Chapter 2

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3.3 Solution. By substituting the data point $(1,2),(2,3),(3,5)$ have

$$Ax = \begin{bmatrix} 1 & e \\ 2 & e^2 \\ 3 & e^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cong \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = b$$

3.7 Solution.

(a) The function $\phi(\boldsymbol{y}) = \|\boldsymbol{b} - \boldsymbol{y}\|_2$ is continuous and coercive on $\mathbb{R}^m$, so ϕ has a minimum on the closed, unbounded set span $(\boldsymbol{A})$, i.e., there is an m-vector $\boldsymbol{y} \in \text{span}(\boldsymbol{A})$ closest to $\boldsymbol{b}$ in the Euclidean norm.

(b) Suppose $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are such solutions, and let $\boldsymbol{z} = \boldsymbol{x_2} - \boldsymbol{x_1}$. Then since $\boldsymbol{Ax_1} = \boldsymbol{y} = \boldsymbol{Ax_2}$, we have $\boldsymbol{Az} = \boldsymbol{0}$. Now if $\boldsymbol{z} \neq \boldsymbol{0} \Leftrightarrow \boldsymbol{x_1} \neq \boldsymbol{x_2}$, then the columns of A must be linearly dependent. We conclude that teh solution to an $m \times n$ least squares problem $\boldsymbol{Ax} \cong \boldsymbol{b}$ is unique if, and only if, $\boldsymbol{A}$ has full column rank, i.e., $\text{rank}(\boldsymbol{A}) = \text{n}$.

3.17 Solution. From definition, we have

$$\alpha = -\text{sign}(a_1)\|\boldsymbol{a}\|_2 = -2$$
$$\boldsymbol{v} = \boldsymbol{a} - \alpha \boldsymbol{e}_1 = [3\ 1\ 1\ 1]^T$$

3.20 Solution.

(a) It's possible to annihilate a_2 with Givens rotation

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) It's not possible, since in elimination matrix calculating, $a_2/a_1 = a_2/0$ is meaningless.

3.28 Solution.

(a) Since $\boldsymbol{q}_k$ is orthogonal, imply $\boldsymbol{q_i^T q_j} = 0, i \neq j$

$$
\begin{aligned}
&\boldsymbol{(I - P_k)(I - P_{k-1}) \dots (I - P_1) = I - \sum P_m + \sum q_i(q_i^T q_j) q_j^T (\dots)} \\
=&\boldsymbol{I - \sum P_m + 0 * \sum q_i q_j^T (\dots) = I - \sum P_m}
\end{aligned}
$$

(b) In the classical Gram-Schmidt procedure

$$
\boldsymbol{q_k} = \boldsymbol{a_k - \sum_j (q_j^T a_k) q_j = a_k - \sum_j q_j (q_j^T a_k) = (I - \sum_j P_j) a_k}
$$

(c) In the modified Gram-Schmidt procedure, assume $M_j(a_k) = a_k - (q_j^T a_k) q_j = \boldsymbol{(I - P_j)} a_k$

$$
\boldsymbol{q_k} = \boldsymbol{M_{k-1}(M_{k-1}(\dots M_1(a_k) \dots)) = (I - P_{k-1})(M_{k-1}(\dots M_1(a_k) \dots)) = (I - P_{k-1}) \dots (I - P_1) a_k}
$$

(d) It's obvious that is same as (a) like

$$
\begin{aligned}
&\boldsymbol{(I - \sum P_i)(I - \sum P_j) = I - 2 * \sum P_m + \sum P_m^2 + \sum q_i (q_i^T q_j) q_j^T (\dots)} \\
=&\boldsymbol{I - 2 * \sum P_m + \sum P_m + 0 * \sum q_i q_j^T (\dots) = I - \sum P_m}
\end{aligned}
$$

4.2 Solution. Since the matrix is upper triangular matrix, the eigenvalues and corresponding eigenvectors are

$$
A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1v_1 \\ 2v_2 \\ 3v_3 \end{bmatrix} = [1\ 2\ 3]^T [v_1\ v_2\ v_3] = e^T [v_1\ v_2\ v_3]
$$

4.14 Solution.

(a) Let $\alpha = 0$ the matrix is lower triangular matrix and eigenvalues is diagonal [1 2 3] which is all real values.

(b) It's impossible that since real matrix eigenvalues has nonzero imaginary part exist as pair. Which is coming from when $a + bi$ and v is eigenvalue and eigenvector, It's easy get $a - bi$ and $\bar{v}$ is another eigenvalue and eigenvector by substituting. However, the matrix have odd eigenvalue that conflicted with all nonzero imaginary part complex eigenvalue.

4.17 Solution. Assuming v is eigenvector of eigenvalue λ. It's easy verify that $A^2 v = A\lambda v = \lambda(Av) = \lambda^2 v$, so λ^2 is A^2's eigenvalue.

4.22 Solution.

(a)

$$
A \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \boldsymbol{O} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} A_{11}\boldsymbol{u} \\ \boldsymbol{0} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{0} \end{bmatrix}
$$

(b)

$$A\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \boldsymbol{O} & A_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A_{11}\boldsymbol{u} + A_{12}\boldsymbol{v} \\ A_{22}\boldsymbol{v} \end{bmatrix} = \lambda\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$$

, we need satisfy $A_{11}\boldsymbol{u} + A_{12}\boldsymbol{v} = \lambda u$, which is equal to $\boldsymbol{u} = (A_{11} - \lambda I)^{-1}A_{12}\boldsymbol{v}$. Since λ is not A_{11}'s eigenvalue, $(A_{11} - \lambda I)^{-1}$ is exist, so λ and $[(A_{11} - \lambda I)^{-1}A_{12}\boldsymbol{v}\ \boldsymbol{v}]^T$ is eigenvalue and eigenvector for A.

(c) By result in (b),

$$\begin{aligned} A_{11}\boldsymbol{u} + A_{12}\boldsymbol{v} &= \lambda u \\ A_{22}\boldsymbol{v} &= \lambda \boldsymbol{v}. \end{aligned}$$

When $\boldsymbol{v} \neq \boldsymbol{0}$ we have $A_{22}\boldsymbol{v} = \lambda\boldsymbol{v}$, When $\boldsymbol{b} = \boldsymbol{0}$, we have $A_{11}\boldsymbol{u} = \lambda\boldsymbol{u}$. So λ is eigenvalue of $A_{11}, \boldsymbol{u}$ or $A_{22}, \boldsymbol{v}$.

(d) The sufficiency follows from (a) and (b) while the necessity follows from (c).

4.32 Solution.

(a) Assume a orthogonal basis contain v is $U = [v, u_0, \ldots, u_k]$, so $v^T u_i = 0$ means

$$\begin{aligned} Hv &= Iv - 2\frac{vv^Tv}{v^Tv} = v - 2v = -v \\ Hu_i &= Iu_i - 2\frac{vv^Tu_i}{v^Tv} = u_i. \end{aligned}$$

And U is a basis, so eigenvalues is -1 with v and 1 with u_i.

(b) The characteristic polynomial of H is

$$p(\lambda) = \begin{vmatrix} c-\lambda & s \\ -s & c-\lambda \end{vmatrix} = \lambda^2 - 2\lambda c + c^2 + s^2.$$

The eigenvalues is solution of characteristic polynomial zero points $c \pm is$.