Chapter 5

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5.1 Problem

Exercise 5.1. Scaled integral of B-spline.

Deduce from the Theorem on derivatives of B-splines that the scaled integral of a B-spline $B_i^n(x)$ over its support is independent its index i even if the spacing of the knots is not uniform.

Solution. By derivatives of B-spline $\frac{d}{dx}B_i^n(x) = \frac{nB_i^{n-1}(x)}{t_{i+n-1}-ti-1} - \frac{nB_{i+1}^{n-1}(x)}{t_{i+n}-ti}$. Intergral both side in $[a,b]$ and $|a|,|b|$ big enough so that $B_i^n(a) = B_i^n(b) = 0$. It follows that

$$\begin{aligned}
0 &= B_i^n(a) - B_i^n(b)\\
&= \int_b^a \frac{d}{dx}B_i^n(x)dx\\
&= \frac{n}{t_{i+n-1}-t_{i-1}}\int_a^b B_i^{n-1}(x)dx + \frac{n}{t_{i+n}-t_i}\int_a^b B_{i+1}^{n-1}(x)dx.
\end{aligned}$$

That's same with $\frac{1}{t_{i+n-1}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n-1}} B_i^{n-1}(x)dx$ is independent with i's value.

Exercise 5.2. Symmetric Polynomials.

We have a theorem on expressing complete symmetric polynomials as divided difference of monomials.

(a) Verify this theorem for $m = 4$ and $n = 2$ by working out the table of divided difference and comparing the result to the dfinition of complete symmetric polynomials.

(b) Prove this theorem by the lemma on the recursive relation on complete symmetric polynomials.

Solution.

(a) Assume $i = 1$, $\tau_2(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$. And

$$\begin{aligned}
[x_1,x_2,x_3]x^4 &= ([x_1,x_2]x^4 - [x_2,x_3]x^4)(x_1 - x_3)\\
&= ((x_1+x_2)(x_1^2+x_2^2) - (x_3+x_2)(x_3^2+x_2^2))/(x_1-x_3)\\
&= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.
\end{aligned}$$

(b) From $\tau_{k+1}(x_1,\dots,x_n,x_{n+1}) = \tau_{k+1}(x_1,\dots,x_n)+x_{n+1}\tau_k(x_1,\dots,x_n,x_{n+1}) = \tau_{k+1}(x_2,\dots,x_{n+1})+x_1\tau_k(x_1,\dots,x_n,x_{n+1})$ And $\forall m, \tau_m(x_i) = [x_i]x^m$. Then proof it by induction. Assuming $n = k+1$ already have $\tau_{m-k+1}(x_i,\dots,x_{i+k-1}) =$

$[x_i, \dots, x_{i+k-1}]x^m$. Calculate τ_{m-k}

$$
\begin{aligned}
\tau_{m-k}(x_i, \dots, x_{i+k}) &= \frac{\tau_{m-k+1}(x_i, \dots, x_{i+k-1}) - \tau_{m-k+1}(x_{i+1}, \dots, x_{i+k})}{x_i - x_{i+k}} \\
&= \frac{[x_i, \dots, x_{i+k-1}]x^m - [x_{i+1}, \dots, x_{i+k}]x^m}{x_i - x_{i+k}} \\
&= [x_i, \dots, x_{i+k}]x^m.
\end{aligned}
$$

So that from induction, It's true for $n \in [0, m]$.

5.2 Program

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. Implement two modules to interpolate f by the quadratic and cubic cardinal B-splines, which corresponds to the two corollaries in the notes on unique interpolation by quadratic B-splines and complete cubic cardinal B-splines, respectively.

(b) Run your subroutines on the function

$$
f(x) = \frac{1}{1+x^2}, \qquad x \in [-5, 5],
$$

using $t_i = i - \frac{11}{2}, i = 1, \dots, 10$ for quadratic B-splines and $t_i = -6 + i, i = 1, \dots, 11$ for complete cubic cardinal B-splines. Plot the polynomials against the exact function.

(c) Define $E_S(x) = |S(x) - f(x)|$ as the interpolation error. For the two cardinal B-spline interpolants, what are the values of $E_S(x)$ at the sites $x = -3.5, -3, -0.5, 0, 0.5, 3, 3.5$? Program in **C++** to output these values. Why are some of the errors close to machine precision? Which of the two B-splines is more accurate?

(d) The roots of the following equation constitute a closed planar curve in the shape of a heart:

$$
x^2 + \left(\frac{3}{2}y - \sqrt{|x|}\right)^2 = 3.
$$

Modify your **C++** subroutines in the previous homework to plot the heart. As discussed in the lectures, the parameter of the curve should be the *cumulative chordal length* defined from n given points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ as

$$
t_i = \begin{cases} 0, & i = 1 \\ t_{i-1} + \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, & i > 1. \end{cases}
$$

Choose $n = 10, 40, 160$ and produce three plots of the heart function.

Solution.

(a) *make run* will complie and run *main.ex*. Output file will be in code/output.

(b) *make print* will plot the quadratic B-spline and the cubic cardinal B-spline. Or using the **matlab** file *p.m* and *plotTruncatedPowerFunc2Bsplines.m* to plot (5.1) (5.2).

(c) *make run* will output $E_S(x)$ for the two cardinal B-spline by function *printerror*. You can Adjustment parameters to computer other points' error. And for quadratic B-spline interpolation, it's error in $-3.5, -0.5, 0.5, 3.5$ is close to machine precision, since these is its interpolate points and B-spline meet $\sum_{i=-\infty}^{+\infty} B_i^n = 1, \forall n \in \mathbb{N}$. As for the same reason, cubic B-spline interpolation's error close to 0 in $-3, 0, 3$.

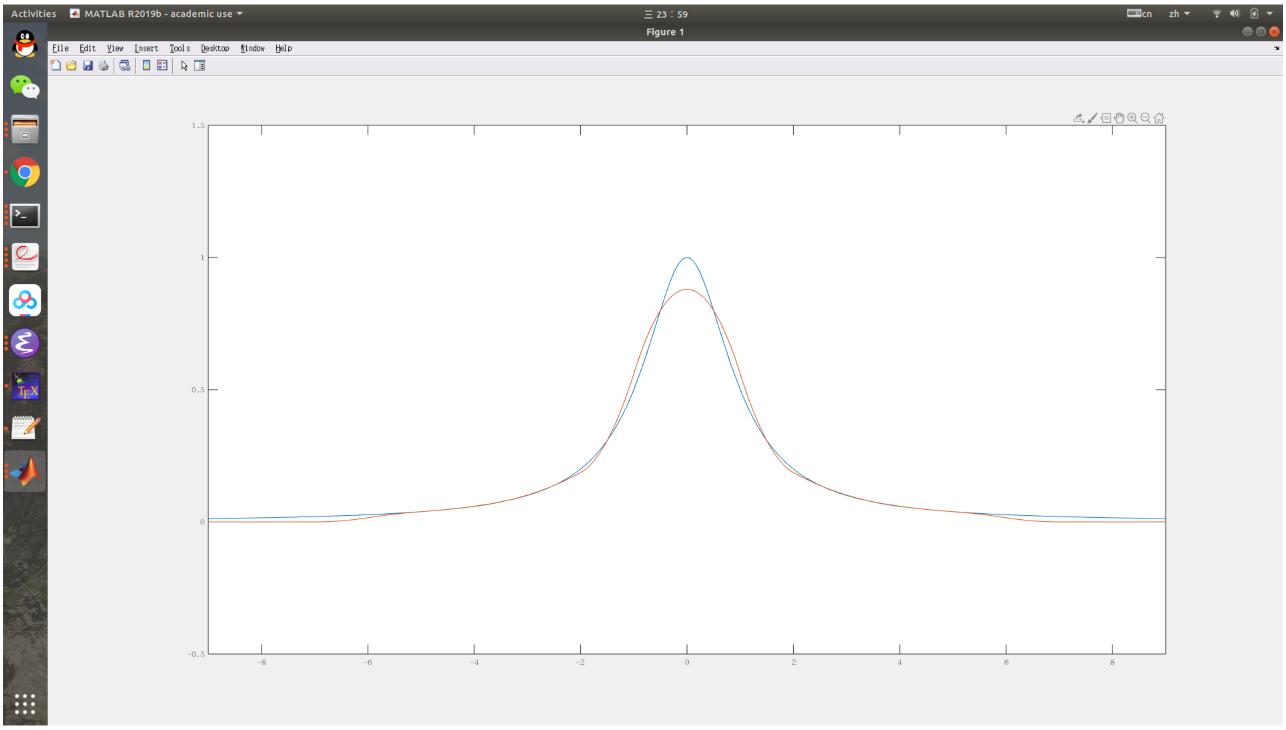

图 5.1: quadratic B-spline

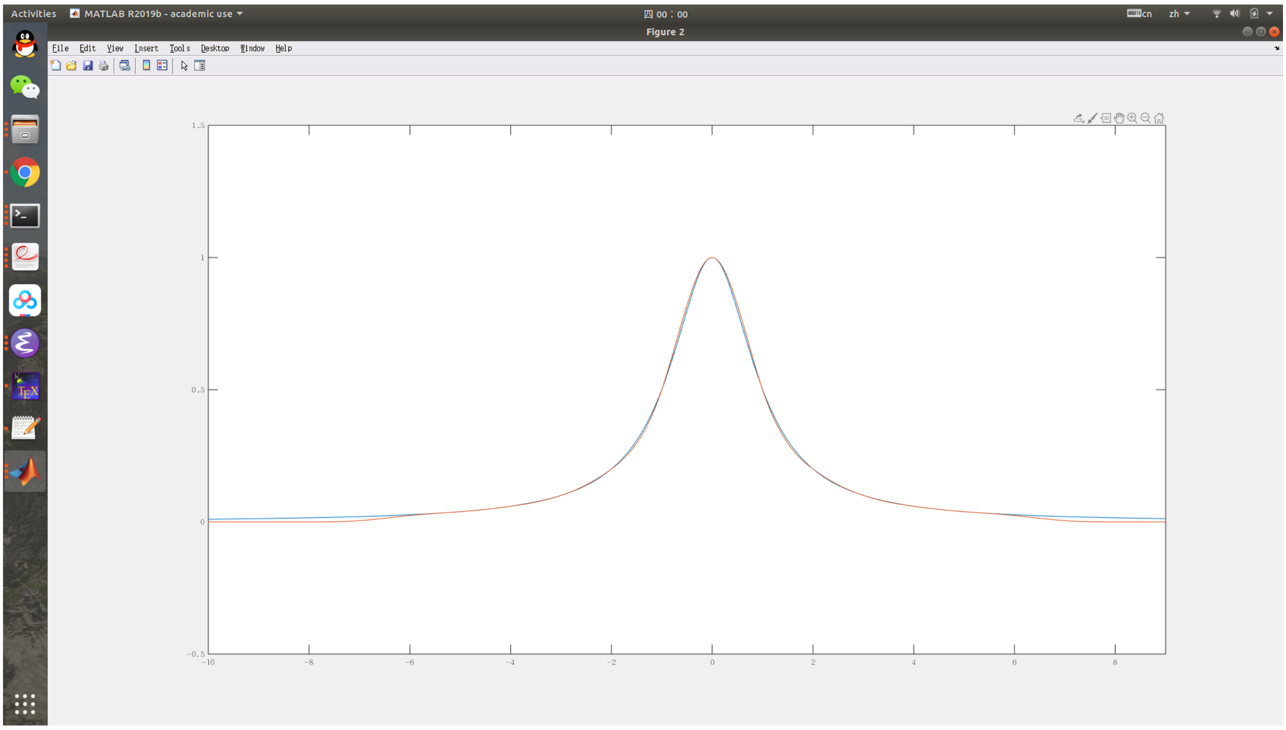

图 5.2: cubic B-spline

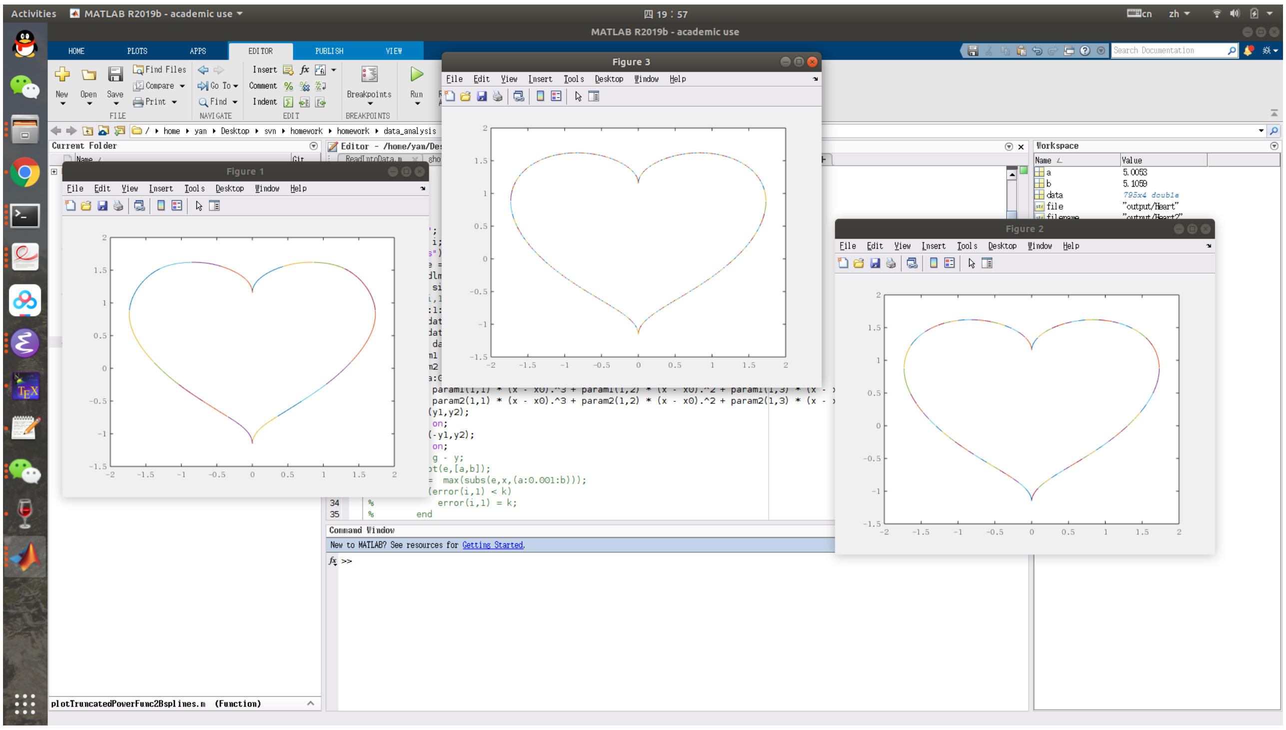

图 5.3: LovingHeart

(d) *make run* will output interpolations in *output/*. Using *Heartplot.m* can plot. Or *make heart*. Figure like (5.3).