Chapter 7

Homework 21935004 谭焱

Exercise 7.1. 9. Consider the problem of minimizing the action $\int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))ds$ over the new admissible class

$$\mathcal{A} := \{\, \mathbf{w}(\cdot) \in C^2([0,t];\mathbb{R}^n) \big|\, \mathbf{w}(t) = x\},$$

where we do not require that $\mathbf{w}(0) = y$.

(a) Show that a minimizer $\mathbf{x}(\cdot) \in \mathcal{A}$ solves the Euler Lagrange equations

$$-\frac{d}{ds}(D_v L(\dot{\mathbf{x}}(s), \mathbf{x}(s))) + D_x L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) = 0 \qquad (0 \le s \le t).$$

(b) Prove that

$$D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) = 0.$$

(c) Suppose now that $\mathbf{x}(\cdot) \in \mathcal{A}$ minimizes the modified action

$$\int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))ds + g(\mathbf{w}(0)).$$

Show that $\mathbf{x}(\cdot)$ solves the usual Euler-Lagrange equations and determine the boundary condition at $s = 0$.

Solution.

(a) Assume $y = \mathbf{x}(0)$, $\mathbf{x}(s)$ is minimizer in $\mathcal{A}$, of course also is minimizer in $\{\mathbf{w}(\cdot) \in C^2([0,t];\mathbb{R}^n)|\, \mathbf{w}(t) = x, \mathbf{w}(0) = y\}$. Which indicate that $-\frac{d}{ds}(D_v L(\dot{\mathbf{x}}(s), \mathbf{x}(s))) + D_x L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) = 0 \qquad (0 \le s \le t)$.

(b) Choose a smooht function $\mathbf{v}\colon [0,t] \to \mathbb{R}^n, \mathbf{v} = (v^1, \ldots, v^n)$, satisfgying $\mathbf{v}(t) = 0$.Let $i(\tau) := I[\mathbf{x}(\cdot) + \tau\mathbf{v}(\cdot)]$. From $\mathbf{x}(\cdot)$ is minimizer in $\mathcal{A}$ know $i'(0) = 0$.Computing this derivative. Observe

$$\begin{aligned}
i'(\tau) &= \int_0^t \sum_{i=1}^n L_{q_i}(\dot{\mathbf{x}} + \tau\dot{\mathbf{v}}, \mathbf{x} + \tau\mathbf{v})\dot{v}^i + L_{x_i}(\dot{\mathbf{x}} + \tau\dot{\mathbf{v}}, \mathbf{x} + \tau\mathbf{v})v^i ds \qquad \text{Set } \tau = 0 \text{ and intergrate by parts} \\
0 = i'(0) &= \sum_{i=1}^n (\int_0^t [-\frac{d}{ds}(L_{q_i}(\dot{\mathbf{x}}, \mathbf{x})) + L_{x_i}(\dot{\mathbf{x}}, \mathbf{x})]v^i ds - L_{q_i}(\dot{\mathbf{x}}, \mathbf{x})v^i\big|_{s=0}) \\
&= \sum_{i=1}^n (\int_0^t [-\frac{d}{ds}(L_{q_i}(\dot{\mathbf{x}}, \mathbf{x})) + L_{x_i}(\dot{\mathbf{x}}, \mathbf{x})]v^i ds) - L_v(\dot{\mathbf{x}}, \mathbf{x}) \cdot \mathbf{v}|_{s=0}
\end{aligned}$$

make $D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0))$ have to be zero, since $\mathbf{v}$ can be any value in $s = 0$.

(c) By calculating above, similarly get

$$0 = i'(0) = \sum_{i=1}^{n} (\int_0^t [-\frac{d}{ds}(L_{q_i}(\dot{\mathbf{x}}, \mathbf{x})) + L_{x_i}(\dot{\mathbf{x}}, \mathbf{x})]v^i ds) - (L_v(\dot{\mathbf{x}}, \mathbf{x}) - Dg(\mathbf{x}(0))) \cdot \mathbf{v}|_{s=0}.$$

$D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) - Dg(\mathbf{x}(0)) = 0$ is boundary condition in $s = 0$.

Exercise 7.2. 10. If $H : \mathbb{R}^n \to \mathbb{R}$ is convex, we write $L = H^*$.

(a) Let $H(p) = \frac{1}{r} |p|^r$, for $1 < r < \infty$. Show

$$L(v) = \frac{1}{s} |v|^s, \qquad \text{where } \frac{1}{r} + \frac{1}{s} = 1.$$

(b) Let $H(p) = \frac{1}{2} \sum_{i,j=1}^n a_{ij} p_i p_j + \sum_{i=1}^n b_i p_i$, where $A = ((a_{ij}))$ is a symmetric, positive definite matrix, $b \in \mathbb{R}^n$. Compute $L(v)$.

Solution.

(a) From definition of $L(v) = \sup_{p = \in \mathbb{R}^n} \{v \cdot p - H(p)\}$, differential inside of left part with p get $v - \frac{p}{|p|^{2-r}}$. It follows that $L(v) = p|p|^{r-2} \cdot p - \frac{1}{r}|p|^r = \frac{1}{s}|p|^r$. In other side, $|v|^2 = p \cdot p\, |p|^{2r-4} = |p|^{2r-2} \Longrightarrow |v|^s = |p|^{rs-s} = |p|^r$.So $L(v) = \frac{1}{s}|p|^r = \frac{1}{s}|v|^s$.

(b) Calculating differential with variable p, assume $L(v) = v \cdot p - H(p)$

$$0 = D_{p_i}(v \cdot p - H(p)) = v_i - \sum_{j=1}^n a_{ij} p_j - b_i$$

$$\begin{aligned} L(v) &= \sum_{i=1}^n (\sum_{j=1}^n a_{ij} p_j p_i + b_i p_i) - H(p) \\ &= 0 \end{aligned}$$

Exercise 7.3. 12. Assume $L_1, L_2 : \mathbb{R}^n \to \mathbb{R}$ are convex, smooth and superlinear. Show that

$$\min_{v \in \mathbb{R}^n} (L_1(v) + L_2(v)) = \max_{p \in \mathbb{R}^n} (-H_1(p) - H_2(-p)),$$

where $H_1 = L_1^*, H_2 = L_2^*$.

Solution. Since $-H_1(p) = -L_1^*(p) = -\max_{v \in \mathbb{R}^n}(p \cdot v - L_1(v)) = \min_{v \in \mathbb{R}^n}(-p \cdot v + L_1(v))$.Similarly get H_2. So $\max_{p \in \mathbb{R}^n}(-H_1(p) - H_2(-p)) \le \max_{p \in \mathbb{R}^n}(\min_{v \in \mathbb{R}^n}(-p \cdot v + L_1(v) + p \cdot v + L_2(v))) = \min_{v \in \mathbb{R}^n}(L_1(v) + L_2(v))$, and $\min_{v \in \mathbb{R}^n}(L_1(v) + L_2(v)) = \min_{v \in \mathbb{R}^n}(\max_{p \in \mathbb{R}^n}(v \cdot p - H_1(p)) + \max_{p \in \mathbb{R}^n}(v \cdot p - H_2(p))) \ge \min_{v \in \mathbb{R}^n}(\max_{p \in \mathbb{R}^n}(v \cdot p - H_1(p) + v \cdot (-p) - H_2(-p))) = \min_{v \in \mathbb{R}^n}(\max_{p \in \mathbb{R}^n}(-H_1(p) - H_2(-p))) = \max_{p \in \mathbb{R}^n}(-H_1(p) - H_2(-p))$.In summary, $\min_{v \in \mathbb{R}^n}(L_1(v) + L_2(v)) = \max_{p \in \mathbb{R}^n}(-H_1(p) - H_2(-p))$.

Exercise 7.4. 13. Prove that the Hopf-Lax formula reads

$$u(x,t) = \min_{y \in \mathbb{R}^n} \{tL(\frac{x-y}{t}) + g(y)\} = \min_{y \in B(x,Rt)} \{tL(\frac{x-y}{t}) + g(y)\}$$

for $R = \sup_{\mathbb{R}^n} |DH(Dg)|, H = L^*$. (This proves *finite propagation speed* for a Hamilton-Jacobi PDE with convex Hamiltonian and Lipschitz continuous initial function g.)

Solution. If $\min_{y \in \mathbb{R}^n}\{tL(\frac{x-y}{t}) + g(y)\} < \min_{y \in B(x,Rt)}\{tL(\frac{x-y}{t}) + g(y)\}$, assume $u(x,t) = tL(\frac{x-y_0}{t}) + g(y_0), m = \frac{x-y_0}{t} > R$.So that it attain that $0 = D_y(tL(\frac{x-y_0}{t}) + g(y_0)) = -DL(\frac{x-y_0}{t}) + Dg(y_0)$, which conclude $DL(m) = Dg(y_0)$.In other side, assume $L(m) = m \cdot v - H(v)$. Also from the minimizer, $D_v L(m) = 0 \Longrightarrow m = DH(v)$. Since $H(v) = v \cdot m - L(m)$, same as above, $v = DL(m)$. Combine with $DL(m) = Dg(y_0)$ and condition $R = \sup_{\mathbb{R}^n} |DH(Dg)|$ induct $D_v L(m) = m - DH(v) = m - DH(DL(m)) = m - DH(Dg(y_0)) > m - R > 0$. It indicate $L(m) = \max_{\mathbb{R}^n}\{m \cdot v - H(v)\}$ doesn't exist, that is a contradiction.

Exercise 7.5. 14. Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \begin{cases} 0 & x \in E \\ +\infty & x \notin E \end{cases} & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

it would give the solution

$$u(x,t) = \frac{1}{4t}\operatorname{dist}(x,E)^2.$$

Solution. Since $L(v) = \sup_{p\in\mathbb{R}^n}(v\cdot p - |p|^2) = \frac{1}{4}|v|^2$.By the Hopf-Lax formula

$$u(x,t) = \min_{y\in\mathbb{R}^n}\{\frac{1}{4t}|x-y|^2 + g(y)\}$$

if $y \notin E$, $u(x,t) = +\infty$ is obvious not the minimizer.So $u(x,t) \geq \frac{1}{4t}\operatorname{dist}(x,E)^2$ and since E is a closed subset, the equal can be satisfy in ∂E. That's mean $u(x,t) = \frac{1}{4t}\operatorname{dist}(x,E)^2$.

Exercise 7.6. 16. Assume u^1, u^2 are two solutions of the initial-value problems

$$\begin{cases} u_t^i + H(Du^i) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^i = g^i & \text{on } \mathbb{R}^n \times \{t=0\} (i=1,2), \end{cases}$$

given by the Hopf-Lax formula. Prove the L^∞-contraction inequality

$$\sup_{\mathbb{R}^n}\left|u^1(\cdot,t) - u^2(\cdot,t)\right| \leq \sup_{\mathbb{R}^n}\left|g^1 - g^2\right| (t>0).$$

Solution. Assume $\sup_{\mathbb{R}^n}|g^1 - g^2| = |g^1(y_0) - g^2(y_0)|, y \in \mathbb{R}^n$, and $\sup_{\mathbb{R}^n}|u^1(\cdot,t) - u^2(\cdot,t)| = u^1(x_0,t) - u^2(x_0,t)$.By Hopf-Lax formula, set y_1 such that $u^2(x_0,t) = tL(\frac{x_0-y_1}{t} + g(y_1))$,Then

$$\begin{aligned} u^1(x_0,t) - u^2(x_0,t) &= \min_{\mathbb{R}^n}\{tL(\frac{x-y}{t} + g^1(y))\} - \min_{\mathbb{R}^n}\{tL(\frac{x-y}{t} + g^2(y))\} \\ &\leq g^1(y_1) - g^2(y_1) \\ &\leq g^1(y_0) - g^2(y_0) \\ &= \sup_{\mathbb{R}^n}\left|g^1 - g^2\right|. \end{aligned}$$

Therefore, $\sup_{\mathbb{R}^n}|u^1(\cdot,t) - u^2(\cdot,t)| \leq \sup_{\mathbb{R}^n}|g^1 - g^2| (t>0)$.