Chapter 14

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In these exercises U always denotes an open subset of $\mathbb{R}^n$, with a smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

Problem 14.1. Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}$$

Solution. Assume $\|u\|_{C^{0,\gamma}(U)} = \sup\limits_{x\in U} |u(x)| + [u]_{C^{0,\gamma}(\bar{U})} = |u(a)| + \frac{|u(x)-u(y)|}{|x-y|^\gamma}$, $\|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} = (\sup\limits_{x\in U} |u(x)| + [u]_{C^{0,\beta}(\bar{U})})^{\frac{1-\gamma}{1-\beta}} \geq (|u(a)| + \frac{|u(x)-u(y)|}{|x-y|^\beta})^{\frac{1-\gamma}{1-\beta}}$, $\|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}} = (\sup\limits_{x\in U} |u(x)| + [u]_{C^{0,1}(\bar{U})})^{\frac{\gamma-\beta}{1-\beta}} \geq (|u(a)| + \frac{|u(x)-u(y)|}{|x-y|})^{\frac{\gamma-\beta}{1-\beta}}$

Let $p_1 = \frac{1-\beta}{1-\gamma}, p_2 = \frac{1-\beta}{\gamma-\beta}$, we have $\frac{1}{p_1} + \frac{1}{p_2} = 1$. By Hölder inequality

$$(|u(a)| + \frac{|u(x)-u(y)|}{|x-y|^\beta})^{\frac{1-\gamma}{1-\beta}} (|u(a)| + \frac{|u(x)-u(y)|}{|x-y|})^{\frac{\gamma-\beta}{1-\beta}} \geq |u(a)| + \frac{|u(x)-u(y)|}{|x-y|^\gamma},$$

which implies that $\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}$

Problem 14.2. Let U be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(U)(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$

Solution. Assume T is the bounded linear operator. Take $u_m = \max\{0, 1 - m\,\mathrm{dist}(x, \partial U)\}$, we have $\forall x \in \partial U, Tu_m(x) = 1$, so $\|Tu_m\|_{L^p(\partial U)} = (\int_{\partial U} 1^p dS)^{\frac{1}{p}} = (S(\partial U))^{\frac{1}{p}} > 0$.

However, let $B_m = \{x \mid x \in \bar{U}, u_m(x) \neq 0\}$, By the definition of u_m, we know $V(B_m) \to 0$, as $m \to +\infty$. So $\|u_m\|_{L^p(\bar{U})} = (\int_U u_m^p dx)^{\frac{1}{p}} = (\int_{B_m} u_m^p dx)^{\frac{1}{p}} \leq (\int_{B_m} 1 dx)^{\frac{1}{p}} = (V(B_m))^{\frac{1}{p}} \to 0$, as $m \to +\infty$.

After all, we have $\|T\| \geq \limsup\limits_{m\to+\infty} (\frac{\|Tu_m\|_{L^p(\partial U)}}{\|u_m\|_{L^p(U)}})^{\frac{1}{p}} = (\frac{S(\partial U)}{0})^{\frac{1}{p}} = +\infty$.That is contradict with T is bounded operator.

Problem 14.3. Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \leq C\|u\|_{L^2}^{1/2} \left\|D^2 u\right\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$ (Hint: Take sequences $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(\bar{U})$converging to u in $H^2(U)$.)

Solution.

1. $u \in C_c^\infty(U)$

$$\begin{aligned}
\|Du\|_{L^2} &= (\int_U |Du|^2 \, dx)^{\frac{1}{2}} \\
&= (\int_U Du \cdot Dudx)^{\frac{1}{2}} \\
&= (-\int_U u\Delta udx)^{\frac{1}{2}} \\
&\leq (\int_U |u|^2 \, dx \int_U |\Delta u|^2 \, dx)^{\frac{1}{4}} \\
&\leq C(\int_U |u|^2 \, d)^{\frac{1}{4}} (\int_U \left|D^2 u\right|^2 dx)^{\frac{1}{4}} \\
&= C\|u\|_{L^2}^{\frac{1}{2}} \|Du\|_{L^2}^{\frac{1}{2}}
\end{aligned}$$

2. $u \in H_0^1(U) \cup H^2(U)$

   Choose $\{v_n\} \in C_c^\infty(U), s.t. v_n \to u \in H_0^1(U), \{w_n\} \in C^\infty(U), s.t. w_n \to u \in H^2(U)$, Therefore, $\lim\limits_{n\to\infty} \|v_n\|_{L^2} = \|u\|$, $\lim\limits_{n\to\infty} \|D^2 w_n\|_{L^2} = \|D^2 u\|_{L^2}$.

$$\begin{aligned}
\int_U Dv_k \cdot Dw_k dx &= -\int_U v_k \Delta w_k dx \\
&\leq C\|v_k\|_{L^2}^{\frac{1}{2}} \|D^2 w_k\|_{L^2}^{\frac{1}{2}}
\end{aligned}$$

   And

$$\begin{aligned}
&\int_U |Du|^2 - Dv_k \cdot Dw_k dx \\
&= \int_U Du(Du - Dv_k) + Dv_k(Du - Dw_k) dx \\
&\leq \|Du\|_{L^2} \| \|Du - Du_k\|_{L^2} + \|Dv_k\|_{L^2} \|Du - Dw_k\|_{L^2} \\
&\to 0 \qquad \text{as } k \to \infty
\end{aligned}$$

   In summary, $\lim\limits_{k\to\infty} \int_U Dv_k \cdot Dw_k dx \leq \lim\limits_{k\to\infty} C\|v_k\|_{L^2}^{\frac{1}{2}} \|D^2 w_k\|_{L^2}^{\frac{1}{2}} \Longrightarrow \|Du\|_{L^2} \leq C\|u\|_{L^2}^{1/2} \|D^2 u\|_{L^2}^{1/2}$.