Chapter 11

第十一次作业

Problem 11.1. $f\colon \mathring{D}^n \to \mathring{D}^n$ 连续映射, 举例说明 f 不一定有不动点.

$\mathring{D}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1^n + x_2^2 + \cdots + x_n^2 < 1\}$.

Solution. 定义 $f\colon \mathring{D}^n \to \mathring{D}^n, \forall (x_1, x_2, \ldots, x_n) \in \mathring{D}^n$,

$$f(x_1, x_2, \ldots, x_n) = (x_1/2, x_2/2, \ldots, x_{n-1}/2, (x_n + 1)/2)$$

已知 $\sum\limits_{i=1}^{n} x_i^2 < 1, x_n < 1$. 所以 $(x_1/2)^2 + (x_2/2)^2 + \cdots (x_{n-1}/2)^2 + ((x_n+1)/2)^2 = \frac{1}{4}(\sum\limits_{i=1}^{n} x_i^2 + 2x_n + 1) < \frac{1}{4}(1+2+1) = 1$, 连续性由定义易得. 所以 f 是将 $\mathring{D}^n$ 映射到 $\mathring{D}^n$ 上. 并且若有不动点 $x, f(x) = x$ 等价于

$$\begin{cases} x_i = x_i/2 & i = 1, 2, \ldots, n-1 \\ x_n = (x_n + 1)/2 \end{cases}$$

解得 $x = (0, 0, \ldots, 0, 1) \notin \mathring{D}^n$. 即 f 没有不动点. 即连续函数不一定有不动点.

Problem 11.2. 设 X 是拓扑空间, 将 $X \times I$ 空间中 $X \times \{0\}$ 等置为一点 S, 又将 $X \times \{1\}$ 等置为一点 N, 所得商空间记 SX, 证明: $H_n(SX) \cong \tilde{H}_{n-1}(X)$

Solution. 设 $X_1 = SX - \{S\}, X_2 = SX - \{N\}$, 则 $SX = \mathring{X}_1 \cup \mathring{X}_2, X \times (0,1) = X_1 \cap X_2$. 且 X_1, X_2 等价于 X 上的圆锥体内部点不妨都写为 $\mathring{CX} = X \times (0,1) \cup \{NS\}$

定义 $F\colon \mathring{CX} \times I \to \mathring{CX}$ 为 $F([x,t], s) = [x, (1-s)t + s]$, 连续性显然. 可以验证 $F([x,t], 0) = [x,t] = Id_{\mathring{CX}}, F([x,t], 1) = [x, 1] = NS$. 所以 $1_{\mathring{CX}}$ 是零伦的, 由定义 $\mathring{CX}$ 是可缩的. 我们有 $H_n(X_1) = H_n(X_2) = 0$.

定义 $f\colon X \to X \times (0,1), g\colon X \times (0,1) \to X$ 为 $f(x) = [x, 1/2], g([x,t]) = x$. 则有 $(g \circ f)(x) = x, (f \circ g)(x,t) = (x, 1/2)$. 显然 $(g \circ f) \simeq 1_X$, 定义 $G\colon (X \times (0,1)) \times I \to (X \times (0,1))$ 为 $G([x,t], i) = [x, (t - 1/2)i + 1/2]$, 连续性显然. 可以验证 $G([x,t], 0) = [x, 1/2] = (f \circ g)(x,t), G([x,t], 1) = [x,t] = 1_{X \times (0,1)}$, 即 $f \circ g \simeq 1_{X_1 \cap X_2}$. 所以 $X_1 \cap X_2, X$ 有相同同伦类型, 所以 $\tilde{H}_n(X_1 \cap X_2) = \tilde{H}_n(X)$

通过 (Mayer-Vietoris for Reduced Homology) 和 $X_1, X_2, SX, X_1 \cap X_2$ 之间的正合序列

$$\cdots \to \tilde{H}_n(X_1 \cap X_2) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to \tilde{H}_n(SX) \to \tilde{H}_{n-1}(X_1 \cap X_2) \to \cdots.$$

和 $\tilde{H}_n(X_1) = \tilde{H}_n(X_2) = 0$. 可知 $H_n(SX) = \tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X_1 \cap X_2) \cong \tilde{H}_{n-1}(X)(n > 0)$.