Chapter 9

Homework 21935004 谭焱

Problem 9.1. Simpson's rule.

(a) Show that on $[-1,1]$ Simpson's rule can be obtained as follows

$$\int_{-1}^{1} y(t)dt = \int_{-1}^{1} p_3(y;-1,0,0,1;t)dt + E^S(y),$$

where $y \in \mathcal{C}^4[-1,1]$ and $p_3(y;-1,0,0,1;t)$ is the interpolation polynomial of y with interpolation conditions $p_3(-1) = y(-1), p_3(0) = y(0), p_3'(0) = y'(0)$, and $p_3(1) = y(1)$.

(b) Derive $E^S(y)$.

(c) Using (a), (b) and a change of variable, derive the composite Simpson's rule and prove the theorem on its error estimation.

Solution.

(a) By interpolating, we can get $p_3(y;-1,0,0,1;t) = y(-1)+(y(0)-y(-1))(x+1)+(y'(0)-y(0)+y(-1))x(x+1)+(\frac{y(1)-y(-1)-2y'(0)}{2})x^2(x+1)$. Intergal p_3 get $\int_{-1}^{1} p_3(y;-1,0,0,1;t)dt = \frac{1}{3}(f(-1)+4f(0) = f(1))$.

(b) Since $y(x) - p_3(y;x) = \frac{y^{(4)}(x)}{4!}(x+1)x^2(x-1)$,therefore

$$\begin{aligned}
E^S(y) &= \int_{-1}^{1} y(t) - p_3(y;-1,0,0,1;t)dt \\
&= \int_{-1}^{1} \frac{y^{(4)}(t)}{4!}(t+1)t^2(t-1)dt \\
&= -\frac{1}{90}f^{(4)}(\zeta).
\end{aligned}$$

(c)

$$\begin{aligned}
E^S(f) &= \int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right] \\
&= \int_a^b f(t)dt - \int_a^b p_3(f;a,\frac{a+b}{2},\frac{a+b}{2},b;t)dt \\
&= \int_a^b \frac{f^{(4)}(t)}{4!}(t-a)(t-\frac{a+b}{2})^2(t-b)dt \\
&= -\frac{(b-a)^5}{2880}f^{(4)}(\zeta).
\end{aligned}$$

Problem 9.2. Estimate the number of subintervals required to approximate $\int_0^1 e^{-x^2}dx$ to 6 correct decimal places, i.e. the absolute error is no greater than 0.5×10^{-6},

(a) by the composite trapezoidal rule.

(b) by the composite Simpson's rule.

Solution.

(a) Let $f(x) = e^{-x^2}$, we have $\left|f^{(2)}(x)\right| = \left|4x^2e^{-x^2} - 2e^{-x^2}\right| < 2$. So by the composite trapezoidal rule,$\left|E_n^T(f)\right| < 0.5 \times 10^{-6} \Longleftarrow \frac{1}{6}h^2 < \frac{1}{2} \times 10^{-6}$. Therefore, the number of subintervals is about $\frac{1}{h} = 578$.

(b) We also have $\left|f^{(4)}(x)\right| < 30$, $\left|E_n^S(f)\right| < 0.5 \times 10^{-6} \Longleftarrow \frac{1}{6}h^4 < \frac{1}{2} \times 10^{-6}$. Therefore, the number of subintervals is about 25.

Problem 9.3. Gauss-Larguerre quadrature formula.

(a) Construct a polynnomial $\pi_2(t) = t^2 + at + b$ that is orthogonal to $\mathbb{P}_1$ with respect to the weight function $\rho(t) = e^{-t}$, *i.e.*

$$\forall p \in \mathbb{P}_1, \qquad \int_0^{+\infty} p(t)\pi_2(t)\rho(t)dt = 0.$$

(hint: $\int_0^{+\infty} t^m e^{-t}dt = m!$)

(b) Derive the two-point Gauss-Larguerre quadrature formula

$$\int_0^{+\infty} f(t)e^{-t}dt = w_1 f(t_1) + w_2 f(t_2) + E_2(f)$$

and express $E_2(f)$ in terms of $f^{(4)}(\tau)$ for some $\tau > 0$.

(c) Apply the formula in (b) to approximate

$$I = \int_0^{+\infty} \frac{1}{1+t}e^{-t}dt.$$

Use the remainder to estimate the error and compare your estimate with the true error. With the true error, identify the unknown quantity τ contained in $E_2(f)$. (*hint*; use the exact value $I = 0.596347361\ldots$)

Solution.

(a) Since $\pi_2(t)$ is orthogonal to $\mathbb{P}_1$ indicate that $\langle \pi_2, 1\rangle = \langle \pi_2, t\rangle = 0$,

$$\begin{aligned}
\langle \pi_2, 1\rangle &= \int_0^{\infty} e^{-t}(t^2 + at + b)dt \\
&= 2! + 1!a + 0!b = 2 + a + b = 0 \\
\langle \pi_2, t\rangle &= \int_0^{\infty} e^{-t}(t^3 + at^2 + b)dt \\
&= 3! + 2!a + 1!b = 6 + 2a + b = 0.
\end{aligned}$$

Solve this get $a = -4, b = 2$.

(b) By (a) get $t_1 = 2 + \sqrt{2}, t_2 = 2 - \sqrt{2}$. And choose $f_1(t) = 1, f_2(t) = t$ respectively give

$$\begin{aligned}
w_1 1 + w_2 1 &= \int_0^{\infty} 1e^{-t}dt = 1 \\
w_1 t_1 + w_2 t_2 &= \int_0^{\infty} te^{-t}dt = 1.
\end{aligned}$$

Solves this equations get $w_1 = \frac{2-\sqrt{2}}{4}, w_2 = \frac{2+\sqrt{2}}{4}$. $E_2(f) = \frac{f^{(4)}(\tau)}{4!}\int_0^{+\infty} e^{-t}(t^2-4t+2)^2 dt = \frac{f^{(4)}(\tau)}{4!}\int_0^{+\infty} e^{-t}(t^4 - 8t^3 + 20t^2 - 16t + 4)dt = \frac{f^{(4)}(\tau)}{4!}(4! - 8\times 3! + 20\times 2! - 16\times 1! + 4\times 0!) = \frac{f^{(4)}(\tau)}{6}$.

(c) The remainder $E_2(\frac{1}{1+t}) = \frac{4}{(1+t)^5}$, the true error $e = I - w_1 f(t_1) - w_2 f(t_2) = 0.0249187896 = \frac{4}{(1+\tau)^5} \Longrightarrow \tau = 1.7612556$.

9.1 Program

Write a **C++** function to perform discrete least square via QR factorization. Your algorithm should take as input the maximum degree of the fitting polynomial, three or more data pairs (x_i, y_i), and output coefficients of the fitted polynomial.

Run your subroutine on the following data.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
y	2.9	2.7	4.8	5.3	7.1	7.6	7.7
x	3.5	4.0	4.5	5.0	5.5	6.0	6.5
y	7.6	9.4	9.0	9.6	10.0	10.2	9.7
x	7.0	7.5	8.0	8.5	9.0	9.5	10.0
y	8.3	8.4	9.0	8.3	6.6	6.7	4.1

In the note, the condition number a matrix A is defined as

$$\mathrm{cond}_A(\mathbf{x}) = \|A\|\|A^{-1}\|.$$

Report the condition number based on the 2-norm of matrix G in the normal equation approach (reuse results of your previous homework!) and that of the matrix R_1 in the QR-factorization approach, verifying that the former is much larger than the latter.

make run will output the polynomial on terminal and use **matlab** calculate the condition number. Matlab will output $A, A^{-1}, \|A\|_2, \|A^{-1}\|_2, \|A\|_2\|A^{-1}\|_2$ respectively. The condition numbers are $1.8981 \times 10^4, 137.7715$, which shows that former is much larger.