PDE Homework #1

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Problem 1. *Find general solutions for spherically symmetric solutions $V = V(\|\mathbf{x}\|)$ such that $\Delta V = 0, \mathbf{x} \in \mathbb{R}^n$ with $n \geq 2$.*

Solution. Define

$$r := \|\mathbf{x}\| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Note for $i = 1, \dots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-1/2} 2x_i = \frac{x_i}{r} \quad (\mathbf{x} \neq \mathbf{0}).$$

We thus have

$$V_{x_i} = V'(r)\frac{x_i}{r}, \quad V_{x_i x_i} = V''(r)\frac{x_i^2}{r^2} + V'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

for $i = 1, \dots, n$, and so

$$\Delta V = V''(r) + \frac{n-1}{r}V'(r).$$

Hence $\Delta V = 0$ if and only if

$$V'' + \frac{n-1}{r}V' = 0.$$

If $V' \neq 0$, we deduce

$$\left(r^{n-1}V'(r)\right)' = r^{n-1}V''(r) + (n-1)r^{n-2}V'(r) = r^{n-1}\left(V''(r) + \frac{n-1}{r}V'(r)\right) = 0,$$

and hence $V'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if $r > 0$, we have

$$V(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3), \end{cases}$$

where b and c are constants.
Therefore

$$V(\mathbf{x}) = \begin{cases} b \log \|\mathbf{x}\| + c & (n = 2) \\ \frac{b}{\|\mathbf{x}\|^{n-2}} + c & (n \geq 3). \end{cases} \tag{1}$$

□

Problem 2 (minimal surface)**.** *Let $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$. For a fixed function f on the boundary of Ω, suppose u is a function such that it minimizes the area*

$$A[u] = \int_\Omega \sqrt{1 + \|\nabla u\|^2} \mathrm{d}\mathbf{x},$$

among all surfaces with $u = f$ on the boundary of Ω. Try to derive a PDE for u.

Solution. Choose a smooth function $v : \Omega \to \mathbb{R}$ satisfying the boundary condition $v|_{\partial\Omega} = 0$ and define for $\tau \in \mathbb{R}$

$$w = u + \tau v.$$

Then w is a surface with $w = f$ on the boundary of Ω, and so

$$A[u] \leq A[w].$$

Thus the real-valued function

$$i(\tau) := A[u + \tau v]$$

has a minimum at $\tau = 0$, and consequently

$$i'(0) = 0 \quad \left(' = \frac{\mathrm{d}}{\mathrm{d}\tau} \right), \tag{2}$$

provided $i'(0)$ exists.
We explicitly compute this derivative. Observe

$$i(\tau) = \int_\Omega \sqrt{1 + \|\nabla(u + \tau v)\|^2} \mathrm{d}\mathbf{x} = \int_\Omega \sqrt{\|\nabla v\|\tau^2 + 2(\nabla u \cdot \nabla v)\tau + \|\nabla u\|^2 + 1} \mathrm{d}\mathbf{x},$$

and so

$$i'(\tau) = \int_\Omega \frac{1}{2} \frac{2\|\nabla v\|\tau + 2(\nabla u \cdot \nabla v)}{\sqrt{\|\nabla v\|\tau^2 + 2(\nabla u \cdot \nabla v)\tau + \|\nabla u\|^2 + 1}} \mathrm{d}\mathbf{x}.$$

Set $\tau = 0$ and remember (2):

$$0 = i'(0) = \int_\Omega \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \cdot \nabla v \mathrm{d}\mathbf{x} = -\int_\Omega v \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) \mathrm{d}\mathbf{x},$$

where the third equality follows from the integration-by-parts formula and the boundary condition of v(i.e., $v|_{\partial\Omega} = 0$).
Since v is an arbitrary smooth function satisfying the boundary condition $v|_{\partial\Omega} = 0$, and so

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) = 0. \tag{3}$$

□

Problem 3. *Solve using characteristics:*

(a) $u_t + xu_x = x, u(0, x) = x^2$.

(b) $u_t + uu_x = 0, u(0, x) = -x$.

Solution. Recall that the characteristic equations of a first-order nonlinear PDE are defined as follows:

Definition 1.

$$\dot{\mathbf{p}}(s) = -D_{\mathbf{x}} F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s); \tag{4a}$$
$$\dot{z}(s) = D_{\mathbf{p}} F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s); \tag{4b}$$
$$\dot{\mathbf{x}}(s) = D_{\mathbf{p}} F(\mathbf{p}(s), z(s), \mathbf{x}(s)). \tag{4c}$$

(a) Substituting

$$F(p_1, p_2; z; x, t) = xp_1 + p_2 - x$$

into (4) yields

$$\dot{x}(s) = x(s), \quad \dot{t}(s) = 1;$$
$$\dot{z}(s) = x(s).$$

Consequently

$$x(s) = x^0 e^s, \quad t(s) = s;$$
$$z(s) = g(x^0) + \int_0^s x^0 e^\tau \mathrm{d}\tau = g(x^0) + x^0(e^s - 1),$$

where $x^0 \in \mathbb{R}, s \geq 0$.
Fix a point $(t, x) \in (0, \infty) \times \mathbb{R}$. We select $s > 0$ and $x^0 \in \mathbb{R}$ so that $(t, x) = (t(s), x(s)) = (s, x^0 e^s)$; that is, $s = t, x^0 = xe^{-s}$. Then

$$u(t, x) = u(t(s), x(s)) = z(s) = g(x^0) + x^0(e^s - 1) = g(xe^{-t}) + xe^{-t}(e^t - 1) = x^2 e^{-2t} - xe^{-t} + x.$$

Therefore

$$u(t, x) = x^2 e^{-2t} - xe^{-t} + x. \tag{5}$$

(b) Substituting

$$F(p_1, p_2; z; x, t) = zp_1 + p_2$$

into (4) yields

$$\dot{x}(s) = z(s), \quad \dot{t}(s) = 1;$$
$$\dot{z}(s) = 0.$$

Consequently

$$x(s) = -x^0 s + x^0, \quad t(s) = s;$$
$$z(s) = g(x^0) = -x^0,$$

where $x^0 \in \mathbb{R}, s \geq 0$.

Fix a point $(t, x) \in (0, \infty) \times \mathbb{R}$. We select $s > 0$ and $x^0 \in \mathbb{R}$ so that $(t, x) = (t(s), x(s)) = (s, -x^0 s + x^0)$; that is, $s = t, x^0 = \frac{x}{1-t}$. Then

$$u(t, x) = u(t(s), x(s)) = z(s) = g(x^0) = g\left(\frac{x}{1-t}\right) = -\frac{x}{1-t}.$$

Therefore

$$u(t, x) = \frac{x}{t-1}. \tag{6}$$

□