Chapter 1

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1. Specify condition on f , and solve the initial boundary value problem of heat equations

$$\begin{cases} u_t = u_{xx} + t\cos x, & x \in [0,1], t > 0. \\ u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = f(x) \end{cases} \tag{1.1}$$

by the method of separation variables.

Solution. Define u_1, u_2 are solution for follow equations separately,

$$\begin{cases} u_t = u_{xx} & u_t = u_{xx} + t\cos x \\ u(x,0) = f(x) & u(x,0) = 0 \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

. So $u = u_1 + u_2$ solve origin equations, assume $u_1 = F(x)G(t)$ get

$$\tfrac{G'}{G} = \tfrac{F''}{F} \quad \Rightarrow \begin{cases} G(t) = e^{-m^2 t}G(0) \\ F(x) = (A_m \cos(2m\pi x) + B_m \sin(2m\pi x)) \end{cases}$$

. By boundary condition,

$$\begin{cases} F(x)G(0) = f(x) \\ F'(0)G(t) = F'(1)G(t) = 0 \end{cases} \quad \Rightarrow \begin{cases} G(0) = 1 \\ A_m = 2\int_0^1 f(x)\cos(m\pi x)dx \\ B_m = 0 \end{cases}$$

. Hence induce $u_1 = A_0/2 + \sum_m A_m \cos(m\pi x)e^{-(m\pi)^2 t}$. Assume w solve

$$\begin{cases} u_t = u_{xx} \\ u(x,\tau) = \tau\cos x \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

. Similarly, $w(x,t,\tau) = A'_0/2 + \sum_m A'_m \cos(m\pi x)e^{-(m\pi)^2 t}$, there is

$$A'_m = 2\int_0^1 \tau e^{(m\pi)^2\tau} \cos x \cos(m\pi x)dx$$

. According to Duhamel's principle, we know

$$u = u_1 + u_2 = u_1 + \int_0^t w(x,t,\tau)d\tau$$

2. Solve the boundary value problem of the Laplace equation in disc:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, & x^2 + y^2 < R^2, \\ u(R\cos\theta, R\sin\theta) = f(\theta) \end{cases} \tag{1.2}$$

by the method of separation of variables (in polar coordinates), in the form

$$u(r,\theta) = \int_0^{2x} G(r,\theta,R,\phi) f(\phi) d\phi.$$

Solution. Rewrite Laplace equation as polar coordinates

$$\Delta u = u_{rr} + u_r/r + u_{\theta\theta}/r^2 = 0$$

. Assume $u = F(r)G(\theta)$ and replace u in previous equation get $(r^2F'' + rF')/F = -G''/G$. By G is period in 2π since polar coordinates induce

$$G(\theta) = A_m \cos(mx) + B_m \sin(mx).$$
$$F(r) = r^m \text{or } F(r) = r^{-m} \qquad m \neq 0$$
$$F(r) = 0 \text{ or } F(r) = \log r \qquad m = 0$$

Since we find meaning $u(0,\theta)$ solution, only $F(r) = r^m$ remain, then by boundary condition

$$u(R,\theta) = A_0/2 + \sum_m (A_m R^m \cos m\theta + B_m R^m \sin m\theta) = f(\theta),$$

therefore,

$$A_m = \frac{1}{\pi R^m} \int_0^{2\pi} f(\tau) \cos m\tau d\tau, B_m = \frac{1}{\pi R^m} \int_0^{2\pi} f(\tau) \sin m\tau d\tau$$

In summary,

$$u(r,\theta) = \frac{1}{\pi} \int_0^{2\pi} \left(1/2 + \sum_m r^m \cos m(\theta - \tau)/R^m \right) d\tau$$

3. Solve the initial boundary value problem of the equation

$$\begin{cases} \partial_t^2 u = u_{xx} + u_{yy}, & (x,y) \in [0,1] \times [0,\pi], \\ u(0,y,t) = u(1,y,t) = u_y(x,0,t) = u_y(x,\pi,t) = 0 \\ u(x,y,0) = f(x,y), & u(x,y,0) = 0. \end{cases} \tag{1.3}$$

Solution. Assume $u = F(x)G(y)K(t)$, replace u obtain $FGK'' = F''GK + FG''K \to K''/K = F''/F + G''/G =: \lambda_1 + \lambda_2$. When $F''/F = \lambda_1$, $\lambda_1 > 0$ will be conflict with $u(0,y,t) = u(1,y,t) = 0 \Rightarrow F(0) = F(1) = 0$, and $\lambda = 0$ get $u = 0$, thus $\lambda_1 < 0$. Similarly, $\lambda_2 < 0$. Combining with boundary condition, we have

$$F(x) = \sin(n\pi x), G(y) = \cos(my)$$

. That means $K''/K = -(n\pi)^2 - m^2 =: \lambda_3$ and

$$K(t) = A_{n,m} \cos\sqrt{((n\pi)^2 + m^2)}t + B_{n,m} \sin\sqrt{((n\pi)^2 + m^2)}t$$

. The coefficients $A_{n,m}, B_{n,m}$ get from double Fourier series

$$A_{n,m} = \frac{4}{\pi} \int_0^1 \int_0^\pi f(x,y) \sin(n\pi x) \cos(my) dy dx$$
$$B_{n,m} = \frac{4}{\pi} \int_0^1 \int_0^\pi 0 * \sin(n\pi x) \cos(my) dy dx = 0$$
$$\Rightarrow u(x,y,t) = F(x)G(y)K(t) = \sum_n \sum_m \sin(n\pi x) \cos(my) A_{n,m} \cos\sqrt{((n\pi)^2 + m^2)}t$$