

Chapter 12

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Problem 12.1. Utilize Lemma 2 in §4.5.3 to discuss the sense in which u defined by formula (20) in §4.3.1 converges to the initial data g as $t \rightarrow 0^+$.

Solution. From Lemma 2 in §4.5.3 know

$$\frac{1}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i}{2\varepsilon} y \cdot Ay} a(y) dy = \frac{e^{i\frac{\pi}{4} \operatorname{sgn} A}}{|\det A|^{1/2}} (a(0) + O(\varepsilon)).$$

Consider the formula (20) as $t \rightarrow 0$ gives

$$\begin{aligned} \lim_{t \rightarrow 0^+} u(x, t) &= \lim_{t \rightarrow 0^+} \frac{1}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} g(y) dy \quad \text{take } 2t = \varepsilon, z = y - x \\ &= \lim_{t \rightarrow 0^+} \left(\frac{e^{i\frac{n\pi}{4}}}{(i)^{n/2}} (g(x) + O(2t)) \right) \\ &= g(x). \end{aligned}$$

Problem 12.2. Let $n = 1$ and suppose that u^ε solves the problem

$$\begin{cases} -(a(\frac{x}{\varepsilon})u_x^\varepsilon)_x = f & \text{in } (0, 1) \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases}$$

where a is a smooth, positive function that is 1-periodic. Assume also that $f \in L^2(0, 1)$.

(a) Show that $u^\varepsilon \rightharpoonup u$ weakly in $H_0^1(0, 1)$, where u solves

$$\begin{cases} -\bar{a}u_{xx} = f & \text{in } (0, 1) \\ u(0) = u(1) = 0, \end{cases}$$

for $\bar{a} := (\int_0^1 a(y)^{-1} dy)^{-1}$.

(b) Check that this answer agrees with the conclusions (73), (74) in §4.5.4.

Solution.

(b) Simplify the first equation in formula (73) as $i, j = 1$ gives

$$\begin{aligned}
 -\left(\int_Q a(y) - a(y)\chi_y(y)dy\right)u_{xx} &= -\left(\int_Q a(y) - a(y)\chi_y(y)dy\right)\left(\int_0^1 a(y)^{-1}dy\right)f \\
 &= \frac{1}{\varepsilon} \int_Q (1 - \chi_y)dy f \quad \text{Since } \chi \text{ is } Q\text{-periodic} \\
 &= \frac{1}{\varepsilon}(\varepsilon - 0)f \\
 &= f.
 \end{aligned}$$

the second equation in formula (73) is obvious. So this answer agrees the conclusions.

Problem 12.3. (Variational principles in homogenization) Let $A(y) = ((a^{ij}(y)))$ be symmetric, positive definite and Q -periodic. Recall from §4.5.4 the expression (74) for the corresponding homogenized coefficients $\bar{A} = ((\bar{a}^{ij}))$.

(a) Derive for each $\xi \in \mathbb{R}^n$ the variational formula

$$\xi \cdot \bar{A}\xi = \min_w \left\{ \int_Q Dw \cdot A(y)Dw dy \mid w = y \cdot \xi + v, v \text{ } Q\text{-periodic} \right\}.$$

(Hint: The minimum is attained by $w = y \cdot \xi - \sum_{i=1}^n \xi_i \chi^i$, for the correctors χ^i introduced in §4.5.4.)

(b) Derive also the dual variational formula

$$\eta \cdot \bar{A}^{-1}\eta = \min_{\sigma} \left\{ \int_Q \sigma \cdot A(y)^{-1}\sigma dy \mid \int_Q \sigma dy = \eta, \operatorname{div} \sigma = 0, \sigma \text{ } Q\text{-periodic} \right\}.$$

(c) Show that therefore

$$\left(\int_Q A(y)^{-1}dy \right)^{-1} \leq \bar{A} \leq \int_Q A(y)dy.$$

(Remember from §A.1 that for symmetric matrices $R \geq S$ means $R - S$ is nonnegative definite.)

Solution.

(a) we have

$$\begin{aligned}
 \int_Q Dw \cdot A(y)Dw dy &= \int_{\partial Q} w(A(y)Dw) \cdot \nu dS - \int_Q w \operatorname{div}(A(y)Dw) dx \quad \text{Since } A(y), Dw \text{ is } Q\text{-periodic} \\
 &= \int_{\partial Q} (y \cdot \xi)(A(y)Dw) \cdot \nu dS - \int_Q (y \cdot \xi + v) \operatorname{div}(A(y)Dw) dx.
 \end{aligned}$$

However, if $\operatorname{div}(A(y)Dw) \neq 0$, we can let $v' = v + C$, where C is a constant, so that $-\int_Q (y \cdot \xi + v') \operatorname{div}(A(y)Dw) dx < -\int_Q (y \cdot \xi + v) \operatorname{div}(A(y)Dw) dx$. It means that minimum doesn't exist, so $\operatorname{div}(A(y)Dw) = 0$. Solves the equations get $w = y \cdot \xi - \sum_i \xi_i \chi^i$. Substitute w with w_0 get $\min_w \left\{ \int_Q Dw \cdot A(y)Dw dy \right\} = \xi \cdot \bar{A}\xi$.

(b) Let $\sigma = A(y)Dw$, From (a) we already know that $\operatorname{div} \sigma = \operatorname{div}(A(y)Dw) = 0$, $\int_Q \sigma dy = \bar{A}\xi$. Then from (a) know $\min_{\sigma} \left\{ \int_Q \sigma \cdot A(y)^{-1}\sigma dy \right\} = \eta \cdot \bar{A}\eta$.

(c) Let $w = y \cdot \xi$, we have $\xi \cdot \left(\int_Q A(y)dy - \bar{A} \right) \xi \geq 0$. Without specifically assume $\int_Q \eta = \eta$, then from (b) know $\eta \cdot \left(\int_Q A(y)^{-1} - \bar{A}^{-1} \right) \eta \geq 0$. From the definition of matrixs' compare get the inequality.