Chapter 5

Cell Complexes

5.1 Attaching spaces

Definition 5.1. Let $\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces, which may or may not be disjoint. Let E be the set that is the union of the disjoint spaces,

$$E = \cup_{\alpha\in J} E_\alpha, \quad E_\alpha := X_\alpha \times \{\alpha\}. \tag{5.1}$$

If we topologize E by declaring U to be open in E if and only if $U \cap E_\alpha$ is open in E_α for each α, then E is said to be the *topological sum* of the spaces E_α. (Sometimes we abuse terminology to say that E is the topological sum of X_α).

Definition 5.2. Given two spaces X, Y, and a continuous map $q : B \to A$ where $A \subset X$ and $B \subset Y$, the *attaching space* $X \sqcup_q Y$ of X and Y by the *attaching map* q is the quotient space of the topological sum $X \sqcup Y$ by identifying each $a \in A$ with every $b \in B$ such that $q(b) = a$.

Example 5.1. Let $X = A$ be the space of one point. Let $Y = \mathbb{D}^2$ and $B = \partial Y$. For the attaching map $q : B \to A$ that collapses B to the single point in A, the attaching space $X \sqcup_q Y$ is homeomorphic to $\mathbb{S}^2$.

Example 5.2. A *robot's glove* is made by removing the interior of an embedded disk from the torus. A *double torus* can be made by jointing two robot's gloves along their boundaries. This process can be continued to make an *n-ple torus* with genus n.

Example 5.3. Let $X = A = \mathbb{S}^1$, $Y = \mathbb{D}^2$, $B = \partial Y = \mathbb{S}^1$. Let $q : B \to A$ be the map that sends B twice around A, i.e., $q(z) = z^2$, where we think of our circles as the set of complex numbers of modulus 1. The attaching space $X \sqcup_q Y$ is homeomorphic to the projective plane $\mathbb{P}^2$. The key to understand this is the behavior of the surjective map $q : \mathbb{S}^1 \to \mathbb{S}^1$, which covers the image with both the subspaces

$$\{e^{i\theta} : \theta \in [0, \pi]\}, \quad \{e^{i\theta} : \theta \in [\pi, 2\pi]\}$$

and thus identify the antipodal points on the domain $\mathbb{S}^1$. Hence, the projective plane can be regarded as the resulting space when you glue the Mobius band to the unit disk.

Exercise 5.4. Let $q : B \to A$ be the same map as that in Example 5.3 and let $X = A = \mathbb{S}^1$, $Y = \mathbb{S}^1 \times [0, 1]$, $B = \mathbb{S}^1 \times \{0\}$. Show that the attaching space $X \sqcup_q Y$ is homeomorphic to the Mobius band.

5.2 Constructing cell complexes

Definition 5.3. A (closed) *n-cell* a^n is a space homeomorphic to the n-dimensional ball $\mathbb{D}^n$. An *open n-cell* $\dot{a}^n$ is a space homeomorphic to an open n-ball, i.e., $\mathbb{D}^n \setminus \partial\mathbb{D}^n$. In particular, we adopt the convention that $a^0 = \dot{a}^0$ and that $\partial a^0 \simeq \mathbb{S}^{-1} = \emptyset$.

Definition 5.4. A *(finite) cell complex* K of dimension n is an attaching space constructed from a list of closed cells

$$\begin{array}{llll} a_1^0, & a_2^0, & \ldots, & a_{k_0}^0, \\ a_1^1, & a_2^1, & \ldots, & a_{k_1}^1, \\ & & \cdots & \\ a_1^n, & a_2^n, & \ldots, & a_{k_n}^n, (k_n > 0) \end{array} \tag{5.2}$$

and corresponding attaching maps $q_1, q_2, \ldots, q_n$ as follows.

(CWC-1) The *0-skeleton* is a disjoint sum of the 0-cells,

$$K^{(0)} := a_1^0 \sqcup a_2^0 \sqcup \ldots \sqcup a_{k_0}^0. \tag{5.3}$$

(CWC-2) Suppose $K^{(m-1)}$ has been constructed. Orchestrate the attaching map $q_m : \partial C^m \to K^{(m-1)}$,

$$C^m := a_1^m \sqcup a_2^m \sqcup \ldots \sqcup a_{k_m}^m;$$
$$\partial C^m := \partial a_1^m \sqcup \partial a_2^m \sqcup \ldots \sqcup \partial a_{k_m}^m,$$

to obtain the *m-skeleton* as

$$K^{(m)} := C^m \sqcup_{q_m} K^{(m-1)}, \tag{5.4}$$

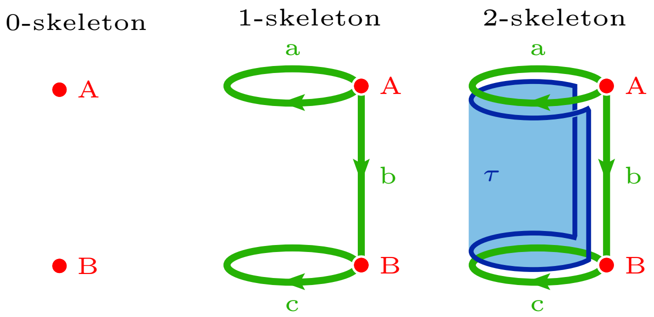
where $q_m(\partial C^m)$ must equal the union of a number of $(m-1)$-cells in $K^{(m-1)}$ and $q_m|_{\dot{a}_j^m}$ must be a homeomorphism for each $j = 1, \ldots, k_m$.

(CWC-3) The n-skeleton is taken to be the cell complex K, i.e., $K := K^{(n)}$.

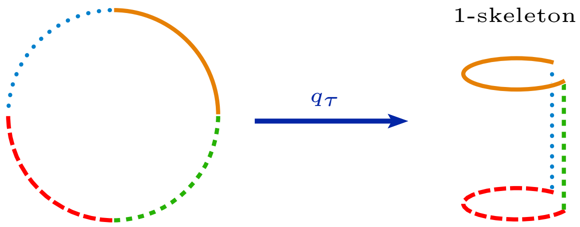
Lemma 5.5. A cell complex K is a union of open cells,

$$K = \cup_{j,m} \dot{a}_j^m. \tag{5.5}$$

Example 5.5. We construct the cylinder as a cell complex.



Suppose $a = b = c := [0, 1]$. Then for the attaching map q_1, we have $q_1|_a(0) = q_1|_a(1) = A$, $q_1|_c(0) = q_1|_c(1) = B$, and $q_1|_b(0) = A, q_1|_b(1) = B$. Similarly, set $C^2 = \tau := I^2$. The following plot illustrates $q_2 : \partial\tau \to K^{(1)}$.



Example 5.6. The n-dimensional sphere $\mathbb{S}^n$ is a cell complex consisting of one a^0 and one closed n-cell a^n with the attaching map $q_n : \partial\mathbb{S}^n \to a^0$ so that $\mathbb{S}^n = a^0 \sqcup_{q_n} a^n$.

Exercise 5.7. Represent the sphere as a cell complex with two 2-cells. List all cells and sketch the gluing maps.

Example 5.8. The real projective plane $\mathbb{P}^2$ consists of one 0-cell a^0, one closed 1-cell a^1, and one closed 2-cell a^2

$$\mathbb{P}^2 = \left(a^0 \sqcup_{q_1} a^1\right) \sqcup_{q_2} a^2,$$

where one choice of $q_2 : \partial a^2 \simeq \mathbb{S}^1 \to \left(a^0 \sqcup_{q_1} a^1\right) \simeq \mathbb{S}^1$ is $q_2(z) = z^2$.

Example 5.9. The torus consists of one 0-cell, two closed 1-cells, and one closed 2-cell,

$$\mathbb{T}^2 = \left(a^0 \sqcup_{q_1} (a_1^1 \sqcup a_2^1)\right) \sqcup_{q_2} a^2,$$

where q_1 and q_2 are clear from the plots in Theorem 4.111.

Exercise 5.10. Represent the double torus as a cell complex. What about the n-ple torus?

Exercise 5.11. Show that the product of cell complexes $X \times Y$ of cell complexes X and Y is again a cell complex.

5.3 The CW topology

Definition 5.6. The *weak topology* of a cell complex K is such that a set $A \subset K$ is open or closed iff $A \cap K^{(m)}$ is open or closed for each m.

Lemma 5.7. For any finite cell complex K, the weak topology of K is the same as the quotient topology induced from the attaching maps.

Proof. We only show that the weak topology is finer than the quotient topology. If A is open in $K = K^{(n)}$, the definition of the quotient topology on $K^{(n)}$ with the attaching map q_n implies that $q_n^{-1}(A)$ is open in $C^n \sqcup K^{(n-1)}$; see (5.4) with $n = m$. Then the subspace topology implies that $q_n^{-1}(A) \cap K^{(n-1)}$ is open in $K^{(n-1)}$. Furthermore, the conditions on q_n in (CWC-2) of Definition 5.4 imply

$$q_n^{-1}(A) \cap K^{(n-1)} = A \cap K^{(n-1)}.$$

An easy induction on the dimension of the cells completes the proof. □

Definition 5.8. The *characteristic map* of an m-cell a_j^m of a cell complex K is the composite of three maps,

$$\phi_j^m := \iota \circ \pi \circ i : a_j^m \to K, \tag{5.6}$$

where $i : a_j^m \to C^m$ is the natural inclusion, $\pi : C^m \to K^{(m)}$ the gluing map, and $\iota : K^{(m)} \to K$ the natural inclusion.

Exercise 5.12. What is the connection between the characteristic map and the attaching map?

Exercise 5.13. Show that a set $A \subset K$ is open (or closed) in K if and only if $(\phi_j^m)^{-1}(A)$ is open (or closed) in a_j^m for each characteristic map ϕ_j^m.

Theorem 5.9. A cell complex is compact.

Proof. The union of a finite number of compact spaces is compact. The rest follows from the theorem that the quotient space of a compact space is compact. □

Definition 5.10. A finite cell complex is also called a *CW complex*, where "C" stands for compactness in Theorem 5.9 and "W" the weak topology in Definition 5.6.

5.4 Compact surfaces from I^2

Lemma 5.11. The kth homology of a cell complex is fully determined by its $(k + 1)$-skeleton.

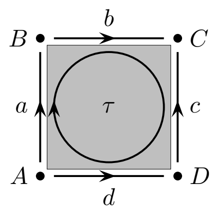
Proof. By definition, we have

$$H_k = Z_k/B_k = \ker \partial_k / \mathrm{Im}\, \partial_{k+1},$$

where the boundary operators ∂_k and ∂_{k+1} are completely determined by the attaching maps q_k and q_{k+1} in Definition 5.4, respectively. □

Example 5.14. The unit square I^2 can be regarded as a cell complex with the ingredients as

- 0-cells: A, B, C, D;
- 1-cells: a, b, c, d;
- 2-cells: τ;
- the attaching maps: $q_2(\partial\tau) = a + b - c - d$, $q_1(a) = B - A$, $q_1(b) = C - B$, $q_1(c) = C - D$, $q_1(c) = D - A$.



The attaching maps determine the boundary operators,

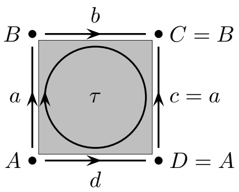
$$\begin{aligned}\partial\tau &= a + b - c - d,\\ \partial a &= B - A,\ \partial b = C - B,\\ \partial c &= C - D,\ \partial d = D - A,\end{aligned} \tag{5.7}$$

which then determine the groups Z_k, B_k, and H_k. In other words, when we identify edges of a square, we change the attaching maps q_2 and q_1, and consequently change the boundary operators.

Example 5.15 (Cylinder from the unit square)**.** Example 5.5 can interpreted as identifying in the cell complex of the unit square the cells

$$a \sim c; A \sim D, B \sim C.$$

The cell complex is then as follows.



Then the boundary operators (5.7) change to

$$\begin{aligned}&\partial\tau = a + b - a - d = b - d,\\ &\partial a = B - A,\ \partial b = C - B = 0,\ \partial d = D - A = 0.\end{aligned} \tag{5.8}$$

The chain complex is thus

$$C_2 = \langle\tau\rangle \xrightarrow{\partial_2=?} C_1 = \langle a, b, d\rangle \xrightarrow{\partial_1=?} C_0 = \langle A, B\rangle$$

The matrix of the boundary operator is

$$\partial_1 = \begin{bmatrix} -1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix};$$
$$\partial_2 = [0, 1, -1]^T,$$

and the homology groups in dimensions 0 and 1 are

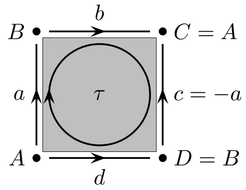
$$\begin{aligned}Z_0 &:= \ker\partial_0 = \langle A, B\rangle;\\ B_0 &:= \operatorname{Im}\partial_1 = \langle B - A\rangle\\ H_0 &:= Z_0/B_0 = \langle [A]\rangle \cong \mathbb{Z};\\ Z_1 &:= \ker\partial_1 = \langle b, d\rangle;\\ B_1 &:= \operatorname{Im}\partial_2 = \langle b - d\rangle;\\ H_1 &:= Z_1/B_1 = \langle b\rangle \cong \mathbb{Z}.\end{aligned}$$

Note that the generators of the group $\ker\partial_1$ can be easily deduced from the columns in the matrix of ∂_1.

Exercise 5.16 (Mobius band from the unit square)**.** The Mobius band can interpreted as identifying in the cell complex of the unit square the cells

$$a \sim -c; A \sim D, B \sim C.$$

The cell complex is then as follows.

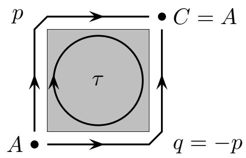


Compute all homology groups of the Mobius band and compare the results with those in Example 5.15. What can you say from your comparison?

Example 5.17 (Projective plane from the unit square)**.** The projective plane can interpreted as identifying in the cell complex of the unit square the cells

$$a \sim -c, b \sim -d; A \sim C, B \sim D.$$

Further simplifications yield a cell complex as follows.



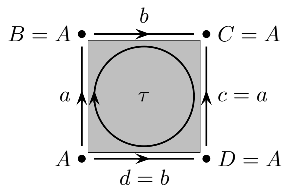
Follow the steps of the above examples and we have

$$\begin{aligned}Z_0 &:= \ker\partial_0 = \langle A\rangle;\\ B_0 &:= \operatorname{Im}\partial_1 = 0\\ H_0 &:= Z_0/B_0 = \langle [A]\rangle \cong \mathbb{Z};\\ Z_1 &:= \ker\partial_1 = \langle p\rangle;\\ B_1 &:= \operatorname{Im}\partial_2 = \langle 2p\rangle;\\ H_1 &:= Z_1/B_1 \cong \mathbb{Z}_2.\end{aligned}$$

Example 5.18 (Torus from the unit square)**.** The torus can interpreted as identifying in the cell complex of the unit square the cells

$$a \sim c, b \sim d;\ A \sim B \sim C \sim D.$$

Then we have a cell complex as follows.



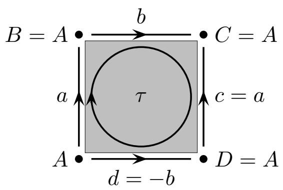
Follow the steps of the above examples and we have

$$\begin{aligned}Z_0 &:= \ker\partial_0 = \langle A\rangle;\\ B_0 &:= \operatorname{Im}\partial_1 = 0\\ H_0 &:= Z_0/B_0 = \langle [A]\rangle \cong \mathbb{Z};\\ Z_1 &:= \ker\partial_1 = \langle a, b\rangle;\\ B_1 &:= \operatorname{Im}\partial_2 = 0;\\ H_1 &:= Z_1/B_1 = \langle a, b\rangle \cong \mathbb{Z}\times\mathbb{Z};\\ Z_2 &:= \ker\partial_2 = \langle\tau\rangle;\\ B_2 &:= \operatorname{Im}\partial_3 = 0;\\ H_2 &:= Z_2/B_2 = \langle\tau\rangle \cong \mathbb{Z}.\end{aligned}$$

Exercise 5.19 (Klein bottle from the unit square)**.** The Klein bottle can interpreted as identifying in the cell complex of the unit square the cells

$$a \sim c, d \sim -b;\ A \sim B \sim C \sim D.$$

The cell complex is then as follows.



Compute all homology groups of the Klein bottle and compare the results with those in Example 5.18. What can you say from your comparison?

Exercise 5.20 (Sphere from the unit square)**.** The sphere can be interpreted as identifying cells in a cell complex of the unit square. Give three different calculations.

Exercise 5.21. Connect two squares by four clockwise twisted strips.

(a) Find its cell complex representation.

(b) To what is it homeomorphic?

(c) What if some strips were twisted the opposite way?

Answer these questions for two triangles connected by three twisted strips.

Exercise 5.22. Find a cell complex with the following homology groups:

$$H_0 = \mathbb{Z}, H_1 = \mathbb{Z}_2, H_2 = \mathbb{Z} \oplus \mathbb{Z}.$$

5.5 Quotients of chain complexes

Example 5.23 (Cylinder from the unit square: quotient version)**.** As an alternative to Example 5.15, we can compute the homology of the cylinder from a quotient of chain complexes that consist of quotient chain groups and quotient boundary operators. Given the cell complex K of the unit square and the identification

$$a \sim c; A \sim D, B \sim C,$$

we can construct quotient chain groups

$$\begin{aligned}
&C_2(K)/\sim\, = \langle [\tau] \rangle\,;\\
&C_1(K)/\sim\, = \langle a, b, c, d \rangle \,/\, \langle a - c \rangle = \langle [a], [b], [d] \rangle\,;\\
&C_0(K)/\sim\, = \langle A, B, C, D \rangle \,/\, \langle A - D, B - C \rangle = \langle [A], [B] \rangle
\end{aligned}$$

and define quotient boundary operators as

$$[\partial_n] : C_n(K)/\sim \,\to C_{n-1}(K)/\sim\,. \tag{5.9}$$

As discussed in Lemma 2.61, the quotient of an operator is well defined only if the operator maps equivalence classes to equivalence classes, i.e. the following diagram commutes

$$\begin{array}{ccc}
C_n(K) & \xrightarrow{\partial_n} & C_{n-1}(K)\\
\downarrow q & & \downarrow q\\
C_n(K)/\sim & \xrightarrow{[\partial_n]} & C_{n-1}(K)/\sim
\end{array}$$

where q is the attaching map. The quotient boundary operators are as follows

$$\begin{aligned}
&[\partial_2]([\tau]) = [\partial_2 \tau] = [a + b - c - d] = [b - d],\\
&[\partial_1]([a]) = [\partial_1(a)] = [B - A],\\
&[\partial_1]([b]) = [C - B] = 0,\\
&[\partial_1]([d]) = [D - A] = 0.
\end{aligned}$$

The nonzero homology groups are the same as those in Example 5.15:

$$\begin{aligned}
Z_0 &:= \ker[\partial_0] = \langle [A], [B] \rangle\,;\\
B_0 &:= \operatorname{Im}[\partial_1] = \langle [B - A] \rangle\,;\\
H_0 &:= Z_0/B_0 = \langle [A] \rangle \cong \mathbb{Z};\\
Z_1 &:= \ker[\partial_1] = \langle [b], [d] \rangle\,;\\
B_1 &:= \operatorname{Im}[\partial_2] = \langle [b - d] \rangle\,;\\
H_1 &:= Z_1/B_1 = \langle [b] \rangle \cong \mathbb{Z}.
\end{aligned}$$

Similar to the calculation in Example 5.15, the way we come up with Z_1 is to read the generators $[b]$, $[d]$ off the actions of $[\partial_1]$, i.e., $[\partial_1]([b]) = [\partial_1]([d]) = 0$.

Exercise 5.24. Perform a homology computation for the Mobius band from quotients of chain complexes.

Exercise 5.25. Perform a homology computation for the double banana from quotients of chain complexes.

Exercise 5.26. Perform a homology computation for the book with n pages from quotients of chain complexes.

5.6 Boundary-relative homology

Example 5.27 (The homology of the disk relative to its boundary)**.** The cell complex of the disk is

- 0-cell: A;
- 1-cell: a;
- 2-cell: τ;
- the attaching map: $q(\partial\tau) = A$.

We collapse the boundary of τ to one point and thus obtain the quotient chain groups as

$$\begin{aligned}
&C_2(K, \partial K) := C_2(K)/\sim = \langle \tau \rangle\,;\\
&C_1(K, \partial K) := C_1(K)/\sim = \langle a \rangle \,/\, \langle a \rangle = 0;\\
&C_0(K, \partial K) := C_0(K)/\sim = \langle A \rangle\,.
\end{aligned}$$

The relative chain complex is then

$$0 \xrightarrow{[\partial_3]=0} \langle \tau \rangle \xrightarrow{[\partial_2]=0} 0 \xrightarrow{[\partial_1]=0} \langle A \rangle \xrightarrow{[\partial_0]=0} 0$$

and the homology groups relative to boundary is thus

$$
\begin{aligned}
Z_0 &:= \ker[\partial_0] = \langle [A] \rangle ;\\
B_0 &:= \mathrm{Im}\,[\partial_1] = 0;\\
H_0 &:= Z_0/B_0 = \langle [A] \rangle \cong \mathbb{Z};\\
Z_1 &:= \ker[\partial_1] = 0;\\
B_1 &:= \mathrm{Im}\,[\partial_2] = 0;\\
H_1 &:= Z_1/B_1 = 0;\\
Z_2 &:= \ker[\partial_2] = \langle [\tau] \rangle ;\\
B_2 &:= \mathrm{Im}\,[\partial_3] = 0;\\
H_2 &:= Z_2/B_2 = \langle [\tau] \rangle \cong \mathbb{Z}.
\end{aligned}
$$

These homology groups are the same as those of the sphere.

Exercise 5.28. Compute the homology of the cylinder relative to its boundary. You should obtain homology groups of the sphere as in Example 5.27.

Exercise 5.29. Compute the homology of the Mobius band relative to its boundary. You should obtain homology groups of the projective plane as in Example 5.17.