Chapter 13

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Problem 13.1. Suppose $k \in \{0, 1, \ldots\}, 0 < \gamma \leq 1$. Prove $C^{k,\gamma}(\bar{U})$ is a Banach space.

Solution.

- Real linear space

  If $u_1, u_2 \in C^{k,\gamma}(\bar{U})$. So $u = u_1 + u_2$ fill that $u \in C^k(\bar{U})$,since $u_1, u_2 \in C^k(\bar{U})$. And

$$\begin{aligned}
\|u\|_{C^{k,\gamma}(\bar{U})} &= \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})} \\
&\leq \sum_{|\alpha| \leq k} \|D^\alpha u_1\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u_1]_{C^{0,\gamma}(\bar{U})} + \sum_{|\alpha| \leq k} \|D^\alpha u_2\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u_2]_{C^{0,\gamma}(\bar{U})}.
\end{aligned}$$

  So $u \in C^{k,\gamma}(\bar{U})$.

- $\|\cdot\|$ is a norm

  (i) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in C^{k,\gamma}(\bar{U})$ can be got from above.

  (ii) $\|\lambda u\| = |\lambda| \|u\|$ for all $u \in C^{k,\gamma}(\bar{U}), \lambda \in \mathbb{R}$.

$$\begin{aligned}
\|\lambda u\| &= \sum_{|\alpha| \leq k} \|\lambda D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [\lambda D^\alpha u]_{C^{0,\gamma}(\bar{U})} \\
&= |\lambda| \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})} \right) \\
&= |\lambda| \|u\|
\end{aligned}$$

  (iii) $\|u\| = 0$ if and only if $u = 0$.

  Since $\|u\| = 0$ is equal to

$$\begin{cases} \|D^\alpha u\|_{C(\bar{U})} & |\alpha| \leq k \\ [D^\alpha u]_{C^{0,\gamma}(\bar{U})} & |\alpha| = k. \end{cases}$$

  From definition of $\|\cdot\|_{C(\bar{U})}, [\cdot]_{C^{0,\gamma}(\bar{U})}$ know that $D^\alpha u = 0, \forall |\alpha| \leq k$. Take $|\alpha| = 0$ get $u = 0$. While $u = 0$, it's clear that $\|u\| = 0$.

- Complete

  We already know that $C(\bar{U}), C^{0,\gamma}(\bar{U})$ is complete. And For each Cauchy sequence $\{u_k\}_{k=1}^{\infty}$, from the definition of $C^{k,\gamma}(\bar{U})$ know that $\{D^\alpha u_k\}_{k=1}^{\infty}$ also is a Cauchy sequence in $C(\bar{U})$. So they will converge to $u^\alpha \in C(\bar{U})$ respectively. Exspecially, Let u denote u^0. Then prove that $u \in C^{k,\gamma}(\bar{U})$. Since $D^\alpha u = D^\alpha \lim_{k=1}^{\infty} u_k = \lim_{k=1}^{\infty} D^\alpha u_k = u^\alpha$. Therefore $u \in C^{k\gamma}(\bar{U})$.

Problem 13.2. Denote by U the open square $\left\{x \in \mathbb{R}^2 \middle| |x_1| < 1, |x_2| < 1\right\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, \quad |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0, \quad |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0, \quad |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0, \quad |x_1| < -x_2 \end{cases}$$

For which $1 \le p \le \infty$ does u belong to $W^{1,p}(U)$?

Solution. Define v as

$$v(x) = \begin{cases} -1 & \text{if } x_1 > 0, \quad |x_2| < x_1 \\ 1 & \text{if } x_1 < 0, \quad |x_2| < -x_1 \\ -1 & \text{if } x_2 > 0, \quad |x_1| < x_2 \\ 1 & \text{if } x_2 < 0, \quad |x_1| < -x_2 \end{cases}$$

Denote U_1, U_2, U_3, U_4 as the four part of U divide by two line $x_1 = x_2, x_1 = -x_2$. So that $\forall \phi \in C_c^\infty(U)$,

$$\begin{aligned} \int_U uD\phi + v\phi dx &= \int_{U_1} uD\phi + v\phi dx + \int_{U_2} uD\phi - v\phi dx + \int_{U_3} uD\phi + v\phi dx + \int_{U_4} uD\phi + v\phi dx \\ &= \int_{\partial U_1} u\phi\boldsymbol{\nu} dS + \int_{\partial U_2} u\phi\boldsymbol{\nu} dS + \int_{\partial U_3} u\phi\boldsymbol{\nu} dS + \int_{\partial U_4} u\phi\boldsymbol{\nu} dS \\ &= \int_{\partial U} u\phi\boldsymbol{\nu} dS \\ &= 0. \end{aligned}$$

That's mean v is weak derivative of u. And it's easy to see that $u, v \in L^p(U), 1 \le p < \infty$. Therefore, u belong to $W^{1,p}(U)$.

Problem 13.3. Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \le p < \infty$
(a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0,1)$
(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \le |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, dt\right)^{1/p}$$

for a.e. $x, y \in [0,1]$

Solution.

(a) Assume v is the weak derivative of u. So $u, v \in L^p(0,1)$ and

$$\int_0^1 uD\phi dx = -\int_0^1 v\phi dx \qquad \forall \phi \in C_c^\infty(0,1).$$

Let $\phi = 1$ know that $w(x) = \int_0^x v dz$ exist and is absolutely continuous in$(0,1)$. Then

$$\begin{aligned}
\int_0^1 (w-u)D\phi dx &= \int_0^1 \int_0^x v(z)dzD\phi(x)dx - \int_0^1 u(x)D\phi(x)dx \\
&= \int_0^1 \int_z^1 D\phi(x)dxv(z)dz - \int_0^1 u(x)D\phi(x)dx \\
&= -\int_0^1 \phi(z)v(z)dz - \int_0^1 u(x)D\phi(x)dx \\
&= \int_0^1 u(x)D\phi dx - \int_0^1 u(x)D\phi(x)dx \\
&= 0.
\end{aligned}$$

From the arbitrary of ϕ know $u = w + C$. Therefore, u is absolutely continuous and $u' = v \in L^p(0,1)$.

(b)

$$\begin{aligned}
&|x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, dt \right)^{1/p} \\
&\geq \left(\int_x^y 1^{\frac{p}{p-1}} dz \right)^{1-\frac{1}{p}} \left(\int_x^y |u'|^p \, dz \right)^{\frac{1}{p}} \\
&\geq \int_x^y |u'| \, dz \\
&\geq |u(x) - u(y)| \, .
\end{aligned}$$

Problem 13.4. Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on $V, \zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_w.)

Solution. Since $V \subset\subset U$, assume a closure fill $V \subset\subset W \subset\subset U$.
First, choose a function u maintain that $\forall x \in W, u(x) = 1, u(x) = 0$,otherwise. Take $\varepsilon = \frac{1}{3}\min\{\operatorname{dist}(\bar{V}, W), \operatorname{dist}(W, \bar{U})\}$. Then use the standard mollifier η_ε define $u^\varepsilon := \eta_\varepsilon * u$. It's follows that u^ε is smooth, $u^\varepsilon(x) = 1, \forall x \in V$ and $u^\varepsilon(x) = 0, \forall x \in \partial U$.

Problem 13.5. Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exist C^∞ functions $\zeta_i (i = 1, \ldots, N)$ such that

$$\begin{cases} 0 \leq \zeta_i \leq 1, \operatorname{spt} \zeta_i \subset V_i (i = 1, \ldots, N) \\ \sum_{i=1}^N \zeta_i = 1 \quad \text{ on } U \end{cases}$$

The functions $\{\zeta_i\}_{i=1}^N$ form a partition of unity.

Solution. Without loss of generality, we may assume that the V_i are open, for if they are not, we can replace V_i by V_i^c. We note that, since U is bounded, $\bar{U}$ is compact. Each $x \in U$ has a compact neighbourhood N_x contained in V_i for some i. Then $\{N_x^c\}$ is an open cover of the $\bar{U}$, which then has a finite subcover $N_{x_1}^c, \ldots, N_{x_k}$. We now let F_i be the union of the N_{x_i} contained in V_i. F_i is the compact since it is the finite union of compact sets. Therefore, F_i, V_i fill (Problem 13.4) since $U \subset\subset \bigcup_{i=1}^N V_i$, so there are smooth function $\{\xi_i \mid \xi_i = 1 \text{ on } F_i, \operatorname{spt} \xi \subset V_i\}$. Since F_i cover $U, U \subset \{x \mid \sum_{i=1}^n \xi_i(x) > 0\}$ and we can use result of (Problem 13.4) again know exist $\omega(x) = 1, \forall x \in \bar{U}$ and $\operatorname{spt} \omega \subset \{x \mid \sum_{i=1}^n \xi_i(x) > 0\}$. Now, we let $\xi_{N+1} = 1 - \omega$, so $\sum_{i=1}^{N+1} \xi_i > 0$ everywhere. We then take

$$\zeta_i = \frac{\xi_i}{\sum_{j=1}^{N+1} \xi_j}$$

as our partiation of unity. Therefore $\operatorname{spt} \zeta_i = \operatorname{spt} \xi_i \subset V_i, \forall y \in \bar{U}, \sum_{i=1}^N \zeta_i = \frac{\sum_{j=1}^{N+1} \xi_j - \xi_{N+1}}{\sum_{j=1}^{N+1} \xi_j} = 1$ and $\zeta_i \in [0,1]$.

Problem 13.6. Assume that U is bounded and there exists a smooth vector field $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \geq 1$ along ∂U, where $\boldsymbol{\nu}$ as usual denotes the outward unit normal. Assume $1 \leq p < \infty$
Apply the Gauss-Green Theorem to $\int_{\partial U} |u|^p \boldsymbol{\alpha} \cdot \boldsymbol{\nu} dS$, to derive a new proof of the trace inequality

$$\int_{\partial U} |u|^p dS \leq C \int_U |Du|^p + |u|^p dx$$

for all $u \in C^1(\bar{U})$

Solution. Since $\boldsymbol{\alpha}$ is smooth and U is bounded, $\exists M \in \mathbb{R}, s.t. \operatorname{div} \boldsymbol{\alpha} < M, |\boldsymbol{\alpha}| < M$.

$$\begin{aligned}
&\int_{\partial U} |u|^p \, dS \\
&\leq \int_{\partial U} |u|^p \, \boldsymbol{\alpha} \cdot \boldsymbol{\nu} dS \\
&= \int_U D\, |u|^p \cdot \boldsymbol{\alpha} + |u|^p \operatorname{div} \boldsymbol{\alpha} dx \\
&\leq \int_U Mp\, |u|^{p-1} |Du| + M\, |u|^p \, dx \\
&\leq \int_U M((p-1)\, |u|^p + |Du|^p) + M\, |u|^p \, dx \\
&\leq Mp \int_U |Du|^p + |u|^p dx.
\end{aligned}$$

Take $C = Mp$ gives the trace inequality.