Exercise 1.5.2

For leap-frog scheme, we have

$$\begin{aligned}(\tau_{LF})_j^n =& \frac{u_j^{n+1}-u_j^{n-1}}{2k} - D_0 u_j^n \\ =& u_t(x_j,t_n) + \frac{k^2}{6}u_{ttt}(x_j,t_n) + O(k^4) - \left(u_x(x_j,t_n) + \frac{h^2}{6}u_{xxx}(x_j,t_n) + O(h^4)\right) \\ =& \frac{k^2-h^2}{6}u_{xxx}(x_j,t_n) + O(k^4+h^4).\end{aligned}$$

For Crank-Nicholson scheme, we have

$$\begin{aligned}(\tau_{CN})_j^n =& \frac{u_j^{n+1}-u_j^n}{k} - D_0\left(\frac{u_j^{n+1}+u_j^n}{2}\right) \\ =& \left(u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt}\right)_{(x_j,t_n)} - \frac{1}{2}\left(u_x + \frac{h^2}{6}u_{xxx}\right)_{(x_j,t_n)} - \frac{1}{2}\left(u_x + \frac{h^2}{6}u_{xxx}\right)_{(x_j,t_{n+1})} \\ &+O(k^3+h^4) \\ =& \left(u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt}\right)_{(x_j,t_n)} - \frac{1}{2}\left(u_x + \frac{h^2}{6}u_{xxx} + u_x + ku_{xt} + \frac{k^2}{2}u_{xtt} + \frac{h^2}{6}u_{xxx}\right)_{(x_j,t_n)} \\ &+O(h^2k+k^3+h^4) \\ =& -\frac{k^2+2h^2}{12}u_{xxx}(x_j,t_n) + O(h^3+k^3).\end{aligned}$$

Therefore both schemes are accurate of $(2,2)$.

By analyzing the principal term for each τ_j^n, it can be expected that the leap-frog scheme is more accurate than the Crank-Nicholson scheme. The CFL condition ensures that $\lambda = \frac{k}{h} < 1$, and we have $\alpha_{LF} = \left|\frac{k^2-h^2}{6}\right| = \frac{h^2}{6} - \frac{k^2}{6}$ and $\alpha_{CN} = \left|-\frac{k^2+2h^2}{12}\right| = \frac{h^2}{6} + \frac{k^2}{12}$. Since $\alpha_{LF} < \alpha_{CN}$, the leap-frog scheme is expected to introduce less pointwise error than the Crank-Nicholson scheme and thus be more accurate.

Exercise 1.6.3

Consider the LTE of the θ scheme:

$$
\begin{aligned}
\tau_j^{n+\frac{1}{2}} &= \frac{u^{j+1} - u^j}{k} - \frac{\theta}{h^2} D_+ D_- u_j^{n+1} - \frac{1-\theta}{h^2} D_+ D_- u_j^n \\
&= \left(u_t + \frac{k^2}{24} u_{ttt} \right)_{(x_j, t_n + k/2)} - (1-\theta) \left(u_{xx} + \frac{h^2}{12} u_{xxxx} \right)_{(x_j, t_n)} - \theta \left(u_{xx} + \frac{h^2}{12} u_{xxxx} \right)_{(x_j, t_{n+1})} \\
&\quad + O(k^4 + h^4) \\
&= \left(\left(\frac{1}{2} - \theta \right) k - \frac{h^2}{12} \right) u_{xxxx}(x_j, t_n + k/2) + \frac{1}{12} \left(\frac{1}{2} - \theta \right) k h^2 u_{xxxxxx}(x_j, t_n + k/2) \\
&\quad + O(k^2 + h^4).
\end{aligned}
$$

For backward-Euler scheme, $\theta = 1$ and $\tau_{BE} = O(h^2 + k)$, while for Crank-Nicholson, $\theta = \frac{1}{2}$ and $\tau_{CN} = O(h^2 + k^2)$.

However, Crank-Nicholson scheme shows oscillatory error in comparison to backward-Euler scheme for the given example. This is because, for large σ, $\widehat{Q}_{CN}$ is close to -1 for $\xi \neq 0$ and there's little damping. In contrast, $\widehat{Q}_{BE}$ tends to zero as σ approaches infinity.

Therefore, as $h \to 0$, we have $\sigma = O(\frac{1}{h}) \to \infty$, and $\widehat{Q}_{BE}$ provides better damping than $\widehat{Q}_{CN}$. This leads to a reduction in the oscillating error produced by the backward-Euler scheme compared to the Crank-Nicholson scheme.

Exercise 1.7.1

We investigate the convection-diffusion equation $u_t + a u_x = \eta u_{xx}$ with initial condition $f(x) = \sin(x) + \sin(4x)$, where $a = \eta = 1$.

To test the numerical stability, we choose three space step sizes $h = 2\pi/N$ with $N \in \{10, 20, 40\}$ and different time step sizes k that satisfy either $\alpha = 1$ or $|\lambda| = 1$. The results of these tests are presented below.

We conclude that both the parabolic stability condition $\alpha \leq 1$ and the CFL condition $|\lambda| \leq 1$ are necessary for numerical solution stability. As shown in the figures, the left subfigures exhibit natural behavior since both conditions are satisfied, whereas the right subfigures display unbounded oscillations due to the violation of the parabolic stability condition.

In our tests, where $\eta = \Theta(a)$, the parabolic term provides adequate damping when $\alpha \leq 1$. Additionally, $\alpha \leq 1$ implies $k = O(h^2/\eta)$, which leads to

$$\lambda = ak/h = \Theta(\eta k/h) = O(h) \to 0.$$

Hence, when $\eta = \Theta(a)$, the parabolic stability condition is sufficient (but not necessary) for the CFL condition.

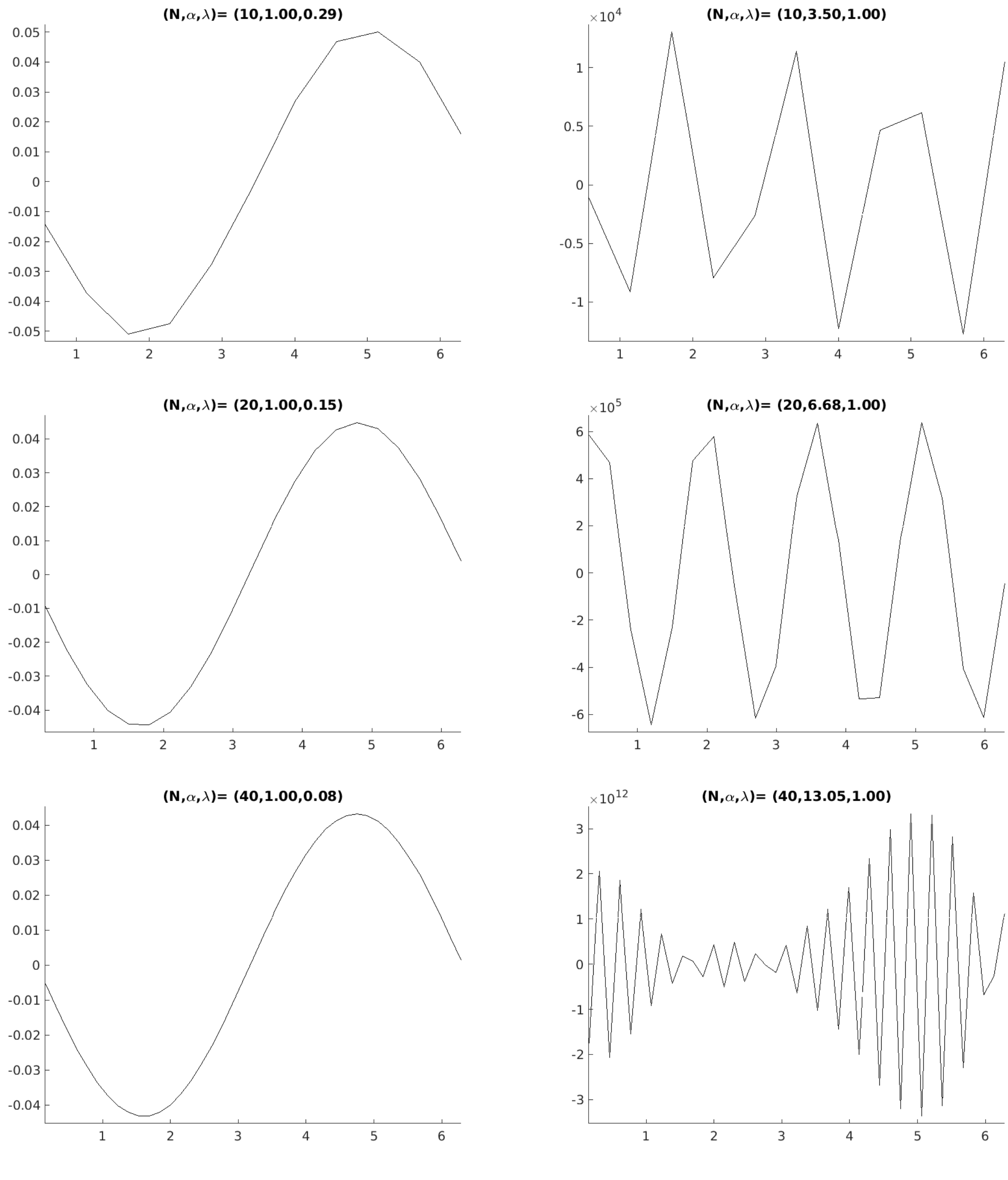

Figure 1: The behavior of numerical solutions, left subfigures fixed $\alpha = 1$ and right subfigures fixed $|\lambda| = 1$.

1.7.1

针对方程 $u_t + au_x = \eta u_{xx}$，取 $h = \dfrac{2\pi}{N}, a = -1, \eta = 0.5$，终止时刻 $T = 2\pi$，初值函数 $f(x) = \sin(x)$。当我们固定 $\lambda = \dfrac{ak}{h} = -0.5$ 时，计算结果如下图所示。

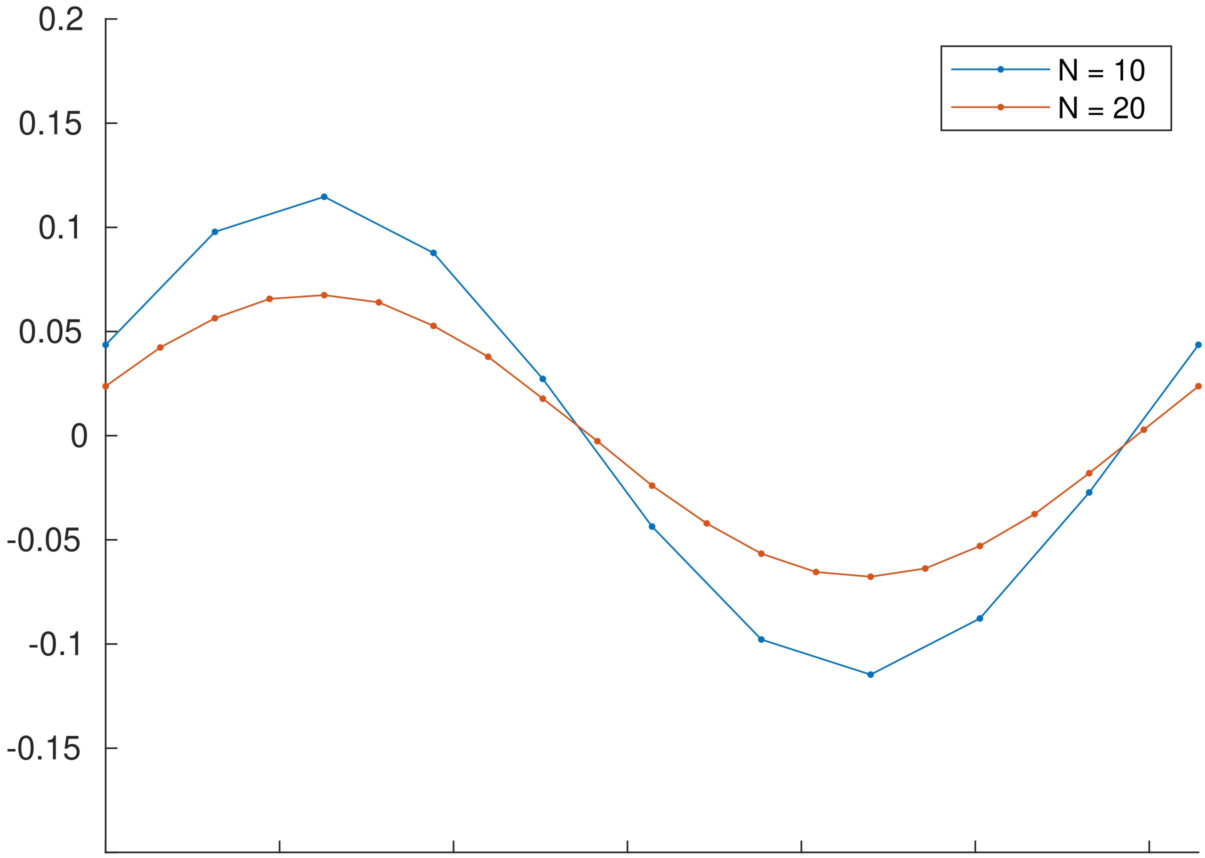

图 1: convection-diffusion 方程，初值函数 $f(x) = \sin(x)$，终止时刻 $T = 2\pi$ 时的计算结果。

在 $k = O(h)$ 的条件下，$N = 10, 20$ 时可以得到合理的结果，但当 $N = 40$ 时结果溢出。出现该现象的原因是稳定性条件 $2\eta \leq h^2/k$ 不再满足。

上述结果可以通过运行 `convection_diffusion_err1.m` 文件得到。

当我们固定 $\alpha = \dfrac{2\eta k}{h^2} = 0.5$ 时，计算结果如下图所示。

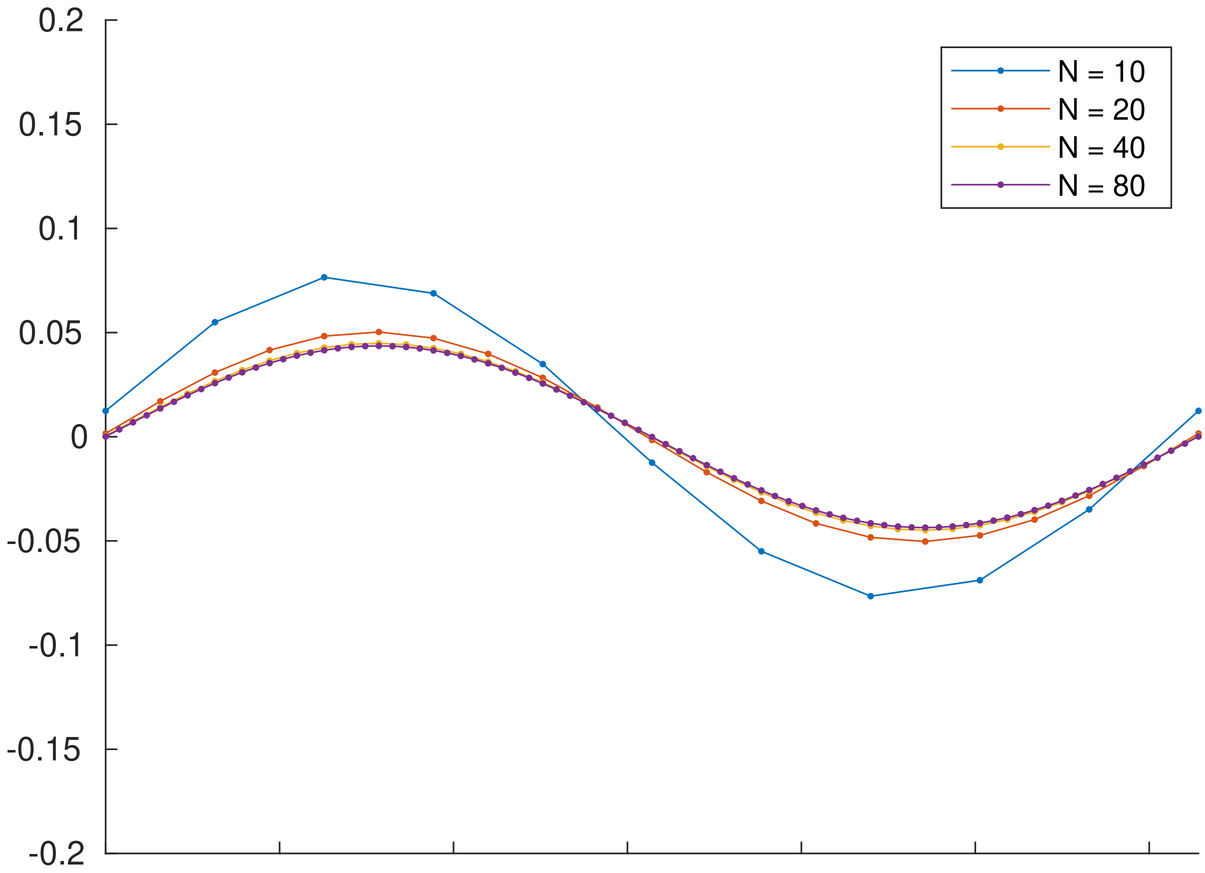

图 2: convection-diffusion 方程，初值函数 $f(x) = \sin(x)$，终止时刻 $T = 2\pi$ 时的计算结果。

在 $k = O(h^2)$ 的条件下，$N = 10, 20, 40, 80$ 时都可以得到较好的结果。这是因为稳定性条件 $2\eta \le h^2/k$ 始终被满足。

上述结果可以通过运行 `convection_diffusion_err2.m` 文件得到。