Chapter 16

Homework 21935004 谭焱

In these exercises U always denotes an open subset of $\mathbb{R}^n$, with a smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

Problem 16.1. Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant C depending only on n and α, such that

$$\int_U u^2 dx \le C \int_U |Du|^2 dx$$

provided

$$|\{x \in U | u(x) = 0\}| \ge \alpha, \quad u \in H^1(U)$$

Proof. Take $A = \{x \in U \mid u(x) = 0\}$. Then

$$\begin{aligned}
\int_U u^2 dx &= \int_{U-A} (u - (u)_U + (u)_U)^2 dx \\
&\le 2 \int_{U-A} (u - (u)_U)^2 dx + \int_{U-A} (u)_U^2 dx \\
&\le C_1 \|Du\|^2_{L^2(U)} + |U - A| \, (u)_U^2 \qquad C_1 \in (0, +\infty) \\
&= C_1 \|Du\|_{L^2(U)} + \frac{|U-A|}{|U|^2} (\int_{U-A} u dx)^2 \\
&\le C_1 \|Du\|_{L^2(U)} + \frac{|U-A|^2}{|U|^2} \int_{U-A} u^2 dx \\
\implies \qquad \int_U u^2 dx &\le \frac{C_1}{1 - \frac{|U-A|^2}{|U|^2}} \|Du\|_{L^2(U)}.
\end{aligned}$$

Since $|A| \ge \alpha > 0$, $1 - \frac{|U-A|^2}{|U|^2} > 0$. Therefore $C = \frac{C_1}{1 - \frac{|U-A|^2}{|U|^2}} \in (0, +\infty)$ is the constant.

Problem 16.2. (Variant of Hardy's inequality) Show that for each $n \ge 3$ there exists a constant C so that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \le C \int_{\mathbb{R}^n} |Du|^2 dx$$

for all $u \in H^1(\mathbb{R}^n)$

(Hint: $\left| Du + \lambda \frac{x}{|x|^2} u \right|^2 \ge 0$ for each $\lambda \in \mathbb{R}$.)

Proof. Take $u \in C_c^\infty(\mathbb{R}^n), F(x) = \frac{x}{|x|^2}$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} u^2 \operatorname{div} F dx &= -\int_{\mathbb{R}^n} D(u^2) \cdot F(x) dx \\
&= -2\int_{\mathbb{R}^n} Du \cdot uF dx \\
\Longrightarrow \qquad \left|\int_{\mathbb{R}^n} u^2 \operatorname{div} F dx\right|^2 &= 4\left|\int_{\mathbb{R}^n} Du \cdot uF dx\right|^2 \\
&\leq 4\|Du\|_{L^2}^2 \|uF\|_{L^2}^2.
\end{aligned}$$

However, we have $\operatorname{div} F = \frac{n2}{|x|^2}, |F(x)|^2 = \frac{1}{|x|^2}$. So

$$\begin{aligned}
\frac{(n-2)^2}{4}(\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx)^2 &\leq \int_{\mathbb{R}^n} |Du|^2 \, dx \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \\
\Longrightarrow \qquad \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx &\leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |Du|^2 \, dx.
\end{aligned}$$

Since $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$, there is $\exists u_k \in C_c^\infty(\mathbb{R}^n), s.t. \lim_{k\to\infty} u_k = u \ \text{ in } H^1(\mathbb{R}^n)$. So that

$$\int_{\mathbb{R}^n} |Du_k|^2 \, dx \longrightarrow \int_{\mathbb{R}^n} |Du|^2 \, dx$$

Above we already get

$$\int_{\mathbb{R}^n} \frac{u_k^2}{|x|^2} dx \leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |Du_k|^2 \, dx.$$

It's indicate $\frac{u_k}{|x|} \in L^2(\mathbb{R}^n)$, and $L^2(\mathbb{R}^n)$ is complete, in other side $u_k \in H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ there is $u \in L^2(\mathbb{R}^n)$ such that as $k \to \infty, u_k \to u, a.e.$, and u maintains $\frac{u_k}{|x|} \to \frac{u}{|x|}, a.e.$.

After all,

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{u_k^2}{|x|^2} dx \leq \lim_{k\to\infty} \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |Du_k|^2 \, dx = C \int_{\mathbb{R}^n} |Du|^2 \, dx.$$

Problem 16.3. Provide details for the following alternative proof that if $u \in H^1(U)$ then $Du = 0$ a.e. on the set $\{u = 0\}$ Let ϕ be a smooth, bounded and nondecreasing function, such that ϕ' is bounded and $\phi(z) = z$ if $|z| \leq 1$. Set

$$u^\epsilon(x) := \epsilon\phi(u/\epsilon)$$

Show that $u^\epsilon \to 0$ weakly in $H^1(U)$ and therefore

$$\int_U Du^\epsilon \cdot Du dx = \int_U \phi'(u/\epsilon)|Du|^2 dx \to 0$$

Employ this observation to finish the proof.

Proof.

- $u^\varepsilon \to 0$ weakly in $L^2(U)$

  For $\forall \varphi \in C_c^\infty(U)$,

$$\begin{aligned}
\int_U u^\varepsilon \varphi dx &= \varepsilon \int_U \phi(u/\varepsilon)\varphi dx \\
&\leq \varepsilon \|\phi\|_{L^\infty(U)} \|\varphi\|_{L^1(U)} \\
&\leq \varepsilon \|\phi\|_{L^\infty(U)} \|\varphi\|_{L^\infty(U)} |U| \\
&\to 0 \qquad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

However, by the same way, we have $\|u^\varepsilon\|_{L^2(U)}^2 = \varepsilon^2 \int_U |\phi(u/\varepsilon)|^2 \, dx \le \varepsilon^2 \|\phi\|_{L^\infty(U)}^2 |U|$ is bounded. So $u^\varepsilon \in L^2(U)$.

Combining with $C_c^\infty(U)$ is a Dense Set of $(L^2(U))^* = L^2(U)$ and $\forall \varphi \in C_c^\infty(U), \langle u^\varepsilon, \varphi\rangle = \int_U u^\varepsilon \varphi dx \to 0$, we know $u^\varepsilon \to 0$ weakly in $L^2(U)$.

- $Du^\varepsilon \to 0$ weakly in $L^2(U)$

  By the same way as above,

$$\|\partial_i u^\varepsilon\|_{L^2(U)} = \int_U |\partial_i u^\varepsilon|^2 \, dx = \int_U |\phi'(u/\varepsilon)\partial_i u|^2 \, dx \le \|\phi'\|_{L^\infty}^2 \|\partial_i u\|_{L^2(U)}^2.$$

  Since ϕ' is bounded and $u \in H^1(U)$ imply $\|Du\|_{L^2(U)}$ is bounded, $\|\partial_i u^\varepsilon\|_{L^2(U)}$ is uniform boundedness.

  In other side, $\forall \varphi \in C_c^\infty(U)$,

$$\langle \partial_i u^\varepsilon, \varphi\rangle = \int_U \partial_i u^\varepsilon \varphi dx = -\int_U u^\varepsilon \partial_i \varphi dx \le \|u^\varepsilon\|_{L^2(U)} \|\partial_i \varphi\|_{L^2(U)}.$$

  Since $\|u^\varepsilon\|_{L^2(U)} \to 0$ as $\varepsilon \to 0$ and $\|\partial_i \varphi\|_{L^2(U)}$ is bounded. $\langle \partial_i u^\varepsilon, \varphi\rangle \to 0$ as $\varepsilon \to 0$. Therefore, $Du^\varepsilon \to 0$ weakly in $L^2(U)$.

In summay, $u^\varepsilon \to 0$ weakly in $H^1(U)$.

Take $A = \{u = 0\} \in U$, $\int_A Du^\varepsilon \cdot Du dx = \sum_{i=1}^n \int_A \partial_i u^\varepsilon \partial_i u dx \le \sum_{i=1}^n \|\partial_i u^\varepsilon\|_{L^2(A)} \|\partial_i u\|_{L^2(A)} \to 0$ as $\varepsilon \to 0$, since $\|\partial_i u^\varepsilon\|_{L^2(A)} \to 0$ and $\|\partial_i u\|_{L^2(A)}$ is bounded. Then

$$\begin{aligned}
\int_A Du^\varepsilon \cdot Du dx &= \sum_{i=1}^n \int_A \partial_i u^\varepsilon \partial u dx \\
&= \sum_{i=1}^n \int_A \phi'(u/\varepsilon)(\partial_i u)^2 dx \\
&= \int_A |Du|^2 \, \phi'(u/\varepsilon) dx.
\end{aligned}$$

Combining with $u = 0, \phi'(0) = 1$, so $Du = 0, a.e.$.

Problem 16.4. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$ then $u \in L^\infty(\mathbb{R}^n)$, with the bound

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n

Proof. Since $u \in H^s(\mathbb{R}^n)$ implies

$$\left(\int_{\mathbb{R}^n} (1 + |y|^s)^2 \, |\hat{u}|^2 \, dx\right)^{1/2} \le C_1 \|u\|_{H^s(\mathbb{R}^n)}.$$

It follows $\hat{u} \in L^1(\mathbb{R}^n)$ from

$$\begin{aligned}
\|\hat{u}\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{u}| \, dx &= \int_{\mathbb{R}^n} \left|(1 + |x|^s)(1 + |x|^s)^{-1} \hat{u}\right| dx \\
&\le C_1 \|u\|_{H^s(\mathbb{R}^n)} \left((1 + |x|^s)^{-2} dx\right)^{1/2} \\
&\le C(s, n) \|u\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, $u(x) = \int_{\mathbb{R}^n} \hat{u}(\xi) e^{2\pi i x \cdot \xi} d\xi$, which shows that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le \|\hat{u}\|_{L^1(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}.$$

Problem 16.5. Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $uv \in H^s(\mathbb{R}^n)$ and

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C(s,n)\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}$$

the constant C depending only on s and n

Proof. First of all,

$$\|u\|_{H^s(\mathbb{R}^n)} = C_1\|(1+|x|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}, \|v\|_{H^s(\mathbb{R}^n)} = C_2\|(1+|x|^s)\hat{v}\|_{L^2(\mathbb{R}^n)},$$

$C_1, C_2 > 0$. And

$$\begin{aligned}
\|uv\|_{H^s(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}(1+|x|^s)^2(\widehat{uv})^2dx\right)^{1/2} \\
&= (2\pi)^{n/2}\left(\int_{\mathbb{R}^n}((1+|x|^s)\hat{u}*\hat{v})^2\right)^{1/2} \\
&= (2\pi)^{n/2}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(1+|x|^s)\hat{u}(x-y)\hat{v}(y)dy\right)^2dx\right)^{1/2} \\
&\leq C(s)\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}((1+|x-y|^s)+(1+|y|^s))\hat{u}(x-y)\hat{v}(y)dy\right)^2dx\right)^{1/2} \\
&= C(s)\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(1+|x-y|^s)\hat{u}(x-y)\hat{v}(y)dy\right)^2dx + \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(1+|y|^s)\hat{u}(x-y)\hat{v}(y)dy\right)^2dx\right)^{1/2} \\
&= C(s)\left(\int_{\mathbb{R}^n}((1+|y|^s)u(y)*v(y))^2+(u(y)*(1+|y|^s)v(y))^2dx\right)^{1/2} \\
&\leq C(s)(\|u(y)*(1+|y|^s)v(y)\|_{L^2(\mathbb{R}^n)}+\|(1+|y|^s)u(y)*v(y)\|_{L^2(\mathbb{R}^n)}) \\
&\leq C(s)(\|(1+|y|^s)v(y)\|_{L^2}\|u(y)\|_{L^1}+\|(1+|y|^s)u(y)\|_{L^2}\|v(y)\|_{L^1}) \qquad \text{by Young inequality} \\
&\leq C(s,n)\,|u|_{H^s(\mathbb{R}^n)}\,|v|_{H^s(\mathbb{R}^n)} \qquad \text{by Problem 16.4.}
\end{aligned}$$