

Chapter 1

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1.1 1.1.1

Exercise 1.1. Derive estimates for

$$\left| \left(D - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right|$$

. where $D = D_+^3, D_- D_+^2, D_-^2 D_+, D_0 D_+ D_-$.

Solution. By

$$\begin{aligned} E^3 e^{i\omega x} &= (1 + 3i\omega h - \frac{9}{2}\omega^2 h^2 - \frac{27}{6}i\omega^3 h^3 + \frac{81}{24}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4)) e^{i\omega x} \\ E^2 e^{i\omega x} &= (1 + 2i\omega h - \frac{4}{2}\omega^2 h^2 - \frac{8}{6}i\omega^3 h^3 + \frac{16}{24}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4)) e^{i\omega x} \\ E e^{i\omega x} &= (1 + i\omega h - \frac{1}{2}i\omega 2h^2 - \frac{1}{6}i\omega^3 h^3 + \frac{1}{24}\omega^4 h^4 \mathcal{O}(\omega^4 h^4)) e^{i\omega x} \\ E^{-1} e^{i\omega x} &= (1 - i\omega h - \frac{1}{2}i\omega 2h^2 + \frac{1}{6}i\omega^3 h^3 + \frac{1}{24}\omega^4 h^4 \mathcal{O}(\omega^4 h^4)) e^{i\omega x} \\ E^{-2} e^{i\omega x} &= (1 - 2i\omega h - \frac{4}{2}i\omega 2h^2 + \frac{8}{6}i\omega^3 h^3 + \frac{16}{24}\omega^4 h^4 \mathcal{O}(\omega^4 h^4)) e^{i\omega x} \\ E^3 e^{i\omega x} &= (1 - 3i\omega h - \frac{9}{2}\omega^2 h^2 + \frac{27}{6}i\omega^3 h^3 + \frac{81}{24}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4)) e^{i\omega x} \end{aligned}$$

1. $D = D_+^3$

$$\begin{aligned} \left| \left(D_+^3 - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| &= |(E^3 - 3E^2 + 3E - E_0 + i(\omega h)^3) e^{i\omega x} / h^3| \\ &= \left| \left(\frac{3}{2}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4) \right) e^{i\omega x} \right| = \mathcal{O}(\omega^3 h^3) \end{aligned}$$

2. $D = D_- D_+^2$

$$\begin{aligned} \left| \left(D_- D_+^2 - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| &= |(E^2 - 3E + 3E_0 - E^{-1} + i(\omega h)^3) e^{i\omega x} / h^3| \\ &= \left| \left(\frac{1}{2}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4) \right) e^{i\omega x} \right| = \mathcal{O}(\omega^3 h^3) \end{aligned}$$

3. $D = D_-^2 D_+$

$$\begin{aligned} \left| \left(D_-^2 D_+ - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| &= |(E - 3E_0 + 3E^{-1} - E^{-2} + i(\omega h)^3) e^{i\omega x} / h^3| \\ &= \left| \left(-\frac{1}{2}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4) \right) e^{i\omega x} \right| = \mathcal{O}(\omega^3 h^3) \end{aligned}$$

4. $D = D_0 D_- D_+$

$$\begin{aligned} \left| \left(D_0 D_- D_+ - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| &= |(E^2 - 2E + 2E^{-1} + E^{-2} + i(\omega h)^3) e^{i\omega x} / h^3| \\ &= |(\mathcal{O}(\omega^4 h^4)) e^{i\omega x}| = \mathcal{O}(\omega^4 h^4) \end{aligned}$$

1.2 1.1.3

Exercise 1.2. Compute $\|D_+ D_-\|_h$.

Solution. Firstly, we have

$$\|D_+ D_-\|_h \leq \|D_+\|_h \|D_-\|_h = \frac{4}{h^2}$$

Then, let $u_j = (-1)^j$, consider

$$\begin{aligned} \|D_+ D_- u\|_h^2 &= \sum_j ((E - 2E_0 + E^{-1})u_j)^2 / h^4 \\ &= \sum_j 16/h^4 = \sum_j 16/h^4 \|u\|_h^2 \end{aligned}$$

. That means

$$\|D_+ D_-\|_h \geq \sqrt{\frac{\|D_+ D_- u\|_h^2}{\|u\|_h^2}} = \frac{2}{h^2}$$

. In summary, $\|D_+ D_-\|_h = \frac{2}{h^2}$.