PDE Homework #4

李阳 11935018

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Problem 1. *A linear form u on $\mathcal{D}(\Omega)$ is continuous ($u(\phi_j) \to 0$ for every sequence $\phi_j \in \mathcal{D}(\Omega)$ converging to 0) iff it satisfies the following property: for any compact set $K \subset \Omega$ there exist an integer k and a constant $C = C_{K,k}$ such that*

$$|\langle u, \phi\rangle| \le C p_{K,k}(\phi), \quad \forall \phi \in \mathcal{C}_c^\infty(K).$$

Proof.

- Sufficiency: Let $\phi_j \in \mathcal{D}(\Omega)$ be a sequence converging to 0, then the definition of the topology of $\mathcal{C}_c^\infty(\Omega)$ yields
  (1) there exists a compact set $K \subset \Omega$, $\mathrm{supp}\phi_j \subset K$, for all $j \ge 1$.
  (2) for any k,
  $$p_{K,k}(\phi_j) := \sup_{|\alpha|\le k}\sup_{x\in K} |\partial^\alpha \phi_j(x)| \to 0 \text{ as } j \to \infty.$$
  By assumption
  $$|u(\phi_j)| = |\langle u, \phi_j\rangle| \le C p_{K,k}(\phi_j) \to 0 \text{ as } j \to \infty.$$
  Therefore u is continuous.
- Necessity: We argue by contradiction. If not, then
  $$\exists K \subset \Omega, \forall k > 0 \text{ s.t. } \exists \phi_k \in \mathcal{C}_c^\infty(K) \text{ s.t. } u(\phi_k) \ge k^2 P_{K,k_0}(\phi_k).$$
  Let $\Phi_k = \frac{\phi_k}{kP_{K,k_0}(\phi_k)} \in \mathcal{C}_c^\infty(K)$, we have
  $$P_{K,k_0}(\Phi_k) \le k^{-1} \to 0 \text{ as } k \to \infty \Rightarrow \Phi_k \to 0 \text{ in } \mathcal{C}_c^\infty(K).$$
  However, by the construction of Φ_k,
  $$u(\Phi_k) = u\left(\frac{\phi_k}{kP_{K,k_0}(\phi_k)}\right) \ge k,$$
  which contradicts the continuity of u.

□

Problem 2. *Let $g \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} g\mathrm{d}\mathbf{x} = 1$, then $g_\epsilon(\mathbf{x}) = \epsilon^{-n}g(\epsilon^{-1}\mathbf{x})$ converges to δ as $\epsilon \to 0^+$, in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. By the change of variable $\mathbf{x} \to \epsilon\mathbf{x}$ we see that $\int_{\mathbb{R}^n} g_\epsilon(\mathbf{x})\mathrm{d}\mathbf{x} = 1$ for all $\epsilon > 0$. Hence $\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\langle g_\epsilon, \phi\rangle - \langle \delta, \phi\rangle &= \int_{\mathbb{R}^n} g_\epsilon(\mathbf{x})\phi(\mathbf{x})\mathrm{d}\mathbf{x} - \phi(\mathbf{0}) = \int_{\mathbb{R}^n} \epsilon^{-n} g(\epsilon^{-1}\mathbf{x})\phi(\mathbf{x})\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^n} g(\mathbf{x})\phi(\mathbf{0})\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^n} g(\mathbf{x})\phi(\epsilon\mathbf{x})\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^n} g(\mathbf{x})\phi(\mathbf{0})\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^n} g(\mathbf{x})\left(\phi(\epsilon\mathbf{x}) - \phi(\mathbf{0})\right)\mathrm{d}\mathbf{x} \to 0 \text{ as } \epsilon \to 0^+
\end{aligned}$$

by the dominated convergence theorem. Therefore g_ϵ converges to δ as $\epsilon \to 0^+$, in $\mathcal{D}'(\mathbb{R}^n)$. □

Problem 3. *Let $f = \ln|x| \in \mathcal{D}'(\mathbb{R})$, check that $f' \in \mathcal{D}'(\mathbb{R})$ is given by*

$$\left\langle \mathrm{pv}\frac{1}{x}, \phi\right\rangle = \lim_{\epsilon\to 0^+}\int_{|x|\ge\epsilon} \frac{\phi(x)}{x}\mathrm{d}x, \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

Proof.

$$\begin{aligned}\int_{|x|\geq\epsilon}\frac{\phi(x)}{x}\mathrm{d}x &= \int_{-\infty}^{-\epsilon}\frac{\phi(x)}{x}\mathrm{d}x + \int_{\epsilon}^{\infty}\frac{\phi(x)}{x}\mathrm{d}x \\ &= \phi(-\epsilon)\ln\epsilon - \int_{-\infty}^{-\epsilon}\phi'(x)\ln|x|\mathrm{d}x - \phi(\epsilon)\ln\epsilon - \int_{\epsilon}^{\infty}\phi'(x)\ln|x|\mathrm{d}x \\ &= -(\phi(\epsilon)-\phi(-\epsilon))\ln\epsilon - \int_{|x|\geq\epsilon}\phi'(x)\ln|x|\mathrm{d}x,\end{aligned}$$

since $(\phi(\epsilon)-\phi(-\epsilon))\ln\epsilon \to 0$ (Taylor expansion), therefore

$$\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\frac{\phi(x)}{x}\mathrm{d}x = -\int_{-\infty}^{\infty}\phi'(x)\ln|x|\mathrm{d}x,$$

which completes the proof. □

Problem 4. *Define* $\hat{K}_\delta = |\xi|^{-2}e^{-\delta|\xi|^2}$, *show that*

$$\mathcal{F}^{-1}\hat{K}_\delta = \frac{1}{8\pi^{3/2}r}\int_{|\lambda|\geq r/\sqrt{\delta}} e^{-\frac{\lambda^2}{4}}\mathrm{d}\lambda \to \frac{(4\pi)^{1/2}}{8\pi^{3/2}r} = \frac{1}{4\pi r}.$$

Proof.

$$\begin{aligned}\mathcal{F}^{-1}\hat{K}_\delta &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\mathbf{x}\cdot\xi}|\xi|^{-2}e^{-\delta|\xi|^2}\mathrm{d}\xi = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-\frac{|\mathbf{x}|^2}{4\delta}}|\xi|^{-2}e^{-\left|\sqrt{\delta}\xi+\frac{i\mathbf{x}}{2\sqrt{\delta}}\right|^2}\mathrm{d}\xi \\ &= \frac{1}{8\pi^{3/2}r}\int_{|\lambda|\geq r/\sqrt{\delta}} e^{-\frac{\lambda^2}{4}}\mathrm{d}\lambda \to \frac{1}{8\pi^{3/2}r}\int_{\mathbb{R}} e^{-\frac{\lambda^2}{4}}\mathrm{d}\lambda \to \frac{(4\pi)^{1/2}}{8\pi^{3/2}r} = \frac{1}{4\pi r}.\end{aligned}$$

□

Problem 5. *Prove that the following holds in the sense of tempered distributions* $\mathcal{S}'(\mathbb{R}^2)$.

$$(|x|^{-\epsilon}-1)\epsilon^{-1} = [\exp(\epsilon\ln|x|^{-1})-1]\epsilon^{-1} \to \ln(|x|^{-1}).$$

Proof. We have

$$\forall x\in\mathbb{R}^2, \quad [\exp(\epsilon\ln|x|^{-1})-1]\epsilon^{-1} - \ln(|x|^{-1}) \to 0 \text{ as } \epsilon\to 0^+.$$

Therefore, by the dominated convergence therem, $\forall\phi\in\mathcal{S}(\mathbb{R}^2)$,

$$\left\langle[\exp(\epsilon\ln|x|^{-1})-1]\epsilon^{-1},\phi\right\rangle - \left\langle\ln(|x|^{-1}),\phi\right\rangle = \int_{\mathbb{R}^2}\left([\exp(\epsilon\ln|x|^{-1})-1]\epsilon^{-1} - \ln(|x|^{-1})\right)\phi(x)\mathrm{d}x \to 0 \text{ as } \epsilon\to 0^+,$$

which gives the desired result. □