Chapter 12

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Problem 12.1. Utilize Lemma 2 in §4.5.3 to discuss the sense in which u defined by formula (20) in §4.3.1 converges to the initial data g as $t \to 0^+$.

Solution. From Lemma 2 in §4.5.3 know

$$\frac{1}{(2\pi\varepsilon)^{n/2}}\int_{\mathbb{R}^n} e^{\frac{i}{2\varepsilon}y\cdot Ay}a(y)dy = \frac{e^{i\frac{\pi}{4}\operatorname{sgn} A}}{|\det A|^{1/2}}(a(0)+O(\varepsilon)).$$

Consider the formula (20) as $t \to 0$ gives

$$\begin{aligned}\lim_{t\to 0^+} u(x,t) &= \lim_{t\to 0^+}\frac{1}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}}g(y)dy \qquad \text{take } 2t=\varepsilon, y=x-y\\ &= \lim_{t\to 0^+}\left(\frac{e^{i\frac{n\pi}{4}}}{(i)^{n/2}}(g(x)+O(2t))\right)\\ &= g(x).\end{aligned}$$

Problem 12.2. Let $n=1$ and suppose that u^ε solves the problem

$$\begin{cases}-(a(\frac{x}{\varepsilon})u^\varepsilon_x)_x = f & \text{in } (0,1)\\ u^\varepsilon(0)=u^\varepsilon(1)=0,\end{cases}$$

where a is a smooth, positive function that is 1-periodic. Assume also that $f \in L^2(0,1)$.

(a) Show that $u^\epsilon \rightharpoonup u$ weakly in $H^1_0(0,1)$, where u solves

$$\begin{cases}-\bar{a}u_{xx} = f & \text{in } (0,1)\\ u(0)=u(1)=0,\end{cases}$$

for $a := (\int_0^1 a(y)^{-1}dy)^{-1}$.

(b) Check that this answer agrees with the conclusions (73), (74) in §4.5.4.

Solution.

(b) Simplify the first equation in formula (73) as $i,j=1$ gives

$$\begin{aligned}
-(\int_Q a(y)-a(y)\chi_y(y)dy)u_{xx} &= -(\int_Q a(y)-a(y)\chi_y(y)dy)(\int_0^1 a(y)^{-1}dy)f \\
&= \frac{1}{\varepsilon}\int_Q (1-\chi_y)dyf \qquad \text{Since } \chi \text{ is } Q-\text{periodic} \\
&= \frac{1}{\varepsilon}(\varepsilon-0)f \\
&= f.
\end{aligned}$$

the second equation in formula (73) is obvious. So this answer agrees the conclusions.

Problem 12.3. (Variational principles in homogenization) Let $A(y)=((a^{ij}(y)))$ be symmetric, positive definite and Q-periodic. Recall from §4.5.4 the expression (74) for the corresponding homogenized coeffixients $\bar{A}=((\bar{a}^{ij}))$.

(a) Derive for each $\xi\in\mathbb{R}^n$ the variational formula

$$\xi\cdot\bar{A}\xi=\min_w\{\int_Q Dw\cdot A(y)Dwdy \ \ |w=y\cdot\xi+v, v\ Q\text{-periodic}\}.$$

(Hint: The minimum is attained by $w=y\cdot\xi-\sum_{i=1}^n\xi_i\chi^i$, for the correctors χ^i introduced in §4.5.4.)

(b) Derive also the dual variational formula

$$\eta\cdot\bar{A}^{-1}\eta=\min_\sigma\{\int_Q \sigma\cdot A(y)^{-1}\sigma dy \ \ |\int_Q\sigma dy=\eta, \operatorname{div}\sigma=0, \sigma\ Q\text{-periodic}\}.$$

(c) Show that therefore

$$\left(\int_Q A(y)^{-1}dy\right)^{-1}\le\bar{A}\le\int_Q A(y)dy.$$

(Remember from §A.1 that for symmetric matrices $R\ge S$ means $R-S$ is nonnegative definite.)

Solution.

(a) we have

$$\begin{aligned}
\int_Q Dw\cdot A(y)Dwdy &= \int_{\partial Q} w(A(y)Dw)\cdot\nu dS-\int_Q w\operatorname{div}(A(y)Dw)dx \qquad \text{Since } A(y), Dw \text{ is } Q-\text{periodic} \\
&= \int_{\partial Q}(y\cdot\xi)(A(y)Dw)\cdot\nu dS-\int_Q(y\cdot\xi+v)\operatorname{div}(A(y)Dw)dx.
\end{aligned}$$

However, if $\operatorname{div}(A(y)Dw)\ne 0$,we can let $v'=v+C$, where C is a constant, so that $-\int_Q(y\cdot\xi+v')\operatorname{div}(A(y)Dw)dx< -\int_Q(y\cdot\xi+v)\operatorname{div}(A(y)Dw)dx$.It's means that minimum doesn't exist, so $\operatorname{div}(A(y)Dw)=0$ Solves the equations get $w=y\cdot\xi-\sum_i\xi_i\chi^i$.Subtitute w with w_0 get $\min_w\{\int_Q Dw\cdot A(y)Dwdy\}=\xi\cdot\bar{A}\xi$.

(b) Let $\sigma=A(y)Dw$, From (a) we already know that $\operatorname{div}\sigma=\operatorname{div}(A(y)Dw)=0$,$\int_q\sigma dy=\bar{A}\xi$. Then from (a) know $\min_\sigma\{\int_Q\sigma\cdot A(y)^{-1}\sigma dy\}=\eta\cdot\bar{A}\eta$.

(c) Let $w=y\cdot\xi$, we have $\xi\cdot(\int_Q A(y)dy-\bar{A})\xi\ge 0$. Without specifically assume $\int_Q\eta=\eta$, then from (b) know $\eta\cdot(\int_Q A(y)^{-1}-\bar{A}^{-1})\eta\ge 0$. From the definition of matrixs' compare get the inequality.