Chapter 1

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1.1 1.10.2

Exercise 1.1. Derive the stability condition for the Euler approximation to $u_t = u_{xx} + u_{yy} + u_{zz}$. Prove that the DuFort-Frankel method is unconditionally stable for the same equation.
Solution.

- Euler approximation is

$$v_j^{n+1} = (I + k(D_{+x}D_{-x} + D_{+y}D_{-y} + D_{+y}D_{-y}))v_j^n.$$

By already know $kD_+D_-e^{i\omega x_j} = -4\sigma \sin^2 \frac{\xi}{2} e^{i\omega x_j}$, where $\sigma = k/h^2, \xi = \omega h$. Get the symbol

$$\hat{Q} = 1 - 4\sigma(\sin^2 \frac{\xi_x}{2} + \sin^2 \frac{\xi_y}{2} + sin^2 \frac{\xi_z}{2}).$$

Stable condition $|\hat{Q}| < 1$ imply

$$\frac{k}{h^2} = \sigma < \frac{1}{6}.$$

- DuFort-Frankel method is

$$v_j^{n+1} = \frac{1}{1+6\sigma}(2\sigma(D_{+x} + D_{-x} + D_{+y} + D_{-y} + D_{+y} + D_{-y})v_j^n + (1 - 6\sigma)v_j^{n-1}).$$

Now the characteristic equation is

$$z^2 - \frac{4\sigma}{1+6\sigma}(\cos\xi_1 + \cos\xi_2 + \cos\xi_3)z - \frac{1-6\sigma}{1+6\sigma} = 0.$$

that is,

$$z_{1,2} = \frac{2\sigma}{1+6\sigma}(\sum \cos\xi) \pm \frac{1}{1+6\sigma}\sqrt{A},$$

where $A = 4\sigma^2(\sum \cos\xi)^2 + 1 - 36\sigma^2$. If $|A| \geq 0$, then $|A| \leq 1$, and

$$|z_{1,2}| \leq \frac{6\sigma}{1+6\sigma} + \frac{1}{1+6\sigma} = 1.$$

If $|A| \leq 0$, then we write

$$z_{1,2} = \frac{1}{1+6\sigma}(2\sigma(\sum \cos\xi) \pm i\sqrt{-A}).$$

and get

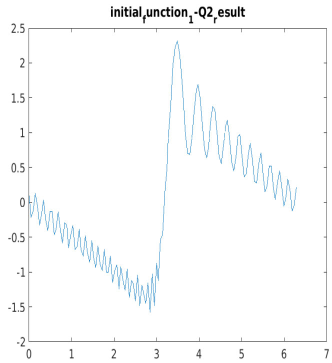
$$|z_{1,2}| = \frac{36\sigma^2 - 1}{(1+6\sigma)^2} = \frac{6\sigma - 1}{6\sigma + 1} \leq 1.$$

That means DuFort-Frankel method is unconditionally stable.

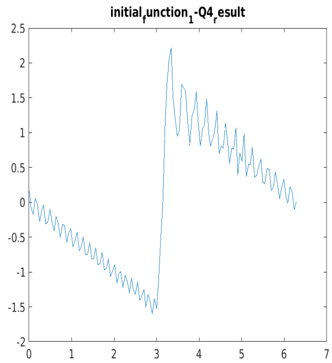
1.2 Program

运行 matlab/leapFrogPeriod.m 复现程序结果如下

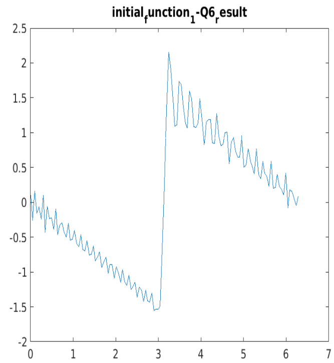
图 1.1: initial function 1



(a) Q_2

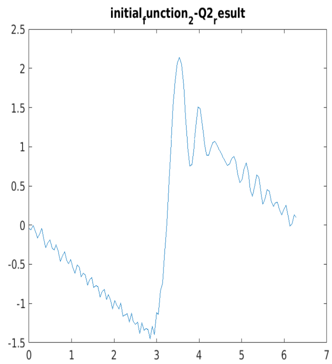


(b) Q_4

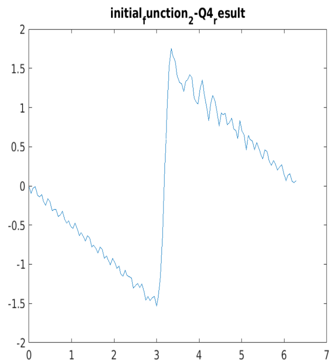


(c) Q_6

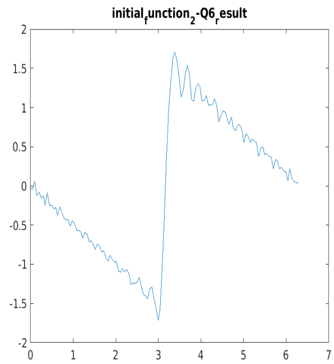
图 1.2: initial function 2



(a) Q_2

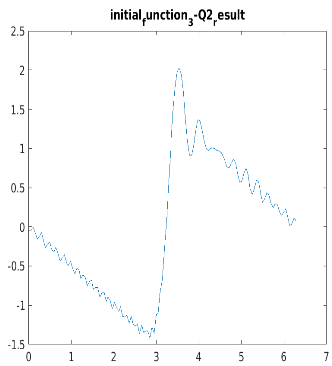


(b) Q_4

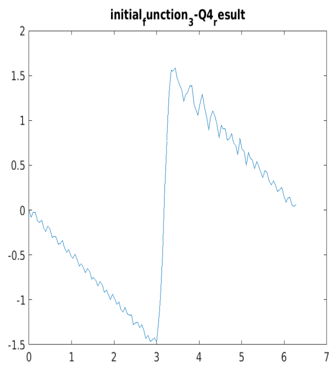


(c) Q_6

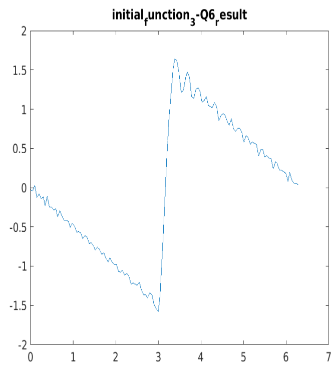
图 1.3: initial function 3



(a) Q_2



(b) Q_4



(c) Q_6