Project I: Boolean Algebra on Regular Regions in $\mathbb{R}^2$

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1 Definitions

In a topological space $\mathcal{X}$, the *complement* of a subset $\mathcal{P} \subseteq \mathcal{X}$, written $\mathcal{P}'$, is the set $\mathcal{X} \setminus \mathcal{P}$. The *closure* of a set $\mathcal{P} \subseteq \mathcal{X}$, written $\mathcal{P}^-$, is the intersection of all closed supersets of $\mathcal{P}$. The *interior* of $\mathcal{P}$, written $\mathcal{P}^\circ$, is the union of all open subsets of $\mathcal{P}$. The *exterior* of $\mathcal{P}$, written $\mathcal{P}^\perp := \mathcal{P}'^\circ := (\mathcal{P}')^\circ$, is the interior of its complement. By the identity $\mathcal{P}^- = \mathcal{P}'^{\circ\prime}$ we have $\mathcal{P}^\perp = \mathcal{P}^{-\prime}$. A point $\mathbf{x} \in \mathcal{X}$ is a *boundary point* of $\mathcal{P}$ if $\mathbf{x} \notin \mathcal{P}^\circ$ and $\mathbf{x} \notin \mathcal{P}^\perp$. The *boundary* of $\mathcal{P}$, written $\partial\mathcal{P}$, is the set of all boundary points of $\mathcal{P}$. It can be shown that $\mathcal{P}^\circ = \mathcal{P} \setminus \partial\mathcal{P}$ and $\mathcal{P}^- = \mathcal{P} \cup \partial\mathcal{P}$. An open set $\mathcal{P} \subseteq \mathcal{X}$ is *regular* if it coincides with the interior of its own closure, i.e. if $\mathcal{P} = \mathcal{P}^{-\circ}$.

Definition 1. A *Boolean algebra* is an algebra of the form

$$\mathbf{B} := (\mathcal{B},\ \vee,\ \wedge,\ ',\ \hat{0},\ \hat{1}), \tag{1}$$

where the binary operations $\vee$, $\wedge$ called "meet" and "join," the unary operation $'$ called complementation, and the nullary operations $\hat{0}$, $\hat{1}$ satisfy

(BA-1) the identity laws: $x \wedge \hat{1} = x$, $x \vee \hat{0} = x$,

(BA-2) the complement laws: $x \wedge x' = \hat{0}$, $x \vee x' = \hat{1}$,

(BA-3) the commutative laws: $x \vee y = y \vee x$, $x \wedge y = y \wedge x$,

(BA-4) the distributive laws: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2. A set $\mathcal{S} \subseteq \mathbb{R}^{\mathsf{D}}$ is *semianalytic* if there exist a finite number of analytic functions $g_i : \mathbb{R}^{\mathsf{D}} \to \mathbb{R}$ such that $\mathcal{S}$ is in the universe of a finite Boolean algebra formed from the sets

$$\mathcal{X}_i = \left\{\mathbf{x} \in \mathbb{R}^{\mathsf{D}} : g_i(\mathbf{x}) \geq 0\right\}. \tag{2}$$

The g_i's are called the *generating functions* of $\mathcal{S}$. In particular, a semianalytic set is *semialgebraic* if all of its generating functions are polynomials.

Recall that a function is *analytic* if and only if its Taylor series at $\mathbf{x}_0$ converges to the function in some neighborhood for every $\mathbf{x}_0$ in its domain.

Definition 3. A *Yin set* $\mathcal{Y} \subseteq \mathbb{R}^2$ is a regular open semianalytic set whose boundary is bounded. The class of all such Yin sets form the *Yin space* $\mathbb{Y}$.

The Yin sets form a model of physically meaningful regions in the plane.

Theorem 4. The algebra $\mathbf{Y} := (\mathbb{Y},\ \cup^{\perp\perp},\ \cap,\ ^\perp,\ \emptyset,\ \mathbb{R}^2)$ is a Boolean algebra, where $A \cup^{\perp\perp} B := \left((A \cup B)^\perp\right)^\perp$.

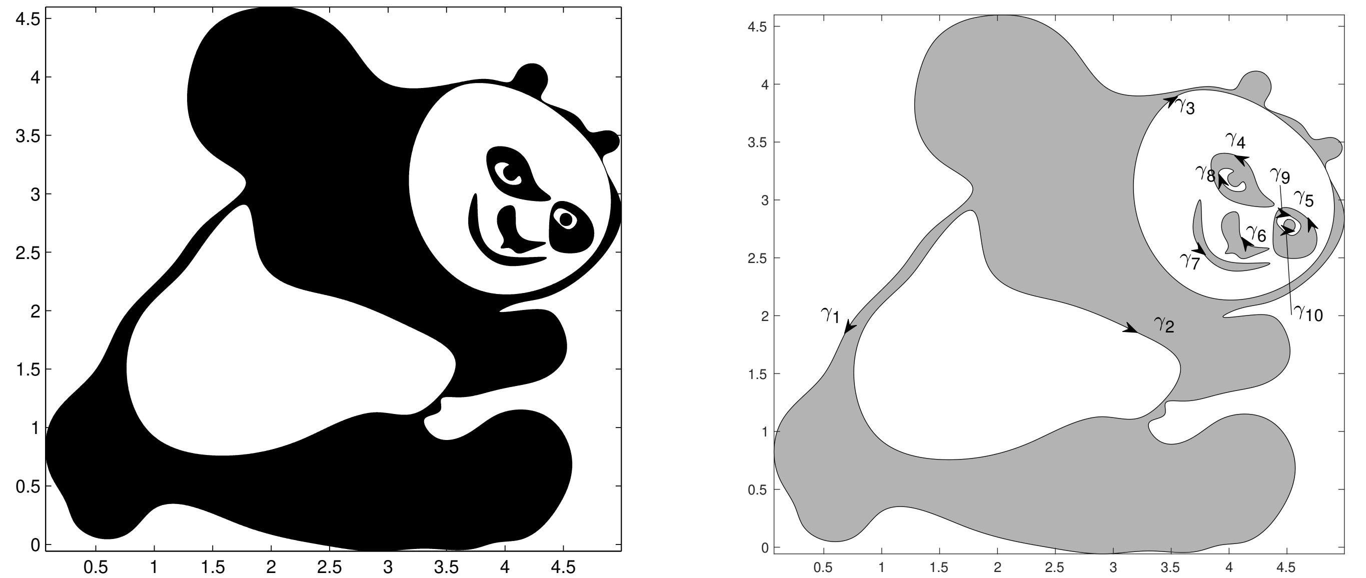
A *planar curve* is a continuous map $\gamma : (0,1) \to \mathbb{R}^2$. It is *smooth* if the map is smooth. It is *simple* if the map is injective; otherwise it is *self-intersecting*. Although strictly speaking a curve γ is a map, we also use γ to refer to its image. Two piecewise smooth curves γ_1 and γ_2 *intersect* at q if there exist $s_1, s_2 \in (0,1)$ such that $\gamma_1(s_1) = \gamma_2(s_2) = q$. Then q is the *intersection* of γ_1 and γ_2. For an open ball $\mathcal{N}_r(q)$ with sufficiently small radius r, $\mathcal{N}_r(q) \setminus \gamma_1$ consists of two disjoint connected regular open sets. If $\gamma_2 \setminus q$ is entirely contained in one of these two sets, q is an *improper intersection*; otherwise it is a *proper intersection*. Two curves are *disjoint* if they have neither proper intersections nor improper ones. Suppose upon its extension to a path, a simple curve γ further satisfies $\gamma(0) = \gamma(1)$, then γ is a *simple closed curve* or *Jordan curve*.

Theorem 5 (Jordan Curve Theorem)**.** The complement of a Jordan curve γ in the plane $\mathbb{R}^2$ consists of two components, each of which has γ as its boundary. One component is bounded and the other is unbounded; both of them are open and path-connected.

Definition 6. The *interior* of an oriented Jordan curve γ, denoted by $\mathrm{int}(\gamma)$, is the component of the complement of γ that always lies to the left when an observer traverses the curve in the increasing direction of its parameterization.

A Jordan curve is said to be *positively oriented* if its interior is the bounded component of its complement; otherwise it is *negatively oriented*.

Definition 7. Two Jordan curves are *almost disjoint* if they have no proper intersections and at most a finite number of improper intersections.



(a) a panda modeled as a Yin set $\mathcal{P}$

(b) representing $\mathcal{P}$ via Theorem 8

Figure 1: A Yin set with complex topology and geometry. In subplot (b), the Jordan curves γ_1, γ_4, γ_5, γ_6, γ_7, and γ_{10} are positively oriented while the others are negatively oriented.

2 Boolean Algebra on Yin Sets in $\mathbb{R}^2$

The following theorem furnishes a representation of Yin sets; see Figure 1 for an example.

Theorem 8. Each Yin set $\mathcal{Y} \neq \emptyset, \mathbb{R}^2$ can be uniquely expressed as

$$\mathcal{Y} = \bigcup_j^{\perp\perp} \bigcap_i \mathrm{int}\,(\gamma_{j,i})\,, \tag{3}$$

where j is the index of connected components of $\mathcal{Y}$ and $\gamma_{j,i}$'s are oriented Jordan curves that are pairwise almost disjoint.

Before you read the mathematical answers of this project, think about the following problems.

(A) Prove Theorem 8.

(B) Find a Boolean algebra

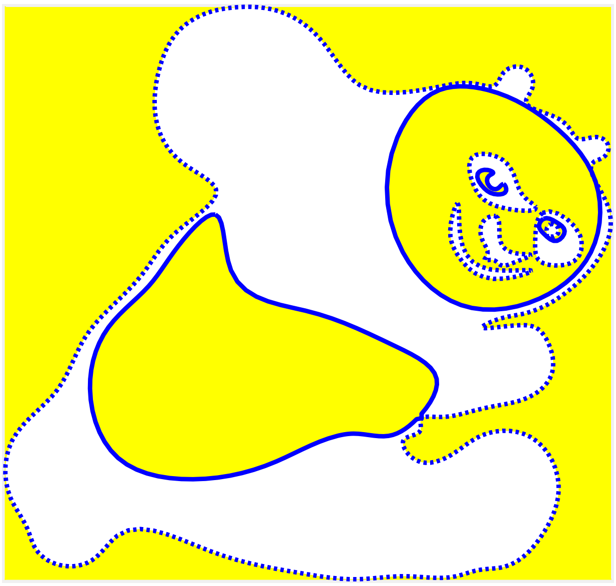
$$(\mathbb{J}, \vee, \wedge, ', \hat{0}, \hat{1}), \tag{4}$$

where $\mathbb{J}$ contains sets of pairwise almost disjoint oriented Jordan curves.

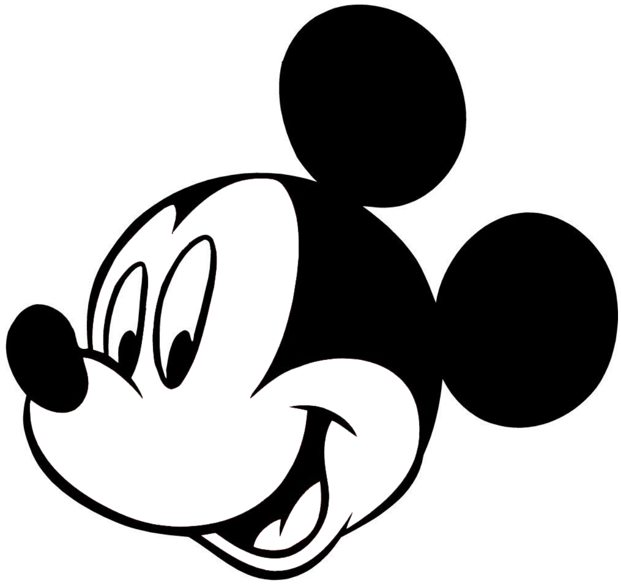
(C) Find an isomomorphism $\rho : \mathbb{J} \to \mathbb{Y}$ such that the Boolean algebras $(\mathbb{J}, \vee, \wedge, ', \hat{0}, \hat{1})$ and $(\mathbb{Y}, \cup^{\perp\perp}, \cap, ^{\perp}, \emptyset, \mathbb{R}^2)$ are isomorphic. Prove your conclusion.

(D) Design an algorithm for Boolean algebra on $\mathbb{Y}$, in such a way that first the operands and operation are sent to $\mathbb{J}$, then the operation is performed in $\mathbb{J}$, and finally the results in $\mathbb{J}$ are mapped back to $\mathbb{Y}$. You can approximate Jordan curves with linear polygons.

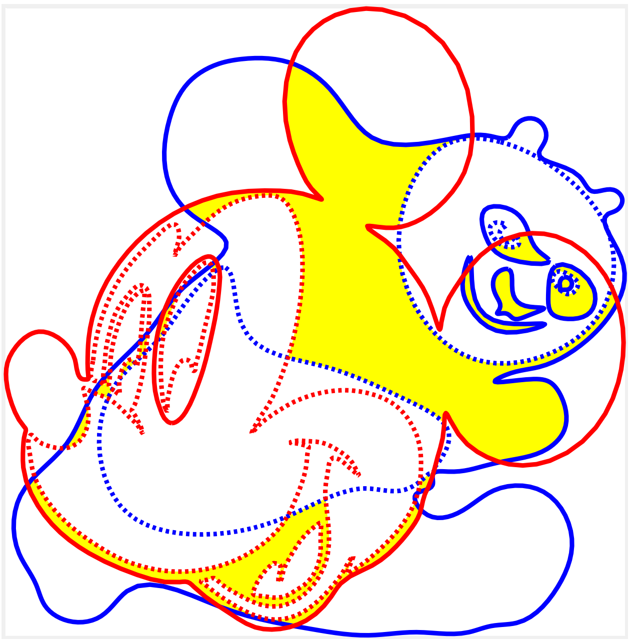
(E) Design an algorithm to calculate in $O(1)$ time the Betti numbers of any Yin set $\mathcal{Y}$, i.e. its number of components and the number of holes in each components.



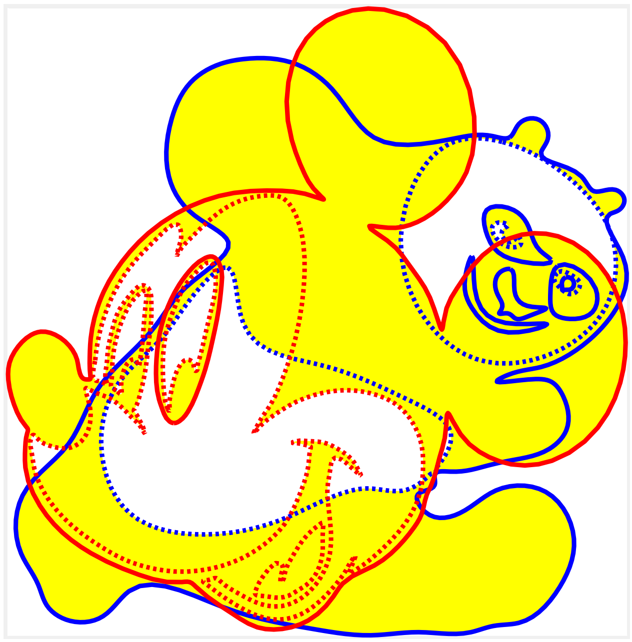
(a) $\mathcal{P}^{\perp}$: exterior of the panda $\mathcal{P}$.



(b) a Mickey mouse modeled as a Yin set $\mathcal{M}$



(c) $\mathcal{M} \cap \mathcal{P}$: intersection of $\mathcal{M}$ and $\mathcal{P}$.



(d) $\mathcal{M} \cup^{\perp\perp} \mathcal{P}$: regularized union of $\mathcal{M}$ and $\mathcal{P}$.

Figure 2: Testing Boolean algorithms on Yin sets with complex topology and geometry. In subplots (a), (c), and (d), a solid line represents a positively oriented Jordan curve, a dotted line a negatively oriented Jordan curve, and a shaded region the result of a Boolean operation. Make sure you have improper intersections in your tests.