中期考核答辩

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2021年11月3日

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课程成绩

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- 英语六级成绩: 623

全国大学英语六级考试(CET6)成绩详情

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准考证号: 330011202201207

总　分: 623

听　力: 220

阅　读: 230

写作和翻译: 173

考试时间: 2020年12月

研究内容：不可压 Navier-Stokes 方程的高阶数值方法

- 无滑移边界下不可压 Navier-Stokes 方程的无量纲形式：

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \boldsymbol{g} - \nabla p + \nu\Delta\boldsymbol{u} & \quad \text{in } \Omega_T, \\ \nabla\cdot\boldsymbol{u} = 0 & \quad \text{in } \Omega_T, \\ \boldsymbol{u} = \boldsymbol{0} & \quad \text{on } \partial\Omega_T, \end{aligned} \tag{1}$$

其中 $\Omega_T = \Omega\times(0,T]$, $\partial\Omega_T = \partial\Omega\times(0,T]$.

- (1) 式描述了粘性不可压缩流体(如海洋流动)的普遍运动规律，对其进行高精度数值模拟对于科学研究和工程应用具有重大意义.

前期准备①: 物理$\longrightarrow$ 数学

- 由质量, 动量, 能量守恒定律出发严格推导不可压 Navier-Stokes 方程.

$$
\begin{aligned}
(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &\Leftrightarrow \text{对流项},\\
\Delta\boldsymbol{u} &\Leftrightarrow \text{扩散项},\\
\nabla\cdot\boldsymbol{u}=0 &\Leftrightarrow \text{不可压条件}.
\end{aligned}
$$

- Helmholtz 分解定理:

$$\boldsymbol{u}^* = \mathscr{P}\boldsymbol{u}^* + \nabla\phi,$$

其中

$$\nabla\cdot\mathscr{P}\boldsymbol{u}^* = 0 \text{ in } \Omega, \qquad \boldsymbol{n}\cdot\mathscr{P}\boldsymbol{u}^* = 0 \text{ on } \partial\Omega.$$

- ...

- 将所学内容整理为讲义，构建知识体系.

1.11 The incompressible Navier-Stokes equations

Theorem 1.149. The *Navier-Stokes equations* for a Newtonian fluid are

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\Delta\mathbf{u} + \rho\mathbf{b}. \tag{1.97}$$

Proof. This follows from (1.86), (1.92), and the identity

$$\nabla\cdot(\lambda(\nabla\cdot\mathbf{u})\mathbf{I} + 2\mu\mathbf{D}) = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\Delta\mathbf{u}. \qquad \square$$

Theorem 1.150. The *Navier-Stokes equations for an incompressible homogeneous ($\rho = \rho_0 =$ constant) flow* are

$$\frac{D\mathbf{u}}{Dt} = -\nabla p' + \nu\Delta\mathbf{u} + \mathbf{b}, \tag{1.98a}$$
$$\nabla\cdot\mathbf{u} = 0, \tag{1.98b}$$

with no-slip boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ on solid walls at rest,} \tag{1.99}$$

where $\nu = \mu/\rho_0$ is the *coefficient of kinematic viscosity*, and $p' = p/\rho_0$.

Proof. Proposition 1.38 and Theorem 1.149. $\square$

Theorem 1.151. The *dimensionless form of the Navier-Stokes equations for an incompressible homogeneous ($\rho = \rho_0 =$ constant) flow* is

$$\frac{\partial\mathbf{u}'}{\partial t'} + (\mathbf{u}'\cdot\nabla')\mathbf{u}' = -\nabla' p' + \frac{1}{\mathrm{Re}}\Delta'\mathbf{u}' + \mathbf{b}', \tag{1.100a}$$
$$\nabla'\cdot\mathbf{u}' = 0, \tag{1.100b}$$

where

$$\begin{aligned}&\mathbf{u}' = \frac{\mathbf{u}}{U},\quad \mathbf{x}' = \frac{\mathbf{x}}{L},\quad t' = \frac{t}{T} = \frac{t}{L/U},\\ &p' = \frac{p}{\rho_0 U^2},\quad \mathbf{b}' = \frac{\mathbf{b}}{U^2/L},\end{aligned} \tag{1.101}$$

and

$$\mathrm{Re} = \frac{LU}{\nu} \tag{1.102}$$

is the dimensionless *Reynolds number*. L is called the *characteristic length* and U the *characteristic velocity*.

Proof. The component form of (1.98a) is

$$\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_j} + \nu\frac{\partial^2 u_j}{\partial x_i\partial x_i} + b_j.$$

The change of variables produces

$$\begin{aligned}&\frac{\partial(Uu_j')}{\partial t'}\frac{\partial t'}{\partial t} + (Uu_i')\frac{\partial(Uu_j')}{\partial x_i'}\frac{\partial x_i'}{\partial x_i}\\ =& -\frac{1}{\rho_0}\frac{\partial(\rho_0 U^2 p')}{\partial x_j'}\frac{\partial x_j'}{\partial x_j} + \nu\frac{\partial^2(Uu_j')}{\partial(Lx_i')\partial(Lx_i')} + \frac{U^2}{L}b_j'.\end{aligned}$$

Hence

$$\frac{U^2}{L}\left(\frac{\partial u_j'}{\partial t'} + u_i'\frac{\partial u_j'}{\partial x_i'}\right) = -\frac{U^2}{L}\frac{\partial p'}{\partial x_j'} + \frac{\nu U}{L^2}\frac{\partial^2 u_j'}{\partial x_i'\partial x_i'} + \frac{U^2}{L}b_j',$$

dividing both sides by U^2/L gives the desired result. $\square$

Definition 1.152. Two flows with the same geometry and the same Reynolds number are called *similar*.

Example 1.153. The significance of the similarity of flows can be seen as follows. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two flows on regions D_1 and D_2 that are related by a scalar factor λ so that $L_1 = \lambda L_2$. Let choice of U_1 and U_2 be made for each flow, and let the viscosities be ν_1 and ν_2 respectively. If

$$\mathrm{Re}_1 = \mathrm{Re}_2,\ \text{i.e.,}\ \frac{L_1U_1}{\nu_1} = \frac{L_2U_2}{\nu_2},$$

then the dimensionless velocity fields $\mathbf{u}_1'$ and $\mathbf{u}_2'$ satisfy exactly the same equation on the same region. Thus, we can conclude that $\mathbf{u}_1$ may be obtained from a suitably rescaled solution $\mathbf{u}_2$.

Example 1.154. The idea of the similarity of flows is used in the design of experimental models. For example, suppose we are contemplating a new design for an aircraft wing and we wish to know the behavior of a fluid flow around it. Rather than build the wing itself, it may be faster and more economical to perform the initial tests on a scaled-down version. We design our model so that it has the same geometry as the full-scale wing and we choose values for the undisturbed velocity, coefficient of viscosity, and so on, such that the Reynolds number for the flow in our experiment matches that of the actual flow. We can then expect the results of our experiment to be relevant to the actual flow over the full-scale wing.

Notation 9. In the following, we shall simply write the incompressible Navier-Stokes equations (INSE) as

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b}, \tag{1.103a}$$
$$\nabla\cdot\mathbf{u} = 0. \tag{1.103b}$$

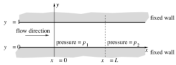
Definition 1.155. In the momentum equation (1.103a), we call

$$\frac{1}{\mathrm{Re}}\Delta\mathbf{u},\quad \text{the } \textit{diffusion} \text{ or } \textit{dissipation} \text{ term,}$$

and

$$(\mathbf{u}\cdot\nabla)\mathbf{u},\quad \text{the } \textit{inertial} \text{ or } \textit{convective} \text{ term.}$$

Example 1.156. Consider the steady viscous incompressible flow between two stationary plates located at $y = 0$ and $y = 1$.



1.11.1 The Leray-Helmholtz Projection $\mathscr{P}$

Theorem 1.157. The Neumann boundary value problem

$$\Delta p = f \text{ in } D,\quad \mathbf{n}\cdot\nabla p = g \text{ on } \partial D \tag{1.104}$$

has a solution unique up to an additive constant if and only if

$$\int_D f\mathrm{d}V = \int_{\partial D} g\mathrm{d}A. \tag{1.105}$$

Proof. Not required. The proof can be found in the book *Partial Differential Equations I* by M. E. Taylor, 2nd edition, Springer, 2011, pp 408-409. $\square$

Remark 1.102. The uniqueness can be easily proved by the identity

$$\int_D (p\Delta p + \nabla p\cdot\nabla p)\,\mathrm{d}V = \int_{\partial D} p\frac{\partial p}{\partial\mathbf{n}}\mathrm{d}A.$$

Lemma 1.158. Suppose the vector field $\mathbf{u}$ has zero divergence and is parallel to ∂D, i.e.,

$$\nabla\cdot\mathbf{u} = 0 \text{ in } D \text{ and } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial D, \tag{1.106}$$

then we have the orthogonality relation

$$\int_D \mathbf{u}\cdot\nabla p\mathrm{d}V = 0 \tag{1.107}$$

for any scalar field p on D.

Proof.

$$\begin{aligned}\int_D \mathbf{u}\cdot\nabla p\mathrm{d}V &= \int_D \nabla\cdot(p\mathbf{u})\mathrm{d}V - \int_D p\nabla\cdot\mathbf{u}\mathrm{d}V\\ &= \int_{\partial D} p\mathbf{u}\cdot\mathbf{n}\mathrm{d}A = 0,\end{aligned}$$

where the first equality follows from Lemma 1.28, the second from the divergence theorem and (1.106), and the last from (1.106). $\square$

Theorem 1.159 (Helmholtz-Hodge decomposition)**.** A vector field $\mathbf{w}$ on D can be uniquely decomposed in the form

$$\mathbf{w} = \mathbf{u} + \nabla p, \tag{1.108}$$

where $\mathbf{u}$ has zero divergence and is parallel to ∂D.

Proof. By Theorem 1.157, the Neumann problem

$$\Delta p = \nabla\cdot\mathbf{w} \text{ in } D,\quad \mathbf{n}\cdot\nabla p = \mathbf{w}\cdot\mathbf{n} \text{ on } \partial D,$$

has a solution unique up to an additive constant for each given $\mathbf{w}$ on D. If we define

$$\mathbf{u} = \mathbf{w} - \nabla p, \tag{1.109}$$

then $\mathbf{u}$ clearly has the desired property (1.106), this completes the proof of existence.

Now we prove the uniqueness of $\mathbf{u}$. Suppose that $\mathbf{w} = \mathbf{u}_1 + \nabla p_1$ and $\mathbf{w} = \mathbf{u}_2 + \nabla p_2$, then

$$\mathbf{0} = \mathbf{u}_1 - \mathbf{u}_2 + \nabla(p_1 - p_2),$$

taking inner product with $\mathbf{u}_1 - \mathbf{u}_2$ and integrating, we get

$$\begin{aligned}0 &= \int_D \left(\|\mathbf{u}_1 - \mathbf{u}_2\|^2 + (\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla(p_1 - p_2)\right)\mathrm{d}V\\ &= \int_D \|\mathbf{u}_1 - \mathbf{u}_2\|^2\mathrm{d}V,\end{aligned}$$

by the orthogonality relation (1.107). It follows that $\mathbf{u}_1 = \mathbf{u}_2$, and so, $\nabla p_1 = \nabla p_2$ (which is the same thing as $p_1 = p_2 +$ constant). $\square$

Remark 1.103. A continuously differentiable solenoidal vector field $\mathbf{u}$ has the following three equivalent characteristics:

(i) $\nabla\cdot\mathbf{u} = 0$;

(ii) $\int_S \mathbf{u}\cdot\mathbf{n}\mathrm{d}A = 0$ for any closed surface S;

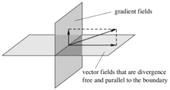
(iii) $\mathbf{u} = \nabla\times\mathbf{w}$, where $\mathbf{w}$ is itself solenoidal.

Definition 1.160. The *Leray-Helmholtz projection operator* $\mathscr{P}$ maps a vector field $\mathbf{w}$ onto its divergence-free part $\mathscr{P}\mathbf{w}$ given by the Helmholtz-Hodge decomposition theorem, that is,

$$\mathbf{w} = \mathscr{P}\mathbf{w} + \nabla p, \tag{1.110}$$

where

$$\nabla\cdot(\mathscr{P}\mathbf{w}) = 0 \text{ in } D \text{ and } \mathbf{n}\cdot(\mathscr{P}\mathbf{w}) = 0 \text{ on } \partial D. \tag{1.111}$$



Proposition 1.161. If $\mathbf{u}$ has zero divergence and is parallel to ∂D, then

$$\mathscr{P}\mathbf{u} = \mathbf{u}. \tag{1.112}$$

Proof. Theorem 1.159 and Definition 1.160. $\square$

Proposition 1.162. The Leray-Helmholtz projection operator $\mathscr{P}$ annihilates gradient fields, i.e.,

$$\mathscr{P}\nabla p = \mathbf{0}, \tag{1.113}$$

for any scalar field p.

Proof. Theorem 1.159 and Definition 1.160. $\square$

Proposition 1.163. $\mathscr{P}$ is a linear operator and

$$\mathscr{P}^2 = \mathscr{P}. \tag{1.114}$$

Proof. Theorem 1.159, Definition 1.160, and Proposition 1.161. $\square$

前期准备②: 数学→ 数值方法

- 阅读相关经典文献, 撰写文献综述.

物理学报 *Acta Phys. Sin.* Vol. 70, No. 13 (2021) 130201

青年科学评述

不可压 Navier-Stokes 方程的投影方法*

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(2021 年 2 月 3 日收到; 2021 年 3 月 9 日收到修改稿)

不可压 Navier-Stokes 方程是流体力学的基本控制方程, 其高精度数值模拟具有重要的科学意义. 本综述性文章回顾了求解 Navier-Stokes 方程的投影方法, 重点介绍了时空一致四阶精度的 GePUP 方法. 该方法用一个广义投影算子对不可压 Navier-Stokes 方程进行了重新表述, 使得投影流速的散度由一个热方程控制, 保持了 UPPE 方法的优点. 与 UPPE 方法不同的是, GePUP 方法的推导不依赖于 Leray-Helmholtz 投影算子的各种性质, 并且 GePUP 表述中的演化变量无需满足散度为零的条件, 因此数值近似 Leray-Helmholtz 投影算子的误差对精度和稳定性的影响非常透明. 在 GePUP 方法中, 时间积分和空间离散是完全解耦的, 因此对这两个模块都能以"黑匣子"的方式自由替换. 时间积分模块的灵活性实现了时间上的高阶精度, 并使得 GePUP 算法能同时适用于低雷诺数流体和高雷诺数流体. 空间离散模块的灵活性使得 GePUP 算法能很好地适应不规则边界. 理论分析和数值测试结果都显示, 相对于二阶投影方法, GePUP 方法无论在精度上还是效率上都具有巨大优势.

- 数值求解不可压 Navier-Stokes 方程的主要困难: 如何高效地处理不可压约束条件 $\nabla \cdot \boldsymbol{u} = 0$?
- 一阶投影方法:

$$\begin{aligned} \frac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\delta t} &= -\boldsymbol{C}\left(\boldsymbol{u}^*, \boldsymbol{u}^n\right) + \boldsymbol{g}^n + \nu \Delta \boldsymbol{u}^*, \\ \boldsymbol{u}^{n+1} &= \boldsymbol{P}\boldsymbol{u}^*, \end{aligned}$$

其中 $\boldsymbol{P} \approx \mathscr{P}$.

优势:

- 效率高: 流速和压强在计算域内解耦, 每个时间步仅需求解若干个关于压强或流速的椭圆边值问题;
- 可推广到二阶精度.

劣势:

- 辅助变量 $\boldsymbol{u}^*$ 不具有物理意义, 其边界条件依赖于具体时间积分方法的细节;
- 无法直接将求解常微分方程组的高阶方法作为“黑匣子”使用.

GePUP 表述:

$$
\begin{aligned}
\frac{\partial \mathbf{w}}{\partial t} &= \mathbf{g} - (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla q + \nu\Delta\mathbf{w} && \text{in} \quad \Omega_T, && \text{(2a)}\\
\mathbf{w} &= \mathbf{0} && \text{on} \quad \partial\Omega_T, && \text{(2b)}\\
\mathbf{u} &= \mathscr{P}\mathbf{w} && \text{in} \quad \Omega_T, && \text{(2c)}\\
\mathbf{u}\cdot\mathbf{n} &= 0 && \text{on} \quad \partial\Omega_T, && \text{(2d)}\\
\Delta q &= \nabla\cdot\left(\mathbf{g} - (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) && \text{in} \quad \Omega_T, && \text{(2e)}\\
\mathbf{n}\cdot\nabla q &= \mathbf{n}\cdot\left(\mathbf{g} + \nu\Delta\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u}\right) && \text{on} \quad \partial\Omega_T. && \text{(2f)}
\end{aligned}
$$

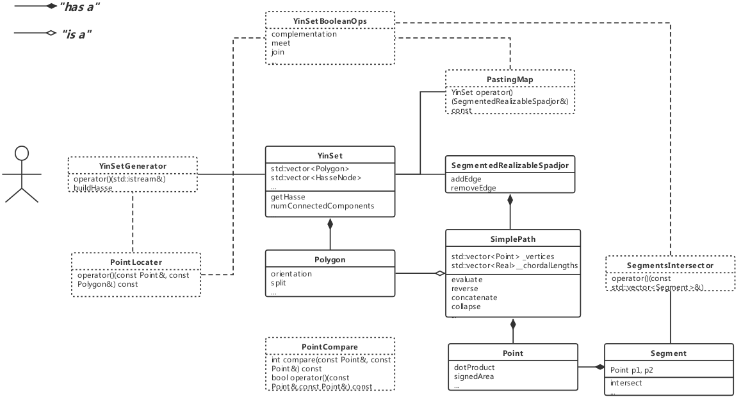
基于该表述构造数值方法具有如下优势:

- 时空一致四阶精度;
- 最优求解复杂度: 单个时间步为 $O(N)$, 其中 N 是控制体数目;
- 时间积分与空间离散完全解耦;
- $\mathbf{w}$ 的散度由一个热方程控制:

$$\frac{\partial\left(\nabla\cdot\mathbf{w}\right)}{\partial t} = \nu\Delta\left(\nabla\cdot\mathbf{w}\right).$$

前期准备③: 数值方法→ 计算机编程实现

- 系统学习 C++ 编程语言;
- 熟悉 Linux 操作系统下的常用工具, 如 emacs, auctex, make, git 等;
- 基本 C++ 编程训练:
  - 多重网格方法的程序设计;
  - 平面上具有任意复杂拓扑结构集合的高效表示及布尔代数运算.



研究进展: 求解不可压 Navier-Stokes 方程的能量稳定算法

定理 (不可压 Navier-Stokes 方程的能量耗散性质)

考虑粘性不可压流体且外力为保守力的情况下, 以下动能耗散式成立:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{kinetic} = -\nu \int_\Omega \|\nabla \boldsymbol{u}\|^2 \, \mathrm{d}\boldsymbol{x} \leq 0, \tag{3}$$

其中

$$E_{kinetic} := \frac{1}{2} \int_\Omega \|\boldsymbol{u}\|^2 \, \mathrm{d}\boldsymbol{x}. \tag{4}$$

- 目标: 设计具有类似能量耗散性质的数值方法.

GePUP-SAV 能量稳定算法

引入标量辅助变量 $r(t) \equiv 1$ 满足

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0 = \int_{\Omega} ((\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}) \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x}, \tag{5}$$

将 GePUP 表述等价地写为

$$\begin{aligned}
\frac{\partial \boldsymbol{w}}{\partial t} &= \boldsymbol{g} - r(t)\,(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \nabla q + \nu \Delta \boldsymbol{w} && \text{in } \Omega_T, && \text{(6a)}\\
\boldsymbol{w} &= \boldsymbol{0} && \text{on } \partial\Omega_T, && \text{(6b)}\\
\frac{\mathrm{d}r}{\mathrm{d}t} &= \int_{\Omega} ((\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}) \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x}, && && \text{(6c)}\\
\boldsymbol{u} &= \mathscr{P} \boldsymbol{w} && \text{in } \Omega_T, && \text{(6d)}\\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega_T, && \text{(6e)}\\
\Delta q &= \nabla \cdot (\boldsymbol{g} - r(t)\,(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}) && \text{in } \Omega_T, && \text{(6f)}\\
\boldsymbol{n} \cdot \nabla q &= \boldsymbol{n} \cdot (\boldsymbol{g} + \nu \Delta \boldsymbol{u} - \nu \nabla\nabla \cdot \boldsymbol{u}) && \text{on } \partial\Omega_T. && \text{(6g)}
\end{aligned}$$

应用 Crank-Nicolson 方法作时间离散得以下 GePUP-SAV-CN 半离散格式:

$$
\begin{aligned}
\frac{\mathbf{w}^{n+1}-\mathbf{w}^{n}}{\delta t} &= \mathbf{g}^{n+\frac{1}{2}} - r^{n+\frac{1}{2}}\widetilde{\mathbf{u}}^{n+1/2}\cdot\nabla\widetilde{\mathbf{u}}^{n+1/2} - \nabla q^{n+1/2} + \nu\Delta\mathbf{w}^{n+\frac{1}{2}} && \text{in } \Omega,\\
\mathbf{w}^{n+1} &= \mathbf{0} && \text{on } \partial\Omega,\\
\frac{r^{n+1}-r^{n}}{\delta t} &= \int_{\Omega}\left(\widetilde{\mathbf{u}}^{n+1/2}\cdot\nabla\widetilde{\mathbf{u}}^{n+1/2}\right)\cdot\mathbf{u}^{n+\frac{1}{2}}\,\mathrm{d}\mathbf{x}, &&\\
\mathbf{u}^{n+1} &= \mathscr{P}\mathbf{w}^{n+1} && \text{in } \Omega,\\
\mathbf{u}^{n+1}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega,\\
\Delta q^{n+1/2} &= \nabla\cdot\left(\mathbf{g}^{n+\frac{1}{2}} - r^{n+\frac{1}{2}}\widetilde{\mathbf{u}}^{n+1/2}\cdot\nabla\widetilde{\mathbf{u}}^{n+1/2}\right) && \text{in } \Omega,\\
\mathbf{n}\cdot\nabla q^{n+1/2} &= \mathbf{n}\cdot\left(\mathbf{g}^{n+\frac{1}{2}} + \nu\Delta\widetilde{\mathbf{u}}^{n+1/2} - \nu\nabla\nabla\cdot\widetilde{\mathbf{u}}^{n+1/2}\right) && \text{on } \partial\Omega,
\end{aligned}
$$

其中上标 $^{n+\frac{1}{2}}$ 表示时间步 n 和 $n+1$ 的平均值, $\widetilde{\mathbf{u}}^{n+1/2}$ 由外插公式得到.

- 对 $\mathbf{w}^{n+1}$, $\mathbf{u}^{n+1}$, 和 $q^{n+1/2}$ 作分解后, 可实现对上述格式的高效求解.

定理 (GePUP-SAV-CN 半离散格式的修正能量耗散性质)

$$\frac{1}{\delta t}\left(\mathcal{E}\left(t^{n+1}\right)-\mathcal{E}\left(t^{n}\right)\right)\leq-\nu\int_{\Omega}\left\|\nabla \boldsymbol{u}^{n+\frac{1}{2}}\right\|^{2}\mathrm{d}\boldsymbol{x} \tag{7}$$

其中

$$\mathcal{E}\left(t^{n}\right):=\frac{1}{2}\int_{\Omega}\left\|\boldsymbol{u}^{n}\right\|^{2}\mathrm{d}\boldsymbol{x}+\frac{1}{2}\left|r^{n}\right|^{2}. \tag{8}$$

- 时间离散方法选为向后微分公式 (BDF) 或 Gauss-Legendre Runge-Kutta 方法, 仍有类似离散能量耗散结论成立.

后期研究计划

- 在现有 C++ 程序框架下实现 GePUP-SAV 能量稳定算法.

敬请各位老师批评指正!