Chapter 2

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Exercise 2.1. For $f \in \mathcal{C}[x_0, x_1]$ and $x \in (x_0, x_1)$, linear interpolation of f at x_0 and x_1 yield

$$f(x) - p_1(f; x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1). \tag{2.1}$$

Consider the case $f(x) = \frac{1}{x},\ x_0 = 1,\ x_1 = 2$.

- Determine $\xi(x)$ explicitly.
- For $x \in [x_0, x_1]$, find $\max \xi(x),\ \min \xi(x),$ and $f''(\xi(x))$.

Solution.

$$\begin{aligned}
&p_1(f; x) = \frac{1}{2} \cdot \frac{x - x_0}{x_1 - x_0} + 1 \cdot \frac{x - x_1}{x_0 - x_1} = -\frac{1}{2}(x - 1) + 1 \\
&f(x) - p_1(f; x) = \frac{x^2 - 3x + 2}{2x} \\
&\frac{1}{2\xi^3(x)} = f''(\xi(x)) = \frac{2(f(x) - p_1(f; x))}{(x - 1)(x - 2)} = \frac{1}{x} \\
&\xi(x) = (\frac{x}{2})^{\frac{1}{3}}.
\end{aligned}$$

So for $x \in [x_0, x_1]$ $\max \xi(x) = 1,\ \min \xi(x) = \frac{1}{2^{1/3}}$ and $f''(\xi(x)) = \frac{1}{x}$

Exercise 2.2. Let $\mathcal{P}_m^+$ be the set of all polynomials of degree $\leq m$ that are non-negative on the real line,

$$\mathbb{P}_m^+ = \{p : p \in \mathbb{P}_m, \forall x \in \mathbb{R}, p(x) \geq 0\}. \tag{2.2}$$

Find $p \in \mathbb{P}_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$ where $f_i \geq 0$ and x_i are distinct points on $\mathbb{R}$.

Solution. Define $l(x) = \prod_{j=0}^n (x - x_j)^2, l_i(x) = \prod_{j \neq i}(x - x_j)^2$,

$$p(x) = \sum_{i=0}^n f_i \frac{l(x)}{l_i(x_i)(x - x_j)^2}.$$

It follows that $p(x_i) = f_i$, $p(x) \geq 0$ and $\deg p(x) \leq 2n$. Therefore, $p(x) \in \mathbb{P}_{2n}^+$ satisfy condition.

Exercise 2.3. Consider $f(x) = e^x$.

- Prove by induction that

$$\forall t \in \mathbb{R}, \qquad f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!} e^t.$$

- From Corollary 3.17 we know

$$\xi \in (0, n) \text{ s.t. } f[0, 1, \dots, n] = \frac{1}{n!} f^{(n)}(\xi).$$

  Determine ξ from the above two equations. Is ξ located to the left or to the right of the midpoint $n/2$?

Solution. When $n = 0$, The equation is trivial. Then set $\forall t \in \mathbb{R}, f[t, t+1, \dots, t+n-1] = \frac{(e-1)^{n-1}}{(n-1)!} e^t, \forall t \in \mathbb{R}, f[t+1, t+2, \dots, t+n] = \frac{(e-1)^{n-1}}{(n-1)!} e^{t+1}$.

$$\begin{aligned}
& f[t, t+1, \dots, t+n] \\
&= (f[t+1, t+2, \dots, t+n] - f[t, t+1, \dots, t+n-1])/(t+n-t) \\
&= \frac{(e-1)^{n-1}}{(n-1)!} e^t (e-1)/(n) \\
&= \frac{(e-1)^n}{(n)!} e^t.
\end{aligned}$$

By induction, $\forall t \in \mathbb{R}, f[t, t+1, \dots, t+n] = \frac{(e-1)^n}{n!} e^t$. Thus,

$$\begin{aligned}
& e^\xi = f^{(n)}(\xi) = (n!) f[0, 1, \dots, n] = (e-1)^n \iff \\
& \xi = n \ln(e-1) > 1/2.
\end{aligned}$$

Exercise 2.4. Consider $f(0) = 5, f(1) = 3, f(3) = 5, f(4) = 12$.

- Use the Newton formula to obtain $p_3(f; x)$;
- The data suggest that f has a minimum in $x \in (1, 3)$. Find an approximate value for the location $x_{\min}$ of the minimum.

Solution.

$$\begin{array}{c|cccc}
0 & 5 & & & \\
1 & 3 & -2 & & \\
3 & 5 & 1 & 1 & \\
4 & 12 & 7 & 2 & 1/4
\end{array}$$

Thus $p_3(f; x) = 5 - 2(x-0) + (x-0)(x-1) + \frac{1}{4}(x-0)(x-1)(x-3)$.Differential $p_3(f; x)$ get $x_{\min} = \sqrt[2]{3}$, Therefore $f(\sqrt[2]{3}) = -\frac{3\sqrt[2]{3}}{2} + 5$.

Exercise 2.5. Condiser $f(x) = x^7$.

- Compute $f[0, 1, 1, 1, 2, 2]$.
- We know that this divided differene is expressible in terms of the 5th derivative of f evaluated at some $\xi \in (0, 2)$. Determine ξ.

Solution.

$$\begin{array}{c|ccc}
 & \mathrm{f}(0) & & \\
1 & \mathrm{f}(1) & & \\
1 & \mathrm{f}(1) & \mathrm{f}'(1) & \\
1 & \mathrm{f}(1) & \mathrm{f}'(1) & \mathrm{f}''(1) \\
2 & \mathrm{f}(2) & & \\
2 & \mathrm{f}(2) & \mathrm{f}'(2) &
\end{array}$$

Replace $f(x) with x^7$,

$$\begin{array}{c|cccccc} 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 7 & 6 \\ 1 & 1 & 7 & 42 & 36 \\ 2 & 128 & 127 & 120 & 78 & 21 \\ 2 & 128 & 448 & 321 & 201 & 123 & 51 \end{array}$$

So $f[0,1,1,1,2,2] = 51$. $\xi = \sqrt[2]{\frac{51}{21}}$

Exercise 2.6. f is a function on $[0,3]$ for which one knows that

$$f(0) = 1, f(1) = 2, f'(1) = -1, f(3) = f'(3) = 0.$$

- Estimate $f(2)$ using Hermite interpolation.
- Estimate the maximum possible error of the above answer if one knows, in addition, that $f \in \mathcal{C}^5[0,3]$ and $|f^{(5)}(x)| \le M$ on $[0,3]$. Express the answer in terms of M.

Solution. By Hermite interpolation,

$$f(x) = 1\frac{(x-1)^2(x-3)^2}{(0-1)^2(0-3)^2} + 2\frac{(x-0)(x-3)^2}{(1-0)(1-3)^2} + (-1)\frac{(x-0)(x-1)(x-3)^2}{(1-0)(1-3)^2}$$

Then, $f(2) = 1/9 + 1 - 1/2 = \frac{11}{18}$.$\mathbf{R}_5(f;x) = \frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2 < \frac{3^5}{5!}M$.

Exercise 2.7. Define forward difference by

$$\Delta f(x) = f(x+h) - f(x),$$

$$\Delta^{k+1} f(x) = \Delta\Delta^k f(x) = \Delta^k f(x+h) - \Delta^k f(x)y$$

and backward difference by

$$\nabla f(x) = f(x+h) - f(x).$$

$$\nabla^{k+1} f(x) = \nabla\nabla^k f(x) = \nabla^k f(x) - \nabla^k f(x-h).$$

Prove

$$\Delta^k f(x) = k!h^k f[x_0, x_1, \dots, x_k],$$

$$\nabla^k f(x) = k!h^k f[x_0, x_{-1}, \dots, x_{-k}],$$

where $x_j = x + jh$.

Solution. By induction, When $k = 1, \Delta f(x) = f(x+h) - f(x) = hf[x_0, x_1]$. Assuming $\Delta^k f(x) = k!h^k f[x_0, x_1, \dots, x_k]$,

$$\begin{aligned} &\Delta^{k+1} f(x) \\ &= \Delta^k f(x_1) - \Delta^k f(x_0) \\ &= k!h^k(f[x_1, x_2, \dots, x_{k+1}] - f[x_0, x_1, \dots, x_k]) \\ &= (k+1)!h^{k+1} f[x_0, x_1, \dots, x_{k+1}]. \end{aligned}$$

In summary, $\Delta^k f(x) = k!h^k f[x_0, x_1, \dots, x_k]$.Similarly, $\nabla^k f(x) = k!h^k f[x_0, x_1, \dots, x_k]$.

Exercise 2.8. Assume f is differentiable at x_0. Prove

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \dots, x_n] = f[x_0, x_0, x_1, \dots, x_n].$$

What about the partial derivative with respect to one of the other variables?

Solution. By Definition, $\frac{\partial}{\partial_{x_0}}f[x_0] = f[x_0, x_0]$. Assuming $\frac{\partial}{\partial x_0}f[x_0, x_1, \ldots, x_{n-1}] = f[x_0, x_0, x_1, \ldots, x_{n-1}]$,

$$\begin{aligned}
&\frac{\partial}{\partial x_0}f[x_0, x_1, \ldots, x_n] \\
&= \frac{\frac{\partial}{\partial x_0}f[x_0, x_1, \ldots, x_{n-2}, x_n] - \frac{\partial}{\partial x_0}f[x_0, x_1, \ldots, x_{n-2}, x_{n-1}]}{x_n - x_{n-1}} \\
&= \frac{f[x_0, x_0, x_1, \ldots, x_{n-2}, x_n] - f[x_0, x_0, x_1, \ldots, x_{n-2}, x_{n-1}]}{x_n - x_{n-1}} \\
&= f[x_0, x_0, x_1, \ldots, x_n].
\end{aligned}$$

Therefore, $\forall n \in \mathbb{Z}^+ \frac{\partial}{\partial x_0}f[x_0, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n]$.And since divided difference is order-independent, $\forall n \in \mathbb{Z}^+ \frac{\partial}{\partial x_j}f[x_0, x_1, \ldots, x_n] = f[x_0, x_1, \ldots, x_j, x_j, \ldots, x_n]$.