Stirring by anisotropic squirming

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Abstract

We consider a fluid stirred by the locomotion of submerged swimming bodies. We generalise the model proposed by Thiffeault and Childress [1] and study swimmers moving in anisotropically random directions. The stirring-induced, anisotropic effective diffusion tensor is calculated from the classical Itô theory in 2D and in axisymmetrically 3D, respectively. Then we define two dimensionless, scalar indices to quantify the anisotropy in the diffusion process. Further, we found that given different probability distribution, the different anisotropic stirring may cause same diffusion property. With compared analytical result of diffusion tenor of particle and swimmer, we find a number which can measure the difference between them. Then, we study the relationship between anisotropy of swimming and anisotropy of diffusion coefficient, and proposed two dimensionless numbers which measure the two anisotropy respectively. From those numbers we can easily judge the anisotropy of them.

Keywords: biomixing, diffusion tensor, anisotropic, distribution

1. Introduction

In the past two decades, the study of biogenic mixing has attracted a plethora of research effort. For scientists and engineers working with swimming organisms submerged in fluid environments of various scales, such as bacteria, algal cells, krill, fish, and etc., a common and intriguing question frequently arises: Does the collective swimming motion of aquatic species enhance passive scalar mixing? Especially, this was been suggested as a significant mechanism in ocean mixing [2, 3] although its efficiency is still controversial [4, 5, 6, 7]. Biomixing in biochemical suspensions of small organisms has also been studied [8, 9, 10, 11] for different experimentally controlled concentrations.

Among the different approaches trying to tackle the problem, Thiffeault and Childress[1] developed a microscopic framework that combines classical results in hydrodynamics and stochastic modeling and produces a rigorous formula for effective diffusivity. The central idea involved is called Darwinian drift, studied in the pioneering work by Maxwell [12] and Darwin [13]. It is the mass displacement due to a moving body and it

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was suggested by Katija [14] as the dominant effect in the swimming-induced mixing. Several studies measures drift volume to directly evaluate the transport ability of an individual swimmer [15, 16, 17]. Alternatively, in this paper we follow the methodology by Thiffeault and Childress since the effective diffusivity computed by averaging multiple drifts is a natural characterisation for the mixing effect of an ensemble of random swimmers similar to a Brownian motion. Furthermore, it enables us to compare the mixing behavior of a passive scalar under different physical scenarios documented in a vast spectrum of literature.

Thiffeault,Childress[1] derived the effective diffusivity of an randomly distributed non-interacting swimmers in potential flow, and showed that it is due to the repeated displacements induced by a swimmer on a particle of fluid. Zhi lin et al [18] developed this model into stokes flow, given the drift caused by one swimmer, an effective diffusivity coefficient could be computed. And Those model has been tested in physical experiments and numerical simulation [19, 20]. Kasyap *et. al.*[21] studied slender-body bacteria affect the passive tracers with similar model and analysis the dimensionless number like peclet number and diffusion coefficient.

The model above focus on the isotropic case, but the anisotropic is more ubiquitous in the nature. Study anisotropic case has much application, like designing requisite nanomotors [22], manipulating microscale structures which is an unsolved challenge in micro engineering and microtechnologyby and herding bacteria, making it moving in certain direction, is one way to achieve that[23]. Also the interest in the dynamical properties of interacting, self-propelled organisms which has tropism movement under anisotropic fluid [23] or external field have been studied [24, 8, 25]. Furthermore, there has significant different result induce by anisotropic squirming, which we will discuss in the following sections.

Our paper is constructed as follow. In section 2, We generalize the model proposed by Thiffeault, Childress [26] into anisotropic case. In section 3, we proposed a general steps to calculate the effective diffusion tensor of particle by applying *Itô* diffusion theory. In section 4, we give a theoretical analysis about the difference between diffusion tensor of particle and diffusion tensor of swimmer. In section 5, diffusivity in any given direct in 2 and 3 dimension is calculated. In section 6 We proposal the indicator of anisotropic distribution and indicator of diffusion coefficient in 2 dimension. Section 7 is devoted to comparing our predictions to the results of numerical simulations.

2. stochastic stirring model

The setting of our problem is a large volume V that contains a number of swimmers N, also typically large. The swimmers move a distance λ independently of each other in random directions and change the directions randomly. We assume the distribution of swimmers are dilute enough so that the velocity field of one swimmer is not significantly affected by the others. Hence a fluid particle (not too near the edges of the domain), will be displaced by the cumulative action of the swimmers.

For simplicity, we assume the swimmer start from the original point, move along straight paths λ at a fixed speed $\boldsymbol{U}$. The velocity field induced at point x by a swimmer is $\boldsymbol{u}\,(\boldsymbol{x} - \boldsymbol{U}t)$. For a fluid particle which initially at $\boldsymbol{x} = \boldsymbol{\eta}$ which has uniform distribution in the space affected by a single swimmer described above, the drift of it is $\boldsymbol{\Delta}_\lambda(\boldsymbol{\eta}, \boldsymbol{U})$:

$$
\begin{aligned}
\boldsymbol{\Delta}_\lambda(\boldsymbol{\eta}, \boldsymbol{U}) &= \int_0^{\lambda \backslash U} \boldsymbol{u}(\boldsymbol{x}(s) - \boldsymbol{U}s) ds \\
\dot{\boldsymbol{x}} &= \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{U}t) \\
\boldsymbol{x}(0) &= \boldsymbol{\eta} \\
U &= |\boldsymbol{U}|
\end{aligned} \tag{1}
$$

Actually, under those equations above, $\boldsymbol{\Delta}_\lambda$ do not depend on the magnitude of U. Hence, to introduce a unit vector $\boldsymbol{a} = \frac{\boldsymbol{U}}{U}$, which is characterizing the orientation of the swimmer motion. Let the PDF of swimmers' moving direction be $f(\boldsymbol{a})$. By the definition of $f(\boldsymbol{a})$, we have $f(\boldsymbol{a}) = f(c\boldsymbol{a})$ for any positive constant. We obtain the probability density of displacement:

$$
p_{\boldsymbol{R}_\lambda^1}(\boldsymbol{r}) = \int_{S^d} f(\boldsymbol{a}) \int_{\Omega} \delta(\boldsymbol{r} - \boldsymbol{\Delta}_\lambda(\boldsymbol{\eta}, \boldsymbol{U})) \frac{d\boldsymbol{\eta}}{V} d\boldsymbol{a} \tag{2}
$$

Here Ω represent the region and V is area of it, $\boldsymbol{R}_\lambda^1$ is a random variable that gives the displacement of the particle from its initial position after being affected by a single swimmer with path length λ. We denote by $p_{\boldsymbol{R}_\lambda^1}(\boldsymbol{r})$ the pdf of $\boldsymbol{R}_\lambda^1$, S^d is unit sphere in d dimensions. $\boldsymbol{R}_\lambda^N$ represent the displacement of the particle from its initial position after being affected by N swimmers with path length λ. Obviously $\boldsymbol{R}_\lambda^N = \sum_{i=1}^N \boldsymbol{R}_\lambda^1$, according to the dilute assumption.

3. Effective diffusion tensor

Isotropic case is a kind of perfect state. However, in reality, anisotropic stirring is more common, like the self-propelled organisms which has tropism movement under anisotropic fluid [23], chemical gradients [24] or external field [8, 25], like the anisotropic swimming cause by stratified structure of ocean [27]. The anisotropic stirring make diffusion tensor anisotropic, that is, different "stirring" effects in different directions. Obviously, if the probability of swimmers moving along x direction is more than those of y direction, the effective diffusivity along x direction should be also bigger than that of y direction, which are always equal in isotropic case.

There is no doubt that the trajectory of swimmer is equivalent to brownian motion on large spatiotemporal scales. But the trajectory of particle is not exactly too. Those long jump in the figure 1 is due to heavy tail of pdf. Many article indicated that the pdf has the feature of Gaussian core and exponential tail [26, 9], in the experimental and theoretical. On the other hand, the variance of this process is still proportional to the square of time[28], which is as same as brownian motion. In addition, many article use the effective diffusivity to measure the effect of stirring [1, 29]. Therefore, we still use the effective diffusivity to measure the effect of stirring.

As we discussed above, in the anisotropic case, "stirring" effects in different directions is different. Hence it is not enough to describe the all diffusion ability with one number. We need use the diffusion tensor instead of it. we use *Itô* diffusion process to

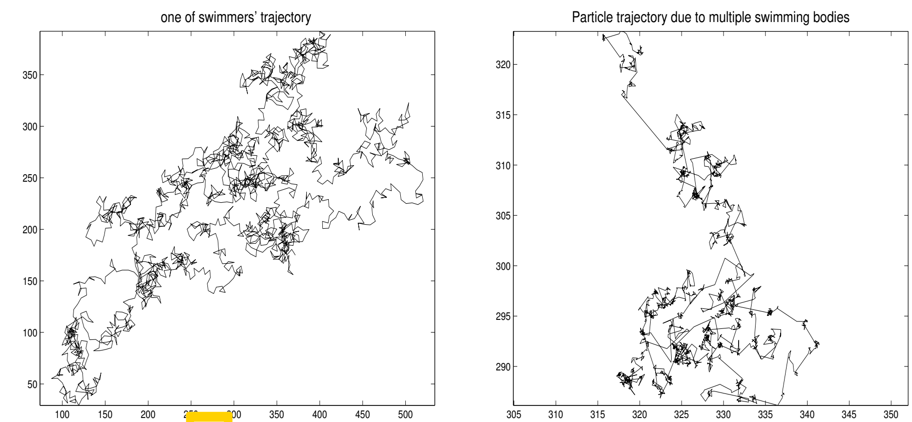

Figure 1: The plot show the [illegible] of numeric[illegible]ulation. One particle is stirred by multiple swimmers, which do the run-and- tumble motion. [illegible]ajectory of one of those swimmers, which is ordinary random walks and is equivalent to brownian motion on large spatiotemporal scales. (b) Particle trajectory due to multiple [illegible]ming bodies, which exhibits local clustering interspersed with long distance jumps. That is feature o[illegible]vy tail of pdf.

describe the trajectory of particle. Now we motion. *Itô* diffusion process is the solution to the following stochastic differential equation:

$$d\boldsymbol{R}_\lambda(t) = \boldsymbol{b}\,(t, \boldsymbol{R}_\lambda(t))\,dt + \boldsymbol{\sigma}\,(t, \boldsymbol{R}_\lambda(t))\,d\boldsymbol{B}_t, \tag{3}$$

where $\boldsymbol{R}_\lambda(t) \in R^d, \boldsymbol{b}\,(t,x) \in R^d, \boldsymbol{\sigma}\,(t,x) \in R^{d\times m}$, $\boldsymbol{B}_t$ is m-dimensional Brownian motion. $\boldsymbol{b}$ is so-call[illegible]he drift coefficient and $\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\sigma}^T$ is the diffusion coefficient or called diffusion tens[illegible] The displacement is so small compare to λ that we can regard $\boldsymbol{b}, \boldsymbol{\sigma}$ as approximately invariants that have no correlation with time and space. By straightforward calculating ,the expression of the diffusion tensor show

$$DT = \frac{1}{2}\sigma\sigma^T = \frac{\left(\mathrm{cov}\left(\boldsymbol{R}_{\lambda,i}(t), \boldsymbol{R}_{\lambda,j}(t)\right)\right)_{d\times d}}{2t}, \tag{4}$$

where $\boldsymbol{R}_{\lambda,i}(t)$ is the i-th coordinate component process of total process $\boldsymbol{R}_\lambda($[illegible]ithin one period time $t = \frac{\lambda}{U}$, the displacement induced by all swimmers in the region is $\boldsymbol{R}_\lambda^N$. When λ is large enough the displacement induced in each period can be regard as independence and $\boldsymbol{R}_\lambda^N = \sum\limits_{i=1}^{N} \boldsymbol{R}_\lambda^1$, diffusion tensor become

$$\begin{aligned} DT &= \frac{U\left(\mathrm{cov}\left(\boldsymbol{R}_{\lambda,i}^1, \boldsymbol{R}_{\lambda,j}^1\right)\right)_{d\times d}}{2\lambda}\{DT_{i,j}\}_{i,j} \\ &= \frac{nU\int_\Omega \Delta_\lambda(\boldsymbol{\eta})^2 d\boldsymbol{\eta}}{2\lambda}\{E\left(r_{k,i}^1 r_{k,j}^1\right) - \rho E\left(r_{k,i}^1\right)E\left(r_{k,j}^1\right)\}_{i,j} \end{aligned} \tag{5}$$

Here $r^1_{\lambda,i}$ is the i-th component of unite vector which represent the direction of $\boldsymbol{R}^1_\lambda$, and $\rho = \frac{\left(\int_\Omega \Delta_\lambda(\boldsymbol{\eta})d\boldsymbol{\eta}\right)^2}{V\int_\Omega \Delta_\lambda(\boldsymbol{\eta})^2 d\boldsymbol{\eta}}$. With the fact that $\rho \to 0$, when $V \to \infty$, the (5) turn out to be

$$DT = \frac{nU\int_\Omega \Delta_\lambda(\boldsymbol{\eta})^2 d\boldsymbol{\eta}}{2\lambda}\{E\left(r^1_{k,i}r^1_{k,j}\right)\}_{i,j} \tag{6}$$

The effective mean diffusivity κ, which we often discussed in the isotropic case, is average of trace of diffusion tensor. And because of $\sum_{i=1}^d E(r^1_{k,i}r^1_{k,i}) = E(\sum_{i=1}^d r^1_{k,i}r^1_{k,i}) = E(1) = 1$:

$$\kappa = \frac{1}{d}tr\,(DT) = \frac{nU\int_\Omega \Delta_\lambda(\boldsymbol{\eta})^2 d\boldsymbol{\eta}}{2d\lambda}, \tag{7}$$

From equation (5), we know that in finite domain effective mean diffusivity may smaller when swimmer has orientational drift. On the contrast, from equation (7) when the domain is infinite, the pdf do not influence the value of effective mean diffusivity. Follow those steps in this section we made, suppose the pdfs of each random reorientations of swimmers are different but independent, the effective mean diffusivity still will not change. In other words, under our assumption of model, the mechanism of random reorientations is independent of the effective mean diffusivity. Although the mechanism of random reorientations do not change the effective mean diffusivity, but it will change the effective diffusivity in given directions. That is what we will discuss in the next section.

4. diffusion of particle and diffusion of swimmers.

Swimmers are power source of particle. So the diffusion tensor of particle should has the relationship with the diffusion tensor of swimmer. It is easy to get the diffusion tensor of swimmer follow the same steps of the previous section.

$$DTS = NU\lambda\{E\left(a_i a_j\right)\}_{i,j} = NU\lambda\{\int_{S^d} f(\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{e}_i)(\boldsymbol{a}\cdot\boldsymbol{e}_j)d\boldsymbol{a}\}_{i,j} \tag{8}$$

$\boldsymbol{a}$ is the moving direction of swimmer, and a_i is the i-th component of $\boldsymbol{a}$.

If the direction of swimmer motion and the drift of particle induced by it are same, that is, $\boldsymbol{r}$ and $\boldsymbol{a}$ are collinear. DT, DTS should be proportional. But the relationship $\boldsymbol{r} = C\boldsymbol{a}$ is not always true, see figure 2.

In this section, We will calculate DT, and then analyze the difference between DT and DTS. Two concrete swimming models will be analyzed below, stokes flow and planar potential flow. In stokes flow, we choose a specific instance of an axially symmetric squirmer of radius l, swimming along the positive z-axis at a constant speed U. Its free-space, steady, axisymmetric streamfunction in a comoving reference frame is:

$$\psi\,(\rho, z) = \frac{1}{2}\rho^2 U\left(-1 + \frac{l^3}{r^3} + \frac{3\beta l^2 z}{2r^3}\left(\frac{l^2}{r^2} - 1\right)\right) \tag{9}$$

in cylindrical coordinates where $r = \rho^2 + z^2 = x^2 + y^2 + z^2$. When $\beta = 0$, the flow become the 3D potential flow. In the planar potential flow case, swimmer of radius l

[0]

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