Chapter 3

Cubical Complexes

3.1 Basic building blocks

3.1.1 Cells and faces

Definition 3.1. In the N-dimensional unit grid $\mathbb{Z}^N$, an *elementary cube*, or simple a *cube*, is the subset of $\mathbb{R}^N$ given by a finite product of N components,

$$P := I_1 \times \cdots \times I_N, \tag{3.1}$$

such that its ith component is either an edge $I_i = [m_i, m_i + 1]$ or a vertex $I_i = \{m_i\}$ in the ith ordinate axis of the grid. N is called the *embedding number* of P and each I_i an *elementary interval*.

Definition 3.2. A cube P with n edges and $N - n$ vertices is called a *n-dimensional cube* or *a cubical cell of dimension n* or simply an *n-cell*. We write $\dim P = n$.

Example 3.1. For $N = 2$, *cubical cells* are

- a vertex, or a 0-cell, or $\{n\} \times \{m\}$,
- an edge, or a 1-cell, or $\{n\} \times [m, m + 1]$ or $[n, n + 1] \times \{m\}$,
- a square, or a 2-cell, or $[n, n + 1] \times [m, m + 1]$,

where $n, m \in \mathbb{Z}$.

Definition 3.3. An *open n-cell* is an n-cell of which each edge is replaced with the corresponding open interval.

Example 3.2. For $N = 2$, open cells are

- a vertex or $\{n\} \times \{m\}$,
- an open edge or $\{n\} \times (m, m+1)$ or $(n, n+1) \times \{m\}$,
- the inside of a square or $(n, n + 1) \times (m, m + 1)$,

where $n, m \in \mathbb{Z}$.

Exercise 3.3. Show that the intersections of the elements of a basis $\mathcal{B}$ of the Euclidean space $\mathbb{R}^N$ with an open cell c form its basis. What about the closed cells?

Definition 3.4. A *face* Q of P is a cube such that $Q \subset P$. A face Q of P is a *proper face* of P if $\dim Q < \dim P$; it is a *primary face* if $\dim Q = \dim P - 1$.

Notation 3. The set of all unit cubes of dimension k (or all k-cells) embedded in $\mathbb{R}^N$ is denoted by $\mathcal{R}^N_k$. The set of all cells is denoted by

$$\mathcal{R}^N := \cup_{k=0}^N \mathcal{R}^N_k. \tag{3.2}$$

Corollary 3.5. A proper face of a k-cell is an ℓ-cell with $\ell < k$.

Proof. This follows from Definitions 3.2 and 3.4. □

Example 3.4. List the faces of the unitary 3-cube $[0, 1]^3$.

Exercise 3.5. For a k-dimensional cube, determine its numbers of vertices, edges, primary faces, proper faces, and all faces.

3.1.2 Boundaries of cubical cells

Example 3.6. The boundary of the 2-cell $[1, 2] \times [1, 2]$ is

$$\begin{aligned}&\partial\left([1,2]\times[1,2]\right)\\ =&[1,2]\times\{1\}+\{2\}\times[1,2]+[1,2]\times\{2\}+\{1\}\times[1,2]\\ =&[1,2]\times\partial[1,2]+\partial[1,2]\times[1,2],\end{aligned}$$

which motivates the following definition.

Definition 3.6. The *boundary of a cubical cell P* in $\mathcal{R}^N$

$$P = I_1 \times \cdots \times I_{j-1} \times I_j \times I_{j+1} \times \cdots \times I_N$$

is given by

$$\partial P := \sum_{j=1}^{n} I_1 \times \cdots \times I_{j-1} \times \partial I_j \times I_{j+1} \times \cdots \times I_N \tag{3.3}$$

where $\partial(A) = 0$ for a vertex A and $\partial(AB) = A + B$ for an edge AB. We also write $\partial_k P$ if P is a k-cell.

Lemma 3.7. Any $(k-1)$-dimensional face Q of a $(k+1)$-dimensional cube P is a common face of exactly two k-dimensional faces of P.

Proof. Let $1 \le i, j \le N$ be the indices of P and Q such that I_i, I_j are edges in P and vertices in Q, By Definition 3.1, we write

$$\begin{aligned} P &= I_1 \times \cdots \times I_i \times \cdots \times I_j \times \cdots \times I_N \\ &= I_1 \times \cdots \times [m_i, m_i+1] \times \cdots \times [m_j, m_j+1] \times \cdots \times I_N. \end{aligned}$$

By Definitions 3.6 and 3.4, each k-dimensional face of P that have Q as one of its faces must be one of the following,

$$\begin{aligned} f_1 &:= I_1 \times \cdots \times \{m_i\} \times \cdots \times I_j \times \cdots \times I_N, \\ f_2 &:= I_1 \times \cdots \times \{m_i+1\} \times \cdots \times I_j \times \cdots \times I_N, \\ f_3 &:= I_1 \times \cdots \times I_i \times \cdots \times \{m_j\} \times \cdots \times I_N, \\ f_4 &:= I_1 \times \cdots \times I_i \times \cdots \times \{m_j+1\} \times \cdots \times I_N. \end{aligned}$$

The rest of this constructive proof is an enumeration of all possibility of $Q := I_1 \times \cdots \times \{x\} \times \cdots \times \{y\} \times \cdots \times I_N$.

x	y	faces sharing Q as a common face
m_i	m_j	f_1, f_3
m_i+1	m_j	f_2, f_3
m_i	m_j+1	f_1, f_4
m_i+1	m_j+1	f_2, f_4

□

Exercise 3.7. Show that if P is an m-cell, then every nonzero term in ∂P is the sum of two $(m-1)$-cells that are a pair of opposite faces of P.

Definition 3.8. A point x in a n-cell a is an interior point of a if it has a neighborhood homeomorphic to $\mathbb{R}^n$; the rest are *boundary points*. The *boundary* ∂a is the set of all boundary points of a.

Exercise 3.8. Are the boundaries defined in Definitions 3.8 and 3.6 equivalent?

Exercise 3.9. Fix $N = 2$. What are the interior, frontier and closure of an open 0-cell P, an open 1-cell a, and an open 2-cell σ?

Exercise 3.10. Prove that a homeomorphism of a n-cell a to the n-ball $\mathbb{D}^n$ in $\mathbb{R}^n$ maps the boundary points of a to the frontier of $\mathbb{D}^n$.

Proof. See Lemma 4.7 for the more general case. □

3.1.3 Cubical complexes

Definition 3.9. A *cubical complex* K is a subset of $\mathcal{R}^N$ such that a cell $c \in K$ implies that all faces of c are in K. It is *finite* if it has a finite number of cells.

Example 3.11. Draw the figure for the following cubical complex K given by a list of all dimensions.

- 0D: $\{0\} \times \{0\}$, $\{0\} \times \{1\}$, $\{1\} \times \{0\}$, $\{1\} \times \{1\}$, $\{2\} \times \{0\}$, and $\{2\} \times \{1\}$.
- 1D: $\{0\} \times [0,1]$, $\{1\} \times [0,1]$, $[0,1] \times \{0\}$, $[0,1] \times \{1\}$, $[1,2] \times \{0\}$, $[1,2] \times \{1\}$, and $\{2\} \times [0,1]$,
- 2D: $[0,1] \times [0,1]$.

Verify that the boundary of each cell is the sum of some sets on the list.

Exercise 3.12. Give a list representation of the complex of a unit cube.

Definition 3.10. The *dimension of a cubical complex* K, written $\dim K$, is the highest among the dimensions of its cells.

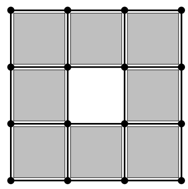
Definition 3.11. For a given n, the *n-skeleton of a cubical complex* K, denoted by $K^{(n)}$, is the collection of all k-cells of K with $k \le n$.

Exercise 3.13. Draw the 0-, 1-, and 2-skeleta of the cubical complex in Example 3.11.

Corollary 3.12. The skeleta are also cubical complexes.

Proof. The follows from Definitions 3.9 and 3.11. □

Exercise 3.14. Build the following cubical complex from that of a unit cube.



Definition 3.13. The *realization of a cubical complex* K is the union of the cells in K.

Definition 3.14. A set $X \subset \mathbb{R}^N$ is *cubical* if X can be written as a finite union of elementary cubes.

Exercise 3.15. Show that the realization of a cubical complex K remains unchanged if we replace "cells" with "open cells" in Definition 3.13.

Lemma 3.15. The realization of a cubical complex is a closed subset of $\mathbb{R}^N$.

Proof. The conclusion is obvious for a finite cubical complex. If it is infinite, it is still locally finite in the sense that the union does not produce new accumulation points. □

Lemma 3.16. Any planar graph can be represented as a one-dimensional cubical complex.

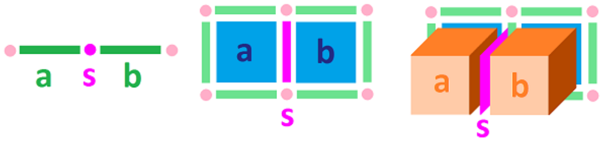
Exercise 3.16. Give an example of a graph that cannot be represented by a one-dimensional cubical complex.

3.2 Homology without orientation

3.2.1 k-chains as binary groups

Definition 3.17. A *k-chain* is a formal sum of k-cells. In particular, *0 is a k-chain for any k.*

Definition 3.18. A *binary k-chain* is a k-chain of which each cell x satisfies the binary arithmetic $x + x = 0$.



Corollary 3.19. In a binary chain, no cell is oriented.

Proof. The binary arithmetic says $x = -x$. □

Definition 3.20. The *kth chain group of a cubical complex $K \subset \mathcal{R}^N$*, written $C_k(K)$, is the set of all binary k-chains whose k-cells are in K. In particular, the *total kth chain group* is $C_k := C_k(\mathcal{R}^N)$.

Lemma 3.21. The chain group $C_k(K)$ of a cubical complex K is the subgroup of C_k generated by the k-cells in K.

3.2.2 The boundary operator

Definition 3.22. The *kth boundary operator on the chain group of a cubical complex K*, $\partial_k^K : C_k(K) \to C_{k-1}(K)$, is given as

$$\forall a = \sum_{\sigma_i \in K} s_i\sigma_i, \quad \partial_k^K(a) := \sum_i s_i \partial_k(\sigma_i), \tag{3.4}$$

where $s_i \in \mathbb{Z}_2$ and each σ_i is a k-cell.

Lemma 3.23. The boundary operator in (3.4) is a homomorphism.

Proof. The group $C_k(K)$ is a free abelian group with its basis as K. The rest of the proof follows from (3.4). □

Exercise 3.17. Why is Definition 3.9 defined that way?

Theorem 3.24 (Double boundary identity)**.** The composition of two consecutive boundary operators $\partial_k\partial_{k+1} : C_{k+1}(K) \to C_{k-1}(K)$ is the trivial homomorphism, i.e.,

$$\forall k = 0, 1, \ldots, \qquad \partial_k\partial_{k+1} = 0. \tag{3.5}$$

Proof. Since K is a basis of the free abelian group $C_{k+1}(K)$, it suffices to show that for each $(k+1)$-cell in K, we have $\partial_k\partial_{k+1}(c) = 0$. This indeed holds because of Lemma 3.7 and Definitions 3.18 and 3.22. □

Exercise 3.18. Compute the boundary of the boundary of a 2-cell,

$$\partial_1\partial_2\left([n, n+1] \times [m, m+1]\right),$$

to verify Theorem 3.24.

3.2.3 Cycles and boundaries

Definition 3.25. Let K be a given cubical complex. A k-chain of K is called a *k-boundary* of K if it is the boundary of a $(k+1)$-chain; and the *kth boundary group* is a subgroup of $C_k(K)$ given by

$$B_k(K) := \operatorname{Im}\partial_{k+1}. \tag{3.6}$$

Definition 3.26. A *k-cycle* of a cubical complex K is a k-chain of K with zero boundary. The *kth cycle group* is a subgroup of $C_k(K)$ given by

$$Z_k(K) := \ker \partial_k. \tag{3.7}$$

Corollary 3.27. Every boundary is a cycle, i.e.,

$$\forall k = 0, 1, \ldots, \qquad B_k \subseteq Z_k \subseteq C_k \tag{3.8}$$

or equivalently

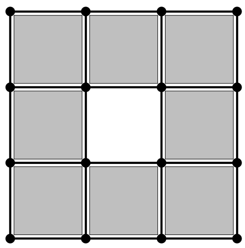
$$\forall k = 0, 1, \ldots, \qquad \operatorname{Im}\partial_{k+1} \subseteq \ker\partial_k \subseteq \operatorname{dom}\partial_k. \tag{3.9}$$

Proof. The second relation holds from Definition 3.26. By Theorem 3.24, the boundary of any boundary is zero, and hence every boundary must be a cycle. More precisely,

$$\begin{aligned} &\forall b \in B_k, \exists c \in C_{k+1}, \text{ s.t. } b = \partial_{k+1}c \\ \Rightarrow &\partial_k b = \partial_k\partial_{k+1}c = 0 \Rightarrow b \in \ker\partial_k. \end{aligned}$$ □

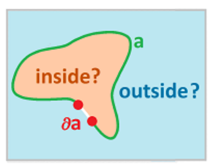
Definition 3.28. Two k-cycles are equivalent or *homologous* if they form the boundary of a $(k+1)$-chain.

Exercise 3.19. Give examples of homologous and non-homologous 1-cycles on the cubical complex as below.



Lemma 3.29. If a 0-chain $c \in C_0$ consists of an even number of vertices, then it is the boundary of some $s \in C_1$; otherwise c is not a boundary.

Proof. This follows from connecting each pair of two vertices with a 1-chain. The second clause holds because Definition 3.22 dictates that the boundary of a 1-chain always contains an even number of vertices. □



Example 3.20. Can you associate the two cases on the relation between cycles and boundaries with theorems in complex analysis?

3.2.4 The chain complex

Definition 3.30. A *chain complex* is a sequence of abelian groups and homomorphisms

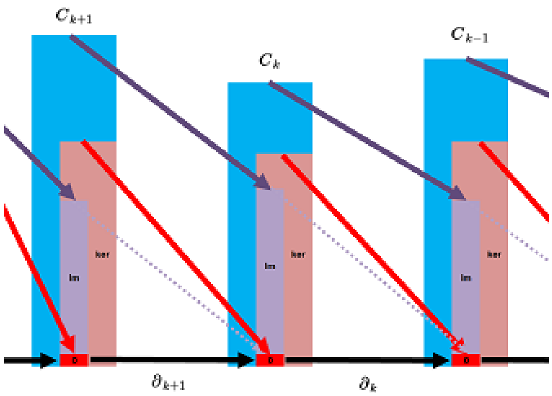
$$\cdots \longrightarrow G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \longrightarrow \cdots, \tag{3.10}$$

such that $\partial_n \partial_{n+1} = 0$ for each $n \in \mathbb{Z}$. The homomorphism ∂_n is called the *differentiation of degree* n, and G_n is called the *term of degree* n.

Definition 3.31. The *chain complex of a cubical complex* K in the Euclidean space $\mathbb{R}^N$ is the sequence of homomorphisms and finitely generated abelian groups

$$0 \xrightarrow{\partial_{N+1}} C_N(K) \xrightarrow{\partial_N} \cdots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0\ , \tag{3.11}$$

where both the start and the end are the zero group, and both ∂_{N+1} and ∂_0 are the trivial homomorphism. In particular, it is called the *total chain complex* for $K = \mathcal{R}^N$.



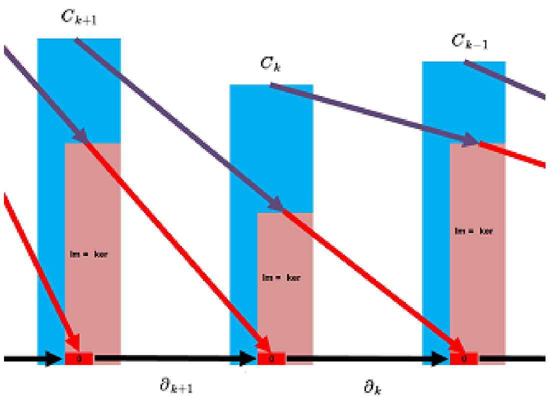
Example 3.21. For a graph, the chain complex becomes

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0\ .$$

Since $\partial_2 = \partial_0 = 0$, we only have to deal with the boundary operator ∂_1. Two other consequences are that every 0-chain is a cycle, i.e. $Z_0 = C_0$, and that the only 1-boundary is 0, i.e. $B_1 = \{0\}$.

Exercise 3.22. In the total chain complex, which ∂_k is surjective? Which ∂_k is injective?

Definition 3.32. A chain complex in the form of (3.10) is *exact at the term of degree* n if $\operatorname{Im}\partial_{n+1} = \ker \partial_n$. It is an *exact sequence* or an *exact complex* if it is exact at each of its terms.



Lemma 3.33. An exact sequence in the form of (3.11) satisfies

$$\sum_{k=0}^{N}(-1)^k \operatorname{rank} C_k = 0. \tag{3.12}$$

Proof. A finitely generated abelian group can be regarded as a linear space over $\mathbb{Z}$. The counting theorem (the fundamental thoerem of linear maps) and Definition 3.32 yield

$$\begin{aligned}\dim C_k &= \dim\ker\partial_k + \dim\operatorname{range}\partial_k \\ &= \dim\operatorname{range}\partial_{k+1} + \dim\operatorname{range}\partial_k.\end{aligned}$$

The rest of the proof follows from $\dim\operatorname{range}\partial_{N+1} = 0$ and $\dim\operatorname{range}\partial_0 = 0$. □

Exercise 3.23. How to make Z_0 unexceptional for Euclidean spaces? In other words, give a redefinition of it so that every cycle in it is indeed the boundary of a 1-chain?

Example 3.24. Determine G_2 and f_2 so that the following sequence is exact.

$$0 \xrightarrow{f_3=0} G_2 \xrightarrow{f_2=?} \mathbb{R}^n \xrightarrow{f_1} \mathbb{R}^m \xrightarrow{f_0=0} 0$$

All homomorphisms are projections.

Exercise 3.25. Determine G_0 and f_1 so that the following sequence is an exact sequence?

$$0 \xrightarrow{f_3=0} 2\mathbb{Z} \xrightarrow{f_2=\mathrm{Id}} \mathbb{Z} \xrightarrow{f_1=?} G_0 \xrightarrow{f_0=0} 0$$

where $2\mathbb{Z}$ is the set of even integers.

3.2.5 Homology groups

Definition 3.34. The *kth homology group*, $k = 0, 1, 2, \ldots$, of a cubical complex K is the quotient group of the kth cycle group by the kth boundary group, i.e.,

$$H_k(K) := Z_k(K)/B_k(K). \tag{3.13}$$

Exercise 3.26. Prove that $H_m(K) = H_m(K^{(m+1)})$ and give an example to show that replacing $K^{(m+1)}$ with $K^{(m)}$ fails.

Lemma 3.35. For a finite cubical complex K, each of the groups $C_k(K)$, $Z_k(K)$, $B_k(K)$, $H_k(K)$ is a direct sum of finitely many copies of $\mathbb{Z}_2$.

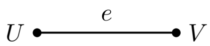
Proof. Due to the binary arithmetic, the order of each element is 2. The rest of the proof follows from the fundamental theorem of finitely generated abelian groups. □

Definition 3.36. The number of k-dimensional topological features in a cubical complex K, known as the *kth Betti number*, is the dimension of $H_k(K)$,

$$\beta_k(K) := \dim H_k(K). \tag{3.14}$$

Example 3.27. For a cubical complex only consisting of two isolated vertices, $K = \{U, V\}$, compute the chain complex, the groups in Lemma 3.35, and the Betti numbers of K.

Example 3.28. For a cubical complex only consisting of a single edge, $K = \{U, V, e = UV\}$, compute the chain complex, the groups in Lemma 3.35, and the Betti numbers of K.



Lemma 3.37. Suppose K and L are cubical complexes such that $L = K \cup \{e\}$ where the edge $e = UV \notin K$ and $U, V \in K$ are two vertices. Then

$$\begin{cases} U \sim V \text{ in } K \Rightarrow & \begin{cases} \beta_0(L) = \beta_0(K); \\ \beta_1(L) = \beta_1(K) + 1, \end{cases} \\ U \not\sim V \text{ in } K \Rightarrow & \begin{cases} \beta_0(L) = \beta_0(K) - 1; \\ \beta_1(L) = \beta_1(K), \end{cases} \end{cases} \tag{3.15}$$

Proof. By Definitions 3.34 and 3.36, we have

$$\begin{aligned} \beta_0 &= \dim Z_0 - \dim B_0 = \dim \ker \partial_0 - \dim \operatorname{Im} \partial_1, \\ \beta_1 &= \dim Z_1 - \dim B_1 = \dim \ker \partial_1 - \dim \operatorname{Im} \partial_2. \end{aligned}$$

$\dim \ker \partial_0$ equals the number of vertices and hence remains unchanged since there are no new vertices added. Consider $\dim \operatorname{Im} \partial_1$. If $U \sim V$, the new column added to the matrix of ∂_1 can be expressed as a linear combination of other columns; otherwise it would contradict $U \sim V$. Therefore $\dim \operatorname{Im} \partial_1$ remains unchanged, as each vector in $\operatorname{Im} \partial_1$ is a linear combination of the columns in ∂_1. If $U \not\sim V$, the new column added to the matrix of ∂_1 cannot be a linear combination of other columns, and hence $\dim \operatorname{Im} \partial_1$ increases by one. The above arguments prove the first and the third case in (3.15).

The condition $e \notin K$ implies that any 2-cell that has e as a face is not in K. Therefore, adding this edge does not change $\dim \operatorname{Im} \partial_2$. If $U \sim V$, the new column added to the matrix of ∂_1 can be expressed as a linear combination of other columns, which implies the presence of a new 1-cycle in L. Therefore $\dim \ker \partial_1$ increases by one. This proves the second case in (3.15). If $U \not\sim V$, adding the edge does not change $\dim \ker \partial_1$ and the fourth case is thus proved. □

Exercise 3.29. Augment the definition of a graph to allow self loops but not directed edges. Let β_0 and β_1 denote the first two Betti numbers of such a graph G. Consider a graph map $f = (f_N, f_E)$ on G with f_E being the identity map. Let β_0' and β_1' denote the first two Betti numbers of $K := f(G)$. Show that

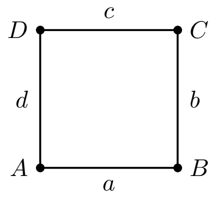
$$\dim \ker f_N = (\beta_1' - \beta_1) - (\beta_0' - \beta_0). \tag{3.16}$$

Give several examples to verify (3.16). Can you use (3.16) to deduce Lemma 3.37? If so, prove it. If not, give a counter example.

Example 3.30. For a hollow square

$$K = \{A, B, C, D, a, b, c, d\}$$

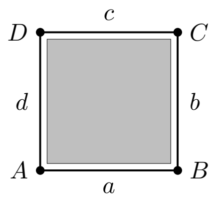
with $a = AB$, $b = BC$, $c = CD$, and $d = DA$, compute the chain complex, the groups in Lemma 3.35, and the Betti numbers of K.



Example 3.31. For a solid square

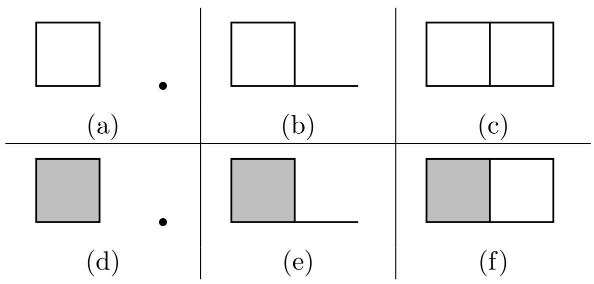
$$K = \{A, B, C, D, a, b, c, d, \tau\}$$

with $\partial\tau = a + b + c + d$ and $a = AB$, $b = BC$, $c = CD$, $d = DA$, compute the chain complex, the groups in Lemma 3.35, and the Betti numbers of K.



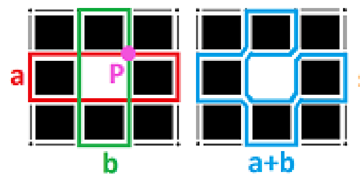
Exercise 3.32. Represent the sets below as realizations of cubical complexes. In order to demonstrate that you understand the algebra, for each of them:

- find the chain groups and find the boundary operator as a matrix;
- using only algebra, find Z_k, B_k, H_k for all k, including the generators.



Exercise 3.33. Compute the homology of a "train" with n cars.

3.3 Homology with orientation



3.3.1 Orientation of a real vector space

Definition 3.38. A *permutation* of the sequence $(1, 2, \ldots, n)$ is a function that reorders this sequence. The set of all such permutations, denoted by S_n, is known as the *symmetric group* on n elements, with function composition as the binary operation.

Definition 3.39. The *signature of a permutation* σ, denoted by $\mathrm{sgn}(\sigma)$, is $+1$ whenever the reordering given by σ can be achieved by successively interchanging two entries an even number of times, and -1 whenever it can be achieved by an odd number of such interchanges.

Example 3.34. Suppose a sequence $(1, 2, 3)$ is reordered to $\sigma = (2, 3, 1)$. Then $\mathrm{sgn}(\sigma) = +1$, $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$.

Definition 3.40 (Leibniz formula of determinants)**.** The *determinant of a square matrix* $A \in \mathbb{R}^{n\times n}$ is

$$\det A = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{3.17}$$

where the sum is over all permutations σ of the sequence $(1, 2, \ldots, n)$ and $a_{i,\sigma(i)}$ is the element of A at the ith row and the $\sigma(i)$th column.

Definition 3.41. Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of a vector space V. Any vector $\mathbf{v} \in V$ can be uniquely expressed as

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{b}_i = M_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}, \tag{3.18}$$

where $M_{\mathcal{B}} := [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ is the matrix of the basis and the column vector $[\mathbf{v}]_{\mathcal{B}} = (a_1, a_2, \ldots, a_n)^T$ is called the *coordinate vector of* $\mathbf{v}$ *with respect to the basis* $\mathcal{B}$.

Example 3.35. For the special case of $V = \mathbb{R}^n$ and $\mathcal{B}$ being the standard basis, a column vector $\mathbf{v} \in V$ and its coordinate vector are the same.

Definition 3.42. Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ be two bases of V. The *change-of-basis matrix* from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \left[[\mathbf{b}_1]_{\mathcal{C}}, \quad \ldots, \quad [\mathbf{b}_n]_{\mathcal{C}}\right]. \tag{3.19}$$

Lemma 3.43. $\forall \mathbf{v} \in V$, $[\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$.

Proof. Definition 3.41 yields

$$\begin{aligned} M_{\mathcal{C}}[\mathbf{v}]_{\mathcal{C}} = \mathbf{v} = M_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} &= [\mathbf{b}_1, \ldots, \mathbf{b}_n][\mathbf{v}]_{\mathcal{B}} \\ &= [M_{\mathcal{C}}[\mathbf{b}_1]_{\mathcal{C}}, \ldots, M_{\mathcal{C}}[\mathbf{b}_n]_{\mathcal{C}}][\mathbf{v}]_{\mathcal{B}} \\ &= M_{\mathcal{C}}\left[[\mathbf{b}_1]_{\mathcal{C}}, \ldots, [\mathbf{b}_n]_{\mathcal{C}}\right][\mathbf{v}]_{\mathcal{B}} \\ &= M_{\mathcal{C}} P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}, \end{aligned}$$

where the second line follows from applying (3.18) to $\mathbf{b}_i$'s. Multiplying the first and last terms with $M_{\mathcal{C}}^{-1}$ completes the proof. □

Lemma 3.44. For any bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ of a vector space V, we have

(i) $P_{\mathcal{B}\leftarrow\mathcal{B}} = I$,
(ii) $P_{\mathcal{C}\leftarrow\mathcal{B}} = P^{-1}_{\mathcal{B}\leftarrow\mathcal{C}}$,
(iii) $P_{\mathcal{D}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{C}} P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Definition 3.45. Two bases $\mathcal{B}$ and $\mathcal{C}$ define the *same orientation* of V iff $\det P_{\mathcal{C}\leftarrow\mathcal{B}} > 0$; they define the *opposite orientation* of V iff $\det P_{\mathcal{C}\leftarrow\mathcal{B}} < 0$.

Definition 3.46. An *orientation* of a vector space V is an equivalence class of bases of V under the equivalence relation of "defining the same orientation" of V. Any basis in an equivalence class indicates that orientation of V.

In particular, an orientation for a 0-dimensional vector space is a choice of sign ± 1.

Lemma 3.47. Any vector space has exactly two orientations.

Definition 3.48. The *standard orientation* of $\mathbb{R}^n$ is the orientation given by the standard basis of $\mathbb{R}^n$. The *non-standard orientation* of $\mathbb{R}^n$ is the orientation opposite to the standard orientation.

In particular, the *standard orientation* of $\mathbb{R}^0$ is $+1$.

Corollary 3.49. A basis $\mathcal{B}$ of $\mathbb{R}^n$ with $n \in \mathbb{N}^+$ gives the standard orientation iff $\det M_{\mathcal{B}} > 0$, i.e., the determinant of its matrix is positive.

Proof. The determinant of the matrix of a standard basis is positive. By Example 3.35, $[\mathbf{b}_i]_{\mathcal{C}} = \mathbf{b}_i$. Then we have $P_{\mathcal{B}\leftarrow\mathcal{C}} = M_{\mathcal{B}}$. The rest follows from Definition 3.45. □

Example 3.36. In $\mathbb{R}$, the standard basis consists of one scalar $+1$, and hence the standard orientation of $\mathbb{R}$ is from left to right. Any non-zero scalar b is a basis, and it gives the standard orientation if it has the same sign of $+1$.

Example 3.37. In $\mathbb{R}^2$, the standard basis consists of $\mathbf{e}_1 = [1, 0]^T, \mathbf{e}_2 = [0, 1]^T$. The (shortest) rotation from $\mathbf{e}_1$ to $\mathbf{e}_2$ is *counter-clockwise*, and this is the standard orientation of $\mathbb{R}^2$. A basis $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ gives the standard orientation iff the shortest rotation from $\mathbf{b}_1$ to $\mathbf{b}_2$ is counter-clockwise.

Example 3.38. Give a discussion on $\mathbb{R}^3$ similar to that in Example 3.37.

Lemma 3.50. Let $\sigma \in S_n$ be a permutation. Two bases

$$\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) \text{ and } \mathcal{S} := (\mathbf{b}_{\sigma(1)}, \mathbf{b}_{\sigma(2)}, \ldots, \mathbf{b}_{\sigma(n)}) \tag{3.20}$$

give the same orientation if and only if $\mathrm{sgn}(\sigma) = +1$.

Proof. By Definition 3.40 and the definition of the two bases, the determinant of the matrix of $\mathcal{S}$ is

$$\begin{aligned} \det M_{\mathcal{S}} &= \sum_{\tau\in S_n} \mathrm{sgn}(\tau) \prod_{i=1}^{n} b_{i,\tau\circ\sigma(i)}, \\ &= \sum_{\mathrm{sgn}(\tau)=1} \prod_{i=1}^{n} b_{i,\tau\circ\sigma(i)} - \sum_{\mathrm{sgn}(\tau)=-1} \prod_{i=1}^{n} b_{i,\tau\circ\sigma(i)} \\ &= \begin{cases} \sum_{\tau\in S_n} \mathrm{sgn}(\tau) \prod_{i=1}^{n} b_{i,\tau(i)} & \text{if } \mathrm{sgn}(\sigma) = +1; \\ -\sum_{\tau\in S_n} \mathrm{sgn}(\tau) \prod_{i=1}^{n} b_{i,\tau(i)} & \text{if } \mathrm{sgn}(\sigma) = -1. \end{cases} \\ &= \begin{cases} \det M_{\mathcal{B}}, & \text{if } \mathrm{sgn}(\sigma) = +1; \\ -\det M_{\mathcal{B}}, & \text{if } \mathrm{sgn}(\sigma) = -1, \end{cases} \end{aligned}$$

where in the third step we have used the fact that the subgroups of even permutations and odd permutations partition S_n. The rest of the proof follows from Definition 3.45 and elementary properties of determinants. □

3.3.2 Orientation of manifolds

Notation 4. $\mathbb{R}^0_{--} = \mathbb{R}^0_{-} = \mathbb{R}^0 := \{0\}$. For $n \in \mathbb{N}^+$, $\mathbb{R}^n_{-} = \{(x_1, \ldots, x_n) : x_1 \le 0\}$. $\mathbb{R}^n_{--} = \{(x_1, \ldots, x_n) : x_1 < 0\}$.

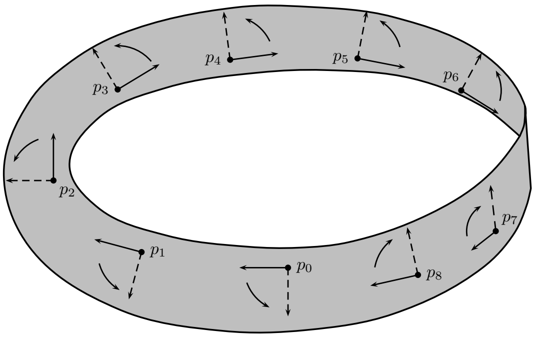
Definition 3.51. An *n-manifold with boundary* $\mathcal{M}$ is a Hausdorff topological space such that every point $p \in \mathcal{M}$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n_{-}$.

Example 3.39. A simple curve $\gamma : [0, 1] \to \mathbb{R}^2$ with $\gamma(0) \ne \gamma(1)$ is a 1-manifold with boundary.

Definition 3.52. An *orientation of a point* $p \in \mathbb{R}^3$ is an assignment of sign (either + or −) to p. An *orientation of a curve* $\gamma \in \mathbb{R}^3$ is an assignment of a direction in which γ is traversed. An *orientation of a smooth surface* $S \in \mathbb{R}^3$ is a continuous assignment of a direction of rotation to the tangent space at each point of S. An *orientation of a smooth 3-manifold* in $\mathbb{R}^3$ is a continuous choice of handedness to the tangent space at each point of it.

Definition 3.53. A manifold $\mathcal{M}$ that can be oriented is called an *orientable manifold*; and $\mathcal{M}$ together with a choice of orientation is called an *oriented manifold*.

Example 3.40. The Mobius band $\mathcal{M}$ is not orientable, i.e., for some point $p \in \mathcal{M}$ there does not exist a continuous assignment of rotation to the tangent space at p.



Definition 3.54. Let $\mathcal{M}$ be an oriented manifold with boundary and $\partial\mathcal{M}$ be its boundary. The *induced orientation* on $\partial\mathcal{M}$ is defined by the condition that the outward-pointing normal to $\mathcal{M}$ at $\partial\mathcal{M}$, followed by the orientation of $\partial\mathcal{M}$, agrees with the orientation of $\mathcal{M}$.

3.3.3 Oriented cells and chains

Definition 3.55. An *oriented k-cell* is a k-cell together with a choice of orientation for $\mathbb{R}^k$.

Definition 3.56. An *oriented k-chain of a cubical complex* K is a formal linear combination of oriented k-cells in K,

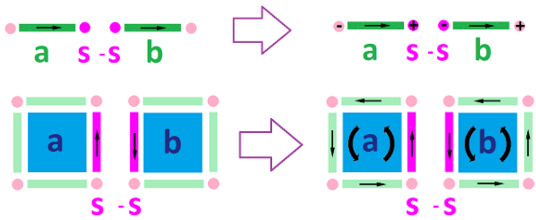
$$S = \sum_i r_i a_i, \tag{3.21}$$

where each a_i is an oriented k-cell and r_i is some scalar either in $\mathbb{Z}$ or in $\mathbb{R}$.

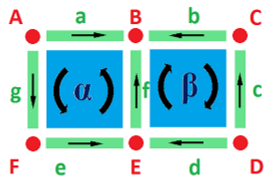
Example 3.41. We interpret oriented k-chains as follows,

- chain x traverses the cell x once in a direction that agrees with its orientation,
- chain $-2x$ traverses the cell x twice in a direction that disagrees with its orientation,
- chain $x + 2y + 5z$ visits cell x once, cell y twice, and cell z five times, in no particular order.

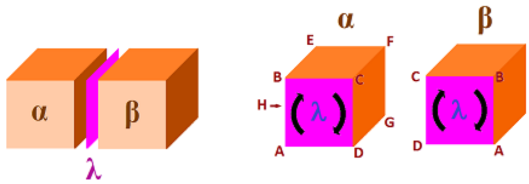
Example 3.42. The addition of orientations into the game aligns with the requirement that interior faces cancel for adjacent cells, as shown below.



Exercise 3.43. As a ubiquitous principle, the homology on oriented cubical complexes should be independent on the choices of how to orient individual k-cells. For the following two 2-cells with their faces oriented randomly, show that the canceling of adjacent cells with different orientations does not depend on the random orientations.



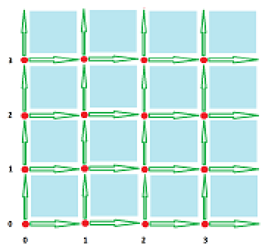
Example 3.44. How would the principle of free orientation work for $\mathbb{R}^k$ with $k > 2$?



As the core difficulty, the orientation of such a k-cell no longer gives a natural *total order* (the "direction" in Example 3.41) on the set of its proper faces!

3.3.4 The oriented boundary operator

Rule 3.57. We choose the orientation of all 1-cells in $\mathcal{R}^N$ as the standard orientation of $\mathbb{R}$.



Definition 3.58. The *boundary of an elementary interval* $E \subset \mathbb{R}$ is defined as

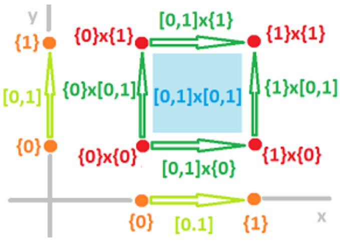
$$\partial E = \begin{cases} 0, & \text{if } E = \{m\}; \\ \{m+1\} - \{m\}, & \text{if } E = [m, m+1], \end{cases} \tag{3.22}$$

where $-\{m\}$ is interpreted as the vertex $\{m\}$ with negative orientation.

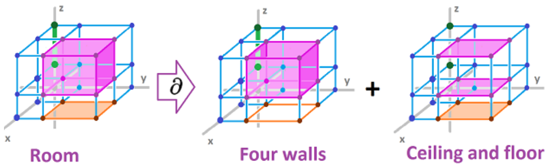
Definition 3.59. Let $Q^k \subset \mathbb{R}^n$ be a k-cube and E be an elementary interval. Then $Q^k \times E \subset \mathbb{R}^n \times \mathbb{R}$ is a k- or a $(k+1)$-cube, and its boundary is defined as

$$\partial(Q^k \times E) := \partial Q^k \times E + (-1)^k Q^k \times \partial E. \tag{3.23}$$

Example 3.45. Express the boundary of the 2-cell $Q^2 = [0,1]^2$ in terms of its proper faces.



Example 3.46. Reuse your result in Example 3.45 to express the boundary of the 3-cell $Q^3 = [0,1]^3$ in terms of its proper faces.



Identify which faces are the four walls and which faces are the ceiling and the floor.

Definition 3.60. The *boundary of an oriented k-chain S* as in Definition 3.56 is give by

$$\partial S = \partial \left(\sum_i r_i a_i \right) := \sum_i r_i \partial a_i. \tag{3.24}$$

Lemma 3.61. Suppose $Q^k \subset \mathbb{R}^n$ is an oriented k-chain, $E \subset \mathbb{R}$ is a 0- or 1-chain, and $Q^k \times E \subset \mathbb{R}^n \times \mathbb{R}$ is a k- or $(k+1)$-chain. Then

$$\partial(Q^k \times E) := \partial Q^k \times E + (-1)^k Q^k \times \partial E. \tag{3.25}$$

Proof. In the first step, we set E as a vertex or an edge to prove an intermediate equality using the linearity of the boundary operator in (3.24). Then (3.25) follows from this intermediate equality by a linear combination. □

Theorem 3.62 (Product fomula for boundaries)**.** Denote by $C_k(\mathcal{R}^n)$ the group of oriented k-chains. For $a \in C_i(\mathcal{R}^n)$, $b \in C_j(\mathcal{R}^m)$, and $a \times b \in C_{i+j}(\mathcal{R}^{m+n})$, we have

$$\partial(a \times b) = \partial a \times b + (-1)^i a \times \partial b. \tag{3.26}$$

Proof. We prove by induction on the dimension of b. Lemma 3.61 is the induction basis. The induction step is as follows.

$$\begin{aligned} & \partial(Q^k \times E \times A) \\ =& \partial(Q^k \times E) \times A + (-1)^{k+1}(Q^k \times E) \times \partial A \\ =& \left(\partial Q^k \times E + (-1)^k Q^k \times \partial E\right) \times A \\ & + (-1)^{k+1}(Q^k \times E) \times \partial A. \\ =& \partial Q^k \times E \times A + (-1)^k Q^k \times (\partial E \times A - E \times \partial A) \\ =& \partial Q^k \times (E \times A) + (-1)^k Q^k \times \partial(E \times A). \end{aligned}$$

□

Theorem 3.63 (Double boundary identity)**.** Oriented chains of cubical complexes have

$$\partial\partial = 0. \tag{3.27}$$

Proof. Represent a $(k+1)$-cube as the product of a k-cube and an edge,

$$Q^{k+1} = Q^k \times E,$$

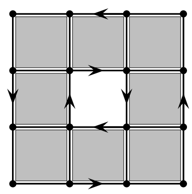
and we prove (3.27) by an induction on the dimension of a chain. The induction basis is $\partial\partial E = 0$ for any 1-chain, which follows from Definition 3.58. Suppose (3.27) holds for any Q^k, i.e., $\partial\partial Q^k = 0$. Then the induction step is as follows.

$$\begin{aligned} & \partial\partial(Q^k \times E) \\ =& \partial(\partial Q^k \times E + (-1)^k Q^k \times \partial E) \\ =& \partial(\partial Q^k \times E) + (-1)^k \partial(Q^k \times \partial E) \\ =& \partial\partial Q^k \times E + (-1)^{k-1}\partial Q^k \times \partial E \\ & + (-1)^k(\partial Q^k \times \partial E + (-1)^k Q^k \times \partial\partial E) \\ =& (-1)^{k-1}\left(\partial Q^k \times \partial E - \partial Q^k \times \partial E\right) \\ =& 0, \end{aligned}$$

where the second last step follows from the induction basis and the induction hypothesis. □

3.3.5 Homology groups

Example 3.47. In Definition 3.28, two k-cycles are defined to be homologous if they form the boundary of a $(k+1)$-chain. For oriented cycles, the meaning is much more interesting: compare the following figure with that in Exercise 3.19.



The two homologous 1-cycles are

$$\sum_{i=0}^{2}[i, i+1] \times (\{0\} - \{3\}) + \sum_{i=0}^{2}(\{3\} - \{0\}) \times [i, i+1],$$
$$-\partial[1,2]^2 = [1,2] \times (\{2\} - \{1\}) + (\{1\} - \{2\}) \times [1,2].$$

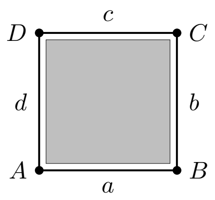
Definition 3.64. The *kth homology group*, $k = 0, 1, 2, \ldots$, of an oriented cubical complex K is the quotient group of the kth cycle group by the kth boundary group, i.e.,

$$H_k(K; \mathtt{R}) := Z_k(K; \mathtt{R})/B_k(K; \mathtt{R}), \tag{3.28}$$

where the scalar ring $\mathtt{R}$ may be $\mathbb{Z}_n$ for any $n > 1$, $\mathbb{Z}$, or $\mathbb{R}$.

Example 3.48. Clearly, the choice of $\mathtt{R} = \mathbb{Z}_2$ reduces Definition 3.64 to Definition 3.34. Hence Definition 3.64 is more generic than Definition 3.34.

Example 3.49. Redo Example 3.31 using Definition 3.64 with $\mathtt{R} = \mathbb{Z}$.



What if we remove the 2-cell?

Exercise 3.50. Redo Exercise 3.32 with $\mathtt{R} = \mathbb{Z}$.

Example 3.51. Compute the homology groups of the oriented cubical complex in Example 3.47.

Exercise 3.52. Compute the homology groups of the oriented cubical complex below.