Appendix E

Fourier Analysis of Linear PDEs

Definition E.1. The L^2-*norm* of a Lebesgue-measurable function $u : \mathbb{R} \to \mathbb{C}$ is a nonnegative or infinite real number

$$\|u\| = \left[\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x\right]^{\frac{1}{2}}. \tag{E.1}$$

Notation 16. Denote by L^2 the set of all functions whose L^2-norms are finite, i.e.,

$$L^2 = \{u : \|u\| < \infty\}. \tag{E.2}$$

Similarly, L^1 and L^∞ respectively denote the sets of functions with finite L^1- and L^∞- norms,

$$\|u\|_1 = \int_{-\infty}^{\infty} |u(x)|\mathrm{d}x, \quad \|u\|_\infty = \sup_{-\infty<x<\infty} |u(x)|. \tag{E.3}$$

Since the L^2 norm is the norm used in most applications, we have reserved the symbol $\|\cdot\|$ without a subscript for it.

E.1 Fourier transform

Definition E.2. The *Fourier transform* of a function $u \in L^2$ is the function $\hat{u} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\mathrm{d}x. \tag{E.4}$$

Remark E.1. The quantity ξ is known as the *wave number*, the spatial analog of frequency. For many functions $u \in L^2$, this integral converges in the usual sense for all $\xi \in \mathbb{R}$, but there are situations where this is not true, and in these cases one must interpret the integral as a limit in a certain L^2-norm sense of integrals $\int_{-M}^{M}$ as $M \to \infty$.

Remark E.2. In (E.4), $\hat{u}(\xi)$ measures the correlation of $u(x)$ with the function $e^{-i\xi x}$. The fundamental idea is to interpret $u(x)$ as a superposition of monochromatic waves $e^{i\xi x}$ with various wave numbers ξ, and $\hat{u}(\xi)$ represents the complex amplitude density of u at wave number ξ. Conversely, u can be reconstructed from its spectrum of this amplitude density, The following inverse Fourier transform expresses the synthesis of $u(x)$ as a superposition of its components $e^{i\xi x}$, each multiplied by the appropriate factor $\hat{u}(\xi)$.

Theorem E.3. If $u \in L^2$, then the Fourier transform

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\mathrm{d}x$$

belongs to L^2 also, and u can be recovered from $\hat{u}$ by the *inverse Fourier transform*

$$u(x) = (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{u}(\xi)\mathrm{d}\xi. \tag{E.5}$$

Remark E.3. The following theorem collects some useful properties of the Fourier transform.

Theorem E.4. Let $u, v \in L^2$ have Fourier transforms $\hat{u} = \mathcal{F}u, \hat{v} = \mathcal{F}v$. Then

(a) Linearity. If $c \in \mathbb{R}$, then

$$\mathcal{F}\{u+v\}(\xi) = \hat{u}(\xi) + \hat{v}(\xi), \tag{E.6}$$
$$\mathcal{F}\{cu\}(\xi) = c\hat{u}(\xi). \tag{E.7}$$

(b) Translation. If $x_0 \in \mathbb{R}$, then

$$\mathcal{F}\{u(x+x_0)\} = e^{i\xi x_0}\hat{u}(\xi). \tag{E.8}$$

(c) Modulation. If $\xi_0 \in \mathbb{R}$, then

$$\mathcal{F}\{e^{i\xi_0 x}u(x)\}(\xi) = \hat{u}(\xi - \xi_0). \tag{E.9}$$

(d) Dilation. If $c \in \mathbb{R}$ with $c \neq 0$, then

$$\mathcal{F}\{u(cx)\}(\xi) = \frac{1}{|c|}\hat{u}\left(\frac{\xi}{c}\right). \tag{E.10}$$

(e) Conjugation.

$$\mathcal{F}\{\bar{u}\}(\xi) = \overline{\hat{u}(-\xi)}. \tag{E.11}$$

(f) Differentiation. If $u_x \in L^2$, then

$$\mathcal{F}\{u_x\}(\xi) = i\xi\hat{u}(\xi). \tag{E.12}$$

(g) Inversion.

$$\mathcal{F}^{-1}\{u\}(\xi) = \hat{u}(-\xi). \tag{E.13}$$

Proof. Most of the conclusions follow direction from the definition (E.4). We only show (E.12) by

$$\begin{aligned}\widehat{u_x}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} u_x(x)\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} (-i\xi) e^{-i\xi x} u(x)\mathrm{d}x = i\xi \hat{u}(\xi),\end{aligned}$$

where we have applied integration by parts with assumptions that $u(x)$ is smooth and decays at ∞. □

Remark E.4. In particular, taking $c = -1$ in part (d) above gives $\mathcal{F}\{u(-x)\} = \hat{u}(-\xi)$. Combining this result with part (e) leads to the following elementary but useful results.

Definition E.5. A function $u(x)$ is *even, odd, real, or imaginary* if $u(x) = u(-x)$, $u(x) = -u(-x)$, $u(x) = \overline{u(x)}$, or $u(x) = -\overline{u(x)}$, respectively; $u(x)$ is *Hermitian or skew-Hermitian* if $u(x) = \overline{u(-x)}$ or $u(x) = -\overline{u(-x)}$, respectively.

Theorem E.6. Let $u \in L^2$ have Fourier transform $\hat{u} = \mathcal{F}u$. Then

(a) $u(x)$ is even (odd) $\Leftrightarrow$ $\hat{u}(\xi)$ is even (odd).

(b) $u(x)$ is real (imaginary) $\Leftrightarrow$ $\hat{u}(\xi)$ is hermitian (skew-hermitian) and therefore

(c) $u(x)$ is real and even $\Leftrightarrow$ $\hat{u}(\xi)$ is real and even.

(d) $u(x)$ is real and odd $\Leftrightarrow$ $\hat{u}(\xi)$ is imaginary and odd.

(e) $u(x)$ is imaginary and even $\Leftrightarrow$ $\hat{u}(\xi)$ is imaginary and even.

(f) $u(x)$ is imaginary and odd $\Leftrightarrow$ $\hat{u}(\xi)$ is real and odd.

Theorem E.7. The L^2-norms of u and $\hat{u}$ are related by *Parseval's equality,*

$$\|\hat{u}\| = \|u\|. \tag{E.14}$$

Remark E.5. The square of its L^2 norm is a measure of the energy of a signal. Thus (E.14), is a statement of energy conservation: the L^2 energy of any signal $u(x)$ is equal to the sum of energies of its component vibrations.

Definition E.8. The *convolution* of two functions u, v is the function $u * v$ defined by

$$\begin{aligned}&(u * v)(x) = (v * u)(x) \\ =& \int_{-\infty}^{\infty} u(x-y)v(y)\mathrm{d}y = \int_{-\infty}^{\infty} u(y)v(x-y)\mathrm{d}y,\end{aligned}$$

assuming these integrals exist.

Remark E.6. One way to think of $u * v$ is as a weighted moving average of values $u(x)$ with weights defined by $v(x)$, or vice versa. As a mnemonic device, the two variables in the parenthese inside the integral always sum up to x.

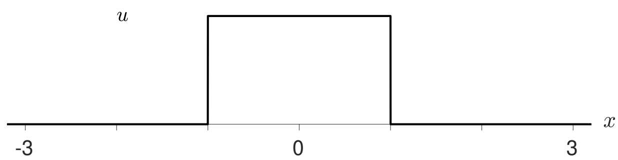
Theorem E.9. If $u \in L^2$ and $v \in L^1$ (or vice versa), then $u * v \in L^2$, and $\widehat{u * v}$ satisfies

$$\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi). \tag{E.15}$$

Remark E.7. In (E.15), the left side, $\widehat{u * v}(\xi)$, represents the strength of the wave number ξ component that results when u is convolved with v— in other words, the degree to which u and v beat in and out of phase with each other at wave number ξ when multiplied together in reverse order with a varying offset. Such beating is caused by a quadratic interaction of the wave number component ξ in u with the same component of v—hence the right-hand side $\hat{u}(\xi)\hat{v}(\xi)$.

Example E.10 (B-splines)**.** For the function

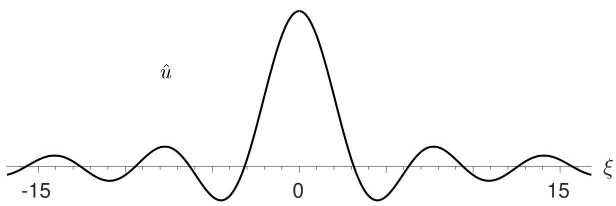
$$u(x) = \begin{cases} \frac{1}{2}, & \text{for } -1 \le x \le 1; \\ 0, & \text{otherwise,} \end{cases} \tag{E.16}$$



(E.1) yields $\|u\| = 1/\sqrt{2}$, and (E.4) gives

$$\hat{u}(\xi) = \frac{1}{2}\int_{-1}^{1} e^{-i\xi x}\mathrm{d}x = \left.\frac{e^{-i\xi x}}{-2i\xi}\right|_{-1}^{1} = \frac{\sin\xi}{\xi}, \tag{E.17}$$

which is known as the *sinc function.*



From (E.1) and the indispensable identity

$$\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\mathrm{d}s = \pi, \tag{E.18}$$

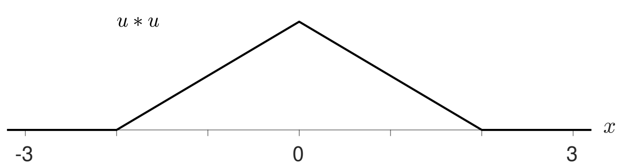
which can be derived by complex contour integration, we calculate $\|\hat{u}\| = \sqrt{\pi}$, which confirms (E.14).

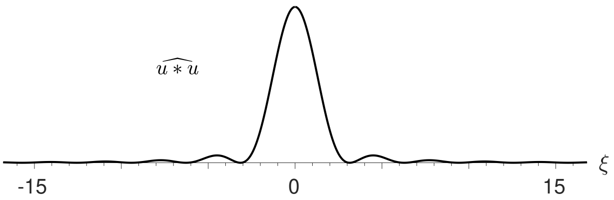
By Definition E.8, it is readily verified that

$$(u * u)(x) = \begin{cases} \frac{1}{2}(1 - |x|/2), & \text{for } -2 \le x \le 2, \\ 0, & \text{otherwise,} \end{cases} \tag{E.19}$$

and

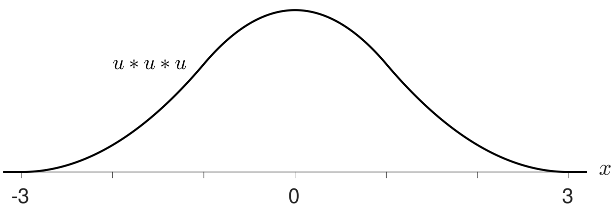
$$(u * u * u)(x) = \begin{cases} \frac{3}{4} - \frac{1}{4}x^2, & \text{for } -1 \le x \le 1, \\ \frac{1}{8}(9 - 6|x| + x^2), & \text{for } 1 \le |x| \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

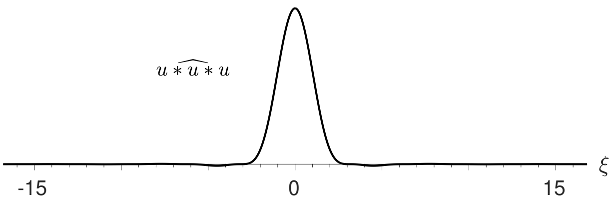




By (E.15) and (E.17), their Fourier transforms are

$$\widehat{u * u}(\xi) = \frac{\sin^2 \xi}{\xi^2}, \quad \widehat{u * u * u}(\xi) = \frac{\sin^3 \xi}{\xi^3}. \tag{E.20}$$





In general, a convolution $u_{(p)}$ of p copies of u has the Fourier transform

$$\widehat{u_{(p)}}(\xi) = \mathcal{F}\{u * u * \cdots * u\}(\xi) = \left(\frac{\sin \xi}{\xi}\right)^p.$$

Remark E.8. Whenever $u_{(p)}$ or any other function is convolved with the function u of (E.16), it becomes smoother, since the convolution amounts to a local moving average. In particular, u itself is piecewise continuous, $u*u$ is continuous and has a piecewise continuous first derivative, $u * u * u$ has a continuous derivative and a piecewise continuous second derivative, and so on. In general $u_{(p)}$ is a piecewise polynomial of degree $p-1$ with a continuous $(p-2)$nd derivative and a piecewise continuous $(p-1)$st derivative, and is known as a B-spline.

Thus convolution with u makes a function smoother, while the effect on the Fourier transform is to multiply it by $\sin \xi / \xi$ and thereby make it decay more rapidly $\xi \to \infty$. This relationship is evident in the plots of Example E.10.

Remark E.9. For numerical methods for PDEs, there are two properties of the Fourier transform that are most important. One is equation (E.15): the Fourier transform converts convolution into multiplication; the other is (E.12), i.e., the Fourier transform converts differentiation into multiplication by $i\xi$. This result is rigorously valid for any absolutely continuous function $u \in L^2$ whose derivative belongs to L^2. Note that differentiation makes a function less smooth, so the fact that it makes the Fourier transform decay less rapidly fits the pattern illustrated above for convolution. In Example E.11, (E.21) is the derivative of (E.19).

Example E.11. The function

$$u(x) = \begin{cases} \frac{1}{4}, & \text{for } -2 \le x < 0, \\ -\frac{1}{4}, & \text{for } 0 < x \le 2, \\ 0, & \text{otherwise,} \end{cases} \tag{E.21}$$

has Fourier transform

$$\begin{aligned} \hat{u}(\xi) &= \frac{1}{4}\int_{-2}^{0} e^{-i\xi x}\mathrm{d}x - \frac{1}{4}\int_{0}^{2} e^{-i\xi x}\mathrm{d}x \\ &= \frac{1}{-4i\xi}(1 - e^{2i\xi} - e^{-2i\xi} + 1) \\ &= \frac{1}{4i\xi}(e^{i\xi} - e^{-i\xi})^2 = \frac{i \sin^2 \xi}{\xi}, \end{aligned}$$

which is $i\xi$ times the Fourier transform (E.20) of the triangular hat function (E.19).

Remark E.10. We have observed the following relationships between the smoothness of a function and the decay of its Fourier transform:

$u(x)$	$\hat{u}(\xi)$
smooth	decays rapidly as $\lvert\xi\rvert \to \infty$
decays rapidly as $\lvert x\rvert \to \infty$	smooth

Of course since the Fourier transform is essentially the same as the inverse Fourier transform, by Theorem E.4(g), the two rows of this summary are equivalent. The intuitive explanation is that if a function is smooth, then it can be accurately represented as a superposition of slowly-varying waves, so one does not need much energy in the high wave number components. Conversely, a non-smooth function requires a considerable amplitude of high wave number components to be represented accurately. These relationships are the bedrock of analog and digital signal processing, where all kinds of smoothing operations are effected by multiplying the Fourier transform by a "windowing function" that decays suitably rapidly.

Definition E.12. A function u defined on $\mathbb{R}$ is said to have *bounded variation* if there is a constant M such that for any finite m and any points $x_0 < x_1 < \cdots < x_m$,

$$\sum_{j=1}^{m} |u(x_j) - u(x_{j-1})| \le M. \tag{E.22}$$

Remark E.11. The following theorem makes the connections between u and $\hat{u}$ precise. It may appear forbidding at first, but it is worth studying carefully. Each of the four parts of the theorem makes a stronger smoothness assumption on u than the last, and reaches a correspondingly stronger conclusion about the rate of decay of $\hat{u}(\xi)$ as $|\xi| \to \infty$. Parts (c) and (d) are known as the *Paley-Wiener theorems.*

Theorem E.13. Let u be a function in L^2.

(a) If u has $p-1$ continuous derivatives in L^2 for some $p \ge 0$, and a pth derivative in L^2 that has bounded variation, then

$$\hat{u}(\xi) = O(|\xi|^{-p-1}) \quad \text{as } |\xi| \to \infty. \tag{E.23}$$

(b) If u has infinitely many continuous derivatives in L^2, then

$$\hat{u}(\xi) = O(|\xi|^{-M}) \quad \text{as } |\xi| \to \infty \text{ for all } M, \tag{E.24}$$

and conversely.

(c) If u can be extended to an analytic function of $z = x + iy$ in the complex strip $|\mathrm{Im}z| < a$ for some $a > 0$, with $\|u(x+iy)\| \le$ const uniformly for each constant $-a < y < a$, then

$$e^{a|\xi|}\hat{u}(\xi) \in L^2, \tag{E.25}$$

and conversely.

(d) If u can be extended to an entire function of $z = x+iy$ with $|u(z)| = O(e^{a|z|})$ as $|z| \to \infty (z \in \mathbb{C})$ for some $a > 0$, then $\hat{u}$ has compact support contained in $[-a, a]$, i.e.,

$$\hat{u}(\xi) = 0 \quad \text{for all } |\xi| > a, \tag{E.26}$$

and conversely.

Remark E.12. A function of the kind described in (d) is said to be *band-limited*, since only a finite band of wave numbers are represented in it.

Since the Fourier transform and its inverse are essentially the same, by Theorem E.4g, Theorem E.13 also applies if the roles of $u(x)$ and $\hat{u}(\xi)$ are interchanged.

Example E.14. The square wave u of Example E.10 satisfies condition (a) of Theorem E.13 with $p = 0$, so its Fourier transform should satisfy

$$|\hat{u}(\xi)| = O(|\xi|^{-1}),$$

as is verified by (E.17). On the other hand, suppose we interchange the role of u and $\hat{u}$ and apply the theorem again. The function $u(\xi) = \sin\xi/\xi$ is entire, and since $\sin(\xi) = (e^{i\xi} - e^{-i\xi})/2i$, it satisfies

$$u(\xi) = O(e^{|\xi|}) \quad \text{as } |\xi| \to \infty$$

(with ξ now taking complex values). By part (d) of Theorem E.13, it follows that $u(x)$ must have compact support contained in $[-1, 1]$, as indeed it does.

Repeating the example for $u * u$, condition (a) now applies with $p = 1$, and the Fourier transform (E.20) is indeed of magnitude $O(|\xi|^{-2})$, as required. Interchanging u and $\hat{u}$, we note that $\sin^2\xi/\xi^2$ is an entire function of magnitude $O(e^{2|\xi|})$ as $|\xi| \to \infty$, and $u * u$ has support contained in $[-2, 2]$.

E.2 Fourier analysis

Lemma E.15. For a linear PDE of the form

$$\frac{\partial u}{\partial t} + \sum_{n=1}^{N} a_n \frac{\partial^n u}{\partial x^n} = 0, \tag{E.27}$$

the evolution of a single Fourier mode of wave number ξ satisfies the ODE

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + \sum_{n=1}^{N} a_n (i\xi)^n \hat{u}(\xi, t) = 0. \tag{E.28}$$

Proof. Differentiating (E.5) with respect to t and x yields

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}_t(\xi, t)e^{i\xi x}\mathrm{d}\xi,\\ \frac{\partial^n u(x,t)}{\partial x^n} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\xi, t)(i\xi)^n e^{i\xi x}\mathrm{d}\xi.\end{aligned}$$

Plug these equations into (E.27) and we have (E.28). □

Lemma E.16. The solution to the linear PDE (E.27) with initial condition $u(x, 0) = \eta(x)$ is

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\eta}(\xi)e^{i(\xi x - \omega t)}\mathrm{d}\xi, \tag{E.29}$$

where $w := \sum_{n=1}^{N} a_n \xi^n i^{n-1}$.

Proof. Rewrite (E.28) as

$$\frac{\partial \hat{u}(\xi, t)}{\partial t} = -i\omega\hat{u}(\xi, t),$$

and we have from Duhamel's principle

$$\hat{u}(\xi, t) = e^{-i\omega t}\hat{\eta}(\xi).$$

Then the inverse Fourier transform yields

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\xi, t)e^{i\xi x}\mathrm{d}\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\eta}(\xi)e^{i(\xi x - \omega t)}\mathrm{d}\xi.\end{aligned}$$ □

Definition E.17. The *dispersion relation* of a PDE or a wave problem is the relation between the frequence ω and the wave number ξ, i.e.,

$$\omega = \omega(\xi). \tag{E.30}$$

Example E.18. The beam equation

$$\varphi_{tt} + \gamma^2\varphi_{xxxx} = 0 \tag{E.31}$$

is characterized by its dispersion relation $\omega = \pm\gamma\xi^2$.

Example E.19. The linear Korteweg-deVries equation

$$\varphi_t + c_0\varphi_x + \nu\varphi_{xxx} = 0 \tag{E.32}$$

is characterized by its dispersion relation $\omega = c_0\xi - \nu\xi^3$.

Definition E.20. The system (E.27) is said to be *hyperbolic* if the PDE is hyperbolic; it is *dissipative* if ω is purely imaginary; it is *dispersive* if $\omega(\xi)$ is real and $\omega'(\xi)$ is not a constant.

Remark E.13. By Lemma E.16,

$$\begin{aligned}u(x,t) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\eta}(\xi)e^{-i\omega t}e^{i\xi x}\mathrm{d}\xi\\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\eta}(\xi)\underbrace{e^{(a_2\xi^2 - a_4\xi^4 + \cdots)t}}_{\text{dissipation}}\\ &\underbrace{e^{i\xi[x - (a_1 - a_3\xi^3 + \cdots)t]}}_{\text{dispersion}}\mathrm{d}\xi.\end{aligned}$$

In general, (E.27) has both dissipative part and dispersive part.

Definition E.21. The *phase velocity* of a monochromatic wave with wave number ξ is

$$C_p(\xi) := \frac{\omega(\xi)}{\xi}. \tag{E.33}$$

Remark E.14. If $\omega'(\xi)$ is not a constant, the phase speed ω/ξ is not the same for all ξ, and hence Fourier modes with different ξ will propagate at different speeds. Consequently, these modes disperse.

Definition E.22. The *group velocity* of a monochromatic wave with wave number ξ is

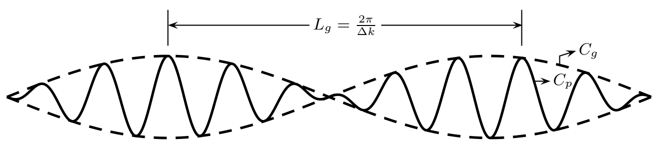
$$C_g(\xi) := \frac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}. \tag{E.34}$$

Example E.23. For the linear PDE (E.27), the phase velocity of a single Fourier mode is

$$C_p(\xi) = \sum_{n=1}^{N} a_n \xi^{n-1} i^{n-1}$$

while the group velocity is

$$C_g(\xi) = \sum_{n=1}^{N} n a_n \xi^{n-1} i^{n-1}.$$



Remark E.15. The energy of a train of waves travels with the group velocity. To see this, consider two monochromatic waves with the same height but with slightly different frequencies and wave numbers.

$$\varphi(x,t) = \varphi_1 + \varphi_2 = \frac{H}{2}\cos(\xi_1 x - \omega_1 t) + \frac{H}{2}\cos(\xi_2 x - \omega_2 t)$$

where

$$\xi_1 = \xi - \frac{1}{2}\Delta\xi, \qquad \omega_1 = \omega - \frac{1}{2}\Delta\omega;$$
$$\xi_2 = \xi + \frac{1}{2}\Delta\xi, \qquad \omega_2 = \omega + \frac{1}{2}\Delta\omega.$$

Trignometric identities yield

$$\varphi(x,t) = H\cos(\xi x - \omega t)\cos\left[\frac{1}{2}\Delta\xi\left(x - \frac{\Delta\omega}{\Delta\xi}t\right)\right].$$

In the above plot of the superimposed profile $\varphi(x,t)$, the wave forms that move with the phase velocity $C_p = \frac{\omega}{\xi}$ are modulated by an "envelope" that propagates with speed $\frac{\Delta\omega}{\Delta\xi}$. Since the wave energy is proportial to the square of wave height, no energy can propagate past a node of zero wave height. Therefore the energy must travel at the speed with which the envelope travels. In the limit of $\Delta\xi \to 0$, we recover (E.34) for a single Fourier mode.

Remark E.16. In a nonuniform media, the Fourier analysis does not apply. Hence it would be useful to free the interpretation of the group velocity from the Fourier analysis. Assume that a nonuniform wave is approximately described in the form

$$\varphi(x,t) = a(x,t)\cos\theta(x,t) = a(x,t)\cos(\xi(x,t)x - \omega(x,t)t),$$

where $a(x,t)$ is the amplitude and $\theta(x,t)$ is the phase function that measures the piont in the cycle of $\cos\theta$ between its extreme values of ± 1. $\xi(x,t)$ is the *local* wave number that measures the density of waves – the number of wave crests per unit length,

$$\xi(x,t) = \frac{\partial\theta}{\partial x},$$

and $\omega(x,t)$ is the *local* frequency that measures the flux of waves – the number of wave crests crossing the position x per unit time,

$$\omega(x,t) = -\frac{\partial\theta}{\partial t}.$$

Hence we have a scalar conservation law that describes the conservation of wave numbers,

$$\frac{\partial\xi}{\partial t} + \frac{\partial\omega}{\partial x} = 0,$$

which can be rewritten as

$$\frac{\partial\xi}{\partial t} + C_g(\xi)\frac{\partial\xi}{\partial x} = 0.$$

Therefore, an observer sees the same local wave number and frequency if she moves with the group velocity. In contrast, the observer sees the same local wave number and frequency changing if she follows any particular crest, i.e., if she moves with the local phase velocity. In the first case the crests keep passing her while in the second case the crests get farther away. We see that hyperbolic phenomena are hidden in dspersive waves.

Example E.24. For the advection equation, we have $a_1 = a$ and $a_i = 0$ for all $i > 1$. Consequently, we have

$$\omega = a\xi, \qquad C_p = a = C_g,$$

hence Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}(\xi)e^{i\xi(x-at)}\mathrm{d}\xi = \eta(x-at).$$

Thus all wave modes that constitute the initial data $\eta(x)$ move at the same phase speed, which is also the moving speed of energy.

Example E.25. For the heat equation

$$u_t = \nu u_{xx},$$

we have $a_2 = -\nu < 0$, $a_1 = a_3 = a_4 = \ldots = 0$, and thus

$$\omega = a_2\xi^2 i, \qquad C_p = a_2\xi i, \quad C_g = 2a_2\xi i.$$

Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}(\xi)e^{i\xi x}e^{a_2\xi^2 t}\mathrm{d}\xi,$$

where the term "$e^{i\xi x}$" denotes the initial mode ξ that does not move while "$e^{-\nu\xi^2 t}$" represents the exponential decay with respect to time.

Example E.26. For the dispersion equation

$$u_t = u_{xxx},$$

we have $a_3 = -1$, $a_1 = a_2 = a_4 = a_5 = \ldots = 0$,

$$\omega = \xi^3, \qquad C_p = \xi^2, \quad C_g = 2\xi^2.$$

Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta} e^{i\xi(x-\xi^2 t)} \mathrm{d}\xi,$$

thus there is no damping, but different phases move with different speed ξ^2.

Example E.27. For the equation

$$u_t + au_x + bu_{xxx} = 0,$$

we have $a_1 = a$, $a_3 = b$, $a_2 = a_4 = a_5 = \ldots = 0$, and

$$\omega = a\xi - b\xi^3, \qquad C_p = a - b\xi^2, \quad C_g = a - 3b\xi^2.$$

Lemma E.16 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi(x-(a-b\xi^2)t)} \mathrm{d}\xi.$$