

# Chapter 4

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**Exercise 4.1.** 13. Assume  $n = 1$  and  $u(x, t) = v(\frac{x}{\sqrt{t}})$ . (b)

(a) Show

$$u_t = u_{xx}$$

if and only if

$$(*) \quad v'' + \frac{z}{2}v' = 0$$

Show that the general solution of  $(*)$  is

$$v(z) = c \int_0^z e^{-s^2/4} ds + d. \quad (4.1)$$

(b) Differential  $u(x, t) = v(\frac{x}{\sqrt{t}})$  with respect to  $x$  and select the constant  $c$  properly, to obtain the fundamental solution  $\Phi$  for  $n = 1$ . Explain why this procedure produce produces the fundamental solution.

**Solution.**

(a) Since

$$u_t = \partial_t v(\frac{x}{\sqrt{t}}) = v' \frac{-x}{2t^{3/2}} \quad (4.2)$$

$$u_{xx} = \partial_x(v'(\frac{x}{\sqrt{t}}) \frac{1}{\sqrt{t}}) = v'' \frac{1}{t}. \quad (4.3)$$

Combining (4.2),(4.3) and  $u_t = u_{xx}$  get  $(*)$ .

Then

$$\begin{aligned} v'' + \frac{z}{2}v' &= 0 \\ \Rightarrow \ln(v') &= -\frac{z^2}{4} + c \\ \Rightarrow v(z) &= c \int_0^z e^{-s^2/4} ds + d. \end{aligned}$$

means (4.1) is general solution of  $(*)$ .

$$u_x(x, t) = ce^{-x^2/4t} * \frac{1}{\sqrt{t}}$$

Since  $\int e^{-x^2/4t} = 2\sqrt{\pi t}$ . Set  $c = \frac{1}{2\sqrt{\pi}}$  get the fundamental solution  $\frac{1}{2\sqrt{\pi}}e^{-x^2/4t}$ . It solves heat equation, satisfies Dirac measure

**Exercise 4.2.** 15. Given  $g : [0, \infty) \rightarrow \mathbb{R}$ , with  $g(0) = 0$ , derive the formula

$$u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s) ds \quad (4.4)$$

for a solution of the initial, boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0, \infty). \end{cases} \quad (4.5)$$

**Solution.** Let

$$v(x, t) = \begin{cases} u(x, t) - g(t) & \text{in } \mathbb{R}_+ \times (0, \infty) \\ 0 & \text{on } \{x = 0\} \times (0, \infty) \\ -u(-x, t) + g(t) & \text{in } \mathbb{R}_- \times (0, \infty) \end{cases} \quad (4.6)$$

Then

$$v_t - v_{xx} = \begin{cases} -g_t(t) & \text{in } \mathbb{R}_+ \times (0, \infty) \\ g_t(t) & \text{in } \mathbb{R}_- \times (0, \infty) \end{cases}$$

and  $v = g(0) = 0$  on  $\mathbb{R}_+ \times \{t = 0\}$ . Thus, from solution

of nonhomogeneous problem get

$$\begin{aligned}
v(x, t) &= \int_{\mathbb{R}} \Phi(x - y, t) 0 dy + \\
&\quad \int_0^t \int_{-\infty}^0 \Phi(x - y, t - s) g'(s) dy ds + \\
&\quad \int_0^t \int_0^\infty \Phi(x - y, t - s) (-g'(s)) dy ds \\
&= \int_0^t \int_{-\infty}^\infty \Phi(x - y, t - s) (-g'(s)) dy ds + \\
&\quad 2 \int_0^t \int_{-\infty}^0 \Phi(x - y, t - s) g'(s) dy ds \\
&= \int_0^t (-g'(s)) ds + \\
&\quad 2 \int_0^t \int_{-\infty}^0 \Phi(x - y, t - s) dy g'(s) ds \\
&= -g(t) + 2 \int_{-\infty}^0 \Phi(x - y, t - s) g(s) \Big|_{s=0}^t + \\
&\quad 2 \int_0^t \partial_s \left( \int_{-\infty}^0 \Phi(x - y, t - s) dy \right) g(s) ds.
\end{aligned} \tag{4.7}$$

Calculating the second term

$$\begin{aligned}
&\partial_s \left( \int_{-\infty}^0 \Phi(x - y, t - s) dy \right) \\
&= \partial_s \left( \int_{-\infty}^0 \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{(4\pi(t-s))^{1/2}} dy \right) \\
&= \partial_s \left( - \int_{-\infty}^{\frac{x}{2(t-s)^{1/2}}} \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{\pi}} d\left(\frac{x-y}{2(t-s)^{1/2}}\right) \right) \\
&= \partial_s \left( - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2(t-s)^{1/2}}} e^{-z^2} dz \right) \\
&= \frac{x}{4\sqrt{\pi}(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}}.
\end{aligned}$$

Therefore, the second term is equal to (4.4). It follows that  $u(x, t) = v(x, t) + g(t) = (4.8)$ .

**Exercise 4.3.** 16. Give a direct proof that if  $U$  is bounded and  $u \in C_1^2(U_T) \cap C(\bar{U}_T)$  solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u. \tag{4.8}$$

**Solution.** Set  $u_\epsilon := u - \epsilon t$   $\epsilon > 0$ . From  $u$  solves the heat equation conclude

$$\partial_t u_\epsilon - \Delta_x u_\epsilon = -\epsilon < 0$$

if  $u_\epsilon$  have max point  $(x_0, t_0)$  in  $U_T$ , from max point char-

acter

$$\partial_t u_\epsilon(x_0, t_0) = 0$$

$$\Delta_x u_\epsilon(x_0, t_0) = 0$$

However it contradict with above  $\partial_t u_\epsilon - \Delta_x u_\epsilon < 0$ . So  $u_\epsilon$  don't have max point in  $U_T$ . If  $u$  have max point  $(x_0, t_0)$  so that  $u(x_0, t_0) > u(x, t) \forall (x, t) \in \Gamma_T \iff \exists \epsilon > 0 \ u(x_0, t_0) > u(x, t) + \epsilon T \iff u_\epsilon(x_0, t_0) > u(x, t) \forall (x, t) \in \Gamma_T \iff u_\epsilon$  has max point in  $U_T$ , which is proved impossible above. So  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$ .

**Exercise 4.4.** 17. We say  $v \in C_1^2(U_T)$  is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T \tag{4.9}$$

(a) Prove for a subsolution  $v$  that

$$v(x, t) \leq \frac{1}{4r^n} \iint_{E(x, t; r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds \tag{4.10}$$

for all  $E(x, t; r) \subset U_T$ .

(b) Prove that therefore  $\max_{\bar{U}_T} = \max_{\Gamma_T} v$ .

(c) Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be smooth and convex. Assume  $u$  solves the heat equation and  $v := \phi(u)$ . Prove  $v$  is a subsolution.

(d) Prove  $v := |Du|^2 + u_t^2$  is a subsolution, whenever  $u$  solves the heat equation.

**Solution.**

(a) Define  $\phi(r) := \frac{1}{r^n} \iint_{E(r)} v(y, s) \frac{|y|^2}{s^2} dy ds = \iint_{E(1)} v(ry, r^2 s) \frac{|y|^2}{s^2} dy ds$ , According to Mean-value property for the heat equation and set  $\psi := -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r$ , calculate  $\phi'$

$$\begin{aligned}
&\phi'(r) \\
&\geq \frac{1}{r^{n+1}} \iint_{E(r)} -4n \Delta v \psi - \frac{2n}{s} \sum_{i=1}^n v_{y_i} y_i dy ds \\
&= \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} -4n v_{y_i} \psi_{y_i} - \frac{2n}{s} v_{y_i} y_i dy ds \\
&= 0
\end{aligned}$$

Thus  $\phi$  is monotonically increasing, and therefore

$$\begin{aligned}
\phi(r) &> \lim_{t \rightarrow 0} \phi(t) \\
&= v(0, 0) \lim_{t \rightarrow 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dy ds \\
&= 4v(0, 0).
\end{aligned}$$

consider  $(x, t)$  is similar with  $(0, 0)$ . So that  $v(x, t) \leq \frac{1}{4r^n} \iint_{E(x, t; r)} v(y, s) \frac{|x-y|^2}{(t-s)^2} dy ds \quad \forall E(x, t; r) \subset U_T$

(b) Assume  $v$  has max point  $(x_0, t_0) \in U_T$ , For all  $E(x_0, t_0; r) \subset U_T$ , we employ the mean-value property to deduce

$$\begin{aligned} v(x_0, t_0) &\leq \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} v(y, s) \frac{|x-y|^2}{(t-s)^2} dy ds \\ &\leq v(x_0, t_0) \frac{1}{4r^n} \iint_{E(x, t; r)} \frac{|x-y|^2}{(t-s)^2} dy ds \\ &= v(x_0, t_0), \end{aligned}$$

that indicate that  $\forall (x, t) \in E(x_0, t_0; r) \quad v(x, t) = M$ . Then the same process of proving the strong maximum principle for the heat equation get  $v(x, t) = v(x_0, t_0) \quad \forall (x, t) \in U_T$ . It follows that  $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$ .

(c) Differential  $v$

$$v_t = \phi_t(u) = \phi' u_t$$

$$\begin{aligned} \Delta v &= \Delta(\phi(v)) = \sum_{i=1}^n \partial_{x_i} (\phi' u_{x_i}) \\ &= \sum_{i=1}^n (\phi'' u_{x_i}^2 + \phi' u_{x_i x_i}) \\ &\geq \phi' \Delta u. \end{aligned}$$

In summary  $v_t - \Delta v \leq 0$ , that indicates that  $v$  is a subsolution.

(d) Differential  $v$

$$\begin{aligned} v_t &= \partial_t (|Du|^2 + u_t^2) \\ &= \sum_{i=1}^n (2u_{x_i} u_{x_i t}) + 2u_t u_{tt} \\ &= 2u_{x_i} (u_{x_i})_t + 2u_t (u_t)_t \end{aligned}$$

$$\begin{aligned} \Delta v &= \Delta (|Du|^2 + u_t^2) \\ &= \sum_{i=1}^n \partial_{x_i} (2u_{x_i} u_{x_i x_i} + 2u_t u_{tx_i}) \\ &= \sum_{i=1}^n (2(u_{x_i x_i})^2 + 2u_{x_i} u_{x_i x_i x_i} + 2(u_{tx_i})^2 + 2u_t u_{tx_i x_i}) \\ &\geq 2u_{x_i} \Delta u_{x_i} + 2u_t \Delta u_t. \end{aligned}$$

Then  $v_t - \Delta v \leq 2u_{x_i} ((u_{x_i})_t - \Delta u_{x_i}) + 2u_t ((u_t)_t - \Delta u_t) = 0$ . The second step come from when  $u$  is heat equation solution,  $u_t$  and  $u_{x_i}$  too. After all,  $v$  is a subsolution, whenever  $u$  solves the heat equation.