Chapter 15

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In these exercises U always denotes an open subset of $\mathbb{R}^n$, with a smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

Problem 15.1. (a) Integrate by parts to prove

$$\|Du\|_{L^p} \leq C\|u\|_{L^p}^{1/2} \left\|D^2u\right\|_{L^p}^{1/2}$$

for $2 \leq p < \infty$ and all $u \in C_c^\infty(U)$

$$\left(\text{Hint: } \int_U |Du|^p dx = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^{p-2} dx.\right)$$

(b) Prove

$$\|Du\|_{L^{2p}} \leq C\|u\|_{L^\infty}^{1/2} \left\|D^2u\right\|_{L^p}^{1/2}$$

for $1 \leq p < \infty$ and all $u \in C_c^\infty(U)$

Solution.

(a)

$$\begin{aligned}
\|Du\|_{L^p}^p &= \int_U |Du|^p \, dx \\
&= \sum_{i=1}^n \int_U u_{x_i} u_{x_i} \left|Du\right|^{p-2} dx \\
&= -\sum_{i=1}^n \int_U u u_{x_i x_i} \left|Du\right|^{p-2} + (p-2) u u_{x_i} \left|Du\right|^{p-4} Du \cdot Du_{x_i} dx \\
&= -\int_U u \Delta u \left|Du\right|^{p-2} + (p-2) u Du \cdot D^2 u Du^T \left|Du\right|^{p-4} dx.
\end{aligned}$$

And, by Hölder inequality

$$\begin{aligned}
\int_U -u\Delta u \left|Du\right|^{p-2} dx &\leq \|u\|_{L^p} \|\Delta u\|_{L^p} \|Du\|_{L^p}^{p-2} \\
&\leq C\|u\|_{L^p} \|D^2 u\|_{L^p} \|Du\|_{L^p}^{p-2}. \\
\int_U -(p-2) u u_{x_i} \left|Du\right|^{p-3} \operatorname{sgn}(Du) Du_{x_i} dx &\leq \int_U (p-2) \left|u\right| \left|Du\right| \left|D^2 u\right| \left|Du\right| \left|Du\right|^{p-4} dx \\
&= (p-2) \int_U \left|u\right| \left|D^2 u\right| \left|Du\right|^{p-2} dx \\
&\leq (p-2) C \|u\|_{L^p} \|D^2 u\|_{L^p} \|Du\|_{L^p}^{p-2}.
\end{aligned}$$

Finally, we have $\|Du\|_{L^p}^p \leq (p-1)C\|u\|_{L^p}\|D^2u\|_{L^p}\|Du\|_{L^p}^{p-2} \implies \|Du\|_{L^p} \leq (p-1)C\|u\|_{L^p}^{\frac{1}{2}}\|D^2u\|_{L^p}^{\frac{1}{2}}$

(b) by Hölder inequality

$$
\begin{aligned}
\|Du\|_{L^{2p}}^{2p} &= \int_U |Du|^{2p}\, dx \\
&= \int_U Du \cdot Du\, |Du|^{2p-2}\, dx \\
&= -\int_U u\Delta u\, |Du|^{2p-2}\, dx \\
&\leq \|u\|_{L^\infty} \left| \int_U \Delta u\, |Du|^{2p-2}\, dx \right| \\
&\leq \|u\|_{L^\infty} \|\Delta u\|_{L^p} \|Du\|_{L^{2p}}^{2p-2} \\
\implies \qquad \|Du\|_{L^{2p}} &\leq C\|u\|_{L^\infty}^{\frac{1}{2}}\|D^2u\|_{L^p}^{\frac{1}{2}}.
\end{aligned}
$$

Problem 15.2. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U$$

Prove u is constant a.e. in U

Solution. For any bounded subset $V \in U$,we have $u \in W^{1,p}(V)$ satisfies $Du = 0 \quad$ a.e. in V, assume $\int_V 1dx = vol(V) < \infty$. And we have $u \in C^\infty(V)$,such that $u_m \to u$ in $W^{1,p}(V)$.

therefore, $\forall \varepsilon > 0, \exists m, s.t. \int_V |Du_m - Du|^p\, dx < \varepsilon$. Let $C_m = u_m(x), x \in V$, Then

$$
\begin{aligned}
\int_V |u - C_m|^p\, dx &\leq C \int_V |u - u_m|^p + |u_m - C_m|\, dx \\
&\leq C(\varepsilon + \varepsilon vol(V)) \\
&\to 0 \qquad \text{as } m \to \infty.
\end{aligned}
$$

However,

$$
\begin{aligned}
\|u_m - u_n\|_{W^{1,p}} &\geq \int_V |u_m - u_n| \\
&\geq |C_m - C_n|\, vol(V) - \int_V \int_V |Du_m| + |Du_n|\, dxdy \\
&\geq |C_m - C_n|\, vol(V) - 2vol(V)^2\varepsilon
\end{aligned}
$$

, so $C_m - C_n \to 0$, as $m, n \to \infty$. Take $C_m \to C$, and $v(x) = C$ satisfies $v(x) \in W^{1,p}(V)$. By $\lim_{m\to\infty} \|u - C_m\|_{W^{1,p}(V)} = \lim_{m\to\infty} \|u - v(x)\|_{W^{1,p}(V)} \to 0$ and $W^{1,p}(V)$ is Banach space, that means $u = v = C$ a.e. in V.Since V is any bounded subset of U, we get $u = C$ a.e. in U.

Problem 15.3. Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$ such that u is not Lipschitz continuous on U. (Hint: Take U to be the open unit disk in $\mathbb{R}^2$, with a slit removed.)

Solution. Choose $U = \overset{o}{B}(0,1) - \{(x,y) \in \overset{o}{B}(0,1) \mid x \geq 0, y = 0\}$, $u(x) = \begin{cases} x^2 & x \geq 0, y > 0 \\ 0 & otherwise \end{cases}$. Then the weak derivative of u is $Du(x,y) = \begin{cases} (2x, 0) & x \geq 0, y > 0 \\ (0,0) & otherwise \end{cases}$., since $\forall \phi \in C_c^\infty(U) \to \forall (x,y) \in \{(x,y) \mid x \geq$

$0, y = 0\} \cup \partial B(0,1), \phi(x,y) = 0$

$$\begin{aligned}
&\int_U uD\phi + Du\phi dx \\
&= \int_{\partial B} u\phi dx + \int_0^1 \lim_{y\to 0^+} u(x,y)\phi(x,y)dx - \int_0^1 \lim_{y\to o^-} u(x,y)\phi(x,y)dx \\
&= 0.
\end{aligned}$$

u, Du is bounded implies $u \in W^{1,\infty}(U)$.However u is not Lipschitz continuous, since

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{|u(1/2,\varepsilon) - u(1/2), -\varepsilon|}{|\varepsilon + \varepsilon|} \\
&= \lim_{\varepsilon\to 0} \frac{1}{8|\varepsilon|} \\
&\to +\infty.
\end{aligned}$$

Problem 15.4. Verify that if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$

Solution. It's solved with two parts.

- $Du \in L^n(U)$.

$$\begin{aligned}
\partial_i u(x) &= \frac{1}{\log(1+\frac{1}{|x|})} \frac{1}{1+\frac{1}{|x|}} \frac{1}{|x|^2} \frac{x_i}{|x|} \\
&= -\frac{1}{\log(1+\frac{1}{|x|})} \frac{1}{|x|+1} \frac{x_i}{|x|^2} \\
\implies \qquad & |Du| \le C \frac{1}{|x|} \frac{1}{\log(1+\frac{1}{|x|})} \\
\implies \qquad & \int_{B^0(0,1)} |Du|^n \, dx \le C \int_0^1 \left(\frac{1}{\log(1+\frac{1}{|x|})}\right)^n \cdot \frac{1}{\rho^n} \cdot \rho^{n-1} d\rho \\
&\le C \int_{\log 2}^{\infty} \frac{1}{\delta^n} d\delta \qquad \text{take } \delta = \frac{1}{\log(1+\frac{1}{\rho})} \\
&\le +\infty.
\end{aligned}$$

- $u \in L^n(U)$.

$$
\begin{aligned}
\int_{B^0(0,1)} |u|^n \, dx &= \int_0^1 \left| \log\log(1+\frac{1}{\rho}) \right|^n \rho^{n-1} d\rho \\
&\leq \int_0^{\frac{1}{e-1}} \left| \log\log(1+\frac{1}{\rho}) \right|^n \rho^{n-1} d\rho + \int_{\frac{1}{e-1}}^1 \left| \log\log(1+\frac{1}{\rho}) \right|^n \rho^{n-1} d\rho \\
&:= I_1 + I_2. \\
I_2 &\leq \int_{\frac{1}{e-1}}^1 \left(\log(1+\frac{1}{\rho}) \right)^n \rho^{n-1} d\rho \\
&\leq \int_{\frac{1}{e-1}}^1 (\log 2)^n \rho^{n-1} d\rho \\
&\leq +\infty. \\
I_1 &= \int_{e-1}^{\infty} (\log\log(1+z))^n \frac{1}{z^{n+1}} dz \qquad z = \frac{1}{\rho} \\
&\leq_{e-1}^{\infty} \frac{1}{z^n} \\
&\leq +\infty.
\end{aligned}
$$