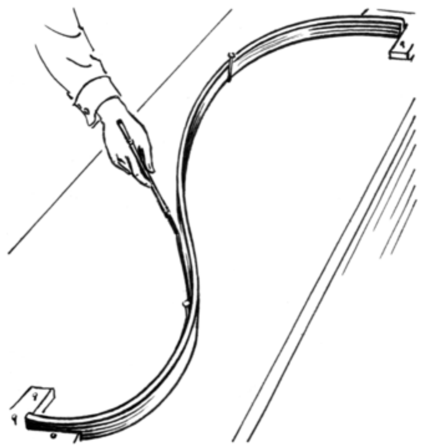
Chapter 4

Splines



4.1 Piecewise-polynomial splines

Definition 4.1. Given nonnegative integers n, k, and a strictly increasing sequence $\{x_i\}$ that partitions $[a, b]$,

$$a = x_1 < x_2 < \cdots < x_N = b, \tag{4.1}$$

the set of *spline functions of degree n and smoothness class k* relative to the partition $\{x_i\}$ is

$$\mathbb{S}_n^k = \left\{s : \ s \in \mathcal{C}^k[a,b]; \ \forall i \in [1, N-1], \ s\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_n\right\}. \tag{4.2}$$

The x_i's are called *knot*s of the spline.

Notation 1. In Section 3, the polynomial degree is denoted by n for all methods. Here we use N to denote the number of knots for a spline.

Example 4.2. As an extreme, $\mathbb{S}_n^n = \mathbb{P}_n$, i.e. all the pieces of $s \in \mathbb{S}_n^n$ belong to a single polynomial. On the other end, $\mathbb{S}_1^0$ is the class of piecewise linear interpolating functions. The most popular splines are the cubic splines in $\mathbb{S}_3^2$.

Lemma 4.3. Denote $m_i = s'(f; x_i)$ for $s \in \mathbb{S}_3^2$. Then, for each $i = 2, 3, \ldots, N-1$, we have

$$\lambda_i m_{i-1} + 2m_i + \mu_i m_{i+1} = 3\mu_i f[x_i, x_{i+1}] + 3\lambda_i f[x_{i-1}, x_i], \tag{4.3}$$

where

$$\mu_i = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}, \qquad \lambda_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}. \tag{4.4}$$

Proof. Denote $p_i(x) = s\big|_{[x_i, x_{i+1}]}$ and $K_i = f[x_i, x_{i+1}]$. The table of divided difference for the Hermite interpolation problem $p_i(x_i) = f_i$, $p_i(x_{i+1}) = f_{i+1}$, $p_i'(x_i) = m_i$, $p_i'(x_{i+1}) = m_{i+1}$ is

$$\begin{array}{c|cccc}
x_i & f_i & & & \\
x_i & f_i & m_i & & \\
x_{i+1} & f_{i+1} & K_i & \frac{K_i - m_i}{x_{i+1} - x_i} & \\
x_{i+1} & f_{i+1} & m_{i+1} & \frac{m_{i+1} - K_i}{x_{i+1} - x_i} & \frac{m_i + m_{i+1} - 2K_i}{(x_{i+1} - x_i)^2}
\end{array}$$

Then the Newton formula yields

$$\begin{aligned} p_i(x) =& f_i + (x - x_i)m_i + (x - x_i)^2 \frac{K_i - m_i}{x_{i+1} - x_i} \\ &+ (x - x_i)^2 (x - x_{i+1}) \frac{m_i + m_{i+1} - 2K_i}{(x_{i+1} - x_i)^2}, \end{aligned} \tag{4.5}$$

or equivalently

$$\begin{cases} p_i(x) &= c_{i,0} + c_{i,1}(x - x_i) + c_{i,2}(x - x_i)^2 + c_{i,3}(x - x_i)^3, \\ c_{i,0} &= f_i, \\ c_{i,1} &= m_i, \\ c_{i,2} &= \frac{3K_i - 2m_i - m_{i+1}}{x_{i+1} - x_i}, \\ c_{i,3} &= \frac{m_i + m_{i+1} - 2K_i}{(x_{i+1} - x_i)^2}. \end{cases} \tag{4.6}$$

$s \in \mathcal{C}^2$ implies that $p_{i-1}''(x_i) = p_i''(x_i)$, i.e.

$$3c_{i-1,3}(x_i - x_{i-1}) = c_{i,2} - c_{i-1,2}.$$

The substitution of the coefficients $c_{i,j}$ into the above equation yields (4.3). □

Definition 4.4. The method of *dynamic programming*, or *dynamic optimization*, solves a complex problem by breaking it down into a collection of simpler sub-problems, solving each of those sub-problems just once, and storing their solutions. When the same sub-problem occurs, instead of recomputing its solution, one simply looks up the previously computed solution, thereby saving computation time at the expense of an increase in storage space.

Lemma 4.5. Denote $M_i = s''(f; x_i)$ for $s \in \mathbb{S}_3^2$. Then, for each $i = 2, 3, \ldots, N-1$, we have

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}] \tag{4.7}$$

where μ_i and λ_i are the same as those in (4.4).

Proof. Taylor expansion of $s(x)$ at x_i yields

$$s(x) = f_i + s'(x_i)(x - x_i) + \frac{M_i}{2}(x - x_i)^2 + \frac{s'''(x_i)}{6}(x - x_i)^3, \tag{4.8}$$

where $x \geq x_i$ and the derivatives should be interpreted as the right-hand derivatives. Differentiate (4.8) twice, set $x = x_{i+1}$, and we have

$$s'''(x_i) = \frac{M_{i+1} - M_i}{x_{i+1} - x_i}. \tag{4.9}$$

Substitute (4.9) into (4.8), set $x = x_{i+1}$, and we have

$$s'(x_i) = f[x_i, x_{i+1}] - \frac{1}{6}(M_{i+1} + 2M_i)(x_{i+1} - x_i). \tag{4.10}$$

Differentiate (4.8) twice, set $x = x_{i-1}$, and we have $s'''(x_i) = \frac{M_{i-1}-M_i}{x_{i-1}-x_i}$. Its substitution into (4.8) yields

$$s'(x_i) = f[x_{i-1}, x_i] - \frac{1}{6}(M_{i-1} + 2M_i)(x_{i-1} - x_i). \tag{4.11}$$

The subtraction of (4.10) and (4.11) yields (4.7). □

Definition 4.6 (Types of splines)**.**

- A *complete cubic spline* $s \in \mathbb{S}_3^2$ satisfies boundary conditions $s'(f;a) = f'(a)$ and $s'(f;b) = f'(b)$.
- A *cubic spline with specified second derivatives at its end points*: $s''(f;a) = f''(a)$ and $s''(f;b) = f''(b)$.
- A *natural cubic spline* $s \in \mathbb{S}_3^2$ satisfies boundary conditions $s''(f;a) = 0$ and $s''(f;b) = 0$.
- A *not-a-knot cubic spline* $s \in \mathbb{S}_3^2$ satisfies that $s'''(f;x)$ exists at $x = x_2$ and $x = x_{N-1}$.
- A *periodic cubic spline* $s \in \mathbb{S}_3^2$ is obtained from replacing $s(f;b) = f(b)$ with $s(f;b) = s(f;a)$, $s'(f;b) = s'(f;a)$, and $s''(f;b) = s''(f;a)$.

Lemma 4.7. For a complete cubic spline $s \in \mathbb{S}_3^2$, denote $M_i = s''(f;x_i)$ and we have

$$2M_1 + M_2 = 6f[x_1, x_1, x_2], \tag{4.12}$$
$$M_{N-1} + 2M_N = 6f[x_{N-1}, x_N, x_N]. \tag{4.13}$$

Proof. As for (4.12), the cubic polynomial on $[x_1, x_2]$ can be written as

$$\begin{aligned} s_1(x) =& f[x_1] + f[x_1, x_1](x - x_1) \\ &+ \frac{M_1}{2}(x - x_1)^2 + \frac{s_1'''(x_1)}{6}(x - x_1)^3. \end{aligned}$$

Differentiate the above equation twice, replace x with x_2, and we have $s_1'''(x_1) = \frac{M_2 - M_1}{x_2 - x_1}$, which implies

$$\begin{aligned} s_1(x) =& f[x_1] + f[x_1, x_1](x - x_1) \\ &+ \frac{M_1}{2}(x - x_1)^2 + \frac{M_2 - M_1}{6(x_2 - x_1)}(x - x_1)^3. \end{aligned} \tag{4.14}$$

Set $x = x_2$, divide both sides by $x_2 - x_1$, and we have

$$f[x_1, x_2] = f[x_1, x_1] + \left(\frac{M_1}{2} + \frac{M_2 - M_1}{6}\right)(x_2 - x_1),$$

which yields (4.12). (4.13) can be proven similarly. □

Theorem 4.8. For a given function $f : [a, b] \to \mathbb{R}$, there exists a unique complete/natural/periodic cubic spline $s(f;x)$ that interpolates f.

Proof. We only prove the case of complete cubic splines since the other cases are similar.

By the proof of Lemma 4.3, s is uniquely determined if all the m_i's are uniquely determined on all intervals. For a complete cubic spline we already have $m_1 = f'(a)$ and $m_N = f'(b)$. Assemble (4.3) into a linear system

$$\begin{bmatrix} 2 & \mu_2 & & & & & \\ \lambda_3 & 2 & \mu_3 & & & & \\ & & \ddots & & & & \\ & & \lambda_i & 2 & \mu_i & & \\ & & & & \ddots & & \\ & & & & \lambda_{N-2} & 2 & \mu_{N-2} \\ & & & & & \lambda_{N-1} & 2 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \\ \vdots \\ m_i \\ \vdots \\ m_{N-2} \\ m_{N-1} \end{bmatrix} = \mathbf{b}, \tag{4.15}$$

where the vector $\mathbf{b}$ is determined from the known information. (4.4) implies that the matrix in the above equation is strictly diagonally dominant. Therefore its determinant is nonzero and the m_i's can be uniquely determined.

Alternatively, a complete cubic spline can be uniquely determined from Lemmas 4.5 and 4.7, following arguments similar to the above. □

Example 4.9. Construct a complete cubic spline $s(x)$ on points $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 6$ from the function values of $f(x) = \ln(x)$ and its derivatives at x_1 and x_5. Approximate $\ln(5)$ by $s(5)$.

From the given conditions, we set up the table of divided differences as follows.

x_i	$f[x_i]$		
1	0		
1	0	1	
2	0.6931	0.6931	−0.3069
3	1.0986	0.4055	−0.1438
4	1.3863	0.2877	−0.05889
6	1.7918	0.2027	−0.02831
6	1.7918	0.1667	−0.01803

All values of λ_i and μ_i are $\frac{1}{2}$ except that

$$\lambda_4 = \frac{2}{3},\ \mu_4 = \frac{1}{3}.$$

Then Lemma 4.5 and Lemma 4.7 yield a linear system

$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 6 & 2 \\ & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} \approx \begin{bmatrix} -1.84112 \\ -1.72610 \\ -0.70670 \\ -0.50967 \\ -0.10820 \end{bmatrix},$$

where elements in the RHS vector are obtained from the last column of the table of divided differences by multiplying $6, 12, 12, 18$, and 6. Why? Solve the linear system and we have all the M_i's. Then we derive an expression of the spline on the last interval following the procedures similar to those for (4.14). After this expression is obtained, we then evaluate it and obtain $s(5) \approx 1.60977$. In comparison, $\ln(5) \approx 1.60944$.

4.2 The minimum properties

Theorem 4.10 (Minimum bending energy)**.** For any function $g \in \mathcal{C}^2[a,b]$ that satisfies $g'(a) = f'(a)$, $g'(b) = f'(b)$, and $g(x_i) = f(x_i)$ for each $i = 1, 2, \ldots, N$, the complete cubic spline $s = s(f; x)$ satisfies

$$\int_a^b [s''(x)]^2 \mathrm{d}x \le \int_a^b [g''(x)]^2 \mathrm{d}x, \tag{4.16}$$

where the equality holds only when $g(x) = s(f; x)$.

Proof. Define $\eta(x) = g(x) - s(x)$. From the given conditions we have $\eta \in \mathcal{C}^2[a,b]$, $\eta'(a) = \eta'(b) = 0$, and $\forall i = 1, 2, \ldots, N$, $\eta(x_i) = 0$. Then

$$\begin{aligned}
&\int_a^b [g''(x)]^2 \mathrm{d}x = \int_a^b [s''(x) + \eta''(x)]^2 \mathrm{d}x \\
=&\int_a^b [s''(x)]^2 \mathrm{d}x + \int_a^b [\eta''(x)]^2 \mathrm{d}x + 2\int_a^b s''(x)\eta''(x) \mathrm{d}x.
\end{aligned}$$

From

$$\begin{aligned}
&\int_a^b s''(x)\eta''(x)\mathrm{d}x = \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} s''(x)\mathrm{d}\eta' \\
=&\sum_{i=1}^{N-1} s''(x)\eta'(x)|_{x_i}^{x_{i+1}} - \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} \eta'(x)s'''(x)\mathrm{d}x \\
=&s''(b)\eta'(b) - s''(a)\eta'(a) - \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} s'''(x)\mathrm{d}\eta \\
=&-\sum_{i=1}^{N-1} s'''(x)\eta(x)|_{x_i}^{x_{i+1}} + \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} \eta(x)s^{(4)}(x)\mathrm{d}x \\
=&0,
\end{aligned}$$

we have

$$\int_a^b [g''(x)]^2 \mathrm{d}x = \int_a^b [s''(x)]^2 \mathrm{d}x + \int_a^b [\eta''(x)]^2 \mathrm{d}x,$$

which completes the proof. □

Theorem 4.11 (Minimum bending energy)**.** For any function $g \in \mathcal{C}^2[a,b]$ with $g(x_i) = f(x_i)$ for each $i = 1, 2, \ldots, N$, the natural cubic spline $s = s(f; x)$ satisfies

$$\int_a^b [s''(x)]^2 \mathrm{d}x \le \int_a^b [g''(x)]^2 \mathrm{d}x, \tag{4.17}$$

where the equality holds only when $g(x) = s(f; x)$.

Proof. The proof is similar to that of Theorem 4.10. Although $\eta'(a) = \eta'(b) = 0$ does not hold, we do have $s''(a) = s''(b) = 0$. □

Lemma 4.12. Suppose a $\mathcal{C}^2$ function $f : [a,b] \to \mathbb{R}$ is interpolated by a complete cubic spline or a cubic spline with specified second derivatives at its end points. Then

$$\forall x \in [a,b], \qquad |s''(x)| \le 3 \max_{x\in[a,b]} |f''(x)|. \tag{4.18}$$

Proof. Since $s''(x)$ is linear on $[x_i, x_{i+1}]$, $|s''(x)|$ attains its maximum at x_j for some j. If $j = 2, \ldots, N-1$, it follows from Lemma 4.5 and Corollary 3.22 that

$$\begin{aligned}
&2M_j = 6f[x_{j-1}, x_j, x_{j+1}] - \mu_j M_{j-1} - \lambda_j M_{j+1} \\
\Rightarrow\ &2|M_j| \le 6\,|f[x_{j-1}, x_j, x_{j+1}]| + (\mu_j + \lambda_j)|M_j| \\
\Rightarrow\ &\exists \xi \in [x_{j-1}, x_{j+1}] \ \text{ s.t. } |M_j| \le 3\,|f''(\xi)| \\
\Rightarrow\ &|s''(x)| \le 3 \max_{x\in[a,b]} |f''(x)|.
\end{aligned} \tag{4.19}$$

If $|s''(x)|$ attains its maximum at x_1 or x_N, (4.19) clearly holds for a cubic spline with specified second derivatives at these end points. Due to symmetry, it suffices to prove (4.19) for the complete spline when $|s''(x)|$ attains its maximum at x_1. Since the first derivative $f'(a) = f[x_1, x_1]$ is specified, $f[x_1, x_1, x_2]$ is a constant. By (4.12), we have

$$2|M_1| \le 6|f[x_1, x_1, x_2]| + |M_2| \le 6|f[x_1, x_1, x_2]| + |M_1|$$

which, together with Corollary 3.22, implies

$$\exists \xi \in [x_1, x_2] \text{ s.t. } |M_1| \le 3|f''(\xi)|.$$

This completes the proof. □

4.3 Error analysis

Theorem 4.13. Suppose a $\mathcal{C}^4$ function $f : [a,b] \to \mathbb{R}$ is interpolated by a complete cubic spline or a cubic spline with specified second derivatives at its end points. Then

$$\forall j = 0, 1, 2, \quad \left|f^{(j)}(x) - s^{(j)}(x)\right| \le c_j h^{4-j} \max_{x\in[a,b]} \left|f^{(4)}(x)\right|, \tag{4.20}$$

where $c_0 = \frac{1}{16}$, $c_1 = c_2 = \frac{1}{2}$, and $h = \max_{i=1}^{N-1} |x_{i+1} - x_i|$.

Proof. Our plan is to first prove the case of $j = 2$, then utilize the conclusion to prove the conclusion for other cases.

Consider an auxiliary function $\hat{s} \in \mathcal{C}^2[a,b]$ that satisfies

$$\forall i = 1, 2, \ldots, N-1, \qquad \hat{s}|_{[x_i, x_{i+1}]} \in \mathbb{P}_3, \ \hat{s}''(x_i) = f''(x_i).$$

We can obtain such an $\hat{s}$ by interpolating $f''(x)$ with some $\tilde{s} \in \mathbb{S}_1^0$ and integrating $\tilde{s}$ twice. Then the theorem of Cauchy remainder (Theorem 3.7) implies

$$\begin{aligned}
&\exists \xi_i \in [x_i, x_{i+1}], \quad \text{s.t. } \forall x \in [x_i, x_{i+1}], \\
&\quad |f''(x) - \tilde{s}(x)| \le \frac{1}{2}\left|f^{(4)}(\xi_i)\right| |(x - x_i)(x - x_{i+1})|,
\end{aligned}$$

hence we have

$$|f''(x) - \hat{s}''(x)|_{x\in[x_i,x_{i+1}]} \le \frac{1}{8} \max_{x\in[x_i,x_{i+1}]} \left|f^{(4)}(x)\right| (x_{i+1} - x_i)^2$$

and thus

$$|f''(x) - \hat{s}''(x)| \le \frac{h^2}{8} \max_{x\in[a,b]} |f^{(4)}(x)|. \tag{4.21}$$

Now consider interpolating $f(x) - \hat{s}(x)$ with a cubic spline. Since $\hat{s}(x) \in \mathbb{S}_3^2$, the interpolant must be $s(x) - \hat{s}(x)$. Then Lemma 4.12 yields

$$\forall x \in [a,b], \qquad |s''(x) - \hat{s}''(x)| \le 3 \max_{x\in[a,b]} |f''(x) - \hat{s}''(x)|,$$

which, together with (4.21), leads to (4.20) for $j = 2$:

$$\begin{aligned}|f''(x) - s''(x)| &\le |f''(x) - \hat{s}''(x)| + |\hat{s}''(x) - s''(x)| \\ &\le 4 \max_{x\in[a,b]} |f''(x) - \hat{s}''(x)| \\ &\le \frac{1}{2}h^2 \max_{x\in[a,b]} \left|f^{(4)}(x)\right|. \end{aligned} \tag{4.22}$$

For $j = 0$, we have $f(x) - s(x) = 0$ for $x = x_i, x_{i+1}$. Then Rolle's theorem C.50 implies $f'(\xi_i) - s'(\xi_i) = 0$ for some $\xi_i \in [x_i, x_{i+1}]$. It follows from the second fundamental theorem of calculus (Theorem C.73) that

$$\forall x \in [x_i, x_{i+1}], \qquad f'(x) - s'(x) = \int_{\xi_i}^{x} (f''(t) - s''(t))\,\mathrm{d}t,$$

which, together with the integral mean value theorem C.71 and (4.22), yields

$$\begin{aligned}|f'(x) - s'(x)|_{x\in[x_i,x_{i+1}]} &= |x - \xi_i|\,|f''(\eta_i) - s''(\eta_i)| \\ &\le \frac{1}{2}h^3 \max_{x\in[a,b]} \left|f^{(4)}(x)\right|.\end{aligned}$$

This proves (4.20) for $j = 1$. Finally, consider interpolating $f(x)-s(x)$ with some linear spline $\bar{s} \in \mathbb{S}_1^0$. The interpolation conditions dictate $\forall x \in [a, b]$, $\bar{s}(x) \equiv 0$. Hence

$$\begin{aligned}&|f(x) - s(x)|_{x\in[x_i,x_{i+1}]} = |f(x) - s(x) - \bar{s}|_{x\in[x_i,x_{i+1}]} \\ \le&\frac{1}{8}(x_{i+1} - x_i)^2 \max_{x\in[x_i,x_{i+1}]} |f''(x) - s''(x)| \\ \le&\frac{1}{16}h^4 \max_{x\in[a,b]} |f^{(4)}(x)|,\end{aligned}$$

where the second step follows from Theorem 3.7 and the third step from (4.22). □

Exercise 4.14. Verify Theorem 4.13 using the results in Example 4.9.

4.4 B-Splines

Notation 2. In the notation $\mathbb{S}_n^{n-1}(t_1, t_2, \cdots, t_N)$, t_i's in the parentheses represent knots of a spline. When there is no danger of ambiguity, we also use the shorthand notation $\mathbb{S}_{n,N}^{n-1} := \mathbb{S}_n^{n-1}(t_1, t_2, \cdots, t_N)$ or simply $\mathbb{S}_n^{n-1}$.

Theorem 4.15. The set of splines $\mathbb{S}_n^{n-1}(t_1, t_2, \cdots, t_N)$ is a linear space with dimension $n + N - 1$.

Proof. It is easy to verify from (4.2) and Definition B.2 that $\mathbb{S}_n^{n-1}(t_1, t_2, \cdots, t_N)$ is indeed a linear space. Note that the additive identity is the zero function not the number zero. One polynomial of degree n is determined by $n + 1$ coefficients. The $N - 1$ intervals lead to $(N - 1)(n + 1)$ coefficients. At each of the $N - 2$ interval knots, the smoothness condition requires that the 0th, 1st, ..., $(n - 1)$th derivatives of adjacent polynomials match. Hence the dimension is $(N - 1)(n + 1) - n(N - 2) = n + N - 1$. □

Example 4.16. The cubic splines in Definition 4.6, have $n = 3$ and hence the dimension of $\mathbb{S}_3^2$ is $N + 2$. Apart from the N interpolation conditions at the knots, we need to impose two other conditions at the ends of the interpolating interval to obtain a unique spline, this leads to different types of cubic splines in Definition 4.6.

4.4.1 Truncated power functions

Definition 4.17. The *truncated power function* with exponent n is defined as

$$x_+^n = \begin{cases} x^n & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{4.23}$$

Example 4.18. According to Definition 4.17, we have

$$\forall t \in [a, b], \quad \int_a^b (t - x)_+^n \mathrm{d}x = \int_a^t (t - x)^n \mathrm{d}x = \frac{(t - a)^{n+1}}{n + 1}. \tag{4.24}$$

Lemma 4.19. The following is a basis of $\mathbb{S}_n^{n-1}(t_1, \ldots, t_N)$,

$$1, x, x^2, \ldots, x^n, (x-t_2)_+^n, (x-t_3)_+^n, \ldots, (x-t_{N-1})_+^n. \tag{4.25}$$

Proof. $\forall i = 2, 3, \ldots, N - 1$, $(x - t_i)_+^n \in \mathbb{S}_{n,N}^{n-1}$. Also, $\forall i = 0, 1, \ldots, n$, $x^i \in \mathbb{S}_{n,N}^{n-1}$. Suppose

$$\sum_{i=0}^{n} a_i x^i + \sum_{j=2}^{N-1} a_{n+j}(x - t_j)_+^n = \mathbf{0}(x). \tag{4.26}$$

To satisfy (4.26) for all $x < t_2$, a_i must be 0 for each $i = 0, 1, \cdots, n$. To satisfy (4.26) for all $x \in (t_2, t_3)$, a_{n+2} must be 0. Similarly, all a_{n+j}'s must be zero. Hence, the functions in (4.25) are linearly independent by Definition B.25. The proof is completed by Theorem 4.15, Lemma B.41, and the fact that there are $n + N - 1$ functions in (4.25). □

Corollary 4.20. Any $s \in \mathbb{S}_{n,N}^{n-1}$ can be expressed as

$$s(x) = \sum_{i=0}^{n} a_i(x - t_1)^i + \sum_{j=2}^{N-1} a_{n+j}(x - t_j)_+^n, \qquad x \in [t_1, t_N]. \tag{4.27}$$

Proof. By Lemma 4.19, it suffices to point out that $\mathrm{span}\{1, x, \ldots, x^n\} = \mathrm{span}\{1, (x - t_1), \ldots, (x - t_1)^n\}$. □

Example 4.21. (4.27) with $n = 1$ is the linear spline interpolation. Imagine a plastic rod that is initially straight. Place one of its end at (t_1, f_1) and let it go through (t_2, f_2). In general (t_3, f_3) will be off the rod, but we can bend the rod at (t_2, f_2) to make the rod go through (t_3, f_3). This "bending" process corresponds to adding the first truncated power function in (4.27).

4.4.2 The local support of B-splines

Definition 4.22. The *hat function* at t_i is

$$\hat{B}_i(x) = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in (t_{i-1}, t_i], \\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

Theorem 4.23. The hat functions form a basis of $\mathbb{S}_1^0$.

Proof. By Definition 4.22, we have

$$\hat{B}_i(t_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.29}$$

Suppose $\sum_{i=1}^N c_i \hat{B}_i(x) = \mathbf{0}(x)$. Then we have $c_i = 0$ for each $i = 1, 2, \cdots, N$ by setting $x = t_j$ and applying (4.29). Hence by Definition B.25 the hat functions are linearly independent. It suffices to show that $\text{span}\{\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_N\} = \mathbb{S}_1^0$, which is true because

$$\forall s(x) \in \mathbb{S}_1^0, \quad \exists s_B(x) = \sum_{i=1}^N s(t_i)\hat{B}_i(x) \text{ s.t. } s(x) = s_B(x).$$

On each interval $[t_i, t_{i+1}]$, (4.29) implies $s_B(t_i) = s(t_i)$ and $s_B(t_{i+1}) = s(t_{i+1})$. Hence $s_B(x) \equiv s(x)$ because they are both linear. Then Definition B.32 completes the proof. $\square$

Definition 4.24. *B-spline*s are defined recursively by

$$B_i^{n+1}(x) = \frac{x - t_{i-1}}{t_{i+n} - t_{i-1}} B_i^n(x) + \frac{t_{i+n+1} - x}{t_{i+n+1} - t_i} B_{i+1}^n(x). \tag{4.30}$$

The recursion base is the B-spline of degree zero,

$$B_i^0(x) = \begin{cases} 1 & \text{if } x \in (t_{i-1}, t_i], \\ 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

Example 4.25. The hat functions in Definition 4.22 are clearly the B-splines of degree one:

$$B_i^1 = \hat{B}_i. \tag{4.32}$$

In (4.30), B-splines of higher degrees are defined by generalizing the idea of hat functions.

Example 4.26. The quadratic B-splines $B_i^2(x) =$

$$\begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i]; \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}]; \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}]; \\ 0, & \text{otherwise.} \end{cases} \tag{4.33}$$

Definition 4.27. The *support* of a function $f : X \to \mathbb{R}$ is

$$\text{supp}(f) = \text{closure}\{x \in X \mid f(x) \neq 0\}. \tag{4.34}$$

Lemma 4.28. For $n \in \mathbb{N}^+$, the interval of support of B_i^n is $[t_{i-1}, t_{i+n}]$. Also, $\forall x \in (t_{i-1}, t_{i+n})$, $B_i^n(x) > 0$.

Proof. This is an easy induction by (4.31) and (4.30). $\square$

Definition 4.29. Let X be a vector space. For each $x \in X$ we associate a unique real (or complex) number $L(x)$. If $\forall x, y \in X$ and $\forall \alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$), we have

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y), \tag{4.35}$$

then L is called a *linear functional* over X.

Example 4.30. $X = \mathcal{C}[a, b]$, then the elements of X are functions continuous over $[a, b]$.

$$L(f) = \int_a^b f(x)\mathrm{d}x, \qquad L(f) = \int_a^b x^2 f(x)\mathrm{d}x$$

are both linear functionals over X.

Notation 3. We have used the notation $f[x_0, \ldots, x_k]$ for the kth divided difference of f, inline with considering $f[x_0, \ldots, x_k]$ as a generalization of the Taylor expansion. Hereafter, for analyzing B-splines, it is both semantically and syntactically better to use the notation $[x_0, \ldots, x_k]f$, inline with considering the *procedures* of a divided difference as a linear functional over $\mathcal{C}[x_0, x_k]$.

Theorem 4.31 (Leibniz formula)**.** For $k \in \mathbb{N}$, the kth divided difference of a product of two functions satisfies

$$[x_0, \ldots, x_k]fg = \sum_{i=0}^k [x_0, \ldots, x_i]f \cdot [x_i, \ldots, x_k]g. \tag{4.36}$$

Proof. The induction basis $k = 0$ holds because (4.36) reduces to $[x_0]fg = f(x_0)g(x_0)$. Now suppose (4.36) holds. For the induction step, we have from Theorem 3.17 that

$$[x_0, \ldots, x_{k+1}]fg = \frac{[x_1, \ldots, x_{k+1}]fg - [x_0, \ldots, x_k]fg}{x_{k+1} - x_0}.$$

By the induction hypothesis, we have

$$[x_1, \ldots, x_{k+1}]fg = \sum_{i=0}^k [x_1, \ldots, x_{i+1}]f \cdot [x_{i+1}, \ldots, x_{k+1}]g$$

$$= S_1 + \sum_{i=0}^k [x_0, \ldots, x_i]f \cdot [x_{i+1}, \ldots, x_{k+1}]g, \text{ where}$$

$$S_1 = \sum_{i=0}^k (x_{i+1} - x_0) \cdot [x_0, \ldots, x_{i+1}]f \cdot [x_{i+1}, \ldots, x_{k+1}]g$$
$$= \sum_{i=1}^{k+1} (x_i - x_0) \cdot [x_0, \ldots, x_i]f \cdot [x_i, \ldots, x_{k+1}]g.$$

$$[x_0, \ldots, x_k]fg = \sum_{i=0}^k [x_0, \ldots, x_i]f \cdot [x_i, \ldots, x_k]g$$

$$= -S_2 + \sum_{i=0}^k [x_0, \ldots, x_i]f \cdot [x_{i+1}, \ldots, x_{k+1}]g, \quad \text{where}$$

$$S_2 = \sum_{i=0}^k [x_0, \ldots, x_i]f \cdot (x_{k+1} - x_i) \cdot [x_i, \ldots, x_{k+1}]g.$$

In the above derivation, we have applied Theorem 3.17 to go from the kth divided difference to the $(k+1)$th. Then

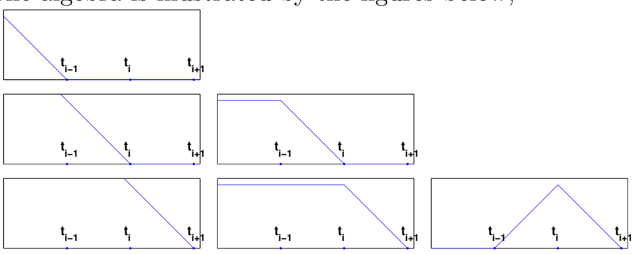
$$\begin{aligned}[x_0,\ldots,x_{k+1}]fg &= \frac{S_1+S_2}{x_{k+1}-x_0}\\ &= \sum_{i=0}^{k+1}[x_0,\ldots,x_i]f\cdot[x_i,\ldots,x_{k+1}]g,\end{aligned}$$

which completes the inductive proof. □

Example 4.32. There exists a relation between B-splines and truncated power functions, e.g.,

$$\begin{aligned}&(t_{i+1}-t_{i-1})[t_{i-1},t_i,t_{i+1}](t-x)_+\\ =&[t_i,t_{i+1}](t-x)_+-[t_{i-1},t_i](t-x)_+\\ =&\frac{(t_{i+1}-x)_+-(t_i-x)_+}{t_{i+1}-t_i}-\frac{(t_i-x)_+-(t_{i-1}-x)_+}{t_i-t_{i-1}}\\ =&B_i^1=\begin{cases}\frac{x-t_{i-1}}{t_i-t_{i-1}} & x\in(t_{i-1},t_i],\\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x\in(t_i,t_{i+1}],\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The algebra is illustrated by the figures below,



The significance is that, by applying divided difference to truncated power functions we can "cure" their drawback of non-local support. This idea is made precise in the next Theorem.

Theorem 4.33 (B-splines as divided difference of truncated power functions)**.** For any $n\in\mathbb{N}$, we have

$$B_i^n(x)=(t_{i+n}-t_{i-1})\cdot[t_{i-1},\ldots,t_{i+n}](t-x)_+^n. \tag{4.37}$$

Proof. For $n=0$, (4.37) reduces to

$$\begin{aligned}B_i^0(x)&=(t_i-t_{i-1})\cdot[t_{i-1},t_i](t-x)_+^0\\ &=(t_i-x)_+^0-(t_{i-1}-x)_+^0\\ &=\begin{cases}0 & \text{if } x\in(-\infty,t_{i-1}],\\ 1 & \text{if } x\in(t_{i-1},t_i],\\ 0 & \text{if } x\in(t_i,+\infty),\end{cases}\end{aligned}$$

which is the same as (4.31). Hence the induction basis holds. Now assume the induction hypothesis (4.37) hold.

By Definition 4.17, $(t-x)_+^{n+1}=(t-x)(t-x)_+^n$. Then the application of Theorem 4.31 with $f=(t-x)$ and $g=(t-x)_+^n$ yields

$$\begin{aligned}&[t_{i-1},\ldots,t_{i+n}](t-x)_+^{n+1}\\ =&(t_{i-1}-x)\cdot[t_{i-1},\ldots,t_{i+n}](t-x)_+^n\\ &+[t_i,\ldots,t_{i+n}](t-x)_+^n.\end{aligned} \tag{4.38}$$

Definition 4.24 and the induction hypothesis yield

$$\begin{aligned}&B_i^{n+1}(x)=\beta(x)+\gamma(x),\ \text{with}\\ &\beta(x)=\frac{x-t_{i-1}}{t_{i+n}-t_{i-1}}B_i^n(x)\\ &\quad=(x-t_{i-1})\cdot[t_{i-1},\ldots,t_{i+n}](t-x)_+^n\\ &\quad=[t_i,\ldots,t_{i+n}](t-x)_+^n-[t_{i-1},\ldots,t_{i+n}](t-x)_+^{n+1},\end{aligned}$$

where the last step follows from (4.38). Similarly,

$$\begin{aligned}\gamma(x)=&\frac{t_{i+n+1}-x}{t_{i+n+1}-t_i}B_{i+1}^n(x)\\ =&(t_{i+n+1}-x)\cdot[t_i,\ldots,t_{i+n+1}](t-x)_+^n\\ =&(t_{i+n+1}-t_i)\cdot[t_i,\ldots,t_{i+n+1}](t-x)_+^n\\ &+(t_i-x)\cdot[t_i,\ldots,t_{i+n+1}](t-x)_+^n\\ =&[t_{i+1},\ldots,t_{i+n+1}](t-x)_+^n-[t_i,\ldots,t_{i+n}](t-x)_+^n\\ &+[t_i,\ldots,t_{i+n+1}](t-x)_+^{n+1}\\ &-[t_{i+1},\ldots,t_{i+n+1}](t-x)_+^n\\ =&[t_i,\ldots,t_{i+n+1}](t-x)_+^{n+1}-[t_i,\ldots,t_{i+n}](t-x)_+^n,\end{aligned}$$

where the second last step follows from Theorem 3.17 and (4.38). The above arguments yield

$$\begin{aligned}B_i^{n+1}(x)=&[t_i,\ldots,t_{i+n+1}](t-x)_+^{n+1}\\ &-[t_{i-1},\ldots,t_{i+n}](t-x)_+^{n+1}\\ =&(t_{i+n+1}-t_{i-1})\cdot[t_{i-1},\ldots,t_{i+n+1}](t-x)_+^{n+1},\end{aligned}$$

which completes the inductive proof. □

4.4.3 Integrals and derivatives

Corollary 4.34 (Integrals of B-splines)**.** The average of a B-spline over its support only depends on its degree,

$$\frac{1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}}B_i^n(x)\mathrm{d}x=\frac{1}{n+1}. \tag{4.39}$$

Proof. The left-hand side (LHS) of (4.39) is

$$\begin{aligned}&\frac{1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}}B_i^n(x)\mathrm{d}x\\ =&\int_{t_{i-1}}^{t_{i+n}}[t_{i-1},\ldots,t_{i+n}](t-x)_+^n\mathrm{d}x\\ =&[t_{i-1},\ldots,t_{i+n}]\int_{t_{i-1}}^{t_{i+n}}(t-x)_+^n\mathrm{d}x\\ =&[t_{i-1},\ldots,t_{i+n}]\frac{(t-t_{i-1})^{n+1}}{n+1}\\ =&\frac{1}{n+1},\end{aligned}$$

where the first step follows from Theorem 4.33, the second step from the commutativity of integration and taking divided difference, the third step from (4.24), and the last step from Corollary 3.22. □

Theorem 4.35 (Derivatives of B-splines)**.** For $n \geq 2$, we have, $\forall x \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^n(x) = \frac{nB_i^{n-1}(x)}{t_{i+n-1} - t_{i-1}} - \frac{nB_{i+1}^{n-1}(x)}{t_{i+n} - t_i}. \tag{4.40}$$

For $n = 1$, (4.40) holds for all x except at the three knots t_{i-1}, t_i, and t_{i+1}, where the derivative of B_i^1 is not defined.

Proof. We first show that (4.40) holds for all x except at the knots t_j. By (4.32), (4.28), and (4.31), we have

$$\begin{aligned}&\forall x \in \mathbb{R} \setminus \{t_{i-1}, t_i, t_{i+1}\},\\ &\frac{\mathrm{d}}{\mathrm{d}x} B_i^1(x) = \frac{1}{t_i - t_{i-1}} B_i^0(x) - \frac{1}{t_{i+1} - t_i} B_{i+1}^0(x).\end{aligned}$$

Hence the induction basis holds. Now suppose (4.40) holds $\forall x \in \mathbb{R} \setminus \{t_{i-1}, \ldots, t_{i+n}\}$. Differentiate (4.30), apply the induction hypothesis (4.40), and we have

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^{n+1}(x) = \frac{B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i} + nC(x), \tag{4.41}$$

where $C(x)$ is

$$\begin{aligned}
&\frac{x - t_{i-1}}{t_{i+n} - t_{i-1}} \left[\frac{B_i^{n-1}(x)}{t_{i+n-1} - t_{i-1}} - \frac{B_{i+1}^{n-1}(x)}{t_{i+n} - t_i} \right] \\
&+ \frac{t_{i+n+1} - x}{t_{i+n+1} - t_i} \left[\frac{B_{i+1}^{n-1}(x)}{t_{i+n} - t_i} - \frac{B_{i+2}^{n-1}(x)}{t_{i+n+1} - t_{i+1}} \right] \\
&= \frac{1}{t_{i+n} - t_{i-1}} \left[\frac{(x - t_{i-1})B_i^{n-1}(x)}{t_{i+n-1} - t_{i-1}} + \frac{(t_{i+n} - x)B_{i+1}^{n-1}(x)}{t_{i+n} - t_i} \right] \\
&- \frac{1}{t_{i+n+1} - t_i} \left[\frac{(x - t_i)B_{i+1}^{n-1}(x)}{t_{i+n} - t_i} + \frac{(t_{i+n+1} - x)B_{i+2}^{n-1}(x)}{t_{i+n+1} - t_{i+1}} \right] \\
&= \frac{B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i},
\end{aligned}$$

where the last step follows from (4.30). Then (4.41) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^{n+1}(x) = \frac{(n+1)B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1} - t_i},$$

which completes the inductive proof of (4.40) except at the knots. Since $B_i^1 = \hat{B}_i$ is continuous, an easy induction with (4.30) shows that B_i^n is continuous for all $n \geq 1$. Hence the right-hand side of (4.40) is continuous for all $n \geq 2$. Therefore, if $n \geq 2$, $\frac{\mathrm{d}}{\mathrm{d}x} B_i^n(x)$ exists for all $x \in \mathbb{R}$. This completes the proof. □

Corollary 4.36 (Smoothness of B-splines)**.** $B_i^n \in \mathbb{S}_n^{n-1}$.

Proof. For $n = 1$, the induction basis $B_i^1(x) \in \mathbb{S}_1^0$ holds because of (4.32). The rest of the proof follows from (4.30) and Theorem 4.35 via an easy induction. □

4.4.4 Marsden's identity

Theorem 4.37 (Marsden's identity)**.** For any $n \in \mathbb{N}$,

$$(t - x)^n = \sum_{i=-\infty}^{+\infty} (t - t_i) \cdots (t - t_{i+n-1}) B_i^n(x), \tag{4.42}$$

where the product $(t - t_i) \cdots (t - t_{i+n-1})$ is defined as 1 for $n = 0$.

Proof. For $n = 0$, (4.42) follows from Definition 4.24. Now suppose (4.42) holds. A linear interpolation of the linear function $f(t) = t - x$ is the function itself,

$$t - x = \frac{t - t_{i+n}}{t_{i-1} - t_{i+n}} (t_{i-1} - x) + \frac{t - t_{i-1}}{t_{i+n} - t_{i-1}} (t_{i+n} - x). \tag{4.43}$$

Hence for the inductive step we have

$$\begin{aligned}
&(t - x)^{n+1} = (t - x) \sum_{i=-\infty}^{+\infty} (t - t_i) \cdots (t - t_{i+n-1}) B_i^n(x) \\
&= \sum_{i=-\infty}^{+\infty} (t - t_i) \cdots (t - t_{i+n}) \frac{t_{i-1} - x}{t_{i-1} - t_{i+n}} B_i^n(x) \\
&\quad + \sum_{i=-\infty}^{+\infty} (t - t_{i-1}) \cdots (t - t_{i+n-1}) \frac{t_{i+n} - x}{t_{i+n} - t_{i-1}} B_i^n(x) \\
&= \sum_{i=-\infty}^{+\infty} (t - t_i) \cdots (t - t_{i+n}) \frac{x - t_{i-1}}{t_{i+n} - t_{i-1}} B_i^n(x) \\
&\quad + \sum_{i=-\infty}^{+\infty} (t - t_i) \cdots (t - t_{i+n}) \frac{t_{i+n+1} - x}{t_{i+n+1} - t_i} B_{i+1}^n(x) \\
&= \sum_{i=-\infty}^{+\infty} (t - t_i) \cdots (t - t_{i+n}) B_i^{n+1}(x),
\end{aligned}$$

where the first step follows from the induction hypothesis, the second step from (4.43), the third step from replacing i with $i + 1$ in the second summation, and the last step from (4.30). □

Corollary 4.38 (Truncated power functions as summation of B-splines)**.** For any $j \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$(t_j - x)_+^n = \sum_{i=-\infty}^{j-n} (t_j - t_i) \cdots (t_j - t_{i+n-1}) B_i^n(x). \tag{4.44}$$

Proof. We need to show that the RHS is $(t_j - x)^n$ if $x \leq t_j$ and 0 otherwise. Set $t = t_j$ in (4.42) and we have

$$(t_j - x)^n = \sum_{i=-\infty}^{+\infty} (t_j - t_i) \cdots (t_j - t_{i+n-1}) B_i^n(x).$$

For each $i = j - n + 1, \ldots, j$, the corresponding term in the summation is zero regardless of x; for each $i \geq j + 1$, Lemma 4.28 implies that $B_i^n(x) = 0$ for all $x \leq t_j$. Hence

$$x \leq t_j \;\Rightarrow\; \sum_{i=-\infty}^{j-n} (t_j - t_i) \cdots (t_j - t_{i+n-1}) B_i^n(x) = (t_j - x)^n.$$

Otherwise $x > t_j$, then Lemma 4.28 implies $B_i^n(x) = 0$ for each $i \leq j - n$. This completes the proof. □

4.4.5 Symmetric polynomials

Definition 4.39. The *elementary symmetric polynomial* of degree k in n variables is the sum of all products of k distinct variables chosen from the n variables,

$$\sigma_k(x_1,\ldots,x_n) = \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}. \tag{4.45}$$

In particular, $\sigma_0(x_1,\ldots,x_n)=1$ and

$$\forall k>n, \qquad \sigma_k(x_1,\ldots,x_n)=0.$$

If the distinctiveness condition is dropped, we have the *complete symmetric polynomial* of degree k in n variables,

$$\tau_k(x_1,\ldots,x_n) = \sum_{1\le i_1\le\cdots\le i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}. \tag{4.46}$$

Example 4.40. $\sigma_2(x_1,x_2,x_3) = x_1x_2+x_1x_3+x_2x_3$. In comparison, $\tau_2(x_1,x_2,x_3) = \sigma_2(x_1,x_2,x_3)+x_1^2+x_2^2+x_3^2$.

Lemma 4.41. For $k\le n$, the elementary symmetric polynomials satisfy a recursion,

$$\begin{aligned}&\sigma_{k+1}(x_1,\ldots,x_n,x_{n+1})\\ =&\sigma_{k+1}(x_1,\ldots,x_n)+x_{n+1}\sigma_k(x_1,\ldots,x_n).\end{aligned} \tag{4.47}$$

Proof. The terms in $\sigma_{k+1}(x_1,\ldots,x_n,x_{n+1})$ can be assorted into two groups: (a) those that contain the factor x_{n+1} and (b) those that do not. By the symmetry in (4.45), group (a) must be $x_{n+1}\sigma_k(x_1,\ldots,x_n)$ and group (b) must be $\sigma_{k+1}(x_1,\ldots,x_n)$. □

Example 4.42. $\sigma_2(x_1,x_2,x_3) = x_1x_2+x_3(x_1+x_2)$.

Definition 4.43. The *generating function for the elementary symmetric polynomials* is

$$g_{\sigma,n}(z) = \prod_{i=1}^{n}(1+x_iz) = (1+x_1z)\cdots(1+x_nz) \tag{4.48}$$

while that for the complete symmetric polynomials is

$$g_{\tau,n}(z) = \prod_{i=1}^{n}\frac{1}{1-x_iz} = \frac{1}{1-x_1z}\cdots\frac{1}{1-x_nz}. \tag{4.49}$$

Lemma 4.44 (Generating elementary and complete symmetric polynomials)**.** The elementary and complete symmetric polynomials are related to their generating functions as

$$g_{\sigma,n}(z) = \sum_{k=0}^{n}\sigma_k(x_1,\ldots,x_n)z^k. \tag{4.50}$$

$$g_{\tau,n}(z) = \sum_{k=0}^{+\infty}\tau_k(x_1,\ldots,x_n)z^k. \tag{4.51}$$

Proof. With Lemma 4.41, we can prove (4.50) by an easy induction. For (4.51), (4.49) and the identity

$$\frac{1}{1-x} = \sum_{k=0}^{+\infty}x^k \tag{4.52}$$

yield

$$\begin{aligned}g_{\tau,n}(z) =& \prod_{i=1}^{n}\sum_{k=0}^{+\infty}x_i^kz^k\\ =&(1+x_1z+x_1^2z^2+\cdots)(1+x_2z+x_2^2z^2+\cdots)\\ &\cdots(1+x_nz+x_n^2z^2+\cdots).\end{aligned}$$

The coefficient of the monomial z^k, is the sum of all possible products of k variables from $x_1,x_2,\ldots,x_n$. Definition 4.39 then completes the proof. □

Example 4.45.

$$\begin{aligned}&(1+x_1z)(1+x_2z)(1+x_3z)\\ =&1+(x_1+x_2+x_3)z\\ &+(x_1x_2+x_1x_3+x_2x_3)z^2+x_1x_2x_3z^3.\end{aligned}$$

Lemma 4.46 (Recursive relations of complete symmetric polynomials)**.** The complete symmetric polynomials satisfy a recursion,

$$\begin{aligned}&\tau_{k+1}(x_1,\ldots,x_n,x_{n+1})\\ =&\tau_{k+1}(x_1,\ldots,x_n)+x_{n+1}\tau_k(x_1,\ldots,x_n,x_{n+1}).\end{aligned} \tag{4.53}$$

Proof. (4.49) implies

$$g_{\tau,n+1} = g_{\tau,n}+x_{n+1}zg_{\tau,n+1}. \tag{4.54}$$

The proof is completed by requiring that the coefficient of z^{k+1} on the LHS equal that of z^{k+1} on the RHS. □

Theorem 4.47 (Complete symmetric polynomials as divided difference of monomials)**.** The complete symmetric polynomial of degree $m-n$ in $n+1$ variables is the nth divided difference of the monomial x^m, i.e.

$$\begin{aligned}&\forall m\in\mathbb{N}^+,\ i\in\mathbb{N},\ \forall n=0,1,\ldots,m,\\ &\tau_{m-n}(x_i,\ldots,x_{i+n}) = [x_i,\ldots,x_{i+n}]x^m.\end{aligned} \tag{4.55}$$

Proof. By Lemma 4.46, we have

$$\begin{aligned}&(x_{n+1}-x_1)\tau_k(x_1,\ldots,x_n,x_{n+1})\\ =&\tau_{k+1}(x_1,\ldots,x_n,x_{n+1})-\tau_{k+1}(x_1,\ldots,x_n)\\ &-x_1\tau_k(x_1,\ldots,x_n,x_{n+1})\\ =&\tau_{k+1}(x_2,\ldots,x_n,x_{n+1})+x_1\tau_k(x_1,\ldots,x_n,x_{n+1})\\ &-\tau_{k+1}(x_1,\ldots,x_n)-x_1\tau_k(x_1,\ldots,x_n,x_{n+1})\\ =&\tau_{k+1}(x_2,\ldots,x_n,x_{n+1})-\tau_{k+1}(x_1,\ldots,x_n).\end{aligned} \tag{4.56}$$

The rest of the proof is an induction on n. For $n=0$, (4.55) reduces to

$$\tau_m(x_i) = [x_i]x^m,$$

which is trivially true. Now suppose (4.55) holds for a nonnegative integer $n<m$. Then (4.56) and the induction hypothesis yield

$$\begin{aligned}&\tau_{m-n-1}(x_i,\ldots,x_{i+n+1})\\ =&\frac{\tau_{m-n}(x_{i+1},\ldots,x_{i+n+1})-\tau_{m-n}(x_i,\ldots,x_{i+n})}{x_{i+n+1}-x_i}\\ =&\frac{[x_{i+1},\ldots,x_{i+n+1}]x^m-[x_i,\ldots,x_{i+n}]x^m}{x_{i+n+1}-x_i}\\ =&[x_i,\ldots,x_{i+n+1}]x^m,\end{aligned}$$

which completes the proof. □

4.4.6 B-splines indeed form a basis

Theorem 4.48. Given any $k \in \mathbb{N}$, the monomial x^k can be expressed as a linear combination of B-splines for any fixed $n \geq k$, in the form

$$\binom{n}{k} x^k = \sum_{i=-\infty}^{+\infty} \sigma_k(t_i, \ldots, t_{i+n-1}) B_i^n(x), \tag{4.57}$$

where $\sigma_k(t_i, \ldots, t_{i+n-1})$ is the elementary symmetric polynomial of degree k in the n variables $t_i, \ldots, t_{i+n-1}$.

Proof. Lemma 4.44 yields

$$(1 + t_i x) \cdots (1 + t_{i+n-1} x) = \sum_{k=0}^{n} \sigma_k(t_i, \ldots, t_{i+n-1}) x^k.$$

Replace x with $-1/t$, multiply both sides with t^n, and we have

$$(t - t_i) \cdots (t - t_{i+n-1}) = \sum_{k=0}^{n} \sigma_k(t_i, \ldots, t_{i+n-1})(-1)^k t^{n-k}.$$

Substituting the above into (4.42) yields

$$\begin{aligned}(t - x)^n &= \sum_{i=-\infty}^{+\infty} \sum_{k=0}^{n} \sigma_k(t_i, \ldots, t_{i+n-1})(-1)^k t^{n-k} B_i^n(x) \\ &= \sum_{k=0}^{n} \left\{ t^{n-k}(-1)^k \sum_{i=-\infty}^{+\infty} \sigma_k(t_i, \ldots, t_{i+n-1}) B_i^n(x) \right\}.\end{aligned}$$

On the other hand, the binomial theorem states that

$$(t - x)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k}(-x)^k = \sum_{k=0}^{n} t^{n-k}(-1)^k \binom{n}{k} x^k.$$

Comparing the last two equations completes the proof. $\square$

Corollary 4.49 (Partition of Unity)**.**

$$\forall n \in \mathbb{N}, \qquad \sum_{i=-\infty}^{+\infty} B_i^n = 1. \tag{4.58}$$

Proof. Setting $k = 0$ in Theorem 4.48 yields (4.58). $\square$

Theorem 4.50. The following list of B-splines is a basis of $\mathbb{S}_n^{n-1}(t_1, t_2, \ldots, t_N)$,

$$B_{2-n}^n(x), B_{3-n}^n(x), \ldots, B_N^n(x). \tag{4.59}$$

Proof. It is easy to verify that

$$\forall t_i \in \mathbb{R}, \ \ (x - t_i)_+^n = (x - t_i)^n - (-1)^n (t_i - x)_+^n. \tag{4.60}$$

Then it follows from Theorem 4.37 and Corollary 4.38 that each truncated power function $(x - t_i)_+^n$ can be expressed as a linear combination of B-splines. By Lemma 4.19, each element in $\mathbb{S}_n^{n-1}(t_1, t_2, \ldots, t_N)$ can be expressed as a linear combination of

$$1, x, x^2, \ldots, x^n, (x - t_2)_+^n, (x - t_3)_+^n, \ldots, (x - t_{N-1})_+^n.$$

Theorem 4.48 states that each monomial x^j can also be expressed as a linear combination of B-splines. Since the domain is restricted to $[t_1, t_N]$, we know from Lemma 4.28 that only those B-splines in the list of (4.59) appear in the linear combination. Therefore, these B-splines form a spanning list of $\mathbb{S}_n^{n-1}(t_1, t_2, \ldots, t_N)$. The proof is completed by Lemma B.40, Theorem 4.15, and the fact that the length of the list (4.59) is also $n + N - 1$. $\square$

4.4.7 Cardinal B-splines

Definition 4.51. The *cardinal B-spline* of degree n, denoted by $B_{i,\mathbb{Z}}^n$, is the B-spline in Definition 4.24 on the knot set $\mathbb{Z}$.

Corollary 4.52. Cardinal B-splines of the same degree are translates of one another, i.e.

$$\forall x \in \mathbb{R}, \qquad B_{i,\mathbb{Z}}^n(x) = B_{i+1,\mathbb{Z}}^n(x + 1). \tag{4.61}$$

Proof. The recurrence relation (4.30) reduces to

$$B_{i,\mathbb{Z}}^{n+1}(x) = \frac{x - i + 1}{n + 1} B_{i,\mathbb{Z}}^n(x) + \frac{i + n + 1 - x}{n + 1} B_{i+1,\mathbb{Z}}^n(x). \tag{4.62}$$

The rest of the proof is an easy induction on n. $\square$

Corollary 4.53. A cardinal B-spline is symmetric about the center of its interval of support, i.e.

$$\forall n > 0, \ \forall x \in \mathbb{R}, \qquad B_{i,\mathbb{Z}}^n(x) = B_{i,\mathbb{Z}}^n(2i + n - 1 - x). \tag{4.63}$$

Proof. The proof is similar with that of Corollary 4.52. $\square$

Example 4.54. For $t_i = i$, the quadratic B-spline in Example 4.26 simplifies to

$$B_{i,\mathbb{Z}}^2(x) = \begin{cases} \frac{(x-i+1)^2}{2}, & x \in (i-1, i]; \\ \frac{3}{4} - \left(x - (i + \frac{1}{2})\right)^2, & x \in (i, i+1]; \\ \frac{(i+2-x)^2}{2}, & x \in (i+1, i+2]; \\ 0, & \text{otherwise.} \end{cases} \tag{4.64}$$

It is straightforward to verify Corollaries 4.52 and 4.53. It also follows from (4.64) that

$$B_{i,\mathbb{Z}}^2(j) = \begin{cases} \frac{1}{2} & j \in \{i, i+1\}; \\ 0, & j \in \mathbb{Z} \setminus \{i, i+1\}. \end{cases} \tag{4.65}$$

Example 4.55. For $t_i = i$, the cubic cardinal B-spline is

$$B_{i,\mathbb{Z}}^3(x) = \begin{cases} \frac{(x-i+1)^3}{6}, & x \in (i-1, i]; \\ \frac{2}{3} - \frac{1}{2}(x - i + 1)(i + 1 - x)^2, & x \in (i, i+1]; \\ B_{i,\mathbb{Z}}^3(2i + 2 - x), & x \in (i+1, i+3); \\ 0, & \text{otherwise.} \end{cases} \tag{4.66}$$

It follows that

$$B_{i,\mathbb{Z}}^3(j) = \begin{cases} \frac{1}{6}, & j \in \{i, i+2\}; \\ \frac{2}{3}, & j = i + 1; \\ 0, & j \in \mathbb{Z} \setminus \{i, i+1, i+2\}. \end{cases} \tag{4.67}$$

This illustrates Corollary 4.52 that cardinal B-splines have the same shape, i.e., they are invariant under integer translations.

Theorem 4.56. The cardinal B-spline of degree n can be explicitly expressed as

$$B^n_{i,\mathbb{Z}}(x) = \frac{1}{n!}\sum_{k=-1}^{n}(-1)^{n-k}\binom{n+1}{k+1}(k+i-x)^n_+. \qquad (4.68)$$

Proof. Theorems 4.33, 3.27, and 3.26 yield

$$\begin{aligned} B^n_{i,\mathbb{Z}}(x) &= (n+1)[i-1,\ldots,i+n](t-x)^n_+ \\ &= \frac{n+1}{(n+1)!}\Delta^{n+1}(i-1-x)^n_+ \\ &= \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}(i-1+k-x)^n_+. \end{aligned}$$

Replacing k with $k+1$ and accordingly changing the summation bounds complete the proof. □

Corollary 4.57. The value of a cardinal B-spline at an integer j is

$$B^n_{i,\mathbb{Z}}(j) = \frac{1}{n!}\sum_{k=j-i+1}^{n}(-1)^{n-k}\binom{n+1}{k+1}(k+i-j)^n \qquad (4.69)$$

for $j \in [i, n+i)$ and is zero otherwise.

Proof. This follows directly from Theorem 4.56 and Definition 4.17. □

Corollary 4.58 (Unique interpolation by complete cubic cardinal B-splines)**.** There is a unique B-spline $S(x) \in \mathbb{S}^2_3$ that interpolates $f(x)$ at $1, 2, \ldots, N$ with $S'(1) = f'(1)$ and $S'(N) = f'(N)$. Furthermore, this B-spline is

$$S(x) = \sum_{i=-1}^{N} a_i B^3_{i,\mathbb{Z}}(x), \qquad (4.70)$$

where

$$a_{-1} = a_1 - 2f'(1), \qquad a_N = a_{N-2} + 2f'(N), \qquad (4.71)$$

and $\mathbf{a}^T = [a_0, \ldots, a_{N-1}]$ is the solution of the linear system $M\mathbf{a} = \mathbf{b}$ with

$$\begin{aligned} \mathbf{b}^T =& [6f(1) + 2f'(1), 6f(2), \\ & \ldots, 6f(N-1), 6f(N) - 2f'(N)], \end{aligned}$$

$$M = \begin{bmatrix} 4 & 2 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 2 & 4 \end{bmatrix}.$$

Proof. By Theorem 4.50 and Lemma 4.28, we have

$$\begin{aligned} &\forall i = 1, 2, \ldots, N, \\ &f(i) = a_{i-2}B^3_{i-2,\mathbb{Z}}(i) + a_{i-1}B^3_{i-1,\mathbb{Z}}(i) + a_i B^3_{i,\mathbb{Z}}(i). \end{aligned}$$

Then (4.67) yields

$$\forall i = 1, 2, \ldots, N, \qquad a_{i-2} + 4a_{i-1} + a_i = 6f(i), \qquad (4.72)$$

which proves the middle $N-2$ equations of $M\mathbf{a} = \mathbf{b}$. By Theorem 4.35, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}B^n_{i,\mathbb{Z}}(x) = B^{n-1}_{i,\mathbb{Z}}(x) - B^{n-1}_{i+1,\mathbb{Z}}(x). \qquad (4.73)$$

Differentiate (4.70), apply (4.73), set $x = 1$, apply (4.65) and we have the first identity in (4.71), which, together with (4.72), yields

$$4a_0 + 2a_1 = 2f'(1) + 6f(1);$$

this proves the first equation of $M\mathbf{a} = \mathbf{b}$. The last equation $M\mathbf{a} = \mathbf{b}$ and the second identity in (4.71) can be shown similarly. The strictly diagonal dominance of M implies a nonzero determinant of M and therefore $\mathbf{a}$ is uniquely determined. The uniqueness of $S(x)$ then follows from (4.71). □

Corollary 4.59. There is a unique B-spline $S(x) \in \mathbb{S}^1_2$ that interpolates $f(x)$ at $t_i = i + \frac{1}{2}$ for each $i = 1, 2, \ldots, N-1$ with end conditions $S(1) = f(1)$ and $S(N) = f(N)$. Furthermore, this B-spline is

$$S(x) = \sum_{i=0}^{N} a_i B^2_{i,\mathbb{Z}}(x), \qquad (4.74)$$

where

$$a_0 = 2f(1) - a_1, \qquad a_N = 2f(N) - a_{N-1}, \qquad (4.75)$$

and $\mathbf{a}^T = [a_1, \ldots, a_{N-1}]$ is the solution of the linear system $M\mathbf{a} = \mathbf{b}$ with

$$\begin{aligned} \mathbf{b}^T =& \left[8f\left(\frac{3}{2}\right) - 2f(1), 8f\left(\frac{5}{2}\right),\right. \\ & \left.\ldots, 8f\left(N-\frac{3}{2}\right), 8f\left(N-\frac{1}{2}\right) - 2f(N)\right], \end{aligned}$$

$$M = \begin{bmatrix} 5 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 5 \end{bmatrix}.$$

Proof. It follows from Lemma 4.28 and Definition 4.51 that there are three quadratic cardinal B-splines, namely $B^2_{i-1,\mathbb{Z}}$, $B^2_{i,\mathbb{Z}}$, and $B^2_{i+1,\mathbb{Z}}$, that have nonzero values at each interpolation site $t_i = i + \frac{1}{2}$. Hence we have

$$f(t_i) = a_{i-1}B^2_{i-1,\mathbb{Z}}(t_i) + a_i B^2_{i,\mathbb{Z}}(t_i) + a_{i+1}B^2_{i+1,\mathbb{Z}}(t_i). \qquad (4.76)$$

Hence the dimension of the space of relevant cardinal B-splines is $N - 1 + 2 = N + 1$, which is different from that in the proof of Theorem 4.50! By Theorem 4.56, we can calculate the values of B-splines as:

$$\begin{aligned} B^2_{0,\mathbb{Z}}(x) &= \frac{1}{2}\sum_{k=-1}^{2}(-1)^{2-k}\binom{3}{k+1}(k-x)^2_+, \\ B^2_{0,\mathbb{Z}}\left(\frac{1}{2}\right) &= \frac{3}{4}, \\ B^2_{0,\mathbb{Z}}\left(-\frac{1}{2}\right) = B^2_{0,\mathbb{Z}}\left(\frac{3}{2}\right) &= \frac{1}{8}, \end{aligned}$$

where for $B^2_{0,\mathbb{Z}}\left(-\frac{1}{2}\right)$ we have used Corollary 4.53. Then Corollary 4.52 and (4.76) yield

$$a_{i-1} + 6a_i + a_{i+1} = 8f(t_i), \tag{4.77}$$

which proves the middle $N-3$ equations in $M\mathbf{a} = \mathbf{b}$. At the end point $x = 1$, only two quadratic cardinal B-splines, $B^2_{0,\mathbb{Z}}(x)$ and $B^2_{1,\mathbb{Z}}$, are nonzero. Then Example 4.26 yields

$$\frac{1}{2}a_0 + \frac{1}{2}a_1 = f(1)$$

and this proves the first identity in (4.75). Also, the above equation and (4.77) with $i = 1$ yield

$$5a_1 + a_2 = 8f\left(\frac{3}{2}\right) - 2f(1),$$

which proves the first equation in $M\mathbf{a} = \mathbf{b}$. The last equation in $M\mathbf{a} = \mathbf{b}$ can be proven similarly. □

4.5 Curve fitting via splines

Definition 4.60. An open *curve* is (the image of) a continuous map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ for some α, β with $-\infty \le \alpha < \beta \le +\infty$. It is *simple* if the map γ is injective.

Definition 4.61. The *tangent vector of a curve* γ is its first derivative

$$\gamma' := \frac{\mathrm{d}\gamma}{\mathrm{d}s} \tag{4.78}$$

and the *unit tangent vector* of γ, denoted by $\mathbf{t}$, is the normalization of its tangent vector.

Definition 4.62. A *unit-speed curve* is a curve whose tangent vector has unit length at each of its points.

Definition 4.63. A point $\gamma(t_0)$ is a *regular point* of γ if $\mathbf{t}(t_0)$ exists and $\mathbf{t}(t_0) \neq \mathbf{0}$ holds; a curve is *regular* if all of its points are regular.

Definition 4.64. The *arc-length* of a curve starting at the point $\gamma(t_0)$ is defined as

$$s_\gamma(t) = \int_{t_0}^{t} \|\gamma'(u)\|_2 \mathrm{d}u. \tag{4.79}$$

Definition 4.65. A map $X \mapsto Y$ is a *homeomorphism* if it is continuous and bijective and its inverse is continuous; then the two sets X and Y are said to be *homeomorphic*.

Definition 4.66. A curve $\tilde{\gamma}(\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a *reparametrization* of another curve $\gamma(\alpha, \beta) \to \mathbb{R}^n$ if there exists a homeomorphism $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ such that $\tilde{\gamma}(\tilde{t}) = \gamma\left(\phi(\tilde{t})\right)$ for each $\tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$.

Lemma 4.67. A reparametrization of a regular curve is unit-speed if and only if it is based on the arc-length.

Definition 4.68. A *closed curve* is (the image of) a continuous map $\mathring{\gamma} : [0, 1] \to \mathbb{R}^2$ that satisfies $\mathring{\gamma}(0) = \mathring{\gamma}(1)$. If the restriction of $\mathring{\gamma}$ to $[0, 1)$ is further injective, then the closed curve is a *simple closed curve* or *Jordan curve*.

Definition 4.69. The *signed unit normal* of a curve, denoted by $\mathbf{n}_s$, is the unit vector obtained by rotating its unit tangent vector counterclockwise by $\frac{\pi}{2}$.

Definition 4.70. For a unit-speed curve γ, its *signed curvature* is defined as

$$\kappa_s := \gamma'' \cdot \mathbf{n}_s. \tag{4.80}$$

Definition 4.71. The *cumulative chordal length*s associated with a sequence of n points

$$\{\mathbf{x}_i \in \mathbb{R}^{\mathrm{D}} : i = 1, 2, \ldots, n\} \tag{4.81}$$

are the n real numbers,

$$t_i = \begin{cases} 0, & i = 1; \\ t_{i-1} + \|\mathbf{x}_i - \mathbf{x}_{i-1}\|_2, & i > 1, \end{cases} \tag{4.82}$$

where $\|\cdot\|_2$ denotes the Euclidean 2-norm.

4.6 Problems

4.6.1 Theoretical questions

I. Consider $s \in \mathbb{S}^2_3$ on $[0, 2]$:

$$s(x) = \begin{cases} p(x) & \text{if } x \in [0, 1], \\ (2-x)^3 & \text{if } x \in [1, 2]. \end{cases}$$

Determine $p \in \mathbb{P}_3$ such that $s(0) = 0$. Is $s(x)$ a natural cubic spline?

II. Given $f_i = f(x_i)$ of some scalar function at points $a = x_1 < x_2 < \cdots < x_n = b$, we consider interpolating f on $[a, b]$ with a quadratic spline $s \in \mathbb{S}^1_2$.

  (a) Why an additional condition is needed to determine s uniquely?

  (b) Define $m_i = s'(x_i)$ and $p_i = s|_{[x_i, x_{i+1}]}$. Determine p_i in terms of f_i, f_{i+1}, and m_i for $i = 1, 2, \ldots, n-1$.

  (c) Suppose $m_1 = f'(a)$ is given. Show how $m_2, m_3, \ldots, m_{n-1}$ can be computed.

III. Let $s_1(x) = 1 + c(x+1)^3$ where $x \in [-1, 0]$ and $c \in \mathbb{R}$. Determine $s_2(x)$ on $[0, 1]$ such that

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in [-1, 0], \\ s_2(x) & \text{if } x \in [0, 1] \end{cases}$$

is a natural cubic spline on $[-1, 1]$ with knots $-1, 0, 1$. How must c be chosen if one wants $s(1) = -1$?

IV. Consider $f(x) = \cos\left(\frac{\pi}{2}x\right)$ with $x \in [-1, 1]$.

  (a) Determine the natural cubic spline interpolant to f on knots $-1, 0, 1$.

  (b) As discussed in the class, natural cubic splines have the minimal total bending energy. Verify this by taking $g(x)$ be (i) the quadratic polynomial that interpolates f at $-1, 0, 1$, and (ii) $f(x)$.

V. The quadratic B-spline $B^2_i(x)$.

(a) Derive the same explicit expression of $B_i^2(x)$ as that in the notes from the recursive definition of B-splines and the hat function.

(b) Verify that $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ is continuous at t_i and t_{i+1}.

(c) Show that only one $x^* \in (t_{i-1}, t_{i+1})$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x^*) = 0$. Express x^* in terms of the knots within the interval of support.

(d) Consequently, show $B_i^2(x) \in [0, 1)$.

(e) Plot $B_1^2(x)$ for $t_i = i$.

VI. Verify Theorem 4.33 algebraically for the case of $n = 2$, i.e.

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t - x)_+^2 = B_i^2.$$

VII. Scaled integral of B-splines.
Deduce from the Theorem on derivatives of B-splines that the scaled integral of a B-spline $B_i^n(x)$ over its support is independent of its index i even if the spacing of the knots is not uniform.

VIII. Symmetric Polynomials.
We have a theorem on expressing complete symmetric polynomials as divided difference of monomials.

(a) Verify this theorem for $m = 4$ and $n = 2$ by working out the table of divided difference and comparing the result to the definition of complete symmetric polynomials.

(b) Prove this theorem by the lemma on the recursive relation on complete symmetric polynomials.

4.6.2 Programming assignments

A. Write a program for cubic-spline interpolation of the function

$$f(x) = \frac{1}{1 + 25x^2}$$

on evenly spaced nodes within the interval $[-1, 1]$ with $N = 6, 11, 21, 41, 81$. Compute for each N the max-norm of the interpolation error vector at mid-points of the subintervals and report the errors and convergence rates with respect to the number of subintervals.

Your algorithm should follow the example of interpolating the natural logarithm in the notes and your program must use an implementation of `lapack`.

Plot the interpolating spline against the exact function to observe that spline interpolation is free of the wide oscillations in the Runge phenomenon.

B. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. Implement two subroutines to interpolate f by the quadratic and cubic cardinal B-splines, which corresponds to Corollaries 4.58 and 4.59, respectively.

C. Run your subroutines on the function

$$f(x) = \frac{1}{1 + x^2}, \qquad x \in [-5, 5],$$

using $t_i = -6 + i$, $i = 1, \ldots, 11$ for Corollary 4.58 and $t_i = i - \frac{11}{2}$, $i = 1, \ldots, 10$ for Corollary 4.59, respectively. Plot the polynomials against the exact function.

D. Define $E_S(x) = |S(x) - f(x)|$ as the interpolation error. For the two cardinal B-spline interpolants, output values of $E_S(x)$ at the sites

$$x = -3.5, -3, -0.5, 0, 0.5, 3, 3.5.$$

Output these values by a program. Why are some of the errors close to machine precision? Which of the two B-splines is more accurate?

E. The roots of the following equation constitute a closed planar curve in the shape of a heart:

$$x^2 + \left(\frac{3}{2}y - \sqrt{|x|}\right)^2 = 3. \tag{4.83}$$

Write a program to plot the heart. The parameter of the curve should be the *cumulative chordal length* defined in (4.82). Choose $n = 10, 40, 160$ and produce three plots of the heart function. (*Hints*: Your knots should include the characteristic points and you should think about (i) how many pieces of splines to use? (ii) what boundary conditions are appropriate?)

F. (*) Write a program to illustrate (4.37) by plotting the truncated power functions for $n = 1, 2$ and build a table of divided difference where the entries are figures instead of numbers. The pictures you generated for $n = 1$ should be the same as those in Example 4.32.