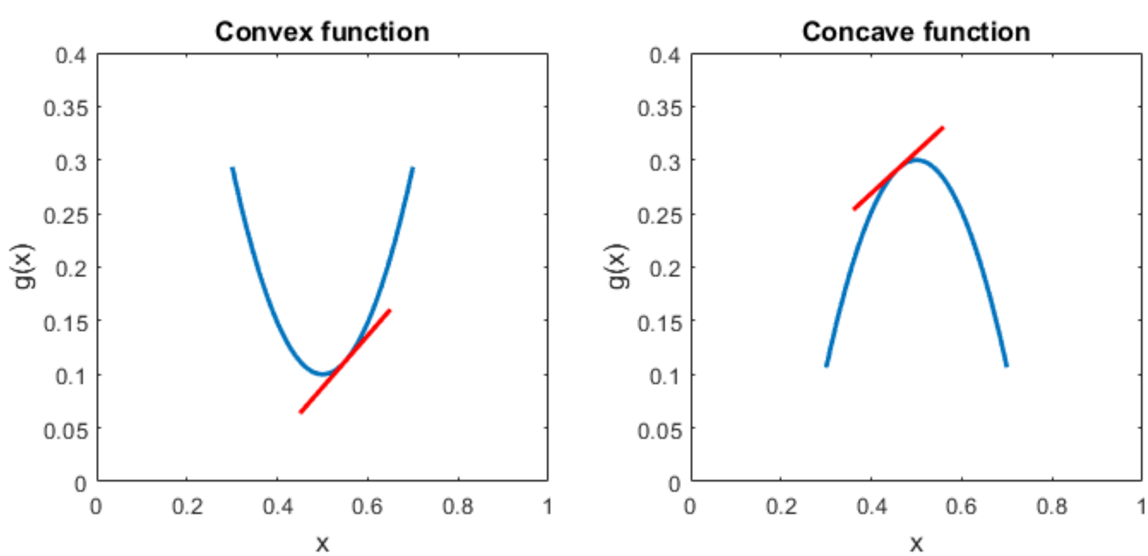
Jensen's inequality

by Marco Taboga, PhD

Jensens's inequality is a probabilistic inequality that concerns the expected value of convex and concave transformations of a random variable.

Convex and concave functions

Jensen's inequality applies to convex and concave functions.



The properties of these functions that are relevant for understanding the proof of the inequality are:

- the tangents of a convex function lie entirely below its graph;
- the tangents of a concave function lie entirely above its graph.

Also remember that a differentiable function is:

- (strictly) convex if its second derivative is (strictly) positive;
- (strictly) concave if its second derivative is (strictly) negative.

Statement

The following is a formal statement of the inequality.

> **Proposition** Let X be an integrable random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function such that
>
> $$Y = g(X)$$
>
> is also integrable. Then, the following inequality, called Jensen's inequality, holds:
>
> $$\mathrm{E}[g(X)] \geq g(\mathrm{E}[X])$$

Proof

If the function g is strictly convex and X is not almost surely constant, then we have a strict inequality:

$$\mathrm{E}[g(X)] > g(\mathrm{E}[X])$$

Proof

If the function g is concave, then

$$\mathrm{E}[g(X)] \leq g(\mathrm{E}[X])$$

Proof

If the function g is strictly concave and X is not almost surely constant, then

$$\mathrm{E}[g(X)] < g(\mathrm{E}[X])$$

Proof

Example

Suppose that a strictly positive random variable X has expected value

$$\mathrm{E}[X] = 1$$

and it is not constant with probability one.

What can we say about the expected value of $\ln(X)$, by using Jensen's inequality?

The natural logarithm is a strictly concave function because its second derivative

$$\frac{d^2}{dx^2}\ln(x) = -x^{-2}$$

is strictly negative on its domain of definition.

As a consequence, by Jensen's inequality, we have

$$\mathrm{E}[\ln(X)] < \ln(\mathrm{E}[X]) = \ln(1) = 0$$

Therefore, $\ln(X)$ has a strictly negative expected value.

Important applications

Jensen's inequality has many applications in statistics. Two important ones are in the proofs of:

- the non-negativity of the Kullback-Leibler divergence;
- the information inequality concerning the expected value of the log-likelihood.

Other inequalities

If you like this page, StatLect has other pages on probabilistic inequalities:

- Markov's inequality;
- Chebyshev's inequality.

Solved exercises

Below you can find some exercises with explained solutions.

Exercise 1

Let X be a random variable having finite mean and variance $\sigma^2 > 0$.

Use Jensen's inequality to find a bound on the expected value of X^2.

Solution

Exercise 2

Let X be a positive integrable random variable.

Find a bound on the mean of $\sqrt{X}$.

Solution

How to cite

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