Chapter 4

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Exercise 4.1. 13. Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show
$$u_t = u_{xx}$$
if and only if
$$(*) \qquad v'' + \frac{z}{2}v' = 0$$
Show that the general solution of $(*)$ is
$$v(z) = c\int_0^z e^{-s^2/4}ds + d. \tag{4.1}$$

(b) Differential $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for $n = 1$. Explain why this procedure produce produces the fundamental solution.

Solution.

(a) Since
$$u_t = \partial_t v(\frac{x}{\sqrt{t}}) = v'\frac{-x}{2t^{3/2}} \tag{4.2}$$
$$u_{xx} = \partial_x(v'(\frac{x}{\sqrt{t}})\frac{1}{\sqrt{t}}) = v''\frac{1}{t}. \tag{4.3}$$
Combining (4.2),(4.3) and $u_t = u_{xx}$ get $(*)$.

Then
$$\begin{aligned} & v'' + \frac{z}{2}v' = 0 \\ \implies \quad & ln(v') = -\frac{z^2}{4} + c \\ \implies \quad & v(z) = c\int_0^z e^{-s^2/4}ds + d. \end{aligned}$$
means (4.1) is general solution of $(*)$.

(b)
$$u_x(x,t) = ce^{-x^2/4t} * \frac{1}{\sqrt{t}}$$
Since $\int e^{-x^2/4t} = 2\sqrt{\pi t}$. Set $c = \frac{1}{2\sqrt{\pi}}$ get the fundamental solution$\frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}$. It solves heat equation, satisfies Dirac measure.Since Fundamental solution have character that integral value is equal to 1, while this procedure is generated from this condition.

Exercise 4.2. 15. Given $g : [0,\infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula
$$u(x,t) = \frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{\frac{-x^2}{4(t-s)}}g(s)ds \tag{4.4}$$
for a solution of the initial, boundary-value problem
$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0,\infty). \end{cases} \tag{4.5}$$

Solution. Let
$$v(x,t) = \begin{cases} u(x,t) - g(t) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ 0 & \text{on } \{x = 0\} \times (0,\infty) \\ -u(-x,t) + g(t) & \text{in } \mathbb{R}_- \times (0.\infty) \end{cases} \tag{4.6}$$
Then
$$v_t - v_{xx} = \begin{cases} -g_t(t) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ g_t(t) & \text{in } \mathbb{R}_- \times (0.\infty) \end{cases}$$

and $v = g(0) = 0 \quad$ on $\mathbb{R}_+ \times \{t = 0\}$.Thus, from solution of nonhomogeneous problem get

$$
\begin{aligned}
v(x,t) = & \int_{\mathbb{R}} \Phi(x-y,t)0dy + \\
& \int_0^t \int_{-\infty}^0 \Phi(x-y,t-s)g'(s)dyds + \\
& \int_0^t \int_0^{\infty} \Phi(x-y,t-s)(-g'(s))dyds \\
= & \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)(-g'(s))dyds + \\
& 2\int_0^t \int_{-\infty}^0 \Phi(x-y,t-s)g'(s)dyds \\
= & \int_0^t (-g'(s))ds + \\
& 2\int_0^t \int_{-\infty}^0 \Phi(x-y,t-s)dyg'(s)ds \\
= & -g(t) + 2\int_{-\infty}^0 \Phi(x-y,t-s)g(s)\Big|_{s=0}^{t} + \\
& 2\int_0^t \partial_s(\int_{-\infty}^0 \Phi(x-y,t-s)dy)g(s)ds.
\end{aligned} \tag{4.7}
$$

Calculating the second term

$$
\begin{aligned}
& \partial_s(\int_{-\infty}^0 \Phi(x-y,t-s)dy) \\
= & \partial_s(\int_{-\infty}^0 \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{(4\pi(t-s))^{1/2}}dy) \\
= & \partial_s(-\int_{\infty}^{\frac{x}{2(t-s)^{1/2}}} \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{\pi}} d(\frac{x-y}{2(t-s)^{1/2}})) \\
= & \partial_s(-\frac{1}{\sqrt{\pi}}\int_{\infty}^{\frac{x}{2(t-s)^{1/2}}} e^{-z^2}dz) \\
= & \frac{x}{4\sqrt{\pi}(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}}.
\end{aligned}
$$

Therefore, the second term is equal to(4.4). It follows that $u(x,t) = v(x,t) + g(t) = (4.8)$.

Exercise 4.3. 16. Five a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$
\max_{\bar{U}_T} u = \max_{\Gamma_T} u. \tag{4.8}
$$

Solution. Set $u_\epsilon := u - \epsilon t \quad \epsilon > 0$. From u solves the heat equation conclude

$$
\partial_t u_\epsilon - \Delta_x u_\epsilon = -\epsilon < 0
$$

if u_ϵ have max point (x_0, t_0) in U_T,from max point character

$$
\begin{aligned}
\partial_t u_\epsilon(x_0,t_0) = 0 \\
\Delta_x u_\epsilon(x_0,t_0) = 0
\end{aligned}
$$

However it contradict with above $\partial_t u_\epsilon - \Delta_x u_\epsilon < 0$.So u_ϵ don't have max point in U_T. If u have max point (x_0,t_0) so that $u(x_0,t_0) > u(x,t) \ \forall (x,t) \in \Gamma_T \iff \exists \epsilon > 0 \ u(x_0,t_0) > u(x,t) + \epsilon T \iff u_\epsilon(x_0,t_0) > u(x,t) \ \forall (x,t) \in \Gamma_T \iff u_\epsilon$ has max point in U_T, which is proved impossible above. So $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$.

Exercise 4.4. 17. We say $v \in C_1^2(U_T)$is a *subsolution* of the heat equation if

$$
v_t - \Delta v \leq 0 \qquad \text{in } U_T \tag{4.9}
$$

(a) Prove for a subsolution v that

$$
v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \tag{4.10}
$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $max_{\bar{U}_T} = max_{\Gamma_T} v$.

(c) Let $\phi : \mathbb{R} \to \phi$ be smooth and convex. Assume u solves the hear equation and $v := \phi(u)$. Prove v is a subsolution.

(d) Prove $v := |Du|^2 + u_t^2$ is a subsolution, whenever u solves the hear equation.

Solution.

(a) Define $\phi(r) := \frac{1}{r^n}\iint_{E(r)} v(y,s)\frac{|y|^2}{s^2}dyds = \iint_{E(1)} v(ry,r^2s)\frac{|y|^2}{s^2}dyds$,According to Mean-value property for the heat equation and set $\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r$,calculate ϕ'

$$
\begin{aligned}
& \phi'(r) \\
\geq & \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta v\psi - \frac{2n}{s}\sum_{i=1}^n v_{y_i} y_i dyds \\
= & \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} -4nv_{y_i}\psi_{y_i} - \frac{2n}{s} v_{y_i} y_i dyds \\
= & 0
\end{aligned}
$$

Thus ϕ is monotonically increasing, and therefore

$$
\begin{aligned}
\phi(r) > & \lim_{t\to 0}\phi(t) \\
& = v(0,0)\lim_{t\to 0}\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2}dyds \\
& = 4v(0,0).
\end{aligned}
$$

consider (x,t) is similar with (0,0). So that $v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \quad \forall E(x,t:r) \subset U_T$

(b) Assume v has max point $(x_0,t_0) \in U_T$, For all $E(x_0,t_0;r) \subset U_T$, we employ the mean-value property to deduce

$$\begin{aligned} v(x_0,t_0) &\leq \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \\ &\leq v(x_0,t_0) \frac{1}{4r^n} \iint_{E(x,t;r)} \frac{|x-y|^2}{(t-s)^2} dyds \\ &= v(x_0,t_0), \end{aligned}$$

that indicate that $\forall (x,t) \in E(x_0,t_0;r)$ $v(x,t) = M$. Then the same process of proving the strong maximum principle for the heat equation get $v(x,t) = v(x_0,t_0)$ $\forall (x,t) \in U_T$. It follows that $max_{\bar{U}_T} = max_{\Gamma_T} v$.

(c) Differential v

$$v_t = \phi_t(u) = \phi' u_t$$

$$\begin{aligned} \Delta v = \Delta(\phi(v)) &= \sum_{i=1}^{n} \partial_{x_i}(\phi' u_{x_i}) \\ &= \sum_{i=1}^{n} (\phi'' u_{x_i}^2 + \phi' u_{x_i x_i}) \\ &\geq \phi' \Delta u. \end{aligned}$$

In summary $v_t - \Delta v \leq 0$, that indicates that v is a subsolution.

(d) Differential v

$$\begin{aligned} v_t &= \partial_t(|Du|^2 + u_t^2) \\ &= \sum_{i=1}^{n} (2u_{x_i} u_{x_i t}) + 2u_t u_{tt} \\ &= 2u_{x_i}(u_{x_i})_t + 2u_t(u_t)_t \end{aligned}$$

$$\begin{aligned} \Delta v &= \Delta(|Du|^2 + u_t^2) \\ &= \sum_{i=1}^{n} \partial_{x_i}(2u_{x_i} u_{x_i x_i} + 2u_t u_{t x_i}) \\ &= \sum_{i=1}^{n} (2(u_{x_i x_i})^2 + 2u_{x_i} u_{x_i x_i x_i} + 2(u_{t x_i})^2 + 2u_t u_{t x_i x_i}) \\ &\geq 2u_{x_i} \Delta u_{x_i} + 2u_t \Delta u_t. \end{aligned}$$

Then $v_t - \Delta v \leq 2u_{x_i}((u_{x_i})_t - \Delta u_{x_i}) + 2u_t((u_t)_t - \Delta u_t) = 0$. The second step come from when u is heat equation solution, u_t and u_{x_i} too.After all, v is a subsolution, whenever u solves the hear equation.