Chapter 8

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Exercise 8.1. 15. Assuming

$$H(\frac{p_1+p_2}{2}) \le \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) - \frac{\theta}{8}\left|p_1 - p_2\right|^2.$$

Prove

$$\frac{1}{2}L(q_1) + \frac{1}{2}L(q_2) \le L(\frac{q_1+q_2}{2}) + \frac{1}{8\theta}\left|q_1 - q_2\right|^2.$$

Solution. From the definition of L, Let $L(q_1) = \max_{p\in\mathbb{R}^n}\{q_1 \cdot p - H(p)\} = q_1 \cdot p_1 - H(p_1), L(q_2) = q_2 \cdot p_2 - H(p_2)$. Then

$$\begin{aligned}
\frac{1}{2}L(q_1) + \frac{1}{2}L(q_2) \le L(\frac{q_1+q_2}{2}) + \frac{1}{8\theta}\left|q_1 - q_2\right|^2 \Longleftrightarrow \\
\frac{1}{2}(p_1q_1 + p_2q_2) + H(\frac{p_1+p_2}{2}) \le \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) + \frac{1}{4}(p_1q_1 + p_1q_2 + p_2q_1 + p_2q_2) + \frac{1}{8\theta}\left|q_1 - q_2\right|^2 \Longleftarrow \\
\frac{1}{4}(p_1 - p_2)(q_1 - q_2) \le \frac{\theta}{8}\left|p_1 - p_2\right|^2 + \frac{1}{8\theta}\left|q_1 - q_2\right|^2 \Longrightarrow \\
\frac{1}{4}(p_1 - p_2)(q_1 - q_2) \le 2\sqrt{\left(\frac{\theta}{8}\left|p_1 - p_2\right|^2 \times \frac{1}{8\theta}\left|q_1 - q_2\right|^2\right)}.
\end{aligned}$$

The last equation is obvious.

Exercise 8.2. 17. Show that

$$u(x,t) := \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x+t^2}\right) & \text{if } 4x + t^2 > 0 \\ 0 & \text{if } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of $u_t + (\frac{u^2}{2})_x = 0$.

Solution. Since while $u(x,t) = 0$ the equations holds trival, and When $u(x,t) \neq 0$

$$\begin{aligned}
u_t + \left(\frac{u^2}{2}\right)_x &= (-\frac{2}{3}t - \frac{2}{3}\sqrt{3x+t^2})_t + \left(\frac{4}{9}t^2 + \frac{2}{3}x + \frac{4}{9}t\sqrt{3x+t^2}\right)_x \\
&= -\frac{2}{3} - \frac{2t}{3\sqrt{3x+t^2}} + \frac{2}{3} + \frac{2t}{3\sqrt{3x+t^2}} \\
&= 0 \\
\frac{F(u_l) - F(u_r)}{u_l - u_r} &= \frac{0 - \left(\frac{-4x}{2}\right)}{0 - (-2\sqrt{-x})} \\
&= -\sqrt{-x} \\
\text{and } x_t &= -t/2 = -\sqrt{-x}
\end{aligned}$$

therefore, it's a intergal solution of $u_t + (\frac{u^2}{2})_x = 0$. Then

$$u(x+z,t) - u(x,t) = \begin{cases} \frac{2}{3}(\sqrt{3x+3z+t^2} - \sqrt{3x+t^2}) = \frac{2z}{\sqrt{3x+3z+t^2}-\sqrt{3x+t^2}} \le \frac{2}{t}z & \text{if } x+z > x > -\frac{t^2}{4} \\ \frac{2}{3}(\sqrt{3x+3z+t^2}) < 0 < z & \text{if } x+z > -\frac{t^2}{4} > x \\ 0 < z & \text{if } -\frac{t^2}{4} > x+z > x \end{cases}.$$

So this is an entropy solution of $u_t + (\frac{u^2}{2})_x = 0$.

Exercise 8.3. 18. Assume $u(x+z) - u(x) \le Ez$ for all $z > 0$. Let $u^\epsilon = \eta_\epsilon * u$, and show

$$u_x^\epsilon \le E.$$

Solution. if $u_x^\epsilon > E$ while $x = x_0$, that's equal to

$$\begin{aligned} u_x^\epsilon(x_0) &= (\int_{B(0,\epsilon)} \eta_\epsilon(y) f(x_0 - y) dy)_x \\ &= \int_{B(0,\epsilon)} \eta_\epsilon(y) f_x(x_0 - y) dy \\ &> E. \end{aligned}$$

It's implies that $x_1 \in B(x_0,\epsilon), f_x(x_1) > E$,Or $\int_{B(0,\epsilon)} \eta_\epsilon(y) f_x(x_0 - y)dy \le \int_{B(0,\epsilon)} \eta_\epsilon(y) E dy = E$. Which is contracdicte with $u_x^\epsilon(x_0) > E$. Since u_x has to continuous, $B(x_1,\epsilon), \forall x \in B(x_1,\epsilon) s.t. u_x(x) > E$. Then let $z \in B(x_1, \frac{\epsilon}{2})$ so that $u(x+z) - u(x) = u_x(\zeta)z, \zeta \in B(x_1,\epsilon) \Rightarrow u(x+z) - u(x) > E * z$. It's contracdicate with $u(x+z) - u(x) \le Ez$.

Exercise 8.4. 19. Assume $F(0) = 0$, u is a continuous integral solution of the conservation law

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

and u has compact support in $\mathbb{R} \times [0,T]$ for each time $T > 0$. Prove

$$\int_{-\infty}^{\infty} u(\cdot,t)dx = \int_{-\infty}^{\infty} g dx$$

for all $t > 0$.

Solution. Derivate left get

$$\begin{aligned} \frac{d}{dt}(\int_{-\infty}^{\infty} u(\cdot,t)dx) &= \int_{-\infty}^{\infty} u_t(\cdot,t)dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dx}(F(u))dx \\ &= F(u)|_{-\infty}^{\infty} \\ &= 0 \end{aligned}$$

The last step from u has compact support and $F(0) = 0$. And it indicate $\int_{-\infty}^{\infty} u(\cdot,t)dx = \int_{-\infty}^{\infty} u(\cdot,0)dx = \int_{-\infty}^{\infty} g dx$.

Exercise 8.5. 20. Compute explicitly the unique entropy solution of

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x. \end{cases}$$

Draw a picture documenting your answer, being sure to illustrate what happens for all times $t > 0$.

Solution. Solve the equations

$$\begin{cases} w_t + \frac{w_x^2}{2} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ w = \int_0^x g(s)ds & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

get $w(x,t) = \min\limits_{y\in\mathbb{R}} t\frac{(x-y)^2}{2t^2} + \int_0^y g(s)ds$.Let $f(y) = \frac{d}{dy}(t\frac{(x-y)^2}{2t^2} + \int_0^y g(s)ds) = -\frac{(x-y)}{t} + g(y)$, Therefore, discussing by situation

1. $x < -1$

$$f(y) = \begin{cases} > 0 & \text{if } y > x - t \\ = 0 & \text{if } y = x - t \\ < 0 & \text{if } y < x - t \end{cases}$$

$$w(x,t) = \frac{t}{2} + \int_0^{x-t} g(s)ds \Rightarrow u(x,t) = 1.$$

2. $-1 < x < 0$

$$f(y) = \begin{cases} \begin{cases} > 0 & \text{if } y > x \\ = 0 & \text{if } y = x \\ < 0 & \text{if } y < x \end{cases} & \text{if } t < 2*(x+1) \\ \begin{cases} > 0 & \text{if } y > x - t \\ = 0 & \text{if } y = x - t \\ < 0 & \text{if } y < x - t \end{cases} & \text{if } t > 2*(x+1) \end{cases}$$

$$w(x,t) = \begin{cases} \int_0^x g(s)ds \Rightarrow u(x,t) = 0 \Rightarrow u(x,t) = g(x) = 0 & \text{if } t < 2*(x+1) \\ \frac{t}{2} + \int_0^{x-t} g(s)ds \Rightarrow u(x,t) = g(x-t) = 1 & \text{if } t > 2*(x+1) \end{cases}$$

3. $0 < x < 1$

$$f(y) = \begin{cases} \begin{cases} > 0 & \text{if } x - t < y < 0 \mid\mid 0 < y \\ < 0 & \text{if } y < x - t \end{cases} & \text{if } x < t - (2t)^{1/2} \\ \begin{cases} > 0 & \text{if } x - t < y < 0 \mid\mid 0 < y \\ < 0 & \text{if } y < x - t \end{cases} & \text{if } t - (2t)^{1/2} < x < 2t \\ \begin{cases} > 0 & \text{if } x - t < y < -1 \mid\mid x - 2t < y \\ < 0 & \text{if } y < x - t \mid\mid -1 < y < x - 2t \end{cases} & \text{if } 2t < x \end{cases}$$

$$w(x,t)=\begin{cases}\min\{\frac{t}{2}+\int_0^{x-t}g(s)ds,\frac{x^2}{2t}+\int_0^0 g(s)ds\}=\frac{t}{2}+\int_0^{x-t}g(s)ds & \\ \Rightarrow u(x,t)=g(x-t),(x-t)<-2\Rightarrow u(x,t)=1 & \text{if } x<t-(2t)^{1/2}\\ \min\{\frac{t}{2}+\int_0^{x-t}g(s)ds,\frac{x^2}{2t}+\int_0^0 g(s)ds\}=\frac{x^2}{2t}+\int_0^0 g(s)ds & \\ \Rightarrow u(x,t)=\frac{x}{t}+g(0)=\frac{x}{t} & \text{if } t-(2t)^{1/2}<x<2t\\ \min\{\frac{t}{2}+\int_0^{x-t}g(s)ds,2t+\int_0^{x-2t}g(s)ds\}=2t+\int_0^{x-2t}g(s)ds & \\ \Rightarrow u(x,t)=g(x-2t)=2 & \text{if } 2t<x\end{cases}$$

4. $1<x<4+2\sqrt{2}$

$$f(y)=\begin{cases}\begin{cases}>0 & \text{if } x-t<y<-1 \mid\mid 0<y<1 \mid\mid x<y\\ <0 & \text{if } y<x-t \mid\mid -1<y<0 \mid\mid 1<y<x\end{cases} & \text{if } x<t-(2t)^{1/2}\\ \begin{cases}>0 & \text{if } (x-t<y<0)\\ & (\text{maybe} x-t<-1 \text{but no influence to anwser})\\ & \mid\mid 0<y<1 \mid\mid x<y\\ <0 & \text{if } y<x-t \mid\mid <y<x\end{cases} & \text{if } t-(2t)^{1/2}<x<2t\&\&x<2t^{1/2}\\ \begin{cases}>0 & \text{if } y<0 \mid\mid x-2t<y\\ <0 & \text{if } y<x-t \mid\mid -1<y<x-2t\end{cases} & \text{if } 2t<x<1+t\\ \begin{cases}>0 & \text{if } y<1 \mid\mid x<y\\ <0 & \text{if } x<y\end{cases} & \text{if } (2t^{1/2}<y\&\&t<1) \mid\mid\\ & (2t<y\&\&t>1)\end{cases}$$

$$w(x,t)=\begin{cases}\min\{\frac{t}{2}+\int_0^{x-t}g(s)ds,\frac{x^2}{2t}+\int_0^0 g(s)ds,\int_0^x g(s)ds\}=\frac{t}{2}+\int_0^{x-t}g(s)ds & \\ \Rightarrow u(x,t)=g(x-t),(x-t)<-1\Rightarrow u(x,t)=1 & \text{if } x<t-(2t)^{1/2}\\ \min\{\frac{t}{2}+\int_0^{x-t}g(s)ds,\frac{x^2}{2t}+\int_0^0 g(s)ds,\int_0^x g(s)ds\}=\frac{x^2}{2t}+\int_0^0 g(s)ds & \\ \Rightarrow u(x,t)=\frac{x}{t}+g(0)=\frac{x}{t} & \text{if } t-(2t)^{1/2}<x<2t\&\&\\ & x<2t^{1/2}\\ \min\{\frac{t}{2}+\int_0^{x-t}g(s)ds,2t+\int_0^{x-2t}g(s)ds\}=2t+\int_0^{x-2t}g(s)ds & \\ \Rightarrow u(x,t)=g(x-2t)=2 & \text{if } 2t<x<1+t\\ \min\{\int_0^x g(s)ds\}=\int_0^x g(s)ds & \\ \Rightarrow u(x,t)=g(x)=0 & \text{if } t<x.\end{cases}$$

5. $4+2\sqrt{2}<x$

$$u(x,t)=\begin{cases}1 & \text{if } 2x-2<t\\ 0 & \text{if } t<2x-2\end{cases}$$

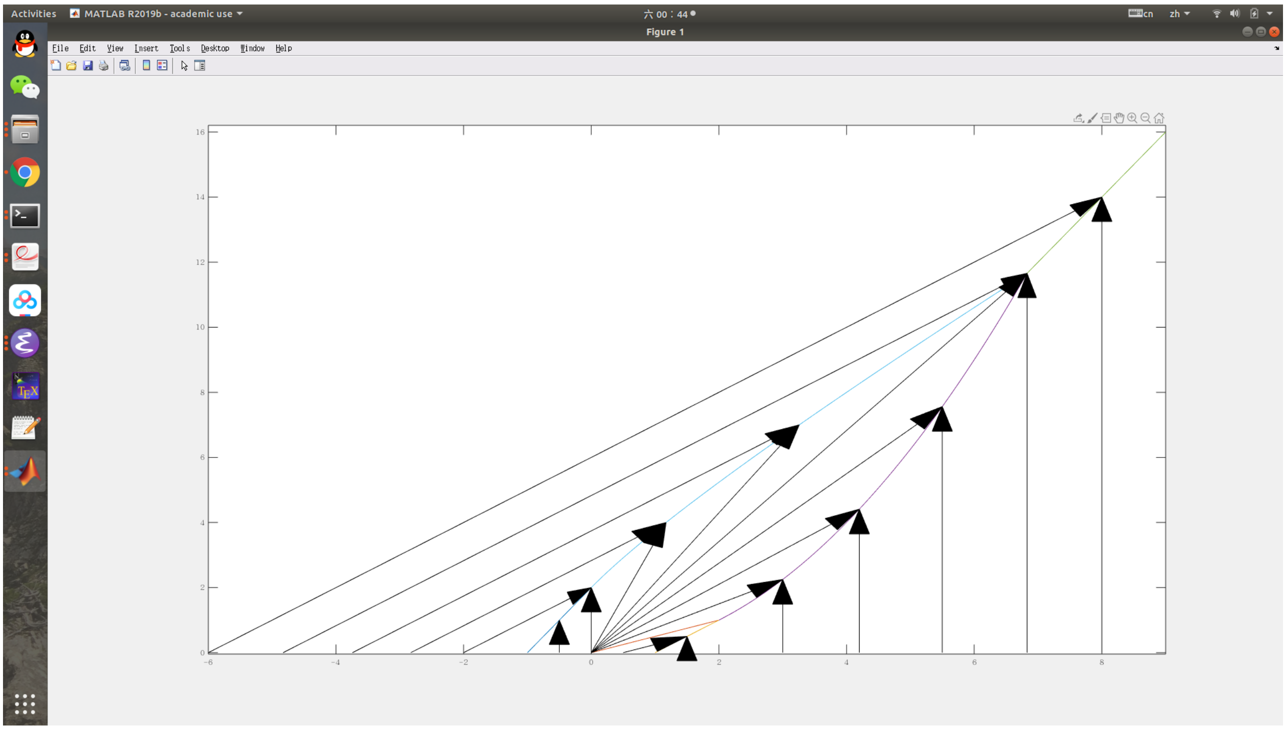

图 8.1: $u(x,t)$

In summary,

$$u(x,t)=\begin{cases} 1 & \text{if } x<0 \;\&\&\; 2x+2<t \\ 0 & \text{if } x<0 \;\&\&\; t<2x+2 \\ 1 & \text{if } 0<x<4+2\sqrt{2} \;\&\&\; x<t-\sqrt{2t} \\ 2 & \text{if } 0<x<2 \;\&\&\; x-1<t<\frac{1}{2}x \\ \frac{x}{t} & \text{if } 0<x<4+2\sqrt{2} \;\&\&\; t-\sqrt{2t}<x \\ & \;\&\&\; \frac{1}{2}<t \text{ while } 0<x\leq 2 \;\&\&\; \frac{1}{4}x^2<t \text{ while } 2<x<4+2\sqrt{2} \\ 1 & \text{if } 4+2\sqrt{2}<x \;\&\&\; 2x-2<t \\ 0 & \text{otherwise.} \end{cases}$$

. And $u(x,t)$ plot in (8)