# PDE Homework #2

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**Problem 1.** *Solve $u_{tt} - c^2u_{xx} = F(t,x)$ for $t > 0$ and $x \in \mathbb{R}$, with data $u(0,x) = 0, u_t(0,x) = 0$ for $x \in \mathbb{R}$.*

*Solution.* First, let's review Duhamel's principle.
Define $u = u(t,x;s)$ to be the solution of

$$\begin{cases} u_{tt}(s;\cdot) - c^2u_{xx}(s;\cdot) = 0 \text{ in } (s,\infty)\times\mathbb{R} \\ u(s;\cdot) = 0, u_t(s;\cdot) = F(\cdot,s) \text{ on } \{t=s\}\times\mathbb{R}. \end{cases}$$

Now set

$$u(t,x) := \int_0^t u(t,x;s)\mathrm{d}s \quad (t \geq 0, x \in \mathbb{R}).$$

Duhamel's principle asserts this is a solution of

$$\begin{cases} u_{tt} - c^2u_{xx} = F \text{ in } (0,\infty)\times\mathbb{R} \\ u = 0, u_t = 0 \text{ on } \{t=0\}\times\mathbb{R}. \end{cases}$$

Apply d'Alembert's formula and Duhamel's principle, and we have

$$u(t,x;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\mathrm{d}y.$$

$$u(t,x) = \int_0^t u(t,x;s)\mathrm{d}s = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\mathrm{d}y\mathrm{d}s.$$

□

**Problem 2.** *Solve $u_{xx} + u_{yy} = 0$ with data $u(0,y) = e^y, u_x(0,y) = 0$, by applying the power series method.*

*Solution.* Let

$$u(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}x^iy^j, \tag{1}$$

substituting (1) into $u(0,y) = e^y$ yields

$$u(0,y) = \sum_{j=0}^{\infty} c_{0,j}y^j = e^y = \sum_{j=0}^{\infty}\frac{1}{j!}y^j \Rightarrow c_{0,j} = \frac{1}{j!}. \tag{2}$$

Substituting (1) into $u_x(0,y) = 0$ yields

$$u_x(0,y) = \sum_{j=0}^{\infty} c_{1,j}y^j = 0 \Rightarrow c_{1,j} = 0. \tag{3}$$

Sustitute (1) into $u_{xx} + u_{yy} = 0$, and we have

$$\begin{aligned} 0 &= \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} i(i-1)c_{i,j}x^{i-2}y^j + \sum_{i=0}^{\infty}\sum_{j=2}^{\infty} j(j-1)c_{i,j}x^iy^{j-2} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+2)(i+1)c_{i+2,j}x^iy^j + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+2)(j+1)c_{i,j+2}x^iy^j \\ \Rightarrow & c_{i+2,j} = -\frac{(j+2)(j+1)}{(i+2)(i+1)}c_{i,j+2}, \quad \forall i,j \geq 0. \end{aligned}$$

Combine with (2) and (3), use mathematical induction, and we can show that

$$c_{2i,j} = \frac{(-1)^i}{(2i)!j!}, \quad c_{2i+1,j} = 0, \quad \forall i, j \geq 0.$$

Therefore

$$\begin{aligned}
u(x,y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}x^i y^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i}{(2i)!j!}x^{2i}y^j \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}x^{2i} \sum_{j=0}^{\infty} \frac{1}{j!}y^j \\
&= \cos(x)e^y.
\end{aligned}$$

□

**Problem 3.** *Prove the uniqueness in Theorem 8, for $(x,t) \in [0,L]^2$. Hint: it could be proved by method of characteristics, or (odd) extension, or idea of Holmgren's uniqueness theorem.*

*Proof.* [Energy methods]
If $\tilde{u}$ is another such solution, then $w := u - \tilde{u}$ solves

$$\begin{cases} w_{tt} - w_{xx} = 0 \text{ in } [0,L] \times (0,L) \\ w(0,t) = w(L,t) = 0 \text{ in } [0,L] \\ w(x,0) = 0, w_t(x,0) = 0 \text{ in } [0,L]. \end{cases}$$

Define the "energy"

$$e(t) := \frac{1}{2}\int_0^L w_t^2(x,t) + w_x^2(x,t)\mathrm{d}x \quad (0 \leq t \leq T).$$

We compute

$$\dot{e}(t) = \int_0^L w_t w_{tt} + w_x(w_x)_t \mathrm{d}x = \int_0^L w_t(w_{tt} - w_{xx})\mathrm{d}x = 0.$$

There is no boundary term since $w(0,t) = w(L,t) = 0$, and hence $w_t(0,t) = w_t(L,t) = 0$ in $[0,L]$. Thus for all $0 \leq t \leq T$, $e(t) = e(0) = 0$, and so $w_t, w_x \equiv 0$ within $[0,L] \times [0,L]$. Since $w(x,0) \equiv 0$ in $[0,L]$, we conclude that $w = u - \tilde{u} \equiv 0$ in $[0,L]^2$. □

**Problem 4.** *Prove the strengthened version of Theorem 8, for $(x,t) \in [0,L]^2$, with $f \in \mathcal{C}^2, g \in \mathcal{C}^1$ instead. [Hint: use the general solutions for wave equations]*

*Proof.* Define $u(x,t)$ by d'Alembert's formula

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\mathrm{d}y \quad (x \in \mathbb{R}, t \geq 0). \tag{4}$$

It's quite trivial to verify that the following statements hold.

(i) $u \in \mathcal{C}^2(\mathbb{R} \times [0,\infty])$;

(ii) $u_{tt} - u_{xx} = 0$ in $\mathbb{R}\infty(,\infty)$;

(iii) $\lim_{(x,t)\to(x^0,0),t>0} u(x,t) = f(x^0)$, $\lim_{x,t\to(x^0,0),t>0} u_t(x,t) = g(x^0)$ for each point $x^0 \in \mathbb{R}$;

(iv) The uniqueness of the solution follows from the energy methods as in Problem 3.

Combining the above completes the proof. □

**Problem 5** (Poisson's formula)**.** *Solve the boundary value problem of the Laplace equation in the disc:*

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < R^2, \\ u(R\cos\theta, R\sin\theta) = f(\theta) \end{cases}$$

*by the method of separation of variables (in polar coordinates). You should finally obtain the celebrated Poisson's formula*

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\mathrm{d}\phi = \frac{R^2 - \|\mathbf{x}\|^2}{2\pi R}\int_{\|\mathbf{y}\|=R} \frac{u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^2}\mathrm{d}S(\mathbf{y}).$$

*Solution.* First we rewrite the Laplacian in polar coordinates. An application of the chain rule gives

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We now multiply both sides by $r^2$, and since $\Delta u = 0$, we get

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}.$$

Separating these variables, and looking for a solution of the form $u(r,\theta) = F(r)G(\theta)$, we find

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}.$$

Since the two sides depend on different variables, they must both be constant, say equal to $\lambda$. We therefore get the following equations:

$$\begin{cases} G''(\theta) + \lambda G(\theta) = 0, \\ r^2F''(r) + rF'(r) - \lambda F(r) = 0. \end{cases}$$

Since $G$ must be periodic of period $2\pi$, this implies that $\lambda \geq 0$ and that $\lambda = m^2$ where $m$ is an integer; hence

$$G(\theta) = \tilde{A}\cos m\theta + \tilde{B}\sin m\theta.$$

With $\lambda = m^2$ and $m \neq 0$, two simple solutions of the equation in $F$ are $F(r) = r^m$ and $F(r) = r^{-m}$. If $m = 0$, then $F(r) = 0$ and $F(r) = \log r$ are two solutions. If $m > 0$, we note that $r^{-m}$ grows unboundedly large as $r$ tends to zero, so $F(r)G(\theta)$ is unbounded at the origin; the same occurs when $m = 0$ and $F(r) = \log r$. We reject these solutions as contrary to our intuition. Therefore, we are left with the following special functions

$$F(r) = r^m, \quad G(\theta) = \tilde{A}\cos m\theta + \tilde{B}\sin m\theta.$$

We now make the important observation that the PDE is linear, and so we may superpose the above special solutions to obtain the presumed general solution:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m r^m \cos m\theta + b_m r^m \sin m\theta).$$

Let $r = R$ and use the boundary condition, and we have

$$f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m R^m \cos m\theta + b_m R^m \sin m\theta).$$

Thus

$$a_m = \frac{1}{\pi R^m}\int_0^{2\pi} f(\phi)\cos m\phi \mathrm{d}\phi, \quad b_m = \frac{1}{\pi R^m}\int_0^{2\pi} f(\phi)\sin m\phi \mathrm{d}\phi.$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\mathrm{d}\phi + \frac{1}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{R}\right)^m \int_0^{2\pi} f(\phi)(\cos m\phi\cos m\theta + \sin m\phi \sin m\theta)\mathrm{d}\phi.$$

Consider $r < \tilde{R} < R$. Since the series converges uniformly there, we can interchange the order of summation and integration, and obtain

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2} + \sum_{m=1}^{\infty}\left(\frac{r}{R}\right)^m \cos m(\theta - \phi)\right]\mathrm{d}\phi. \tag{5}$$

The summation of the infinite series

$$\frac{1}{2} + \sum_{m=1}^{\infty}\left(\frac{r}{R}\right)^m \cos m(\theta - \phi)$$

requires a little side calculation. Define for this purpose $z = \rho e^{i\alpha}$ and evaluate (for $\rho < 1$) the geometric sum

$$\frac{1}{2} + \sum_{m=1}^{\infty} z^m = \frac{1}{2} + \frac{z}{1-z} = \frac{1 - \rho^2 + 2i\rho\sin\alpha}{2(1 - 2\rho\cos\alpha + \rho^2)}.$$

Since $z^m = \rho^m(\cos m\alpha + i\sin m\alpha)$, we conclude upon separating the real and imaginary parts that

$$\frac{1}{2} + \sum_{m=1}^{\infty} \rho^m \cos m\alpha = \frac{1 - \rho^2}{2(1 - 2\rho\cos\alpha + \rho^2)}.$$

Returning to (5) using $\rho = r/R, \alpha = \theta - \phi$, we obtain the Poisson formula

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \mathrm{d}\phi = \frac{R^2 - \|\mathbf{x}\|^2}{2\pi R} \int_{\|\mathbf{y}\|=R} \frac{u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^2} \mathrm{d}S(\mathbf{y}).$$

$\square$