

Chapter 5

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5.1 Problem

Exercise 5.1. Scaled integral of B-spline.

Deduce from the Theorem on derivatives of B-splines that the scaled integral of a B-spline $B_i^n(x)$ over its support is independent its index i even if the spacing of the knots is not uniform.

Solution. By derivatives of B-spline $\frac{d}{dx} B_i^n(x) = \frac{nB_i^{n-1}(x)}{t_{i+n-1}-t_{i-1}} - \frac{nB_{i+1}^{n-1}(x)}{t_{i+n}-t_i}$. Intergral both side in $[a, b]$ and $|a|, |b|$ big enough so that $B_i^n(a) = B_i^n(b) = 0$. It follows that

$$\begin{aligned} 0 &= B_i^n(a) - B_i^n(b) \\ &= \int_b^a \frac{d}{dx} B_i^n(x) dx \\ &= \frac{n}{t_{i+n-1} - t_{i-1}} \int_a^b B_i^{n-1}(x) dx + \frac{n}{t_{i+n} - t_i} \int_a^b B_{i+1}^{n-1}(x) dx. \end{aligned}$$

That's same with $\frac{1}{t_{i+n-1}-t_{i-1}} \int_{t_{i-1}}^{t_{i+n-1}} B_i^{n-1}(x) dx$ is independent with i 's value.

Exercise 5.2. Symmetric Polynomials.

We have a theorem on expressing complete symmetric polynomials as divided difference of monomials.

- (a) Verify this theorem for $m = 4$ and $n = 2$ by working out the table of divided difference and comparing the result to the dfinition of complete symmetric polynomials.
- (b) Prove this theorem by the lemma on the recursive relation on complete symmetric polynomials.

Solution.

- (a) Assume $i = 1$, $\tau_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$. And

$$\begin{aligned} [x_1, x_2, x_3]x^4 &= ([x_1, x_2]x^4 - [x_2, x_3]x^4)(x_1 - x_3) \\ &= ((x_1 + x_2)(x_1^2 + x_2^2) - (x_3 + x_2)(x_3^2 + x_2^2))/(x_1 - x_3) \\ &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3. \end{aligned}$$

- (b) From $\tau_{k+1}(x_1, \dots, x_n, x_{n+1}) = \tau_{k+1}(x_1, \dots, x_n) + x_{n+1}\tau_k(x_1, \dots, x_n, x_{n+1}) = \tau_{k+1}(x_2, \dots, x_{n+1}) + x_1\tau_k(x_1, \dots, x_n, x_{n+1})$
And $\forall m, \tau_m(x_i) = [x_i]x^m$. Then proof it by induction. Assuming $n = k+1$ already have $\tau_{m-k+1}(x_i, \dots, x_{i+k-1}) =$

$[x_i, \dots, x_{i+k-1}]x^m$. Calculate τ_{m-k}

$$\begin{aligned}\tau_{m-k}(x_i, \dots, x_{i+k}) &= \frac{\tau_{m-k+1}(x_i, \dots, x_{i+k-1}) - \tau_{m-k+1}(x_{i+1}, \dots, x_{i+k})}{x_i - x_{i+k}} \\ &= \frac{[x_i, \dots, x_{i+k-1}]x^m - [x_{i+1}, \dots, x_{i+k}]x^m}{x_i - x_{i+k}} \\ &= [x_i, \dots, x_{i+k}]x^m.\end{aligned}$$

So that from induction, It's true for $n \in [0, m]$.

5.2 Program

- (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a given function. Implement two modules to interpolate f by the quadratic and cubic cardinal B-splines, which corresponds to the two corollaries in the notes on unique interpolation by quadratic B-splines and complete cubic cardinal B-splines, respectively.

- (b) Run your subroutines on the function

$$f(x) = \frac{1}{1+x^2}, \quad x \in [-5, 5],$$

using $t_i = i - \frac{11}{2}, i = 1, \dots, 10$ for quadratic B-splines and $t_i = -6 + i, i = 1, \dots, 11$ for complete cubic cardinal B-splines. Plot the polynomials against the exact function.

- (c) Define $E_S(x) = |S(x) - f(x)|$ as the interpolation error. For the two cardinal B-spline interpolants, what are the values of $E_S(x)$ at the sites $x = -3.5, -3, -0.5, 0, 0.5, 3, 3.5$? Program in **C++** to output these values. Why are some of the errors close to machine precision? Which of the two B-splines is more accurate?

- (d) The roots of the following equation constitute a closed planar curve in the shape of a heart:

$$x^2 + \left(\frac{3}{2}y - \sqrt{|x|}\right)^2 = 3.$$

Modify your **C++** subroutines in the previous homework to plot the heart. As discussed in the lectures, the parameter of the curve should be the *cumulative chordal length* defined from n given points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ as

$$t_i = \begin{cases} 0, & i = 1 \\ t_{i-1} + \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, & i > 1. \end{cases}$$

Choose $n = 10, 40, 160$ and produce three plots of the heart function.

Solution.

- (a) *make run* will compile and run *main.ex*. Output file will be in code/output.
- (b) *make print* will plot the quadratic B-spline and the cubic cardinal B-spline. Or using the **matlab** file *p.m* and *plotTruncatedPowerFunc2Bsplines.m* to plot (5.1) (5.2).
- (c) *make run* will output $E_S(x)$ for the two cardinal B-spline by function *prnterror*. You can Adjustment parameters to computer other points' error. And for quadratic B-spline interpolation, it's error in $-3.5, -0.5, 0.5, 3.5$ is close to machine precision, since these is its interpolate points. As for the same reason, cubic B-spline interpolation's error close to 0 in $-3, 0, 3$. Cubic B-spline will be more accurate.
- (d) *make run* will output interpolations in *output/*. Using *Heartplot.m* can plot. Or *make heart*. Figure like (5.3).

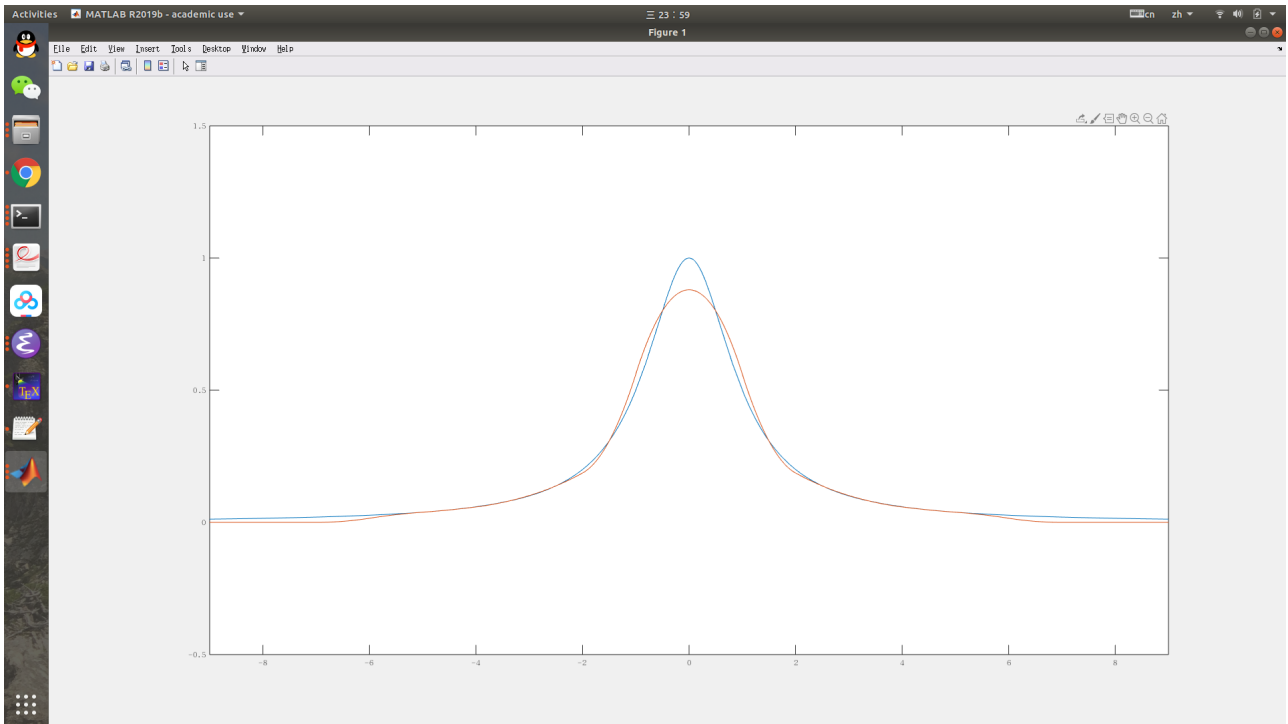


图 5.1: quadratic B-spline

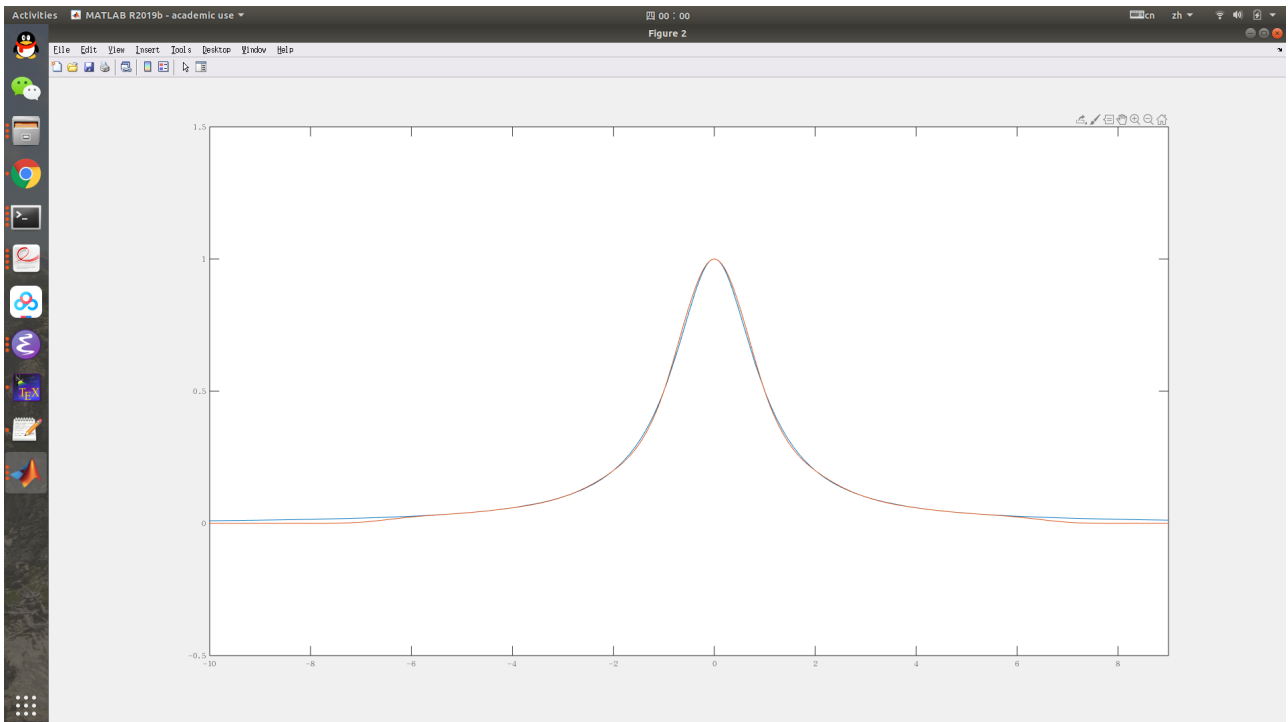


图 5.2: cubic B-spline

