Section 2. Basic formulas and identities in Riemannian geometry

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1. Bianchi identities

The **first and second Bianchi identities** are

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

and

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0.$$

We may prove the above formulas directly. But it would be easy to prove them by using the **Cartan structure equations**. The **twice contracted second Bianchi identity** is

$$g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R.$$

This is equivalent to the Einstein tensor $R_{ij} - \frac{1}{2}Rg_{ij}$ being divergence-free:

$$div\left(Rc - \frac{1}{2}Rg\right) = 0.$$

Theorem 1 (Schur) *Let M be connected and $\dim M \geq 3$. If the sectional curvature of M is constant at each point, i.e.*

$$K(P) = f(x) \quad \text{for all 2-plane } P \subset T_x M,$$

then $f(x) = const.$, and M is a space form. Likewise, if the Ricci curvature is constant at each point, i.e.

$$R_{ik} = c(x)\, g_{ij},$$

then $c(x) = const.$, and M is Einstein.

Proof. By the definition of the sectional curvature, we have

$$R_{ijkl} = f(x)(g_{ik}g_{jl} - g_{il}g_{jk})$$

then by the second Bianchi identity, we have (take normal coordinates at $x \in M$)

$$\begin{aligned} 0 &= R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} \\ &= f_{,m}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + f_{,k}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ &+ f_{,l}(\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm}) \end{aligned}$$

Since we assume $\dim M \geq 3$, for each m, we can find m, i, j, k, l with $i = k, j = l$ and $i \neq m, j \neq m$ and $i \neq j$. Then we have $f_{,m} = 0$. Since M is connected, then $f(x) = const.$. The second claim follows in the same manner. QED

2. Lie derivative.

For fixed point $q \in M$, one seeks a curve through q whose tangent vector at each point coincides with the value of a given vector X at this point, i.e. a curve $c_q(t)$ is always tangent to the vector field X. Now we want to fix t and vary q; we put

$$\varphi_t(q) := c_q(t).$$

Theorem 2. *We have*

$$\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q) \quad \text{if } s, t, s+t \in I_q \tag{2.1}$$

and if φ_t is defined on $U \subset M$, it maps U diffeomorphically onto its image.

Proof. We have

$$\dot{c}_q(t+s) = X(c_q(t+s)),$$

hence

$$c_q(t+s) = c_{c_q(s)}(t).$$

Starting from q, at time s one reaches the point $c_q(s)$, and if one proceeds a time t further, one reaches $c_q(t+s)$. One therefore reaches the same point if one walks from q on the integral curve for a time $t+s$, or if one walks a time t from $c_q(s)$. This shows (2.1). Inserting $t = -s$ into (2.1) for $s \in I_q$, we obtain

$$\varphi_{-s} \circ \varphi_s(q) = \varphi_0(q) = q.$$

Thus the map φ_{-s} is the inverse of φ_s, and the diffeomorphism property follows. QED

Definition 1 *A family $(\varphi_t)_{t\in I}$ (I open interval with $0 \in I$) of diffeomorphisms from M to M satisfying (2.1) is called a local **1-parameter group** of diffeomorphisms.*

In general, a local 1-parameter group need not be extendable to a group, since the maximal interval of definition I_q of c_q need not be all of R. This is already seen by easy example, e.g. $M = R, X(\tau) = \tau^2 \frac{d}{d\tau}$, i.e. $\dot{c}(t) = c^2(t)$ as differential equation.

However, *on a compact differentiable manifold, any vector field generates a 1-parameter group of diffeomorphisms.*

We call a vector field X is **complete** if there is a **1-parameter group** of diffeomorphisms $\{\varphi_t\}_{t\in R}$ generated by X. If M^n is closed, then any smooth vector field is complete. Let α be a tensor of type (r,s) and let X be a complete vector field generating a global 1-parameter group of diffeomorphisms φ_t. The Lie derivative of α with respect to X is defined by

$$L_X\alpha =: \lim_{t\to 0} \frac{1}{t}\left(\alpha - (\varphi_t)_* \alpha\right). \tag{2.2}$$

Here $(\varphi_t)_* : T_pM \to T_{\varphi_t(p)}M$ is the differential of φ_t. It acts on the cotangent bundle by $(\varphi_t)_* = \left(\varphi_t^{-1}\right)^* : T_p^*M \to T^*_{\varphi_t(p)}M^n$. We can naturally extend the action of $(\varphi_t)_*$ to the tensor bundles of M^n, which is used in (2.2).

The Lie derivative, which measures the infinitesimal lack of diffeomorphism invariance of a tensor with respect to a 1-parameter group of diffeomorphism generated by a vector field, has the following properties:

(1) If f is a function, then $L_X f = X(f)$.

(2) If Y is a vector field, then $L_X Y = [X,Y]$.

(3) If α and β are tensors, then $L_X(\alpha\otimes\beta) = (L_X\alpha)\otimes\beta + \alpha\otimes(L_X\beta)$.

(4) If α is an $(0,r)-$tensor, then for any vector fields $X, Y_1, ..., Y_r$

$$\begin{aligned}(L_X\alpha)(Y_1,...,Y_r) &= X(\alpha(Y_1,...,Y_r)) \\ &\quad - \Sigma_{i=1}^r \alpha(Y_1,...,Y_{i-1},[X,Y_i],Y_{i+1},...,Y_r) \\ &= (\nabla_X\alpha)(Y_1,...,Y_r) \\ &\quad + \Sigma_{i=1}^r \alpha(Y_1,...,Y_{i-1},\nabla_{Y_i}X,Y_{i+1},...,Y_r)\end{aligned} \tag{2.3}$$

For example, if α is a 2-tensor, then

$$(L_X\alpha)_{ij} = (\nabla_X\alpha)_{ij} + g^{kl}\left(\nabla_i X_k \alpha_{lj} + \nabla_j X_k \alpha_{il}\right).$$

Recall that the gradient of a function f with respect to the metric g is defined by

$$g\left(grad_g f, X\right) := Xf = df\left(X\right).$$

In other words, $grad_g f$ is the dual of the 1-form df. using (2.3), we may show that the Lie derivative of the metric is given by

$$\left(L_X g\right)\left(Y_1, Y_2\right) = g\left(\nabla_{Y_1} X, Y_2\right) + g\left(Y_1, \nabla_{Y_2} X\right).$$

In local coordinates, this implies

$$\left(L_X g\right)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

In particular, if f is a function, then

$$\left(L_{grad_g f} g\right)_{ij} = 2\nabla_i \nabla_j f.$$

We say that a vector field X on (M^n, g) is **Killing** if $L_X g = 0$. If X is a complete Killing vector field, then a 1-parameter group of diffeomorphisms φ_t that it generates is a 1-parameter group of isometries of (M^n, g).

Definition 2. *We say that a diffeomorphism $\psi : (M^n, g) \to (N^n, h)$ is an isometry if $\psi^* h = g$. Two Riemannian manifolds are said to be isometric if there is an isometry from one to the other.*

3. Differential forms.

The **volume form** $d\mu$ of an oriented riemannian manifold (M^n, g) is defined in terms of a positively oriented orthonormal coframe $\{\omega^i\}_{i=1}^n$ by

$$d\mu = \omega^1 \wedge ... \wedge \omega^n.$$

The volume satisfies $(d\mu)(e_1, ..., e_n) = 1$, where $\{e_i\}_{i=1}^n$ is the orthonormal frame dual to $\{\omega^i\}_{i=1}^n$ (i.e. $\omega^i(e_j) = \delta^i_j$). In a positively oriented local coordinate system $\{x^i\}$, we have

$$d\mu = \sqrt{\det\left(g_{ij}\right)} dx^1 \wedge ... \wedge dx^n.$$

In general, the wedge product of a p-form α and a q-form β is defined by

$$\begin{aligned} &\alpha \wedge \beta\left(X_1, ..., X_{p+q}\right) \\ &=: \frac{(p+q)!}{p!q!} \mathcal{A}\left(\alpha \otimes \beta\right)\left(X_1, ..., X_{p+q}\right) \\ &= \Sigma_{(J,K)} sign\left(J, K\right) \alpha\left(X_{j_1}, ..., X_{j_p}\right) \beta\left(X_{k_1}, ..., X_{k_q}\right), \end{aligned}$$

where $J = (j_1, ..., j_p)$ and $K = (k_1, ..., k_q)$ are multi-indices and sign (J, K) is the sign of the permutation $(1, ..., p+q) \mapsto (j_1, ..., j_p, k_1, ..., k_q)$. $\mathcal{A}_p$ is the operator: $\otimes^p T^*M \to \wedge^p T^*M \subset \otimes^p T^*M$,

$$\mathcal{A}_p(\alpha) = \frac{1}{p!}\Sigma_J (signJ)\, J\alpha.$$

Recall that the **exterior derivative** of a p-form β satisfies

$$\begin{aligned}
&(d\beta)(Y_1, ..., Y_{p+1}) \\
&= \Sigma_{i=1}^{p+1} (-1)^{i-1} Y_i \left(\beta\left(Y_1, ..., \hat{Y}_i, ..., Y_{p+1}\right)\right) \\
&+ \Sigma_{i<j} (-1)^{i+j} \beta\left([Y_i, Y_j], Y_1, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_{p+1}\right).
\end{aligned}$$

Using the product rule and the fact that ∇ is torsion-free, we may express $d\beta$ in terms of covariant derivatives as

$$(d\beta)(Y_1, ..., Y_{p+1}) = \Sigma_{j=1}^{p+1} (-1)^{j-1} \left(\nabla_{Y_j}\beta\right)\left(Y_1, ..., \hat{Y}_j, ..., Y_{p+1}\right). \qquad (2.4)$$

In local coordinates, this is

$$(d\beta)_{i_1 \ldots i_{p+1}} = \Sigma_{j=1}^{p+1} (-1)^{j-1} \beta_{i_1 \ldots \overset{\wedge}{i_j} \ldots i_{p+1}, i_j}$$

where $\beta_{i_1 \ldots i_p} = \beta\left(\frac{\partial}{\partial x^{i_1}}, ..., \frac{\partial}{\partial x^{i_p}}\right)$. For example, if β is a 1-form, then

$$(d\beta)_{ij} = \beta_{j,i} - \beta_{i,j}.$$

If β is a 2-form, then

$$\begin{aligned}
(d\beta)_{ijk} &= \beta_{jk,i} - \beta_{ik,j} + \beta_{ij,k} \\
&= \beta_{jk,i} + \beta_{ki,j} + \beta_{ij,k}.
\end{aligned}$$

Now we define the **divergence** of a $(0, p)$-tensor as

$$(div\alpha)_{i_1 \ldots i_{p-1}} = g^{jk} \alpha_{k i_1 \ldots i_{p-1}, j}.$$

In particular, if X is a 1-form, then

$$div(X) = g^{ij} X_{j,i}.$$

Given a p-form β and a vector field X, we define the **interior product** by

$$(\iota_X\beta)(Y_1,...,Y_{p-1}) =: \beta(X,Y_1,...,Y_{p-1})$$

for all vector fields $Y_1,...,Y_{p-1}$. Recall that for any vector field X and any differential form γ

$$L_X\gamma = (d\circ\iota_X + \iota_X\circ d)\gamma. \tag{2.5}$$

The **inner product** on $\wedge^p T^*M$ is defined by

$$\langle\gamma,\eta\rangle =: g^{i_1j_1}...g^{i_pj_p}\gamma_{i_1...i_p}\eta_{j_1...j_p}.$$

For example,

$$\left\langle \omega^{i_1}\wedge...\wedge\omega^{i_p}, \omega^{j_1}\wedge...\wedge\omega^{j_p}\right\rangle = \frac{1}{p!}\det\left(\delta^{i_kj_l}\right).$$

Recall that given p-forms γ and η, their L^2-**inner product** is defined by

$$(\gamma,\eta)_{L^2} =: \int_{M^n}\langle\gamma,\eta\rangle\, d\mu.$$

The **Hodge star operator** $* : \wedge^pT^*M \to \wedge^{n-p}T^*M$, $p = 0,1,...,n$ is defined by

$$\langle\gamma,\eta\rangle\, d\mu = \gamma\wedge*\eta$$

for any $\gamma,\eta\in\wedge^pT^*M$. For example

$$*\left(\omega^1\wedge...\wedge\omega^p\right) = \omega^{p+1}\wedge...\wedge\omega^n$$

for a positively oriented orthonormal coframe $\{\omega^i\}_{i=1}^n$.

The **adjoint operator** δ of d acting on a p-form α is defined in terms of d and the Hodge star operator by the formula

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha.$$

In terms of covariant derivatives, the adjoint δ is given by

$$(\delta\alpha)(X_1,...,X_{p-1}) = -\Sigma_{i=1}^p(\nabla_{e_i}\alpha)(e_i,X_1,...,X_{p-1}), \tag{2.6}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal frame. That is, $\delta\alpha = -div\alpha$, or in local coordinates,

$$(\delta\alpha)_{i_1...i_{p-1}} = -g^{jk}\alpha_{ki_1...i_{p-1},j}.$$

One easily verifies that

$$(d\beta, \alpha)_{L^2} = (\beta, \delta\alpha)_{L^2}$$

where $\alpha \in \wedge^p T^*M$ and $\beta \in \wedge^{p-1} T^*M$.

The **Hodge Laplacian** acting on differential p-forms is defined by

$$\Delta_d =: -(d\delta + \delta d).$$

Note that Δ_d is a self-adjoint operator. Acting on **functions**, it is the same as the **Laplace-Beltrami operator** defined by

$$\Delta =: div\nabla = g^{ij}\nabla_i\nabla_j = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right).$$

There are other equivalent ways to define Δ, such as

$$\Delta f = \Sigma_{a=1}^n e_a (e_a f) - (\nabla_{e_a} e_a) f,$$

where $\{e_a\}_{a=1}^n$ is an orthonormal frame. Or

$$\begin{aligned} \Delta f &= Trace\,(Hess f) = Trace \nabla\nabla f \\ &= \Sigma_{a=1}^n \nabla^2 f(e_a, e_a), \end{aligned}$$

or

$$\Delta f = \frac{1}{\sqrt{|g|}} \Sigma_{i,j=1}^n \frac{\partial}{\partial x^i}\left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j}\right),$$

where $|g| = \det(g_{ij})$.

More generally, the **(rough) Laplacian operator** acting on tensors is given by

$$\Delta = div\nabla = trace_g \nabla^2 = g^{ij}\nabla_i\nabla_j =: \nabla_i\nabla_i.$$

More explicitly, given an (r,s)-tensor β, $\nabla\nabla\beta$ is an $(r, s+2)$-tensor, which we contract to get

$$\Delta\beta\left(\omega^1, ..., \omega^r, X_1, ..., X_s\right) = \sum_{a=1}^n \nabla\nabla\beta\left(\omega^1, ..., \omega^r, X_1, ..., X_s; e_a, e_a\right)$$

for all covectors $\omega^1, ..., \omega^r$ and vectors $X_1, ..., X_s$.

Lemma 1. *For any function f, we have*

$$\Delta\nabla_i f = \nabla_i \Delta f + R_{ij} f.$$

Proof. This follows from

$$f_{i,jj} = f_{j,ij} = f_{j,ji} + R^k_{jij} f_k.$$

QED

Similarly, the **Bochner formula** for $|\nabla f|^2$ is

$$\Delta |\nabla f|^2 = 2 |\nabla\nabla f|^2 + 2R_{ij}\nabla_i f \nabla_j f + 2\nabla_i f \nabla_i (\Delta f).$$

If X is a 1-form, then

$$\Delta X_i - R_{ij} X_j = \Delta_d X_i.$$

If β is a 2-form, then

$$(\Delta_d \beta)_{ij} = \Delta\beta_{ij} + 2R_{ikjl}\beta_{kl} - R_{ik}\beta_{kj} - R_{jk}\beta_{ik}.$$

Now let α be a differential p-form. Using (2.4) and (2.6), in local coordinates we may write the Hodge Laplacian as

$$\begin{aligned}(\Delta_d \alpha)_{i_1 \dots i_r} &= (-1)^{j+1} g^{kl} \nabla_{i_j} \nabla_k \alpha_{l i_1 \dots \hat{i}_j \dots i_r} + g^{kl} \nabla_k \nabla_l \alpha_{i_1 \dots i_r} \\ &\quad + (-1)^j g^{kl} \nabla_k \nabla_{i_j} \alpha_{l i_1 \dots \hat{i}_j \dots i_r} \\ &= (\Delta\alpha)_{i_1 \dots i_r} + (-1)^j g^{kl} \left(\nabla_k \nabla_{i_j} - \nabla_{i_j} \nabla_k \right) \alpha_{l i_1 \dots \hat{i}_j \dots i_r}.\end{aligned}$$

4. Integration by parts.

Recall that **Stokes' Theorem** says that

Theorem 3. *If α is an $(n-1)$-form on a compact oriented manifold M^n with (possibly empty) boundary ∂M, then*

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

Let (M^n, g) be an oriented Riemannian manifold with boundary ∂M. The orientation on M defines an orientation on ∂M. Locally, on the boundary, choose a positively oriented frame field $\{e_i\}_{i=1}^n$ such that $e_1 = \nu$ is the unit outward normal. Then the frame field $\{e_i\}_{i=2}^n$ is positively oriented on ∂M. Let $\{\omega^i\}_{i=1}^n$ denote the orthonormal coframe field dual to $\{e_i\}_{i=1}^n$. The volume form of M is

$$d\mu = \omega^1 \wedge \dots \wedge \omega^n$$

and the volume form of ∂M is

$$d\sigma = \omega^2 \wedge ... \wedge \omega^n.$$

It is not difficult to see that

$$d\sigma = \iota_\nu (d\mu).$$

Indeed,

$$\iota_\nu (d\mu)(e_2, ..., e_n) = (d\mu)(e_1, e_2, ..., e_n) = 1.$$

The **divergence theorem** says

Theorem 4. *Let (M^n, g) be a compact oriented Riemannian manifold. If X is a vector field, then*

$$\int_M div(X)\, d\mu = \int_{\partial M} \langle X, \nu \rangle\, d\sigma,$$

where $div(X) = X^i{}_{,i}$.

Proof. Define the $(n-1)$-form α by

$$\alpha = \iota_X (d\mu).$$

Using $d^2 = 0$ and (2.5) we compute

$$\begin{aligned} d\alpha &= d \circ \iota_X (d\mu) = (d \circ \iota_X + \iota_X \circ d)(d\mu) \\ &= L_X (d\mu) = div(X)\, d\mu, \end{aligned}$$

where to obtain the last equality, we may compute in an orthonormal frame $e_1, ..., e_n$:

$$\begin{aligned} L_X (d\mu)(e_1, ..., e_n) &= \sum_{i=1}^n d\mu (e_1, ..., \nabla_{e_i} X, ..., e_n) \\ &= div(X)\, d\mu (e_1, ..., e_n). \end{aligned}$$

Now Stokes' Theorem implies that

$$\begin{aligned} \int_M div(X)\, d\mu &= \int_M d\alpha = \int_{\partial M} \alpha \\ &= \int_{\partial M} \iota_X (d\mu) = \int_{\partial M} \langle X, \nu \rangle\, d\sigma. \end{aligned}$$

To verify the last equality, we used

$$\begin{aligned}(\iota_X (d\mu)) (e_2, ..., e_n) &= (d\mu) (X, e_2, ..., e_n) \\ &= \langle X, \nu \rangle (d\mu) (e_1, ..., e_n) \\ &= \langle X, \nu \rangle .\end{aligned}$$

Hence the theorem is proved. QED

Corollary 1. *Let (M^n, g) be a compact oriented Riemannian manifold. If α is an (r, s)-tensor and β is an $(r, s-1)$-tensor, then*

$$\int_M \langle \alpha, \nabla \beta \rangle \, dV = -\int_M \langle div (\alpha), \beta \rangle \, dV.$$

Proof. Let $X_j = \alpha^{k_1...k_r}_{ji_2...i_s} \beta^{k_1...k_r}_{i_2...i_s}$. We compute that

$$divX = \langle div (\alpha), \beta \rangle + \langle \alpha, \nabla \beta \rangle ,$$

and the result follows from the divergence theorem. QED