Chapter 11

Hyperbolic Problems

Definition 11.1. A second-order, constant-coefficient, linear partial differential equation (PDE) of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0 \tag{11.1}$$

is called a *hyperbolic PDE* if its coefficients satisfy

$$B^2 - 4AC > 0. \tag{11.2}$$

Definition 11.2. The *one-dimensional wave equation* is a hyperbolic PDE of the form

$$u_{tt} = a^2 u_{xx}, \tag{11.3}$$

where $a > 0$ is the *wave speed*.

Definition 11.3. The *one-dimensional advection equation* is

$$u_t = -au_x \ \text{ in } \ \Omega := (0,1) \times (0,T), \tag{11.4}$$

where $x \in (0,1)$ is the spatial location and $t \in (0,T)$ the time; the equation has to be supplemented with an *initial condition*

$$u(x,0) = \eta(x), \ \text{ on } \ (0,1) \times \{0\} \tag{11.5}$$

and appropriate boundary conditions at either $\{0\} \times (0,T)$ or $\{1\} \times (0,T)$, depending on the sign of a.

Theorem 11.4. The exact solution of the Cauchy problem (11.4) is

$$u(x,t) = \eta(x - at). \tag{11.6}$$

Definition 11.5. A system of PDEs of the form

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{0} \tag{11.7}$$

is *hyperbolic* if A is diagonalizable and its eigenvalues are all real.

Example 11.6. The Euler equations are

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ u \end{bmatrix} + \begin{bmatrix} 0 & \kappa_0 \\ \frac{1}{\rho_0} & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} p \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{11.8}$$

The equation for the pressure p can be further written as

$$p_{tt} = a^2 p_{xx} \text{ with } a = \pm\sqrt{\kappa_0/\rho_0}.$$

11.1 Classical MOLs

Example 11.7. Discretize the advection equation (11.4) in space at grid point x_j by

$$U_j'(t) = -\frac{a}{2h}\left(U_{j+1}(t) - U_{j-1}(t)\right), \quad 2 \le j \le m, \tag{11.9}$$

where $U_j(t) \approx u(x_j, t)$ for $j = 1, 2, \cdots, m+1$. For periodic boundary conditions

$$u(0,t) = u(0,t) = g_0(t), \tag{11.10}$$

the discretizations of (11.4) at $j = 1$ and $j = m+1$ are

$$U_1'(t) = -\frac{a}{2h}\left(U_2(t) - U_{m+1}(t)\right), \tag{11.11}$$

$$U_{m+1}'(t) = -\frac{a}{2h}\left(U_1(t) - U_m(t)\right). \tag{11.12}$$

Then the semi-discrete system can be written as

$$\mathbf{U}'(t) = A\mathbf{U}(t), \tag{11.13}$$

where

$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ 1 & & & & -1 & 0 \end{bmatrix}, \tag{11.14}$$

and $\mathbf{U}(t) = [U_1(t), U_2(t), \cdots, U_{m+1}(t)]^T$.

Lemma 11.8. The eigenvalues of A in (11.13) are

$$\lambda_p = -\frac{ia}{h}\sin(2\pi ph) \text{ for } p = 1, 2, \ldots, m+1. \tag{11.15}$$

The corresponding eigenvector $\mathbf{w}^p$ has components

$$w_j^p = e^{2\pi ipjh} \text{ for } j = 1, 2, \ldots, m+1. \tag{11.16}$$

11.1.1 The FTCS method

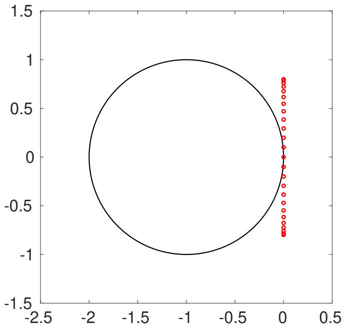
Definition 11.9. The FTCS method for the advection equation (11.4) is

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right), \tag{11.17}$$

or in matrix form

$$\mathbf{U}^{n+1} = (I + kA)\mathbf{U}^n. \tag{11.18}$$

Corollary 11.10. The FTCS method for the advection equation (11.4) is unconditionally unstable for $k = O(h)$.



Lemma 11.11. The FTCS method for the advection equation has Lax-Richtmyer stability for $k = O(h^2)$.

11.1.2 The leapfrog method

Definition 11.12. The *leapfrog method* for the advection equation (11.4) is

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right),$$

or, equivalently

$$U_j^{n+1} = U_j^{n-1} - \frac{ak}{h}\left(U_{j+1}^n - U_{j-1}^n\right). \tag{11.19}$$

11.1.3 Lax-Friedrichs

Definition 11.13. The *Lax-Friedrichs method* for the advection equation (11.4) is

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right). \tag{11.20}$$

Lemma 11.14. Consider the IVP system

$$\mathbf{U}'(t) = A_\epsilon \mathbf{U}(t), \tag{11.21}$$

where

$$A_\epsilon = A + \frac{\epsilon}{h^2}\begin{bmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{bmatrix} \tag{11.22}$$

with A defined in (11.14). The eigenvalues of A_ϵ are

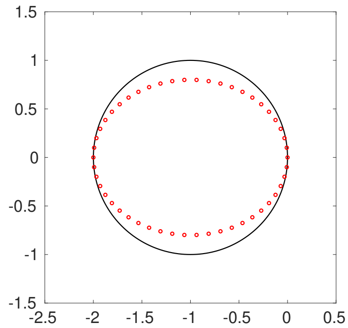
$$\lambda_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\epsilon}{h^2}\left[1 - \cos(2\pi ph)\right] \tag{11.23}$$

for $p = 1, 2, \ldots, m+1$. The corresponding eigenvector $\mathbf{w}^p$ has components

$$w_j^p = e^{2\pi ipjh} \text{ for } j = 1, 2, \ldots, m+1. \tag{11.24}$$

Lemma 11.15. The Lax-Friedrichs method can be considered as the MOL obtained by applying the forward Euler to the semidiscrete system (11.21) with $\epsilon = \frac{h^2}{2k}$.

Theorem 11.16. The Lax-Friedrichs method (11.20) is convergent provided that $\left|\frac{ak}{h}\right| \le 1$.



11.1.4 Lax-Wendroff

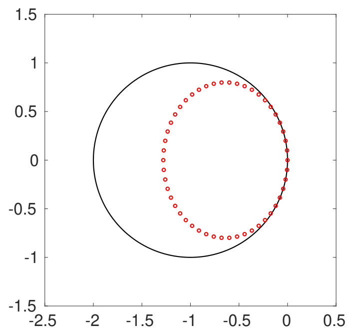
Definition 11.17. The *Lax-Wendroff method* for the advection equation (11.4) is

$$\begin{aligned} U_j^{n+1} =& U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) \\ &+ \frac{k^2a^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right). \end{aligned} \tag{11.25}$$

Lemma 11.18. The Lax-Wendroff method (11.25) is second-order accurate both in space and in time.

Lemma 11.19. The Lax-Wendroff method (11.25) can be considered as the MOL obtained by applying the forward Euler to the semidiscrete system (11.21) with $\epsilon = \frac{1}{2}ka^2$.

Theorem 11.20. The Lax-Wendroff method (11.25) is convergent provided $\left|\frac{ak}{h}\right| \le 1$.



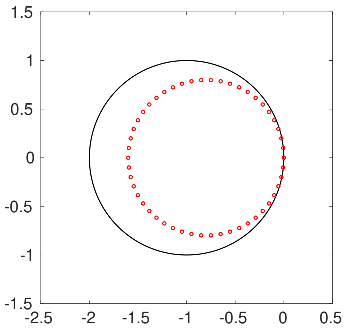
11.1.5 The Upwind method

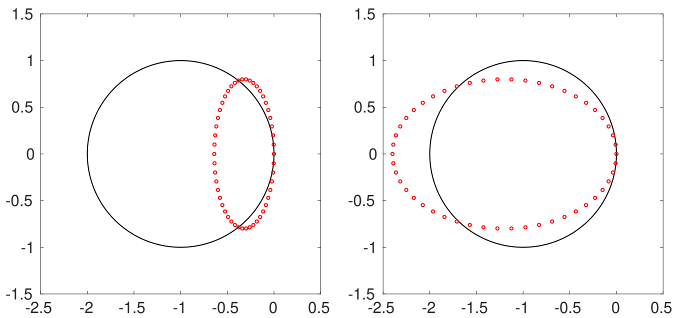
Definition 11.21. The *upwind method* for the advection equation (11.4) is

$$U_j^{n+1} = \begin{cases} U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right) & \text{if } a \geq 0; \\ U_j^n - \frac{ak}{h}\left(U_{j+1}^n - U_j^n\right) & \text{if } a < 0. \end{cases} \tag{11.26}$$

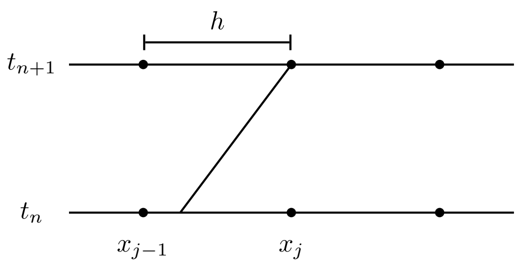
Lemma 11.22. The upwind method (11.25) can be considered as the MOL obtained by applying the forward Euler to the semidiscrete system (11.21) with $\epsilon = \frac{ah}{2}$.

Theorem 11.23. For $a > 0$, the upwind method is convergent if and only if $\frac{ak}{h} \leq 1$; for $a < 0$, the upwind method is convergent if and only if $\frac{ak}{h} \geq -1$.





Corollary 11.24. The upwind method is equivalent to characteristic tracing followed by a linear interpolation.



11.1.6 The Beam-Warming method

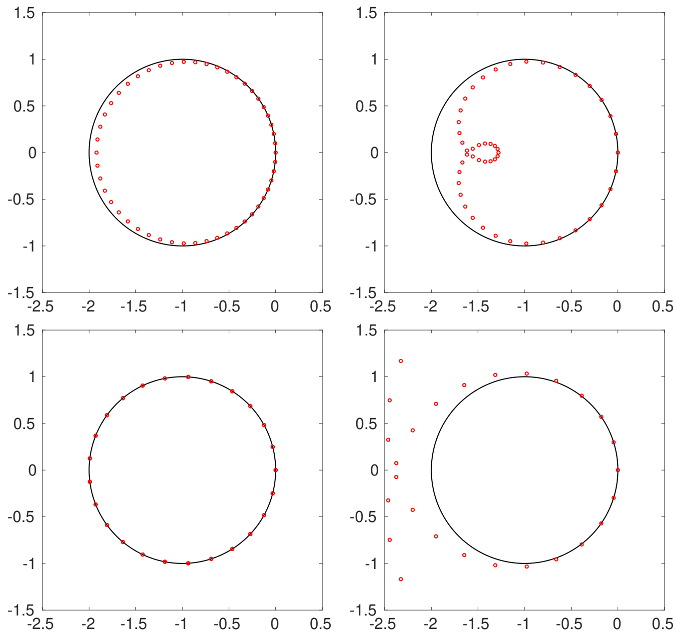
Definition 11.25. The *Beam-Warming method* solves the advection equation (11.4) by

$$\begin{aligned} U_j^{n+1} =& U_j^n - \frac{ak}{2h}\left(3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right) \\ &+ \frac{a^2k^2}{2h^2}\left(U_j^n - 2U_{j-1}^n + U_{j-2}^n\right) \quad \text{if } a \geq 0; \end{aligned} \tag{11.27}$$

$$\begin{aligned} U_j^{n+1} =& U_j^n - \frac{ak}{2h}\left(-3U_j^n + 4U_{j+1}^n - U_{j+2}^n\right) \\ &+ \frac{a^2k^2}{2h^2}\left(U_j^n - 2U_{j+1}^n + U_{j+2}^n\right) \quad \text{if } a < 0. \end{aligned} \tag{11.28}$$

Exercise 11.26. Show that the Beam-Warming method is second-order accurate both in time and in space.

Exercise 11.27. Show that the Beam-Warming methods (11.27) and (11.28) are stable for $\frac{ak}{h} \in [0, 2]$ and $\frac{ak}{h} \in [-2, 0]$, respectively. Reproduce the following plots for $\frac{ak}{h} = 0.8$, 1.6, 2, and 2.4.



11.2 The CFL condition

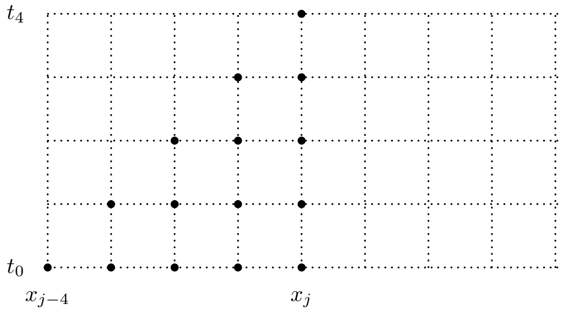
Definition 11.28. For the advection equation (11.4), the *domain of dependence* of a point $(X, T) \in \Omega$ is

$$\mathcal{D}_{\text{ADV}}(X, T) = \{X - aT\}. \tag{11.29}$$

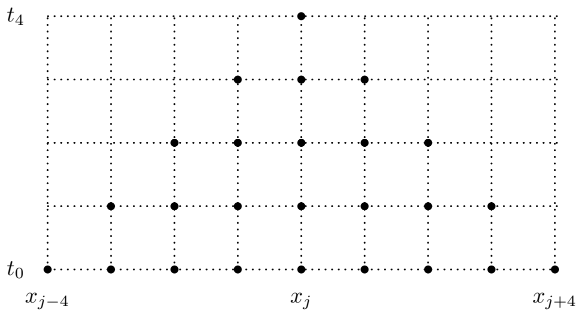
Definition 11.29. The *numerical domain of dependence* of a grid point (x_j, t_n) is the set of all grid points x_i such that U_i^0 at x_i has an effect on U_j^n.

$$\mathcal{D}_N(x_j, t_n) = \{x_i : U_i^0 \text{ affects } U_j^n\}. \tag{11.30}$$

Example 11.30. Numerical domain dependence of a grid point using the upwind method.



Example 11.31. Numerical domain dependence of a grid point using a 3-point explicit method.



Theorem 11.32 (Courant-Friedrichs-Lewy)**.** A numerical method can be convergent only if its numerical domain of dependence contains the domain of dependence of the PDE, at least in the limit of $k, h \to 0$.

Example 11.33. The heat equation

$$\begin{cases} u_t = \nu u_{xx} \\ u(x,0) = \sqrt{\frac{\beta}{\pi}} e^{-\beta(x-\bar{x})^2}, \end{cases} \tag{11.31}$$

has its exact solution as

$$u(x,t) = \frac{1}{\sqrt{4\pi\nu t + \pi/\beta}} e^{-(x-\bar{x})^2/(4\nu t + 1/\beta)}. \tag{11.32}$$

The domain of dependence is the whole line, i.e.,

$$\mathcal{D}_{\text{DIFF}}(X, T) = (-\infty, +\infty) \tag{11.33}$$

because an initial point source

$$\lim_{\beta\to\infty} u(x,0) = \delta(x - \bar{x})$$

instantaneously affect each point on the real line:

$$\lim_{\beta\to\infty} u(x,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(x-\bar{x})^2}{4\nu t}}.$$

This is very much an artifact of the mathematical model rather than the true physics.

11.3 Modified equations

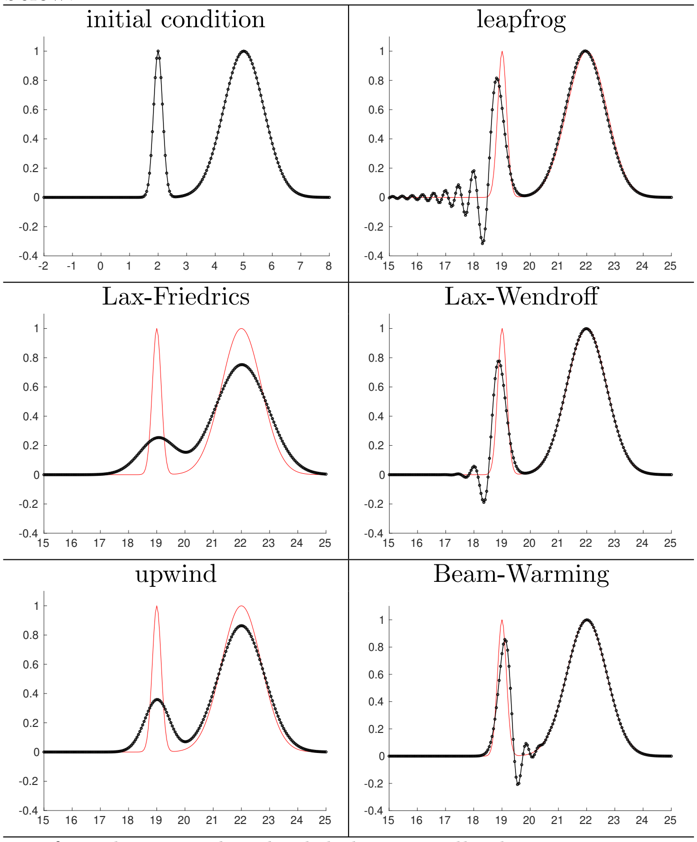
Example 11.34. For the advection equation

$$u_t + u_x = 0$$

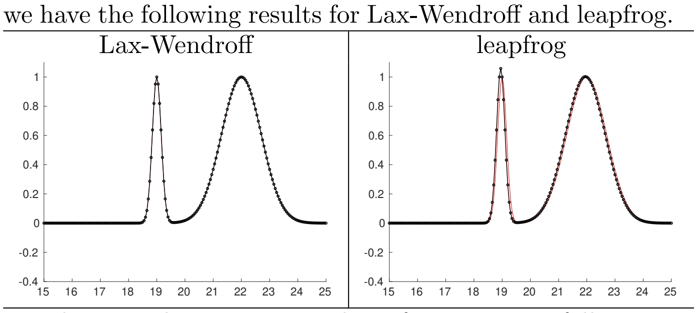
with initial condition

$$u(x,0) = \eta(x) = \exp(-20(x-2)^2) + \exp(-(x-5)^2), \tag{11.34}$$

the exact solution at $t = T$ is simply the initial data shifted by T. We solve this IVP problem with $h = 0.05$ and $k = 0.8h$ using the leapfrog method, the Lax-Friedrichs method, the Lax-Wendroff method, the upwind method, and the Beam-Warming method with the final results at $T = 17$ shown below.



If we change to $k = h$ while keeping all other parameters, we have the following results for Lax-Wendroff and leapfrog.



These results invite a number of questions as follows.

(a) Why are the solutions of Lax-Friedrichs and upwind much smoother than those of the other three methods?

(b) What caused the ripples in the solutions of the three methods in the right column?

(c) Why do the numerical solution of the leapfrog method contains more oscillations than that of the Lax-Wendroff method?

(d) For the Lax-Wendroff method, why do the ripples of numerical solutions lag behind the true crest?

(e) For the Beam-Warming method, why do the ripples of numerical solutions move ahead of the true crest?

(f) Why are numerical results with $k = h$ much better than those with $k = 0.8h$?

These questions concern the physics behind the different features of the results of different methods; they can be answered by the modified equations.

Exercise 11.35. Reproduce all results in Example 11.34.

Definition 11.36. The *modified equation of an MOL for solving a PDE* (the original equation) is a PDE obtained from the formula of the MOL by

(1) replacing U_j^n with a smooth grid function $v(x_j, t^n)$ in the MOL formula,

(2) expanding all terms in Taylor series at (x_j, t^n),

(3) neglecting potentially high-order terms.

Example 11.37. Consider the upwind method for solving the advection equation

$$U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right).$$

The modified equation can be derived as follows.

(1) Replace U_j^n with $v(x_j, t_n)$ and we have

$$v(x, t+k) = v(x,t) - \frac{ak}{h}\left(v(x,t) - v(x-h,t)\right).$$

(2) Expand all terms in Taylor series at (x,t) in a way similar to the calculation of the LTE.

$$\begin{aligned}0 =& \frac{v(x,t+k) - v(x,t)}{k} + \frac{a}{h}\left(v(x,t) - v(x-h,t)\right)\\ =& \left(v_t + \frac{1}{2}kv_{tt} + \frac{1}{6}k^2 v_{ttt} + \cdots\right)\\ &+ a\left(v_x - \frac{1}{2}hv_{xx} + \frac{1}{6}h^2 v_{xxx} + \cdots\right),\end{aligned}$$

and thus

$$v_t + av_x = \frac{1}{2}\left(ahv_{xx} - kv_{tt}\right) - \frac{1}{6}\left(ah^2 v_{xxx} + k^2 v_{ttt}\right) + \cdots,$$

differentiating with respect to t and x gives

$$\begin{aligned}v_{tt} &= -av_{xt} + \frac{1}{2}\left(ahv_{xxt} - kv_{ttt}\right) + \cdots,\\ v_{tx} &= -av_{xx} + \frac{1}{2}\left(ahv_{xxx} - kv_{ttx}\right) + \cdots.\end{aligned}$$

Combining these gives

$$v_{tt} = a^2 v_{xx} + O(k).$$

Therefore we have

$$v_t + av_x = \frac{1}{2}ah\left(1 - \frac{ak}{h}\right)v_{xx} + O(h^2 + k^2),$$

(3) Neglect the high-order terms and we have the modified equation as

$$v_t + av_x = \frac{1}{2}ah\left(1 - \frac{ak}{h}\right)v_{xx} := \beta v_{xx}, \tag{11.35}$$

which is satisfied better by the grid function than the advection equation $v_t + av_x = 0$.

Exercise 11.38. Derive the modified equation of the Lax-Wendroff method for the advection equation as

$$v_t + av_x + \frac{ah^2}{6}\left(1 - \left(\frac{ak}{h}\right)^2\right)v_{xxx} = 0. \tag{11.36}$$

Example 11.39. By Lemma E.16, The solution to the modified equation (11.36) is

$$v(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}(\xi)e^{i\xi(x - C_p t)}\mathrm{d}\xi.$$

By (11.36) and Example E.27, we have for the Lax-Wendroff method

$$\begin{aligned}C_p(\xi) &= a - \frac{ah^2}{6}\left(1 - \left(\frac{ak}{h}\right)^2\right)\xi^3,\\ C_g(\xi) &= a - \frac{ah^2}{2}\left(1 - \left(\frac{ak}{h}\right)^2\right)\xi^3;\end{aligned}$$

each velocity has a magnitude smaller than $|a|$, which explains the fact that the numerical oscillations lag behind the true wave crest.

Exercise 11.40. What if $\left|\frac{ak}{h}\right| = 1$? Discuss this case for both Lax-Wendroff and leapfrog methods.