Contents

Chapter 0

Preliminaries

We collect concepts and results in a coherent manner to form a solid foundation for our study of computational homology. Every math major should master the English glossary as well as the math in this chapter.

0.1 First-order logic

Definition 0.1. A *set* $\mathcal{S}$ is a collection of *distinct* objects x's, often denoted with the following notation

$$\mathcal{S} = \{x \mid \text{ the conditions that } x \text{ satisfies. }\}. \tag{0.1}$$

Notation 1. $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{C}$ denote the sets of real numbers, integers, natural numbers, rational numbers and complex numbers, respectively. $\mathbb{R}^+, \mathbb{Z}^+, \mathbb{N}^+, \mathbb{Q}^+$ the sets of positive such numbers.

Definition 0.2. $\mathcal{S}$ is a *subset* of $\mathcal{U}$, written as $\mathcal{S} \subseteq \mathcal{U}$, if and only if (iff) $x \in \mathcal{S} \Rightarrow x \in \mathcal{U}$. $\mathcal{S}$ is a *proper subset* of $\mathcal{U}$, written as $\mathcal{S} \subset \mathcal{U}$, if $\mathcal{S} \subseteq \mathcal{U}$ and $\exists x \in \mathcal{U}$ s.t. $x \notin \mathcal{S}$.

Definition 0.3 (Statements of first-order logic)**.** A *universal statement* is a logic statement of the form

$$\mathsf{U} = (\forall x \in \mathcal{S},\ \mathsf{A}(x)). \tag{0.2}$$

An *existential statement* has the form

$$\mathsf{E} = (\exists x \in \mathcal{S},\ \text{s.t. } \mathsf{A}(x)), \tag{0.3}$$

where $\forall$ ("for each") and $\exists$ ("there exists") are the *quantifiers*, $\mathcal{S}$ is a set, "s.t." means "such that," and $\mathsf{A}(x)$ is the *formula*.
A statement of *implication/conditional* has the form

$$\mathsf{A} \Rightarrow \mathsf{B}. \tag{0.4}$$

Example 0.1. Universal and existential statements:
$\forall x \in [2, +\infty)$, $x > 1$;
$\forall x \in \mathbb{R}^+$, $x > 1$;
$\exists p, q \in \mathbb{Z}$, s.t. $p/q = \sqrt{2}$;
$\exists p, q \in \mathbb{Z}$, s.t. $\sqrt{p} = \sqrt{q} + 1$.

Definition 0.4. *Uniqueness quantification* or *unique existential quantification*, written $\exists!$ or $\exists_{=1}$, indicates that exactly one object with a certain property exists.

Exercise 0.2. Express the logic statement $\exists! x$, s.t. $\mathsf{A}(x)$ with $\exists$, $\forall$, and $\Leftrightarrow$.

Solution. $\exists x$ s.t. $\forall y, \mathsf{A}(y) \Leftrightarrow x = y$.

Remark 0.1. A logic statement is either true or false. There is no such thing that a logic statement is sometimes true and sometimes false. To prove a universal statement, conceptually we have to verify the statement for all elements in the set. To deny a universal statement, we only need to show a counterexample. To prove an existential statement, we only need to show an instance. To deny an existential statement, conceptually we have to show that the statement holds for none of the elements.

Remark 0.2. In Definition 0.3, the formula $\mathsf{A}(x)$ itself might also be a logic statement. Hence universal and existential statements might be nested. This observation leads to the next definition.

Definition 0.5. A *universal-existential statement* is a logic statement of the form

$$\mathsf{U}_E = (\forall x \in \mathcal{S},\ \exists y \in \mathcal{T} \text{ s.t. } \mathsf{A}(x, y)). \tag{0.5}$$

An *existential-universal statement* has the form

$$\mathsf{E}_U = (\exists y \in \mathcal{T},\ \text{s.t. } \forall x \in \mathcal{S},\ \mathsf{A}(x, y)). \tag{0.6}$$

Example 0.3. True or false:
$\forall x \in [2, +\infty)$, $\exists y \in \mathbb{Z}^+$ s.t. $x^y < 10^5$;
$\exists y \in \mathbb{R}$ s.t. $\forall x \in [2, +\infty)$, $x > y$;
$\exists y \in \mathbb{R}$ s.t. $\forall x \in [2, +\infty)$, $x < y$.

Exercise 0.4 (Translating an English statement into a logic statement)**.** Goldbach's conjecture states *every even natural number greater than 2 is the sum of two primes.*

Solution. Let $\mathbb{P} \subset \mathbb{N}^+$ denote the set of prime numbers. Then Goldbach's conjecture is

$$\forall a \in 2\mathbb{N}^+ + 2, \exists p, q \in \mathbb{P} \text{ s.t. } a = p + q.$$

Theorem 0.6. The existential-universal statement implies the corresponding universal-existential statement, but not vice versa.

Example 0.5 (Translating a logic statement to an English statement)**.** Let $\mathcal{S}$ be the set of all human beings.
U_E =($\forall p \in \mathcal{S}, \exists q \in \mathcal{S}$ s.t. q is p's mom.)
E_U=($\exists q \in \mathcal{S}$ s.t. $\forall p \in \mathcal{S}$, q is p's mom.)
U_E is probably true, but E_U is certainly false.
If E_U were true, then U_E would be true. why?

Axiom 0.7 (First-order negation of logical statements)**.** The negations of the statements in Definition 0.3 are

$$\neg \mathsf{U} = (\exists x \in \mathcal{S}, \text{ s.t. } \neg \mathsf{A}(x)). \tag{0.7}$$
$$\neg \mathsf{E} = (\forall x \in \mathcal{S}, \ \neg \mathsf{A}(x)). \tag{0.8}$$

Rule 0.8. The negation of a more complicated logic statement abides by the following rules:

- switch the type of each quantifier until you reach the last formula without quantifiers;
- negate the last formula.

One might need to group quantifiers of like type.

Exercise 0.6. Write the logic statement for the negation of Goldbach's conjecture.

Solution. $\exists a \in 2\mathbb{N}^+ + 2$ s.t. $\forall p, q \in \mathbb{P}$, $a \neq p + q$.

Axiom 0.9 (Contraposition)**.** A conditional statement is logically equivalent to its contrapositive.

$$(\mathsf{A} \Rightarrow \mathsf{B}) \Leftrightarrow (\neg \mathsf{B} \Rightarrow \neg \mathsf{A}) \tag{0.9}$$

Example 0.7. "If Jack is a man, then Jack is a human being." is equivalent to "If Jack is not a human being, then Jack is not a man."

Exercise 0.8. Draw an Euler diagram of subsets to illustrate Example 0.7.

0.2 Ordered sets

Definition 0.10. The *Cartesian product* $\mathcal{X} \times \mathcal{Y}$ between two sets $\mathcal{X}$ and $\mathcal{Y}$ is the set of all possible ordered pairs with first element from $\mathcal{X}$ and second element from $\mathcal{Y}$:

$$\mathcal{X} \times \mathcal{Y} = \{(x, y) \mid x \in \mathcal{X}, \ y \in \mathcal{Y}\}. \tag{0.10}$$

Axiom 0.11 (Fundamental principle of counting)**.** A task consists of a sequence of k independent steps. Let n_i denote the number of different choices for the i-th step, the total number of distinct ways to complete the task is then

$$\prod_{i=1}^{k} n_i = n_1 n_2 \cdots n_k. \tag{0.11}$$

Example 0.9. Let A, E, D be the set of appetizers, main entrees, desserts in a restaurant. $A \times E \times D$ is the set of possible dinner combos. If $\#A = 10$, $\#E = 5$, $\#D = 6$, $\#(A \times E \times D) = 300$.

Remark 0.3. The menu of a restaurant in Example 0.9 seldom contains all the different combinations explicitly spelled out. You simply pick one entry from each category and call the three choices your own dinner. In math learning, you follow a similar pattern. You learn algebra and topology, then you combine the two to learn algebraic topology. The difference is that, the process of coupling the two is much more difficult and much more powerful.

Definition 0.12. A *binary relation between two sets* $\mathcal{X}$ and $\mathcal{Y}$ is an ordered triple $(\mathcal{X}, \mathcal{Y}, \mathcal{G})$ where $\mathcal{G} \subseteq \mathcal{X} \times \mathcal{Y}$.
A *binary relation on* $\mathcal{X}$ is the relation between $\mathcal{X}$ and $\mathcal{X}$. The statement $(x, y) \in R$ is read "x is R-related to y," and denoted by xRy or $R(x, y)$.

Definition 0.13. An *equivalence relation* "$\sim$" on $\mathcal{A}$ is a binary relation on $\mathcal{A}$ that satisfies $\forall a, b, c \in \mathcal{A}$,

- $a \sim a$ (reflexivity);
- $a \sim b$ implies $b \sim a$ (symmetry);
- $a \sim b$ and $b \sim c$ imply $a \sim c$ (transitivity).

Definition 0.14. A binary relation "$\leq$" on some set $\mathcal{S}$ is a *total order* or *linear order* on $\mathcal{S}$ iff, $\forall a, b, c \in \mathcal{S}$,

- $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry);
- $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity);
- $a \leq b$ or $b \leq a$ (totality).

A set equipped with a total order is a *chain* or *totally ordered set.*

Example 0.10. The real numbers with less or equal.

Example 0.11. The English letters of the alphabet with dictionary order.

Example 0.12. The Cartesian product of a set of totally ordered sets with the *lexicographical order.*

Example 0.13. Sort your book in lexicographical order and save a lot of time. $\log_{26} N \ll N$!

Definition 0.15. A binary relation "$\leq$" on some set $\mathcal{S}$ is a *partial order* on $\mathcal{S}$ iff, $\forall a, b, c \in \mathcal{S}$, antisymmetry, transitivity, and reflexivity ($a \leq a$) hold.
A set equipped with a partial order is a called a *poset.*

Remark 0.4. Change the symmetry condition in Definition 0.13 to antisymmetry condition in Definition 0.15 and you navigate from the equivalence relation to the partial order.

Example 0.14. The set of subsets of a set $\mathcal{S}$ ordered by inclusion "$\subseteq$." In this class we will not distinguish between "$\subseteq$" and "$\subset$."

Example 0.15. The natural numbers equipped with the relation of divisibility.

Example 0.16. The set of stuff you will put on your body every morning with the time ordered: undershorts, pants, belt, shirt, tie, jacket, socks, shoes, watch.

Example 0.17. Inheritance ("is-a" relation) is a partial order. $A \to B$ reads "B is a special type of A".

Example 0.18. Composition ("has-a" relation) is also a partial order. $A \rightsquigarrow B$ reads "B *has an* instance/object of A."

Example 0.19. Implication "$\Rightarrow$" is a partial order on the set of logical statements.

Example 0.20. The set of definitions, axioms, propositions, theorems, lemmas, etc., is a poset with inheritance, composition, and implication. It is helpful to relate them with these partial orderings.
"If syntax sugar does not count, there is nothing left."

0.3 Linear algebra

Definition 0.16. A *field* $\mathbb{F}$ is a set together with two binary operations "+" and "$*$," usually called "addition" and "multiplication," such that the following axioms hold $\forall a, b, c \in \mathbb{F}$,

- commutativity: $a + b = b + a$, $ab = ba$;
- associativity: $a+(b+c) = (a+b)+c$, $a(bc) = (ab)c$;
- identity: $a + 0 = a$, $a1 = a$;
- invertibility: $a + (-a) = 0$, $a \neq 0 \Rightarrow aa^{-1} = 1$;
- distributivity: $a(b + c) = ab + ac$,

where the distinguished members $0, 1 \in \mathbb{F}$ are called the *additive identity* and *multiplicative identity*, respectively.

Exercise 0.21. What is the smallest field?

Solution. It is of course $\{0, 1\}$.

Exercise 0.22. For $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, which of them are fields, which of them are not?

Solution. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are all fields. $\mathbb{Z}$ is not.

0.3.1 Vector spaces

Definition 0.17. A *vector space* or *linear space* over a field $\mathbb{F}$ is a set V together with a binary operation $+ : V\times V \to V$ and another scalar multiplication $\cdot : \mathbb{F}\times V \to V$ respectively called vector addition and scalar multiplication that satisfy the following axioms:

(VSA-1) associativity
$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(VSA-2) additive identity
$\exists \mathbf{0} \in V$, $\forall \mathbf{u} \in V$, s.t. $\mathbf{u} + \mathbf{0} = \mathbf{u}$;

(VSA-3) additive inverse
$\forall \mathbf{u} \in V$, $\exists \mathbf{v} \in V$, s.t. $\mathbf{u} + \mathbf{v} = \mathbf{0}$;

(VSA-4) commutativity
$\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(VSA-5) multiplicative identity
$\forall \mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$;

(VSA-6) compatibility
$\forall \mathbf{u} \in V$, $\forall a, b \in \mathbb{F}$, $(ab)\mathbf{u} = a(b\mathbf{u})$;

(VSA-7) distributive laws

$$\forall \mathbf{u}, \mathbf{v} \in V,\ \forall a, b \in \mathbb{F},\ \left\{ \begin{array}{l} (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}, \\ a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}. \end{array} \right.$$

The elements of V are called *vectors* and the elements of $\mathbb{F}$ are called *scalars*.

Example 0.23. The Euclidean n-space $\mathbb{R}^n$ over the field $\mathbb{R}$ is a vector space. In particular, $\mathbb{R}$ is a vector space where the scalars and vectors are the same set. Then the vector space reduces to a field.

Example 0.24. Let $\mathcal{C}[a, b]$ be the set of functions $\mathbb{R} \to \mathbb{R}$ that are continuous on $[a, b]$. This is also a vector space.

Example 0.25. The set of polynomials $\mathbb{R}[x]$ with their orders less than a fixed integer is a vector space.

Definition 0.18. A *list of length n or n-tuple* is an ordered collection of n objects separated by commas and surrounded by parentheses: $(x_1, \ldots, x_n)$.

Remark 0.5. An n-tuple is different from a set in that repetition of the same objects are allowed.

Definition 0.19. A *linear combination* of a list of vectors $\{\mathbf{v}_i\}$ is a vector of the form $\sum_i a_i\mathbf{v}_i$ where $a_i \in \mathbb{F}$.

Example 0.26. Matrix-vector product can be viewed as a linear combination of the column vectors of the matrix.

Definition 0.20. The *span* of a m-tuple $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is the set of all linear combinations of these vectors,

$$\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \left\{ \sum_{i=1}^{m} a_i\mathbf{v}_i : a_i \in \mathbb{F} \right\}. \tag{0.12}$$

In particular, the span of the empty set is $\{\mathbf{0}\}$. We say that $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ *spans* V if $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$.

Example 0.27. The two vectors $(1, 0)$ and $(0, 1)$ span the whole Euclidean plane.

Definition 0.21. A vector space V is called *finite dimensional* if some finite list of vectors span V; otherwise it is called *infinite dimensional*.

Remark 0.6. Linear algebra concerns finite-dimensional vector spaces and infinite-dimensional vector spaces are the subject of functional analysis.

Definition 0.22. A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ in V is called *linearly independent* iff

$$\sum_{i=1}^{m} a_i\mathbf{v}_i = \mathbf{0} \ \Rightarrow\ \forall i, a_i = 0. \tag{0.13}$$

Otherwise the list of vectors is called *linearly dependent*.

Definition 0.23. A *basis* of a vector space V is a list of vectors in V that is linearly independent and spans V. In particular, the list of vectors

$$\left((1, 0, \cdots, 0)^T,\ (0, 1, 0, \cdots, 0)^T,\ \ldots,\ (0, \cdots, 0, 1)^T\right) \tag{0.14}$$

is called the *standard basis* of $\mathbb{F}^n$.

Definition 0.24. The *dimension* of a finite-dimensional vector space V, denoted $\dim V$, is the length of any basis of the vector space.

Theorem 0.25. If V is finite-dimensional, then every *minimal spanning list* in V, i.e. a spanning list of vectors with length $\dim V$, is a basis of V.

Theorem 0.26. If V is finite-dimensional, then every *maximal linearly independent list* in V, i.e. a linearly independent list of vectors with length $\dim V$, is a basis of V.

0.3.2 Linear maps

Definition 0.27. A *linear map* or *linear transformation* between two vector spaces V and W is a function $T : V \to W$ that satisfies

(LNM-1) additivity
$\forall \mathbf{u}, \mathbf{v} \in V$, $T(\mathbf{u}+\mathbf{v}) = T\mathbf{u} + T\mathbf{v}$;

(LNM-2) homogeneity
$\forall a \in \mathbb{F}$, $\forall \mathbf{v} \in V$, $T(a\mathbf{v}) = a(T\mathbf{v})$,

where $\mathbb{F}$ is the underlying field of V and W. In particular, a linear map is called a *linear operator* if $W = V$.

Notation 2. The set of all linear maps from V to W is denoted by $\mathcal{L}(V, W)$. The set of all linear operators from V to itself is denoted by $\mathcal{L}(V)$.

Definition 0.28. A *linear functional* on V is a linear map from V to $\mathbb{F}$, or, it is an element of $\mathcal{L}(V, \mathbb{F})$.

Example 0.28. The differentiation operator on $\mathbb{R}[x]$ is a linear map $T \in \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$

Example 0.29. $\mathbb{F}^{m\times n} = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space with the additive identity as the zero map $\mathbf{0}$.

Lemma 0.29. The set $\mathcal{L}(V, W)$, equipped with vector addition $(S + T)\mathbf{v} = S\mathbf{v} + T\mathbf{v}$ and scalar multiplication $(aT)\mathbf{v} = a(T\mathbf{v})$ is a vector space.

Proof. The scalar field $\mathbb{F}$ of $\mathcal{L}(V, W)$ is the same as that of V and W. So multiplicative identity is still 1, the same as that of $\mathbb{F}$. However, the additive identity is the zero map $\mathbf{0} \in \mathcal{L}(V, W)$. □

Definition 0.30. The *identity map*, denoted by I, is the function on a vector space that assigns to each element to the same element:

$$I\mathbf{v} = \mathbf{v}. \tag{0.15}$$

Definition 0.31. The *null space* of a linear map $T \in \mathcal{L}(V, W)$ is the subset of V consisting of those vectors that T maps to the additive identity $\mathbf{0}$:

$$\text{null } T = \{\mathbf{v} \in V : T\mathbf{v} = \mathbf{0}\}. \tag{0.16}$$

Example 0.30. The null space of the differentiation map in Example 0.28 is $\mathbb{R}$.

Definition 0.32. The *range* of a linear map $T \in \mathcal{L}(V, W)$ is the subset of W consisting of those vectors that are of the form $T\mathbf{v}$ for some $\mathbf{v} \in V$:

$$\text{range } T = \{T\mathbf{v} : \mathbf{v} \in V\}. \tag{0.17}$$

Example 0.31. The range of $A \in \mathbb{F}^{m\times n}$ is the span of its column vectors.

Theorem 0.33. If V is a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$, then range T is a finite-dimensional subspace of W and

$$\dim V = \dim \text{null } T + \dim \text{range } T. \tag{0.18}$$

Definition 0.34. The *matrix of a linear map* $T \in \mathcal{L}(V, W)$ *with respect to the bases* $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of V and $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ of W, denoted by

$$A_T := M(T, (\mathbf{v}_1, \ldots, \mathbf{v}_n), (\mathbf{w}_1, \ldots, \mathbf{w}_m)), \tag{0.19}$$

is the $m \times n$ matrix $A(T)$ whose entries $a_{i,j} \in \mathbb{F}$ satisfy the linear system

$$\forall j = 1, 2, \ldots, n, \qquad T\mathbf{v}_j = \sum_{i=1}^{m} a_{i,j}\mathbf{w}_i. \tag{0.20}$$

Remark 0.7. There are $m \times n$ equations and $m \times n$ variables in the linear system (0.19). In $\sum_{i=1}^{m} a_{i,j}\mathbf{w}_i$, we index a as $a_{i,j}$, not $a_{j,i}$, why? Because when we write it in matrix product form, $a_{i,j}$ naturally corresponds to the (i, j) entry of the matrix.

Remark 0.8. When $m \neq n$, any matrix $\mathbb{F}^{m\times n}$ cannot be one-to-one and hence V and W cannot be isomorphic. It is well known that V and W are isomorphic if and only if they have the same dimension. Hence if the matrix of the linear operator T is non-singular, T must be an automorphism.

Corollary 0.35. The matrix A_T in (0.19) of a linear map $T \in \mathcal{L}(V, W)$ satisfies

$$T[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]A_T. \tag{0.21}$$

Proof. This follows directly from (0.19). □

Remark 0.9. Definition 0.34 packages an essential idea: if we express any vector as a linear combination of the basis and insist on the conditions in Definition 0.27, the action of a linear map on an arbitrary vector can be characterized by the effects on the two bases via the corresponding matrix of the linear map. More precisely, for $\mathbf{u} = \sum_i c_i\mathbf{v}_i$, we have

$$T\mathbf{u} = \sum_i c_i T\mathbf{v}_i = T[\mathbf{v}_1, \ldots, \mathbf{v}_n]\mathbf{c} = [\mathbf{w}_1, \ldots, \mathbf{w}_m]A_T\mathbf{c}.$$

Finally, a linear map does not depend on the choice of basis while its corresponding matrix does.

0.3.3 Dual vector spaces

Definition 0.36. The *dual space* of a vector space V is the vector space of all linear functionals on V,

$$V' = \mathcal{L}(V, \mathbb{F}). \tag{0.22}$$

Definition 0.37. For a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of a vector space V, its *dual basis* is the list $\varphi_1, \ldots, \varphi_n$ where each $\varphi_j \in V'$ is

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases} \tag{0.23}$$

Exercise 0.32. Show that the dual basis is a basis of the dual space.

Solution. This follows from the definition of linear independence and Definition 0.37.

Lemma 0.38. A finite-dimensional vector space V satisfies

$$\dim V' = \dim V. \tag{0.24}$$

Proof. This follows from Definition 0.36 and the identity $\dim \mathcal{L}(V, W) = \dim(V) \dim(W)$. □

Definition 0.39. The *double dual space* of a vector space V, denoted by V'', is the dual space of V'.

Remark 0.10. A map $\theta \in V''$ has the signature $\theta : V' \to \mathbb{F}$.

Lemma 0.40. The function $\Lambda : V \to V''$ defined as

$$\forall v \in V, \forall \varphi \in V', \qquad (\Lambda v)(\varphi) = \varphi(v) \tag{0.25}$$

is a linear bijection.

Proof. It is easily verified that Λ is a linear map. The rest follows from Definitions 0.36, 0.39, and Lemma 0.38. □

Remark 0.11. If V is finite-dimensional, V' and V are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and is considered more natural.

0.3.4 Dual linear maps

Definition 0.41. The *dual map* of a linear map $T : V \to W$ is the linear map $T' : W' \to V'$ defined as

$$\forall \varphi \in W', \qquad T'(\varphi) = \varphi \circ T. \tag{0.26}$$

Exercise 0.33. Denote by d the linear map $\mathrm{d}p = p'$ on the vector space $\mathcal{P}(\mathbb{R})$ of polynomials with real coefficients. Under the dual map of d, what is the image of the linear functional $\varphi(p) = \int_0^1 p$ on $\mathcal{P}(\mathbb{R})$?

Solution.

$$(\mathrm{d}'\varphi)(p) = (\varphi \circ \mathrm{d})(p) = \varphi(p') = \int_0^1 p' = p(1) - p(0).$$

Theorem 0.42. The matrix of T' is the transpose of the matrix of T.

Proof. Exercise. □

Definition 0.43. The *double dual map* of a linear map $T : V \to W$ is the linear map $T'' : V'' \to W''$ defined as $T'' = (T')'$.

Theorem 0.44. For $T \in \mathcal{L}(V)$ and Λ in (0.25), we have

$$T'' \circ \Lambda = \Lambda \circ T. \tag{0.27}$$

Proof. Definition 0.43 and equation (0.25) yields

$$\begin{aligned} &\forall v \in V,\ \forall \varphi \in V', \\ &(T'' \circ \Lambda)v\varphi = ((T')'\Lambda v)\varphi = (\Lambda v \circ T')\varphi = \Lambda v(T'\varphi) \\ &\qquad = (T'\varphi)(v) = \varphi(Tv) = \Lambda(Tv)(\varphi) \\ &\qquad = (\Lambda \circ T)v\varphi, \end{aligned}$$

where the third step is natural since T' send V' to V'. □

Corollary 0.45. For $T \in \mathcal{L}(V)$ where V is finite-dimensional, the double dual map is

$$T'' = \Lambda \circ T \circ \Lambda^{-1}. \tag{0.28}$$

Proof. This follows directly from Theorem 0.44 and Lemma 0.40. □

Remark 0.12. For $T \in \mathcal{L}(V)$, the linear map Λ in Lemma 0.40 can be used to construct a formula of T''. By the signature of $T'' : V'' \to V''$, the input of T'' is a linear functional $\pi : V'' \to \mathbb{F}$. By sending V to V'', Λ satisfies $\varphi = \pi \circ \Lambda$, as shown below.

$$\begin{tikzcd} V'' \arrow[dr, "\pi"] & \\ V \arrow[u, "\Lambda"] \arrow[r, "\varphi"] & \mathbb{F} \end{tikzcd}$$

The commutative diagram for Theorem 0.44 is as follows.

$$\begin{tikzcd} V'' \arrow[r, "T''"] & V'' \\ V \arrow[u, "\Lambda"] \arrow[r, "T"] & V \arrow[u, "\Lambda"] \end{tikzcd}$$

Flip the direction of the left arrow and we have the diagram for Corollary 0.45.