Chapter 4

hw4 12235005 谭焱

Problem 1. (a) Using the composite midpoint quadrature rule, compute the approximate value ofr the integral $\int_0^1 x^3 dx$, using a mesh size (subinterval length) of $h = 1$ and also using a mesh size of $h = 0.5$.

(b) Based on the two approximate values computed in part a, Use Richardson extrapolation to compute a more accurate approximate to the integral.

(c) Would you expect the extrapolated result computed in part b to exact in this case? Why?

Solution. (a)

$$\begin{aligned} h &= 1, \qquad & M(x^3) &= (1-0)(1/2)^3) = 1/8 \\ h &= 0.5, \qquad & M(x^3) &= 0.5 \times ((1/4)^3 + (3/4)^3) = 7/32 \end{aligned}$$

(b) Since $I(f) = \sum_i f(m_i)h + \frac{f''(m_i)}{24}h^3 + \cdots \Rightarrow F(h) = a_0 + a_1 h^2 + \mathcal{O}(h^4)$ means that $p = 2$ and $r = 4$. Using step $1, 0.5$ obtain

$$F(0) = F(1) + \frac{F(1) - F(0.5)}{2^{-2} - 1} = 1/4.$$

(c) It's not exact that $F(h)$ have $\mathcal{O}(h^4)$ haven't been eliminated.

□

Problem 2. (a) If the integrand f is twice continuously differentiable and $f''(x) \geq 0$ on $[a, b]$, show that the composite midpoint and trapezoid quadrature rules satisfy the bracketing property

$$M_k(f) \leq \int_a^b f(x)dx \leq T_k(f).$$

(b) If the integrand f is convex on $[a, b]$ (see Section 6.2.1), show that the composite midpoint and trapezoid quadrature rules satisfy the bracketing property in part a.

Solution. (a)

$$\begin{aligned} \int_a^b f(x)dx - M_k(f) &= \sum \int_{a_i}^{a_i+h} f(x) - f(a_i + h/2)dx \\ &= \sum \int_0^{h/2} (f'(\chi) - f'(\xi))tdt \qquad \chi \in (a_i, a_i + h/2), \xi \in (a_i + h/2, a_i + h) \\ &\geq 0 \end{aligned}$$

$$\begin{aligned}
\int_a^b f(x)dx - T_k(f) &= \sum \int_{a_i}^{a_i+h} f(x) - (f(a_i) + f(a_i+h))/2dx \\
&= \sum \int_0^{h/2} (f'(\xi) - f'(\chi))tdt \qquad \chi \in (a_i, a_i + h/2), \xi \in (a_i + h/2, a_i + h) \\
&\le 0
\end{aligned}$$

(b)

$$\int_a^b f(x)dx = \sum \int_0^{h/2} (f(a_i+t) + f(a_i+h-t))dt \ge \sum \int_0^{h/2} f(a_i + h/2)dt = M_k(f)$$

$$\int_a^b f(x)dx = \sum \int_0^h f(a_i+t)dt \le \sum \int_0^h tf(a_i) + (1-t)f(a_i+h)dt = T_k(f)$$

□

Problem 3. Let p be a real polynomial of degree n such that

$$\int_a^b p(x)x^k dx = 0, \qquad k = 0, \dots, n-1.$$

(a) Show that the n zeros of p are real, simple , and lie in the open interval (a, b). (Hint: Consider the polynomial $q_k(x) = (x - x_1)(x - x_2)\cdots(x - x_k)$, where $x_i, i = 1, \dots, k$, are the roots of p in $[a, b]$.)

(b) Show that the n-point interpolatory quadrature rule on $[a, b]$ whose nodes are the zeros of p has degree $2n - 1$. (Hint: Consider the quotient and remainder polynomial when a given polynomial is divided by p.)

Solution. (a) Assume $p(x)$ contain m zeros x_i in (a, b) that $m < n$. we have $p(x)(x - x_1)\cdots(x - x_m)$ won't change sign in (a, b), which is conflict with $\int_a^b p(x)(x - x_1)\cdots(x - x_m) = \int_a^b p(x)(\sum_p \alpha_p x^p)dx = 0$.

(b) Suppose x_i, w_i satisfy

$$\sum_i w_i x_i^k = \int_a^b x^k dx, \qquad k = 0, \cdots, n-1$$

and x_i is zeros of $p(x)$. There is exist $\alpha_k(x), \beta_k(x) \in P_{n-1}(x)$ such that $x^{n+k} = p(x)\alpha_k(x) + \beta_k(x), k < n$, and

$$\begin{aligned}
\int_a^b x^{n+k} dx = \int_a^b (p(x)\alpha_k(x) + \beta_k(x))dx &= \int_a^b \beta_k(x)dx \\
= \sum_i w_i \beta_k(x_i) = \sum_i w_i(p(x_i)\alpha_k(x_i) + \beta_k(x_i)) &= \sum_i x_i^{n+k}
\end{aligned}$$

□

Problem 4. The forward difference formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

and the backward difference formula

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

are both first-order accurate approximations to the first derivative of a function $f : \mathbb{R} \to \mathbb{R}$. What order accuracy results if we average these two approximations? Support your answer with an error analysis.

Solution.

$$\frac{f(x+h)-f(x)}{h} - f'(x) = \frac{f''(x)}{2}h - \mathcal{O}(h^2)$$
$$\frac{f(x)-f(x-h)}{h} - f'(x) = -\frac{f''(x)}{2}h - \mathcal{O}(h^2)$$

So they are first-order accurate. Average result get

$$\frac{f(x+h)-2f(x)+f(x-h)}{2h} - f'(x) = \frac{f'''(x)}{6}h^2 - \mathcal{O}(h^3).$$

Therefore it has second-order accurate. □

Problem 5. Suppose that the first-order accurate, forward difference approximation to the derivative of a function at a given point produces the value -0.8333 for $h = 0.2$ and the value -0.9091 for $h = 0.1$ Use Richardson extrapolation to obtain a better approximate value for the derivative.

Solution. Since forward difference formula have $F(h) = a_0 + a_1 h + \mathcal{O}(h^2)$ Which means that $p = 1, r = 2$ in this case. Using step sizes of $h = 0.2$ and $h = 0.1 (q = 2)$, we obtain $F(0.2) = -0.8333, F(0.1) = -0.9091$. The extrapolated value is then given by

$$F(0) = F(0.2) + \frac{F(0.2) - F(0.1)}{(1/2 - 1)} = 2F(0.1) - F(0.2) = -0.9861.$$

□

Problem 6. With an initial value of $y_0 = 1$ at $t_0 = 0$ and a time step of $h = 1$, compute the approximate solution value y_1 at time $t_1 = 1$ for the ODE $y' = -y$ using each of the following two numerical methods. (Your answers should be numbers, not formulas.)

(a) Euler's method

(b) Backward Euler method

Solution. (a)

$$y_1 = y_0 + hf(t_0, y_0) = y_0 - hy_0 = 1 - 1 = 0.$$

(b)

$$y_1 = y_0 + hf(t_1, y_1) = y_0 - hy_1 \Rightarrow y_1 = \frac{y_0}{1+h} = \frac{1}{1+1} = 0.5.$$

□

Problem 7. Consider the IVP

$$y'' = y$$

for $t \geq 0$, with initial values $y(0) = 1$ and $y'(0) = 2$.

(a) Express this second-order ODE as an equivalent system of two first-order ODEs.

(b) What are the corresponding initial conditions for the system of ODEs in part a?

(c) Are solutions of this system stable?

(d) Perform one step of Euler's method for this ODE system using a step size of $h = 0.5$.

(e) Is Euler's method stable for this problem using this step size?

(f) Is the backward Euler method stable for this problem using this step size?

Solution. (a) Define the new unknowns $u_1(t) = y(t)$ and $u_2(t) = y'(t)$, then we have

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(c) The eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are $1(> 0)$ and -1, thus solutions of this system are unstable.

(d)

$$\mathbf{u}_1 = \mathbf{u}_0 + hA\mathbf{u}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix}.$$

(e) The eigenvalues of the matrix $I + hA = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ are $1.5(> 1)$ and 0.5, therefore, Euler's method is unstable for this problem using this step size.

(f) The formula for the backward Euler method is

$$\mathbf{u}_{n+1} = \mathbf{u}_n + hA\mathbf{u}_{n+1} \Rightarrow \mathbf{u}_{n+1} = (I - hA)^{-1}\mathbf{u}_n,$$

the eigenvalues of the matrix $(I - hA)^{-1} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}^{-1}$ are $(1/2)^{-1} = 2(> 1)$ and $2/3$, therefore, backward Euler's method is unstable for this problem using this step size.

□

Problem 8. Applying the midpoint quadrature rule on the interval $[t_k, t_{k+1}]$ leads to the implicit *midpoint method*

$$y_{k+1} = y_k + h_k f(t_k + h_k/2, (y_k + y_{k+1})/2)$$

for solving the ODE $y' = f(t, y)$. Determine the order of accuracy and the stability region of this method.

Solution. Applying Taylor's theorem yields

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + h_k y'(t_k) + \frac{h_k^2}{2} y''(t_k) + \mathcal{O}(h_k^3); \\ f\left(t_k + \frac{h_k}{2}, \frac{y(t_k) + y(t_{k+1})}{2}\right) &= f(t_k, y(t_k)) + \frac{h_k}{2} f_t(t_k, y(t_k)) + \frac{y(t_{k+1}) - y(t_k)}{2} f_y(t_k, y(t_k)) + \mathcal{O}(h_k^2) \\ &= y'(t_k) + \frac{h_k}{2} \left(f_t(t_k, y(t_k)) + f_y(t_k, y(t_k)) y'(t_k)\right) + \mathcal{O}(h_k^2) \\ &= y'(t_k) + \frac{h_k}{2} y''(t_k) + \mathcal{O}(h_k^2). \end{aligned}$$

Therefore

$$
\begin{aligned}
& y(t_{k+1}) - \left[y(t_k) + h_k f\left(t_k + \frac{h_k}{2}, \frac{y(t_k) + y(t_{k+1})}{2}\right)\right] \\
&= y(t_k) + h_k y'(t_k) + \frac{h_k^2}{2} y''(t_k) + \mathcal{O}(h_k^3) \\
&- \left\{ y(t_k) + h_k \left[y'(t_k) + \frac{h_k}{2} y''(t_k) + \mathcal{O}(h_k^2)\right]\right\} \\
&= \mathcal{O}(h_k^3),
\end{aligned}
$$

which shows that the implicit midpoint method is of order 2.

To determine the stability of the implicit midpoint method, we apply it to the scalar test ODE $y' = \lambda y$, obtaining

$$
y_{k+1} = y_k + \frac{\lambda h_k}{2}(y_k + y_{k+1}),
$$

which implies that

$$
y_k = \left(\frac{1 + h_k\lambda/2}{1 - h_k\lambda/2}\right)^k y_0.
$$

Thus, the stability region of the implicit midpoint method is

$$
\mathcal{D} = \left\{ z \in \mathbb{C} : \left|\frac{1+z}{1-z}\right| < 1 \right\}.
$$

□

Problem 9. Consider the two-point BVP for the second-order scalar ODE

$$
u'' = u, \quad 0 < t < b,
$$

with boundary conditions

$$
u(0) = \alpha, \quad u(b) = \beta.
$$

(a) Rewrite the problem as a first-order system of ODEs with separated boundary conditions.

(b) Show that the fundamental solution matrix for the resulting linear system of ODEs is given by

$$
Y(t) = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}.
$$

(c) Are the solutions to this ODE stable?

(d) Determine the matrix $Q \equiv B_0 Y(0) + B_b Y(b)$ for this problem.

(e) Determine the rescaled solution matrix $\Phi(t) = Y(t)Q^{-1}$.

(f) What can you say about the conditioning of Q, the norm of $\Phi(t)$, and the stability of solutions to this BVP as the right endpoint b grows?

Solution. (a) Define the new unknowns $y_1(t) = u(t)$ and $y_2(t) = u'(t)$, then we have

$$
\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},
$$

with separated linear boundary conditions

$$
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(b) \\ y_2(b) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.
$$

(b) Solving $\mathbf{y}' = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, we obtain $\mathbf{y}(t) = \begin{bmatrix} \cosh(t) & \sinh(t) \end{bmatrix}^T$, with $\mathbf{y}(0) = \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $\mathbf{y}(t) = \begin{bmatrix} \sinh(t) & \cosh(t) \end{bmatrix}^T$. Therefore the fundamental solution matrix is

$$Y(t) = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}.$$

(c) The solutions to this ODE are stable, since growth in the solution is limited by the boundary conditions.

(d)

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cosh(b) & \sinh(b) \\ \sinh(b) & \cosh(b) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cosh(b) & \sinh(b) \end{bmatrix}.$$

(e)

$$\begin{aligned} \Phi(t) = Y(t)Q^{-1} &= \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{\cosh(b)}{\sinh(b)} & \frac{1}{\sinh(b)} \end{bmatrix} \\ &= \frac{1}{\sinh(b)} \begin{bmatrix} \sinh(b-t) & \sinh(t) \\ -\cosh(b-t) & \cosh(t) \end{bmatrix} \end{aligned}$$

(f) As b grows, the condition number of Q and the norm of $\Phi(t)$ grow as well, and the stability of solutions to this BVP decreases.

□

Problem 10. Consider the two-point BVP

$$u'' = u^3 + t, \quad a < t < b,$$

with boundary conditions

$$u(a) = \alpha, \quad u(b) = \beta.$$

To use the shooting method to solve this problem, one needs a starting guess for the initial slope $u'(a)$. One way to obtain such a starting guess for the initial slope is, in effect, to do a "preliminary shooting" in which we take a single step of Euler's method with $h = b - a$.

(a) Using this approach, write out the resulting algebraic equation for the initial slope.

(b) What starting value for the inital slope results from this approach?

Solution. (a)

$$u(b) = u(a) + hu'(a) \Rightarrow hu'(a) = u(b) - u(a).$$

(b)

$$u'(a) = \frac{u(b) - u(a)}{h} = \frac{\beta - \alpha}{b - a}.$$

□