Chapter 1

Homework1

1.1 Chapter 1

1.1.1 2. Let k be a positive integer. Show that a smooth function defined on $\mathbb{R}^n$ has in general $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ distinct partial derivatives of order k.

Solution. Think this question as a problem of choose n-1 positions from n+k-1 positions in a queue. Which has $\binom{n+k-1}{n-1}$ possibles to do that.

Then make a map f from the choose way to a derivatives of order k of the smooth function.

The n-1 positions divide the queue into n part that contains 0 position or some positions. Take a$_i$ as the number of position in the i-th part, so $0 \leq a_i \leq k$.

$$f(a_1, a_2..., a_n) = \partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}u$$

It is easy to proof the map f is a bijective mapping. Because partial derivatives of order k also sets the $0 \leq a_i \leq k$ always true.

And $\binom{n+k-1}{k}$ is coming to choose k position and these near positions combine with a part. Then the same map f finish the remain proof part.

1.1.2 3. Prove the Multinomial Theorem: 1.1 where $\binom{|\alpha|}{\alpha} := \frac{|\alpha|}{\alpha}$, $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$, and $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. The sum is taken over all multiindices $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| = k$.

$$(x_1 + ... + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha, \tag{1.1}$$

Solution. Look $(x_1 + ...x_n)^k$ as put x_i in positions set S , and S contain k elements.

So The coefficient of x_α is the number of situations that S contain α_i x_is $\forall i \in \{1, 2, ...k\}$.

According to Combinatorics, $x\alpha$ coefficient of $(x_1 + ...x_n)^k$ is $\frac{k!}{\alpha_1!\alpha_2!...\alpha_n!} = \binom{|\alpha|}{\alpha}$. Finally get 1.1.

1.1.3 4. Prove Leibniz's fomula 1.2. where $u, v : \mathbb{R}^n \to \mathbb{R}$ are smooth, $\binom{\alpha}{\beta} := \frac{\alpha!}{\alpha!(\alpha-\beta)!}$, and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i (i = 1, ..., n)$.

$$D^\alpha(uv) = \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} u, \tag{1.2}$$

Solution. By induction proof.

When $\alpha = 0$ is trivial problem.

Now suppose when satisfy $\alpha \leq k$, equation(0.1) already true.

$$D^{k+1}(uv) = D^\alpha(Duv + uDv) \tag{1.3}$$

using Duv and uDv replace uv in (0.1) get

$$\begin{aligned} D^{k+1}(uv) &= \sum_{\beta<\alpha\le k}(\binom{\alpha-1}{\beta-1}+\binom{\alpha-1}{\beta})D^{\beta}uD^{\alpha-\beta}u \\ &= \sum_{\beta<\alpha\le k+1}\binom{\alpha}{\beta}D^{\beta}uD^{\alpha-\beta}u \end{aligned} \tag{1.4}$$

1.1.4 5. Assume that $f:\mathbb{R}^n\to\mathbb{R}$ are smooth. Prove 1.5 for each $k=1,2,...$ This is *Taylor's formula* in mutiindex notation.

$$f(x)=\sum_{|\alpha|\le k}\frac{1}{\alpha!}D^{\alpha}f(0)x^{\alpha}+O(|x|^{k+1}) \qquad \text{as } x\to 0 \tag{1.5}$$

Solution. Let $g(t)=f(tx)$, $|x|=1$ and consider normal *Taylor's formula.*

$$g(t)=g(0)+\frac{g\prime(0)}{1!}t^1+\frac{g\prime\prime(0)}{2!}t^2+...+\frac{g^{(k)}}{k!}t^k+O(t^{k+1}) \\ \text{as } x\to 0 \tag{1.6}$$

$forall i\in 1,2,...k$ And using 1.1's conclusion we have

$$\begin{aligned} \frac{g^{(i)}(t)}{i!}t^i &= \frac{t^i}{i!}\cdot\partial_t^i f(tx) \\ &= \frac{t^i}{i!}\cdot\sum_{|\alpha|=i}\frac{i!}{\alpha_1!\alpha_2!...\alpha_n!}x^{\alpha}D^{\alpha}f(tx) \\ &= \sum_{|\alpha|=i}\frac{(tx)^{\alpha}}{\alpha!}D^{\alpha}f(tx) \\ &= \sum_{|\alpha|\le k}\frac{1}{\alpha!}D^{\alpha}f(tx)(tx)^{\alpha} \end{aligned} \tag{1.7}$$

Combining g(0) = f(0), $|x|=1$, f is smooth, 1.6 and 1.7 get

$$f(tx)=\sum_{|\alpha|\le k}\frac{1}{\alpha!}D^{\alpha}f(0)x^{\alpha}+O(|tx|^{k+1}) \qquad \text{as } x\to 0 \tag{1.8}$$

This is equal to 1.5.

1.2 Chapter 2

1.2.1 1. Write down an explicit formula for a function u solving the initial-value problem 1.9. Here $c\in\mathbb{R}$ and $b\in\mathbb{R}^n$ are constants

$$\begin{cases} u_t+b\cdot Du+cu=0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u=g & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases} \tag{1.9}$$

Solution. Let $Z(s)=u(t+s,x+sb)\cdot e^{cx}$

Then we have $\partial_s Z(s)=0$ from below

$$\begin{aligned} \partial_s Z(s) &= \partial_s u(t+s,x+sb)\cdot e^{cx} \\ &= \partial_t u(t+s,x+sb)\cdot e^{cx}+ \\ &\quad b\partial_x u(t+s,x+sb)\cdot e^{cx}+cu(t+s,x+sb)\cdot e^{cx} \\ &= u_t+b\cdot Du+cu=0 \end{aligned} \tag{1.10}$$

So

$$\begin{aligned} u(t,x) &= Z(0)\cdot e^{-cx} \\ &= Z(-t)\cdot e^{-cx} \\ &= u(0,x-tb)\cdot e^{-cx} \\ &= g(x-tb)\cdot e^{-cx} \end{aligned} \tag{1.11}$$

1.2.2 3. Modify the proof of the mean-value formulas to show for $n\ge 3$ that 1.12 provided 1.13

$$u(0)=\fint_{\partial B(0,r)}gdS+\frac{1}{n(n-2)\alpha(n)}\int_{B(0,r)}(\frac{1}{|x|^{n-2}}-\frac{1}{r^{n-2}})fdx, \tag{1.12}$$

$$\begin{cases} -\Delta u=f & \text{in } B^0(0,r) \\ u=g & \text{on } \partial B(0,r) \end{cases} \tag{1.13}$$

Solution.

$$\begin{aligned} \int_0^r\frac{\int_{B_r(x)}\Delta Fdx}{|\partial B_r(x)|} &= \int_{B(0,r)}-f\int_x^r\frac{1}{n\alpha(n)y^{1-n}}dydx \\ &= -\frac{1}{n(n-2)\alpha(n)}\int_{B(0,r)}(\frac{1}{|x|^{n-2}}-\frac{1}{r^{n-2}})fdx \end{aligned} \tag{1.14}$$

So can get 1.12

1.3 Therorem

1.3.1 (Intergration-by-parts formula). Let $u, v \in C^1(\bar{U})$. Then get 1.15

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS \quad (i = 1, ..., n) \tag{1.15}$$

Solution.

$$\int_U u_{x_i} v dx + \int_U u v_{x_i} dx = \int_U (uv)_{x_i} dx \tag{1.16}$$

Then according Gauss-Green Theorem and replace u as uv.

$$\int_U (uv)_{x_i} dx = \int_{\partial U} u v \nu^i dS \tag{1.17}$$

1.3.2 (Green's formulas). Let $u, v \in C^2(\bar{U})$. Then have 1.18 1.19 1.20

(i)

$$\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS, \tag{1.18}$$

(ii)

$$\int_U Dv \cdot Du dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS, \tag{1.19}$$

(iii)

$$\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} dS. \tag{1.20}$$

Solution. By Gauss-Green Theorem:

i

$$\begin{aligned}
\int_U \Delta u dx &= \sum_1^n \int_U u_{x_i x_i} dx \\
&= \sum_1^n \int_{\partial U} u_{x_i} \nu^i dS \\
&= \int_{\partial U} \nabla u \cdot \nu dS \\
&= \int_{\partial U} \frac{\partial u}{\partial \nu} dS.
\end{aligned}$$

ii By (Intergration by parts formula) conclusion

$$\begin{aligned}
\int_U Dv \cdot Du dx &= -\int_U u \Delta v dx + \int_U u D v dx \\
&= -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS.
\end{aligned}$$

iii By (ii) conclusion

$$\int_U u \Delta v dx = \int_{\partial U} \frac{\partial v}{\partial \nu} u dS - \int_U Dv \cdot Du dx$$

replace u,v with u,v and v,u respectively. and add together get (iii).