Chapter 0

Homework1

We collect concepts and results in a coherent manner to form a solid foundation for our study of computational homology. Every math major should master the English glossary as well as the math in this chapter.

0.1 Chapter 1

0.1.1 2. Let k be a positive integer. Show that a smooth function defined on $\mathbb{R}^n$ has in general $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ distinct partial derivatives of order k.

Solution. Think this question as a problem of choose n-1 positions from n+k-1 positions in a queue. Which has $\binom{n+k-1}{n-1}$ possibles to do that.

Then make a map f from the choose way to a derivatives of order k of the smooth function.

The n-1 positions divide the queue into n part that contains 0 position or some positions. Take a_i as the number of position in the i-th part, so $0 \leq a_i \leq k$.

$$f(a_1, a_2..., a_n) = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} u$$

It is easy to proof the map f is a bijective mapping. Because partial derivatives of order k also sets the $0 \leq a_i \leq k$ always true.

And $\binom{n+k-1}{k}$ is coming to choose k position and these near positions combine with a part. Then the same map f finish the remain proof part.

0.1.2 4. Prove Leibniz's fomula 0.1. where $u, v : \mathbb{R}^n \to \mathbb{R}$ are smooth, $\binom{\alpha}{\beta} := \frac{\alpha!}{\alpha!(\alpha-\beta)!}$, and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i (i = 1, ..., n)$.

$$D^\alpha(uv) = \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} u, \tag{0.1}$$

Solution. By induction proof.

When $\alpha = 0$ is trivial problem.

Now suppose when satisfy $\alpha \leq k$, equation(0.1) already true.

$$D^{k+1}(uv) = D^\alpha(Duv + uDv) \tag{0.2}$$

using Duv and uDv replace uv in (0.1) get

$$\begin{aligned} D^{k+1}(uv) &= \sum_{\beta < \alpha \leq k} (\binom{\alpha-1}{\beta-1} + \binom{\alpha-1}{\beta}) D^\beta u D^{\alpha-\beta} u \\ &= \sum_{\beta < \alpha \leq k+1} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} u \end{aligned} \tag{0.3}$$

0.2 Chapter 2

0.2.1 1. Write down an explicit formula for a function u solving the initial-value problem 0.4. Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{0.4}$$

Solution. Let $Z(s) = u(t + s, x + sb) \cdot e^{cx}$

Then we have $\partial_s Z(s) = 0$ from below

$$\begin{aligned} \partial_s Z(s) &= \partial_s u(t + s, x + sb) \cdot e^{cx} \\ &= \partial_t u(t + s, x + sb) \cdot e^{cx} + \\ &\quad b\partial_x u(t + s, x + sb) \cdot e^{cx} + cu(t + s, x + sb) \cdot e^{cx} \\ &= u_t + b \cdot Du + cu = 0 \end{aligned} \tag{0.5}$$

So

$$\begin{aligned} u(t, x) &= Z(0) \cdot e^{-cx} \\ &= Z(-t) \cdot e^{-cx} \\ &= u(0, x - tb) \cdot e^{-cx} \\ &= g(x - tb) \cdot e^{-cx} \end{aligned} \tag{0.6}$$

0.2.2 3. Modify the proof of the mean-value formulas to show for $n \geq 3$ that 0.7 provided 0.8

$$u(0) = \fint_{\partial B(0,r)} gdS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f dx, \tag{0.7}$$

$$\begin{cases} -\Delta u = f & \text{in } B^0(0, r) \\ u = g & \text{on } \partial B(0, r) \end{cases} \tag{0.8}$$

Solution.

$$\begin{aligned} \int_0^r \frac{\int_{B_r(x)} \Delta F dx}{|\partial B_r(x)|} &= \int_{B(0,r)} -f \int_x^r \frac{1}{n\alpha(n) y^{1-n}} dy dx \\ &= -\frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f dx \end{aligned} \tag{0.9}$$

So can get 0.7

0.3 Therorem

0.3.1 (Intergration-by-parts formula). Let $u, v \in C^1(\bar{U})$. Then get 0.10

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS \quad (i = 1, ..., n) \tag{0.10}$$

Solution.

$$\int_U u_{x_i} v dx + \int_U u v_{x_i} dx = \int_U (uv)_{x_i} dx \tag{0.11}$$

Then according Gauss-Green Theorem and replace u as uv.

$$\int_U (uv)_{x_i} dx = \int_{\partial U} u v \nu^i dS \tag{0.12}$$

0.3.2 (Green's formulas). Let $u, v \in C^2(\bar{U})$. Then have 0.13 0.14 0.15

(i)

$$\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS, \tag{0.13}$$

(ii)

$$\int_U Dv \cdot Du dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS, \tag{0.14}$$

(iii)

$$\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS. \tag{0.15}$$

Solution. By Gauss-Green Theorem:

i

$$\begin{aligned} \int_U \Delta u dx &= \sum_1^n \int_U u_{x_i x_i} dx \\ &= \sum_1^n \int_{\partial U} u_{x_i} \nu^i dS \\ &= \int_{\partial U} \nabla u \cdot \nu dS \\ &= \int_{\partial U} \frac{\partial u}{\partial \nu} dS. \end{aligned}$$

ii By (Intergration by parts formula) conclusion

$$\begin{aligned} \int_U Dv \cdot Du dx &= -\int_U u \Delta v dx + \int_U u D v dx \\ &= -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS. \end{aligned}$$

iii By (ii) conclusion

$$\int_U u \Delta v dx = \int_{\partial U} \frac{\partial v}{\partial \nu} u dS - \int_U Dv \cdot Du dx$$

replace u,v with u,v and v,u respectively. and add together get (iii).