Chapter 1

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1.1 chapter 2.5

5 We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \qquad \text{in } U. \tag{1.1}$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v \, dy \qquad \text{for all } B(x,t) \subset U. \tag{1.2}$$

(b) Prove that there for $max_{\bar{U}} v = max_{\partial \bar{U}} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Solution.

(a) Already have two equation below

$$\fint_{B(x,r)} v \, dy = \int_0^r (\int_{\partial B(x,r)} u \, dS)/\alpha(n) x^n dx = \fint_{\partial B(x,r)} v \, dS$$

$$\fint_{\partial B(x,r)} v \, dS = \fint_{\partial B(0,1)} Du(x+rz) \cdot z \, dS(z) = \phi(r)$$

and differential $\phi(r)$ get

$$\begin{aligned}
\phi\prime(r) &= \fint_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y) \\
&= \fint_{B(x,r)} \frac{\partial u}{\partial \nu} dS(y) \\
&= \frac{r}{n} \fint_{B(x,r)} \Delta v(y) dy \\
&\geq 0
\end{aligned} \tag{1.3}$$

so $\phi(r) \geq \phi(0)$ indicate that 1.1

(b) Because 1.1.we have $v(x)$ always small than a value in $B(x,r) \in U$,

do $max_{\bar{U}} \in \partial U$ means $max_{\bar{U}} = max_{\partial U} v$.

(c) Differential v get

$$\partial_i v = \partial \phi(u) \cdot \partial_i u$$

then

$$\begin{aligned}
\sum_{i=1}^n \partial_i \partial_i v &= \sum_{i=1}^n \partial^2 \phi(u) \cdot (\partial_i u)^2 + \partial \phi(u) \cdot (\partial_i)^2 u \\
&= \sum_{i=1}^n \partial^2 \phi(u) \cdot (\partial_i u)^2 \\
&\geq 0,
\end{aligned} \tag{1.4}$$

which means v is subharmonic.

(d) The same as (c) have

$$\partial_i v = \partial_i (\sum_{j=1}^n u_j^2) = \sum_{j=1}^n 2u_j \cdot u_{ji}$$

then

$$\begin{aligned}
\sum_{i=1}^n \partial_i \partial_i v &= \sum_{i=1}^n \sum_{j=1}^n (2(u_{ij})^2 + 2u_j u_{jii}) \\
&\geq \sum_{i=1}^n \sum_{j=1}^n 2u_j u_{jii}
\end{aligned} \tag{1.5}$$

u is harmonic conclude $\sum_{i=1}^n u_{ii} = 0$,differential j have $\sum_{i=1}^n u_{iij} = 0$. Combining with above. $\Delta v \geq 0$.

6 Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$max_{\bar{U}}|u| \leq C(max_{\partial U}|g| + max_{\bar{U}}|f|) \qquad (1.6)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ \quad u = g & \text{on } \partial U. \end{cases} \qquad (1.7)$$

Because $\Delta(\frac{|x|^2}{2n}\lambda) = \lambda$ and U is bounded. Let $C > \frac{|x|^2}{xn}$ for $x \in U$ and $\lambda = max_{\bar{U}}|f|$. So set $\phi = u + \frac{|x|^2}{2n}\lambda$ and

$$\Delta\phi = \Delta(u + \frac{|x|^2}{2n}\lambda) = f + \lambda \geq 0 \qquad (1.8)$$

By problem 4, ϕ is subharmonic, and accord to 1.2

$$\phi \leq w(n)r^n max_{\partial U}|g|. \qquad (1.9)$$

r is the biggest r for $B(x.r) \in U$. Let $C > w(n)r^n \quad \forall x \in U$ get 1.6.

10 (Reflection principle)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n|\ |x| < 1,\ x_n > x\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap x_n = 0$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1,...,x_{n-1},-x_n) & \text{if } x_n \leq 0 \end{cases} \qquad (1.10)$$

for $x \in U = B^0(0,1)$. Prove $v \in C^2(U)$ and thus v is harmonic within U.

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U^+})$. Show that v is harmonic within U.

Solution.

(a) u is harmonic in $C^2(U^+)$.calculate v

$$\Delta v = \begin{cases} \Delta u(x) & \text{if } x_n > 0 \\ \Delta - u(x) & \text{if } x_n < 0 \end{cases} = 0 \qquad (1.11)$$

because $u \in C^2(\overline{U^+})$, Δu is continue in x satisfy $x_n = 0$. $\Delta v = 0$ if $x_n = 0$.

Finally, $\Delta v = 0 \ \forall x \in U$. Means v is harmonic within U.

(b) Using Poisson's formula for boundary ∂U. get solution

$$f(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(x,1)} \frac{u(y)}{|x - y|^n} dS(y)$$

It is easy to confirm that $f(x) = 0 = u(x) \ \forall x$ satisfy $x_n = 0$,because of $u(y) = -u(-y)$.

$f(x)$ and $u(x)$ is harmonic in U^+, conclude $f(x) - u(x)$ is harmonic.linking $f(x) - u(x) = 0$ in ∂U^+, so according to strong maximum principle. $f(x) - u(x) = 0 \ \forall x \in \overline{U^+}$.

The same to get $f(x) + u(x) = 0$ in ∂U^+.In summary. $f(x) - v(x) = 0 \ \forall x \in \bar{U}$, which means $v(x)$ is harmonic within U.

11 (Kelvin transform for Laplace's equation)

The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|)|x|^{2-n} \qquad (x \neq 0), \qquad (1.12)$$

where $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.

Solution. Calculation differential and laplace of $\bar{x} = \frac{(x_1,x_2,...,x_n)}{\sum_{i=1}^n x_i^2}$

$$D_{x_i}\bar{x}_j = (\delta_{ij} \cdot |x|^2 - 2x_i x_j)/|x|^4 \qquad (1.13)$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$ so $D\bar{x}$ is a matrix $A = |x|^2 * I - x^T \cdot x$. By product $D_x\bar{x}(D_x\bar{x})^T$ is a matrix $B = AA^T$ satisfy

$$B_{ij}|x|^8 = \sum_{k=1}^n (\delta_{ik} \cdot |x|^2 - 2x_i x_k)(\delta_{jk} \cdot |x|^2 - 2x_j x_k)$$

$$= \begin{cases} |x|^4 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \qquad (1.14)$$

Now calculate $\Delta \bar{u}(x)$

$$\Delta \bar{u}(x) = \sum_{i=1}^{n} \partial_i \partial_i (u(\bar{x})|x|^{2-n})$$

$$= \sum_{i=1}^{n} \partial_i (\sum_{j=1}^{n} \partial_j(u(\bar{x}))\partial_i(\bar{x}_j)|x|^{2-n} + u(\bar{x})\partial_i(|x|^{2-n}))$$

$$= \sum_{i=1}^{n} ((\partial_i)^2(|x|^{2-m})u(\bar{x})|x|^{1-n} +$$

$$2\partial_i(|x|^{2-n})\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i(\bar{x}_j) +$$

$$\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i^2(\bar{x}_j)|x|^{2-n} +$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\partial_k\partial_j(u(\bar{x}))\partial_i(\bar{x}_k)\partial_i(\bar{x}_j)|x|^{2-n}).$$

(1.15)

According to $|x|^{2-n}$ is harmonic, the first of 1.15 is 0. According to 1.14 and 1.13 ,the fourth of 1.15 $= \sum_{i=1}^{n} tr\partial(\bar{x})\partial^2 u\partial(\bar{x})|x|^{2-n} = \sum_{i=1}^{n} \partial_i^2 u|x|^{-2-n} = 0.$

the second of 1.15

$$\sum_{i=1}^{n} 2\partial_i(|x|^{2-n})\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i(\bar{x}_j)$$

$$= 2DuD(\bar{x})D(|x|^{2-n})$$

$$= 2Du|x|^{-2}(I - \frac{xx^T}{|x|^2}) \cdot (2-n)|x|^{1-n}\frac{x^t}{|x|}$$

(1.16)

$$= 2(2-n)|x|^{-2-n}(x - 2x) \cdot Du$$

$$= -2(2-n)|x|^{-2-n}x \cdot Du,$$

the third of 1.15 according to 1.13

$$\partial_i^2(\bar{x}_j)$$

$$= \begin{cases} \frac{(\delta_{ij}(2)x_i|x|^4 - 2x_j|x|^4 - \delta_{ij}(4)x_i|x|^4) + (8)x_i^2 x_j|x|^2}{|x|^8} & \text{if } i \neq j \\ \frac{(\delta_{ij}(2)x_i|x|^4 - 4x_i|x|^4 - \delta_{ij}(4)x_i|x|^4) + (8)x_i^2 x_j|x|^2}{|x|^8} & \text{if } i = j \end{cases}$$

(1.17)

so

$$\sum_{j=1}^{n}\partial_j(u(\bar{x}))\partial_i^2(\bar{x}_j)|x|^{2-n} = Du2(2-n)|x|^{-2-n} \cdot x.$$

(1.18)

In summary, $\Delta \bar{u}(x) = 0$, when$\Delta u(x) = 0$.