Chapter 7

Homework 21935004 谭焱

Problem 7.1. If the bisection method is used in single precision FPNs of IEEE 754 starting with the interval $[128, 129]$, can we compute the root with absolute accuracy $< 10^{-6}$? Why?

Solution. We can't do that. Since the unit roundoff ϵ_u of single precision FPNs of IEEE 754 is $\frac{1}{2} \times 2^{-23} > 1 \times 10^{-8} > \frac{10^{-6}}{128}$.

Problem 7.2. What are the condition numbers of the following functions? Where are they large?

- $(x-1)^\alpha$,
- $\ln x$,
- e^x,
- $\arccos x$.

Solution.

- $C_{(x-1)^\alpha}(x) = \left|\frac{x \times \alpha(x-1)^{\alpha-1}}{(x-1)^\alpha}\right| = \left|\frac{\alpha x}{x-1}\right|$. And while $x \to 1, C_{(x-1)^\alpha}(x) \to +\infty$.
- $C_{\ln x}(x) = \left|\frac{1}{\ln x}\right|$. And while $x \to 1, C_{\ln x}(x) \to +\infty$.
- $C_{e^x}(x) = \left|\frac{xe^x}{e^x}\right| = |x|$. And while $x \to \pm\infty, C_{e^x}(x) \to +\infty$.
- $C_{\arccos x}(x) = \left|\frac{x}{\sqrt{1-x^2}\arccos x}\right|$. And while $x \to \pm 1, C_{\arccos x}(x) \to +\infty$.

Problem 7.3. The last Exercise in Section 1.3.5 in the notes.

Solution. Assume that $\sin x, \cos x$ are computed with relative error within machine roundoff. Since $\cos x >$

$0, x \in (0, \pi/2)$ Then

$$\begin{aligned}
f_A &= \mathrm{fl}\left[\frac{\mathrm{fl}(\sin \mathrm{x})}{\mathrm{fl}(1+\mathrm{fl}(\cos \mathrm{x}))}\right] \\
&= \frac{\sin x(1+\delta_1)}{(1+\cos x(1+\delta_2))(1+\delta_3)}(1+\delta_4) \\
&= f(x)\frac{(1+\cos x)}{1+\cos x(1+\delta_2)}\frac{(1+\delta_1)(1+\delta_4)}{(1+\delta_2)(1+\delta_3)} \\
&\approx f(x)(1+\delta_1-\delta_3+\delta_4+\frac{\cos x}{1+\cos x(1+\delta_2)}\delta_2) \\
&\leq f(x)(1+(3+\frac{\cos x}{1+\cos x(1+\delta)})\epsilon_u) \\
\mathrm{cond}_f(x) &= \left|\frac{x}{\sin x}\right| = \frac{x}{\sin x} \\
\mathrm{cond}_A(x) &\leq \frac{(3+\frac{\cos x}{1+\cos x})\sin x}{x}
\end{aligned}$$

Problem 7.4. Consider the function $f(x) = 1 - e^{-x}$ for $x \in [0, 1]$.

- Show that $\mathrm{cond}_f(x) \leq 1$ for $x \in [0, 1]$.
- Let A be the algorithm that evaluates $f(x)$ for the machine number $x \in \mathbb{F}$. Assume that the exponential function is computed with relative error within machine roundoff. Estimate $\mathrm{cond}_A(x)$ for $x \in [0, 1]$.
- Use **C++** to plot $\mathrm{cond}_f(x)$ and $\mathrm{cond}_A(x)$ as a function of x on $[0, 1]$. Discuss your results.

Solution.

- Calculate $\mathrm{cond}_f(x) = \frac{xe^{-x}}{1-e^{-x}} \leq xe^{-x} \leq x \leq 1, x \in [0, 1]$
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$$\begin{aligned}
f_A &= \mathrm{fl}(1-\mathrm{fl}(e^{-x})) \\
&= (1-e^{-x}(1+\delta_1))(1+\delta_2) \\
&\approx f(x)(1+\frac{e^{-x}}{1-e^{-x}}\delta_1+\delta_2) \\
&\leq f(x)(1+(1+\frac{e^{-x}}{1-e^{-x}})\epsilon_u) \\
\mathrm{cond}_A(x) &\leq \frac{1}{xe^{-x}}.
\end{aligned}$$
- $\mathrm{cond}_f(x)$ from 1 decrease as x increase, and $\mathrm{cond}_A(x)$ decrease from $+\infty$, Since as $x \to 0, f(x) = 1-e^{-x} \to 0$, this will conduct catastrophic cancellation.

Problem 7.5. The math problem of root finding for a polynomial

$$q(x) = \sum_{i=0}^{n} a_i x^i, \qquad a_n = 1, a_0 \neq 0, a_i \in \mathbb{R}$$

can be considered as a vector function $f \colon \mathbb{R}^n \to \mathbb{C}$:

$$r = f(a_0, a_1, \ldots, a_{n-1}).$$

Derive the componentwise condition number of f based on the 1-norm. For the Wilkinson example, compute your condition number, and compare your result with in the Wilkinson Example. What does the comparison tell you?

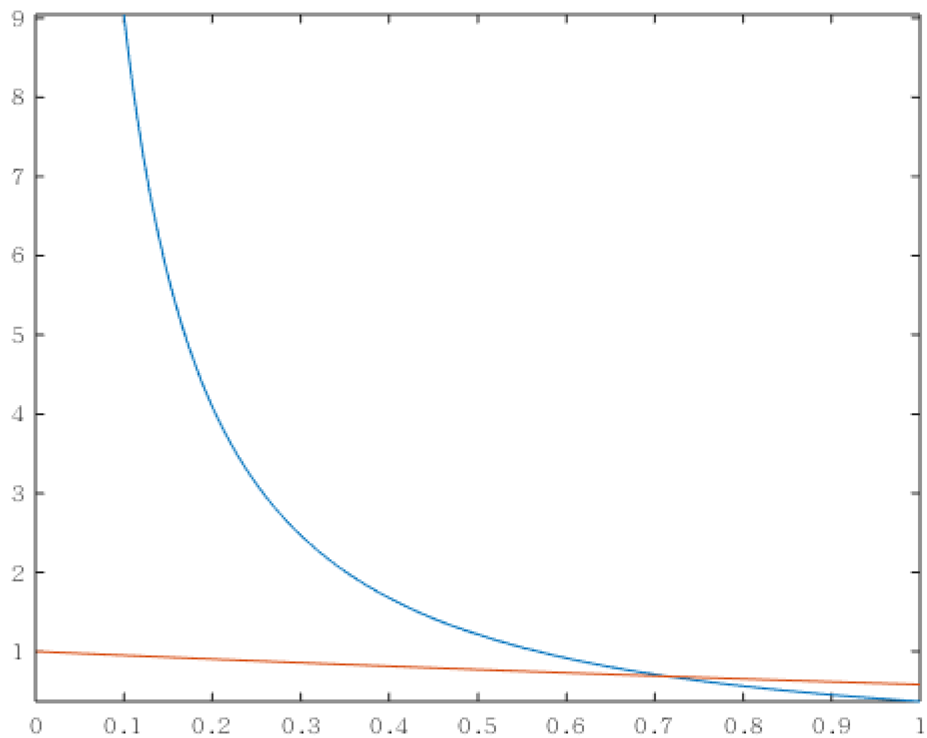

图 7.1: condition function

Solution. By definition of the componentwise condition number

$$a_{ij}(x) = \left| \frac{a_j \frac{\partial f}{\partial a_j}}{f(x)} \right| = \left| \frac{a_j}{r} \frac{dr}{da_j} \right|.$$

Since $q(r) = 0$, So dr, da_j fill the $q_x dr + r^j da_j = 0$. So

$$\begin{aligned} \text{cond}_f(x) &= \|A(x)\|_1 \\ &= \max\{\left| \frac{a_j r^{j-1}}{q_x(r)} \right|\}. \end{aligned}$$

For Wilkinson example, replace $q(x)$ with $f(x) = \prod_{k=1}^{n}(x-k)$ gives

$$\begin{aligned} \text{cond}_f(x) &= \|A(x)\|_1 \\ &= \max\{\left| \frac{a_j n^{j-1}}{(n-1)!} \right|\} \\ &\geq (1+2+\cdots+n)n^{n-2}/(n-1)! \qquad \text{take } j = n-1 \\ &= \frac{n+1}{2n} \frac{n^n}{(n-1)!} \\ &\to +\infty. \end{aligned}$$

The condition number is the same magnitude with Wilkinson Example.