Chapter 1

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1.1 1

Exercise 1.1. Have

$$\xi'' - \frac{2}{x^{3/2}}\xi' = \frac{3}{16x^2} \tag{1.1}$$

prove $\xi \sim -\frac{3\sqrt{x}}{16}$

Solution. As $x \to 0$,

(a) Assuming

$$\xi'' \sim \frac{2}{x^{3/2}}\xi' \gg \frac{3}{16x^2} \tag{1.2}$$

The left solve get

$$\xi'' = \frac{2}{x^{3/2}}e^{-4x^{-1/2}+C} \ll \frac{3}{16x^2}$$

not satisfy assumption 1.2.

(b) Assuming

$$\xi'' \sim \frac{3}{16x^2} \gg \frac{2}{x^{3/2}}\xi' \tag{1.3}$$

The left solve get

$$\xi' \cdot \frac{2}{x^{3/2}} = -\frac{3}{16x} \cdot \frac{2}{x^{3/2}} = \frac{3}{8x^{5/2}} \gg \frac{3}{16x^2}$$

contradict to the assumption **??**1:1:3.

(c) Assuming

$$-\frac{3}{16x^2} \sim \frac{2}{x^{3/2}}\xi' \gg \xi'', \tag{1.4}$$

Solving 1.4 get

$$\xi \sim -\frac{3\sqrt{x}}{16},$$

and it's easy to verify the solution is maintain the assumption 1.4.

Exercise 1.2. If have condition

$$\frac{d^n y}{(dx)^n} = Q(x)y, \qquad x \to x_0 \; is \; i,s,p \; for \; Q(x). \tag{1.5}$$

Find asymptotic approx of y near x_0 .prove answer is $y \sim cQ^{\frac{1-n}{2n}}e^{\int^x Q^{\frac{1}{n}}\,dx}$.

Solution. Assuming $y = e^{s(x)}$. Then 1.5 convert to

$$\begin{aligned}
\partial^n(e^s) &= \partial^(n-1)((s')e^s) \\
&= \partial^(n-2)((s')^2e^s + s''e^s) \\
&\sim \partial^(n-2)((s')^2e^s) \\
&= \partial^(n-3)((s')^3e^s + 2s's''e^s) \\
&\dots \\
&= \partial((s')^(n-1)e^s) \\
&= (s')^ne^s + (n-1)(s')^(n-2)s''e^s \\
&\sim (s')^ne^s = Qe^s
\end{aligned} \tag{1.6}$$

So that $s(x) = \int^x Q^{\frac{1}{n}}dx$.The third and the last step above is because $x \to x_0 \; is \; i,s,p \; for \; Q(x)$.

Continuing assuming $y = e^{\int^x Q^{\frac{1}{n}}\,dx + d(x)}$, and as the same ways above computations get

$$\begin{aligned}
\partial^n(e^{\int^x Q^{\frac{1}{n}}\,dx+d}) &= \partial^(n-1)((Q^{1/n}+d')e^{\int^x Q^{\frac{1}{n}}\,dx+d}) \\
&\dots \\
&= \partial((Q^{1/n}+d')^{n-1}e^{\int^x Q^{\frac{1}{n}}\,dx+d} + \\
&\frac{(n-2)(n-1)}{2n}(Q^{1/n}+d')^{n-3}(Q^{\frac{1-n}{n}}Q' + d'')e^{\int^x Q} \\
&= (Q^{1/n}+d')^ne^{\int^x Q^{\frac{1}{n}}\,dx+d} + \\
&\frac{(n-1)(n)}{2n}(Q^{1/n}+d')^{n-2}(Q^{\frac{1-n}{n}}Q' + d'')e^{\int^x Q^{\frac{1}{n}}\,dx} \\
&= Qe^{\int^x Q^{\frac{1}{n}}\,dx+d}.
\end{aligned} \tag{1.7}$$

After eliminate the e and Qe,

$$nQ^{n-1/n}d' = -\frac{n-1}{2}Q^{-1/n}Q'. \tag{1.8}$$

Solve this get $d = \frac{1-n}{2n}ln(Q) + C$, so that $y \sim cQ^{\frac{1-n}{2n}}e^{\int^x Q^{\frac{1}{n}}dx}$.

Exercise 1.3. Verify:

$$\int_0^\infty \frac{e^{-t}}{1+xt}dt \sim \sum_{n=0}^{\infty}(-1)^n n! x^n \qquad as\ x \to 0. \tag{1.9}$$

Solution. Set $y(x) = \int_0^\infty \frac{e^{-t}}{1+xt}dt$,Then differential y with x get

$$\begin{aligned}\partial y &= \int_0^\infty \frac{(-1)^1 1! \times t^1 e^{-t}}{(1+xt)^2}dt\\ \partial^2 y &= \int_0^\infty \frac{(-1)^2 2! \times t^2 e^{-t}}{(1+xt)^3}dt\\ &\cdots\\ \partial^n y &= \int_0^\infty \frac{(-1)^n n! \times t^n e^{-t}}{(1+xt)^{n+1}}dt\end{aligned} \tag{1.10}$$

Combining with

$$\begin{aligned}\int_0^\infty \frac{t^n e^{-t}}{1}dt &= \int_0^\infty \frac{t^{n-1}e^{-t}}{1}dt + \left(t^n e^{-t}\right)\Big|_0^\infty\\ &\cdots\\ &= \int_0^\infty e^{-t}dt\\ &= 1,\end{aligned} \tag{1.11}$$

and $x \to 0$ get $\int_0^\infty \frac{e^{-t}}{1+xt}dt \sim \sum_{n=0}^\infty (-1)^n n! x^n \qquad as\ x \to 0$.

Exercise 1.4.

$$y'' = y^2 + e^x \tag{1.12}$$

analysis two situation

$$\begin{cases} x \to x_0 \\ x \to -\infty \end{cases}$$

Solution.

(a) $x \to x_0$

Since x_0 is finite, e^{x_0} is finite. But x_0 is singular point implicate $y- > \infty$, means that $y'' \sim y^2$.

Set $y = e^{s(x)}$,replace above equation y obtain

$$\begin{aligned} y'' &\sim y^2\\ (s')^2 e^s + s'' e^s &\sim e^{2s}\\ (s')^2 &\sim e^s\\ s &= -2ln(x-x_0)\end{aligned} \tag{1.13}$$

Set $y = e^{-2ln(x-x_0)+d(x)}$ $d \ll -2ln(x-x_0)\ as\ x \to x_0$, replace y in 1.13 get

$$\frac{6}{(x-x_0)^2} = \frac{1}{(x-x_0)^2}e^d$$
$$d = ln6$$

Set $y = \frac{6}{(x-x_0)^2}(1+c(x)) \quad c \ll 1\ as\ x \to x_0$, replace y in 1.13,

$$\begin{aligned}& y'' \sim y^2\\ \iff & \\ & \frac{24}{(x-x_0)^4}((1+c) - \frac{2}{3}c'(x-x_0) + \frac{1}{6}c''(x-x_0)^2)\\ & = \frac{24}{(x-x_0)^4}(1+2c+c^2)\\ \iff & \\ & c + \frac{2}{3}c'(x-x_0) - \frac{1}{6}c''(x-x_0)^2 = 0\\ \iff & \\ & c = C(x-x_0)^6\end{aligned}$$

$y = 6(x-x_0)^{-2} + 6C(x-x_0)^4$ replace y in 1.12

$$\begin{aligned}\frac{36}{(x-x_0)^{-4}} + 72C(x-x_0)^2 &= \frac{36}{(x-x_0)^{-4}} + 72C(x-x_0)^2 +\\ 36C^2(x-x_0)^8 + e^x &= 0\end{aligned} \tag{1.14}$$

The coefficient of $(x-x_0)^8$ C can't be every complex number so that satisfy 1.14, because that $(x-x_0)^8 \to 0,$ while $e^x > \alpha > 0$ as $x \to x_0$,

(b) $x \to -\infty$

Let $t = e^x$, $dx = dt/t$. From 1.12 get

$$\begin{aligned} & y'' = y^2 + e^x\\ \iff & d^2y/dx = y^2 + e^x\\ \iff & y'' = t^2y^2 + t^3\end{aligned} \tag{1.15}$$

Set $y = \sum_{i=0}^\infty a_i t^i$.From 1.15 set $y'' \sim t^3 \gg t^2y^2 \implies y = \frac{x^5}{20}$. Using 1.15 and Taylor's formula can get y Taylor expansion in $t = 0$.So the answer is analytic.