Chapter 1

Homework 21935004 谭焱

1.1 Proof

Exercise 1.1. A min-max problem.

For $n \in \mathbb{N}^+$, determine

$$\min \max_{x \in [a,b]} |a_0 x^n + a_1 x^{n-1} + \cdots + a_n|, \tag{1.1}$$

where the minimum is taken over all $a_i \in \mathbb{R}$ and $a_0 \neq 0$.

Solution. Let $f(y) = \min \max_{(b+a)/2+y(b-a)/2 \in [a,b]} |a_0((b+a)/2+y(b-a)/2)^n + a_1((b+a)/2+y(b-a)/2)^{n-1} + \cdots + a_n|$, T_n. It's obvious that f's domain is $[-1,1]$. While $p(x)$'s leading coefficient is 1,$\max_{x\in[-1,1]} |p(x)| \leq \frac{1}{2^{n-1}}$.At the same time, $\frac{f(y)}{a_0(\frac{b-a}{2})^n}$'s leading coefficient is 1.Therefore, $|f(y)| \leq a_0 \frac{(b-a)^n}{2^{2n-1}}$.

Exercise 1.2. Imitate the proof of Chebyshev Theorem.

Let $a > 1$ and denote $\mathbb{P}_n^a = \{p \in \mathbb{P}_n : p(a) = 1\}$. Define

$$\hat{p}_n(x) = \frac{T_n(x)}{T_n(a)},$$

where $\mathbf{T}_n$ is the Chebyshev polynomial of degree n. Chearly $\hat{p}_n \in \mathbb{P}_n^a$.Define the *max-norm* of a function $f : \mathbb{R} \to \mathbb{R}$ as

$$\|f\|_\infty = \max_{x\in[-1,1]} |f(x)|.$$

Prove

$$\forall p \in \mathbb{P}_n^a, \qquad \|\hat{p}_n(x)\|_\infty \leq \|p\|_\infty.$$

Solution. Let $x_j = \cos\frac{k}{n}$ are extreme values points of $T_n(x)$. So x_j is $\hat{p}_n(x) = \frac{T_n(x)}{T_n(a)}$'s extreme values points, where $\hat{p}_n(x_j) = (-1)^k \frac{1}{T_n(a)}$. Without loss trivial, assuming $T_n(a) > 0$.

Since x_j is $\hat{p}_n$'s all extreme values between $[-1,1]$, if $\|p\|_\infty < \|\hat{p}_n\| = |\hat{p}_n(x_j)| = \frac{1}{T_n(a)}$. It indicates that $\hat{p}_n(x_j) - p(x_j) \begin{cases} > 0 & \text{j is even} \\ < 0 & \text{j is odd.} \end{cases}$. That follows that (x_{j-1}, x_j) contains at least a root of $\hat{p}_n(x) - p(x)$. So there is at least n roots between $[-1.1]$ and another root a.However, for all polynomials in $\mathbb{P}_n^a \in \mathbb{P}_n$, there is only $\hat{p}_n(x) - p(x) = 0$ has $n+1$ roots in $\mathbb{R}$. So $\|\hat{p}_n(x)\|_\infty \leq \|p\|_\infty$.

1.2 Program

complie and run code Open shell and then "make print",(This way needs have matlab can be run in shell), or "make run" and then run "p.m" in matlab.

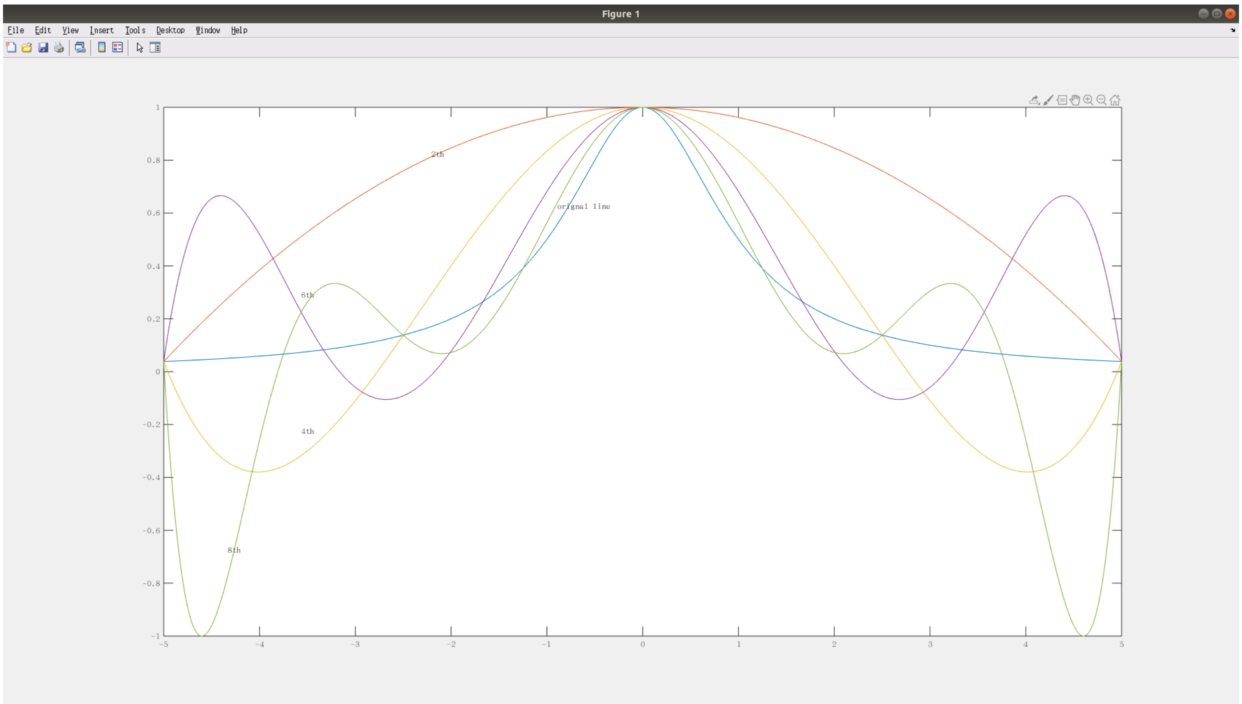

图 1.1: Newton1

Exercise 1.3. Implement the Newton formula in a subroutine that produces the value of the interpolation polynomial $p_n(f; x_0, x_1, \ldots, x_n; x)$ at any real x, where $n \in \mathbb{N}^+, x_i's$ are distinct, and f is a function assumed to be available in the from of a subroutine.

Solution.

Exercise 1.4. Run your routine on the function

$$f(x) = \frac{1}{1+x^2}$$

for $x \in [-5, 5]$ using $x_i = -5 + 10\frac{i}{n}, i = 0, 1, \ldots, n$, and $n = 2, 4, 6, 8$. Plot the polynomials against the exact function to rerproduce the plot in the notes that illustrate the Runge phenomenon.

Solution. Result display in graphic (1.1)

Exercise 1.5. Reuse your subroutine of Newton interpolation to perform Chebyshev interpolation for the function

$$f(x) = \frac{1}{1+25x^2}$$

for $x \in [-1, 1]$ on the zeros of Chebyshev polynomials $\mathbf{T}_n$ with $n = 5, 10, 15, 20$. Clearly the Runge function $f(x)$ is a scaled version of the function in (b). Plot the interpolating polynomials against the exact function to observe that the Chebyshev interpolation is free of the wide oscillations in the previous homework.

Solution. Newton interpolation polynomial plot in (fig:2). Chebyshev interpolation polynomial plot in (fig:3).

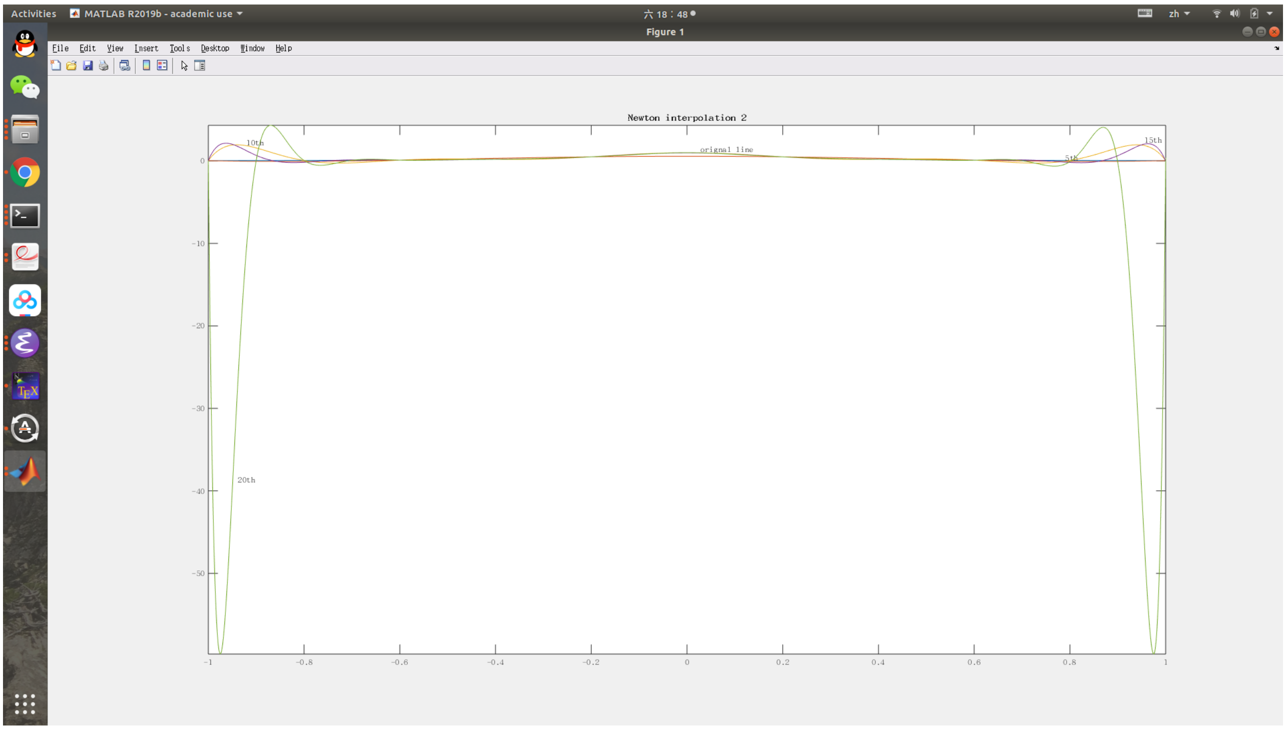

图 1.2: Newton2

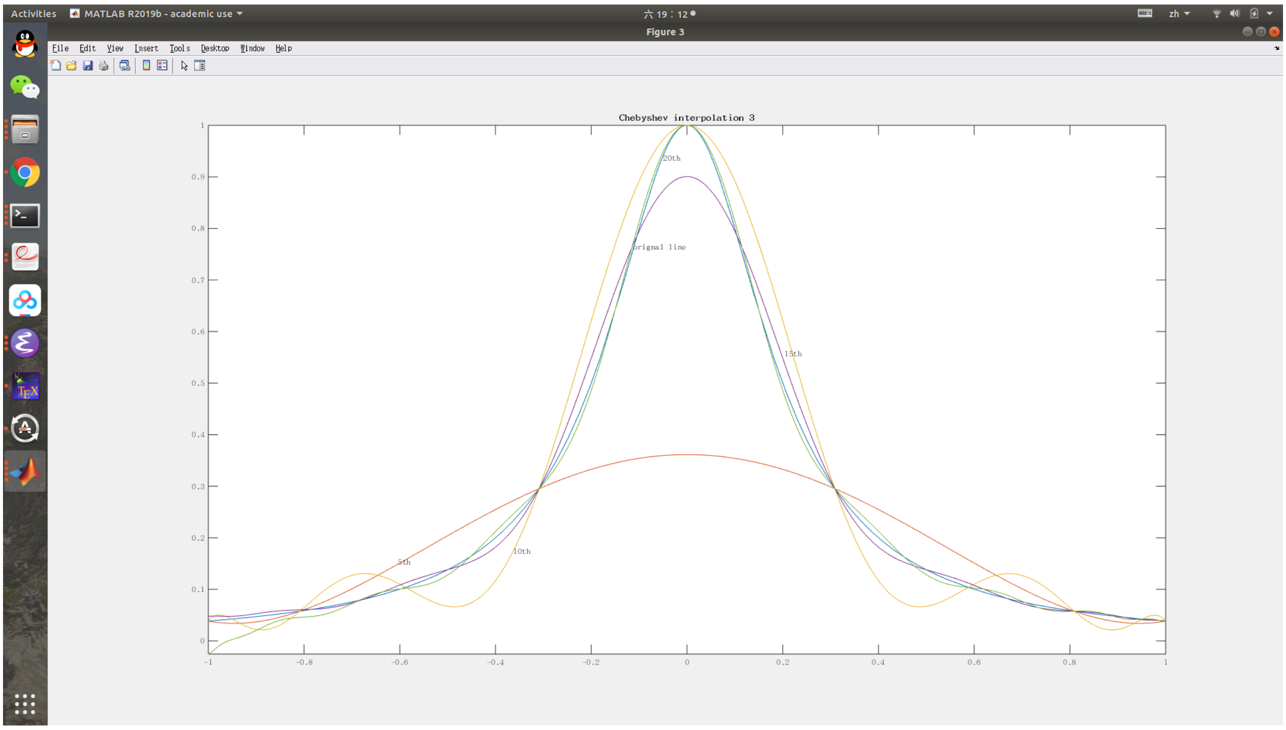

图 1.3: Chebyshev2