Chapter 1

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1.1 homework1

Exercise 1.1. 10. Rewrite each of the following statements and its *negation* into *logical statements* using symbols,quantifiers, and formula.

(a) The only even prime is 2.

(b) Multiplication of integers is associative.

(c) Goldbach's conjecture has at most a finite number os counterexamples.

Solution.

(a) $\exists x \in 2\mathbb{N}, x \in \mathbb{P}, s.t. x = 2$

$\forall x \in 2\mathbb{N}, x \in \mathbb{P}, s.t. x \neq 2$

(b) $\forall a, b, c \in \mathbb{Z}, s.t. (a \times b) \times c = a \times (b \times c)$

$\exists a, b, c \in \mathbb{Z}, s.t. (a \times b) \times c \neq a \times (b \times c)$

(c) $\exists \mathbb{S} \subset 2\mathbb{N}^{+}+2$ and,$|\mathbb{S}| \in \mathbb{N}^{+}, \forall a \in 2\mathbb{N}^{+}+2$ and, $a \notin \mathbb{S}, \exists p, q \in \mathbb{P}, s.t. a = p + q$.

$\forall \mathbb{S} \subset 2\mathbb{N}^{+}+2$ and,$|\mathbb{S}| \in \mathbb{N}^{+}, \exists a \in 2\mathbb{N}^{+}+2$ and, $a \notin \mathbb{S}, \forall p, q \in \mathbb{P}, s.t. a \neq p + q$.

Exercise 1.2. 28. On (a, ∞), $f(x) = \frac{1}{x^2}$ is uniformly continuous if $a > 0$ and is not so if $a = 0$.

Solution. $\forall x \in (a, \infty), f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} = -\frac{2x+\Delta x}{x^2(x+\Delta x)^2}$ as $\Delta x \to 0$, which is equal to $-\frac{2}{x^3} > -\frac{2}{a^3}$.So $f'(s)$ is bounded that is meaning $f(x)$ is uniformly continues.

However, When $a = 0$ $f'(x) = -\frac{2}{x^3}$, Let $x \to 0$,Then $f'(x) -> \infty$. So $f'(x)$ is bounded, equal ot $f(x)$ is not uniformly continues.

Exercise 1.3. 37. Let $\mathcal{X}$ be the set of all bounded and unbounded sequences of complex numbers. Show that the following is a metric on $\mathcal{X}$,

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}, \tag{1.1}$$

where $x = (\xi_j)$ and $y = (\eta_j)$.

Solution. $d(x, y) > 0$ is obvious from $\forall j \in \mathbb{N}, \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} > 0$.When $x = y, \xi_j = \eta_j, \forall j \in \mathbb{N}, \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} = 0 \Longrightarrow d(x, y) = 0$.. $\forall j \in \mathbb{N}, d(x, y) - d(y, x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} - \frac{1}{2^j} \frac{|\eta_j - \xi_j|}{1+|\eta_j - \xi_j|} = \sum_{j=1}^{\infty} 0 = 0$.Means $d(x, y) = d(y.x)$.

Set $x = (x_j), y = (y_j), z = (z_j)$.

$$\begin{aligned}
& d(x, y) + d(y, z) - d(x, z) \\
&= \sum_{j=0}^{\infty} \frac{1}{2^j} \left(\frac{|x_j - y_j|}{1 + |x_j - y_j|} + \frac{|y_j - z_j|}{1 + |y_j - z_j|} - \frac{|x_j - z_j|}{1 + |x_j - z_j|} \right) \\
&> \sum_{j=0}^{\infty} \\
& \begin{cases} \frac{1}{2^j} \left(\frac{|x_j - y_j|}{1+|x_j - y_j|} - \frac{|x_j - z_j|}{1+|x_j - z_j|} \right) > 0 & \text{if } |x_j - y_j| > |x_j - z_j| \\ \frac{1}{2^j} \left(\frac{|x_j - y_j| + |y_j - z_j|}{1+|x_j - z_j|} - \frac{|x_j - z_j|}{1+|x_j - z_j|} \right) = 0 & \text{if } |x_j - y_j| < |x_j - z_j| \end{cases} \\
&= 0
\end{aligned} \tag{1.2}$$

therefore, $d(x, y) + d(y, z) > d(x, z)$. In summary, $d(x, y)$ is a metric on $\mathcal{X}$.

Exercise 1.4. 38. Prove that (0.41) is indeed a metric. Inparticular, prove that (0.41) satisfies the triangular inequality by showing

(a) Lemma 0.61 implies the *Hölder inequality*, i.e., for conjugate exponents p, q and for any $(\xi_j) \in$

$\ell^p, (\eta_j) \in \ell^p$,

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} (\sum_{m=1}^{\infty} |\eta_m|^q)^{1/q} \tag{1.3}$$

(b) The Hölder inequality implies the *Minkowski inequality*, i.e. for any p, q and for any $p \ge 1, (\xi_j) \in \ell^p, (\eta_j) \in \ell^p$,

$$(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{1/p} \le (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} + (\sum_{m=1}^{\infty} |\eta_m|^p)^{1/p} \tag{1.4}$$

(c) The Minkowski inequality implies that the triangular inequality holds for (0.41).

Solution. Since $d(x, y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|)^{1/p} > (\sum_{j=0}^{\infty} 0)^{1/p} > 0$, the non-negativity is hold for $d(x, y)$. And $x = y \iff \xi_j = \eta_j \iff d(x, y) = (\sum_{j=0}^{\infty} 0)^{1/p} = 0$ means identity of indiscernibles true. Symmetry come from $|\xi_j - \eta_j| = |\eta_j - \xi_j|$.

Finally, the triangle inequality. take ln on both side of (1.3) get

$$ln(\sum_{j=1}^{\infty} |\xi_j - \eta_j|) \le \frac{ln(\sum_{k=1}^{\infty} |\xi_k|^p)}{p} + \frac{ln(\sum_{m=1}^{\infty} |\eta_m|^q)}{q} \tag{1.5}$$

Combining Jensen inequality and $\frac{1}{p} + \frac{1}{q} = 1$, The right part of inequality above using Lemma 0.61 have

$$\begin{aligned} Right &\ge ln(\frac{\sum_{k=1}^{\infty} |\xi_k|^p}{p} + \frac{\sum_{m=1}^{\infty} |\eta_m|^q}{q}) \\ &\ge ln(\sum_{j=1}^{\infty} |\xi_j \eta_j) = Left \end{aligned} \tag{1.6}$$

induct (1.3) true. Since $p+q = pq$, multiply $(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{1/q}$ both side of (1.4)

$$\begin{aligned} & Left \cdot (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{\frac{1}{q}} \\ &= \sum_{j=1}^{\infty} |\xi_j + \eta_j|^p \\ &= (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^{\frac{p+q}{q}}) \\ &= (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^{\frac{p}{q}} \cdot \sum_{j=1}^{\infty} |\xi_j + \eta_j|) \\ &\le (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{\frac{1}{q}} \cdot (\sum_{j=1}^{\infty} |\xi_j|^p)^{\frac{1}{p}} + \\ &\quad (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{\frac{1}{q}} \cdot (\sum_{j=1}^{\infty} |\eta_j|^p)^{\frac{1}{p}} \\ &= (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{\frac{1}{q}} \cdot Right, \end{aligned} \tag{1.7}$$

therefor, (1.4) hold.that is imply the triangle inequality is right.

In summary (0.41) is a metric.

Exercise 1.5. 56. Deduce *additivity in the second slot and conjugate homogeneity in the second slot* from Definition 0.87.

Solution.

- additivity in the second slot.
  $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \overline{\langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle} = \overline{\langle \mathbf{v}, \mathbf{u} \rangle} + \overline{\langle \mathbf{w}, \mathbf{u} \rangle} = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
- conjugate homogeneity in the second slot.
  $\langle \mathbf{u}, a\mathbf{v} \rangle = \overline{\langle a\mathbf{v}, \mathbf{u} \rangle} = \overline{a} \overline{\langle \mathbf{v}, \mathbf{u} \rangle} = \overline{a} \langle \mathbf{u}, \mathbf{v} \rangle$

Exercise 1.6. 62. In the case of Euclidean ℓ^p norms, show that the parallelogram law (0.72) holds if and only if $p = 2$.

Solution. Using $\mathbf{u} = (1, 0, 0, ...), \mathbf{v} = (0, 1, 0, 0, ...)$ replace $\mathbf{u}, \mathbf{v}$ in (0.72) get

$$2 + 2 = 2^{2/p}, \tag{1.8}$$

it's trivial that the equation come true if and only if $p = 2$.

Exercise 1.7. 63. Prove That the induced norm (0.63) holds for some inner product $\langle \cdot, \cdot \rangle$ if and only if the parallelogram law (0.72) hods for every pair of $u, v \in \mathcal{V}$.

Solution.

- necessary from Theorem 0.101 already prove.
- sufficient

  Assuming (0.72) is right, then prove that exist a $\langle\cdot,\cdot\rangle$ satisfy $\langle \mathbf{u},\mathbf{u}\rangle = \|\mathbf{u}\|$.

$$
\begin{aligned}
&\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle \\
&= \langle \mathbf{u}+\mathbf{v},\mathbf{v}\rangle - \|\mathbf{v}\|^2 + \langle \mathbf{v},\mathbf{u}\rangle \\
&= \langle \mathbf{u}+\mathbf{v},\mathbf{v}+\mathbf{u}\rangle - \langle \mathbf{u}+\mathbf{v},\mathbf{u}\rangle - \|\mathbf{v}\|^2 + \langle \mathbf{v},\mathbf{u}\rangle \\
&= \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2
\end{aligned}
\tag{1.9}
$$

  Set $\mathbf{u},\mathbf{v} = 0, (0.72) \iff 4\|\mathbf{0}\|^2 = 2\|\mathbf{0}\|^2 \iff \|\mathbf{0}\|^2 = 0$ $\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 \geq 0$ Let $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle = \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2/2$.

  So real posititvity is trivial for root. if $\exists \mathbf{v}, s.t. \langle \mathbf{v},\mathbf{v}\rangle = 0 \iff \|\mathbf{v}\|^2 = 0$

Exercise 1.8. Using the contents in Section 0.4, tell a story about determinanta from the viewpoint of problem-driven abstraction. You get no points unless your story contains all of the following.

(1) Why do we need the concept of a determinant?

(2) What is the geometric meaning of determinant?

(3) How is the development of mathematical abstraction parallel to the geometric meaning of determinants?

(4) How is the sign of the signed volume captured?

(5) What are the partial or linearing orderings of various concepts related to determinants?

Solution. 随着研究中三维空间推广到 n 维向量空间 **S**. 需要定义 n 维空间中的体积是如何计算的，并且保持在二维，三维空间中体积的性质. 于是有了行列式计算. 行列式的值就是 n 维空间以行列式 n 行中每一个 n 维坐标和原点坐标共 n + 1 个点组成的 n 维空间体的体积. 行列式中的加减就是固定 n - 1 个坐标和原点形成的 n - 1 维子空间 **D** 外后，最后一个点到 **D** 的距离加减，此时距离有正负值分别对应于最后一个点在 **D** 的两个方向，在 **D** 上时为 0 ，此时同是产生了负值体积，即有向体积. 定义了初始的加作为两元运算后，可得行列式是交换群. 还可定义乘法为一个两元运算，此时行列式是一个非交换群. 并且非零行列式值的所有矩阵构成矩阵子群. 并且存在一些特殊矩阵行列式构成轨道.