Chapter 1

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1.1 1.1.1

Exercise 1.1. Derive estimates for

$$\left|\left(D-\frac{\partial^3}{\partial x^3}\right)e^{i\omega x}\right|$$

. where $D=D_+^3, D_-D_+^2, D_-^2D_+, D_0D_+D_-$.

Solution. By

$$\begin{aligned}
E^3e^{i\omega x} &= (1+3i\omega h-\frac{9}{2}\omega^2h^2-\frac{27}{6}i\omega^3h^3+\frac{81}{24}\omega^4h^4+\mathcal{O}(\omega^4h^4))e^{i\omega x}\\
E^2e^{i\omega x} &= (1+2i\omega h-\frac{4}{2}\omega^2h^2-\frac{8}{6}i\omega^3h^3+\frac{16}{24}\omega^4h^4+\mathcal{O}(\omega^4h^4))e^{i\omega x}\\
Ee^{i\omega x} &= (1+i\omega h-\frac{1}{2}i\omega 2h^2-\frac{1}{6}i\omega^3h^3+\frac{1}{24}\omega^4h^4\mathcal{O}(\omega^4h^4))e^{i\omega x}\\
E^{-1}e^{i\omega x} &= (1-i\omega h-\frac{1}{2}i\omega 2h^2+\frac{1}{6}i\omega^3h^3+\frac{1}{24}\omega^4h^4\mathcal{O}(\omega^4h^4))e^{i\omega x}\\
E^{-2}e^{i\omega x} &= (1-2i\omega h-\frac{4}{2}i\omega 2h^2+\frac{8}{6}i\omega^3h^3+\frac{16}{24}\omega^4h^4\mathcal{O}(\omega^4h^4))e^{i\omega x}\\
E^3e^{i\omega x} &= (1-3i\omega h-\frac{9}{2}\omega^2h^2+\frac{27}{6}i\omega^3h^3+\frac{81}{24}\omega^4h^4+\mathcal{O}(\omega^4h^4))e^{i\omega x}
\end{aligned}$$

1. $D=D_+^3$

$$\begin{aligned}
\left|\left(D_+^3-\frac{\partial^3}{\partial x^3}\right)e^{i\omega x}\right| &= \left|\left(E^3-3E^2+3E-E_0+i(\omega h)^3\right)e^{i\omega x}/h^3\right|\\
&= \left|(\frac{3}{2}\omega^4h^4+\mathcal{O}(\omega^4h^4))e^{i\omega x}\right| = \mathcal{O}(\omega^3h^3)
\end{aligned}$$

2. $D=D_-D_+^2$

$$\begin{aligned}
\left|\left(D_-D_+^2-\frac{\partial^3}{\partial x^3}\right)e^{i\omega x}\right| &= \left|\left(E^2-3E+3E_0-E^{-1}+i(\omega h)^3\right)e^{i\omega x}/h^3\right|\\
&= \left|(\frac{1}{2}\omega^4h^4+\mathcal{O}(\omega^4h^4))e^{i\omega x}\right| = \mathcal{O}(\omega^3h^3)
\end{aligned}$$

3. $D = D_-^2 D_+$

$$\left|\left(D_-^2 D_+ - \frac{\partial^3}{\partial x^3}\right) e^{i\omega x}\right| = \left|\left(E - 3E_0 + 3E^{-1} - E^{-2} + i(\omega h)^3\right) e^{i\omega x}/h^3\right|$$
$$= \left|(-\frac{1}{2}\omega^4 h^4 + \mathcal{O}(\omega^4 h^4))e^{i\omega x}\right| = \mathcal{O}(\omega^3 h^3)$$

4. $D = D_0 D_- D_+$

$$\left|\left(D_0 D_- D_+ - \frac{\partial^3}{\partial x^3}\right) e^{i\omega x}\right| = \left|\left(E^2 - 2E + 2E^{-1} + E^{-2} + i(\omega h)^3\right) e^{i\omega x}/h^3\right|$$
$$= \left|(\mathcal{O}(\omega^4 h^4))e^{i\omega x}\right| = \mathcal{O}(\omega^4 h^4)$$

1.2 1.3.1

Exercise 1.2. Compute $\|D_+ D_-\|_h$.

Solution. Firstly, we have

$$\|D_+ D_-\|_h \leq \|D_+\|_h \|D_-\|_h = \frac{4}{h^2}$$

Then, let $u_j = (-1)^j$, consider

$$\|D_+ D_- u\|_h^2 = \sum_j ((E - 2E_0 + E^{-1})u_j)^2/h^4$$
$$= \sum_j 16/h^4 = \sum_j 16/h^4 \|u\|_h^2$$

. That means

$$\|D_+ D_-\|_h \geq \sqrt{\frac{\|D_+ D_- u\|_h^2}{\|u\|_h^2}} = \frac{2}{h^2}$$

. In summary, $\|D_+ D_-\|_h = \frac{2}{h^2}$.