Chapter 3

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Exercise 3.1. 7. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0) \tag{3.1}$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Solution. Considering the left part of (3.1). Replace $u(x)$ with Poisson's

$$\begin{aligned}
&r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}\frac{r^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|y|^n}dS(y) \\
&\le \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \iff \\
&\int_{\partial B(0,r)}(\frac{r}{(r+|x|)})^n\frac{g(y)}{|y|^n}-\frac{g(y)}{|x-y|^n}dS(y)\le 0 \iff \\
&|x-y|\le r+|x|
\end{aligned} \tag{3.2}$$

Which is trivial. Since $u(x)$ is positive and continue. $g(x)>0$ means (3.2) second step. $y\in\partial B(0,r)$ implicates the third step. The right part of (3.1) is similar with left part.

Exercise 3.2. 9. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{3.3}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x)=|x|$ for $x\in\partial\mathbb{R}^n_+, |x|\le 1$. Show Du is *not* bounded near $x=0$.

Solution. Computing Du in $x=0$

$$\begin{aligned}
&\frac{u(\lambda e_n)-u(0)}{\lambda} \\
&= \frac{2}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|\lambda e_n - y|^n}dy \\
&= \frac{2}{n\alpha(n)}(\int_{\partial\mathbb{R}^n_+\cap B(0,1)}\frac{g(y)}{|\lambda e_n - y|^n}dy + \int_{\partial\mathbb{R}^n_+-\cap B(0,1)}\frac{g(y)}{|\lambda e_n - y|^n}dy) \\
&= I + J
\end{aligned}$$

Then I

$$\begin{aligned}
I &= \frac{2}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+\cap B(0,1)}\frac{|y|}{|\lambda e_n - y|^n}dy \\
&= 2\int_0^1(\frac{r^{n-1}}{(\lambda^2+r^2)^{n/2}})dr \\
&\ge 2\int_\lambda^1(\frac{1}{r(\frac{\lambda^2}{r^2}+1)^{n/2}})dr \\
&\ge 2^{n-2/2}\int_\lambda^1(\frac{1}{r})dr \\
&\ge -2^{n-2/2}ln(\lambda) \qquad \text{as } \lambda\to 0 \\
&\to \infty
\end{aligned}$$

And J , assuming $|g|<M$

$$\begin{aligned}
J &\le 2M\int_1^\infty(\frac{r^{n-2}}{(\lambda^2+r^2)^{n/2}})dr \\
&\le 2M\int_1^\infty(\frac{1}{r^2})dr \\
&\le 2M.
\end{aligned}$$

In summary, $\lim\limits_{\lambda\to 0}\frac{u(\lambda e_n)-u(0)}{\lambda} = \infty + 2M = \infty$. Therefore Du is unbounded at 0.

Exercise 3.3. 12. Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n\times(0,\infty)$.

(a) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the hear equation as well.

Solution.

(a) Replace u with u_λ in heat equation,

$$\begin{aligned}&(u_\lambda)_t - \Delta u_\lambda(x,t)\\ =& u_t(\lambda x, \lambda^2 t) * \lambda^2 - \Delta u(\lambda x, \lambda^2 t) * \lambda^2 \qquad \text{Let}\end{aligned} \tag{3.4}$$

Since u solves heat equation, set $\bar{x} = \lambda x, \bar{t} = \lambda^2 t$ also maintain heat equation. Therefore, $(u_\lambda)_t - \Delta u_\lambda(x,t) = 0$ means u_λ solves heat equation.

(b) Different u_λ with λ

$$\begin{aligned}&D_\lambda u_\lambda(x,t)\\ =& D_\lambda u(\lambda x, \lambda^2 t)\\ =& x \cdot D_x u(\lambda x, \lambda^2 t) + 2\lambda t D_t u(\lambda x, \lambda^2 t).\end{aligned}$$

According to [(a)] , for every λ, u_λ solves heat equation, therefore, $D_\lambda u_\lambda$ solves heat equation. Set $\lambda = 1$, get $v(x,t) = D_\lambda u_1(x,t)$ is a solution for heat equation.

Exercise 3.4. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty)\\ u = g & \text{on } \mathbb{R}^n \times t = 0, \end{cases} \tag{3.5}$$

where $c \in \mathbb{R}$.

Solution.

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty)\\ u = g & \text{on } \mathbb{R}^n \times t = 0, \end{cases}$$

have solution $u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy$ $(x \in \mathbb{R}^n, t > 0)$.In order to add item cu, multiply e^{-ct} get $\bar{u} = u \times e^{-ct}$ so that $D_t(u \times e^{-ct}) = u_t \times e^{-ct} - cu \times e^{-ct}$ and maintain $\Delta(u \times e^{-ct} = \Delta u \times e^{-ct})$ and $u \times e^{-ct} = u = g$, while $t = 0$.Equal to

$$\begin{cases} \bar{u}_t - \Delta \bar{u} + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty)\\ u = g & \text{on } \mathbb{R}^n \times t = 0. \end{cases}$$

Duhamel's principle donate a solution

$$v(x,t) = \frac{e^{-ct}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy + \\ \int_0^t \int_{\mathbb{R}^n} \frac{e^{-c(t-s)}}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds.$$