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1. Find a solution to the following Dirichlet problem for the Laplace equation, by suing the Fourier transform:

$$(\partial_x^2 + \partial_y^2)u = 0, (x, y) \in \mathbb{R} \times \mathbb{R}_+, u(x, 0) = f(x) \in \mathcal{S}(\mathbb{R})$$

Solution. By taking Fourier transform get

$$(\partial_y^2 - \xi^2)\hat{u} = (\partial_y + |\xi|)(\partial_y - |\xi|)\hat{u} = 0, (\xi, y) \in \mathbb{R} \times \mathbb{R}_+, \hat{u}(\xi, 0) = \hat{f}(\xi) \in \mathcal{S}(\mathbb{R}).$$

The equation's general solutions is $\hat{u} = Ae^{-|\xi|y} + Be^{|\xi|y}$. However, since $\hat{u} \in L^1 \Rightarrow \lim_{y\to\infty} \hat{u} = 0$, $B = 0$. Combining with boundary condition

$$\hat{u}(\xi, 0) = A = \hat{f}(\xi).$$

According to

$$2\pi\mathcal{F}^{-1}(e^{-|\xi|y}) = \int e^{-|\xi|y + ix\cdot\xi} d\xi = \frac{1}{ix - y} - \frac{1}{-ix - y} = \frac{2y}{x^2 + y^2}$$

$$\begin{aligned} u(x, y) &= \mathcal{F}^{-1}\left(\hat{f}e^{-|\xi|y}d\xi\right) = \mathcal{F}^{-1}\left(\hat{f} \cdot \mathcal{F}(\frac{y}{\pi(x^2 + y^2)})\right) \\ &= \mathcal{F}^{-1}\mathcal{F}\left(\int \frac{y}{\pi(\xi^2 + y^2)} f(x - \xi)d\xi\right) = \int \frac{y}{\pi(\xi^2 + y^2)} f(x - \xi)d\xi \end{aligned}$$

□

2. Check that any polynomial $P(x) \in \mathcal{S}'(\mathbb{R}^n)$, however, $f(x) = e^{x^2} \notin \mathcal{S}'(\mathbb{R}), g(x) = e^s \notin \mathcal{S}'(\mathbb{R})$. Hint: you may want to use test functions like $e^{-\sqrt{1+x^2}}$.

Solution.

- For polynomial $P(x)$, without lost general, assuming the highest item is order N, so we have

  $$\int_{\mathbb{R}^n} (1 + \|x\|^2)^{-N-n}|P(x)|dx = C < \infty.$$

  Then

  $$\begin{aligned} \lim_{m\to\infty} |\int_{\mathbb{R}^n} P(x)(\phi_m 0\phi)dx| &\le \int_{\mathbb{R}^n} (1 + \|x\|^2)^{-N-n}|P(x)||(\lim_{m\to\infty}(1 + \|x\|^2)^{N+n}(\phi_m - \phi))dx| \\ &\le C \lim_{m\to\infty} P_{2(N+n)}(\phi_m - \phi) = 0. \end{aligned}$$

  It means $P(x) \in \mathcal{S}'$.

- Take $\phi_m = e^{-\sqrt{m+x^2}}$, we know

  $$\int \phi_m < \int e^{-|x|} = 2, \text{ and } \lim_{m\to\infty} \phi_m = 0.$$

  However take L_m large enough $x^{3/2} > \sqrt{m + x^2}$, $\lim_{m\to\infty} \int e^{x^2}\phi_m dx \le \int_{L_m}^{\infty} e^{x^{1/2}}dx + C = \infty$ indicate $f(x) \notin \mathcal{S}'(\mathbb{R})$.

- Take $\phi_m = e^{-\sqrt{m+|x|}}$, $\lim_{m\to\infty}\int e^x\phi_m dx \le \int e^{x^{1/3}}dx + C = \infty$ have $g(x) \notin \mathcal{S}'(\mathbb{R})$.

$\square$

3. Based on the formula

$$K_t(x) = (4\pi t)^{-1/2}e^{-|x|^2/(4t)}, \mathcal{F}(K_t)(\xi) = e^{-t|\xi|^2}, t > 0, x \in \mathbb{R}.$$

- Prove the formula holds for $t \in \mathbb{C}$ with $\Re t > 0$.
- With $t = \epsilon + i\lambda, \epsilon > 0, \lambda \in \mathbb{R}\backslash\{0\}$, By considering limit in $\mathcal{S}'(\mathbb{R})$ as $\epsilon \to 0+$, calculate $\mathcal{F}(K_{i\lambda})$.

Solution.

- For $t \in \mathbb{C}, \Re t > 0$,

$$\begin{aligned}\mathcal{F}(K_t)(\xi) &= \frac{1}{(4\pi t)^{1/2}}\int e^{-\frac{|x|^2}{4t}-ix\xi}dx = \frac{1}{\pi^{1/2}}\int e^{-y^2-i2t^{1/2}y\xi}dy \\ &= \frac{e^{-t\xi^2}}{\pi^{1/2}}\int e^{-(y-it^{1/2}\xi)^2}dy = e^{-t|\xi|^2} \qquad \text{by } \Re(y-it^{1/2}\xi) = \Re\frac{|x|}{2t^{1/2}} > 0\end{aligned}$$

- The same as above, replace t with $\epsilon + i\lambda$ get

$$\begin{aligned}\lim_{\epsilon\to 0}\mathcal{F}(K_{\epsilon+i\lambda})(\xi) &= \lim_{\epsilon\to 0}\frac{1}{(4\pi(\epsilon+i\lambda))^{1/2}}\int e^{-\frac{|x|^2}{4(\epsilon+i\lambda)}-ix\xi}dx \\ &= \lim_{\epsilon\to 0}\frac{1}{\pi^{1/2}}\int e^{-(\frac{x}{2\sqrt{\epsilon+i\lambda}}+i\sqrt{\epsilon+i\lambda}\xi)^2-i\lambda\xi^2}d(\frac{x}{2\sqrt{\epsilon+i\lambda}}) \\ &= \lim_{\epsilon\to 0}e^{-(\epsilon+i\lambda)|\xi|^2} \qquad \text{by } \Re(\frac{x}{2\sqrt{\epsilon+i\lambda}}+i\sqrt{\epsilon+i\lambda}\xi) > 0 \\ &= e^{-i\lambda|\xi|^2}\end{aligned}$$

$\square$