Chapter 11

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Problem 11.1. Assume that u solves the nonlinear heat equation

$$u_t = \frac{u_{xx}}{u_x^2} \qquad \text{in } \mathbb{R} \times (0, \infty)$$

with $u_x > 0$. Let v denote the inverse function to u in the variable x for each time $t > 0$, so that $y = u(x,t)$ if and only if $x = v(y,t)$. Show that v solves a linear PDE.

Solution. Since the v is the inverse function to u in variable x, we have

$$\begin{aligned} u_x &= \frac{du}{dx} = \frac{dy}{dx} = \frac{dy}{dv} = \frac{1}{v_y} \\ u_t &= \frac{du}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{v_t}{v_y} \\ u_{xx} &= \frac{d\left(\frac{du}{dx}\right)}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dy}\frac{dy}{dx} = \frac{d\left(\frac{1}{v_y}\right)}{dy}\frac{1}{v_y} = -\frac{v_{yy}}{v_y^3}. \end{aligned}$$

In summary, the nonlinear heat equation convert to

$$v_t = -v_{yy}.$$

Problem 11.2. Find a function $f\colon \mathbb{R}^3 \to \mathbb{R}, f = f(z, p_1, p_2)$, so that if u is any solution of the rotated wave equation

$$u_{xt} = 0,$$

then $w := f(u, u_x, u_t)$ solves *Liouville's equation*

$$w_{xt} = e^w.$$

(Hint: Show that f must have the form $f(z, p_1, p_2) = a(z) + b(p_1) + c(p_2)$.)

Solution. Take $f(z, p_1, p_2) = -2\ln z + \ln 2 + \ln p_1 + \ln P_2$. We are coming to verify the function satisfy condition.

With $u_{xt} = 0$,

$$\begin{aligned}
w_{xt} &= f_{xt}(u, u_x, u_t) \\
&= (-2\ln u + \ln 2)_{xt} + (\ln u_x)_{xt} + (\ln u_t)_{xt} \\
&= (2\frac{1}{u^2}u_x u_t - 2\frac{1}{u}u_{xt}) + (-\frac{1}{u_x^2}u_{xx}u_{xt} + \frac{1}{u_x}u_{xxt}) + (-\frac{1}{u_t^2}u_{tx}u_{tt} + \frac{1}{u_t}u_{txt}) \\
&= 2\frac{1}{u^2}u_x u_t \\
&= e^{-2\ln u + \ln 2 + \ln u_x + \ln u_t} \\
&= e^{f(u,u_x,u_t)} = e^w
\end{aligned}$$

Problem 11.3. (Lax pairs) Assume that $\{L(t)\}_{t\geq 0}$ is a family of symmetric linear operators on some real Hilbert space H, satisfying the evolution equation

$$\dot{L} = [B, L] = BL - LB,$$

for some collection of operators $\{B(t)\}_{t\geq 0}$. Suppose also that we have a corresponding family of eigenvalues $\{\lambda(t)\}_{t\geq 0}$ and eigenvectors $\{w(t)\}_{t\geq 0}$:

$$L(t)w(t) = \lambda(t)w(t).$$

Assume that L, B, λ and w all depend smoothly upon the time parameter t. Show that

$$\dot{\lambda} \equiv 0.$$

(Hint: Differentiate the identity $Lw = \lambda w$ with respect to t. Calculate $(\dot{\lambda}w, w)$.)

Solution. Consider the evolution equation, and differentiate the identity $Lw = \lambda w$ gives

$$\begin{aligned}
(Lw)_t &= (\lambda w)_t \\
(\dot{L}w + Lw_t) &= (\dot{\lambda}w + \lambda w_t) \\
\dot{\lambda}w &= BLw - LBw + Lw_t - \lambda w_t
\end{aligned}$$

Then calculate $(\dot{\lambda}w, w)$

$$\begin{aligned}
(\dot{\lambda}w, w) &= w^T \cdot w\dot{\lambda}w \\
&= w^T(BLw - LBw + Lw_t - \lambda w_t) \\
&= w^T B \cdot (Lw) - (Lw)^T \cdot Bw + (Lw)^T \cdot w_t - \lambda w^T \cdot w_t \\
&= \lambda w^T \cdot Bw - \lambda w^T \cdot Bw + \lambda w^T \cdot w_t - \lambda w^T \cdot w_t \\
&= 0.
\end{aligned}$$

From the abitrary of w, λ, $\dot{\lambda} \equiv 0$.

Problem 11.4. (Continuation) Given a function $u = u(x,t)$, define the linear operators $L(t)v := -v_{xx} + uv$ and $B(t)v = -4v_{xxx} + 6uv_x + 3u_x v$. Show that

$$(\dot{L} - [B, L])v = (u_t + u_{xxx} - 6uu_x)v = 0.$$

Consequently, if u solves this form of the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0,$$

then the eigenvalues of the linear operators $L(t)$ do not change with time.

Solution. Calculate $(\dot{L}-[B,L])v$ gives

$$\begin{aligned}(\dot{L}-[B,L])v &= (\dot{L}-BL+LB)v\\ &= (u_t v) - B(-v_{xx}+uv) + L(-4v_{xxx}+6uv_x+3u_x v)\\ &= (u_t v) - (-4(-v_{xx}+uv)_{xxx} + 6u(-v_{xx}+uv)_x + 3u_x(-v_{xx}+uv))\\ &\quad + (-(-4v_{xxx}+6uv_x+3u_x v)_{xx} + u(-4v_{xxx}+6uv_x+3u_x v))\\ &= (u_t + u_{xxx} - 6uu_x)v.\end{aligned}$$

So that if u solves this form of the KdV equation, $(\dot{L}-[B,L])v = (u_t+u_{xxx}-6uu_x)v = 0$ imply $\dot{L} = [B,L]$. From last problem known now the eigenvalues $\lambda(t)$ of the linear operators $L(t)$ maintain $\dot{\lambda} \equiv= 0$, immediately known that do not change with time t.

Problem 11.5. Let u^ϵ and v^ϵ solve the system

$$\begin{cases} u^\epsilon_t + \frac{1}{\epsilon}u^\epsilon_x = \frac{(v^\epsilon)^2-(u^\epsilon)^2}{\epsilon^2}\\ v^\epsilon_t - \frac{1}{\epsilon}v^\epsilon_x = \frac{(u^\epsilon)^2-(v^\epsilon)^2}{\epsilon^2}\end{cases}$$

Suppose we can write

$$u^\epsilon = u_0 + \epsilon u_1 + \epsilon u_2^\epsilon,\ v^\epsilon = v_0 + \epsilon v_1 + \epsilon v_2^\epsilon,$$

where u_0, v_0, u_1, v_1 are smooth, $u_0, v_0 > 0$, and the functions $u_2^\epsilon, v_2^\epsilon$ are bounded, along with their derivatives, uniformly in ϵ. Show that $w := u_0 \equiv v_0$ and w solves the nonlinear heat equation

$$w_t - \frac{1}{4}(\log w)_{xx} = 0.$$

(T. Kurtz, Trans. AMS 186 (1973), 259–272)

Solution. Substitute u^ϵ, v^ϵ with $u_0+\epsilon u_1+\epsilon u_2^\epsilon, v_0+\epsilon v_1+\epsilon v_2^\epsilon$ in the equation system gives

$$\begin{cases} (u_0^2-v_0^2)\epsilon^{-2} + (u_{0x} + 2u_0(u_1+u_2^\epsilon) - 2v_0(v_1+v_2^\epsilon))\epsilon^{-1}\\ +(u_{0t} + u_{1x} + (u_1+u_2^\epsilon)^2 - (v_1+v_2^\epsilon)^2)\epsilon^0 + (u_{1t} + u_2^{\epsilon-1}u_{2x})\epsilon^1 + (u_2^{\epsilon-1}u_{2t})\epsilon^2 = 0\\ (u_0^2-v_0^2)\epsilon^{-2} + (v_{0x} + 2u_0(u_1+u_2^\epsilon) - 2v_0(v_1+v_2^\epsilon))\epsilon^{-1}\\ +(-v_{0t} + v_{1x} + (u_1+u_2^\epsilon)^2 - (v_1+v_2^\epsilon)^2)\epsilon^0 + (-v_{1t} + v_2^{\epsilon-1}v_{2t})\epsilon^1 + (-v_2^{\epsilon-1}v_{2t})\epsilon^2 = 0\end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} u_0 - v_0 = 0\\ u_{0x} + 2u_0(u_1+u_2^\epsilon) - 2v_0(v_1+v_2^\epsilon) = 0\\ u_{0t} + u_{1x} + (u_1+u_2^\epsilon)^2 - (v_1+v_2^\epsilon)^2 = 0\\ u_{1t} + u_2^{\epsilon-1}u_{2x} = 0\\ u_2^{\epsilon-1}u_{2t} = 0\\ u_{0x} - v_{0x} = 0\\ u_{0t} + u_{1x} + v_{0t} - v_{1x} = 0\\ u_{1t} + u_2^{\epsilon-1}u_{2x} + v_{1t} - v_2^{\epsilon-1}v_{2x} = 0\\ v_2^{\epsilon-1}v_{2t} = 0\end{cases}.$$

Therefore, already have $u_0 = v_0 = w$. In other side, $w_t = \frac{1}{4}(\log w)_{xx}$ is equal to $4w_t = w_{xx}w^{-1} - w_x^2 w^{-2}$. From above equation system, we can know

$$
\begin{aligned}
4w_t &= 2(u_{0t} + v_{ot}) = 2(v_{1x} - u_{1x}) \\
\frac{w_x}{w} &= 2(v_1 + v_2^\epsilon - u_1 - u_2^\epsilon) \\
w_{xx} &= (u_{0x})_x = (2v_0(v_1 + v_2^\epsilon) - 2u_0(u_1 + u_2^\epsilon))_x \\
&= 2u_0(\frac{u_{0x}}{u_0}(v_1 + v_2^\epsilon - u_1 - u_2^\epsilon) + (v_{1x} + \epsilon v_2^{\epsilon-1} v_{2x} - u_{1x} + \epsilon u_2^{\epsilon-1} u_{2t})) \\
&= 2w(w_x^2 w^{-2} + v_{1x} - u_{1x}) \qquad \text{Since } \epsilon \to 0, s.t. \epsilon v_2^{\epsilon-1} v_{2x}, \epsilon u_2^{\epsilon-1} u_{2t} \to 0 \\
&= 2w(w_x^2 w^{-2} + 2w_t) \\
\Longrightarrow & \\
4w_t &= w_{xx}w^{-1} - w_x^2 w^{-2}.
\end{aligned}
$$

Which is equal to $w_t - \frac{1}{4}(\log w)_{xx} = 0$.

Problem 11.6. Firstly,$\phi \in C_c^\infty(\mathbb{R}^n)$ and A is a real, nonsingular, symmetric matrix. Then if

$$J_{\delta,A}(y) := \int_{\mathbb{R}^n} e^{ix \cdot Ax - \delta|x|^2 - ix \cdot y} dx$$

Prove

$$\lim_{\delta \to 0} \int_{\mathbb{R}^n} \phi(y) J_{\delta,A}(y) dy = \frac{\pi^{\frac{n}{2}} e^{\frac{i\pi}{4} \operatorname{sgn} A}}{|\det A|^{\frac{1}{2}}} \int_{\mathbb{R}^n} e^{-\frac{i}{4} x \cdot A^{-1} x} \phi(y) dy$$

Solution. Firstly, we can assume A is diagonal:

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \ (\lambda_k \neq 0, k = 1, dots, n).$$

Or we can rotate to new coordinate to diagonalize A, Since A is a real, nonsingular, symmetric matrix. Now for fixed $y, \lambda \in \mathbb{R}$ and $\delta > 0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}} e^{i\lambda x^2 - \delta x^2 - ixy} dx &= e^{\frac{y^2}{4(i\lambda - \delta)}} \int_{\mathbb{R}} e^{(i\lambda - \delta)\left(x - \frac{iy}{2(i\lambda - \delta)}\right)^2} dx \\
&= \frac{e^{\frac{y^2}{4(i\lambda - \delta)}}}{(\delta - i\lambda)^{1/2}} \int_\Gamma e^{-z^2} dz \qquad z = (\delta - i\lambda)^{1/2} \left(x - \frac{iy}{2(i\lambda - \delta)} \right)
\end{aligned}
$$