Chapter 5

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Exercise 5.1. 18. (Stokes's rule) Assume u solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = 0, \; u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{5.1}$$

Show that $v := u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ v = h, \; v_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5.2}$$

This is *Stokes's rule.*

Solution. From (5.1) $v = u_t = h, \; v_t = (u_t)_t = h_t = 0$, since h is constant for variable t. u solves the wave equation. Thus $h \in C^2(\mathbb{R}^n)$, and

$$\begin{aligned} v_{tt} - \Delta v =& (u_t)_{tt} - \Delta u_t \\ =& (u_{tt})_t - (\Delta u)_t \\ =& (u_{tt} - \Delta u)_t \\ =& 0_t \\ =& 0. \end{aligned}$$

So v solves (5.2).

Exercise 5.2. 19.

(a) Show the general solution of the **PDE** $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using th changed of variables $\xi = x + t, \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.

(d) Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?

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Solution.

(a) $u_{xy} = 0 \iff u_x = f(x) \iff u = \int f(x)dx + g(y)$. Let $F(x) = \int f dx, \; G(y) = g(y)$ replace above get $u(x, y) = F(x) + G(y)$.

(b) $u_{tt} - u_{xx} = 0 \iff (\partial_t^2 - \partial_x^2)(u) = 0 \iff (\partial_t + \partial_x)(\partial_t - \partial_x)(u) = 0 \iff -\partial_\xi \partial_\eta(u) = 0 \iff u_{\xi\eta} = 0$.

(c) From (a) and (b) we know the wave equation general solution is $u(x, t) = F(x + t) + G(x - t)$. Combining with inital condition

$$\begin{cases} F(x) + G(x) = g(x) \\ F_t(x) - G_t(x) = h(x). \end{cases}$$

Solving the equations get $u(x, t) = \frac{1}{2}(g(x + t) + g(x - t)) + \int_{x-t}^{x+t} h(y)dy$.

(d) $F(x)$ is left-moving wave, since when t increasing, $x =$ -t move left so that $F(x+t)$ maintain constant. Similarly, $G(y)$ is right-moving wave.

Exercise 5.3. 20. Assume that for some attenuation function $\alpha = \alpha(r)$ and delay function $\beta = \beta(r) \geq 0$, there eixst for *all* profiles ϕ solutions of the wave equation in $(\mathbb{R}^n - 0) \times \mathbb{R}$ having the form

$$u(x, t) = \alpha(t)\phi(r - \beta(r)). \tag{5.3}$$

Here $r = |x|$ and we assume $\beta(0) = 0$.

Show that this is possible only if $n = 1$ or 3, and compute the form of the functions α, β.

Solution. Since $\partial_{x_i} r = \frac{x_i}{r} \implies \Delta r = \frac{n-1}{r}$. Replacing the wave equation with (5.3)

$$\Delta(\alpha(r)\phi(t-\beta(r))) - \partial_t^2(\alpha(r)\phi(t-\beta(r))) = 0 \iff$$
$$\alpha''\phi + \alpha\phi''((\beta')^2 - 1) - 2\alpha'\phi'\beta' +$$
$$\frac{n-1}{r}(\alpha'\phi - \alpha\phi'\beta') - \alpha\phi'\beta'' = 0.$$

It's true for every ϕ means that ϕ, ϕ' and ϕ'''s coefficient is 0 whatever t value. Therefore we have equations

$$\begin{cases} \alpha'' + \frac{n-1}{r}\alpha' = 0 \\ -2\alpha'\beta' - \frac{n-1}{r}\alpha\beta' - \alpha\beta'' = 0 \\ \alpha((\beta')^2 - 1) = 0 \end{cases}$$

The third equation implies that $\beta = \pm x + c$ or $\alpha = 0$, which is meanless. c is a constant value. Others two equations can be translate to

$$\begin{cases} p(p-1) + p(n-1) = 0 \\ 2p + n - 1 = 0 \end{cases}$$

Here p comes from $\alpha = Cr^p$. The equations solution is $p = 0, -1$ correspond $n = 1, 3$.So $\alpha = Cr, Cr^3$.

Exercise 5.4. 21.

(a) Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B},\ \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0 \end{cases} \tag{5.4}$$

Show

$$\mathbf{E}_{tt} - \Delta\mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta\mathbf{B} = 0.$$

(b) Assume that $\mathbf{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0} \quad \text{in } \mathbb{R}^3 \times (0,\infty). \tag{5.5}$$

Show $w := \operatorname{div}\mathbf{u}$ and $\mathbf{w} := \operatorname{curl}\mathbf{u}$ each solve wave equations, but with differing speeds of propagation.

Solution.

(a) $\operatorname{div}\mathbf{E} = 0 \iff E_1^1 + E_2^2 + E_3^3 = 0$, calculating the wave equation

$$\begin{aligned}
\mathbf{E}_{tt} =& (\operatorname{curl}\mathbf{B})_t \\
=& (B_2^3 - B_3^2, B_3^1 - B_1^3, B_1^2 - B_2^1)_t \\
=& ((-E_1^2 + E_2^1)_2 - (-E_3^1 + E_1^3)_3, \\
& (-E_2^3 + E_3^2)_3 - (-E_1^2 + E_2^1)_1, \\
& (-E_3^1 + E_1^3)_1 - (-E_2^3 + E_3^2)_2) \\
=& (E_{11}^1 + E_{22}^1 + E_{33}^1 - (E_1^1 + E_2^2 + E_3^3)_1, \\
& E_{11}^2 + E_{22}^2 + E_{33}^2 - (E_1^1 + E_2^2 + E_3^3)_2, \\
& E_{11}^3 + E_{22}^3 + E_{33}^3 - (E_1^1 + E_2^2 + E_3^3)_3) \\
=& \Delta(E^1, E^2, E^3) = \Delta\mathbf{E}.
\end{aligned}$$

Similarly, get $\mathbf{B}_{tt} - \Delta\mathbf{B} = 0$.

(b)

$$\begin{aligned}
w_{tt} &= \mu\text{div (div (gred }\mathbf{u})) + (\lambda+\mu)\text{div (gred (div }\mathbf{u})) \\
&= (\lambda + 2\mu)(\Delta(\operatorname{div}\mathbf{u})) \\
&= (\lambda + 2\mu)(\Delta w)
\end{aligned}$$

Exercise 5.5. 22. Let u denote the density of particles moving to the right with speed one slong the real line and let v denote the density of particles moving to the left with speed one. If at rate $d > 0$ right-moving, and vice versa, we have the system of PDE

$$\begin{cases} u_t + u_x = d(v-u) \\ v_t - v_x = d(u-v). \end{cases} \tag{5.6}$$

Show that both $w := u$ and $w := v$ solve the telegraph equation

$$w_{tt} + 2dw_t - w_{xx} = 0. \tag{5.7}$$

Solution.

$$\begin{aligned}
u_{xx} &= (d(v-u) - u_t)_x \\
&= dv_x - du_x - u_{xt} \\
&= d(-d(u-v) + v_t) - d(d(v-u) - u_t) - (d(v-u) - u_t)_t \\
&= -d^2u + d^2v + dv_t - d^2v + d^2u + du_t - dv_t + du_t + u_{tt} \\
&= 2du_t + u_{tt}.
\end{aligned}$$

So $w := u$ solves the telegraph equation. $w := v$ is similar.

Exercise 5.6. 24. (Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{5.8}$$

suppose g,h have compact support. The *kinetic energy* is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the *potential energy* is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$.

Prove

(a) $k(t) + p(t)$ is constant in t,

(b) $k(t) = p(t)$ for all large enough times t.

Solution.

(a) Let $f(t) = k(t) + p(t)$. Differential f with t

$$\begin{aligned} \partial_t f &= \frac{1}{2}\int_{-\infty}^{\infty} 2u_x u_{xt} + 2u_t u_{tt} dx \\ &= \frac{1}{2}\int_{\infty}^{\infty} u_t(-u_{xx} + u_{tt})dx \\ &= 0. \end{aligned}$$

The second step comes from that g,h have compact support and Finite propagation speed. So f is constant function for t.

(b) From d'Alembert's rule $u(x,t) = \frac{1}{2}(g(x+t)+g(x-t)) + \int_{x-t}^{x+t} h(y)dy$. calculate $k(t) - p(t)$

$$\begin{aligned} &k(t) - p(t) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (g'(x+t) + g'(x-t) + h(x+t) - h(x-t))^2 \\ &- (g'(x+t) - g'(x-t) + h(x+t) + h(x-t))^2 dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (2g'(x-t) - 2h(x-t))(2g'(x+t) - 2h(x+t)) \end{aligned}$$

Since g,h have compact support, g', h have compact support. It follows that When t is big enough. $g'(x-t) - h(x-t), g'(x+t) - h(x+t)$ can't have non-zero value at same time. In summary, when t big enough $k(t) = p(t) \forall t$.