Chapter 4

Homework 21935004 谭焱

4.1 Problem

Exercise 4.1. Consider $s \in \mathbb{S}_3^2$ on $[0,2]$:

$$s(x) = \begin{cases} p(x) & \text{if } x \in [0,1], \\ (2-x)^3 & \text{if } x \in [1,2]. \end{cases}$$

Determine $p \in \mathbb{P}_3$ such that $s(0) = 0$. Is $s(x)$ a natural cubic spline?

Solution. Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, Then $s(x)$ is natural cubic implicate

$$\begin{cases} 6a_3(0) + 2a_2 = 0 \\ 6a_3(1) + 2a_2 = 6(2-1) \\ 3a_3(1)^2 + 2a_2(1) + a_1 = 3(2-1)^2 \\ a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 = (2-1)^3 \end{cases} \Longrightarrow \begin{cases} a_3 = 1 \\ a_2 = 0 \\ a_1 = 0 \\ a_0 = 0 \end{cases}.$$

So $p(x) = x^3$.

Exercise 4.2. Given $f_i = f(x_i)$ of some scalar function at points $a = x_1 < x_2 < \cdots < x_n = b$, we consider interpolating f on $[a,b]$ with a quadratic spline $s \in \mathbb{S}_2^1$.

(a) Why an additional condition is needed to determine s uniquely?

(b) Define $m_i = s'(x_i)$ and $p_i = s|_{[x_i, x_{i+1}]}$. Determine p_i in terms of f_i, f_{i+1}, and m_i for $i = 1, 2, \ldots, n-1$.

(c) Suppose $m_1 = f'(a)$ is given. Show how $m_2, m_3, \ldots, m_{n-1}$ can be computed.

Solution.

(a) Since there are $3n-3$ variables, however only $3n-4$ functions. Need another propriet function so that have unique solution.

(b) Let $p_i(x) = \frac{(x-x_i)^2}{(x_{i+1}-x_i)^2}(f_{i+1} - f_i) + \frac{(x-x_i)(x-x_{i+1})}{x_i - x_{i+1}} m_i + f_i$. It can be verify that $p_i(x_i) = f_i, p_i(x_{i+1}) = f_{i+1}, p'(x_i) = m_i$.

(c) As define in (b), derivating $p_i(x)$ in point x_{i+1} get

$$m_{i+1} = p_i'(x_{i+1}) = 2\frac{f_{i+1} - f_i}{x_{i+1} - x_i} - m_i$$

Exercise 4.3. Let $s_1(x) = 1 + c(x+1)^3$ where $x \in [-1, 0]$ and $c \in \mathbb{R}$. Determine $s_2(x)$ on $[0, 1]$ such that

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in [-1, 0], \\ s_2(x) & \text{if } x \in [0, 1] \end{cases}$$

is a natural cubic spline on $[-1, 1]$ with knots $-1, 0, 1$. How must c be chosen if one wants $s(1) = -1$?

Solution. Let $s_2(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. As a natural cubic spline have

$$\begin{cases} 6a_3(1) + 2a_2 = 0 \\ 6a_3(0) + 2a_2 = 6c((0) + 1) \\ 3a_3(0)^2 + 2a_2(0) + a_1 = 3c((0) + 1)^2 \\ a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 1 + c((0) + 1)^3 \end{cases} \implies \begin{cases} a_3 = -c \\ a_2 = 3c \\ a_1 = 3c \\ a_0 = 1 + c \end{cases}.$$

So $s_2 = -cx^3 + 3cx^2 + 3cx + 1 + c$. From $s_2(1) = s(1) = -1$, conclude $c = -\frac{1}{3}$.

Exercise 4.4. Consider $f(x) = \cos(\frac{\pi}{2}x)$ with $x \in [-1, 1]$.

(a) Determine the natural cubic spline interpolant to f on knots $-1, 0, 1$.

(b) As discussed in the class, natural cubic splines have the minimal total bending energy. Verify this by taking $g(x)$ be (i) the quadratic polynomial that interpolates f at $-1, 0, 1$, and (ii) $f(x)$.

Solution.

(a) Define $(x_1, x_2, x_3) = (-1, 0, 1), (f_1, f_2, f_3) = (f(x_1), f(x_2), f(x_3)) = (0, 1, 0)$. And let $s(x)$ is natural cubic spline interpolant satisfies $s(x) = s_1(x)\ x \in [-1, 0], s(x) = s_2(x)\ x \in [0, 1]$. Assume $s_i(x) = a_3^ix^3 + a_2^ix^2 + a_1^ix + a_0^i$, Then have equations

$$\begin{cases} 6a_3^1(-1) + 2a_2^1 = 0 \\ 6a_3^2(1) + 2a_2^2 = 0 \\ 6a_3^1(0) + 2a_2^1 = 6a_3^2(0) + 2a_2^2 \\ 3a_3^1(0)^2 + 2a_2^1(0) + a_1^1 = 3a_3^2(0)^2 + 2a_2^2(0) + a_1^2 \\ a_3^1(-1)^3 + a_2^1(-1)^2 + a_1^1(-1) + a_0^1 = 0 \\ a_3^1(0)^3 + a_2^1(0)^2 + a_1^1(0) + a_0^1 = 1 \\ a_3^2(0)^3 + a_2^2(0)^2 + a_1^2(0) + a_0^2 = 1 \\ a_3^2(1)^3 + a_2^2(1)^2 + a_1^2(1) + a_0^2 = 0 \end{cases} \implies \begin{cases} a_3^1 = \frac{1}{3} \\ a_2^1 = -1 \\ a_1^1 = -\frac{1}{3} \\ a_0^1 = 1 \\ a_3^2 = \frac{1}{3} \\ a_2^2 = -1 \\ a_1^2 = -\frac{1}{3} \\ a_0^2 = 1 \end{cases}.$$

Therefore, $s(x) = \frac{1}{3}x^3 - x^2 - \frac{1}{3}x + 1$ is the natural cubic spline interpolant to f on knots $-1, 0, 1$.

(b) Interpolating get $g(x) = -x^2 + 1$. Then calculate bending energy

$$\int_{-1}^{1} s(x)dx = (\frac{1}{12}x^4 - \frac{1}{3}x^3 - \frac{1}{6}x^2 + x)\Big|_{-1}^{1} = \frac{4}{3}$$
$$\int_{-1}^{1} g(x)dx = (-\frac{1}{3}x^3 + x)\Big|_{-1}^{1} = \frac{4}{3}$$
$$\int_{-1}^{1} f(x)dx = (\frac{2}{\pi}\sin(\frac{\pi}{2}x))\Big|_{-1}^{1} = \frac{4}{\pi}.$$

Which meet the natural cubic spline interpolant is the minimal total bending energy.

Exercise 4.5. The quadratic B-spline $B_i^2(x)$.

(a) Derive the same explicit expression of $B_i^2(x)$ as that in the notes from the recursive Definition of B-splines and the hat function.

(b) Verify that $\frac{d}{dx}B_i^2(x)$ is continuous at t_i and t_{i+1}.

(c) Show that only one $x^* \in (t_{i-1}, t_{i+1})$ satisfies $\frac{d}{dx}B_i^2(x^*) = 0$. Express x^* in terms of the knots within the interval of support.

(d) Consequently, show $B_i^2(x) \in [0, 1)$.

(e) Plot $B_i^2(x)$ for $t_i = i$.

Solution.

(a) We already have

$$B_i^1 = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in (t_{i-1}, t_i], \\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], 0 \\ \text{otherwise.} \end{cases}$$

Combining with recursive definition $B_i^{n+1}(x) = \frac{x-t_{i-1}}{t_{i+n}-t_{i-1}}B_i^n(x) + \frac{t_{i+n+1}-x}{t_{i+n+1}-t_i}B_{i+1}^n(x)$, calculate B_i^2

$$B_i^2 = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}}B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i}B_{i+1}^1(x)$$

$$= \begin{cases} 0, & x \in (-\infty, t_{i-1}]; \\ \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}}B_i^1(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}}(\frac{x-t_{i-1}}{t_i-t_{i-1}}) = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i]; \\ \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}}(\frac{t_{i+1}-x}{t_{i+1}-t_i}) + \frac{t_{i+2}-x}{t_{i+2}-t_i}(\frac{x-t_i}{t_{i+1}-t_i}) = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}]; \\ \frac{t_{i+2}-x}{t_{i+2}-t_i}(\frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}) = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}]; \\ 0, & x \in (t_{i+2}, +\infty); \end{cases}$$

(b) From B_i^2 in (a) calculate

$$\lim_{x\to t_i^-}(\frac{(x - t_{i-1})^2 - (t_i - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})(x - t_i)}) = \lim_{x\to t_i^-}(\frac{(x + t_i - 2t_{i-1})}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}) = \frac{2}{(t_{i+1} - t_{i-1})}$$

$$\lim_{x\to t_i^+}(\frac{(x - t_{i-1})(t_{i+1} - x) - (t_i - t_{i-1})(t_{i+1} - t_i)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)(x - t_i)} + \frac{(x - t_i)(t_{i+2} - x) - (t_i - t_i)(t_{i+2} - t_i)}{(t_{i+2} - t_i)(t_{i+1} - t_i)(x - t_i)})$$
$$= \lim_{x\to t_i^+}(\frac{(t_{i+1} + t_{i-1} - t_i - x)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{(t_{i+2} - x)}{(t_{i+2} - t_i)(t_{i+1} - t_i)}) = \frac{2}{(t_{i+1} - t_{i-1})}.$$

So $\frac{d}{dx}B_i^2(x)$ is continuous in t_i, t_{i+1} proved similarly.

(c) It is easy to see that $\frac{d}{dx}B_i^2(x) \neq 0, \forall x \in (t_{i-1}, t_i] \cap [t_{i+1}, t_{i+2})$. So it is left to verify $x \in (t_i, t_{i+1})$.

$$\frac{d}{dx}B_i^2(x) = \frac{(t_{i+1} - x) - (x - t_{i-1})}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{(t_{i+2} - x) - (x - t_i)}{(t_{i+2} - t_i)(t_{i+1} - t_i)}$$

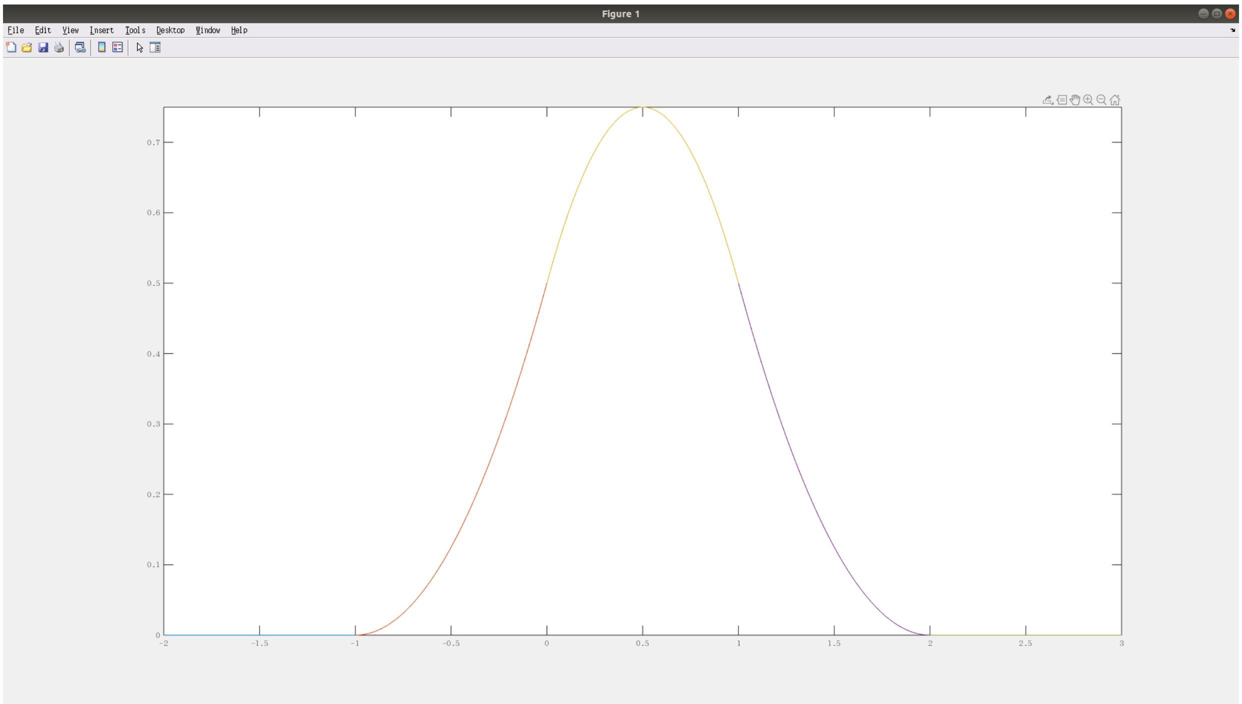


图 4.1: Bspline: $(t_i = i = 0)$

This is a linear function. Take it as $f(x)$. Since $f(t_i) = \frac{2}{(t_{i+1}-t_{i-1})} > 0$ and $f(t_{i+1}) = -\frac{2}{(t_{i+2}-t_i)} < 0$, there is a root x^* of $f(x)$ and it's the only one. Then calculate get $x^* = \frac{t_{i+2}t_{i+1}-t_it_{i-1}}{t_{i+2}+t_{i+1}-t_i-t_{i-1}}$.

(d) Since $\frac{d}{dx}B_i^2(x) > 0, \forall x \in (t_{i-1}, x^*)$ and $\frac{d}{dx}B_i^2(x) < 0, \forall x \in (x^*, t_{i+2})$. It indicates that

$$\begin{aligned}
&B_i^2(x) < B_i^2(x^*)\\
&= (\frac{1}{t_{i+2}+t_{i+1}-t_i-t_{i-1}})^2\\
&(\frac{(t_{i+2}t_{i+1}-t_it_{i-1}-t_{i+2}t_{i-1}-t_{i+1}t_{i-1}+t_it_{i-1}+t_{i-1}^2)(t_{i+1}(t_{i+2}+t_{i+1}-t_i-t_{i-1}}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}\\
&\frac{(t_{i+2}t_{i+1}-t_it_{i-1}-t_i(t_{i+2}+t_{i+1}-t_i-t_{i-1}))(t_{i+2}(t_{i+2}+t_{i+1}-t_i-t_{i-1})-t_{i+2}t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)}\\
&= (\frac{1}{t_{i+2}+t_{i+1}-t_i-t_{i-1}})^2((t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})+(t_{i+2}-t_i)(t_{i+2}-t_{i-1}))\\
&= \frac{t_{i+2}-t_{i-1}}{t_{i+2}+t_{i+1}-t_i-t_{i-1}}\\
&< 1.
\end{aligned}$$

$B_i^2(x) \geq \min B_i^2(t_{i-1}), B_i^2(t_{i+2}) = 0$.

In summary, $B_i^2(x) \in [0, 1)$.

(e) Plot B_0^2 blow (4.1), other i can be geted from translation.

Exercise 4.6. We proved a theorem on constructing B-splines from truncated power functions,

$$B_i^n(x) = (t_{i+n} - t_{i-1}) \cdot [t_{i-1}, \ldots, t_{i+n}](t-x)_+^n.$$

Verify it algebraically for the case of $n = 2$, i.e.

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2.$$

Solution. We already have

$$B_i^2 = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i]; \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}]; \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}]; \\ 0, & \text{otherwise.} \end{cases}$$

And Split the difference format

$$\begin{aligned} B_i^2 &= (t_{i+2}-t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 \\ &= [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2 \\ &= \frac{(t_{i+2}-x)_+^2 - (t_{i+1}-x)_+^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} - \frac{(t_{i+1}-x)_+^2 - (t_i-x)_+^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{(t_{i+1}-x)_+^2 - (t_i-x)_+^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_i-x)_+^2 - (t_{i-1}-x)_+^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} \\ &= \begin{cases} = 0, & x \in (t_{i+2}, +\infty); \\ \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}]; \\ \\ (\frac{(t_{i+2}-x)_+^2-(t_{i+1}-x)_+^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} - \frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{(t_i-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}) - \frac{(t_{i+1}-x)_+^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \\ = (\frac{t_{i+2}+t_{i+1}-2x}{t_{i+2}-t_i} - \frac{t_{i+1}+t_i-2x}{t_{i+2}-t_i}) - \frac{(t_{i+1}-t_{i-1})(t_i-x)^2+(t_{i+2}-t_i)(t_{i+1}-x)^2}{(t_{i+2}-t_i)(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \\ = 1 - \frac{(t_{i+1}-t_{i-1})(t_i-x)^2+(t_{i+2}-t_i)(t_{i+1}-x)^2}{(t_{i+2}-t_i)(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \\ = \frac{(t_{i=1}-t_{i-1})(t_{i+2}-t_i)(x-t_i)-(t_{i+1}-t_{i-1})(t_i-x)^2+(t_{i+2}-t_i)(t_{i+1}-x)(x-t_{i-1})}{(t_{i+2}-t_i)(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \\ = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}]; \\ \\ = (\frac{t_{i+2}+t_{i+1}-2x}{t_{i+2}-t_i} - \frac{t_{i+1}+t_i-2x}{t_{i+2}-t_i}) - (\frac{t_{i+1}+t_i-2x}{t_{i+1}-t_{i-1}} - \frac{t_i+t_{i-1}-2x}{t_{i+1}-t_{i-1}}) + \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} \\ = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}]; \\ \\ = (\frac{t_{i+2}+t_{i+1}-2x}{t_{i+2}-t_i} - \frac{t_{i+1}+t_i-2x}{t_{i+2}-t_i}) - (\frac{t_{i+1}+t_i-2x}{t_{i+1}-t_{i-1}} - \frac{t_i+t_{i-1}-2x}{t_{i+1}-t_{i-1}}) \\ = 1 - 1 = 0, & x \in (-\infty, t_{i-1}]; \end{cases} \end{aligned}$$

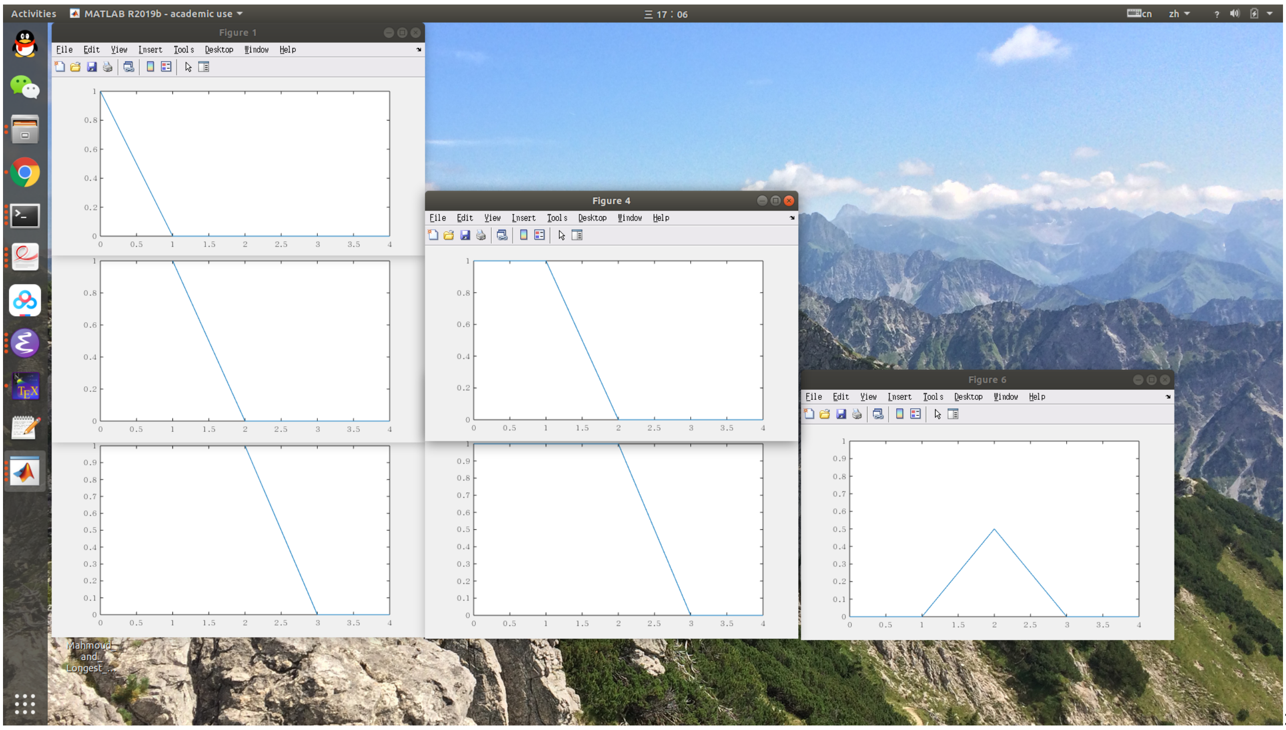
4.2 programming

(a) Write a **C++** subroutine for cubic-spline interpolation of the function

$$f(x) = \frac{1}{1+25x^2}$$

on evenly spaced nodes within the interval $[-1, 1]$ with $N = 6, 11, 21, 41, 81$. Compute the max-norm of interpolation errors at the nodes for each N and report the errors and convergence rates with respect to the number of subintervals.

(b) Write a **Matlab** subroutine to illustrate (1) by ploting the truncated power functions and building a table of divided difference where the entries are figures intread of numbers. See the hinits in Section4.



;

图 4.2: N = 1

Solution.

1.

2. Run program by *make bsp1* and *make bsp2* corresponds to $n = 1$ and $n = 2$. Or using **matlab** to run function *plotTruncatedPowerFunc2Bsplines* in *plotTruncatedPowerFunc2Bsplines.m*. Outcome is similar to figure ()

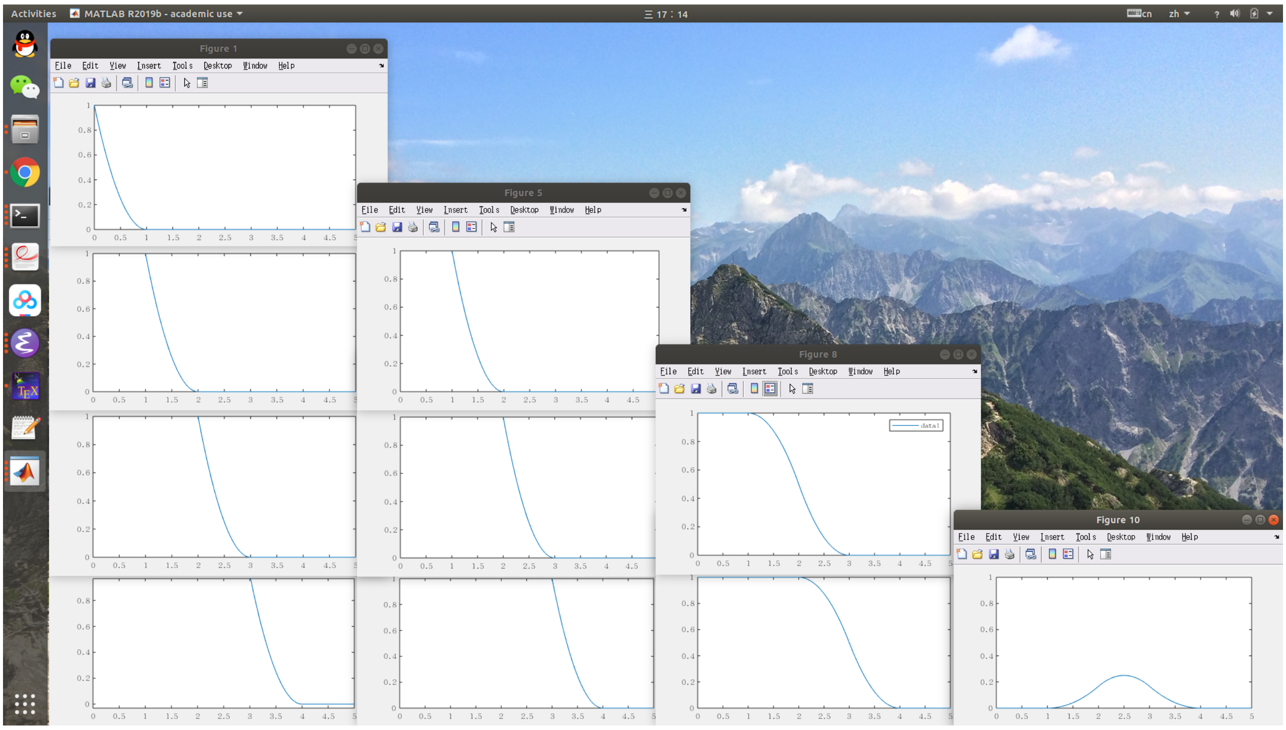

Activities
MATLAB R2019b - academic use
Figure 1
Figure 5
Figure 8
Figure 10

图 4.3: N = 2