Chapter 4

Simplicial Complexes

4.1 Affine and convex hulls

Definition 4.1. The *linear hull* or *linear span* of a set of vectors is the span of these vectors.

Definition 4.2. An *affine combination* of $n+1$ points $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a linear combination of them

$$\mathbf{y} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i, \text{ s.t. } \sum_{i=0}^{n} \lambda_i = 1. \tag{4.1}$$

The *affine hull* of X is the set of all affine combinations of points in it, written $\operatorname{aff}(X)$ or $\operatorname{aff}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Definition 4.3. A subset A of an Euclidean space is called *affine* iff the affine hull of every pair of distinct points is contained in A, i.e.,

$$\forall \mathbf{x}, \mathbf{y} \in A, \mathbf{x} \neq \mathbf{y} \quad \Rightarrow \quad \forall \lambda \in \mathbb{R}, \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in A. \tag{4.2}$$

Definition 4.4. A *convex combination* of $n+1$ points $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine combination of them with each coefficent in $[0,1]$, i.e.

$$\mathbf{y} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i, \text{ s.t. } \lambda_i \in [0,1], \sum_{i=0}^{n} \lambda_i = 1. \tag{4.3}$$

The *convex hull* of X is the set of all convex combinations of it, written $\operatorname{conv}(X)$ or $\operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Definition 4.5. A subset A of an Euclidean space is called *convex* iff the convex hull of every pair of distinct points is contained in A, i.e.,

$$\forall \mathbf{x}, \mathbf{y} \in A, \mathbf{x} \neq \mathbf{y} \quad \Rightarrow \quad \forall \lambda \in [0,1], \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in A. \tag{4.4}$$

Exercise 4.1. Show that $\operatorname{conv}Q = Q$ if and only if Q is convex.

Exercise 4.2. For two distinct vectors on the plane, what are their convex hull, affine hull, and linear hull? What about three distinct vectors on the plane?

Lemma 4.6. The intersection of two convex sets is convex.

Proof. Use Definition 4.5. □

Lemma 4.7. Any bounded, convex, open set U in $\mathbb{R}^n$ satisfies the following.

(A) For any $\mathbf{w} \in U$, each ray emanating from $\mathbf{w}$ intersects $\partial U := \overline{U} \setminus U$ in precisely one point.

(B) There exists a homeomorphism $G : \mathbb{D}^n \to \overline{U}$ such that $G(\mathbb{S}^{n-1}) = \partial U$.

Proof. A ray emanating from $\mathbf{w}$ can be expressed as

$$R = \{\mathbf{w} + t\mathbf{p} : t \in [0, +\infty)\},$$

where $\mathbf{p}$ is a unit vector representing the direction of R. Lemma 4.6 implies that $R \cap U$ is convex, which, together with U being bounded and $R \cap U \subset R$, yields

$$\exists s \in \mathbb{R}^+, \text{ s.t. } \{\mathbf{w} + t\mathbf{p} : t \in [0, s)\} \subseteq R \cap U,$$

where U being open implies $\mathbf{w} + s\mathbf{p} \in \partial U$. In particular, t only ranges in a single interval in the expression of $R \cap U$; otherwise it would contradict the convexity of $R \cap U$. Hence we can replace "$\subseteq$" in the above equation with "=."

WLOG, assume $\mathbf{w} = 0$. The function $f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ defines a continuous function from $\mathbb{R}^n \setminus \{\mathbf{w}\}$ to $\mathbb{S}^{n-1}$. By (A), the restriction $f|_{\partial U}$ is a bijection, and denote its inverse by $g : \mathbb{S}^{n-1} \to \partial U$. We then extend g to $G : \mathbb{D}^n \to \overline{U}$,

$$G(\mathbf{x}) = \begin{cases} \left\| g\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \right\| \mathbf{x} & \text{if } \mathbf{x} \neq \mathbf{w}, \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{w}. \end{cases}$$

It is easy to show that G is bijective and continuous, and the inverse of G is also continuous. □

Definition 4.8. An ordered set $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ is *affine independent* iff $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_m - \mathbf{x}_0\}$ is a linearly independent subset of $\mathbb{R}^n$.

Example 4.3. Any singleton set $\{\mathbf{x}_0\}$ is affine independent because there are no points of the form $\mathbf{x}_i - \mathbf{x}_0$ with $i \neq 0$, and $\emptyset$ is linearly independent.

Example 4.4. A set $\{\mathbf{x}_0, \mathbf{x}_1\}$ is affine independent if $\mathbf{x}_1 \neq \mathbf{x}_0$; a set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ is affine independent if the three points are not collinear; a set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is affine independent if the four points are not coplanar.

Exercise 4.5. Show that any linearly independent subset of $\mathbb{R}^n$ is affine independent, but the converse is not true.

Theorem 4.9. The following conditions on an ordered set of points $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ are equivalent,

(i) X is affine independent;

(ii) if $\{\lambda_0, \lambda_1, \ldots, \lambda_m\} \subset \mathbb{R}$ satisfies $\sum_{i=0}^m \lambda_i \mathbf{x}_i = 0$ and $\sum_{i=0}^m \lambda_i = 0$, then we have $\lambda_0 = \lambda_1 = \cdots = \lambda_m = 0$;

(iii) each $\mathbf{y} \in \mathrm{aff}(X)$ has a unique expression as an affine combination:

$$\mathbf{y} = \sum_{i=0}^m t_i \mathbf{x}_i \text{ where } \sum_{i=0}^m t_i = 1. \tag{4.5}$$

Proof. (i) $\Rightarrow$ (ii).

$$\begin{aligned} 0 = \sum_{i=0}^m \lambda_i \mathbf{x}_i &= \sum_{i=0}^m \lambda_i \mathbf{x}_i - \sum_{i=0}^m \lambda_i \mathbf{x}_0 \\ &= \sum_{i=0}^m \lambda_i (\mathbf{x}_i - \mathbf{x}_0) = \sum_{i=1}^m \lambda_i (\mathbf{x}_i - \mathbf{x}_0) \end{aligned}$$

and the linearly independence of $\{\mathbf{x}_i - \mathbf{x}_0\}$ imply that $\lambda_1 = \cdots = \lambda_m = 0$, which, together with $\sum_{i=0}^m \lambda_i = 0$, implies $\lambda_0 = 0$.

(ii) $\Rightarrow$ (iii). Suppose we have two different affine expression of $\mathbf{y}$,

$$\mathbf{y} = \sum_i t_i \mathbf{x}_i = \sum_i s_i \mathbf{x}_i.$$

Then by (ii), $\sum_i (t_i - s_i) = \sum_i t_i - \sum_i s_i = 1 - 1 = 0$ and $\sum_i (t_i - s_i)\mathbf{x}_i = 0$ yield $(t_i - s_i) = 0$ for each i, which contradicts the assumption.

(iii) $\Rightarrow$ (i). Suppose X is not affine independent. Then there exists t_i's not all zero such that

$$\sum_i t_i (\mathbf{x}_i - \mathbf{x}_0) = 0.$$

Pick an arbitrary $t_j \neq 0$, multiply the above equation with $1/t_j$ so that we have

$$\mathbf{x}_j - \mathbf{x}_0 + \sum_{i \neq j} \frac{t_i}{t_j} (\mathbf{x}_i - \mathbf{x}_0) = 0.$$

However, $\mathbf{x}_j = 1\mathbf{x}_j$ is another different affine combination of $\mathbf{x}_j$ in terms of the vectors in X. This is a contradiction, and hence X must be affine independent. $\square$

Corollary 4.10. The property of affine independence of a set $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ is independent of its given ordering.

Proof. Statements in Theorem 4.9 do not depend on the ordering of the sets. $\square$

Corollary 4.11. For an affine independent set $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$, its affine hull is a translation of an m-dimensional sub-vector-space V of $\mathbb{R}^n$, i.e.

$$\exists \mathbf{y}_0 \in \mathbb{R}^n \text{ s.t. } \mathrm{aff}(X) = V + \mathbf{y}_0. \tag{4.6}$$

Proof. By Theorem 4.9, any $\mathbf{y} \in \mathrm{aff}(X)$ has a unique expression as in (4.5). We rewrite this expression as

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^n t_i (\mathbf{x}_i - \mathbf{x}_0).$$

The proof is completed by choosing $\mathbf{y}_0 = \mathbf{x}_0$ and choosing V to be the vector space spanned by $\mathbf{x}_i - \mathbf{x}_0$'s, c.f. Definition 4.8. $\square$

Definition 4.12. Let $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be an affine independent subset of $\mathbb{R}^N$. For any $\mathbf{y} \in \mathrm{aff}(X)$, its *barycentric coordinates* relative to the ordered set X are entries of the tuple $(t_0, t_1, \ldots, t_m)$ in (4.5).

Definition 4.13. A set of points $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ in $\mathbb{R}^N$ is said to be *in general position* or *in general linear position* if every $N+1$ points of it form an affine independent set.

Definition 4.14. For $n+1$ points $x_0, x_1, \ldots, x_n \in \mathbb{R}$, the associated *Vandermonde matrix* $V \in \mathbb{R}^{(n+1)\times(n+1)}$ is

$$V(x_0, x_1, \ldots, x_n) = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}. \tag{4.7}$$

Lemma 4.15. The determinant of a Vandermonde matrix can be expressed as

$$\det V(x_0, x_1, \ldots, x_n) = \prod_{i>j} (x_i - x_j). \tag{4.8}$$

Proof. Consider the function

$$\begin{aligned} U(x) &= \det V(x_0, x_1, \ldots, x_{n-1}, x) \\ &= \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}. \end{aligned} \tag{4.9}$$

Clearly, $U(x) \in \mathbb{P}_n$ and it vanishes at $x_0, x_1, \ldots, x_{n-1}$ since inserting these values in place of x yields two identical rows in the determinant. It follows that

$$U(x_0, x_1, \ldots, x_{n-1}, x) = A \prod_{i=0}^{n-1} (x - x_i),$$

where A depends only on $x_0, x_1, \ldots, x_{n-1}$. Meanwhile, the expansion of $U(x)$ in (4.9) by minors of its last row implies that the coefficient of x^n is $U(x_0, x_1, \ldots, x_{n-1})$. Hence we have

$$U(x_0, x_1, \ldots, x_{n-1}, x) = U(x_0, x_1, \ldots, x_{n-1}) \prod_{i=0}^{n-1} (x - x_i),$$

and consequently the recursion

$$U(x_0, x_1, \ldots, x_{n-1}, x_n) = U(x_0, x_1, \ldots, x_{n-1}) \prod_{i=0}^{n-1} (x_n - x_i).$$

An induction based on $U(x_0, x_1) = x_1 - x_0$ yields (4.8). $\square$

Example 4.6 (Affine independence in numerical approximation)**.** Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ denote $n+1$ distinct points in $\mathbb{R}^N$, and $\phi_0, \phi_1, \ldots, \phi_n$ denote $n+1$ linearly independent continuous functions $\mathbb{R}^N \mapsto \mathbb{R}$. The *multivariate interpolation problem* seeks $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\forall j = 0, 1, \ldots, n, \qquad \sum_{i=0}^{n} a_i \phi_i(\mathbf{x}_j) = f(\mathbf{x}_j), \tag{4.10}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a given function.

The *sites* $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ of the multivariate interpolation problem are said to be *poised* with respect to the basis functions $\phi_0, \phi_1, \ldots, \phi_n$ iff the *sample matrix*

$$S = \begin{bmatrix} \phi_0(\mathbf{x}_0) & \phi_1(\mathbf{x}_0) & \cdots & \phi_n(\mathbf{x}_0) \\ \phi_0(\mathbf{x}_1) & \phi_1(\mathbf{x}_1) & \cdots & \phi_n(\mathbf{x}_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\mathbf{x}_n) & \phi_1(\mathbf{x}_n) & \cdots & \phi_n(\mathbf{x}_n) \end{bmatrix} \tag{4.11}$$

is nonsingular. In particular, for $N = 1$ and the monomial basis, the sample matrix is the Vandermonde matrix in (4.7). Then Lemma 4.15 implies that poisedness of the points is equivalent to the distinctness of the points. In higher dimensions, similar criteria exist via the so-called quasi-determinants, but they are much more complicated than the one-dimensional case.

Theorem 4.16. For every $n \geq 0$, the Euclidean space $\mathbb{R}^N$ contains n points in general position.

Proof. If $n \leq N+1$, choose the origin together with $n-1$ elements from the standard bases. Hence we can assume $n > N+1$. Select n distinct reals $t_1, t_2, \ldots, t_n$ and for each $i = 1, 2, \ldots, n$, define

$$\mathbf{x}_i = [t_i, t_i^2, \ldots, t_i^N]^T \in \mathbb{R}^N.$$

We claim that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is in general position. If not, by Corollary 4.10 we can assume that the first $N+1$ points $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_N\}$ are not affine independent, and hence $\{\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_N - \mathbf{x}_0\}$ is linearly dependent, i.e.

$$V^* \mathbf{s} = \mathbf{0}$$

where $\mathbf{s}$ is a vector of reals, $V^* = [\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_N - \mathbf{x}_0]$,

$$V^* = \begin{bmatrix} t_1 - t_0 & t_2 - t_0 & \cdots & t_N - t_0 \\ t_1^2 - t_0^2 & t_2^2 - t_0^2 & \cdots & t_N^2 - t_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^N - t_0^N & t_2^N - t_0^N & \cdots & t_N^N - t_0^N \end{bmatrix}.$$

Consequently, we have

$$\det V^* = \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ t_0 & t_1 - t_0 & \cdots & t_N - t_0 \\ t_0^2 & t_1^2 - t_0^2 & \cdots & t_N^2 - t_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_0^N & t_1^N - t_0^N & \cdots & t_N^N - t_0^N \end{bmatrix} = \det V^T,$$

where the last step follows from (4.7). By Lemma 4.15, the condition of pairwise distinct t_i's implies $\det V^T \neq 0$ and $\det V^* \neq 0$. Hence $\mathbf{s} = \mathbf{0}$, which contradicts the hypothesis that $\{\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_N - \mathbf{x}_0\}$ is linearly dependent. □

4.2 Simplexes and affine maps

Definition 4.17. Let $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be an affine independent subset of $\mathbb{R}^N$ with $m \leq N$. The (affine) *simplex* $\mathbf{X}$ with vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m$, written

$$\mathbf{X} = [\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m] := \operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}, \tag{4.12}$$

is the convex hull of X.

Definition 4.18. The *vertex set* of an m-simplex is denoted by

$$\operatorname{Vert}(\mathbf{X}) = X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}. \tag{4.13}$$

The *dimension of an m-simplex* $\mathbf{X}$, written $\dim \mathbf{X}$, is m.

Definition 4.19. Let $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be an affine independent subset of $\mathbb{R}^N$. The *barycenter* of the m-simplex of X is $\frac{1}{m+1}\sum_{i=0}^{m} \mathbf{x}_i$.

Definition 4.20. A *face of a simplex* $\mathbf{X}$ is a simplex $\mathbf{V}$ satisfying $\operatorname{Vert}(\mathbf{V}) \subseteq \operatorname{Vert}(\mathbf{X})$; we write $\mathbf{V} \leq \mathbf{X}$. A *proper face of a simplex* $\mathbf{X}$ is a face $\mathbf{V}$ such that $\operatorname{Vert}(\mathbf{V})$ is a proper subset of $\operatorname{Vert}(\mathbf{X})$; we write $\mathbf{V} < \mathbf{X}$.

Definition 4.21. For an m-simplex $[\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m]$, its $(m-1)$-*face* opposite a vertex $\mathbf{x}_i$, written $[\mathbf{x}_0, \ldots, \hat{\mathbf{x}}_i \ldots, \mathbf{x}_m]$, is the set

$$\left\{ \sum_{j=0}^{m} t_j \mathbf{x}_j : t_j \geq 0, \sum_{j=0}^{m} t_j = 1, t_i = 0 \right\}. \tag{4.14}$$

The *boundary of an m-simplex* is the union of its $(m-1)$-faces.

Definition 4.22. The *standard n-simplex*, denoted by Δ^n, is a subset of $\mathbb{R}^{n+1}$,

$$\Delta^n := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0; \sum_i x_i = 1 \right\}. \tag{4.15}$$

Example 4.7. $\Delta^2 \subset \mathbb{R}^3$ is the intersection of the set $\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$ with the plane $x+y+z = 1$.

Corollary 4.23. An m-simplex is compact in $\mathbb{R}^N$.

Proof. This is a direct consequence of Lemma 4.7. □

Definition 4.24. Let $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be an affine independent subset of $\mathbb{R}^N$. An *affine map* or *affine transformation* is a function $T : \operatorname{aff}(X) \to \mathbb{R}^k$ for some $k \geq 1$ such that

$$\sum_j t_j = 1 \;\Rightarrow\; T\left(\sum_j t_j \mathbf{x}_j\right) = \sum_j t_j T(\mathbf{x}_j). \tag{4.16}$$

The restriction of T to $\operatorname{conv}(X)$ gets the same name.

Lemma 4.25. If $T : \mathbb{R}^N \to \mathbb{R}^k$ is affine, then there exists a fixed $\mathbf{y}_0 \in \mathbb{R}^k$ and a linear map $\lambda : \mathbb{R}^N \to \mathbb{R}^k$ such that $T(\mathbf{x}) = \lambda(\mathbf{x}) + \mathbf{y}_0$.

Proof. Choose $\mathbf{y}_0 = T(\mathbf{0})$ and it is easy to verify that $\lambda(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0})$ is indeed a linear map,

$$\begin{aligned}\lambda\left(\sum_j t_j \mathbf{x}_j\right) &= T\left(\sum_j t_j \mathbf{x}_j\right) - T(\mathbf{0}) \\ &= \sum_j t_j T(\mathbf{x}_j) - T(\mathbf{0}) \sum_j t_j \\ &= \sum_j t_j \left[T(\mathbf{x}_j) - T(\mathbf{0})\right] = \sum_j t_j \lambda(\mathbf{x}_j). \square\end{aligned}$$

Corollary 4.26. Any affine map is equivalent to the composition of a translation and a linear map.

Proof. It is easy to verify that translations and linear maps are both affine maps. Hence their compositions are still affine maps. The converse follows from Lemma 4.25. □

Corollary 4.27. Let $\mathrm{conv}(X)$ and $\mathrm{conv}(Y)$ be an m-simplex and an n-simplex, respectively. Any function $f : X \to \mathrm{conv}(Y)$ can be uniquely extended to an affine map $T : \mathrm{conv}(X) \to \mathrm{conv}(Y)$ such that $T(\mathbf{x}_i) = f(\mathbf{x}_i)$ for all $\mathbf{x}_i \in X$.

Proof. For any convex combination $\sum t_i \mathbf{x}_i$, define $T(\sum t_i \mathbf{x}_i) := \sum t_i f(\mathbf{x}_i)$, which, by (4.16), is clearly an affine map. □

4.3 Simplicial Complexes

Definition 4.28. A *(finite) simplicial complex K* is a (finite) collection of simplexes in some Euclidean space such that

(SPC-1) $\sigma \in K$ implies that all of its faces belong to K;

(SPC-2) $\sigma, \tau \in K$ implies that $\sigma \cap \tau$ is either empty or a single common face of σ and τ.

Exercise 4.8. Give an example of finite collections of simplexes satisfying (SPC-1) but not (SPC-2).

Lemma 4.29. A collection K of simplexes is a simplicial complex if and only if

(a) $\sigma \in K$ implies that all of its faces belongs to K;

(b) any two distinct simplexes in K have disjoint interiors.

Proof. For necessity, we show that, for any two simplexes $\sigma, \tau \in K$, their interiors having a point in common implies $\sigma = \tau$. Let $r = \sigma \cap \tau$. By Definition 4.28, r must be a face of σ. Suppose it is a proper face, then $r \subset \partial\sigma$, which contradicts the condition that r has an interior point of σ. Hence $r = \sigma$. Similar argument shows that $r = \tau$.

For sufficiency, let $\sigma = \mathrm{conv}\{s_0, \ldots, s_m\}$ and $\tau = \mathrm{conv}\{t_0, \ldots, t_n\}$ be two simplexes as in Definition 4.17. If $X := \mathrm{Vert}(\sigma) \cap \mathrm{Vert}(\tau) = \emptyset$, then we have $\sigma \cap \tau = \emptyset$; otherwise it would contradict condition (b). If $X \neq \emptyset$, condition (a) implies that $\mathrm{conv}(X)$ is also a simplex in K and by Definition 4.20 it is a common face of σ and τ. □

Corollary 4.30. For a simplex σ, the collection consisting of σ and all of its proper faces is a simplicial complex.

Proof. This follows from Lemma 4.29 and Lemma 4.7. □

Definition 4.31. The *dimension of a simplicial complex* K, written $\dim K$, is

$$\dim K := \sup_{\sigma \in K}\{\dim \sigma\}. \tag{4.17}$$

Definition 4.32. A *subcomplex L of a simplicial complex* K is a subcollection of K such that L contains all faces of its elements.

Definition 4.33. The *n-skeleton of a simplicial complex* K, written $K^{(n)}$, is the subcomplex of K that contains all simplexes in K of dimension at most n.

Definition 4.34. The *vertex set* of a simplicial complex K, written $\mathrm{Vert}(K)$ or $K^{(0)}$, is the 0-skeleton of K.

Definition 4.35. The *underlying space of a simplicial complex K* (or *polytope of K*), written $|K|$, is the subspace of the ambient Euclidean space,

$$|K| = \cup_{\sigma \in K} \sigma, \tag{4.18}$$

i.e. the union of all simplexes in K.

Corollary 4.36. The polytope of a finite simplicial complex is compact, i.e. closed and bounded.

Proof. If K is finite, then $|K|$ is a finite union of compact subspaces σ, c.f. Corollary 4.23, and hence is compact. □

Definition 4.37. A topological space X is a *polyhedron* if it is homeomorphic to the polytope of some simplicial complex, i.e., if there exists a simplicial complex K and a homeomorphism $h : |K| \to X$. The ordered pair (K, h) is called a *triangulation* of X.

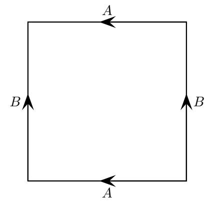
Example 4.9. By Corollary 4.30, every n-simplex σ determines a simplicial complex K, namely, the collection of all faces of σ. Clearly, $|K| = \sigma$ and σ is a polyhedron because the identity map $h : |K| \to \sigma$ is a homeomorphism.

Exercise 4.10. Modify your example (if necessary) in Exercise 4.8 so that although it is not a simplicial complex, it is a polyhedron; also give the simplicial complex homeomorphic to the polyhedron.

Example 4.11. The standard 2-simplex Δ^2 is contained in Euclidean space $\mathbb{R}^3$. Define K to be the collection of all proper faces of Δ^2. Then K is a simplicial complex and $|K|$ is the perimeter of the triangle Δ^2 in $\mathbb{R}^3$. For $X = \mathbb{S}^1$, choose distinct points $a_0, a_1, a_2 \in \mathbb{S}^1$ and define a homeomorphism $h : |K| \to \mathbb{S}^1$ with $h(\mathbf{e}_i) = a_i$ where $\mathbf{e}_i$'s are the standard bases of $\mathbb{R}^3$; also, the injective function h takes each 1-simplex to an arc of $\mathbb{S}^1$. Then (K, h) is a triangulation of $\mathbb{S}^1$ and hence $\mathbb{S}^1$ is a polyhedron.

Example 4.12. If K is the family of all proper faces of an n-simplex σ, then there exists a triangulation (K, h) of $\mathbb{S}^{n-1}$. Hereafter we denote this simplicial complex by $\dot{\sigma}$, which satisfies $\dot{\sigma} \approx \mathbb{S}^{n-1}$.

Exercise 4.13. The square $\mathbf{I} \times \mathbf{I}$ with sides identified as follows would yield the torus $\mathbb{T}^2$.



More precisely, identify $(t, 0)$ with $(t, 1)$ for each $t \in \mathbf{I}$ and we get a cylinder; then identify $(0, r)$ with $(1, r)$ for each $r \in \mathbf{I}$ and we get a torus. Give a triangulation of the torus to show that it is indeed a polyhedron.

4.4 The coherent topology on polyhedra

Definition 4.38. In the *topology of the underlying space* $|K|$ of a simplicial complex K, a subset A of $|K|$ is defined to be closed in $|K|$ iff for each $\sigma \in K$ the subset $A \cap \sigma$ is closed in σ, where σ is equipped with the natural Euclidean topology.

Lemma 4.39. The topology of a polytope $|K|$ in Definition 4.38 is finer than the subspace topology on $|K|$ induced by the Euclidean topology on $\mathbb{R}^N$. However, if K is finite, then the two topologies are the same.

Proof. Suppose $A \subset |K|$ is closed in $|K|$ in the sense of subspace topology. Then by the lemma on the relative closedness in a subspace, there exists a closed subset Y in $\mathbb{R}^N$ such that $A = |K| \cap Y$. Then for each $\sigma \in K$, the intersection $A \cap \sigma$ is closed because Lemma 4.7 implies σ being closed in $\mathbb{R}^N$. The above argument holds regardless of K being finite or not.

For a finite simplicial complex K, suppose $A \subset |K|$ is closed in $|K|$ in the sense of polytope topology. Then $A \cap \sigma_i$ is closed in σ for each $\sigma_i \in K$. Hence there exists a closed subset $Y_i \subset \mathbb{R}^N$ for each σ_i such that $Y_i \cap \sigma_i = A \cap \sigma_i$. Since K is finite, $Y = \cup_i Y_i$ is also closed. Due to $A \subset |K|$, we have

$$\begin{aligned} Y \cap A &= Y \cap [\cup_i (A \cap \sigma_i)] = \cup_i [Y \cap (A \cap \sigma_i)] \\ &= \cup_i [A \cap (Y \cap \sigma_i)] = \cup_i (A \cap \sigma_i) \end{aligned}$$

By the lemma on the relative closedness in a subspace, each $A \cap \sigma_i$ is closed in $\mathbb{R}^N$. Any finite union of closed sets in $\mathbb{R}^N$ is also closed in $\mathbb{R}^N$. The above argument does not hold when K is infinite. □

Example 4.14. Let K be the collection of all 1-simplexes in $\mathbb{R}$ of the form $[m, m+1]$ where $m \in \mathbb{Z} \backslash \{0\}$, along with all simplexes of the form $[\frac{1}{n+1}, \frac{1}{n}]$ where $n \in \mathbb{Z}^+$, along with all faces of these simplexes. Then the *set* $|K|$ is $\mathbb{R}$, but the polytope topology on $|K|$ is different from the standard topology on $\mathbb{R}$: the set $K = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$ is closed in $|K|$ but not closed in $\mathbb{R}$.

Example 4.15. Define 1-simplexes of the form

$$\sigma_n = \operatorname{conv}\{(0, 0), (1, \frac{1}{n})\} \subset \mathbb{R}^2, \; n \in \mathbb{Z}^+$$

and let K be the simplicial complex with

$$\begin{aligned} K^{(0)} &= \{(0, 0)\} \cup \{(1, \frac{1}{n}) : n \in Z^+\}; \\ K^{(1)} &= \{\sigma_n : n \in \mathbb{Z}^+\}. \end{aligned}$$

Then the set

$$|K| \cap \{(x, x^2) : x > 0\}$$

is closed in K but not closed in $\mathbb{R}^2$.

Definition 4.40. The *open m-simplex of an m-simplex* σ is defined as

$$\sigma^\circ := \begin{cases} \sigma & \text{if } m = 0; \\ \operatorname{Int}(\sigma) & \text{otherwise.} \end{cases} \tag{4.19}$$

Exercise 4.16. An open m-simplex σ° is an open subset of the space σ, but it might not be an open subset of $|K|$ for some simplicial complex K with $\sigma \in K$. Give two such examples.

Exercise 4.17. Show that the polytope of a simplicial complex is the disjoint union of its open simplexes.

Definition 4.41. If X is a space and if $\mathscr{C}$ is a collection of subspaces of X whose union is X, the topology of X is said to be *coherent* with the collection $\mathscr{C}$, provided that a set A is closed in X if and only if $A \cap C$ is closed in C for each $C \in \mathscr{C}$. It is equivalent to require that U be open in X if and only if $U \cap C$ is open in C for each $C \in \mathscr{C}$.

Example 4.18. The topology in Definition 4.38 is coherent with the collection of subspaces $\sigma \in K$.

Lemma 4.42. If L is a subcomplex of K, then $|L|$ is a closed subspace of $|K|$. In particular, any $\sigma \in K$ is a closed subspace of $|K|$.

Proof. By definition of a subspace, we show both directions hold.

If B is closed in $|K|$, then Definition 4.38 states that $B \cap \sigma$ is closed for each $\sigma \in K$, and in particular for each $\sigma \in L$. Hence $B \cap |L|$ is closed in L.

Conversely, suppose A is closed in $|L|$. For any $\sigma \in K$, $\sigma \cap |L|$ is the union of those faces s_i of σ that belong to L. Since A is closed in $|L|$, Definition 4.38 states that $A \cap s_i$ is closed in s_i for each s_i. By the corollary on the transitivity of relative closedness, $A \cap s_i$ is closed in σ. Because $A \cap \sigma$ is the finite union of the sets $A \cap s_i$, we conclude that A is closed in K. □

Lemma 4.43. A function $f : |K| \to X$ is continuous if and only if $f|_\sigma$ is continuous for each $\sigma \in K$.

Proof. If f is continuous, so is $f|_\sigma$. Conversely, for any $\sigma \in K$ suppose $f|_\sigma$ is continuous. If C is a closed set in X, then

$$f^{-1}(C) \cap \sigma = (f|_\sigma)^{-1}(C)$$

is closed in σ. Then Definition 4.38 states that $f^{-1}(C)$ is closed in $|K|$. □

Exercise 4.19. Show that Lemma 4.43 holds in the more general context of coherent topology.

Definition 4.44. If $\mathbf{x}$ is a point of the polytope $|K|$, then $\mathbf{x}$ is interior to precisely one simplex of K, whose vertices are (say) $\mathbf{a}_0, \ldots, \mathbf{a}_n$. Then we have

$$\mathbf{x} = \sum_{i=0}^{n} t_i \mathbf{a}_i, \tag{4.20}$$

where $t_i > 0$ for each i and $\sum_i t_i = 1$. For any vertex $\mathbf{v}$ of K, the *barycentric coordinate* $t_{\mathbf{v}}(\mathbf{x})$ *of* $\mathbf{x}$ *with respect to* $\mathbf{v}$ is defined as

$$t_{\mathbf{v}}(\mathbf{x}) = \begin{cases} t_i & \text{if } \mathbf{v} = \mathbf{a}_i; \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

Lemma 4.45. The function $t_{\mathbf{v}} : |K| \to [0, 1]$ in (4.21) is continuous.

Proof. For any fixed σ, $t_{\mathbf{v}}$ is continuous because it is either identically zero or a barycentric coordinate as in Definition 4.12. The rest follows from Lemma 4.43. □

Lemma 4.46. The underlying space of a simplicial complex is Hausdorff.

Proof. For any $p \neq q$ in $|K|$, there is at least one vertex $\mathbf{v}$ such that $t_{\mathbf{v}}(p) \neq t_{\mathbf{v}}(q)$, c.f. Definition 4.44 and the uniqueness of barycentric coordinates in a simplex. Choose r between these two numbers; then the sets $\{\mathbf{x} : t_{\mathbf{v}}(\mathbf{x}) < r\}$ and $\{\mathbf{x} : t_{\mathbf{v}}(\mathbf{x}) > r\}$ are the desired disjoint open sets. □

Lemma 4.47. If a subset A of $|K|$ is compact, then there exists some finite subcomplex K_0 of K such that $A \subset |K_0|$.

Proof. Suppose A is not contained in the polytope of any finite subcomplex of K. For each $\sigma \in K$ satisfying $A \cap \mathrm{Int}(\sigma) \neq \emptyset$, we choose a single point $x_s \in A \cap \mathrm{Int}(\sigma)$. By the assumption, $B = \{x_s\}$ is infinite. By Lemmas 4.29 and 4.46, each x_s is an isolated point of B, and hence B has no accumulation points, which contradicts the Bolzano-Weierstrass Theorem. □

Definition 4.48. The *star of a vertex* p *in a simplicial complex* K, written $\mathrm{st}(p)$, is a subset of $|K|$ defined as

$$\mathrm{st}(p) = \cup_{\sigma \in K, p \in \mathrm{Vert}(\sigma)} \mathrm{Int}(\sigma). \tag{4.22}$$

The *closed star of* p *in* K, written $\overline{\mathrm{st}}(p)$, is the closure of its star. The *link of* p *in* K is the set $\overline{\mathrm{st}}(p) \setminus \mathrm{st}(p)$.

Example 4.20. $\mathrm{st}(p)$ is open.

Proof. $\mathrm{st}(p)$ has all points of $|K|$ satisfying $t_p(x) > 0$. □

Exercise 4.21. Show that $\overline{\mathrm{st}}(p)$ is the union of all simplexes of K that have p as a vertex. Show that the complement of $\mathrm{st}(p)$ is the union of all simplexes of K that do not have p as a vertex.

Definition 4.49. A simplicial complex K is *locally finite* iff each vertex of K belongs only to finitely many simplexes of K.

Corollary 4.50. A complex K is locally finite if and only if each closed star $\overline{\mathrm{st}}(p)$ is the polytope of a finite subcomplex of K.

Proof. This follows from Definitions 4.48 and 4.49. □

Theorem 4.51. A simplicial complex K is locally finite if and only if its underlying space $|K|$ is locally compact.

Proof. Suppose K is locally finite. For any $x \in |K|$, it lies in $\mathrm{st}(v)$ for some vertex v. Since $\overline{\mathrm{st}}(v)$ is compact, $|K|$ is by definition locally compact.

Conversely, suppose $|K|$ is locally compact. For each vertex v of $|K|$, there exists a compact neighborhood A of v in $|K|$. By Lemma 4.47, A is contained in some finite subcomplex of K and hence K is locally finite. □

Example 4.22. The simplicial complex consisting of $\mathbb{Z}$ and $\{[n, n+1] : n \in \mathbb{Z}\}$ is locally compact, but not compact. The simplicial complex K in Example 4.15 is not even locally compact.

4.5 Simplicial maps

Definition 4.52. Let K and L be simplicial complexes. A *simplicial map* $f : K \to L$ is a function $f : K^{(0)} \to L^{(0)}$ such that whenever the vertices $v_0, \ldots, v_n$ of K span a simplex of K, the points $f(v_0), \ldots, f(v_n)$ of L span a simplex of L.

Example 4.23. A constant function $f : K \to L$ is a simplicial map.

Exercise 4.24. Consider simplicial map for two complexes

$$K = \{A, B, C, AB, BC, AC, ABC\},$$
$$L = \{P, Q, R, PQ, QR\}.$$

Is the function $f : K \to L$ defined by $f(A) = P$, $f(B) = Q$, $f(C) = R$ a simplicial map? If not, modify it minimally to get a simplicial map. Give three other possible definitions of a simplicial map.

Lemma 4.53. The composition of two simplicial maps is a simplicial map.

Proof. The composite does not need all information of the second simplicial map to qualify for a simplicial map. □

Lemma 4.54. Any simplicial map f can be *uniquely* extended to a continuous map $g : |K| \to |L|$ such that

$$\sum_{i=0}^{n} t_i = 1 \quad \Rightarrow \quad g\left(\sum_{i=0}^{n} t_i v_i\right) = \sum_{i=0}^{n} t_i f(v_i). \tag{4.23}$$

Proof. For any simplex $\sigma \in K$, Corollary 4.27 implies that $f|_{\mathrm{Vert}(\sigma)}$ can be uniquely extended to an affine map $g|_{(\sigma)}$ in Definition 4.24. $g|_{(\sigma)}$ is continuous because the preimage of any closed sets is closed. The rest of the proof follows from Lemma 4.43.

Note that the extension from f to g is unique, regardless of the fact that the vertices $f(v_0), \ldots, f(v_n)$ are not necessarily distinct; $\sum_i t_i = 1$ holds not only in σ but also τ after collecting the coefficients of $f(v_j) = f(v_k)$. □

Definition 4.55. The function g in Lemma 4.54 is called the *affine map* induced by the *simplicial map* f.

Lemma 4.56. The composition of two affine maps induced by simplicial maps is an affine map.

Proof. Let $g : |K| \to |L|$ and $h : |L| \to |M|$ be two such affine maps. For the vertex set $\mathbf{v} = \{v_0, \ldots, v_n\}$ of $\sigma \in K$ and $x = \sum_i t_i v_i$ with $\sum_i t_i = 1$, Lemma 4.54 yields

$$\begin{aligned} h(g(x)) &= h\left(\sum t_i g(v_i)\right) = \sum t_i (h \circ g)(v_i) \\ &= \sum t_i \left.(h \circ g)\right|_{\mathbf{v}} (v_i). \end{aligned}$$

The possibility of $g(v_0) = g(v_1)$ does not affect the conclusion aftering collect the coefficients of $f(v_j) = f(v_k)$. □

Definition 4.57. A bijective simplicial map is called a *simplicial homeomorphism* or an *isomorphism* of two simplicial complexes.

Lemma 4.58. The induced affine map $g : |K| \to |L|$ from an isomorphism between K and L is a homeomorphism.

Proof. Each simplex $\sigma \in K$ is mapped by g onto a simplex $\tau \in L$ of the same dimension. It suffices to show that the linear map $h : \tau \to \sigma$ induced by the inverse of the simplicial map f^{-1} is the inverse of the map $g : \sigma \to \tau$. By Definition 4.52, $x = \sum_i t_i v_i$ implies $g(x) = \sum_i t_i f(v_i)$. Hence we have

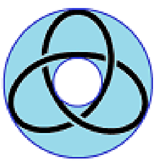
$$\begin{aligned} h(g(x)) &= h\left(\sum_i t_i f(v_i)\right) = \sum t_i f^{-1}(f(v_i)) \\ &= \sum t_i v_i = x. \end{aligned}$$ □

Exercise 4.25. Verify that, for an induced affine map from an isomorphism $f : K \to L$, the barycentric coordinates of a point in K are used as those of the image points in L.

Corollary 4.59. Any finite complex K is isomorphic to a subcomplex of some simplicial complex Δ_σ^N, which consists of an N-simplex σ and its faces.

Proof. Set $N = \dim K$. Let $v_0, \ldots, v_N$ be the vertices of K. By Theorem 4.16, we can choose $x_0, \ldots, x_N$ in $\mathbb{R}^N$ that are in general position. The simplicial map $f(v_i) = x_i$ (c.f. Definition 4.52) induces an isomorphism of K with a subcomplex of Δ_σ^N. □

Exercise 4.26. Provide a simplicial map that represents the function $f : \mathbb{S}^1 \to \mathbb{T}^2$ illustrated as follows.



Exercise 4.27. Show that a simplicial map $f : K \to L$ satisfies

$$\forall v \in K^{(0)}, \quad f(\mathrm{st}(v)) \subseteq \mathrm{st}(f(v)). \tag{4.24}$$

Definition 4.60. A simplicial map $f : K \to L$ is called a *simplicial approximation to a continuous function* $\phi : |K| \to |L|$ iff

$$\forall v \in K^{(0)}, \quad \phi(\mathrm{st}(v)) \subseteq \mathrm{st}(f(v)). \tag{4.25}$$

Example 4.28. A simplicial approximation to the identity function cannot collapse edges.

Exercise 4.29. Show that if $f : K \to L$ is a simplicial approximation to $\phi : |K| \to |L|$, then the affine map g induced from f is homotopic to ϕ.

Theorem 4.61. The class $\mathscr{K}$ consisting of all simplicial complexes and all simplicial maps with usual composition is a category, and the function of taking underlying spaces is a functor $|\cdot| : \mathscr{K} \to \mathbf{Top}$.

Proof. It is straightforward to verify that $\mathscr{K}$ is a category from Definition 8.1. In particular, the identity map on K is the identity $1_K \in \mathrm{Hom}(K, K)$.

We define $|\cdot|$ by assigning $|K|$ to K, which satisfies (FCO-1) in Definition 8.7. We define $|f|$ of $f : K \to L$ to be the affine map in Definition 4.55; then (FCO-2) holds because an affine map is continuous. (FCO-3) holds from arguments similar to those in the proof of Lemma 4.56. Finally, (FCO-4) holds trivially since $|1_K|$ is the identity function on $|K|$. □

4.6 Abstract simplicial complexes

Definition 4.62. Let V be a nonempty set. An *abstract simplicial complex* K is a collection of finite nonempty subsets of V satisfying

$$v \in V \;\Rightarrow\; \{v\} \in K; \tag{4.26a}$$
$$\sigma \in K, \sigma' \subset \sigma \Rightarrow \sigma' \in K. \tag{4.26b}$$

V is called the *vertex set* of K, written $\mathrm{Vert}(K)$. An element in K is called a *simplex*, and it is an *n-simplex* if it has $n + 1$ elements.

Definition 4.63. A *simplicial map of two abstract simplicial complexes* $f : K \to L$ is a function $f : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ such that whenever $\{v_0, \ldots, v_n\}$ is a simplex in K, $\{f(v_0), \ldots, f(v_n)\}$ is a simplex in L (the latter list of vertices may have repetitions).

Theorem 4.64. All abstract simplicial complexes and simplicial maps determine a category, denoted by $\mathscr{K}^a$.

Example 4.30. Every simplicial complex K determines an abstract simplicial complex K^a with the same vertex set V: each simplex $\sigma \in K$ yields a subset of V in K^a as $\mathrm{Vert}(\sigma) \in K^a$. This defines a functor $\mathscr{K} \to \mathscr{K}^a$.

Example 4.31. Let X be a topological space, and let $\mathcal{U}$ be a finite open cover of X. Define an abstract simplicial complex having vertices as the open sets in $\mathcal{U}$ and declare that any open sets $U_0, \ldots, U_n \in \mathcal{U}$ form a simplex if $\cap_{i=0}^n U_i = \emptyset$. This simplicial complex is called the *nerve* of the open cover $\mathcal{U}$.

Theorem 4.65. There exists a functor $u : \mathscr{K}^a \to \mathscr{K}$ such that

$$\begin{cases} \forall K \in \operatorname{obj}(\mathscr{K}),\ K \simeq u(K^a); \\ \forall L^a \in \operatorname{obj}(\mathscr{K}^a),\ L^a \simeq (uL^a)^a. \end{cases} \tag{4.27}$$

Definition 4.66. A *geometric realization of an abstract simplicial complex* L^a is a space homeomorphic to $|u(L^a)|$.

Corollary 4.67. Isomorphic abstract simplicial complexes have homeomorphic geometric realizations.

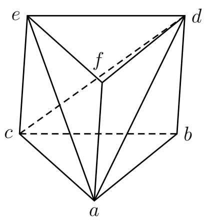
Proof. Every functor preserves equivalences, which is in particular true for the composition $\mathscr{K}^a \to \mathscr{K} \to \mathbf{Top}$. □

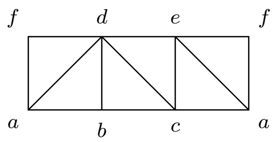
Definition 4.68. A *labeling* of a finite simplicial complex L is a surjective function $f : L^{(0)} \to \mathcal{L}$ where $\mathcal{L}$ is a set of *labels* such as $a, b, c, \ldots$ and so on.

Definition 4.69. An *abstract simplicial complex induced from a labeled finite simplicial complex* L is an abstract simplicial complex whose vertices are the labels and whose simplexes are sets of the form $\{f(v_0), \ldots, f(v_n)\}$, where $v_0, \ldots, v_n$ span a simplex of L.

Definition 4.70. Let $|K|$ be a geometric realization of an abstract simplicial complex induced from a labeled finite simplicial complex L. Then the simplicial map $L^{(0)} \to K^{(0)}$ derived from the labeling f extends to a surjective affine map $g : |L| \to |K|$. We call K the *simplicial complex derived from the labeled complex* L and call g the *associated pasting map.*

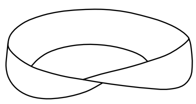
Example 4.32. Suppose we want to have simplicial complex whose underlying space is homeomorphic to the cylinder $\mathbb{S}^1 \times \mathbf{I}$. We can first deform the cyclinder to the union of triangles like the one on the left.



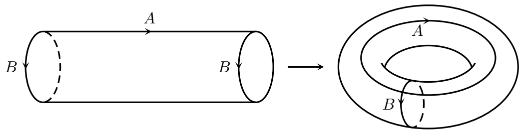


Cutting the surface along the edge af yields the right plot. On the other hand, the cyclinder is often defined as the quotient space obtained from a rectangle by identifying two disjoint edges of it.

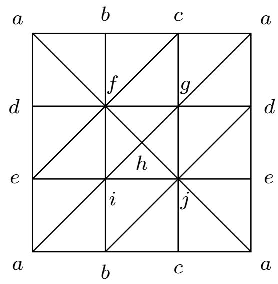
Exercise 4.33. Give a simplicial complex whose underlying space is homeomorphic to the Mobius band below.



Example 4.34. Recall that thetorus $\mathbb{T}^2$ is the quotient space obtained from the unit square by the quotient map that is induced from the equivalence relation $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.



The following is a labeled simplicial complex L, and the geometric realization of the abstract simplicial complex induced from L is homeomorphic to the torus.



Can you prove it?

Definition 4.71. If L is a simplicial complex, a subcomplex L_0 of L is called a *full subcomplex* of L iff

$$\forall \sigma \in L, \operatorname{Vert}(\sigma) \subset L_0^{(0)} \ \Rightarrow\ \sigma \in L_0. \tag{4.28}$$

Lemma 4.72. Let L be a finite simplicial complex, f a labeling of L, and $g : |L| \to |K|$ the associated pasting map. Let L_0 be a full subcomplex of L such that

$$\forall u, v \in L, u \neq v; f(u) = f(v) \ \Rightarrow\ u, v \in L_0. \tag{4.29}$$

Suppose the following condition holds,

$$\forall u, v \in L_0, u \neq v; f(u) = f(v) \ \Rightarrow\ \overline{\mathrm{st}}(u) \cap \overline{\mathrm{st}}(v) = \emptyset \text{ in } L_0. \tag{4.30}$$

Then we have

$$\forall \sigma \in L,\ \dim g(\sigma) = \dim \sigma; \tag{4.31}$$

$$\sigma_1 \neq \sigma_2; g(\sigma_1) = g(\sigma_2) \ \Rightarrow\ \sigma_1, \sigma_2 \in L_0; \sigma_1 \cap \sigma_2 = \emptyset. \tag{4.32}$$

Proof. Suppose $\dim g(\sigma) < \dim \sigma$. Then there must exist two vertices $u, v \in \operatorname{Vert}(\sigma)$ such that $f(u) = f(v)$. Hence (4.31) implies $u, v \in L_0$. Definition 4.71 and $\sigma \in L$ imply that the edge $\operatorname{conv}\{u, v\}$ is in L_0, contradicting (4.30).

As for (4.32), we have

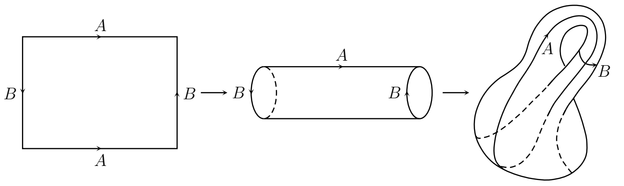
$$\forall u \in \operatorname{Vert}(\sigma_1), \exists v \in \operatorname{Vert}(\sigma_2), \text{ s.t. } f(u) = f(v);$$

otherwise it would contradict the definition of an affine map. Hence (4.29) implies $\operatorname{Vert}(\sigma_1) \cup \operatorname{Vert}(\sigma_1) \subset L_0$. Then Definition 4.71 implies $\sigma_1, \sigma_2 \in L_0$. Also, $\sigma_1 \cap \sigma_2 \neq \emptyset$ would contradict (4.30). □

Example 4.35. In Example 4.34, $|L|$ is the unit square and $|L_0|$ its boundary.

Exercise 4.36. What does the contrapositive of (4.32) mean? Is it useful for judging whether two simplexes would collapse into one?

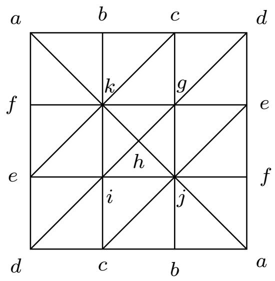
Example 4.37. Recall that theKlein Bottle $\mathbb{K}^2$ is the quotient space obtained from the unit square by the quotient map that is induced from the equivalence relation $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, y)$.



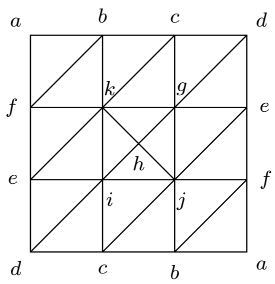
The following is a labeled simplicial complex L, and the geometric realization of the abstract simplicial complex induced from L is homeomorphic to the Klein bottle.

Exercise 4.38. Show $\mathbb{P}^2$ (the quotient space obtained from the unit square by identifying antipodal points) is homeomorphic with the space obtained from $\mathbb{D}^2$ by identifying antipodal points on $\mathbb{S}^1$.

Exercise 4.39. Show that the following labeled complex determines a complex K whose polytope is homeomorphic to $\mathbb{P}^2$.



Exercise 4.40. Describe the space determined by the labeled complex as follows.



4.7 Homology groups

Definition 4.73. An *oriented simplicial complex* K is an (abstract or geometric) simplicial complex with a partial order on $K^{(0)}$ whose restriction to the vertices of any simplex in K is a linear order.

Definition 4.74. A *presentation* of a group G is a definition of G in the form

$$G := \langle S|R\rangle, \tag{4.33}$$

where S is a set of generators so that every element of G can be written as a linear combination of these generators, and R is a set of relations among those generators.

Definition 4.75. The *qth chain group of an oriented simplicial complex* K, written $C_q(K) = \langle S_q|R_q\rangle$, is an abelian group with the following presentation,

- the *generators* S_q are all $(q+1)$-tuples $(v_0,\dots,v_q)$ with $v_0,\dots,v_q \in K^{(0)}$ such that $[v_0,\dots,v_q]$ is a simplex in K;
- the relations R_q are
  - (i) $(v_0,\dots,v_q) = 0$ if any vertex is repeated;
  - (ii) $(v_0,\dots,v_q) = \operatorname{sgn}(\pi)(v_{\pi(0)},\dots,v_{\pi(n)})$ where $\operatorname{sgn}(\pi)$ is the signature of the permutation π as in Definition 3.39.

Lemma 4.76. The following holds for an oriented simplicial complex K with $\dim K = m$.

(i) $C_q(K) = 0$ for all $q > m$.

(ii) Each $C_q(K)$ with $0 \le q \le m$ is a free abelian group with a basis as the set of all symbols $[v_0,\dots,v_q]$, where $v_0,\dots,v_q$ span a q-simplex of K with the ordering $v_0 < v_1 < \cdots < v_q$.

Proof. If $q > m$, each $(q+1)$-tuple must have some repeated vertex in spanning an m-simplex of K, hence $C_q(K) = 0$ follows from 4.74 (i).

Define F_q to be the free abelian group with its basis S_q in Definition 4.74. We would like to show $C_q(K) = F_q/R_q$. By enumerating all possibilities, we have

$$S_q = B_1 \cup B_2 \cup B_3,$$

where B_1 consists of all $(q+1)$-tuples with some repeated vertex, B_2 consists of all $(v_0,\dots,v_q)$ with $v_0 < \ldots < v_q$, and B_3 consists of all terms of the form

$$(v_0,\dots,v_q) - \operatorname{sgn}(\pi)(v_{\pi(0)},\dots,v_{\pi(n)}),$$

where π is a non-identity permutation on $\{0,1,\dots,q\}$. Clearly, $B_1 \cup B_3$ is the basis of a subgroup of F_q and Definition 4.74 yields $C_q(K) = F_q/R_q$. □

Definition 4.77. For an oriented simplicial complex K, its *qth boundary operator*, written $\partial_q : C_q(K) \to C_{q-1}(K)$, is a linear operator defined by $\partial_q(\sigma+\tau) = \partial_q(\sigma) + \partial_q(\tau)$ and

$$\partial_q([v_0,\dots,v_q]) = \sum_{i=0}^{q}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_q], \tag{4.34}$$

where $\hat{v}_i$ means deleting the vertex v_i.

Lemma 4.78. $\partial_{q-1}\partial_q = 0$.

Proof. Definition 4.77 yields

$$\begin{aligned}
&\partial_{q-1}\partial_q[v_0,\dots,v_q] = \sum_{i=0}^{q}\partial_{q-1}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_q] \\
=&\sum_{j<i}\partial_{q-1}(-1)^i(-1)^j[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_q] \\
+&\sum_{j>i}\partial_{q-1}(-1)^i(-1)^{j-1}[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_q] \\
=&0.
\end{aligned}$$

□

Theorem 4.79. For an oriented simplicial complex of dimension m, the following is a *chain complex* in the sense of Definition 3.30,

$$0 \longrightarrow C_m \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \ , \quad (4.35)$$

which we denote by $C_*(K)$.

Definition 4.80. For an oriented simplicial complex K, the *group of simplicial q-cycles* is

$$Z_q(K) := \ker \partial_q, \quad (4.36)$$

the *group of simplicial q-boundaries* is

$$B_q(K) := \operatorname{Im} \partial_{q+1}, \quad (4.37)$$

and the *qth simplicial homology group* is

$$H_q(K) := Z_q/B_q. \quad (4.38)$$

Theorem 4.81. For an oriented simplicial complex K with $\dim K = m \in \mathbb{N}$, we have

(i) $H_q(K)$ is finitely generated for every $q \geq 0$;

(ii) $H_q(K) = 0$ for all $q > m$;

(iii) $H_m(K)$ is free abelian, possibly zero.

Proof. By Lemma 4.76 (ii), $C_q(K)$ is finitely generated, and hence its subgroups and the quotient group are finite generated. (ii) follows from Lemma 4.76 (i).

As for (iii), $C_{m+1}(K) = 0$ implies $B_m(K) = 0$, and hence $H_m(K) = Z_m(K)$. Then (iii) follows from a theorem stating that a subgroup of a free abelian group is also free abelian. □

Definition 4.82. The *Euler-Poincaré characteristic* of a simplicial complex K with dimension m is

$$\chi(K) := \sum_{q=0}^{m} (-1)^q \alpha_q, \quad (4.39)$$

where α_q is the number of q-simplexes in K.

Theorem 4.83. An oriented simplicial complex K of dimension m satisfies

$$\chi(K) = \sum_{q=0}^{m} (-1)^q \text{rank} H_q(K). \quad (4.40)$$

Proof. For $q \geq 0$, construct an exact sequence

$$0 \longrightarrow Z_q(K) \xrightarrow{\text{Id}} C_q(K) \xrightarrow{\partial_q} B_{q-1} \longrightarrow 0 \ .$$

Lemma 3.33 yields

$$\alpha_q = \text{rank} C_q(K) = \text{rank} Z_q(K) + \text{rank} B_{q-1}(K).$$

Then we have

$$\begin{aligned} \chi(K) &= \sum_{q=0}^{m} (-1)^q \text{rank} Z_q(K) + \sum_{q=0}^{m} (-1)^q \text{rank} B_{q-1}(K) \\ &= \sum_{q=0}^{m} (-1)^q \text{rank} Z_q(K) + \sum_{q=0}^{m} (-1)^{q+1} \text{rank} B_q(K) \\ &= \sum_{q=0}^{m} (-1)^q (\text{rank} Z_q(K) - \text{rank} B_q(K)) \\ &= \sum_{q=0}^{m} (-1)^q \text{rank} H_q(K), \end{aligned}$$

where the second line follows from a change of index and the fact $\text{rank} B_{-1} = 0 = \text{rank} B_m(K)$, the last step follows from the definition of the homology groups. □

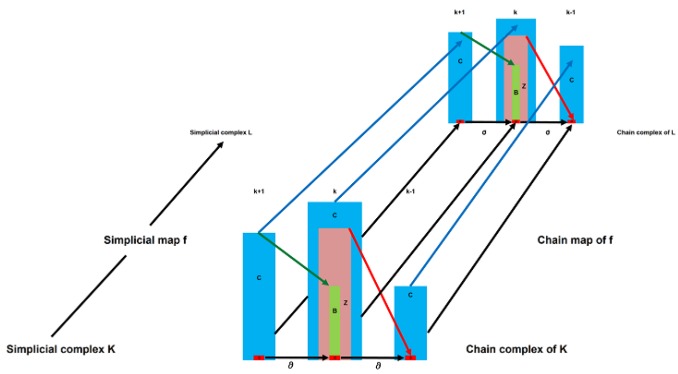
4.8 Chain maps

Definition 4.84. Let K and L be oriented simplicial complexes and let $\varphi : K \to L$ be a simplicial map. The *homomorphism induced from the simplicial map* φ, written $\varphi_\# : C_*(K) \to C_*(L)$, is defined as

$$\forall n \in N, \quad \varphi_\#([v_0, \ldots, v_n]) = [\varphi(v_0), \ldots, \varphi(v_n)]. \quad (4.41)$$

Definition 4.85. A *chain map* is any collection of homomorphisms $\{\phi_q : q \in \mathbb{N}\}$ between elements of two chain complexes such that it commutes with the boundary operators of the chain complexes.

$$\begin{tikzcd} C_q(K) \arrow[r, "\partial_q^K"] \arrow[d, "\phi_q"] & C_{q-1}(K) \arrow[d, "\phi_{q-1}"] \\ C_q(L) \arrow[r, "\partial_q^L"] & C_{q-1}(L) \end{tikzcd} \quad (4.42)$$



Lemma 4.86. If $\varphi : K \to L$ is a simplicial map, then $\varphi_\#$ is a chain map, i.e., it satisfies

$$\phi_{q-1} \partial_q^K = \partial_q^L \phi_q, \quad (4.43)$$

where ϕ_q is the qth component of $\varphi_\#$,

$$\varphi_\# = \{\phi_q : C_q(K) \to C_q(L); q \in \mathbb{N}\} \quad (4.44)$$

Proof. Definitions 4.77 and 4.84 yield

$$\begin{aligned}\phi_q\partial[v_0,\ldots,v_q] &= \phi_q\sum_{i=0}^{q}(-1)^i[v_0,\ldots,\hat{v}_i,\ldots,v_q]\\ &= \sum_{i=0}^{q}(-1)^i\left[\phi_q(v_0),\ldots,\widehat{\phi_q(v)}_i,\ldots,\phi_q(v_q)\right]\\ &= \partial\phi_q[v_0,\ldots,v_q].\end{aligned}$$

□

Corollary 4.87. A chain map takes cycles to cycles, i.e.

$$\varphi_\#(Z_q(K))\subset Z_q(L). \tag{4.45}$$

Proof. Suppose $x\in Z_q(K)$, then $\partial_q x=0$.

$$\partial\phi_q(x)=\phi_{q-1}(\partial x)=0,$$

and hence $\phi_q(x)\in Z_q(L)$. □

Corollary 4.88. A chain map takes boundaries to boundaries, i.e.

$$\varphi_\#(B_q(K))\subset B_q(L). \tag{4.46}$$

Proof. Suppose $x\in B_q(K)$, then there exists $y\in B_{q+1}(K)$ such that $x=\partial_{q+1}y$. It follows that

$$\phi_q(x)=\phi_q(\partial_{q+1}y)=\partial_{q+1}\phi_q(y),$$

and hence $\phi_q(x)\in B_q(L)$. □

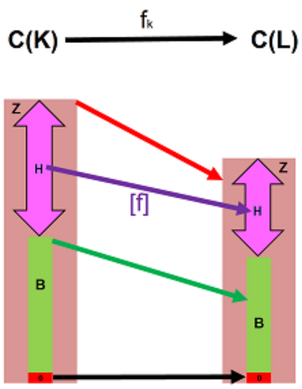
4.9 Homology maps

Definition 4.89. The *homology map* φ_* induced by a chain map $\varphi_\#$ in (4.44) are the linear maps

$$\varphi_*=\{[\phi_q]:H_q(K)\to H_q(L),q\in\mathbb{N}\}$$

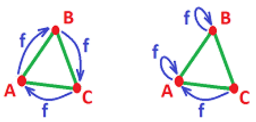
defined as

$$\forall q\in\mathbb{N},\qquad [\phi_q]([x]):=[\phi_q(x)]. \tag{4.47}$$



Example 4.41. Read Example 2.31 again.

Example 4.42. Revisiting Exercise 2.27.



The simplicial maps are obvious from the plots. For the one on the left, we have

$$\begin{aligned}&f_1(AB)=BC,\ f_1(BC)=CA,\ f_1(CA)=AB;\\ &f_1(AB+BC+CA)=BC+CA+AB;\\ &[f_1]([AB+BC+CA])=[AB+BC+CA];\end{aligned}$$

and hence $[f_1]$ is the identity. It is easy to show that $[f_0]$ is also the identity. For the one on the right, we have

$$\begin{aligned}&f_1(AB)=AB,\ f_1(BC)=AB,\ f_1(CA)=0.\\ &f_1(AB+BC+CA)=AB+AB+0=0;\\ &[f_1]([AB+BC+CA])=0;\end{aligned}$$

and hence $[f_1]$ is the trivial homomorphism. What is $[f_0]$?

Lemma 4.90. The homology map induced by the identity chain map (induced by the identity simplicial map) is the identity operator.

Lemma 4.91. The homology map induced by the composition of two chain maps $f_q:C_q(K)\to C_q(L)$ and $g_q:C_q(L)\to C_q(M)$ is the composition of the homology maps induced by these chain maps

$$[g_q\circ f_q]=[g_q]\circ[f_q]. \tag{4.48}$$

Theorem 4.92. For each $q\geq 0$, $H_q:\mathscr{K}\to\mathbf{Ab}$ is a functor.

Proof. (FCO-1) holds by Definition 4.80. As for (FCO-2), the morphism in **Ab** is simply the homology map in Definition 4.89. In other words, we define $\varphi_*:H_q(K)\to H_q(L)$ by

$$\varphi_*:z+B_q(K)\mapsto\varphi_\#(z)+B_q(L).$$

(FCO-3,4) follow from Lemmas 4.91 and 4.90. □

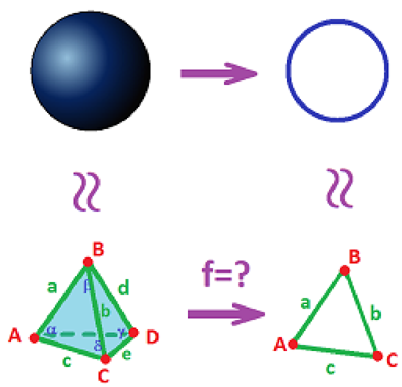
Corollary 4.93. The homology map induced by an isomorphic chain map (which is further induced by a simplicial isomorphism) is an isomorphism.

Proof. This follows from Theorems 8.13 and 4.92. □

Exercise 4.43. Show that the homology groups of the circle $\mathbb{S}^1$ and the sphere $\mathbb{S}^2$ are as follows.

$$H_0(\mathbb{S}^1)=\mathbb{Z},\ H_1(\mathbb{S}^1)=\mathbb{Z},\ H_2(\mathbb{S}^1)=0; \tag{4.49}$$

$$H_0(\mathbb{S}^2)=\mathbb{Z},\ H_1(\mathbb{S}^2)=0,\ H_2(\mathbb{S}^2)=\mathbb{Z}. \tag{4.50}$$



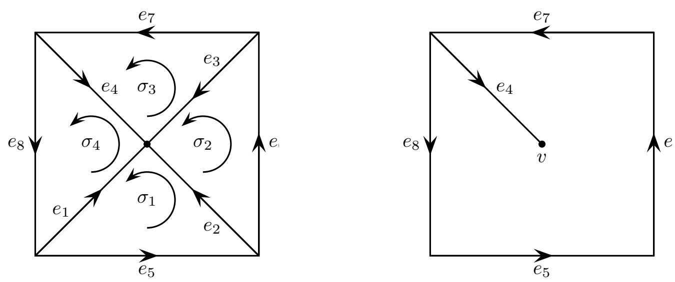
Does there exist a continuous function that transforms $\mathbb{S}^2$ to $\mathbb{S}^1$?

4.10 Homology groups of surfaces

Definition 4.94. For a simplicial complex K, we say that a q-chain c is *carried* by a subcomplex L of K if for any q-simplex outside L, its coefficient in the formal sum of c is zero.

Definition 4.95. Two q-chains c and c' are *homologous* if there exists a $(q+1)$-chain d such that $\partial_{q+1}d = c - c'$. For the particular case of $c = \partial_{q+1}d$, we say c *bounds* or c is *homologous to zero.*

Example 4.44. Consider the complex M below on the left, whose underlying space it the square. Instead of computing the group of 1-cycles straightforwardly, we can simplify it to the complex below on the right. The two are indistinguishable in calculating H_1 for $|M|$. In other words, *for any 1-chain c in M, it is homologous to a chain c_3 carried by the subcomplex in M shown below on the right.*



Given a 1-chain c, let a be the coefficient of e_1 in c. Then the chain

$$c_1 = c + \partial_2(a\sigma_1)$$

is homologous to c, but does not contain e_1 in its formal sum. Similarly, let b be the coefficient of e_2 in c_2. Then the chain

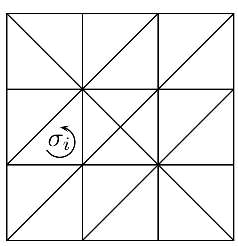
$$c_2 = c_1 + \partial_2(b\sigma_2)$$

is homologous to c_1, but does not contain e_2 in its formal sum. Let d be the coefficient of e_3 in c_2. Then the chain

$$c_3 = c_2 + \partial_2(d\sigma_3)$$

is homologous to c_2, but does not contain e_3 in its formal sum. If c is a cycle, c_3 must also be a cycle, and the coefficient of e_4 in c_3 must be 0; otherwise ∂c_3 would have a nonzero term of v.

Lemma 4.96. Let L be the complex shown below, whose underlying space is a unit square. Let Bd L denote the complex whose underlying space is the boundary of $|L|$.



Orient each 2-simplex $\sigma_i \in L$ counter-clockwise and orient the 1-simplexes arbitrarily. Then

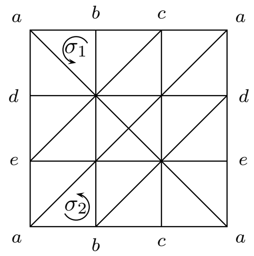
(i) every 1-cycle of L is homologous to a 1-cycle carried by Bd L;

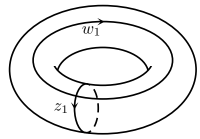
(ii) if τ is a 2-chain of L and if $\partial\tau$ is carried by Bd L, then τ is a multiple of $\sum_i \sigma_i$.

Proof. (i) can be proven by an argument similar to that in Example 4.44. As for (ii), if σ_i and σ_j have an edge e in common, then the coefficient of e in the expression of $\partial\tau$ must be zero, hence the coefficients of σ_i and σ_j must be the same in τ. Continuing the process for all σ_i's finish the process. □

Definition 4.97. A *wedge of two circles* is a topological space homeomorphic to the union of two circles that intersect at a single point.

Theorem 4.98. Let T denote the complex represented by the labeled complex L shown below.





Then $|T|$ is the torus, and

$$H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}, \qquad H_2(T) \simeq \mathbb{Z}. \tag{4.51}$$

Orient each 2-simplex of L counterclockwise and use the induced orientation of the 2-simplexes of T; let γ denote their sum. Define

$$w_1 := [a,b] + [b,c] + [c,a]; \tag{4.52}$$
$$z_1 := [a,d] + [d,e] + [e,a]. \tag{4.53}$$

Then γ generates $H_2(T)$ and w_1 and z_1 represent a basis for $H_1(T)$.

Proof. Let $f : L \to T$ be the labeling and $g : |L| \to |T|$ be the pasting map. The identification only happens in the subcomplex Bd L. By definition of a torus and Lemma 4.72, $|T|$ must be a torus. The subspace $A = g(|\mathrm{Bd}\, L|)$ is homeomorphic to a wedge of two circles. By the given conditions, the two conclusions in Lemma 4.96 hold. We would also like to prove

(a) if c is a 1-cycle of T carried by Bd L, then c is of the form $nw_1 + mz_1$;

(b) $\partial\gamma = 0$.

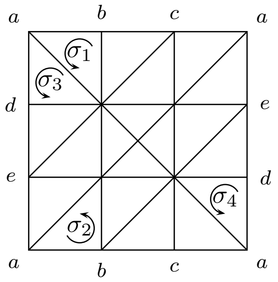
(a) follows directly from Definition 4.97. To prove (b), consider every 1-simplex of T. Any 1-simplex not carried by Bd L must have its coefficient as zero. Any 1-simplex carried by Bd L must also have its coefficient as zero, as exemplified by $[a,b]$ in σ_1 and σ_2 in the picture above.

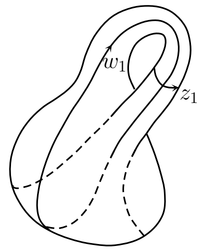
Consider $H_1(T) = Z_1(T)/B_1(T)$. By (a) and Lemma 4.96 (i), every 1-cycle of T is homologous to a 1-cycle of the form $nw_1 + mz_1$. Furthermore, a cycle in T bounds only if it is trivial; for if $c = \partial\tau$ for some τ, then Lemma 4.96

(ii) applies to show that $\tau = r\gamma$ for some $r \in \mathbb{Z}$ and (b) states that $\partial\gamma = 0$, which implies $\partial\tau = 0$. Hence $B_1(T)$ is trivial and $H_1(T) = Z_1(T)$ and w_1 and z_1 represent the two basis of $H_1(T)$.

By (b) and Lemma 4.96 (ii), any 2-cycle of T must be of the form $r\gamma$ for some $r \in \mathbb{Z}$. Since there are no 3-chains, we have $H_2(T) = \{r\gamma : r \in \mathbb{Z}\}$. □

Theorem 4.99. Let K denote the complex represented by the labeled complex L shown below.





Then $|K|$ is the Klein bottle, and

$$H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}_2, \qquad H_2(K) \simeq 0. \tag{4.54}$$

Orient each 2-simplex of L counterclockwise and use the induced orientation of the 2-simplexes of K. Define

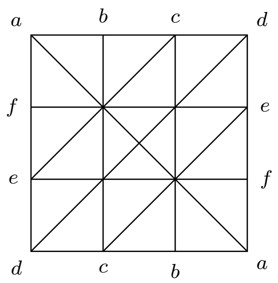
$$w_1 := [a, b] + [b, c] + [c, a]; \tag{4.55}$$
$$z_1 := [a, d] + [d, e] + [e, a]. \tag{4.56}$$

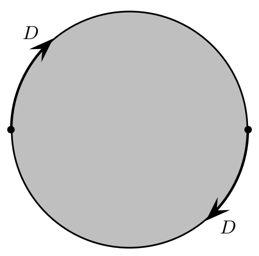
Then the torsion element of $H_1(K)$ is generated by the chain z_1, and a generator for the group $H_1(K)$ modulo torsion is represented by w_1.

Proof. The proof is similar to that of Theorem 4.98. As the crucial difference, the conclusion (b) does not hold. Instead we have $\partial\gamma = 2z_1$, which can be shown by considering $[a, d]$ in σ_3 and σ_4. As the first consequence, $B_1(K)$ is generated by $2z_1$, and hence $Z_1(K)/B_1(K) \simeq \mathbb{Z}\oplus\mathbb{Z}_2$. As the second consequence, $Z_2(K) = 0$ because there are no 2-chains with zero boundary, and hence $H_2(K) \simeq 0$. □

Exercise 4.45. How to interpret of (4.51) and (4.54) topologically?

Theorem 4.100. Let P denote the complex represented by the labeled complex L shown below.





Then $|P|$ is the projective plane, and

$$H_1(P) \simeq \mathbb{Z}_2, \qquad H_2(P) \simeq 0. \tag{4.57}$$

Proof. The proof is similar to that of Theorem 4.98. As the crucial difference, the key observations are

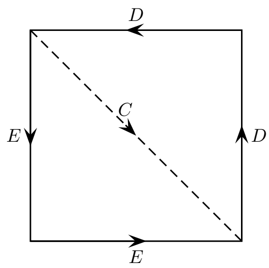
(a) if c is a 1-cycle of T carried by $\mathrm{Bd}\, L$, then c is a multiple of z_1;

(b) $\partial\gamma = -2z_1$.

The rest follows by similar calculations of H_1 and H_2. □

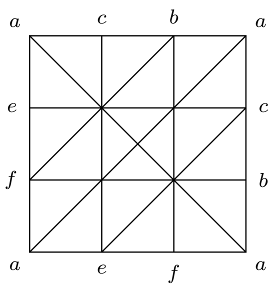
Definition 4.101. The *connected sum* of two projective planes, denoted by $\mathbb{P}^2\#\mathbb{P}^2$, is the space obtained from two copies of projective planes by deleting a small open disc from each, and pasting together the remaining pieces along their free edge.

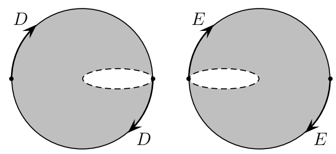
Exercise 4.46. Show that the connected sum of two projective planes is the quotient space obtained from the unit square by identifying its edges as follows.



Theorem 4.102. For the connected sum of projective planes, we have

$$H_1(\mathbb{P}^2\#\mathbb{P}^2) \simeq \mathbb{Z} \oplus \mathbb{Z}_2, \qquad H_2(\mathbb{P}^2\#\mathbb{P}^2) \simeq 0. \tag{4.58}$$





Proof. The proof is similar to that of Theorem 4.98. As the crucial difference, the conclusion (b) does not hold. Instead we have $\partial\gamma = 2(w_1 + z_1)$. As the first consequence, $B_1(K)$ is generated by $2(w_1 + z_1)$, and rewriting the form of Z_1 as

$$Z_1 = \{n(w_1 + z_1) + mz_1 : n, m \in \mathbb{Z}\}$$

implies that $Z_1(K)/B_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$. As the second consequence, $Z_2(K) = 0$ because there are no 2-chains with zero boundary, and hence $H_2(K) \simeq 0$. □

Exercise 4.47. By Theorems 4.99, 4.102, and 8.13, the Klein bottle and the connected sum of two projective planes are homeomorphic. Prove this via their fundamental squares of surface.

Exercise 4.48. Show that the following table contains all compact 2-manifolds you can make from a unit square by identifying its edges.

	$\mathbb{S}^2$	$\mathbb{P}^2$	$\mathbb{T}^2$	$\mathbb{K}^2$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	0	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}$
H_2	$\mathbb{Z}$	0	$\mathbb{Z}$	0