Chapter 10

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Problem 10.1. Find a nonnegative scaling invariant solution having the form

$$u(x,t) = t^{-\alpha} v(xt^{-\beta})$$

for the nonlinear heat equation

$$u_t - \Delta(u^\lambda) = 0,$$

where $\frac{n-2}{n} < \lambda < 1$. Your solution should go to zero algebraically as $|x| \to \infty$.

Solution. Replace u in the nonlinear heat equation with $t^{-\alpha} v(xt^{-\beta})$ gives

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha\lambda+2\beta)} \Delta(v^\lambda)(y) = 0$$

where $y = t^{-\beta} x$. In order to eliminate t, let us require

$$\alpha + 1 = \alpha\lambda + 2\beta.$$

Then, we have

$$\alpha v + \beta y \cdot Dv + \Delta(v^\lambda) = 0.$$

Then simplify further by supposing v is radial; that is, $v(y) = w(r), r = |y|\, w \colon \mathbb{R} \to \mathbb{R}$. Then, above equation becomes

$$\alpha w + \beta r w' + (w^\lambda)'' + \frac{n-1}{r}(w^\lambda)' = 0,$$

set $\alpha = n\beta$

$$\begin{aligned} & (r^{n-1}(w^\lambda)')' + \beta(r^n w)' = 0 \\ \implies \quad & r^{n-1}(w^\lambda)' + \beta r^n w = a \end{aligned}$$

for some constant a, Since $\lim_{r\to\infty} w = 0$ and assume $\lim_{r\to\infty} w' = 0$, we conclude $a = 0$. After all get the *Barenblatt-Kompaneetz-Zeldovich solution*

$$u(x,t) = \frac{1}{t^{n\beta}} \left(b - \frac{\lambda - 1}{2\lambda} \beta \frac{|x|^2}{t^{2\beta}} \right)_+^{\frac{1}{\lambda-1}}$$

Problem 10.2. Find a solution of

$$-\Delta u + u^{\frac{n+2}{n-2}} = 0 \qquad \text{in } B(0,1)$$

having the form $u = \alpha(1-|x|^2)^{-\beta}$ for positive constants α, β. This example shows that a solution of a nonlinear PDE can be finite within a region and yet approach infinity everywhere on its boundary.

Solution. u can be write as $\alpha(1-\sum x_i^2)^{-\beta}$, Then calculate Laplace operator in u

$$\begin{aligned}\Delta u &= (2n(1-\sum x_i^2) + 4(\beta+1)\sum x_i^2)\alpha\beta(1-\sum x_i^2)^{-(\beta+2)} \\ &= \alpha^{\frac{n+2}{n-2}}(1-\sum x_i^2)^{-\beta\frac{n+2}{n-2}} = u^{\frac{n+2}{n-2}}.\end{aligned}$$

In order to maintain the same order of $(1-|x|^2)$, Let $\beta = \frac{n-2}{2}$, then the equation becomes

$$\begin{aligned} & 2n\alpha\beta(1-|x|^2)^{-\frac{n+2}{2}} = \alpha^{\frac{n+2}{n-2}}(1-|x|^2)^{-\frac{n+2}{2}} \\ \Longrightarrow \qquad & \alpha = (n(n-2))^{\frac{n-2}{4}}.\end{aligned}$$

Finally, get the solution $u = (n(n-2))^{\frac{n-2}{4}}(1-|x|^2)^{-\frac{n-2}{2}}$.

Problem 10.3. Consider the viscous conservation law

$$(*) \qquad u_t + F(u)_x - au_{xx} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty),$$

where $a > 0$ and F is uniformly convex.

(a) Show u solves $(*)$ if $u(x,t) = v(x-\sigma t)$ and v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z)-\sigma z + b} dz \qquad (s \in \mathbb{R}),$$

where b and c are constants.

(b) Demonstrate that we can find a traveling wave satisfying

$$\lim_{s\to-\infty} v(s) = u_l, \ \lim_{s\to\infty} v(s) = u_r$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l)-F(u_r)}{u_l - u_r}.$$

(c) Let u^ε denote the above traveling wave solution of $(*)$ for $a = \varepsilon$, with $u^\varepsilon(0,0) = \frac{u_l+u_r}{2}$. Compute $\lim_{\varepsilon\to 0} u^\varepsilon$ and explain your answer.

Solution.

(a) Substitute u with v in the viscous conservation law gives

$$-\sigma v' + F'(v)v' - av'' = 0.$$

In other side, differential the formula twice like

$$\begin{aligned} & s = \int_c^{v(s)} \frac{a}{F(z)-\sigma z+b} dz \\ \Longrightarrow \qquad & 1 = v'\frac{a}{F(v)-\sigma v + b} \\ \Longrightarrow \qquad & F(v) - \sigma v + b - av' = 0 \\ \Longrightarrow \qquad & F'(v)v' - \sigma v' - av'' = 0.\end{aligned}$$

Which is same with the viscous conservation law after substituting. So the v defined implicitly solves the $(*)$.

(b)
- Necessary, assume $\lim_{s\to-\infty} v(s) = u_l,\ \lim_{s\to\infty} v(s) = u_r$, we can conclude

$$\int_c^{u_l} \frac{a}{F(z)-\sigma z+b} dz = -\infty,$$

which imply that $F(u_l)-\sigma u_l+b=0$. Similarly, $F(u_r)-\sigma u_r+b=0$. Eliminate b gives $\sigma=\frac{F(u_l)-F(u_r)}{u_l-u_r}$.
- Sufficient, assume $\sigma=\frac{F(u_l)-F(u_r)}{u_l-u_r}$. Let $b=-F(u_l)+\sigma u_l$, It's obvious that $F(z)-\sigma z-F(u_l)+\sigma u_l$ have two root at most and u_l, u_r is exactly two root, Since $F(x)$ is convex so that $F(z)-\sigma z-F(u_l)+\sigma u_l$ is convex that means at most two root. And $F(z)-\sigma z-F(u_l)+\sigma u_l<0, z\in(u_r,u_l)$, it indicates $\lim_{s\to-\infty} v(s) = u_l,\ \lim_{s\to\infty} v(s) = u_r$.

(c) $u^\varepsilon(0,0)=\frac{u_l+u_r}{2}$ imply that $s=\int_{\frac{u_l+u_r}{2}}^{v(s)} \frac{\varepsilon}{F(z)-\sigma z+b}dz$.Then $\lim_{\varepsilon\to0} v^\varepsilon(0)=\lim_{\varepsilon\to0} u^\varepsilon(0,0)=\frac{u_l+u_r}{2}$, and assume $\forall c<0, \lim_{\varepsilon\to0} v^\varepsilon(c)=u_c\in(\frac{u_l+u_r}{2},u_l)$. Since F is convex, $\exists M, M>F(z)-\sigma z+b, z\in[\frac{u_l+u_r}{2},u_c]$, so that

$$\begin{aligned} c &= \int_{\frac{u_l+u_r}{2}}^{u_c} \frac{\varepsilon}{F(z)-\sigma z+b}dz \\ &> s = \int_{\frac{u_l+u_r}{2}}^{u_c} \frac{\varepsilon}{M}dz \\ &= \frac{\varepsilon(u_c-\frac{u_l+u_r}{2})}{M} \\ &> c \qquad \text{while } \varepsilon \text{ sufficient small.} \end{aligned}$$

That is contradict, so $\lim_{\varepsilon\to0} u_c = u_l$. Similarly, while $c>0$, $\lim_{\varepsilon\to0} u_c=u_r$.

$$\lim_{\varepsilon\to0} u(x,t) = \begin{cases} u_l & x-\frac{F(u_l)-F(u_r)}{u_l-u_r}t<0 \\ \frac{u_l+u_r}{2} & x-\frac{F(u_l)-F(u_r)}{u_l-u_r}t=0 \\ u_r & x-\frac{F(u_l)-F(u_r)}{u_l-u_r}t>0 \end{cases}.$$

Problem 10.4. Prove that if u is the solution of problem (23) for *Schrödinger*'s equation in §4.3 given by formula (20), then

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{(4\pi|t|)^{n/2}}\|g\|_{L^1(\mathbb{R}^n)}$$

for each $t\neq0$.

Solution. From the formula (20) and Young inequality knowns

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\le \|\frac{1}{(4\pi i|t|)^{n/2}} e^{\frac{i}{4t}|x-y|^2}\|_{L^\infty(\mathbb{R}^n)}\cdot\|g\|_{L^1(\mathbb{R}^n)} \\ &= \frac{1}{(4\pi|t|)^{n/2}}\|g\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$