Econ 7710
Assignment 4

The due date for this assignment is Friday October 20th

1. Suppose that $X \sim N(0,1)$ and $Y \sim N(0,1)$ and X and Y are independent. Find the distribution of random variable $Z = X/Y$ and determine which moments of this random variable exist.

2. Suppose that $\{X_n\}_{n=1}^{\infty}$ is such that $X_n \xrightarrow{d} X$, where $X \sim N(0,1)$. Suppose that $Y_n = X_n$ for all $n \geq 1$.

   (a) Find the distribution limit of Y_n

   (b) Consider the distribution limit of Y_n, $Y_n \xrightarrow{d} Y$. Prove or disprove that $X_n + Y_n \xrightarrow{d} X + Y$. Comment your findings.

3. The *median* of the distribution of random variable X is the number $q_{.5}$ that solves

$$\inf_q \left\{ P(X \leq q) \geq \frac{1}{2} \right\}$$

   Suppose that for the sequence of random variables X_n there exists a numeric sequence a_n such that $X_n - a_n \xrightarrow{p} 0$. Let $q_{.5}^n$ be the median of the distribution of X_n.

   (a) Prove that $\lim_{n\to\infty} (q_{0.5}^n - a_n) = 0$.

   (b) Prove or disprove that $\lim_{n\to\infty} (E[X_n] - a_n) = 0$

4. Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables and $X_n \xrightarrow{p} X$. Prove that X has a degenerate distribution.

1. Suppose that $X \sim N(0,1)$ and $Y \sim N(0,1)$ and X and Y are independent. Find the distribution of random variable $Z = X/Y$ and determine which moments of this random variable exist.

Let $U = Z = X/Y$
& $V = |Y|$

Let $A_1 = \{(x,y) : y > 0\}$
$A_2 = \{(x,y) : y < 0\}$
$A_0 = \{(x,y) : y = 0\}$

A_0, A_1, A_2 form partition of $A = \mathbb{R}^2$

$\Rightarrow$ $B = \{(u,v) : v > 0\}$ is the image of both A_1 & A_2.

$\Rightarrow$ Inverse transformation from B to A_1 & A_2 :-

$x = h_{11}(u,v) = uv$ | $x = h_{12}(u,v) = -uv$
$y = h_{21}(u,v) = v$ | $y = h_{22}(u,v) = -v$

$\Rightarrow$ $f_{XY}(x,y) = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}$

$f_{U,V}(u,v) = \frac{v}{\pi} e^{-(u^2+1)v^2/2}$, $-\infty < u < \infty$, $0 < v < \infty$

$f_U(u) = \int_0^\infty \frac{v}{\pi} e^{-(u^2+1)v^2/2} \, dv$

$= \frac{1}{\pi(u^2+1)}$ $-\infty < u < \infty$

No moments exist.

2. Suppose that $\{X_n\}_{n=1}^{\infty}$ is such that $X_n \xrightarrow{d} X$, where $X \sim N(0,1)$. Suppose that $Y_n = X_n$ for all $n \geq 1$.

(a) Find the distribution limit of Y_n

(b) Consider the distribution limit of Y_n, $Y_n \xrightarrow{d} Y$. Prove or disprove that $X_n + Y_n \xrightarrow{d} X + Y$. Comment your findings.

(a) If for the distⁿ function of r.v. X_n & X,
$F_n \Rightarrow F$, then $X_n \xrightarrow{d} X$.

We know for $\{X_n\}_{n=1}^{\infty}$ $X_n \xrightarrow{d} X$

& $Y_n = \{X_n\}$ $\forall$ $n \geq 1$

$\Rightarrow$ $\{Y_n\}_{n=1}^{\infty} = \{X_n\}_{n=1}^{\infty}$

$Y_n \xrightarrow{d} X$

(b) $Y_n \xrightarrow{d} Y$

$X_n + Y_n \xrightarrow{d} X + Y$

This is false.

Let $Y_n = -X_n$

then, $X_n + Y_n \not\xrightarrow{d} X + Y$

3. The *median* of the distribution of random variable X is the number $q_{.5}$ that solves

$$\inf_q \left\{ P(X \le q) \ge \frac{1}{2} \right\}$$

Suppose that for the sequence of random variables X_n there exists a numeric sequence a_n such that $X_n - a_n \xrightarrow{p} 0$. Let $q^n_{.5}$ be the median of the distribution of X_n.

(a) Prove that $\lim_{n\to\infty} (q^n_{0.5} - a_n) = 0$.

(b) Prove or disprove that $\lim_{n\to\infty} (E[X_n] - a_n) = 0$

$X_n - a_n \xrightarrow{P} 0$

Let $q^n_{.5}$ be the median of the dist^n of X_n

(a) Prove that $\lim_{n\to\infty} (q^n_{0.5} - a_n) = 0$

$\exists$ exists a_n s.t.

$$X_n - a_n \xrightarrow{P} 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n - a_n| > \varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} P\left(\left| \frac{X_n}{2} + \frac{X_n}{2} - a_n \right| > \varepsilon \right) = 0$$

$$\Rightarrow \lim_{n\to\infty} P\left(\left| 2\frac{X_n}{2} - a_n \right| > \varepsilon \right) = 0$$

$$\Rightarrow \lim_{n\to\infty} P\left(\left| 2q^n_{0.5} - a_n \right| > \varepsilon \right) = 0$$

$$\Rightarrow \lim_{n\to\infty} (2q^n_{0.5} - a_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} (q^n_{0.5} - a_n) = 0$$

$\square$

(b)

(b) Prove or disprove that $\lim_{n\to\infty} (E[X_n] - a_n) = 0$

$$\lim_{n\to\infty} \left(E\left[X_n\right] - a_n\right) = 0$$

The above is not true.

We know that $X_n - a_n \xrightarrow{P} 0$. Here, what matters is the where the majority of the mass is located.

But in the case of $\lim_{n\to\infty} \left(E(X_n] - a_n\right)$, each mass pt. matters as it gets multiplied with its probability to find the expectation.

4. Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables and $X_n \xrightarrow{p} X$. Prove that X has a degenerate distribution.

A degenerate distribution has all the probability mass at one pt.

$$X_n \xrightarrow{p} X$$

$$\Rightarrow X_n \xrightarrow{d} X$$

$$\Rightarrow \phi_{X_n} \longrightarrow \phi_X$$

If $\{x_i\}$ & $\{x_j\}$ are sub-sequences s.t. $x_i + x_j \to X$

$$\Rightarrow x_i \xrightarrow{p} X$$

$$\Rightarrow x_j \xrightarrow{p} X$$

$$\Rightarrow \phi_{x_i + x_j}(t) = \phi_x(t)$$

$$\rightarrow \lim_{n\to\infty} \phi_{x_i + x_j}(t) = \phi_{2x}(t) = E[e^{i2tx}] \quad -(1)$$

And, $\phi_{x_i + x_j}(t) = \phi_{x_i}(t) \cdot \phi_{x_j}(t)$

$$\lim_{n\to\infty} [\phi_{x_i}(t) \cdot \phi_{x_j}(t) = [\phi_x(t)]^2$$

$$= E[e^{itx}]^2 \quad -(2)$$

from (1) & (2)

$$\Rightarrow E[e^{i2tx}] = E[e^{itx}]^2$$

$$\Rightarrow E[(e^{itx})^2] - E[e^{itx}]^2 = 0$$

$$\Rightarrow Var(e^{itx}) = 0$$

$\rightarrow \phi_X = e^{itc} = P(X=c) = 1$ where c is a constant

$\square$