# Appendix E

# Fourier Analysis of Linear PDEs

**Definition E.1.** The $L^2$-*norm* of a Lebesgue-measurable function $u : \mathbb{R} \to \mathbb{C}$ is a nonnegative or infinite real number

$$\|u\| = \left[\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x\right]^{\frac{1}{2}}. \tag{E.1}$$

**Notation 16.** Denote by $L^2$ the set of all functions whose $L^2$-norms are finite, i.e.,

$$L^2 = \{u : \|u\| < \infty\}. \tag{E.2}$$

Similarly, $L^1$ and $L^\infty$ respectively denote the sets of functions with finite $L^1$- and $L^\infty$- norms,

$$\|u\|_1 = \int_{-\infty}^{\infty} |u(x)| \mathrm{d}x, \quad \|u\|_\infty = \sup_{-\infty<x<\infty} |u(x)|. \tag{E.3}$$

Since the $L^2$ norm is the norm used in most applications, we have reserved the symbol $\|\cdot\|$ without a subscript for it.

## E.1 Fourier transform

**Definition E.2.** The *Fourier transform* of a function $u \in L^2$ is the function $\hat{u} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x) \mathrm{d}x. \tag{E.4}$$

**Theorem E.3.** If $u \in L^2$, then the Fourier transform

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x) \mathrm{d}x$$

belongs to $L^2$ also, and $u$ can be recovered from $\hat{u}$ by the *inverse Fourier transform*

$$u(x) = (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{u}(\xi) \mathrm{d}\xi. \tag{E.5}$$

**Theorem E.4.** Let $u, v \in L^2$ have Fourier transforms $\hat{u} = \mathcal{F}u, \hat{v} = \mathcal{F}v$. Then

(a) Linearity. If $c \in \mathbb{R}$, then

$$\mathcal{F}\{u + v\}(\xi) = \hat{u}(\xi) + \hat{v}(\xi), \tag{E.6}$$
$$\mathcal{F}\{cu\}(\xi) = c\hat{u}(\xi). \tag{E.7}$$

(b) Translation. If $x_0 \in \mathbb{R}$, then

$$\mathcal{F}\{u(x + x_0)\} = e^{i\xi x_0} \hat{u}(\xi). \tag{E.8}$$

(c) Modulation. If $\xi_0 \in \mathbb{R}$, then

$$\mathcal{F}\{e^{i\xi_0 x} u(x)\}(\xi) = \hat{u}(\xi - \xi_0). \tag{E.9}$$

(d) Dilation. If $c \in \mathbb{R}$ with $c \neq 0$, then

$$\mathcal{F}\{u(cx)\}(\xi) = \frac{1}{|c|} \hat{u}\left(\frac{\xi}{c}\right). \tag{E.10}$$

(e) Conjugation.

$$\mathcal{F}\{\bar{u}\}(\xi) = \overline{\hat{u}(-\xi)}. \tag{E.11}$$

(f) Differentiation. If $u_x \in L^2$, then

$$\mathcal{F}\{u_x\}(\xi) = i\xi \hat{u}(\xi). \tag{E.12}$$

(g) Inversion.

$$\mathcal{F}^{-1}\{u\}(\xi) = \hat{u}(-\xi). \tag{E.13}$$

**Definition E.5.** A function $u(x)$ is *even, odd, real, or imaginary* if $u(x) = u(-x)$, $u(x) = -u(-x)$, $u(x) = \overline{u(x)}$, or $u(x) = -\overline{u(x)}$, respectively; $u(x)$ is *Hermitian or skew-Hermitian* if $u(x) = \overline{u(-x)}$ or $u(x) = -\overline{u(-x)}$, respectively.

**Theorem E.6.** Let $u \in L^2$ have Fourier transform $\hat{u} = \mathcal{F}u$. Then

(a) $u(x)$ is even (odd) $\Leftrightarrow$ $\hat{u}(\xi)$ is even (odd).

(b) $u(x)$ is real (imaginary) $\Leftrightarrow$ $\hat{u}(\xi)$ is hermitian (skew-hermitian) and therefore

(c) $u(x)$ is real and even $\Leftrightarrow$ $\hat{u}(\xi)$ is real and even.

(d) $u(x)$ is real and odd $\Leftrightarrow$ $\hat{u}(\xi)$ is imaginary and odd.

(e) $u(x)$ is imaginary and even $\Leftrightarrow$ $\hat{u}(\xi)$ is imaginary and even.

(f) $u(x)$ is imaginary and odd $\Leftrightarrow$ $\hat{u}(\xi)$ is real and odd.

**Definition E.7.** The *convolution* of two functions $u, v$ is the function $u * v$ defined by

$$(u * v)(x) = (v * u)(x) \\ = \int_{-\infty}^{\infty} u(x-y)v(y)\mathrm{d}y = \int_{-\infty}^{\infty} u(y)v(x-y)\mathrm{d}y,$$

assuming these integrals exist.

**Theorem E.8.** If $u \in L^2$ and $v \in L^1$ (or vice versa), then $u * v \in L^2$, and $\widehat{u * v}$ satisfies

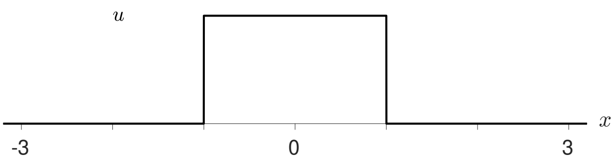
$$\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi). \tag{E.14}$$

**Theorem E.9.** The $L^2$-norms of $u$ and $\hat{u}$ are related by *Parseval's equality*, a.k.a the *Plancherel theorem*,

$$\forall u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \qquad \|\hat{u}\| = \|u\|. \tag{E.15}$$

**Example E.10** (B-splines)**.** For the function

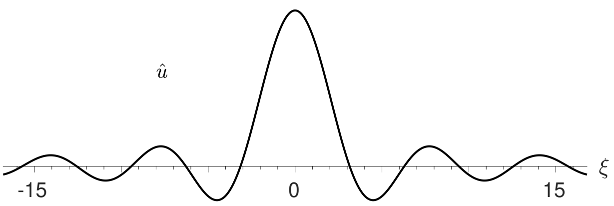
$$u(x) = \begin{cases} \frac{1}{2}, & \text{for } -1 \le x \le 1; \\ 0, & \text{otherwise,} \end{cases} \tag{E.16}$$




(E.1) yields $\|u\| = 1/\sqrt{2}$, and (E.4) gives

$$\hat{u}(\xi) = \frac{1}{2}\int_{-1}^{1} e^{-i\xi x}\mathrm{d}x = \left.\frac{e^{-i\xi x}}{-2i\xi}\right|_{-1}^{1} = \frac{\sin\xi}{\xi}, \tag{E.17}$$

which is known as the *sinc function.*




From (E.1) and the indispensable identity

$$\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\mathrm{d}s = \pi, \tag{E.18}$$

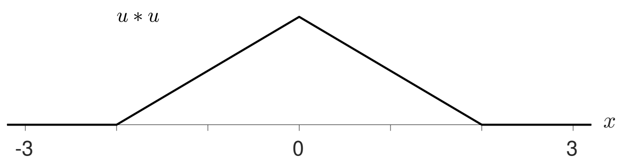
which can be derived by complex contour integration, we calculate $\|\hat{u}\| = \sqrt{\pi}$, which confirms (E.15).

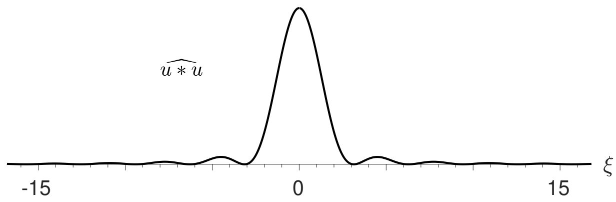
By Definition E.7, it is readily verified that

$$(u * u)(x) = \begin{cases} \frac{1}{2}(1 - |x|/2), & \text{for } -2 \le x \le 2, \\ 0, & \text{otherwise,} \end{cases} \tag{E.19}$$

and

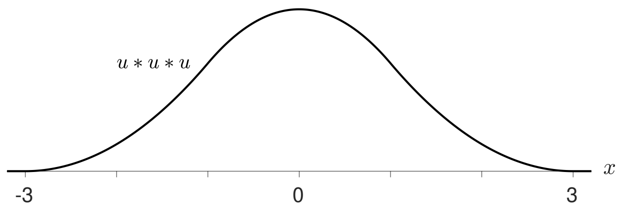
$$(u * u * u)(x) = \begin{cases} \frac{3}{4} - \frac{1}{4}x^2, & \text{for } -1 \le x \le 1, \\ \frac{1}{8}(9 - 6|x| + x^2), & \text{for } 1 \le |x| \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

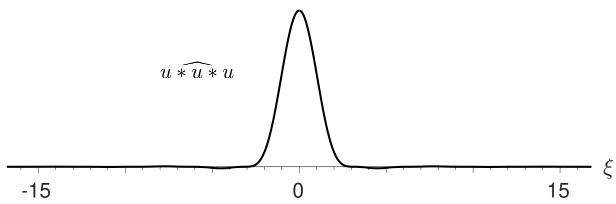



By (E.14) and (E.17), their Fourier transforms are

$$\widehat{u * u}(\xi) = \frac{\sin^2\xi}{\xi^2}, \quad \widehat{u * u * u}(\xi) = \frac{\sin^3\xi}{\xi^3}. \tag{E.20}$$




In general, a convolution $u_{(p)}$ of $p$ copies of $u$ has the Fourier transform

$$\widehat{u_{(p)}}(\xi) = \mathcal{F}\{u * u * \cdots * u\}(\xi) = \left(\frac{\sin\xi}{\xi}\right)^p.$$

**Example E.11.** The function

$$u(x) = \begin{cases} \frac{1}{4}, & \text{for } -2 \le x < 0, \\ -\frac{1}{4}, & \text{for } 0 < x \le 2, \\ 0, & \text{otherwise,} \end{cases} \tag{E.21}$$

has Fourier transform

$$\begin{aligned} \hat{u}(\xi) &= \frac{1}{4}\int_{-2}^{0} e^{-i\xi x}\mathrm{d}x - \frac{1}{4}\int_{0}^{2} e^{-i\xi x}\mathrm{d}x \\ &= \frac{1}{-4i\xi}(1 - e^{2i\xi} - e^{-2i\xi} + 1) \\ &= \frac{1}{4i\xi}(e^{i\xi} - e^{-i\xi})^2 = \frac{i\sin^2\xi}{\xi}, \end{aligned}$$

which is $i\xi$ times the Fourier transform (E.20) of the triangular hat function (E.19).

**Definition E.12.** A function $u$ defined on $\mathbb{R}$ is said to have *bounded variation* if there is a constant $M$ such that for any finite $m$ and any points $x_0 < x_1 < \cdots < x_m$,

$$\sum_{j=1}^{m} |u(x_j) - u(x_{j-1})| \le M. \tag{E.22}$$

**Theorem E.13.** Let $u$ be a function in $L^2$.

(a) If $u$ has $p-1$ continuous derivatives in $L^2$ for some $p \geq 0$, and a $p$th derivative in $L^2$ that has bounded variation, then

$$\hat{u}(\xi) = O(|\xi|^{-p-1}) \quad \text{as } |\xi| \to \infty. \tag{E.23}$$

(b) If $u$ has infinitely many continuous derivatives in $L^2$, then

$$\hat{u}(\xi) = O(|\xi|^{-M}) \quad \text{as } |\xi| \to \infty \text{ for all } M, \tag{E.24}$$

and conversely.

(c) If $u$ can be extended to an analytic function of $z = x + iy$ in the complex strip $|\mathrm{Im}z| < a$ for some $a > 0$, with $\|u(x+iy)\| \leq$ const uniformly for each constant $-a < y < a$, then

$$e^{a|\xi|}\hat{u}(\xi) \in L^2, \tag{E.25}$$

and conversely.

(d) If $u$ can be extended to an entire function of $z = x + iy$ with $|u(z)| = O(e^{a|z|})$ as $|z| \to \infty (z \in \mathbb{C})$ for some $a > 0$, then $\hat{u}$ has compact support contained in $[-a, a]$, i.e.,

$$\hat{u}(\xi) = 0 \quad \text{for all } |\xi| > a, \tag{E.26}$$

and conversely.

**Example E.14.** The square wave $u$ of Example E.10 satisfies condition $(a)$ of Theorem E.13 with $p = 0$, so its Fourier transform should satisfy

$$|\hat{u}(\xi)| = O(|\xi|^{-1}),$$

as is verified by (E.17). On the other hand, suppose we interchange the role of $u$ and $\hat{u}$ and apply the theorem again. The function $u(\xi) = \sin\xi/\xi$ is entire, and since $\sin(\xi) = (e^{i\xi} - e^{-i\xi})/2i$, it satisfies

$$u(\xi) = O(e^{|\xi|}) \quad \text{as } |\xi| \to \infty$$

(with $\xi$ now taking complex values). By part $(d)$ of Theorem E.13, it follows that $u(x)$ must have compact support contained in $[-1, 1]$, as indeed it does.

Repeating the example for $u * u$, condition $(a)$ now applies with $p = 1$, and the Fourier transform (E.20) is indeed of magnitude $O(|\xi|^{-2})$, as required. Interchanging $u$ and $\hat{u}$, we note that $\sin^2\xi/\xi^2$ is an entire function of magnitude $O(e^{2|\xi|})$ as $|\xi| \to \infty$, and $u * u$ has support contained in $[-2, 2]$.

## E.2 Fourier analysis

**Lemma E.15.** For a linear PDE of the form

$$\frac{\partial u}{\partial t} + \sum_{n=1}^{N} a_n \frac{\partial^n u}{\partial x^n} = 0, \tag{E.27}$$

the evolution of a single Fourier mode of wave number $\xi$ satisfies the ODE

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + \sum_{n=1}^{N} a_n (i\xi)^n \hat{u}(\xi, t) = 0. \tag{E.28}$$

**Lemma E.16.** The solution to the linear PDE (E.27) with initial condition $u(x, 0) = \eta(x)$ is

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i(\xi x - \omega t)} \mathrm{d}\xi, \tag{E.29}$$

where $w := \sum_{n=1}^{N} a_n \xi^n i^{n-1}$.

**Definition E.17.** The *dispersion relation* of a PDE or a wave problem is the relation between the frequence $\omega$ and the wave number $\xi$, i.e.,

$$\omega = \omega(\xi). \tag{E.30}$$

**Example E.18.** The beam equation

$$\varphi_{tt} + \gamma^2 \varphi_{xxxx} = 0 \tag{E.31}$$

is characterized by its dispersion relation $\omega = \pm\gamma\xi^2$.

**Example E.19.** The linear Korteweg-deVries equation

$$\varphi_t + c_0\varphi_x + \nu\varphi_{xxx} = 0 \tag{E.32}$$

is characterized by its dispersion relation $\omega = c_0\xi - \nu\xi^3$.

**Definition E.20.** The system (E.27) is said to be *hyperbolic* if the PDE is hyperbolic; it is *dissipative* if $\omega$ is purely imaginary; it is *dispersive* if $\omega(\xi)$ is real and $\omega'(\xi)$ is not a constant.

**Definition E.21.** The *phase velocity* of a monochromatic wave with wave number $\xi$ is

$$C_p(\xi) := \frac{\omega(\xi)}{\xi}. \tag{E.33}$$

**Definition E.22.** The *group velocity* of a monochromatic wave with wave number $\xi$ is

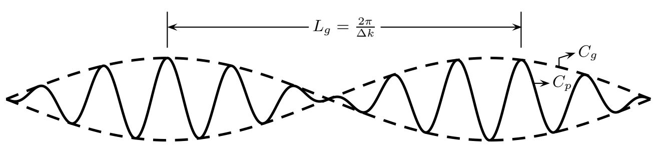
$$C_g(\xi) := \frac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}. \tag{E.34}$$

**Example E.23.** For the linear PDE (E.27), the phase velocity of a single Fourier mode is

$$C_p(\xi) = \sum_{n=1}^{N} a_n \xi^{n-1} i^{n-1}$$

while the group velocity is

$$C_g(\xi) = \sum_{n=1}^{N} n a_n \xi^{n-1} i^{n-1}.$$



**Example E.24.** For the advection equation, we have $a_1 = a$ and $a_i = 0$ for all $i > 1$. Consequently, we have

$$\omega = a\xi, \qquad C_p = a = C_g,$$

hence Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi(x-at)} \mathrm{d}\xi = \eta(x-at).$$

Thus all wave modes that constitute the initial data $\eta(x)$ move at the same phase speed, which is also the moving speed of energy.

**Example E.25.** For the heat equation

$$u_t = \nu u_{xx},$$

we have $a_2 = -\nu < 0$, $a_1 = a_3 = a_4 = \ldots = 0$, and thus

$$\omega = a_2\xi^2 i, \qquad C_p = a_2\xi i, \quad C_g = 2a_2\xi i.$$

Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi x} e^{a_2\xi^2 t} \mathrm{d}\xi,$$

where the term "$e^{i\xi x}$" denotes the initial mode $\xi$ that does not move while "$e^{-\nu\xi^2 t}$" represents the exponential decay with respect to time.

**Example E.26.** For the dispersion equation

$$u_t = u_{xxx},$$

we have $a_3 = -1$, $a_1 = a_2 = a_4 = a_5 = \ldots = 0$,

$$\omega = \xi^3, \qquad C_p = \xi^2, \quad C_g = 2\xi^2.$$

Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta} e^{i\xi(x-\xi^2 t)} \mathrm{d}\xi,$$

thus there is no damping, but different phases move with different speed $\xi^2$.

**Example E.27.** For the equation

$$u_t + au_x + bu_{xxx} = 0,$$

we have $a_1 = a$, $a_3 = b$, $a_2 = a_4 = a_5 = \ldots = 0$, and

$$\omega = a\xi - b\xi^3, \qquad C_p = a - b\xi^2, \quad C_g = a - 3b\xi^2.$$

Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi(x-(a-b\xi^2)t)} \mathrm{d}\xi.$$