Chapter 6

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Exercise 6.1. 2. Compute the envelopes of the family of lines

$$x_1 + a^2 x_2 - 2a = 0 \qquad (a \in \mathbb{R}) \tag{6.1}$$

in $\mathbb{R}^2$ and of the family of planes

$$2a_1 x_1 + 2a_2 x_x - x_3 + a_1^2 + a_2^2 = 0 \qquad (a_1, a_2 \in \mathbb{R}) \tag{6.2}$$

in $\mathbb{R}^3$. Draw pictures illustrating the geometric meaning of the enveplopes.

Solution. Let $u(x, a) = x_1 + a^2 x_2 - 2a$, from definition of envelopes

$$\begin{aligned} &D_a u = 2ax_2 - 2 = 0 \Longrightarrow \\ &a = \frac{1}{x_2}. \end{aligned}$$

So envelopes is $x_1 - \frac{1}{x_2} = 0$.

Similarly,

$$\begin{aligned} &2x_1 + 2a_1 = 0 \Longrightarrow a_1 = -x_1 \\ &2x_2 + 2a_2 = 0 \Longrightarrow a_2 = -x_2 \\ &\Longrightarrow \mathbf{a} = -\mathbf{x} \end{aligned}$$

So envelopes is $x_1^2 + x_2^2 - x_3 = 0$.

Above envelopes in $\mathbb{R}^2$ ploted blow envelopes is all point x satisfy that exist a line u in the family such that $u(x) = 0$.

Exercise 6.2. 4.

(a) Write down the characteristic equations for the PDE

$$(*) \qquad u_t + b \cdot Du = f \qquad \text{in } \mathbb{R}^n \times (0, \infty),$$

where $b \in \mathbb{R}^n$, $f = f(x, t)$.

(b) Use the characteristic ODE to solve $(*)$ subject to the initial condition

$$u = g \qquad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{6.3}$$

Make sure your answer agrees with formula (5) in 2.1.2.

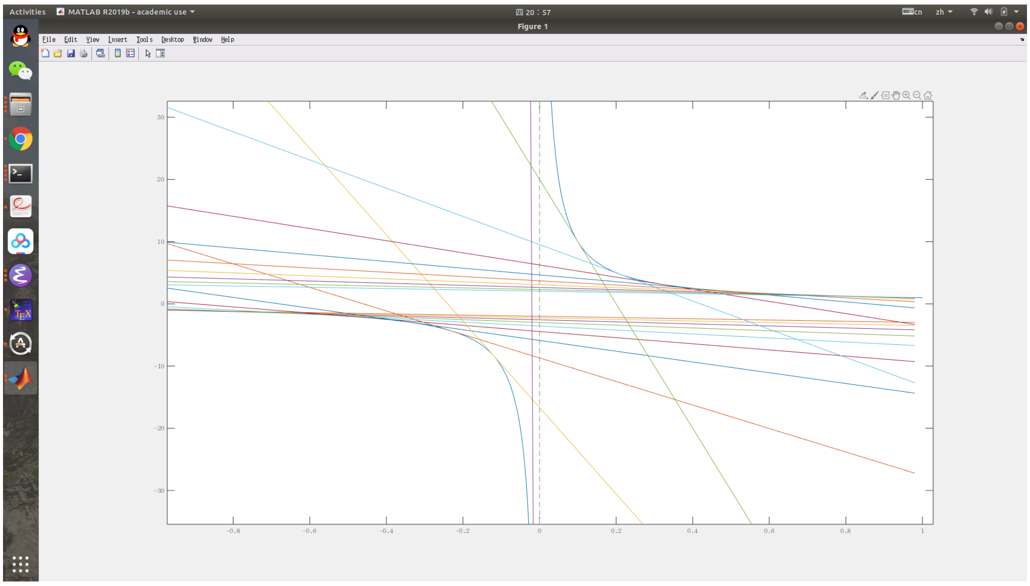

图 6.1: envelopes

Solution.

(a) Transfer (*) to $\mathbf{F}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = (b, 1) \cdot \mathbf{p}(s) - f(\mathbf{x}) = 0$, then

$$\begin{aligned}
\dot{\mathbf{p}}(s) &= -(D_x f(\mathbf{x}, t) + D_t f(\mathbf{x}, t)) \\
\dot{z}(s) &= (b, 1) \cdot \mathbf{p}(s) \\
\dot{\mathbf{x}}(s) &= (b, 1)
\end{aligned}$$

(b) Solving second and third of the equations get

$$\begin{aligned}
\mathbf{x}(s) &= (bs, s) + (x_0, 0) \\
\dot{z}(s) &= f(\mathbf{x}) \Longrightarrow \\
z(s) &= \int_0^t f((bs + x_0, s))ds + g(x_0)
\end{aligned}$$

Since $\mathbf{x}(s) = (bs, s) + (x_0, 0)$, eliminating the x_0 in last equaltion get the formula (5).

Exercise 6.3. 5. Solve using characteristic:

(a)

$$x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \qquad u(x_1, x_2, 0) = g(x_1, x_2). \tag{6.4}$$

(b)

$$u u_{x_1} + u_{x_2} = 1, \qquad u(x_1, x_1) = \frac{1}{2} x_1. \tag{6.5}$$

Solution.

(a) Writing characterstic equations as follow

$$
\begin{aligned}
&F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = (x_1, 2x_2, 1) \cdot \mathbf{p}(s) - 3z(s) = 0 \\
&\dot{z}(s) = 3z(s) \\
&\dot{\mathbf{x}}(s) = (x_1, 2x_2, 1)
\end{aligned}
$$

Solving equations get $\mathbf{x}(s) = (x_1^0 e^s, x_2^0 e^{2s}, s), z(s) = z^0 e^{3s}$, eliminating x_1^0, x_2^0, z^0 to obtain $u(x_1, x_2, x_3) = g(\frac{x_1}{e^{x_3}}, \frac{x_2}{e^{x_3}})e^{3s}$.

(b)

$$
\begin{aligned}
&F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = (z(s), 1) \cdot \mathbf{p}(s) - 1 = 0 \\
&\dot{z}(s) = 1 \\
&\dot{\mathbf{x}}(s) = (z(s), 1).
\end{aligned}
$$

Solving equations get $z(s) = s + z_0, \mathbf{x}(s) = (\frac{1}{2}s^2 + z_0 s + x_1^0, s + x_2^0)$. By $u(x_1, x_1) = \frac{1}{2}x_1 \Longrightarrow z_0 = \frac{1}{2}x_1^0 = \frac{1}{2}x_2^0$, eliminating z_0, x_1^0, x_2^0 obtain $u(x_1, x_2) = \frac{1}{2}x_2 + \frac{x_2 - x_1}{2 - x_2}$.

Exercise 6.4. 6. Given a smooth vector field $\mathbf{b}$ on $\mathbb{R}^n$, let $\mathbf{x}(s) = \mathbf{x}(s, x, t)$ solve the ODE

$$
\begin{cases}
\dot{\mathbf{x}} = \mathbf{b}(x) & (s \in \mathbb{R}) \\
\mathbf{x}(t) = x.
\end{cases}
$$

(a) Define the Jacobian

$$
J(s, x, t) := \det D_x \mathbf{x}(s, x, t) \tag{6.6}
$$

and derive *Euler's formula*:

$$
J_s = \operatorname{div} \mathbf{b}(\mathbf{x}) J. \tag{6.7}
$$

(b) Demonstrate that

$$
u(x, t) := g(\mathbf{x}(0, x, t)) J(0, x, t) \tag{6.8}
$$

solves

$$
\begin{cases}
u_t + \operatorname{div}(u\mathbf{b}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
u = g & \text{on } \mathbb{R}^n \times \{t = 0\},
\end{cases} \tag{6.9}
$$

(Hint: Show $\frac{\partial}{\partial s}(u(\mathbf{x}, s)J) = 0$.)

Solution.

(a) Unfold J and differential J

$$
\begin{aligned}
J(s, x, t) &= \sum \prod_{i=1}^{n} (-1)^\sigma x^i_{\sigma(i)} \\
J_s &= \sum \sum_{j=1}^{n} (\prod_{i \neq j} (-1)^\sigma x^i_{\sigma(i)}) x^j_{\sigma(j)s} \\
&= \sum \sum_{j=1}^{n} (\prod_{i=1}^{n} (-1)^\sigma x^i_{\sigma(i)}) (\mathbf{b}^j(x)(s - t) + x)_{\sigma(j)s} \\
&= \sum \sum_{j=1}^{n} (\prod_{i=1}^{n} (-1)^\sigma x^i_{\sigma(i)}) \mathbf{b}^j_j(x) x^j_{\sigma(j)} \\
&= \operatorname{div} \mathbf{b}(x) J.
\end{aligned}
$$

(b) Boundary condition is obvious, since $J(0,x,0) = \det D_x x(0,x,0) = \det D_x x = 1$ and $x(0,x,0) = x$.And $g(\mathbf{x})J$ means that transfer coordinate$\mathbf{x}(s,x,t)$ to $\mathbf{x}$. So it value $u(x,t)$ won't change while $\mathbf{x}$ is constant.It follows that $0 = \frac{\partial}{\partial s}(u(\S,s)J) = u_t(x,s)J + u(x,s)J_s = u_t(x,s)J + u(x,s)\mathrm{div}\,(\mathbf{b}J) = (u_t + \mathrm{div}\,(u\mathbf{b}))J = 0$.We know that J won't be zero when $t = s$, therefore (6.9) is solved by $u(x,t)$.

Exercise 6.5. 8. Confirm that the formula $u = g(x - t\mathbf{F}'(u))$ from 3.2.5 provides an implicit solution for the conservation law

$$u_t + \mathbf{F}(u)_x = 0. \tag{6.10}$$

Solution. The initial condition is trival, then confirm the function

$$\begin{aligned}
&u_t + \mathbf{F}(u)_x = 0 \iff \\
&u_t = Dg \cdot (x - t\mathbf{F}'(u))(-\mathbf{F}'(u) - t\mathbf{F}''(u)u_t) \text{ While } 1 + t\mathbf{F}''(u) \neq 0 \\
&u_t = -\frac{Dg \cdot \mathbf{F}'(u)}{1 + Dg \cdot t\mathbf{F}''(u)} \\
&\mathbf{F}(u)_x = D\mathbf{F}(u) = \mathbf{F}'(u) \cdot Du \\
&\frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i} g(x - t\mathbf{F}'(u)) \\
&\qquad = g_{x_i}(1 - t\mathbf{F}''(u) \cdot u_{x_i}) \text{ need the same requirement above so that} \\
&Du = \frac{Dg}{1 + t\mathbf{F}''(u)} \\
&u_t - \mathbf{F}'(u) \cdot Du = 0
\end{aligned}$$