3.1 PDEhw4 12235005 谭焱

1. Prove the following Proposition.

Proposition 3.1. *A linear form u on $\mathcal{D}(\Omega)$ is continuous ($u(\phi_j) \to 0$ for every sequence $\phi_j \in \mathcal{D}(\Omega)$ converging to 0) iff it verifies the following property: for any compact set $K \subset \Omega$ there exists an integer k and a constant $C = C_{K,k}$ such that*

$$|\langle u, \phi\rangle| \leq C p_{K,k}(\phi), \forall \phi \in C_c^\infty(K).$$

Solution.

- Sufficiency: Let $\phi_j \in \mathcal{D}(\Omega)$ be a sequence converging to 0, then the definition of the topology of $\mathcal{C}_c^\infty(\Omega)$ yields there is a compact set $K \subset \Omega$, supp $\phi_j \subset K$, for all $j \geq 1$. and for any k,

$$p_{K,k} := \sup_{|\alpha|\leq k} \sup_{x\in K} |\partial^\alpha \phi_j(x)| \to 0 \text{ as } j \to \infty.$$

  Combining with assumption

$$|u(\phi_j)| = |\langle u, \phi_j\rangle| \leq C p_{K,k}(\phi_j) \to 0 \text{ as } j \to \infty.$$

- Necessity: Assuming u is continuous and

$$\exists K \subset \Omega, \forall k > 0, \exists \phi_j \in \mathcal{C}_c^\infty(K) s.t. u(\phi_k) > C p_{K,k}.$$

  Choosing $C = j^2$ and $\Phi_j = \frac{\phi_j}{j p_{K,k}(\phi_j)} \in \mathcal{C}_c^\infty(K)$. Then

$$p_{K,k}(\Phi_j) = \frac{p_{K,k}(\phi_j)}{j p_{K,k}(\phi_j)} = \frac{1}{j} \to 0, \text{ as } j \to \infty$$
$$u(\Phi_j) = \frac{u(\phi_j)}{j p_{K,k}(\phi_j)} \geq \frac{C p_{K,k}(\phi_j)}{j p_{k,k}(\phi_j)} = j$$

  Which is conflict with u is continuous.

□

2. Prove the following lemma

Lemma 3.2. *Let $g \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} g dx = 1$, then $g_\epsilon(x) = \epsilon^{-1} g(\epsilon^{-1}x)$ converges to δ as $\epsilon \to 0+$, in $\mathcal{D}'(\mathbb{R})$.*

Solution. Since $\int_{\mathbb{R}} g(x)dx = \epsilon^{-1}\int_{\mathbb{R}} g(\epsilon^{-1}(\epsilon x))d(\epsilon x) = \int_{\mathbb{R}} g_\epsilon(x)dx$. By definition of converges, considering

$$\begin{aligned}
\langle g_\epsilon - \delta, \phi\rangle &= \int_{\mathbb{R}} g_\epsilon(x)\phi(x)dx - \int_{\mathbb{R}} g(x)\phi(0)dx = \int_{\mathbb{R}} g(x)(\phi(\epsilon x) - \phi(0))dx \\
&\leq \left(\int_{\mathbb{R}} |g(x)|^2 dx \int_{\mathbb{R}} |\phi(\epsilon x) - \phi(0)|^2 dx\right)^{1/2} \\
&= C\left(\int_{\mathbb{R}} |\phi(\epsilon x) - \phi(0)|\right)^{1/2} \to 0 \qquad \text{as } \epsilon \to 0+.
\end{aligned}$$

Therefore, $g_\epsilon(x)$ converges to δ. □

3. As $\delta \to 0+$,

$$K_\delta(\xi) = |\xi|^{-2} e^{-\delta|\xi|} \to |\xi|^{-2}, \text{ in } S'(\mathbb{R}^3).$$

Solution. By definition of converges, considering

$$\begin{aligned}
\langle K_\delta(\xi) - |\xi|^{-2}, \phi(\xi)\rangle &= \int_{\mathbb{R}^3} (e^{-\delta|\xi|} - 1)|\xi|^{-2}\phi(\xi)d\xi = \int_0^{2\pi}\int_0^{2\pi}\int_0^\infty (-r\delta + \mathcal{O}(\delta^2))\phi d\theta_1 d\theta_2 dr \\
&\to 0 \qquad \text{as } \delta \to 0+.
\end{aligned}$$

□