Chapter 13

FV Methods for the Incompressible Navier-Stokes Equations (INSE)

Definition 13.1. A *domain* is a connected and bounded regular open subset $\Omega \subset \mathbb{R}^{\mathrm{D}}$ with $\mathrm{D} = 2, 3$.

Definition 13.2. The incompressible Navier-Stokes equations (INSE) is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - \nabla p + \nu \Delta \mathbf{u}, \tag{13.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{13.1b}$$

where $t \in [0, +\infty)$ is time, $\mathbf{x} \in \mathbb{R}^{\mathrm{D}}$ ($\mathrm{D} = 2, 3$) the spatial location, $\mathbf{g}$ the external forcing term, p the pressure, $\mathbf{u}$ the velocity, ν the dynamic viscosity.

Definition 13.3. The *Eulerian accelerations* of the INSE are vectors of time derivatives

$$\mathbf{a} := \frac{\partial \mathbf{u}}{\partial t}, \qquad \mathbf{a}^* := -\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{g} + \nu \Delta \mathbf{u}. \tag{13.2}$$

Definition 13.4. The *pressure Poisson equation* (PPE) is an elliptic equation that describes the relation between the pressure and the velocity in the INSE,

$$\Delta p = \nabla \cdot (\mathbf{g} - \mathbf{u} \cdot \nabla \mathbf{u}) \qquad \text{in } \Omega, \tag{13.3a}$$

$$\mathbf{n} \cdot \nabla p = \mathbf{n} \cdot (\mathbf{a}^* - \mathbf{a}) \qquad \text{on } \partial\Omega. \tag{13.3b}$$

13.1 Leray-Helmholtz Projection

Definition 13.5. A *projection* $\overline{\mathbf{P}}$ is a linear transformation from a vector space to itself such that the idempotent condition holds

$$\overline{\mathbf{P}}^2 = \overline{\mathbf{P}}. \tag{13.4}$$

Definition 13.6. The *Leray-Helmholtz projection* on a domain $\Omega \subset \mathbb{R}^{\mathrm{D}}$ is a projection $\mathscr{P} : \mathcal{C}^1(\Omega) \to \mathcal{C}^1(\Omega)$

$$\mathscr{P}\mathbf{v}^* := \mathbf{v} = \mathbf{v}^* - \nabla \phi \tag{13.5}$$

such that $\mathbf{v}, \mathbf{v}^* : \Omega \to \mathbb{R}^{\mathrm{D}}$ are vector fields, $\mathbf{v}$ satisfies $\nabla \cdot \mathbf{v} = 0$, and $\phi : \Omega \to \mathbb{R}$ is a scalar function.

Definition 13.7. The *no-penetration condition* for a vector field $\mathbf{v}$ on a domain Ω is the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0, \tag{13.6}$$

where $\mathbf{n}$ is the outward normal of the domain boundary $\partial\Omega$.

Lemma 13.8. On domains with periodic or no-penetration conditions, the Leray-Helmholtz projection is well defined and can be expressed as

$$\mathscr{P} = 1 - \nabla(\Delta_n)^{-1}\nabla \cdot, \tag{13.7}$$

where $(\Delta_n)^{-1}$ denotes solving Poisson's equation with pure Neumann conditions.

Proof. By Helmholtz's theorem, a sufficiently continuous vector field $\mathbf{v}^*$ on regular compact domain $\overline{\Omega}$ in $\mathbb{R}^{\mathrm{D}}$ can be uniquely decomposed into a divergence-free part $\mathbf{v}$ and a curl-free part $\nabla\phi$:

$$\mathbf{v}^* = \mathbf{v} + \nabla\phi,$$

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \times \nabla\phi = \mathbf{0}.$$

This is achieved by solving the Poisson's equation with pure Neumann boundary conditions:

$$\Delta\phi = \nabla \cdot \mathbf{v}^* \qquad \text{in } \Omega, \tag{13.8a}$$

$$\mathbf{n} \cdot \nabla\phi = \mathbf{n} \cdot (\mathbf{v}^* - \mathbf{v}) \qquad \text{on } \partial\Omega, \tag{13.8b}$$

where $\mathbf{n}$ denotes the outward normal of the domain boundary $\partial\Omega$. The above arguments justifies (13.7) and it remains to show that the BVP (13.8) admits a unique solution.

Periodic conditions imply $\oint_{\partial\Omega} \mathbf{n} \cdot \mathbf{v} = 0$. As for no-penetration conditions, Gauss theorem and $\nabla \cdot \mathbf{v} = 0$ yield

$$0 = \int_{\Omega} \nabla \cdot \mathbf{v} = \oint_{\partial\Omega} \mathbf{n} \cdot \mathbf{v},$$

thus the solvability of (13.8) holds. Since ϕ in (13.8) is determined up to an additive constant, $\nabla\phi$ is unique, which further implies the uniqueness of $\mathbf{v}$ in (13.5). □

Lemma 13.9. The Leray-Helmholtz projection $\mathscr{P}$ satisfies

$$\mathscr{P}^2 = \mathscr{P}, \quad \nabla \cdot \mathscr{P}\mathbf{v}^* = 0, \quad \mathscr{P}\nabla\phi = \mathbf{0}. \tag{13.9}$$

Proof. These identities follow from Definition 13.6 and Lemma 13.8. □

13.2 The approximate projection

Definition 13.10. The fourth-order *approximate projection* associated with the Leray-Helmholtz projection is the discrete operator

$$\mathbf{P} = \mathbf{I} - \mathbf{G}\mathbf{L}^{-1}\mathbf{D}, \tag{13.10}$$

where $\mathbf{I}$ is the identity operator and the other operators are the same as those in Definition 12.6.

Exercise 13.11. For periodic domains, express $\mathbf{DG}$ as a linear combination of cell averages to verify that $\mathbf{DG} \neq \mathbf{L}$. What is the one-dimensional stencil of this operator? Can you say anything about this stencil?

Lemma 13.12. The approximate projection $\mathbf{P}$ is not a projection.

Proof. Exercise 13.11 implies $\mathbf{P}^2 \neq \mathbf{P}$, and the conclusion follows from Definition 13.5. □

Exercise 13.13. Show that the discrete operator

$$\mathbf{P}_E = \mathbf{I} - \mathbf{G}(\mathbf{DG})^{-1}\mathbf{D} \tag{13.11}$$

is indeed a projection.

Definition 13.14. The *FV scalar inner product* and the *FV vector inner product* on a domain Ω are respectively

$$\begin{aligned} \langle \phi, \psi \rangle_S &= h^{\mathrm{D}} \textstyle\sum_{\mathbf{i}} \langle \phi \rangle_{\mathbf{i}} \langle \psi \rangle_{\mathbf{i}}, \\ \langle \mathbf{v}, \mathbf{w} \rangle_V &= h^{\mathrm{D}} \textstyle\sum_{\mathbf{i}} \langle \mathbf{v} \rangle_{\mathbf{i}} \cdot \langle \mathbf{w} \rangle_{\mathbf{i}}, \end{aligned} \tag{13.12}$$

where $\phi, \psi : \Omega \to \mathbb{R}$ are scalar functions and $\mathbf{v}, \mathbf{w} : \Omega \to \mathbb{R}^{\mathrm{D}}$ are vector functions.

Lemma 13.15. On periodic domains, the linear maps $\mathbf{G}$ and $-\mathbf{D}$ in (12.18) and (12.19) are adjoint in the sense that

$$\langle \mathbf{D}\mathbf{u}, \phi \rangle_S = -\langle \mathbf{u}, \mathbf{G}\phi \rangle_V. \tag{13.13}$$

The corresponding matrices satisfy $\mathbf{G} = -\mathbf{D}^T$.

Proof. It suffices to show

$$\begin{aligned} &\langle \mathbf{D}\mathbf{u}, \phi \rangle_S + \langle \mathbf{u}, \mathbf{G}\phi \rangle_V \\ =& h^{\mathrm{D}} \sum_{\mathbf{i}} \left(\langle \phi \rangle_{\mathbf{i}} \mathbf{D} \langle \mathbf{u} \rangle_{\mathbf{i}} + \langle \mathbf{u} \rangle_{\mathbf{i}} \cdot \mathbf{G} \langle \phi \rangle_{\mathbf{i}} \right) = 0. \end{aligned}$$

Consider all possible terms containing $\langle \phi \rangle_{\mathbf{i}}$. For dimension d, $\sum_{\mathbf{j}} \left(\langle \phi \rangle_{\mathbf{j}} \mathbf{D} \langle \mathbf{u} \rangle_{\mathbf{j}} \right)$ expands to

$$\langle \phi \rangle_{\mathbf{i}} \left(8 \langle u_d \rangle_{\mathbf{i}+\mathbf{e}^d} - \langle u_d \rangle_{\mathbf{i}+2\mathbf{e}^d} - 8 \langle u_d \rangle_{\mathbf{i}-\mathbf{e}^d} + \langle u_d \rangle_{\mathbf{i}-2\mathbf{e}^d} \right).$$

Similarly, $\sum_{\mathbf{j}} \left(\langle \mathbf{u} \rangle_{\mathbf{j}} \cdot \mathbf{G} \langle \phi \rangle_{\mathbf{j}} \right)$ expands to

$$\langle \phi \rangle_{\mathbf{i}} \left(-8 \langle u_d \rangle_{\mathbf{i}+\mathbf{e}^d} + \langle u_d \rangle_{\mathbf{i}+2\mathbf{e}^d} + 8 \langle u_d \rangle_{\mathbf{i}-\mathbf{e}^d} - \langle u_d \rangle_{\mathbf{i}-2\mathbf{e}^d} \right).$$

In the former, all the terms are contributed by $\langle \phi \rangle_{\mathbf{i}} \mathbf{D} \langle \mathbf{u} \rangle_{\mathbf{i}}$; in the latter, no terms come from $\langle \mathbf{u} \rangle_{\mathbf{i}} \cdot \mathbf{G} \langle \phi \rangle_{\mathbf{i}}$, e.g., $-8 \langle \phi \rangle_{\mathbf{i}} \langle u_d \rangle_{\mathbf{i}+\mathbf{e}^d}$ is contributed by $\langle \mathbf{u} \rangle_{\mathbf{i}+\mathbf{e}^d} \cdot \mathbf{G} \langle \phi \rangle_{\mathbf{i}+\mathbf{e}^d}$. Because of periodicity, all five multi-indices are well defined for the cells to remain inside the domain, hence these above terms cancel. The same argument also applies to the terms containing $\langle u_d \rangle_{\mathbf{i}}$ for all d. □

Corollary 13.16. On periodic domains, the corresponding matrix of the approximate projection operator $\mathbf{P}$ defined in (13.10) is symmetric, i.e. $\mathbf{P}^T = \mathbf{P}$.

Proof. The symmetry of $\mathbf{L}$ and Lemma 13.15 yield

$$\mathbf{P}^T = (\mathbf{I} - \mathbf{G}\mathbf{L}^{-1}\mathbf{D})^T = \mathbf{I} - \mathbf{D}^T\mathbf{L}^{-1}\mathbf{G}^T = \mathbf{I} - \mathbf{G}\mathbf{L}^{-1}\mathbf{D} = \mathbf{P}.$$

□

Lemma 13.17. On periodic domains, the discrete gradient $\mathbf{G}$ in (12.18) and discrete Laplacian $\mathbf{L}$ in (12.20) commute,

$$\mathbf{GL} = \mathbf{LG}. \tag{13.14}$$

Consequently, the discrete Laplacian and the approximate projection commute,

$$\mathbf{PL} = \mathbf{LP}. \tag{13.15}$$

Proof. For a 1D periodic domain, let $\bar{\mathbf{G}}$ and $\bar{\mathbf{L}}$ denote the matrices of the discrete gradient operator and Laplacian operator scaled by $12h$ and $12h^2$, respectively. From (12.18) and (12.20), we have

$$\bar{\mathbf{G}}_{i,j} = \begin{cases} \pm 8, & j = \mathrm{mod}(i \pm 1, m) \\ \mp 1, & j = \mathrm{mod}(i \pm 2, m) \\ 0, & \text{otherwise} \end{cases}, \tag{13.16}$$

$$\bar{\mathbf{L}}_{i,j} = \begin{cases} -30, & j = i \\ 16, & j = \mathrm{mod}(i \pm 1, m) \\ -1, & j = \mathrm{mod}(i \pm 2, m) \\ 0, & \text{otherwise} \end{cases}, \tag{13.17}$$

where m is the number of cells. To avoid clustering of notation, I drop "mod" in the indices of matrix entries to use the cyclic shorthands "$i \pm \cdot$ " for "$\mathrm{mod}(i \pm \cdot, m)$," "$k \pm \cdot$" for "$\mathrm{mod}(k \pm \cdot, m)$," and so on. It follows that

$$\begin{aligned} (\bar{\mathbf{G}}\bar{\mathbf{L}})_{k,\ell} &= \textstyle\sum_{j=k-2}^{k+2} \bar{\mathbf{G}}_{k,j} \bar{\mathbf{L}}_{j,\ell} \\ &= -\bar{\mathbf{L}}_{k+2,\ell} + 8\bar{\mathbf{L}}_{k+1,\ell} - 8\bar{\mathbf{L}}_{k-1,\ell} + \bar{\mathbf{L}}_{k-2,\ell}, \\ (\bar{\mathbf{L}}\bar{\mathbf{G}})_{k,\ell} &= \textstyle\sum_{j=k-2}^{k+2} \bar{\mathbf{L}}_{k,j} \bar{\mathbf{G}}_{j,\ell} \\ &= -\bar{\mathbf{L}}_{k,\ell-2} + 8\bar{\mathbf{L}}_{k,\ell-1} - 8\bar{\mathbf{L}}_{k,\ell+1} + \bar{\mathbf{L}}_{k,\ell+2}. \end{aligned}$$

Since $\bar{\mathbf{L}}$ is a Toeplitz matrix, we have $\bar{\mathbf{L}}_{k+2,\ell} = \bar{\mathbf{L}}_{k,\ell-2}$, $\bar{\mathbf{L}}_{k+1,\ell} = \bar{\mathbf{L}}_{k,\ell-1}$, and so on. It follows that

$$\bar{\mathbf{G}}\bar{\mathbf{L}} = \bar{\mathbf{L}}\bar{\mathbf{G}}. \tag{13.18}$$

For a 2D domain, the matrices of the discrete operators can be expressed as Kronecker products of their 1D counterparts and identity matrices

$$12h^2\mathbf{L} = \bar{\mathbf{L}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{L}}, \quad 12h\mathbf{G} = \left(\bar{\mathbf{G}} \otimes \mathbf{I},\ \mathbf{I} \otimes \bar{\mathbf{G}} \right)^T. \tag{13.19}$$

In order to prove (13.14), it suffices to show that $\mathbf{L}$ commutes with each subblock of $\mathbf{G}$, which follows from (13.18) and the mixed-product property of Kronecker products, i.e. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. For three and higher dimensions, (13.14) can be proved by a straightforward induction based on the 1D and 2D arguments.

Finally, (13.15) follows directly from (13.10), (13.14), and Lemma 13.15. □

Theorem 13.18. On periodic domains, both the spectral radius and the Euclidean 2-norm of the approximate projection operator are one,

$$\rho(\mathbf{P}) = \|\mathbf{P}\|_2 = 1. \tag{13.20}$$

Furthermore, $\mathbf{P}$ is a fourth-order approximation to the Leray-Helmholtz projection,

$$\mathbf{P}\langle\mathbf{u}\rangle_{\mathbf{i}} - \frac{1}{h^{\mathrm{D}}}\int_{\mathcal{C}_{\mathbf{i}}} \left(\mathbf{I} - \nabla(\Delta_n^{-1})\nabla\cdot\right)\mathbf{u} = O(h^4). \tag{13.21}$$

Proof. It follows from Corollary 13.16 that $\rho(\mathbf{P}) = 1$ implies $\|\mathbf{P}\|_2 = 1$, hence we only need to show the former. For the projection $\mathbf{P}_E$ in (13.11), we have $\lambda(\mathbf{P}_E) \in \{0,1\}$ since its *minimal polynomial* is $\mathbf{P}_E^2 - \mathbf{P}_E = 0$. Let $\mathbf{P}_E = \mathbf{I} - \mathbf{Q}_E$, $\mathbf{P} = \mathbf{I} - \mathbf{Q}$. Clearly, $\lambda(\mathbf{Q}_E) \in \{0,1\}$, and we only need to show $0 \le \lambda(\mathbf{Q}) \le 1$. By Lemma 13.17, $\mathbf{G}$ commutes with both $\mathbf{L}^{-1}$ and $\mathbf{DG}$. Thus we have

$$\begin{aligned}\mathbf{Q} =&\mathbf{G}\mathbf{L}^{-1}(\mathbf{DG})(\mathbf{DG})^{-1}\mathbf{D}\\ =&\mathbf{L}^{-1}(\mathbf{DG})\mathbf{G}(\mathbf{DG})^{-1}\mathbf{D}\\ =&\mathbf{L}^{-1}\mathbf{DG}\mathbf{Q}_E,\end{aligned}$$

and it suffices to shows

$$0 \le \lambda\left(\mathbf{L}^{-1}\mathbf{DG}\right) \le 1.$$

Using discrete Fourier analysis, we can define the shift operator as

$$s_d\langle\phi\rangle_{\mathbf{i}} = \langle\phi\rangle_{\mathbf{i}+\mathbf{e}^d}, \tag{13.22}$$

whose eigenvectors are the single Fourier modes with the eigenvalues $e^{\mathrm{i}\beta_d}$, where $\beta_d = \kappa_d\frac{\pi}{N}$, $\kappa_d = 1, \cdots, N-1$, and N the total number of points in dimension d. It follows from (12.18), (12.19), and (12.20) that for a given Fourier component,

$$\lambda(\mathbf{DG}) = -\frac{4}{h^2}\sum_{d=1}^{\mathrm{D}}\sin^2\frac{\beta_d}{2}\left(1-\sin^2\frac{\beta_d}{2}\right)\left(1+\frac{2}{3}\sin^2\frac{\beta_d}{2}\right)^2;$$

$$\mathbf{L} = \frac{1}{12h^2}\sum_{d=1}^{\mathrm{D}}\left(16s_d + \frac{16}{s_d} - 30 - s_d^2 - \frac{1}{s_d^2}\right)$$

$$\Rightarrow \lambda(\mathbf{L}) = -\frac{4}{h^2}\sum_{d=1}^{\mathrm{D}}\sin^2\frac{\beta_d}{2}\left(1+\frac{1}{3}\sin^2\frac{\beta_d}{2}\right).$$

$\lambda\left(\mathbf{L}^{-1}\mathbf{DG}\right) \ge 0$ follows from the negative definiteness of $\lambda(\mathbf{L})$ and $\lambda(\mathbf{DG})$. $\lambda\left(\mathbf{L}^{-1}\mathbf{DG}\right) \le 1$ holds because

$$\eta(1-\eta)\left(1+\tfrac{2}{3}\eta\right)^2 - \eta\left(1+\tfrac{1}{3}\eta\right) = -\tfrac{4}{9}\eta^2(2\eta+\eta^2) \le 0,$$

for $\eta = \sin^2\frac{\beta_d}{2} \in [0,1]$.

Finally, (13.21) follows directly from (12.24a), (12.24b), and (12.24c) in Lemma 12.7; it can also be proved via considering the Taylor expansions of the symbols of individual operators for any *fixed* Fourier mode. ☐

13.3 The INSE on periodic domains

Theorem 13.19. The following ODE is a fourth-order approximation to the INSE (13.1) on periodic domains:

$$\frac{\mathrm{d}\langle\mathbf{u}\rangle}{\mathrm{d}t} = \mathbf{P}\Big(-\mathbf{D}\langle\mathbf{uu}\rangle + \langle\mathbf{g}\rangle + \nu\mathbf{L}\langle\mathbf{u}\rangle\Big). \tag{13.23}$$

Proof. This follows from applying the Leray-Helmholtz projection to the INSE (13.1), taking average of the resulting equation, using (13.21) and Lemma 12.7. ☐

Algorithm 13.20. A *fourth-order FV method for solving the INSE on periodic domains* is obtained by directly applying the ERK-ESDIRK IMEX algorithm (12.35) to the ODE system (13.23) with

$$\mathbf{X}^{[\mathrm{E}]}\langle\mathbf{u}\rangle = -\mathbf{D}\langle\mathbf{uu}\rangle + \langle\mathbf{g}\rangle, \qquad \mathbf{X}^{[\mathrm{I}]}\langle\mathbf{u}\rangle = \nu\mathbf{L}\langle\mathbf{u}\rangle. \tag{13.24}$$

More precisely, the algorithmic steps are

$$\langle\mathbf{u}\rangle^{(1)} = \langle\mathbf{u}\rangle^n \approx \langle\mathbf{u}(t^n)\rangle, \tag{13.25a}$$

for $s = 2, 3, \ldots, n_{\mathrm{s}}$,

$$\begin{aligned}(\mathbf{I}-k\nu\gamma\mathbf{L})\langle\mathbf{u}\rangle^{(s)} =& \langle\mathbf{u}\rangle^n + k\sum_{j=1}^{s-1}a_{s,j}^{[\mathrm{E}]}\mathbf{P}\mathbf{X}^{[\mathrm{E}]}\langle\mathbf{u}\rangle^{(j)}\\ &+ k\nu\sum_{j=1}^{s-1}a_{s,j}^{[\mathrm{I}]}\mathbf{L}\langle\mathbf{u}\rangle^{(j)},\end{aligned} \tag{13.25b}$$

$$\begin{aligned}\langle\mathbf{u}^*\rangle^{n+1} =& \langle\mathbf{u}\rangle^n + k\sum_{s=1}^{n_{\mathrm{s}}}b_s^{[\mathrm{E}]}\mathbf{X}^{[\mathrm{E}]}\langle\mathbf{u}\rangle^{(s)}\\ &+ k\nu\sum_{s=1}^{n_{\mathrm{s}}}b_s^{[\mathrm{I}]}\mathbf{L}\langle\mathbf{u}\rangle^{(s)},\end{aligned} \tag{13.25c}$$

$$\langle\mathbf{u}\rangle^{n+1} = \mathbf{P}\langle\mathbf{u}^*\rangle^{n+1}. \tag{13.25d}$$

At any time instant, the pressure can be extracted from the velocity by solving the discrete pressure Poisson equation

$$\mathbf{L}\langle p\rangle = \mathbf{D}\big(\langle\mathbf{g}\rangle - \mathbf{D}\langle\mathbf{uu}\rangle + \nu\mathbf{L}\langle\mathbf{u}\rangle\big) \tag{13.26}$$

with periodic boundary conditions.

Theorem 13.21. The solution $\langle\mathbf{u}\rangle$ produced by Algorithm 13.20 evolves in a vector space that is solenoidal with fourth-order accuracy, i.e. $\mathbf{D}\langle\mathbf{u}\rangle = O(h^4)$.

Proof. It suffices to point out that the velocity at each intermediate stage satisfies $\mathbf{D}\langle\mathbf{u}\rangle^{(s)} = O(h^4)$ because

- the initial velocity $\langle\mathbf{u}\rangle^n$ satisfies $\mathbf{D}\langle\mathbf{u}\rangle^n = O(h^4)$,
- $\mathbf{DPX}^{[\mathrm{E}]}\langle\mathbf{u}\rangle = O(h^4)$,
- $\mathbf{L}$ is linear and commutes with $\mathbf{P}$. ☐