-1)}$$

that is

$$\eta(0) = \xi(0) > \lambda^{1/(p-1)},\ (4\pi s)^{-n/2}\int g(x)e^{-\frac{|x|^2}{4s}}dx > (\frac{n}{2s})^{1/(p-1)},$$

$$(4\pi)^{-n/2}\int g(x)e^{-\frac{|x|^2}{4s}}dx > (\frac{n}{2})^{1/(p-1)}s^{n/2-1/(p-1)}$$

Notice that $n/2 - 1/(p-1) < 0$, RHS tend to infinity, while the LHS is uniformly bounded, which gives us the desired result, by choosing s large enough.

8.5.2. existence of global solutions. $u_t - \Delta u = |u|^p$ $p > 1 + 2/n$ small data global solutions? Yes.

Actually, we could use $L^1 - L^1$ bound and $L^1 - L^\infty$ decay to prove global well-posedness for small $L^1 \cap L^\infty$ data, by using Picard's iteration and Banach's contraction mapping principle.

Recall that (5.14) and (5.15), we have

$$\|u(t)\|_{L^1} \leq \|f\|_{L^1}, \|u(t)\|_{L^\infty} \leq \|f\|_{L^\infty} \tag{8.19}$$

$$\|u(t)\|_{L^\infty} \leq (4\pi t)^{-\frac{n}{2}}\|f\|_{L^1} \tag{8.20}$$

and so

$$\|u(t)\|_{L^\infty} \leq g(t)\|f\|_{L^\infty\cap L^1}$$

where $g(t) = (4\pi t \vee 1)^{-n/2} \leq 1$.

Let $\|f\|_{L^\infty\cap L^1} = \epsilon \leq \epsilon_0$ with ϵ_0 to be decided later. If $\partial_t u - \Delta u = |v|^p$, $u(0) = f$, and assuming

$$\|v(t)\|_{L^\infty} \leq 2g(t)\epsilon, \|v(t)\|_{L^1} \leq 2\epsilon, \tag{8.21}$$

we have

$$\|u(t)\|_{L^\infty} \leq g(t)\|f\|_{L^\infty\cap L^1} + \int_0^t g(t-s)\||v(s)|^p\|_{L^\infty\cap L^1}ds \leq g(t)\epsilon + 2^p\int_0^t g(t-s)g(s)^{p-1}\epsilon^p ds \leq 2g(t)\epsilon$$

$$\|u(t)\|_{L^1} \le \|f\|_{L^1} + \int_0^t \||v(s)|^p\|_{L^1} ds \le \epsilon + 2^p \int_0^t g(s)^{p-1}\epsilon^p ds \le 2\epsilon$$

provided $\epsilon \le \epsilon_0$.

Thus $T : v \to u$ is a map in

$$X = \{u \in C_t([0,\infty); L^1), \|v(t)\|_{L^\infty} \le 2g(t)\epsilon_0, \|v(t)\|_{L^1} \le 2\epsilon_0\ .\}$$

With possibly smaller $\epsilon_0 > 0$, we have

$$\|Tu - Tv(t)\|_{L^1} \le \int_0^t \||u|^p - |v|^p\|_{L^1} ds \le \int_0^t (4g(s)\epsilon_0)^{p-1}\|u - v(s)\|_{L^1} ds \le \frac{1}{2}\|u - v\|_{C_t L^1}$$

REMARK 8.14. *How about the critical case: still no global solutions in general, see Weissler 1981.*

8.6. Harmonic functions*

8.6.1. Harmonic functions: mean value property, maximum principle, uniqueness/stability.

DEFINITION 8.3. *$u \in C^2(\Omega)$ is said to be harmonic if $\Delta u = 0$.*

As Δ is a hypoelliptic operator, it is equivalent, although looks much weaker, to define $u \in \mathcal{D}'(\Omega)$ to be harmonic if $\Delta u = 0$ in $\mathcal{D}'(\Omega)$.

THEOREM 8.15 (Mean value property). *Suppose u is harmonic on an open set Ω. If $x \in \Omega$ and $r > 0$ so that $\overline{B_r(x)} \subset \Omega$, then*

$$u(x) = \fint_{B_r(x)} u(y)dy = \fint_{S_r(x)} u(y)dy.$$

For it's proof, by polar coordinates centering x_0, $x = x_0 + r\omega$, we get, for $n \ge 2$ and any $R > 0$,

$$0 = \int_{B_R(x_0)} \Delta u dx = \int_{\mathbb{S}^{n-1}} \int_0^R \partial_r r^{n-1}\partial_r u dr d\omega = \int_{\mathbb{S}^{n-1}} R^{n-1}\partial_r u(x_0 + r\omega)|_{r=R} d\omega$$

which tells us that

$$\int_{\mathbb{S}^{n-1}} \partial_r u(x_0 + r\omega)d\omega = 0$$

for any $r > 0$. This means that $F(r) = \int_{\mathbb{S}^{n-1}} u(x_0 + r\omega)d\omega$ is independent of $r > 0$. As u is continuous, we see that $F(r) = F(0) = |\mathbb{S}^{n-1}|u(x_0)$, which is exactly the mean value property.

REMARK 8.16. *It follows also from the Poisson formula (2.34), with $r = 0$, when $n = 2$.*

THEOREM 8.17 (Maximum principle). *Suppose Ω is a connected open bounded set. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, u is real-valued and $\Delta u = 0$, then :*

1. *(Strong maximum principle) If $\exists x_0 \in \Omega$, $u(x_0) = \max_{\overline{\Omega}} u$, then $u \equiv C$.*
2. *(Weak maximum principle) $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.*

In particular, if $u|_{\partial\Omega} = 0$, then we deduce that for $u \equiv 0$. By linearity, considering $u = u_1 - u_2$, this gives uniqueness for solutions of the Dirichlet problem for Poisson's equation:

COROLLARY 8.18 (Uniqueness theorem). *Suppose $\overline{\Omega}$ is compact. There is at most one solutions for*

$$\Delta u = f, x \in \Omega, \ u = \phi, x \in \partial\Omega \ .$$

We also get the following stability result: suppose that

COROLLARY 8.19 (Stability). *Suppose $\overline{\Omega}$ is compact. If $u_1, u_2 \in C(\overline{\Omega})$ solves*

$$\Delta u_j = f, x \in \Omega, \ u_j = \phi_j, x \in \partial\Omega \ ,$$

then

$$\min_{y\in\partial\Omega} \phi_1(y) - \phi_2(y) \le u_1(x) - u_2(x) \le \max_{y\in\partial\Omega} \phi_1(y) - \phi_2(y)$$

for any $x \in \Omega$, which particularly implies

$$\max_{\overline{\Omega}} |u_1 - u_2| \le \max_{y\in\partial\Omega} |\phi_1(y) - \phi_2(y)| \ .$$

REMARK 8.20. *The boundedness of the domain is essential, since it guarantees the existence of the maximum and minimum of u in $\bar{\Omega}$. The uniqueness may not hold if the domain is unbounded. For example, $\Omega = \mathbb{R}^n \backslash B_1$, then the nontrivial functions $u = \ln |x|$ for $n = 2$ and $u = 1 - |x|^{2-n}$ for $n \ge 3$ solves the Dirichlet problem*

$$\Delta u = 0, x \in \Omega, u = 0, \partial\Omega \ .$$

8.6.2. Basic properties. In the following, we record some of the important results concerning harmonic functions, see, e.g., Folland [**9**, Chapter 2] for more details.

THEOREM 8.21. *Suppose $u \in C(\Omega)$ and satisfies Mean value property in Ω, then $u \in C^\infty$ and u is harmonic on Ω.*

COROLLARY 8.22. *If u is harmonic on Ω, then $u \in C^\infty(\Omega)$.*

(Trivial in view of hypoelliptic operator, but here we can avoid the knowledge of the fundamental solution)

THEOREM 8.23 (Liouville's Theorem). *If u is harmonic and bounded on $\mathbb{R}^n$, then u is constant.*

THEOREM 8.24 (Harnack inequality). *Suppose u is harmonic on Ω, and $u \ge 0$. Then for any $K \subseteq \Omega$ and K is compact, there exists constant $C = C(K, \Omega)$, such that*

$$\frac{1}{C} u(y) \le u(x) \le C u(y), \qquad \forall x, y \in K.$$

8.6.3. More about harmonic functions. Suppose Ω is a neighborhood of $x_0 \in \mathbb{R}^n$. If u is a harmonic function on $\Omega\backslash\{x_0\}$, u is said to have a removable singularity at x_0 if u can be defined at x_0 so as to be harmonic on Ω. The following theorem says that any singularity which is weaker than that of the fundamental solution is removable.

THEOREM 8.25 (removable singularity). *Suppose u is a harmonic function on $\Omega\backslash\{x_0\}$. If $u(x) = o(K_{x_0}(x))$, more specifically,*

$$u(x) = \begin{cases} o(|x - x_0|^{2-n} & n \ge 3 \\ o(\ln(1/|x - x_0|) & n = 2 \end{cases}$$

as $x \to x_0$, then u has a removable singularity at x_0.

Remark 8.26. *It is a very weak assumption, as we know from the result that u is bounded near x_0.*

Proof. WLOG, assume u is real valued, $x_0 = 0$ and $B_{2\epsilon} \subset \Omega$ for some $\epsilon \in (0, 1]$. As illustration, we consider only the case $n = 2$. As $u \in C(\Omega\backslash\{x_0\}) \subset C(\partial B_\epsilon)$, we know from the Poisson formula (2.34) that there is $v \in C(\bar{B}_\epsilon)$ such that

$$\Delta v = 0, x \in B_\epsilon,\ v = u, x \in \partial B_\epsilon\ .$$

We aim at proving $u = v$ in $B_\epsilon\backslash\{0\}$, so we could remove the singularity by settting $u(0) = v(0)$.

To achieve this, we introduce

$$w(x) = u - v + \delta \ln(|x|/\epsilon)\ ,$$

in $B_\epsilon\backslash\{0\}$, which is harmonic and real. Notice also that $w = 0$ on ∂B_ϵ. By assumption, $|u(x)| = o(\ln(1/|x|))$, we know that there exists $\epsilon_1 \in (0, \epsilon)$ such that for any x with $|x| \in (0, \epsilon)$, we have

$$u(x) \le \frac{\delta}{4}\ln(\epsilon/|x|),\ |v(x)| \le \frac{\delta}{4}\ln(\epsilon/|x|)\ ,$$

$$w(x) = u - v + \delta \ln(|x|/\epsilon) \le \frac{\delta}{2}\ln(|x|/\epsilon) < 0\ .$$

Thus an application of maximum principle, Theorem 8.17, gives us

$$w(x) < 0, \forall x \in B_\epsilon\backslash\{0\}\ .$$

Letting $\delta \to 0$, we obtain

$$u(x) - v(x) \le 0, \forall x \in B_\epsilon\backslash\{0\}\ .$$

Similarly, we know that $u(x) - v(x) \ge 0$ and thus $u(x) - v(x) = 0$ on $B_\epsilon\backslash\{0\}$. ■

With the help of this result, let us present another explicit example of Subsection 5.5.2, which is copied from Q. Han's book, Example 4.4.4.

Example 8.2. *Let $B_1 \subset \mathbb{R}^2$, $f(0) = 0$ and*

$$f = \frac{y^2 - x^2}{x^2 + y^2}\left(2\ln^{-1/2}(1/r) + \frac{1}{4}\ln^{-3/2}(1/r)\right)$$

with $r = \sqrt{x^2 + y^2} > 0$. Then $f \in C(B_1) \cap C^\infty(B_1\backslash\{0\})$.

Let $u(x, y) = (x^2 - y^2)(-\ln r)^{1/2} \in C(B_1) \cap C^\infty(B_1\backslash\{0\})$. We see that, for $r \ne 0$, $\partial_x(-\ln r)^{1/2} = -\frac{x}{2r^2}(-\ln r)^{-1/2}$,

$$\partial_x^2(-\ln r)^{1/2} = -\frac{1}{2r^2}(-\ln r)^{-1/2} + \frac{x^2}{r^4}(-\ln r)^{-1/2} - \frac{x^2}{4r^4}(-\ln r)^{-3/2}$$

$$\partial_x^2 u = 2(-\ln r)^{1/2} + 2(2x)\partial_x(-\ln r)^{1/2} + (x^2 - y^2)\partial_x^2(-\ln r)^{1/2}$$

$$= 2(-\ln r)^{1/2} - 2\frac{x^2}{r^2}(-\ln r)^{-1/2} + (x^2 - y^2)(-\frac{1}{2r^2}(-\ln r)^{-1/2} + \frac{x^2}{r^4}(-\ln r)^{-1/2} - \frac{x^2}{4r^4}(-\ln r)^{-3/2})$$

$$-\partial_y^2 u = 2(-\ln r)^{1/2} - 2\frac{y^2}{r^2}(-\ln r)^{-1/2} + (y^2 - x^2)(-\frac{1}{2r^2}(-\ln r)^{-1/2} + \frac{y^2}{r^4}(-\ln r)^{-1/2} - \frac{y^2}{4r^4}(-\ln r)^{-3/2})$$

$$\Delta u = 2\frac{y^2 - x^2}{r^2}(-\ln r)^{-1/2} + (x^2 - y^2)(-\frac{1}{4r^2}(-\ln r)^{-3/2}) = f\ .$$

Suppose there exists $v \in C^2(B_1)$ such that $\Delta v = f$, then $w = u - v \in C(B_1)$ satisfies $\Delta w = 0$ in $B_1\backslash\{0\}$. Then, by Theorem 8.25, we know that w has removable singularity at 0, which gives us that $w \in C^\infty(B_1)$ and so $u = w + v \in C^2(B_1)$.

However, as we see, for $r > 0$, we have

$$\partial_x^2 u(x,y) = 2(-\ln r)^{1/2} + \mathcal{O}(1) \to \infty$$

as $r \to 0$.

Next, we would like to investigate the behavior of harmonic functions at infinity. To obtain these, we first need a formula for the Laplace operator Δ in general coordinates, which is of independent interest.

8.6.4. Laplace operator in general coordinates. Let $T : X \to Y$ be a diffeomorphism (bijection with both T and T^{-1} being smooth). Viewing $y \in Y$ as coordinates of X, we would like to express $\Delta u(x)$ in terms of y.

Let $y = T(x)$, then $J_T = (\frac{\partial y_j}{\partial x_k})$ and $J_{T^{-1}} = (\frac{\partial x_j}{\partial y_k})$ are the Jacobian matrices of T and T^{-1}.

THEOREM 8.27. *Let $u \in C^2(X)$ and $U(y) = u(x) = u(T^{-1}(y))$, then*

$$\Delta u(x) = \sum_{jkl} |J_T| \frac{\partial}{\partial y^j}(|J_{T^{-1}}| \frac{\partial y^j}{\partial x^l}\frac{\partial y^k}{\partial x^l}\frac{\partial U}{\partial y^k}) \ . \tag{8.22}$$

Proof. Let $w \in \mathcal{D}(X)$ and $W(y) = w(T^{-1}(y))$. Then

$$\begin{aligned}
\int w\Delta u dx &= -\sum \int \partial_{x^j} w \partial_{x^j} u dx \\
&= -\sum \int \frac{\partial y^k}{\partial x^j}\partial_{y^k} W \frac{\partial y^l}{\partial x^j}\partial_{y^l} U |J_{T^{-1}}| dy \\
&= \sum \int W \partial_{y^k}(\frac{\partial y^k}{\partial x^j}\frac{\partial y^l}{\partial x^j}|J_{T^{-1}}|\partial_{y^l} U)(y) dy \\
&= \sum \int w \partial_{y^k}(\frac{\partial y^k}{\partial x^j}\frac{\partial y^l}{\partial x^j}|J_{T^{-1}}|\partial_{y^l} U)(Tx)|J_T| dx \ .
\end{aligned}$$

This is true for any w, so we obtain (8.22). ∎

Geometrically, it is of course just a reminder of the invariance of the Laplace-Beltrami operator in various coordinates.

To write (8.22) more geometrically. Let us introduce the metric

$$ds^2 = \sum_l (dx^l)^2 = \sum_l (\sum_j \frac{\partial x^l}{\partial y^j} dy^j)(\sum_k \frac{\partial x^l}{\partial y^k} dy^k) = \sum_{jk} g_{jk}(y) dy^j dy^k$$

where

$$g_{jk}(y) = \sum_l \frac{\partial x^l}{\partial y^j}\frac{\partial x^l}{\partial y^k} = (J^t_{T^{-1}} J_{T^{-1}})_{jk} \ .$$

Let $g = \det(g_{jk}) = |J_{T^{-1}}|$ and (g^{jk}) the the inverse of (g_{jk}), we get the conventional expression of the Laplace-Beltrami operator,

$$\Delta u(x) = \sum_{jkl} g^{-1/2}\partial_{y^j}(g^{1/2} g^{jk}\partial_{y^k} U) := (\Delta_g U)(y) \ .$$

8.6.5. Kelvin transform. Recall that, in complex analysis, we know that $f(z) = z^{-1}$ is analytic (and also diffeomorphisom) in $\mathbb{C}\backslash\{0\}$, which tells us that $g(f(z))$ is analytic if $g(z)$ is analytic in $\mathbb{C}\backslash\{0\}$. This gives us a relation for analytic functions near infinity and analytic functions near 0. As we know, analytic functions are harmonic. This suggests the possible way to study the harmonic functions near infinity: just do the inversion

$$x \to y = T(x) = \frac{x}{|x|^2} \ ,$$

for $x \in \mathbb{R}^n\backslash\{0\}$.

Here, we will use the Einstein summation convention for repeated indices. Let $r = \sqrt{y^j y^k \delta_{jk}}$, $\omega^j = y^j/r$ and $x^j = r^{-2}y^j = r^{-1}\omega^j$. The metric in terms of y could be derived from the following

$$dx^j = \frac{\delta^{jk} - 2\omega^j\omega^k}{r^2} dy^k = \frac{l^{jk}}{r^2} dy^k$$
$$ds^2 = \delta_{jk} dx^j dx^k = r^{-4} h_{jk} dy^j dy^k$$

where

$$\begin{aligned} h_{jk} = l^{ji} l^{kl} \delta_{il} &= (\delta^{ji} - 2\omega^j\omega^i)\delta_{il}(\delta^{lk} - 2\omega^l\omega^k) = (\delta^{jl} - 2\omega^j\omega^l)(\delta^{lk} - 2\omega^l\omega^k) \\ &= \delta^{jk} - 2\omega^j\omega^k - 2\omega^j\omega^k + 4\omega^j\omega^k\omega^l\omega^l = \delta^{jk} \ . \end{aligned}$$

Thus, $g_{jk} = r^{-4}\delta_{jk}$, $g = r^{-4n}$, and we obtain

$$\Delta u(x(y)) = \Delta_g U = r^{2n}\partial_{y^j}(r^{-2n}r^4\delta^{jk}\partial_{y^k}U) = r^4\Delta U + (4-2n)r^3\nabla r \cdot \nabla U \ , \tag{8.23}$$

which is of the form $r^4(\Delta U + 2(2-n)r^{-1}\nabla r \cdot \nabla U$. Recall $(fg)'' = f''g + 2f'g' + fg''$, we write

$$(2-n)r^{-1}\nabla r = r^{n-2}\nabla r^{2-n}$$

to obtain that

$$\nabla \cdot \nabla + 2r^{n-2}\nabla r^{2-n} \cdot \nabla = r^{n-2}\nabla \cdot \nabla r^{2-n} - r^{n-2}(\Delta r^{2-n}) \ .$$

As we have $y \neq 0$, we $\Delta r^{2-n} = 0$ and so

$$\Delta_x u(|y|^{-2}y) = |y|^{n+2}\Delta_y(|y|^{2-n}U) \ , \ U(y) = u(|y|^{-2}y) \ , \ \forall y \neq 0 \ . \tag{8.24}$$

Based on this formula, for $u \in C^2(\Omega)$ with $\Omega \subset \mathbb{R}^n\backslash\{0\}$, we could define its Kelvin transform

$$\tilde{u}(x) = |x|^{2-n}u(|x|^{-2}x) \ ,$$

on

$$\Omega' = \{x : |x|^{-2}x \in \Omega\} \ .$$

With this definition, we see that $|y|^{2-n}U(y) = \tilde{u}(y)$ and so (8.24) reads as follows

$$\Delta_x u(|y|^{-2}y) = |y|^{n+2}\Delta_y(\tilde{u}(y)) \ , \ \forall y \neq 0 \ . \tag{8.25}$$

THEOREM 8.28. *Let u be harmonic in $\Omega \subset \mathbb{R}^n\backslash\{0\}$, then its Kelvin transform is harmonic in Ω'.*

Now suppose u is harmonic outside some compact set, then its Kelvin transform $\tilde{u}$ is harmonic in a punctured neighborhood of the origin. We say that u is harmonic at infinity if $\tilde{u}$ has a removable singularity at 0. As an immediate corollary of Theorem 8.25, we have

THEOREM 8.29 (Harmonic at infinity). *Let K be compact in $\mathbb{R}^n$. Suppose u is a harmonic function on $\Omega = \mathbb{R}^n\backslash K$. the following statements are equivalent:*

(1) *u is harmonic at infinity.*
(2) *$u = \mathcal{O}(|x|^{2-n})$ as $x \to \infty$*
(3) *as $x \to \infty$,*

$$u(x) = \begin{cases} o(1) & n \geq 3 \\ o(\ln|x|) & n = 2 \end{cases} .$$

Proof. (1)$\Rightarrow$ (2). By definition, we have $\tilde{u}$ has a removable singularity at 0, and thus $\tilde{u} = \mathcal{O}(1)$ as $x \to 0$, which gives us that $u(x) = |x|^{2-n}\tilde{u}(x/|x|^2)\mathcal{O}(|x|^{2-n})$ as $x \to \infty$. (2)$\Rightarrow$ (3). It is trivial.

For the last part, (3)$\Rightarrow$ (1). By assumption, we have

$$\tilde{u}(x) = |x|^{2-n}u(x/|x|^2) = \begin{cases} o(|x|^{2-n} & n \geq 3 \\ o(\ln(1/|x|) & n = 2 \end{cases}$$

Also, Theorem 8.28 tells us that $\tilde{u}$ is harmonic in $B_\epsilon \backslash \{0\}$ for some $\epsilon > 0$. By Theorem 8.25, we know $\tilde{u}$ has a removable singularity at 0, which gives us the conclusion: u is harmonic at infinity. ∎

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