Chapter 3

hw3 12235005 谭焱

Problem 1. Consider the nonlinear equation

$$f(x) = x^2 - 2 = 0.$$

(a) With $x_0 = 1$ as a starting point, what is the value of x_1 if you use Newton's method for solving this problem?

(b) With $x_0 = 1$ and $x_1 = 2$ as starting points, what is the value of x_2 if you use the secant method for the same problem?

Solution. (a)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1-2}{2} = 1.5.$$

(b)

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 2 - 2\frac{2-1}{2-(-1)} = \frac{4}{3}.$$

□

Problem 2. Newton's method is sometimes used to implement the built-in root function on a computer, with the initial guess supplied by a lookup table.

(a) What is the Newton iteration for computing the square root of a positive number y (i.e., for solving the equation $f(x) = x^2 - y = 0$, given y)?

Solution.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - y}{2x_n} = \frac{x_n^2 + y}{2x_n}.$$

□

Problem 3. Express the Newton iteration for solving each of the following systems of nonlinear equations.

(b)

$$x_1^2 + x_1 x_2^3 = 9,$$
$$3x_1^2 x_2 - x_2^3 = 4.$$

Solution. The Jacobian matrix is

$$J(x_1,x_2)=\begin{bmatrix}2x_1+x_2^3 & 3x_1x_2^2\\ 6x_1x_2 & 3x_1^2-3x_2^2.\end{bmatrix}$$

We have

$$J(x_1^{(n)},x_2^{(n)})\begin{bmatrix}s_1^{(n)}\\ s_2^{(n)}\end{bmatrix}=\mathbf{f}(x_1^{(n)},x_2^{(n)}).$$

So

$$\begin{bmatrix}x_1^{(n+1)}\\ x_2^{(n+1)}\end{bmatrix}=\begin{bmatrix}x_1^{(n)}\\ x_2^{(n)}\end{bmatrix}+\begin{bmatrix}s_1^{(n)}\\ s_2^{(n)}\end{bmatrix}.=\begin{bmatrix}x_1^{(n)}\\ x_2^{(n)}\end{bmatrix}+\begin{bmatrix}2x_1+x_2^3 & 3x_1x_2^2\\ 6x_1x_2 & 3x_1^2-3x_2^2.\end{bmatrix}^{-1}\begin{bmatrix}x_1^{(n)}\\ x_2^{(n)}\end{bmatrix}$$

□

Problem 4. Suppose you are using the secant method to find a root x^* of a nonlinear equation $f(x)=0$. Show that if at any iteration it happens to be the case that either $x_k=x^*$ or $x_{k-1}=x^*$ (but not both), then it will also be true that $x_{k+1}=x^*$.

Solution.

- When $x_k=x^*$, knowing

$$x_{k+1}=x_k-f(x_k)\frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})}=x^*-f(x^*)\frac{x^*-x_{k-1}}{f(x^*)-f(x_{k-1})}=x^*-0=x^*.$$

- When $x_{k-1}=x^*$, knowing

$$\begin{aligned}x_{k+1}&=x_k-f(x_k)\frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})}=x_k-f(x_k)\frac{x_k-x^*}{f(x_k)-f(x^*)}\\&=x_k-f(x_k)\frac{x_k-x^*}{f(x_k)}=x_k-(x_k-x^*)\\&=x^*.\end{aligned}$$

□

Problem 5. Consider the system of equations

$$\begin{aligned}x_1-1&=0,\\ x_1x_2-1&=0.\end{aligned}$$

For what starting point or points, if any, will Newton's method for solving this system fail? Why?

Solution. The Jacobian matrix is

$$J(x_1,x_2)=\begin{bmatrix}1 & 0\\ x_2 & x_1\end{bmatrix}$$

When $x_1=0$, it is singular, and Newton's method will fail. □

Problem 6. Given the three data points $(-1,1),(0,0),(1,1)$, determine the interpolating polynomial of degree two:

(a) Using the monomial basis

(b) Using the Lagrange basis

(c) Using the Newton basis

Show that the three representations give the same polynomial.

Solution. (a) Solving the following system of linear equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

get $x_1 = 0, x_2 = 0, x_3 = 1$, means the interpolating polynomial is

$$p_2(t) = t^2.$$

(b) By Lagrange interpolation polynomial, we have

$$\begin{aligned} p_2(t) &= 1 \cdot \frac{(t-0)(t-1)}{(-1-0)(-1-1)} + 0 \cdot \frac{(t+1)(t-1)}{(0+1)(0-1)} + 1 \cdot \frac{(t+1)(t-0)}{(1+1)(1-0)} \\ &= \frac{t(t-1)}{2} + \frac{t(t+1)}{2} = t^2. \end{aligned}$$

(c) From the table of divided differences

t	y		
-1	1		
0	0	-1	
1	1	1	1

, and by Newton's formula we have polynomial

$$p_2(t) = 1 - (t+1) + (t+1)t = t^2.$$

It's obvious that the above three representations give the same polynomial.

□

Problem 7. Write a formal algorithm for evaluating a polynomial at a given argument using Horner's nested evaluation scheme

(a) For a polynomial expressed in terms of the monomial basis

(b) For a polynomial expressed in Newton form

Solution. (a)

算法 1 Evaluate a polynomial expressed in monomial basis form

```
y ← 0
for i = n to 1 do
    y ← ty + a_i
end for
```

(b)

□

Problem 8. Use Lagrange interpolation to derive the formulas given in Section 5.5.5 for inverse quadratic interpolation.

算法 2 Evaluate polynomial expressed in Newton form

```
y ← 0
for i = n to 1 do
    y ← (t − t_i)y + a_i
end for
```

Solution. The interpolation condition is

$$p_2(f_a) = a, \quad p_2(f_b) = b, \quad p_2(f_c) = c.$$

Applying the Lagrange interpolation polynomial gets

$$p_2(y) = a\frac{(y-f_b)(y-f_c)}{(f_a-f_b)(f_a-f_c)} + b\frac{(y-f_a)(y-f_c)}{(f_b-f_a)(f_b-f_c)} + c\frac{(y-f_a)(y-f_b)}{(f_c-f_a)(f_c-f_b)},$$

Replace with $y = 0$ gives

$$\begin{aligned} p_2(0) &= a\frac{f_bf_c}{(f_a-f_b)(f_a-f_c)} + b\frac{f_af_c}{(f_b-f_a)(f_b-f_c)} + c\frac{f_af_b}{(f_c-f_a)(f_c-f_b)} \\ &= b + \frac{v(w(u-w)(c-b)-(1-u)(b-a))}{(w-1)(u-1)(v-1)} \qquad \text{by definecolor of } p, q \\ &= b + p/q, \end{aligned}$$

where

$$u = \frac{f_b}{f_c}, \quad v = \frac{f_b}{f_a}, \quad w = \frac{f_a}{f_c}.$$

□

Problem 9. Prove that the formula using divided differences given in Section 7.3.3,

$$x_j = f[t_1, t_2, \ldots, t_j],$$

indeed gives the coefficient of the jth basis function in the Newton polynomial interpolant.

证明. We utilize a mathematical induction on j.

- For $j = 1$, the interpolating polynomial is

$$p_1(t) = x_1 = f(t_1).$$

- Suppose the conclusion is true for all integers less than n, we show that it holds for $n+1$ as well.

  By our inductive hypothesis, we know that the polynomial interpolating

$$p_n(t_1) = f(t_1), \quad p_n(t_2) = f(t_2), \quad p_n(t_n) = f(t_n)$$

  is given by

$$p_n(t) = \sum_{i=1}^{n} x_i \prod_{j=1}^{i-1}(t-t_j) = \sum_{i=1}^{n} f[t_1, \ldots, t_i] \prod_{j=1}^{i-1}(t-t_j).$$

  Similarly,

$$q_n(t_2) = f(t_2), \quad q_n(t_3) = f(t_3), \quad q_n(t_{n+1}) = f(t_{n+1})$$

  is given by

$$q_n(t) = \sum_{i=1}^{n} x_i \prod_{j=2}^{i}(t-t_j) = \sum_{i=1}^{n} f[t_2, \ldots, t_{i+1}] \prod_{j=2}^{i}(t-t_j).$$

Therefore from the uniqueness of the interpolating polynomial, we know that the polynomial for interpolating

$$p_{n+1}(t_1) = f(t_1), \quad p_{n+1}(t_2) = f(t_2), \quad p_{n+1}(t_n) = f(t_n), \quad p_{n+1}(t_{n+1}) = f(t_{n+1})$$

is given by

$$p_{n+1}(t) = \frac{t - t_{n+1}}{t_1 - t_{n+1}} p_n(t) + \frac{t - t_1}{t_{n+1} - t_1} q_n(t)$$

Comparing the coefficient of the highest-order term of the above two polynomials gives

$$x_{n+1} = \frac{f[t_2, t_3, \ldots, t_{n+1}] - f[t_1, t_2, \ldots, t_n]}{t_{n+1} - t_1} = f[t_1, t_2, \ldots, t_{n+1}],$$

where the second equality follows from the definition of divided differences. Therefore we have shown that the conclusion holds for $n+1$, which completes the inductive proof.

□

Problem 10. Verify the properties of B-splines enumerated in Section 7.4.3.

Solution. 1 By mathematical induction

- For $k = 0$, $B_i^0(t) = 0$ is obvious.
- Assuming for $k = 0, \ldots n$, $t < t_i$ or $t > t_{i+k+1}$ imply $B_i^k = 0$. By induction hypothesis, we have $B_i^n(t) = 0, B_{i+1}^n(t) = 0$, so

$$B_i^{n+1}(t) = v_i^{n+1}(t) B_i^n(t) + v_{i+1}^{n+1}(t) = 0.$$

  Therefore we have $B_i^k(t) = 0, t < t_i$ or $t > t_{i+k+1}$.

2 Similarly assuming $B_i^k(t) > 0$ as in 1 in different domain, we have $B_i^k(t) > 0, t \in [t_i, t_{i+k+1}]$.

3
- For $k = 0, \forall t$, $\sum_{i--\infty}^{\infty} B_i^0(t) = 1$ is obvious.
- Assuming for $k = n, \forall t$, the assumption is right.

$$\begin{aligned}\sum_{i--\infty}^{\infty} B_i^{n+1}(t) &= \sum_{i--\infty}^{\infty} (v_i^{n+1}(t) B_i^n(t) + v_{i+1}^{n+1}(t) B_{i+1}^n(t)) \\ &= \sum_{i--\infty}^{\infty} B_i^n(t)(v_i^{n+1}(t) + (1 - v_i^{n+1})) = \sum_{i--\infty}^{\infty} B_i^n(t) = 1\end{aligned}$$

4 We prove the following theorem:

Theorem 3.1. For $k \geq 2$, we have, $\forall t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} B_i^k(t) = \frac{k B_i^{k-1}(t)}{t_{i+k} - t_i} - \frac{k B_{i+1}^{k-1}(t)}{t_{i+k+1} - t_{i+1}}. \tag{3.1}$$

For $k = 1$, (3.1) holds for all t except at the three knots t_i, t_{i+1}, t_{i+2}, where the derivative of B_i^1 is not defined.

Proof of Theorem. We first show that (3.1) holds for all t except at the knots t_j.

$$\forall t \in \mathbb{R} \backslash \{t_i, t_{i+1}, t_{i+2}\}, \quad \frac{\mathrm{d}}{\mathrm{d}t} B_i^1(t) = \frac{1}{t_{i+1} - t_i} B_i^0(t) - \frac{1}{t_{i+2} - t_{i+1}} B_{i+1}^0(t).$$

Hence the induction hypothesis holds. Now suppose (3.1) holds $\forall t \in \mathbb{R}\backslash\{t_i, \ldots, t_{i+k+1}\}$. apply the induction hypothesis (3.1), and we have

$$\frac{\mathrm{d}}{\mathrm{d}t} B_i^{k+1}(t) = \frac{B_i^k(t)}{t_{i+k+1} - t_i} - \frac{B_{i+1}^k(t)}{t_{i+k+2} - t_{i+1}} + kC(t) \tag{3.2}$$

where

$$\begin{aligned} C(t) &= \frac{t - t_i}{t_{i+k+1} - t_i} \left[\frac{B_i^{k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1}^{k-1}(t)}{t_{i+k+1} - t_{i+1}} \right] + \frac{t_{i+k+2} - t}{t_{i+k+2} - t_{i+1}} \left[\frac{B_{i+1}^{k-1}(t)}{t_{i+k+1} - t_{i+1}} - \frac{B_{i+2}^{k-1}(t)}{t_{i+k+2} - t_{i+2}} \right] \\ &= \frac{1}{t_{i+k+1} - t_i} \left[\frac{(t - t_i) B_i^{k-1}(t)}{t_{i+k} - t_i} + \frac{(t_{i+k+1} - t) B_{i+1}^{k-1}(t)}{t_{i+k+1} - t_{i+1}} \right] \\ &\quad - \frac{1}{t_{i+k+2} - t_{i+1}} \left[\frac{(t - t_{i+1}) B_{i+1}^{k-1}(t)}{t_{i+k+1} - t_{i+1}} + \frac{(t_{i+k+2} - t) B_{i+2}^{k-1}(t)}{t_{i+k+2} - t_{i+2}} \right]] \\ &= \frac{B_i^k(t)}{t_{i+k+1} - t_i} - \frac{B_{i+1}^k(t)}{t_{i+k+2} - t_{i+1}}, \end{aligned}$$

Then (3.2) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} B_i^{k+1}(t) = \frac{(k+1) B_i^k(t)}{t_{i+k+1} - t_i} - \frac{(k+1) B_{i+1}^k(t)}{t_{i+k+2} - t_{i+1}},$$

which completes the inductive proof of (3.1) except at the knots. Since $B_i^1(t)$ is continuous, an easy induction shows that B_i^k is continuous for all $k \geq 1$. Hence the right-hand side of (3.1) is continuous for all $k \geq 2$. Therefore, if $k \geq 2$, $\frac{\mathrm{d}}{\mathrm{d}t} B_i^k(t)$ exists for all $t \in \mathbb{R}$. This completes the proof of the theorem. □

The proof follows from the above theorem and a simple induction on k.

5 By Theorem 2.1 and assume there is c_i such that

$$\sum_{i=-\infty}^{\infty} c_i B_i^k(t) = 0.$$

We have

$$\frac{d}{dt} \sum_{i=-\infty}^{\infty} c_i B_i^k(t) = \sum_{i=-\infty}^{\infty} \left(\frac{c_i - c_{i-1}}{t_{i+k} - t_i} B_i^{k-1}(t)\right) = 0.$$

Combining with mathematical induction and obvious situation $k = 0$, which means B_i^{k-1} is linearly independent. So we have the coefficients $\forall i$, $\frac{c_i - c_{i-1}}{t_{i+k} - t_i} = 0 \Rightarrow c_i = C$. Then

$$\sum_{i=-\infty}^{\infty} C B_i^{k-1}(t) = 0$$

is contradiction with property 3.

6 Since there is $(k+1)*2$ coefficient unknowns in two spline functions have $k-1$ times continuous differentiable. And in other side there is $k-1$ continuous differentiable condition equations and $k+2$ independent B-splines coefficients(By property 1, 5), At last there is one value condition in t_i. So $(k+1)*2 = (k-1)+(k+2)+1$ imply there always is a unique solution. Which also means $B_i^k, i \in N$ is a basis span the $k-1$ times continuous differentiable spline function.

□