Chapter 2

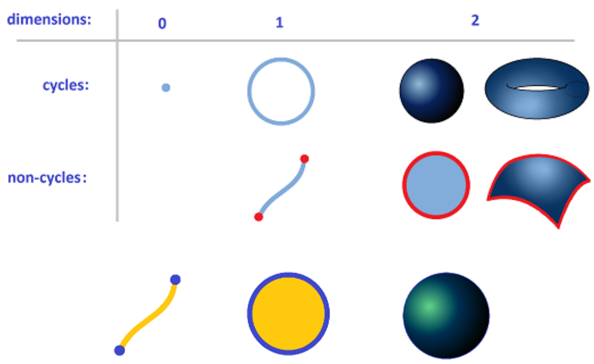
Cycles

2.1 Introduction

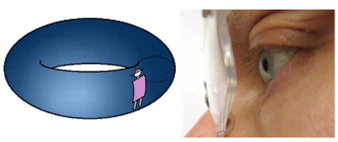
Math is the science of patterns. In particular, topology is the science of features that occurs repeatedly in everyday or abstract objects. More precisely, *we study in topology the spatial properties that are preserved under continuous transformations.*

2.1.1 Topology around us

2.1.2 Topological features

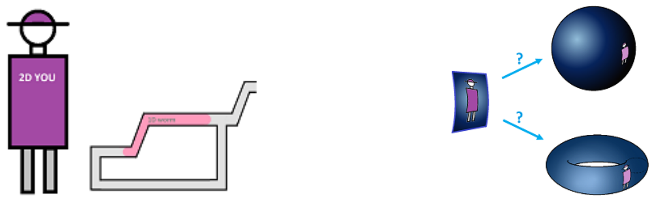


2.1.3 Guiding principles of our study



Definition 2.1. An *intrinsic property* of an object is one that you can detect from within the object.

Example 2.1. If a worm crawls in a tube system and return to the same spot, it should realize that the system contains a 1-cycle.



Rule 2.2 (The guiding principles of topology)**.** Our study should

(TPP-1) be verifiable from *inside the object* X so that the features are independent of how the space X fits into a larger object.

(TPP-2) include *no measuring* so that the features are preserved under homeomorphisms.

(TPP-3) be quantitative so that what's left is *counting.*

Example 2.2. Hundreds of years ago, sailors noticed that the surface of the earth is not flat, but is curved. This can be found by traveling around or doing some experiments on curvature. Similarly, the universe we live in is not the flat world of Euclid and Newton, but the curved world of Riemann and Einstein.

2.2 Homology classes

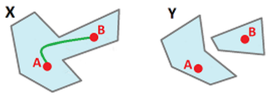
2.2.1 Define and count 0-cycles

Definition 2.3. A *path* in a topological space X is a continuous map $\gamma : [0,1] \to X$.

Definition 2.4. Two points $A, B \in X$ are *path-connected* in X if there exists a path γ such that $\gamma(0) = A$ and $\gamma(1) = B$.

Theorem 2.5. The path-connectedness in Definition 2.4 is an equivalence relation.

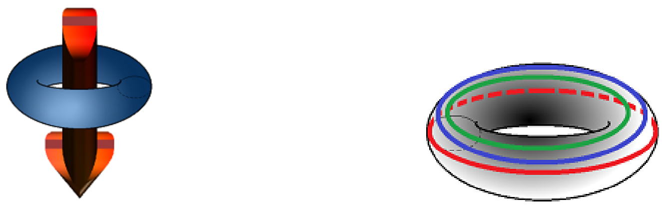
Proof. Exercise. □



Definition 2.6. The *connected components* of X, or the *0-dimensional homology classes* on X, are the equivalence classes induced by Definition 2.4.

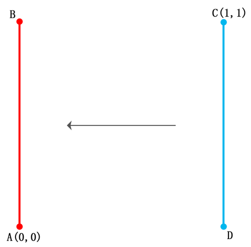
2.2.2 Define and count 1-cycles

Definition 2.7. A *loop* or *1-cycle* is a path whose end-points coincide.

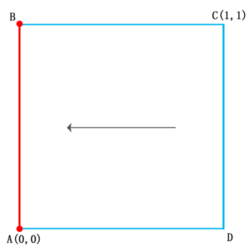


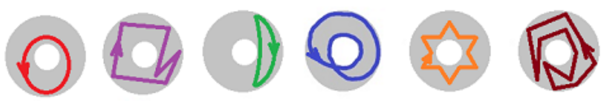
Definition 2.8. A *free homotopy* between two paths γ_1 and γ_2 in X is a continuous function $H_f : [0,1]^2 \to X$ such that $H_f(0,s) = \gamma_1(s)$ and $H_f(1,s) = \gamma_2(s)$ for all s. Then γ_1 and γ_2 are said to be *freely homotopic* in X.

Example 2.3. The function for the free homotopy from $\overline{CD}$ to $\overline{BA}$ as follows is simply $H_f(t,s) = (1-t,s)$.



Exercise 2.4. Give a function for the following homotopy from $\overline{ADCB}$ to $\overline{AB}$ as follows

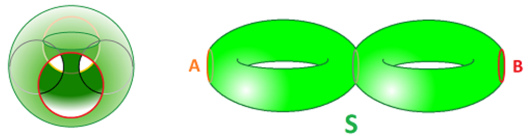




Definition 2.9. A topological space X with all loops homotopic to each other is *simply connected.*

Example 2.5. A sphere is simply connected.

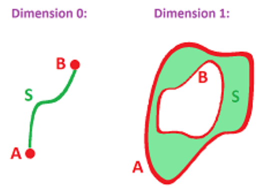
Example 2.6. The following two spaces are not simply connected.



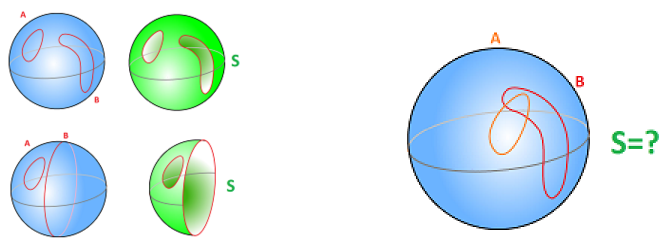
Exercise 2.7. For the right figure in Example 2.6, how do you detect its 1-dimensional topological features, or, how do you count the number of its 1-cycles?

2.2.3 Homology as equivalence relations

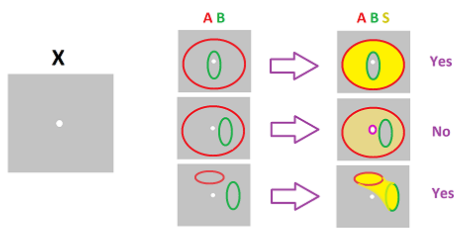
Definition 2.10. Two closed curves A and B in X are *homologous* if there exists a surface in X such that A and B form its boundary.



Example 2.8. Definition 2.10 can also be exemplified on the sphere as follow. For the intersecting loops, we can still find a surface, albeit a self-intersecting one. Note that the signature of a surface patch $\sigma : [0,1]^2 \to \mathbb{R}^3$ is the same as the signature of a free homotopy in Definition 2.8!



Example 2.9. Are the following two curves A and B homologous in the punctured plane?

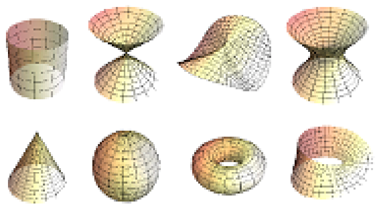


In the last case, the surface that connects the two loops is flattened by being "crushed" into the plane.

Exercise 2.10. Discuss the homology of surfaces by sketching two pairs of examples for each of the surface below

(a) two homologous closed curves;

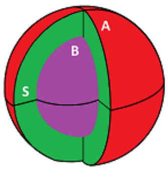
(b) two closed curves that are not homologous, if possible.



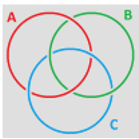
Definition 2.11. Two manifolds of the same dimension are *cobordant* if their disjoint union is the boundary of a compact manifold one dimension higher. A *cobordism* between manifolds M and N is a compact manifold W whose boundary is the disjoint union of M and N:

$$\partial W = M \sqcup N. \tag{2.1}$$

Example 2.11. For example, two closed surfaces A and B in X are homologous if there exists a solid S in X such that A and B form its boundary.



Exercise 2.12. Is the homology class of A in $X = \mathbb{R}^3 \setminus (B \cup C)$ trivial or non-trivial? Justify your answer for the two cases of defining homology via homotopy and cobordism.



2.3 Topology of graphs

2.3.1 Graph realization

Definition 2.12. A *graph* is an ordered pair $G = (N, E)$ where N is the set of *nodes* or *vertices* and E the set of *edges*, each edge being an *unordered* pair of distinct nodes.

Definition 2.13. A *realization of a graph* G, denoted by $|G|$, is a subset of the Euclidean space that is the union of the following two subsets of the space:

- a collection of points $|N|$, one for each node in N, and
- a collection of pairwise disjoint simple open curves $|E|$ with their endpoints in $|N|$, one for each edge in E.

Theorem 2.14. Every finite graph can be realized in $\mathbb{R}^3$.

Proof. We show that every complete graph can be realized in $\mathbb{R}^3$ by two steps. First we can arrange the nodes in $\mathbb{R}^3$ such that no four points are on the same plane by the following constructive steps.

- Choose arbitrary positions for the first three points in $\mathbb{R}^3$ and delete from $\mathbb{R}^3$ the plane P.
- Choose an arbitrary position for the next point in $\mathbb{R}^3 \setminus P$ and delete from $\mathbb{R}^3 \setminus P$ all planes that can be formed from current points set.
- Repeat the previous step until all nodes in the graph are set in $\mathbb{R}^3$.

Because the number of nodes in a finite graph is finite, the number of planes removed from $\mathbb{R}^3$ is also finite. Hence the above construction finishes with the invariant that no four points are on the same plane.

In the second step, we connect each pair of these distinct points by line segments and we claim that no two distinct line segments intersect because otherwise this is a contradiction to the above invariant that no four points are on the same plane. □

Definition 2.15. A *path* or an *edge-path* is a graph P of the form

$$\begin{aligned} V(P) &= \{v_0, v_1, \ldots, v_\ell\}, \\ E(P) &= \{v_0v_1, v_1v_2, \ldots, v_{\ell-1}v_\ell\}. \end{aligned} \tag{2.2}$$

In particular, a *cycle* is a graph of the form $P + v_0v_\ell$ where P is a path and $\ell \geq 2$.

Definition 2.16. A graph is *connected* or *edge-connected* if for any pair of nodes there is a path between them.

Theorem 2.17. A graph is edge-connected if and only if its realization $|G|$ is path-connected.

Definition 2.18. Two nodes of a graph are *homologous* iff there exists an edge-path between them.

Theorem 2.19. The homology in Definition 2.18 is an equivalence relation on the set of nodes of the graph.

Definition 2.20. The equivalence classes derived from Definition 2.18 are called *edge-components* or *components* of G.

Theorem 2.21. For any graph G, the number of its edge components equals the number of path-components of $|G|$.

2.3.2 The Euler characteristic

Definition 2.22. The *Euler characteristic* of a graph G is the number of its nodes minus that of its edges,

$$\chi(G) = \#N_G - \#E_G. \tag{2.3}$$

Lemma 2.23. If a simple curve is a realization of a graph, this graph is a sequence of $n \geq 1$ consecutive edges.

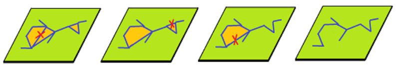
Definition 2.24. A *tree* is an edge-connected graph with no cycles. The *root node of a tree* can be any node of the graph. A *leaf node* is a node other than the root node such that it is incident to only one edge of the graph.

Theorem 2.25. Any tree T has $\chi(T) = 1$.

Proof. The removal of a leaf node and its incident edge does not change the Euler characteristic of the tree. It is an easy induction to show that keep doing this will reduce any tree to the trivial graph that only contains the root node. □

Corollary 2.26. If a graph G consists of n disjoint trees, then $\chi(G) = n$.

Corollary 2.27. An edge-connected G has $\chi(T) \leq 1$.



Lemma 2.28. Every connected graph has a tree subgraph that contains all of its vertices.

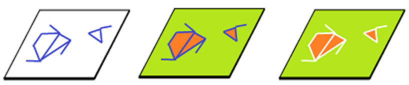
Proof. If the graph G his no cycles, the conclusion holds by Definition 2.24. Otherwise we pick a cycle and remove an arbitrary edge from E_G. The number of cycles is clearly reduced, but according to Definition 2.15 G is still edge-connected. Keep doing this until G has no cycles. □

Definition 2.29. A *planar graph* is a graph that can be realized or embedded in the Euclidean plane.

Theorem 2.30 (Jordan curve)**.** The complement of a simple closed curve in the plane has two path-components, one bounded and one unbounded.

Lemma 2.31. If a simple closed curve is a realization of a graph, then this graph is a cycle of $n \geq 3$ consecutive edges.

Definition 2.32. A *hole* of a planar graph G is a cycle in G that corresponds to the boundary of a bounded path-connected component of $\mathbb{R}^2 \setminus |G|$, where $|G|$ is a realization or embedding of G in $\mathbb{R}^2$.

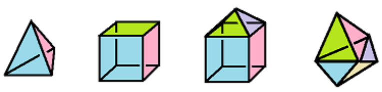


Lemma 2.33. Any edge-connected planar graph with n holes satisfies

$$\chi(G) = 1 - n. \tag{2.4}$$

Proof. Similar to that of Lemma 2.28. □

Definition 2.34. A *convex polyhedron* is the convex set of a finite number of points in the Euclidean space.



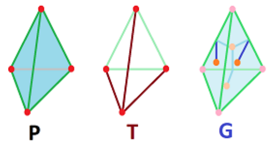
Definition 2.35. A *polyhedron* is a solid in the Euclidean space with flat polygonal faces, straight edges, and sharp corners (vertices) such that

- it is a path-connected subset of $\mathbb{R}^3$;
- its surface is cut into two disjoint pieces by any loop on it.

Definition 2.36. The *Euler characteristic of a polyhedron P* is defined as

$$\chi(P) := \# \text{ of vertices} - \# \text{ of edges} + \# \text{ of faces}. \tag{2.5}$$

2.3.3 The Euler-Poincaré formula



Theorem 2.37 (Euler-Poincaré formula)**.** Any polyhedron P has $\chi(P) = 2$.

Proof. By Lemma 2.28, there exists a tree T that contains all the vertices. We also construct another *dual graph* G of T such that

(i) N_G is the set of middle points of all faces of P,

(ii) each edge of P not in T is an edge in E_G connecting the middle point of the two adjacent faces.

It can be shown that G is also a tree. Then we have

$$\begin{aligned}\chi(P) &= \#N_T - (\#E_T + \#E_G) + \#N_G \\ &= \#N_T - \#E_T + \#N_G - \#E_G \\ &= 1 + 1 = 2,\end{aligned}$$

where the first line follows from Definition 2.36 and the construction of T and G. The third line follows from Theorem 2.25 and the fact that both T and G are trees. □

Exercise 2.13. Fill the gap in the proof of Theorem 2.37 by showing that the dual graph G is a tree.

Exercise 2.14. Is the converse of Theorem 2.25 true?

Definition 2.38. A polyhedron is a *regular polyhedron* iff its faces have the same number of edges and its vertices have the same number of incident edges.

Definition 2.39. The *Platonic solids* are the tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron.

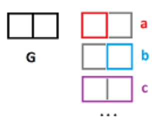
Exercise 2.15. Show that Platonic solids are the only regular polyhedra.

2.4 Homology groups of graphs

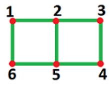
2.4.1 Chains of nodes/edges

Definition 2.40. A *chain of nodes or edges* is a formal sum of nodes or edges satisfying the *cancellation rule*, i.e. $x + x = 0$. In particular, a chain of edges can neither include doubles such as AA nor include edges with opposite directions (we assume $AB = BA$).

Example 2.16. The following graph has three cycles.



They are linearly dependent with respect to the formal sum in Definition 2.40.



$$
\begin{aligned}
a &= 12 + 25 + 56 + 61 \\
b &= 23 + 34 + 45 + 52 \\
a + b &= 12 + 25 + 56 + 61 + 23 + 34 + 45 + 52 \\
&= 12 + 23 + 34 + 45 + 56 + 61 = c.
\end{aligned}
$$

Definition 2.41. The *set of chains of nodes*, or *0-chains*, is

$$C_0(G) := \left\{ \sum_{A \in Q} A : Q \subset N_G \right\} \cup \{0\}. \tag{2.6}$$

The *set of chains of edges*, or *1-chains*, is

$$C_1(G) := \left\{ \sum_{AB \in P} AB : P \subset E_G \right\} \cup \{0\}. \tag{2.7}$$

Lemma 2.42. Let $+$ denote the formal sum with the cancellation rule. Then both $(C_0(G), +)$ and $(C_1(G), +)$ are Abelian groups generated by N_G and E_G, respectively.

$$C_0(G) = \langle N_G \rangle, \tag{2.8}$$
$$C_1(G) = \langle E_G \rangle. \tag{2.9}$$

Proof. Exercise. □

Lemma 2.43. Both $C_0(G)$ and $C_1(G)$ are vector spaces over the field $\mathbb{Z}_2$.

Proof. Let $(G, +)$ be a group with binary arithmetic, i.e. $\forall x \in G$, $x + x = 0$. Then G can be considered as a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$ with $+$ as the addition. The integer multiplication is naturally derived from the binary arithmetic, or counting. □

2.4.2 The boundary operator

Definition 2.44. The *boundary operator of a graph* G, $\partial_G : C_1(G) \to C_0(G)$, is defined inductively as

$$\forall AB \in E_G,\ \partial(AB) = A + B, \tag{2.10a}$$
$$\forall x, y \in E_G,\ \partial(x + y) = \partial x + \partial y. \tag{2.10b}$$

Lemma 2.45. The boundary operator ∂_G is a homomorphism.

Proof. Exercise. □

Lemma 2.46. The boundary operator ∂_G is a linear map between the vector spaces $C_1(G)$ and $C_0(G)$.

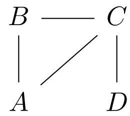
Proof. Exercise. □

Definition 2.47. A 1-chain is a *1-cycle* if its boundary is zero. The *group of 1-cycles of a graph* G is

$$Z_1(G) := \ker \partial_G = \{x \in C_1(G) : \partial_G x = 0\}, \tag{2.11}$$

wher Z stands for "Zyklus", "cycle" in German.

Example 2.17. Consider the following graph.



From the above definitions, we have

$$
\begin{aligned}
C_0 &= \langle N_G \rangle = \langle \{A, B, C, D\} \rangle; \\
C_1 &= \langle E_G \rangle = \langle \{AB, BC, CA, CD\} \rangle;
\end{aligned}
$$

Represent A, B, C, D as the column vectors of the 4-by-4 identity matrix. Do the same for the four edges AB, BC, CA, and CD. Then $\partial(AB) = A + B$ translates to

$$\partial\left([1, 0, 0, 0]^T\right) = [1, 1, 0, 0]^T,$$

which implies that the first column of the matrix M_∂ is the RHS vector. Therefore we have

$$M_\partial = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the null space of M_∂, we have

$$Z_1(G) = \ker \partial_G = \{0, AB + BC + CA\}$$

and $\mathrm{rank} Z_1 = 1$, implying that there is only one hole in G.

Exercise 2.18. Repeat the algebra in Example 2.17 to the figure-8 graph in Example 2.16. This time you should get $\mathrm{rank} Z_1 = 2$, suggesting that there are only two independent cycles.

Lemma 2.48. The number of holes in a planar graph equals the number of its independent cycles, which is the rank of the kernel of the boundary operator.

Definition 2.49. A 0-chain is called a *0-boundary* if it is the boundary of some 1-chain. The *group of 0-boundaries* of a graph G is the image (or range) of its boundary operator, $B_0(G) = \mathrm{Im}\, \partial_G$,

$$\mathrm{Im}\, \partial_G := \{P \in C_0(G) : \exists w \in C_1(G) \text{ s.t. } \partial w = P\}. \tag{2.12}$$

Lemma 2.50. The group of 0-boundaries is a subgroup of C_0.

Example 2.19. For the graph in Example 2.17, we have

$$
\begin{aligned}
B_0 &= \langle \{A + B, B + C, C + A, C + D\} \rangle \\
&= \langle \{A + B, B + C, C + D\} \rangle
\end{aligned}
$$

and hence $\mathrm{rank} B_0 = 3$. By Definition 2.22, Theorem 2.25, and Lemma 2.28, $\mathrm{rank} C_0 - \mathrm{rank} B_0 = 1$ implies that there is only one component in G: the generators of C_0 are precisely the vertices of the graph and those of B_0 are precisely the edges of the graph. See Theorem 2.54.

2.4.3 Homology as quotient groups

Definition 2.51. Two chains of nodes $P, Q \in C_0(G)$ are *homologous*, denoted as $P \sim Q$, if there is a chain of edges $w \in C_1(G)$ such that $\partial w = P + Q$.

Exercise 2.20. Show that $P \sim Q$ implies $P + Q \sim 0$.

Definition 2.52. The 0*th homology group* of a graph G is

$$H_0(G) := C_0(G)/B_0(G) \tag{2.13}$$

and the *1st homology group* of G is

$$H_1(G) := Z_1(G) = \ker \partial_G. \tag{2.14}$$

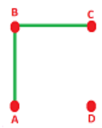
Lemma 2.53. A finitely-generated Abelian group L and its subgroup M satisfies

$$\text{rank}L/M = \text{rank}L - \text{rank}M. \tag{2.15}$$

Theorem 2.54. The number of components of a graph G, i.e. the rank of $H_0(G)$, equals rank$C_0(G)-$ rankIm ∂_G.

Proof. This follows directly from Lemma 2.53. $\square$

Example 2.21. We summarize the contents in this section by considering the following graph.



Following Example 2.17, we have

$$\begin{aligned} C_0 &= \langle \{A, B, C, D\} \rangle, \\ C_1 &= \langle \{AB, BC\} \rangle, \\ M_\partial &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

M_∂ is of full rank and the group of cycles is then

$$Z_1 = \ker \partial = \{0\}.$$

Hence there is no cycles in G. The group of boundaries is

$$B_0 = \text{Im}\,\partial = \langle \{A + B, B + C\} \rangle = \{0, A+B, B+C, C+A\}$$

and rank$B_0 = 2$. Therefore the number of components is rankC_0-rank$B_0 = 2$.

We can verify that the cosets of B_0 partitions C_0:

$$\begin{aligned} [A] := A + B_0 = \{A, B, A + B + C, C\} &= [B] = [C], \\ [D] &= \{D, A + B + D, B + C + D, C + A + D\}, \\ [A + D] &= \{A + D, B + D, A + B + C + D, C + D\}. \end{aligned}$$

Exercise 2.22. Repeat the algebra in Example 2.21 for the graph in Example 2.16.

Exercise 2.23. Compute the homology groups of the graph of n edges arranged in (a) a string, (b) a circle, and (c) a star.

Exercise 2.24. Compute the homology groups of the graph of edges arranged in an $n \times m$ grid.

2.5 Maps of graphs

2.5.1 Graph maps

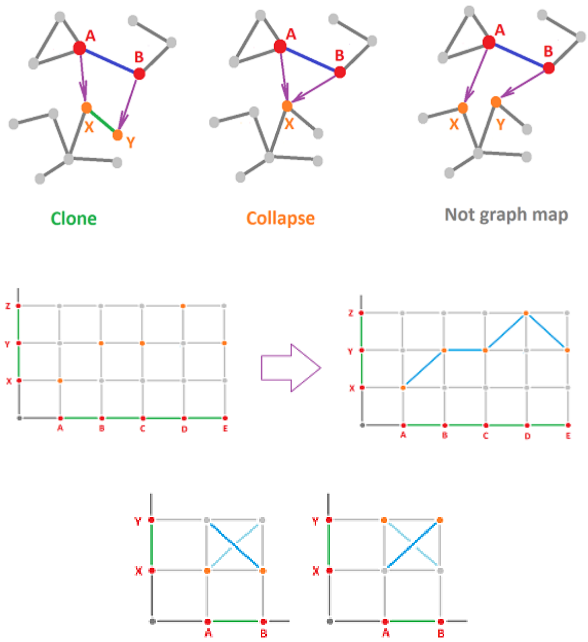
Definition 2.55. A *graph map* is a function of graphs $f : G \to J$

$$f = \{f_N : N_G \to N_J,\ f_E : E_G \to E_J \cup N_J\} \tag{2.16}$$

that satisfies the *discrete continuity condition*

$$f_E(AB) = \begin{cases} f_N(A)f_N(B) & \text{if } f_N(A) \neq f_N(B); \\ f_N(A) & \text{if } f_N(A) = f_N(B). \end{cases} \tag{2.17}$$

Example 2.25. Two graph functions are as follows.



2.5.2 Chain maps

Definition 2.56. The *chain map* generated by a graph map $f : G \to J$ is a pair of homomorphisms $f_\Delta := \{f_0, f_1\}$,

$$f_0 : C_0(G) \to C_0(J); \qquad f_1 : C_1(G) \to C_1(J), \tag{2.18}$$

generated by f_N and f_E, respectively.

Theorem 2.57 (Algebraic continuity condition)**.** The chain map of any graph map $f : G \to J$ satisfies

$$\forall e \in E_G, \qquad \partial_J f_1(e) = f_0(\partial_G e), \tag{2.19}$$

which is often written in the compact form

$$\partial f = f \partial, \tag{2.20}$$

and is often represented by the commutative diagram

$$\begin{tikzcd} C_1(G) \arrow[r, "f_1"] \arrow[d, "\partial_G"] \arrow[dr] & C_1(J) \arrow[d, "\partial_J"] \\ C_0(G) \arrow[r, "f_0"] & C_0(J) \end{tikzcd}$$

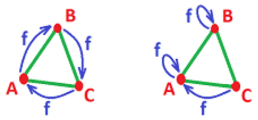
Proof. For an edge $e := AB \in E_G$ that satisfies $f_N(A) = f_N(B)$, (2.20) holds from our design of $f_1(e) = 0$. Otherwise we have

$$\begin{aligned}\partial_J f_1(AB) &= \partial_J(f_N(A)f_N(B)) = f_N(A) + f_N(B)\\ &= f_0(A) + f_0(B) = f_0(A+B) = f_0(\partial_G e),\end{aligned}$$

where the second and the last steps follows from Definition 2.44, and all other steps follow from Definition 2.56. □

Example 2.26. For a graph map $f : G \to J$ that collapses all edges, its chain map f_0 assigns a constant for all nodes in a connected component of G, while f_1 is the *trivial homomorphism*, i.e. the constant zero.

Exercise 2.27. Find the matrices of the chain maps as follows.



Corollary 2.58. For a chain map $f_\Delta = \{f_0, f_1\}$, f_0 takes boundaries to boundaries and f_1 takes cycles to cycles.

Proof. By Theorem 2.57, we have

$$\begin{aligned}x = \partial_G(y) &\Rightarrow f_0(x) = f_0(\partial_G(y)) = \partial_J(f_1(y));\\ \partial_G(y) = 0 &\Rightarrow \partial_J(f_1(y)) = f_0(\partial_G(y)) = f_0(0) = 0.\end{aligned}$$ □

2.5.3 Homology maps

Definition 2.59. The *map of cycles* $f_{1z} : Z_1(G) \to Z_1(J)$ is obtained by restricting the domain and range of f_1 to $Z_1(G)$ and $Z_1(J)$, respectively.

Corollary 2.60. The map of cycles is well-defined.

Proof. This follows from Corollary 2.58. □

Exercise 2.28. What are the maps of cycles for the graph maps $G \to G$ in Exercise 2.27?

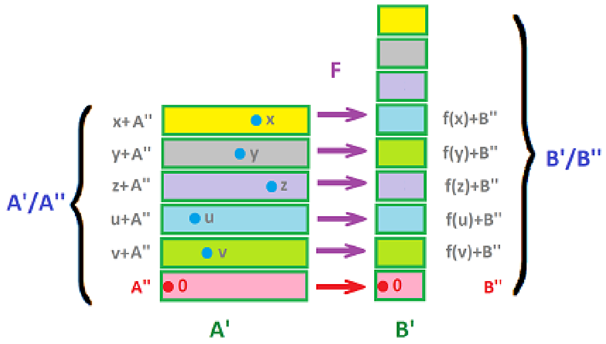
Lemma 2.61. Consider two Abelian groups A, B and a homomorphism between them $F : A \to B$. Suppose two pairs of subgroups $A'' \subset A' \subset A$ and $B'' \subset B' \subset B$ satisfy

$$F(A') \subset B', \qquad F(A'') \subset B''. \tag{2.21}$$

Then the following *quotient map* $[F] : A'/A'' \to B'/B''$ is well defined with $[F](0) = 0$,

$$\forall x \in A', \qquad [F]([x]) := [F(x)]. \tag{2.22}$$

Proof. The subgroup conditions justify the quotient groups $A'/A'' \to B'/B''$. The first condition in (2.21) guarantees that $F(x) \in B'$ so that $[F(x)]$ is indeed an element of B'/B''. Similarly, $[x]$ is an element of A'/A'' and the cosets of $[x]$ partitions A', i.e. for each $[x] \in A'/A''$, $[F]$ associates to it the unique element $[F(x)]$ in B'/B''. Finally, by the second condition in (2.21), the zero element of A'/A'' is mapped to the zero element of B'/B''. □



Example 2.29. Let $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. Find all homomorphisms $f : K \to K$ such that their quotient maps

$$[f] : K/\langle(1,1)\rangle \to K/\langle(1,0)\rangle$$

are well defined.

Exercise 2.30. Consider linear operators $f : \mathbb{R} \to \mathbb{R}^3$ and the quotient $\mathbb{R}^3/\mathbb{R}^2$. Which reflections and rotations have well-defined quotient maps with respect to this quotient group?

Corollary 2.62. Let $[z] := z + B_0(G)$ denote the homology class of a 0-cycle z. Then the following *map of components* $[f_0] : H_0(G) \to H_0(J)$ is well defined,

$$[f_0]([z]) = [f_0(z)]. \tag{2.23}$$

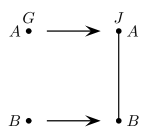
Proof. By Corollary 2.58, $f_0(B_0(G)) \subset B_0(J)$. By (2.6) and Definition 2.56, we also have $f_0(C_0(G)) \subset C_0(J)$. The rest of the proof follows from (2.12), Lemma 2.61, and Definition 2.52. □

Example 2.31. Consider a two-node graph G and a single-edge graph J

$$N_G = N_J = \{A, B\}, \qquad E_G = \emptyset,\ E_J = \{AB\}.$$

Merging of components can be illustrated by the graph map:

$$f_N(A) = A,\ f_N(B) = B.$$



It follows that

$$\begin{aligned}&C_0(G) = H_0(G) = \langle [A], [B]\rangle;\\ &C_0(J) = C_0(G),\ H_0(J) = \langle [A] = [B]\rangle;\\ &[f_0]([A]) = [f_0]([B]) = [A].\end{aligned}$$

Definition 2.63. The *homology map of the graph map* f is the pair $f_* = \{[f_0], f_{1z}\}$.

Theorem 2.64. The homology maps preserves identity, composition, and inverses, i.e.

$$\begin{aligned}
[(\mathrm{Id}_G)_0] &= \mathrm{Id}_{H_0(G)}, & (\mathrm{Id}_G)_{1z} &= \mathrm{Id}_{H_1(G)};\\
[(fg)_0] &= [f_0][g_0], & (fg)_{1z} &= f_{1z}g_{1z};\\
[(f^{-1})_0] &= [f_0]^{-1}, & (f^{-1})_{1z} &= f_{1z}^{-1};
\end{aligned}$$

Proof. Exercise. □

2.6 Binary calculus on graphs

2.6.1 The dual of a group/space

Definition 2.65. The *dual of a group* $(L, \cdot)$ over $\mathbb{Z}_2$, denoted L^*, is the set of all homomorphisms on L,

$$L^* = \{s : L \to \mathbb{Z}_2 \mid s(xy) = s(x) + s(y)\}. \tag{2.24}$$

Lemma 2.66. The dual of $\mathbb{Z}_2$ is a group isomorphic to $\mathbb{Z}_2$.

Proof. Any homomorphism h must satisfy $h(0) = 0$, hence according to $h(1)$ we have only two homomorphisms,

$$1^* = \{0 \mapsto 0, 1 \mapsto 1\} \text{ or } 0^* = \{0 \mapsto 0, 1 \mapsto 0\},$$

i.e. h is either the identity 1^* or the constant zero 0^*. Consider the map $f : \mathbb{Z}_2 \to \mathbb{Z}_2^*$ defined by $f(x) = x^*$. We have

$$\begin{aligned}
f(0+0) &= f(0) = 0^* = 0^* + 0^*;\\
f(0+1) &= f(1) = 1^* = 0^* + 1^*;\\
f(1+1) &= f(0) = 0^* = 1^* + 1^*,
\end{aligned}$$

where the last equality in each equation can be verified by actions of the homomorphism on all elements in its domain. For example, $0^* = 1^* + 1^*$ is verified by

$$\begin{aligned}
0^*(0) &= 0 = 0 + 0 = 1^*(0) + 1^*(0);\\
0^*(1) &= 0 = 1 + 1 = 1^*(1) + 1^*(1).
\end{aligned}$$

The proof is completed by f being bijective. □

Lemma 2.67. The dual group L^* is also a ring.

Proof. The additional binary operation is function composition, which is associative and distributes over addition. □

Theorem 2.68. The group $L := (\mathbb{Z}_2)^n$ satisfies

$$L^* \cong L. \tag{2.25}$$

Proof. Each element $x \in L$ can be written as

$$x = (x_1, \ldots, x_n), \quad x_i \in \mathbb{Z}_2.$$

Then the corresponding element of L^* is

$$x^* = (x_1^*, \ldots, x_n^*), \quad x_i^* \in \mathbb{Z}_2^*.$$

The rest of the proof follows from Lemma 2.66. □

Exercise 2.32. Define and find the dual over $\mathbb{Z}_2$ of the group $\mathbb{Z}_n$ where $n = 3$. What about the case $n > 3$?

2.6.2 Cochains of nodes/edges

Definition 2.69. A *k-cochain* on graph G is a homomorphism $s : C_k(G) \to \mathbb{Z}_2$.

Lemma 2.70. The 0-cochains and 1-cochains on a graph G, respectively denoted by $C^0(G)$ and $C^1(G)$, are dual groups of $C_0(G)$ and $C_1(G)$, respectively.

Proof. The sum of two homomorphisms is still a homomorphism, the rest follows from Definition 2.65. □

Definition 2.71. A *basic chain* of a graph G is a node or an edge of G. The *basic cochain of a basic chain* x, denoted by $x^* : C_k \to \mathbb{Z}_2$, is defined as

$$x^*(t) = \begin{cases} 1 & \text{if } t = x,\\ 0 & \text{if } t \neq x. \end{cases} \tag{2.26}$$

Lemma 2.72. The groups of 0- and 1-cochains of a graph G can be expressed as

$$C^0(G) = \left\{ \sum_{A\in Q} A^* : Q \subset N_G \right\} \cup \{0^*\}; \tag{2.27}$$

$$C^1(G) = \left\{ \sum_{AB\in P} AB^* : P \subset E_G \right\} \cup \{0^*\}, \tag{2.28}$$

where A^* and AB^* are the basic cochains in Definition 2.71.

Proof. By Definition 2.69, $s(x + y) = s(x) + s(y)$ holds for a k-cochain s. Hence the value of s on any k-chain is completely determined by values of s on individual nodes/edges. In other words, a 0-cochain/1-cochain assigns a number, 0 or 1, to each node/edge in the graph, and this completely characterizes all 0-cochains and 1-cochains.

Consider a 0-cochain that assigns 1 to each node in an arbitrary nonempty subset $Q \subset N_G$, and assigns 0 to each node in $N_G \setminus Q$. It is easily verified that $\sum_{A\in Q} A^*$ is a representation of this 0-cochain.

$$\forall B \in Q, \quad \left(\sum_{A\in Q} A^*\right)(B) = B^*(B) = 1;$$

$$\forall C \notin Q, \quad \left(\sum_{A\in Q} A^*\right)(C) = 0.$$

In particular, the 0-cochain that assigns 0 to all nodes is represented by 0^*. Similar arguments on 1-cochains complete the proof. □

Lemma 2.73. C^k is the dual space of C_k.

Proof. Exercise. □

Lemma 2.74. For any graph $G = (N, E)$, we have

$$C_0 \cong C^0 \cong (\mathbb{Z}_2)^n, \qquad C_1 \cong C^1 \cong (\mathbb{Z}_2)^m, \tag{2.29}$$

where $n = \#N$ and $m = \#E$.

Proof. A vector space and its dual space have the same dimension. The first isomorphism then follows from Lemma 2.73. In addition, it is easy to verify that the following map $f : C_0 \to C^0$ is an isomorphism,

$$\forall A_i \in N_G, \forall x = \sum_i A_i \in C_0, \qquad f(x) = \sum_i A_i^*. \quad (2.30)$$

The second isomorphism follows from the proof of Lemma 2.72 where each 0-cochain is uniquely identified with one possibility of assigning 1 or 0 to each node of the graph. The third and fourth isomorphisms are proved similarly. □

Lemma 2.75. The value of a cochain s at a given chain a is the dot-product:

$$s(a) = \langle s, a \rangle . \quad (2.31)$$

Proof. Use Lemma 2.74. □

2.6.3 The dual homomorphism/map

Definition 2.76. The *dual homomorphism* of a group homomorphism $h : L \to K$ is the homomorphism $h^* : K^* \to L^*$ defined as

$$\forall x \in L, \forall y^* \in K^*, \qquad h^*(y^*)(x) = (y^* \circ h)(x). \quad (2.32)$$

Example 2.33. For $L = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $K = \mathbb{Z}_2$. Provide a formula for h^* for every homomorphism $h : L \to K$.

Exercise 2.34. How many homomorphisms $(\mathbb{Z}_2)^n \to \mathbb{Z}_2$ are there? Prove your conclusion.

Exercise 2.35. For $L = \mathbb{Z}_2$ and $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. Provide a formula for h^* for every homomorphism $h : L \to K$.

2.6.4 Maps of cochains

Definition 2.77. The *k-cochain map* of a graph map $f : G \to J$ is the homomorphism $f^k : C^k(J) \to C^k(G)$,

$$\forall a \in C_k(G),\ \forall t \in C^k(J), \qquad f^k(t)(a) := t f_k(a). \quad (2.33)$$

Corollary 2.78. The k-cochain map is the dual homomorphism of the k-chain map for $k = 0, 1$.

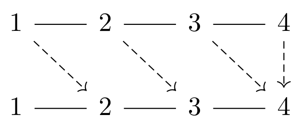
Proof. This follows directly from Definitions 2.56, 2.69, and 2.76. □

Corollary 2.79. The k-cochain map is the dual map of the k-chain map for $k = 0, 1$.

Proof. This follows from Definitions 2.56 and 2.69. □

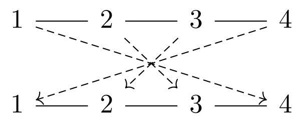
Example 2.36. Consider a graph $G = (N, E)$ and a graph map f of three shifts and one collapse:

$$N := \{1, 2, 3, 4\}, \qquad E := \{12, 23, 34\};$$
$$f(1) = 2,\ f(2) = 3,\ f(3) = 4,\ f(4) = 4.$$



Derive the matrices of chain maps and cochain maps.

Exercise 2.37. For the graph in Example 2.36, compute the cochain maps of the following graph map.



2.6.5 The coboundary operator

Definition 2.80. The *coboundary operator of a graph G* is a homomorphism $\mathrm{d} : C^0(G) \to C^1(G)$,

$$\forall a \in C_1(G), \forall Q \in C^0(G), \qquad \mathrm{d}(Q)(a) := Q\partial(a). \quad (2.34)$$

Corollary 2.81. The coboundary operator is the dual homomorphism of the boundary operator.

Proof. This follows from Definitions 2.44, 2.76, and 2.80.

$$\begin{tikzcd} C_1(G) \arrow[d, "\partial"] \arrow[dr, "s"] & \\ C_0(G) \arrow[r, "Q"] & \mathbb{Z}_2 \end{tikzcd}$$

The above commutative diagram might be helpful. □

Corollary 2.82. The coboundary operator is the dual map of the boundary operator $\partial : C_1 \to C_0$.

Proof. This follows from Definition 2.80 and the definition of dual maps. □

Example 2.38. Consider a graph $G = (N_G, E_G)$ with

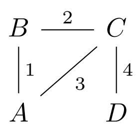
$$N_G = \{A, B, C, D, E, F\},$$
$$E_G = \{AB, AC, AD, AE, BE, DF\}.$$

Derive $\mathrm{d}A^*$.

Lemma 2.83. The coboundary operator maps a basic 0-cochain A^* to the sum of all basic 1-cochains of edges incident to node A. More precisely, for any given graph $G = (N, E)$ we have

$$\forall A \in N, \qquad \mathrm{d}A^* = \sum_{AY \in E} (AY)^*. \quad (2.35)$$

Exercise 2.39. Derive the matrix M_d of the coboundary operator for the graph below in Example 2.17,



and show that M_d is the transpose of M_∂, verifying the theorem that the matrix of the dual map of a linear map T is the transpose of the matrix of T.