Chapter 1

General Topology

1.1 Continuous functions

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a* iff

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ s.t. } |x - a| < \delta \ \Rightarrow \ |f(x) - f(a)| < \epsilon. \quad (1.1)$$

Definition 1.2. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* $\mathbf{x} = a$ iff

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ s.t. } f(B(a, \delta)) \subset B(f(a), \epsilon), \quad (1.2)$$

where the n-dimensional open ball $B(p, r)$ is

$$B(p, r) = \{x \in \mathbb{R}^n : \|x - p\|_2 < r\}. \quad (1.3)$$

Definition 1.3. A function $f : X \to Y$ with $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is *continuous at* $\mathbf{x} = a$ iff

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ s.t. } f(V_a) \subset U_a, \quad (1.4)$$

where the two sets associated with a are

$$V_a := B(a, \delta) \cap X, \qquad U_a := B(f(a), \epsilon) \cap Y. \quad (1.5)$$

Definition 1.4. A function f is *continuous* if it is continuous at every point of its domain.

Definition 1.5. A function $f : X \to Y$ with $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is *continuous* iff

$$\forall U_a \in \gamma_Y, \exists V_a \in \gamma_X \text{ s.t. } f(V_a) \subset U_a, \quad (1.6)$$

where γ_X and γ_Y are sets of intersections of the open balls to X and Y, respectively,

$$\gamma_X := \{B(a, \delta) \cap X : a \in X, \delta \in \mathbb{R}^+\};$$
$$\gamma_Y := \{B(f(a), \epsilon) \cap Y : f(a) \in Y, \epsilon \in \mathbb{R}^+\}.$$

Example 1.1. Is the function $x \mapsto \frac{1}{x}$ continuous? It depends on whether its domain includes the origin. But it is indeed continuous on domains such as $(0, 1]$, $\mathbb{R} \setminus \{0\}$, and $[1, 2]$. Note that definitions of the one-sided continuity in calculus are nicely incorporated in Definition 1.3.

Definition 1.6. A *basis of neighborhoods* (or a *basis*) on a set X is a collection $\mathcal{B}$ of subsets of X such that

- covering: $\cup \mathcal{B} = X$, and
- refining:

$$\forall U, V \in \mathcal{B}, \forall x \in U \cap V, \ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset (U \cap V).$$

Definition 1.7. For two sets X, Y with bases of neighborhoods $\mathcal{B}_X$, $\mathcal{B}_Y$, a surjective function $f : X \to Y$ is *continuous* iff

$$\forall U \in \mathcal{B}_Y \ \exists V \in \mathcal{B}_X \text{ s.t. } f(V) \subset U. \quad (1.7)$$

Lemma 1.8. If a surjective function $f : X \to Y$ is continuous in the sense of Definitions 1.3 and 1.4, then it is continuous in the sense of Definition 1.7.

Proof. By Definition 1.6, the following collections are bases of $X \subseteq \mathbb{R}^m$ and $Y = f(X) \subseteq \mathbb{R}^n$, respectively,

$$\mathcal{B}_X = \{B(a, \delta) \cap X : a \in X, \delta > 0\};$$
$$\mathcal{B}_Y = \{B(b, \epsilon) \cap Y : b \in Y, \epsilon > 0\}.$$

The rest follows from Definitions 1.7 and 1.3. □

Example 1.2. The *right rays*

$$\mathcal{B}_{RR} = \{\{x : x > s\} : s \in \mathbb{R}\} \quad (1.8)$$

form a basis of $\mathbb{R}$.

Exercise 1.3. Prove that the set of all right half-intervals in $\mathbb{R}$ is a basis of neighborhoods:

$$\mathcal{B} = \{[a, b) : a < b\}. \quad (1.9)$$

Example 1.4. A basis on $\mathbb{R}^2$ is the set of all quadrants

$$\mathcal{B}_q = \{Q(r, s) : r, s \in \mathbb{R}\}, \quad (1.10)$$
$$Q(r, s) = \{(x, y) \in \mathbb{R}^2 : x > r, y > s\}. \quad (1.11)$$

Exercise 1.5. Prove that the collection of all open squares in $\mathbb{R}^2$ is a basis of $\mathbb{R}^2$,

$$\mathcal{B}_s = \{S((a, b), d) : (a, b) \in \mathbb{R}^2, d > 0\},$$

where $S((a, b), d) = \{(x, y) : \max(|x - a|, |y - b|) < d\}$.

Exercise 1.6. Show that the *closed balls* $(r > 0)$

$$\bar{B}(p,r) = \{x \in \mathbb{R}^n : \|x - p\|_2 \le r\} \tag{1.12}$$

do not form a basis of $\mathbb{R}^n$. However, the following collection is indeed a basis:

$$\mathcal{B}_p = \{\bar{B}(a,r) : a \in \mathbb{R}^n, r \ge 0\}, \tag{1.13}$$

which is the union of all closed balls and all singleton sets.

1.2 Open subsets: from bases to topologies

Definition 1.9. A subset U of X is *open* (with respect to a given basis of neighborhoods $\mathcal{B}$ of X) iff

$$\forall x \in U\ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U. \tag{1.14}$$

Lemma 1.10. Each neighborhood in the basis $\mathcal{B}$ is open.

Proof. This follows from $B \subset B \in \mathcal{B}$ and Definition 1.9. □

Exercise 1.7. What are the open subsets of $\mathbb{R}$ with respect to the right rays in (1.8)?

Lemma 1.11. The intersection of two open sets is open.

Proof. Let U_1 and U_2 be two open sets and fix a point $x \in U_1 \cap U_2$. By Definition 1.9, there exists $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then Definition 1.6 implies that there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Then the proof is completed by Definition 1.9 and x being arbitrary. □

Lemma 1.12. The union of two open sets is open.

Lemma 1.13. The union of any collection of open sets is open.

Definition 1.14. *The topology of X generated by a basis* $\mathcal{B}$ is the collection $\mathcal{T}$ of all open subsets of X in the sense of Definition 1.9.

Definition 1.15. The *standard topology* is the topology generated by the *standard Euclidean basis*, which is the collection of all open balls in $X = \mathbb{R}^n$.

Theorem 1.16. The topology of X generated by a basis satisfies

- $\emptyset, X \in \mathcal{T}$;
- $\alpha \subset \mathcal{T} \ \Rightarrow\ \cup_{U\in\alpha} U \in \mathcal{T}$;
- $U, V \in \mathcal{T} \ \Rightarrow\ U \cap V \in \mathcal{T}$.

Proof. The first item follows from Definition 1.9. The others follows from Lemmas 1.11 and 1.13. □

Example 1.8. The largest basis on a set X is the set of all subsets of X,

$$\mathcal{B}_d(X) = \{A \subset X\} = 2^X, \tag{1.15}$$

and the topology it generates is called *the discrete topology*, which coincides with the basis. This topology is more economically generated by the basis of all singletons,

$$\mathcal{B}_s(X) = \big\{\{x\} : x \in X\big\}. \tag{1.16}$$

The smallest basis on X is simply $\{X\}$ and the topology it generates is called the *trivial/anti-discrete/indiscrete topology* $\mathcal{T}_a = \{\emptyset, X\}$.

Exercise 1.9. Show that if U is open with respect to a basis $\mathcal{B}$, then $\mathcal{B} \cup \{U\}$ is also a basis.

1.3 Topological spaces: from topologies to bases

Definition 1.17. For an arbitrary set X, a collection $\mathcal{T}$ of subsets of X is called a *topology on* X iff it satisfies the following conditions,

(TPO-1) $\emptyset, X \in \mathcal{T}$;

(TPO-2) $\alpha \subset \mathcal{T} \ \Rightarrow\ \cup\alpha \in \mathcal{T}$;

(TPO-3) $U, V \in \mathcal{T} \ \Rightarrow\ U \cap V \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets*.

Corollary 1.18. The topology of X generated by a basis $\mathcal{B}$ as in Definition 1.14 is indeed a topology in the sense of Definition 1.17.

Proof. This follows directly from Theorem 1.16. □

Theorem 1.19. A topology generated by a basis $\mathcal{B}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. Given a collection of elements of $\mathcal{B}$, Lemma 1.10 states that each of them belongs to $\mathcal{T}$. Since $\mathcal{T}$ is a topology, (TPO-2) implies that all unions of these elements are also in $\mathcal{T}$. Conversely, given an open set $U \in \mathcal{T}$, we can choose for each $x \in U$ an element $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Hence $U = \cup_{x\in U} B_x$ and this completes the proof. □

Corollary 1.20. Let $\mathcal{T}$ be a topology on X generated by the basis $\mathcal{B}$. Then every open set $U \in \mathcal{T}$ is a union of some basis neighborhoods in $\mathcal{B}$. (In particular, the empty set is the union of "empty collections" of elements of $\mathcal{B}$.)

Lemma 1.21. Let $(X, \mathcal{T})$ be a topological space. Suppose a collection of open sets $\mathcal{C} \subset \mathcal{T}$ satisfies

$$\forall U \in \mathcal{T}, \forall x \in U, \exists C \in \mathcal{C} \text{ s.t. } x \in C \subset U. \tag{1.17}$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Proof. We first show that $\mathcal{C}$ is a basis. The covering relation holds trivially by setting $U = X$ in (1.17). As for the refining condition, let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since $C_1 \cap C_2$ is open, (1.17) implies that there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Hence $\mathcal{C}$ is a basis by Definition 1.6.

Then we show the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals $\mathcal{T}$. On one hand, for any $U \in \mathcal{T}$ and any $x \in U$, by (1.17) there exists $C \in \mathcal{C}$ such that $x \in C \subset U$. By Definitions 1.9 and 1.14, we have $U \in \mathcal{T}'$. On the other hand, it follows from Corollary 1.20 that any $W \in \mathcal{T}'$ is a union of elements of $\mathcal{C}$. Since each element of $\mathcal{C}$ is in $\mathcal{T}$, we have $W \in \mathcal{T}$. □

Example 1.10. The following countable collection

$$\mathcal{B} = \{(a, b) : a < b, a \text{ and } b \text{ are rational}\} \tag{1.18}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Lemma 1.22. A collection of subsets of X is a topology on X if and only if it generates itself.

Proof. The necessity holds trivially since (TPO-1) implies the covering condition and (TPO-3) implies the refining condition. As for the sufficiency, suppose $U, V \in \mathcal{T}$. By Definition 1.9, $U \cup V$ is also open, hence $U \cup V \in \mathcal{T}$. This argument holds for the union of an arbitrary number of open sets. □

1.4 Continuous functions revisited

Definition 1.23. For a function $f : X \to Y$, the *preimage of a set* $U \subset Y$ *under* f is

$$f^{-1}(U) := \{x \in X : f(x) \in U\}. \tag{1.19}$$

Definition 1.24. A function $f : X \to Y$ is *continuous* iff the preimage of each open set $U \subset Y$ is open in X.

Lemma 1.25. For a continuous function $f : X \to Y$ and an open subset $U \subset Y$, we have

$$f^{-1}(U) = \cup_{y \in U} V_y, \tag{1.20}$$

where the set V_y is a basis element of X containing some x such that $f(x) = y$ and $f(V_y) \subset U$.

Proof. If $x \in f^{-1}(U)$, then $y := f(x) \in U$ and by the covering condition V_y exists. Hence $f^{-1}(U) \subset \cup_{y \in U} V_y$. Conversely, any V_y is a subset of $f^{-1}(U)$ because of $f(V_y) \subset U$ and Definition 1.23. Hence $f^{-1}(U) \supset \cup_{y \in U} V_y$. □

Theorem 1.26. Definitions 1.7 and 1.24 are equivalent for surjective functions.

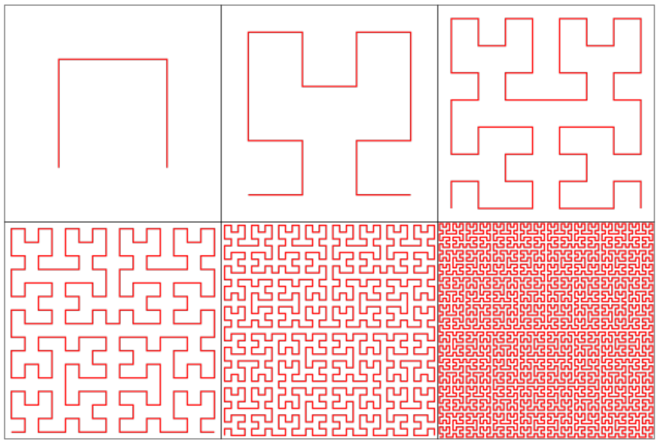
Proof. We first show that Definition 1.24 yields Definition 1.7. Consider $U \in \mathcal{B}_Y$. By Lemma 1.10, U is open and then Definition 1.24 implies that $f^{-1}(U)$ is open in X. The surjectivity of f implies that $f^{-1}(U)$ is not empty. Then by Definition 1.9 we have

$$\forall x \in f^{-1}(U), \exists V \in \mathcal{B}_X \text{ s.t. } x \in V \subset f^{-1}(U),$$

hence $f(V) \subset f(f^{-1}(U)) = U$.

Conversely, Definition 1.7 yields Definition 1.24: by Lemma 1.25 the preimage of any open subset U of Y can be expressed as the RHS of (1.20), and $f^{-1}(U)$ is open because of Lemma 1.13 and the fact that each V_y is open (by Lemma 1.10). □

Example 1.11. A continuous function is not necessarily "well behaved," as exemplified by the following space-filling *Hilbert curve*.



Theorem 1.27. The composition of continuous functions is continuous.

Proof. Suppose we have continuous functions $f = X \to Y$ and $g = Y \to Z$. Let $h = gf : X \to Z$ be their composition. Then for any open set $U \in Z$,

$$h^{-1}(U) = (gf)^{-1}(U) = f^{-1}(g^{-1}(U))$$

is open due to the continuity of g and f and Definition 1.24. □

1.5 Closed sets

Definition 1.28. A subset of X is called *closed* if its complement is open.

Example 1.12. The set $K = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$ is neither open nor closed. In comparison, $K \cup \{0\}$ is closed.

Theorem 1.29. The set σ of all closed subsets of X satisfies the following conditions:

(TPC-1) $\emptyset, X \in \sigma$;

(TPC-2) $\alpha \subset \sigma \;\Rightarrow\; \cap\alpha \in \sigma$;

(TPC-3) $U, V \in \sigma \;\Rightarrow\; U \cup V \in \sigma$.

Example 1.13. The following example shows that infinite intersections of open sets might not be open and infinite unions of closed sets might not be closed.

$$\bigcap\left\{\left(-\frac{1}{n}, \frac{1}{n}\right) : n = 1, 2, \ldots\right\} = \{0\};$$
$$\bigcup\left\{\left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] : n = 1, 2, \ldots\right\} = (-1, 1).$$

Lemma 1.30. A function $f : X \to Y$ is continuous if and only if the preimage of any closed set is closed.

Proof. By Definition 1.23, we have

$$f^{-1}(U) = f^{-1}(Y \setminus (Y \setminus U)) = X \setminus f^{-1}(Y \setminus U).$$

The rest follows from Definitions 1.24 and 1.28. □

Definition 1.31. The *graph of a function* $f : X \to Y$ is the set $\{(x, y) \in X \times Y : y = f(x)\}$.

Lemma 1.32. The graph of a continuous function $f : [a, b] \to \mathbb{R}$ is closed in the space $[a, b] \times \mathbb{R}$.

Exercise 1.14. Give an example of the graph of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ being not closed in $\mathbb{R}^2$. Give another example of the graph of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ being closed in $\mathbb{R}^2$.

Exercise 1.15. Let X be a topological space.

(a) For a continuous function $f : X \to \mathbb{R}$, show that the set $\{x \in X : f(x) = r\}$, i.e. the solution set of any equation with respect to f for some $r \in \mathbb{R}$, is closed.
(b) Show that this fails for a general continuous function $f : X \to Y$ where Y is an arbitrary topological space.
(c) What condition on Y would guarantee that the conclusion holds?

1.6 Homeomorphisms

Definition 1.33. A function $f : X \to Y$ between topological spaces X and Y is called a *homeomorphism* iff f is bijective and both f and f^{-1} are continuous. Then X and Y are said to be *homeomorphic* or *topologically equivalent*, written $X \approx Y$.

Lemma 1.34. If two spaces X and Y are homeomorphic, then

$$\forall a \in X,\ \exists b \in Y \text{ s.t. } X \setminus \{a\} \approx Y \setminus \{b\}. \tag{1.21}$$

Exercise 1.16. Show that the function $f : \{A, B\} \to \{C\}$ given by $f(A) = f(B) = C$ is continuous, but not a homeomorphism. Hence a necessary condition for homeomorphism is the number of connected components.

Example 1.17. Consider X the letter "T" and Y a line segment. They are not homeomorphic because removing the junction point in T would result in three pieces while removing any point in the line segment yields at most two connected components.

Exercise 1.18. Classify the following symbols of the standard computer keyboard by considering them as 1-dimensional topological spaces.

```
‘ 1 2 3 4 5 6 7 8 9 0 - =
q w e r t y u i o p [ ] \
a s d f g h j k l ; ’
z x c v b n m , . /
~ ! @ # $ % ^ & * ( ) _ +
Q W E R T Y U I O P { } |
A S D F G H J K L : "
Z X C V B N M < > ?
```

Exercise 1.19. Consider the identity function $f = \mathrm{Id}_X : (X, \mathcal{T}) \to (X, \kappa)$ where κ is the anti-discrete topology and $\mathcal{T}$ is not. Show that f^{-1} is not continuous and hence f is not a homeomorphism.

Exercise 1.20. Give an example of a continuous bijection $f : X \to Y$ that isn't a homeomorphism; this time both X and Y are subspaces of $\mathbb{R}^2$.

Exercise 1.21. For a continuous function $f : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(x) = (x, f(x))$. Prove that g is continuous and its image, the graph of f, is homeomorphic to $\mathbb{R}$.

Lemma 1.35. All closed intervals of a non-zero, finite length are homeomorphic.

Lemma 1.36. All open intervals, including infinite ones, are homeomorphic.

Proof. The tangent function gives you a homeomorphism between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(-\infty, +\infty)$. □

Lemma 1.37. An open interval is not homeomorphic to a closed interval (nor half-open).

Definition 1.38. The *n-sphere* is a subset in $\mathbb{R}^{n+1}$,

$$\mathbb{S}^n := \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}. \tag{1.22}$$

Its north pole is denoted by $N = (0, 0, \cdots, 0, 1)$.

Definition 1.39. The *stereographic projection*

$$P : \mathbb{S}^n \setminus N \to \mathbb{R}^n$$

is given by

$$P(\mathbf{x}) := \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right). \tag{1.23}$$

Lemma 1.40. The stereographic projection is a homeomorphism with its inverse as

$$P^{-1}(\mathbf{y}) = \frac{1}{1 + \|\mathbf{y}\|^2} \left(2y_1, 2y_2, \ldots, 2y_n, \|\mathbf{y}\|^2 - 1 \right). \tag{1.24}$$

Exercise 1.22. Show that the 2-sphere and the hollow cube are homeomorphic by using the *radial projection* f,

$$f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{1.25}$$

Theorem 1.41. Homeomorphisms form an equivalence relation on the set of all topological spaces.

Proof. For a homeomorphism $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, we can define a function $f_{\mathcal{T}} : \mathcal{T}_X \to \mathcal{T}_Y$ by setting $f_{\mathcal{T}}(V) := f(V)$. It is easy to show that $f_{\mathcal{T}}$ is also a bijection from Definition 1.33. □

1.7 The subbasis topology

Definition 1.42. A *subbasis* $\mathcal{S}$ on X is a collection of subsets of X such that the covering condition in Definition 1.6 holds.

Example 1.23. The set of all open balls with their radii no less than a given $h > 0$, written $\mathcal{B}_h$, is a subbasis but not a basis.

Definition 1.43. *The topology of X generated by a subbasis $\mathcal{S}$* is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Exercise 1.24. Check that the topology generated by a subbasis $\mathcal{S}$ as in Definition 1.43 is indeed a topology in the sense of Definition 1.17.

Proof. By Definition 1.43 and Theorem 1.19, we only need to show that all finite intersections of elements of $\mathcal{S}$ form a basis of X in the sense of Definition 1.6. The covering condition holds trivially. As for the covering condition, it suffices to note that the intersection of two sets of the form $B_1 = \cap_{i=1}^m S_i$ and $B_2 = \cap_{j=1}^n S_j$ is still a finite intersection of elements in $\mathcal{S}$. □

1.8 The metric topology

Definition 1.44. A *metric* is a function $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ that satisfies, for all $x, y, z \in \mathcal{X}$,

(1) non-negativity: $d(x, y) \geq 0$;

(2) identity of indiscernibles: $x = y \Leftrightarrow d(x, y) = 0$;

(3) symmetry: $d(x, y) = d(y, x)$;

(4) triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.45. For a metric d on X, the number $d(x, y)$ is called the *distance* between x and y. Given $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\} \tag{1.26}$$

is called the *ϵ-ball centered at x*.

Lemma 1.46. If d is a metric on X, then the collection of all ϵ-balls is a basis on X.

Definition 1.47. The topology on X generated by the basis of all ϵ-balls is called the *metric topology* induced by the metric d.

Lemma 1.48. A set U is open in the metric topology induced by d if and only if

$$\forall x \in U, \exists \delta > 0 \text{ s.t. } B_d(x, \delta) \subset U.$$

Definition 1.49. A topological space X is said to be *metrizable* if there exists a metric d on X that induces the topology of X. A *metric space* is a metrizable topological space together with a specific metric d that gives the topology of X.

1.9 Hierarchy of topologies

Definition 1.50. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is *finer/larger* than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer/strictly larger* than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser/smaller*, or *strictly coarser/strictly smaller*, in these two respective situations. We say $\mathcal{T}$ and $\mathcal{T}'$ are *comparable* if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T}' \subset \mathcal{T}$.

Lemma 1.51. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if

$$\forall x \in X, \forall B \in \mathcal{B} \text{ with } x \in B, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B. \tag{1.27}$$

Proof. The sufficiency $U \in \mathcal{T} \Rightarrow U \in \mathcal{T}'$ follows directly from (1.27) and Definition 1.50.

As for the necessity, we start with given $x \in X$ and $B \in \mathcal{B}$ with $x \in B$. By Lemma 1.10, B is open, i.e. $B \in \mathcal{T}$. Then by hypothesis $B \in \mathcal{T}'$. Definition 1.9 implies that there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, which completes the proof. □

Exercise 1.25. The bounded complements of all non-degenerate Jordan curves form a basis of neighborhoods. Is the topology generated by this basis finer than that generated by the open balls?

Definition 1.52. The *finite complement topology* on X is

$$\mathcal{T} = \{U \subset X : U = \emptyset \text{ or } X \setminus U \text{ is finite}\}. \tag{1.28}$$

The *countable complement topology* on X is

$$\mathcal{T} = \{U \subset X : U = \emptyset \text{ or } X \setminus U \text{ is countable}\}. \tag{1.29}$$

The *particular point topology* on X is

$$\mathcal{T} = \{U \subset X : U = \emptyset \text{ or } p \in U\}. \tag{1.30}$$

The *excluded point topology* on X is

$$\mathcal{T} = \{U \subset X : U = X \text{ or } p \notin U\}. \tag{1.31}$$

Exercise 1.26. Show that each of the topologies in Definition 1.52 is indeed a topology.

Exercise 1.27. For a three-element set $X = \{a, b, c\}$, enumerate all possible topologies up to the permutation isomorphism.

Exercise 1.28. Which topology in the answer of Exercise 1.27 has a basis other than itself?

Exercise 1.29. Define a directed graph $G = (V, E)$ where the vertex set V contains the topologies in Example 1.27 and E contains an edge $\mathcal{T}_1 \to \mathcal{T}_2$ iff $\mathcal{T}_2$ is strictly finer than $\mathcal{T}_1$. Plot the graph G.

Definition 1.53. The *lower limit topology* $\mathcal{T}_\ell$ on $\mathbb{R}$ is the topology generated by all half-open intervals of the form $[a, b)$ with $a < b$. The space $\mathbb{R}$ endowed with $\mathcal{T}_\ell$ is denoted by $\mathbb{R}_\ell$.

Definition 1.54. The *K-topology* $\mathcal{T}_K$ on $\mathbb{R}$ is the topology generated by all open intervals (a,b) and all sets of the form $(a,b) \setminus K$ where

$$K = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}. \tag{1.32}$$

The space $\mathbb{R}$ endowed with $\mathcal{T}_K$ is denoted by $\mathbb{R}_K$.

Lemma 1.55. The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof. For any $x \in (a,b)$, we can always find $[x,b) \in \mathcal{T}_\ell$ and $(a,b) \in \mathcal{T}_K$ such that $x \in [x,b) \subset (a,b)$ and $x \in (a,b) \subset (a,b)$. On the other hand, for any $x \in \mathbb{R}$ and any neighborhood $[x,b) \in \mathbb{R}_\ell$, no open interval in the standard topology simultaneously contains x and is a subset of $[x,b)$. Similarly, for $0 \in \mathbb{R}$ and $B_K := (-1,1)\setminus K \supset \{0\}$, no open interval simultaneously contains 0 and is a subset of B_K. Hence $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$.

To show that $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable, it suffices to give two examples. For any $x \in K \subset \mathbb{R}$ and any neighborhood $[x,b) \in \mathcal{T}_\ell$, no open sets in $\mathcal{T}_K$ simultaneously contains x and is a subset of $[x,b)$. Conversely, for $0 \in \mathbb{R}$ and the above B_K, no interval $[a,b) \in \mathcal{T}_\ell$ simultaneously contains 0 and is a subset of B_K. $\square$

Exercise 1.30. The topologies on $\mathbb{R}^2$ generated by the open balls and the open squares are the same topology.

Exercise 1.31. Show that the collection

$$\mathcal{C} = \{[a,b) : a < b, a \text{ and } b \text{ are rational}\} \tag{1.33}$$

is a basis that generates a topology $\mathcal{T}_Q$ different from the lower limit topology $\mathcal{T}_\ell$ on $\mathbb{R}$. Compare this to Example 1.10.

1.10 The order topology

Definition 1.56. Let X be a totally ordered set with more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a,b) in X;

(2) All half-open intervals of the form $[a_0,b)$ where a_0 is the smallest element (if any) of X;

(3) All half-open intervals of the form $(a,b_0]$ where b_0 is the largest element (if any) of X.

The *order topology* on X is the topology generated by the basis $\mathcal{B}$.

Exercise 1.32. Show that $\mathcal{B}$ is indeed a basis of X in Definition 1.56.

Example 1.33. The standard topology on $\mathbb{R}$ as in Definition 1.15 is the same as the order topology derived from the usual order on $\mathbb{R}$. This is due to the fact that there exists in $\mathbb{R}$ neither the smallest element nor the largest element.

Definition 1.57. The *dictionary order* or *lexicopgraphical order* on $\mathbb{R} \times \mathbb{R}$ is a total order defined as

$$(a,b) < (c,d) \;\Leftrightarrow a < c \text{ or } a = c, b < d. \tag{1.34}$$

Example 1.34. A basis for the order topology on $\mathbb{R} \times \mathbb{R}$ with the dictionary order is the collection of all open intervals of the form (1) in Definition 1.56.

Example 1.35. The order topology of positive integers $\mathbb{Z}^+$ is the same as the discrete topology. For $n > 1$, take the basis interval $(n-1, n+1)$; for $n = 1$, take the interval $[1,2)$.

Exercise 1.36. Show that the order topology derived from the dictionary order on the set $X = \{1,2\} \times \mathbb{Z}^+$ is not the discrete topology.

Exercise 1.37. Show that Definition 1.56 does not generalize to posets.

Definition 1.58. Let X be an ordered set and $a \in X$. The four subsets of X are called the *rays* determined by a:

$$(a,+\infty) := \{x : x > a\}; \tag{1.35a}$$
$$(-\infty,a) := \{x : x < a\}; \tag{1.35b}$$
$$[a,+\infty) := \{x : x \geq a\}; \tag{1.35c}$$
$$(-\infty,a] := \{x : x \leq a\}. \tag{1.35d}$$

The first two are *open rays* while the last two *closed rays*.

Exercise 1.38. Show that the open rays form a subbasis for the order topology on X.

Definition 1.59. The set $[0,1] \times [0,1]$ in the dictionary order topology is called the *ordered square*, denoted by I_o^2.

1.11 The product topology: $X \times Y$

Definition 1.60. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by the basis

$$\bar{\gamma}_{X\times Y} := \{B_1 \times B_2 : B_1 \in \mathcal{T}_X, B_2 \in \mathcal{T}_Y\}, \tag{1.36}$$

where $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on X and Y, respectively.

Exercise 1.39. Check that $\bar{\gamma}_{X\times Y}$ in (1.36) is indeed a basis.

Exercise 1.40. Give an example that $\bar{\gamma}_{X\times Y}$ is not a topology.

Theorem 1.61. Let X and Y be topological spaces with bases γ_X and γ_Y, respectively. Then the set

$$\gamma_{X\times Y} := \{B_1 \times B_2 : B_1 \in \gamma_X, B_2 \in \gamma_Y\}, \tag{1.37}$$

is a basis for the topology of $X \times Y$.

Proof. Both the covering and refining conditions hold trivially. $\square$

Definition 1.62. For topological spaces X and Y, the functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given by

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y \tag{1.38}$$

are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Lemma 1.63. The product topology on $X \times Y$ is the same as the topology generated by the subbasis

$$\mathcal{S} := \left\{\pi_1^{-1}(U) : U \in \mathcal{T}_X\right\} \cup \left\{\pi_2^{-1}(V) : V \in \mathcal{T}_Y\right\}. \tag{1.39}$$

Proof. Let $\mathcal{T}$ denote the product topology in Definition 1.60 and $\mathcal{T}'$ denote the topology generated by the subbasis (1.39). Every element in $\mathcal{S}$ belongs to $\mathcal{T}$, so do any unions of finite intersections of elements of $\mathcal{S}$. Hence Definition 1.42 yields $\mathcal{T}' \subset \mathcal{T}$.

Conversely, each element in the basis of $\mathcal{T}$ is an intersection of elements in $\mathcal{S}$,

$$B_1 \times B_2 = \pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2),$$

hence $B_1 \times B_2 \in \mathcal{T}'$ and thus $\mathcal{T} \subset \mathcal{T}'$. □

Corollary 1.64. The projections in Definition 1.62 are continuous (with respect to the product topology).

Proof. Consider $\pi_1 : X \times Y \to X$. For each open set $U \in \mathcal{T}_X$, Lemma 1.63 and Definition 1.43 imply that its preimage under π_1 is open in the product topology. □

Theorem 1.65 (Maps into products)**.** Let $f : A \to X \times Y$ be given by $f_1 : A \to X$ and $f_2 : A \to Y$ as

$$f(a) := (f_1(a), f_2(a)). \tag{1.40}$$

Then f is continuous if and only if both f_1 and f_2 are continuous.

Proof. Write $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. The necessity follows from Corollary 1.64 and Theorem 1.27.

As for the sufficiency, we need to show that the preimage $f^{-1}(U \times V)$ of any basis element $U \times V$ is open. By Definition 1.60, $U \times V \in \mathcal{B}_{X\times Y}$ implies that $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. By Definition 1.23, any point $a \in f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, which, by (1.40), is equivalent to $f_1(a) \in U$ and $f_2(a) \in V$. Hence, we have

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V).$$

The rest of the proof follows from the conditions of both f_1 and f_2 being continuous. □

Example 1.41. A parametrized curve $\gamma(t) = (x(t), y(t))$ is continuous if and only if both x and y are continuous.

1.12 The subspace topology

Lemma 1.66. Consider a subset A of a topological space X. Suppose γ_X is a basis of neighborhoods of X. Then

$$\gamma_A := \{W \cap A : W \in \gamma_X\} \tag{1.41}$$

is a basis of neighborhoods of A.

Proof. The covering condition for A holds because the covering condition of X holds. As for the refining condition, for any $U, V \in \gamma_A$ and any $x \in U \cap V$, there exists $U', V' \in \gamma_X$ such that $U = U' \cap A$, $V = V' \cap A$, and $W' \subset U' \cap V'$ for some $W' \in \gamma_X$. Setting $W := W' \cap A$ and we have

$$x \in W \subset (U' \cap V') \cap A = (U' \cap A) \cap (V' \cap A) = U \cap V,$$

which completes the proof. □

Definition 1.67. The topology generated by γ_A in (1.41) is called the *relative topology* or *subspace topology* on A generated by the basis γ_X of X.

Lemma 1.68. Consider a subset A of a topological space X. Suppose $\mathcal{T}_X$ is a topology on X. Then

$$\mathcal{T}_A := \{W \cap A : W \in \mathcal{T}_X\} \tag{1.42}$$

is a topology on A.

Proof. For (TPO-1), we choose $W = \emptyset, A$. For (TPO-2),

$$\bigcup_{W\in\alpha} (W \cap A) = (\bigcup_{W\in\alpha} W) \cap A,$$

where $\bigcup_{W\in\alpha} W$ is a subset of X. For (TPO-3),

$$(U \cap A) \cap (V \cap A) = (U \cap V) \cap A,$$

where $U \cap V$ is a subset of X. □

Definition 1.69. Given a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, the topological space $(A, \mathcal{T}_A)$ is called a *subspace* of X and the topology $\mathcal{T}_A$ in (1.42) is called the *subspace topology or relative topology induced by* X.

Theorem 1.70. Let γ_X be a basis that generates the topology $\mathcal{T}_X$ on a topological space X. Then the subspace topology on A induced by $\mathcal{T}_X$ is equivalent to the subspace topology generated by γ_X. In other words, $\mathcal{T}_A$ is generated by γ_A.

$$\begin{tikzcd} \gamma_X \arrow[r, "\text{open}"] \arrow[d, "\cap A"] & \mathcal{T}_X \arrow[d, "\cap A"] \\ \gamma_A \arrow[r, "\text{open}"] & \mathcal{T}_A \end{tikzcd}$$

Proof. We first show that U is open with respect to (w.r.t.) γ_A for any given $U \in \mathcal{T}_A$. By Lemma 1.68, there exists $U' \in \mathcal{T}_X$ such that $U = U' \cap A$. The condition of γ_X being a basis of X yields

$$\forall y \in U', \ \exists B' \in \gamma_X \text{ s.t. } y \in B' \subset U',$$

which implies

$$\forall x \in U \subset U', \ \exists B := (B' \cap A) \in \gamma_A \text{ s.t. } x \in B \subset U.$$

It remains to show that any set U that is open w.r.t. γ_A is in $\mathcal{T}_A$, i.e., we need to find $U' \in \mathcal{T}_X$ such that $U = U' \cap A$. Since U is open w.r.t. γ_A, Definition 1.9 yields

$$\forall x \in U, \ \exists N_x \in \gamma_A \text{ s.t. } x \in N_x \subset U,$$

where $N_x = N'_x \cap A$ for some $N'_x \in X$. We then choose

$$U' := \bigcup_{x\in U} N'_x.$$

Theorem 1.19 implies that U' is open and $U = U' \cap A$. □

Lemma 1.71. Let A be a subspace of X. If U is open in A and A is open in X, then U is open in X.

Proof. Since U is open in A, Definition 1.69 yields

$$\exists U' \in X \text{ open s.t. } U = U' \cap A,$$

the rest of the proof follows from A being open in X. $\square$

Lemma 1.72. Let A be a subspace of X. Then a set V is closed in A if and only if it equals the intersection of a closed subset of X with A.

Proof. Suppose V is closed in A. Then

$$\exists V' \subset A \text{ s.t. } V \cup V' = A, V' \in \mathcal{T}_A.$$

Since A is a subspace of X, we have from Definition 1.69

$$\exists U' \subset X, \text{ s.t. } V' = U' \cap A, U' \in \mathcal{T}_X.$$

Hence the set $U := X \setminus U'$ is closed in X and

$$\begin{aligned} A \cap U &= A \cap (X \setminus U') = A \setminus (X \setminus (X \setminus U')) \\ &= A \setminus U' = A \setminus (U' \cap A) = A \setminus V' = V. \end{aligned}$$

Conversely, suppose

$$\exists U \in X \text{ s.t. } (X \setminus U) \in \mathcal{T}_X, V = U \cap A.$$

Define $V' := (X \setminus U) \cap A$ and we know from Definition 1.69 that V' is open in A. The proof is then completed by

$$V \cup V' = (U \cap A) \cup ((X \setminus U) \cap A) = A,$$

where the last step follows from the condition $A \subset X$. $\square$

Corollary 1.73. Let A be a subspace of X. If V is closed in A and A is closed in X, then V is closed in X.

Proof. This follows directly from Lemma 1.72 by using $V = V \cap A$. $\square$

1.13 Interior–Frontier–Exterior

Definition 1.74. A point $x \in X$ is an *interior point* of A if there is a neighborhood W of x that lies entirely in A. The set of interior points of a set U is called its *interior* and denoted by $\mathrm{Int}(U)$.

Lemma 1.75. $\mathrm{Int}(A)$ is open for any A.

Proof. Exercise. $\square$

Example 1.42. The interior of a closed ball is the corresponding open ball.

Definition 1.76. A point $x \in X$ is an *exterior point* of A if there is a neighborhood W of x that lies entirely in $X \setminus A$. The set of exterior points of a set U is called its *exterior* and denoted by $\mathrm{Ext}(U)$.

Example 1.43. The exterior of the set K in (1.32) is $\mathbb{R} \setminus K \setminus \{0\}$. Why not 0?

Definition 1.77. A point x is a *closure point* of A if each neighborhood of x contains some point in A.

Example 1.44. Any point in the set K in (1.32) is a closure point of K, so is 0.

Definition 1.78. A point x is an *accumulation point* of A if each neighborhood of x contains some point $p \in A$ with $p \neq x$.

Example 1.45. The only accumulation point of the set K in (1.32) is 0.

Example 1.46. Each point in $\mathbb{R}$ is an accumulation point of $\mathbb{Q}$.

Definition 1.79. A point x in a set A is *isolated* if there exists a neighborhood of x such that x is the only point of A in this neighborhood.

Example 1.47. Every point of the set K in (1.32) is isolated.

Definition 1.80. A point x is a *frontier point* of a set A iff it is a closure point for both A and its complement. The set of all frontier points is called *the frontier* $\mathrm{Fr}(A)$ of A.

Theorem 1.81. For any set A in X, its interior, its frontier, and its exterior form a partition of X.

Proof. Consider an arbitrary point $a \in X$. If there exists a neighborhood $\mathcal{N}_a$ of a such that $\mathcal{N}_a \subset A$, then Definition 1.74 implies $a \in \mathrm{Int}(A)$. If $\mathcal{N}_a \subset X \setminus A$, then Definition 1.76 implies $a \in \mathrm{Ext}(A)$. Otherwise, for all neighborhoods of a we have $\mathcal{N}_a \not\subset A$ and $\mathcal{N}_a \not\subset X \setminus A$, which implies that any $\mathcal{N}_a$ contains points both from A and $X \setminus A$. The rest follows from Definition 1.80. $\square$

Definition 1.82. The closure of A, written $\mathrm{Cl}(A)$ or $\overline{A}$, is the set of all closure points of A.

Lemma 1.83. $\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A)$.

Lemma 1.84. $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Fr}(A)$.

Theorem 1.85. The closure of a set A is the smallest closed set containing A:

$$\mathrm{Cl}(A) = \cap\{G : A \subset G, G \text{ is closed in } X\}. \tag{1.43}$$

Proof. Write $\alpha := \{G : A \subset G, G \text{ is closed in } X\}$ and $A^- := \cap\alpha$ and we need to show

- $A^- \subset \mathrm{Cl}(A)$;
- $A^- \supset \mathrm{Cl}(A)$.

We only prove the first part and leave the other as an exercise. Consider $x \notin \mathrm{Cl}(A)$. Then by Definitions 1.77 and 1.82 there exists an open neighborhood $\mathcal{N}_x$ of x such that $\mathcal{N}_x \cap A = \emptyset$. Hence the set $P := X \setminus \mathcal{N}_x$ contains A. P is also closed because $\mathcal{N}_x$ is open. Therefore $P \in \alpha$ and $x \notin A^-$. $\square$

Exercise 1.48. Prove $\mathrm{Cl}(A \cap B) \subset \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$. What if we have infinitely many sets?

Theorem 1.86. The interior of a set A is the largest open set contained in A,

$$\mathrm{Int}(A) = \cup\{U : U \subset A, \ U \text{ is open in } X\}. \tag{1.44}$$

1.14 Convergence of sequences

Definition 1.87. Suppose X is a set with a basis of neighborhoods γ. Let $\{x_n : n = 1, 2, \ldots\}$ be a sequence of elements of X and $a \in X$. Then we say the sequence *converges* to a, written

$$\lim_{n\to\infty} x_n = a, \text{ or } x_n \to a \text{ as } n \to \infty,$$

iff

$$\forall U \in \gamma \text{ with } a \in U,\ \exists N \in \mathbb{N}^+ \text{ s.t. } n > N \ \Rightarrow\ x_n \in U. \tag{1.45}$$

Exercise 1.49. Prove that the definition remains equivalent if we replace "basis γ" with "topology $\mathcal{T}$."

Exercise 1.50. Show that if a sequence converges with respect to a basis γ, it also converges with respect to any basis equivalent to γ.

Theorem 1.88. Continuous functions preserve convergence, i.e., for a continuous $f : X \to Y$, $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof. This follows from Definitions 1.87 and 1.7. □

Exercise 1.51. A sequence $\alpha = \{x_n : n = 1, 2, \ldots\}$ in a topological space X can be viewed as a subset of X, $A = \{x_n : n \in \mathbb{N}^+\}$. Compare the meanings of the closure points of A and the accumulation points of A. What about the limit of α?

Exercise 1.52. For metric topology, show that a function $f : X \to Y$ is continuous if and only if the function commutes with limits for any convergent sequence in X.

Example 1.53. When do we have $x_n \to a$ for discrete topology?

Example 1.54. When do we have $x_n \to a$ for anti-discrete topology?

Definition 1.89. A topological space $(X, \mathcal{T})$ is called a *Hausdorff space* iff

$$\forall a, b \in X,\ a \neq b, \exists U, V \in \mathcal{T} \text{ s.t. } a \in U, b \in V, U \cap V = \emptyset. \tag{1.46}$$

Lemma 1.90. Every subset of finite points in a Hausdorff space is closed.

Proof. By (TPC-3) in Theorem 1.29, it suffices to show that every singleton set is closed. Consider $X \setminus \{x_0\}$. For any $x \neq x_0$, Definition states that there exists $U \supset x$, $V \supset x_0$ such that $U \cap V = \emptyset$, hence $x_0 \notin U$ and $U \in X \setminus \{x_0\}$. Therefore $X \setminus \{x_0\}$ is open. □

Exercise 1.55. Does there exist a topological space X that is not Hausdorff but in which every finite point set is closed?

Definition 1.91. A topological space is called a *T1 space* iff every finite subset is closed in it.

Theorem 1.92. Let X be a T1 space and A a subset of X. A point x is an accumulation point of A if and only if every neighborhood of x intersects with infinitely many points of A.

Proof. The sufficiency follows directly from Definition 1.78. As for the necessity, suppose there exists a neighborhood U of x such that $(A \setminus \{x\}) \cap U = \{x_1, x_2, \ldots, x_m\}$. Then by Definition 1.91 we know

$$U \cap (X \setminus \{x_1, x_2, \ldots, x_m\}) = U \cap (X \setminus (A \setminus \{x\}))$$

is an open set containing x, yet it does not contain any points in A other than x. This contradicts the condition of x being an accumulation point of A. □

Theorem 1.93. A sequence of points in a Hausdorff space X converges to at most one point in X.

Proof. By Definition 1.87, a convergence to two points in X would be a contradiction to Definition 1.89. □

Definition 1.94. An *n-manifold* $\mathcal{M}$ is a Hausdorff space such that every point $p \in \mathcal{M}$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Example 1.56. Any finite set of points is a 0-manifold. Both a simple closed curve and a trefoil knot are 1-manifolds. The unit sphere is a 2-manifold.

1.15 New maps from old ones

Theorem 1.95. Suppose X is a topological space and $f, g : X \to \mathbb{R}$ are continuous functions. Then $f + g$, $f - g$, and $f \cdot g$ are continuous; f/g is also continuous if $g(x) \neq 0$ for all x.

Proof. By Theorem 1.65, the function $h : X \to \mathbb{R}^2$ given by $h(x) = (f(x), g(x))$ is continuous. We also know that the function $+ : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Hence the function $f + g = + \circ h$ is continuous. □

Definition 1.96. Let X be a topological space and A a subset of X. The *inclusion* $i_A : A \hookrightarrow X$ is given by

$$\forall x \in A, \qquad i_A(x) = x. \tag{1.47}$$

Definition 1.97. Let X and Y be topological spaces and A a subset of X. The *restriction of a function* $f : X \to Y$ to A is a function given as

$$\forall x \in A, \qquad f|_A(x) := f(x). \tag{1.48}$$

Theorem 1.98 (Restricting the domain)**.** Any restriction of a continuous function is continuous.

Proof. For any open set U in Y, we have $i_A^{-1}(U) = U \cap A$. The rest follows from the relative topology. □

Exercise 1.57. Let $i_A : A \hookrightarrow X$ be an inclusion. Suppose the set A is given a topology such that, for every topological space Y and every function $f : Y \to A$, f is continuous if and only if the composition $(i_A \circ f) : Y \to X$ is continuous. Prove that this topology of A is the same as the relative topology of A in X.

Definition 1.99. An injective function $f : X \to Y$ is called an *embedding* if it yields a homeomorphism

$$g_f : X \to f(X), \qquad \forall x,\ g_f(x) := f(x). \tag{1.49}$$

Lemma 1.100 (Restricting the range)**.** If $f : X \to Y$ is a continuous function, so is g_f given by (1.49).

Proof. Of course the topology of $f(X)$ is understood as the subspace topology of Y. The rest follows from Definition 1.69. □

Lemma 1.101 (Expanding the range)**.** Let $f : X \to Y$ be a continuous function and Y a subspace of Z. Then the function $g : X \to Z$ given by $g(x) := f(x)$ for all $x \in X$ is continuous.

Proof. Write $g = i_Y \circ f$. □

Lemma 1.102 (Pasting lemma)**.** Let A, B be two closed subsets of a topological space X such that $X = A \cup B$. Suppose $f_A : A \to Y$ and $f_B : B \to Y$ are continuous functions

$$\forall x \in A \cap B, \qquad f_A = f_B. \tag{1.50}$$

Then the following function $f : X \to Y$ is continuous,

$$f(x) := \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B. \end{cases} \tag{1.51}$$

Proof. Define $W := f_A(A) \cup f_B(B)$. Then for any $V \subset Y$, (1.51) and the condition (1.50) yields

$$V = (V \cap W) \cup (V \setminus W) = (V \cap f_A(A)) \cup (V \cap f_B(B)) \cup (V \setminus W).$$

If V is closed in Y, then its preimage is

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V \cap f_A(A)) \cup f^{-1}(V \cap f_B(B)) \\ &= f_A^{-1}(V \cap f_A(A)) \cup f_B^{-1}(V \cap f_B(B)) \\ &= g_A^{-1}(V \cap f_A(A)) \cup g_B^{-1}(V \cap f_B(B)), \end{aligned}$$

where g_A and g_B are defined in Lemma 1.100. We claim that $f^{-1}(V)$ is also closed in X with arguments as follows.

(i) Since V is closed, Lemma 1.72 implies that $V \cap f_A(A)$ and $V \cap f_B(B)$ are closed in $f_A(A)$ and $f_B(B)$, respectively.

(ii) By Lemma 1.100, both g_A and g_B are continuous. Hence the two sets to be unioned in the last line of the above equation are closed in A and B, respectively.

(iii) By Corollary 1.73, both sets in the last step are closed in X.

The rest of the proof follows from Lemma 1.30. □

Exercise 1.58. Show that Lemma 1.102 fails if A and B are not closed.

Exercise 1.59. Formulate the pasting lemma in terms of open sets and prove it.

Exercise 1.60. What is the counterpart of the pasting lemma in complex analysis?

Definition 1.103 (Expanding the domain)**.** For $A \subset X$ and a given function $f : A \to Y$, a function $F : X \to Y$ is called an *extension* of f if $F|_A = f$.

Exercise 1.61. State and prove the sufficient and necessary conditions for existence of the extension of a continuous function $f : (a, b) \to \mathbb{R}$ to another continuous function $F : (a, b] \to \mathbb{R}$.

Exercise 1.62. State and prove the sufficient and necessary condition for existence of a *linear* extension of a linear operator $f : A \to W$ to $f : V \to W$ where V, W are vector spaces and A is a subspace of V.

1.16 Connectedness

Definition 1.104. Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. A topological space is *connected* if there does not exist a separation of X.

Exercise 1.63. Why do we define the separation as a pair of disjoint *open* sets? Can we define separation using closed sets?

Example 1.64. A space X with indiscrete topology is connected, since there exists no separation of X.

Lemma 1.105. For a subspace Y of X, a separation of Y is a pair of disjoint nonempty sets A and B such that $A \cup B = Y$ and neither of them contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof. Suppose first that A and B form a separation of Y. By Definition 1.104, A and B are both open in Y. Furthermore, A is also closed since its complement B is open in Y. Thus $\overline{A} = A$ and $B \cap \overline{A} = \emptyset$.

Conversely, suppose $A \cup B = Y$, $A \cap \overline{B} = \emptyset$, and $B \cap \overline{A} = \emptyset$. Then we have

$$\overline{A} \cap Y = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A.$$

Similarly, $\overline{B} \cap Y = B$. Both A and B are closed in Y, hence they are both open in Y and form a separation of Y. □

Example 1.65. Let $Y = [-1, 1]$ be a subspace of $X = \mathbb{R}$. The sets $[-1, 0]$ and $(0, 1]$ are disjoint and nonempty, but they do not form a separation of Y because $[-1, 0]$ is not open in Y. Alternatively, one can use Lemma 1.105 to say that $[-1, 0]$ contains a limit point 0 of the other set $(0, 1]$.

Example 1.66. Let $Y = [-1, 0) \cup (0, 1]$. Each of the sets $[-1, 0)$ and $(0, 1]$ is nonempty and open in Y; therefore, they form a separation of Y. Again, an alternative argument utilizes Lemma 1.105.

Example 1.67. The rationals $\mathbb{Q}$ are not connected. The only connected subspaces are the one-point spaces: for $Y = \{p, q\} \subset \mathbb{Q}$ with $p < q$, choose an irrational number $a \in (p, q)$ and write

$$Y = (Y \cap (-\infty, p)) \cup (Y \cap (q, +\infty)).$$

According to Definition 1.104, this separation implies that Y is not connected.

Theorem 1.106. Connectedness is preserved by continuous functions; i.e., the image of a connected space under a continuous map is connected.

Proof. Let X be a connected space and $f : X \to Y$ a continuous function. We show that the image space $Z := f(X)$ is connected. Suppose Z is not connected. Then there exists disjoint nonempty open sets U, V such that $Z = U \cup V$. By Definition 1.24, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets and $X = f^{-1}(U) \cup f^{-1}(V)$, which contradicts the condition of X being connected. □

Theorem 1.107 (Intermediate value theorem (generalized))**.** Let $f : X \to Y$ be a continuous function where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Definition 1.108. A topological space Q is *path-connected* iff for any $A, B \in Q$, there exists a continuous function $p : [0, 1] \to Q$ such that

$$p(0) = A, \qquad p(1) = B.$$

Exercise 1.68. Prove that if $[a, b]$ is path-connected, so are (a, b) and $[a, b)$.

Theorem 1.109. Path-connectedness is preserved by continuous functions; i.e., the image of a path-connected space under a continuous function is path-connected.

Proof. Let X be a connected space and $f : X \to Y$ a continuous function. We show that the image space $Z := f(X)$ is connected. Any $C, D \in Z$ have their preimages $A = f^{-1}(C) \in X$ and $B = f^{-1}(D) \in X$. The path-connectedness of X implies that there exists a continuous function $q : [0, 1] \to X$ such that $q(0) = A$ and $q(1) = B$. By Theorem 1.27, the composition $p = fq$ is continuous, $p(0) = f(q(0)) = f(A) = C$, and $p(1) = f(q(1)) = f(B) = D$. Hence Z is path-connected by Definition 1.108. □

Lemma 1.110. Every path-connected space is connected.

Proof. Suppose a topological space X is not connected but path-connected. Then there exists a separation U, V of X such that $X = U \cup V$. Consider an arbitrary path $f : [0, 1] \to X$. Since $f([0, 1])$ is a continuous image of a connected set, we know from Theorem 1.106 that $f([0, 1])$ is connected, hence it must lie entirely in either U or V. Consequently, there is no path in X joining a point of A to a point of B, contradicting the condition of X being path-connected. □

Example 1.69. A connected space is not necessarily path-connected, c.f. the *topologist's sine curve*. The space

$$S = \left\{ \left(x, \sin\frac{1}{x} \right) : x \in (0, 1] \right\}. \tag{1.52}$$

is connected because it is the image of the connected space $(0, 1]$ under a continuous map. Hence the closure of S

$$\overline{S} = S \cup \{(0, y) : y \in [-1, 1]\}. \tag{1.53}$$

is also connected in $\mathbb{R}^2$. But $\overline{S}$ is not path-connected. Can you prove it?

Proof. Suppose there is a path $f : [a, c] \to \overline{S}$ beginning at the origin and ending at a point of S. The set of those t for which $f(t) \in V := \overline{S} \setminus S$ is closed, so it has a largest element b. Then $f : [b, c] \to \overline{S}$ is a path that maps b into the vertical interval $V = \{0\} \times [-1, 1]$ and maps $(b, c]$ to S.

Without loss of generality (WLOG), replace $[b, c]$ by $[0, 1]$ and let $f(t) = (x(t), y(t))$. Then $x(0) = 0$, $x(t) > 0$ and $y(t) = \sin\frac{1}{x(t)}$ for all $t > 0$. We show there is a sequence of points $t_n \to 0$ such that $y(t_n) = (-1)^n$, which implies that the sequence $y(t_n)$ does not converge to $y(0)$ (whatever its value is in $[-1, 1]$). This contradicts the continuity of $y(t)$ on $[0, 1]$, and consequently the continuity of f.

t_n is given as follows. For any $n \in \mathbb{N}^+$, we choose $u \in (0, x(\frac{1}{n}))$ such that $\sin\frac{1}{u} = (-1)^n$. Then the intermediate value theorem implies the existence of $t_n \in (0, \frac{1}{n})$ such that $x(t_n) = u$. □

Theorem 1.111 (Intermediate value theorem from calculus)**.** Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then, for any c between $f(a)$ and $f(b)$, there exists a number $d \in [a, b]$ such that $f(d) = c$.

Exercise 1.70. Deduce Theorem 1.111 from Theorem 1.109.

Theorem 1.112 (Fixed points in one dimension)**.** Every continuous function $f : [-1, 1] \to [-1, 1]$ has a fixed point.

Proof. If $f(-1) = -1$ or $f(1) = 1$, we are done; otherwise we have $f(-1) = a > -1$ and $f(1) = b < 1$. Hence none of the following two disjoint sets is empty,

$$A := \{(x, f(x)) : f(x) > x\}, \;\; B := \{(x, f(x)) : f(x) < x\}.$$

By Theorems 1.65 and 1.106, the graph of f,

$$G := \{(x, f(x)) : x \in [-1, 1]\},$$

is path-connected.

Suppose no x^* satisfies $f(x^*) = x^*$, then $G = A \cup B$. In the topological space G, both A and B are open with respect to a subspace topology of the standard topology. By Definition 1.104 and Lemma 1.110, this is a contradiction to G being path connected. □

Exercise 1.71. Prove Theorem 1.112 via connectedness.

Definition 1.113. The equivalence classes resulting from connectedness and path-connectedness are called *components* and *path components*, respectively.

Example 1.72. The topologist's sine curve $\overline{S}$ in Example 1.69 has only one component, but has two path components S and $V := \overline{S} \setminus S$. Note that S is open in $\overline{S}$ but not closed, while V is closed in $\overline{S}$ but not open.

If one forms a space from $\overline{S}$ by deleting all points of V having rational second coordinate, one obtains a space that has only one component but uncountably many path components.

Definition 1.114. A space X is called *locally connected at* x iff for every neighborhood U of x, there exists a connected neighborhood V of x contained in U. X is *locally connected* iff it is locally connected at each of its points.

Example 1.73. $\mathbb{Q}$ is neither connected nor locally connected; the subspace $[-1,0)\cup(0,+1]$ is not connected but locally connected; the topologist's sine curve is connected but not locally connected; each interval and each ray in the real line is both connected and locally connected.

Definition 1.115. A space X is called *locally path-connected at* x iff for every neighborhood U of x, there exists a path-connected neighborhood V of x contained in U. X is *locally path-connected* iff it is locally path-connected at each of its points.

Theorem 1.116. A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Theorem 1.117. A space X is locally path-connected if and only if for every open set U of X, each path component of U is open in X.

Theorem 1.118. Each path component of a topological space X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proof. The first statement follows from Lemma 1.110. Let C be a component of X, P be a path-component of X. If there is a point $x\in P$ and $x\in C$, we have $P\subset C$. Suppose $P\neq C$. Let Q be the union of all other path components of X, each of which intersects C and thus lies in C. Hence we have $C=P\cup Q$. By Theorem 1.117 and the local path-connectedness of X, each path component of X must be open in X. Thus P and Q constitute a separation of X, contradicting the connectedness of C. ☐

1.17 Compactness

Theorem 1.119 (Extreme values)**.** A continuous function attains its extreme values on closed bounded intervals. In other words, if f is continuous on $[a,b]$, there exist $c,d\in[a,b]$ such that

$$f(c)=\max_{x\in[a,b]} f(x), \qquad f(d)=\min_{x\in[a,b]} f(x). \tag{1.54}$$

Definition 1.120. A collection α of subsets of a topological space X is said to *cover* X, or to be a covering of X, if the union of all elements of α equals X; it is an *open covering* of X if each element of α is an open subset of X.

Definition 1.121. An *(open) cover of a subset* X *in a topological space* Y is a collection α of (open) subsets in Y such that $X\subset\cup\alpha$. A subcover of X is a subcollection of a cover that also covers X.

Example 1.74. Consider K in (1.32) and $X=K\cup\{0\}$. An open cover of K in $\mathbb{R}$ is $\{U_n : n\in\mathbb{N}^+\}$ where

$$U_n=\left(\frac{1}{n}-\epsilon_n,\frac{1}{n}+\epsilon_n\right), \qquad \epsilon_n:=\frac{1}{n(n+1)};$$

elements of this open cover are pairwise disjoint for all $n>1$. An open cover of X in $\mathbb{R}$ is $\{U_n : n\in\mathbb{N}^+\}\cup(-\epsilon,\epsilon)$ with $\epsilon:=\frac{1}{N}$ for some $N\in\mathbb{N}^+$.

Example 1.75. Consider K in Example 1.74 as a space with relative topology induced from $\mathbb{R}$. Each singleton set

$$s_n:=\left\{\frac{1}{n}\right\}$$

is open in K since $s_n=U_n\cap K$ and U_n is open in $\mathbb{R}$. Hence $\{s_n : n\in\mathbb{N}^+\}$ is an infinite open cover of K.

Exercise 1.76. Consider X in Example 1.74 as a space with relative topology induced from $\mathbb{R}$. Is the collection

$$\{\{0\}\}\cup\{s_n : n\in\mathbb{N}^+\}$$

an open cover of X? If not, can you find an infinite open cover of X whose elements are pairwise disjoint for sufficiently large n? If not, can you give a finite open cover of X?

Exercise 1.77. What is the crucial difference between K and X in the space $\mathbb{R}$ in terms of covers and subcovers?

Proof. For any open cover U of X, there exists an element of U containing all but finite many of the points $1/n$. Hence, we have a finite subcover in U for X. This is not true for K. ☐

Definition 1.122. A *compact topological space* is a topological space X where every open cover of X has a finite subcover.

Lemma 1.123. A subspace Y of a topological space X is *compact* if and only if every open cover of Y contains a finite subcover of Y.

Lemma 1.124. If X is a compact subset of a space Y, then X is compact in relative topology.

Theorem 1.125 (Bolzano-Weierstrass)**.** In a compact space, every infinite subset has an accumulation point.

Proof. Suppose there exists an infinite subset A that does not have an accumulation point. Then we can construct an open cover of X

$$\alpha=\{U_x : x\in X\}$$

such that there is at most one element of A in an element of α. By compactness, α contains a finite subcover α' that covers X. However, since each element in the finite set α' only has one element of A and α' covers A, A must be finite, which contradicts the condition of A being infinite. ☐

Corollary 1.126. In a compact space, every sequence has a convergent subsequence.

Definition 1.127. A topological space is said to be *locally compact at* x iff there is some compact subspace C of X that contains a neighborhood of x; it is *locally compact* iff it is locally compact at each of its points.

Example 1.78. The real line $\mathbb{R}$ is not compact, but locally compact. The subspace $\mathbb{Q}$ is not locally compact.

Theorem 1.128. A topological space X is locally compact Hausdorff if and only if there exists a compact Hausdorff space Y such that X is a subspace of Y and $Y \setminus X$ consists of a single point.

Proof. Munkres p. 183. □

Definition 1.129. If Y is a compact Hausdorff space and X is a proper subspace of Y such that $\overline{X} = Y$, then Y is said to be a *compactification* of X. In particular, if $Y \setminus X$ is a singleton set, then Y is called the *one-point compactification* of X.

Example 1.79. In Example 1.74, X is the one-point compactification of K.

Example 1.80. The one-point compactification of the real line $\mathbb{R}$ is homeomorphic with the circle. Similarly, the one-point compactification of the complex plane is homeomorphic with the sphere $\mathbb{S}^2$. The *Riemann sphere* is the space $\mathbb{C} \cup \{\infty\}$.

Theorem 1.130. Let X be a Hausdorff space. Then X is locally compact if and only if, given $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Corollary 1.131. Let X be locally compact Hausdorff and let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Corollary 1.132. A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

1.18 Quotient spaces

Definition 1.133. Denote by

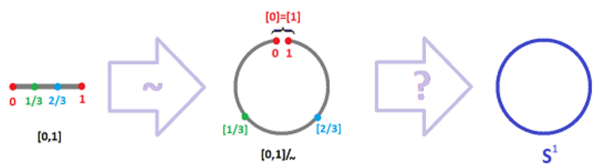
$$X/\sim := \{[x] : x \in X\} \tag{1.55}$$

the quotient set associated with a set X and an equivalence relation $\sim$. An *identification function* on X, with its signature as $q : X \to X/\sim$, takes each point in X to its equivalence class,

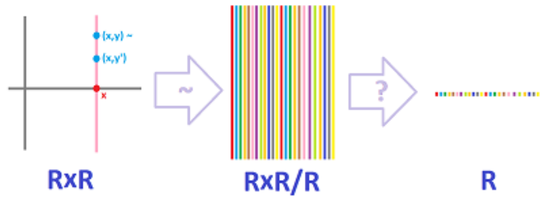
$$q(x) = [x]. \tag{1.56}$$

Example 1.81 (Gluing an interval into a circle)**.** This process is described in the Language of Definition 1.133.

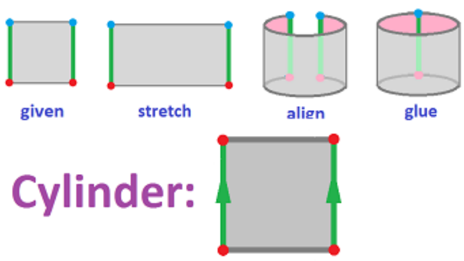
$$X = [0, 1]$$
$$0 \sim 1;\ \forall x \in \mathrm{Int}(X), x \sim x$$
$$q : X \to \mathbb{S}^1;\ q(t) = (\cos(t), \sin(t)).$$



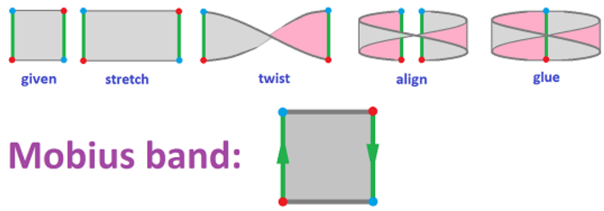
Exercise 1.82. Describe the following process in the Language of Definition 1.133.



Exercise 1.83 (Gluing the square into the cylinder)**.** What is the equivalence relation for the following process?



Exercise 1.84 (Gluing the square into the Mobius band)**.** What is the equivalence relation for the following process?



Definition 1.134. The *quotient space* with respect to a topological space X and an equivalence relation $\sim$ on X is the topological space $(X/\sim, \mathcal{T}_{X/\sim})$ such that

$$\mathcal{T}_{X/\sim} = \{U : q^{-1}(U) \in \mathcal{T}_X\}, \tag{1.57}$$

where the identification function q is also called a *gluing map*, an *attaching map*, a *pasting map*, or a *quotient map*.

Exercise 1.85. Prove that (1.57) is indeed a topology.

Theorem 1.135. If X is path-connected, then so is $X/\sim$ for any equivalence relation.

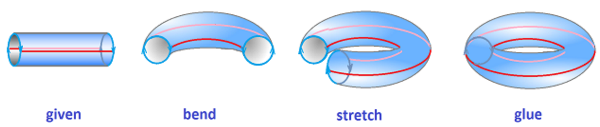
Theorem 1.136. If X is compact, then so is $X/\sim$ for any equivalence relation.

Theorem 1.137. Any function defined on a topological space can be thought of as an identification map.

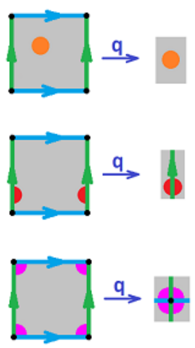
Proof. Let the topological space be X, define the range of the function f to be $f(X)$. Then define the topology of $f(X)$ as that in (1.57). □

Definition 1.138. The *torus*, denoted by $\mathbb{T}^2$, is the quotient space obtained from the unit square by the quotient map that is induced from the equivalence relation $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.

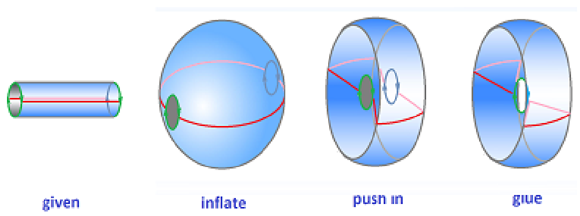
Example 1.86. The torus can be constructed from a cylinder by the equivalence relation $(x, 0) \sim (x, 1)$.



As the torus is a quotient space of the square, there are three types of neighborhoods created by gluing.

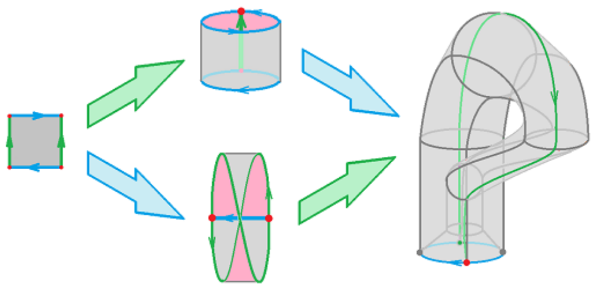


Note that there is a different way of forming the torus.

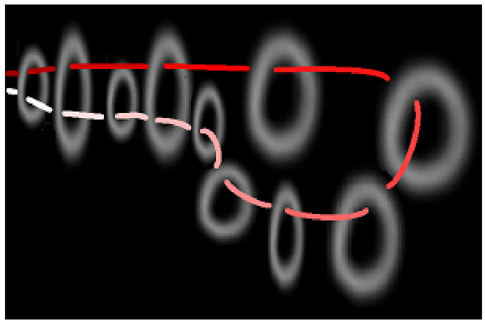


Definition 1.139. The *Klein Bottle*, denoted by $\mathbb{K}^2$, is the quotient space obtained from the unit square by the quotient map that is induced from the equivalence relation $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, y)$.

Example 1.87. The Klein bottle can be constructed from a cylinder by the equivalence relation $(x, 0) \sim (1 - x, 1)$. To bring them together, we need to "cut" through the cyclinder's side. Similar to constructing the torus, there are two ways to do this.

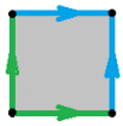


To understand that there is really no self-intersection, one can think of the Klein bottle as a circle moving through space. Imagine a smoke ring that leaves the mouth, floats forward, shrinks, and then floats back in.



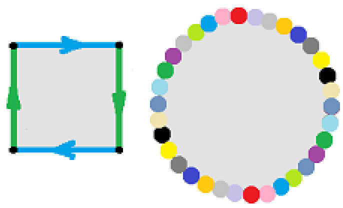
Exercise 1.88. Why is the Klein bottle a 2-manifold?

Exercise 1.89. What is the quotient space resulting from the following equivalence relation?



Definition 1.140. The *antipodal point* of a point on $\mathbb{S}^n$ is the point on $\mathbb{S}^n$ diametrically opposite to it such that a line drawn from the one to the other passes through the center of the sphere and forms a diameter.

Definition 1.141. The *projective plane*, denoted by $\mathbb{P}^2$, is the quotient space obtained from the unit square by identifying antipodal points.



Exercise 1.90. Show that we can also define the projective plane as the quotient space obtained from $\mathbb{R}^3 \setminus \mathbf{0}$ by choosing the lines through the origin $\mathbf{0}$ as the equivalence classes.

Solution. This alternative definition yields the same set of equivalence classes.

Exercise 1.91. Why is the projecive plane a 2-manifold?

Exercise 1.92. The projective plane contains the Mobius band. Find it.

Solution. Delete a disc from the projective plane and we have a Mobius band. To see this, delete in the unit disk the points satisfying $|y| > \frac{1}{2}$.

Definition 1.142. A *fundamental polygon of surface* is an oriented polygon with an even number of sides by pairwise identification of its edges.

Exercise 1.93. How many different fundamental squares of surface are there? Draw their diagrams.