Chapter 1

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1.1 chapter 2.5

5 We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \le 0 \qquad \text{in } U. \tag{1.1}$$

(a) Prove for subharmonic v that

$$v(x) \le ⨍_{B(x,r)} v dy \qquad \text{for all } B(x,t) \subset U. \tag{1.2}$$

(b) Prove that there for $max_{\bar{U}} v = max_{\partial\bar{U}} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Solution.

(a) Already have two equation below

$$⨍_{B(x,r)} vdy = \int_0^r (\int_{\partial B(x,r)} udS)/\alpha(n)x^n dx = ⨍_{\partial B(x,r)} vdS$$

$$⨍_{\partial B(x,r)} vdS = ⨍_{\partial B(0,1)} Du(x+rz)\cdot zdS(z) = \phi(r)$$

and differential $\phi(r)$ get

$$\begin{aligned} \phi\prime(r) &= ⨍_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y) \\ &= ⨍_{B(x,r)} \frac{\partial u}{\partial \nu} dS(y) \\ &= \frac{r}{n} ⨍_{B(x,r)} \Delta v(y) dy \\ &\ge 0 \end{aligned} \tag{1.3}$$

so $\phi(r) \ge \phi(0)$ indicate that (1.1)

(b) Because 1.1.we have $v(x)$ always small than a value in $B(x,r) \in U$,

do $max_{\bar{U}} \in \partial U$ means $max_{\bar{U}} = max_{\partial U} v$.

(c) Differential v get

$$\partial_i v = \partial\phi(u) \cdot \partial_i u$$

then

$$\begin{aligned} \sum_{i=1}^n \partial_i \partial_i v &= \sum_{i=1}^n \partial^2\phi(u) \cdot (\partial_i u)^2 + \partial\phi(u) \cdot (\partial_i)^2 u \\ &= \sum_{i=1}^n \partial^2\phi(u) \cdot (\partial_i u)^2 \\ &\ge 0, \end{aligned} \tag{1.4}$$

which means v is subharmonic.

(d) The same as (c) have

$$\partial_i v = \partial_i (\sum_{j=1}^n u_j^2) = \sum_{j=1}^n 2u_j \cdot u_{ji}$$

then

$$\begin{aligned} \sum_{i=1}^n \partial_i \partial_i v &= \sum_{i=1}^n \sum_{j=1}^n (2(u_{ij})^2 + 2u_j u_{jii}) \\ &\ge \sum_{i=1}^n \sum_{j=1}^n 2u_j u_{jii} \end{aligned} \tag{1.5}$$

u is harmonic conclude $\sum_{i=1}^n u_{ii} = 0$,differential j have $\sum_{i=1}^n u_{iij} = 0$. Combining with above. $\Delta v \ge 0$.

6 Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C , depending only on U, such that

$$max_{\bar{U}}|u| \leq C(max_{\partial U}|g| + max_{\bar{U}}|f|) \tag{1.6}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \tag{1.7}$$

Because $\Delta(\frac{|x|^2}{2n}\lambda) = \lambda$ and U is bounded .Let $C > \frac{|x|^2}{xn}$ for $x \in U$ and $\lambda = max_{\bar{U}}|f|$ .So set $\phi = u + \frac{|x|^2}{2n}\lambda$ and

$$\Delta\phi = \Delta(u + \frac{|x|^2}{2n}\lambda) = f + \lambda \geq 0 \tag{1.8}$$

By problem 4 , ϕ is subharmonic, and accord to 1.2

$$\phi \leq w(n)r^n max_{\partial U}|g|. \tag{1.9}$$

r is the biggest r for $B(x.r) \in U$. Let $C > w(n)r^n \quad \forall x \in U$ get 1.6.

10 (Reflection principle)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n | \ |x| < 1,\ x_n > x\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+ , with $u = 0$ on $\partial U^+ \cap x_n = 0$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n \leq 0 \end{cases} \tag{1.10}$$

for $x \in U = B^0(0,1)$. Prove $v \in C^2(U)$ and thus v is harmonic within U .

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U^+})$. Show that v is harmonic within U.

Solution.

(a) u is harmonic in $C^2(U^+)$.calculate v

$$\Delta v = \begin{cases} \Delta u(x) & \text{if } x_n > 0 \\ \Delta - u(x) & \text{if } x_n < 0 \end{cases} = 0 \tag{1.11}$$

because $u \in C^2(\overline{U^+})$, Δu is continue in x satisfy $x_n = 0$. $\Delta v = 0$ if $x_n = 0$.

Finally, $\Delta v = 0\ \forall x \in U$.Means$v$ is harmonic within U.

(b) Using Poisson's formula for boundary ∂U.get solution

$$f(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(x,1)} \frac{u(y)}{|x-y|^n} dS(y)$$

It is easy to confirm that $f(x) = 0 = u(x) \quad \forall x$ satisfy $x_n = 0$,because of $u(y) = -u(-y)$.

$f(x)$ and $u(x)$ is harmonic in U^+, conclude $f(x) - u(x)$ is harmonic.linking $f(x) - u(x) = 0$ in ∂U^+, so according to strong maximum principle. $f(x) - u(x) = 0\ \forall x \in \overline{U^+}$.

The same to get $f(x) + u(x) = 0$ in ∂U^+.In summary. $f(x) - v(x) = 0\ \forall x \in \bar{U}$, which means $v(x)$ is harmonic within U.

11 (Kelvin transform for Laplace's equation) The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|)|x|^{2-n} \qquad (x \neq 0), \tag{1.12}$$

where $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.

Solution. Calculation differential and laplace of $\bar{x} = \frac{(x_1, x_2, \ldots, x_n)}{\sum_{i=1}^n x_i^2}$

$$D_{x_i}\bar{x}_j = (\delta_{ij} \cdot |x|^2 - 2x_i x_j)/|x|^4 \tag{1.13}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$ so $D\bar{x}$ is a matrix $A = |x|^2 * I - x^T \cdot x$. By product $D_x\bar{x}(D_x\bar{x})^T$ is a matrix $B = AA^T$ satisfy

$$\begin{aligned} B_{ij}|x|^8 &= \sum_{k=1}^n (\delta_{ik} \cdot |x|^2 - 2x_i x_k)(\delta_{jk} \cdot |x|^2 - 2x_j x_k) \\ &= \begin{cases} |x|^4 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned} \tag{1.14}$$

Now calculate $\Delta\bar{u}(x)$

$$
\begin{aligned}
\Delta\bar{u}(x) =& \sum_{i=1}^{n} \partial_i\partial_i(u(\bar{x})|x|^{2-n}) \\
=& \sum_{i=1}^{n} \partial_i(\sum_{j=1}^{n} \partial_j(u(\bar{x}))\partial_i(\bar{x}_j)|x|^{2-n} + u(\bar{x})\partial_i(|x|^{2-n})) \\
=& \sum_{i=1}^{n}((\partial_i)^2(|x|^{2-m})u(\bar{x})|x|^{1-n}+ \\
& 2\partial_i(|x|^{2-n})\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i(\bar{x}_j)+ \\
& \sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i^2(\bar{x}_j)|x|^{2-n} + \\
& \sum_{j=1}^{n}\sum_{k=1}^{n}\partial_k\partial_j(u(\bar{x}))\partial_i(\bar{x}_k)\partial_i(\bar{x}_j)|x|^{2-n}).
\end{aligned}
\tag{1.15}
$$

According to $|x|^{2-n}$ is harmonic,the first of 1.15 is 0. According to 1.14 and 1.13 ,the fourth of 1.15 $= \sum_{i=1}^{n} tr\partial(\bar{x})\partial^2 u\partial(\bar{x})|x|^{2-n} = \sum_{i=1}^{n}\partial_i^2 u|x|^{-2-n} = 0.$ the second of 1.15

$$
\begin{aligned}
& \sum_{i=1}^{n} 2\partial_i(|x|^{2-n})\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i(\bar{x}_j) \\
&= 2DuD(\bar{x})D(|x|^{2-n}) \\
&= 2Du|x|^{-2}(I - \frac{xx^T}{|x|^2})\cdot(2-n)|x|^{1-n}\frac{x^t}{|x|} \\
&= 2(2-n)|x|^{-2-n}(x-2x)\cdot Du \\
&= -2(2-n)|x|^{-2-n}x\cdot Du,
\end{aligned}
\tag{1.16}
$$

the third of 1.15 according to 1.13

$$
\partial_i^2(\bar{x}_j) = \begin{cases} \frac{(\delta_{ij}(2)x_i|x|^4 - 2x_j|x|^4 - \delta_{ij}(4)x_i|x|^4) + (8)x_i^2 x_j|x|^2}{|x|^8} & \text{if } i \neq j \\ \frac{(\delta_{ij}(2)x_i|x|^4 - 4x_i|x|^4 - \delta_{ij}(4)x_i|x|^4) + (8)x_i^2 x_j|x|^2}{|x|^8} & \text{if } i = j \end{cases}
\tag{1.17}
$$

so

$$
\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i^2(\bar{x}_j)|x|^{2-n} = Du2(2-n)|x|^{-2-n}\cdot x.
\tag{1.18}
$$

In summary, $\Delta\bar{u}(x) = 0$,when$\Delta u(x) = 0$.