Chapter 9

Homework 21935004 谭焱

Problem 9.1. Use separation of variables to find a nontrivial solution u of the PDE

$$u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2} = 0 \qquad \text{in } \mathbb{R}^2.$$

(G. Aronsson, Maunscripta Math. 47 (1984), 133–151)

Solution. By separation of variables, Let $u(x_1, x_2) = a(x_1) \cdot b(x_2)$ and substitute u with ab, Since a, b is symmetric,

$$\begin{aligned} & a''(a')^2b^3 + (a')^2(b')^2ab = 0 \\ \implies \quad & -\frac{a''}{a} = \left(\frac{b'}{b}\right)^2 \\ \implies \quad & b = e^{x_2\sqrt{-\frac{a''}{a}}+C}. \end{aligned}$$

Since $\frac{b''}{b} = -\frac{a''}{a}$, combine with symmetric give b is a implicit function gived by $e^{x_2\sqrt{\frac{b''}{b}}+C} - b = 0$. So does a, therefore $u = ab$, a, b is gived by the impicit function.

Problem 9.2. Consider Laplace's equaiton $\Delta u = 0$ in $\mathbb{R}^2$, taken with the Cauchy data

$$u = 0, \frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1) \qquad \text{on } \{x_2 = 0\}.$$

Employ separation of variables to derive the solution

$$u = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2).$$

What happens to u as $n \to \infty$? Is the Cauchy problem for Laplace's equation well-posed? (This example is due to Hadamard.)

Solution. By separation of variables, Let $u(x_1, x_2) = a(x_1) \cdot b(x_2)$ and substitute u with ab gives

$$\begin{aligned} & a''b + ab'' = 0 \\ \implies \quad & \frac{a''}{a} = -\frac{b''}{b} = \lambda \end{aligned}$$

Assume $\lambda = -1$ and solves equation get $a = \frac{1}{n_1}\sin(n_1x_1) + C_1, b = \frac{1}{n_2}\sinh(n_2x_2) + C_2$ or cosh. In the other side, $u = 0, \frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1)$ while $x_2 = 0$, it's means that b is sinh, $C_1 = 0, C_2 = 0$ and $n_1 = n$. Let $n_2 = n$ get the solution $u = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2)$.

While $n \to \infty$, u still oscillation with the same period and amplitude come to infinity.

Problem 9.3. Find explicit formulas for v and σ, so that $u(x,t) := v(x-\sigma t)$ is a traveling wave solution of the nonlinear diffusion equation

$$u_t - u_{xx} = f(u),$$

where

$$f(z) = -2z^3 + 3z^2 - z.$$

Assume $\lim_{s\to\infty} v = 1, \lim_{s\to-\infty} v = 0, \lim_{s\to\pm\infty} v' = 0$. (Hint: Multiply the equation $v'' + \sigma v' + f(v) = 0$ by v' and intergrate, to determine the value of σ.)

Solution. In the nonlinear diffusion equation, substitute u with v gives

$$\begin{aligned}
& \sigma v' + v'' = 2v^3 - 3v^2 + v \\
\implies \quad & \sigma v'v' + v'v'' = (2v^3 - 3v^2 + v)v' \\
\implies \quad & \sigma \int (v')^2 dz + \frac{(v')^2}{2} = \frac{1}{2}v^4 - v^3 + \frac{1}{2}v^2 \\
& \textit{Since } \lim_{s\to\infty} v = 1, \lim_{s\to-\infty} v = 0, \lim_{s\to\pm\infty} v' = 0 \\
\implies \quad & \sigma \int_{-\infty}^{\infty} (v')^2 dz = 0 \\
\implies \quad & \sigma = 0 \\
& \text{Combining with the third equation} \\
\implies \quad & \frac{(v')^2}{v^2(v-1)^2} = 1 \\
\implies \quad & v = \frac{e^{x+C}}{1+e^{x+C}}
\end{aligned}$$

Problem 9.4. If we look for a radial solution $u(x) = v(r)$ of the nonlinear elliptic equation

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^n,$$

where $r = |x|$ and $p > 1$, we are led to the nonautonomous ODE

$$(*) \qquad v'' + \frac{n-1}{r}v' + v^p = 0.$$

Show that the *Emden-Fowler transformation*

$$t := \log r, x(t) := c^{\frac{2t}{p-1}} v(c^t)$$

converts $(*)$ into an autonomous ODE for the new unknown $x = x(t)$.

Solution. First, substitute r in $(*)$ with t gives

$$v''(e^t) + \frac{n-1}{e^t}v'(e^t) + v^p(e^t) = 0.$$

Calculate derivatives of $x(t)$ gives

$$\begin{aligned}
x(t) &= e^{\frac{2t}{p-1}\ln c} v(e^{t\ln c}) \\
x'(t) &= (v\frac{2\ln c}{p-1} + v'e^{t\ln c}\ln c)e^{\frac{2t}{p-1}\ln c} \\
x''(t) &= (v(\frac{2\ln c}{p-1})^2 + 2v'e^{t\ln c}\frac{2\ln^2 c}{p-1} + v''e^{2t\ln c}\ln^2 c + v'e^{t\ln c}\ln^2 c)e^{\frac{2t}{p-1}\ln c}
\end{aligned}$$

So that

$$
\begin{aligned}
v(e^{t\ln c}) &= e^{-\frac{2t}{p-1}\ln c}x \\
v'(e^{t\ln c}) &= e^{-\frac{2t}{p-1}\ln c}(x' - x\frac{2\ln c}{p-1})\frac{1}{e^{t\ln c}\ln c} \\
v''(e^{t\ln c}) &= e^{-\frac{2t}{p-1}\ln c}(x'' + x(\frac{4\ln^2 c}{(p-1)^2} + \frac{2\ln^2 c}{p-1}) - x'(\ln c + \frac{4\ln c}{p-1}))\frac{1}{e^{2t\ln c}\ln^2 c}
\end{aligned}
$$

Assume $c = e$, get

$$x'' + (n - 1 + \frac{3+p}{1-p})x' + (\frac{2(n-1)}{1-p} + \frac{2(1+p)}{(1-p)^2})x + x^p = 0.$$