Chapter 1

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1.1 Problem

Problem 1.1. Use Frobenius method to find the complete asymptotic series expansion for the 2nd-order *modified bessel Differential Equation of order ν*:

$$y'' + \frac{1}{x}y' \mp \left(1 \pm \frac{\nu^2}{x^2}\right) y = 0$$

near $x = 0$. How many independent solutions can be found as a Frobenius series?

Hint: Disscuss different root scenarios of the indicial polynomial

$$P(\alpha) = \alpha^2 - \nu^2$$

Solution. Replace y with Frobenius series $\sum_{n=0}^{\infty} a_n x^{\alpha+n}$ gives

$$\sum_{n=0}^{\infty}(\alpha+n)(\alpha+n-1)a_n x^{\alpha+n-2} + \sum_{n=0}^{\infty}(\alpha+n)a_n x^{\alpha+n-2} \mp \sum_{n=0}^{\infty} a_n x^{\alpha+n} - \nu^2 \sum_{n=0}^{\infty} a_n x^{\alpha+n-2} = 0.$$

Change all x power to $\alpha + n - 2$, then since equal to zero, every coefficients of powers of x equal to zero gives

$$\begin{cases} x^{\alpha-2}: & (\alpha^2 - \nu^2)a_0 = 0, \\ x^{\alpha-1}: & [(\alpha+1)^2 - \nu^2]a_1 = 0, \\ x^{\alpha+n-2}: & [(\alpha+n)^2 - \nu^2]a_n = \mp a_{n-2}, n = 2, 3 \dots \end{cases}$$

$_0 \neq 0 \Rightarrow P(\alpha) = 0 \Rightarrow \alpha = \pm\nu$. Let $\alpha_1 = |\nu|, \alpha_2 = -|\nu|$. And it's easy to see that $\alpha = \alpha_1$ will recursively decide a Frobenius solution $y = \sum\limits_{i=0}^{\infty} a_i x^{i+\alpha-2}, a_i = \begin{cases} 0 & i \text{ is odd} \\ \dfrac{a_0\Gamma(\nu+1)}{2^{2n}n!\Gamma(\nu+n+1)} & i \text{ is even} \end{cases}$, if $\alpha_2 - \alpha_1 = 2|\nu|$ isn't integer, α_2 also will decide another solution.

Or $2|\nu| = N$ is a integer. From the coefficient of powers of x know

$$0 * a_N = \mp a_{N-2}$$

There are two situation, N is odd or even. As above already know that if $N-2$ is even, $0 * a_N = a_{N-2} \neq 0$, that can't happen (a_{N-2} will be recursively calculated from a_0 or $N = 0$ which is no different, and can't be zero). So at this situation, $\alpha = \alpha_2$ won't decide a solution.

If N is odd, a_N can be any value ($a_{N-2} = 0$ can be recursively define from a_1 or $N = 1$ that no influence), so there is another linearly independent solution.

In summary, equation will have two independent solutions as Frobenius series unless 2ν is even integer, and this situation there is only one Frobenius series solution.

Problem 1.2. Identify the drastic change in the behavior of the solution to the ODE

$$\varepsilon y'' + \left(x^2 - \frac{1}{4}\right) y' - e^{2x-1} y = 0, 0 < x < 1$$

with $y(0) = 2$ and $y(1) = 3$ with the method of matched asymptotic expansions. Find the leading order, composite expansion of the exact solution.

Solution. Calculate outer solution gives

$$y_{outer} = a_0 e^{\int_0^x \frac{e^{2t-1}}{t^2-1/4} dt}$$

However, this is discontinuous in $x = 1/2$. Therefore, assuming the laryer is inner laryer and in $x = 1/2$. So that

$$y_{outer} = \begin{cases} 2e^{\int_0^x \frac{e^{2s-1}}{s^2-1/4} ds} & 0 < x < \frac{1}{2} \\ 3e^{-\int_x^1 \frac{e^{2s-1}}{s^2-1/4} ds} & \frac{1}{2} < x < 1 \end{cases}$$

and let $\bar{x} = \frac{x-1/2}{\varepsilon^\alpha}$, then $\frac{d}{dx} = \varepsilon^{-\alpha} \frac{d}{d\bar{x}}$, substitute this into the ODE gives

$$\varepsilon^{1-2\alpha} Y'' + \bar{x}(1 + \varepsilon^\alpha \bar{x}) Y' - e^{2x-1} Y = 0$$

Let $Y_0 = y_0(\bar{x}) + \varepsilon^\gamma y_1(\bar{x}) + \cdots$. Combining with above equation and take $\alpha = 1/2$ get coefficient of $O(1) = 0$ is

$$y_0'' + \bar{x} y_0' - e^{2x-1} y = 0.$$

Solve this ODE get

$$y_0(\bar{x}) = A\bar{x} + B[e^{-\bar{x}^2/2} + \bar{x} \int_0^{\bar{x}} e^{-s^2/2} ds].$$

Then coming to matching, consider $x_\eta = \frac{x-1/2}{\varepsilon^\kappa}, 0 < \kappa < \alpha$, and rewrite y_{outer} with x_η

$$y_{outer} = \begin{cases} 2e^{\int_0^{\varepsilon^\kappa x_\eta + 1/2} \frac{x^{2s-1}}{s^2-1/4} ds} & x_\eta < 0 \\ 3e^{-\int_{\varepsilon^\kappa x_\eta + 1/2}^1 \frac{x^{2s-1}}{s^2-1/4} ds} & 0 < x_\eta \end{cases}$$

While x close to $1/2$,

$$\begin{aligned} \int_0^x \frac{e^{2s-1}}{s^2-1/4} ds &= \int_0^x \left(\frac{e^{2s-1}}{s-1/2} - \frac{e^{2s-1}}{s+1/2}\right) ds \\ &= e^{\zeta_l} (\ln(1/2 - x) - \ln(1/2) - \ln(1/2 + x) + \ln(1/2)) \\ &\sim e^{\zeta_l} \ln(1/2 - x), \zeta_l \in (-1, 0). \\ &= \sim \ln(1/2 - x) \\ \int_x^1 \frac{e^{2s-1}}{s^2-1/4} ds &= \int_x^1 \left(\frac{e^{2s-1}}{s-1/2} - \frac{e^{2s-1}}{s+1/2}\right) ds \\ &= e^{\zeta_r} (\ln(1/2) - \ln(x - 1/2) - \ln(3/2) + \ln(x + 1/2)) \\ &\sim e^{\zeta_r} (\ln(x - 1/2) + \ln(1/3)), \zeta_r \in (0, 1). \\ &\sim \ln((x - 1/2)/3). \end{aligned}$$

The last steps comes from that $C_l = e^{\zeta_l} \to 1$, and $C_r = e^{\zeta_r} \to 1$ as $x \to 1/2$. Then we have

$$y_{outer} = \begin{cases} 2e^{\int_0^x \frac{e^{2s-1}}{s^2-1/4}ds} \sim 2e^{\ln(1/2-x)} = -2\varepsilon^\kappa x_\eta & x_\eta < 0 \\ 3e^{-\int_x^1 \frac{e^{2s-1}}{s^2-1/4}ds} \sim 3e^{\ln((x-1/2)/3)} = \varepsilon^\kappa x_\eta & 0 < x_\eta \end{cases}.$$

In other side,

$$y_0 \sim \begin{cases} \varepsilon^{\kappa-1/2} x_\eta (A - B\sqrt{\frac{\pi}{2}}) & x_\eta < 0 \\ \varepsilon^{\kappa-1/2} x_\eta (A + B\sqrt{\frac{\pi}{2}}) & 0 < x_\eta \end{cases}.$$

However, y_0 and y_{outer} can't matched except $y_0 = 0$. By the same step will get

$$y_1 \sim \begin{cases} \varepsilon^{\kappa+\gamma-1/2} x_\eta (A - B\sqrt{\frac{\pi}{2}}) & x_\eta < 0 \\ \varepsilon^{\kappa+\gamma-1/2} x_\eta (A + B\sqrt{\frac{\pi}{2}}) & 0 < x_\eta \end{cases}.$$

It's follows that $\gamma = \frac{1}{2}$ and

$$\begin{cases} A - B\sqrt{\frac{\pi}{2}} = -2 \\ A + B\sqrt{\frac{\pi}{2}} = 1 \end{cases},$$

from which comes out $A = -1/2$ and $B = 3/\sqrt{2\pi}$.

Finally, coming to composite expansion.

$$y \sim \begin{cases} 2e^{\int_0^x \frac{e^{2s-1}}{s^2-1/4}ds} + \varepsilon^{\frac{1}{2}}(-\frac{1}{2}\bar{x} + \frac{3}{\sqrt{2\pi}}[e^{-\bar{x}^2/2} + \bar{x}\int_0^{\bar{x}} e^{-s^2/2}ds]) + 2\varepsilon^\kappa x_\eta & 0 < x < \frac{1}{2} \\ 3e^{-\int_x^1 \frac{e^{2s-1}}{s^2-1/4}ds} + \varepsilon^{\frac{1}{2}}(-\frac{1}{2}\bar{x} + \frac{3}{\sqrt{2\pi}}[e^{-\bar{x}^2/2} + \bar{x}\int_0^{\bar{x}} e^{-s^2/2}ds]) - \varepsilon^\kappa x_\eta & \frac{1}{2} < x < 1 \end{cases}.$$

Problem 1.3. Derive the leading order asymptotic behavior of the solution to the ODE

$$y'' + k^2(\varepsilon t)y = 0, 0 < t$$

where $\varepsilon \ll 1$ and

$$y(0) = a, y'(0) = b.$$

Try solving with the method of multiple scales.

Solution. Choose time scale $t_1 = f(t,\varepsilon) = \int_0^t k(\varepsilon s)ds, t_2 = \varepsilon t \ll t_1$. Therefore,

$$\frac{\partial}{\partial t} = f_t \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}$$

$$\begin{aligned} \frac{\partial(\frac{\partial}{\partial t})}{\partial t} = \frac{\partial(\frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2})}{\partial t} &= f_{tt}\frac{\partial}{\partial t_1} + f_t^2 \frac{\partial^2}{\partial t_1^2} + 2\varepsilon f_t \frac{\partial^2}{\partial t_1 \partial t_2} + \varepsilon^2 \frac{\partial^2}{\partial t_2^2} \\ &= k^2(\varepsilon t)\frac{\partial^2}{\partial t_1^2} + \varepsilon(k'(\varepsilon t)\frac{\partial}{\partial t_1} + 2k(\varepsilon t)\frac{\partial^2}{\partial t_1 \partial t_2}) + \varepsilon^2 \frac{\partial^2}{\partial t_2^2}. \end{aligned}$$

Subtitute this and $y \sim y_0(t_1,t_2) + \varepsilon y_1(t_1,t_2) + \cdots$ into the equation gives

$$(k^2(\varepsilon t)\frac{\partial^2}{\partial t_1^2} + \varepsilon(k'(\varepsilon t)\frac{\partial}{\partial t_1} + 2k(\varepsilon t)\frac{\partial^2}{\partial t_1 \partial t_2}) + \varepsilon^2\frac{\partial^2}{\partial t_2^2})(y_0(t_1,t_2) + \varepsilon y_1(t_1,t_2) + \cdots) + k^2(t_2)(y_0(t_1,t_2) + \varepsilon y_1(t_1,t_2) + \cdots) = 0.$$

From coefficients of $\rceil$ must be zero and initial condition get

$$\begin{cases} k^2(\varepsilon t)(\frac{\partial^2}{\partial t_1^2} + 1)y_0(t_1,t_2) = 0 \\ y_0(0,0) = a, k(0)\frac{\partial}{\partial t_1}y_0(0,0) = b. \end{cases}$$

Solve this get $y_0(t_1,t_2) = a_0(t_2)\cos(t_1) + b_0(t_2)\sin(t_1), a_0(0) = a, b_0(0) = b/k(0)$. Consider next coefficient

$$\begin{cases} (k'(\varepsilon t)\frac{\partial}{\partial t_1} + 2k(\varepsilon t)\frac{\partial^2}{\partial t_1 \partial t_2})y_0(t_1,t_2) + k^2(\varepsilon t)(\frac{\partial^2}{\partial t_1^2}+1)y_1(t_1,t_2) = 0 \\ k^2(\varepsilon t)(\frac{\partial^2}{\partial t_1^2}+1)y_1(t_1,t_2) = -(k'a_0 + 2ka_0')(\cos(t_1)) - (k'b_0 + 2kb_0')(-\sin(t_1)). \end{cases}$$

In order to clear secular terms, secular terms' coefficient have to be zero

$$\begin{cases} k'a_0 + 2ka_0' = 0, \\ k'b_0 + 2kb_0' = 0, \\ a_0(0) = a, b_0(0) = b/k(0). \end{cases}$$

Solve the equations get

$$a_0(t_2) = \frac{a\sqrt{k(0)}}{\sqrt{k(t_2/\varepsilon)}}$$
$$b_0(t_2) = \frac{b}{\sqrt{k(0)k(t_2/\varepsilon)}}$$

In summary,

$$y \sim y_0(t_1,t_2) = \frac{a\sqrt{k(0)}}{\sqrt{k(t)}}\cos(\int_0^t k(\varepsilon s)ds) + \frac{b}{\sqrt{k(0)k(t)}}\sin(\int_0^t k(\varepsilon s)ds).$$