

## Chapter 6

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**Exercise 6.1.** 2. Compute the envelopes of the family of lines

$$x_1 + a^2 x_2 - 2a = 0 \quad (a \in \mathbb{R}) \quad (6.1)$$

in  $\mathbb{R}^2$  and of the family of planes

$$2a_1 x_1 + 2a_2 x_2 - x_3 + a_1^2 + a_2^2 = 0 \quad (a_1, a_2 \in \mathbb{R}) \quad (6.2)$$

in  $\mathbb{R}^3$ . Draw pictures illustrating the geometric meaning of the envelopes.

**Solution.** Let  $u(x, a) = x_1 + a^2 x_2 - 2a$ , from definition of envelopes

$$\begin{aligned} D_a u &= 2ax_2 - 2 = 0 \implies \\ a &= \frac{1}{x_2}. \end{aligned}$$

So envelopes is  $x_1 - \frac{1}{x_2} = 0$ .

Similarly,

$$\begin{aligned} 2x_1 + 2a_1 &= 0 \implies a_1 = -x_1 \\ 2x_2 + 2a_2 &= 0 \implies a_2 = -x_2 \\ \implies \mathbf{a} &= -\mathbf{x} \end{aligned}$$

So envelopes is  $x_1^2 + x_2^2 - x_3 = 0$ .

Above envelopes in  $\mathbb{R}^2$  plotted blow envelopes is all point  $x$  satisfy that exist a line  $u$  in the family such that  $u(x) = 0$ .

**Exercise 6.2.** 4.

(a) Write down the characteristic equations for the PDE

$$(*) \quad u_t + b \cdot Du = f \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

where  $b \in \mathbb{R}^n, f = f(x, t)$ .

(b) Use the characteristic ODE to solve  $(*)$  subject to the initial condition

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \quad (6.3)$$

Make sure your answer agrees with formula (5) in 2.1.2.

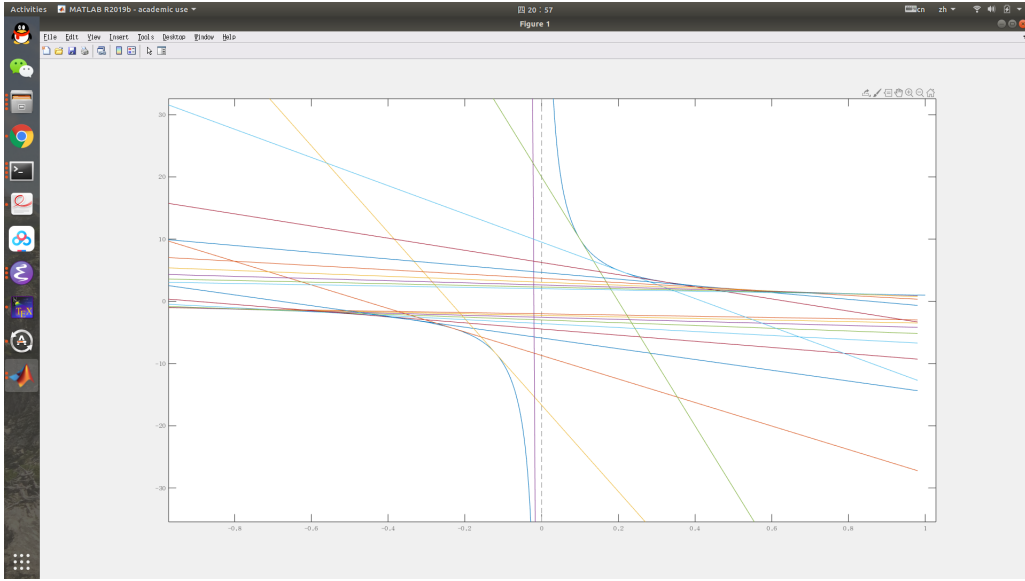


图 6.1: envelopes

**Solution.**

(a) Transfer (\*) to  $\mathbf{F}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = (b, 1) \cdot \mathbf{p}(s) - f(\mathbf{x}) = 0$ , then

$$\dot{\mathbf{p}}(s) = -(D_x f(\mathbf{x}, t) + D_t f(\mathbf{x}, t))$$

$$\dot{z}(s) = (b, 1) \cdot \mathbf{p}(s)$$

$$\dot{\mathbf{x}}(s) = (b, 1)$$

(b) Solving second and third of the equations get

$$\mathbf{x}(s) = (bs, s) + (x_0, 0)$$

$$\dot{z}(s) = f(\mathbf{x}) \implies$$

$$z(s) = \int_0^s f((bs + x_0, s)) ds + g(x_0)$$

Since  $\mathbf{x}(s) = (bs, s) + (x_0, 0)$ , eliminating the  $x_0$  in last equation get the formula (5).

**Exercise 6.3.** 5. Solve using characteristic:

(a)

$$x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2). \quad (6.4)$$

(b)

$$u u_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2} x_1. \quad (6.5)$$

**Solution.**

(a) Writing characteristic equations as follow

$$F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = (x_1, 2x_2, 1) \cdot \mathbf{p}(s) - 3z(s) = 0$$

$$\dot{z}(s) = 3z(s)$$

$$\dot{\mathbf{x}}(s) = (x_1, 2x_2, 1)$$

Solving equations get  $\mathbf{x}(s) = (x_1^0 e^s, x_2^0 e^{2s}, s)$ ,  $z(s) = z^0 e^{3s}$ , eliminating  $x_1^0, x_2^0, z^0$  to obtain  $u(x_1, x_2, x_3) = g(\frac{x_1}{e^{x_3}}, \frac{x_2}{e^{x_3}})e^{3x_3}$ .

(b)

$$F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = (z(s), 1) \cdot \mathbf{p}(s) - 1 = 0$$

$$\dot{z}(s) = 1$$

$$\dot{\mathbf{x}}(s) = (z(s), 1).$$

Solving equations get  $z(s) = s + z_0$ ,  $\mathbf{x}(s) = (\frac{1}{2}s^2 + z_0 s + x_1^0, s + x_2^0)$ . By  $u(x_1, x_1) = \frac{1}{2}x_1 \implies z_0 = \frac{1}{2}x_1^0 = \frac{1}{2}x_2^0$ , eliminating  $z_0, x_1^0, x_2^0$  obtain  $u(x_1, x_2) = \frac{1}{2}x_2 + \frac{x_2 - x_1}{2 - x_2}$ .

**Exercise 6.4.** 6. Given a smooth vector field  $\mathbf{b}$  on  $\mathbb{R}^n$ , let  $\mathbf{x}(s) = \mathbf{x}(s, x, t)$  solve the ODE

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{b}(\mathbf{x}) & (s \in \mathbb{R}) \\ \mathbf{x}(t) = x. \end{cases} \quad (6.6)$$

(a) Define the Jacobian

$$J(s, x, t) := \det D_x \mathbf{x}(s, x, t) \quad (6.7)$$

and derive *Euler's formula*:

$$J_s = \operatorname{div} \mathbf{b}(\mathbf{x})J. \quad (6.8)$$

(b) Demonstrate that

$$u(x, t) := g(\mathbf{x}(0, x, t))J(0, x, t) \quad (6.9)$$

solves

$$\begin{cases} u_t + \operatorname{div}(u\mathbf{b}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \quad (6.10)$$

(Hint: Show  $\frac{\partial}{\partial s}(u(\mathbf{x}, s)J) = 0$ .)

**Solution.**

(a) Unfold  $J$  and differential  $J$

$$\begin{aligned} J(s, x, t) &= \sum \prod_{i=1}^n (-1)^\sigma x_{\sigma(i)}^i \\ J_s &= \sum_{j=1}^n \sum_{i \neq j}^n \left( \prod_{i \neq j} (-1)^\sigma x_{\sigma(i)}^i \right) x_{\sigma(j)s}^j \\ &= \sum_{j=1}^n \sum_{i=1}^n \left( \prod_{i=1}^n (-1)^\sigma x_{\sigma(i)}^i \right) (\mathbf{b}^j(x)(s - t) + x)_{\sigma(j)s} \\ &= \sum_{j=1}^n \sum_{i=1}^n \left( \prod_{i=1}^n (-1)^\sigma x_{\sigma(i)}^i \right) \mathbf{b}_j^j(x) x_{\sigma(j)}^j \\ &= \operatorname{div} \mathbf{b}(x)J. \end{aligned}$$

(b) Boundary condition is obvious, since  $J(0, x, 0) = \det D_x x(0, x, 0) = \det D_x x = 1$  and  $x(0, x, 0) = x$ .

$$\frac{\partial}{\partial s}(u(\mathbf{x}, s)J) = u_t(\mathbf{x}, s)J + D_x u \cdot \mathbf{x}_s + u(x, s)J_s = J(u_t + D_x u \cdot \mathbf{b}(x) + u \operatorname{div} \mathbf{b}(x)) = J(u_t(x, s) + u(x, s) \operatorname{div} (\mathbf{b}J)) = (u_t + \operatorname{div} (u\mathbf{b}))J = 0.$$

$$u(\mathbf{x}, s)J|_{s=t} = u(\mathbf{x}, s)J|_{s=0} \iff u(x, t) = u(\mathbf{x}(t, x, t), t)J(t, x, t) = u(\mathbf{x}(0, x, t), 0)J(0, x, t) = g(\mathbf{x}(0, x, t))J(0, x, t),$$

therefore (6.10) is solved by  $u(x, t)$ .

**Exercise 6.5.** 8. Confirm that the formula  $u = g(x - t\mathbf{F}'(u))$  from §3.2.5 provides an implicit solution for the conservation law

$$u_t + \mathbf{F}(u)_x = 0. \quad (6.11)$$

**Solution.** The initial condition is trivial, then confirm the function

$$u_t + \mathbf{F}(u)_x = 0 \iff$$

$$u_t = Dg \cdot (x - t\mathbf{F}'(u))(-\mathbf{F}'(u) - t\mathbf{F}''(u)u_t) \text{ While } 1 + t\mathbf{F}''(u) \neq 0$$

$$u_t = -\frac{Dg \cdot \mathbf{F}'(u)}{1 + Dg \cdot t\mathbf{F}''(u)}$$

$$\mathbf{F}(u)_x = D\mathbf{F}(u) = \mathbf{F}'(u) \cdot Du$$

$$\frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i} g(x - t\mathbf{F}'(u))$$

$$= g_{x_i}(1 - t\mathbf{F}''(u) \cdot u_{x_i}) \text{ need the same requirement above so that}$$

$$Du = \frac{Dg}{1 + t\mathbf{F}''(u)}$$

$$u_t - \mathbf{F}'(u) \cdot Du = 0$$