The Overlapping Generations Model

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The Overlapping Generations Model

- Neoclassical growth model has a representative agent
- No way to discuss implications of heterogeneity
- Life-cycle / generations are important examples of heterogeneity
- OLG model allows discussion these issues
- More generally, allows discussion of issues that arise with
  - Heterogeneity
  - Infinite number of agents
- Seminal papers: Samuelson (1958), Diamond (1965)

Specific issues we will discuss:

- Dynamic efficiency (i.e., over-accumulation of capital)
- Social Security (i.e., old age pension systems)
- Public debt
- Money / Bubbles

Basic Structure

- Two generations: Young and Old
- Each lives for two periods (discrete time)
- Young work, consume, save
- Old consume and dissave (do not work)
- Common extensions:
  - Many generations
  - Perpetual youth model (Blanchard, 1985)
- Two generation version particularly simple because it precludes intertemporal trade (no one meets twice)

Basic Structure

- L_t individuals are born at time t
- Exogenous population growth at rate n:

$$L_{t+1} = (1+n)L_t$$

- Each young agent supplies 1 unit of labor
- "Youth" need not be due to birth. Could be immigration or the binding of a borrowing constraint.

Production

- Production function:

$$Y_t = F(K_t, A_t L_t)$$

- Exogenous productivity growth:

$$A_{t+1} = (1+g)A_t$$

- Perfect competition in factor markets yields:

$$r_t = f'(k_t) \qquad\qquad w_t = f(k_t) - k_t f'(k_t)$$

(See Ramsey model lecture for details)

  - r_t is the return on savings held from period $t-1$ to t
  - w_t is the wage per effective unit of labor

Households

- Preferences of households born at t:

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

- Budget constraints:

$$C_{1t} + s_t = w_t A_t$$

$$C_{2t+1} = (1 + r_{t+1}) s_t$$

  - s_t is savings of young at time t
  - Old consume both interest and principle
  - We are assuming no depreciation of capital (for simplicity)

Household Optimization

- We can plug budget constraints into U_t to get

$$U_t = \frac{(w_t A_t - s_t)^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{((1+r_{t+1})s_t)^{1-\theta}}{1-\theta}$$

- Differentiating with respect to s_t yields:

$$-(w_t A_t - s_t)^{-\theta} + \frac{1+r_{t+1}}{1+\rho}((1+r_{t+1})s_t)^{-\theta} = 0$$

- Rearranging and using budget constraints again:

$$C_{1t}^{-\theta} = \frac{1+r_{t+1}}{1+\rho} C_{2t+1}^{-\theta}$$

- This is the consumption Euler equation (same as Ramsey model)

Household Consumption Function

- Combining the budget constraints:

$$C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} = A_t w_t$$

  this is called the intertemporal budget constraint

- Rearranging Euler equation:

$$C_{2t+1} = \left(\frac{1 + r_{t+1}}{1 + \rho} \right)^{1/\theta} C_{1t}$$

- Combining these two:

$$C_{1t} + \frac{(1 + r_{t+1})^{(1-\theta)/\theta}}{(1 + \rho)^{1/\theta}} C_{1t} = A_t w_t$$

Consumption and Saving

- Solving for C_{1t} yields:

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)\theta}} A_t w_t$$

- Young spend some fraction of labor income on time 1 consumption

- Savings:

$$s_t = A_t w_t - C_{1t} = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)\theta}} A_t w_t$$

- Young save a complementary fraction of their labor income

Savings: Comparative Statics

$$s_t = \frac{(1 + r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1 + r_{t+1})^{(1-\theta)/\theta}} A_t w_t$$

- Savings unambiguously increase in wage income

  (Both C_{1t} and C_{2t+1} are normal goods)

- Effect of a change in r_{t+1} is ambiguous

- Change in r_{t+1} both and income effect and a substitution effect
  - Increase in r_{t+1} decreases price of C_{2t+1} (which increases savings)
  - Increase in r_{t+1} increases feasible consumption set

    (which decreases savings)

Savings: Comparative Statics

$$s_t = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} A_t w_t$$

- Savings increase in r_{t+1} if $(1+r_{t+1})^{(1-\theta)/\theta}$ is increasing in r_{t+1}

$$\frac{d}{dr}(1+r)^{(1-\theta)/\theta} = \frac{1-\theta}{\theta}(1+r)^{(1-\theta)/\theta}$$

- Savings increase in r_{t+1} if $\theta < 1$, i.e., if IES > 1
- If IES > 1, substitution effect is strong and overwhelms income effect
- If IES = 1 (log utility) saving is unaffected by r_{t+1}

Evolution of Capital Stock

- Savings of young at time t become capital stock at time $t+1$:

$$K_{t+1} = s_t L_t$$

- Using notation from Romer (2019): $s_t = s(r_{t+1}) A_t w_t$

$$K_{t+1} = s(r_{t+1}) A_t w_t L_t$$

- Dividing through by $A_{t+1} L_{t+1}$ yields:

$$k_{t+1} = \frac{s(r_{t+1}) w_t}{(1+n)(1+g)}$$

where $k_t = K_t/(A_t L_t)$

Evolution of Capital Stock

- Plugging in for w_t and r_{t+1}:

$$k_{t+1} = \frac{s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]}{(1+n)(1+g)}$$

- Implicitly defines k_{t+1} as a function of k_t
- Let's call this function the "savings locus"
- Steady state when $k_{t+1} = k_t$

Evolution of Capital Stock

$$k_{t+1} = \frac{s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]}{(1+n)(1+g)}$$

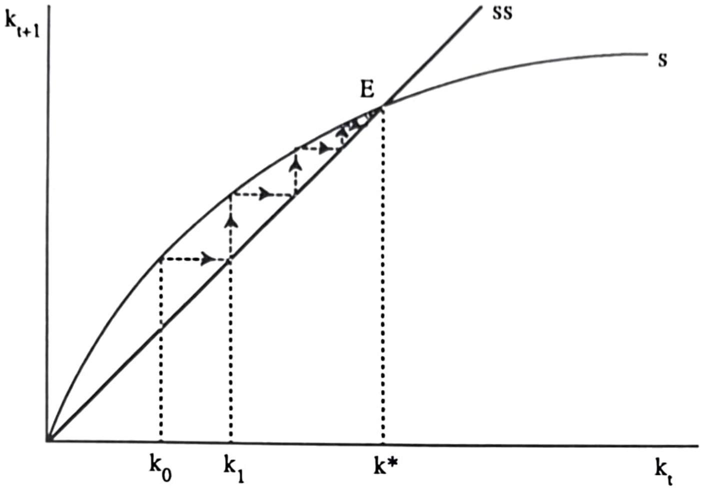
- Let's start by considering special case:
  - Logarithmic utility (i.e., $\theta = 1$)
  - Cobb-Douglas production function ($y = k^\alpha$)
- In this case:

$$s(r_{t+1}) = \frac{1}{2+\rho} \qquad \text{and} \qquad f(k) - kf'(k) = k^\alpha - \alpha k^\alpha = (1-\alpha)k^\alpha$$

- So, we have:

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} k_t^\alpha$$

Evolution of Capital in Special Case



Source: Blanchard and Fischer (1989)

Evolution of Capital in Special Case

- In this special case:
  - There is a single steady state (with positive capital)
  - The steady state is locally stable
- What is it that makes the steady state locally stable?

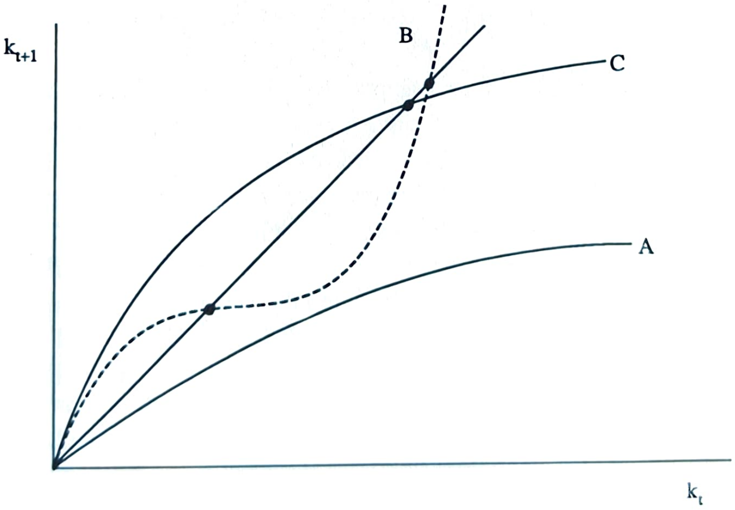
$$\left.\frac{dk_{t+1}}{dk_t}\right|_{ss} < 1$$

Evolution of Capital More Generally

$$k_{t+1} = \frac{s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]}{(1+n)(1+g)}$$

- More generally, the savings locus can take many different shapes
- This can lead to various types of pathologies
  - No steady state with positive capital
  - Multiple steady states with positive capital
  - Multiple equilibria

EVOLUTION OF CAPITAL



Source: Blanchard and Fischer (1989)

Evolution of Capital

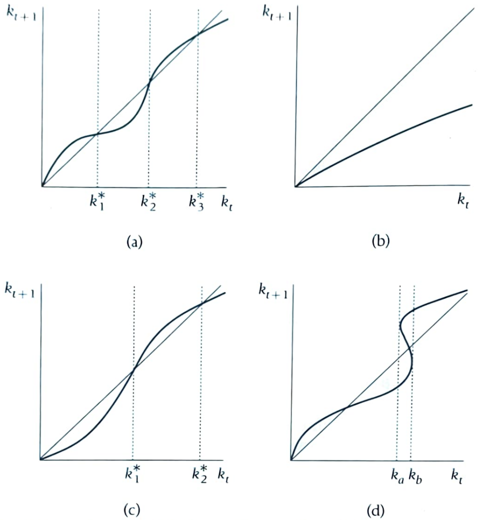


FIGURE 2.12 Various possibilities for the relationship between k_t and k_{t+1}

Source: Romer (2019)

Evolution of Capital

$$k_{t+1} = \frac{s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]}{(1+n)(1+g)}$$

- We can rewrite this as follows:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \underbrace{s(r_{t+1})}_{\text{savings rate}} \underbrace{\frac{f(k_t) - k_t f'(k_t)}{f(k_t)}}_{\text{labor share}} \underbrace{f(k_t)}_{\text{output per person}}$$

- $f(k)$ concave (diminishing returns)
- With log utility $s(r)$ constant, with Cobb-Douglas labor share constant
- Multiple steady states: need sharply rising savings rate or labor share

Welfare in the OLG Model

- Common in macro to compare market outcome to outcome from "planner's problem"
- Conceptually simple in a model with a representative agent
  (planner will maximize that agent's welfare)
- Not as simple in model with heterogeneous agents such as OLG model
- How should planner weight the welfare of different generations?
- However, Pareto optimality is still unambiguous

Welfare in OLG Model

- Is market outcome Pareto optimal in OLG model?
- Turns out this is not necessarily the case
- Economy may accumulate “too much” capital
- If so, it is possible to make everyone better off

Golden Rule Capital

- Let's consider log-utility, Cobb-Douglas production case
- Let's also assume $g = 0$ for simplicity and focus on steady state
- Golden Rule capital stock:
  - Capital stock that yields the highest steady state consumption per effective unit of labor
- Never makes sense to have more capital than Golden Rule capital
  - In this case, less capital would give more consumption
  - "the economy staggers under the weight of the need to maintain the per capita capital stock constant." (Blanchard and Fischer, 1989)

Resource Constraint

- Economy's resource constraint:

$$K_t + F(K_t, A_t L_t) = K_{t+1} + C_{1t} L_t + C_{2t} L_{t-1}$$

- Divide through by $A_t L_t$

$$k_t + f(k_t) = (1+n) k_{t+1} + A_t^{-1} c_t$$

where $c_t = C_{1t} + (1+n)^{-1} C_{2t}$ (weighted average of young and old consumption)

- In steady state with $g = 0$:

$$A^{-1} c = f(k) - nk$$

Golden Rule Capital

- In steady state with $g = 0$

$$A^{-1}c = f(k) - nk$$

- c is maximized when

$$f'(k_{GK}) = n$$

which implicitly gives the Golden Rule capital stock

OLG Market Steady State

- OLG savings locus:

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} k_{t+1}^{\alpha}$$

- With $g = 0$ and in steady state:

$$k^* = \frac{(1-\alpha)}{(1+n)(2+\rho)} k^{*\alpha}$$

which simplifies to

$$k^* = \left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{1/(1-\alpha)}$$

OLG Market Steady State

- If

$$k^* = \left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{1/(1-\alpha)}$$

  then

$$f'(k^*) = \alpha k^{*\alpha-1} = \frac{\alpha}{1-\alpha}(1+n)(2+\rho)$$

- We have ignored depreciation. If $f(k) = k^\alpha - \delta k$:

$$f'(k^*) = \frac{\alpha}{1-\alpha}(1+n)(2+\rho) - \delta$$

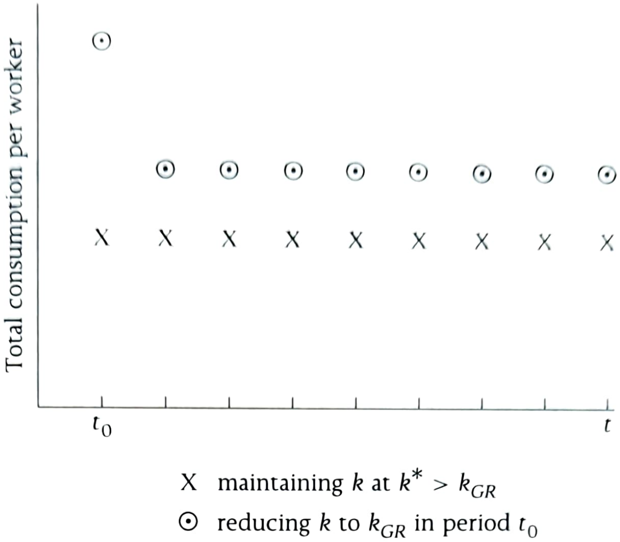
- Recall that $r = f'(k)$. So, we have

$$r^* = \frac{\alpha}{1-\alpha}(1+n)(2+\rho) - \delta$$

Dynamic Inefficiency

- If
$$r^* < n$$
economy has more capital than Golden Rule capital
- This outcome is Pareto inefficient
- Economy is said to be dynamically inefficient
- Suppose in some period t_0, social planner cuts capital to k_{GK}
  - In period t_0: More resources available for consumption due to cut
  - In periods $t > t_0$: More resources available for consumption because nk falls more than $f(k)$
- This policy change can thus make everyone better off

DYNAMIC INEFFICIENCY



Source: Romer (2019)

Dynamic Inefficiency

- Only technology available to households to transfer resources from when they are young to when they are old is capital accumulation
- At the margin, the return on this technology is

$$r = f'(k)$$

- If households are patient enough, they will accumulate capital to the point where

$$r < n$$

- They have no private reason to pay any attention to n

Dynamic Inefficiency

- Society (the government) has another technology for transferring resources from the young to the old

- The government can simply:
  - Take d units from each young
  - Give $(1 + n)d$ units to each old

- Notice that the "return" on this technology is n

  (because the old generation is less populous than the young)

- Must be repeated forever to be a Pareto improvement

- If $r < n$, this "government technology" is better than what is available to people "in the market" (i.e., through saving or bilateral trade)

Dynamic Inefficiency

- With growth in output per person ($g \neq 0$) we get
  - Economy is dynamically efficient if
$$r^* > g + n$$
  - Economy is dynamically inefficient if
$$r^* < g + n$$
- This suggests a way to test dynamic efficiency
- Complication: Which interest rate to use?

  (More on this later.)

Why Inefficiency?

- It may seem puzzling that the market equilibrium in inefficient
- What is the failure of the First Welfare Theorem?
  - All markets are competitive
  - All agents are rational
  - Property rights are well defined and costlessly enforced
- Isn't this enough?

Problem with Infinity

- Things can get complicated when there are an infinite number of agents
- Consider "government technology" discussed above:
  - Take 1 from each young and give $1 + n$ to each old
    (Recall that the young generation is more populous)
  - Do this again next period, and so on
  - If return to saving is less than n, this makes everyone better off
- This scheme only works if there are infinite number of generations
- FWT holds with infinite agents if present value of endowments is finite
  (which does not hold if economy is dynamically inefficient)

Public Debt

- When $r < n$, government can issue debt at no cost
- Suppose government borrows B from each young person
- Next period it owes $(1 + r)B$ to each old.
- Suppose it again borrows B from each young
- Since there are $(1 + n)$ young for each old, it borrows $(1 + n)B$ for each $(1 + r)B$ that it owes
- System is self-financing as long as $r < n$!!
- With growth, relevant issue is perhaps debt-to-GDP ratio. Relevant condition is then $r < g$

More Public Debt, Anyone?

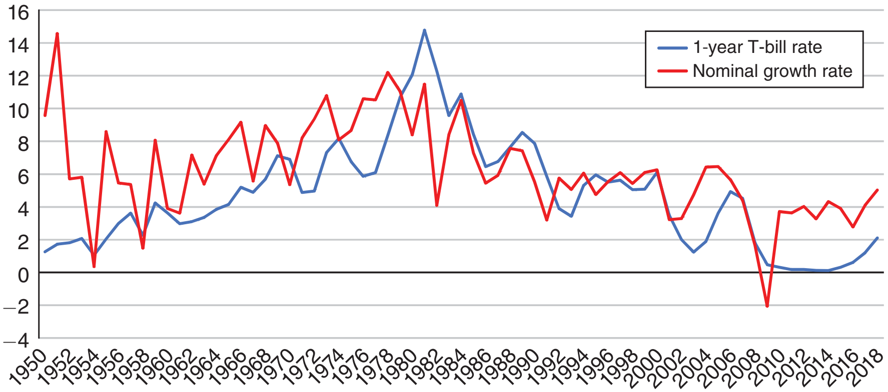


Figure 1. Nominal GDP Growth Rate and 1-Year T-Bill Rate, 1950–2018

Source: Blanchard (2019)

Should We Issue More Public Debt?

- Looks like $r < g$ much of the time
- So, looks like public debt is a “free lunch”
- Does this mean we should issue more?
- Well, public debt “crowds out” private capital
- But with $r < g$, isn’t there overaccumulation of capital?
- Not so fast! Relevant r for dynamic efficiency is not necessarily the same as for debt sustainability

Blanchard (2019):

- Two types of welfare effects of more debt:
  - Lower capital accumulation
  - Induced changes in returns to labor and capital
- Relevant interest rate for first of these:
  - Safe rate because safe rate is the “risk adjusted” rate of return on capital
- Relevant interest rate for second of these:
  - Average (risky) marginal return on capital
- Welfare effects of more debt ambiguous

OLG Economy without Capital/Production

Consider the following simpler setting:

- Two generation OLG model: young and old
- Population growth: $L_t = (1+n)^t$
- No production / **No capital**
- Each young individual endowed with 1 unit of consumption good
- Old receive no endowment
- Consumption good is perishable
- Individuals have standard utility function $U(C_{1t}, C_{2t+1})$

Society's Consumption Possibilities

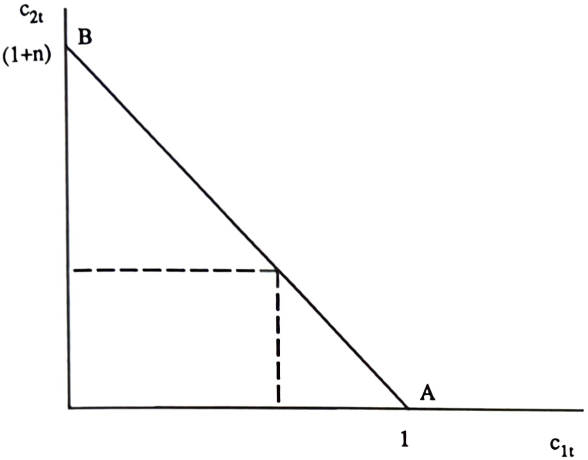


Figure 4.1
Society's consumption possibilities in period *t*

Source: Blanchard and Fischer (1989)

INDIVIDUAL'S LIFETIME C POSSIBILITIES

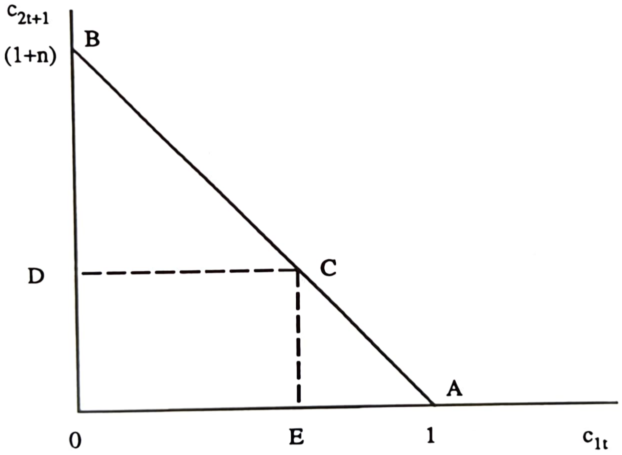


Figure 4.2
Lifetime consumption possibilities for an individual

Source: Blanchard and Fischer (1989)

Barter Equilibrium

- Given this set of possibilities, individual would choose an “interior” point (e.g., C on last slide)
- However, this is not attainable through bilateral trade
- Initial old have nothing to offer
- Initial young would like to exchange goods today for goods next period, but next period’s young not yet born
- No trade possible!!
- “Market outcome” is A on last slide, which is highly Pareto inefficient

Shadow Interest Rate

- Intertemporal trade not possible. So no actual interest rate
- But we can define a “shadow interest rate”
- I.e., interest rate that would make young happy not to trade
- For “normal preferences”, this interest rate would be -100%
  (i.e., if $U'(C) \to \infty$ as $C \to 0$)
- So, this simple case is clearly a case of

$$r < n + g$$

Pay-as-You-Go Government Pension System

- Suppose the government transferred an amount $d < 1$ from young to old from period t onward

- Initial old obviously much better off

- Young and all future generations also better off
    - No longer destitute in old age.

- For moderate d, an increase in d is a Pareto improvement
    - Marginal cost: $U'(1-d)$
    - Marginal benefit $(1+n)U'((1+n)d)(1+\rho)^{-1}$

- Increase in d is a Pareto improvement as long as

$$(1+n)\frac{U'((1+n)d)}{(1+\rho)} > U'(1-d) \quad => \quad 1+n > 1+r$$

(Recall that $(1+r)^{-1} = U'(C_{t+1})/(U'(C_t)(1+\rho))$)

Two kinds of Government Pension Systems

1. Fully Funded
   - Government forces young to save (buy capital)
   - No effect on capital accumulation if people are fully rational (and forced saving is not too large)
   - Increases capital accumulation if people are myopic
2. Pay-as-You-Go
   - Government taxes young and gives proceeds to current old
   - Reduces capital accumulation if people are fully rational
   - Welfare improving even with rational agents if economy is dynamically inefficient ($r < n + g$)

(See Blanchard and Fischer (1989, ch. 3.2))

Intergenerational Risk Sharing

- We have ignored risk up until now
- Risk introduces another source of inefficiency in OLG models
- Efficient intergenerational risk sharing is not possible
- Suppose there is a shock at time t:
  - Efficient to smooth the shock over infinite future
  - This will not happen in an OLG model
- Gov. pension system can help bring about efficient risk sharing
- Ball and Mankiw (2007) take a “first stab” at this

Pure Fiat Money

- Consider again the simple barter economy
- Suppose at $t = 0$ government gives old H units of (completely divisible) inherently useless green pieces of paper
- Let's call these pieces of paper money
- Suppose the old and every future generation believe they will be able to exchange goods for money at price P_t in period t
- P_t is the price level in this economy
- If this is an equilibrium, individuals can trade:
  - Buy money for goods when young
  - Sell money for goods when old

HOUSEHOLD PROBLEM

- Maximize

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

  subject to

$$P_t(1 - C_{1t}) = M_t^d$$

$$P_{t+1}C_{2t+1} = M_t^d$$

- Plugging constraints into objective, differentiating, setting result to zero, and rearranging yields:

$$\frac{M_t^d}{P_t} = \frac{1}{1 + (1+\rho)^{1/\theta}\Pi_{t+1}^{(\theta-1)/\theta}} \qquad \text{where} \qquad \Pi_{t+1} = \frac{P_{t+1}}{P_t}$$

- This is the money demand function, also the savings function

Money Demand

$$\frac{M_t^d}{P_t} = \frac{1}{1 + (1+\rho)^{1/\theta} \Pi_{t+1}^{(\theta-1)/\theta}}$$

- Π_{t+1} is the (inverse of the) rate of return on money
- Effect of an increase in Π_{t+1} on money demand ambiguous
  - If $\theta > 1$, higher Π_{t+1} leads to lower money demand (substitution effect dominates)
  - If $\theta < 1$, higher Π_{t+1} leads to higher money demand (income effect dominates)
- Let's denote money demand function:

$$\frac{M_t^d}{P_t} = L(\Pi_{t+1})$$

Equilibrium with Money

- Money demand equal to money supply:

$$(1+n)^t M_t^d = H$$

- Also true in period $t+1$

$$(1+n)^t M_t^d = (1+n)^{t+1} M_{t+1}^d$$

- Dividing by P_t on both sides:

$$\frac{M_t^d}{P_t} = (1+n)\frac{P_{t+1}}{P_t}\frac{M_{t+1}^d}{P_{t+1}}$$

- Plugging in for money demand:

$$L(\Pi_{t+1}) = (1+n)\Pi_{t+1} L(\Pi_{t+2})$$

$$L(\Pi_t) = (1+n)\Pi_t L(\Pi_{t+1})$$

- Consider a steady state where

$$\Pi_t = \Pi_{t+1} = \bar{\Pi}$$

- Then we have that

$$L(\bar{\Pi}) = (1+n)\bar{\Pi} L(\bar{\Pi})$$

- This simplifies to

$$\bar{\Pi} = (1+n)^{-1}$$

Equilibrium with Money

- This means that there is an equilibrium of the model with a constant inflation rate equal to $(1+n)^{-1}$
- Return on holding money is Π^{-1}
- In equilibrium with constant inflation rate, return on holding money is

$$\bar{\Pi}^{-1} = (1+n)$$

- This is the “golden rule” return on assets in this economy
- Money allows economy to reach efficient equilibrium

Consumption Possibilities with Money

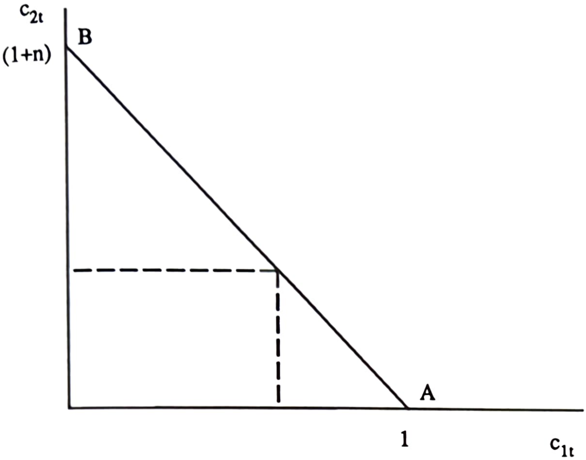


Figure 4.1
Society's consumption possibilities in period t

Source: Blanchard and Fischer (1989)

Fiat Money in OLG Model

- Money is intrinsically worthless in this model
- Yet, it is valued in equilibrium
- Valued because everyone believes it will continue to be valued
- Not just valued, it allows economy to reach Pareto efficient outcome!

Fiat Money and Time Horizon

- For money to be valued, economy must go on forever
- If world ends at time T, money will not be valued in period T
- If money not valued in period T, also not valued in period $T-1$

- Many other equilibria including one were money is not valued
- If people don't believe money will be valued tomorrow, it will not be valued today
- Lots of equilibria in between

Fragility of Monetary Equilibrium

- In simple economy $r < n$
- In economy with assets with $r > n$, there is no monetary equilibrium (Blanchard and Fischer, 1989, ch. 4.1)
- Monetary equilibrium only exists when economy is dynamically inefficient
- Money plays the same role as government pension system

Money and OLG Model

- In OLG model, money is only valued if it is not dominated in rate of return
- In reality, money is dominated in rate of return
- In OLG model, money is a store of value
- In reality, money is a unit of account (and medium of exchange)
- OLG model doesn't capture some crucial features of money

Rational Bubbles

- In OLG model, money can be valued even though it pays no dividends
- Example of a "rational bubble"
- Bubble: Asset that has a higher price than discounted value of future dividends
- Bubbles cannot arise in Ramsey model
- Bubbles can arise in OLG model
  (Tirole, 1985; Blanchard and Fischer, 1989, ch. 5)
- Bubbles can arise in some other settings as well
  (Santos and Woodford, 1997)