Chapter 1

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Section 1.1

(2) Find general solutions for spherically symmetric functions $V = V(|x|)$ such that $\Delta V = 0, x \in \mathbb{R}^n \backslash \{0\}$ with $n \geq 2$.

(3) (minimal surface) Let $u : \Omega \to \mathbb{R}$ with $\Omega \in \mathbb{R}^2$. For fixed function f on the boundary of Ω, suppose u is a function such that it minimizes the area

$$A[u] = \int_\Omega \sqrt{1 + |\nabla u|^2} d\mathbf{x},$$

among all surface with $u = f$ on the boundary of $Omega$. Try to derive a PDE for u.

Solution.

(2) Since $|x| = (\sum x_i^2)^{1/2}$, we have $\partial |x| / \partial x = x/|x|$.

So,

$$V' = V'(|x|) \partial |x| / \partial x = V' x / |x|,$$

and

$$\Delta V = V_{x_i x_i} + V_{x_i} (n-1)/|x|.$$

That means

$$\Delta V = 0 \Longleftrightarrow V'' + (n-1) V'/|x| = 0.$$

Differential $r^{n-1} V'$ get

$$(r^{n-1} V')' = r^{n-1} \left(V'' + (n-1) V'/|x| \right) = 0.$$

hence,

$$V' = a / r^{n-1}.$$

Therefore

$$V = \begin{cases} a \log |x| + b & (n = 2) \\ a/|x|^{n-2} + b & (n \geq 3) \end{cases}.$$

(3) Assume u match the condition, For any smooth function $v : \Omega \to \mathbb{R}$ and Real δ. Define $w(\delta) = u + \delta v$ satisfy $w(\delta) = f$ on the $\partial \Omega$. So $A[u] \leq A[w(\delta)]$.

Hence

$$w'(\delta) = \int_\Omega \frac{(|\nabla v|^2 \delta + \nabla u \cdot \nabla v)}{\sqrt{1 + |\nabla u|^2}} d\mathbf{x}.$$

Set $\delta = 0$,

$$0 = w'(0) = \int_\Omega \frac{\Delta u \cdot \Delta v}{\sqrt{1+|\nabla u|^2}} d\mathbf{x} = \int_\Omega v\Delta \cdot \left(\frac{\Delta u}{\sqrt{1+|\nabla u|^2}} \right) d\mathbf{x} \qquad \text{Since } v = 0, \text{ on } \partial\Omega$$

.

By random of v, we know

$$\Delta \cdot \left(\frac{\Delta u}{\sqrt{1+|\nabla u|^2}} \right) = 0$$

Section 1.3

(3) Solve $u_t + uu_x = 0, u(0, x) = -x$.

Solution.

(3) Firstly,

$$\det \begin{pmatrix} \partial_x g^1 & 1 \\ \partial_x g^2 & u \end{pmatrix} = -1 \neq 0.$$

We obtain

$$t = s, x = -x_0 s + x_0, u = -x_0$$

. means $\forall (t, x) \in \mathbb{R}^2, u(t, x) = -x_0$,

and since

$$x_0 = \frac{x}{1-s} = \frac{x}{1-t}$$

. So

$$u = \frac{x}{t-1}$$

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Section 1.5

(5) Solve $u_t + xu_x = x$ with data $u(0, x) = 2x$, by applying the power series method.

Solution.

(5) Assume

$$u(t, x) = \sum \frac{c_{j,k}}{j!k!} t^j x^k + 2x.$$

So $c_{0,1} = 0$, substitute into $u_t + xu_x = x$ know

$$\sum_{j \neq 0} (\frac{c_{j+1,k} + c_{j,k}}{j!k!}) t^j x^k + \sum \frac{c_{1,k}}{k!} x^k = -x.$$

Which imply

$$c_{1,k} = \begin{cases} -1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$
$$c_{j+1,k} = -c_{j,k}, \qquad j \neq 0.$$

So

$$c_{j,k} = \begin{cases} (-1)^j & k = 1 \\ 0 & k \neq 1 \end{cases}.$$

Therefore,

$$u(t, x) = \sum_{j \neq 0} \frac{(-t)^j x}{j!} + 2x$$