Chapter 1

The Equations of Motion

Reference books:

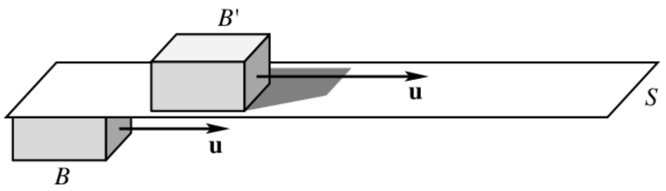
1) *A Mathematical Introduction to Fluid Mechanics* by A. J. Chorin and J. E. Marsden, third edition, Springer, 1993.

2) *Fluid Mechanics* by P. K. Kundu, I. M. Cohen and D. R. Dowling, sixth edition, Academic Press, 2016.

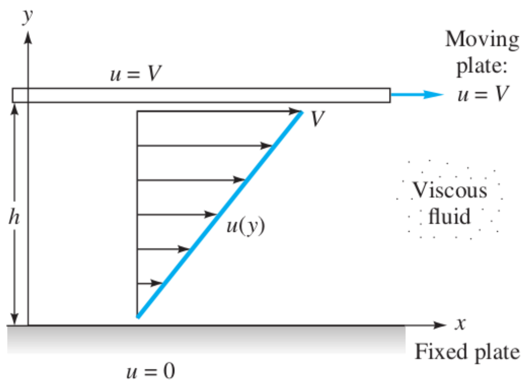
Definition 1.1. *Fluid mechanics* is the science that deals with the behavior of fluids at rest (*fluid statics*) or in motion (*fluid dynamics*), and the interaction of fluids with solids or other fluids at the boundaries.

Definition 1.2. A *fluid* is defined as a substance that deforms continuously under the influence of a shear stress, no matter how small that shear stress may be.

Example 1.3. *Viscosity* is an important property of fluids, which can be thought of as the internal stickiness of a fluid. When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer. Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases.



Example 1.4. The following picture shows the viscous flow induced between a fixed lower plate and an upper plate moving steadily at velocity $\mathbf{V}$.



Experiments show that the shear stress τ (shear force per unit area) satisfies *Newton's law of friction*

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y},$$

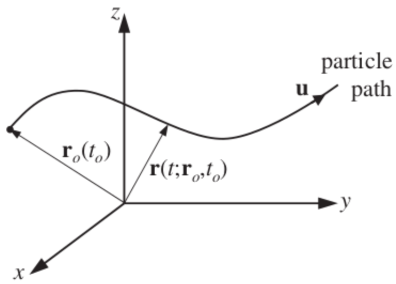
where μ is the dynamic viscosity.

Assumption 1.5 (Continuum assumption)**.** In explaining macroscopic phenomena of flow problems, the discrete molecular structure of matter may be ignored and replaced by a continuous distribution, called a *continuum.* In a continuum, fluid properties like density ρ, pressure p, velocity $\mathbf{u}$, and temperature T are defined at every point in space, and these properties are known to be appropriate averages of molecular characteristics in a small region surrounding the point of interest.

1.1 Description of Fluid Motion

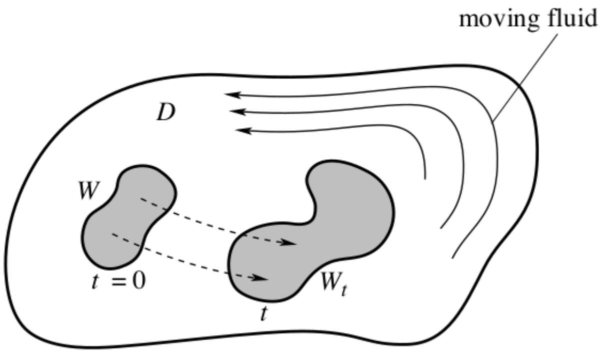
Definition 1.6. In the *Lagrangian description* of fluid motions, we follow individual fluid particles as they move about and determine how the fluid properties associated with these particles change as a function of time.

Definition 1.7. A *pathline* (or *trajectory*) of a given fluid particle is the line traced out by it as it flows from one point to another.



Definition 1.8. A *system* is a collection of matter of fixed identity (always the same atoms or fluid particles), which may move, flow, and interact with its surroundings.

Notation 1. In the following, we shall use W_t to denote a system, i.e., $W_t = \boldsymbol{\varphi}_t(W_0)$, where W_0 is the initial position of the system.



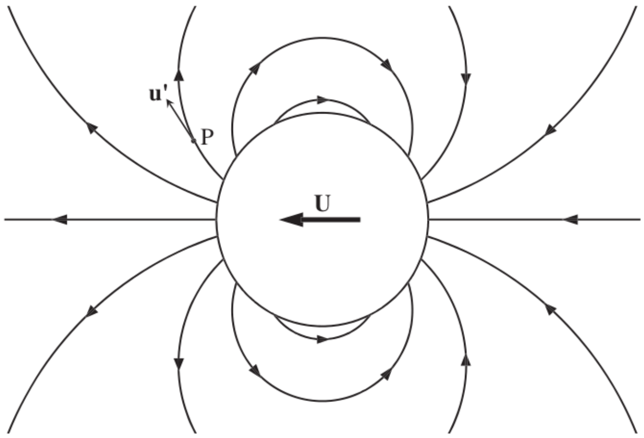
Definition 1.9. In the *Eulerian description* of fluid motions, we observe the physical properties (e.g. velocity) of particles passing each point in space. Thus the flow properties are functions of both space and time.

Notation 2. In the following, $\rho(\mathbf{x},t)$, $\mathbf{u}(\mathbf{x},t)$, and $p(\mathbf{x},t)$ will denote the density, velocity and pressure of the particle of fluid that is moving through $\mathbf{x}$ at time t, respectively. For simplicity, we shall assume that these quantities are smooth enough so that standard operations of calculus may be performed on them.

Definition 1.10. A *streamline* $\mathbf{x}(s)$ at a fixed time t is a line that is everywhere tangent to the velocity field $\mathbf{u}(\mathbf{x},t)$, i.e.,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}(s),t), \quad t \text{ fixed.} \tag{1.1}$$

Example 1.11. The following figure shows the streamlines of an unsteady flow of a nominally quiescent ideal incompressible fluid around a moving long circular cylinder with its axis perpendicular to the page. Here the cylinder velocity $\mathbf{U}$ is shown inside the cylinder, and the fluid velocity $\mathbf{u}'$ at point P is caused by the presence of the moving cylinder alone.



Definition 1.12. A flow is *steady* (or *stationary*) if the velocity $\mathbf{u}$ at any given point in space does not vary with time, i.e., $\partial_t \mathbf{u} = \mathbf{0}$.

Exercise 1.13. Show that if the fluid flow is steady, then the streamlines and pathlines coincide.

Definition 1.14. A *streakline* at a particular time of interest through a fixed point $\mathbf{p}$ is the line consisting of all particles in a flow that have previously passed through $\mathbf{p}$.

Example 1.15. Consider a simple plane flow

$$\mathbf{u}(x,y,t) = \left(\frac{x}{1+t}, y\right).$$

(i) Given the fluid particle at position $(1,1)$ at time $t=0$, its pathline $\mathbf{x}(t) = (x(t), y(t))$ is given by

$$\begin{cases} x'(t) = \frac{x(t)}{1+t}, \\ y'(t) = y(t), \\ x(0) = y(0) = 1. \end{cases} \Rightarrow \begin{cases} x(t) = 1+t, \\ y(t) = e^t. \end{cases}$$

(ii) The streamline $\mathbf{x}(s)$ at $t=1$ passing through $(1,1)$ is given by

$$\begin{cases} x'(s) = \frac{x(s)}{2}, \\ y'(s) = y(s), \\ x(0) = y(0) = 1. \end{cases} \Rightarrow \begin{cases} x(s) = e^{s/2}, \\ y(s) = e^s. \end{cases}$$

(iii) The streakline at $t=1$ through $\mathbf{p} = (1,1)$ is obtained as follows. Consider a fluid particle at position $\mathbf{p}$ at some instant τ, its pathline is given by

$$\begin{cases} x'(t;\tau,\mathbf{p}) = \frac{x(t;\tau,\mathbf{p})}{1+t}, \\ y'(t;\tau,\mathbf{p}) = y(t;\tau,\mathbf{p}), \\ x(\tau;\tau,\mathbf{p}) = y(\tau;\tau,\mathbf{p}) = 1. \end{cases} \Rightarrow \begin{cases} x(t;\tau,\mathbf{p}) = \frac{1+t}{1+\tau}, \\ y(t;\tau,\mathbf{p}) = e^{t-\tau}. \end{cases}$$

Letting $t=1$ gives the streakline

$$\begin{cases} x(\tau) = \frac{2}{1+\tau}, \\ y(\tau) = e^{1-\tau}. \end{cases}$$

Exercise 1.16. Consider the plane flow

$$\mathbf{u}(x,y,t) = (x(1+2t), y).$$

(i) Find the pathline of the fluid particle at position $(1, 1)$ at time $t = 0$.

(ii) Find the streamline at $t = 1$ passing through $(1, 1)$.

(iii) Find the streakline at $t = 1$ through $\mathbf{p} = (1, 1)$.

Notation 3. In the following, D will denote a region in two- or three-dimensional space filled with fluid particles.

Definition 1.17. Let $\mathbf{x} \in D$ and $\boldsymbol{\varphi}(\mathbf{x}, t) : D \times \mathbb{R} \to D$ be the trajectory followed by the particle that is at point $\mathbf{x}$ at time $t = 0$, i.e.,

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\varphi}(\mathbf{x}, t), t), \\ \boldsymbol{\varphi}(\mathbf{x}, 0) = \mathbf{x}. \end{cases} \tag{1.2}$$

$\boldsymbol{\varphi}$ is called the *fluid flow map.* With fixed t, let $\boldsymbol{\varphi}_t$ denote the map $\mathbf{x} \mapsto \boldsymbol{\varphi}(\mathbf{x}, t)$, i.e., $\boldsymbol{\varphi}_t$ advances each fluid particle from its position at $t = 0$ to its position at time t.

Proposition 1.18. If the velocity field $\mathbf{u}(\mathbf{x}, t)$ is Lipschitz continuous in space and continuous in time, then the fluid flow map $\boldsymbol{\varphi}(\mathbf{x}, t)$ is injective for each fixed t.

Proof. From the theory of ODEs, we know that the IVP (1.2) admits a unique solution $\boldsymbol{\varphi}(\mathbf{x}, t)$. If for some t_0, $\boldsymbol{\varphi}_{t_0}$ is not injective, i.e.,

$$\exists \mathbf{x}_1, \mathbf{x}_2 \in D \text{ s.t. } \mathbf{x}_1 \neq \mathbf{x}_2 \text{ and } \boldsymbol{\varphi}(\mathbf{x}_1, t_0) = \boldsymbol{\varphi}(\mathbf{x}_2, t_0) = \mathbf{x}_0.$$

If we now consider the IVP

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t}(\mathbf{x}_0, t) = -\mathbf{u}(\boldsymbol{\psi}(\mathbf{x}_0, t), t_0 - t), \\ \boldsymbol{\psi}(\mathbf{x}_0, 0) = \mathbf{x}_0, \end{cases} \tag{1.3}$$

then a direct computation shows that $\boldsymbol{\psi}_1(\mathbf{x}_0, t) = \boldsymbol{\varphi}(\mathbf{x}_1, t_0 - t)$ and $\boldsymbol{\psi}_2(\mathbf{x}_0, t) = \boldsymbol{\varphi}(\mathbf{x}_2, t_0 - t)$ are both solutions and $\boldsymbol{\psi}_1(\mathbf{x}_0, t) \neq \boldsymbol{\psi}_2(\mathbf{x}_0, t)$ since $\boldsymbol{\psi}_1(\mathbf{x}_0, t_0) = \mathbf{x}_1 \neq \mathbf{x}_2 = \boldsymbol{\psi}_2(\mathbf{x}_0, t_0)$, this, however, is a contradiction since (1.3) is clearly well-posed (the RHS function $-\mathbf{u}(\mathbf{x}, t_0 - t)$ is Lipschitz continuous in space and continuous in time). □

1.2 Material Derivative

Proposition 1.19. Let $\mathbf{x}(t) = (x(t), y(t), z(t))$ be the path followed by a fluid particle, then the velocity of the particle is

$$\mathbf{u}(x(t), y(t), z(t), t) = (x'(t), y'(t), z'(t)), \tag{1.4}$$

i.e.,

$$\mathbf{u}(\mathbf{x}(t), t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t), \tag{1.5}$$

and the acceleration is

$$\mathbf{a}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t) = (\partial_t + \mathbf{u} \cdot \nabla)\,\mathbf{u}. \tag{1.6}$$

Proof.

$$\begin{aligned} \mathbf{a}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(x(t), y(t), z(t), t) \\ &= \frac{\partial \mathbf{u}}{\partial x}x'(t) + \frac{\partial \mathbf{u}}{\partial y}y'(t) + \frac{\partial \mathbf{u}}{\partial z}z'(t) + \frac{\partial \mathbf{u}}{\partial t} \\ &= (\partial_t + \mathbf{u} \cdot \nabla)\,\mathbf{u}, \end{aligned}$$

where the third equality follows from the chain rule. □

Definition 1.20. $\partial_t \mathbf{u}$ is called the *local acceleration* and represents effects of the unsteadiness of the flow while $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is termed the *convective acceleration* and represents the fact that a flow property associated with a fluid particle may vary because of the motion of the particle from one point in space where the parameter has one value to another point in space where its value is different. This contribution is due to the convection, or motion, of the particle through space in which there is a gradient in the parameter value.

Definition 1.21. The *material derivative* (or *particle derivative*) is

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla = \partial_t + u_i \partial_{x_i}, \tag{1.7}$$

where we have employed the Einstein summation convention.

Exercise 1.22. If $f(\mathbf{x}, t)$ is any (scalar or vector) function of position $\mathbf{x}$ and time t, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t), t) = \frac{Df}{Dt}(\mathbf{x}(t), t).$$

Exercise 1.23. Prove the following properties of the material derivative D/Dt.

(i) Linearity:

$$\frac{D}{Dt}(\alpha f + \beta g) = \alpha \frac{Df}{Dt} + \beta \frac{Dg}{Dt}.$$

(ii) Leibniz or product rule:

$$\frac{D}{Dt}(fg) = f\frac{Dg}{Dt} + g\frac{Df}{Dt}.$$

(iii) Chain rule:

$$\frac{D}{Dt}(g \circ f) = (g' \circ f)\frac{Df}{Dt}.$$

1.3 Reynolds Transport Theorem

Theorem 1.24 (Change-of-variables)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^n$ is differentiable at every point of Ω. If f is a real-valued function defined on $\boldsymbol{\sigma}(\Omega)$, then

$$\int_{\boldsymbol{\sigma}(\Omega)} f(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\Omega} f(\boldsymbol{\sigma}(\mathbf{x}))\,|\det \boldsymbol{\sigma}'(\mathbf{x})|\,\mathrm{d}\mathbf{x}.$$

Lemma 1.25. Let A be an $n \times n$ matrix, then the characteristic polynomial of A satisfies

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) \\ &= \lambda^n - \mathrm{trace}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A). \end{aligned} \tag{1.8}$$

Lemma 1.26. Let $J(\mathbf{x}, t)$ be the Jacobian determinant of $\boldsymbol{\varphi}_t$, then

$$\frac{\partial}{\partial t} J(\mathbf{x}, t) = (\nabla \cdot \mathbf{u})\,(\boldsymbol{\varphi}(\mathbf{x}, t), t)\,J(\mathbf{x}, t). \tag{1.9}$$

Proof. Applying the Taylor expansion of $\boldsymbol{\varphi}(\mathbf{x}, k)$ and the definition of the velocity yields

$$\boldsymbol{\varphi}(\mathbf{x}, k) = \mathbf{x} + k\mathbf{u}(\mathbf{x}, 0) + O(k^2) \text{ as } k \to 0,$$

which, in component form, is

$$\varphi_i(\mathbf{x}, k) = x_i + k u_i(\mathbf{x}, 0) + O(k^2) \text{ as } k \to 0.$$

Differentiating with respect to x_j gives the ij-th element of the Jacobian matrix,

$$\partial_j \varphi_i(\mathbf{x}, k) = \delta_{ij} + k\frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) + O(k^2) = \delta_{ij} + kA_{ij},$$

where

$$A_{ij} = \frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) + O(k) \text{ as } k \to 0. \tag{1.10}$$

Hence, the Jacobian determinant is

$$\begin{aligned} J(\mathbf{x}, k) &= \det(I + kA) = (-k)^3 p_A\left(-\frac{1}{k}\right) \\ &= 1 + k \operatorname{trace}(A) + \cdots + k^3 \det(A) \\ &= 1 + k\frac{\partial u_i}{\partial x_i}(\mathbf{x}, 0) + O(k^2) \\ &= 1 + k(\nabla \cdot \mathbf{u})(\mathbf{x}, 0) + O(k^2), \end{aligned}$$

where the third equality follows from (1.8) and the fourth from (1.10) and we have used the Einstein summation convention, this further yields

$$\frac{\mathrm{d}J}{\mathrm{d}t}(\mathbf{x}, 0) = (\nabla \cdot \mathbf{u})(\mathbf{x}, 0). \tag{1.11}$$

The volume $V(t)$ of a system W_t satisfies

$$V(t) = \int_{W_t} 1\mathrm{d}\mathbf{x}' = \int_{W_0} J(\mathbf{x}, t)\mathrm{d}\mathbf{x}, \tag{1.12}$$

which follows from the change-of-variables formula. Differentiate (1.12) with respect to t, set $t = 0$, apply (1.11), and we have

$$\frac{\mathrm{d}V(0)}{\mathrm{d}t} = \int_{W_0} (\nabla \cdot \mathbf{u})(\mathbf{x}, 0)\mathrm{d}\mathbf{x},$$

this result is invariant under translation of the origin of time, i.e.,

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \int_{W_t} (\nabla \cdot \mathbf{u})(\mathbf{x}', t)\mathrm{d}\mathbf{x}' \\ &= \int_{W_0} (\nabla \cdot \mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t)J(\mathbf{x}, t)\mathrm{d}\mathbf{x}, \end{aligned}$$

where the second step follows from the change-of-variables formula. On the other hand, differentiating (1.12) with respect to t yields

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \int_{W_0} \frac{\partial J}{\partial t}(\mathbf{x}, t)\mathrm{d}\mathbf{x}.$$

Since the last two equations are both true for arbitrary W_0, we conclude that

$$\frac{\partial J}{\partial t}(\mathbf{x}, t) = (\nabla \cdot \mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t)J(\mathbf{x}, t). \qquad \square$$

Example 1.27 (An alternative proof of Lemma 1.26)**.** Write the components of $\boldsymbol{\varphi}$ as $\xi(\mathbf{x}, t), \eta(\mathbf{x}, t)$, and $\zeta(\mathbf{x}, t)$. First, observe that

$$\frac{\partial}{\partial t}\boldsymbol{\varphi}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\varphi}(\mathbf{x}, t), t),$$

by definition of the velocity field of the fluid.

The determinant $J(\mathbf{x}, t)$ can be differentiated by recalling that the determinant of a matrix is multilinear in the rows. Thus, holding $\mathbf{x}$ fixed throughout, we have

$$\begin{aligned} \frac{\partial}{\partial t}J = &\begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{vmatrix} + \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial}{\partial t}\frac{\partial \eta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{vmatrix} \\ &+ \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} \end{vmatrix}. \end{aligned}$$

Now write

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}u(\boldsymbol{\varphi}(\mathbf{x}, t), t), \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} &= \frac{\partial}{\partial y}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial y}u(\boldsymbol{\varphi}(\mathbf{x}, t), t), \\ &\vdots \\ \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} &= \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z}w(\boldsymbol{\varphi}(\mathbf{x}, t), t). \end{aligned}$$

The components u, v, and w of $\mathbf{u}$ in this expression are functions of x, y, and z through $\boldsymbol{\varphi}(\mathbf{x}, t)$; therefore,

$$\begin{aligned} \frac{\partial}{\partial x}u(\boldsymbol{\varphi}(\mathbf{x}, t), t) &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x}, \\ &\vdots \\ \frac{\partial}{\partial z}w(\boldsymbol{\varphi}(\mathbf{x}, t), t) &= \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial z}. \end{aligned}$$

When these are substituted into the above expression for $\partial J/\partial t$, one gets for the respective terms

$$\frac{\partial u}{\partial x}J + \frac{\partial v}{\partial y}J + \frac{\partial w}{\partial z}J = (\nabla \cdot \mathbf{u})J.$$

Lemma 1.28.

$$\nabla \cdot (f\mathbf{F}) = \mathbf{F} \cdot \nabla f + f\nabla \cdot \mathbf{F}. \tag{1.13}$$

Proof.

$$\nabla \cdot (f\mathbf{F}) = \frac{\partial (fF_i)}{\partial x_i} = F_i\frac{\partial f}{\partial x_i} + f\frac{\partial F_i}{\partial x_i} = \mathbf{F} \cdot \nabla f + f\nabla \cdot \mathbf{F}. \ \square$$

Theorem 1.29. Let $\mathbf{u}$ be the velocity field of a fluid flow, W_t a system, and $f(\mathbf{x}, t)$ a scalar function of position $\mathbf{x}$ and time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} f \mathrm{d}V = \int_{W_t}\left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\right)\mathrm{d}V. \tag{1.14}$$

Proof.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} f(\mathbf{x},t)\mathrm{d}V \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_0} f(\boldsymbol{\varphi}(\mathbf{x},t),t)J(\mathbf{x},t)\mathrm{d}V \\
=&\int_{W_0}\frac{\mathrm{d}}{\mathrm{d}t}\left(f(\boldsymbol{\varphi}(\mathbf{x},t),t)\right)J(\mathbf{x},t)\mathrm{d}V \\
&+\int_{W_0} f(\boldsymbol{\varphi}(\mathbf{x},t),t)\frac{\partial J}{\partial t}(\mathbf{x},t)\mathrm{d}V \\
=&\int_{W_0}\frac{Df}{Dt}(\boldsymbol{\varphi}(\mathbf{x},t),t)J(\mathbf{x},t)\mathrm{d}V \\
&+\int_{W_0} f(\boldsymbol{\varphi}(\mathbf{x},t),t)(\nabla\cdot\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x},t),t)J(\mathbf{x},t)\mathrm{d}V \\
=&\int_{W_t}\frac{Df}{Dt}\mathrm{d}V + \int_{W_t} f\nabla\cdot\mathbf{u}\mathrm{d}V \\
=&\int_{W_t}\left(\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f + f\nabla\cdot\mathbf{u}\right)\mathrm{d}V \\
=&\int_{W_t}\left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\right)\mathrm{d}V,
\end{aligned}$$

where the first and fourth step follow from the change-of-variables formula, the third step from Exercise 1.22 and Lemma 1.26 and the last from Lemma 1.28. □

Theorem 1.30 (Reynolds transport theorem)**.**

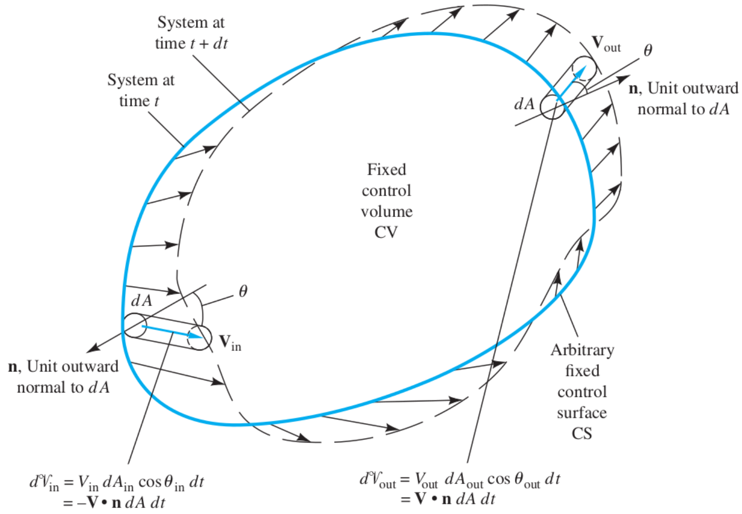
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} f(\mathbf{x},t)\mathrm{d}V = \int_{W_t}\frac{\partial f}{\partial t}(\mathbf{x},t)\mathrm{d}V + \int_{\partial W_t} f\mathbf{u}\cdot\mathbf{n}\mathrm{d}A. \tag{1.15}$$

Proof. Theorem 1.29 and the divergence theorem. □

Definition 1.31. A *control volume* is a *fixed* volume in space through which fluid may flow, and a *control surface* is the surface of a control volume.

Notation 4. In the following, we shall use W to denote a control volume.

Example 1.32. The following figure shows a fixed control volume with an arbitrary flow pattern passing through.



Examining the figure, we see three sources of changes relating to the control volume:

(i) A change within the control volume

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{CV}} f\mathrm{d}V\right) = \int_{\mathrm{CV}}\frac{\partial f}{\partial t}\mathrm{d}V,$$

(ii) Outflow from the control volume

$$\int_{\mathrm{CS}} fV\cos\theta\mathrm{d}A_{\mathrm{out}},$$

(iii) Inflow to the control volume

$$\int_{\mathrm{CS}} fV\cos\theta\mathrm{d}A_{\mathrm{in}},$$

The notations CV and CS refer to the control volume and control surface, respectively. Therefore in the limit as $dt \to 0$, the instantaneous change of f in the system is the sum of the change within, plus the outflow, minus the inflow:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{SYS}} f\mathrm{d}V = \int_{\mathrm{CV}}\frac{\partial f}{\partial t}\mathrm{d}V + \int_{\mathrm{CS}} fV\cos\theta\mathrm{d}A_{\mathrm{out}} \\
-\int_{\mathrm{CS}} fV\cos\theta\mathrm{d}A_{\mathrm{in}}.
\end{aligned}$$

Theorem 1.33 (Transport theorem)**.** Let f be a (scalar or vector) function of position $\mathbf{x}$ and time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho f\mathrm{d}V = \int_{W_t}\rho\frac{Df}{Dt}\mathrm{d}V. \tag{1.16}$$

Exercise 1.34. Prove Theorem 1.33.

1.4 Incompressibility

Definition 1.35. A flow is *incompressible* if it's volume preserving, i.e., for any system W_t,

$$\mathrm{volume}(W_t) = \int_{W_t}\mathrm{d}V = \text{ constant in } t, \tag{1.17}$$

where $\mathrm{d}V$ is the volume element in space.

Example 1.36. The flow of liquids is typically incompressible.

Example 1.37. Gas flows can often be approximated as incompressible if the *Mach number* Ma is less than 0.3, which is defined as

$$\mathrm{Ma} = \frac{V}{c} = \frac{\text{Speed of flow}}{\text{Speed of sound}}.$$

Thus, when analyzing rockets, spacecraft, and other systems that involve high-speed gas flows, compressibility effects should never be neglected.

Proposition 1.38. The following statements are equivalent:

(i) a fluid is incompressible;

(ii) $J \equiv 1$;

(iii) $\nabla \cdot \mathbf{u} = 0$.

Proof. (i)⇒(ii):

$$\begin{aligned}\int_{W_0} (J-1)\mathrm{d}V &= \int_{W_0} J\mathrm{d}V - \int_{W_0} \mathrm{d}V \\ &= \int_{W_t} \mathrm{d}V - \int_{W_0} \mathrm{d}V = 0,\end{aligned}$$

by the incompressibility condition (1.17), and thus $J \equiv 1$ since W_0 is arbitrary.

(ii)⇒(iii): By Lemma 1.26, we have

$$0 = \frac{\partial}{\partial t} J(\mathbf{x}, t) = J(\mathbf{x}, t)(\nabla\cdot\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t) = (\nabla\cdot\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t).$$

(iii)⇒(i):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} \mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{W_0} J\mathrm{d}V = \int_{W_0} \frac{\partial J}{\partial t}\mathrm{d}V \\ &= \int_{W_0} (\nabla\cdot\mathbf{u})J\mathrm{d}V = \int_{W_t} (\nabla\cdot\mathbf{u})\mathrm{d}V = 0,\end{aligned}$$

where the first equality follows from the change-of-variables formula and the third from Lemma 1.26, and hence volume(W_t) is constant in t. □

1.5 Conservation of Mass

Proposition 1.39. The mass of fluid in a region W at time t is

$$m(W, t) = \int_W \rho(\mathbf{x}, t)\mathrm{d}V, \tag{1.18}$$

where $\mathrm{d}V$ is the area element in the plane or the volume element in space.

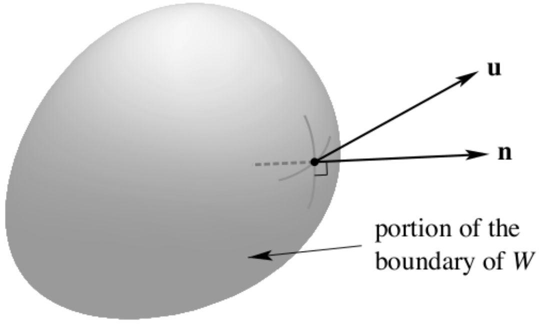
Principle 1.40 (Conservation of mass)**.** Mass is neither created nor destroyed.

Theorem 1.41. From the law of conservation of mass, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} \rho(\mathbf{x}, t)\mathrm{d}V = 0. \tag{1.19}$$

Theorem 1.42. The law of conservation of mass can be more precisely stated as follows: the rate of increase of mass in a region W equals the rate at which mass is crossing the boundary ∂W in the inward direction, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} m(W, t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathrm{d}V = -\int_{\partial W} \rho\mathbf{u}\cdot\mathbf{n}\mathrm{d}A. \tag{1.20}$$



Theorem 1.43. The integral form of the law of conservation of mass is

$$\int_W \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right)\mathrm{d}V = 0. \tag{1.21}$$

Proof.

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathrm{d}V + \int_{\partial W} \rho\mathbf{u}\cdot\mathbf{n}\mathrm{d}A \\ &= \int_W \frac{\partial\rho}{\partial t}\mathrm{d}V + \int_W \nabla\cdot(\rho\mathbf{u})\mathrm{d}V \\ &= \int_W \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right)\mathrm{d}V,\end{aligned}$$

where we have used (1.20) in obtaining the first equality and the second equality follows from the divergence theorem. □

Theorem 1.44. The differential form of the law of conservation of mass is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.22}$$

which is known as the *continuity equation.*

Proof. This follows directly from (1.21) since W is arbitrary. □

Exercise 1.45. Deduce the differential form of the law of conservation of mass from Theorem 1.41 using the Reynolds transport theorem.

Corollary 1.46. The continuity equation (1.22) is equivalent to

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0. \tag{1.23}$$

Proof.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u},$$

where the first equality follows from Lemma 1.28 and the second from Definition 1.21. □

Corollary 1.47. A fluid is incompressible iff

$$\frac{D\rho}{Dt} = 0, \tag{1.24}$$

that is, the mass density is constant following the fluid.

Proof. Corollary 1.46 and Proposition 1.38. □

Definition 1.48. A fluid is *homogeneous* if its density ρ is constant in space at any given time.

Corollary 1.49. A homogeneous flow is incompressible iff its density ρ is constant in time.

Proof. The density ρ of a homogeneous flow satisfies $\nabla\rho = \mathbf{0}$, and hence by the definition of the material derivative and Corollary 1.47, we have

$$\frac{\partial\rho}{\partial t} = \frac{D\rho}{Dt} - (\mathbf{u}\cdot\nabla)\rho = 0.$$ □

Theorem 1.50. Another form of mass conservation:

$$\rho(\boldsymbol{\varphi}(\mathbf{x},t),t)J(\mathbf{x},t) = \rho(\mathbf{x},0). \tag{1.25}$$

Proof.

$$\int_{W_0}\rho(\mathbf{x},0)\mathrm{d}V = \int_{W_t}\rho(\mathbf{x},t)\mathrm{d}V = \int_{W_0}\rho(\boldsymbol{\varphi}(\mathbf{x},t),t)J(\mathbf{x},t)\mathrm{d}V,$$

where the first equality is the law of conservation of mass and the second follows from the change-of-variables formula. Now the proof is completed since W_0 is arbitrary. □

Corollary 1.51. An incompressible fluid that is homogeneous at $t = 0$ will remain homogeneous.

Proof. Theorem 1.50 and Proposition 1.38. □

Example 1.52. The converse is not true. Consider

$$\boldsymbol{\varphi}((x,y,z),t) = ((1+t)x, y, z),$$

where

$$J((x,y,z),t) = 1+t,$$

so the flow is not incompressible, yet for

$$\rho((x,y,z),t) = \frac{1}{1+t},$$

one has mass conservation and homogeneity for all time.

1.6 Euler's Equations

1.6.1 Conservation of Momentum

Proposition 1.53. The momentum of fluid in a region W at time t is

$$\int_W \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\mathrm{d}V. \tag{1.26}$$

Principle 1.54 (Conservation of momentum)**.** The rate of change of momentum of a portion of the fluid equals the force applied to it, which is also known as Newton's second law (force = mass × acceleration).

Example 1.55. For any continuum, forces acting on a piece of material are of two types:

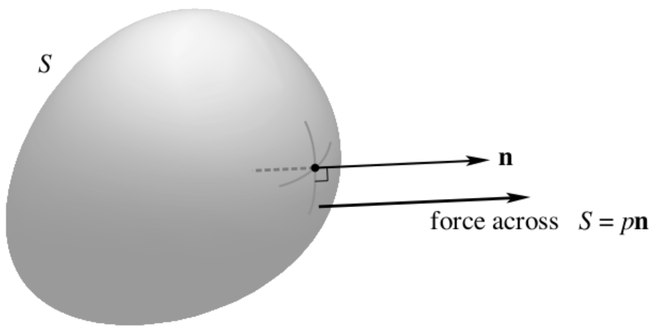
1. forces of *stress*, whereby the piece of material is acted on by forces across its surface by the rest of the continuum;

2. *external*, or *body*, forces such as gravity or magnetic field, which exert a force per unit volume on the continuum.

Definition 1.56. An *ideal fluid* is a fluid with the following property: for any motion of the fluid, there is a function $p(\mathbf{x},t)$ called the *pressure* such that if S is a surface in the fluid with a chosen unit normal $\mathbf{n}$, the force of stress exerted across the surface S per unit area at $\mathbf{x}\in S$ at time t is $p(\mathbf{x},t)\mathbf{n}$, i.e.,

$$\text{force across } S \text{ per unit area} = p(\mathbf{x},t)\mathbf{n}.$$

Note that the force is in the direction $\mathbf{n}$ and that the force acts orthogonally to the surface S, that is, there is no tangential forces.



Assumption 1.57. In this and the next subsection, the fluids that we consider are assumed to be ideal.

Lemma 1.58. The total force exerted on the fluid inside W by means of stress on its boundary is

$$\mathbf{S}_{\partial W} = -\int_W \nabla p\mathrm{d}V. \tag{1.27}$$

Proof. By Definition 1.56, we have

$$\mathbf{S}_{\partial W} = -\int_{\partial W} p\mathbf{n}\mathrm{d}A.$$

Let $\mathbf{e}$ be a *fixed* vector in space, then

$$\begin{aligned}\mathbf{e}\cdot\mathbf{S}_{\partial W} &= -\int_{\partial W} p\mathbf{e}\cdot\mathbf{n}\mathrm{d}A = -\int_W \nabla\cdot(p\mathbf{e})\mathrm{d}V \\ &= -\int_W \mathbf{e}\cdot\nabla p\mathrm{d}V = -\mathbf{e}\cdot\int_W \nabla p\mathrm{d}V,\end{aligned}$$

where the second step follows from the divergence theorem and the third from Lemma 1.28. The proof is completed since $\mathbf{e}$ is arbitrary. □

Proposition 1.59. Let $\mathbf{b}(\mathbf{x},t)$ denote the given body force per unit mass, then the total body force is

$$\mathbf{B}_W = \int_W \rho\mathbf{b}\mathrm{d}V. \tag{1.28}$$

Proposition 1.60. On any piece of fluid material,

$$\text{force per unit volume } = -\nabla p + \rho \mathbf{b}. \tag{1.29}$$

Proof. Lemma 1.58 and Proposition 1.59. □

Theorem 1.61. The differential form of the law of conservation of momentum is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b}. \tag{1.30}$$

Proof. Newton's second law, Propositions 1.19 and 1.60. □

Theorem 1.62. The integral form of the law of conservation of momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho \mathbf{u} \mathrm{d}V = -\int_{\partial W} (p\mathbf{n} + \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n}))\,\mathrm{d}A + \int_W \rho \mathbf{b} \mathrm{d}V, \tag{1.31}$$

where the quantity $p\mathbf{u} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})$ is the *momentum flux per unit area crossing ∂W*.

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}(\rho \mathbf{u}) &= \frac{\partial \rho}{\partial t}\mathbf{u} + \rho \frac{\partial \mathbf{u}}{\partial t} \\ &= -\nabla\cdot(\rho\mathbf{u})\mathbf{u} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \rho\mathbf{b}, \end{aligned}$$

where the second step follows from the continuity equation (1.22) and (1.30). Let $\mathbf{e}$ be a *fixed* vector in space, then

$$\begin{aligned} &\mathbf{e}\cdot\frac{\partial}{\partial t}(\rho\mathbf{u}) \\ =& -\nabla\cdot(\rho\mathbf{u})(\mathbf{u}\cdot\mathbf{e}) - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{e} - (\nabla p)\cdot\mathbf{e} + \rho\mathbf{b}\cdot\mathbf{e} \\ =& -\nabla\cdot(p\mathbf{e} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})) + \rho\mathbf{b}\cdot\mathbf{e}, \end{aligned}$$

where the second equality follows from Lemma 1.28. Therefore, the rate of change of momentum in direction $\mathbf{e}$ in W is

$$\begin{aligned} &\mathbf{e}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathbf{u}\mathrm{d}V = \int_W \mathbf{e}\cdot\frac{\partial}{\partial t}(\rho\mathbf{u})\mathrm{d}V \\ =& -\int_W \nabla\cdot(p\mathbf{e} + \rho\mathbf{u}(\mathbf{e}\cdot\mathbf{u}))\cdot\mathbf{n}\mathrm{d}V + \int_W \rho\mathbf{b}\cdot\mathbf{e}\mathrm{d}V \\ =& -\int_{\partial W}(p\mathbf{e}\cdot\mathbf{n} + \rho\mathbf{e}\cdot\mathbf{u}(\mathbf{u}\cdot\mathbf{n}))\mathrm{d}A + \int_W \rho\mathbf{e}\cdot\mathbf{b}\mathrm{d}V, \end{aligned}$$

where we have used the divergence theorem in obtaining the last equality. The proof is completed since $\mathbf{e}$ is arbitrary. □

Example 1.63. The "primitive" integral form of balance of momentum states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho\mathbf{u}\mathrm{d}V = \mathbf{S}_{\partial W_t} + \int_{W_t}\rho\mathbf{b}\mathrm{d}V, \tag{1.32}$$

that is, the rate of change of momentum of a moving piece of fluid equals the total force (surface stresses plus body forces) acting on it.

Exercise 1.64. Derive the differential form of the law of conservation of momentum from (1.32).

1.6.2 Conservation of Energy

Principle 1.65 (Conservation of energy)**.** Energy is neither created nor destroyed.

Definition 1.66. The *kinetic energy* contained in W is

$$E_{\text{kinetic}} = \frac{1}{2}\int_W \rho\|\mathbf{u}\|^2\mathrm{d}V. \tag{1.33}$$

Assumption 1.67. Assume the total energy of the fluid can be written as

$$E_{\text{total}} = E_{\text{kinetic}} + E_{\text{internal}}, \tag{1.34}$$

where E_{internal} is the *internal energy*, which is energy we cannot "see" on a macroscopic scale, and derives from sources such as intermolecular potentials and internal molecular vibrations. If energy is pumped into the fluid or if we allow the fluid to do work, E_{total} will change.

Lemma 1.68.

$$\frac{1}{2}\frac{D}{Dt}\|\mathbf{u}\|^2 = \mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}.$$

Proof.

$$\begin{aligned} \frac{D}{Dt}\|\mathbf{u}\|^2 &= \left(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}\right)(u_j u_j) \\ &= 2u_j\frac{\partial u_j}{\partial t} + 2u_j u_i\frac{\partial u_j}{\partial x_i} \\ &= 2u_j\left(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}\right)u_j \\ &= 2\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}. \end{aligned}$$

□

Proposition 1.69. The rate of change of kinetic energy of a moving portion W_t of fluid is

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{kinetic}} = \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V. \tag{1.35}$$

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_{\text{kinetic}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{W_t}\rho\|\mathbf{u}\|^2\mathrm{d}V\right) \\ &= \frac{1}{2}\int_{W_t}\rho\frac{D\|\mathbf{u}\|^2}{Dt}\mathrm{d}V \\ &= \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V, \end{aligned}$$

where the second step follows from the transport theorem and the last from Lemma 1.68. □

Theorem 1.70. Assume the fluid is incompressible and that all the energy is kinetic, then the integral form of the law of conservation of energy is

$$\int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V = \int_{W_t}\mathbf{u}\cdot(-\nabla p + \rho\mathbf{b})\mathrm{d}V. \tag{1.36}$$

Proof. The principle of conservation of energy can be more precisely stated as: the rate of change of energy in a portion of the fluid equals the rate at which the pressure and body forces do work, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{total}} = -\int_{\partial W_t} p\mathbf{u}\cdot\mathbf{n}\mathrm{d}A + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b}\mathrm{d}V, \tag{1.37}$$

next we compute the right-hand side as

$$\begin{aligned}\text{RHS} &= -\int_{W_t}\left(\nabla\cdot(p\mathbf{u}) - \rho\mathbf{u}\cdot\mathbf{b}\right)\mathrm{d}V\\ &= -\int_{W_t}\left(\mathbf{u}\cdot\nabla p + p\nabla\cdot\mathbf{u} - \rho\mathbf{u}\cdot\mathbf{b}\right)\mathrm{d}V\\ &= -\int_{W_t}\left(\mathbf{u}\cdot\nabla p - \rho\mathbf{u}\cdot\mathbf{b}\right)\mathrm{d}V,\end{aligned}$$

where the first equality follows from the divergence theorem, the second from Lemma 1.28 and the last from the incompressibility and Proposition 1.38. Combining with (1.35) and that $E_{\text{total}} = E_{\text{kinetic}}$ gives the desired result. ☐

Example 1.71. Note that the integral form of the law of conservation of energy (1.36) for an incompressible fluid where all energy is kinetic is also a consequence of conservation of momentum (1.30). If we assume $E_{\text{total}} = E_{\text{kinetic}}$ and $p \neq 0$, then the fluid must be incompressible.

Theorem 1.72. The *Euler's equations for an incompressible fluid* are

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b}, \tag{1.38a}$$

$$\frac{D\rho}{Dt} = 0, \tag{1.38b}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.38c}$$

with no-penetration boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial D. \tag{1.39}$$

Proof. Theorem 1.61, Corollary 1.47 and Proposition 1.38. ☐

Definition 1.73. A compressible flow is *isentropic* if there is a function w, called the *enthalpy*, such that

$$\nabla w = \frac{1}{\rho}\nabla p. \tag{1.40}$$

Notation 5. In thermodynamics we have the following basic quantities, each of which is a function of position $\mathbf{x}$ and time t depending on a given flow:

$$\begin{aligned} p &= \text{ pressure}\\ \rho &= \text{ density}\\ T &= \text{ temperature}\\ s &= \text{ entropy}\\ w &= \text{ enthalpy (per unit mass)}\\ \epsilon = w - p/\rho &= \text{ internal energy (per unit mass)}\end{aligned}$$

Principle 1.74 (First law of thermodynamics)**.**

$$\mathrm{d}w = T\mathrm{d}s + \frac{1}{\rho}\mathrm{d}p, \tag{1.41}$$

or equivalently,

$$\mathrm{d}\epsilon = T\mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho. \tag{1.42}$$

Proposition 1.75. If the flow is isentropic and the pressure p is a function of ρ only, then the entropy s is a constant and

$$w = \int^{\rho}\frac{p'(\lambda)}{\lambda}\mathrm{d}\lambda, \quad \epsilon = \int^{\rho}\frac{p(\lambda)}{\lambda^2}\mathrm{d}\lambda.$$

Proof. Definition 1.73, Principle 1.74, and the fundamental theorem of calculus. ☐

Lemma 1.76. If the fluid is isentropic and the pressure p is a function of ρ only, then

$$\rho\frac{D\epsilon}{Dt} = -p\nabla\cdot\mathbf{u}. \tag{1.43}$$

Proof.

$$\begin{aligned}\rho\frac{D\epsilon}{Dt} &= \rho\left(\frac{\partial\epsilon}{\partial t} + u_i\frac{\partial\epsilon}{\partial x_i}\right)\\ &= \rho\left(\frac{\partial\epsilon}{\partial\rho}\frac{\partial\rho}{\partial t} + u_i\frac{\partial\epsilon}{\partial\rho}\frac{\partial\rho}{\partial x_i}\right)\\ &= \frac{p}{\rho}\frac{D\rho}{Dt} = -p\nabla\cdot\mathbf{u},\end{aligned}$$

where the third equality follows from (1.42) and the last from (1.23). ☐

Theorem 1.77. Assume the fluid is isentropic and that the pressure p is a function of ρ only, then the integral form of the law of conservation of energy is

$$\int_{W_t}\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V = \int_{W_t}\mathbf{u}\cdot(-\nabla w + \mathbf{b})\mathrm{d}V. \tag{1.44}$$

Proof. The principle of conservation of energy can be more precisely stated as: the rate of change of energy in a portion of the fluid equals the rate which work is done on it, i.e.,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{total}} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\left(\frac{1}{2}\rho\|\mathbf{u}\|^2 + \rho\epsilon\right)\mathrm{d}V\\ &= -\int_{\partial W_t} p\mathbf{u}\cdot\mathbf{n}\mathrm{d}A + \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}\mathrm{d}V \qquad (1.45)\\ &= -\int_{W_t}\nabla\cdot(p\mathbf{u})\mathrm{d}V + \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}\mathrm{d}V,\end{aligned}$$

where the third step follows from the divergence theorem. Next we compute

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\left(\frac{1}{2}\rho\|\mathbf{u}\|^2 + \rho\epsilon\right)\mathrm{d}V\\ &= \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V + \int_{W_t}\rho\frac{D\epsilon}{Dt}\mathrm{d}V\\ &= \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V - \int_{W_t}p\left(\nabla\cdot\mathbf{u}\right)\mathrm{d}V\\ &= \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V - \int_{W_t}\nabla\cdot(p\mathbf{u})\mathrm{d}V + \int_{W_t}\mathbf{u}\cdot(\nabla p)\mathrm{d}V\\ &= \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V - \int_{W_t}\nabla\cdot(p\mathbf{u})\mathrm{d}V + \int_{W_t}\rho\mathbf{u}\cdot(\nabla w)\mathrm{d}V,\end{aligned}$$

where the first step follows from (1.35) and the transport theorem, the second from (1.43), the third from Lemma 1.28, and the last from (1.40). Combining the above two equations completes the proof. □

Theorem 1.78. The *Euler's equations for isentropic flows* are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla w + \mathbf{b}, \tag{1.46a}$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.46b}$$

with no-penetration boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial D, \tag{1.47}$$

or

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{V}\cdot\mathbf{n} \text{ if } \partial D \text{ is moving with velocity } \mathbf{V}. \tag{1.48}$$

Proof. Theorems 1.77 and 1.44. □

Example 1.79. Gases can often be viewed as isentropic with

$$p = A\rho^\gamma,$$

where ρ and γ are constants and $\gamma \geq 1$. Here

$$w = \int^\rho \frac{\gamma A s^{\gamma-1}}{s}\mathrm{d}s = \frac{\gamma A\rho^{\gamma-1}}{\gamma-1} \text{ and } \epsilon = \frac{A\rho^{\gamma-1}}{\gamma-1}.$$

Lemma 1.80.

$$\frac{1}{2}\nabla\left(\|\mathbf{u}\|^2\right) = (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}).$$

Proof. The ℓ-th component of the RHS is

$$\begin{aligned}
&((\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}))_\ell \\
=& u_m\frac{\partial u_\ell}{\partial x_m} + \epsilon_{\ell mn}u_m(\nabla\times\mathbf{u})_n \\
=& u_m\frac{\partial u_\ell}{\partial x_m} + \epsilon_{\ell mn}u_m\epsilon_{npq}\frac{\partial u_q}{\partial x_p} \\
=& u_m\frac{\partial u_\ell}{\partial x_m} + \epsilon_{\ell mn}\epsilon_{npq}u_m\frac{\partial u_q}{\partial x_p} \\
=& u_m\frac{\partial u_\ell}{\partial x_m} + (\delta_{\ell p}\delta_{mq} - \delta_{\ell q}\delta_{mp})u_m\frac{\partial u_q}{\partial x_p} \\
=& u_m\frac{\partial u_\ell}{\partial x_m} + u_m\frac{\partial u_m}{\partial x_\ell} - u_m\frac{\partial u_\ell}{\partial x_m} \\
=& u_m\frac{\partial u_m}{\partial x_\ell} = \left(\frac{1}{2}\nabla\left(\|\mathbf{u}\|^2\right)\right)_\ell.
\end{aligned}$$

□

Theorem 1.81 (Bernoulli's theorem)**.** In stationary homogeneous incompressible flows and in the absence of external forces, the quantity

$$\frac{1}{2}\|\mathbf{u}\|^2 + p/\rho \tag{1.49}$$

is constant along streamlines. The same conclusion remains true if a force $\mathbf{b}$ is present and is conservative, i.e.,

$$\mathbf{b} = -\nabla\varphi \text{ for some function } \varphi,$$

with p/ρ replaced by $p/\rho + \varphi$.

Proof. Since the flow is stationary and there is no external force, we have

$$\begin{aligned}
\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho}\right) &= (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) + \frac{\nabla p}{\rho} \\
&= \mathbf{u}\times(\nabla\times\mathbf{u}),
\end{aligned}$$

where the first equality follows from Lemma 1.80 and Corollary 1.49 and the second from (1.38a). Let $\mathbf{x}(s)$ be a streamline, then

$$\begin{aligned}
\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho}\right)\Bigg|_{s_1}^{s_2} &= \int_{s_1}^{s_2}\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho}\right)\cdot\mathbf{x}'(s)\mathrm{d}s \\
&= \int_{s_1}^{s_2}(\mathbf{u}\times(\nabla\times\mathbf{u}))\cdot\mathbf{x}'(s)\mathrm{d}s \\
&= \int_{s_1}^{s_2}(\mathbf{u}\times(\nabla\times\mathbf{u}))\cdot\mathbf{u}\mathrm{d}s \\
&= 0,
\end{aligned}$$

where the first equality follows from the fundamental theorem of calculus and the chain rule. □

Theorem 1.82. In stationary isentropic flows and in the absence of external forces, the quantity

$$\frac{1}{2}\|\mathbf{u}\|^2 + w \tag{1.50}$$

is constant along streamlines. The same conclusion remains true if a force $\mathbf{b}$ is present and is conservative, i.e.,

$$\mathbf{b} = -\nabla\varphi \text{ for some function } \varphi,$$

with w replaced by $w + \varphi$.

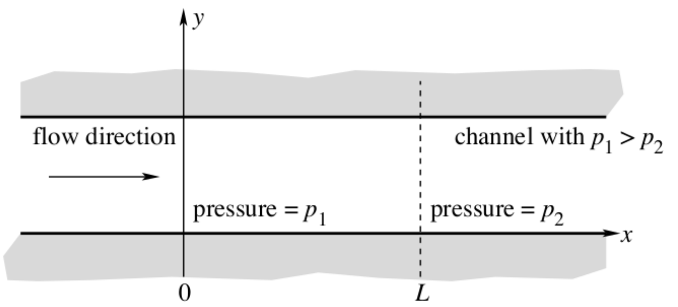
Exercise 1.83. Prove Theorem 1.82.

Exercise 1.84. Consider an isentropic flow without any body forces. Show that for a *fixed* volume W in space (*not* moving with the flow).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_W\left(\frac{1}{2}\rho\|\mathbf{u}\|^2 + \rho\epsilon\right)\mathrm{d}V = -\int_W\rho\left(\frac{1}{2}\|\mathbf{u}\|^2 + w\right)\mathbf{u}\cdot\mathbf{n}\mathrm{d}A.$$

Use this to justify the term *energy flux vector* for the vector function $\rho\mathbf{u}\left(\frac{1}{2}\|\mathbf{u}\|^2 + w\right)$ and compare with Bernoulli's theorem.

Example 1.85. Consider the fluid-filled channel.



Suppose that the pressure p_1 at $x = 0$ is larger than that at $x = L$ so the fluid is pushed from left to right. We seek a solution of Euler's incompressible homogeneous equations in the form

$$\mathbf{u}(x, y, t) = (u(x, t), 0) \text{ and } p(x, y, t) = p(x).$$

Incompressibility implies $\partial_x u = 0$. Thus, Euler's equations become $\rho_0 \partial_t u = -\partial_x p$. This implies that $\partial_x^2 p = 0$, and so a linear interpolation gives

$$p(x) = p_1 - \left(\frac{p_1 - p_2}{L}\right) x.$$

Substitution into $\rho_0 \partial_t u = -\partial_x p$ and integration yields

$$u = \frac{p_1 - p_2}{\rho_0 L} t + \text{ constant}.$$

This solution suggests that the velocity in channel flow with a constant pressure gradient increases indefinitely. Of course, this cannot be the case in a real flow; however, in our modeling, we have not yet taken friction into account. This situation will be remedied in the Navier Stokes equations.

1.7 Rotation and Vorticity

Definition 1.86. If the velocity field of a fluid is $\mathbf{u} = (u, v, w)$, then its curl

$$\boldsymbol{\xi} = \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{1.51}$$

is called the *vorticity field* of the flow.

Lemma 1.87. Let

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \text{ and } \mathbf{S} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix},$$

then

$$\mathbf{Sh} = \boldsymbol{\xi} \times \mathbf{h}, \quad \forall \mathbf{h} \in \mathbb{R}^3.$$

Theorem 1.88. Let $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ be the velocity field of a fluid, then

$$\mathbf{u}(\mathbf{x}+\mathbf{h}) = \mathbf{u}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\mathbf{h} + \frac{1}{2}\boldsymbol{\xi}(\mathbf{x}) \times \mathbf{h} + O(\|\mathbf{h}\|^2) \text{ as } \mathbf{h} \to \mathbf{0}, \tag{1.52}$$

where

$$\mathbf{D}(\mathbf{x}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \tag{1.53}$$

is a symmetric 3×3 matrix called the *deformation tensor* and $\boldsymbol{\xi}$ is the vorticity field of $\mathbf{u}$.

Proof. Let $\nabla \mathbf{u}$ denote the Jacobian matrix of $\mathbf{u}$. By Taylor's expansion, we have

$$\begin{aligned} \mathbf{u}(\mathbf{x}+\mathbf{h}) &= \mathbf{u}(\mathbf{x}) + \nabla \mathbf{u}(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &= \mathbf{u}(\mathbf{x}) + (\mathbf{D}(\mathbf{x}) + \mathbf{S}(\mathbf{x}))\,\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &= \mathbf{u}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\mathbf{h} + \frac{1}{2}\boldsymbol{\xi}(\mathbf{x}) \times \mathbf{h} + O(\|\mathbf{h}\|^2), \end{aligned}$$

where in the second step we have decomposed $\nabla \mathbf{u}(\mathbf{x})$ into a symmetric and an antisymmetric part, i.e.,

$$\mathbf{D}(\mathbf{x}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \text{ and } \mathbf{S}(\mathbf{x}) = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^T\right),$$

and the third step follows from Lemma 1.87. □

Proposition 1.89. If the velocity $\mathbf{u}(\mathbf{x})$ at position $\mathbf{x}$ can be represented as the product of a symmetric matrix $\mathbf{D}$ with $\mathbf{x}$, i.e.,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}) = \mathbf{D}\mathbf{x}, \tag{1.54}$$

then the motion is locally a deformation.

Proof. The symmetry of $\mathbf{D}$ implies that there is an orthonormal basis $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ and $\tilde{\mathbf{e}}_3$ with respect to which $\mathbf{D}$ is diagonal, i.e.,

$$(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)^T \mathbf{D} (\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3) = \operatorname{diag}(d_1, d_2, d_3).$$

Define

$$\tilde{\mathbf{x}} = (\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)^T \mathbf{x},$$

then

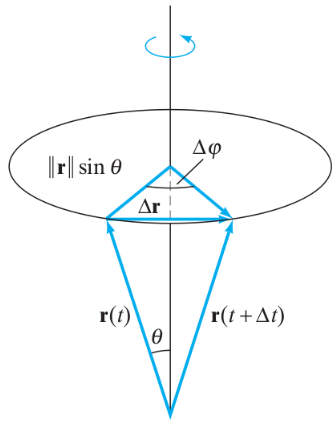
$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \operatorname{diag}(d_1, d_2, d_3)\tilde{\mathbf{x}},$$

which decouples to three linear ODEs

$$\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}t} = d_i \tilde{x}_i, \quad i = 1, 2, 3.$$

Therefore the rate of change of a unit length along the $\tilde{\mathbf{e}}_i$ axis at $t = 0$ is thus d_i, which shows that the vector field $\mathbf{Dx}$ is merely expanding or contracting along each of the axes $\tilde{\mathbf{e}}_i$. □

Example 1.90. Consider a rigid body rotation with angular velocity vector $\boldsymbol{\omega} = \omega \mathbf{n}$, where $\omega = \|\boldsymbol{\omega}\|$ is the angular speed (measured in radians per unit time) and $\mathbf{n}$ is a unit vector representing the axis of rotation.



Then the velocity $\mathbf{u}$ at $\mathbf{r}(t)$ is given by

$$\mathbf{u} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t},$$

when Δt is small, the magnitude of $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ is approximately $\|\mathbf{r}(t)\|\sin\theta\Delta\varphi$. Thus,

$$\begin{aligned}
\|\mathbf{u}\| &= \lim_{\Delta t\to 0}\left\|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right\| \\
&= \lim_{\Delta t\to 0}\|\mathbf{r}(t)\|\sin\theta\frac{\Delta\varphi}{\Delta t} \\
&= \|\mathbf{r}(t)\|\sin\theta\lim_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t} \\
&= \|\mathbf{r}(t)\|\sin\theta\|\boldsymbol{\omega}\| \\
&= \|\boldsymbol{\omega}\times\mathbf{r}\|,
\end{aligned}$$

clearly $\mathbf{u}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, and therefore

$$\mathbf{u} = \boldsymbol{\omega}\times\mathbf{r}.$$