Trade Comprehension Check 3

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If we ignore sectoral assignments and agglomeration economies and restrict attention to a single city, the locational assignment problem of Davis and Dingel - The Comparative Advantage of Cities is a familiar model from urban economics. See Kevin Murphy's lectures on "Chapter 8: Location Choice" on YouTube.

Using the Davis and Dingel notation, suppose that there is a single city in which skill ω earns income $G(\omega)$ and the total population is L. Let $A(c) = 1$ for this city. The city is monocentric with linear commuting costs: $T(\tau) = d_1 - d_2\tau$ and $S(\tau) = \pi\tau^2$. For simplicity, let $G(\omega) = g\omega$, where g is a constant, and assume skill is uniformly distributed, $\omega \sim U(\underline{\omega}, \bar{\omega})$.

Framework

Consumers

$$\begin{aligned}
\text{Demand for Final Good: } & U(c,\tau,\sigma;\omega) = A(c)T(\tau)H(\omega,\sigma)p(\sigma) - r(c,\tau) \\
\text{Individual income: } & \underbrace{A(c)T(\tau)H(\omega,\sigma)}_{:=q(c,\tau,\sigma;\omega)}p(\sigma) \\
\text{Fractions of Workers by Type: } & L \times f(\omega, c, \tau\sigma) \\
\text{Sectoral choice: } & M(\omega) = \text{argmax}_\sigma\, H(\omega,\sigma)p(\sigma) \\
& G(\omega) \equiv H(\omega, M(\omega))p(M(\omega)) \\
\text{Locational choice: } & \gamma = A(c)T(\tau) \\
& \max_\gamma \gamma G(\omega) - r_\Gamma(\gamma)
\end{aligned}$$

Firms

$$\begin{aligned}
\text{Final good aggregator: } & Q = \left\{ \int_{\sigma\in\Sigma} B(\sigma)[Q(\sigma)]^{\frac{\varepsilon-1}{\varepsilon}} d\sigma \right\}^{\frac{\varepsilon}{\varepsilon-1}} \\
\text{Final good profit: } & \Pi = Q - \int_{\sigma\in\Sigma} p(\sigma)Q(\sigma)d\sigma \\
\text{Equilibrium intermediate demand: } & Q(\sigma) = I\left(\frac{p(\sigma)}{B(\sigma)}\right)^{-\varepsilon}
\end{aligned}$$

Market Clearing

$$\text{Location market clearing: } r(c,\tau) \times \left(S'(\tau) - L \int_{\sigma\in\Sigma}\int_{\omega\in\Omega} f(\omega,c,\tau,\sigma)d\omega d\sigma\right) = 0 \forall c \forall \tau$$

Show that $\bar{\tau}$ and $\underline{\gamma}$ depend only on the exogenous parameters L, d_1, d_2.

In the Davis and Dingel paper, they define $\bar{\tau}(c) \equiv \sup_\tau\{\tau : f(\omega,c,\tau,\sigma) > 0\}$. and further say that $\bar{\tau}(c)$ is determined by $L(c) = S(\bar{\tau}(c))$ Note that we can ignore any functions depending on c since we are in a single-city context. Then, using hte function for $S(\tau)$ given in the problem, we have:

$$L = S(\bar{\tau}) = \pi\bar{\tau}^2$$
$$\sqrt{\frac{L}{\pi}} = \bar{\tau}$$

Next, we know from the definition of $\underline{\gamma}$ in Lemma 1 that $\underline{\gamma} \equiv A(C)T(\bar{\tau}(C))$. In this lemma, C is defined as the index of positive-population cities, which we can again ignore in the single-city context. We can solve as follows:

$$\begin{aligned}
\underline{\gamma} &= T(\bar{\tau}) \\
&= d_1 - d_2\bar{\tau} \\
&= d_1 - d_2\sqrt{\frac{L}{\pi}}
\end{aligned}$$

The rent at the city edge $\bar{\tau}$ is zero. What is the rent at the center of the city, $\tau = 0$?

From Lemma 3 of the paper, we are given that $r_\Gamma(\gamma) = \int_{\underline{\gamma}}^{\gamma} G(K(x))dx$. Further, since $A(c) = 1$, then we know that $\gamma = T(\tau) = d_1 - d_2\tau$, so γ is strictly decreasing in τ. So if we care about the rent at $\tau = 0$, this is equivalent to finding the rent at $\bar{\gamma}$. We need to evaluate the expression $r_\Gamma(\bar{\gamma}) = \int_{\underline{\gamma}}^{\bar{\gamma}} G(K(x))dx$, which is equivalent to:

$$\int_{\underline{\gamma}}^{\bar{\gamma}} gF^{-1}\left(\frac{L - S_\Gamma(x)}{L}\right)dx$$

To evaluate this further, I focus on the functions inside the integral. First let us examine $S_\Gamma(\gamma)$. This function is equivalent to:

$$\begin{aligned}
S_r(\gamma) &= S\left(T^{-1}\left(\frac{\gamma}{A(c)}\right)\right) \\
&= S\left(T^{-1}(\gamma)\right)
\end{aligned}$$

Since we know the functional form for $T(\tau)$, we can find $T^{-1}(y)$:

$$T(\tau) := y = d_1 - d_2\tau$$
$$\frac{d_1 - y}{d_2} = \tau = T^{-1}(y)$$

Then the equation for supply becomes

$$
\begin{aligned}
S_\Gamma(\gamma) &= S\left(\frac{d_1 - y}{d_2}\right) \\
&= \pi\left(\frac{d_1 - y}{d_2}\right)^2
\end{aligned}
$$

Next we can evaluate the $K(\gamma)$ function. We know that $\omega \sim U[\underline{\omega}, \bar{\omega}]$. Using the CDF of the uniform distribution, we can find the inverse CDF:

$$
\begin{aligned}
F(x) := y &= \frac{x - \underline{\omega}}{\bar{\omega} - \underline{\omega}} \\
y(\bar{\omega} - \underline{\omega}) + \underline{\omega} &= F^{-1}(y)
\end{aligned}
$$

Then plugging in all the values we found into the equation for $G(K(\gamma))$, we get:

$$
\int_{\underline{\gamma}}^{\bar{\gamma}} G(K(\gamma)) = \int_{\underline{\gamma}}^{\bar{\gamma}} g\left[\left(\frac{L - \pi\left(\frac{d_1 - x}{d_2}\right)^2}{L}\right)(\bar{\omega} - \underline{\omega}) + \underline{\omega}\right] dx \tag{1}
$$

Then solving out the integral we get:

$$
\begin{aligned}
&\int_{\underline{\gamma}}^{\bar{\gamma}} g\bar{\omega}dx - \int_{\underline{\gamma}}^{\bar{\gamma}} g\frac{\pi\left(\frac{d_1 - x}{d_2}\right)^2}{L}(\bar{\omega} - \underline{\omega})\, dx \\
&= g\bar{\omega}x\Big|_{\underline{\gamma}}^{\bar{\gamma}} - \frac{g\pi(\bar{\omega} - \underline{\omega})}{d_2^2 L}\int_{\underline{\gamma}}^{\bar{\gamma}} (d_1 - x)^2 dx \\
&= g\bar{\omega}x\Big|_{\underline{\gamma}}^{\bar{\gamma}} - \frac{g\pi(\bar{\omega} - \underline{\omega})}{d_2^2 L}\int_{d_1 - \underline{\gamma}}^{d_1 - \bar{\gamma}} (y)^2 dy \\
&= g\bar{\omega}x\Big|_{\underline{\gamma}}^{\bar{\gamma}} - \frac{g\pi(\bar{\omega} - \underline{\omega})}{d_2^2 L}\frac{1}{3}y^3\Big|_{d_1 - \underline{\gamma}}^{d_1 - \bar{\gamma}}
\end{aligned}
$$

Note that $\underline{\gamma} = d_1 - d_2\sqrt{\frac{L}{\pi}}$ and $\bar{\gamma} = d_1$ since it is associated with $\tau = 0$. Plugging in these values, we get:

$$
\begin{aligned}
&= g\bar{\omega}\left(d_2\sqrt{\frac{L}{\pi}}\right) - \frac{g\pi(\bar{\omega} + \underline{\omega})}{3d_2^2 L}\left(d_2\sqrt{\frac{L}{\pi}}\right)^3 \\
&= g\bar{\omega}\left(d_2\sqrt{\frac{L}{\pi}}\right) - \frac{g\pi d_2(\bar{\omega} + \underline{\omega})}{3}\sqrt{\frac{L}{\pi}}
\end{aligned}
$$

> Suppose that g increases. What happens to the rent schedule? What happens to the equilibrium utility of each skill level?

If g increases, the whole equilibrium rent schedule increases by g. However the rent at the edge of the city doesn't change since we found $\underline{\omega} = 0$. As a result, the rent schedule will have a proportional shift or rotation upwards.

The equilibrium utility for skill level ω is:

$$\begin{aligned} U(\tau;\omega) &= T(\tau)G(\omega) - r(c,\tau) \\ &= (d_1 - d_2\tau)\, g\omega - \int_{\underline{\gamma}}^{\gamma} g\left[\left(\frac{L - \pi\left(\frac{d_1-x}{d_2}\right)^2}{L}\right)(\bar{\omega} - \underline{\omega}) + \underline{\omega})\right] dx \\ U(\omega) &= \gamma^*(\omega) g\omega - \int_{\underline{\gamma}}^{\gamma^*(\omega)} g\left[\left(\frac{L - \pi\left(\frac{d_1-x}{d_2}\right)^2}{L}\right)(\bar{\omega} - \underline{\omega}) + \underline{\omega})\right] dx \end{aligned}$$

Note $\gamma^*(\omega)$ is the optimal value of γ for a given skill level ω as defined by the relationship $K^{-1}(\omega) = \gamma^*$. We can calculate $K^{-1}(\omega)$ as follows:

$$\begin{aligned} \omega &= F^{-1}\left(\frac{L - \pi\left(\frac{d_1-\gamma}{d_2}\right)^2}{L}\right) \\ \gamma^* &= d_1 - d_2\sqrt{\frac{L - F(\omega)L}{\pi}} \\ &= d_1 - d_2\sqrt{\frac{L - \frac{\omega - \underline{\omega}}{\bar{\omega} - \underline{\omega}}L}{\pi}} \end{aligned}$$

Since γ^* doesn't depend on g, then we know that increasing g will increase the equilibrium utility level for all values of ω.

> What happens to the equilibrium utility of $\bar{\omega}$ if the value of $\underline{\omega}$ increases? (skill compression)

The utility function for $\bar{\omega}$ is:

$$\begin{aligned} U(\bar{\omega}) &= \gamma^*(\bar{\omega}) g\bar{\omega} - \int_{\underline{\gamma}}^{\gamma^*(\bar{\omega})} g\left[\left(\frac{L - \pi\left(\frac{d_1-x}{d_2}\right)^2}{L}\right)(\bar{\omega} - \underline{\omega}) + \underline{\omega}\right] dx \\ &= d_1 g\bar{\omega} - \int_{\underline{\gamma}}^{d_1} g\left[\left(\frac{L - \pi\left(\frac{d_1-x}{d_2}\right)^2}{L}\right)(\bar{\omega} - \underline{\omega}) + \underline{\omega}\right] dx \end{aligned}$$

Note that since the highest skilled people live at $\tau = 0$, we get that $\gamma^*(\bar{\omega}) = d_1$. So if $\underline{\omega}$ increases, we only need to evaluate the change in the rent schedule to see what happens to utility. The rent schedule at each value of γ changes by:

$$-(1 - \lambda(x))\underline{\omega} + \underline{\omega} = \lambda(x)\underline{\omega}$$

where $\lambda(x) = \frac{\pi\left(\frac{d_1-x}{d_2}\right)^2}{L}$. We need to determine the sign of $\lambda(x)$. We know that $S(\tau)$ is strictly increasing so $\pi > 0$. Additionally $L > 0$, which means that $\lambda(x) > 0$ for all x. So the rent schedule increases at each point. The bounds of integration remain unaffected because $\underline{\gamma} = d_1 - d_2\sqrt{\frac{L}{\pi}}$, which is not dependent on $\underline{\omega}$. So equilibrium utility for $\bar{\omega}$ decreases.

> What happens to the equilibrium utility of $\underline{\omega}$ if the value of $\bar{\omega}$ increases? (skill dilation)

As shown in the previous question, the lowest skilled people will have $\gamma^*(\underline{\omega}) = \underline{\gamma} = d_1 - d_2\sqrt{\frac{L}{\pi}}$. Now the utility can be written as:

$$U(\underline{\omega}) = \gamma^*(\underline{\omega})g\underline{\omega}$$

We can see that $\bar{\omega}$ only enters utility through the rent equation and for the person with skill $\underline{\omega}$, rents will be 0 since we are integrating the rent equation from $\underline{\gamma}$ to $\underline{\gamma}$. So there should be no change in the equilibrium utility level for the $\underline{\omega}$ worker.