Chapter 8

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Problem 8.1. Prove Theorem 6.3 by assuming that $\forall x \in (a, b)$ the weight function $\rho(x) > 0$.

Solution.

(a) $L^2_\rho[a, b]$ is a vector space

- commutativity
  $\forall u, v \in L^2_\rho[a, b], u + v = u(x) + v(x) = v(x) + u(x) = v + u.$
- associativity
  $\forall u, v, w \in L^2_\rho[a, b], (u + v) + w = (u(x) + v(x)) + w(x) = u(x) + (v(x) + w(x)) = u + (v + w).$
- compatibility
  $\forall u \in L^2_\rho[a, b], a, b \in \mathbb{F}, (ab)u = abu(x) = a(bu(x)) = a(bu).$
- additive identity
  $\forall u \in L^2_\rho[a, b]$. Let $0(x) = x, \forall x \in [a, b]$, we have $(u + 0)(x) = u(x) + 0(x) = u(x) = u$.
- additive inverse
  $\forall u \in L^2_\rho[a, b]$. Let $v(x) = -u(x). \forall x \in [a, b]$, we have $(u + v)(x) = u(x) + v(x) = 0 = 0(x)$.
- multiplicative identity
  $\forall u \in L^2_\rho[a, b], \exists 1 \in \mathbb{F}, s.t. 1u = 1u(x) = u(x) = u.$
- distributive laws
  $\forall u, v \in L^2_\rho[a, b], \forall a, b \in \mathbb{F}, (a + b)(u)(x) = (a + b)(u(x)) = au(x) + bu(x) = au + bu, a(u + v)(x) = au(x) + av(x) = au + av.$

(b) It satisfies inner product

- real positivity
  $\forall v \in L^2_\rho[a, b], \langle v, v\rangle = \int_a^b \rho(t)v(t)\overline{v(t)}dt = \int_a^b \rho(t)\left|v(t)\right|^2 dt > 0$, since $\rho(t) > 0, \left|v(t)\right|^2 > 0$.
- definiteness
  $\langle v, v\rangle = 0 \iff \int_a^b \rho(t)\left|v(t)\right|^2 dt = 0$, since $\rho > 0$, $\left|v(t)\right|^2 = 0$ $a.e.$, so $v(t) = 0$ $a.e.$.
- additivity in the first slot
  $\forall u, v, w \in L^2_\rho[a, b], \langle u + v, w\rangle = \int_a^b \rho(t)(u(t) + v(t))\overline{w(t)}dt = \int_a^b \rho(t)u(t)\overline{w(t)}dt + \int_a^b \rho(t)v(t)\overline{w(t)}dt = \langle u, w\rangle + \langle v, w\rangle.$

- homogeneity in the first slot

  $\forall a \in \mathbb{R}, \forall v, w \in L^2_\rho[a,b], \langle av, w\rangle = \int_a^b \rho(t)av(t)\overline{w(t)}dt = a\int_a^b \rho(t)v(t)\overline{w(t)}dt = a\langle v, w\rangle.$
- conjugate symmetry

  $\forall v, w \in L^2_\rho[a,b], \langle v, w\rangle = \int_a^b \rho(t)v(t)\overline{w(t)}dt == \int_a^b \rho(t)\overline{\overline{v(t)}w(t)}dt = \overline{\int_a^b \rho(t)w(t)\overline{v(t)}dt} = \overline{\langle w, v\rangle}.$

(c) Induced norm

Since $\|u\|_2 = \left(\int_a^b \rho(t)\,|v(t)|^2\right)^{\frac{1}{2}} = \sqrt{\langle v, v\rangle}$

Problem 8.2. Consider the Chebyshev polynomials of the first kind.

(a) Show that they are orthogonal on $[-1,1]$ with respect to the inner product in Theorem 6.3 with the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$.

(b) Normalize the first three Chebyshev polynomials to arrive at an orthonormal system.

Solution.

(a) It's coming to prove $\langle \cos(i\arccos x), \cos(j\arccos x)\rangle = 0, i \neq j$. And

$$\begin{aligned}
&\langle \cos(i\arccos x), \cos(j\arccos x)\rangle \\
=& \int_{-1}^{1} \frac{\cos(i\arccos x)\cos(j\arccos x)}{\sqrt{1-x^2}} \\
=& \int_0^\pi \cos(i\arccos x)\cos(j\arccos x)d(-\arccos x) \\
=& \int_0^\pi \cos(iy)\cos(jy)dy \\
=& \int_0^\pi (\cos((i-j)y) - \cos((i+j)y))/2dy \\
=& 0 \qquad \text{while } i-j \neq 0.
\end{aligned}$$

(b) The first three terms is $T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1$. Then from above equation knows $\|T_i\| = \begin{cases} \pi & i = 0 \\ \pi/2 & i > 0 \end{cases}$. Therefore, after orthonormalize, $T_o^*(x) = \frac{1}{\sqrt{\pi}}, T_1^*(x) = \frac{x}{\sqrt{\pi/2}}, T_2^*(x) = \frac{2x^2-1}{\sqrt{\pi/2}}$.

Problem 8.3. Least-square approximate of a continuous function. Approximate the circular arc given by the equation $y(x) = \sqrt{1-x^2}$ for $x \in [-1,1]$ by a quadratic polynomial with respect to the inner product in Theorem 6.3.

(a) $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with Fourier expansion.

(b) $\rho(s) = \frac{1}{\sqrt{1-x^2}}$ with normal equations.

Solution.

(a) Use above orthonormal system $T_0^*(x), T_1^*(x), T_2^*(x)$. Since

$$\begin{aligned}
\langle y, T_0^*\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{\pi}}dx = \frac{2}{\sqrt{\pi}} \\
\langle y, T_1^*\rangle &= \int_{-1}^{1} \frac{x}{\sqrt{\pi/2}}dx = 0 \\
\langle y, T_2^*\rangle &= \int_{-1}^{1} \frac{2x^2-1}{\sqrt{\pi/2}}dx = \frac{2}{3}\sqrt{\frac{2}{\pi}}.
\end{aligned}$$

The quadratic polynomial $\hat{y}(x) = \frac{2}{\sqrt{\pi}} \times \frac{1}{\sqrt{\pi}} + \frac{2}{3}\sqrt{\frac{2}{\pi}} \times \frac{2x^2-1}{\sqrt{\pi/2}} = \frac{10}{3\pi} - \frac{8x^2}{3\pi}$.

(b) Calculate the Gram matrix $G(1, x, x^2)$ and the $c = (\langle y, 1\rangle, \langle y, x\rangle, \langle y, x^2\rangle)^T$. And we have

$$
\begin{aligned}
&\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\pi} 1 dy = \pi \\
&\int_{-1}^{1} \frac{x^i}{\sqrt{1-x^2}} dx = 0 \qquad i \text{ is odd} \\
&\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos^2 y dy = \frac{\pi}{2} \\
&\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos^4 y dy = \frac{3\pi}{8} \\
&\int_{-1}^{1} 1 dx = 2 \\
&\int_{-1}^{1} x dx = 0 \\
&\int_{-1}^{1} x^2 dx = \frac{2}{3}.
\end{aligned}
$$

After all, solve the equations get $\hat{y}(x) = \frac{10}{3\pi} - \frac{8x^2}{3\pi}$.

Problem 8.4. Discete least square via orthonormal polynomials. Consider the example on the table of sales record in the notes.

(a) Starting from the independent list $(1, x, x^2)$, construct orthonormal polynomials by the GramSchmidt process using

$$\langle u(t), v(t)\rangle = \sum_{i=1}^{N} \rho(t_i) u(t_i) v(t_i)$$

as the inner product with $N = 12$ and $\rho(x) = 1$.

(b) Find the best approximation $\hat{\varphi} = \sum_{i=0}^{2} a_i x^i$ such that $\|y - \hat{\varphi}\| \le \|y - \sum_{i=0}^{2} b_i x^i\|$ for all $b_i \in \mathbb{R}$. Verify that $\hat{\varphi}$ is the same as that of the example on the table of sales record in the notes.

(c) Suppose there are other tables of sales record in the same format as that in the example. Values of N and x_i's are the same, but the values of y_i's are different. Which of the above calculations can be reused? Which cannot be reused? What advantage of orthonormal polynomials over normal equations does this reuse imply?

Solution.

(a) normalize $(1, x, x^2)$,by definition of GramSchmidt,

$$
\begin{aligned}
v_1 &= u_1 = 1 \\
u_1^* &= v_1/\|v_1\| = \frac{1}{\sqrt{12}} \\
v_2 &= u_2 - \langle u_2, u_1^*\rangle u_1^* = x - \frac{12}{2} \\
u_2^* &= v_2/\|v_2\| = \frac{2x - 13}{2\sqrt{143}} \\
v_3 &= u_3 - \langle u_3, u_2^*\rangle u_2^* - \langle u_3, u_1^*\rangle u_1^* = x^2 - 13x + \frac{91}{3} \\
u_3^* &= v_3/\|v_3\| = \frac{3x^2 - 39x + 91}{\sqrt{12012}}.
\end{aligned}
$$

(b) Calculate the coefficients of Fourier expansion $\langle y, u_1^* \rangle = 277\sqrt{3}, \langle y, u_2^* \rangle = \frac{589}{\sqrt{143}}, \langle y, u_3^* \rangle = \frac{36204}{\sqrt{12012}}$. Therefore $\hat{y}(x) = \sum_{i=1}^{3} \langle y, u_i^* \rangle u_i^* \approx 9.04x^2 - 113.43x + 386.00$.

(c) The normalize system can be reused while y_i's are different and x_i are the same, Since that's indenpedent with y_i. All of the Fourier expansion's coefficients have to be calculated again. The advantage is haven't to solve the equations.

8.1 Program

Write a **C++** function to perform discrete least square via normal equations. Your subroutin should take two arrays x and y as the input and output three coefficients a_0, a_1, a_2 that determines a quadratic polynomial as the best fitting polynomial in the sense of least squares with the weight function $\rho = 1$.

Run your subroutine on the following data.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
y	2.9	2.7	4.8	5.3	7.1	7.6	7.7
x	3.5	4.0	4.5	5.0	5.5	6.0	6.5
y	7.6	9.4	9.0	9.6	10.0	10.2	9.7
x	7.0	7.5	8.0	8.5	9.0	9.5	10.0
y	8.3	8.4	9.0	8.3	6.6	6.7	4.1

make run will output the function that get from normal equations.