Expenditure Minimization

Expenditure Minimization Problem (EMP)

$\min \; p \cdot x$

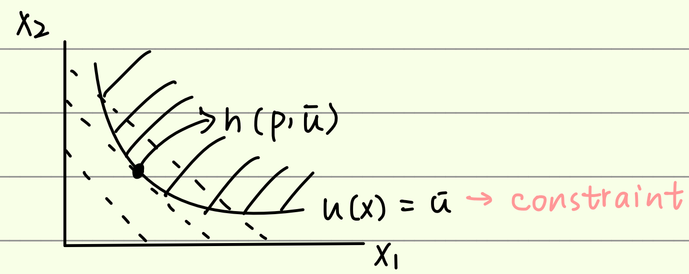
s.t. $u(x) \geq \bar{u}$ → target utility

exactly isomorphic to $\min \; w \cdot x$ s.t. $f(x) \geq y$

$w \Leftrightarrow p$, $u(\cdot) \Leftrightarrow f(\cdot)$, $y \Leftrightarrow \bar{u}$

Solution Functions

- $h(p, \bar{u}) = \arg\min \; p \cdot x$ ← Hicksian demand
  s.t. $u(x) \geq \bar{u}$
- $e(p, \bar{u}) = \min \; p \cdot x = p \cdot h(p, \bar{u})$
  s.t. $u(x) \geq \bar{u}$
  value function, expenditure function



Properties of $e(p, \bar{u})$

1) hd-1 in p
2) nondecreasing in p & $\bar{u}$
3) concave in p

Properties of $h(p, \bar{u})$

1) hd-0 in p: $h(\alpha p, \bar{u}) = h(p, \bar{u})$
2) $\succsim$ convex ⇒ $h(p, \bar{u})$ is convex set
   $\succsim$ strictly convex ⇒ $h(p, \bar{u})$ is unique

Shepard's Lemma

- $\partial c / \partial w_i = x_i(w, y)$
  For the EMP: $\partial e(p, u) / \partial p_i = h_i(p, u)$
- Substitution matrix

$$Dh(p, u) = \begin{bmatrix} \partial h_1 / \partial p_1 & \cdots & \partial h_1 / \partial p_n \\ \vdots & \ddots & \vdots \\ \partial h_n / \partial p_1 & \cdots & \partial h_n / \partial p_n \end{bmatrix}$$

$$= \begin{bmatrix} \partial^2 e(p,\bar{u})/\partial p_1^2 & \cdots & \partial^2 e(p,\bar{u})/\partial p_1 \partial p_n \\ \vdots & & \\ \partial^2 e(p,\bar{u})/\partial p_1 \partial p_n & \cdots & \partial^2 e(p,\bar{u})/\partial p_n^2 \end{bmatrix}$$

- Theorem:

  The substitution matrix $Dh(p,\bar{u})$ is:

  1) negative semidefinite ~ (differential) law of (compensated) demand, $\frac{\partial h_i}{\partial p_i} \le 0$
  2) symmetric
  3) satisfies $Dh(p,\bar{u}) \cdot p = 0$

Duality

$\min \; p_1 x_1 + p_2 x_2$

s.t. $\alpha \log x_1 + \beta \log x_2 \geq \bar{u}$

$\Rightarrow \mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda(\bar{u} - \alpha \log(x_1) - \beta \log(x_2))$

- FOC:

  ① $p_1 = \lambda\alpha / x_1$

  ② $p_2 = \lambda\beta / x_2$

  $\lambda(\underbrace{\bar{u} - \alpha\log(x_1) - \beta\log(x_2))}_{\text{will bind, b/c of locally non-satiated}}$] CSC

  $\Rightarrow \bar{u} = \alpha\log(x_1) + \beta\log(x_2)$ ③

- Solve:

  - ① / ②: $\frac{x_2}{x_1}\frac{\alpha}{\beta} = \frac{p_1}{p_2}$

    $\Rightarrow x_2 = \left(\frac{\beta}{\alpha}\frac{p_1}{p_2}\right) x_1$

  - Plug into ③: $\alpha\log(x_1) + \beta\log(cx_1) = \bar{u}$

    $\alpha\log(x_1) + \beta\log(x_1) + \beta\log(c) = \bar{u}$

    $\log(x_1) + \log c^{\beta} = \bar{u}$

    $x_1 = e^{\bar{u}} e^{-\log c^{\beta}}$

    $x_1 = e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha-1} \left[\frac{p_2}{\beta}\right]^{\beta}$

    $\Rightarrow h_1(p,\bar{u}) = e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha-1} \left[\frac{p_2}{\beta}\right]^{\beta}$

    $h_2(p,\bar{u}) = e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha} \left[\frac{p_2}{\beta}\right]^{\beta-1}$

    $\Rightarrow e(p,\bar{u}) = p_1 h_1(p,\bar{u}) + p_2 h_2(p,\bar{u})$

    $= \left[e^{\bar{u}} \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta}\right]\alpha$

$$+ [e^{\bar{u}} (\tfrac{p_1}{\alpha})^\alpha (\tfrac{p_2}{\beta})^\beta]\beta$$
$$= e^{\bar{u}} (\tfrac{p_1}{\alpha})^\alpha (\tfrac{p_2}{\beta})^\beta$$

- $x_1(p,m) = \alpha m / p_1$

  $x_2(p,m) = \beta m / p_2$

  What if we set $m = e(p,\bar{u})$?

  $$x_1(p, e(p,\bar{u})) = \alpha e^{\bar{u}} (\tfrac{p_1}{\alpha})^\alpha (\tfrac{p_2}{\beta})^\beta / p_1$$
  $$= e^{\bar{u}} (\tfrac{p_1}{\alpha})^{\alpha-1} (\tfrac{p_2}{\beta})^\beta$$
  $$= h_1(p,\bar{u})$$! Duality b/t UMP & EMP

Theorem: Assume $u(\cdot)$ is continuous & LNS.

1) If x^* solves $\max u(x)$ s.t. $px \leq m$, $x_i \geq 0$,
   x^* also solves $\min p\cdot x$ s.t. $u(x) \geq u(x^*)$, $x_i \geq 0$
   Further, $e(p, u(x^*)) = m$.
2) If x^* solves $\min p \cdot x$ s.t. $u(x) \geq \bar{u}$, $x_i \geq 0$,
   x^* also solves $\max u(x^*)$ s.t. $p \cdot x \leq p \cdot x^*$, $x_i \geq 0$.
   Further, $v(p, p\cdot x^*) = \bar{u}$

Proof of 1):

- Say x^* solves UMP but not EMP at $\bar{u} = u(x^*)$
  $\Rightarrow \exists$ some y s.t. $u(y) \geq u(x^*)$ and $p \cdot y < p \cdot x^* = m$
- By LNS, $\exists$ another y' s.t. $p\cdot y' < m$ and $u(y') > u(y)$
  $\Rightarrow u(y') > u(x^*)$
  $\Rightarrow x^*$ is not optimal in UMP W

Proof of 2):

- Say x^* solves EMP but not UMP
  $\Rightarrow \exists$ some y s.t. $u(y) > u(x^*)$ and $py \leq p \cdot x^*$
- Let $y' = \alpha y$ for $\alpha \in (0,1)$
- By continuity, for $\alpha \approx 1$, $u(y') > u(x^*)$ and $p \cdot y' < p \cdot x^*$
  $\Rightarrow x^*$ is not optimal in EMP W

4 "Duality Identities"

1) $e(p, v(p,m)) = m$

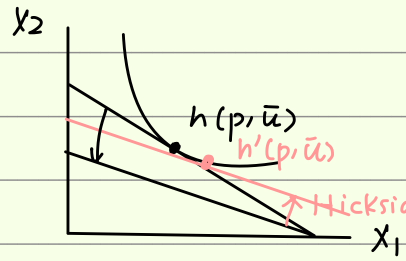
2) $v(p, e(p,\bar{u})) = \bar{u}$

3) $x_i(p, e(p,\bar{u})) = h_i(p,\bar{u})$

4) $h_i(p, v(p,m)) = x_i(p,m)$

Hicksian Demand

- Why is it called "compensated" demand?



With an increase in p_2,

$h(p,\bar{u}) = x(p, e(p,\bar{u}))$

both change

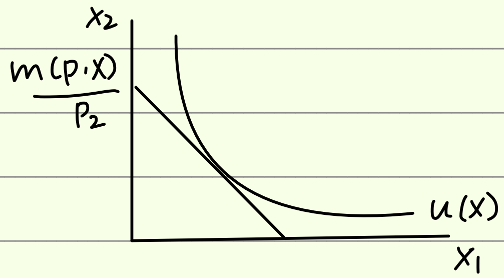
Hicksian compensation to achieve same utility even when price changes

Money Metric Utility

Bundle x, $u(x)$, fix price at p.

Q: How much money does the consumer need to achieve utility $u(x)$ at p?

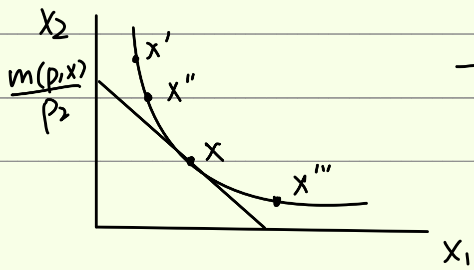
- It's NOT the price of x.
- Should be $m(p,x) = e(p, u(x))$



- Fix p, view as a function of x

  $m(p,\cdot) = e(p,\cdot)$

  As a function of x, $m(p,x)$ is a utility function that represents some preferences as $u(\cdot)$



- $m(p,x) = m(p,x') = m(p,x'') = \cdots$

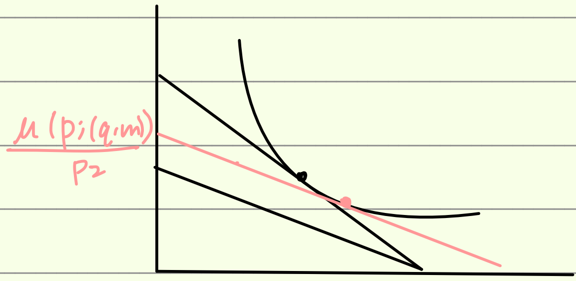
  $u(x) = u(x') = u(x'') = \cdots$

  since you are indifferent with $x, x', x'', x''', \ldots$

- Also, money metric <u>indirect</u> utility function:

  $\mu(p; (q,m)) = e(p; v(q,m))$

  └ answers question: How much money you need at prices p, to be just as well off as you were at prices q and income m?



- Example

  $u(x) = \alpha \log(x_1) + \beta \log(x_2) \rightarrow$ Cobb-Douglas

  $e(p,\bar{u}) = e^{\bar{u}} (\frac{p_1}{\alpha})^{\alpha} (\frac{p_2}{\beta})^{\beta}$

  $m(p,x) = e(p, u(x))$

  $= e^{\alpha \log x_1 + \beta \log x_2} (\frac{p_1}{\alpha})^{\alpha} (\frac{p_2}{\beta})^{\beta}$

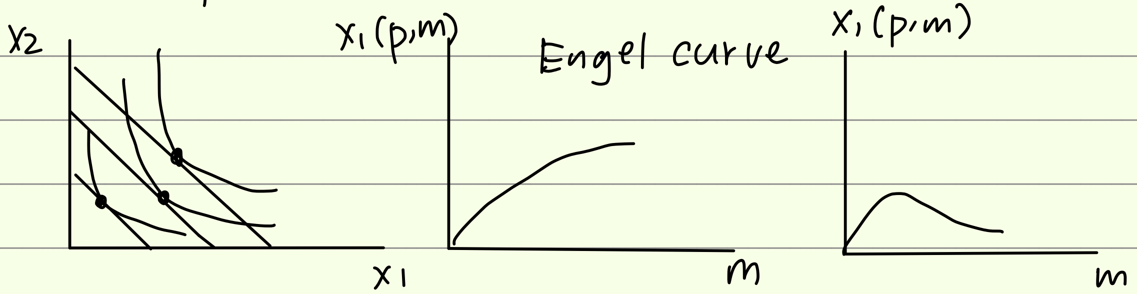
  $= x_1^{\alpha} x_2^{\beta} \underbrace{(\frac{\beta_1}{\alpha})^{\alpha} (\frac{p_2}{\beta})^{\beta}}_{\text{constant}}$

  $\Rightarrow m(p,x)$ is a monotonic transformation of original CD-utility

  measure util by money?

Comparative Statics of Demand

$x(p,m)$ — income effect: varying m
 — price effect: varying p



- If $\partial x_1(p,m) / \partial m \geq 0$: normal good

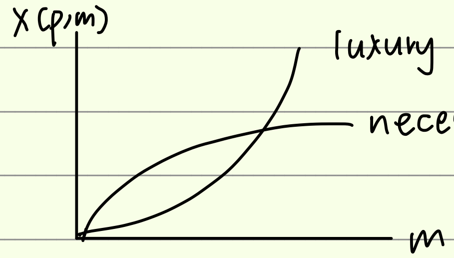
  If $\partial x_i(p,m) / \partial m < 0$: inferior good

- To measure the curvature, we use the income elasticity.

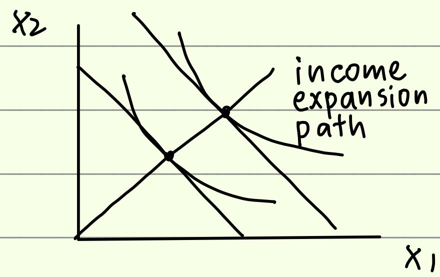
$$\varepsilon_{im} = \frac{d\log(x_i(p,m))}{d\log(m)} = \frac{m}{x_i(p,m)} \frac{dx_i(p,m)}{dm}$$

$0 < \varepsilon_{im} < 1$: necessary good

$\varepsilon_{im} > 1$: luxury good



- Special case: homothetic preferences



$u(x)$ is hd-1

- More formally, if u is hd-1, then $x(p,m) = \alpha x(p,m)$

Proof: Let $x \in x(p,m)$

$p \cdot x \le m$

$\alpha p x \le \alpha m$

$p(\alpha x) \le \alpha m$

Take any y s.t. $p \cdot y \le \alpha m$

$p \cdot (\frac{1}{\alpha} y) \le m$

So, $\frac{1}{\alpha} y$ is affordable at m, so

$u(\frac{1}{\alpha} y) \le u(x)$

$\alpha u(\frac{1}{\alpha} y) \le \alpha u(x)$

$u(y) \le u(\alpha x)$ (by hd-1)

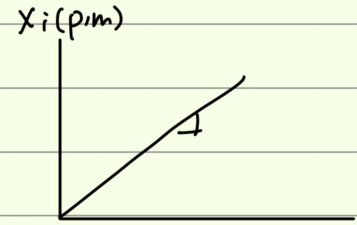
$\Rightarrow \alpha x$ is optimal at income αm

so, $x(p, \alpha m) = \alpha x(p,m)$

set $\alpha = \frac{1}{m}$

$x(p,1) = \frac{1}{m} x(p,m)$

$x(p,m) = m x(p,1)$



$$\varepsilon_{im} = \frac{d \log x_i(p,m)}{d \log m}$$

$\log x_i(p,m) = \log m + \log x_i(p,1)$

$\varepsilon_{im} = 1$

For CD, $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$

$x_1(p,m) = \alpha m / p_1$

Engel curve linear, slope α / p_1

- Price effect
  - Hotelling : $y_i(p) = \partial \pi / \partial p_i$

    $\partial y_i / \partial p_i = \partial^2 \pi / \partial p_i^2$
  - $\pi(p)$ convex $\Rightarrow Dy(p) = D^2 \pi(p)$ is p.s.d., $\frac{\partial^2 \pi}{\partial p_i^2} \geq 0$

    $\Rightarrow \frac{\partial y_i}{\partial p_i} \geq 0$: Law of supply
  - The analogous result for $x_i(p,m)$ would be

    $\frac{\partial x_i(p,m)}{\partial p_1} \leq 0$ "law of demand"

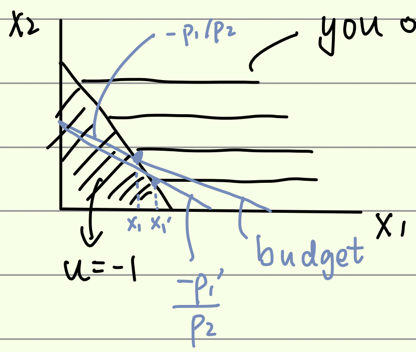
    Unfortunately this is not true

    $v(p,m)$ only quasiconvex so ② doesn't work
  - example: meat (x_2) & potatoes (x_1)

    minimum calarie requirement : c

    For any bundle (x_1, x_2), $c = 2x_1 + x_2$

    $$u(x_1, x_2) \begin{cases} x_2 & , 2x_1 + x_2 \geq c \\ -1 & , 2x_1 + x_2 < c \end{cases}$$



p_1 rises to $p_1' > p_1$

giffen good $\leftarrow$ $x_1' > x_1$ so $\partial x_1 / \partial p_1 > 0$

Law of demand fails

$\partial x_1 / \partial p_1 \leq 0$: ordinary

"Law of demand" not true (in general) for $x_i(p,m)$.

What about $h_i(p,\bar{u})$?

- In the EMP, $e(p,\bar{u})$ is convex in p
- So here: $D\, e(p,\bar{u})$ is a n.s.d. matrix
- Shepard's Lemma: $h_i(p,\bar{u}) = \frac{\partial e(p,\bar{u})}{\partial p_i}$

$$\frac{\partial h_i(p,\bar{u})}{\partial p_i} = \frac{\partial^2 e(p,\bar{u})}{\partial p_i^2} \leq 0$$

Law of compensated demand

$$h_i(p,\bar{u}) = x_i(p, e(p,\bar{u}))$$

- Differentiate wrt p_k

$$\frac{\partial h_i(p,\bar{u})}{\partial p_k} = \frac{\partial x_i(p,e(p,\bar{u}))}{\partial p_k} + \frac{\partial x_i(p,e(p,\bar{u}))}{\partial m} \underbrace{\cancel{\frac{\partial e(p,\bar{u})}{\partial p_k}}}_{h_k(p,\bar{u})}$$

(Shepard's)

① $m = e(p,\bar{u})$

② $\bar{u} = v(p,m)$

③ $h_k(p,\bar{u}) = x_k(p; e(p,\bar{u})) = x_k(p,m)$

- Slutsky Equation:

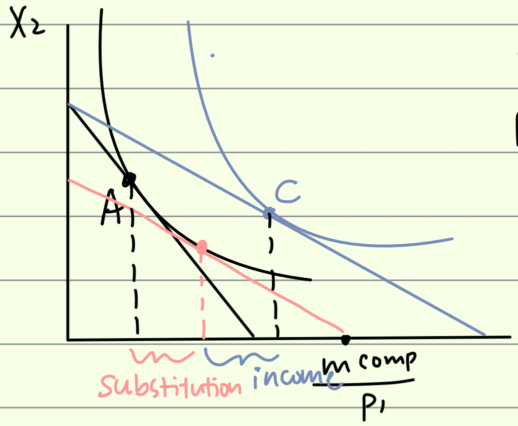
$$\frac{\partial x_i(p,m)}{\partial p_k} = \underbrace{\frac{\partial h_i(p, v(p,m))}{\partial p_k}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_i(p,m)}{\partial m} x_k(p,m)}_{\text{income effect}}$$

$\partial h_i / \partial p_k \geq 0$: i, k are substitutes

$\partial h_i / \partial p_k \leq 0$: i, k are complements

$\partial x_i / \partial p_k \geq 0$: i, k are gross substitutes

$\partial x_i / \partial p_k \leq 0$: i, k are gross complements



$p_1' < p_1$

$p_2' = p_2$

A: $x(p,m) = h(p, u_{old})$

C: $x(p',m) = h(p', u_{new})$

B: $h(p', u_{old})$

$m^{comp} = e(p', u^{old})$

$$\underbrace{x_1(p',m) - x_1(p,m)}_{\text{total effect}} = [x(p', m^{comp}) - x(p,m)] + [x(p',m) - x_1(p', m^{comp})]$$

$$= [h(p', u^{old}) - h(p, u^{old})] + [x(p',m) - x(p', m^{comp})]$$

$$= \underset{\substack{\downarrow \\ \text{substitution effect}}}{[B - A]} + \underset{\substack{\downarrow \\ \text{income effect}}}{[C - B]}$$