Chapter 1

The Equations of Motion

Reference books:

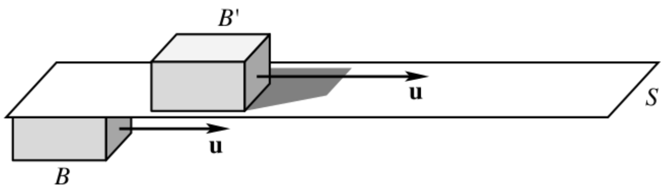
1) *A Mathematical Introduction to Fluid Mechanics* by A. J. Chorin and J. E. Marsden, third edition, Springer, 1993.

2) *Fluid Mechanics* by P. K. Kundu, I. M. Cohen and D. R. Dowling, sixth edition, Academic Press, 2016.

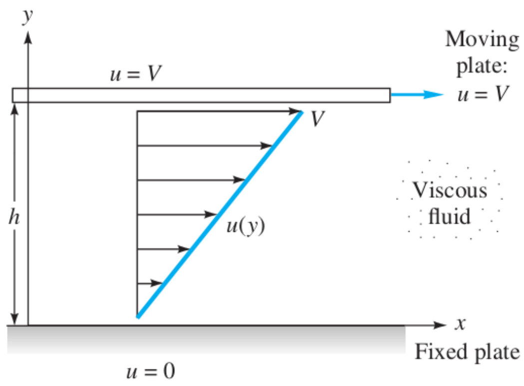
Definition 1.1. *Fluid mechanics* is the science that deals with the behavior of fluids at rest (*fluid statics*) or in motion (*fluid dynamics*), and the interaction of fluids with solids or other fluids at the boundaries.

Definition 1.2. A *fluid* is a substance that deforms continuously under the influence of a shear stress, no matter how small that shear stress may be.

Example 1.3. *Viscosity* is an important property of fluids, which can be thought of as the internal stickiness of a fluid. When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer. Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases.



Example 1.4. The following figure shows the viscous flow induced between a fixed lower plate and an upper plate moving steadily at speed V.



Experiments show that the shear stress τ (shear force per unit area) satisfies *Newton's law of friction*

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y},$$

where μ is the dynamic viscosity.

Assumption 1.5 (Continuum assumption)**.** In explaining macroscopic phenomena of flow problems, the discrete molecular structure of matter may be ignored and replaced by a continuous distribution, called a *continuum*. In a continuum, fluid properties like density ρ, pressure p, velocity $\mathbf{u}$, and temperature T are defined at every point in space, and these properties are known to be appropriate averages of molecular characteristics in a small region surrounding the point of interest.

1.1 Description of Fluid Motion

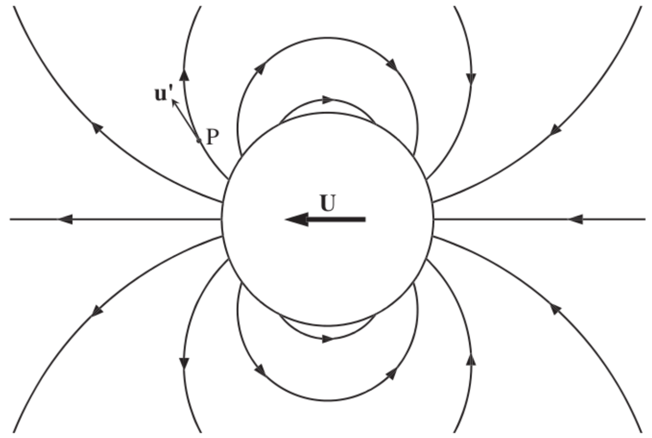
Definition 1.6. In the *Eulerian description* of fluid motions, we observe the physical properties (e.g. velocity) of fluid particles passing each *fixed* point in space. Thus the flow properties are functions of both space and time.

Notation 1. In the following, $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, and $p(\mathbf{x}, t)$ will denote the density, velocity and pressure of the fluid particle that is moving through $\mathbf{x}$ at time t, respectively. For simplicity, we shall assume that these quantities are smooth enough so that standard operations of calculus may be performed on them.

Definition 1.7. A *streamline* $\mathbf{x}(s)$ at a fixed time t passing through $\mathbf{p}$ is a curve that is everywhere tangent to the velocity field $\mathbf{u}(\mathbf{x}, t)$, i.e.,

$$\begin{cases} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}(s), t), & t \text{ fixed}, \\ \mathbf{x}(0) = \mathbf{p}. \end{cases} \tag{1.1}$$

Example 1.8. The following figure shows the streamlines of an unsteady flow of a nominally quiescent ideal incompressible fluid around a moving long circular cylinder with its axis perpendicular to the page. Here the cylinder velocity $\mathbf{U}$ is shown inside the cylinder, and the fluid velocity $\mathbf{u}'$ at point P is caused by the presence of the moving cylinder alone.



Notation 2. In the following, D will denote a region in two- or three-dimensional space filled with fluid particles, and its boundary ∂D is assumed to be smooth.

Theorem 1.9. Suppose the velocity field $\mathbf{u}(\mathbf{x}, t)$ is Lipschitz continuous in space and continuous in time, then the initial value problem

$$\begin{cases} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t), \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{1.2}$$

admits a unique solution $\mathbf{x}(t)$.

Definition 1.10. The *fluid flow map* $\boldsymbol{\varphi} : D \times \mathbb{R} \times \mathbb{R} \to D$ is the map that takes the initial position $\mathbf{x}_0$ of a Lagrangian particle $\mathbf{x}$, the initial time t_0 and the time increment k, and returns $\mathbf{x}(t_0 + k)$, the position of $\mathbf{x}$ at the final time $t_0 + k$:

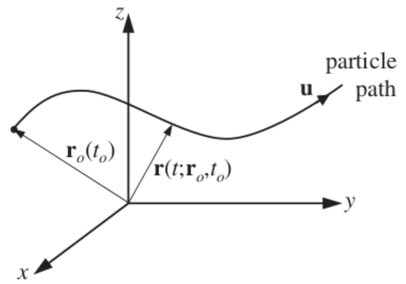
$$\boldsymbol{\varphi}(\mathbf{x}_0, t_0, k) := \mathbf{x}(t_0 + k) = \mathbf{x}_0 + \int_{t_0}^{t_0+k} \mathbf{u}(\mathbf{x}(t), t)\mathrm{d}t, \tag{1.3}$$

With fixed t, let $\boldsymbol{\varphi}_t$ denote the map $\mathbf{x} \mapsto \boldsymbol{\varphi}(\mathbf{x}, 0, t)$, i.e., $\boldsymbol{\varphi}_t$ advances each fluid particle from its position at $t = 0$ to its position at time t.

Definition 1.11. In the *Lagrangian description* of fluid motions, we follow individual fluid particles as they move about and determine how the fluid properties associated with these particles change as a function of time.

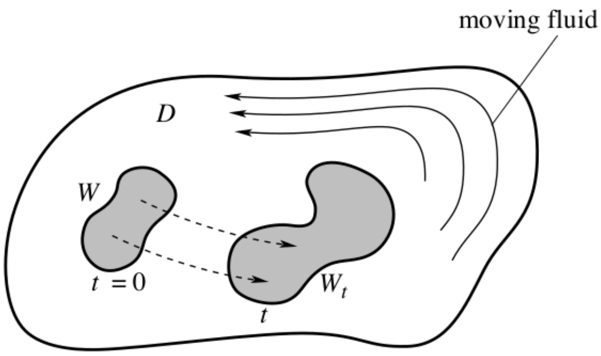
Definition 1.12. A *pathline* (or *trajectory*) of a given fluid particle $\mathbf{x}_0$ at time t_0 is the curve traced out by it in a time interval $(0, k)$, i.e.,

$$\Phi_{t_0}^k(\mathbf{x}_0) = \{\boldsymbol{\varphi}(\mathbf{x}_0, t_0, \tau) : \tau \in (0, k)\}. \tag{1.4}$$



Definition 1.13. A *system* is a collection of matter of fixed identity (always the same fluid particles), which may move, flow, and interact with its surroundings.

Notation 3. In the following, we shall use W_t to denote a system, i.e., $W_t = \boldsymbol{\varphi}_t(W_0)$, where W_0 is the initial position of the system.



Definition 1.14. A *streakline* (or *dye line*) at a particular time t of interest through a fixed point $\mathbf{p}$ in a time interval $(0, k)$ is the curve consisting of all particles in a flow that have previously passed through $\mathbf{p}$, i.e.,

$$\Psi_t^k(\mathbf{p}) := \{\boldsymbol{\varphi}(\mathbf{p}, t - \tau, \tau) : \tau \in (0, k)\}. \tag{1.5}$$

Definition 1.15. A fluid flow is *steady* (or *stationary*) if the velocity $\mathbf{u}$ at any given point in space does not vary with time, i.e., $\partial_t \mathbf{u} = \mathbf{0}$.

Exercise 1.16. Show that if the fluid flow is steady, then the streamlines, pathlines and streaklines coincide.

Example 1.17. Consider a simple plane flow

$$\mathbf{u}(x, y, t) = \left(\frac{x}{1+t}, y\right).$$

The streamline $\mathbf{x}(s)$ at $t = 1$ passing through $(1, 1)$ is given by

$$\begin{cases} x'(s) = \frac{x(s)}{2}, \\ y'(s) = y(s), \\ x(0) = y(0) = 1. \end{cases} \Rightarrow \begin{cases} x(s) = e^{s/2}, \\ y(s) = e^s. \end{cases}$$

Solving the IVP

$$x'(t) = \frac{x(t)}{1+t}, \quad x(t_0) = x_0;$$
$$y'(t) = y(t), \quad y(t_0) = y_0,$$

we obtain

$$x(t) = x_0 \frac{1+t}{1+t_0}, \quad y(t) = y_0 e^{t-t_0}.$$

Therefore the fluid flow map $\boldsymbol{\varphi}$ is

$$\boldsymbol{\varphi}(\mathbf{x}_0, t_0, k) = \left(x_0 \frac{1+t_0+k}{1+t_0}, y_0 e^k \right).$$

(i) Given the fluid particle at position $\mathbf{x}_0 = (1,1)$ at time $t_0 = 0$, its pathline in a time interval $(0,k)$ is given by

$$\begin{aligned}\Phi^k_{t_0}(\mathbf{x}_0) &= \{\boldsymbol{\varphi}(\mathbf{x}_0, t_0, \tau) : \tau \in (0,k)\} \\ &= \{(1+\tau, e^\tau) : \tau \in (0,k)\}.\end{aligned}$$

(ii) The streakline at $t = 1$ through $\mathbf{p} = (1,1)$ in a time interval $(0,k)$ is

$$\begin{aligned}\Psi^k_t(\mathbf{p}) &= \{\boldsymbol{\varphi}(\mathbf{p}, t-\tau, \tau) : \tau \in (0,k)\} \\ &= \left\{ \left(\frac{2}{2-\tau}, e^\tau \right) : \tau \in (0,k) \right\}.\end{aligned}$$

Exercise 1.18. Consider the plane flow

$$\mathbf{u}(x,y,t) = (x(1+2t), y).$$

(i) Find the streamline at $t = 1$ passing through $(1,1)$.

(ii) Given the fluid particle at position $\mathbf{x}_0 = (1,1)$ at time $t_0 = 0$, find its pathline in a time interval $(0,k)$.

(iii) Find the streakline at $t = 1$ through $\mathbf{p} = (1,1)$ in a time interval $(0,k)$.

Notation 4. In the following, we shall assume that the initial time $t_0 = 0$ and omit the dependence of $\boldsymbol{\varphi}$ on t_0, i.e., we simply write $\boldsymbol{\varphi}(\mathbf{x}_0, t)$ instead of $\boldsymbol{\varphi}(\mathbf{x}_0, 0, t)$.

Proposition 1.19. If the velocity field $\mathbf{u}(\mathbf{x}, t)$ is Lipschitz continuous in space and continuous in time, then $\boldsymbol{\varphi}_t$ is injective for each fixed t.

Proof. From the theory of ODEs, we know that the IVP

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\varphi}(\mathbf{x}, t), t), \\ \boldsymbol{\varphi}(\mathbf{x}, 0) = \mathbf{x} \end{cases} \tag{1.6}$$

admits a unique solution $\boldsymbol{\varphi}(\mathbf{x}, t)$. If for some t_0, $\boldsymbol{\varphi}_{t_0}$ is not injective, i.e.,

$$\exists \mathbf{x}_1, \mathbf{x}_2 \in D \text{ s.t. } \mathbf{x}_1 \neq \mathbf{x}_2 \text{ and } \boldsymbol{\varphi}(\mathbf{x}_1, t_0) = \boldsymbol{\varphi}(\mathbf{x}_2, t_0) = \mathbf{x}_0.$$

If we now consider the IVP

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t}(\mathbf{x}_0, t) = -\mathbf{u}(\boldsymbol{\psi}(\mathbf{x}_0, t), t_0 - t), \\ \boldsymbol{\psi}(\mathbf{x}_0, 0) = \mathbf{x}_0, \end{cases} \tag{1.7}$$

then a direct computation shows that $\boldsymbol{\psi}_1(\mathbf{x}_0, t) = \boldsymbol{\varphi}(\mathbf{x}_1, t_0 - t)$ and $\boldsymbol{\psi}_2(\mathbf{x}_0, t) = \boldsymbol{\varphi}(\mathbf{x}_2, t_0 - t)$ are both solutions and $\boldsymbol{\psi}_1(\mathbf{x}_0, t) \neq \boldsymbol{\psi}_2(\mathbf{x}_0, t)$ since $\boldsymbol{\psi}_1(\mathbf{x}_0, t_0) = \mathbf{x}_1 \neq \mathbf{x}_2 = \boldsymbol{\psi}_2(\mathbf{x}_0, t_0)$, this, however, is a contradiction since (1.7) is clearly well-posed (the RHS function $-\mathbf{u}(\mathbf{x}, t_0 - t)$ is Lipschitz continuous in space and continuous in time). ◻

1.2 Material Derivative

Proposition 1.20. Let $\mathbf{x}(t) = (x(t), y(t), z(t))$ be the path followed by a fluid particle, then the velocity of the particle is

$$\mathbf{u}(x(t), y(t), z(t), t) = (x'(t), y'(t), z'(t)), \tag{1.8}$$

i.e.,

$$\mathbf{u}(\mathbf{x}(t), t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t), \tag{1.9}$$

and the acceleration is

$$\mathbf{a}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t) = (\partial_t + \mathbf{u} \cdot \nabla)\, \mathbf{u}. \tag{1.10}$$

Proof.

$$\begin{aligned}\mathbf{a}(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(x(t), y(t), z(t), t) \\ &= \frac{\partial \mathbf{u}}{\partial x}x'(t) + \frac{\partial \mathbf{u}}{\partial y}y'(t) + \frac{\partial \mathbf{u}}{\partial z}z'(t) + \frac{\partial \mathbf{u}}{\partial t} \\ &= (\partial_t + \mathbf{u} \cdot \nabla)\, \mathbf{u},\end{aligned}$$

where the third equality follows from the chain rule and the last from (1.9). ◻

Definition 1.21. The *material derivative* (or *particle derivative*) is

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla = \partial_t + u_i \partial_{x_i}, \tag{1.11}$$

where we have employed the Einstein summation convention.

Exercise 1.22. If $f(\mathbf{x}, t)$ is any (scalar or vector) function of position $\mathbf{x}$ and time t, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t), t) = \frac{Df}{Dt}(\mathbf{x}(t), t).$$

Exercise 1.23. Prove the following properties of the material derivative D/Dt.

(i) Linearity:

$$\frac{D}{Dt}(\alpha f + \beta g) = \alpha \frac{Df}{Dt} + \beta \frac{Dg}{Dt}.$$

(ii) Leibniz or product rule:

$$\frac{D}{Dt}(fg) = f \frac{Dg}{Dt} + g \frac{Df}{Dt}.$$

(iii) Chain rule:

$$\frac{D}{Dt}(g \circ f) = (g' \circ f) \frac{Df}{Dt}.$$

1.3 Reynolds Transport Theorem

Lemma 1.24 (Change-of-variables)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^n$ is differentiable at every point of Ω. If f is a real-valued function defined on $\boldsymbol{\sigma}(\Omega)$, then

$$\int_{\boldsymbol{\sigma}(\Omega)} f(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\Omega} f(\boldsymbol{\sigma}(\mathbf{x}))\left|\det D\boldsymbol{\sigma}(\mathbf{x})\right|\mathrm{d}\mathbf{x}, \tag{1.12}$$

where $D\boldsymbol{\sigma}$ is the Jacobian matrix of $\boldsymbol{\sigma}$.

Lemma 1.25. Let A be an $n \times n$ matrix, then the characteristic polynomial of A satisfies

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) \\ &= \lambda^n - \operatorname{trace}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A). \end{aligned} \tag{1.13}$$

Lemma 1.26. Let $J(\mathbf{x}, t)$ be the Jacobian determinant of $\boldsymbol{\varphi}_t$, then

$$\frac{\partial}{\partial t} J(\mathbf{x}, t) = (\nabla \cdot \mathbf{u})\left(\boldsymbol{\varphi}(\mathbf{x}, t), t\right) J(\mathbf{x}, t). \tag{1.14}$$

Proof. Applying the Taylor expansion of $\boldsymbol{\varphi}(\mathbf{x}, k)$ and the definition of the velocity yields

$$\boldsymbol{\varphi}(\mathbf{x}, k) = \mathbf{x} + k\mathbf{u}(\mathbf{x}, 0) + O(k^2) \text{ as } k \to 0,$$

which, in component form, is

$$\varphi_i(\mathbf{x}, k) = x_i + k u_i(\mathbf{x}, 0) + O(k^2) \text{ as } k \to 0.$$

Differentiating with respect to x_j gives the ij-th element of the Jacobian matrix,

$$\partial_j \varphi_i(\mathbf{x}, k) = \delta_{ij} + k\frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) + O(k^2) = \delta_{ij} + kA_{ij},$$

where

$$A_{ij} = \frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) + O(k) \text{ as } k \to 0. \tag{1.15}$$

Hence, the Jacobian determinant is

$$\begin{aligned} J(\mathbf{x}, k) = \det(I + kA) &= (-k)^3 p_A\left(-\frac{1}{k}\right) \\ &= 1 + k \operatorname{trace}(A) + \cdots + k^3 \det(A) \\ &= 1 + k\frac{\partial u_i}{\partial x_i}(\mathbf{x}, 0) + O(k^2) \\ &= 1 + k(\nabla \cdot \mathbf{u})(\mathbf{x}, 0) + O(k^2), \end{aligned}$$

where the third equality follows from (1.13) and the fourth from (1.15) and we have used the Einstein summation convention, this further yields

$$\frac{\mathrm{d}J}{\mathrm{d}t}(\mathbf{x}, 0) = (\nabla \cdot \mathbf{u})(\mathbf{x}, 0). \tag{1.16}$$

The volume $V(t)$ of a system W_t satisfies

$$V(t) = \int_{W_t} 1\mathrm{d}\mathbf{x}' = \int_{W_0} J(\mathbf{x}, t)\mathrm{d}\mathbf{x}, \tag{1.17}$$

which follows from the change-of-variables formula. Differentiate (1.17) with respect to t, set $t = 0$, apply (1.16), and we have

$$\frac{\mathrm{d}V(0)}{\mathrm{d}t} = \int_{W_0} (\nabla \cdot \mathbf{u})(\mathbf{x}, 0)\mathrm{d}\mathbf{x},$$

this result is invariant under translation of the origin of time, i.e.,

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \int_{W_t} (\nabla \cdot \mathbf{u})(\mathbf{x}', t)\mathrm{d}\mathbf{x}' \\ &= \int_{W_0} (\nabla \cdot \mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t) J(\mathbf{x}, t)\mathrm{d}\mathbf{x}, \end{aligned}$$

where the second step follows from the change-of-variables formula. On the other hand, differentiating (1.17) with respect to t yields

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \int_{W_0} \frac{\partial J}{\partial t}(\mathbf{x}, t)\mathrm{d}\mathbf{x}.$$

Since the last two equations are both true for an arbitrary W_0, we conclude that

$$\frac{\partial J}{\partial t}(\mathbf{x}, t) = (\nabla \cdot \mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t) J(\mathbf{x}, t). \qquad \square$$

Example 1.27 (An alternative proof of Lemma 1.26)**.** Write the components of $\boldsymbol{\varphi}$ as $\xi(\mathbf{x}, t), \eta(\mathbf{x}, t)$, and $\zeta(\mathbf{x}, t)$. First, observe that

$$\frac{\partial}{\partial t}\boldsymbol{\varphi}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\varphi}(\mathbf{x}, t), t),$$

by definition of the velocity field of the fluid.

The determinant $J(\mathbf{x}, t)$ can be differentiated by recalling that the determinant of a matrix is multilinear in the rows. Thus, holding $\mathbf{x}$ fixed throughout, we have

$$\begin{aligned} \frac{\partial}{\partial t} J = &\begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{vmatrix} + \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial}{\partial t}\frac{\partial \eta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{vmatrix} \\ &+ \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} \end{vmatrix}. \end{aligned}$$

Now write

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x} u(\boldsymbol{\varphi}(\mathbf{x}, t), t), \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} &= \frac{\partial}{\partial y}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial y} u(\boldsymbol{\varphi}(\mathbf{x}, t), t), \\ &\vdots \\ \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} &= \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z} w(\boldsymbol{\varphi}(\mathbf{x}, t), t). \end{aligned}$$

The components u, v, and w of $\mathbf{u}$ in this expression are functions of x, y, and z through $\boldsymbol{\varphi}(\mathbf{x}, t)$; therefore by the chain rule,

$$\frac{\partial}{\partial x} u(\boldsymbol{\varphi}(\mathbf{x}, t), t) = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x},$$
$$\vdots$$
$$\frac{\partial}{\partial z} w(\boldsymbol{\varphi}(\mathbf{x}, t), t) = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial z}.$$

When these are substituted into the above expression for $\partial J/\partial t$, one gets for the respective terms

$$\frac{\partial u}{\partial x}J + \frac{\partial v}{\partial y}J + \frac{\partial w}{\partial z}J = (\nabla \cdot \mathbf{u})J.$$

Lemma 1.28.

$$\nabla \cdot (f\mathbf{F}) = \mathbf{F} \cdot \nabla f + f \nabla \cdot \mathbf{F}. \tag{1.18}$$

Proof.

$$\nabla \cdot (f\mathbf{F}) = \frac{\partial (fF_i)}{\partial x_i} = F_i \frac{\partial f}{\partial x_i} + f\frac{\partial F_i}{\partial x_i} = \mathbf{F} \cdot \nabla f + f \nabla \cdot \mathbf{F}. \quad \square$$

Notation 5. From now on, the volume element and the area element will be denoted by $\mathrm{d}V$ and $\mathrm{d}A$, respectively.

Theorem 1.29. Let $\mathbf{u}$ be the velocity field of a fluid flow, W_t a system, and $f(\mathbf{x}, t)$ a scalar function of position $\mathbf{x}$ and time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} f \mathrm{d}V = \int_{W_t}\left(\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{u})\right)\mathrm{d}V. \tag{1.19}$$

Proof.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} f(\mathbf{x}, t)\mathrm{d}V \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_0} f(\boldsymbol{\varphi}(\mathbf{x}, t), t)J(\mathbf{x}, t)\mathrm{d}V \\
=&\int_{W_0}\frac{\mathrm{d}}{\mathrm{d}t}\left(f(\boldsymbol{\varphi}(\mathbf{x}, t), t)\right)J(\mathbf{x}, t)\mathrm{d}V \\
&+ \int_{W_0} f(\boldsymbol{\varphi}(\mathbf{x}, t), t)\frac{\partial J}{\partial t}(\mathbf{x}, t)\mathrm{d}V \\
=&\int_{W_0}\frac{Df}{Dt}(\boldsymbol{\varphi}(\mathbf{x}, t), t)J(\mathbf{x}, t)\mathrm{d}V \\
&+ \int_{W_0} f(\boldsymbol{\varphi}(\mathbf{x}, t), t)(\nabla \cdot \mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t)J(\mathbf{x}, t)\mathrm{d}V \\
=&\int_{W_t}\frac{Df}{Dt}\mathrm{d}V + \int_{W_t} f\nabla \cdot \mathbf{u}\mathrm{d}V \\
=&\int_{W_t}\left(\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f + f\nabla\cdot\mathbf{u}\right)\mathrm{d}V \\
=&\int_{W_t}\left(\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{u})\right)\mathrm{d}V,
\end{aligned}$$

where the first and fourth step follow from the change-of-variables formula, the third step from Exercise 1.22 and Lemma 1.26 and the last from Lemma 1.28. $\square$

Theorem 1.30 (Reynolds transport theorem)**.** Let $\mathbf{u}$ be the velocity field of a fluid flow, W_t a system, and $f(\mathbf{x}, t)$ a scalar function of position $\mathbf{x}$ and time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} f(\mathbf{x}, t)\mathrm{d}V = \int_{W_t}\frac{\partial f}{\partial t}(\mathbf{x}, t)\mathrm{d}V + \int_{\partial W_t} f\mathbf{u}\cdot\mathbf{n}\mathrm{d}A. \tag{1.20}$$

Proof. Theorem 1.29 and the divergence theorem. $\square$

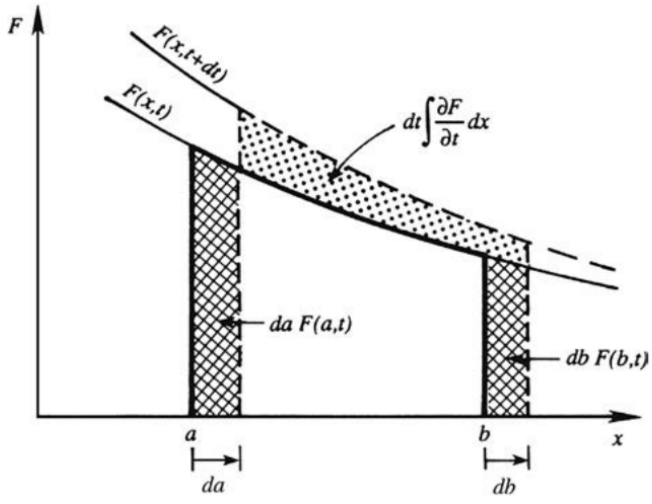
Example 1.31. The Reynolds transport theorem is the three-dimensional generalization of Leibniz's formula,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(x, t)\mathrm{d}x = &\int_{a(t)}^{b(t)}\frac{\partial}{\partial t}F(x, t)\mathrm{d}x + F(b(t), t)\frac{\mathrm{d}b}{\mathrm{d}t} \\ &- F(a(t), t)\frac{\mathrm{d}a}{\mathrm{d}t},\end{aligned}$$

which can be derived by considering

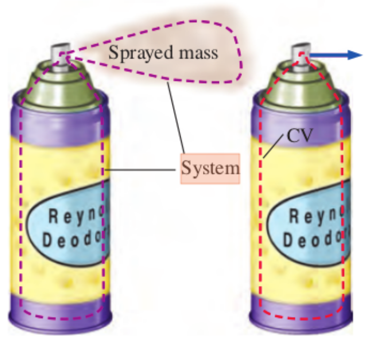
$$G(x, t) = \int_0^x F(y, t)\mathrm{d}y,$$

and applying the fundamental theorem of calculus.



Definition 1.32. A *control volume* is a *fixed* volume in space through which fluid may flow, and a *control surface* is the surface of a control volume.

Example 1.33. An illustration of a system and a control volume.



Notation 6. In the following, we shall use W to denote a control volume.

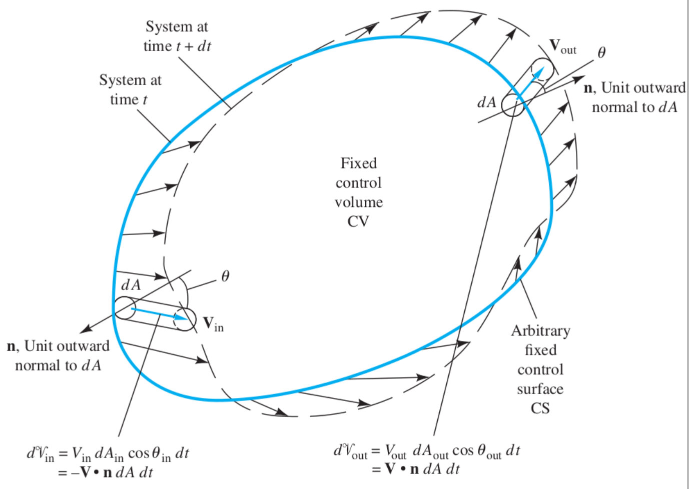
Example 1.34. Graphical illustration of the Reynolds transport theorem. The first term on the RHS of (1.20)

$$\int_{W_t} \frac{\partial f}{\partial t}(\mathbf{x},t)\mathrm{d}V$$

represents the rate of change of f within the control volume as the fluid flows through it while the second term

$$\int_{\partial W_t} f\mathbf{u}\cdot\mathbf{n}\mathrm{d}A$$

represents the net flow rate of f across the entire control surface.



1.4 Incompressibility

Definition 1.35. A flow is *incompressible* if it's volume preserving, i.e., for any system W_t,

$$\text{volume}(W_t) = \int_{W_t} \mathrm{d}V = \text{ constant in } t. \tag{1.21}$$

Example 1.36. The flow of liquids is typically incompressible.

Example 1.37. Gas flows can often be approximated as incompressible if the *Mach number* Ma is less than 0.3, which is defined as

$$\mathrm{Ma} = \frac{V}{c} = \frac{\text{Speed of flow}}{\text{Speed of sound}}.$$

c is 346 m/s in air at room temperature at sea level. Thus, when analyzing rockets, spacecraft, and other systems that involve high-speed gas flows, compressibility effects should never be neglected.

Proposition 1.38. The following statements are equivalent:

(i) a fluid is incompressible;

(ii) $J \equiv 1$;

(iii) $\nabla\cdot\mathbf{u} = 0$.

Proof. (i)⇒(ii):

$$\begin{aligned}\int_{W_0}(J-1)\mathrm{d}V &= \int_{W_0} J\mathrm{d}V - \int_{W_0}\mathrm{d}V \\ &= \int_{W_t}\mathrm{d}V - \int_{W_0}\mathrm{d}V = 0,\end{aligned}$$

by the incompressibility condition (1.21), and thus $J \equiv 1$ since W_0 is arbitrary.

(ii)⇒(iii): By Lemma 1.26, we have

$$0 = \frac{\partial}{\partial t}J(\mathbf{x},t) = J(\mathbf{x},t)(\nabla\cdot\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x},t),t) = (\nabla\cdot\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x},t),t).$$

(iii)⇒(i):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\mathrm{d}V = \int_{W_t}\nabla\cdot\mathbf{u}\mathrm{d}V = 0,$$

where the first step follows from (1.19) and setting $f = 1$, and hence volume(W_t) is constant in t. □

1.5 Conservation of Mass

Proposition 1.39. The mass of fluid in a region W at time t is

$$m(W,t) = \int_W \rho(\mathbf{x},t)\mathrm{d}V, \tag{1.22}$$

where $\mathrm{d}V$ is the area element in the plane or the volume element in space.

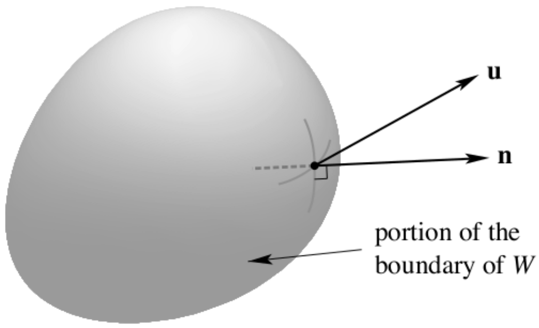
Principle 1.40 (Conservation of mass)**.** Mass is neither created nor destroyed.

Theorem 1.41. From the law of conservation of mass, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho(\mathbf{x},t)\mathrm{d}V = 0. \tag{1.23}$$

Theorem 1.42. The law of conservation of mass can be more precisely stated as follows: the rate of increase of mass in a region W equals the rate at which mass is crossing the boundary ∂W in the inward direction, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}m(W,t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathrm{d}V = -\int_{\partial W}\rho\mathbf{u}\cdot\mathbf{n}\mathrm{d}A. \tag{1.24}$$



Proof. This follows directly from the Reynolds transport theorem (Theorem 1.30), Theorem 1.41, and letting the control volume W coincide with the system W_t at time t. □

Theorem 1.43. The integral form of the law of conservation of mass is

$$\int_W \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right) \mathrm{d}V = 0. \tag{1.25}$$

Proof. Theorems 1.29 and 1.41. □

Theorem 1.44. The differential form of the law of conservation of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.26}$$

which is known as the *continuity equation*.

Proof. This follows directly from (1.25) since W is arbitrary. □

Corollary 1.45. The continuity equation (1.26) is equivalent to

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{1.27}$$

Proof.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u},$$

where the first equality follows from Lemma 1.28 and the second from Definition 1.21. □

Theorem 1.46 (Transport theorem)**.** Let f be a (scalar or vector) function of position $\mathbf{x}$ and time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{W_t} \rho f \mathrm{d}V = \int_{W_t} \rho \frac{Df}{Dt} \mathrm{d}V. \tag{1.28}$$

Exercise 1.47. Prove Theorem 1.46.

Corollary 1.48. A fluid is incompressible iff

$$\frac{D\rho}{Dt} = 0, \tag{1.29}$$

that is, the mass density is constant following the fluid.

Proof. Corollary 1.45 and Proposition 1.38. □

Definition 1.49. A fluid is *homogeneous* if its density ρ is constant in space at any given time.

Example 1.50. Problems involving inhomogeneous incompressible flow occur, for example, in oceanography.

Corollary 1.51. A homogeneous flow is incompressible iff its density ρ is constant in time.

Proof. The density ρ of a homogeneous flow satisfies $\nabla \rho = \mathbf{0}$, and hence by the definition of the material derivative and Corollary 1.48, we have

$$\frac{\partial \rho}{\partial t} = \frac{D\rho}{Dt} - (\mathbf{u} \cdot \nabla)\rho = 0. \qquad \square$$

Theorem 1.52. Another form of mass conservation:

$$\rho(\boldsymbol{\varphi}(\mathbf{x}, t), t) J(\mathbf{x}, t) = \rho(\mathbf{x}, 0). \tag{1.30}$$

Proof.

$$\int_{W_0} \rho(\mathbf{x}, 0) \mathrm{d}V = \int_{W_t} \rho(\mathbf{x}, t) \mathrm{d}V = \int_{W_0} \rho(\boldsymbol{\varphi}(\mathbf{x}, t), t) J(\mathbf{x}, t) \mathrm{d}V,$$

where the first equality is the law of conservation of mass and the second follows from the change-of-variables formula. Now the proof is completed since W_0 is arbitrary. □

Corollary 1.53. An incompressible fluid that is homogeneous at $t = 0$ will remain homogeneous.

Proof. Theorem 1.52 and Proposition 1.38. □

Example 1.54. The converse is not true, i.e., a homogeneous fluid is not necessarily incompressible. Consider

$$\boldsymbol{\varphi}((x, y, z), t) = ((1 + t)x, y, z),$$

where

$$J((x, y, z), t) = 1 + t,$$

so the flow is not incompressible, yet for

$$\rho((x, y, z), t) = \frac{1}{1 + t},$$

one has mass conservation and homogeneity for all time.

1.6 Euler's Equations

1.6.1 Conservation of Momentum

Principle 1.55 (Newton's first law)**.** When the net external force acting on an object is zero, the object is in equilibrium and has zero acceleration, i.e., if the object is initially at rest, it remains at rest; if it is initially in motion, it continues to move with constant velocity.

Principle 1.56 (Newton's second law)**.** The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass, i.e.,

$$\mathbf{F} = m\mathbf{a}. \tag{1.31}$$

Principle 1.57 (Conservation of momentum)**.** The rate of change of the momentum of a portion of the fluid is equal to the net force acting on it.

Principle 1.58 (Newton's third law)**.** When two objects interact, they exert forces on each other that are equal in magnitude and opposite in direction.

Proposition 1.59. The (linear) momentum of fluid in W at time t is

$$\int_W \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t) \mathrm{d}V. \tag{1.32}$$

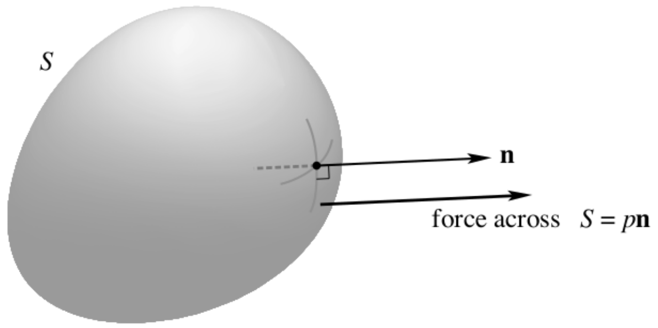
Example 1.60. For any continuum, forces acting on a piece of material are of two types:

1. forces of *stress*, whereby the piece of material is acted on by forces across its surface by the rest of the continuum;

2. *external*, or *body*, forces such as gravity or magnetic field, which exert a force per unit volume on the continuum.

Definition 1.61. An *ideal fluid* is a fluid with the following property: for any motion of the fluid, there is a function $p(\mathbf{x},t)$ called the *pressure* such that the force of stress exerted across a surface S (with unit outward normal $\mathbf{n}$) per unit area at $\mathbf{x}\in S$ at time t is $p(\mathbf{x},t)\mathbf{n}$, i.e.,

$$\text{force across } S \text{ per unit area} = p(\mathbf{x},t)\mathbf{n}.$$



Assumption 1.62. In this and the next subsection, we shall deal with ideal fluids.

Lemma 1.63. The total force exerted on the fluid inside W by means of stress on its boundary is

$$\mathbf{S}_{\partial W} = -\int_W \nabla p \mathrm{d}V. \tag{1.33}$$

Proof. By Definition 1.61, we have

$$\mathbf{S}_{\partial W} = -\int_{\partial W} p\mathbf{n}\mathrm{d}A.$$

Let $\mathbf{e}$ be a *fixed* vector in space, then

$$\begin{aligned}\mathbf{e}\cdot\mathbf{S}_{\partial W} &= -\int_{\partial W} p\mathbf{e}\cdot\mathbf{n}\mathrm{d}A = -\int_W \nabla\cdot(p\mathbf{e})\mathrm{d}V\\ &= -\int_W \mathbf{e}\cdot\nabla p\mathrm{d}V = -\mathbf{e}\cdot\int_W \nabla p\mathrm{d}V,\end{aligned}$$

where the second step follows from the divergence theorem and the third from Lemma 1.28. The proof is completed since $\mathbf{e}$ is arbitrary. $\square$

Proposition 1.64. Let $\mathbf{b}(\mathbf{x},t)$ denote the given body force per unit mass, then the total body force exerted on the fluid inside W is

$$\mathbf{B}_W = \int_W \rho\mathbf{b}\mathrm{d}V. \tag{1.34}$$

Proposition 1.65. On any piece of fluid material,

$$\text{force per unit volume } = -\nabla p + \rho\mathbf{b}. \tag{1.35}$$

Proof. Lemma 1.63 and Proposition 1.64. $\square$

Theorem 1.66. The differential form of the law of conservation of momentum is

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b}. \tag{1.36}$$

Proof. Newton's second law, Propositions 1.20 and 1.65. $\square$

Theorem 1.67. The integral form of the law of conservation of momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathbf{u}\mathrm{d}V = -\int_{\partial W}(p\mathbf{n}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}))\,\mathrm{d}A + \int_W \rho\mathbf{b}\mathrm{d}V, \tag{1.37}$$

where the quantity $p\mathbf{u}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})$ is the *momentum flux per unit area crossing* ∂W.

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}(\rho\mathbf{u}) &= \frac{\partial\rho}{\partial t}\mathbf{u} + \rho\frac{\partial\mathbf{u}}{\partial t}\\ &= -\nabla\cdot(\rho\mathbf{u})\mathbf{u} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \rho\mathbf{b},\end{aligned}$$

where the second step follows from the continuity equation (1.26) and (1.36). Let $\mathbf{e}$ be a *fixed* vector in space, then

$$\begin{aligned}&\mathbf{e}\cdot\frac{\partial}{\partial t}(\rho\mathbf{u})\\ =&-\nabla\cdot(\rho\mathbf{u})(\mathbf{u}\cdot\mathbf{e}) - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{e} - (\nabla p)\cdot\mathbf{e} + \rho\mathbf{b}\cdot\mathbf{e}\\ =&-\nabla\cdot(p\mathbf{e}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})) + \rho\mathbf{b}\cdot\mathbf{e},\end{aligned}$$

where the second equality follows from Lemma 1.28. Therefore, the rate of change of momentum in direction $\mathbf{e}$ in W is

$$\begin{aligned}&\mathbf{e}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathbf{u}\mathrm{d}V = \int_W \mathbf{e}\cdot\frac{\partial}{\partial t}(\rho\mathbf{u})\mathrm{d}V\\ =&-\int_W \nabla\cdot(p\mathbf{e}+\rho\mathbf{u}(\mathbf{e}\cdot\mathbf{u}))\mathrm{d}V + \int_W \rho\mathbf{b}\cdot\mathbf{e}\mathrm{d}V\\ =&-\int_{\partial W}(p\mathbf{e}\cdot\mathbf{n}+\rho\mathbf{e}\cdot\mathbf{u}(\mathbf{u}\cdot\mathbf{n}))\mathrm{d}A + \int_W \rho\mathbf{e}\cdot\mathbf{b}\mathrm{d}V,\end{aligned}$$

where we have used the divergence theorem in obtaining the last equality. The proof is completed since $\mathbf{e}$ is arbitrary. $\square$

Theorem 1.68. The "primitive" integral form of balance of momentum states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho\mathbf{u}\mathrm{d}V = \mathbf{S}_{\partial W_t} + \int_{W_t}\rho\mathbf{b}\mathrm{d}V, \tag{1.38}$$

that is, the rate of change of momentum of a moving piece of fluid equals the total force (surface stresses plus body forces) acting on it.

Example 1.69. The differential form of the law of conservation of momentum can also be derived from (1.38) and the transport theorem as follows. By (1.28), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho\mathbf{u}\mathrm{d}V = \int_{W_t}\rho\frac{D\mathbf{u}}{Dt}\mathrm{d}V,$$

combining with (1.38) and (1.33), we obtain

$$\int_{W_t}\rho\frac{D\mathbf{u}}{Dt}\mathrm{d}V = \int_{W_t}(-\nabla p+\rho\mathbf{b})\,\mathrm{d}V,$$

which yields (1.36) since W_t is arbitrary.

1.6.2 Conservation of Energy

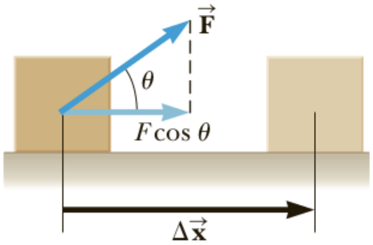
Principle 1.70 (Conservation of energy)**.** Energy is neither created nor destroyed.

Theorem 1.71 (Work-energy theorem)**.** When forces act on a particle while it undergoes a displacement, the particle's kinetic energy $K = \frac{1}{2}mv^2$ changes by an amount equal to the total work W done on the particle by all the forces.

$$W_{\text{tot}} = \Delta K. \tag{1.39}$$

Definition 1.72. The *work* W done on an object by a constant force $\mathbf{F}$ during a linear displacement $\Delta\mathbf{x}$ is

$$W = \mathbf{F} \cdot \Delta\mathbf{x}. \tag{1.40}$$



Definition 1.73. The *work* W done on an object by a varying force $\mathbf{F}$ following a path Γ is the line integral

$$W = \int_\Gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}. \tag{1.41}$$

Principle 1.74 (The first law of thermodynamics)**.** When heat Q is added to a system while the system does work W, the internal energy U of the system changes by an amount equal to $Q - W$, i.e.,

$$\Delta U = Q - W, \tag{1.42}$$

or expressing for an infinitesimal process, we have

$$\mathrm{d}U = \mathrm{d}Q - \mathrm{d}W. \tag{1.43}$$

Principle 1.75 (The second law of thermodynamics)**.** The entropy of the Universe increases in all natural processes.

Definition 1.76. The *kinetic energy* contained in W is

$$E_{\text{kinetic}} = \frac{1}{2}\int_W \rho\|\mathbf{u}\|^2 \mathrm{d}V. \tag{1.44}$$

Assumption 1.77. Assume the total energy of the fluid can be written as

$$E_{\text{total}} = E_{\text{kinetic}} + E_{\text{internal}}, \tag{1.45}$$

where E_{internal} is the *internal energy*, which is energy we cannot "see" on a macroscopic scale, and derives from sources such as intermolecular potentials and internal molecular vibrations.

Lemma 1.78.

$$\frac{1}{2}\frac{D}{Dt}\|\mathbf{u}\|^2 = \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt}.$$

Proof.

$$\begin{aligned}\frac{D}{Dt}\|\mathbf{u}\|^2 &= \left(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}\right)(u_j u_j) \\ &= 2u_j\frac{\partial u_j}{\partial t} + 2u_j u_i \frac{\partial u_j}{\partial x_i} \\ &= 2u_j\left(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}\right)u_j \\ &= 2\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt}.\end{aligned}$$

□

Proposition 1.79. The rate of change of kinetic energy of a moving portion W_t of fluid is

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{kinetic}} = \int_{W_t} \rho\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt}\mathrm{d}V. \tag{1.46}$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{kinetic}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{W_t}\rho\|\mathbf{u}\|^2\mathrm{d}V\right) \\ &= \frac{1}{2}\int_{W_t}\rho\frac{D\|\mathbf{u}\|^2}{Dt}\mathrm{d}V \\ &= \int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V,\end{aligned}$$

where the second step follows from the transport theorem and the last from Lemma 1.78. □

Theorem 1.80. Assume the fluid is incompressible and that all the energy is kinetic, then the integral form of the principle of conservation of energy is

$$\int_{W_t}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\mathrm{d}V = \int_{W_t}\mathbf{u}\cdot(-\nabla p + \rho\mathbf{b})\mathrm{d}V. \tag{1.47}$$

Proof. The principle of conservation of energy can be more precisely stated as: the rate of change of energy in a portion of the fluid equals the rate at which the pressure and body forces do work, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{total}} = -\int_{\partial W_t} p\mathbf{u}\cdot\mathbf{n}\mathrm{d}A + \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}\mathrm{d}V, \tag{1.48}$$

next we compute the right-hand side as

$$\begin{aligned}\text{RHS} &= -\int_{W_t}\left(\nabla\cdot(p\mathbf{u}) - \rho\mathbf{u}\cdot\mathbf{b}\right)\mathrm{d}V \\ &= -\int_{W_t}\left(\mathbf{u}\cdot\nabla p + p\nabla\cdot\mathbf{u} - \rho\mathbf{u}\cdot\mathbf{b}\right)\mathrm{d}V \\ &= -\int_{W_t}\left(\mathbf{u}\cdot\nabla p - \rho\mathbf{u}\cdot\mathbf{b}\right)\mathrm{d}V,\end{aligned}$$

where the first equality follows from the divergence theorem, the second from Lemma 1.28 and the last from the incompressibility assumption and Proposition 1.38. Combining with (1.46) and that $E_{\text{total}} = E_{\text{kinetic}}$ gives the desired result. □

Example 1.81. Note that the integral form of the principle of conservation of energy (1.47) for an incompressible fluid where all energy is kinetic is also a consequence of conservation of momentum (1.36). If we assume $E_{\text{total}} = E_{\text{kinetic}}$ and $p \neq 0$, then the fluid must be incompressible.

Theorem 1.82. The *Euler's equations for an incompressible fluid* are

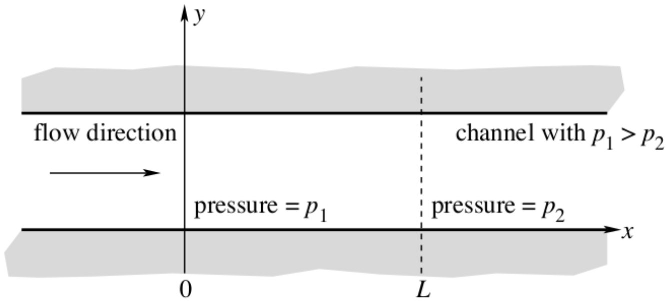
$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b}, \tag{1.49a}$$
$$\frac{D\rho}{Dt} = 0, \tag{1.49b}$$
$$\nabla\cdot\mathbf{u} = 0, \tag{1.49c}$$

with no-penetration boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial D. \tag{1.50}$$

Proof. Theorem 1.66, Corollary 1.48 and Proposition 1.38. □

Example 1.83. Consider a fluid-filled channel.



Suppose that the pressure p_1 at $x = 0$ is larger than that at $x = L$ so the fluid is pushed from left to right. We seek a solution of Euler's incompressible homogeneous equations (1.49) in the form

$$\mathbf{u}(x,y,t) = (u(x,t),0) \text{ and } p(x,y,t) = p(x).$$

Incompressibility implies $\partial_x u = 0$. Thus, Euler's equations become $\rho_0\partial_t u = -\partial_x p$. This implies that $\partial_x^2 p = 0$, and so a linear interpolation gives

$$p(x) = p_1 - \left(\frac{p_1 - p_2}{L}\right)x.$$

Substitution into $\rho_0\partial_t u = -\partial_x p$ and integration yield

$$u = \frac{p_1 - p_2}{\rho_0 L}t + \text{ constant}.$$

This solution suggests that the velocity in a channel flow with a constant pressure gradient increases indefinitely. Of course, this cannot be the case in a real flow; however, in our modeling, we have not yet taken friction into account. This situation will be remedied in the Navier-Stokes equations.

Lemma 1.84.

$$\frac{1}{2}\nabla\left(\|\mathbf{u}\|^2\right) = (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}).$$

Proof. The ℓ-th component of the RHS is

$$\begin{aligned}
&((\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}))_\ell\\
=&u_m\frac{\partial u_\ell}{\partial x_m} + \varepsilon_{\ell mn}u_m(\nabla\times\mathbf{u})_n\\
=&u_m\frac{\partial u_\ell}{\partial x_m} + \varepsilon_{\ell mn}u_m\varepsilon_{npq}\frac{\partial u_q}{\partial x_p}\\
=&u_m\frac{\partial u_\ell}{\partial x_m} + \varepsilon_{\ell mn}\varepsilon_{npq}u_m\frac{\partial u_q}{\partial x_p}\\
=&u_m\frac{\partial u_\ell}{\partial x_m} + (\delta_{\ell p}\delta_{mq} - \delta_{\ell q}\delta_{mp})u_m\frac{\partial u_q}{\partial x_p}\\
=&u_m\frac{\partial u_\ell}{\partial x_m} + u_m\frac{\partial u_m}{\partial x_\ell} - u_m\frac{\partial u_\ell}{\partial x_m}\\
=&u_m\frac{\partial u_m}{\partial x_\ell} = \left(\frac{1}{2}\nabla\left(\|\mathbf{u}\|^2\right)\right)_\ell.
\end{aligned}$$

□

Theorem 1.85 (Bernoulli's theorem)**.** In steady homogeneous incompressible flows and in the absence of external forces, the quantity

$$\frac{1}{2}\|\mathbf{u}\|^2 + p/\rho \tag{1.51}$$

is constant along streamlines. The same conclusion remains true if a force $\mathbf{b}$ is present and is conservative, i.e.,

$$\mathbf{b} = -\nabla\varphi \text{ for some function } \varphi,$$

with p/ρ replaced by $p/\rho + \varphi$.

Proof. Since the flow is steady and there is no external force, we have

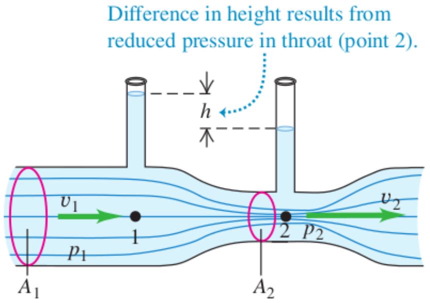
$$\begin{aligned}
\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho}\right) &= (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) + \frac{\nabla p}{\rho}\\
&= \mathbf{u}\times(\nabla\times\mathbf{u}),
\end{aligned}$$

where the first equality follows from Lemma 1.84 and Corollary 1.51 and the second from (1.49a). Let $\mathbf{x}(s)$ be a streamline, then

$$\begin{aligned}
\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho}\right)\Big|_{s_1}^{s_2} &= \int_{s_1}^{s_2}\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho}\right)\cdot\mathbf{x}'(s)\mathrm{d}s\\
&= \int_{s_1}^{s_2}(\mathbf{u}\times(\nabla\times\mathbf{u}))\cdot\mathbf{x}'(s)\mathrm{d}s\\
&= \int_{s_1}^{s_2}(\mathbf{u}\times(\nabla\times\mathbf{u}))\cdot\mathbf{u}\mathrm{d}s\\
&= 0,
\end{aligned}$$

where the first equality follows from the fundamental theorem of calculus and the chain rule and the third from the definition of the velocity field. □

Example 1.86. The following figure shows a Venturi meter, used to measure the flow speed in a pipe.



As an application of the Bernoulli's theorem, we use it to derive the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2.

Assume the flow is steady and that the fluid is homogeneous incompressible and has negligible internal friction.

Points 1 and 2 have the same vertical coordinate, so Bernoulli's theorem says

$$\frac{p_1}{\rho} + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2.$$

From the continuity equation,

$$\rho A_1 v_1 = \rho A_2 v_2 \Rightarrow v_2 = \frac{A_1 v_1}{A_2}.$$

Substituting this and rearranging, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2 \left(\left(\frac{A_1}{A_2} \right)^2 - 1 \right).$$

The pressure difference $p_1 - p_2$ is also equal to $\rho g h$. Substituting this and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}.$$

1.7 Cartesian Tensors

Notation 7. In this section, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ denote two orthonormal bases of $\mathbb{R}^3$ and

$$C_{ij} = \langle \mathbf{e}_i, \mathbf{e}'_j \rangle \tag{1.52}$$

is the *change of coordinate matrix*.

Proposition 1.87. The change of coordinate matrix C has the following properties:

(i)
$$\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} C,$$

(ii)
$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} C^T,$$

(iii) C is an orthogonal matrix, i.e.,

$$C^T C = C C^T = I.$$

Exercise 1.88. Prove Proposition 1.87.

Example 1.89. Let x_i and x'_i be the components of the position vector $\mathbf{x} \in \mathbb{R}^3$ with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, respectively, i.e.,

$$\mathbf{x} = x_i \mathbf{e}_i = x'_j \mathbf{e}'_j,$$

then by Proposition 1.87, we have

$$\mathbf{x} = x_i \mathbf{e}_i = x_i \mathbf{e}'_j C_{ij} = x_i C_{ij} \mathbf{e}'_j,$$

and therefore

$$x'_j = C_{ij} x_i. \tag{1.53}$$

Example 1.90. Let A and A' be the matrices of a linear operator $T \in \mathcal{L}(\mathbb{R}^3)$ with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, respectively. Hence

$$a_{ij} = \langle \mathbf{e}_i, T\mathbf{e}_j \rangle \text{ and } a'_{ij} = \langle \mathbf{e}'_i, T\mathbf{e}'_j \rangle,$$

then by Proposition 1.87, we have

$$\begin{aligned} a'_{ij} &= \langle \mathbf{e}'_i, T\mathbf{e}'_j \rangle = \langle \mathbf{e}_k C_{ki}, T(\mathbf{e}_l C_{lj}) \rangle = \langle C_{ki}\mathbf{e}_k, C_{lj} T\mathbf{e}_l \rangle \\ &= C_{ki} C_{lj} \langle \mathbf{e}_k, T\mathbf{e}_l \rangle = C_{ki} C_{lj} a_{kl}, \end{aligned} \tag{1.54}$$

or in matrix form,

$$A' = C^T A C. \tag{1.55}$$

Definition 1.91. A *(Cartesian) tensor of order* n is a mathematical object with n suffixes, $T_{i_1 i_2 \cdots i_n}$, which obeys the transformation law

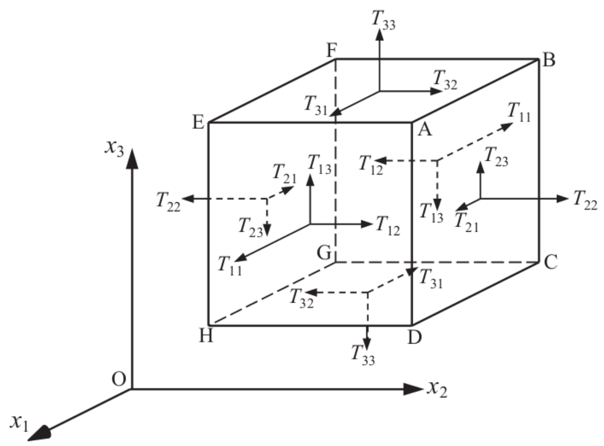
$$T'_{j_1 j_2 \cdots j_n} = C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_n j_n} T_{i_1 i_2 \cdots i_n}. \tag{1.56}$$

Definition 1.92. A *scalar* is a zero-order tensor and a *vector* is a first-order tensor.

Example 1.93. Typical scalars in fluid mechanics include the pressure p, density ρ, temperature T, etc.

Example 1.94. Typical vectors in fluid mechanics include the position $\mathbf{x}$, velocity $\mathbf{u}$, acceleration $\mathbf{a}$ of a fluid particle and the force $\mathbf{f}$, etc.

Example 1.95. The surface stresses (surface forces per unit area) at a point in the flow are described by a second-order tensor T_{ij} called the *stress tensor*, where the first (i) index of T_{ij} denotes the direction of the surface normal, and the second (j) index denotes the force component direction.



The diagonal components of the stress tensor, T_{11}, T_{22}, and T_{33}, are called *normal stresses*; they are composed of pressure (which always acts inwardly normal) and viscous stresses. The off-diagonal components, T_{12}, T_{31}, etc., are called *shear stresses*; since pressure can act only normal to a surface, shear stresses are composed entirely of viscous stresses.

Proposition 1.96. The stresses are locally in equilibrium, i.e., suppose V is a volume of given shape with characteristic dimension d, and the volume of V is proportional to d^3 and the area of ∂V to d^2, with the proportionality constants depending only on the shape, then

$$\lim_{d\to 0} \frac{1}{d^2}\int_{\partial V} \mathbf{T}(\mathbf{n})\mathrm{d}A = \mathbf{0}, \tag{1.57}$$

where $\mathbf{T}(\mathbf{n})$ is the stress force per unit area exerted on V by the material outside ∂V and in the limit we let V shrink to a point but preserve its shape.

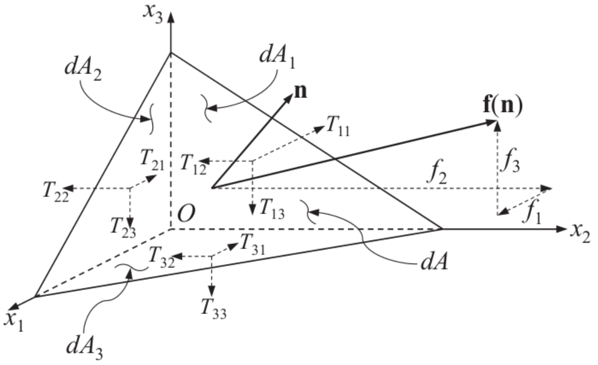
Proof. The law of conservation of momentum states

$$\int_V \rho\frac{D\mathbf{u}}{Dt}\mathrm{d}V = \int_V \rho\mathbf{g}\mathrm{d}V + \int_{\partial V}\mathbf{T}(\mathbf{n})\mathrm{d}A,$$

now let V shrink to a point but preserve its shape, then the first two integrals in the above equation will decrease as d^3 but the last will be as d^2. It follows that

$$\lim_{d\to 0} \frac{1}{d^2}\int_{\partial V} \mathbf{T}(\mathbf{n})\mathrm{d}A = \mathbf{0}.$$ □

Example 1.97. Suppose the nine components, T_{ij}, of the stress tensor with respect to a given set of Cartesian coordinates $O123$ are given, and we want to find the force per unit area, $\mathbf{f}(\mathbf{n})$ with components f_i, on an arbitrary oriented surface element with normal $\mathbf{n}$.



Consider a small tetrahedral element, by Proposition 1.96, the net force f_1 on the element in the first direction produced by the stresses T_{ij} is:

$$f_1\mathrm{d}A = T_{11}\mathrm{d}A_1 + T_{21}\mathrm{d}A_2 + T_{31}\mathrm{d}A_3.$$

The geometry of the tetrahedron requires: $\mathrm{d}A_i = n_i\mathrm{d}A$, where n_i are the components of the surface normal vector $\mathbf{n}$. Thus, the net force equation can be rewritten:

$$f_1\mathrm{d}A = T_{11}n_1\mathrm{d}A + T_{21}n_2\mathrm{d}A + T_{31}n_3\mathrm{d}A.$$

Dividing by $\mathrm{d}A$ then produces $f_1 = T_{j1}n_j$, or for any component of $\mathbf{f}$:

$$f_i = T_{ji}n_j \text{ or } \mathbf{f} = \mathbf{n}\cdot\mathbf{T}. \tag{1.58}$$

Example 1.98. The gradient $\partial u_i/\partial x_j$ of the velocity vector $\mathbf{u}$ is a second-order tensor.

Example 1.99. The *Kronecker delta*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{1.59}$$

is a second-order tensor.

Example 1.100. The *alternating tensor*

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of (1,2,3)}, \\ 0 & \text{any of } i,j,k \text{ are equal}, \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of (1,2,3)} \end{cases} \tag{1.60}$$

is a third-order tensor.

Lemma 1.101 (Epsilon delta relation)**.**

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{1.61}$$

Proof. First we prove a generalization of (1.61):

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}. \tag{1.62}$$

Both sides clearly vanish if any of i, j, k are equal; or if any of l, m, n are. Now take $i = l = 1, j = m = 2, k = n = 3$: both sides are clearly 1. Finally the equality still holds when swapping any two indices.

Contracting k and n in (1.62), we have

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{klm} =& \varepsilon_{ijk}\varepsilon_{lmk} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \delta_{jk} \\ \delta_{kl} & \delta_{km} & \delta_{kk} \end{vmatrix} \\ =& \delta_{kl}(\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm}) - \delta_{km}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \\ &+ \delta_{kk}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ =& \delta_{jl}\delta_{im} - \delta_{il}\delta_{jm} - \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} \\ &+ 3(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ =& \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \end{aligned}$$

where the third equality follows from the Laplace expansion theorem and the fourth from $\delta_{kk} = 3, \delta_{kl}\delta_{jk} = \delta_{jl}, \delta_{kl}\delta_{ik} = \delta_{il}$, etc. □

Exercise 1.102. Prove the following identities:

$$\mathbf{a}\times\mathbf{b} = \varepsilon_{ijk}a_jb_k, \tag{1.63}$$

$$\nabla\times\mathbf{u} = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}, \tag{1.64}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3}. \tag{1.65}$$

Note that $\varepsilon_{ijk}a_jb_k$ denotes the vector whose ith component is $\varepsilon_{ijk}a_jb_k$!

Proposition 1.103. If $T_{i_1 i_2 \cdots i_n}$ is a tensor of order n, then

$$C_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_{l-1} i_{l+1} \cdots i_n} = T_{i_1 \cdots i_{k-1} \ell i_{k+1} \cdots i_{l-1} \ell i_{l+1} \cdots i_n} \tag{1.66}$$

is a tensor of order $n-2$, which is termed a *contraction* of $T_{i_1 i_2 \cdots i_n}$.

Exercise 1.104. Prove Proposition 1.103.

Example 1.105. Let T_{ij} be a second-order tensor, then the scalar $T_{\ell\ell}$ is a contraction of T_{ij}, known as the *trace* of T_{ij}, and is independent of the coordinate system. In other words, $T_{\ell\ell}$ is invariant under the transformation of coordinate systems.

Proposition 1.106. If $A_{i_1 i_2 \cdots i_n}$ and $B_{i_1 i_2 \cdots i_n}$ are two tensors of order n, and α, β are scalars, then

$$T_{i_1 i_2 \cdots i_n} = \alpha A_{i_1 i_2 \cdots i_n} + \beta B_{i_1 i_2 \cdots i_n} \tag{1.67}$$

is also a tensor of order n.

Proof.

$$\begin{aligned}
&T'_{j_1 \cdots j_n} = \alpha' A'_{j_1 \cdots j_n} + \beta' B'_{j_1 \cdots j_n} \\
=&\alpha C_{i_1 j_1} \cdots C_{i_n j_n} A_{i_1 \cdots i_n} + \beta C_{i_1 j_1} \cdots C_{i_n j_n} B_{i_1 \cdots i_n} \\
=&C_{i_1 j_1} \cdots C_{i_n j_n} (\alpha A_{i_1 \cdots i_n} + \beta B_{i_1 \cdots i_n}) \\
=&C_{i_1 j_1} \cdots C_{i_n j_n} T_{i_1 \cdots i_n},
\end{aligned}$$

where the second step follows from the definition of Cartesian tensors. □

Proposition 1.107. If $A_{i_1 i_2 \cdots i_m}$ and $B_{j_1 j_2 \cdots j_n}$ are tensors of order m and n, respectively, then $A_{i_1 i_2 \cdots i_m} B_{j_1 j_2 \cdots j_n}$ is a tensor of order $m+n$.

Exercise 1.108. Prove Proposition 1.107.

Example 1.109. If $\mathbf{a}$ and $\mathbf{b}$ are vectors, then the outer product T_{ij} defined by

$$T_{ij} = a_i b_j$$

is a second-order tensor.

Lemma 1.110. Let $T_{i_1 i_2 \cdots i_n}$ be a tensor of order n, then

$$T_{i_1 i_2 \cdots i_n} = C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_n j_n} T'_{j_1 j_2 \cdots j_n}. \tag{1.68}$$

Proof. By the definition of Cartesian tensors, we have

$$T'_{j_1 j_2 \cdots j_n} = C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_n j_n} T_{i_1 i_2 \cdots i_n},$$

multiplying both sides by $C_{k_1 j_1}$ and summing over j_1, we obtain

$$\begin{aligned}
C_{k_1 j_1} T'_{j_1 j_2 \cdots j_n} &= C_{k_1 j_1} C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_n j_n} T_{i_1 i_2 \cdots i_n} \\
&= \delta_{k_1 i_1} C_{i_2 j_2} \cdots C_{i_n j_n} T_{i_1 i_2 \cdots i_n},
\end{aligned}$$

where the second equality holds since C is an orthogonal matrix, and hence

$$C_{k_1 j_1} T'_{j_1 j_2 \cdots j_n} = C_{i_2 j_2} \cdots C_{i_n j_n} T_{k_1 i_2 \cdots i_n},$$

continue the above procedure and we have the desired result. □

Theorem 1.111 (The quotient rule)**.** Let A be a mathematical or physical entity which, when it is associated with any set of rectangular Cartesian coordinate axes, may be represented by an ordered set of 3^{m+n} scalars $a_{i_1 \cdots i_m j_1 \cdots j_n}$. Suppose that for all tensors $v_{j_1 \cdots j_n}$ of order n,

$$u_{i_1 \cdots i_m} = a_{i_1 \cdots i_m j_1 \cdots j_n} v_{j_1 \cdots j_n} \tag{1.69}$$

is a tensor of order m. Then A is a tensor of order $m+n$.

Proof.

$$\begin{aligned}
&a'_{i_1 \cdots i_m j_1 \cdots j_n} v'_{j_1 \cdots j_n} = u'_{i_1 \cdots i_m} = C_{k_1 i_1} \cdots C_{k_m i_m} u_{k_1 \cdots k_m} \\
=&C_{k_1 i_1} \cdots C_{k_m i_m} a_{k_1 \cdots k_m \ell_1 \cdots \ell_n} v_{\ell_1 \cdots \ell_n} \\
=&C_{k_1 i_1} \cdots C_{k_m i_m} a_{k_1 \cdots k_m \ell_1 \cdots \ell_n} C_{\ell_1 j_1} \cdots C_{\ell_n j_n} v'_{j_1 \cdots j_n},
\end{aligned}$$

where the last step follows from Lemma 1.110, and hence

$$\begin{aligned}
(a'_{i_1 \cdots i_m j_1 \cdots j_n} - C_{k_1 i_1} \cdots C_{k_m i_m} C_{\ell_1 j_1} \cdots C_{\ell_n j_n} a_{k_1 \cdots k_m \ell_1 \cdots \ell_n}) \\
v'_{j_1 \cdots j_n} = 0,
\end{aligned}$$

since $v'_{j_1 \cdots j_n}$ is arbitrary, this further implies that

$$a'_{i_1 \cdots i_m j_1 \cdots j_n} = C_{k_1 i_1} \cdots C_{k_m i_m} C_{\ell_1 j_1} \cdots C_{\ell_n j_n} a_{k_1 \cdots k_m \ell_1 \cdots \ell_n},$$

therefore $a_{i_1 \cdots i_m j_1 \cdots j_n}$ is a tensor of order $m+n$ by Definition 1.91. □

Example 1.112. The Kronecker delta δ_{ij} is a second-order tensor since for any vector $\mathbf{a}$, we have

$$a_i = \delta_{ij} a_j.$$

Example 1.113. The alternating tensor ε_{ijk} is a third-order tensor since for any second-order tensor $a_j b_k$, the cross product of $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a_j b_k$$

is a vector.

1.7.1 Symmetric and Anti-symmetric Tensors

Definition 1.114. A tensor $T_{i_1 i_2 \cdots i_n}$ is *symmetric* in a pair of indices i_k and i_l if

$$T_{i_1 \cdots i_k \cdots i_l \cdots i_n} = T_{i_1 \cdots i_l \cdots i_k \cdots i_n},$$

or *anti-symmetric* in i_k and i_l if

$$T_{i_1 \cdots i_k \cdots i_l \cdots i_n} = -T_{i_1 \cdots i_l \cdots i_k \cdots i_n}.$$

Example 1.115. δ_{ij} is a symmetric second-order tensor and ε_{ijk} is anti-symmetric in any pair of indices.

Theorem 1.116. Any second-order tensor can be uniquely expressed as the sum of a symmetric and an anti-symmetric tensor:

$$T_{ij} = S_{ij} + A_{ij}, \tag{1.70}$$

where $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$.

Proof. Swapping i and j in T_{ij}, we obtain

$$T_{ji} = S_{ji} + A_{ji} = S_{ij} - A_{ij},$$

therefore

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right), \quad A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right). \qquad \square$$

Example 1.117. The velocity gradient tensor $\partial u_i/\partial x_j$ can be decomposed into symmetric, D_{ij}, and anti-symmetric, R_{ij}, tensors:

$$\frac{\partial u_i}{\partial x_j} = D_{ij} + \frac{1}{2}R_{ij}, \tag{1.71}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad R_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}. \tag{1.72}$$

Here, D_{ij} is called the *deformation* (or *strain rate*) *tensor*, and R_{ij} is the *rotation tensor*.

Lemma 1.118. Suppose that S_{ij} is a symmetric tensor and A_{ij} an anti-symmetric tensor, then

$$S_{ij}A_{ij} = 0.$$

Proof.

$$S_{ij}A_{ij} = -S_{ji}A_{ji} = -S_{ij}A_{ij} \Rightarrow S_{ij}A_{ij} = 0. \qquad \square$$

Theorem 1.119. Any anti-symmetric second-order tensor A_{ij} can be expressed in terms of a vector $\boldsymbol{\omega}$ (sometimes known as the *dual vector*) such that

$$A_{ij} = -\varepsilon_{ijk}\omega_k. \tag{1.73}$$

Proof. Define $\boldsymbol{\omega}$ by

$$\omega_k = -\frac{1}{2}\varepsilon_{klm}A_{lm}.$$

Then

$$\begin{aligned}\varepsilon_{ijk}\omega_k &= -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}A_{lm} = -\frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)A_{lm} \\ &= -\frac{1}{2}\left(A_{ij} - A_{ji}\right) = -A_{ij},\end{aligned}$$

where the second step follows from the epsilon delta relation (1.61). $\square$

Definition 1.120. The *vorticity field* $\boldsymbol{\omega}$ of a flow is the curl of the velocity field $\mathbf{u}$, i.e.,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{1.74}$$

Example 1.121. The rotation tensor R_{ij} and the vorticity field $\boldsymbol{\omega}$ are related by

$$R_{ij} = -\varepsilon_{ijk}\omega_k. \tag{1.75}$$

Theorem 1.122. Let $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ be the velocity field of a fluid flow, then

$$\mathbf{u}(\mathbf{x}+\mathbf{h}) = \mathbf{u}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\mathbf{h} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{h} + O(\|\mathbf{h}\|^2) \text{ as } \mathbf{h} \to \mathbf{0}, \tag{1.76}$$

where $\mathbf{D}(\mathbf{x})$ is the deformation tensor and $\boldsymbol{\omega}$ the vorticity field of the flow.

Proof. Let $\nabla\mathbf{u}$ denote the Jacobian matrix of $\mathbf{u}$. By Taylor's expansion, we have

$$\begin{aligned}\mathbf{u}(\mathbf{x}+\mathbf{h}) &= \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &= \mathbf{u}(\mathbf{x}) + \left(\mathbf{D}(\mathbf{x}) + \frac{1}{2}\mathbf{R}(\mathbf{x})\right)\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &= \mathbf{u}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\mathbf{h} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{h} + O(\|\mathbf{h}\|^2),\end{aligned}$$

where the second step follows from (1.71) and the third equality holds since $\mathbf{R}(\mathbf{x})\mathbf{h} = \boldsymbol{\omega}(\mathbf{x}) \times \mathbf{h}$, which can be verified as follows:

$$R_{ij}h_j = -\varepsilon_{ijk}\omega_k h_j = \varepsilon_{ikj}\omega_k h_j. \qquad \square$$

Example 1.123. The decomposition (1.76) suggests that in a small neighborhood of each point of the fluid, the fluid velocity $\mathbf{u}$ is *the sum of a (rigid) translation, a deformation* (Proposition 1.124), and *a (rigid) rotation with rotation vector* $\boldsymbol{\omega}/2$ (Example 1.126).

Proposition 1.124. If the velocity $\mathbf{u}(\mathbf{x})$ at position $\mathbf{x}$ can be represented as the product of a symmetric matrix $\mathbf{D}$ with $\mathbf{x}$, i.e.,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}) = \mathbf{D}\mathbf{x}, \tag{1.77}$$

then the motion is locally a deformation.

Proof. The symmetry of $\mathbf{D}$ implies that there is an orthonormal basis $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ and $\tilde{\mathbf{e}}_3$ of $\mathbb{R}^3$ with respect to which $\mathbf{D}$ is diagonal, i.e.,

$$\left(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\right)^T \mathbf{D}\left(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\right) = \mathrm{diag}(d_1, d_2, d_3).$$

Define

$$\tilde{\mathbf{x}} = \left(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\right)^T \mathbf{x},$$

then

$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \mathrm{diag}(d_1, d_2, d_3)\tilde{\mathbf{x}},$$

which decouples to three linear ODEs

$$\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}t} = d_i\tilde{x}_i, \quad i = 1, 2, 3.$$

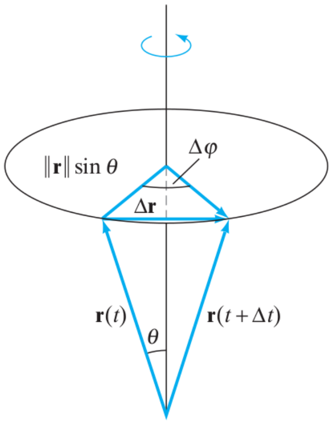
Therefore the rate of change of a unit length along the $\tilde{\mathbf{e}}_i$ axis at $t = 0$ is d_i, which shows that the vector field $\mathbf{D}\mathbf{x}$ is merely expanding or contracting along each of the axes $\tilde{\mathbf{e}}_i$. $\square$

Example 1.125. Proposition 1.124 makes clear the physical meaning of $\nabla \cdot \mathbf{u}$. The rate of change of the volume of a box with sides of length $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ parallel to the $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ axes is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) &= \left[\frac{\mathrm{d}\tilde{h}_1}{\mathrm{d}t}\right]\tilde{h}_2\tilde{h}_3 + \tilde{h}_1\left[\frac{\mathrm{d}\tilde{h}_2}{\mathrm{d}t}\right]\tilde{h}_3 + \tilde{h}_1\tilde{h}_2\left[\frac{\mathrm{d}\tilde{h}_3}{\mathrm{d}t}\right] \\ &= (d_1 + d_2 + d_3)\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) \\ &= \mathrm{trace}(\mathbf{D})\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) \\ &= \mathrm{trace}\left(\frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\right)\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) \\ &= (\nabla \cdot \mathbf{u})\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right),\end{aligned}$$

where the third step follows from the fact that the trace of a matrix is invariant under orthogonal transformations. This confirms the fact proved in Lemma 1.26 that volume elements change at a rate proportional to $\nabla \cdot \mathbf{u}$.

Example 1.126. Consider a rigid body rotation with angular velocity vector $\boldsymbol{\omega} = \omega \mathbf{n}$, where $\omega = \|\boldsymbol{\omega}\|$ is the angular speed (measured in radians per unit time) and $\mathbf{n}$ is a unit vector representing the axis of rotation.



Then the velocity $\mathbf{u}$ at $\mathbf{r}(t)$ is given by

$$\mathbf{u} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t},$$

when Δt is small, the magnitude of $\mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ is approximately $\|\mathbf{r}(t)\| \sin\theta \Delta\varphi$. Thus,

$$\begin{aligned}
\|\mathbf{u}\| &= \lim_{\Delta t \to 0} \left\| \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \\
&= \lim_{\Delta t \to 0} \|\mathbf{r}(t)\| \sin\theta \frac{\Delta\varphi}{\Delta t} \\
&= \|\mathbf{r}(t)\| \sin\theta \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} \\
&= \|\mathbf{r}(t)\| \sin\theta \|\boldsymbol{\omega}\| \\
&= \|\boldsymbol{\omega} \times \mathbf{r}\|,
\end{aligned}$$

clearly $\mathbf{u}$ is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$, and therefore

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}.$$

1.7.2 Isotropic Tensors

Definition 1.127. A tensor is *isotropic* if its components are identical in all Cartesian coordinate systems, i.e.,

$$T'_{i_1 i_2 \cdots i_n} = T_{i_1 i_2 \cdots i_n}. \tag{1.78}$$

Example 1.128. All scalars are isotropic, since the tensor transformation law states that $T' = T$ for tensors of order zero.

Example 1.129. There are no non-zero isotropic first-order tensors (vectors).

Example 1.130. The Kronecker delta δ_{ij} is an isotropic second-order tensor.

Theorem 1.131. An isotropic second-order tensor T_{ij} must be of the form

$$T_{ij} = \lambda \delta_{ij}, \tag{1.79}$$

where λ is a scalar.

Proof. Consider a rotation about the $\mathbf{e}_3$ axis through 90°, where the new axes are given by

$$\mathbf{e}'_1 = \mathbf{e}_2, \quad \mathbf{e}'_2 = -\mathbf{e}_1, \quad \mathbf{e}'_3 = \mathbf{e}_3.$$

The change of coordinate matrix corresponding to this rotation is

$$C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Using (1.55), we obtain

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = C^T T C = \begin{bmatrix} T_{22} & -T_{21} & T_{23} \\ -T_{12} & T_{11} & -T_{13} \\ T_{32} & -T_{31} & T_{33} \end{bmatrix}.$$

But, because T is isotropic, $T'_{ij} = T_{ij}$. Hence, comparing matrix entries, we have

$$T_{11} = T_{22};$$

$$T_{13} = T_{23} = -T_{13} \Rightarrow T_{13} = T_{23} = 0;$$

$$T_{31} = T_{32} = -T_{31} \Rightarrow T_{31} = T_{32} = 0.$$

Similarly, considering a rotation about $\mathbf{e}_2$ through 90°, we find that $T_{11} = T_{33}$ and that $T_{12} = T_{32} = 0, T_{21} = T_{23} = 0$. Therefore all off-diagonal elements of T are zero, and all diagonal elements are equal to a scalar λ. Thus $T_{ij} = \lambda \delta_{ij}$. □

Example 1.132. The alternating tensor ε_{ijk} is an isotropic third-order tensor.

Theorem 1.133. An isotropic third-order tensor T_{ijk} must be of the form

$$T_{ijk} = \lambda \varepsilon_{ijk}, \tag{1.80}$$

where λ is a scalar.

Exercise 1.134. Prove Theorem 1.133.

Example 1.135. $\delta_{ij}\delta_{kl}$ and $\varepsilon_{ijk}\varepsilon_{klm}$ are isotropic fourth-order tensors.

Theorem 1.136. An isotropic fourth-order tensor T_{ijkl} must be of the form

$$T_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}, \tag{1.81}$$

where λ, μ, and γ are three scalars.

Proof. The main idea is to apply some special rotations and utilize the isotropy of T_{ijkl} as in the proof of Theorem 1.131. For more details, see the book *Vectors, Tensors, and the Basic Equations of Fluid Mechanics* by R. Aris, Dover edition, 1962, pp 30–33. □

1.8 Cauchy's Equation of Motion

Assumption 1.137. From now on, assume that

$$\text{force on } S \text{ per unit area} = -p(\mathbf{x},t)\mathbf{n} + \mathbf{n}\cdot\boldsymbol{\sigma}(\mathbf{x},t), \quad (1.82)$$

where $\boldsymbol{\sigma}$ is the *(deviatoric) stress tensor* and $\mathbf{n}$ is the unit outward normal of S.

Definition 1.138. The *divergence of a second-order tensor* $\mathbf{T}$ is the vector whose jth component is

$$(\nabla\cdot\mathbf{T})_j = \frac{\partial T_{ij}}{\partial x_i}. \quad (1.83)$$

Lemma 1.139. The total force exerted on the fluid inside W by means of stress on its boundary is

$$\mathbf{S}_{\partial W} = \int_W (-\nabla p + \nabla\cdot\boldsymbol{\sigma})\,\mathrm{d}V. \quad (1.84)$$

Proof. By Assumption 1.137, we have

$$\begin{aligned}\mathbf{S}_{\partial W} &= \int_{\partial W}(-p\mathbf{n} + \mathbf{n}\cdot\boldsymbol{\sigma})\mathrm{d}A \\ &= \int_W (-\nabla p + \nabla\cdot\boldsymbol{\sigma})\mathrm{d}V,\end{aligned}$$

where the second equality follows from the divergence theorem. □

Theorem 1.140. The "primitive" integral form of the balance of momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho\mathbf{u}\mathrm{d}V = \int_{W_t}(-\nabla p + \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b})\mathrm{d}V. \quad (1.85)$$

Proof. The law of conservation of momentum (Principle 1.57) and Lemma 1.139. □

Theorem 1.141. The differential form of the balance of momentum is

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b}, \quad (1.86)$$

which is known as *Cauchy's equation of motion.*

Proof. This follows directly from Theorem 1.140, the transport theorem (Theorem 1.46), and the arbitrariness of W_t. □

1.9 Conservation of Angular Momentum

Principle 1.142 (Conservation of angular momentum)**.** The rate of change of angular momentum of a system is equal to the net torque acting on the system, i.e.,

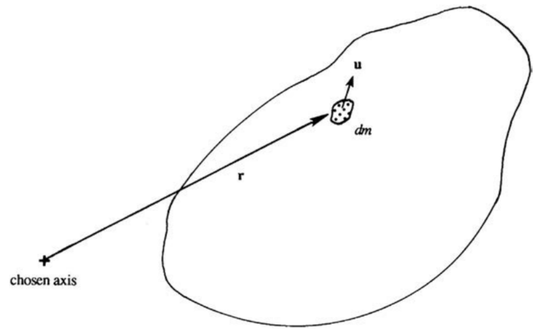
$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}, \quad (1.87)$$

where $\boldsymbol{\tau}$ is the torque of all external forces on the system about any chosen axis, and $\mathrm{d}\mathbf{L}/\mathrm{d}t$ is the rate of change of angular momentum of the system about the same axis.

Proposition 1.143. The angular momentum of the fluid in W at time t about a chosen axis is

$$\int_W \rho(\mathbf{x},t)\,(\mathbf{r}\times\mathbf{u})\,\mathrm{d}V, \quad (1.88)$$

where $\mathbf{r}$ is the position vector from the chosen axis.



Notation 8. In the following, $\mathbf{r}$ is chosen to coincide with the position vector $\mathbf{x}$, and therefore

$$\int_W \rho(\mathbf{x},t)(\mathbf{r}\times\mathbf{u})\mathrm{d}V = \int_W \rho(\mathbf{x},t)(\mathbf{x}\times\mathbf{u})\mathrm{d}V.$$

Lemma 1.144.

$$(\mathbf{u}\cdot\nabla)(\mathbf{x}\times\mathbf{u}) = \mathbf{x}\times(\mathbf{u}\cdot\nabla)\mathbf{u}. \quad (1.89)$$

Proof.

$$\begin{aligned} u_p\frac{\partial(\varepsilon_{ijk}x_ju_k)}{\partial x_p} &= u_p\varepsilon_{ijk}\delta_{jp}u_k + \varepsilon_{ijk}x_ju_p\frac{\partial u_k}{\partial x_p} \\ &= \varepsilon_{ijk}u_ju_k + \varepsilon_{ijk}x_ju_p\frac{\partial u_k}{\partial x_p} \\ &= \varepsilon_{ijk}x_ju_p\frac{\partial u_k}{\partial x_p},\end{aligned}$$

where we have used the fact that $\mathbf{u}\times\mathbf{u} = \mathbf{0}$ in obtaining the third equality. □

Theorem 1.145. The stress tensor $\boldsymbol{\sigma}$ is symmetric.

Proof. From the law of conservation of angular momentum (Principle 1.142) and Assumption 1.137, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho(\mathbf{x}\times\mathbf{u})\mathrm{d}V = \int_{\partial W_t}\mathbf{x}\times(-p\mathbf{n}+\mathbf{n}\cdot\boldsymbol{\sigma})\mathrm{d}A + \int_{W_t}\mathbf{x}\times(\rho\mathbf{b})\mathrm{d}V, \quad (1.90)$$

where the two terms on the right are the torque produced by surface stresses and body forces, respectively.

Applying the transport theorem (Theorem 1.46) to the left-hand side of (1.90) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho(\mathbf{x}\times\mathbf{u})\mathrm{d}V &= \int_{W_t}\rho\frac{D(\mathbf{x}\times\mathbf{u})}{Dt}\mathrm{d}V \\ &= \int_{W_t}\rho\frac{\partial(\mathbf{x}\times\mathbf{u})}{\partial t}\mathrm{d}V + \int_{W_t}\rho(\mathbf{u}\cdot\nabla)(\mathbf{x}\times\mathbf{u})\mathrm{d}V \\ &= \int_{W_t}\rho\mathbf{x}\times\frac{\partial\mathbf{u}}{\partial t}\mathrm{d}V + \int_{W_t}\rho\mathbf{x}\times(\mathbf{u}\cdot\nabla)\mathbf{u}\mathrm{d}V \\ &= \int_{W_t}\rho\mathbf{x}\times\frac{D\mathbf{u}}{Dt}\mathrm{d}V,\end{aligned}$$

where the second and the last step follow from the definition of the material derivative and the third equality from Lemma 1.144 and

$$\begin{aligned}\frac{\partial(\mathbf{x}\times\mathbf{u})}{\partial t} &= \frac{\partial \mathbf{x}}{\partial t}\times\mathbf{u}+\mathbf{x}\times\frac{\partial \mathbf{u}}{\partial t}\\ &= \mathbf{u}\times\mathbf{u}+\mathbf{x}\times\frac{\partial \mathbf{u}}{\partial t}\\ &= \mathbf{x}\times\frac{\partial \mathbf{u}}{\partial t}.\end{aligned}$$

We compute the ith component of $\int_{\partial W_t}\mathbf{x}\times(\mathbf{n}\cdot\boldsymbol{\sigma})\mathrm{d}A$ as

$$\begin{aligned}&\int_{\partial W_t}\varepsilon_{ijk}x_j(n_p\sigma_{pk})\mathrm{d}A\\ &=\int_{\partial W_t}\varepsilon_{ijk}x_j\sigma_{pk}n_p\mathrm{d}A=\int_{W_t}\varepsilon_{ijk}\frac{\partial(x_j\sigma_{pk})}{\partial x_p}\mathrm{d}V\\ &=\int_{W_t}\varepsilon_{ijk}\delta_{jp}\sigma_{pk}\mathrm{d}V+\int_{W_t}\varepsilon_{ijk}x_j\frac{\partial\sigma_{pk}}{\partial x_p}\mathrm{d}V\\ &=\int_{W_t}\left(\varepsilon_{ijk}\sigma_{jk}+\varepsilon_{ijk}x_j\frac{\partial\sigma_{pk}}{\partial x_p}\right)\mathrm{d}V,\end{aligned}$$

where the second step follows from the divergence theorem. The integrand

$$\varepsilon_{ijk}\sigma_{jk}+\varepsilon_{ijk}x_j\frac{\partial\sigma_{pk}}{\partial x_p}$$

is the ith component of the vector

$$\boldsymbol{\sigma}_\times+\mathbf{x}\times(\nabla\cdot\boldsymbol{\sigma}),$$

where $\boldsymbol{\sigma}_\times$ is the vector whose ith component is $\varepsilon_{ijk}\sigma_{jk}$.

Rearranging the terms in (1.90), we obtain

$$\begin{aligned}\int_{W_t}\boldsymbol{\sigma}_\times\mathrm{d}V &= \int_{W_t}\mathbf{x}\times\left(\rho\frac{D\mathbf{u}}{Dt}+\nabla p-\nabla\cdot\boldsymbol{\sigma}-\rho\mathbf{b}\right)\mathrm{d}V\\ &=\mathbf{0},\end{aligned}$$

where the second equality follows from Cauchy's equation of motion (1.86). Since W_t is arbitrary, we have

$$\boldsymbol{\sigma}_\times\equiv\mathbf{0}.$$

However, the components of $\boldsymbol{\sigma}_\times$ are $(\sigma_{23}-\sigma_{32}), (\sigma_{31}-\sigma_{13})$, and $(\sigma_{12}-\sigma_{21})$ and the vanishing of these implies

$$\sigma_{ij}=\sigma_{ji},$$

so that $\boldsymbol{\sigma}$ is symmetric. □

1.10 Constitutive Equation for a Newtonian Fluid

Definition 1.146. A *Newtonian fluid* is a fluid for which the stress tensor $\boldsymbol{\sigma}$ in (1.82) has the following properties:

(NFP-I) $\boldsymbol{\sigma}$ depends linearly on the velocity gradients $\nabla\mathbf{u}$ at each point, that is, there exists a fourth-order tensor c_{ijkl} such that

$$\sigma_{ij}=c_{ijkl}\frac{\partial u_k}{\partial x_l}. \tag{1.91}$$

(NFP-II) The fluid is isotropic (the material properties of the fluid at any given point are the same in all directions), and therefore c_{ijkl} is an isotropic tensor.

The relationship between the stress tensor $\boldsymbol{\sigma}$ and deformation in a continuum is called a *constitutive equation.*

Example 1.147. Most common fluids such as water, air, gasoline, and oils are Newtonian fluids. Blood and liquid plastics are examples of non-Newtonian fluids.

Theorem 1.148. The constitutive equation for a Newtonian fluid is

$$\boldsymbol{\sigma}=\lambda(\nabla\cdot\mathbf{u})\mathbf{I}+2\mu\mathbf{D}, \tag{1.92}$$

where $\mathbf{I}$ is the identity, $\mathbf{D}$ is the deformation tensor, and λ,μ are constants.

Proof. Since c_{ijkl} is an isotropic fourth-order tensor, we have, by Theorem 1.136,

$$c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}, \tag{1.93}$$

where λ,μ, and γ are three scalars. In addition, σ_{ij} is symmetric in i and j by Theorem 1.145, so (1.91) requires that c_{ijkl} also be symmetric in i and j, which, together with (1.93) and

$$\begin{aligned}c_{jikl}&=\lambda\delta_{ji}\delta_{kl}+\mu\delta_{jk}\delta_{il}+\gamma\delta_{jl}\delta_{ik}\\ &=\lambda\delta_{ij}\delta_{kl}+\gamma\delta_{ik}\delta_{jl}+\mu\delta_{il}\delta_{jk},\end{aligned}$$

yields

$$\gamma=\mu.$$

Therefore

$$c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{1.94}$$

it is easily seen that c_{ijkl} is also symmetric in k and l.

The velocity gradient $\nabla\mathbf{u}$ can be decomposed into a symmetric and anti-symmetric part (Examples 1.117 and 1.121), i.e.,

$$\frac{\partial u_k}{\partial x_l}=d_{kl}-\frac{1}{2}\varepsilon_{klm}\omega_m,$$

and thus

$$\sigma_{ij}=c_{ijkl}d_{kl}-\frac{1}{2}c_{ijkl}\varepsilon_{klm}\omega_m. \tag{1.95}$$

Next we show that the deviatoric stress tensor $\boldsymbol{\sigma}$ is only related to the symmetric part of $\nabla\mathbf{u}$. In fact,

$$c_{ijkl}\varepsilon_{klm}\omega_m=c_{ijlk}(-\varepsilon_{lkm})\omega_m=-c_{ijkl}\varepsilon_{klm}\omega_m,$$

where the first step holds since c_{ijkl} is symmetric in k and l and the alternating tensor ε_{klm} is anti-symmetric in k and l, and the second step follows from swapping indices k and l, hence

$$c_{ijkl}\varepsilon_{klm}\omega_m=0. \tag{1.96}$$

Use (1.96), substitute (1.94) into (1.95), and we have

$$\begin{aligned}\sigma_{ij}=c_{ijkl}d_{kl}&=\lambda\delta_{ij}\delta_{kl}d_{kl}+\mu\delta_{ik}\delta_{jl}d_{kl}+\mu\delta_{il}\delta_{jk}d_{kl}\\ &=\lambda d_{kk}\delta_{ij}+2\mu d_{ij}.\end{aligned}$$

□

1.11 The Incompressible Navier-Stokes Equations

Theorem 1.149. The *Navier-Stokes equations* for a Newtonian fluid are

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u} + \rho \mathbf{b}. \tag{1.97}$$

Proof. This follows from (1.86), (1.92), and the identity

$$\nabla \cdot (\lambda(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu \mathbf{D}) = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u}. \qquad \square$$

Theorem 1.150. The *Navier-Stokes equations for an incompressible homogeneous ($\rho = \rho_0 =$ constant) flow* are

$$\frac{D\mathbf{u}}{Dt} = -\nabla p' + \nu \Delta \mathbf{u} + \mathbf{b}, \tag{1.98a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.98b}$$

with no-slip boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ on solid walls at rest,} \tag{1.99}$$

where $\nu = \mu/\rho_0$ is the *coefficient of kinematic viscosity*, and $p' = p/\rho_0$.

Proof. Proposition 1.38 and Theorem 1.149. $\square$

Theorem 1.151. The *dimensionless form of the Navier-Stokes equations for an incompressible homogeneous ($\rho = \rho_0 =$ constant) flow* is

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}' \cdot \nabla')\mathbf{u}' = -\nabla p' + \frac{1}{\mathrm{Re}}\Delta' \mathbf{u}' + \mathbf{b}', \tag{1.100a}$$

$$\nabla' \cdot \mathbf{u}' = 0, \tag{1.100b}$$

where

$$\begin{aligned} &\mathbf{u}' = \frac{\mathbf{u}}{U}, \quad \mathbf{x}' = \frac{\mathbf{x}}{L}, \quad t' = \frac{t}{T} = \frac{t}{L/U}, \\ &p' = \frac{p}{\rho_0 U^2}, \quad \mathbf{b}' = \frac{\mathbf{b}}{U^2/L}, \end{aligned} \tag{1.101}$$

and

$$\mathrm{Re} = \frac{LU}{\nu} \tag{1.102}$$

is the dimensionless *Reynolds number*. L is called the *characteristic length* and U the *characteristic velocity*.

Proof. The component form of (1.98a) is

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i} + b_j.$$

The change of variables produces

$$\begin{aligned} &\frac{\partial (U u_j')}{\partial t'}\frac{\partial t'}{\partial t} + (U u_i')\frac{\partial (U u_j')}{\partial x_i'}\frac{\partial x_i'}{\partial x_i} \\ &= -\frac{1}{\rho_0}\frac{\partial(\rho_0 U^2 p')}{\partial x_j'}\frac{\partial x_j'}{\partial x_j} + \nu \frac{\partial^2 (U u_j')}{\partial (L x_i')\partial (L x_i')} + \frac{U^2}{L} b_j'. \end{aligned}$$

Hence

$$\frac{U^2}{L}\left(\frac{\partial u_j'}{\partial t'} + u_i' \frac{\partial u_j'}{\partial x_i'}\right) = -\frac{U^2}{L}\frac{\partial p'}{\partial x_j'} + \frac{\nu U}{L^2}\frac{\partial^2 u_j'}{\partial x_i' \partial x_i'} + \frac{U^2}{L} b_j',$$

dividing both sides by U^2/L gives the desired result. $\square$

Definition 1.152. Two flows with the same geometry and the same Reynolds number are called *similar*.

Example 1.153. The significance of the similarity of flows can be seen as follows. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two flows on regions D_1 and D_2 that are related by a scalar factor λ so that $L_1 = \lambda L_2$. Let choice of U_1 and U_2 be made for each flow, and let the viscosities be ν_1 and ν_2 respectively. If

$$\mathrm{Re}_1 = \mathrm{Re}_2, \text{ i.e., } \frac{L_1 U_1}{\nu_1} = \frac{L_2 U_2}{\nu_2},$$

then the dimensionless velocity fields $\mathbf{u}_1'$ and $\mathbf{u}_2'$ satisfy exactly the same equation on the same region. Thus, we can conclude that $\mathbf{u}_1$ may be obtained from a suitably rescaled solution $\mathbf{u}_2$.

Example 1.154. The idea of the similarity of flows is used in the design of experimental models. For example, suppose we are contemplating a new design for an aircraft wing and we wish to know the behavior of a fluid flow around it. Rather than build the wing itself, it may be faster and more economical to perform the initial tests on a scaled-down version. We design our model so that it has the same geometry as the full-scale wing and we choose values for the undisturbed velocity, coefficient of viscosity, and so on, such that the Reynolds number for the flow in our experiment matches that of the actual flow. We can then expect the results of our experiment to be relevant to the actual flow over the full-scale wing.

Notation 9. In the following, we shall simply write the incompressible Navier-Stokes equations (INSE) as

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta \mathbf{u} + \mathbf{b}, \tag{1.103a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.103b}$$

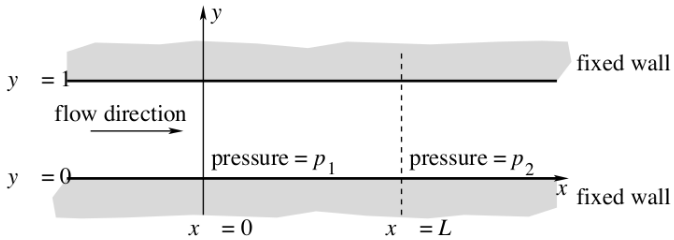
Definition 1.155. In the momentum equation (1.103a), we call

$$\frac{1}{\mathrm{Re}}\Delta \mathbf{u}, \quad \text{the } \textit{diffusion} \text{ or } \textit{dissipation} \text{ term,}$$

and

$$(\mathbf{u} \cdot \nabla)\mathbf{u}, \quad \text{the } \textit{inertial} \text{ or } \textit{convective} \text{ term.}$$

Example 1.156. Consider the steady viscous incompressible flow between two stationary plates located at $y = 0$ and $y = 1$.



We seek a solution for which $\mathbf{u}(x, y) = (u(x, y), 0)$ and p is only a function of x, with $p_1 = p(0), p_2 = p(L)$, and $p_1 > p_2$, so the fluid is "pushed" in the positive x direction. The incompressible Navier-Stokes equations are

$$\partial_x u = 0$$

and

$$0 = -u\partial_x u - \partial_x p + \frac{1}{\mathrm{Re}}\left(\partial_x^2 u + \partial_y^2 u\right)$$

with no-slip boundary conditions $u(x, 0) = u(x, 1) = 0$. Because $\partial_x u = 0$, u is only a function of y and thus, writing $u(x, y) = u(y)$, we obtain

$$p'(x) = \frac{1}{\mathrm{Re}} u''(y).$$

Because each side depends on different variables,

$$p' = \text{constant}, \quad \frac{1}{\mathrm{Re}} u'' = \text{constant}.$$

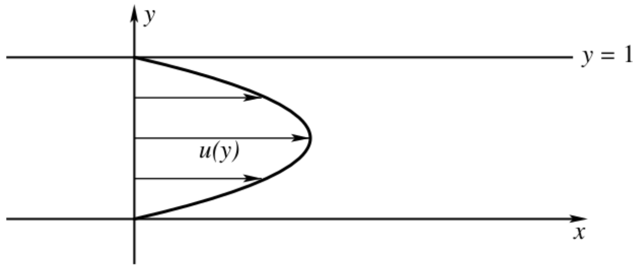
Integration gives

$$p(x) = p_1 - \frac{\Delta p}{L} x, \quad \Delta p = p_1 - p_2,$$

and

$$u(y) = y(1-y)\mathrm{Re}\frac{\Delta p}{2L}.$$

Notice that the velocity profile is a parabola.



The presence of viscosity allows the pressure forces to be balanced by the term $\frac{1}{\mathrm{Re}} u''(y)$ and allows the fluid to achieve a steady state. We saw that this was not possible for ideal flows (Example 1.83).

1.11.1 The Leray-Helmholtz Projection $\mathscr{P}$

Theorem 1.157. The Neumann boundary value problem

$$\Delta p = f \text{ in } D, \quad \mathbf{n} \cdot \nabla p = g \text{ on } \partial D \tag{1.104}$$

has a solution unique up to an additive constant if and only if

$$\int_D f \mathrm{d}V = \int_{\partial D} g \mathrm{d}A. \tag{1.105}$$

Proof. Not required. The proof can be found in the book *Partial Differential Equations I* by M. E. Taylor, 2nd edition, Springer, 2011, pp 408-409. □

Lemma 1.158. Suppose the vector field $\mathbf{u}$ has zero divergence and is parallel to ∂D, i.e.,

$$\nabla \cdot \mathbf{u} = 0 \text{ in } D \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D, \tag{1.106}$$

then we have the orthogonality relation

$$\int_D \mathbf{u} \cdot \nabla p \mathrm{d}V = 0 \tag{1.107}$$

for any scalar field p on D.

Proof.

$$\begin{aligned}\int_D \mathbf{u} \cdot \nabla p \mathrm{d}V &= \int_D \nabla \cdot (p\mathbf{u}) \mathrm{d}V - \int_D p \nabla \cdot \mathbf{u} \mathrm{d}V \\ &= \int_{\partial D} p\mathbf{u} \cdot \mathbf{n} \mathrm{d}A = 0,\end{aligned}$$

where the first equality follows from Lemma 1.28, the second from the divergence theorem and (1.106), and the last from (1.106). □

Theorem 1.159 (Helmholtz-Hodge decomposition)**.** A vector field $\mathbf{w}$ on D can be uniquely decomposed in the form

$$\mathbf{w} = \mathbf{u} + \nabla p, \tag{1.108}$$

where $\mathbf{u}$ has zero divergence and is parallel to ∂D.

Proof. By Theorem 1.157, the Neumann problem

$$\Delta p = \nabla \cdot \mathbf{w} \text{ in } D, \quad \mathbf{n} \cdot \nabla p = \mathbf{w} \cdot \mathbf{n} \text{ on } \partial D,$$

has a solution unique up to an additive constant for each given $\mathbf{w}$ on D. If we define

$$\mathbf{u} = \mathbf{w} - \nabla p, \tag{1.109}$$

then $\mathbf{u}$ clearly has the desired property (1.106), this completes the proof of existence.

Now we prove the uniqueness of $\mathbf{u}$. Suppose that $\mathbf{w} = \mathbf{u}_1 + \nabla p_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \nabla p_2$, then

$$\mathbf{0} = \mathbf{u}_1 - \mathbf{u}_2 + \nabla(p_1 - p_2),$$

taking inner product with $\mathbf{u}_1 - \mathbf{u}_2$ and integrating, we get

$$\begin{aligned}0 &= \int_D \left(\|\mathbf{u}_1 - \mathbf{u}_2\|^2 + (\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla(p_1 - p_2)\right) \mathrm{d}V \\ &= \int_D \|\mathbf{u}_1 - \mathbf{u}_2\|^2 \mathrm{d}V,\end{aligned}$$

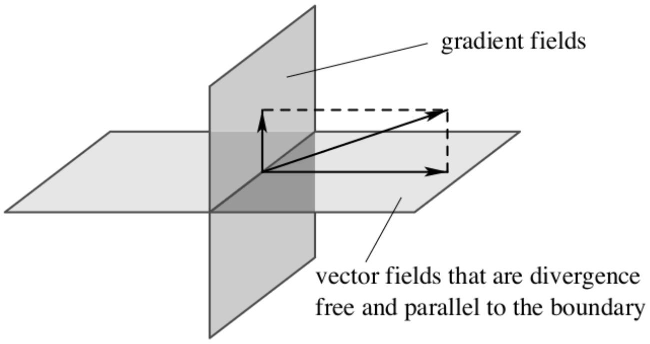
by the orthogonality relation (1.107). It follows that $\mathbf{u}_1 = \mathbf{u}_2$, and so, $\nabla p_1 = \nabla p_2$ (which is the same thing as $p_1 = p_2 + \text{constant}$). □

Definition 1.160. The *Leray-Helmholtz projection operator* $\mathscr{P}$ maps a vector field $\mathbf{w}$ onto its divergence-free part $\mathscr{P}\mathbf{w}$ given by the Helmholtz-Hodge decomposition theorem, that is,

$$\mathbf{w} = \mathscr{P}\mathbf{w} + \nabla p, \tag{1.110}$$

where

$$\nabla \cdot (\mathscr{P}\mathbf{w}) = 0 \text{ in } D \text{ and } \mathbf{n} \cdot (\mathscr{P}\mathbf{w}) = 0 \text{ on } \partial D. \tag{1.111}$$



Proposition 1.161. If $\mathbf{u}$ has zero divergence and is parallel to ∂D, then

$$\mathscr{P}\mathbf{u} = \mathbf{u}. \tag{1.112}$$

Proof. Theorem 1.159 and Definition 1.160. □

Proposition 1.162. The Leray-Helmholtz projection operator $\mathscr{P}$ annihilates gradient fields, i.e.,

$$\mathscr{P}\nabla p = \mathbf{0}, \tag{1.113}$$

for any scalar field p.

Proof. Theorem 1.159 and Definition 1.160. □

Proposition 1.163. $\mathscr{P}$ is a linear operator and

$$\mathscr{P}^2 = \mathscr{P}. \tag{1.114}$$

Proof. Theorem 1.159, Definition 1.160, and Proposition 1.161. □

Proposition 1.164. The Leray-Helmholtz projection operator $\mathscr{P}$ may be written symbolically as

$$\mathscr{P} = \mathcal{I} - \nabla(\nabla\cdot\nabla)^{-1}\nabla\cdot, \tag{1.115}$$

where $\mathcal{I}$ is the identity operator.

Proof. Taking divergence of both sides of (1.110) and utilizing (1.111), we obtain

$$\nabla\cdot\mathbf{w} = \nabla\cdot\mathscr{P}\mathbf{w} + \nabla\cdot(\nabla p) = \nabla\cdot(\nabla p),$$

and therefore

$$p = (\nabla\cdot\nabla)^{-1}\nabla\cdot\mathbf{w}.$$

Now taking gradient of both sides gives

$$\nabla p = \nabla(\nabla\cdot\nabla)^{-1}\nabla\cdot\mathbf{w} = \mathbf{w} - \mathscr{P}\mathbf{w},$$

and hence

$$\mathscr{P}\mathbf{w} = \left(\mathcal{I} - \nabla(\nabla\cdot\nabla)^{-1}\nabla\cdot\right)\mathbf{w}.$$ □

Example 1.165. Rewrite the momentum equation (1.103a) to a form that resembles (1.108),

$$\partial_t\mathbf{u} + \nabla p = -(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b},$$

applying the Leray-Helmholtz projection operator $\mathscr{P}$ to both sides, we obtain

$$\mathscr{P}\left(\partial_t\mathbf{u} + \nabla p\right) = \mathscr{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b}\right).$$

Because $\mathbf{u}$ is divergence-free and vanishes on the boundary, the same is true of $\partial_t\mathbf{u}$ (if $\mathbf{u}$ is smooth enough). Thus, by Proposition 1.161, $\mathscr{P}\partial_t\mathbf{u} = \partial_t\mathbf{u}$. Because $\mathscr{P}\nabla p = \mathbf{0}$, we get

$$\partial_t\mathbf{u} = \mathscr{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b}\right). \tag{1.116}$$

Note that although $\Delta\mathbf{u}$ is divergence free, it need not be parallel to the boundary and so we cannot simply write $\mathscr{P}\Delta\mathbf{u} = \mathbf{0}$. This form (1.116) of the INSE eliminates the pressure and expresses $\partial_t\mathbf{u}$ in terms of $\mathbf{u}$ alone. The pressure can then be recovered as the gradient part of

$$-(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b}.$$

This form (1.116) of the equations is not only of practical interest, shedding light on the role of the pressure, but is of practical interest for numerical algorithms.

Example 1.166. Suppose the Reynolds number Re is very small. If we write the equations in the form $\partial_t\mathbf{u} = \mathscr{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b}\right)$, we see that they are approximated by

$$\partial_t\mathbf{u} = \mathscr{P}\left(\frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b}\right),$$

that is,

$$\partial_t\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{b} \text{ and } \nabla\cdot\mathbf{u} = 0,$$

which are the *Stokes' equations* for incompressible flows. These are linear equations of "parabolic" type. For small Re (i.e., slow velocity, large viscosity, or small bodies), the solution of the Stokes' equation provides a good approximation to the solution of the Navier-Stokes equations.

In practice, we shall mostly be interested in flows with large Re; for these the inertial term is important and in some cases is dominant. We hesitate and say "in some sense" because no matter how small $(1/\mathrm{Re})\Delta\mathbf{u}$ may be, it still produces a large effect, namely, the change in boundary conditions from $\mathbf{u}\cdot\mathbf{n} = 0$ when $(1/\mathrm{Re})\Delta\mathbf{u}$ is absent to $\mathbf{u} = \mathbf{0}$ when it is present.

1.11.2 Energy Dissipation

Lemma 1.167 (Gauss-Green theorem)**.** Suppose $u \in \mathcal{C}^1(\bar{D})$. Then

$$\int_D \frac{\partial u}{\partial x_i}\mathrm{d}V = \int_{\partial D} u n_i \mathrm{d}A,$$

where n_i is the ith component of the unit outward normal $\mathbf{n}$ of ∂D.

Lemma 1.168 (Integration-by-parts formula)**.** Let $u, v \in \mathcal{C}^1(\bar{D})$. Then

$$\int_D \frac{\partial u}{\partial x_i} v\mathrm{d}V = -\int_D u\frac{\partial v}{\partial x_i}\mathrm{d}V + \int_{\partial D} uvn_i\mathrm{d}A, \tag{1.117}$$

where n_i is the ith component of the unit outward normal $\mathbf{n}$ of ∂D.

Proof. Apply Gauss-Green theorem to uv. □

Lemma 1.169 (Green's formula)**.** Let $u, v \in \mathcal{C}^2(\bar{U})$. Then

$$\int_D \nabla u \cdot \nabla v \mathrm{d}V = -\int_D u \Delta v \mathrm{d}V + \int_{\partial D} u \frac{\partial v}{\partial \mathbf{n}} \mathrm{d}A. \tag{1.118}$$

Proof. Employ the integration-by-parts formula with $v = \partial v / \partial x_i$ and sum over i. □

Theorem 1.170. For an incompressible homogeneous viscous flow with no body forces ($\mathbf{b} = \mathbf{0}$), assume that the density is normalized to $\rho = 1$, then we have the energy dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\text{kinetic}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_D \|\mathbf{u}\|^2 \mathrm{d}V\right) = -\nu \int_D \|\nabla \mathbf{u}\|^2 \mathrm{d}V \leq 0. \tag{1.119}$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_{\text{kinetic}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_D \|\mathbf{u}\|^2\right) \mathrm{d}V = \int_D \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} \mathrm{d}V \\
&= -\int_D \mathbf{u} \cdot \nabla p \mathrm{d}V - \int_D \mathbf{u} \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u})\, \mathrm{d}V + \nu \int_D \mathbf{u} \cdot (\Delta \mathbf{u}) \mathrm{d}V,
\end{aligned}$$

where the last equality follows from (1.98a). The orthogonality relation (1.107) gives

$$\int_D \mathbf{u} \cdot \nabla p \mathrm{d}V = 0.$$

Next we compute the inertial term as

$$\begin{aligned}
&\int_D \mathbf{u} \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}) \mathrm{d}V = \int_D u_i \left(u_j \frac{\partial u_i}{\partial x_j}\right) \mathrm{d}V \\
=&\frac{1}{2}\int_D u_j \frac{\partial u_i^2}{\partial x_j} \mathrm{d}V = -\frac{1}{2}\int_D u_i^2 \frac{\partial u_j}{\partial x_j} \mathrm{d}V + \frac{1}{2}\int_{\partial D} u_i^2 u_j n_j \mathrm{d}A \\
=& -\frac{1}{2}\int_D \|\mathbf{u}\|^2 (\nabla \cdot \mathbf{u}) \mathrm{d}V = 0,
\end{aligned}$$

where the third step follows from the integration-by-parts formula, the fourth from the no-slip boundary condition of $\mathbf{u}$, and the last from the incompressibility. The diffusion term can be computed by

$$\begin{aligned}
\int_D \mathbf{u} \cdot (\Delta \mathbf{u}) \mathrm{d}V &= \int_D u_i \Delta u_i \mathrm{d}V \\
&= -\int_D \|\nabla \mathbf{u}\|^2 \mathrm{d}V + \int_{\partial D} u_i \frac{\partial u_i}{\partial \mathbf{n}} \mathrm{d}A \\
&= -\int_D \|\nabla \mathbf{u}\|^2 \mathrm{d}V,
\end{aligned}$$

where the second step follows from Green's formula and the last from the no-slip boundary condition of $\mathbf{u}$.

Combining the above results, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\text{kinetic}} = -\nu \int_D \|\nabla \mathbf{u}\|^2 \mathrm{d}V \leq 0.$$

□

Exercise 1.171. Under the assumptions of Theorem 1.170 with "viscous" replaced by "ideal", we have the energy conservation for Euler's equations, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\text{kinetic}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_D \|\mathbf{u}\|^2 \mathrm{d}V\right) = 0. \tag{1.120}$$

1.12 Dimensional Analysis

Example 1.172. Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiment or approximated by computational fluid dynamics (CFD). These results are typically reported as experimental or numerical data points and smoothed curves. Such data have much more generality if they are expressed in compact, economical form. This is the motivation for dimensional analysis.

Definition 1.173. A *dimension* is a measure of a physical quantity (without numerical values).

Example 1.174. In fluid mechanics, the four *fundamental* (also called *basic* or *primary*) dimensions are usually taken to be

$$\text{mass } M,\ \text{length } L,\ \text{time } T,\ \text{and temperature } \theta. \tag{1.121}$$

An alternative system uses

$$\text{force } F,\ \text{length } L,\ \text{time } T,\ \text{and temperature } \theta. \tag{1.122}$$

All other derived dimensions can be formed by some combination of the fundamental dimensions.

Notation 10. We use the square bracket as a shorthand for "the dimension of ... is", e.g.,

$$\begin{aligned}
[\text{density}] &= \frac{[\text{mass}]}{[\text{volume}]} = ML^{-3}, \\
[\text{velocity}] &= \frac{[\text{length}]}{[\text{time}]} = LT^{-1}, \\
[\text{acceleration}] &= \frac{[\text{length}]}{[\text{time}]^2} = LT^{-2}, \\
[\text{force}] &= [\text{mass}] \times [\text{acceleration}] = MLT^{-2}, \\
[\text{pressure}] &= \frac{[\text{force}]}{[\text{area}]} = ML^{-1}T^{-2}, \\
[\text{dynamic viscosity}] &= \frac{[\text{stress}]}{[\text{velocity-gradient}]} \\
&= \frac{[\text{force}]/[\text{area}]}{[\text{velocity}]/[\text{length}]} = ML^{-1}T^{-1}.
\end{aligned}$$

Principle 1.175 (Principle of dimensional homogeneity)**.** All physically meaningful equations are dimensionally homogeneous, that is, every additive term in an equation must have the same dimension.

Example 1.176. Consider the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant},$$

each additive term has the same dimension L^2T^{-2}.

1.12.1 Buckingham Π Theorem

Definition 1.177. A *dimensionless* physical quantity is a quantity to which no physical dimension is assigned.

Example 1.178. The Reynolds number

$$\mathrm{Re} = \frac{\rho L U}{\mu} = \frac{LU}{\nu}$$

and the Mach number

$$\mathrm{Ma} = \frac{V}{c}$$

are both dimensionless.

Notation 11. Denote the dimension of a dimensionless variable q by 1, i.e.,

$$[q] = 1. \tag{1.123}$$

Definition 1.179. The physical quantities $q_1, q_2, \ldots, q_n$ are called *dimensionally independent* if

$$\prod_{i=1}^{n} [q_i]^{\alpha_i} = 1 \Rightarrow \alpha_i = 0 \text{ for } i = 1, 2, \ldots, n. \tag{1.124}$$

Example 1.180. The physical quantities density ($[\rho] = ML^{-3}$), velocity ($[U] = LT^{-1}$), and force ($[f] = MLT^{-2}$) are dimensionally independent.

Example 1.181. The physical quantities density ($[\rho] = ML^{-3}$), velocity ($[U] = LT^{-1}$), and pressure ($[p] = ML^{-1}T^{-2}$) are not dimensionally independent since

$$[\rho][U]^2[p]^{-1} = 1.$$

Definition 1.182. Given a system of n physical variables $q_1, q_2, \ldots, q_n$ in k fundamental dimensions $d_1, d_2, \ldots, d_k$ with

$$[q_j] = \prod_{i=1}^{k} d_i^{a_{ij}}, \tag{1.125}$$

the *dimensional matrix* A of $q_1, q_2, \ldots, q_n$ with respect to $d_1, d_2, \ldots, d_k$ is defined by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix}. \tag{1.126}$$

Theorem 1.183 (Buckingham Π theorem)**.** Let $q_1, q_2, \ldots, q_n$ be n variables and parameters involved in a particular physical problem and there exists a functional relationship of the form

$$f(q_1, q_2, \ldots, q_n) = 0, \tag{1.127}$$

then $q_1, q_2, \ldots, q_n$ can be expressed in terms of r independent physical quantities, and (1.127) can be restated as

$$\varphi(\Pi_1, \Pi_2, \ldots, \Pi_{n-r}) = 0 \text{ or } \Pi_1 = \psi(\Pi_2, \ldots, \Pi_{n-r}), \tag{1.128}$$

where Π_i are $n-r$ dimensionless parameters constructed from the q_i — the so-called Π-*groups* (or *dimensionless groups*).

Proof. Let $d_1, d_2, \ldots, d_k$ be k fundamental dimensions involved in this particular physical problem. Denote the dimensional matrix of $q_1, q_2, \ldots, q_n$ with respect to $d_1, d_2, \ldots, d_k$ by A, as in Definition 1.182.

Suppose $\mathrm{rank}(A) = r$, i.e., A has r linearly independent columns, we may as well assume these are the first r columns, corresponding to the variables $q_1, q_2, \ldots, q_r$ (If not, we just renumber the variables to make it thus).

Now we show that $q_1, \ldots, q_r$ are dimensionally independent. Assume there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_r$ such that

$$\prod_{j=1}^{r} [q_j]^{\alpha_j} = 1,$$

which, by definition of the dimensional matrix, becomes

$$\prod_{j=1}^{r} [q_j]^{\alpha_j} = \prod_{j=1}^{r} \left(\prod_{i=1}^{k} d_i^{a_{ij}} \right)^{\alpha_j} = \prod_{i=1}^{k} d_i^{\sum_{j=1}^{r} a_{ij}\alpha_j},$$

and thus

$$\sum_{j=1}^{r} a_{ij}\alpha_j = 0, \quad 1 \le i \le k,$$

which implies $\alpha_j = 0$ for all j, since the first r columns of the dimensional matrix A are linearly independent. This completes the proof that $q_1, q_2, \ldots, q_r$ are dimensionally independent.

For each j satisfying $r+1 \le j \le n$, the jth column of A can be expressed as a linear combination of the first r columns, i.e., there exist constants $\beta_m^j (1 \le m \le r)$ such that

$$a_{ij} = \sum_{m=1}^{r} \beta_m^j a_{im},$$

using these constants, we can construct $n-r$ dimensionless parameters

$$\Pi_{j-r} = q_j q_1^{-\beta_1^j} q_2^{-\beta_2^j} \cdots q_r^{-\beta_r^j}, \quad r+1 \le j \le n,$$

the relation (1.127) now reads

$$f\left(q_1, \ldots, q_r, q_1^{\beta_1^{r+1}} \cdots q_r^{\beta_r^{r+1}} \Pi_1, \ldots, q_1^{\beta_1^n} \cdots q_r^{\beta_r^n} \Pi_{n-r}\right) = 0,$$

the fact that this can be further reduced to (1.128) is a consequence of the principle of dimensional homogeneity. □

1.12.2 The Method of Repeating Variables

Example 1.184. We illustrate the general steps of *the method of repeating variables* for performing dimensional analysis. Consider the example of determining the functional dependence of the pressure difference Δp between two locations in a round pipe carrying a flowing viscous fluid.

(i) *select variables and parameters*

- Δp: pressure difference (solution variable);
- Δx: distance between the pressure measurement locations;
- d: inside diameter of the pipe;

- ε: average height of the pipe's wall roughness;
- U: average flow velocity;
- ρ: fluid density;
- μ: fluid viscosity.

The resulting functional dependence between these seven parameters can be stated as

$$f(\Delta p, \Delta x, d, \varepsilon, U, \rho, \mu) = 0. \tag{1.129}$$

(ii) *create the dimensional matrix*

Express the above variables in terms of basic dimensions, e.g., $[\Delta p] = [\text{force}]/[\text{area}] = MLT^{-2}/L^2 = ML^{-1}T^{-2}$, $[\mu] = [\text{density}][\text{length}][\text{velocity}] = (ML^{-3})L(LT^{-1}) = ML^{-1}T^{-1}$.

	Δp	Δx	d	ϵ	U	ρ	μ
M	1	0	0	0	0	1	1
L	−1	1	1	1	1	−3	−1
T	−2	0	0	0	−1	0	−1

(1.130)

(iii) *determine the rank of the dimensional matrix*

The rank of the dimensional matrix in (1.130) is $r = 3$.

(iv) *determine the number of dimensionless groups*

The number of dimensionless groups is $n - r = 7 - 3 = 4$.

(v) *construct the dimensionless groups*

Select r parameters from the dimensional matrix as *repeating parameters*: the determinant of the dimensional matrix formed from these r parameters must be nonzero. (For many fluid-flow problems, a characteristic velocity, a characteristic length, and a fluid property involving mass are ideal repeating parameters).

Choose U, d, and ρ as the repeating parameters.

Construct the dimensionless group

$$\Pi_1 = \Delta p U^a d^b \rho^c$$

involving Δp.

$$\begin{aligned} M^0L^0T^0 &= [\Pi_1] = [\Delta p U^a d^b \rho^c] \\ &= (ML^{-1}T^{-2})(LT^{-1})^a(L^b)(ML^{-3})^c \\ &= M^{c+1}L^{a+b-3c-1}T^{-a-2}. \end{aligned}$$

Equating exponents between the two extreme ends of this extended equality produces three algebraic equations that are readily solved to find $a = -2, b = 0, c = -1$, so

$$\Pi_1 = \frac{\Delta p}{\rho U^2}.$$

A similar procedure with Δp replaced by the other unused variables $(\Delta x, \varepsilon, \mu)$ produces

$$\Pi_2 = \frac{\Delta x}{d}, \quad \Pi_3 = \frac{\varepsilon}{d}, \quad \Pi_4 = \frac{\mu}{\rho U d}.$$

(vi) *state the dimensionless relationship*

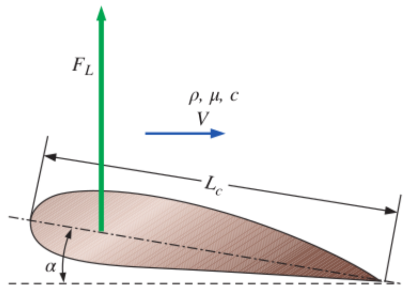
The dimensionless relationship corresponding to (1.129) expressed in the dimensionless groups Π_i constructed in Step (v) is

$$\frac{\Delta p}{\rho U^2} = \varphi\left(\frac{\Delta x}{d}, \frac{\varepsilon}{d}, \frac{\mu}{\rho U d}\right), \tag{1.131}$$

where φ is an undetermined function.

(vii) *use physical reasoning or additional knowledge to simplify the dimensionless relationship*

Exercise 1.185. Consider the lift on a wing,



nondimensionalize the functional form

$$f(F_L, V, L_c, \rho, \mu, c, \alpha) = 0. \tag{1.132}$$

Here,

- F_L: lift force on the wing (solution variable);
- V: fluid velocity;
- L_c: chord length of the wing;
- ρ: fluid density;
- μ: fluid viscosity;
- c: speed of sound in the fluid;
- α: angle of attack of the wing.