Solutions Manual

for

Microeconomic Theory

Mas-Colell, Whinston, and Green

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New York Oxford OXFORD UNIVERSITY PRESS 1997

INTRODUCTION.

We could never overestimate the amount of work which needed to be done to complete this solution book, but the satisfaction in seeing the finished product more than makes up for the many hours of work. We have tried to be both concise and exhaustive, and we hope that these two objectives do not conflict too often in this solution book. On rare occasions, we refer the reader to a book or article containing a well-presented solution.

Chiaki Hara has done a lion's share of the work, preparing solutions for parts I and IV. Ilya Segal has provided the solutions for chapters 10 and 22. Steve Tadelis has prepared the solutions for chapters 11, 13, 14, 21 and 23, and has completed the work on part II. Finally, Ilya and Steve have together prepared solutions for chapter 12. Some of the early work on the solutions had been done by Marc Nachman, and we thank him for laying down foundations for solutions for part II and chapter 12.

We would like to thank Andreu Mas-Colell and Mike Whinston for many hours of discussions. We are also thankful to the many teaching fellows who taught Ec2010a/b, the Harvard graduate sequence in microeconomic theory, for their input over the years when earlier versions of the textbook had been used for instruction. The list of these teaching fellows is too long to include here. While some errors surely remain, we hope that both students and teachers will benefit from our solutions, which make an excellent textbook even more useful.

Finally, some personal thanks: Chiaki Hara thanks his parents and colleagues at various places where he has spent the last four years. Ilya Segal thanks Olga, for her unwavering support during all times. Steve Tadelis thanks Irit, for always being there to encourage and support.

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June 1996

CHAPTER 1

1.B.1 Since $y \succ z$ implies $y \succsim z$, the transitivity implies that $x \succsim z$. Suppose that $z \succsim x$. Since $y \succsim z$, the transitivity then implies that $y \succsim x$. But this contradicts $x \succ y$. Thus we cannot have $z \succsim x$. Hence $x \succ z$.

1.B.2 By the completeness, $x \succsim x$ for every $x \in X$. Hence there is no $x \in X$ such that $x \succ x$. Suppose that $x \succ y$ and $y \succ z$, then $x \succ y \succsim z$. By (iii) of Proposition 1.B.1, which was proved in Exercise 1.B.1, we have $x \succ z$. Hence $\succ$ is transitive. Property (i) is now proved.

As for (ii), since $x \succsim x$ for every $x \in X$, $x \sim x$ for every $x \in X$ as well. Thus $\sim$ is reflexive. Suppose that $x \sim y$ and $y \sim z$. Then $x \succsim y$, $y \succsim z$, $y \succsim x$, and $z \succsim y$. By the transitivity, this implies that $x \succsim z$ and $z \succsim x$. Thus $x \sim z$. Hence $\sim$ is transitive. Suppose x that $\sim$ y. Then $x \succsim y$ and $y \succsim x$. Thus $y \succsim x$ and $x \succsim y$. Hence $y \sim x$. Thus $\sim$ is symmetric Property (ii) is now proved.

1.B.3 Let $x \in X$ and $y \in X$. Since $u(\cdot)$ represents $\succsim$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Since $f(\cdot)$ is strictly increasing, $u(x) \geq u(y)$ if and only if $v(x) \geq v(y)$. Hence $x \succsim y$ if and only if $v(x) \geq v(y)$. Therefore $v(\cdot)$ represents $\succsim$.

1.B.4 Suppose first that $x \succsim y$. If, furthermore, $y \succsim x$, then $x \sim y$ and hence $u(x) = u(y)$. If, on the contrary, we do not have $y \succsim x$, then $x \succ y$. Hence $u(x) > u(y)$. Thus, if $x \succsim y$, then $u(x) \geq u(y)$.

Suppose conversely that $u(x) \geq u(y)$. If, furthermore, $u(x) = u(y)$, then

$x \sim y$ and hence $x \succsim y$. If, on the contrary, $u(x) > u(y)$, then $x \succ y$, and hence $x \succsim y$. Thus, if $u(x) \geq u(y)$, then $x \succsim y$. So $u(\cdot)$ represents $\succsim$.

1.B.5 First, we shall prove by induction on the number N of the elements of X that, if there is no indifference between any two different elements of X, then there exists a utility function. If $N = 1$, there is nothing to prove: Just assign any number to the unique element. So let $N > 1$ and suppose that the above assertion is true for $N - 1$. We will show that it is still true for N. Write $X = \{x_1,\dots,x_{N-1},x_N\}$. By the induction hypothesis, $\succsim$ can be represented by a utility function $u(\cdot)$ on the subset $\{x_1,\dots,x_{N-1}\}$. Without loss of generality we can assume that $u(x_1) > u(x_2) > \dots > u(x_{N-1})$.

Consider the following three cases:

Case 1: For every $i < N$, $x_N \succ x_i$.

Case 2: For every $i < N$, $x_i \succ x_N$.

Case 3: There exist $i < N$ and $j < N$ such that $x_i \succ x_N \succ x_j$.

Since there is no indifference between two different elements, these three cases are are exhaustive and mutually exclusive. We shall now show how the value of $u(x_N)$ should be determined, in each of the three cases, for $u(\cdot)$ to represent $\succsim$ on the whole X.

If Case 1 applies, then take $u(x_N)$ to be larger than $u(x_1)$. If Case 2 applies, take $u(x_N)$ to be smaller than $u(x_{N-1})$. Suppose now that Case 3 applies. Let $I = \{i \in \{1,\dots, N - 1\}: x_i \succ x_{N+1}\}$ and $J = \{j \in \{1,\dots, N - 1\}: x_{N+1} \succ x_j\}$. Completeness and the assumption that there is no indifference implies that $I \cup J = \{1,\dots, N - 1\}$. The transitivity implies that both I and J are "intervals," in the sense that if $i \in I$ and $i' < i$, then $i' \in I$; and if $j \in J$ and $j' > j$, then $j' \in J$. Let $i^* = \max I$, then $i^* + 1 = \min J$. Take

$u(x_N)$ to lie in the open interval $(u(x_{i^*+1}),u(x_{i^*}))$. Then it is easy to see that $u(\cdot)$ represents $\succsim$ on the whole X.

Suppose next that there may be indifference between some two elements of $X = \{x_1,\ldots, x_N\}$. For each $n = 1,\ldots,N$, define $X_n = \{x_m \in X: x_m \sim x_n\}$. Then, by the reflexivity of $\sim$ (Proposition 1.B.1(ii)), $\cup_{n=1}^{N} X_n = X$. Also, by the transitivity of $\sim$ (Proposition 1.B.1(ii)), if $X_n \neq X_m$, then $X_n \cap X_m \neq \emptyset$. So let M be a subset of $\{1,\ldots,N\}$ such that $X = \cup_{m\in M} X_m$ and $X_m \neq X_n$ for any $m \in M$ and any $n \in M$ with $m \neq n$. Define an relation $\succsim^*$ on $\{X_m: m \in M\}$ by letting $X_m \succsim^* X_n$ if and only if $x_m \succsim x_n$. In fact, by the definition of M, there is no indifference between two different elements of $\{X_m: m \in M\}$. Thus, by the preceding result, there exists a utility function $u^*(\cdot)$ that represents $\succsim^*$. Then define $u: X \to \mathbb{R}$ by $u(x_n) = u^*(X_m)$ if $m \in M$ and $x_n \in X_m$. It is easy to show that, by the transitivity, $u(\cdot)$ represents $\succsim$.

1.C.1 If $y \in C(\{x,y,z\})$, then the WA would imply that $y \in C(\{x,y\})$. But contradicts the equality $C(\{x,y\}) = \{x\}$. Hence $y \notin C(\{x,y,z\})$. Thus $C(\{x,y,z\}) \in \{\{x\},\{z\},\{x,z\}\}$.

1.C.2 The property in the question are equivalent to the following property: If $B \in \mathcal{B}$, $B' \in \mathcal{B}$, $x \in B$, $y \in B$, $x \in B'$, $y \in B'$, $x \in C(B)$, and $y \in C(B')$, then $x \in C(B')$ and $y \in C(B)$. We shall thus prove the equivalence between this property and the Weak Axiom.

Suppose first that the Weak Axiom is satisfied. Assume that $B \in \mathcal{B}$, $B' \in \mathcal{B}$, $x \in B$, $y \in B$, $x \in B'$, $y \in B'$, $x \in C(B)$, and $y \in C(B')$. If we apply the Weak Axiom twice, we obtain $x \in C(B')$ and $y \in C(B)$. Hence the above property is also satisfied.

Suppose conversely that the above property is satisfied. Let $B \in \mathcal{B}$, $x \in B$, $y \in B$, $x \in B'$, and $x \in C(B)$. Furthermore, let $B' \in \mathcal{B}$, $x \in B'$, $y \in B'$, and $y \in C(B')$. Then the above condition implies that $x \in C(B')$ (and $y \in C(B)$). Thus the Weak Axiom is satisfied.

1.C.3 (a) Suppose that $x \succ^* y$, then there is some $B \in \mathcal{B}$ such that $x \in B$, $y \in B$, $x \in C(B)$, and $y \notin C(B)$. Thus $x \succsim^* y$. Suppose that $y \succsim^* x$, then there exists $B \in \mathcal{B}$ such that $x \in B$, $y \in B$ and $x \in C(B)$. But the Weak Axiom implies that $y \in C(B)$, which is a contradiction. Hence if $x \succ^* y$, then we cannot have $y \succsim^* x$. Hence $x \succ^{**} y$.

Converseiy, suppose that $x \succ^{**} y$, then $x \succsim^* y$ but not $y \succsim^* x$. Hence there is some $B \in \mathcal{B}$ such that $x \in B$, $y \in B$, $x \in C(B)$ and if $x \in B'$ and $y \in B'$ for any $B' \in \mathcal{B}$, then $y \notin C(B')$. In particular, $x \in C(B)$ and $y \notin C(B)$. Thus $x \succ^* y$.

The equality of the two relation is not guaranteed without the WA. As can be seen from the above proof, the WA is not necessary to guarantee that if $x \succ^{**} y$, then $x \succ^* y$. But the converse need not be true, as shown by the following example. Define $X = \{x,y,z\}$, $\mathcal{B} = \{\{x,y\},\{x,y,z\}\}$, $C(\{x,y\}) = \{x\}$, and $C(\{x,y,z\}) = \{y\}$. Then $x \succ^* y$ and $y \succ^* x$. But neither $x \succ^* y$ nor $y \succ^* x$.

(b) The relation $\succ^*$ need not be transitive, as shown by the following example. Define $X = \{x,y,z\}$, $\mathcal{B} = \{\{x,y\},\{y,z\}\}$, $C(\{x,y\}) = \{x\}$ and $C(\{y,z\}) = \{y\}$. Then $x \succ^* y$ and $y \succ^* z$. But we do not have $x \succsim^* z$ (because neither of the two sets in $\mathcal{B}$ includes $\{x,z\}$) and hence we do not have $x \succ^* z$ either.

(c) According to the proof of Proposition 1.D.2, if $\mathcal{B}$ includes all three-element subset of X, then $\succsim^*$ is transitive. By Proposition 1.B.1(i), $\succ^{**}$ is transitive. Since $\succ^*$ is equal to $\succ^{**}$, $\succ^*$ is also transitive.

An alternative proof is as follows: Let $x \in X$, $y \in X$, $z \in X$, $x \succ^* y$, and $y \succ^* z$. Then $\{x,y,z\} \in \mathcal{B}$ and, by (a), $x \succ^{**} y$, and $y \succ^{**} z$. Hence we have neither $y \succsim^* x$ nor $z \succsim^* y$. Since $\succsim^*$ rationalizes $(\mathcal{B},C(\cdot))$, this implies that $y \notin C(\{x,y,z\})$ and $z \notin C(\{x,y,z\})$. Since $C(\{x,y,z\}) \neq \varnothing$, $C(\{x,y,z\}) = \{x\}$. Thus $x \succ^* z$.

1.D.1 The simplest example is $X = \{x,y\}$, $\mathcal{B} = \{\{x\},\{y\}\}$, $C(\{x\}) = \{x\}$, $C(\{y\}) = \{y\}$. Then any rational preference relation of X rationalizes $C(\cdot)$.

1.D.2 By Exercise 1.B.5, let $u(\cdot)$ be a utility representation of $\succsim$. Since X is finite, for any $B \subset X$ with $B \neq \varnothing$, there exists $x \in B$ such that $u(x) \geq u(y)$ for all $y \in B$. Then $x \in C^*(B,\succsim)$ and hence $C^*(B,\succsim) \neq \varnothing$. (A direct proof with no use of utility representation is possible, but it is essentially the same as the proof of Exercise 1.B.5.)

1.D.3 Suppose that the Weak Axiom holds. If $x \in C(X)$, then $x \in C(\{x,z\})$, which contradicts the equality $C(\{x,z\}) = \{z\}$. If $y \in C(X)$, then $y \in C(\{x,y\})$, which contradicts $C(\{x,y\}) = \{x\}$. If $z \in C(X)$, then $z \in C(\{y,z\})$, which contradicts $C(\{x,z\}) = \{y\}$. Thus $(\mathcal{B},C(\cdot))$ must violate the Weak Axiom.

1.D.4 Let $\succsim$ rationalize $C(\cdot)$ relative to $\mathcal{B}$. Let $x \in C(B_1 \cup B_2)$ and $y \in C(B_1) \cup C(B_2)$, then $x \succsim y$ because $B_1 \cup B_2 \supset C(B_1) \cup C(B_2)$. Thus $x \in C(C(B_1) \cup C(B_2))$.

Let $x \in C(C(B_1) \cup C(B_2))$ and $y \in B_1 \cup B_2$, then there are four cases:

Case 1. $x \in C(B_1)$, $y \in B_1$.

Case 2. $x \in C(B_1)$, $y \in B_2$.

Case 3. $x \in C(B_2)$, $y \in B_1$.

Case 4. $x \in C(B_2)$, $y \in B_2$.

If either Case 1 or 4 is true, then $x \succsim y$ follows directly from rationalizability. If Case 2 is true, then pick any $z \in C(B_2)$. Then $z \succsim y$. Since $x \in C(C(B_1) \cup C(B_2))$, $x \succsim z$. Hence, by the transitivity, $x \succsim y$. If Case 3 is true, then pick any $z \in C(B_1)$ and do the same argument as for Case 2.

1.D.5 (a) Assign probability 1/6 to each of the six possible preferences, which are $x \succ y \succ z$, $x \succ z \succ y$, $y \succ x \succ z$, $y \succ z \succ x$, $z \succ x \succ y$, and $z \succ y \succ x$.

(b) If the given stochastic choice function were rationalizable, then the probability that at least one of $x \succ y$, $y \succ z$, and $z \succ x$ holds would be at most $3 \times (1/4) = 3/4$. But, in fact, at least one of the three relations always holds, because, if the first two do not hold, then $y \succ x$ and $z \succ y$. Hence the transitivity implies the third. Thus, the given stochastic choice function is not rationalizable.

(c) The same argument as in (b) can be used to show that $\alpha \geq 1/3$. Since $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\alpha,\ 1 - \alpha)$ is equivalent to $C(\{y,x\}) = C(\{z,y\}) = C(\{x,z\}) = (1 - \alpha,\ \alpha)$, if we apply the same argument as in (b) to $y \succ x$, $z \succ y$, and $x \succ z$, then we can establish $1 - \alpha \geq 1/3$, that is, $\alpha \leq 2/3$. Thus, in order for the given stochastic choice function is rationalizable, it is necessary that $\alpha \in [1/3,2/3]$. Moreover, this condition is actually sufficient: For any $\alpha \in [1/3,2/3]$, assign probability $\alpha - 1/3$ to each of $x \succ$

$y \succ z$, $y \succ z \succ x$, and $z \succ x \succ y$; assign probability $2/3 - \alpha$ to each of $x \succ z \succ y$, $y \succ x \succ z$, and $z \succ y \succ x$. Then we obtain the given stochastic choice function.

CHAPTER 2

2.D.1 Let p_2 be the price of the consumption good in period 2, measured in units of the consumption good in period 1. Let x_1, x_2 be the consumption levels in periods 1 and 2, respectively. Then his lifetime Walrasian budget set is equal to $\{x \in \mathbb{R}^2_+ : x_1 + p_2x_2 \le w\}$.

2.D.2 $\{(x,h) \in \mathbb{R}^2_+ : h \le 24,\ px + h \le 24\}$.

2.D.3 (a) No. In fact, the budget set consists of the two points, each of which is the intersection of the budget line and an axis.

(b) Let $x \in B_{p,w}$, $x' \in B_{p,w}$, and $\lambda \in [0,1]$. Write $x'' = \lambda x + (1 - \lambda)x'$. Since X is convex, $x'' \in X$. Moreover, $p \cdot x'' = \lambda(p \cdot x) + (1 - \lambda)(p \cdot x') \le \lambda w + (1 - \lambda)w = w$. Thus $x'' \in B_{p,w}$.

2.D.4 It follows from a direct calculation that consumption level M can be attained by $(8 + (M - 8s)/s')$ hours of labor. It follows from the definition that $(24,0)$ and $(16 - (M - 8s)/s', M)$ are in the budget set. But their convex combination of these two consumption vectors with ratio

$$\left(\frac{\frac{M - 8s}{s'}}{8 + \frac{M - 8s}{s'}},\ \frac{8}{8 + \frac{M - 8s}{s'}} \right)$$

is not in the budget set: the amount of leisure of this combination equals to 16 (so the labor is eight hours), but the amount of the consumption good is

$$M \frac{8}{8 + \frac{M - 8s}{s'}} > M \frac{8}{8 + \frac{M - 8s}{s}} = M \frac{8}{M/s} = 8s.$$

2.E.1 The homogeneity can be checked as follows:

$$x_1(\alpha p,\alpha w) = \frac{\alpha p_2}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_1} = \frac{p_2}{p_1 + p_2 + p_3}\frac{w}{p_1} = x_1(p,w),$$

$$x_2(\alpha p,\alpha w) = \frac{\alpha p_3}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_2} = \frac{p_3}{p_1 + p_2 + p_3}\frac{w}{p_2} = x_2(p,w),$$

$$x_3(\alpha p,\alpha w) = \frac{\alpha p_1}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_3} = \frac{p_1}{p_1 + p_2 + p_3}\frac{w}{p_3} = x_3(p,w).$$

To see if the demand function satisfies Walras' law, note that

$$p\cdot x(p,w) = \frac{\beta p_1 + p_2 + p_3}{p_1 + p_2 + p_3}w.$$

Hence $p\cdot x(p,w) = w$ if and only if $\beta = 1$. Therefore the demand function satisfies Walras' law if and only if $\beta = 1$.

2.E.2 Multiply by p_k/w both sides of (2.E.4), then we obtain

$$\textstyle\sum_{\ell=1}^{L}(p_\ell x_\ell(p,w)/w)(\partial x_\ell(p,w)/\partial p_k)(p_k/x_\ell(p,w)) + p_k x_k(p,w)/w = 0.$$

Hence $\sum_{\ell=1}^{L} b_\ell(p,w)\varepsilon_{\ell k}(p,w) + b_k(p,w) = 0$.

By (2.E.6), $\sum_{\ell=1}^{L}(p_\ell x_\ell(p,w)/w)(\partial x_\ell(p,w)/\partial w)(w/x_\ell(p,w)) = 1$. Hence $\sum_{\ell=1}^{L} b_\ell(p,w)\varepsilon_{\ell w}(p,w) = 1$.

2.E.3 There are two ways to verify that $p\cdot D_p x(p,w)p = -w$.

One way is to post-multiply (2.E.5) by p, then $p\cdot D_p x(p,w)\,p + w = 0$ by Walras' law.

The other way is to pre-multiply (2.E.1) by p^T, then $p\cdot D_p x(p,w)p + p\cdot D_w x(p,w)w = 0$. By Proposition 2.E.3, this is equal to $p\cdot D_p x(p,w)p + w = 0$.

An interpretation is that, when all prices are doubled, in order for the consumer to stay at the same consumption, it is necessary to increase his wealth by w.

2.E.4 By differentiating the equation $x(p,\alpha w) = \alpha x(p,w)$ with respect to α and evaluating at $\alpha = 1$, we obtain $wD_w x(p,w) = x(p,w)$. Hence $D_w x(p,w) = (1/w)x(p,w)$. Hence $\varepsilon_{\ell w} = (\partial x_\ell(p,w)/\partial w)(w/x_\ell(p,w)) = 1$. This means that an one-percent increase in wealth will increase the consumption level for all goods by one percent.

Since $(1/w)x(p,w) = x(p,1)$ by the homogeneity assumption, $D_w x(p,w)$ is a function of p only. The assumption also implies that the wealth expansion path, $E_p = \{x(p,w): w > 0\}$, is a ray going through $x(p,1)$.

2.E.5 Since $x(p,w)$ is homogeneous of degree one with respect to w, $x(p,\alpha w) = \alpha x(p,w)$ for every $\alpha > 0$. Thus $x_\ell(p,w) = x_\ell(p,1)w$. Since $\partial x_\ell(p,1)/\partial p_k = \partial \varphi_\ell(p)/\partial p_k = 0$ whenever $k \neq \ell$, $x_\ell(p,1)$ is actually a function of p_ℓ alone. So we can write $x_\ell(p,w) = x_\ell(p_\ell)$. Since $x(p,w)$ is homogeneous of degree zero, $x_\ell(p_\ell)$ must be homogeneous of degree - 1 (in p_ℓ). Hence there exists $\alpha_\ell > 0$ such that $x_\ell(p_\ell) = \alpha_\ell/p_\ell$. By Walras' law, $\sum_\ell p_\ell(\alpha_\ell/p_\ell)w = w\sum_\ell \alpha_\ell = w$. We must thus have $\sum_\ell \alpha_\ell = 1$.

2.E.6 When $\alpha = 1$, Walras' law and homogeneity hold. Hence the conclusions of Propositions 2.E.1 - 2.E.3 hold.

2.E.7 By Walras' law,

$$x_2 = (w - p_1x_1)/p_2 = w/p_2 - (p_1/p_2)(\alpha w/p_1) = (1 - \alpha)w/p_2.$$

This demand function is thus homogeneous of degree zero.

2.E.8 For the first part, note that

$$\ln x_\ell(p,w) = \ln x_\ell(\exp(\ln p_1),\ldots,\exp(\ln p_L),\exp(\ln w)).$$

Thus, by the chain rule,

$$\frac{d(\ln x_\ell(p,w))}{d(\ln p_k)} = \frac{\frac{\partial x_\ell}{\partial p_k}(p,w)\cdot\exp(\ln p_k)}{x_\ell(p,w)} = \frac{\frac{\partial x_\ell}{\partial p_k}(p,w)\cdot p_k}{x_\ell(p,w)} = \varepsilon_{\ell k}(p,w).$$

Similarly,

$$\frac{d(\ln x_\ell(p,w))}{d(\ln w)} = \frac{\frac{\partial x_\ell}{\partial w}(p,w)\cdot\exp(\ln w)}{x_\ell(p,w)} = \frac{\frac{\partial x_\ell}{\partial w}(p,w)w}{x_\ell(p,w)} = \varepsilon_{\ell w}(p,w).$$

Since $\alpha_1 = d(\ln x_\ell(p,w))/d(\ln p_1)$, $\alpha_2 = d(\ln x_\ell(p,w))/d(\ln p_2)$, and $\alpha_3 = d(\ln x_\ell(p,w))/d(\ln w)$, the assertion is established.

2.F.1 We proved in Exercise 1.C.2 that Definition 1.C.1 and the property in the exercise is equivalent. It is easy to see that the latter is equivalent to the following property: For every $B \in \mathcal{B}$ and $B' \in \mathcal{B}$, if $C(B) \cap B' \neq \emptyset$ and $B \cap C(B') \neq \emptyset$, then $C(B) \cap B' \subset C(B')$ and $B \cap C(B') \subset C(B)$. If $C(\cdot)$ is single-valued, then this property is equivalent to the following one: For every $B \in \mathcal{B}$ and $B' \in B$, if $C(B) \subset B'$ and $B \subset C(B')$, then $C(B) = C(B')$. In the context of Walrasian demand functions, this can be restated as follows: For any (p,w) and (p',w'), if $p\cdot x(p',w') \leq w$ and $p'\cdot x(p,w) \leq w'$, then $x(p,w) = x(p',w')$. But this is the contraposition of the property stated in Definition 2.F.1. Hence Definitions 1.C.1 and 2.F.1 are equivalent.

2.F.2 It is straightforward to check that the Weak Axiom holds. In fact, if $p^i\cdot x^j \leq 8$ and $i \neq j$, then $p^j\cdot x^i = 9$. Since $p^2\cdot x^1 = 8$, x^2 is revealed preferred to x^1. Similarly, since $p^1\cdot x^3 = 8$, x^1 is revealed preferred to x^3.

But, since $p^3 \cdot x^2 = 8$, x^3 is revealed preferred to x^2.

2.F.3 [First printing errata: Add the sentence "Assume that the weak axiom is satisfied." in (b) and (c).] Denote the demand for good 2 in year 2 by y.

(a) His behavior violates the weak axiom if

$$100 \cdot 120 + 100y \le 100 \cdot 100 + 100 \cdot 100$$

and

$$100 \cdot 100 + 80 \cdot 100 \le 100 \cdot 120 + 80y.$$

That is, the Weak Axiom is violated if $y \in [75,80]$.

(b) The bundle in year 1 is revealed preferred if

$$100 \cdot 120 + 100y \le 100 \cdot 100 + 100 \cdot 100$$

and

$$100 \cdot 100 + 80 \cdot 100 > 100 \cdot 120 + 80y,$$

that is, $y < 75$.

(c) The bundle in year 2 is revealed preferred if

$$100 \cdot 100 + 80 \cdot 100 \le 100 \cdot 120 + 80y$$

and

$$100 \cdot 120 + 100y > 100 \cdot 100 + 100 \cdot 100,$$

that is, $y > 80$.

(d) For any value of y, we have sufficient information to justify exactly one of (a), (b), and (c).

(e) We shall prove that if $y < 75$, then good 1 is an inferior good. So suppose that $y < 75$. Then

$$100\cdot 120 + 100y \le 100\cdot 100 + 100\cdot 100$$

and

$$100\cdot 100 + 80\cdot 100 > 100\cdot 120 + 80y.$$

Hence the real wealth decreases from year 1 to 2. Also the relative price of good 1 increases. But the demand for good 2, y, decreases because $y < 75 < 100$. This means that the wealth effect on good 1 must be negative. Hence it is an inferior good.

(f) We shall prove that if $80 < y < 100$, then good 2 is an inferior good. So suppose that $80 < y < 100$. Then

$$100\cdot 100 + 80\cdot 100 \le 100\cdot 120 + 80y$$

and

$$100\cdot 120 + 100y > 100\cdot 100 + 100\cdot 100.$$

Hence the real wealth increases from year 1 to 2. Also the relative price of good 2 decreases. But the demand for good 2, y, decreases because $y < 100$. This means that the wealth effect on good 2 must be negative. Hence it is an inferior good.

2.F.4 (a) If $L_Q < 1$, then $(p_0\cdot x_1)/(p_0\cdot x_0) < 1$ and hence $p_0\cdot x_1 < p_0\cdot x_0$. Thus the consumer has a revealed preference for x_0 over x_1.

(b) If $P_Q > 1$, then $(p_1\cdot x_1)/(p_1\cdot x_0) > 1$ and hence $p_1\cdot x_1 > p_1\cdot x_0$. Thus the consumer has a revealed preference for x_1 over x_0.

(c) If $p_2 = \lambda p_1$ and $w_2 = \lambda w_1$, and $x_1 = x_2$, then $E_Q = \lambda$. Hence, by taking λ larger or smaller than one, we can make E_Q larger or smaller than one. But this obviously does not have any revealed preference relationship.

2.F.5 We shall first prove the discrete version. By the homogeneity of degree one with respect to wealth, it is enough to show that

$$(p' - p)\cdot(x(p',1) - x(p,1)) \le 0 \text{ for every p and p'}.$$

Since

$$x(p',1) - x(p,1) = \frac{1}{p'\cdot x(p,1)}(x(p',p'\cdot x(p,1)) - x(p,1)) + (x(p, \frac{1}{p'\cdot x(p,1)}) - x(p,1)),$$

it is sufficient to show that

$$(p' - p)\cdot(x(p',p'\cdot x(p,1)) - x(p,1)) \le 0,$$

and

$$(p' - p)\cdot(x(p, \frac{1}{p'\cdot x(p,1)}) - x(p,1)) \le 0.$$

For the first inequality, note that

$$(p' - p)\cdot(x(p',p'\cdot x(p,1)) - x(p,1)) = - p\cdot x(p',p'\cdot x(p,1)) + 1.$$

If $x(p',p'\cdot x(p,1)) = x(p,1)$, then the value is equal to zero. If $x(p',p'\cdot x(p,1)) \ne x(p,1)$, then the weak axiom implies that $p\cdot x(p',p'\cdot x(p,1)) > 1$. Hence the above value is negative.

As for the second inequality,

$$\begin{aligned}
&(p' - p)\cdot(x(p, \frac{1}{p'\cdot x(p,1)}) - x(p,1)) \\
&= p'\cdot x(p, \frac{1}{p'\cdot x(p,1)}) - p'\cdot x(p,1) - \frac{1}{p'\cdot x(p,1)} + 1. \\
&= 2 - (p'\cdot x(p,1) + \frac{1}{p'\cdot x(p,1)}) \\
&\le 2 - 2\sqrt{(p'\cdot x(p,1))(\frac{1}{p'\cdot x(p,1)})} \\
&= 2 - 2 = 0.
\end{aligned}$$

The infinitesimal version goes as follows. By differentiating $x(p,\alpha w) = \alpha x(p,w)$ with respect to α and evaluating at $\alpha = 1$, we obtain $D_w x(p,w)w = x(p,w)$. Hence

$$S(p,w) = D_p x(p,w) + D_w x(p,w)x(p,w)^T = D_p x(p,w) + (1/w)x(p,w)x(p,w)^T.$$

Thus

$$D_p x(p,w) = S(p,w) - (1/w)x(p,w)x(p,w)^T.$$

By Proposition 2.F.2, $S(p,w)$ is negative semidefinite. Moreover, since $v\cdot(x(p,w)x(p,w)^T)v = -(v\cdot x(p,w))^2$, the matrix $-(1/w)x(p,w)x(p,w)^T$ is also negative semidefinite. Thus $D_p x(p,w)$ is negative semidefinite.

2.F.6 Clearly the weak axiom implies that there exists $w > 0$ such that for every p, p', and w', if $p\cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$, then $p'\cdot x(p,w) > w$.

Conversely, suppose that such a $w > 0$ exists and that $p\cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$. Let $\alpha = w'/w$. Then $x(p',w') = x(p',\alpha w) = x(\alpha^{-1}p',w)$ by the homogeneity assumption, and $p\cdot x(\alpha^{-1}p',w) \le w$ and $x(\alpha^{-1}p',w) \ne x(p,w)$. But this implies that $(\alpha^{-1}p')\cdot x(p,w) > w$, or, equivalently, $p'\cdot x(p,w) > \alpha w = w'$. Thus the weak axiom holds.

2.F.7 By Propositions 2.E.2 and 2.E.3,

$$p\cdot S(p,w) = p\cdot D_p x(p,w) + p\cdot D_w x(p,w)x(p,w)^T = p\cdot D_p x(p,w) + x(p,w)^T = 0$$

By Proposition 2.E.1 and Walras' law,

$$S(p,w)p = D_p x(p,w)p + D_w x(p,w)x(p,w)^T p = D_p x(p,w)p + D_w x(p,w)w = 0.$$

2.F.8
$$\begin{aligned}\hat{s}_{\ell k}(p,w) &= \frac{p_k}{x_\ell(p,w)} s_{\ell k}(p,w) \\ &= \frac{p_k}{x_\ell(p,w)}\frac{\partial x_\ell}{\partial p_k}(p,w) + \frac{p_k}{x_\ell(p,w)}\frac{\partial x_\ell}{\partial w}(p,w)x_k(p,w) \\ &= \varepsilon_{\ell k}(p,w) + \frac{w}{x_\ell(p,w)}\frac{\partial x_\ell}{\partial w}(p,w)\frac{p_k x_k(p,w)}{w} \\ &= \varepsilon_{\ell k}(p,w) + \varepsilon_{\ell w}(p,w)b_k(p,w).\end{aligned}$$

2.F.9 (a) Since $x^T A^T x = (x^T A x)^T = x^T A x$, a matrix A is negative definite if and only if $x^T A x + x^T A^T x < 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$. Since $x^T A x + x^T A^T x = x^T(A + A^T)x$, this is equivalent to the negative definiteness of $A + A^T$. Thus A is negative definite if and only if so is $A + A^T$. The case of negative definiteness can be proved similarly.

The following examples shows that the determinant condition is not sufficient for the nonsymmetric case. Let $A = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}$, then $A_{11} = -1$ and $A_{22} = 1$. But $(1, 1)\begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$. Hence A is not negative semidefinite.

(b) Let $S(p,w)$ be a substitution matrix. By Proposition 2.F.3, $S(p,w)p = 0$ and hence $s_{12}(p,w) = (-p_1/p_2)s_{11}(p,w)$. Also $p \cdot S(p,w) = 0$ and hence $s_{21}(p,w) = (-p_1/p_2)s_{11}(p,w)$. Thus $s_{22}(p,w) = (p_1^2/p_2^2)s_{11}(p,w)$. Thus, for every $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

$$(*) \qquad v \cdot S(p,w)v = s_{11}(p,w)(v_1^2 - (2p_1/p_2)v_1v_2 + (p_1^2/p_2^2)v_2^2)$$
$$= s_{11}(p,w)(v_1 - (p_1/p_2)v_2)^2.$$

Now, suppose that $S(p,w)$ is negative semidefinite and of rank one. According to (*), the negative semidefiniteness implies that $s_{11}(p,w) \le 0$. Being of rank one implies that $s_{11}(p,w) \ne 0$. Hence $s_{11}(p,w) < 0$. Thus $s_{22}(p,w) < 0$. Conversely, let $s_{11}(p,w) < 0$, then, by (*), $v \cdot S(p,w)v \le 0$ for every v.

2.F.10 (a) If $p = (1,1,1)$ and $w = 1$, then, by a straightforward calculation, we obtain

$$S(p,w) = (1/3)\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Hence $S(p,w)$ is not symmetric. Note that

$$(v_1, v_2)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_1^2 + v_1v_2 - v_2^2 = -(v_1 - v_2/2)^2 - 3v_2^2/4.$$

Hence $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ is negative definite. Thus, by Proposition 2.F.3 and Theorem M.D.4(iii), $S(p,w)$ is negative semidefinite.

(b) Let $p = (1,1,\varepsilon)$ and $w = 1$. Let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column. By a straightforward calculation, we obtain

$$\hat{S}(p,w) = (2 + \varepsilon)^{-2}\begin{bmatrix} -2-\varepsilon & 1 + 2\varepsilon \\ 0 & -3\varepsilon \end{bmatrix}.$$

Thus,

$$(1,\ 4,\ 0)S(p,w)\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = (1,\ 4)\hat{S}(p,w)\begin{bmatrix} 1 \\ 4 \end{bmatrix} = (2 + \varepsilon)^{-2}(2 - 41\varepsilon) > 0,$$

if $\varepsilon > 0$ is sufficiently small. Then $S(p,w)$ is not negative semidefinite and hence the demand function in Exercise 2.E.1 does not satisfy the Weak Axiom.

2.F.11 By Proposition 2.F.3, $S(p,w)p = 0$ and hence $s_{12}(p,w) = (-p_1/p_2)s_{11}(p,w)$. Also $p\cdot S(p,w) = 0$ and hence $s_{21}(p,w) = (-p_1/p_2)s_{11}(p,w)$. (We saw this in the answer for Exercise 2.F.9 as well.) Thus $s_{12}(p,w) = s_{21}(p,w)$.

2.F.12 By applying Proposition 1.D.1 to the Walrasian choice structure, we know that $x(p,w)$ satisfies the weak axiom in the sense of Definition 1.C.1. By Exercise 2.F.1, this implies that $x(p,w)$ satisfies the weak axiom in the sense of Definition 2.F.1.

2.F.13 [First printing errata: In the last part of condition (*) of (b), the inequality $p\cdot x > w$ should be $p'\cdot x > w'$. Also, in the last part of (c), the relation $x' \in x(p,w)$ should be $x' \notin x(p,w)$.]

(a) We say that a Walrasian demand correspondence satisfies the weak axiom if the following condition is satisfied: For any (p,w) and (p',w'), if $x \in x(p,w)$, $x' \in x(p',w')$, $p'\cdot x \le w'$, and $p\cdot x' \le w$, then $x' \in x(p,w)$. Or equivalently, for any (p,w) and (p',w'), if $x \in x(p,w)$, $x' \in x(p',w')$, $p\cdot x' \le w$, and $x' \notin x(p,w)$, then $p'\cdot x > w'$.

(b) If $x \in x(p,w)$, $x' \in x(p',w')$, and $p\cdot x' < w$, then $x' \notin x(p,w)$ by Walras' law. Thus $p'\cdot x > w'$.

(c) If $x \in x(p,w)$, $x' \in x(p',w')$, and $p'\cdot x = w'$, then $(p' - p)\cdot(x' - x) = w - p\cdot x'$. If, furthermore, $x' \in x(p,w)$, then Walras' law implies that $p\cdot x' = w$. Hence $(p' - p)\cdot(x' - x) = 0$. If, on the contrary, $x' \notin x(p,w)$, then the generalized weak axiom implies that $p\cdot x' > w$. Hence $(p' - p)\cdot(x' - x) < 0$.

(d) It can be shown in the same way as in the small-type discussion of the proof of Proposition 2.F.1 that, in order to verify the assertion, it is sufficient to show that the generalized weak axiom holds for all compensated price changes. So suppose that $x \in x(p,w)$, $x' \in x(p',w')$, $p'\cdot x = w'$, and $p\cdot x' \le w$. Then $(p' - p)\cdot(x' - x) = w - p\cdot x' \ge 0$. Hence, by the generalized compensated law of Demand, we must have $(p' - p)\cdot(x' - x) = 0$ and $x' \in x(p,w)$.

2.F.14 Let $p >> 0$, $w \ge 0$, and $\alpha > 0$. Since $p\cdot x(p,w) \le w$ and $(\alpha p)\cdot x(\alpha p,\alpha w) \le \alpha w$, we have $\alpha p\cdot x(p,w) \le \alpha w$ and $p\cdot x(\alpha p,\alpha w) \le w$. The weak axiom now implies that $x(p,w) = x(\alpha p, \alpha w)$.

2.F.15 Since $\partial x_\ell(p,w)/\partial w = 0$ for both $\ell = 1,2$, we have $s_{\ell k}(p,w) = \partial x_\ell(p,w)/\partial p_k$ for both $\ell = 1,2$ and $k = 1,2$. Hence, let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column, then $\hat{S}(p,w) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. This matrix is negative definite because

$$(v_1, v_2)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_1^2 + v_1 v_2 - v_2^2 = -(v_1 - v_2/2)^2 - 3v_2^2/4.$$

(We saw this in the answer to Exercise 2.F.10(a).) Hence, by Theorem M.D.4(iii), $v \cdot S(p,w)v < 0$ for all v not proportional to p. Since $\hat{S}(p,w)$ is not symmetric, $S(p,w)$ is not symmetric either.

2.F.16 (a) The homogeneity can be checked as follows:

$$x_1(\alpha p, \alpha w) = \alpha p_2/\alpha p_3 = p_2/p_3 = x_1(p,w),$$

$$x_2(\alpha p, \alpha w) = -\alpha p_1/\alpha p_3 = -p_1/p_3 = x_2(p,w),$$

$$x_3(\alpha p, \alpha w) = \alpha w/\alpha p_3 = w/p_3 = x_3(p,w).$$

As for Walras' law,

$$p_1 x_1(p,w) + p_2 x_2(p,w) + p_3 x_3(p,w) = (p_1 p_2 - p_2 p_1 + p_3 w)/p_3 = w.$$

(b) Let $p = (1,2,1)$, $w = 1$, $p' = (1,1,1)$, and $w' = 2$, then $x(p,w) = (2, -1, 1)$ and $x(p',w') = (1, -1, 2)$. Thus $p' \cdot x(p,w) = 2 = w'$ and $p \cdot x(p',w') = 1 = w$. Hence the Weak Axiom is violated.

(c) Denote by $D\tilde{x}(p,w)$ the 2×2 submatrix of the Jacobian matrix $Dx(p,w)$ obtained by deleting the last row and column, then

$$D\tilde{x}(p,w) = (1/p_3)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column, then $\hat{S}(p,w) = D\hat{x}(p,w) = (1/p_3)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, because $\partial x_1(p,w)/\partial w = \partial x_2(p,w)/\partial w = 0$. Note that $\hat{v}\cdot\hat{S}(p,w)\hat{v} = 0$ for every $\hat{v} \in \mathbb{R}^2$. Now let $v \in \mathbb{R}^3$. Note that $v = (v - (v_3/p_3)p) + (v_3/p_3)p$ and the third coordinate of $v - (v_3/p_3)p$ is equal to zero. So denote its first two coordinates by $\hat{v} \in \mathbb{R}^2$. Then, by Proposition 2.F.3, $v\cdot S(p,w)v = \hat{v}\cdot\hat{S}(p,w)\hat{v} = 0$.

2.F.17 (a) Yes. In fact, $x_k(\alpha p,\alpha w) = \alpha w/(\sum_\ell \alpha p_\ell) = w/(\sum_\ell p_\ell) = x_k(p,w)$.

(b) Yes. In fact, $p\cdot x(p,w) = \sum_k p_k x_k(p,w) = \sum_k p_k w/(\sum_\ell p_\ell) = w$.

(c) Suppose that $p'\cdot x(p,w) \le w'$ and $p\cdot x(p',w') \le w$. The first inequality implies that $(\sum_\ell p'_\ell)w/(\sum_\ell p_\ell) \le w'$, that is, $w/(\sum_\ell p_\ell) \le w'/(\sum_\ell p'_\ell)$. The second inequality implies similarly that $(\sum_\ell p_\ell)w'/(\sum_\ell p'_\ell) \le w$, that is, $w'/(\sum_\ell p'_\ell) \le w/(\sum_\ell p_\ell)$. Therefore $w/(\sum_\ell p_\ell) = w'/(\sum_\ell p'_\ell)$. Hence $x(p,w) = x(p',w')$. Thus the weak axiom holds.

(d) By calculation, we obtain

$$D_p x(p,w) = (-\, w/(\textstyle\sum_\ell p_\ell)^2)\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix},$$

$$D_w x(p,w) = (1/\textstyle\sum_\ell p_\ell)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = x(p,w).$$

Hence $S(p,w) = 0$. It is symmetric, negative semidefinite, but not negative definite.

CHAPTER 3

3.B.1 (a) Assume that $\succsim$ is strongly monotone and $x \gg y$. Then $x \geq y$ and $x \neq y$. Hence $x \succ y$. Thus $\succsim$ is monotone.

(b) Assume that $\succsim$ is monotone, $x \in X$, and $\varepsilon > 0$. Let $e = (1,\ldots,1) \in \mathbb{R}^L$ and $y = x + (\varepsilon/\sqrt{L})e$. Then $\|y - x\| \leq \varepsilon$ and $y \succ x$. Thus $\succsim$ is locally nonsatiated.

3.B.2 Suppose that $x \gg y$. Define $\varepsilon = \mathrm{Min}\,\{x_1 - y_1, \ldots, x_L - y_L\} > 0$, then, for every $z \in X$, if $\|y - z\| < \varepsilon$, then $x \gg z$. By the local nonsatiation, there exists $z^* \in X$ such that $\|y - z^*\| < \varepsilon$ and $z^* \succ y$. By $x \gg z^*$ and the weak monotonicity, $x \succsim z^*$. By Proposition 1.B.1(iii) (which is implied by the transitivity), $x \succ y$. Thus $\succsim$ is monotone.

3.B.3 Following is an example of a convex, locally nonsatiated preference relation that is not monotone in $\mathbb{R}^2_+$. For example, $x \gg y$ but $y \succ x$.

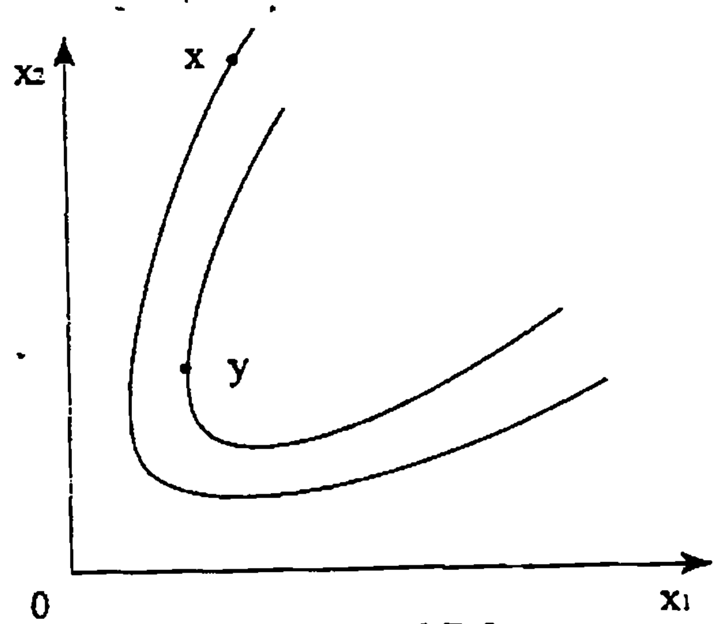


Figure 3.B.3

3.C.1 Let $\succsim$ be a lexicographic ordering. To prove the completeness, suppose that we do not have $x \succsim y$. Then "$y_1 \geq x_1$" and "$x_1 \neq y_1$ or $y_2 > x_2$". Hence either "$y_1 > x_1$" or "$y_1 \geq x_1$ and $y_2 > x_2$". Thus $y \succsim x$.

To prove the transitivity, suppose that $x \succsim y$ and $y \succsim z$. Then $x_1 \geq y_1$ and $y_1 \geq z_1$. Hence $x_1 \geq z_1$. If $x_1 > z_1$, then $x \succsim z$. If $x_1 = z_1$, then $x_1 = y_1 = z_1$. Thus $x_2 \geq y_2$ and $y_2 \geq z_2$. Hence $x_2 \geq z_2$. Thus $x \succsim z$.

To show that the strong monotonicity, suppose that $x \geq y$ and $y \neq x$. This implies either that $x_1 > y_1$ and $x_2 \geq y_2$, or that $x_1 = y_1$ and $x_2 > y_2$. In either case $x \succ y$.

To show the strict convexity, suppose that $y \succsim x$, $z \succsim x$, $y \neq z$, and $\alpha \in (0,1)$. Without loss of generality, assume that $x \neq y$. By the definition of the lexicographic ordering, we have either "$y_1 > x_1$" or "$y_1 = x_1$ and $y_2 > x_2$". On the other hand, since $z \succsim x$, we have either "$z_1 > x_1$" or "$z_1 = x_1$ and $x_2 \geq y_2$." Hence, we have either "$\alpha y_1 + (1 - \alpha)z_1 > x_1$," or "$\alpha y_1 + (1 - \alpha)z_1 = x_1$ and $\alpha y_2 + (1 - \alpha)z_2 > x_2$." Thus $\alpha y + (1 - \alpha)z \succ x$.

3.C.2 Take a sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ such that $x^n \succsim y^n$ for all n, $x^n \to x$, and $y^n \to y$. Then $u(x^n) \geq u(y^n)$ for all n, and the continuity of $u(\cdot)$ implies that $u(x) \geq u(y)$. Hence $x \succsim y$. Thus $\succsim$ is continuous.

3.C.3 One way to prove the assertion is to assume that $\succsim$ is monotone and notice that the proof actually make use only of the closedness of upper and lower contour sets. Then the proposition is applicable to $\succsim$, implying that it has a continuous utility function. Thus, by Exercise 3.C.2, $\succsim$ is continuous.

A more direct proof (without assuming monotonicity or using a utility function) goes as follows. Suppose that there exist two sequences $\{x^n\}$ and

$\{y^n\}$ in X such that $x^n \succsim y^n$ for every n, $x^n \to x \in X$, $y^n \to y \in X$, and $y \succ x$. Since $\{z: y \succ z\}$ is open, there exists a positive integer N_1 such that $y \succ x^n$ for every $n > N_1$. Since $\{z: z \succ x\}$ is open there exists a positive integer N_2 such that $y^n \succ x$ for every $n > N_2$. Conceivably, there are two cases on the sequence $\{y^n\}$:

Case 1: There exists a positive integer N_3 such that $y^n \succsim y$ for every $n > N_3$.
Case 2: There exists a subsequence $(y^{k(n)})$ such that $y \succ y^{k(n)}$ for every n.
If Case 1 applies, then, by Proposition 1.B.1(iii), we have $y^n \succ x^n$ for every $n > \text{Max}\ \{N_1, N_3\}$. This is a contradiction. If Case 2 applies, then there exists a positive integer m such that $k(m) > N_2$. Since $\{z: z \succ y^{k(m)}\}$ is open, there exists a positive integer N_4 such that $y^n \succ y^{k(m)}$ for every $n > N_4$. By $x^n \succsim y^n$ and Proposition 1.B.1(iii), $x^n \succ y^{k(m)}$ for every $n > N_4$. Since $\{z: z \succsim y^{k(m)}\}$ is closed, $x \succsim y^{k(m)}$. But, since $k(m) > N_2$, this is a contradiction.

3.C.4 We provide two examples. The first one is simpler, but the second one satisfies monotonicity, which the first does not.

<u>Example 1</u>. Let $X = \mathbb{R}_+$ and define $u(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}$ by letting $u(x) = 0$ for $x < 1$, $u(x) = 1$ for $x > 1$, and $u(1)$ be any number in $[0,1]$. Denote by $\succsim$ the preference relation represented by $u(\cdot)$. We shall now prove that $\succsim$ is not continuous. In fact, if $u(1) > 0$, then consider a sequence $\{x^n\}$ with $x^n = 1 - 1/n$ for every n. Although $x^n \sim 0$ for every n and $x^n \to 1$, we have $1 \succ 0$. If $u(1) < 1$, then consider a a sequence $\{x^n\}$ with $x^n = 1 + 1/n$ for every n. Although $x^n \sim 2$ for every n and $x^n \to 1$, we have $2 \succ 1$. Note that if $u(x) = 0$, then all lower contour sets are closed. If $u(1) = 1$, then all upper contour sets are closed.

Example 2. Take $X = \mathbb{R}^2_+$ and define a utility function $u(\cdot)$: $\mathbb{R}^2_+ \to \mathbb{R}$ by the following rule:

Case 1. If $x_1 + x_2 \le 2$ and $x \ne (1,1)$, then $u(x) = x_1 + x_2$.

Case 2. If $\min\{x_1,x_2\} \ge 1$ and $x \ne (1,1)$, then $u(x) = \min\{x_1,x_2\} + 2$.

Case 3. If $x_1 + x_2 > 2$, $\min\{x_1,x_2\} < 1$, and $x_1 > x_2$, then

$$u(x) = 3 - (1 - x_2)/(x_1 - 1).$$

Case 4. If $x_1 + x_2 > 2$, $\min\{x_1,x_2\} < 1$, and $x_1 < x_2$, then

$$u(x) = 3 - (1 - x_1)/(x_2 - 1).$$

Case 5. $u(1,1) \in [2,3]$.

The indifference curves of the preference relation $\succsim$ represented by $u(\cdot)$ are described in the following picture:

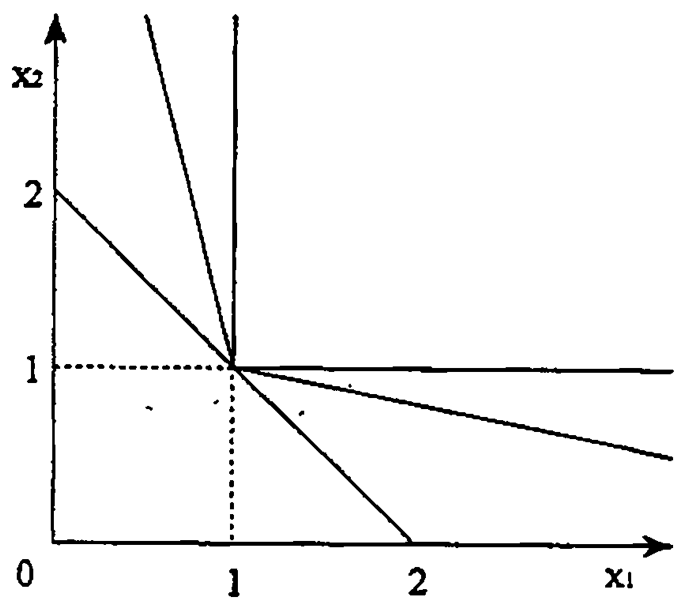


Figure 3.C.4

It follows from this construction that $u(\cdot)$ is continuous at every $x \ne (1,1)$. The preference $\succsim$ is convex and monotone. But, whatever the choice of the value of $u(1,1)$ is, it cannot be continuous at $(1,1)$. In fact, $(1 - 1/n, 1 - 1/n) \to (1,1)$ and $(1 + 1/n, 1 + 1/n) \to (1,1)$ as $n \to \infty$, and

$$u(1 - 1/n, 1 - 1/n) = 2 - 2/n \to 2;$$

$$u(1 + 1/n, 1 + 1/n) = 1 + 1/n + 2 \to 3.$$

Hence, if $2 < u(1,1)$, then $(2,0) \succsim (1 - 1/n, 1 - 1/n)$ but $(1,1) \succ (2,0)$; if $u(1,1) < 3$, then $(1 + 1/n, 1 + 1/n) \succsim (2,1)$ but $(2,1) \succ (1,1)$. If $u(1,1) = 3$, then all upper contour sets of $\succsim$ are closed; if $u(1,1) = 2$, then all lower contour sets of $\succsim$ are closed.

3.C.5 (a) Suppose first that $u(\cdot)$ is homogeneous of degree one and let $\alpha \geq 0$, $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$, and $x \sim y$. Then $u(x) = u(y)$ and hence $\alpha u(x) = \alpha u(y)$. By the homogeneity, $u(\alpha x) = u(\alpha y)$. Thus $\alpha x \sim \alpha y$.

Suppose conversely that $\succsim$ is homothetic. We shall prove that the utility function constructed in the proof of Proposition 3.C.1 is homogeneous of degree one. Let $x \in \mathbb{R}^L_+$ and $\alpha > 0$, then $u(x)e \sim x$ and $u(\alpha x)e \sim \alpha x$. Since $\succsim$ is homothetic, $\alpha u(x)e \sim \alpha x$. By the transitivity of $\sim$ (Proposition 1.B.1(ii)), $u(\alpha x)e \sim \alpha u(x)e$. Thus $u(\alpha x) = \alpha u(x)$.

(b) Suppose first that $\succsim$ is represented by a utility function of the form $u(x) = x_1 + \phi(x_2,\dots,x_L)$. Let $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$, and $x \sim y$. Then $u(x) = u(y)$ and hence $u(x) + \alpha = u(y) + \alpha$. By the functional form,

$$u(x) + \alpha = (\alpha + x_1) + \phi(x_2,\dots,x_L) = u(x + \alpha e_1),$$

$$u(y) + \alpha = (\alpha + y_1) + \phi(y_2,\dots,y_L) = u(y + \alpha e_1),$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^L_+$. Hence $u(x + \alpha e_1) = u(y + \alpha e_1)$, or $x + \alpha e_1 \sim y + \alpha e_1$.

Suppose conversely that $\succsim$ is quasilinear with respect to the first commodity. The idea of the proof of this direction is the same as in (a) or Proposition 3.C.1, in that we reduce comparison of commodity bundles on a line by fining out indifferent bundles and then assigning utility levels along the line. But this proof turns out to exhibit more intricacies, partly because it depends crucially on the connectedness of $\mathbb{R}_+^{L-1}$, which appears in $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$. (Connectedness was mentioned in the first small-type discussion in the proof of Proposition 3.C.1.) The proof will be done in a series of steps. First, we show that comparison of bundles can be reduced to a line parallel to e_1. Then we show that the quasilinearity of $\succsim$ implies the given functional form.

Let $\succsim$ be a quasilinear preference and a utility function $\tilde{u}(\cdot)$ represent $\succsim$. The existence of such a $\tilde{u}(\cdot)$ is guaranteed by Proposition 3.C.1, but, of course, it need not be of the quasilinear form. For each $\hat{x} \in \mathbb{R}_+^{L-1}$, define $I(\hat{x}) = \{u(x_1,\hat{x}) \in \mathbb{R}: x_1 \in \mathbb{R}\}$, then $I(\hat{x})$ is a nonempty open interval, by the continuity and the strong monotonicity of $\succsim$ along e_1.

Step 1: For every $\hat{x} \in \mathbb{R}_+^{L-1}$ *and* $\hat{y} \in \mathbb{R}_+^{L-1}$, *if* $I(\hat{x}) \neq I(\hat{y})$, *then* $I(\hat{x}) \cap I(\hat{y}) = \varnothing$.

Proof: Suppose that $I(\hat{x}) \neq I(\hat{y})$. Without loss of generality, we can assume that there exists $u \in I(\hat{x})$ such that $u \notin I(\hat{y})$. Then either $u \geq \sup I(\hat{y})$ or $u \leq \inf I(\hat{y})$. Suppose that $u \geq \sup I(\hat{y})$. (The other case can be treated similiarly.) Then let $x_1^* \in \mathbb{R}$ satisfy $u = \tilde{u}(x_1^*,\hat{x})$, then, for every $y_1 \in \mathbb{R}$, $(x_1^*,\hat{x}) \succ (y_1,\hat{y})$. In particular, for every $x_1 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$, $(x_1^*,\hat{x}) \succ (y_1 - x_1 + x_1^*, \hat{y})$. By the quasilinearity, this implies that $(x_1,\hat{x}) \succ (y_1,\hat{y})$. Thus $\tilde{u}(x_1,\hat{x}) > \tilde{u}(y_1,\hat{y})$. Hence $I(\hat{x}) \cap I(\hat{y}) = \varnothing$.

For each $\hat{x} \in \mathbb{R}_+^{L-1}$, define $E(\hat{x}) = \{\hat{y} \in \mathbb{R}_+^{L-1}: I(\hat{x}) = I(\hat{y})\}$.

Step 2: For every $\hat{x} \in \mathbb{R}_+^{L-1}$, $E(\hat{x})$ *is open in* $\mathbb{R}_+^{L-1}$.

Proof: Let $\hat{x} \in \mathbb{R}_+^{L-1}$, $x_1 \in \mathbb{R}$, and $u = \tilde{u}(x_1,\hat{x}) \in I(\hat{x})$. Let $\varepsilon > 0$ satisfy $(u - \varepsilon, u + \varepsilon) \subset I(\hat{x})$. Since $\tilde{u}(\cdot)$ is continuous, there exists $\delta > 0$ such that if $\hat{y} \in \mathbb{R}_+^{L-1}$ and $\|\hat{x} - \hat{y}\| < \delta$, then $|\tilde{u}(x_1,\hat{x}) - \tilde{u}(x_1,\hat{y})| < \varepsilon$. Hence

$$I(\hat{x}) \cap I(\bar{y}) \supset (u - \varepsilon, u + \varepsilon) \cap I(\bar{y}) \neq \emptyset.$$

Thus, by Step 1, $I(\hat{x}) = I(\hat{y})$, or $\hat{y} \in E(\hat{x})$. Hence $\{\hat{y} \in \mathbb{R}_+^{L-1}: \|\hat{x} - \hat{y}\| < \delta\}$ $\subset E(\hat{x})$. Thus $E(\hat{x})$ is open.

Step 3: For every $\hat{x} \in \mathbb{R}_+^{L-1}$, $E(\hat{x}) = \mathbb{R}_+^{L-1}$.

Proof: It is sufficient to show that for every $\hat{x} \in \mathbb{R}_+^{L-1}$ and $\hat{y} \in \mathbb{R}_+^{L-1}$, we have $E(\hat{x}) = E(\hat{y})$. Suppose not, then there exist $\hat{x} \in \mathbb{R}_+^{L-1}$ and $\hat{y} \in \mathbb{R}_+^{L-1}$ such that $E(\hat{x}) \neq E(\hat{y})$, then the complement $\mathbb{R}_+^{L-1}\setminus E(\hat{x})$ is nonempty. By Step 1, $\mathbb{R}_+^{L-1}\setminus E(\hat{x})$ is equal to the union of those $E(\hat{y})$ for which $\hat{y} \in \mathbb{R}_+^{L-1}\setminus E(\hat{x})$. By Step 2, this implies that $\mathbb{R}_+^{L-1}\setminus E(\hat{x})$ is open. Hence we have obtained a partition $(E(\hat{x}), \mathbb{R}_+^{L-1}\setminus E(\hat{x}))$ of $\mathbb{R}_+^{L-1}$, both of whose elements are nonempty and open. This contradicts the connectedness of $\mathbb{R}_+^{L-1}$. Hence $E(\hat{x}) = E(\hat{y})$ for every $\hat{x} \in \mathbb{R}_+^{L-1}$ and $\hat{y} \in \mathbb{R}_+^{L-1}$.

By Step 3, $I(\hat{x}) = I(0)$ for every $\hat{x} \in \mathbb{R}_+^{L-1}$. Thus for every $\hat{x} \in \mathbb{R}_+^{L-1}$, there exists a unique $\alpha \in \mathbb{R}$ such that $\alpha e_1 \sim (0,\hat{x})$. Define ϕ: $\mathbb{R}_+^{L-1} \to \mathbb{R}$ by $\phi(\hat{x})e_1 \sim (0,\hat{x})$ for every $\hat{x} \in \mathbb{R}_+^{L-1}$. Define u: $X \to \mathbb{R}$ by $u(x) = x_1 + \phi(x_2,\ldots,x_L)$ for every $x \in X$.

Step 4: The function $u(\cdot)$ represents $\succsim$.

Proof: Suppose that $x \in X$, $y \in X$, and $x \succsim y$. By the quasilinearity, this is equivalent to $(x_1 - y_1, x_2,\ldots, x_L) \succsim (0,y_2,\ldots,y_L)$. By the definition of $\phi(\cdot)$, this is equivalent to $(x_1 - y_1, x_2,\ldots, x_L) \succsim \phi(y_2,\ldots,y_L)e_1$. Again by the quasilinearity, this is equivalent to

$$(0,x_2,\ldots, x_L) \succsim (\phi(y_2,\ldots,y_L) + y_1 - x_1)e_1.$$

Again by the definition of $\phi(\cdot)$, this is equivalent to

$$\phi(x_2,\ldots, x_L)e_1 \succsim (\phi(y_2,\ldots,y_L) + y_1 - x_1)e_1.$$

Hence $\phi(x_2,\ldots, x_L) \geq \phi(y_2,\ldots,y_L) + y_1 - x_1$, that is, $u(x) \geq u(y)$.

These properties of $u(\cdot)$ are cardinal, because they are not preserved under some monotone transformation, such as $f(u(x)) = u(x)^3$.

3.C.6 (a) For $\rho = 1$, we have $u(x) = \alpha_1 x_1 + \alpha_2 x_2$. Thus the indifference curves are linear.

(b) Since every monotonic transformations of a utility function represents the same preference, we shall consider

$$\tilde{u}(x) = \ln u(x) = (1/\rho)\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho).$$

By L'Hopital's rule,

$$\begin{aligned} &\lim_{\rho\to 0} \tilde{u}(x) \\ &= \lim_{\rho\to 0} (\alpha_1 x_1^\rho \ln x_1 + \alpha_2 x_2^\rho \ln x_2)/(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho) \\ &= (\alpha_1 \ln x_1 + \alpha_2 \ln x_2)/(\alpha_1 + \alpha_2). \end{aligned}$$

Since $\exp((\alpha_1 + \alpha_2)\tilde{u}(x)) = x_1^{\alpha_1} x_2^{\alpha_2}$, we have obtained a Cobb-Douglas utility function.

There is an alternative proof to this proposition: Since both the CES and the Cobb-Douglas utility functions are continuously differentiable and homothetic, it is sufficient to check the convergence of the marginal rate of substitution at every point. The marginal rate of substitution at (x_1,x_2) with respect to the CES utility function is equal to $\alpha_1 x_1^{\rho-1}/\alpha_2 x_2^{\rho-1}$. The marginal rate of substitution at (x_1,x_2) with respect to the Cobb-Douglas utility function is equal to $\alpha_1 x_1/\alpha_2 x_2$. Note that $\alpha_1 x_1^{\rho-1}/\alpha_2 x_2^{\rho-1} \to \alpha_1 x_1/\alpha_2 x_2$ as $\rho \to 1$. (In fact, $\alpha_1 x_1^{\rho-1}/\alpha_2 x_2^{\rho-1}$ is well defined for every ρ and is equal to $\alpha_1 x_1/\alpha_2 x_2$ when $\rho = 1$.) The proof is thus completed.

Strictly speaking, there is a missing point in both proofs: We proved the convergence of preferences on the strictly positive orthant $\{x \in \mathbb{R}^2: x >> 0\}$, but we did not prove the convergence on the horizontal and vertical axes. In fact, the convergence on the axes are obtained in such a way that all vectors there tend to be indifferent. To be more specific, compare, for example, $x = (x_1, 0)$ and $y = (y_1, 0)$ with $x_1 > y_1 > 0$. According to the CES utility function, x is preferred to y, regardless of the values of ρ. But, according to the Cobb-Douglas utility function, x and y are indifferent. Futhermore, the following is true: If x is in the strictly positive orthant and y is on an axis, then x is preferred to y for every ρ sufficiently close to 0. To see this, simply note that if $x = (x_1, 0)$ with $x_1 > 0$ and $y \gg 0$, then $\alpha_1 x_1^\rho \to \alpha_1$ and $\alpha_1 y_1^\rho + \alpha_2 y_2^\rho \to \alpha_1 + \alpha_2$. The implication of this fact is that, as $\rho \to 0$, every vector in the strictly positive orthant becomes preferred to all vectors on the axes. That is, unconditional preference towards strictly positive vectors tends to hold, as it is true for the Cobb-Douglas utility function.

(c) Suppose that $x_1 \le x_2$. We want to show that

$$x_1 = \lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho}.$$

Let $\rho < 0$. Since $x_1 \ge 0$ and $x_2 \ge 0$, we have $\alpha_1 x_1^\rho \le \alpha_1 x_1^\rho + \alpha_2 x_2^\rho$. Thus $\alpha_1^{1/\rho} x_1 \ge (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho}$. On the other hand, since $x_1 \le x_2$,

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \le \alpha_1 x_1^\rho + \alpha_2 x_1^\rho = (\alpha_1 + \alpha_2) x_1^\rho.$$

Hence $(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho} \ge (\alpha_1 + \alpha_2)^{1/\rho} x_1$. Therefore,

$$\alpha_1^{1/\rho} x_1 \ge (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho} \ge (\alpha_1 + \alpha_2)^{1/\rho} x_i.$$

Letting $\rho \to -\infty$, we obtain $\lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho} = x_1$, because $\lim_{\rho \to -\infty} \alpha_i^{1/\rho} x_1 = \lim_{\rho \to -\infty} (\alpha_1 + \alpha_2)^{1/\rho} x_i = x_i$.

3.D.1 To check condition (i),

$$x_1(\lambda p, \lambda w) = \alpha(\lambda w)/(\lambda p_1) = \alpha w/p_1 = x_1(p, w),$$

$$x_2(\lambda p, \lambda w) = (1 - \alpha)(\lambda w)/(\lambda p_2) = (1 - \alpha)w/p_2 = x_2(p, w).$$

To check condition (ii),

$$p_1 x_1(p, w) + p_2 x_2(p, w) = p_1 \alpha w/p_1 + p_2(1 - \alpha)w/p_2 = w.$$

Condition (iii) is obvious.

3.D.2 To check condition (i),

$$\begin{aligned}
&v(\lambda p, \lambda w) \\
&= \alpha\ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln\lambda w - \alpha\ln\lambda p_1 - (1 - \alpha)\ln\lambda p_2 \\
&= \alpha\ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln\lambda + \ln w \\
&\qquad - \alpha\ln\lambda - \alpha\ln p_1 - (1 - \alpha)\ln\lambda - (1 - \alpha)\ln p_2 \\
&= \alpha\ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln w - \alpha\ln p_1 - (1 - \alpha)\ln p_2 \\
&= v(p, w).
\end{aligned}$$

To check condition (ii),

$$\begin{aligned}
&\partial v(p, w)/\partial w = 1/w > 0, \\
&\partial v(p, w)/\partial p_1 = -\alpha/p_1 < 0, \\
&\partial v(p, w)/\partial p_2 = -(1 - \alpha)/p_2 < 0.
\end{aligned}$$

Condition (iv) follows the functional form of $v(\cdot)$.

In order to verify (iii), by property (i), it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^L_{++}: v(p, w) \le v\}$ is convex. Since the logarithmic function is concave, the set

$$\{(p_1, p_2) \in \mathbb{R}^2_{++}: -\alpha\ln p_1 - (1 - \alpha)\ln p_2 \le v\}$$

is convex for every $v \in \mathbb{R}$. Since the other terms,

$$\alpha\ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln w,$$

do not depend on p, this implies that the set $\{p \in \mathbb{R}^L_{++}: v(p, w) \le v\}$ is convex.

3.D.3 (a) We shall prove that for every $p \in \mathbb{R}^L_{++}$, $w \ge 0$, $\alpha \ge 0$, and $x \in \mathbb{R}^L_+$, if

$x = x(p,w)$, then $\alpha x = x(p,\alpha w)$. Note first that $p\cdot(\alpha x) \le \alpha w$, that is, αx is affordable at $(p,\alpha w)$. Let $y \in \mathbb{R}^L_+$ and $p\cdot y \le \alpha w$. Then $p\cdot(\alpha^{-1}y) \le w$. Hence $u(\alpha^{-1}y) \le u(x)$. Thus, by the homogeneity, $u(y) \le u(\alpha x)$. Hence $\alpha x = x(p,\alpha w)$.

By this result,

$$v(p,\alpha w) = u(x(p,\alpha w)) = u(\alpha x(p,w)) = \alpha u(x(p,w)) = \alpha v(p,w).$$

Thus the indirect utility function is homogeneous of degree one in w.

Given the above results, we can write $x(p,w) = wx(p,1) = w\tilde{x}(p)$ and $v(p,w) = wv(p,1) = w\tilde{v}(p)$. Exercise 2.E.4 showed that the wealth expansion path $\{x(p,w): w > 0\}$ is a ray going through $\tilde{x}(p)$. The wealth elasticity of demand $\varepsilon_{\ell w}$ is equal to 1.

(b) We first prove that for every $p \in \mathbb{R}^L_{++}$, $w \ge 0$, and $\alpha \ge 0$, we have $x(p,\alpha w) = \alpha x(p,w)$. In fact, since $v(\cdot,\cdot)$ is homogeneous of degree one in w, $\nabla_p v(p,\alpha w) = \alpha\nabla_p v(p,w)$ and $\nabla_w v(p,\alpha w) = \nabla_w v(p,w)$. Thus, by Roy's identity, $x(p,\alpha w) = \alpha x(p,w)$.

Now let $x \in \mathbb{R}^L_+$, $x' \in \mathbb{R}^L_+$, $u(x) = u(x')$, and $\alpha \ge 0$. Since $u(\cdot)$ is strictly quasiconcave, by the supporting hyperplane theorem (Theorem M.G.3), there exist $p \in \mathbb{R}^L_{++}$, $p' \in \mathbb{R}^L_{++}$, $w \ge 0$, and $w' \ge 0$ such that $x = x(p,w)$ and $x' = x(p',w')$. Then $u(x) = v(p,w)$ and $u(x') = v(p',w')$. Hence $v(p,w) = v(p',w')$. Thus, by the homogeneity, $v(p,\alpha w) = v(p',\alpha w')$. But as we saw above, $x(p,\alpha w) = \alpha x$ and $x(p',\alpha w') = \alpha x'$. Hence $v(p,\alpha w) = u(\alpha x)$ and $v(p',\alpha w') = u(\alpha x')$. Thus $u(\alpha x) = u(\alpha x')$. Therefore $u(x)$ is homogeneous of degree one.

3.D.4 (a) Let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^L$. We shall prove that for every $p \in \mathbb{R}^L_{++}$, $w \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $x \in (-\infty,\infty) \times \mathbb{R}^{L-1}_+$, if $x = x(p,w)$, then $x + \alpha e_1 = x(p, w + \alpha)$. Note first that, by $p_1 = 1$, $p\cdot(\alpha x + e_1) \le \alpha w$, that is, $x + \alpha e_1$ is affordable at $(p, w + \alpha e_1)$. Let $y \in \mathbb{R}^L_+$ and $p\cdot y \le w + \alpha$. Then $p\cdot(y - \alpha e_1)$

$\le w$. Hence $x \succsim y - \alpha e_1$. Thus, by the quasilinearity, $x + \alpha e_1 \succsim y$. Hence $x + \alpha e_1 = x(p, w + \alpha)$.

Therefore, for every $\ell \in \{2,\dots,L\}$, $w \in \mathbb{R}$, and $w' \in \mathbb{R}$, $x_\ell(p,w) = x_\ell(p,w')$. That is, the Walrasian demand functions for goods $2,\dots,L$ are independent of wealth. As for good 1, we have $\partial x(p,w)/\partial w = 1$ for every (p,w). That is, any additional amount of money is spent on good 1.

(b) Define $\phi(p) = u(x(p,0))$. Since $x(p,w) = x(p,0) + we_1$ and the preference relation can be represented by a utility function of the quasilinear form $u(x) = x_1 + \tilde{u}(x_2,\dots,x_L)$ (Exercise 3.C.5), we have

$$\begin{aligned} v(p,w) &= u(x(p,w)) \\ &= x_1(p,w) + \tilde{u}(x_2(p,w),\dots,x_L(p,w)). \\ &= w + x_1(p,0) + \tilde{u}(x_2(p,0),\dots,x_L(p,0)) \\ &= w + u(x(p,0)) = w + \phi(p). \end{aligned}$$

(c) The non-negativity constraint is binding if and only if $p_2x_2(p,0) > w$. Note that $x_2(p,0) = (\eta')^{-1}(p_2)$, because $p_1 = 1$. Hence the constraint is binding if and only if $p_2(\eta')^{-1}(p_2) > w$. If so, the Walrasian demand is given by $x(p,w) = (0,w/p_2)$. Thus, as w changes, the consumption level of the first good is unchanged and the consumption of the second good changes at rate $1/p_2$ with w until the non-negativity constraint no longer binds.

3.D.5 (a) Since any monotone transformation of a utility function represents the same preference relation, we may as well choose

$$\tilde{u}(x) = \rho u(x)^\rho = \rho(x_1^\rho + x_2^\rho).$$

By the first-order condition of the UMP with $\tilde{u}(\cdot)$,

$$x(p,w) = (w/(p_1^\delta + p_2^\delta))(p_1^{\delta-1}, p_2^{\delta-1}),$$

where $\delta = \rho/(\rho - 1) \in (-\infty, 1)$. Plug this into $u(\cdot)$, then we obtain

$$v(p,w) = w/(p_1^{\delta} + p_2^{\delta})^{1/\delta}.$$

(b) To check the homogeneity of the demand function,

$$\begin{aligned} x(\alpha p,\alpha w) &= (\alpha w/((\alpha p_1)^{\delta} + (\alpha p_2)^{\delta})((\alpha p_1)^{\delta-1},(\alpha p_2)^{\delta-1}) \\ &= (\alpha\cdot\alpha^{\delta-1}/\alpha^{\delta})(w/(p_1^{\delta} + p_2^{\delta}))(p_1^{\delta-1},p_2^{\delta-1}) \\ &= x(p,w). \end{aligned}$$

To check Walras' law,

$$p\cdot x(p,w) = (w/(p_1^{\delta} + p_2^{\delta}))(p_1\cdot p_1^{\delta-1} + p_2\cdot p_2^{\delta-1}) = w.$$

The uniqueness is obvious.

To check the homogeneity of the indirect utility function,

$$\begin{aligned} &v(\alpha p,\alpha w) \\ &= \alpha w/((\alpha p_1)^{\delta} + (\alpha p_2)^{\delta})^{1/\delta} = \alpha w/\alpha^{\delta\cdot 1/\delta}(p_1^{\delta} + p_2^{\delta})^{1/\delta} = w/(p_1^{\delta} + p_2^{\delta})^{1/\delta} \\ &= v(p,w) \end{aligned}$$

To check the monotonicity,

$$\begin{aligned} \partial v(p,w)/\partial w &= 1/(p_1^{\delta} + p_2^{\delta})^{1/\delta} > 0, \\ \partial v(p,w)/\partial p_{\ell} &= -\, wp_{\ell}^{\delta-1}/(p_1^{\delta} + p_2^{\delta})^{1/\delta+1} < 0. \end{aligned}$$

The continuity follows immediately from the derived functional form.

In order to prove the quasiconvexity, by property the homogeneity, it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^2: v(p,w) \le v\}$ is convex. If $\delta = 0$, then the utility function is a Cobb-Douglas one, and the quasiconcavity was already established in Exercise 3.D.2. So we consider two cases, $\delta \in (0,1)$ and $\delta < 0$. In either case, define $f(p) = (p_1^{\delta} + p_2^{\delta})^{1/\delta}$.

If $\delta \in (0,1)$, then $f(p)^{\delta} = p_1^{\delta} + p_2^{\delta}$ is a concave function. Hence $\{p \in \mathbb{R}^2: f(p) = (f(p)^{\delta})^{1/\delta} \ge v\}$ is convex for every v. Since $v(p,w) = w/f(p)$, this implies that $\{p \in \mathbb{R}^2: v(p,w) \le v\}$ is convex for every v and w.

If $\delta < 0$, then $f(p)^{\delta} = p_1^{\delta} + p_2^{\delta}$ is a convex function. Hence $\{p \in \mathbb{R}^2$: $1/f(p) = (f(p)^{\delta})^{1/(-\delta)} \le v\}$ is convex for every v. Since $v(p,w) = w/f(p)$, this implies that $\{p \in \mathbb{R}^2: v(p,w) \le v\}$ is convex for every v and w.

(c) For the linear indifference curves, we have

$$x(p,w) = \begin{cases} (w/p_1,\ 0) & \text{if } p_1 < p_2, \\ (0,\ w/p_2) & \text{if } p_1 > p_2, \\ \{(w/p_1)(\lambda,\ 1-\lambda): \lambda \in [0,1]\} & \text{if } p_1 = p_2; \end{cases}$$

$$v(p,w) = \max\{w/p_1, w/p_2\}.$$

For the Leontief preference,

$$x_1(p,w) = (w/(p_1 + p_2))(1,1);$$

$$v(p,w) = w/(p_1 + p_2).$$

As for the limit argument with respect to ρ. First consider the case with $\rho < 1$ and $\rho \to 1$. Then $\delta = \rho/(\rho - 1) \to -\infty$ as $\rho \to 1$.

Case 1. $p_1 < p_2$.

Since $p_2/p_1 > 1$, we have $(p_2/p_1)^{\delta} \to 0$. Thus

$$\lim_{\delta\to-\infty} p_1^{\delta-1} w/(p_1^{\delta} + p_2^{\delta}) = \lim_{\delta\to-\infty} \frac{w/p_1}{1 + (p_2/p_1)^{\delta}} = w/p_1.$$

Since $p_1/p_2 < 1$, we have $(p_1/p_2)^{\delta} \to \infty$. Thus

$$\lim_{\delta\to-\infty} p_2^{\delta-1} w/(p_1^{\delta} + p_2^{\delta}) = \lim_{\delta\to-\infty} \frac{w/p_2}{(p_1/p_2)^{\delta} + 1} = 0.$$

Thus the CES Walrasian demands converge to the Walrasian demand of the linear indifference curves. As for the indirect utility functions, we showed in the answer to Exercise 3.C.6(c) that $(p_1^{\delta} + p_2^{\delta})^{1/\delta} \to p_1$ for $p_1 \le p_2$. Hence the CES indirect utilities converge to the indirect utility of the linear indifference curves.

Case 2. $p_1 > p_2$.

Do the same argument as in the Case 1.

Case 3. $p_1 = p_2$.

In this case, $(w/(p_1^\delta + p_2^\delta))(p_1^{\delta-1}, p_2^{\delta-1}) = (w/(p_1^\delta + p_1^\delta))(p_1^{\delta-1}, p_1^{\delta-1}) = (w/2p_1)(1,1)$. This consumption bundle belongs to the set of the Walrasian demands of the linear indifference curves when $p_1 = p_2$. As for the indirect utility functions, we showed in the answer to Exercise 3.C.6(c) that $(p_1^\delta + p_2^\delta)^{1/\delta} \to p_1$ for $p_1 \le p_2$.

Let's next consider the case $\rho \to -\infty$. Note that $\delta = \rho/(\rho - 1) \to 1$ as $\rho \to 1$. So just plug $\delta = 1$ into the CES Walrasian demand functions and the indirect utility functions. We then get the Walrasian demand function and the indirect utility function of the Leontief preference.

(d) From the calculation of the Walrasian demand functions in (a) we get

$$x_1(p,w)/x_2(p,w) = (p_1/p_2)^{\delta-1},$$
$$(x_1(p,w)/x_2(p,w))/(p_1/p_2) = (p_1/p_2)^{\delta-2},$$
$$d[x_1(p,w)/x_2(p,w)]/d[p_1/p_2] = (\delta - 1)(p_1/p_2)^{\delta-2}.$$

Thus $\xi_{12}(p,w) = -(\delta - 1) = 1/(1 - \rho)$. Hence, $\xi_{12}(p,w) = \infty$ for the linear, $\xi_{12}(p,w) = 0$ for the Leontief, and $\xi_{12}(p,w) = 1$ for the Cobb-Douglas utility functions.

3.D.6 (a) Define $\bar{u}(x) = u(x)^{1/(\alpha+\beta+\gamma)} = (x_1 - b_1)^{\alpha'}(x_2 - b_2)^{\beta'}(x_3 - b_3)^{\gamma'}$, with $\alpha' = \alpha/(\alpha + \beta + \gamma)$, $\beta' = \beta/(\alpha + \beta + \gamma)$, $\gamma' = \gamma/(\alpha + \beta + \gamma)$. Then $\alpha' + \beta' + \gamma' = 1$ and $\tilde{u}(\cdot)$ represents the same preferences as $u(\cdot)$, because the function $u \to u^{1/(\alpha+\beta+\gamma)}$ is a monotone transformation. Thus we can assume without loss of generality that $\alpha + \beta + \gamma = 1$.

(b) Use another monotone transformation of the given utility function,

$$\ln u(x) = \alpha\ln(x_1 - b_1) + \beta\ln(x_2 - b_2) + \gamma\ln(x_3 - b_3).$$

The first-order condition of the UMP yields the demand function

$$x(p,w) = (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3),$$

where $p\cdot b = p_1b_1 + p_2b_2 + p_3b_3$. Plug this demand function to $u(\cdot)$, then we obtain the indirect utility function

$$v(p,w) = (w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma}.$$

(c) To check the homogeneity of the demand function,

$$\begin{aligned} x(\lambda p,\lambda w) &= (b_1,b_2,b_3) + (\lambda w - \lambda p\cdot b)(\alpha/\lambda p_1,\beta/\lambda p_2,\gamma/\lambda p_3) \\ &= (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3) = x(p,w). \end{aligned}$$

To check Walras law,

$$\begin{aligned} p\cdot x(p,w) &= p\cdot b + (w - p\cdot b)(p_1\alpha/p_1 + p_2\beta/p_2 + p_3\gamma/p_3) \\ &= p\cdot b + (w - p\cdot b)(\alpha + \beta + \gamma) = w. \end{aligned}$$

The uniqueness is obvious.

To check the homogeneity of the indirect utility function,

$$\begin{aligned} v(\lambda p,\lambda w) &= (\lambda w - \lambda p\cdot b)(\alpha/\lambda p_1)^{\alpha}(\beta/\lambda p_2)^{\beta}(\gamma/\lambda p_3)^{\gamma} \\ &= \lambda^{1-(\alpha+\beta+\gamma)}(w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} \\ &= (w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} = v(p,w). \end{aligned}$$

To check the monotonicity,

$$\partial v(p,w)/\partial w = (\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} > 0,$$

$$\partial v(p,w)/\partial p_1 = v(p,w)\cdot(-\ \alpha/p_1) < 0,$$

$$\partial v(p,w)/\partial p_2 = v(p,w)\cdot(-\ \beta/p_2) < 0,$$

$$\partial v(p,w)/\partial p_3 = v(p,w)\cdot(-\ \gamma/p_3) < 0.$$

The continuity follows directly from the given functional form. In order to prove the quasiconvexity, it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^3: v(p,w) \le v\}$ is convex. Consider

$$\ln v(p,w) = \alpha\ln\alpha + \beta\ln\beta + \gamma\ln\gamma + \ln(w - p\cdot b) - \alpha\ln p_1 - \beta\ln p_2 - \gamma\ln p_3.$$

Since the logarithmic function is concave, the set

$$\{p \in \mathbb{R}^3: \ln(w - p\cdot b) - \alpha\ln p_1 - \beta\ln p_2 - \gamma\ln p_3 \leq v\}$$

is convex for every $v \in \mathbb{R}$. Since the other terms, $\alpha\ln\alpha + \beta\ln\beta + \gamma\ln\gamma$, do not depend on p, this implies that the set $\{p \in \mathbb{R}^3: \ln v(p,w) \leq v\}$ is convex. Hence so is $\{p \in \mathbb{R}^3: v(p,w) \leq v\}$

3.D.7 (a) Since $p^1\cdot x^0 < w^1$ and $x^1 \neq x^0$, the weak axiom implies $p^0\cdot x^1 > w^0$. Thus x^1 has to be on the bold line in the following figure.

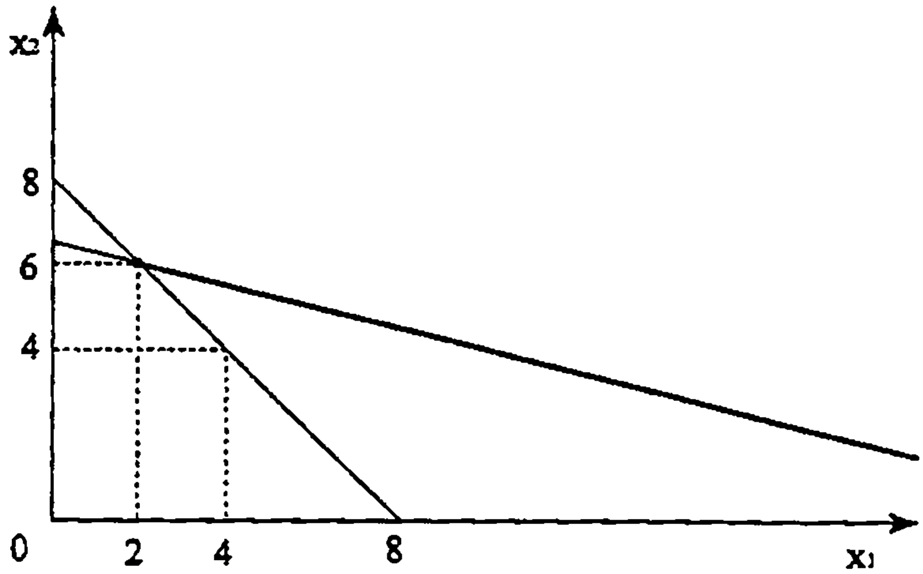


Figure 3.D.7(a)

In the following four question, we assume the given preference can be a differentiable utility function $u(\cdot)$.

(b) If the preference is quasilinear with respect to the first good, then we can take a utility function $u(\cdot)$ so that $\partial u(x)/\partial x_1 = 1$ for every x (Exercise 3.C.5(b)). Hence the first-order condition implies $\partial u(x^t)/\partial x_2^t = p_2^t/p_1^t$ for each $t = 0,1$. Since $p_2^0/p_1^0 < p_2^1/p_1^1$ and $u(\cdot)$ is concave, $x_2^0 > x_2^1$. Thus x^1 has to be on the bold line in the following figure.

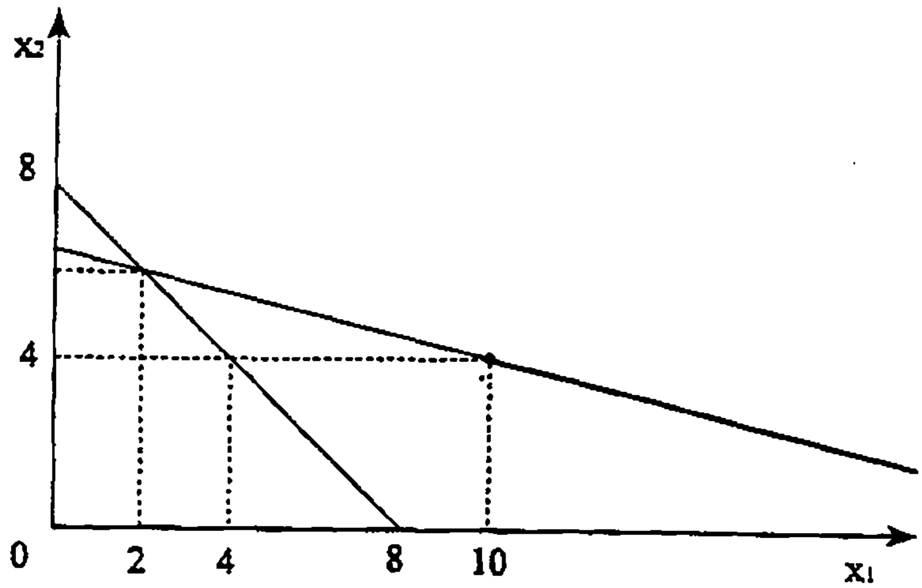


Figure 3.D.7(b)

(c) If the preference is quasilinear with respect to the second good, then then we can take a utility function $u(\cdot)$ so that $\partial u(x)/\partial x_2 = 1$ for every x (Exercise 3.C.5(b)). Hence the first-order condition implies $\partial u(x^t)/\partial x_1^t = p_1^t/p_2^t$ for each $t = 0,1$. Since $p_1^0/p_2^0 > p_1^1/p_2^1$ and $u(\cdot)$ is concave, we must have $x_1^0 < x_1^1$. Thus x^1 has to be on the bold line in the following figure.

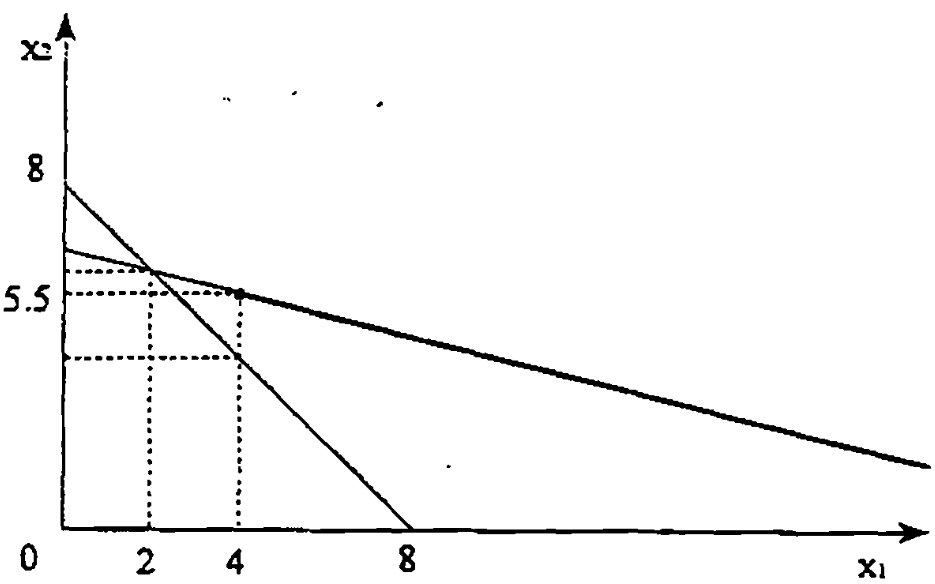


Figure 3.D.7(c)

(d) Since $p^1 \cdot x^0 < w^1$ and the relative price of good 1 decreased, x_1^t has to increase if good 1 is normal. If good 2 is normal, then the wealth effect (positive) and the substitution effect (negative) go in opposite direction which gives us no additional information about x_2. Thus x^1 has to be on the bold line in the following figure.

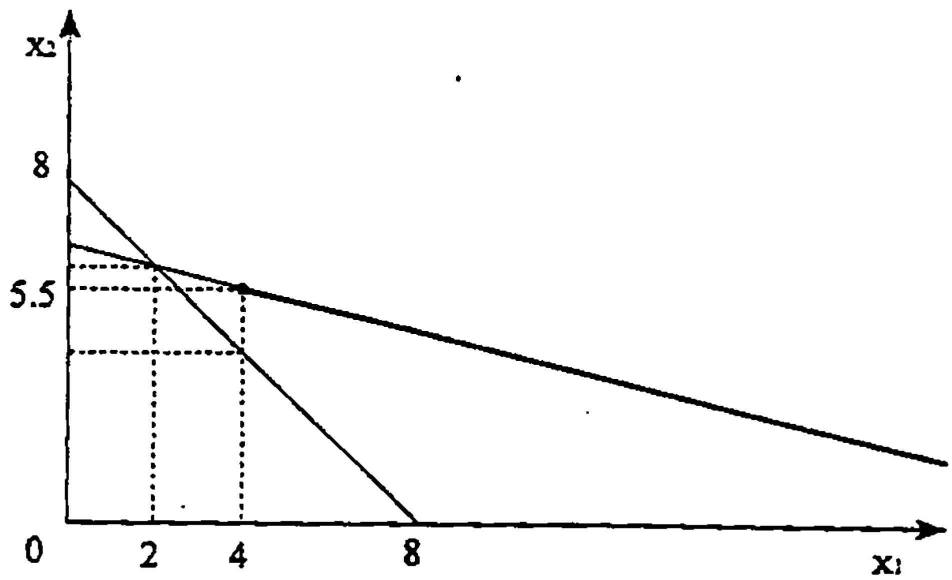


Figure 3.D.7(d)

(e) If the preference is homothetic, the the marginal rates of substitution at all vectors on a ray are the same, and they becomes less steep as the ray becomes flatter. By the first-order conditions and $p_1^0/p_2^0 > p_1^1/p_2^1$, x^1 has to be on the right side of the ray that goes through x^0. Thus x^1 has to be on the bold line in the following figure.

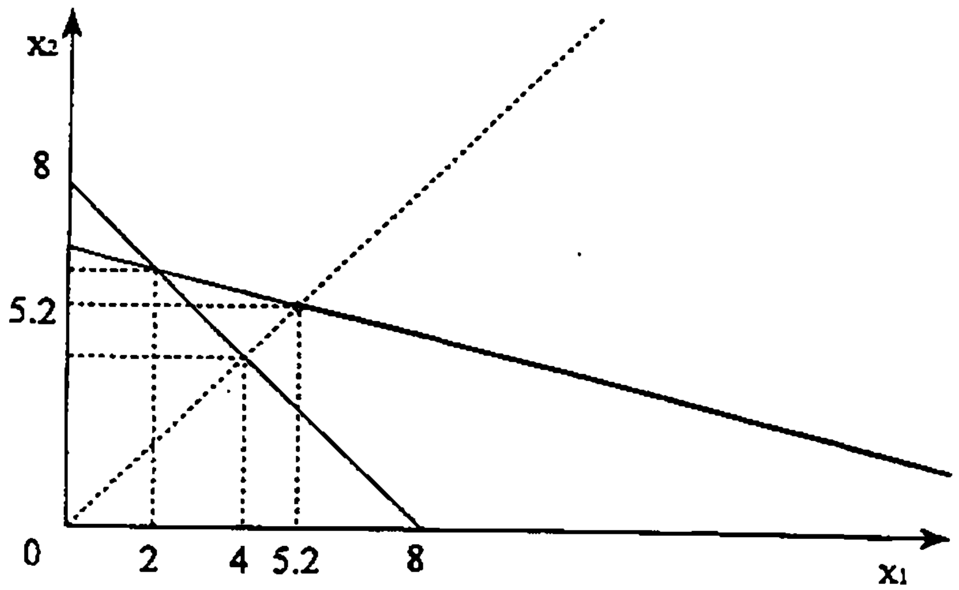


Figure 3.D.7(e)

3.D.8 By Proposition 3.D.3(i), $v(\alpha p, \alpha w) = v(p,w)$ for all $\alpha > 0$. By differentiating this equality with respect to α and evaluating at $\alpha = 1$, we obtain $\nabla_p v(p,w)\cdot p + w\partial v(p,w)/\partial w = 0$. Thus $w\partial v(p,w)/\partial w = -\nabla_p v(p,w)\cdot p$.

3.E.1 The EMP is equivalent to the following maximization problem:

$$\text{Max} \quad -p\cdot x \quad \text{s.t.} \quad u(x) \ge u \text{ and } x \ge 0.$$

The Kuhn-Tucker condition (Theorem M.K.2) implies that the first-order conditions are that there exists $\lambda > 0$ and $\mu \in \mathbb{R}^L_+$ such that $p = \lambda\nabla u(x^*) + \mu$ and $\mu\cdot x^* = 0$. That is, for some $\lambda > 0$, $p \le \lambda\nabla u(x^*)$ and $x^*\cdot(p - \lambda\nabla u(x^*)) = 0$. This is the same as that of the UMP.

3.E.2 To check the homogeneity of the expenditure function,

$$\begin{aligned} e(\lambda p,u) &= \alpha^{-\alpha}(1-\alpha)^{\alpha-1}(\lambda p_1)^{\alpha}(\lambda p_2)^{1-\alpha}u \\ &= \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\lambda^{\alpha+1-\alpha}p_1^{\alpha}p_2^{1-\alpha}u = e(p,u). \end{aligned}$$

To check the monotonicity,

$$\partial e(p,u)/\partial u = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha} > 0,$$
$$\partial e(p,u)/\partial p_1 = \alpha^{1-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha-1}p_2^{1-\alpha} > 0,$$
$$\partial e(p,u)/\partial p_2 = \alpha^{\alpha}(1-\alpha)^{\alpha}p_1^{\alpha}p_2^{-\alpha} > 0.$$

To check the concavity, it is easy to actually calculate $D_p^2 e(p,u)$ and then apply the condition in Exercise 2.F.9 to show that $D_p^2 e(p,u)$ is negative semidefinite. An alternative way is to only calculate

$$\partial^2 e(p,u)/\partial p_1^2 = -\alpha^{1-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha-2}p_2^{1-\alpha} < 0.$$

Then note that the homogeneity implies that $D_p^2 e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2 e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$h_1(\lambda p,u) = \left(\frac{\alpha\lambda p_2}{(1-\alpha)\lambda p_1}\right)^{1-\alpha} u = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} u = h_1(p,u),$$
$$h_2(\lambda p,u) = \left(\frac{(1-\alpha)\lambda p_1}{\alpha\lambda p_2}\right)^{\alpha} u = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u = h_2(p,u).$$

To check the no excess utility,

$$u(h(p,u)) = \left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} u\right)^{\alpha}\left(\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u\right)^{1-\alpha}$$
$$= \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)\alpha-\alpha(1-\alpha)} u^{\alpha+(1-\alpha)} = u.$$

The uniqueness is obvious.

3.E.3 Let $\bar{x} \in \mathbb{R}^L_+$ and $u(\bar{x}) \geq u$. Define $A = \{x \in \mathbb{R}^L_+ : p\cdot x \leq p\cdot\bar{x} \text{ and } u(x) \geq u\}$. Then $A \neq \emptyset$ by $\bar{x} \in A$. Furthermore, A is compact: The closedness follows from that of $\{x \in \mathbb{R}^L : u(x) \geq u\}$ and of $\mathbb{R}^L_+$; the boundedness follows from the inclusion

$$A \subset \{x \in \mathbb{R}^L : 0 \leq x_\ell \leq p\cdot\bar{x}/p_\ell \text{ for every } \ell = 1,\ldots,L\}.$$

Now consider the truncated EMP:

$$\text{Min } p \cdot x \text{ s.t. } x \in A.$$

Since $p \cdot x$ is a continuous function and A is a compact set, this problem has a solution, denoted by $x^* \in A$. We shall show that this is also a solution to the original EMP. Let $x \in \mathbb{R}^L_+$ and $u(x) \geq u$. If $x \in A$, then $p \cdot x \geq p \cdot x^*$ because x^* is a solution to the truncated EMP. If $x \notin A$, then $p \cdot x > p \cdot \bar{x}$ and hence $p \cdot x > p \cdot x^*$. Thus x^* is a solution of the original EMP.

3.E.4 Suppose first that $\succsim$ is convex and that $x \in h(p,u)$ and $x' \in h(p,u)$. Then $p \cdot x = p \cdot x'$ and $u(x) \geq u$, $u(x') \geq u$. Let $\alpha \in [0,1]$ and define $x'' = \alpha x + (1 - \alpha)x'$. Then $p \cdot x'' = \alpha p \cdot x + (1 - \alpha)p \cdot x' = p \cdot x = p \cdot x'$ and, by the convexity of $\succsim$, $u(x'') \geq u$. Thus $x'' \in h(p,u)$.

Suppose next that $\succsim$ is strictly convex and that $x \in h(p,u)$, $x' \in h(p,u)$, $x \neq x'$, and $u(x) \geq u(x') \geq u$. By the argument above, $x'' = \alpha x + (1 - \alpha)x'$ with $\alpha \in (0,1)$ satisfies $p \cdot x'' = p \cdot x = p \cdot x'$ and, by the strict convexity of $\succsim$, we have $x'' \succ x'$. Since $\succsim$ is continuous, $\beta x'' \succ x'$ for any $\beta \in (0,1)$ close enough to 1. But this implies that $p \cdot (\beta x'') < p \cdot x$ and $u(\beta x'') > u(x') \geq u$, which contradicts the fact that x is a solution of the EMP. Hence $h(p,u)$ must be a singleton.

3.E.5 [First printing errata: The equality $h(p,u) = \tilde{h}(p)u$ should be $h(p,u) = u\tilde{h}(p)$, because u is a scalar and $\tilde{h}(p)$ is a vector.] We shall first prove that, for every $p >> 0$, $u \geq 0$, $\alpha \geq 0$, and $x \geq 0$, if $x = h(p,u)$, then $\alpha x = h(p,\alpha u)$. In fact, note that $u(\alpha x) = \alpha u(x) \geq \alpha u$, that is, αx satisfies the constraint of the EMP for αu. Let $y \in \mathbb{R}^L_+$ and $u(y) \geq \alpha u$. Then $u(\alpha^{-1}y) \geq u$. Hence $p \cdot (\alpha^{-1}y) \geq p \cdot x$. Thus $p \cdot y \geq p \cdot (\alpha x)$. Hence $\alpha x = h(p,\alpha u)$. Therefore

$h(p,u)$ is homogeneous of degree one in u.

By this result,

$$e(p,\alpha u) = p\cdot h(p,\alpha u) = p\cdot(\alpha h(p,u)) = \alpha(p\cdot h(p,u)) = \alpha e(p,u).$$

Thus the expenditure function is homogeneous of degree one in u.

Now define $\tilde{h}(p) = h(p,1)$ and $\tilde{e}(p) = e(p,1)$, then $h(p,u) = u\tilde{h}(p)$ and $e(p,u) = u\tilde{e}(p)$.

3.E.6 Define $\delta = \rho/(\rho - 1)$, then the expenditure function and the Hicksian demand function are derived from the first-order conditions of the EMP and they are as follows:

$$h(p,u) = u(p_1^{\delta} + p_2^{\delta})^{(1-\delta)/\delta}(p_1^{\delta-1}, p_2^{\delta-1}),$$
$$e(p,u) = u(p_1^{\delta} + p_2^{\delta})^{1/\delta}.$$

To check the homogeneity of the expenditure function,

$$e(\alpha p,u) = u((\alpha p_1)^{\delta} + (\alpha p_2)^{\delta})^{1/\delta} = \alpha^{\delta\cdot 1/\delta}u(p_1^{\delta} + p_2^{\delta})^{1/\delta} = \alpha e(p,u).$$

To check the monotonicity,

$$\partial e(p,u)/\partial u = (p_1^{\delta} + p_2^{\delta})^{1/\delta} > 0,$$
$$\partial e(p,u)/\partial p_\ell = up_\ell^{\delta-1}(p_1^{\delta} + p_2^{\delta})^{1/\delta-1} > 0.$$

To check the concavity, it is a bit lengthy but easy to actually calculate $D_p^2 e(p,u)$ and then apply the condition in Exercise 2.F.9 to show that $D_p^2 e(p,u)$ is negative semidefinite. An alternative way is to only calculate

$$\begin{aligned}
&\partial^2 e(p,u)/\partial p_1^2 \\
&= u(\delta - 1)p_1^{\delta-2}(p_1^{\delta} + p_2^{\delta})^{1/\delta-1} + up_1^{\delta-1}(p_1^{\delta} + p_2^{\delta})^{1/\delta-2}(1/\delta - 1)\delta p_1^{\delta-1} \\
&= u(1 - \delta)p_1^{\delta-2}(p_1^{\delta} + p_2^{\delta})^{1/\delta-2}(p_1^{\delta} - (p_1^{\delta} + p_2^{\delta})) < 0,
\end{aligned}$$

by $\delta < 1$. Then note that the homogeneity implies that $D_p^2 e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2 e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$h(\alpha p,u) = u((\alpha p_1)^{\delta} + (\alpha p_2)^{\delta})^{(1-\delta)/\delta}((\alpha p_1)^{\delta-1},(\alpha p_2)^{\delta-1})$$

$$= \alpha^{(\delta(1-\delta)/\delta)+(\delta-1)}u(p_1^{\delta} + p_2^{\delta})^{(1-\delta)/\delta}(p_1^{\delta-1},p_2^{\delta-1})$$

$$= h(p,u).$$

To check no excess utility,

$$u(h(p,u)) = u(p_1^{\delta} + p_2^{\delta})^{(1-\delta)/\delta}(p_1^{(\delta-1)\rho} + p_2^{(\delta-1)\rho})^{1/\rho}$$

Since $(\delta - 1)/\delta = -1/\rho$, we obtain $u(h(p,u)) = u$. The uniqueness is obvious.

3.E.7 In Exercise 3.C.5(b), we showed that every quasilinear preference with respect to good 1 can be represented by a utility function of the form $u(x) = x_1 + \tilde{u}(x_2,\dots,x_L)$. Let $e_1 = (1,0,\dots,0) \in \mathbb{R}^L$. We shall prove that for every $p \gg 0$ with $p_1 = 1$, $u \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $x \in (-\infty, \infty) \times \mathbb{R}_+^{L-1}$, if $x = h(p,u)$, then $x + \alpha e_1 = h(p, u + \alpha)$. Note first that $u(x + \alpha e_1) \geq u + \alpha$, that is, $x + \alpha e_1$ satisfies the constraint of the EMP for $(p, u + \alpha)$. Let $y \in \mathbb{R}_+^L$ and $u(y) \geq u + \alpha$. Then $u(y - \alpha e_1) \geq u$. Hence $p\cdot(y - \alpha e_1) \geq p\cdot x$. Thus $p\cdot y \geq p\cdot(x + \alpha e_1)$. Hence $x + \alpha e_1 = h(p, u + \alpha)$.

Therefore, for every $\ell \in \{2,\dots,L\}$, $u \in \mathbb{R}$, and $u' \in \mathbb{R}$, $h_\ell(p,u) = h_\ell(p,u')$. That is, the Hicksian demand functions for goods $2,\dots,L$ are independent of utility levels. Thus, if we define $\tilde{h}(p) = h(p,0)$, then $h(p,u) = \tilde{h}(p) + ue_1$.

Since $h(p, u + \alpha) = h(p,u) + \alpha e_1$, we have $e(p, u + \alpha) = e(p,u) + \alpha$. Thus, if we define $\tilde{e}(p) = e(p,0)$, then $e(p,u) = \tilde{e}(p) + u$.

3.E.8 We use the utility function $u(x) = x_1^{\alpha}x_2^{(1-\alpha)}$. To prove (3.E.1),

$$e(p,v(p,w)) = \alpha^{-\alpha}(1 - \alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}(\alpha^{\alpha}(1 - \alpha)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}w) = w,$$

$$(p,e(p,u)) = \alpha^{\alpha}(1 - \alpha)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}(\alpha^{-\alpha}(1 - \alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u) = u.$$

To prove (3.E.3),

$$x(p,e(p,u)) = (\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u)(\alpha/p_1,\ (1-\alpha)/p_2)$$

$$= \left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}u,\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}u\right) = h(p,u),$$

$$h(p,v(p,w)) = \alpha^{\alpha}(1-\alpha)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}w\left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha},\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\right)$$

$$= w(\alpha/p_1,\ (1-\alpha)/p_2) = x(p,w).$$

3.E.9 First, we shall prove that Proposition 3.D.3 implies Proposition 3.E.2 via (3.E.1). Let $p \gg 0$, $p' \gg 0$, $u \in \mathbb{R}$, $u' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) Homogeneity of degree one in p: Let $\alpha > 0$. Define $w = e(p,u)$, then $u = v(p,w)$ by the second relation of (3.E.1). Hence

$$e(\alpha p,u) = e(\alpha p,v(p,w)) = e(\alpha p,v(\alpha p,\alpha w)) = \alpha w = \alpha e(p,u),$$

where the second equality follows from the homogeneity of $v(\cdot,\cdot)$ and the third from the first relation of (3.E.1).

(ii) Monotonicity: Let $u' > u$. Define $w = e(p,u)$ and $w' = e(p,u')$, then $u = v(p,w)$ and $u' = v(p,w')$. By the monotonicity of $v(\cdot,\cdot)$ in w, we must have $w' > w$, that is, $e(p',u) > e(p,u)$.

Next let $p' \geq p$. Define $w = e(p,u)$ and $w' = e(p',u)$, then, by the second relation of (3.E.1), $u = v(p,w) = v(p',w')$. By the monotonicity of $v(\cdot,\cdot)$, we must have $w' \geq w$, that is, $e(p',u) \geq e(p,u)$.

(iii) Concavity: Let $\alpha \in [0,1]$. Define $w = e(p,u)$ and $w' = e(p',u)$, then $u = v(p,w) = v(p',w)$. Define $p'' = \alpha p + (1-\alpha)p''$ and $w'' = \alpha w + (1-\alpha)w'$. Then, by the quasiconvexity of $v(\cdot,\cdot)$, $v(p'',w'') \leq u$. Hence, by the monotonicity of $v(\cdot,\cdot)$ in w and the second relation of (3.E.1), $w'' \leq e(p'',u)$. that is,

$$e(\alpha p + (1-\alpha)p',\ u) \geq \alpha e(p,u) + (1-\alpha)e(p',u).$$

(iv) Continuity: It is sufficient to prove the following statement: For any sequence $\{(p^n,u^n)\}_{n=1}^{\infty}$ with $(p^n,u^n) \to (p,u)$ and any w, if $e(p^n,u^n) \leq w$ for every n, then $e(p,u) \leq w$; if $e(p^n,u^n) \geq w$ for every n, then $e(p,u) \geq w$. Suppose that $e(p^n,u^n) \leq w$ for every n. Then, by the monotonicity of $v(\cdot,\cdot)$ in w, and the second relation of (3.E.1), we have $u^n \leq v(p^n,w)$ for every n. By the continuity of $v(\cdot,\cdot)$, $u \leq v(p,w)$. By the second relation of (3.E.1) and the monotonicity of $v(\cdot,\cdot)$ in w, we must have $e(p,u) \leq w$. The same argument can be applied for the case with $e(p^n,u^n) \geq w$ for every n.

Let's next prove that Proposition 3.E.2 implies Proposition 3.D.3 via (3.E.1). Let $p \gg 0$, $p' \gg 0$, $w \in \mathbb{R}$, $w' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) Homogeneity: Let $\alpha > 0$. Define $u = v(p,w)$. Then, by the first relation of (3.E.1), $e(p,u) = w$. Hence

$$v(\alpha p,\alpha w) = v(\alpha p,\alpha e(p,w)) = v(\alpha p,e(\alpha p,u)) = u = v(p,w),$$

where the second equality follows from the homogeneity of $e(\cdot,\cdot)$ and the third from the second relation of (3.E.1).

(ii) Monotonicity: Let $w' > w$. Define $u = v(p,w)$ and $u' = v(p,w')$, then $e(p,u) = w$ and $e(p,u') = w'$. By the monotonicity of $e(\cdot,\cdot)$ and $w' > w$, we must have $u' > u$, that is, $v(p,w') > v(p,w)$.

Next, assume that $p' \geq p$. Define $u = v(p,w)$ and $u' = v(p',w)$, then $e(p,u) = e(p',u') = w$. By the monotonicity of $e(\cdot,\cdot)$ and $p' \geq p$, we must have $u' \leq u$, that is, $v(p,w) \geq v(p',w)$.

(iii) Quasiconvexity: Let $\alpha \in [0,1]$. Define $u = v(p,w)$ and $u' = v(p',w')$. Then $e(p,u) = w$ and $e(p,u') = w'$. Without loss of generality, assume that $u' \geq u$. Define $p'' = \alpha p + (1 - \alpha)p'$ and $w = \alpha w + (1 - \alpha)w'$. Then

$$\begin{aligned} &e(p'',u') \\ &\geq \alpha e(p,u') + (1 - \alpha)e(p',u') \end{aligned}$$

$$\geq \alpha e(p,u) + (1-\alpha)e(p',u')$$

$$= \alpha w + (1-\alpha)w' = w'',$$

where the first inequality follows from the concavity of $e(\cdot,u)$, the second from the monotonicity of $e(\cdot,\cdot)$ in u and $u' \geq u$. We must thus have $v(p'',w'') \leq u'$.

(iv) Continuity: It is sufficient to prove the following statement. For any sequence $\{(p^n,w^n)\}_{n=1}^{\infty}$ with $(p^n,w^n) \to (p,w)$ and any u, if $v(p^n,w^n) \leq u$ for every n, then $v(p,w) \leq u$; if $v(p^n,w^n) \geq u$ for every n, then $v(p,w) \geq u$. Suppose that $v(p^n,w^n) \leq u$ for every n. Then, by the monotonicity of $e(\cdot,\cdot)$ in u and the first relation of (3.E.1), we have $w^n \leq e(p^n,u)$ for every n. By the continuity of $e(\cdot,\cdot)$, $w \leq e(p,u)$. We must thus have $v(p,w) \leq u$. The same argument can be applied for the case with $v(p^n,w^n) \geq u$ for every n.

An alternative, simpler way to show the equivalence on the concavity/quasiconvexity and the continuity uses what is sometimes called the epigraph.

For the concavity/quasiconvexity, the concavity of $e(\cdot,u)$ is equivalent to the convexity of the set $\{(p,w): e(p,u) \geq w\}$ and the quasi-convexity of $v(\cdot)$ is equivalent to the convexity of the set $\{(p,w): v(p,w) \leq u\}$ for every u. But (3.E.1) and the monotonicity imply that $v(p,w) \leq u$ if and only if $e(p,u) \geq w$. Hence the two sets coincide and the quasiconvexity of $v(\cdot)$ is equivalent to the concavity of $e(\cdot,u)$.

As for the continuity, the function $e(\cdot)$ is continuous if and only if both $\{(p,w,u): e(p,u) \leq w\}$ and $\{(p,w,u): e(p,u) \geq w\}$ are closed sets. The function $v(\cdot)$ is continuous if and only if both $\{(p,w,u): v(p,w) \geq u\}$ and $\{(p,w,u): v(p,w) \leq u\}$ are closed sets. But, again by (3.E.1) and the monotonicity,

$\{(p,w,u): e(p,u) \le w\} = \{(p,w,u): v(p,w) \ge u\}$;

$\{(p,w,u): e(p,u) \ge w\} = \{(p,w,u): v(p,w) \le u\}$.

Hence the continuity of $e(\cdot)$ is equivalent to that of $v(\cdot)$.

3.E.10 [First printing errata: Proposition 3.E.4 should be Proposition 3.E.3.]

Let's first prove that Proposition 3.D.2 implies Proposition 3.E.3 via the relations of (3.E.1) and (3.E.4). Let $p \in \mathbb{R}^L_{++}$ and $u \in \mathbb{R}$.

(i) Homogeneity: Let $\alpha > 0$. Define $w = e(p,u)$, then $u = v(p,w)$ by the second relation of (3.E.1). Hence

$$h(\alpha p,u) = x(\alpha p,e(\alpha p,u)) = x(\alpha p,\alpha e(p,u)) = x(p,e(p,u)) = h(p,u),$$

where the first equality follows from by the first relation of (3.E.4), the second from the homogeneity of $e(\cdot,u)$, the third from the homogeneity of $x(\cdot,\cdot)$, and the last from by the first relation of (3.E.4).

(ii) No excess utility: Let (p,u) be given and $x \in h(p,u)$. Then $x \in x(p,e(p,u))$ by the first relation of (3.E.4). Thus $u(x) = v(p,e(p,u)) = u$ by the second relation of (3.E.1).

(iii) Convexity/Uniqueness: Obvious.

Let's first prove that Proposition 3.E.3 implies Proposition 3.D.2. via the relations of (3.E.1) and (3.E.4). Let $p \in \mathbb{R}^L_{++}$ and $w \in \mathbb{R}$.

(i) Homogeneity: Let $\alpha > 0$ and define $w = e(p,u)$, then $v(p,w) = u$. Hence

$$x(\alpha p,\alpha w) = h(\alpha p,v(\alpha p,\alpha w)) = h(\alpha p,v(p,w)) = h(p,v(p,w)) = x(p,w),$$

where the first equality follows from the second relation of (3.E.4), the second from the homogeneity of $v(\cdot)$, the third from the homogeneity of $h(\cdot)$ in p, and the last from the first relation of (3.E.4).

(ii) Walras' law: Let (p,w) be given and $x \in x(p,w)$. Then $x \in h(p,v(p,w))$ by the second relation of (3.E.4). Thus $p \cdot x = e(p, v(p,w)) = w$ by the definition of the Hicksian demand and the first relation of (3.E.1).

(iii) Convexity/Uniqueness: Obvious.

3.F.1 Denote by A the intersection of the half spaces that includes K, then clearly $A \supset K$. To show the inverse inclusion, let $\bar{x} \notin K$, then, since K is a closed convex set, the separating hyperplane theorem (Theorem M.G.2) implies that there exists a $p \neq 0$ and c, such that $p \cdot \bar{x} < c < p \cdot x$ for every $x \in K$. Then $\{z \in \mathbb{R}^L : p \cdot z \geq c\}$ is a half space that includes K but does not contain x. Hence $\bar{x} \notin A$. Thus $K \supset A$.

3.F.2 If K is not a convex set, then there exists $x \in K$ and $y \in K$ such that $(1/2)x + (1/2)y \notin K$, as depicted in the figure below. The intersection of all the half-spaces containing K (which also means containing x and y) will contain the point $(1/2)x + (1/2)y$, since half-spaces are convex and the intersection of convex sets is convex. Therefore, the point $(1/2)x + (1/2)y$ cannot be separated from K.

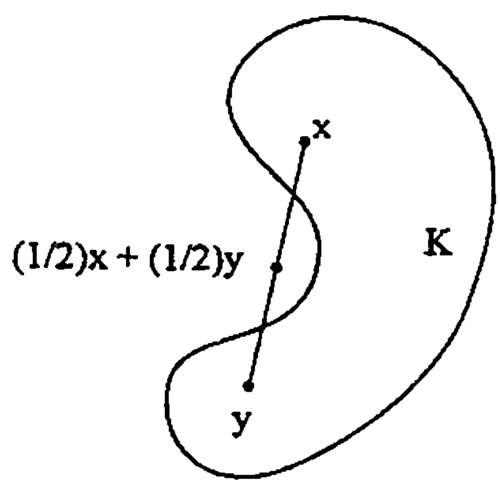


Figure 3.F.2

As for the second statement, if K is not convex, then there exist $x \in K$, $y \in K$, and $\alpha \in [0,1]$ such that $\alpha x + (1 - \alpha)y \notin K$. Since every half space that includes K also contains $\alpha x + (1 - \alpha)y$, it cannot be separated from K.

3.G.1 Since the identity $v(p,e(p,u)) = u$ holds for all p, differentiation with respect to p yields

$$\nabla_p v(p,e(p,u)) + (\partial v(p,e(p,u))/\partial w)\nabla_p e(p,u) = 0.$$

By Roy's identity,

$$(\partial v(p,e(p,u))/\partial w)(- x(p,e(p,u)) + \nabla_p e(p,u)) = 0.$$

By $\partial v(p,e(p,u))/\partial w > 0$ and $h(p,u) = x(p,e(p,u))$, we obtain $h(p,u) = \nabla_p e(p,u)$.

3.G.2 From Examples 3.D.1 and 3.E.1, for the utility function $u(x) = x_1^{\alpha}x_2^{1-\alpha}$, we obtain

$$D_w x(p,w) = \begin{bmatrix} \alpha/p_1 \\ (1 - \alpha)/p_2 \end{bmatrix},$$

$$D_p x(p,w) = \begin{bmatrix} - \alpha w/p_1^2 & 0 \\ 0 & - (1 - \alpha)w/p_2^2 \end{bmatrix},$$

$$\nabla e(p,u) = u(p_1/\alpha)^{\alpha}(p_2/(1 - \alpha))^{1-\alpha} \begin{bmatrix} \alpha/p_1 \\ (1 - \alpha)/p_2 \end{bmatrix},$$

$$\begin{aligned} D_p e(p,u) &= D_p h(p,u) \\ &= u(p_1/\alpha)^{\alpha}(p_2/(1 - \alpha))^{1-\alpha} \begin{bmatrix} - \alpha(1 - \alpha)/p_1^2 & \alpha(1 - \alpha)/p_1p_2 \\ \alpha(1 - \alpha)/p_1p_2 & - \alpha(1 - \alpha)/p_2^2 \end{bmatrix}. \end{aligned}$$

The indirect utility function for $u(x) = x_1^{\alpha}x_2^{1-\alpha}$ is

$$v(p,w) = (p_1/\alpha)^{\alpha}(p_2/(1 - \alpha))^{1-\alpha}w.$$

(Note here that the indirect utility function obtained in Example 3.D.2 is for the utility function $u(x) = \alpha \ln x_1 + (1 - \alpha)\ln x_2$.) Thus

$$\nabla_p v(p,w) = v(p,w)(- \alpha/p_1, - (1 - \alpha)/p_2),$$

$$\nabla_w v(p,w) = v(p,w)/w.$$

Hence:

$h(p,u) = \nabla_p e(p,u)$,

$D_p^2 e(p,u) = D_p h(p,u)$, which is negative semidefinite and symmetric,

$D_p h(p,u)p = 0$,

$D_p h(p,u) = D_p x(p,w) + D_w x(p,w)x(p,w)^T$,

$x_\ell(p,w) = -(\partial v(p,u)/\partial p_\ell)/(\partial v(p,u)/\partial w)$.

3.C.3 (a) Suppose that $\alpha + \beta + \gamma = 1$. Note that

$$\ln u(x) = \alpha\ln(x_1 - b_1) + \beta\ln(x_2 - b_2) + \gamma\ln(x_3 - b_3).$$

By the first-order condition of the EMP,

$$h(p,u) = (b_1,b_2,b_3) + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\alpha/p_1,\beta/p_2,\gamma/p_3).$$

Plug this into $p\cdot h(p,u)$, then we obtain the expenditure function

$$e(p,u) = p\cdot b + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma,$$

where $p\cdot b = p_1b_1 + p_2b_2 + p_3b_3$.

To check the homogeneity of the expenditure function,

$$e(\lambda p,u) = \lambda p\cdot b + u(\lambda p_1/\alpha)^\alpha(\lambda p_2/\beta)^\beta(\lambda p_3/\gamma)^\gamma$$
$$= \lambda p\cdot b + \lambda u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma = \lambda e(p,u).$$

To check the monotonicity, assume $b_1 \geq 0$, $b_2 \geq 0$, and $b_3 \geq 0$. Then

$$\partial e(p,u)/\partial u = (p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma > 0,$$

$$\partial e(p,u)/\partial p_1 = b_1 + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\alpha/p_1) > 0,$$

$$\partial e(p,u)/\partial p_2 = b_2 + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\beta/p_2) > 0,$$

$$\partial e(p,u)/\partial p_3 = b_3 + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\gamma/p_3) > 0.$$

To check the concavity, we can show that $D_p^2 e(p,u)$ is equal to

$$u(p_1/\alpha)^{\alpha}(p_2/\beta)^{\beta}(p_3/\gamma)^{\gamma}\begin{bmatrix} -\alpha(1-\alpha)/p_1^2 & \alpha\beta/p_1p_2 & \alpha\gamma/p_1p_3 \\ \alpha\beta/p_1p_2 & -\beta(1-\beta)/p_2^2 & \beta\gamma/p_2p_3 \\ \alpha\gamma/p_1p_3 & \beta\gamma/p_2p_3 & -\gamma(1-\gamma)/p_3^2 \end{bmatrix}.$$

and then apply the condition in Exercise 2.F.9 to show that $D_p^2 e(p,u)$ is negative semidefinite. An alternative way is to only calculate the 2×2 submatrix obtained from $D_p^2 e(p,u)$ by deleting the last row and the last column and apply the condition in Exercise 2.F.9 to show that it is negative definite. Then note that the homogeneity implies that $D_p^2 e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2 e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$\begin{aligned} h(\lambda p,u) &= (b_1,b_2,b_3) + u(\lambda p_1/\alpha)^{\alpha}(\lambda p_2/\beta)^{\beta}(\lambda p_3/\gamma)^{\gamma}(\alpha/\lambda p_1,\beta/\lambda p_2,\gamma/\lambda p_3) \\ &= (b_1,b_2,b_3) + u\lambda^{\alpha+\beta+\gamma-1}(p_1/\alpha)^{\alpha}(p_2/\beta)^{\beta}(p_3/\gamma)^{\gamma}(\alpha/p_1,\beta/p_2,\gamma/p_3) \\ &= h(p,u). \end{aligned}$$

To check no excess utility,

$$u(h(p,u)) = u(p_1/\alpha)^{\alpha}(p_2/\beta)^{\beta}(p_3/\gamma)^{\gamma}(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} = u.$$

The uniqueness is obvious.

(b) We calculated the derivatives $\partial e(p,u)/\partial p_\ell$ in (a). If we compare them with $h_\ell(p,u)$, then we can immediately see $\partial e(p,u)/\partial p_\ell = h_\ell(p,u)$.

(c) By (b), $D_p h(p,u) = D_p^2 e(p,u)$. In (a), we calculated $D_p^2 e(p,u)$. In Exercise 3.D.6, we showed

$$x(p,w) = (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3)$$

and hence $D_w x(p,w) = (\alpha/p_1,\beta/p_2,\gamma/p_3)$ and

$$D_p x(p,w) = -(w - p\cdot b)\begin{bmatrix} \alpha/p_1^2 & 0 & 0 \\ 0 & \beta/p_2^2 & 0 \\ 0 & 0 & \gamma/p_3^2 \end{bmatrix} - \begin{bmatrix} \alpha/p_1 \\ \beta/p_2 \\ \gamma/p_3 \end{bmatrix}(b_1,b_2,b_3).$$

Using these results, we can verify the Slutsky equation.

(d) Use $D_p h(p,u) = D_p^2 e(p,u)$ and the explicit expression of $D_p^2 e(p,u)$ in (a).

(e) This follows from $S(p,u) = D_p h(p,u) = D_p^2 e(p,u)$ and (a), in which we showed that $D_p^2 e(p,u)$ is negative semidefinite and has rank 2.

3.G.4 (a) Let $a > 0$ and $b \in \mathbb{R}$. Define $\tilde{u}\colon \mathbb{R}_+^L \to \mathbb{R}$ by $\tilde{u}(x) = au(x) + b$ and, for each ℓ, $\tilde{u}_\ell\colon \mathbb{R}_+ \to \mathbb{R}$ by $\tilde{u}_\ell(x_\ell) = au_\ell(x_\ell) + b/L$. Then

$$\tilde{u}(x) = a\textstyle\sum_\ell u_\ell(x_\ell) + b = \sum_\ell (au_\ell(x_\ell) + b/L) = \sum_\ell \tilde{u}_\ell(x_\ell).$$

Thus any linear (to be exact, affine) transformation of a separable utility function is again separable.

Next, we prove that if a monotone transformation of a separable utility function is again separable, then the monotone transformation must be linear (affine). To do this, let's assume that each $u_\ell(\cdot)$ is continuous and strongly monotone. Then, for each ℓ, the range $u_\ell(\mathbb{R}_+)$ is a half-open interval. So let $u_\ell(\mathbb{R}_+) = [a_\ell, b_\ell)$, where b_ℓ may be $+\infty$. Define $c_\ell = b_\ell - a_\ell > 0$, $a = \sum_\ell a_\ell$, $b = \sum_\ell b_\ell$, and $c = \sum_\ell c_\ell$. (If some b_ℓ is equal to $+\infty$, then b and c are $+\infty$ as well.) Then $u(\mathbb{R}_+^L) = [a,b)$. Suppose that $f\colon [a,b) \to \mathbb{R}$ is strongly monotone and the utility function $\tilde{u}(\cdot)$ defined by $\tilde{u}(x) = f(u(x))$ is separable. To simplify the proof, let's assume that $f(\cdot)$ is differentiable. Define $g\colon [0,c) \to \mathbb{R}$ by

$$g(v) = f(v + a) - f(a),$$

then $g(0) = 0$, $g(\cdot)$ is differentiable, and

$$g(u(x) - u(0)) = f(u(x)) - f(u(0))$$

for every $x \in \mathbb{R}^L_+$. Thus, in order to prove that $f(\cdot)$ is linear (affine), it is sufficient to prove that $g(\cdot)$ is linear. For this, it is sufficient to show that the first-order derivatives $g'(v)$ do not depend on the choice of $v \in [0,c)$.

To this end, we shall first prove that if $v_\ell \in [0,c_\ell)$ for each ℓ, then $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$. For this, it is sufficient to prove that

$$g(u(x) - u(0)) = \textstyle\sum_\ell g(u_\ell(x_\ell) - u_\ell(0))$$

for every $x \in \mathbb{R}^L_+$. In fact, by the separability assumption, for each ℓ, there exists a monotone utility function $\tilde{u}_\ell(\cdot)$ such that $\tilde{u}(x) = \sum_\ell \tilde{u}_\ell(x_\ell)$ for every $x \in \mathbb{R}^L_+$. Fix an $x \in \mathbb{R}^L_+$ and, for each ℓ, define $y^\ell \in \mathbb{R}^L_+$ by $y^\ell_\ell = x_\ell$ and $y^\ell_k = 0$ for any $k \neq \ell$. Since $\tilde{u}(y^\ell) = f(u(y^\ell))$,

$$\tilde{u}_\ell(x_\ell) + \textstyle\sum_{k\neq\ell}\tilde{u}_k(0) = f(u_\ell(x_\ell) + \sum_{k\neq\ell} u_k(0)).$$

Subtracting $\sum_{k=1}^L \tilde{u}_k(0) = \tilde{u}(0) = f(u(0))$ from both sides and noticing that $u_\ell(x_\ell) + \sum_{k\neq\ell} u_k(0) = u_\ell(x_\ell) - u_\ell(0) + \sum_{k=1}^L u_k(0)$, we obtain

$$\tilde{u}_\ell(x_\ell) - \tilde{u}_\ell(0) = g(u_\ell(x_\ell) - u_\ell(0)).$$

Summing over ℓ, we obtain

$$\textstyle\sum_\ell \tilde{u}_\ell(x_\ell) - \sum_\ell \tilde{u}_\ell(0) = \sum_\ell g(u_\ell(x_\ell) - u_\ell(0)).$$

Since

$$\textstyle\sum_\ell \tilde{u}_\ell(x_\ell) - \sum_\ell \tilde{u}_\ell(0) = \tilde{u}(x) - \tilde{u}(0) = f(u(x)) - f(u(0)) = g(u(x) - u(0)),$$

we have

$$g(u(x) - u(0)) = \textstyle\sum_\ell g(u_\ell(x_\ell) - u(0)).$$

We have thus proved that $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$.

To prove that the $g'(v)$ do not depend on the choice of $v \in [0,c)$, note first that if $v_\ell \in [0,c_\ell)$ for each ℓ and $v = \sum_\ell v_\ell \in [0,c)$, then $g'(v) = g'(v_\ell)$

for each ℓ. This can be established by differentiating both sides of $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$ with respect to v_ℓ.

So let $v \in [0,c)$ and $v' \in [0,c)$, then, for each ℓ, there exist $v_\ell \in [0,c_\ell)$ and $v'_\ell \in [0,c_\ell)$ such that $v = \sum_\ell v_\ell$ and $v' = \sum_\ell v'_\ell$. Then $g'(v) = g'(v_1)$ and $g'(v') = g'(v'_1)$. Now, for some $\bar{v}_2 \in [0,c_2)$, consider $v_1 + \bar{v}_2 \in [0,c)$ and $v'_1 + \bar{v}_2 \in [0,c)$. Then

$$g'(v_1) = g'(v_1 + \bar{v}_2) = g'(\bar{v}_2),$$
$$g'(v'_1) = g'(v'_1 + \bar{v}_2) = g'(\bar{v}_2).$$

Thus $g'(v_1) = g'(v'_1)$ and hence $g'(v) = g'(v')$.

Note that the above proof by means of derivatives is underlain by the cardinal property of additively separable utility function, which is that, when moving from one commodity vector to another, if the loss in utility from some commodity is exactly compensated by the gain in utility from another, then this must be the case for any of its monotone transformations resulting in another additively separable utility function. (This fact is often much more shortly put into as: utility differences matter.) For example, consider $x \in \mathbb{R}^L_+$ and $x' \in \mathbb{R}^L_+$ such that $u_1(x_1) - u_1(x'_1) = u_2(x'_2) - u_2(x_2) > 0$ and $x_\ell = x'_\ell$ for every $\ell \geq 3$. Since $u_1(x_1) + u_2(x_2) = u_1(x'_1) + u_2(x'_2)$, the separability implies that $u(x) = u(x')$. By the equality $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$,

$g(u_1(x_1) - u_1(0)) + g(u_2(x_2) - u_2(0)) = g(u_1(x'_1) - u_1(0)) + g(u_2(x'_2) - u_2(0)).$

Hence

$g(u_1(x_1) - u_1(0)) - g(u_1(x'_1) - u_1(0)) = g(u_2(x'_2) - u_2(0)) - g(u_2(x_2) - u_2(0)).$

We have shown that, under the differentiability assumption, if this holds for every x_1, x_2, x'_1, and x'_2, then $g(\cdot)$ must be linear.

(b) Define $S = \{1,\ldots,L\}$ and let T be a subset of S. The commodity vectors for those in S are represented by $z_1 = (z_\ell)_{\ell\in T} \in \mathbb{R}_+^{\#T}$ and the like, and the commodity vectors for those outside S are represented by $z_2 = (z_\ell)_{\ell\notin T} \in \mathbb{R}_+^{L-\#T}$ and the like. We shall prove that for every $z_1 \in \mathbb{R}_+^{\#T}$, $z_1' \in \mathbb{R}_+^{\#T}$, $z_2 \in \mathbb{R}_+^{L-\#T}$, and $z_2' \in \mathbb{R}_+^{L-\#T}$, $(z_1,z_2) \succsim (z_1',z_2)$ if and only if $(z_1,z_2') \succsim (z_1',z_2')$. In fact, since $u(\cdot)$ represents $\succsim$, $(z_1,z_2) \succsim (z_1',z_2)$ if and only if

$$\textstyle\sum_{\ell\in T}u_\ell(z_\ell) + \sum_{\ell\notin T}u_\ell(z_\ell) \geq \sum_{\ell\in T}u_\ell(z_\ell') + \sum_{\ell\notin T}u_\ell(z_\ell).$$

Likewise, $(z_1,z_2') \succsim (z_1',z_2')$ if and only if

$$\textstyle\sum_{\ell\in T}u_\ell(z_\ell) + \sum_{\ell\notin T}u_\ell(z_\ell') \geq \sum_{\ell\in T}u_\ell(z_\ell') + \sum_{\ell\notin T}u_\ell(z_\ell').$$

But both of these two inequalities are equivalent to

$$\textstyle\sum_{\ell\in T}u_\ell(z_\ell) \geq \sum_{\ell\in T}u_\ell(z_\ell').$$

Hence they are equivalent to each other.

(c) Suppose that the wealth level w increases and all prices remain unchanged. Then the demand for at least one good (say, good ℓ) has to increase by the Walras' law. From (3.D.4) we know that $u_k'(x_k(p,w)) = (p_k/p_\ell)u_\ell'(x_\ell(p,w))$ for every $k = 1,\dots,L$. Since $x_\ell(p,w)$ increased and $u_\ell(\cdot)$ is strictly concave, the right hand side will decrease. Hence, again since $u_k(\cdot)$ is strictly concave, $x_k(p,w)$ will have to increase. Thus all goods are normal.

(d) The first-order condition of the UMP can be written as

$$\lambda(p,w)p_\ell = \hat{u}'(x_\ell(p,w)),$$

where the Lagrange multiplier $\lambda(p,w)$ is a differentiable function of (p,w): This can be easily seen in the proof of the differentiability of Walrasian demand functions, which is contained in the Appendix.

By differentiating the above first-order condition with respect to p_ℓ, we obtain

$$(\partial\lambda(p,w)/\partial p_\ell)p_\ell + \lambda(p,w) = \hat{u}''(x_\ell(p,w))(\partial x_\ell(p,w)/\partial p_\ell).$$

By differentiating the above first-order condition with respect to p_k ($k \neq \ell$), we obtain

$$(*) \qquad (\partial\lambda(p,w)/\partial p_k)p_\ell = \hat{u}''(x_\ell(p,w))(\partial x_\ell(p,w)/\partial p_k).$$

Thus

$$\begin{aligned}
&d[p\cdot x(p,w)]/dp_k \\
&= d[\textstyle\sum_\ell p_\ell x_\ell(p,w)]/dp_k \\
&= x_k(p,w) + p_k(\partial x_k(p,w)/\partial p_k) + \textstyle\sum_{\ell\neq k} p_\ell(\partial x_\ell(p,w)/p_k). \\
&= x_k(p,w) + \frac{(\partial\lambda(p,w)/\partial p_k)p_k^2 + \lambda(p,w)p_k}{\hat{u}''(x_k(p,w))} + \sum_{\ell\neq k}\frac{(\partial\lambda(p,w)/\partial p_k)p_\ell^2}{\hat{u}''(x_\ell(p,w))} \\
&= x_k(p,w) + \frac{\lambda(p,w)p_k}{\hat{u}''(x_k(p,w))} + (\partial\lambda(p,w)/\partial p_k)\sum_\ell \frac{p_\ell^2}{\hat{u}''(x_\ell(p,w))}.
\end{aligned}$$

By the first-order condition, $\lambda(p,w)p_k = \hat{u}'(x_k(p,w))$ and hence this equals

$$\frac{\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))}\left(\frac{x_k(p,w)\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} + 1\right) + (\partial\lambda(p,w)/\partial p_k)\sum_\ell \frac{p_\ell^2}{\hat{u}''(x_\ell(p,w))}.$$

By Walras' law, this equals zero. By the strong monotonicity and the strict concavity, $\dfrac{\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} < 0$ and $\sum_\ell \dfrac{p_\ell^2}{\hat{u}''(x_\ell(p,w))} < 0$. By the assumption on $\hat{u}(\cdot)$, $\dfrac{x_k(p,w)\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} + 1 > 0$. Hence

$$\frac{\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))}\left(\frac{x_k(p,w)\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} + 1\right) < 0.$$

We must thus have $\partial\lambda(p,w)/\partial p_k < 0$. Hence, by (*), $\partial x_\ell(p,w)/\partial p_k > 0$.

3.G.5 (a) We shall show the following two statements: First, if (x^*,z^*) is a solution to

$$\text{Max}_{(x,z)}\ \tilde{u}(x,z)\quad \text{s.t.}\ p\cdot x + \alpha z \le w,$$

then there exists y^* such that $q\cdot y^* \le z^*$ and (x^*,y^*) is a solution to

$$\text{Max}_{(x,y)}\ u(x,y)\quad \text{s.t.}\ p\cdot x + (\alpha q_0)\cdot y \le w;$$

second, if (x^*,y^*) is a solution to

$$\text{Max}_{(x,y)}\ u(x,y)\quad \text{s.t.}\ p\cdot x + (\alpha q_0)\cdot y \le w,$$

then $(x^*,q_0\cdot y^*)$ is a solution to

$$\text{Max}_{(x,z)}\ \tilde{u}(x,z)\quad \text{s.t.}\ p\cdot x + \alpha z \le w.$$

Suppose first that (x^*,z^*) is a solution to $\text{Max}_{(x,z)}\ \tilde{u}(x,z)$ s.t. $p\cdot x + \alpha z \le w$. Then, by the definition of $\tilde{u}(\cdot)$, there exists y^* such that $q_0\cdot y^* \le z^*$ and $u(x^*,y^*) = \tilde{u}(x^*,z^*)$. Let (x,y) satisfy $p\cdot x + (\alpha q_0)\cdot y \le w$. Then

$$u(x,y) \le \tilde{u}(x,q_0\cdot y) \le u(x^*,z^*) = u(x^*,y^*),$$

where the first inequality follows from the definition of $\tilde{u}(\cdot)$ and the second inequality follows from $p\cdot x + \alpha(q_0\cdot y) = p\cdot x + (\alpha q_0)\cdot y \le w$ and the definition of (x^*,z^*). The first statement is thus established.

As for the second one, suppose that (x^*,y^*) is a solution to $\text{Max}_{(x,y)}$ $u(x,y)$ s.t. $p\cdot x + (\alpha q_0)\cdot y \le w$. For every y, if $q_0\cdot y \le q_0\cdot y^*$, then

$$p\cdot x^* + (\alpha q_0)\cdot y = p\cdot x^* + \alpha(q_0\cdot y) \le p\cdot x^* + \alpha(q_0\cdot y^*) \le w.$$

Hence $u(x^*,y) \le u(x^*,y^*)$. Thus $u(x^*,y^*) = \tilde{u}(x^*,q_0\cdot y^*)$. Now let (x,z) satisfy $p\cdot x + \alpha z \le w$. Then there exists y such that $q_0\cdot y \le z$ and $u(x,y) = \tilde{u}(x,z)$. Thus $p\cdot x + (\alpha q_0)\cdot y = p\cdot x + \alpha(q_0\cdot y) \le p\cdot x + \alpha z \le w$. Hence

$$\tilde{u}(x,z) = u(x,y) \le u(x^*,y^*) = \tilde{u}(x^*,q_0\cdot y^*).$$

(b) (c) These are immediate consequences of the fact that the Walrasian demand functions and the Hicksian demand functions are derived in the standard way, by taking $\tilde{u}(\cdot)$ to be the (primitive) direct utility function.

3.G.6 (a) By applying Walras' law, we obtain $x_3 = (w - x_1p_1 - x_2p_2)/p_3$.

(b) Yes. In fact, for every $\lambda > 0$,

$$100 - 5\lambda p_1/\lambda p_3 + \beta\lambda p_2/\lambda p_3 + \delta\lambda w/\lambda p_3 = 100 - 5p_1/p_3 + \beta p_2/p_3 + \delta w/p_3,$$

$$\alpha + \beta\lambda p_1/\lambda p_3 + \gamma\lambda p_2/\lambda p_3 + \delta\lambda w/\lambda p_3 = \alpha + \beta p_1/p_3 + \gamma p_2/p_3 + \delta w/p_3.$$

(c) By Proposition 3.G.2 and 3.G.3, the Slutsky substitution matrix is symmetric. Thus

$$\beta/p_3 + (\delta/p_3)(\alpha + \beta p_1/p_3 + \gamma p_2/p_3 + \delta w/p_3)$$
$$= \beta/p_3 + (\delta/p_3)(100 - 5p_1/p_3 + \beta p_2/p_3 + \delta w/p_3).$$

Hence by putting $p_3 = 1$ and rearranging terms, we obtain

$$(\beta + \alpha\delta) + \beta\delta p_1 + \gamma\delta p_2 + \delta^2 w = (\beta + 100\delta) - 5\delta p_1 + \beta\delta p_2 + \delta^2 w.$$

Since this equality must hold for all p_1, p_2 and w, we have

$$\beta + \alpha\delta = \beta + 100\delta,\ \beta\delta = -5\delta,\ \gamma\delta = \beta\delta.$$

Hence $\alpha = 100$, $\beta = -5$, and $\gamma = -5$. Therefore,

$$x_1 = x_2 = 100 - 5p_1/p_3 + 5p_2/p_3 + \delta w/p_3.$$

Recall also that all diagonal entries of the Slutsky matrix must be nonpositive. We shall now derive from this fact that $\delta = 0$. Let $p_3 = 1$, then the first diagonal element is equal to

$$-5 + \delta(100 - 5p_1 + 5p_2) + \delta^2 w.$$

If $\delta \neq 0$, then $\delta^2 > 0$ and hence we can always find (p_1,p_2,w) such that the above value is positive. We must thus have $\delta = 0$. In conclusion,

$$x_1 = x_2 = 100 - 5p_1/p_3 + 5p_2/p_3.$$

(d) Since $x_1 = x_2$ for all prices, the consumer's indifference curves in the (x_1,x_2)-plane must be L-shaped ones, with kinks on the diagonal. (So the restricted preference is the Leontief one.) They are depicted in the following figure.

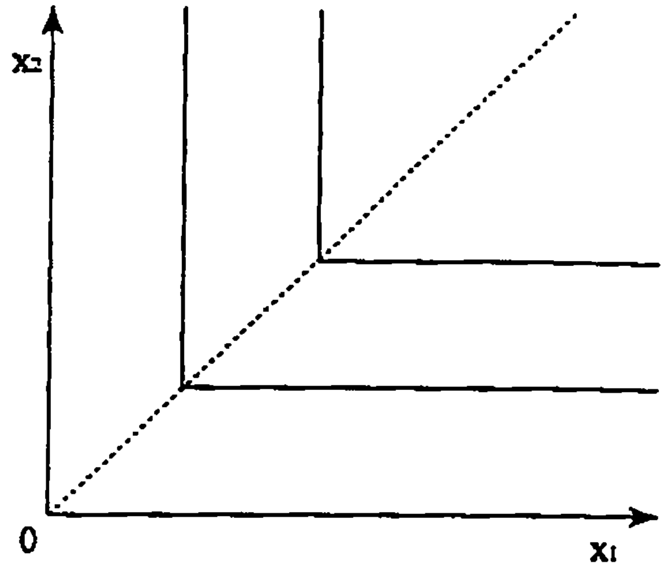


Figure 3.G.6(d)

(e) By (d), for a fixed x_3, the preference for goods 1 and 2 can be represented by $\min\{x_1,x_2\}$. Moreover, there is no wealth effect on the demands for goods 1 and 2. We must thus have

$$u(x_1,x_2,x_3) = \min\{x_1,x_2\} + x_3$$

or a monotone transformation of this.

3.G.7 By the first-order condition of the UMP, there exists $\lambda > 0$ such that $\lambda g(x) = \nabla u(x)$. Premultiply both sides by x, then $\lambda x \cdot g(x) = x \cdot \nabla u(x)$. By Walras' law, $x \cdot g(x) = 1$ and hence $\lambda = x \cdot \nabla u(x)$. Thus

$$g(x) = \lambda^{-1}\nabla u(x) = \frac{1}{x \cdot \nabla u(x)} \nabla u(x)$$

By Exercise 3.D.8, we have $\partial v(p,1)/\partial w = - p \cdot \nabla_p v(p,1)$. By Proposition 3.G.4, $x(p) = - \frac{1}{\partial v(p,1)/\partial w} \nabla_p v(p,1) = \frac{1}{p \cdot \nabla_p v(p,1)} \nabla_p v(p,1)$.

3.G.8 Differentiate the equality $v(p,\alpha w) = v(p,w) + \ln\alpha$ with respect to α and evaluate at $\alpha = 1$, then we obtain $(\partial v(p,w)/\partial w)w = 1$. Hence $\partial v(p,1)/\partial w = 1$. By Proposition 3.G.4, $x(p,1) = - \nabla_p v(p,1)$.

3.G.9 Let $p \gg 0$ and $w > 0$. All the functions and derivatives below are evaluated at (p, w).

By differentiating Roy's identity with respect to p_k, we obtain

$$\partial x_\ell/\partial p_k = \frac{(\partial^2 v/\partial p_\ell \partial p_k)(\partial v/\partial w) - (\partial v/\partial p_\ell)(\partial^2 v/\partial p_\ell \partial w)}{(\partial v/\partial w)^2}.$$

Or, in the matrix notation,

$$D_p x = -\frac{1}{(\nabla_w v)^2}(\nabla_w v D_p^2 v - \nabla_p v D_p \nabla_w v) \in \mathbb{R}^{L \times L}.$$

(Recall that $\nabla_p v$ is a column vector of $\mathbb{R}^L$, and $D_p v$ and $D_p \nabla_w v$ are row vectors of $\mathbb{R}^L$ (Section M.A).) By differentiating Roy's identity with respect to w, we obtain

$$\partial x_\ell/\partial w = \frac{(\partial^2 v/\partial p_\ell \partial w)(\partial v/\partial w) - (\partial v/\partial p_\ell)(\partial^2 v/\partial w^2)}{(\partial v/\partial w)^2}.$$

Or, in matrix notation,

$$D_w x = -\frac{1}{(\nabla_w v)^2}(\nabla_w v \nabla_p \nabla_w v - \nabla_w^2 v \nabla_p v) \in \mathbb{R}^L.$$

Hence

$$S = -\frac{1}{(\nabla_w v)^2}(\nabla_w v D_p^2 v - \nabla_p v D_p \nabla_w v - (\nabla_w v \nabla_p \nabla_w v)(\frac{1}{\nabla_w v} D_p v) + \nabla_w^2 v \; \nabla_p v(\frac{1}{\nabla_w v} D_p v$$

$$= -\frac{1}{(\nabla_w v)^2}(\nabla_w v D_p^2 v - (\nabla_p v D_p \nabla_w v + \nabla_p \nabla_w v D_p v) + \frac{\nabla_w^2 v}{\nabla_w v} \nabla_p v D_p v).$$

It is noteworthy that we can know directly from Roy's identity and this equality that the Slutsky matrix S has all the properties stated in Proposition 3.G.2. To see this, note first that both $\nabla_p v(p,w)$ and $\nabla_w v(p,w)$ are homogeneous of degree -1 (in (p,w)) by Theorem M.B.1. Hence $x(p,w)$ is homogeneous of degree 0 (where we are regarding Roy's identity as defining $x(p,w)$ from $\nabla_p v(p,w)$ and $\nabla_w v(p,w)$). Thus (2.E.2) follows, as proved in Proposition 2.E.1. On the other hand, by Exercise 3.D.8, $p \cdot x(p,w) = w$.

Hence, as proved in Propositions 2.E.2 and 2.E.3, this implies (2.E.5) and (2.E.7). Now, as proved in Exercise 2.F.7, (2.E.2), (2.E.5), and (2.E.7) together imply that $S(p,w)p = 0$ and $p\cdot S(p,w) = 0$.

The matrix $S(p,w)$ is symmetric because $D_p^2 v$, $\nabla_p v D_p \nabla_w v + \nabla_p \nabla_w v D_p v$, and $\nabla_p v D_p v$ are symmetric. The negative semidefiniteness can be shown in the following way. Since $v(\cdot)$ is quasiconvex, for every price-wealth pair (q,b), if $D_p vq + \nabla_w vb = 0$, then $(q,b)\cdot D^2 v\,(q,b) \geq 0$ (Theorem M.C.4). But $D_p vq + \nabla_w vb = 0$ if and only if $b = -D_p vq/\nabla_w v$. Plug this into $(q,b)\cdot D^2 v(q,b)$, then

$$q\cdot D_p^2 vq + 2b(D_p\nabla_w vq) + \nabla_w^2 v\, b^2$$

$$= q\cdot D_p^2 vq - \frac{2D_p\nabla_w vq}{\nabla_w v} D_p vq + \nabla_w^2 v \frac{(D_p vq)^2}{(\nabla_w v)^2}$$

$$= \frac{1}{\nabla_w v}((\nabla_w v)(q\cdot D_p^2 vq) - 2(D_p\nabla_w vq)(D_p vq) + \nabla_w^2 v \frac{(D_p vq)^2}{(\nabla_p v)^2}).$$

Hence the quasiconvexity of $v(\cdot)$ implies that the above expression is nonnegative for every q. On the other hand,

$$q\cdot S(p,w)q = -\frac{1}{(\nabla_w v)^2}((\nabla_w v)(q\cdot D_p^2 vq) - 2(D_p\nabla_w vq)(D_p vq) + \nabla_w^2 v \frac{(D_p vq)^2}{(\nabla_w v)^2}).$$

Thus $q\cdot S(p,w)q \leq 0$. Therefore $S(p,w)$ is negative semidefinite.

3.G.10 We shall prove that $a(p)$ is a constant function and $b(p)$ is homogeneous of degree -1, quasiconvex, and satisfies $b(p) \geq 0$ and $\nabla b(p) \leq 0$ for every $p \gg 0$.

We shall first prove that $a(p)$ must be homogeneous of degree zero. Since $v(p,w)$ is homogeneity of degree zero, we must have $a(\lambda p) + b(\lambda p)\lambda w = a(p) + b(p)w$ for all $p \gg 0$, $w \geq 0$, and $\lambda > 0$. If there are p and λ for which $a(\lambda p) \neq a(p)$, then this constitutes an immediate contradiction with $w = 0$. Thus

$a(p)$ is homogeneous of degree zero.

Next, we show by contradiction that $\nabla a(p) \leq 0$ for every $p \gg 0$. Since $v(p,w)$ is nonincreasing in p, we must have $\nabla a(p) + \nabla b(p)w \leq 0$ for every $p \gg 0$ and $w \geq 0$. If there are $p \gg 0$ and ℓ for which $\partial a(p)/\partial p_\ell > 0$, then this constitutes an immediate contradiction with $w = 0$. Thus $a(p) \leq 0$ for every $p \gg 0$.

We shall now prove that the only function $a(p)$ that is homogeneous of degree zero and satisfies $\nabla a(p) \leq 0$ for every $p \gg 0$ is a constant function. In fact, by differentiating both sides of $a(\lambda p) = a(p)$ with respect to λ and evaluating them at $\lambda = 1$, we obtain $\nabla a(p)p = 0$. Since $p \gg 0$ and $\nabla a(p) \leq 0$, we must have $\nabla a(p) = 0$. Hence $a(p)$ must be constant.

Given this result, the homogeneity of $v(p,w)$ implies that $b(p)$ is homogeneous of degree -1. Since $v(p,w)$ is quasiconvex, so is $a(p)$ as a function of p. Finally, since $\nabla_p v(p,w) = \nabla b(p)w \leq 0$ and $\nabla_w v(p,w) = b(p)$, we must have $b(p) \geq 0$ and $\nabla b(p) \leq 0$ for every $p \gg 0$.

This result implies that, up to a constant, $v(p,w) = b(p)w$ and hence, if the underlying utility function is quasiconcave, then it must be homogeneous of degree one. On the other hand, according to Exercise 3.D.4(b), if the underlying utility function is quasilinear with respect to good 1, then, for all w and $p \gg 0$ with $p_1 = 1$, $v(p,w)$ can be written in the form $\phi(p_2,\dots,p_L) + w$. You will thus wonder why we have ended up excluding this quasilinear case. The reason is that, when we derived $v(p,w) = \phi(p) + w$, we assumed that the consumption set is $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$. Thus $x_1(p,w)$ can be negative and, if so, then $\partial v(p,w)/\partial p_1$ is positive, which we excluded at the beginning of our analysis, following Proposition 3.D.3. (Note that, when we established $\partial v(p,w)/\partial p_\ell \leq 0$ in Proposition 3.D.3, we assumed that the consumption set is

$\mathbb{R}^L_+$.) As $x_1(p,w)$ is positive for sufficiently large $w > 0$, the quasilinear case could be accommodated in our analysis if we assume that the inequality $\partial v(p,w)/\partial p_\ell \le 0$ applies only for sufficiently large $w > 0$. (In this case, we can show that $a(p)$ is homogeneous of degree zero, and $b(p)$ is homogeneous of degree -1, quasiconvex, and satisfies $b(p) \ge 0$ and $\nabla b(p) \le 0$ for every $p \gg 0$.)

3.G.11 Suppose that $v(p,w) = a(p) + b(p)w$. By Roy's identity,

$$x(p,w) = -(1/b(p))\nabla_p a(p) - (w/b(p))\nabla_p b(p).$$

Thus the wealth expansion path is linear in the direction of $\nabla_p b(p)$ and intercept $(-1/b(p))\nabla_p a(p)$.

3.G.12 Note first that, according to Exercise 3.G.11, the wealth expansion path is linear in the direction of $\nabla_p b(p)$ and intercept $(-1/b(p))\nabla_p a(p)$.

If the underlying preference is homothetic, then $(-1/b(p))\nabla_p a(p) = 0$. Hence $a(p)$ must be a constant function. If the underlying utility function is homogeneous of degree one in w, then $v(p,w)$ must be homogeneous of degree one in w by Exercise 3.D.3(a). Hence $a(p) = 0$ for every $p \gg 0$.

If the preference is quasilinear in good 1, then please first go back to the proviso given at the end of the answer to Exercise 3.G.10. After doing so, note that, since the demand for goods $2,\dots,L$ do not depend on w,

$$(-1/b(p))\nabla_p b(p) = (-1/p_1, 0, \dots, 0),$$

or $(\partial b(p)/\partial p_1)/b(p) = 1/p_1$ and $\partial b(p)/\partial p_\ell = 0$ for every $\ell > 1$. Hence $b(p) = \beta p_1^\rho + \gamma$ for some $\beta \ne 0$, $\rho \ne 0$, and $\gamma \in \mathbb{R}$. But, by Exercise 3.G.10, $b(p)$ must be homogeneous of degree -1, positive, and nonincreasing. Hence $\rho = -1$, $\gamma = 0$, and $\beta > 0$. That is, $b(p) = \beta/p_1$ with $\beta > 0$. If the underlying utility function is in the quasilinear form $x_1 + \tilde{u}(x_2,\ldots,x_L)$, then, by Exercise 3.D.4(b), $v(p,w)$ must be written in the form $\phi(p_2,\ldots,p_L) + w$ for all $p \gg 0$ with $p_1 = 1$. Thus $\beta = 1$.

3.G.13 For each $i \in \{0,1,\ldots,n\}$, let a_i be a differentiable function defined on the strictly positive orthant $\{p \in \mathbb{R}^L: p \gg 0\}$. Let $v(p,w) = \sum_{i=0}^{n} a_i(p)w^i$ be an indirect utility function. Denoting the corresponding Walrasian demand function by $x(p,w)$. By Roy's identity,

$$x(p,w) = -\frac{1}{\nabla_w v(p,w)}\nabla_p v(p,w)$$

$$= \frac{1}{\sum_{i=1}^{n} i a_i(p)w^{i-1}}\sum_{i=0}^{n} w^i \nabla a_i(p) = \sum_{i=0}^{n}\frac{w^i}{\sum_{j=1}^{n} j a_j(p)w^{j-1}}\nabla a_i(p).$$

Hence, for any fixed p, the wealth expansion path is contained the linear subspace of $\mathbb{R}^L$ that is spanned by $\nabla a_0(p),\ldots,\nabla a_n(p)$.

As for the interpretation, recall from Exercise 3.G.11 that, an indirect utility function in the Gorman form exhibits linear wealth expansion curves. But the Gorman form is a polynomial of degree one on w and a linear wealth expansion curve is contained in a linear subspace of dimension two. Hence the above result implies that the indirect utility functions that are polynomials on w is a natural extension of the Gorman form.

3.G.14 Define a, b, c, d, e, and f so that

$$\begin{bmatrix} -10 & a & b \\ c & -4 & d \\ 3 & e & f \end{bmatrix}.$$

Since the substitution matrix is symmetric, we know $b = 3$, $a = c$, and $e = d$. By Propositions 2.F.3, $p \cdot S(p,w) = 0$. Hence $p_1(-10) + p_2 c + p_3 3 = 0$. Thus $c = -4$ and $a = c = -4$. For the second column, $p_1(-4) + p_2(-4) + p_3 e = 0$.

Hence e = 2 and d = e = 2. Finally, for the third column, we have $p_1 3 + p_2 2 + p_3 f = 0$. Thus $f = -7/6$. Hence we have

$$\begin{bmatrix} -10 & -4 & 3 \\ -4 & -4 & 2 \\ 3 & 2 & -7/6 \end{bmatrix}.$$

The matrix has all the properties of a substitution matrix, which are symmetry, negative semidefiniteness, $S(p,w)p = 0$, and $p \cdot S(p,w) = 0$. (For negative semidefiniteness, apply the determinant test of Exercise 2.F.10 and Theorem M.D.4(iii).)

3.G.15 (a) $x(p_1,p_2,w) = \left(\dfrac{p_2 w}{p_1 p_2 + 4p_1^2}, \dfrac{4p_1 w}{4p_1 p_2 + p_2^2}\right)$.

(b) $h(p_1,p_2,u) = \left(\left(\dfrac{p_2 u}{2(4p_1 + p_2)}\right)^2, \left(\dfrac{p_1 u}{4p_1 + p_2}\right)^2\right)$.

(c) $e(p_1,p_2,u) = \dfrac{p_1 p_2 u^2}{4(4p_1 + p_2)}$. It is then easy to show that $\nabla_p e(p_1,p_2,u) = h(p_1,p_2,u)$.

(d) $v(p_1,p_2,w) = 2(w/p_1 + 4w/p_2)^{1/2}$. To verify Roy's identity, use

$$\partial v(p_1,p_2,w)/\partial p_1 = (w/p_1 + 4w/p_2)^{-1/2}(-w/p_1^2),$$

$$\partial v(p_1,p_2,w)/\partial p_2 = (w/p_1 + 4w/p_2)^{-1/2}(-4w/p_2^2),$$

$$\partial v(p_1,p_2,w)/\partial w = (w/p_1 + 4w/p_2)^{-1/2}(1/p_1 + 4/p_2).$$

3.G.16 (a) It is easy to check that

$$\partial e(p,u)/\partial p_k = e(p,u)(\alpha_k + u\beta_k(\Pi_\ell p_\ell^{\beta_\ell}))/p_k.$$

Since $e(p,u)$ is nondecreasing in p, this must be nonnegative for all (p,u). But, if $\alpha_k < 0$ and $\|p\|$ is sufficiently small, then this becomes negative. Also, if $\beta_k < 0$ and $\|p\|$ is sufficiently big, then this becomes negative.

Therefore

(1) $$\alpha_k \geq 0 \text{ and } \beta_k \geq 0 \text{ for all } k.$$

It is a little bit manipulation to show that

$$e(\lambda p,u) = \lambda^{\sum\alpha_\ell}\exp((\sum_\ell \alpha_\ell \ln p_\ell) + \lambda^{\sum\beta_\ell}u(\Pi_\ell p_\ell^{\beta_\ell})),$$

$$\lambda e(p,u) = \lambda\exp((\Pi_\ell \alpha_\ell \ln p_\ell) + u(\Pi_\ell p_\ell^{\beta_\ell})).$$

Since $e(p,u)$ is homogeneous of degree one with respect to p, they must be equal for every (p,u) and $\lambda > 0$. Take, for example, $p = (1,\ldots,1)$ and $u = 1$. Then

$$\log e(p,u) = (\sum_\ell \alpha_\ell)\log\lambda + \lambda^{\sum\beta_\ell},$$

$$\log \lambda e(p,u) = \log\lambda + 1.$$

They must be equal for every $\lambda > 0$. Therefore

(2) $$\sum_\ell \alpha_\ell = 1,\ \sum_\ell \beta_\ell = 0.$$

Thus

$$\sum_\ell \alpha_\ell = 1,\ \alpha_\ell \geq 0,\ \beta_\ell = 0.$$

Hence the expenditure function now takes the simplified form:

(3) $$e(p,u) = (\exp u)(\Pi_\ell p_\ell^{\alpha_\ell});\ \sum_\ell \alpha_\ell = 1,\ \alpha_\ell \geq 0.$$

This is increasing with respect to u and concave in p.

(b) By equation (3.E.1), $w = (\exp v(p,w))(\Pi_\ell p_\ell^{\alpha_\ell})$. Hence

(4) $$v(p,w) = \log w - \sum_\ell \alpha_\ell \log p_\ell.$$

(c) By differentiating $e(p,u)$ with respect to p, we obtain

$$h(p,u) = e(p,u)(\alpha_1/p_1,\ldots,\alpha_L/p_L).$$

Since $x(p,w) = h(p,v(p,w))$ and $e(p,v(p,w)) = w$,

(5) $$x(p,w) = w(\alpha_1/p_1,\ldots,\alpha_L/p_L).$$

Use equations (3), (4), and (5) and follows the same method as in the answer to Exercise 3.G.2 to verify Roy's identity and the Slutsky equation.

3.G.17 [First printing errata: The minus sign at the beginning of the right-hand side of the indirect utility function should be deleted. That is, it should be

$$v(p,w) = (w/p_2 + b^{-1}(ap_1/p_2 + a/b + c))\exp(-bp_1/p_2).$$

Also, in (b), the minus sign in front of the first term of the right-hand side of the expenditure function should be deleted. That is, it should be

$$e(p,u) = p_2 u\exp(bp_1/p_2) - (1/b)(ap_1 + ap_2/b + p_2 c).$$

Finally, in (c), the minus sign in front of the first term of the right-hand side of the Hicksian function should be deleted. That is, it should be

$$h(p,u) = ub\exp(bp_1/p_2) - a/b.$$

(a) Use

$$\partial v(p,w)/\partial p_1 = -p_2^{-1}(ap_1/p_2 + bw/p_2 + c)\exp(-bp_1/p_2),$$
$$\partial v(p,w)/\partial w = p_2^{-1}\exp(-bp_1/p_2),$$

and apply Roy's formula.

(b) According to (3.E.1), we can obtain the expenditure function by solving

$$u = (e(p,u)/p_2 + b^{-1}(ap_1/p_2 + a/b + c))\exp(-bp_1/p_2).$$

(c) Apply Proposition 3.G.1 to obtain the given Hicksian demand function for the first good.

3.G.18 We prove the assertion by contradiction. Suppose that there exist $\ell \in \{1,\ldots,L\}$ and $k \in \{1,\ldots,L\}$ such that there is no chain of substitutes connecting ℓ and k. Define $J = \{\ell\} \cup \{j \in \{1,\ldots,L\}$: there is a chain of substitutes connecting ℓ and j$\}$. Since $\ell \in J$ and $k \notin J$, both J and its complement $\{1,\dots,L\}\setminus J$ are nonempty. Moreover, for any $j \in J$ and any $j' \notin J$ $\partial h_j(p,u)/\partial p_{j'} < 0$, because, otherwise, $j' \in J$.

Let u: $\mathbb{R}^L_+ \to \mathbb{R}$ be the underlying utility function. Following the hint, as in Exercise 3.G.5, define $\tilde{u}$: $\mathbb{R}^2_+ \to \mathbb{R}$ by

$$\tilde{u}(y_1,y_2) = \text{Max}\{u(x)\colon x \in \mathbb{R}^L_+, \textstyle\sum_{j\in J} p_j x_j \le y_1, \sum_{j\notin J} p_j x_j \le y_2\}.$$

Let $\tilde{h}$: $\mathbb{R}^2_{++} \times \tilde{u}(\mathbb{R}^2_+) \to \mathbb{R}^2_+$ be the Hicksian demand function derived from $\tilde{u}(\cdot)$. That is, $\tilde{h}(\alpha_1,\alpha_2,u) \in \mathbb{R}^2_+$ is the solution to

$$\text{Min}_{(y_1,y_2)} \ \alpha_1 y_1 + \alpha_2 y_2$$

$$\text{s.t.} \qquad \tilde{u}(y_1,y_2) \ge u.$$

Define $p(\alpha_1,\alpha_2) \gg 0$ by $p_j(\alpha_1,\alpha_2) = \alpha_1 p_j$ if $j \in J$ and $p_j(\alpha_1,\alpha_2) = \alpha_2 p_j$ if $j \notin J$. We shall prove that

$$\tilde{h}(\alpha_1,\alpha_2,u) = (\textstyle\sum_{j\in J} p_j h_j(p(\alpha_1,\alpha_2),u), \sum_{j\notin J} p_j h_j(p(\alpha_1,\alpha_2),u)).$$

Write $x^* = h(p(\alpha_1,\alpha_2),u)$. Then $\tilde{u}(\sum_{j\in J} p_j x^*_j, \sum_{j\notin J} p_j x^*_j) \ge u(x^*) = u$. Hence the constraint of the cost minimization problem is satisfied. Suppose that $(y_1,y_2) \in \mathbb{R}^2_+$ and $\tilde{u}(y_1,y_2) \ge u$, then (assuming strong monotonicity) there exists $x \in \mathbb{R}^L_+$ such that $\sum_{j\in J} p_j x_j = y_1$, $\sum_{j\notin J} p_j x_j = y_2$, and $u(x) = \tilde{u}(y_1,y_2)$. Thus $u(x) \ge u$ and hence, by the cost minimization of x^*, $p(\alpha_1,\alpha_2)\cdot x \ge p(\alpha_1,\alpha_2)\cdot x^*$. This is equivalent to saying that

$$\alpha_1 y_1 + \alpha_2 y_2 \ge \alpha_1(\textstyle\sum_{j\in J} p_j x^*_j) + \alpha_2(\sum_{j\notin J} p_j x^*_j).$$

Thus $\tilde{h}(\alpha_1,\alpha_2,u) = (\sum_{j\in J} p_j x^*_j, \sum_{j\notin J} p_j x^*_j)$.

By this equality and the chain rule (M.A.1),

$$\begin{aligned}\partial \tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_2 &= \textstyle\sum_{j\in J} p_j(\sum_{k\notin J}(\partial h_j(p(\alpha_1,\alpha_2),u)/\partial p_k)p_k) \\ &= \textstyle\sum_{j\in J}\sum_{k\notin J} p_j p_k(\partial h_j(p(\alpha_1,\alpha_2),u)/\partial p_k).\end{aligned}$$

We now derive a contradiction from this equality evaluated at $(\alpha_1,\alpha_2) = (1,1)$. On the one hand, since $\partial h_j(p,u)/\partial p_k < 0$ for every $j \in J$ and every $k \notin J$, we must have $\partial \tilde{h}_1(1,1,u)/\partial\alpha_2 < 0$. On the other hand, note that $\tilde{h}(\cdot)$ is the

Hicksian demand function of $\tilde{u}(\cdot)$ for two (composite) goods. Since $\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_1 \le 0$ by the negative semidefiniteness, and

$$(\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_1)p_1 + (\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_2)p_2 = 0,$$

we must have $\partial\tilde{h}_1(1,1,u)/\partial\alpha_2 \ge 0$. We have thus obtained a contradiction.

3.H.1 By Proposition 3.H.1, $e(p,u) = \text{Min}\{p\cdot x: x \in V_u\}$. Thus, to complete the proof, it is sufficient to show that $V_u = \{x: u(x) \ge u\}$. That is, $x \in V_u$ if and only if $\text{Sup}\ \{t: x \in V_t\} \ge u$.

Clearly, if $x \in V_t$ then $\text{Sup}\ \{t: x \in V_t\} \ge u$.

Assume that $\text{Sup}\ \{t: x \in V_t\} \ge u$. Define $u^* = \text{Sup}\ \{t: x \in V_t\}$. If $u^* > u$, then there exists $t \in (u,u^*]$ such that $x \in V_t$. Since $e(\cdot)$ is increasing in utility levels, $V_u \supset V_t$ and hence $x \in V_u$. If $u^* = u$, then, for every $n \in \mathbb{N}$, there exists $u_n \in (u - 1/n, u)$ such that $x \in V_{u_n}$, that is, $p\cdot x \ge e(p,u_n)$ for all p. Let $n \to \infty$, then $u_n \to u$ and, by continuity of $e(p,u)$, $p\cdot x \ge e(p,u)$ for all p. Thus $x \in V_u$.

3.H.2 We show the contrapositive of the assertion. If a preference is not convex, then there exists at least one nonconvex upper contour set. Let $u \in \mathbb{R}$ be its corresponding utility level. We can choose a price vector p so that $h(p,u)$ consists of more than one elements, as the following figure shows. According to Proposition 3.F.1, $e(\cdot)$ is not differential at (p,u).

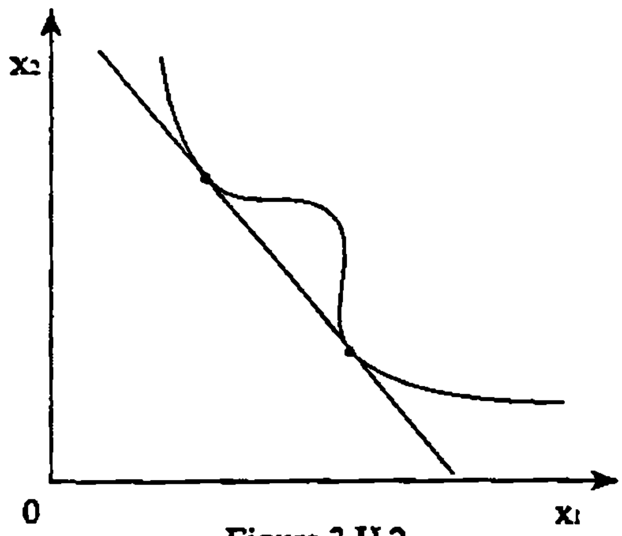


Figure 3.H.2

3.H.3 By (3.E.1), for each p, take the inverse of $e(p,u)$ with respect to u.

3.H.4 The following method is analogous to that of "Recovering the Expenditure Function from Demand" for L = 2.

Pick an arbitrary consumption vector x^0 and assign a utility value u^0 to x^0. We will now recover the indifference curve $\{x: u(x) = u^0\}$ going through x^0. Assuming strong mono tonicity, this is equivalent to finding a function $\xi(\cdot,u^0)$: $(0, \infty) \to (0, \infty)$ such that $u(x_1,\xi(x_1)) = u^0$ for every $x_1 > 0$. Differentiate

both sides of $u(\bar{x}_1,\xi(\bar{x}_1)) = u^0$ with respect to $\bar{x}_1$, then we obtain

$$\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_1 + (\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_2)\xi'(\bar{x}_1) = 0.$$

Hence

$$\xi'(\bar{x}_1) = -\frac{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_1}{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_2}.$$

Since $\dfrac{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_1}{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_2} = \dfrac{g_1(\bar{x}_1,\xi(\bar{x}_1))}{g_2(\bar{x}_1,\xi(\bar{x}_1))}$, we have $\xi'(\bar{x}_1) = -\dfrac{g_1(\bar{x}_1,\xi(\bar{x}_1))}{g_2(\bar{x}_1,\xi(\bar{x}_1))}$.

or, by replacing $\bar{x}_1$ by x_1, we obtain

$$\xi'(x_1) = -\frac{g_1(x_1,\xi(x_1))}{g_2(x_1,\xi(x_1))}.$$

By solving this differential equation, we obtain the indifference curve going through x^0.

3.H.5 By (3.E.1), we can recover the expenditure function by simply inverting the indirect utility function.

To recover the direct utility function, define $u\colon \mathbb{R}^L_+ \to \mathbb{R}$ by $u(x) = \mathrm{Min}\{v(p,w)\colon p\cdot x \le w\}$. We shall prove that $u(x)$ is the direct utility function that generates $v(p,w)$. So let $x^*(p,w)$ be the demand function and $v^*(p,w)$ be the indirect utility function generated by $u(x)$. It is sufficient to show that $v^*(p,w) = v(p,w)$ for all $p \gg 0$ and $w \ge 0$.

Let $p \gg 0$ and $w \ge 0$, then $p\cdot x^*(p,w) = w$ and hence $v^*(p,w) = u(x^*(p,w)) \le v(p,w)$. It thus remains to show that $v^*(p,w) \ge v(p,w)$. Define

$$x = -\frac{1}{\nabla_w v(p,w)}\nabla_p v(p,w).$$

Then $x \in \mathbb{R}^L_+$ by the monotonicity. Since $\nabla_p v(p,w)\cdot p + \nabla_w v(p,w)w = 0$ by the homogeneity, $p\cdot x = w$. It is thus sufficient to show that $u(x) \ge v(p,w)$. So let $p' \gg 0$ and $w' \ge 0$ satisfy $p'\cdot x = w'$. Then $(p' - p)\cdot x = w' - w$, or, by the definition of x, $\nabla_p v(p,w)\cdot(p' - p) + \nabla_w v(p,w)(w' - w) = 0$. Hence, by the quasiconvexity of $v(p,w)$, $v(p',w') \ge v(p,w)$. Thus $u(x) \ge v(p,w)$.

3.H.6 Let's first prove that the symmetry condition on $S(p,w)$ is satisfied. By equation (3.H.2), $\partial e(p,u)/\partial p_\ell = \alpha_\ell e(p,u)/p_\ell$ for every ℓ. By differentiating both sides with respect to p_k, we obtain

$$\partial^2 e(p,u)/\partial p_\ell \partial p_k = (\alpha_\ell/p_\ell)(\partial e(p,u)/\partial p_k) = (\alpha_\ell\alpha_k/p_\ell p_k)e(p,u).$$

On the other hand, by differentiating both sides of $\partial e(p,u)/\partial p_k = \alpha_k e(p,u)/p_k$

with respect to p_ℓ, we obtain

$$\partial^2 e(p,u)/\partial p_k \partial p_\ell = (\alpha_k/p_k)(\partial e(p,u)/\partial p_\ell) = (\alpha_\ell \alpha_k/p_\ell p_k)e(p,u).$$

Hence the symmetry condition is satisfied.

We can thus apply the iterative method explained in the small-type discussion at the end of Section 3.H to derive the expenditure function. First, we shall prove by induction that, for every ℓ, there exists a function $f_\ell(p_{\ell+1},\ldots,p_L,u)$ such that

$$\ln e(p,u) = \sum_{k\le\ell}\alpha_k \ln p_k + f_\ell(p_{\ell+1},\ldots,p_L,u).$$

Suppose first that $\ell = 1$. Since $(\partial e(p,u)/\partial p_1)/e(p,u) = \alpha_1/p_1$. Hence, by integrating both sides with respect to p_1, we obtain

$$\ln e(p,u) = \alpha_1 \ln p_1 + f_1(p_2,\ldots,p_L,u).$$

Thus the equality is verified for $\ell = 1$. Suppose next that $\ell > 1$ and the equality holds for $\ell - 1$. By differentiating both sides of

$$\ln e(p,u) = \sum_{k\le\ell-1}\alpha_k \ln p_k + f_{\ell-1}(p_\ell,\ldots,p_L,u)$$

with respect to p_ℓ, we obtain

$$\partial e(p,u)/\partial p_\ell = \partial f_{\ell-1}(p_\ell,\ldots,p_L,u)/\partial p_\ell.$$

Since $\partial e(p,u)/\partial p_\ell = \alpha_\ell/p_\ell$, this is equivalent to

$$\alpha_\ell/p_\ell = \partial f_{\ell-1}(p_\ell,\ldots,p_L,u)/\partial p_\ell.$$

Hence, by integrating both sides with respect to p_ℓ, we know that there exists $f_\ell(p_{\ell+1},\ldots,p_L,u)$ such that

$$\alpha_\ell \ln p_\ell = f_{\ell-1}(p_\ell,\ldots,p_L,u) - f_\ell(p_{\ell+1},\ldots,p_L,u)$$

By plugging this into

$$\ln e(p,u) = \sum_{k\le\ell-1}\alpha_k \ln p_k + f_{\ell-1}(p_\ell,\ldots,p_L,u),$$

we obtain

$$\ln e(p,u) = \sum_{k\le\ell}\alpha_k \ln p_k + f_\ell(p_{\ell+1},\ldots,p_L,u).$$

If $\ell = L$, then this equality becomes $\ln e(p,u) = \sum_{k\le L}\alpha_k \ln p_k + f_L(u)$. Or, equivalently, $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$.

In what follows, for every increasing function $f_L(u)$, we shall find the utility function that generates the expenditure function $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$. To start, consider the utility function $u^*(x) = \Pi_\ell x_\ell^{\alpha_\ell}$, which appeared in Example 3.E.1 for the case of L = 2. Denote its expenditure function by $e^*(p,u)$, then

$$e^*(p,u^*) = (\Pi_\ell \alpha_\ell^{-\alpha_\ell})(\Pi_\ell p_\ell^{\alpha_\ell})u^*.$$

(We considered a similar expenditure function in Exercise 3.G.16. Note that, these similarities incidentally show that, for every increasing function $f_L(u)$, the expenditure function $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$ has all the properties of expenditure functions in Proposition 3.E.2, because it corresponds one of the monotone transformations of $u^*(x) = \Pi_\ell x_\ell^{\alpha_\ell}$.) Let $g(u^*)$ be an monotone transformation and denote by $e_g(p,u)$ the expenditure function of the utility function $(g \circ u^*)(x)$. Then

$$e_g(p,u) = e^*(p,g^{-1}(u)) = (\Pi_\ell \alpha_\ell^{-\alpha_\ell})(\Pi_\ell p_\ell^{\alpha_\ell})g^{-1}(u).$$

By comparing this with $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$, we know that $(g \circ u^*)(x) = u(x)$ if

$$(\Pi_\ell \alpha_\ell^{-\alpha_\ell})(\Pi_\ell p_\ell^{\alpha_\ell})g^{-1}(u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u).$$

This equality is equivalent to $g^{-1}(u) = (\Pi_\ell \alpha_\ell^{\alpha_\ell})\exp f_L(u)$. Letting $u^* = g^{-1}(u)$ and solving this with respect to u^*, we obtain

$$g(u^*) = f_L^{-1}(\ln u^* - \textstyle\sum_\ell \alpha_\ell \ln \alpha_\ell).$$

Thus

$$u(x) = g(u^*(x)) = f_L^{-1}(\ln u^*(x) - \textstyle\sum_\ell \alpha_\ell \ln \alpha_\ell) = f_L^{-1}(\textstyle\sum_\ell \alpha_\ell(\ln x_\ell - \ln \alpha_\ell)).$$

Of course, two possible utility functions are $u(x) = \Pi_\ell x_\ell^{\alpha_\ell}$ (corresponding to $f_L(u) = \ln u - \sum_\ell \alpha_\ell \ln \alpha_\ell$) and $u(x) = \sum_\ell \alpha_\ell \ln x_\ell$ (corresponding to $f_L(u) = u -$

$\sum_\ell \alpha_\ell \ln\alpha_\ell)$.

3.H.7 (a) Let $\bar{p} = (1,\ldots,1)$. Since $x(\bar{p},L) = (1,\ldots,1)$ and $u(1,\ldots,1) = 1$, according to Propositions 3.E.1 and 3.G.1, $D_p e(\bar{p},1) = h(\bar{p},1) = (1,\ldots,1)$. Hence $e(\bar{p},1) = L$.

On the other hand, $S(q,w) = D_p^2 e(q,w) = 0$ for every $q \gg 0$ by Exercise 2.F.17(d). Hence

$$e(p,u) - e(q,u) = D_p e(q,u)(p - q).$$

for every $q \gg 0$ and $p \gg 0$. Now take $q = \bar{p}$, then $e(p,1) - L = \bar{p}\cdot(p - \bar{p})$. Thus $e(p,1) = \sum_{\ell=1}^{L} p_\ell$.

(b) The upper contour set is equal to

$$\{x \in \mathbb{R}_+^L : p\cdot x \geq \sum_{\ell=1}^{L} p_\ell \text{ for every } p \gg 0\} = \{x \in \mathbb{R}_+^L : x \geq (1,\ldots,1)\}.$$

3.I.1 By the same method as deriving equation (3.I.3), we obtain

$$EV(p^0,p^1,w) = e(p^0,u^1) - e(p^1,u^1)$$

$$= e(p^0,u^1) - e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^1) + e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^1) - e(p^1,u^1)$$

$$= \int_{p_1^0}^{p_1^1} h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^1)\,dp_1 + \int_{p_2^0}^{p_2^1} h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^1)\,dp_2;$$

$$CV(p^0,p^1,w) = e(p^0,u^0) - e(p^1,u^0)$$

$$= e(p^0,u^0) - e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^0) + e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^0) - e(p^1,u^0)$$

$$= \int_{p_1^0}^{p_1^1} h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^0)\,dp_1 + \int_{p_2^0}^{p_2^1} h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^0)\,dp_2.$$

If there is no wealth effect for either good, then, by the first relation of (3.E.4),

$$h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^1) = h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^0) \text{ for every } p_1 > 0,$$

$$h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^1) = h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^0) \text{ for every } p_2 > 0.$$

Thus $EV(p^0,p^1,w) = CV(p^0,p^1,w)$.

3.I.2 Denote the deadweight loss given in equation (3.I.5) by $DW_1(t)$ and that in equation (3.I.6) by $DW_0(t)$. Then

$$DW_1'(t) = h(p_1^0 + t,\ \bar{p}_{-1},\ u^1) - (h(p_1^0 + t,\ \bar{p}_{-1},\ u^1) + t\cdot\partial h(p_1^0 + t,\ \bar{p}_{-1},\ u^1)/\partial p_1)$$
$$= -\ t\cdot\partial h(p_1^0 + t,\ \bar{p}_{-1},\ u^1)/\partial p_1.$$

Thus $DW_1'(0) = 0$ and, if $\partial h(p_1,\bar{p}_{-1},u^1)/\partial p_1 > 0$ for every $p_1 > 0$, then $DW_1'(t) > 0$ for every $t > 0$. It can be similarly shown that $DW_0'(0) = 0$ and, if $\partial h(p_1,\bar{p}_{-1},u^0)/\partial p_1 > 0$ for every $p_1 > 0$, then $DW_0'(t) > 0$ for every $t > 0$.

A possible interpretation of this result is that the first-order derivatives of the deadweight loss at $t = 0$ may be a bit misleading approximation. In fact, their being zero means that, approximately, there is no deadweight loss. On the other hand, since those derivatives are positive at every $t > 0$, $DW_1(t) = \int_0^t DW_1'(\tau)\,d\tau > 0$ and $DW_0(t) = \int_0^t DW_0'(\tau)\,d\tau > 0$. Hence the deadweight losses are in fact positive.

3.I.3 Write $u^0 = v(p^0,w)$ and $u^1 = v(p^1,w)$, then $u^0 < u^1$ because $p^0 \geq p^1$ and $p^0 \neq p^1$. Thus

$$e(p_\ell,p_{-\ell}^0,u^0) < h_\ell(p_\ell,p_{-\ell}^0,u^1)$$

for every $p_\ell > 0$. Since good ℓ is inferior,

$$x_\ell(p_\ell,p_{-\ell}^0,e(p_\ell,p_{-\ell}^0,u^0)) > x_\ell(p_\ell,p_{-\ell}^0,e(p_\ell,p_{-\ell}^0,u^1)).$$

By the first relation of (3.E.4), this is equivalent to

$$h_\ell(p_\ell,p_{-\ell}^0,u^0) > h_\ell(p_\ell,p_{-\ell}^0,u^1).$$

Hence, by (3.I.3), (3.I.4), and $p_\ell^0 < p_\ell^1$, we have $CV(p^0,p^1,w) > EV(p^0,p^1,w)$.

3.I.4 We shall give two examples, both of which have two commodities. The first one is simpler, while the second one is more illustrative.

In the first example, we consider a preference with "L-shaped" indifference curves such that the vectors (1,1), (4,2), and (5,3) are kinks of indifference curves. Let $u(1,1) = 1$. Note that if one of the two prices is equal to zero, then the demand is not a singleton. We thus need to consider a demand correspondence $x(p,w)$. But this does not essentially change our argument because we are working on expenditure functions, which is single-valued by its definition.

Let $p^0 = (1,1)$, $p^1 = (1/2,0)$, $p^2 = (0,2/3)$, and $w = 2$. Then $x(p^0,w) \ni (1,1)$, $x(p^1,w) \ni (4,2)$, $x(p^2,w) \ni (5,3)$, and $v(p^2,w) > v(p^1,w)$. But $e(p^1,1) = 1/2$ and $e(p^2,1) = 2/3$. Thus

$$CV(p^0,p^1,w) = 2 - 1/2 = 3/2,$$

$$CV(p^0,p^2,w) = 2 - 2/3 = 4/3.$$

Hence $CV(p^0,p^1,w) > CV(p^0,p^2,w)$.

It is worthwhile to remark that, although the given preference is neither smooth, strongly monotone, nor strictly convex, it can be approximated by such one.

In the second example, we consider a utility function $u(x)$ which is quasilinear with respect to the first commodity. Let $v(p,w)$ be the corresponding indirect utility function. Starting from $p^0 = (1,1)$ and $w > 0$, we consider two other price vectors $p^1 = (p_1^1,1)$ and $p^2 = (1,p_2^2)$ such that $0 < p_1^1 < 1$, $0 < p_2^2 < 1$, and $v(p^1,w) = v(p^2,w)$. Write $u^0 = v(p^0,w)$ and $u^1 = v(p^1,w) = v(p^2,w)$. Then $EV(p^0,p^1,w) = EV(p^0,p^2,w)$.

We shall now show that $CV(p^0,p^1,w) < CV(p^0,p^2,w)$. By $p_1^1 < 1$, $CV(p^0,p^1,w) < EV(p^0,p^1,w)$. Also, by the quasilinearity, $CV(p^0,p^2,w) = EV(p^0,p^2,w)$ (Exercise 3.I.5). Hence $CV(p^0,p^1,w) < CV(p^0,p^2,w)$.

It is worthwhile to remark that, although $EV(p^0,p^1,w) = EV(p^0,p^2,w)$, we can obtain the strict reverse inequality $EV(p^0,p^1,w) > EV(p^0,p^2,w)$, while preserving $CV(p^0,p^1,w) < CV(p^0,p^2,w)$, by decreasing p_2^2 only slightly.

3.I.5 According to Exercise 3.E.7, we can write the expenditure function $e(p,u) = \tilde{e}(p_2,\ldots,p_L) + u$ for $p_1 = 1$. Hence

$$\begin{aligned}EV(p^0,p^1,w) &= e(p^0,u^1) - e(p^0,u^0)\\ &= (\tilde{e}(p_2^0,\ldots,p_L^0) + u^1) - (\tilde{e}(p_2^0,\ldots,p_L^0) + u^0)\\ &= u^1 - u^0.\\ CV(p^0,p^1,w) &= e(p^1,u^1) - e(p^1,u^0)\\ &= (\tilde{e}(p_2^1,\ldots,p_L^1) + u^1) - (\tilde{e}(p_2^1,\ldots,p_L^1) + u^0)\\ &= u^1 - u^0.\end{aligned}$$

Hence $EV(p^0,p^1,w) = CV(p^0,p^1,w)$.

3.I.6 Let $u_i^0 = v_i(p^0,w_i)$. If $\sum_i CV_i(p^0,p^1,w_i) \geq 0$, then $\sum_i w_i \geq \sum_i e_i(p^1,u_i^0)$. So define $w_i' = e_i(p^1,u_i^0)$, then $\sum_i w_i' \leq \sum_i w_i$ and $v_i(p^1,w_i') = u_i^0 = v_i(p^0,w_i)$.

3.I.7 (a) By applying Walras' law and the homogeneity of degree zero, we can obtain the demand functions for all three good defined over the whole domain $\{(p,w) \in \mathbb{R}^3 \times \mathbb{R}: p \gg 0\}$. Thus we can obtain the whole 3×3 Slutsky matrix as well from the demand function. The 2×2 submatrix of the Slutsky matrix that is obtained by deleting the last row and the last column is equal to $(1/p_3)\begin{bmatrix} b & c \\ e & g \end{bmatrix}$. By the homogeneity and Walras' law, the 3×3 Slutsky matrix is symmetric if and only if this 2×2 matrix is symmetric. Moreover, just as in the proof of Theorem M.D.4(iii), we can show that the 3×3 Slutsky matrix is negative semidefinite (on T_p, and hence on the whole $\mathbb{R}^3$) if and only if the

2×2 matrix is negative semidefinite. Hence, utility maximization implies that $c = e$, $b \le 0$, $g \le 0$, and $bg - c^2 \ge 0$.

(b) First, we verify that the corresponding Hicksian demand functions for the first two commodities are independent of utility levels and, as functions of the prices of the first two commodities alone, they are equal to the given Walrasian demand functions. Let p be any price vector and u, u' be any two utility levels. By (3.E.4), $h_\ell(p,u) = x_\ell(p,e(p,u))$ and $h_\ell(p,u') = x_\ell(p,e(p,u'))$ for $\ell = 1,2$. Since the $x_\ell(\cdot)$ do not depend on wealth, $x_\ell(p,e(p,u)) = x_\ell(p,e(p,u'))$. Hence $h_\ell(p,u) = h_\ell(p,u')$. Thus the $h_\ell(p,u)$ do not depend on utility level and they are the same as the $x_\ell(p,w)$.

If the prices change following the path $(1,1) \to (2,1) \to (2,2)$, then the equivalent variation is

$$\begin{aligned}
&\int_1^2 h^1(p^1,1,u)dp^1 + \int_1^2 h^2(2,p^2,u)dp^2 \\
&= \int_1^2 x^1(p^1,1,w)dp^1 + \int_1^2 x^2(2,p^2,w)dp^2 \\
&= (a + (3/2)b + c) + (d + 2e + (3/2)g).
\end{aligned}$$

If the prices change following the path $(1,1) \to (1,2) \to (2,2)$, then the equivalent variation is

$$\begin{aligned}
&\int_1^2 h^2(1,p^2,u)dp^2 + \int_1^2 h^1(p^1,2,u)dp^1 \\
&= \int_1^2 x^2(1,p^2,w)dp^2 + \int_1^2 x^1(p^1,2,w)dp^1 \\
&= (d + e + (3/2)g) + (a + (3/2)b + 2c).
\end{aligned}$$

These two equivalent variations are the same if and only if $c = e$.

(c) As we saw above,

$$\begin{aligned}
EV_1 &= \int_1^2 x^1(p^1,1,w)dp^1 = a + (3/2)b + c, \\
EV_2 &= \int_1^2 x^2(1,p^2,w)dp^2 = d + e + (3/2)g = d + c + (3/2)g, \\
EV\ \ &= (a + (3/2)b + c) + (d + 2e + (3/2)g)
\end{aligned}$$

$= a + (3/2)b + 3c + d + (3/2)g.$

Hence $EV - (EV_1 + EV_2) = c$.

The sum $EV_1 + EV_2$ does not contain the effect on equivalent variation due to the shift of the graph of the demand function for the second commodity when p^1 goes up to 2 (or equivalently, the shift of the graph of the demand function for the first commodity when p^2 goes up to 2). Graphically, letting $c = e > 0$, EV contains the shaded area below but $EV_1 + EV_2$ does not:

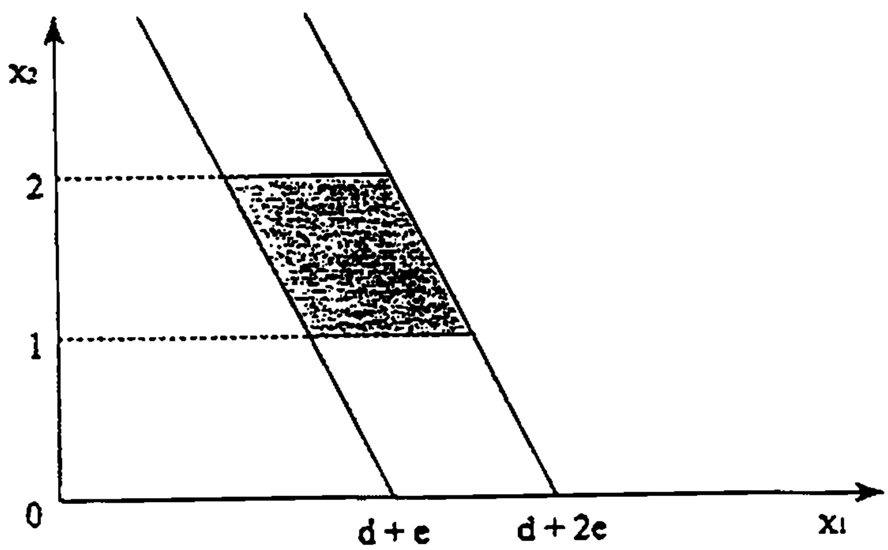


Figure 3.I.7(c)

(d) Since $x_1(2,1,w) = a + 2b + c$, the tax revenue from the first good is equal to this. Thus $DW_1 = (a + (3/2)b + c) - (a + 2b + c) = - b/2$.

Since $x_2(1,2,w) = d + e + 2g$, the tax revenue from the second good is equal to this. Thus $DW_2 = (d + e + (3/2)g) - (d + e + 2g) = - g/2$.

Since $x_1(2,2,w) = a + 2b + 2c$ and $x_2(2,2,w) = d + 2e + 2g$, the tax revenue from both commodities is

$$(a + 2b + 2c) + (d + 2e + 2g) = a + 2b + 4c + d + 2g.$$

Thus

$$DW = (a + (3/2)b + 3c + d + (3/2)g) - (a + 2b + 4c + d + 2g).$$

$$= - b/2 - c - g/2.$$

Hence $DW - (DW_1 + DW_2) = - c$.

(e) Our problem is

$$\text{Min}_{(t_1,t_2)}\ DW(t_1,t_2)$$

$$\text{s.t. } \textstyle\sum_{\ell=1}^{2} h_\ell(1 + t_1,\ 1 + t_2,\ u)t_\ell \geq R.$$

Here,

$$DW(t_1,t_2) = EV(t_1,t_2) - TR(t_1,t_2)$$

$$= e(1 + t_1,\ 1 + t_2,\ u) - e(1,1,u) - \textstyle\sum_{\ell=1}^{2} h_\ell(1 + t_1,\ 1 + t_2,\ u)t_\ell.$$

Set up the Lagrangean by $L(t_1,t_2,\lambda) = DW(t_1,t_2) + \lambda(R - TR(t_1,t_2))$. Then the first-order condition with respect to t_ℓ is $\partial DW(t_1,t_2)/\partial t_\ell - \lambda\partial TR(t_1,t_2)/\partial t_\ell = 0$. But,

$$\partial DW(t_1,t_2)/\partial t_\ell = \partial e(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell - h_\ell(1 + t_1,\ 1 + t_2,\ u)$$

$$- \textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k$$

$$= - \textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k$$

by $\partial e(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell = h_\ell(1 + t_1,\ 1 + t_2,\ u)$, and

$$\partial TR(t_1,t_2)/\partial t_\ell = h_\ell(1 + t_1,\ 1 + t_2,\ u) + \textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k.$$

Hence the first-order condition is written as

$$\textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k(1 + \lambda)$$

$$+ \lambda h_\ell(1 + t_1,\ 1 + t_2,\ u) = 0$$

for both $\ell = 1,2$. From this and $R = \sum_{\ell=1}^{2} h_\ell(1 + t_1,\ 1 + t_2,\ u)t_\ell$, we obtain

$$- \lambda = \frac{bt_1 + ct_2}{a + b(1 + 2t_1) + c(1 + 2t_2)} = \frac{ct_1 + gt_2}{a + c(1 + 2t_1) + g(1 + 2t_2)},$$

$$(a + b(1 + t_1) + c(1 + t_2))t_1 + (d + c(1 + t_1) + g(1 + t_2))t_2 = R.$$

3.I.8 (a) Quasilinear utility functions: $u(x_1,x_2,x_3) = \tilde{u}(x_1,x_2) + x_3$.

(b) As in Exercise 3.I.7(a), the symmetry implies

$$c_1 + d_1p_1 = b_2 + d_2p_2 \text{ for all } p_1 > 0 \text{ and } p_2 > 0.$$

Thus $c_1 = b_2$, $d_1 = d_2 = 0$. Then the negative semidefiniteness implies that

$$b_1 \le 0,\ c_2 \le 0,\ b_1c_2 - c_1^2 \le 0.$$

(c) Since the Walrasian demand functions and Hicksian demand functions are the same as we saw in Exercise 3.I.7(b), we can define

$$CV = \int_{p_1}^{p_1'} x_1(q,p_2,w)dq + \int_{p_2}^{p_2'} x_1(p_1',q,w)dq,$$

or, equivalently,

$$CV = \int_{p_2}^{p_2'} x_2(p_1,q,w)dq + \int_{p_1}^{p_1'} x_1(q,p_2',w)dq.$$

(d) By the same calculation as in Exercise 3.I.7(c), we obtain

$$EV_1 = 1/2,\ EV_2 = 1/2,\ EV_3 = 3/2.$$

In this case, $EV \neq EV_1 + EV_2$. In the general case in which the conditions in (b) hold,

$$EV_1 = a_1 + (3/2)b_1 + c_1,$$
$$EV_2 = a_2 + b_2 + (3/2)c_2,$$
$$EV_3 = a_1 + (3/2)b_1 + c_1 + a_2 + 2b_2 + (3/2)c_2.$$

Hence $EV_1 + EV_2 = EV_3$ if and only if $b_2 = c_1 = 0$. This condition is equivalent to saying that any change in the price of one good does not have any (cross) effect on the demand for the other.

3.I.9 Let $e_\ell \in \mathbb{R}^L$ be the vector of which the ℓ-th component is one and all the other components are zero. For each t, define $p(t) = p + te_\ell$. Then the after-rebate income $w(t)$ with tax t satisfies $w(t) = w + x_\ell(p(t),w(t))t$.

Hence $p \cdot x(p(t),w(t)) = (p(t) - te_\ell) \cdot x(p(t),w(t)) = w(t) - x_\ell(p(t),w(t))t = w$. Therefore $x(p(t),w(t))$ is at most as good as $x(p,w)$. In order to prove that $x(p(t),w(t))$ is strictly less preferred to $x(p,w)$, it is sufficient to prove that these two are different, because the demand function is assumed to be single-valued.

Now suppose that there exists a $t > 0$ such that $x(p(t),w(t)) = x(p,w)$. Let $u = v(p,w)$. Then we have $h(p(t),u) = h(p,u)$. In particular, $h_\ell(p(t),u) = h_\ell(p,u)$. Since the Hicksian demand function $s \to h_\ell(p(s),u)$ is nonincreasing, this equality implies that $h_\ell(p(s),u) = h_\ell(p,u)$ for every $s \in [0,t]$. But $d[h_\ell(p(s),u)]/ds = \partial h_\ell(p(s),u)/\partial p_\ell$. Evaluating at $s = 0$, we have $\partial h_\ell(p,u)/\partial p_\ell = s_{\ell\ell}(p,w) = 0$. This violates the assumption that $s_{\ell\ell}(p,w) < 0$. Hence $h(p(t),w(t)) \neq h(p,w)$ for every $t > 0$.

3.I.10 We consider an example of a consumer who face the choices over two goods and whose preference $\succsim$ and demand function $x(p_1,p_2,w)$ satisfy the following condition:

For every $p_1 \in [1,2]$, $x(p_1,1,2) = ((1 - \varepsilon)/p_1,\ 1 + \varepsilon)$;

for every $p_2 \in [1,2]$, $x(1,p_2,2) = (1 - \varepsilon,\ (1 + \varepsilon)/p_2)$;

$((1 - \varepsilon)/2,\ 1 + \varepsilon) \succ (1 - \varepsilon,\ (1 + \varepsilon)/2)$.

By using the figure below, you can convince yourself, perhaps with some application of the weak axiom, that there actually exists such a preference.

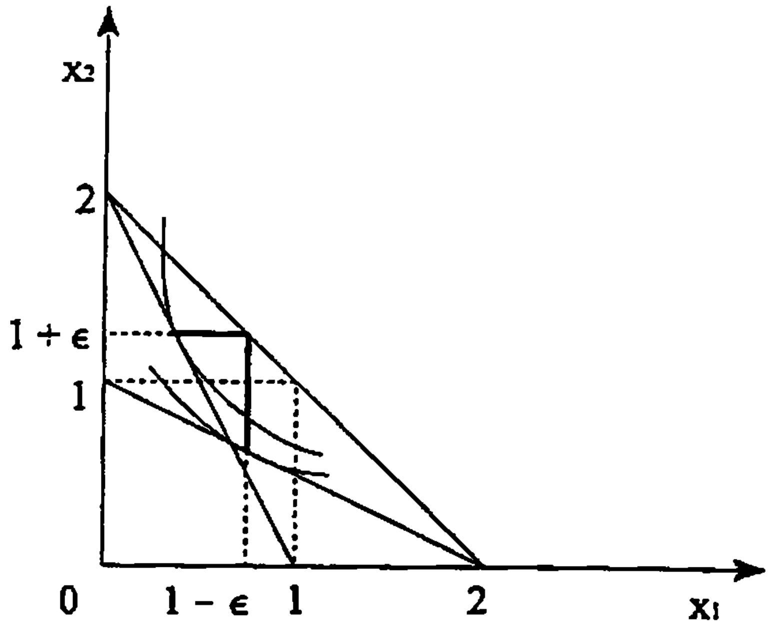


Figure 3.I.10

Define $p^0 = (2,1)$ and $p^1 = (1,2)$. Then

$$x(p^0,2) = ((1-\varepsilon)/2,\ 1+\varepsilon),$$

$$x(p^1,2) = (1-\varepsilon,\ (1+\varepsilon)/2).$$

Thus $x(p^0,2) \succ x(p^1,2)$. However, the area variation measure following the price-change path $p^0 \to (1,1) \to p^1$ is

$$\begin{aligned} AV(p^0,p^1,2) &= \int_2^1 x_1(p_1,1,2)dp_1 + \int_1^2 x_2(1,p_2,2)dp_2 \\ &= -\int_1^2 (1-\varepsilon)/p_1\, dp_1 + \int_1^2 (1+\varepsilon)/p_2\, dp_2 \\ &= -[(1-\varepsilon)\ln p_1]_{p_1=1}^2 + [(1+\varepsilon)\ln p_2]_{p_1=1}^2 \\ &= 2\varepsilon\ln 2 > 0. \end{aligned}$$

Hence the area variation measure ranks p^1 over p^0.

3.I.11 If $(p^1 - p^0)\cdot x^1 > 0$, then $w > p^0\cdot x^1$. The local non-satiation implies that x^0 is preferred to x^1. Hence the consumer must be worse off at (p^1,w).

As for the interpretation in term of the first-order approximation, since $e(p,u)$ is concave in p,

$$e(p^0,u^1) \le e(p^1,u^1) + \nabla e(p^1,u^1)\cdot(p^0 - p^1).$$

Since $\nabla e(p^1,u^1)\cdot(p^0 - p^1) < 0$, $e(p^0,u^1) < e(p^1,u^1) = w$. Thus $u^0 = v(p^0,w) > u^1$.

Finally, $(p^1 - p^0)\cdot x^1 > 0$ if and only if $w > p^0\cdot x^1$, which, in turn, is equivalent to $p^0\cdot(x^1 - x^0) < 0$. This test is depicted in the picture below:

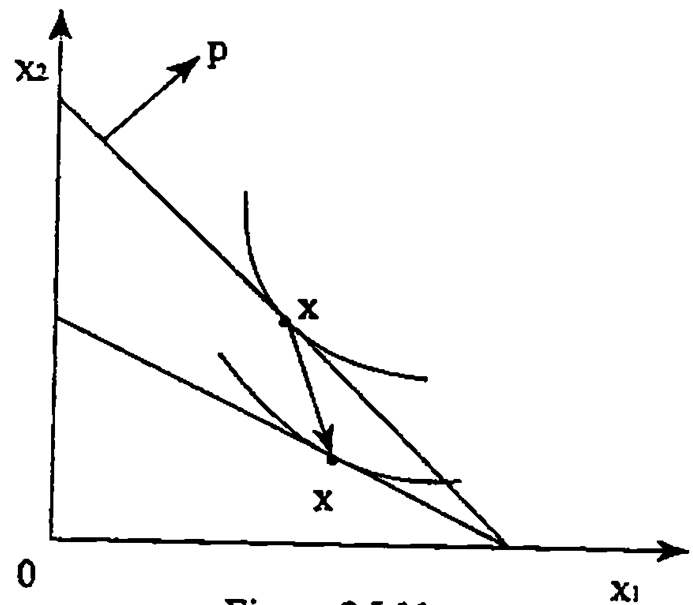


Figure 3.I.11

3.I.12 Let $u^0 = v(p^0,w^0)$ and $u^1 = v(p^1,w^1)$. Then we define

$$EV(p^0,w^0;p^1,w^1) = e(p^0,u^1) - e(p^0,u^0) = e(p^0,u^1) - w^0,$$

$$CV(p^0,w^0;p^1,w^1) = e(p^1,u^1) - e(p^1,u^0) = w^1 - e(p^1,u^0).$$

The "partial information" test can be extended as follows: If $p^1\cdot x^0 < w^1$, then the consumer is better off at (p^1,w^1). This can be proved in three ways. The first one is the same revealed-preference argument as in the proof of Proposition 3.I.1.

The second way is to use the indirect utility function. Since $v(p,w)$ is quasiconvex, if

$$(p^1 - p^0)\cdot\nabla_p v(p^0,w^0) + (w^1 - w^0)\partial v(p^0,w^0)/\partial w > 0,$$

then we can conclude that $v(p^1,w^1) > v(p^0,w^0)$. But, by Roy's identity, this sufficient condition is equal to

$$- (p^1 - p^0)\cdot(\partial v(p^0,w^0)/\partial w)x(p^0,w^0) + (w^1 - w^0)(\partial v(p^0,w^0)/\partial w)$$

$$= (\partial v(p^0,w^0)/\partial w)(- p^1\cdot x^0 + w^0 + w^1 - w^0)$$

$$= (\partial v(p^0,w^0)/\partial w)(w^1 - p^1\cdot x^0) > 0.$$

Hence, if $p^1\cdot x^0 < w^1$, then $v(p^1,w^1) > v(p^0,w^0)$.

The third way is to use the expenditure function. $v(p^1,w^1) > v(p^0,w^0)$ if and only if $e(p^1,v(p^1,w^1)) > e(p^1,v(p^0,w^0))$. But $e(p^1,v(p^1,w^1)) = w^1$ and $e(p^1,v(p^0,w^0)) \le p^1\cdot x^0$. Hence, if $p^1\cdot x^0 < w^1$, then we can conclude that $v(p^1,w^1) > v(p^0,w^0)$.

3.J.1 [First printing errata: The difficulty level should probably be B.] It follows immediately from the definition that if $x(p,w)$ satisfies the strong axiom, then it satisfied the weak axiom. Conversely, if $x(p,w)$ satisfies the weak axiom (in addition to the homogeneity of degree zero and Walras' law), then the Slutsky matrix is negative semidefinite and, by Exercise 2.F.11, symmetric. Hence $x(p,w)$ is integrable, implying that there exists a preference relation that generates $x(p,w)$. Thus $x(p,w)$ satisfies the strong axiom as well.

3.AA.1 If $(p,w) = (1,1,1)$, then $x(p,w) = (0,1)$. The locally cheaper condition is not satisfied since $B_{p,w} = \{x \in \mathbb{R}^L_+ : x_1 + x_2 = 1\}$ and there is no y such that $p\cdot y < w$, as depicted in the following figure.

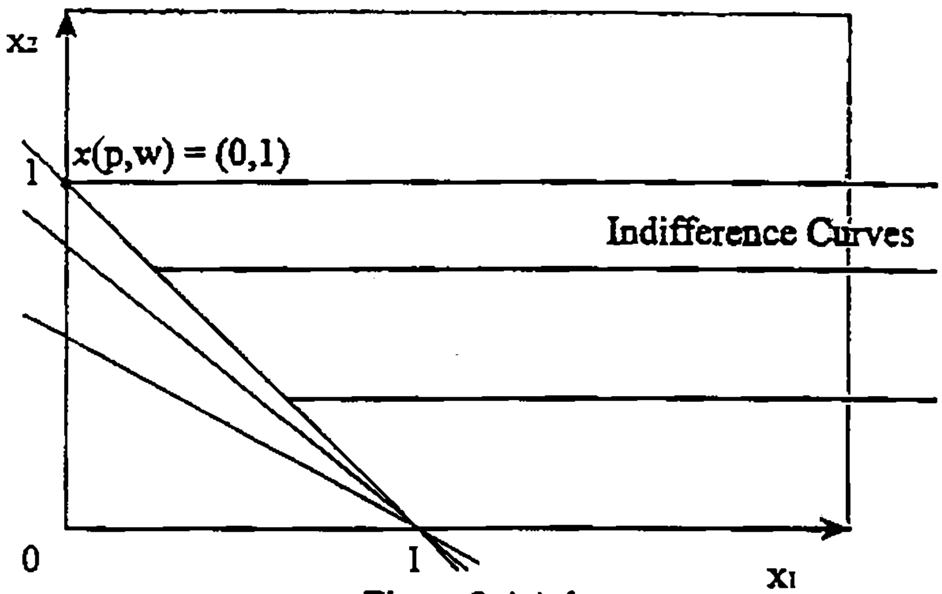


Figure 3.AA.1

To check that the demand function is not continuous at (1,1,1), consider the sequence $(p^n, w^n) = (1 - 1/n,\ 1,\ 1 - 1/n)$. Then $(p^n, w^n) \to (1,1,1)$ and $x(p^n, w^n) = (1,0)$, but $x(1,1,1) = (0,1)$. This discontinuous change in demands arises because the budget set B_{p^n,w^n} consists of (1,0) for every n, but $B_{(1,1),1} = \{x \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$, so that the commodity bundle (0,1) becomes available suddenly at p = (1,1).

3.AA.2 [First printing errata: The upper hemicontinuity of $h(p,u)$ cannot be guaranteed at $p \geq 0$, because the local boundedness condition in the definition of upper hemicontinuity need not be satisfied. Hence the clause in the bracket "even if we replace minimum by infimum and allow $p \geq 0$" should be understood as concerning only with $e(p,u)$.] We shall first prove that $h(p,u)$ is upper hemicontinuous. Let B be a compact subset of the domain of $h(p,u)$ (which is, in turn, a subset of $\{p \in \mathbb{R}^L : p \gg 0\} \times \mathbb{R}$). Then there exists a $(\bar{p},\bar{u}) \in B$ such that $\bar{u} \geq u$ for every $(p,u) \in B$. Let $\bar{x} \in h(\bar{p},\bar{u})$, then $u(\bar{x}) \geq \bar{u} \geq u$ for every $(p,u) \in B$. For each ℓ, define

$$\bar{y}_\ell = \text{Max}\{p\cdot\bar{x}/p_\ell \in \mathbb{R}_+ : (p,u) \in B\}$$

and $\bar{y} = (\bar{y}_1,\ldots,\bar{y}_L) \in \mathbb{R}^L_+$. We now show that, for every $(p,u) \in B$ and $x \in h(p,u)$, we have $\bar{y} \geq x$. If fact, since $u(\bar{x}) \geq u$, $p\cdot\bar{x} \geq p\cdot x$. Since $p \gg 0$ and $x \in \mathbb{R}^L_+$, $p_\ell x_\ell \leq p\cdot x$. Thus $p\cdot\bar{x} \geq p_\ell x_\ell$. Divide both side by p_ℓ, then we obtain $p\cdot\bar{x}/p_\ell \geq x_\ell$ and hence $\bar{y}_\ell \geq x_\ell$. We have therefore established the local boundedness condition of upper hemicontinuity. Next, let $((p^n,u^n))_n$ be a sequence of pairs of price vectors and utility levels, converging to (p,u). Let $(x^n)_n$ be a sequence in $\mathbb{R}^L_+$, $x^n \in h(p^n,u^n)$ for every n, and $x^n \to x$. It is sufficient to prove that $x \in h(p,u)$. Since $u(x^n) \geq u^n$ and $u(x^n) \to u(x)$ by the continuity, we obtain $u(x) \geq u$. Hence x satisfies the constraint of the EMP at (p,u). To show that it is cost-minimizing, let $y \in \mathbb{R}^L_+$ and $u(y) \geq u$. If $u(y) > u$, then $u(y) > u^n$ for any sufficiently large n. Hence $p^n\cdot y \geq p^n\cdot x^n$ for such n. By taking the limit as $n \to \infty$, we obtain $p\cdot y \geq p\cdot x$. Suppose then that $u(y) = u$. By the local nonsatiation, there exists a sequence $(y^n)_n$ in $\mathbb{R}^L_+$ such that $u(y^n) > u(x)$ for every n and $y^n \to x$. Hence there exists a subsequence $((p^{k(n)},u^{k(n)}))_n$ of $((p^n,u^n))_n$ such that $u(y^n) \geq u^{k(n)}$. Hence $p^{k(n)}\cdot y^n \geq p^{k(n)}\cdot x^{k(n)}$. By taking the limit as $n \to \infty$, we obtain $p\cdot y \geq p\cdot x$. Hence x is cost-minimizing.

We now turn to the continuity of $e(p,u)$. In fact, its continuity at every $p \gg 0$ can be derived immediately the continuity of $h(p,u)$, as the latter is well defined at every $p \gg 0$. Thus the essential part of the following proof is the case of nonnegative, but not strictly positive, price vectors. We shall establish the continuity with respect to p and that with respect to u separately.

Let $u \in \mathbb{R}$ be a utility level, $p \in \mathbb{R}^L_+$ be a price vector, and (p^n) be a sequence of price vectors in $\mathbb{R}^L_+$ converging to p. We need to prove that

$e(p^n,u) \to e(p,u)$. As a preliminary result, let's first prove that if the sequence $\{e(p^n,u)\}$ in $\mathbb{R}_+$ converges, then it must do so to $e(p,u)$. Let w be the limit of $\{e(p^n,u)\}$. Let $x \in \mathbb{R}^L_+$ and $u(x) \geq u$. Then $p^n \cdot x \geq e(p^n,u)$ for every n. Taking the limit as $n \to \infty$, we obtain $p \cdot x \geq w$. Since this holds for every $x \in \mathbb{R}^L_+$ with $u(x) \geq u$, we have $e(p,u) \geq w$. To prove the reverse inequality $e(p,u) \leq w$, we use the concavity of $e(p,u)$ in $p \in \mathbb{R}^L_+$. Take a subsequence $\{p^{k(n)}\}$ of $\{p^n\}$ such that $(p^{k(n)}_\ell - p_\ell)(p^{k(m)}_\ell - p_\ell) \geq 0$ for all $\ell \in \{1,\dots,L\}$ and positive integers n and m. That is, for each $\ell \in \{1,\dots,L\}$, we require the sign of $p^{k(n)}_\ell - p_\ell$ along the subsequence to be constant (including zero). Such a subsequence does actually exist because each $p^n - p$ has one of at most 2^L sign patterns. Now, for each $\ell \in \{1,\dots,L\}$, let $v_\ell = 1$ if $p^{k(n)}_\ell - p_\ell \geq 0$ for every n; and $v_\ell = -1$ if $p^{k(n)}_\ell - p_\ell \leq 0$ for every n. Then,

$$p^{k(n)}_\ell = p_\ell + |p^{k(n)}_\ell - p_\ell| v_\ell.$$

So let $z^n_\ell = |p^{k(n)}_\ell - p_\ell| \geq 0$ and define $v^\ell \in \mathbb{R}^L$ by letting $v^\ell_\ell = v_\ell$ and $v^\ell_k = 0$ for every $k \neq \ell$, then

$$p^{k(n)} = p + \textstyle\sum_\ell z^n_\ell v^\ell.$$

Now, define $z^n_0 = 1 - \sum^L_{\ell=1} z^n_\ell$, then

$$p^{k(n)} = z^n_0 p + \textstyle\sum_\ell z^n_\ell (p + v^\ell).$$

Since $p^{k(n)} \to p$, $z^n_\ell \to 0$ for every $\ell \in \{1,\dots,L\}$. Thus $z^n_0 \to 1$. Hence, for every sufficiently large n, $z^n_0 > 0$ and $p^{k(n)}$ is a convex combination of z^n_0, $z^n_1, \dots, z^n_L$. Therefore, by the concavity,

$$e(p^{k(n)},u) \leq z^n_0 e(p,u) + \textstyle\sum_\ell z^n_\ell e(p + v^\ell, u).$$

Since $e(p^n,u) \to w$, $e(p^{k(n)},u) \to w$. The right-hand side converges to $e(p,u)$. Therefore $w \leq e(p,u)$.

We have thus proved our preliminary fact that if the sequence $\{e(p^n,u)\}$ in $\mathbb{R}_+$ converges, then it must do so to $e(p,u)$. Let's now prove by contradiction that this implies that $e(p^n,u) \to e(p,u)$. So suppose not, then there are a $\delta > 0$ and a subsequence $\{p^{k(n)}\}$ of $\{p^n\}$ such that

$$|e(p^{k(n)},u) - e(p,u)| \geq \delta$$

for all n. Since the subsequence $\{e(p^{k(n)},u)\}$ is bounded, it has a further, convergent subsequence. On the one hand, the limit can never be $e(p,u)$, because $|e(p^{k(n)},u) - e(p,u)| \geq \delta$ for all n. On the other hand, our preliminary result implies that the limit must be $e(p,u)$. This is a contradiction. We must thus have $e(p^n,u) \to e(p,u)$.

Let's now turn to the continuity of $e(p,u)$ with respect to u. Let $p \in \mathbb{R}^L_+$ be a price vector, $u \in \mathbb{R}$ be a utility level, and $\{u^n\}$ be a sequence of utility levels in $\mathbb{R}$ converging to u. We need to prove that $e(p,u^n) \to e(p,u)$. Just as before, it is sufficient to prove that if the sequence $\{e(p,u^n)\}$ in $\mathbb{R}_+$ converges, then it must do so to $e(p,u)$. Let w be the limit of $\{e(p,u^n)\}$. Let $\varepsilon > 0$, $x \in \mathbb{R}^L_+$, $u(x) \geq u$, and $p \cdot x < e(p,u) + \varepsilon$. By the local nonsatiation, we can make $u(x) > u$ while preserving $p \cdot x < e(p,u) + \varepsilon$. Then there exists a positive integer N such that $u(x) > u^n$ for every $n > N$. Thus, for such n, $p \cdot x \geq e(p,u^n)$. Take the limit as $n \to \infty$, then $p \cdot x \geq w$. Thus $e(p,u) + \varepsilon > w$. Since this holds for every $\varepsilon > 0$, we must have $e(p,u) \geq w$. To show the reverse inequality $e(p,u) \leq w$, we can assume that $u^n \leq u$ for every n. (The reason is as follows: If there is a subsequence such that $u^n \leq u$ for every n in the subsequence, then we can apply this case to subsequence. If there is no such subsequence, then there is a subsequence such that $u^n \geq u$ for every n in the subsequence. Hence $e(p,u) \leq e(p,u^n)$ for such n. Taking the limit, we obtain $e(p,u) \leq w$.) Now let $\bar{x} \in \mathbb{R}^L_+$ and $u(\bar{x}) \geq u$. Define $B = \{x \in \mathbb{R}^L_+ : \bar{x} \geq x\}$, then B is compact. This and $u^n \leq u$ implies that the truncated EMP

$$\text{Min } p \cdot x \quad \text{s.t } u(x) \geq u^n$$

has a solution, denoted by $x^n \in B$. Then $x^n \in h(p,u^n)$, that is, x^n a solution to the original, untruncated EMP, because $p \in \mathbb{R}^L_+$. Since B is compact, there is a convergent subsequence $\{x^{k(n)}\}$ of $\{x^n\}$. Denote its limit by x. Since $u(\cdot)$ is continuous, $u(x) \geq u$ and hence $p \cdot x \geq e(p,u)$. Moreover, $p \cdot x^{k(n)} = e(p,u^{k(n)})$ and $p \cdot x^{k(n)} \to p \cdot x$. Thus $w = p \cdot x$ and hence $w \geq e(p,u)$.

Suppose that $u(x)$ is strictly quasiconcave, twice continuously differentiable and that $\nabla u(x) \neq 0$ for all x. Then we know that $h(p,u)$ is a function and the Lagrange multiplier λ of the EMP must be positive. The first-order condition for the EMP can be considered as a system of L + 1 equations and L + 1 unknowns:

$$p - \lambda \nabla u(x) = 0$$

$$u(x) - u = 0$$

By the implicit function theorem (Theorem M.E.1), the solution $h(p,u)$ as a function of the parameters (p,u) of the system is differentiable if the Jacobian of this system has a nonzero determinant

$$\begin{vmatrix} -D^2u(x) & -p \\ \nabla u(x)^T & 0 \end{vmatrix} \neq 0$$

at (p,x) satisfying the above two equations. But, then, $p = \lambda \nabla u(x)$ and hence this condition is equivalent to

$$\begin{vmatrix} -D^2u(x) & -\nabla u(x) \\ \nabla u(x)^T & 0 \end{vmatrix} \neq 0,$$

that is,

$$\begin{vmatrix} D^2u(x) & \nabla u(x) \\ \nabla u(x)^T & 0 \end{vmatrix} \neq 0.$$

By Theorem M.D.3(i), this inequality holds if $D^2u(x)$ is negative definite on $\{y \in \mathbb{R}^L: \nabla u(x) \cdot y = 0\}$. This sufficient condition is a stronger differential version of quasiconcavity, as the latter is equivalent to the condition that

$D^2u(x)$ is negative semidefinite on $\{y \in \mathbb{R}^L : \nabla u(x) \cdot y = 0\}$.

CHAPTER 4

4.B.1 By Roy's identity (Proposition 3.G.4) and $v_i(p,w_i) = a_i(p) + b(p)w_i$,

$$x_i(p,w_i) = -\frac{1}{\nabla_{w_i} v_i(p,w_i)} \nabla_p v_i(p,w_i) = -\frac{1}{b(p)} \nabla_p a_i(p) - \frac{w_i}{b(p)} \nabla_p b(p).$$

Thus $\nabla_{w_i} x_i(p,w_i) = -\frac{1}{b(p)} \nabla_p b(p)$ for all i. Since the right-hand side is identical for every i, the set of consumers exhibit parallel, straight expansion paths.

As for the second part, by (3.E.1),

$$e_i(p,u_i) = (u_i - a_i(p))/b(p).$$

Hence, by letting $c(p) = 1/b(p)$ and $d_i(p) = -a_i(p)/b(p)$, we obtain $e_i(p,u_i) = c(p)u_i + d_i(p)$.

4.B.2 (a) Let $p \in \mathbb{R}^L$ be a price vector and $w \geq 0$ be an aggregate wealth. Consider two consumers, i and j. Consider two wealth distributions $(w_1,\ldots,w_I)$ and $(w'_1,\ldots,w'_I)$ such that $w_i = w'_j = w \geq 0$, $w_k = 0$ for any $k \neq i$, and $w'_k = 0$ for any $k \neq j$. Since the preferences are homothetic, $x(p,0,s_k) = 0$ for every k. Thus the aggregate demand with $(w_1,\ldots,w_I)$ is $x(p,w,s_i)$ and the aggregate demand with $(w'_1,\ldots,w'_I)$ is $x(p,w,s_j)$. Since aggregate demand depends only on prices and aggregate wealth, we have $x(p,w,s_i) = x(p,w,s_j)$. Since p and w were arbitrarily chosen, this means that i and j have the same demand function. Hence they have the same preference. Since i and j were arbitrarily chosen, we conclude that all consumers have the same preference.

(b) By analogy to the Gorman form, consider the following form of indirect utility functions:

$$v_i(p, w_i, s_i) = a_i(p) + b(p)w_i + c(p)s_i.$$

Note that $b(p)$ and $c(p)$ do not depend on i. By this and Roy's identity (Proposition 3.G.4),

$$x(p, w_i, s_i) = -\frac{1}{\nabla_{w_i} v_i(p, w_i)} \nabla_p v_i(p, w_i)$$
$$= -\frac{1}{b(p)} \nabla a_i(p) - \frac{w_i}{b(p)} \nabla b(p) - \frac{s_i}{b(p)} \nabla c(p)$$

Thus

$$\sum_i x(p, w_i, s_i) = -\frac{1}{b(p)} \sum_i \nabla a_i(p) - \frac{\sum_i w_i}{b(p)} \nabla b(p) - \frac{\sum_i s_i}{b(p)} \nabla c(p).$$

Thus the aggregate demand depends only on $\sum_i w_i$ and $\sum_i s_i$ (and p).

4.C.1 By the definition of a directional partial derivative,

$$D_p x(p,w)dp = \lim_{\varepsilon \to 0} (1/\varepsilon)(x(p + \varepsilon dp, w) - x(p,w)).$$

Hence

$$dp \cdot D_p x(p,w)dp = dp \cdot (\lim_{\varepsilon \to 0} (1/\varepsilon)(x(p + \varepsilon dp, w) - x(p,w)))$$
$$= \lim_{\varepsilon \to 0} (1/\varepsilon) dp \cdot (x(p + \varepsilon dp, w) - x(p,w))$$

But the ULD property implies that $dp \cdot (x(p + \varepsilon dp, w) - x(p,w)) \geq 0$ for all $\varepsilon > 0$. Hence, by taking the limit $\varepsilon \to 0$, we obtain $dp \cdot D_p x(p,w)dp \leq 0$. Thus $D_p x(p,w)$ is negative semidefinite.

We shall prove the converse by contradiction. Suppose that the Jacobian $D_p x(p,w)$ is negative definite for all (p,w) and that there exist $p \in \mathbb{R}^L$, $p' \in \mathbb{R}^L$, and $w_i \in \mathbb{R}$ such that $x_i(p,w_i) \neq x_i(p',w_i)$ and

$$(p' - p) \cdot (x_i(p',w_i) - x_i(p,w_i)) \geq 0.$$

Let $\bar{\lambda} > 1$ be sufficiently close to 1 that for every $\lambda \in [0,\bar{\lambda}]$, demand is well defined at $(1 - \lambda)p + \lambda p'$. (If demand is well defined at strictly positive price vectors, $\bar{\lambda}$ is determined so that $(1 - \lambda)p + \lambda p' >> 0$ for every $\lambda \in [0,\bar{\lambda}]$.) Define $p(\lambda) = (1 - \lambda)p + \lambda p'$ and

$$w_i(\lambda) = (p' - p)\cdot(x_i(p(\lambda),w_i) - x_i(p,w_i)).$$

Then the function $w(\cdot)$ is differentiable, $w_i(0) = 0$, $w_i(1) \geq 0$, and

$$w_i'(\lambda) = (p' - p)\cdot D_p x_i(p(\lambda),w_i)(p' - p).$$

We consider two cases:

Case 1: $w_i(\lambda) \leq 0$ for every $\lambda \in [0,\bar{\lambda}]$.

Then $w_i(1) = 0$ and it is a maximum. Thus $w_i'(1) = 0$, that is,

$$(p' - p)\cdot D_p x(p',w)(p' - p) = 0.$$

This is a contradiction to the negative definiteness.

Case 2: $w_i(\lambda) > 0$ for some $\lambda \in [0,\bar{\lambda}]$.

Then, by the mean-value theorem, there exists $\lambda^* \in (0,\lambda)$ such that $w_i(\lambda) - w_i(0) = w_i'(\lambda^*)(\lambda - 0)$. By $w_i(0) = 0$ and $w_i(\lambda) > 0$, $w_i'(\lambda^*) > 0$. That is,

$$(p' - p)\cdot D_p x(p(\lambda^*),w)(p' - p) > 0.$$

This is a contradiction to the negative definiteness. Our proof is thus completed.

4.C.2 If $D_p x_i(p,\alpha_i w)$ is negative definite on the whole $\mathbb{R}^L$ for every i, then the sum $\sum_i D_p x_i(p,\alpha_i w)$ is negative definite on the whole $\mathbb{R}^L$. Since $D_p x(p,w) = \sum_i D_p x_i(p,\alpha_i w)$, $D_p x(p,w)$ is negative definite on the whole $\mathbb{R}^L$, implying that $x(p,w)$ satisfies the ULD property. To establish the WA, one way is simply to notice that the ULD property implies the WA, as the latter considers only compensated price changes.

Another way is to prove the given differential sufficient condition. Let's assume that $w > 0$. Define $H = \{v \in \mathbb{R}^L: v\cdot x(p,w) = 0\}$, that is, H is the hyperplane with normal $x(p,w)$ that goes through the origin. Then $p \notin H$ because $p\cdot x(p,w) = w > 0$. Thus, if $v \in \mathbb{R}^L$ and v is not proportional to p, then there exist $v_1 \in \mathbb{R}^L$ and $v_2 \in \mathbb{R}^L$ such that $v_1 \in H$, $v_1 \neq 0$, v_2 is proportional to p, and $v = v_1 + v_2$. Since $S(p,w)v_2 = 0$ and $v_2 \cdot S(p,w) = 0$ by Proposition 2.F.3, we have

$$\begin{aligned} v \cdot S(p,w)v &= (v_1 + v_2) \cdot S(p,w)(v_1 + v_2) \\ &= v_1 \cdot S(p,w)v_1 + v_1 \cdot S(p,w)v_2 + v_2 \cdot S(p,w)v_1 + v_2 \cdot S(p,w)v_2 \\ &= v_1 \cdot S(p,w)v_1. \end{aligned}$$

But here, by $v_1 \in H$,

$$S(p,w)v_1 = (D_p x(p,w) + D_w x(p,w)x(p,w)^T)v_1 = D_p x(p,w)v_1.$$

Hence $v_1 \cdot S(p,w)v_1 = v_1 \cdot D_p x(p,w)v_1 < 0$ because $v_1 \neq 0$ and $D_p x(p,w)$ is negative definite. Thus the WA holds.

4.C.3 A Giffen good will be a most familiar example. In the figure below, good 1 is a Giffen good.

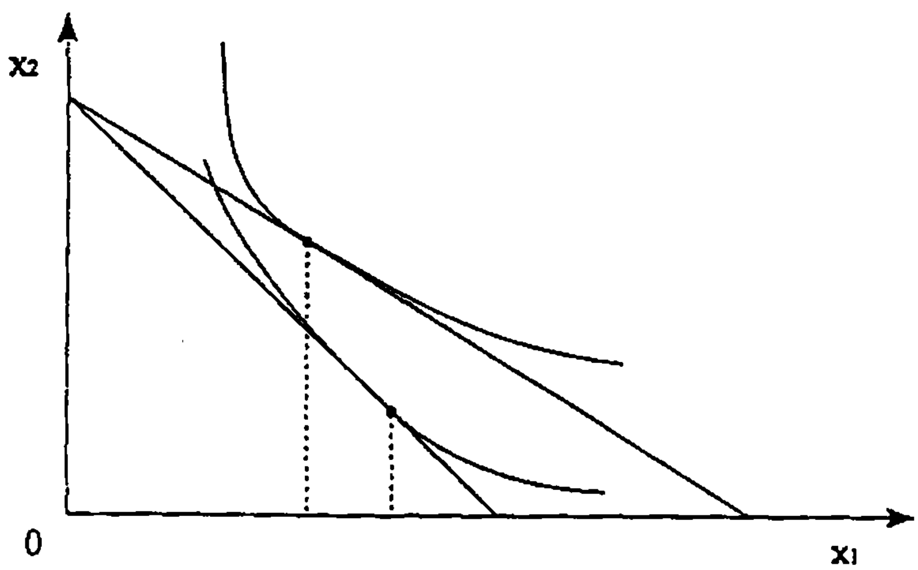


Figure 4.C.3

This example shows that the ULD property is actually not derived from the utility maximization. It is a restriction on preferences.

4.C.4 The L-shaped indifference curves imply that for every strictly positive price vector, the consumer's demand is always be at the corner of a upper contour set. Hence no compensated price change will change the demand. Thus $S_i(p,w_i) = 0$ for every (p,w_i) and $D_p x_i(p,w_i) = D_{w_i} x_i(p,w_i) x_i(p,w_i)^T$. Suppose that there exists (p,w_i) such that $D_{w_i} x_i(p,w_i) \neq (1/w_i) x_i(p,w_i)$. Since $p \cdot D_{w_i} x_i(p,w_i) = p \cdot (1/w_i) x(p,w_i) = 1$, this implies that $D_{w_i} x_i(p,w_i)$ and $x_i(p,w_i)$ are not proportional. Hence there exists a $v \in \mathbb{R}^2$ such that $v \cdot D_{w_i} x_i(p,w_i) < 0$ and $v \cdot x_i(p,w_i) > 0$, as illustrated in the figure below:

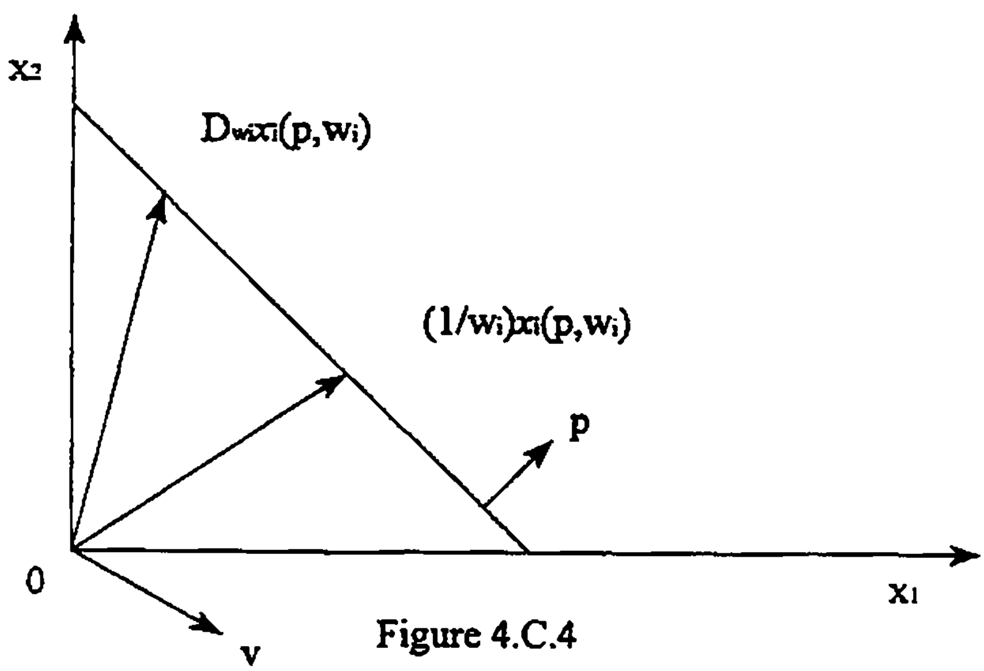


Figure 4.C.4

Thus

$$v \cdot D_p x_i(p,w_i) v = - v \cdot D_{w_i} x_i(p,w_i) x_i(p,w_i)^T v = - (v \cdot D_{w_i} x_i(p,w_i))(v \cdot x_i(p,w_i)) > 0,$$

which implies that the ULD property is not satisfied. Thus, if it is in fact satisfied, then we must have $D_{w_i} x_i(p,w_i) = (1/w_i) x_i(p,w_i)$ for every (p,w_i). Thus the unique wealth expansion path, which is the set of the corners of the upper contour sets, is a ray going through the origin. Hence the preference is homothetic.

4.C.5 Following the hint, we fix $w = 1$ and write $x_i(p) = x_i(p,1)$. Consider the indirect demand function $g_i(x) = \frac{1}{x\cdot\nabla u_i(x)}\nabla u_i(x)$. Since $x_i(p) = x$ if and only if $g_i(x) = p$, the ULD property of $x_i(p)$ is equivalent to the following property: if $x \neq y$, then $(g_i(x) - g_i(y))\cdot(x - y) < 0$. For this latter property, it is sufficient to show that $D^2g_i(x)$ is negative definite. We shall now establish this.

By the chain rule (Appendix M.A),

$$Dg_i(x) = (x\cdot\nabla u_i(x))^{-2}((x\cdot\nabla u_i(x))D^2u_i(x) - \nabla u_i(x)\nabla u_i(x)^T - \nabla u_i(x)x^TD^2u_i(x))$$

Let $q = \nabla u_i(x)$ and $C = D^2u_i(x)$, then this can be rewritten as

$$Dg_i(x) = (x\cdot q)^{-2}((x\cdot q)C - qq^T - qx^TC).$$

We need to show that $v\cdot Dg_i(x)v < 0$ for every $v \neq 0$. If $v\cdot q = 0$, then $v\cdot Dg_i(x)v = (x\cdot q)^{-1}v\cdot Cv < 0$. (This property is equivalent to the negative definiteness of the bordered Hessian of $u_i(\cdot)$ and used to guarantee the differentiability of the demand function, as explained in the Appendix to Chapter 3). So suppose that $v\cdot q \neq 0$. By multiplying a scalar to v if necessary, we can assume that $v\cdot q = x\cdot q$. Then

$$v\cdot Dg_i(x)v = (x\cdot q)^{-1}(v\cdot Cv - v\cdot q - x\cdot Cv).$$

By $x\cdot q > 0$, we need to show that $v\cdot Cv - v\cdot q - x\cdot Cv < 0$. Since C is symmetric,

$$v\cdot Cv - x\cdot Cv = (v - (1/2)x)\cdot C(v - (1/2)x) - (1/4)x\cdot Cx.$$

Since $u_i(\cdot)$ is concave, C is negative semidefinite and the first term in the above expression is non-positive. Thus,

$$v\cdot Cv - x\cdot Cv \leq -(1/4)x\cdot Cx.$$

Hence

$$v\cdot Cv - v\cdot q - x\cdot Cv \leq -(1/4)x\cdot Cx - q\cdot x.$$

Since $-\frac{x\cdot Cx}{q\cdot x} < 4$, the right-hand side is negative. Hence so is the left-hand side.

4.C.6 By differentiating both sides of $u(\lambda x) = \lambda u(x)$ with respect to λ and taking $\lambda = 1$, we obtain $\nabla u(x)\cdot x = u(x)$. Then by differentiating both sides of this equality with respect to x, we obtain $D^2u(x)x + \nabla u(x) = \nabla u(x)$. Thus $D^2u(x)x = 0$ and hence $\sigma(x) = 0$.

4.C.7 Suppose that the distribution of wealth has a differentiable, nonincreasing density function $f(\cdot)$ over the interval $[0,\bar{w}]$. Let $v \in \mathbb{R}^L$ and $v \neq 0$, then, just as in the proof of Proposition 4.C.4, we have

$$v\cdot Dx(p)v = \int_0^{\bar{w}}(v\cdot S(p,w)v)f(w)dw - \int_0^{\bar{w}}(v\cdot D_w\tilde{x}(p,w))(v\cdot\tilde{x}(p,w))f(w)dw.$$

Here, the first term is negative, unless v is proportional to p. (This property is equivalent to the negative definiteness of the bordered Hessian of $u_i(\cdot)$ and used to guarantee the differentiability of the demand function, as explained in the Appendix to Chapter 3). As for the second term, just as in the proof of Proposition 4.C.4,

$$\int_0^{\bar{w}}(v\cdot D_w\tilde{x}(p,w))(v\cdot\tilde{x}(p,w))f(w)dw = (1/2)\int_0^{\bar{w}} \frac{d(v\cdot\tilde{x}(p,w))^2}{dw}f(w)dw.$$

By integration by parts and $\tilde{x}(p,0) = 0$, this is equal to

$$(1/2)(v\cdot\tilde{x}(p,\bar{w}))^2f(\bar{w}) - (1/2)\int_0^{\bar{w}}(v\cdot x(p,w))^2f'(w)dw$$

The first part of this is always nonnegative, and it is positive when v is proportional to p. The integral of the second part is nonpositive because $f'(w) \le 0$. Thus $(1/2)\int_0^{\bar{w}} \frac{d(v\cdot\tilde{x}(p,w))^2}{dw}f(w)dw \ge 0$. Hence $v\cdot Dx(p)v < 0$.

To see that there are unimodal density functions for which the conclusions of this propositions do not hold, recall that

$$v\cdot Dx(p)v = \int_0^{\bar{w}}(v\cdot S(p,w)v)f(w)dw - \int_0^{\bar{w}}(v\cdot D_w\tilde{x}(p,w))(v\cdot\tilde{x}(p,w))f(w)dw.$$

To be specific, let $v = (1,0,\ldots,0) \in \mathbb{R}^L$, then

$$D_1x_1(p) = \int_0^{\bar{w}} s_{11}(p,w)f(w)dw - \int_0^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw.$$

Suppose also that the graph of the function $w \to \tilde{x}_1(p,w)$ is as shown below.

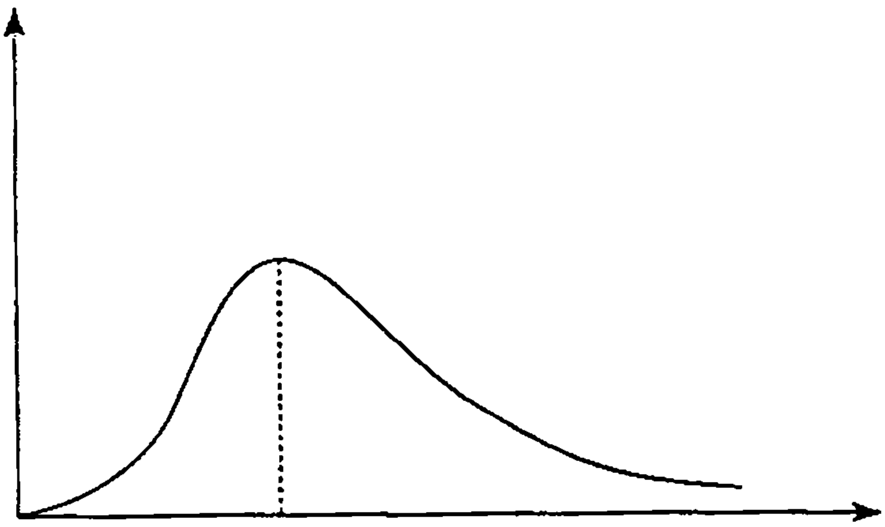


Figure 4.C.7

Then $D_w\tilde{x}_1(p,w) \geq 0$ for every $w \in [0,w^*]$ and $D_w\tilde{x}_1(p,w) \leq 0$ for every $w \in [w^*,\bar{w}]$. Thus $\int_0^{w^*} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw \geq 0$ and $\int_{w^*}^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw \leq 0$. If the density function $f(\cdot)$ is nonincreasing, the weight placed on the interval $[0,w^*]$ is sufficient to dominate the negative effect of the interval $[w^*,\bar{w}]$, implying that $\int_0^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw \geq 0$. If the distribution function is not nonincreasing, then the weight on the interval $[w^*,\bar{w}]$ may dominate that on the interval $[0,w^*]$, so that $\int_0^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw < 0$. It could even dominate the substitution effect, in which case we have

$$v \cdot Dx(p)v = \int_0^{\bar{w}} (v \cdot S(p,w)v)f(w)dw - \int_0^{\bar{w}} (v \cdot D_w\tilde{x}(p,w))(v \cdot \tilde{x}(p,w))f(w)dw > 0.$$

4.C.8 By substituting (4.C.6) into (4.C.5), we obtain

$$S(p,w) = \sum_i S_i(p,\alpha_i w) - \sum_i D_{w_i} x_i(p,\alpha_i w)x_i(p,\alpha_i w) + (\sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w))x(p,w)^T.$$

On the other hand, the right-hand side of (4.C.7) equals to

$$\sum_i S_i(p,\alpha_i w) - \sum_i D_{w_i} x_i(p,\alpha_i w)x_i(p,\alpha_i w) + (\sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w))x(p,w)^T$$

$$+ D_w x(p,w)(\sum_i x_i(p,\alpha_i w))^T - (\sum_i \alpha_i) D_w x(p,w)) x(p,w)^T$$

$$= \sum_i S_i(p,\alpha_i w) - \sum_i D_{w_i} x_i(p,\alpha_i w) x_i(p,\alpha_i w) + (\sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w)) x(p,w)^T.$$

We have thus proved (4.C.8).

4.C.9 The homotheticity implies that $D_{w_i} x_i(p,\alpha_i w) = (1/\alpha_i w) x_i(p,\alpha_i w)$ and hence $D_w x(p,w) = \sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w) = (1/w) x(p,w)$. Thus

$$C(p,w) = \sum_i \alpha_i ((1/\alpha_i w) x_i(p,\alpha_i w) - (1/w) x(p,w))((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w))^T$$

$$= \sum_i (\alpha_i / w)((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w))((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w))^T.$$

Thus, for every $v \in \mathbb{R}^L$,

$$v \cdot C(p,w) v = \sum_i (\alpha_i / w)((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w) \cdot v)^2 \geq 0$$

Therefore, $C(p,w)$ is positive semidefinite.

4.C.10 Let's start by formulating our continuum-of-consumers situation. We take the interval [0,2] as the set of (the names of) the consumers. The population density is equal to 1/2 uniformly on [0,2]. We assume that the proportion of each consumer's wealth to the average wealth (which is, in this situation, a counterpart of the aggregate wealth) is constant regardless of the amount of the aggregate wealth. We further assume that when the "average" wealth is equal to $\bar{w} > 0$, the wealth of consumer $\eta \in [0,2]$ is equal to $\eta\bar{w}$. Since $\int_0^2 \eta\bar{w}(1/2)d\eta = \bar{w}$, the term "average" is justified.

The average demand is then defined by $x(p,\bar{w}) = \int_0^2 \tilde{x}(p,\eta\bar{w})(1/2)d\eta$ and the Slutsky matrix $S(p,\bar{w})$ is defined as in (4.C.4). The Slutsky matrix of consumer η is denoted by $\tilde{S}(p,\eta\bar{w})$. We then define $C(p,\bar{w})$ by

$$C(p,\bar{w}) = \int_0^2 \tilde{S}(p,\eta\bar{w})(1/2)d\eta - S(p,\bar{w}).$$

Just as in the small-type discussion following the proof of Proposition 4.C.4, we have

$$C(p,\bar{w}) = \int_0^2 (\eta/2)(D_w\tilde{x}(p,\eta\bar{w}) - D_w x(p,\bar{w}))((1/\eta)\tilde{x}(p,\eta\bar{w}) - x(p,\bar{w}))^T d\eta$$

$$= \int_0^2 (1/2)D_w\tilde{x}(p,\eta\bar{w})\tilde{x}(p,\eta\bar{w})d\eta - D_w x(p,\bar{w})x(p,\bar{w})^T$$

$$= \int_0^2 (1/2)D_w\tilde{x}(p,\eta\bar{w})\tilde{x}(p,\eta\bar{w})d\eta - (\int_0^2(\eta/2)D_w\tilde{x}(p,\eta\bar{w})d\eta)x(p,\bar{w})^T.$$

Thus, for each $v \in \mathbb{R}^L$,

$$v\cdot C(p,\bar{w})v$$

$$= \int_0^2 (1/2)(v\cdot D_w\tilde{x}(p,\eta\bar{w}))(v\cdot\tilde{x}(p,\eta\bar{w}))d\eta - (\int_0^2(\eta/2)v\cdot D_w\tilde{x}(p,\eta\bar{w})d\eta)(v\cdot x(p,\bar{w})).$$

For the first term, we know that

$$\int_0^2 (1/2)(v\cdot D_w\tilde{x}(p,\eta\bar{w}))(v\cdot\tilde{x}(p,\eta\bar{w}))d\eta = (1/4\bar{w})(v\cdot\tilde{x}(p,2\bar{w}))^2.$$

As for the second term, by integration by parts,

$$\int_0^2 (\eta/2)v\cdot D_w\tilde{x}(p,\eta\bar{w})d\eta$$

$$= [(\eta/2)(v\cdot\tilde{x}(p,\eta\bar{w}))(1/\bar{w})]_{\eta=0}^{\eta=2} - \int_0^2(1/2)(v\cdot\tilde{x}(p,\eta\bar{w}))(1/\bar{w})d\eta$$

$$= (1/\bar{w})(v\cdot\tilde{x}(p,2\bar{w})) - (1/\bar{w})(v\cdot x(p,\bar{w})).$$

Hence

$$v\cdot C(p,\bar{w})v = (1/\bar{w})((1/4)(v\cdot\tilde{x}(p,2\bar{w}))^2 - (v\cdot\tilde{x}(p,2\bar{w}))(v\cdot x(p,\bar{w})) + (v\cdot x(p,\bar{w}))^2)$$

$$= (1/\bar{w})((1/2)(v\cdot\tilde{x}(p,2\bar{w})) - v\cdot x(p,\bar{w}))^2 \ge 0.$$

Thus $C(p,\bar{w})$ is positive semidefinite.

4.C.11 (a) When deriving individual demands from the first-order conditions of utility maximization, we will neglect the nonnegativity constraints (which is investigated in Exercise 3.D.4(c)). In fact, we will later see that, for prices and wealths under consideration, the demands are always in the interior of the nonnegative orthant.

It follows directly from the first-order conditions that

$$x_1(p,w/2) = (x_{11}(p,w/2),x_{21}(p,w/2)) = (w/2p_1 - 4p_1/p_2,\ 4p_1^2/p_2^2),$$

$$x_2(p,w/2) = (x_{12}(p,w/2),x_{22}(p,w/2)) = (4p_2^2/p_1^2,\ w/2p_2 - 4p_2/p_1),$$

Hence

$$x(p,w) = x_1(p,w/2) + x_2(p,w/2)$$
$$= (w/2p_1 - 4p_1/p_2 + 4p_2^2/p_1^2,\ w/2p_2 - 4p_2/p_1 + 4p_1^2/p_2^2).$$

(b) Denote the (ℓ,k) entry of the Slutsky matrix $S_i(p,w)$ of consumer i by $s_{\ell ki}(p,w)$. Since $\partial x_{21}(p,w/2)/\partial w_1 = 0$, $s_{221}(p,w/2) = \partial x_{21}(p,w/2)/\partial p_2 = -8p_1^2/p_2^3$. Hence by Proposition 2.F.3, $s_{211}(p,w/2) = s_{121}(p,w/2) = 8p_1/p_2^2$, and hence $s_{111}(p,w/2) = -8/p_2$. Thus

$$S_1(p,w/2) = \begin{bmatrix} -8/p_2 & 8p_1/p_2^2 \\ 8p_1/p_2^2 & -8p_1^2/p_2^3 \end{bmatrix}.$$

Similarly, we can show that

$$S_2(p,w/2) = \begin{bmatrix} -8p_1^2/p_2^3 & 8p_2/p_1^2 \\ 8p_2/p_1^2 & -8/p_1 \end{bmatrix}.$$

We can also apply Proposition 2.F.3 to derive the Slutsky matrix $S(p,w)$ of the aggregate demand function:

$$S(p,w) = \begin{bmatrix} -w/4p_1^2 - 6/p_2 - 6p_2^2/p_1^3 & w/4p_1p_2 + 6p_1/p_2^2 + 6p_2/p_1^2 \\ w/4p_1p_2 + 6p_1/p_2^2 + 6p_2/p_1^2 & -w/4p_2^2 - 6/p_1 - 6p_1^2/p_2^3 \end{bmatrix}$$

By Exercise 2.F.9(b) (and $S(p,w)p = 0$), if $dp \in \mathbb{R}^2$, $dp \neq 0$, and dp is not proportional to p, then $dp \cdot S(P,w)dp < 0$. Thus, according to the small-type discussion after Proposition 2.F.3, the aggregate demand function $x(p,w)$ satisfies the WA.

(c) By substituting $p = (1,1)$, we obtain

$$C(p,w) = \textstyle\sum_i S_i(p,w/2) - S(p,w) = \begin{bmatrix} w/4 - 4 & 4 - w/4 \\ 4 - w/4 & w/4 - 4 \end{bmatrix}.$$

Thus, if $w > 16$, then it is positive semidefinite, and if $8 < w < 16$, then it is negative semidefinite. (This can be shown by applying Theorem M.D.2(ii), or by noticing that $C(p,w)p = 0$ and $p \cdot C(p,w) = 0$ and then applying the argument in the proof of Exercise 2.F.9(b).) For example, if $w = 12$ and $v = (1,0)$, then $v \cdot C(p,w)v = -1$. Thus $C(p,w)$ is not positive semidefinite. We saw in (b) that the aggregate demand function satisfies the WA. We can thus conclude that in order for an aggregate demand function to satisfy the WA, it is not necessary that the matrix $C(p,w)$ is positive semidefinite.

(d) Here is a figure depicting the wealth expansions paths of the two consumers for $p = (1,1)$. They intersect each other at (4,4) because $x_1(1,1,8) = x_2(1,1,8) = (4,4)$. Note that if $8 < w < 16$, the Engel curves resemble those of figure 4.C.2(b); if $w > 16$, the Engel curves resemble those of figure 4.C.2(a).

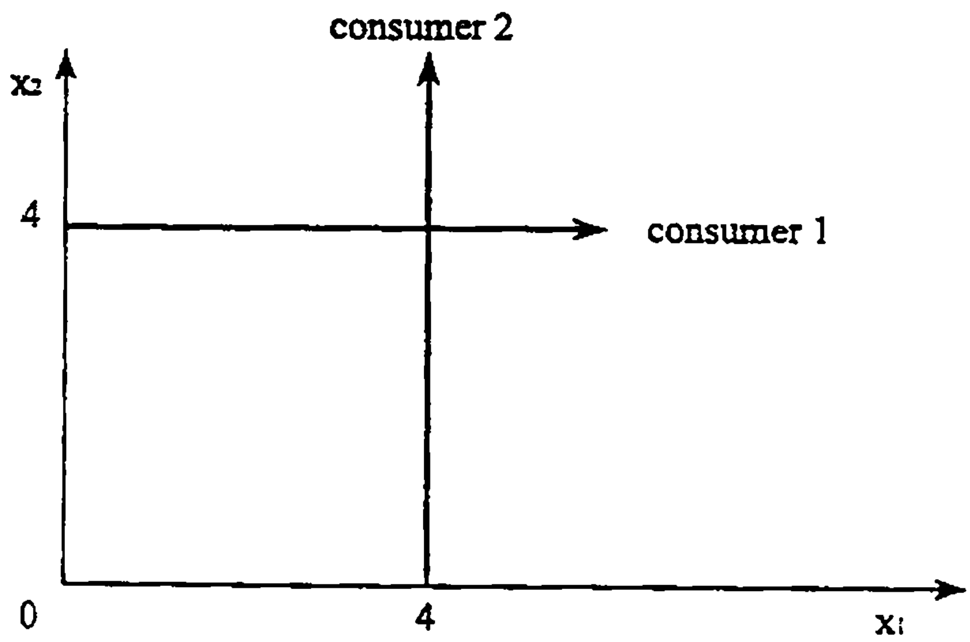


Figure 4.C.11(d)

4.C.12 As suggested in the hint, our example is a two-commodity, two-consumer economy in which the two consumers have the same preference and the wealth distribution rule is such that when the aggregate wealth is equal to 4, the wealth of consumer 1 is equal to 1 and the wealth of consumer 2 is equal to 3. The example here is essentially the same as Example 4.C.1 (which is illustrated in Figure 4.C.1). Example 4.C.1 is not directly applicable to the present context, because the two consumers have the same wealth but the different demands. However, if wealth levels are different between the two consumers, then it is possible that they have the same preference and yet demands similar to those in Figure 4.C.1, yielding a violation of the WA in the aggregate. This is illustrated in the figure below:

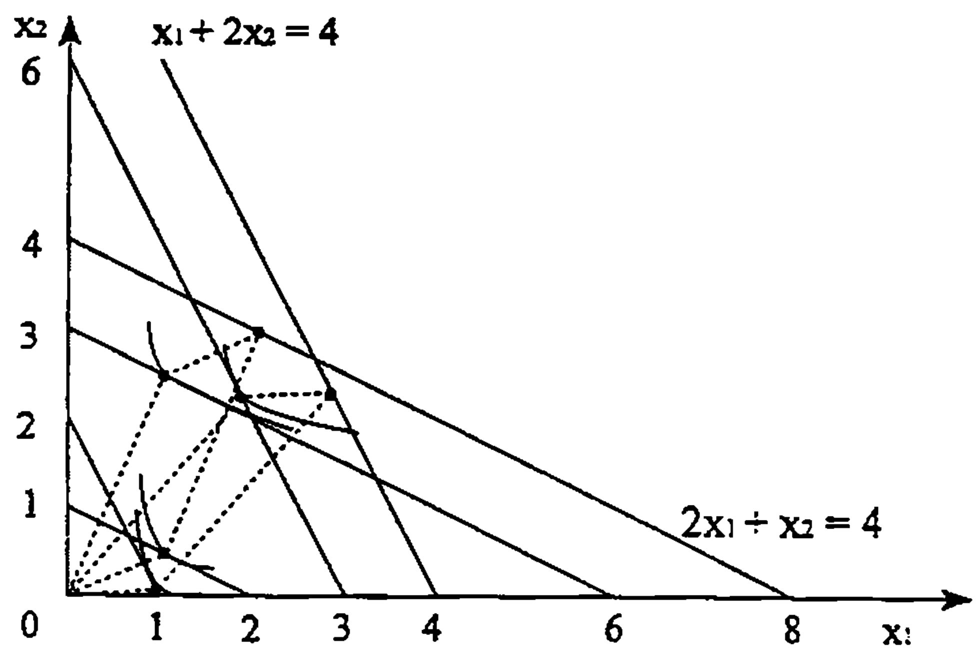


Figure 4.C.12

4.C.13 We consider a two-commodity, two-consumer economy with the given wealth distribution rule $w_1(p,w) = wp_1/(p_1 + p_2)$ and $w_2(p,w) = wp_2/(p_1 + p_2)$. The preferences of the consumers are represented by the following utility functions:

$$u_1(x_1) = \text{Min } \{x_{11}, 2x_{21}\},$$

$$u_2(x_2) = \text{Min } \{2x_{12}, x_{22}\}.$$

So the preferences are homothetic and have L-shaped indifference curves. The unique wealth expansion paths are depicted in the figure below:

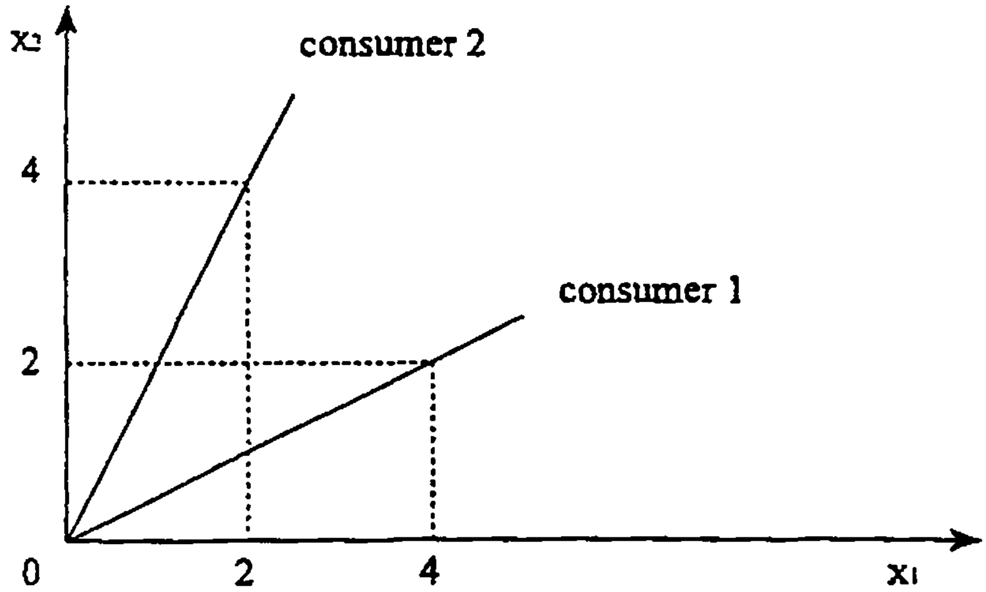


Figure 4.C.13

Just as we saw in the answer to Exercise 3.D.5(c), the individual demand functions are

$$x_1(p,w_1) = (2w_1/(2p_1 + p_2),\ w_1/(2p_1 + p_2)),$$

$$x_2(p,w_2) = (w_2/(p_1 + 2p_2),\ 2w_2/(p_1 + 2p_2)).$$

The aggregate demand function is given by

$$x(p,w) = x_1(p,w_1(p,w)) + x_2(p,w_2(p,w)).$$

We claim that the aggregate demand function does not satisfy the WA. To see this, define $\Delta = \{p \in \mathbb{R}^2_{++}: p_1 + p_2 = 1\}$ and our restrict attention to $p \in \Delta$ and $w = 1$. Then $w_i(p,w) = p_i$ for both i and hence

$$\lim_{p\to(1,0)} x(p,1) = \lim_{p\to(1,0)} x_1(p,w_1(p,1)) = (1,1/2),$$

$$\lim_{p\to(0,1)} x(p,1) = \lim_{p\to(0,1)} x_2(p,w_2(p,1)) = (1/2,1).$$

Thus, if $p \in \Delta$, $q \in \Delta$, and p_2 and q_1 are sufficiently small, then $q\cdot x(p,1) < 1$ and $p\cdot x(q,1) < 1$. Hence the aggregate demand function does not satisfy the WA.

Proposition 4.C.1 does not apply to this example because the wealth distribution rule of this example depends on prices, while the proposition does not allow this.

4.D.1 It is easy to check from the budget constraints that the distribution $(x_1(p,w_1(p,w)),\ldots,x_I(p,w_I(p,w)))$ of commodity bundles satisfies the constraints of the maximization problem in this exercise, and, from the definition of the indirect utility functions, that it attains the value $v(p,w)$. It thus remains to show that if $(x_1,\ldots,x_I)$ satisfies the constraints of the maximization problem in this exercise, then $W(u_1(x_1),\ldots,u_I(x_I)) \le v(p,w)$. For each i, define $w_i = p\cdot x_i$, then $\sum_i w_i = \sum_i p\cdot x_i = p\cdot(\sum_i x_i) \le w$, that is, $(w_1,\ldots,w_I)$ satisfies the constraint of the maximization problem of (4.D.1). Hence, by the definition of $v(p,w)$,

$$W(v_1(p,w_1),\ldots,v_I(p,w_I)) \le v(p,w).$$

On the other hand, by the definition of the indirect utility functions, $u_i(x_i) \le v_i(p,w_i)$ for every i. This and the increasingness of $W(\cdot)$ imply that

$$W(u_1(x_1),\ldots,u_I(x_I)) \le W(v_1(p,w_1),\ldots,v_I(p,w_I)).$$

Hence $W(u_1(x_1),\ldots,u_I(x_I)) \le v(p,w)$. This completes our proof.

4.D.2 To check that $v(p,w)$ is increasing in w, let p be a price vector, w and w' be two aggregate wealth levels with $w \le w'$, and $(w_1,\ldots,w_I)$ be a solution to the social welfare maximization problem of (p,w). Then

$$v(p,w) = W(v_1(p,w_1),\ldots,v_I(w_I)).$$

Also, $\sum_i w_i \le w$ and hence $\sum_i w_i \le w'$. Thus, by the definition of $v(p,w')$,

$$W(v_1(p,w_1),\ldots,v_I(p,w_I)) \le v(p,w').$$

Hence $v(p,w) \le v(p,w')$.

To check that $v(p,w)$ is nonincreasing in p, let w be an aggregate wealth level, p and p' be two price vectors with and $p' \ge p$, and let $(w_1,\ldots,w_I)$ be a solution to the social welfare maximization problem of (p',w). Then

$$v(p',w) = W(v_1(p',w_1),\ldots,v_I(p',w_I)).$$

Also, $v_i(p',w_i) \leq v_i(p,w_i)$ for every i because $p' \geq p$. Since $W(\cdot)$ is increasing, this implies that

$$W(v_1(p',w_1),\ldots, v_I(p',w_I)) \leq W(v_1(p,w_1),\ldots,v_I(p,w_I)).$$

By the definition of $v(p,w)$,

$$W(v_1(p,w_1),\ldots,v_I(p,w_I)) \leq v(p,w).$$

Hence $v(p',w) \leq v(p,w)$.

To verify the homogeneity of degree zero and the quasiconvexity, we apply the equivalence of the two maximization problems established in Exercise 4.D.1. For any (p,w) and $\lambda > 0$, the two price-wealth pairs (p,w) and $(\lambda p,\lambda w)$ give the same constraints to the maximization problem of Exercise 4.D.1. Hence $v(p,w) = v(\lambda p,\lambda w)$, implying the homogeneity. As for the quasiconvexity, let $v \in \mathbb{R}$. Let (p,w) and (p',w') be two price-wealth pairs such that $v(p,w) \leq v$ and $v(p',w') \leq v$. Let $\lambda \in [0,1]$ and define $p'' = \lambda p + (1 - \lambda)p'$ and $w'' = \lambda w + (1 - \lambda)w'$. Let $(x_1,\ldots,x_I)$ be a solution for (p'',w''). Then $p''\cdot(\sum_i x_i) \leq w''$ and hence

$$\lambda p\cdot(\textstyle\sum_i x_i) + (1 - \lambda)p'\cdot(\sum_i x_i) \leq \alpha w + (1 - \alpha)w'.$$

We must thus have either $w \geq p\cdot(\sum_i x_i)$ or $w' \geq p'\cdot(\sum_i x_i)$. Hence we must have either $W(u_1(x_1),\ldots,u_I(x_I)) \leq v(p,w)$ or $W(u_I(x_1),\ldots,u_I(x_I)) \leq v(p',w')$. In either case, we have $W(u_1(x_1),\ldots,u_I(x_I)) \leq v$. Hence $v(p,w)$ is quasiconvex.

4.D.3 The welfare maximization problem is now rewritten with nonnegativity constraints:

$$\max_{(w_1,\ldots,w_I)} W(v_1(p,w_1),\ldots,v_I(p,w_I))$$

$$\text{s.t.} \quad \textstyle\sum_i w_i \leq w,$$

$$w_i \geq 0 \text{ for all } i.$$

We assume that $x_i(p,0) = 0$ for every i and every $p \gg 0$. This assumption is satisfied [illegible]ple, the consumption sets are all $\mathbb{R}^L_+$. Then $\nabla_p v_i(p,0) = 0$ for ever[illegible] $p >> 0$. The Kuhn-T[illegible] conditions for the social wel[illegible]ation (Theorem M.K.2) are that there exist $\lambda > 0$ and $\mu_i \geq 0$ such [illegible]

$$[illegible]/\partial w_i) - \lambda + \mu_i = 0 \text{ for every } i = 1,\ldots,I,$$

[illegible]

$$[illegible]\,\mu_i w_i = 0 \quad \text{for every } i = 1,\ldots,I,$$

where all [illegible] are evaluated at the solution

$$(w_1,\ldots,w_I) = (w_1(p,w),\ldots,w_I(p,w)).$$

The Envelo[illegible]Theorem M.L.1) implies that

[illegible]

$$[illegible](\partial W/\partial u_i)(\partial v_i/\partial p_\ell).$$

Now defin[illegible](p,w) > 0\}. Since $\mu_i = 0$ for every $i \in J$, $\partial v/\partial w = (\partial W/\partial u_i)(\partial[illegible]$ every $i \in J$. Since $\partial v_i/\partial p_\ell = 0$ for any $i \notin J$, $\partial v/\partial p_\ell = \sum_{i\in J}(\partial W/\partial u[illegible]$. Thus,

$$-\frac{\partial v[illegible]}{\partial v[illegible]} \sum_{i\in I} \frac{(\partial W/\partial u_i)(\partial v_i/\partial p_\ell)}{\partial v/\partial w} = -\sum_{i\in J} \frac{(\partial W/\partial u_i)(\partial v_i/\partial p_\ell)}{(\partial W/\partial u_i)(\partial v_i/\partial w_i)}$$

$$= -\sum_{i\in I} \frac{\partial v_i/\partial p_\ell}{\partial v_i/\partial w_i} = \sum_{i\in J} x_i(p,w_i(p,w)) = \sum_i x_i(p,w_i(p,w)).$$

Hence the [illegible]and function derived from $v(p,w)$ equals $\sum_i x_i(p,w_i(p,w))$ and the pr[illegible]ted.

4.D.4 (a) [illegible] and $W(\cdot)$ are monotone, then so is $u(\cdot)$.

If the[illegible]$(\cdot)$ are continuous, then the composite map $(x_1,\ldots,x_I) \to W(u_1(x_1),$[illegible] also continuous. Thus, by the Theorem of the Maximum ([illegible]6), $u(\cdot)$ is also contin[illegible].

If the[illegible]ncave and $W(\cdot)$ are monotone and concave, then $u(\cdot)$ is concave. This can be proved as follows. Let $x \in \mathbb{R}^L$, $x' \in \mathbb{R}^L$, and $\lambda \in [0,1]$. Define $x'' = \lambda x + (1 - \lambda)x' \in \mathbb{R}^L$. Let $(x_1,\dots,x_I) \in \mathbb{R}^{LI}$, $(x_1',\dots,x_I') \in \mathbb{R}^{LI}$, $\sum_i x_i = x$, $\sum_i x_i' = x'$, $u(x) = W(u_1(x_1),\dots,u_I(x_I))$, and $u(x') = W(u_1(x_1'),\dots,u_I(x_I'))$. Define $(x_1'',\dots,x_I'') \in \mathbb{R}^{LI}$ by $x_i'' = \lambda x_i + (1 - \lambda)x_i'$ for each i. Then $\sum_i x_i'' \le \lambda x + (1 - \lambda)x'$ and hence

$$u(\lambda x + (1 - \lambda)x') \ge W(u_1(x_1''),\dots,u_I(x_I'')).$$

By the concavity of the $u_i(\cdot)$ and the monotonicity of $W(\cdot)$,

$$W(u_1(x_1''),\dots,u_I(x_I''))$$
$$\ge W(\lambda u_1(x_1) + (1 - \lambda)u_1(x_1'),\ \dots,\ \lambda u_I(x_I) + (1 - \lambda)u_I(x_I')).$$

By the concavity of $W(\cdot)$,

$$W(\lambda u_1(x_1) + (1 - \lambda)u_1(x_1'),\ \dots,\ \lambda u_I(x_I) + (1 - \lambda)u_I(x_I'))$$
$$\ge \lambda u(x) + (1 - \lambda)u(x').$$

Hence $u(\lambda x + (1 - \lambda)x') \ge \lambda u(x) + (1 - \lambda)u(x')$. Hence $u(\cdot)$ is concave.

It is worthwhile to point out that the quasiconcavity of $W(\cdot)$ and the $u_i(\cdot)$ does not imply that of $u(\cdot)$. As an example, let $L = 2$, $I = 2$, $u_1(x_1) = x_{11}^2$, $u_2(x_2) = x_{22}^2$, and $W(u_1,u_2) = u_1 + u_2$. Then

$$W(2,0) = W(u_1(2,0),u_2(0,0)) = W(4,0) = 4,$$

$$W(0,2) = W(u_1(0,0),u_2(0,2)) = W(0,4) = 4,$$

$$W(1,1) = W(u_1(1,0),u_2(0,1)) = W(1,1) = 2.$$

(b) We shall first prove that, for every $x \in \mathbb{R}^L$, if there exists $(x_1,\dots,x_I) \in \mathbb{R}^{LI}$ that satisfies $\sum_i x_i \le x$ and is a solution to the maximization problem of Exercise 4.D.1, then x is a solution to the maximization problem of this part. In fact, suppose that $(x_1,\dots,x_I) \in \mathbb{R}^{LI}$, $\sum_i x_i \le x$, and $(x_1,\dots,x_I)$ is a solution to the maximization problem of Exercise 4.D.1. Let $x' \in \mathbb{R}^L$ and $p \cdot x' \le w$. Let $(x_1',\dots,x_I') \in \mathbb{R}^{LI}$, $\sum_i x_i' \le x'$, and $u(x') = W(u_1(x_1'),\dots,u_I(x_I'))$. Then $p \cdot (\sum_i x_i') \le w$. Hence $W(u_1(x_1'),\dots,u_I(x_I')) \le W(u_1(x_1),\dots,u_I(x_I))$. Since

$\sum_i x_i \le x$, $W(u_1(x_1),\dots,u_I(x_I)) \le u(x)$. Thus $u(x') \le u(x)$. Hence x is a solution to the maximization problem of this part.

By Exercise 4.D.1, $(x_1(p,w_1(p,w)),\dots,x_I(p,w_I(p,w))$ is a solution to its maximization problem. Thus, by the above result, $\sum_i x_i(p,w_i(p,w))$ is a solution to the maximization problem of this part. Hence the Walrasian demand function generated from it is equal to the aggregate demand function.

4.D.5 There is no positive representative consumer if the WA is violated. Example 4.C.1 thus serves as an example for this exercise.

4.D.6 The social welfare maximization problem is now written as

$$\text{Max}_{(w_1,\dots,w_I)} \ \sum_i \alpha_i \ln v_i(p,w_i)$$
$$\text{s.t.} \qquad \sum_i w_i \le w.$$

The first-order conditions are that there exists $\lambda > 0$ such that $(\partial W/\partial u_i)(\partial v_i/\partial w_i) = \lambda$ for every i, where all derivatives are evaluated at a solution $(w_1,\dots,w_I)$. By the definition of $W(\cdot)$, $\partial W/\partial u_i = \alpha_i/v_i(p,w_i)$. By Exercise 3.D.3(b), $v_i(p,w_i)$ is homogeneous of degree one in w_i and hence $\partial v_i(p,w_i)/\partial w_i = v_i(p,w_i)/w_i$. Hence the left-hand sides of the above first-order conditions equal $(\alpha_i/v_i(p,w_i))(v_i(p,w_i)/w_i) = \alpha_i/w_i$ for every i. Thus $w_i = \alpha_i/\lambda$. Since $\sum_i \alpha_i = 1$ and $\sum_i w_i = w$, $w = 1/\lambda$. Hence $w_i = \alpha_i w$. Thus $w_i(p,w) = \alpha_i w$.

4.D.7 The social welfare maximization problem is now written as

$$\text{Max}_{(w_1,\dots,w_I)} \ \sum_i a_i(p) + b(p)(\sum_i w_i)$$
$$\text{s.t.} \qquad \sum_i w_i \le w.$$

Thus any $(w_1,\dots,w_I)$ with $\sum_i w_i = w$ is a solution to this problem and $v(p,w) =$

$e(p) + b(p)w$.

Suppose that (p', w') passes the potential compensation test over (p, w). Then there exists $(w_1', \ldots, w_I') \in \mathbb{R}^I$ such that $\sum_i w_i' \le w$ and $v_i(p', w_i') > v_i(p, w))$. Since $W(\cdot)$ is increasing, $W(v_1(p', w_1'), \ldots, v_I(p', w_I')) > v(p, w)$. By the definition of $v(p', w')$, $v(p', w') \ge W(v_1(p', w_1'), \ldots, v_I(p', w_I'))$. Hence $v(p', w') > v(p, w)$.

Define $A_i = \{x_i : u_i(x_i) \ge u_i(\bar{x}_i)\}$, then $A = \sum_i A_i$. Then, for every $p \gg 0$, $x \in \mathbb{R}^L$ is a solution to

$$\operatorname{Min}_{y \in A} \; p \cdot y$$

if and only if there exists $(x_1, \ldots, x_I) \in A_1 \times \ldots \times A_I$ such that $\sum_i x_i = x$ and for every i, x_i is a solution to

$$\operatorname{Min}_{y_i \in A_i} \; p \cdot x_i.$$

Hence $\operatorname{Min}\{p \cdot y : y \in A\} = \sum_i \operatorname{Min}\{p \cdot y_i : y_i \in A_i\}$. By the definition,

$$\operatorname{Min}\{p \cdot y_i : y_i \in A_i\} = e_i(p, u_i(\bar{x}_i))$$

for every i. Hence

$$\operatorname{Min}\{p \cdot y : y \in A\} = \sum_i e_i(p, u_i(\bar{x}_i)).$$

On the other hand, by the definition,

$$\operatorname{Min}\{p \cdot y : y \in B\} = e(p, u(\bar{x})).$$

Since $A \subset B$, $\operatorname{Min}\{p \cdot y : y \in A\} \ge \operatorname{Min}\{p \cdot y : y \in B\}$. Hence

$$g(p) = e(p, u(\bar{x})) - \sum_i e_i(p, u_i(\bar{x}_i)) \le 0$$

for every p. By the definition, $g(\bar{p}) = 0$. Hence, by the second-order necessary condition for a maximum (Section M.K), $D^2 g(\bar{p})$ is negative semidefinite. Since $D^2 g(\bar{p}) = S(\bar{p}, u(\bar{x})) - \sum_i S_i(p, u_i(\bar{x}_i))$ by Propositions 3.G.3 and 3.G.3, $S(p, u(\bar{x})) - \sum_i S_i(p, u_i(\bar{x}_i))$ is negative semidefinite.

4.D.10 As we saw in the answer to Exercise 4.C.11(b), for every (p,w), $S(p,w)$ is symmetric, and if $dp \in \mathbb{R}^2$, $dp \neq 0$, and dp is not proportional to p, then $dp \cdot S(P,w)dp < 0$. Thus, according to Section 3.H, there exists a positive representative consumer. But according to Exercise 4.C.11(c), if $8 < w < 16$, then the matrix $C(p,w)$ is not positive semidefinite. Hence, according to the small-type discussion in Section 4.D, there is no normative representative consumer.

4.D.11 We shall give an example in which $L = 3$ and $C_{12}(p,\bar{w}) \neq C_{21}(p,\bar{w})$. By the definition,

$$C_{12}(p,\bar{w}) = \int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta - (\int_0^2(\eta/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)d\eta)\tilde{x}_2(p,\eta\bar{w}).$$

Here, by integration by parts,

$$\int_0^2(\eta/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)d\eta$$
$$= \int_0^2(\eta/2\bar{w})(d\tilde{x}_1(p,\eta\bar{w})/d\eta)d\eta$$
$$= [(\eta/2\bar{w})\tilde{x}(p,\eta\bar{w})]_{\eta=0}^{\eta=2} - \int_0^2(1/2\bar{w})\tilde{x}_1(p,\eta\bar{w})d\eta$$
$$= (1/\bar{w})\tilde{x}_1(p,2\bar{w}) - (1/2\bar{w})x_1(p,\bar{w}).$$

Hence

$$C_{12}(p,\bar{w}) = \int_0^2(1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$
$$- (1/\bar{w})\tilde{x}_1(p,2\bar{w})x_2(p,\bar{w}) + (1/2\bar{w})x_1(p,\bar{w})x_2(p,\bar{w}).$$

Similarly,

$$C_{21}(p,\bar{w}) = \int_0^2(1/2)\tilde{x}_1(p,\eta\bar{w})(\partial\tilde{x}_2(p,\eta\bar{w})/\partial w)d\eta$$
$$- (1/\bar{w})x_1(p,\bar{w})\tilde{x}_2(p,2\bar{w}) + (1/2\bar{w})x_1(p,\bar{w})x_2(p,\bar{w}).$$

Again by integration by parts,

$$\int_0^2(1/2)\tilde{x}_1(p,\eta\bar{w})(\partial\tilde{x}_2(p,\eta\bar{w})/\partial w)d\eta$$

$$= \int_0^2 (1/2)\tilde{x}_1(p,\eta\bar{w})(1/\bar{w})(d\tilde{x}_2(p,\eta\bar{w})/d\eta)d\eta$$

$$= [(1/2\bar{w})\tilde{x}_1(p,\eta\bar{w})\tilde{x}_2(p,\eta\bar{w})]_{\eta=0}^{\eta=2} - \int_0^2 (1/2\bar{w})(d\tilde{x}_1(p,\eta\bar{w})/d\eta)\tilde{x}_2(p,\eta\bar{w})d\eta$$

$$= (1/2\bar{w})\tilde{x}_1(p,2\bar{w})\tilde{x}_2(p,2\bar{w}) - \int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$

It is thus sufficient to obtain an example in which

$$\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta - (1/\bar{w})\tilde{x}_1(p,2\bar{w})x_2(p,\bar{w})$$

$$\neq (1/2\bar{w})\tilde{x}_1(p,2\bar{w})\tilde{x}_2(p,2\bar{w}) - \int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$

$$- (1/\bar{w})x_1(p,\bar{w})\tilde{x}_2(p,2\bar{w}),$$

or, equivalently,

$$\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$

$$\neq (1/2\bar{w})\tilde{x}_1(p,2\bar{w})\tilde{x}_2(p,2\bar{w}) + (1/\bar{w})\tilde{x}_1(p,2\bar{w})x_2(p,\bar{w}) - (1/\bar{w})x_1(p,\bar{w})\tilde{x}_2(p,2\bar{w}).$$

So consider a preference, a price vector p, and the average wealth $\bar{w}$ such that:

$$\tilde{x}_\ell(p,2\bar{w}) = 2 \text{ for both } \ell = 1,2,$$

$$x_\ell(p,\bar{w}) = \int_0^2 (1/2)\tilde{x}_\ell(p,\eta\bar{w})d\eta = 1 \text{ for both } \ell = 1,2.$$

(Another restriction will be given shortly. The demand for good 3 is determined by Walras' law.) Then the right-hand side of the above inequality is equal to $2/\bar{w}$. It is then sufficient to show that

$$\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta \neq 2/\bar{w}.$$

But if the graphs of the functions $\eta \to \tilde{x}_\ell(p,\eta\bar{w})$ are as in the figure below, then the first term can be made as close to zero as needed. The above inequality can thus be established.

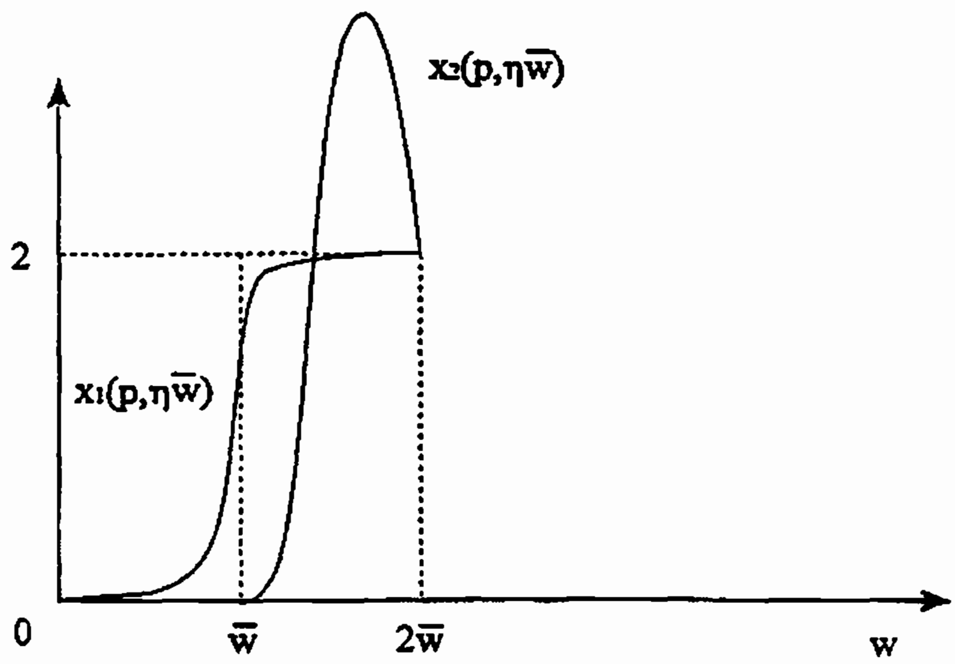


Figure 4.D.11

CHAPTER 5

5.B.1 The first example violates irreversibility and the second one satisfies this property.

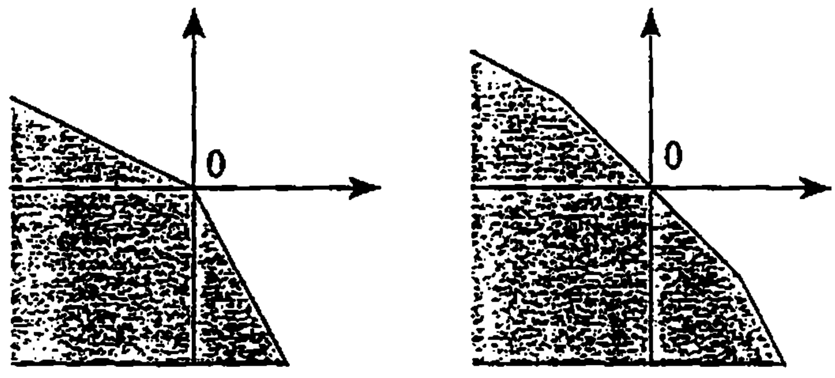


Figure 5.B.1

5.B.2 Suppose first that Y exhibits constant returns to scale. Let $z \in \mathbb{R}_+^{L-1}$ and $\alpha > 0$. Then $(-z, f(z)) \in Y$. By the constant returns to scale, $(-\alpha z, \alpha f(z)) \in Y$. Hence $\alpha f(z) \le f(\alpha z)$. By applying this inequality to αz in place of z and α^{-1} in place of α, we obtain $\alpha^{-1} f(\alpha z) \le f(\alpha^{-1}(\alpha z)) = f(z)$, or $f(\alpha z) \le \alpha f(z)$. Hence $f(\alpha z) = \alpha f(z)$. The homogeneity of degree one is thus obtained.

Suppose conversely that $f(\cdot)$ is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha \ge 0$, then $q \le f(z)$ and hence $\alpha q \le \alpha f(z) = f(\alpha z)$. Since $(-\alpha z, f(\alpha z)) \in Y$, we obtain $(-\alpha z, \alpha q) \in Y$. The constant returns to scale is thus obtained.

5.B.3 Suppose first that Y is convex. Let $z, z' \in \mathbb{R}_+^{L-1}$ and $\alpha \in [0,1]$, then

$(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$. By the convexity,

$$(-(\alpha z + (1-\alpha)z),\ \alpha f(z) + (1-\alpha)f(z)) \in Y.$$

Thus, $\alpha f(z) + (1-\alpha)f(z) \le f(\alpha z + (1-\alpha)z)$. Hence $f(z)$ is concave.

Suppose conversely that $f(z)$ is concave. Let $(q, -z) \in Y$, $(q', -z') \in Y$, and $\alpha \in [0,1]$, then $q \le f(z)$ and $q' \le f(z')$. Hence

$$\alpha q + (1-\alpha)q' \le \alpha f(z) + (1-\alpha)f(z').$$

By the concavity,

$$\alpha f(z) + (1-\alpha)f(z') \le f(\alpha z + (1-\alpha)z').$$

Thus

$$\alpha q + (1-\alpha)q' \le f(\alpha z + (1-\alpha)z').$$

Hence

$$(-(\alpha z + (1-\alpha)z'),\ \alpha q + (1-\alpha)q') = \alpha(-z, q) + (1-\alpha)(-z', q') \in Y.$$

Therefore Y is convex.

5.B.4 Note first that if Y itself is additive, then $Y^+ = Y$ by the definition of the additive closure. This applies to Figures 5.B.4, 5.B.6(a), 5.B.6(b), 5.B.7, and 5.B.8. So we shall not depict them.

Let's now take up the cases in which the production set is not additive. Note first that Y^+ is equal to the set of vectors of $\mathbb{R}^L$ that can be represented as the sum of finitely many vectors of Y. If the production set is convex, as in Figures 5.B.1, 5.B.2(a), 5.B.2(b), 5.B.3(a), 5.B.3(b), and 5.B.5(a) (which is the same as 5.B.2(b)), then we have the following stronger property: Y^+ consists of all "multiplied" production plans. To be more precise, for each positive integer n, define $nY \subset \mathbb{R}^L$ by $nY = \{ny \in \mathbb{R}^L: y \in Y\}$. We then claim that $Y^+ = \cup_{n=1}^{\infty} nY$. In fact, if $y \in nY$ for some n, then $(1/n)y \in Y$. By the definition, $(1/n)y \in Y^+$ and hence $y = n((1/n)y) \in Y^+$. Thus $Y^+ \supset \cup_{n=1}^{\infty} nY$. To prove that $Y^+ \subset \cup_{n=1}^{\infty} nY$, it is sufficient to show that $\cup_{n=1}^{\infty} nY$ is additive, by the definition of Y^+. So let $y \in \cup_{n=1}^{\infty} nY$ and $y' \in \cup_{n=1}^{\infty} nY$. Then there exist positive integers n and n' such that $y \in nY$ and $y' \in n'Y$. Thus $(1/n)y \in Y$ and $(1/n')y' \in Y$. Since Y is convex and $n/(n + n') + n'/(n + n') = 1$,

$$(n/(n + n'))((1/n)y) + (n'/(n + n'))((1/n')y') \in Y.$$

That is, $(1/(n + n'))(y + y') \in Y$. Thus $y + y' \in (n + n')Y$.

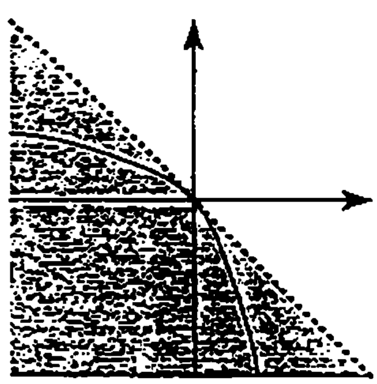
If the production set is not convex, as in Figure 5.B.5(b), then we no longer have $Y^+ = \cup_{n=1}^{\infty} nY$. As we can see in the above proof, while we still have $Y^+ \supset \cup_{n=1}^{\infty} nY$, we need not have $Y^+ \subset \cup_{n=1}^{\infty} nY$. That is, it may be true that some production plans in Y^+ can be attained only by allocating different inputs to different production units. This point can be formulated as follows. Let (y_1^*, y_2^*) be the production plan at which the average return is maximized. That is, $y_2^*/|y_1^*| > y_2/|y_1|$ for any other $y = (y_1, y_2) \in Y$. Assume that the function that associates each $y_1 < 0$ to the average return at y_1 is quasiconcave. (This appears to be true from the figure. It is equivalent to saying that the average return function is single-peaked.) Let $y_1 < 0$ be an (aggregate) input level and n be the positive integer such that $ny_1^* < y_1 \leq (n - 1)y_1^*$. Let $(y_{11}, \ldots, y_{1J})$ be an output-maximizing allocation of aggregate input y_1. (The number J of production units to be used is also being optimized here.) We shall prove that one of the following three cases must then apply:

(i) $J = n - 1$ and $y_{1j} = y_1/(n - 1)$ for every j.

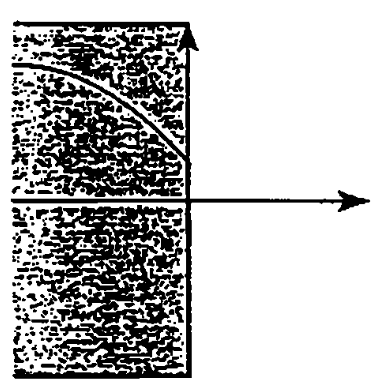
(ii) $J = n$ and $y_{1j} = y_1/n$ for every j.

(iii) $J = n$ and there exists $k \in \{1, \ldots, J\}$ such that $y_{1k} = y_1 - (n - 1)y_1^*$ and $y_{1j} = y_1^*$ for any $j \neq k$.

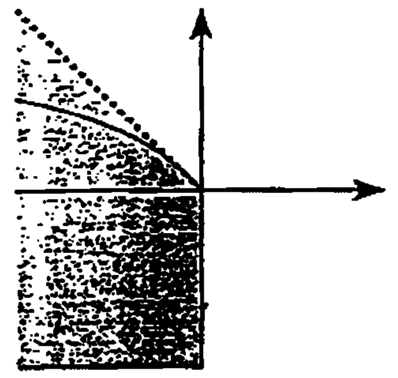
To prove this, note first that we must have either $y_{1j} \geq y_1^*$ for every j, or $y_{1j} \leq y_1^*$ for every j. In fact, if neither of these applies, then a small input reassignment from a production unit with $y_{1j} < y_1^*$ to one with $y_{1j} > y_1^*$ increases the (aggregate) output level, because the average return increases. If $y_{1j} \le y_1^*$ for every j, then the average return is decreasing at every y_{1j} and hence the y_{1j} must be all the same. By the quasiconcavity, J must be the maximum integer that satisfies $y_1^* = \sum_j y_{1j}$ (and $y_{1j} \le y_1^*$). Hence J = n - 1 and (i) applies. If $y_{1j} \ge y_1^*$ for every j, then the average return is increasing at every y_{1j}. If the y_{1j} are all the same, then, by the quasiconvexity, J must be the minimum integer that satisfies $y_1^* = \sum_j y_{1j}$ (and $y_{1j} \ge y_1^*$). Hence J = n and (ii) applies. So suppose that some of the y_{1j} are different. If there exist two production units for which $y_{1j} > y_1^*$, then a small input reassignment from one to the other would increase the output level, because the average return is increasing. Hence there must exist at most one k for which $y_{1k} > y_1^*$. By the quasiconcavity, $y_{1j} = y_1^*$ for any $j \ne k$. Thus (iii) applies. This last case happens when the average return decreases very fast after the input level goes beyond y_1^*.



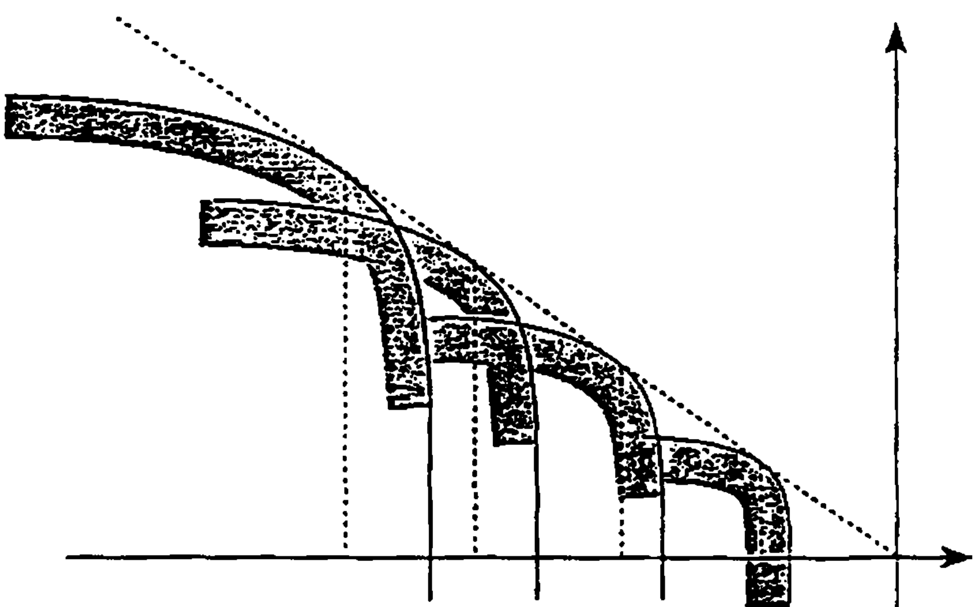
For Figure 5.B.1



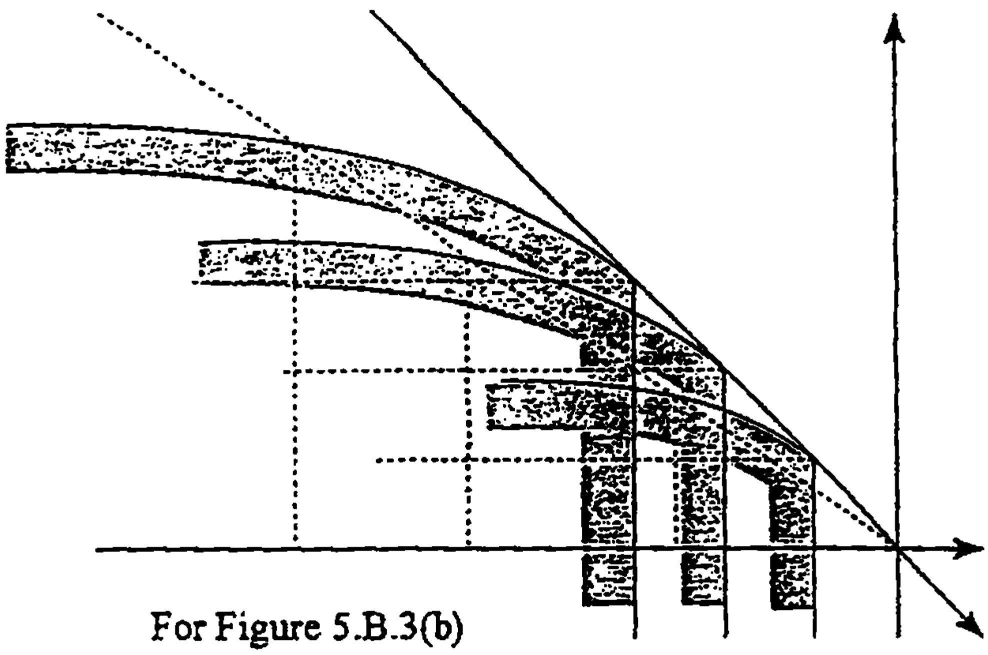
For Figure 5.B.2(a)



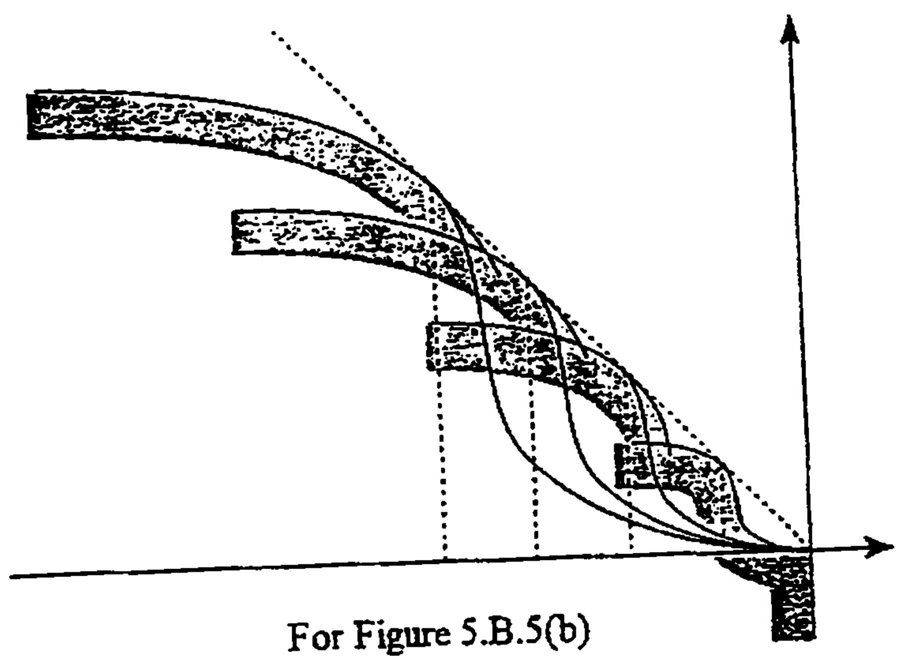
For Figure 5.B.2(b)



For Figure 5.B.3(a)



For Figure 5.B.3(b)



For Figure 5.B.5(b)

5.B.5 Let $y \in Y$ and $v \in -\mathbb{R}^L_+$. Then, for every $n \in \mathbb{N}$, $nv \in -\mathbb{R}^L_+$ and hence $nv \in Y$ by $Y \subset -\mathbb{R}^L_+$. Since Y is convex,

$$(1 - 1/n)y + (1/n)(nv) = (1 - 1/n)y + v \in Y.$$

Since Y is closed, $y + v = \lim_{n\to\infty} ((1 - 1/n)y + v) \in Y$.

5.B.6 (a) From the given functions $\phi_i(\cdot)$ (i = 1,2), the production set is defined as $Y = \{(y_1,y_2,q)$: there exist $q_1 \geq 0$ and $q_2 \geq 0$ such that $q_1 + q_2 \geq q$ and $-y_i \geq \phi_i(q_i)$ for both i$\}$. A three-dimensional production set is depicted in the following picture, assuming that the $\phi_i(\cdot)$ are convex.

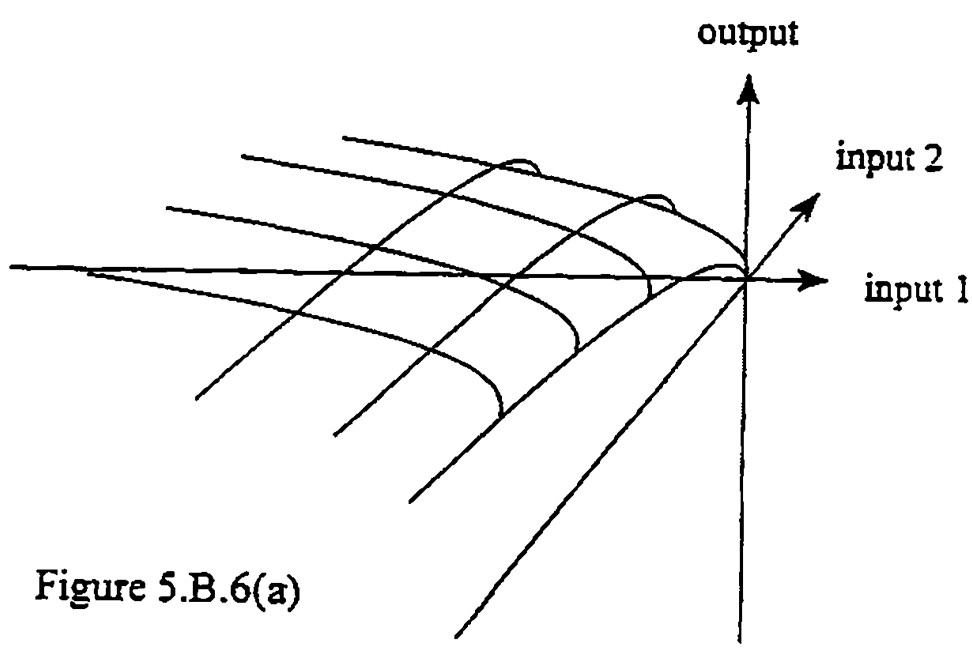


Figure 5.B.6(a)

(b) We claim that the condition that

$$\phi_i(q_i + q_i') \leq \phi_i(q_i) + \phi_i(q_i') \text{ for all } q_i \geq 0,\ q_i' \geq 0, \text{ and } i = 1,2,$$

is sufficient for additivity. In fact, let $(y_1,y_2,q) \in Y$ and $(y_1',y_2',q') \in Y$. Then there exist (q_1,q_2) such that $q_1 + q_2 \geq q$ and $-y_i \geq \phi_i(q_i)$, and (q_1', q_2') such that $q_i' + q_2' \geq q'$ and $-y_i' \geq \phi_i(q_i')$. Then

$$(q_1 + q_1') + (q_2 + q_2') \geq q + q',$$

and

$$-(y_i + y_i') \geq \phi_i(q_i) + \phi_i(q_i') \geq \phi_i(q_i + q_i').$$

Thus $(y_1 + y'_1, y_2 + y'_2, q + q') \in Y$, establishing the additivity.

(c) Let the output price be p. The first-order necessary conditions for profit maximization are that, for both i, $w_i\phi'_i(q_i) \le p$, with equality if $q_i > 0$. The interpretation is that marginal cost (in monetary term) due to a unit increase of output level must be smaller than or equal to the output price, and the former must be equal to the latter if the output level is positive.

If both $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are convex (so that the corresponding production set is convex), then, according to Theorem M.K.3, these first-order necessary conditions are also sufficient.

(d) Let $q > 0$. By renumbering $i = 1,2$ if necessary, we can assume that $w_1\phi_1(q) \le w_2\phi_2(q)$. In order to prove the statement, it is sufficient to show that, for any $q_1 > 0$ and $q_2 > 0$ with $q_1 + q_2 = q$, we have $w_1\phi_1(q_1) + w_2\phi_2(q_2) > w_1\phi_1(q)$. In fact, since the $\phi_i(\cdot)$ are strictly concave (and $\phi_i(0) = 0$) and $q_1/q + q_2/q = 1$,

$$w_1\phi_1(q_1) + w_2\phi_2(q_2) > w_1(q_1/q)\phi_1(q) + w_2(q_2/q)\phi_2(q)$$
$$= (q_1/q)w_1\phi_1(q) + (q_2/q)w_2\phi_2(q) \ge w_1\phi_1(q).$$

The statement is thus proved. The strict concavity of the $\phi_i(\cdot)$ is interpreted as the increasing returns to scale, which makes the statement quite plausible: Under the increasing returns to scale, it is better to concentrate on one technique.

The isoquants of the input use is drawn in the following figure. Note that the additive separability imposes the same restriction on the isoquants as that alluded to in Exercise 3.G.4(b).

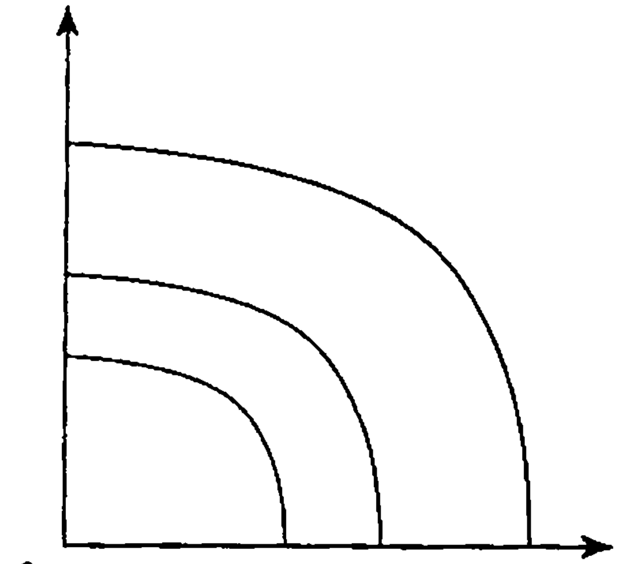


Figure 5.B.6(d)

5.C.1 If there is a production plan $y \in Y$ with $p \cdot y > 0$, then, by using $\alpha y \in Y$ with a large $\alpha > 1$, it is possible to attain any sufficiently large profit level. Hence $\pi(p) = \infty$. If, on the contrary, $p \cdot y \le 0$ for all $y \in Y$, then $\pi(p) \le 0$. Thus we have either $\pi(p) = +\infty$ or $\pi(p) \le 0$.

5.C.2 Let $p \gg 0$, $p' \gg 0$, $\alpha \in [0,1]$, and $y \in y(\alpha p + (1 - \alpha)p')$, then $p \cdot y \le \pi(p)$ and $p' \cdot y \le \pi(p')$. Thus,

$$(\alpha p + (1 - \alpha)p') \cdot y = \alpha p \cdot y + (1 - \alpha)p' \cdot y \le \alpha\pi(p) + (1 - \alpha)\pi(p').$$

Since $(\alpha p + (1 - \alpha)p') \cdot y = \pi(\alpha p + (1 - \alpha)p')$,

$$\pi(\alpha p + (1 - \alpha)p') \le \alpha\pi(p) + (1 - \alpha)\pi(p').$$

Hence $\pi(\cdot)$ is convex.

5.C.3 The homogeneity of $c(\cdot)$ in q is implied by that of $z(\cdot)$. We shall thus prove this latter homogeneity only. Let $w \gg 0$, $q \ge 0$, and $\alpha > 0$. Let $z \in z(w,q)$. Since $f(\cdot)$ is homogeneous of degree one, $f(\alpha z) = \alpha f(z) \ge \alpha q$. For every $z' \in \mathbb{R}_+^{L-1}$, if $f(z') \ge \alpha q$, then $f(\alpha^{-1}z') = \alpha^{-1}f(z') \ge q$. Thus, by $z \in$

$z(w,q)$, $w\cdot(\alpha^{-1}z') \geq w\cdot z$. Hence $w\cdot z' \geq w\cdot(\alpha z)$. Thus $\alpha z \in z(w,\alpha q)$. So $\alpha z(w,q) \subset z(w,\alpha q)$. By applying this inclusion to α^{-1} in place of α and αq in place of q, we obtain $\alpha^{-1}z(w,\alpha q) \subset z(w,\alpha^{-1}(\alpha q))$, or $z(w,\alpha q) \subset \alpha z(w,q)$ and thus conclude that $z(w,\alpha q) = \alpha z(w,q)$.

We next prove property (viii). Let $w \in \mathbb{R}^{L-1}_{++}$, $q \geq 0$, $q' \geq 0$, and $\alpha \in [0,1]$. Let $z \in z(w,q)$ and $z' \in z(w,q')$. Then $f(z) \geq q$, $f(z') \geq q'$, $c(w,q) = w\cdot z$, and $c(w,q') = w\cdot z'$. Hence

$$\alpha c(w,q) + (1 - \alpha)c(w,q') = \alpha(w\cdot z) + (1 - \alpha)(w\cdot z') = w\cdot(\alpha z + (1 - \alpha)z').$$

Since $f(\cdot)$ is concave,

$$f(\alpha z + (1 - \alpha)z') \geq \alpha f(z) + (1 - \alpha)f(z') \geq \alpha q + (1 - \alpha)q'.$$

Thus $w\cdot(\alpha z + (1 - \alpha)z') \geq c(w, \alpha q + (1 - \alpha)q')$. That is,

$$\alpha c(w, q) + (1 - \alpha)c(w, q') \geq c(w, \alpha q + (1 - \alpha)q').$$

5.C.4 [<u>First printing errata</u>: When there are multiple outputs, the function $f(z)$ need not be well defined because it is conceivably possible to produce different combinations of outputs from a single combination of inputs. Assuming that the first L - M commodities are inputs and the last M commodities are outputs, we should thus understand the set $\{z \geq 0: f(z) \geq q\}$ as $\{z \in \mathbb{R}^{L-M}_{+}: (-z, q) \in Y\}$.] For each $q \geq 0$, define

$$Y(q) = \{z \in \mathbb{R}^{L-M}_{+}: (-z, q) \in Y\} = \{z \in \mathbb{R}^{L-M}_{+}: f(z) \geq q\}.$$

Then $c(\cdot,q)$ is the support function of $Y(q)$ for every q. Hence property (ii) follows from the discussion of Section 3.F. Moreover, according to Exercise 3.F.1, if $Y(q)$ is closed and convex, then

$$Y(q) = \{z \in \mathbb{R}^{L-M}_{+}: w\cdot z \geq c(w,q) \text{ for all } w >> 0\}.$$

Since $Y = \{(-z, q): q \geq 0 \text{ and } z \in Y(q)\}$, this implies property (iii).

To prove property (iv), let $w >> 0$, $q \geq 0$, $\alpha > 0$, $z \in z(w,q)$, and $z' \in Y(q)$. Then $w\cdot z \leq w\cdot z'$. Hence $(\alpha w)\cdot z \leq (\alpha w)\cdot z'$. Thus $z \in z(\alpha w,q)$. Therefore

$z(w,q) \subset z(\alpha w,q)$. By applying this inclusion to αw in place of w and α^{-1} in place of α, we obtain $z(\alpha w,q) \subset z(\alpha^{-1}(\alpha w),q) = z(w,q)$. Property (iv) thus follows.

Property (iv) implies the homogeneity of degree one of $c(\cdot)$ in w, which is the first part of property (i). As for its second part, let $w >> 0$, $q \geq 0$, $q' \geq 0$, and $q' \geq q$. Then $Y(q') \subset Y(q)$. Since $c(\cdot,q)$ and $c(\cdot,q')$ are the support functions, this inclusion implies that $c(w,q') \geq c(w,q)$. Hence $c(\cdot)$ is nondecreasing in q.

As for property (v), note that $z(w,q) = Y(q) \cap \{z \in \mathbb{R}^L: w\cdot z = c(w,q)\}$ for every $w >> 0$ and $q \geq 0$. Since both of the two sets on the right-hand side is convex, so is the intersection, and hence so is $z(w,q)$. As for the single-valuedness, let $q \geq 0$, $w >> 0$, $z \in z(w,q)$, $z' \in z(w,q)$, and $z \neq z'$. Also suppose that $Y(q)$ is strictly convex. By the convexity of $z(w,q)$,

$$(1/2)z + (1/2)z' \in z(w,q).$$

By the strict convexity of $Y(q)$, there exists a $z'' \in Y(q)$ such that

$$(1/2)z + (1/2)z' >> z''.$$

Hence $w\cdot((1/2)z + (1/2)z') > w\cdot z''$, which contradicts $(1/2)z + (1/2)z' \in z(w,q)$. Thus $z(w,q)$ must be single-valued.

Property (vi) follows from the fact that $c(\cdot,q)$ is the support function of $Y(q)$ and the duality theorem (Proposition 3.F.1).

Property (vi) implies that if $z(\cdot,q)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w},q) = D_w(\nabla_w c(\bar{w},q)) = D_w^2 c(\bar{w},q)$. As a Hessian matrix, this is symmetric. By property (ii), it is negative semidefinite. By property (iv), $D_w z(\bar{w},q)\bar{w} = 0$. Property (vii) is thus established.

5.C.5 If the production function $f(\cdot)$ is quasiconcave, then the set $Y(q) = \{z \in \mathbb{R}_+^{L-1}: f(z) \geq q\}$ is convex for any q, and thus property (iii) holds. (Note that we used the convexity of $Y(q)$, not of Y.)

If there is a single input and the production function is given by $f(z) = z^2$, then it is quasiconcave but the corresponding production set exhibits increasing returns to scale. Quasiconcavity is thus compatible with increasing returns.

5.C.6 Throughout the following answers, the input prices are denoted by $w \gg 0$ and the output price is denoted by $p > 0$. For convenience, we denote by $z(p,w)$ the input demands at prices (p,w). As a preliminary result, by using the implicit function theorem (Theorem M.E.1), we shall prove that $z(\cdot)$ is a continuously differentiable function and give its derivatives in terms of $f(\cdot)$. (To be rigorous, we need to assume that $f(\cdot)$ is twice continuously differentiable and the input demands are always strictly positive, so that the nonnegativity constraints never bind.)

Since $D^2f(z)$ is negative definite for all z, the first-order necessary and sufficient condition (Theorems M.K.2 and M.K.3) for profit maximization is then that z is an input demand vector at prices (p,w) if and only if

$$p\nabla f(z) - w = 0.$$

If we regard the left-hand side as defining the function defined over (p,w,z), then the function is continuously differentiable and its derivative with respect to z is equal to $pD^2f(z)$. It is negative definite, and hence has the inverse matrix (at every z). Thus, by the implicit function theorem (Theorem M.E.1), for each (p,w), there is a unique z for which $p\nabla f(z) - w = 0$ and the mapping from (p,w) to z is continuously differentiable. This is equivalent to saying that $z(\cdot)$ is a continuously differentiable function. The implicit function theorem also tells us that

$$\partial z(p,w)/\partial p = (-1/p)D^2f(z(p,w))^{-1}\nabla f(z(p,w)),$$

$$\nabla_w z(p,w) = (1/p)D^2 f(z(p,w))^{-1}.$$

Note here that, since $D^2 f(z(p,w))$ is negative definite, so is its inverse $D^2 f(z(p,w))^{-1}$ by Theorem M.D.1(iii).

(a) By the chain rule (Section M.A),

$$\frac{d}{dp}[f(z(p,w))] = \nabla f(z(p,w))\,\frac{\partial z}{\partial p}(p,w)$$
$$= (-\,1/p)\nabla f(z(p,w))\cdot D^2 f(z(p,w))^{-1}\nabla f(z(p,w))$$

Since $D^2 f(z(p,w))^{-1}$ is negative definite, $d[f(z(p,w))]/dp > 0$.

(b) Since $\partial f(z)/\partial z_\ell \geq 0$ for all ℓ and $d[f(z(p,w))]/dp > 0$, as the output price increases, the demand for some input must increase.

(c) Since $\nabla_w z(p,w) = (1/p)D^2 f(z(p,w))^{-1}$, $\partial z(p,w)/\partial p_\ell$ is equal to the ℓth diagonal entry of $(1/p)D^2 f(z(p,w))^{-1}$. Since $D^2 f(z(p,w))^{-1}$ is negative definite, the diagonal entry is negative. Hence so is $\partial z(p,w)/\partial p_\ell$.

5.C.7 [<u>First printing errata</u>: The condition $\partial^2 f(z)/\partial z_\ell \partial z_k < 0$ should be $\partial^2 f(z)/\partial z_\ell \partial z_k > 0$. That is, all inputs are complementary to one another.] As we saw in the answer to Exercise 5.C.6,

$$\frac{\partial z}{\partial w}(p,w) = (1/p)[D^2 f(z(p,w))]^{-1}$$

and

$$\frac{\partial z}{\partial p}(p,w) = -\,(1/p)[D^2 f(z(p,w))]^{-1}\nabla f(z(p,w)).$$

Hence, in order to prove that $\partial z_\ell(p,w)/\partial w_k < 0$ for all $k \neq \ell$ and $\dfrac{\partial z_\ell}{\partial p}(p,w) > 0$ for all ℓ, it is sufficient to show that all entries of $[D^2 f(z(p,w))]^{-1}$ are negative. This is an immediate consequence of the celebrated Hawkins-Simon condition, which can be found, for example, in "Convex Structures and Economic Theory" by Hukukane Nikaido. Here, instead of simplemindedly quoting the Hawkins-Simon condition, we shall provide a direct proof that relies on the symmetry of $D^2 f(z(p,w))$ (which is not assumed in the Hawkins-Simon condition).

Write $H = D^2 f(z(p,w))$. To show that all entries of H^{-1} are negative, it is sufficient to prove that, for every $v \in \mathbb{R}^{L-1}$, if $Hv \geq 0$ and $Hv \neq 0$, then $v \ll 0$. In fact, then, for each ℓ, we can choose $v \in \mathbb{R}^{L-1}$ so that Hv is the vector whose ℓth coordinate is equal to one and the other coordinates are equal to zero. Then $H^{-1}(Hv)$ is equal to the ℓth column of H^{-1}. Of course, it is also equal to v, which is claimed to be strictly negative. Thus every column of H^{-1} is strictly negative. Hence all entries of H^{-1} are negative.

For this property, in turn, it is sufficient to prove that for every $v \in \mathbb{R}^{L-1}$, if $Hv \geq 0$, then $v \leq 0$. (That is, weak inequalities suffice.) In fact, if there exists a $v \in \mathbb{R}^{L-1}$ such that $Hv \geq 0$, $Hv \neq 0$, $v \leq 0$, and $v_\ell = 0$ for some ℓ. Then $v \neq 0$ and hence

$$\textstyle\sum_k (\partial z_\ell(p,w)/\partial w_k) v_k = \sum_{k \neq \ell} (\partial z_\ell(p,w)/\partial w_k) v_k < 0,$$

which contradicts $Hv \geq 0$.

We shall now prove by contradiction that for every $v \in \mathbb{R}^{L-1}$, if $Hv \geq 0$, then $v \leq 0$. Then there exists $v \in \mathbb{R}^{L-1}$ such that $Hv \geq 0$, and $v_\ell > 0$ for some ℓ. By re-ordering the inputs if necessary, we can assume that the first M entries of v are positive and the last $L - 1 - M$ entries are nonpositive Write write $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1 \in \mathbb{R}^M$, $x_1 \gg 0$, $x_2 \in \mathbb{R}^{L-1-M}$, and $x_2 \leq 0$. Also, write $H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$, where H_1 is an $M \times M$ matrix, H_2 is $M \times (L - 1 - M)$ matrix whose entries are all positive, H_3 is an $(L - 1 - M) \times M$ matrix whose entries are all positive, and H_4 is an $(L - 1 - M) \times (L - 1 - M)$ matrix. Let $Hv = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where $y_1 \in \mathbb{R}^M$, $y_1 \geq 0$, $y_2 \in \mathbb{R}^{L-1-M}$, and $y_2 \geq 0$. Then

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} H_1x_1 + H_2x_2 \\ H_3x_1 + H_4x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence $H_1x_1 = y_1 - H_2x_2 \geq 0$ because $y_1 \geq 0$, $x_2 \leq 0$, and all entries of H_2 are positive. Thus, by $x_1 >> 0$, we obtain $x_1 \cdot H_1 x_1 \leq 0$, which is a contradiction to the negative definiteness of H, and hence that of H_1.

5.C.8 The cost that Al incurred in month 95 is $2 \cdot 55 + 2 \cdot 40 = 190$, but it could attain the same output level with a lower cost by using the input combination of month 3: $2 \cdot 40 + 2 \cdot 50 = 180$. Thus the problem we will encounter is that, perhaps due to mis-observation and/or some restrictions that Al faced outside the market, the profit-maximizing production plans are not observed to have been used and it is impossible to use those observations in order to recover its technology based on Proposition 5.C.2(iii) or 5.C.1(iii).

5.C.9. To find $\pi(\cdot)$ and $y(\cdot)$ for (a), the first-order condition (5.C.2) is not very useful, because one of the nonnegativity constraint binds. Also, to find $\pi(\cdot)$ and $y(\cdot)$ for (b), it is not even applicable because $f(\cdot)$ is not differentiable. In both cases, however, because of the nature of the production functions, it is quite easy to solve their CMP (which is similar to those in Exercise 5.C.10.), and the cost functions $c(\cdot)$ turn out to be differentiable with respect to output levels q. We can thus apply the first-order condition (5.C.6) (which requires only the differentiability of the cost function with respect to output levels) to find profit maximizing production levels, and hence the profit functions and supply correspondences.

Throughout the answer, the output price is fixed to be equal to one.

(a) $\pi(w) = \begin{cases} 1/4w_1 & \text{if } w_1 \leq w_2; \\ 1/4w_2 & \text{if } w_1 > w_2. \end{cases}$

$$y(w) = \begin{cases} \{(-1/4w_1^2,\ 0,\ 1/2w_1)\} & \text{if } w_1 < w_2; \\ \{(-z_1,\ -z_2,\ 1/2w_1): z_1 \geq 0,\ z_2 \geq 0,\ z_1 + z_2 = 1/4w_1^2\} & \text{if } w_1 = w_2; \\ \{(0,\ -1/4w_2^2,\ 1/2w_1)\} & \text{if } w_1 > w_2. \end{cases}$$

(b) $\pi(w) = 1/4(w_1 + w_2)$.

$y(w) = (-1/4(w_1 + w_2)^2, -1/4(w_1 + w_2)^2, 1/2(w_1 + w_2))$.

(c) Note first that this production function exhibits constant returns to scale. Moreover, if $\rho < 1$, then the nonnegativity constraint does not bind. If $\rho = 1$, then this production function gives rise to the same isoquants as that of (a), and hence one of the nonnegativity constraints binds. It is thus easy to apply (5.C.6).

If $\rho < 1$, then

$$\pi(w) = \begin{cases} \infty & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} < 1; \\ 0 & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \ge 1. \end{cases}$$

$$y(w) = \begin{cases} \emptyset & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} < 1, \\ \{\alpha(-w_1^{1/(\rho-1)}, -w_2^{1/(\rho-1)}, (w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{1/\rho}): \alpha \ge 0\} & \\ & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} = 1; \\ \{0\} & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} > 1. \end{cases}$$

If $\rho = 1$, then

$$\pi(w) = \begin{cases} 0 \text{ if } \mathrm{Min}\{w_1, w_2\} \ge 1 \\ \infty \text{ if } \mathrm{Min}\{w_1, w_2\} < 1 \end{cases}$$

$$y(w) = \begin{cases} \{0\} & \text{if } \mathrm{Min}\{w_1, w_2\} \ge 1; \\ \emptyset & \text{if } \mathrm{Min}\{w_1, w_2\} < 1; \\ \{\alpha(-1, 0, 1): \alpha \ge 0\} & \text{if } 1 = w_1 < w_2; \\ \{\alpha(0, -1, 1): \alpha \ge 0\} & \text{if } w_1 > w_2 = 1. \\ \{\alpha(-z_1, -z_2, 1): \alpha \ge 0, z_1 \ge 0, z_2 \ge 0, z_1 + z_2 = 1\} & \\ & \text{if } w_1 = w_2 = 1; \end{cases}$$

5.C.10

(a) $$c(w,q) = \begin{cases} qw_1 & \text{if } w_1 \le w_2; \\ qw_2 & \text{if } w_1 > w_2. \end{cases}$$

$$z(w,q) = \begin{cases} (q,0) & \text{if } w_1 < w_2; \\ \{(z_1,z_2) \in \mathbb{R}^2_+ : z_1 + z_2 = q\} & \text{if } w_1 = w_2; \\ (0,q) & \text{if } w_1 > w_2. \end{cases}$$

(b) $c(w,q) = (w_1 + w_2)q$. $z(w,q) = (q,q)$.

(c) $c(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(1-1/\rho)}$.

$z(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(-1/\rho)}(w_1^{1/(\rho-1)}, w_2^{1/(\rho-1)})$.

5.C.11 Assume that $c(\cdot)$ is twice continuously differentiable. By Proposition 5.C.2(vi), $z(\cdot)$ is continuously differentiable and

$$\partial z_\ell(w,q)/\partial q = (\partial/\partial q)(\partial c(w,q)/\partial w_\ell) = (\partial/\partial w_\ell)(\partial c(w,q)/\partial q).$$

Hence $\partial z_\ell(w,q)/\partial q > 0$ if and only if $(\partial/\partial w_\ell)(\partial C(w,q)/\partial q) > 0$, that is, marginal cost is increasing in w_ℓ.

5.C.12 Suppose first that $y \in Y$ maximizes profit at p, then $(p,\pi(p))\cdot(y, -1) = 0$. Also, for every $\alpha \geq 0$ and $y' \in Y$, $(p,\pi(p))\cdot(\alpha(y', -1)) = \alpha(p\cdot y' - \pi(p)) \leq 0$. Thus $(y', -1)$ maximizes profit in Y' at prices $(p,\pi(p))$.

Conversely, if $y \in Y$ and $(y, -1)$ maximizes profit in Y' at (p,p_{L+1}), then $(p,p_{L+1})\cdot(y, -1) = p\cdot y - p_{L+1} = 0$ by the constant returns to scale. Also, for every $y' \in Y$, $(p,p_{L+1})\cdot(y', -1) = p\cdot y' - p_{L+1} \leq 0$. Hence y maximizes profit in Y at prices p and $\pi(p) = p\cdot y = p_{L+1}$.

5.C.13 Denote the production function of the firm by $f(\cdot)$, then its optimization problem is

$$\text{Max}_{(z_1,z_2)\geq 0}\ pf(z_1,z_2) \quad \text{s.t.} \quad w_1z_1 + w_2z_2 \leq C.$$

This is analogous to the utility maximization problem in Section 3.D and the function $R(\cdot)$ corresponds to the indirect utility function. Hence, analogously to Roy's identity (Proposition 3.G.4), the input demands are obtained as

$$-\frac{1}{\nabla_C R(p,w,C)}\nabla_w R(p,w,C) = (\alpha C/w_1,\ (1-\alpha)C/w_2).$$

5.D.1 We shall use the differentiability of $C(\cdot)$ only at $\bar{q}$. The everywhere differentiability is not necessary. By the definition,

$$AC'(\bar{q}) = \frac{d}{dq}\left[\frac{C(\bar{q})}{\bar{q}}\right] = \frac{C'(\bar{q})\bar{q} - C(\bar{q})}{\bar{q}^2}.$$

Thus, if the average cost is minimized at $q = \bar{q}$, then $AC'(\bar{q}) = 0$ and hence $C'(\bar{q}) = C(\bar{q})/\bar{q} = AC(\bar{q})$.

5.D.2

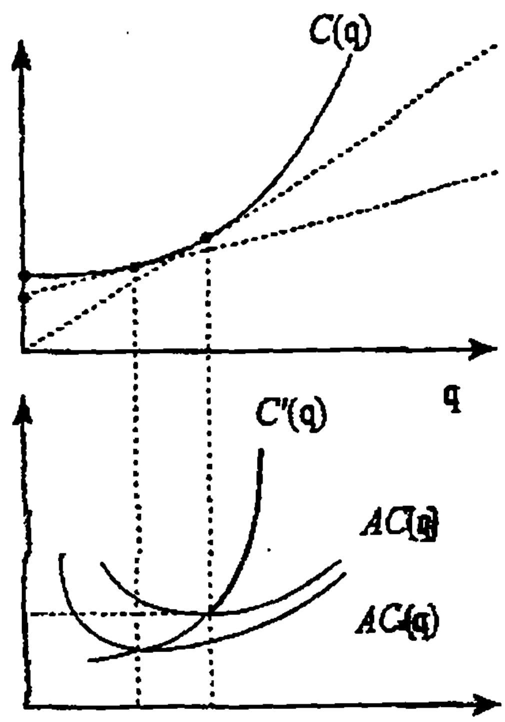


Figure 5.D.2

5.D.3 Let $q(w,p)$ be the profit-maximizing output level when the input price are w and the output price is p. Let $\bar{w}$ be the initial input prices, $\bar{p}$ be the initial output price, $\bar{z}$ be the initial long-run input demand at $(\bar{w},\bar{p})$, and $\bar{q}$ be the initial long-run output level at $(\bar{w},\bar{p})$. By (5.C.6), $\partial c(\bar{w},q(\bar{w},p))/\partial q = p$ for every p (assuming that $q(\bar{w},p) > 0$). Thus, by differentiating both si with respect to p and then evaluating at $p = \bar{p}$, we obtain

$$(\partial^2 c(\bar{w}, q(\bar{w},\bar{p}))/\partial q^2)(\partial q(\bar{w},\bar{p})/\partial p) = 1,$$

that is, $\partial q(\bar{w},\bar{p})/\partial p = (\partial^2 c(\bar{w},\bar{q})/\partial^2 q)^{-1}$.

On the other hand, define the short-run cost function function $c_s(w,q|\bar{z}_1)$ and the short-run output function $q_s(w,q|\bar{z}_1)$ as suggested in the hint. Just as we did above, we can show that $\partial q_s(\bar{w},\bar{p}|\bar{z}_1)/\partial p = (\partial c_s^2(\bar{w},\bar{q}|\bar{z}_1)/\partial^2 q)^{-1}$ (assuming that $q_s(\bar{w},\bar{p}|\bar{z}_1) > 0$).

Now, by the definition, $c(\bar{w},q) \le c_s(\bar{w},q|\bar{z}_1)$ for all q and $c(\bar{w},\bar{q}) = c_s(\bar{w},\bar{q}|\bar{z}_1)$. Hence the function $g(q) = c(\bar{w},q) - c_s(\bar{w},q|\bar{z}_1)$ is maximized at $q = \bar{q}$ and, by the second-order necessary condition (see Section M.K), $g''(\bar{q}) \le 0$, that is, $\partial c^2(\bar{w},\bar{q})/\partial^2 q \le \partial c_s^2(\bar{w},\bar{q}|\bar{z}_1)/\partial^2 q$. Thus $\partial q(\bar{w},\bar{p})/\partial p \ge \partial q_s(\bar{w},\bar{p}|\bar{z}_1)/\partial p$.

5.D.4 (a) Suppose that $q = \sum_{j=1}^{J} q_j$. By the decreasing average costs (and $C(0) = 0$), $(q_j/q)C(q) \le C(q_j)$. By summing over j, we obtain $C(q) \le \sum_{j=1}^{J} C(q_j)$. Hence there is no way to break up the production of q among multiple firms and lower the cost of production. Hence $C(\cdot)$ is subadditive.

(b) Let $M = 2$ and define $C(q) = \sqrt{\min\{q_1, q_2\}}$, then $C(\cdot)$ exhibits decreasing ray average cost. But let $q_1 = (1,8)$, $q_2 = (8,1)$, and $q = q_1 + q_2 = (9,9)$. Then $C(q_1) = C(q_2) = 1$ and $C(q) = 3$. Hence $C(q) > C(q_1) + C(q_2)$. Hence $C(\cdot)$ is not subadditive.

(c) We shall first prove that if $q = \sum_j q_j$ and $q_j \gg 0$ for every j, then $C(q) \le \sum_j C(q_j)$. In fact, then, for every j, there exists $\gamma_j > 0$ such that $\gamma_j q_j \gg q$. By the increasingness, $C(\gamma_j q_j) > C(q)$. Thus, by the continuity and $C(q) > C(q_j)$, there exists $\alpha_j \in (1, \gamma_j]$ such that $C(\alpha_j q_j) = C(q)$. Define $\beta = \sum_j 1/\alpha_j$, then $\sum_j 1/\beta\alpha_j = 1$ and $\sum_j (1/\beta\alpha_j)\alpha_j q_j = (1/\beta)q$. Thus, by the quasiconvexity, $C((1/\beta)q) \le C(q)$. By the increasingness, $\beta \ge 1$. Hence, by the decreasing ray average cost,

$$\sum_j C(q_j) \geq \sum_j (1/\alpha_j) C(\alpha_j q_j) = \sum_j (1/\alpha_j) C(q) = \beta C(q) \geq C(q).$$

For the general case in which some $q_j \geq 0$ need not be strictly positive, apply the above result to the $q_j + \varepsilon e$, where $\varepsilon > 0$ and $e = (1,1,\ldots,1) \in \mathbb{R}^M$, and then take the limit as $\varepsilon \to 0$. The continuity of $C(\cdot)$ then implies that $\sum_{j=1}^{J} C(q_j) \geq C(q)$.

5.D.5 (a) The production function $f(\cdot)$ exhibits increasing returns if and only if $f(\lambda z) \geq \lambda f(z)$ for all z and all $\lambda \geq 1$. Hence, if $z' \geq z > 0$, then

$$(1/z')f(z') = (1/z')f((z'/z)z) \geq (1/z')(z'/z)f(z) = (1/z)f(z).$$

Thus the average product is nondecreasing. The marginal product may however be decreasing on some region of output levels, as the following example shows:

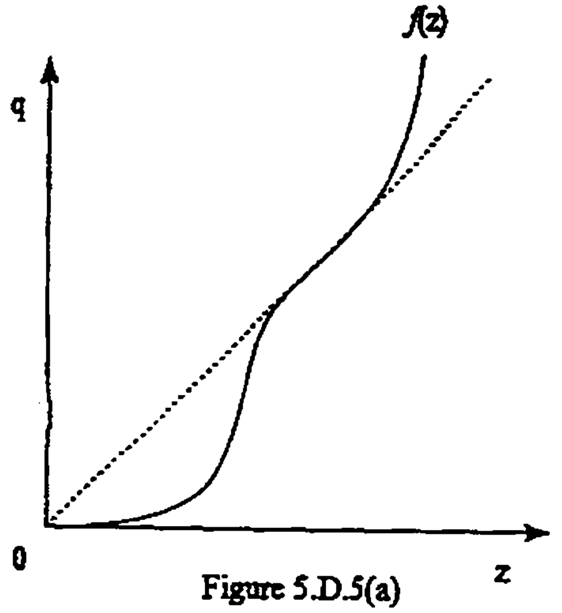


Figure 5.D.5(a)

(b) Mathematically, the consumer's maximization problem is

$$\max_{z \geq 0} \; u(f(z)) - z$$

The first-order necessary condition is $u'(f(z))f'(z) = 1$, which can be rewritten as $u'(f(z)) = f'(z)^{-1}$. Since the cost function is given by $z = f^{-1}(q)$, the marginal cost is equal to $f'(z)^{-1}$ and the equality of marginal cost and marginal utility is thus a necessary condition for a maximum.

Economically, the consumer will choose the output level at which the marginal utility of an extra unit of the output is exactly equal to the disutility incurred by giving up the necessary amounts of input to produce it. But the latter is nothing but the marginal cost. Hence the marginal utility is equal to the marginal cost.

(c) This assertion is wrong. As the following figure shows, even if marginal cost and marginal utility are equal at an input level, there may be another input level at which the consumer attains higher utility:

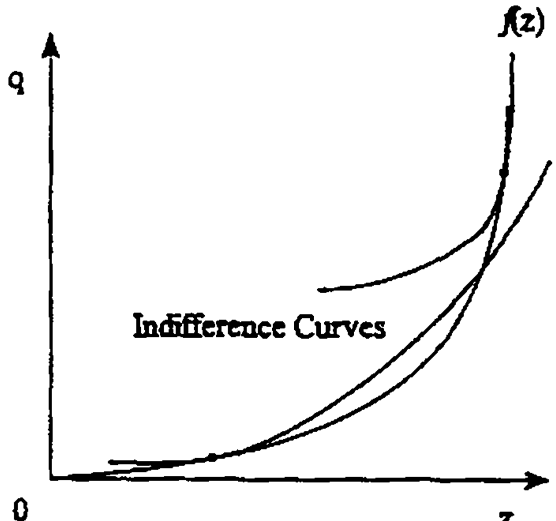


Figure 5.D.5(c)

The reason for suboptimality is that the first-order necessary conditions are not sufficient when the production function exhibits increasing returns (which gives rise to nonconvexity of the feasible set).

5.E.1 By applying Hotelling's lemma (Proposition 5.C.1(vi)) twice, we get

$$y^*(p) = \nabla\pi^*(p) = \nabla(\textstyle\sum_j \pi_j(p)) = \sum_j \nabla\pi_j(p) = \sum_j y_j(p).$$

5.E.2 This is just a matter of going through the proof of Proposition 5.E.1

and checking that convexity was never used. Its interpretation was given before the statement of the proposition (p. 148). It is a consequence of the very definition of the aggregate production set, that is, it is the sum of the J firms' production sets. It is thus independent of convexity or any other properties of the firms' production sets.

5.E.3 [First printing errata: We should assume that there is a $p^* >> 0$ and $y^* \in \sum_j Y_j$ such that $p^* \cdot y^* \geq p^* \cdot y$ for every $y \in \sum_j Y_j$. Otherwise, denoting the profit function of $\sum_j Y_j$ by $\pi^*(\cdot)$, the set $\{y \in \mathbb{R}^L: p \cdot y \leq \pi^*(p)$ for all $p >> 0\}$ may be empty, and all we can obtain is the equality between $\sum_j Y_j$ and $\{y \in \mathbb{R}^L: p \cdot y \leq \pi^*(p)$ for all $p \geq 0\}$. This assumption is also necessary for the validity of Proposition 5.C.1(iii). In fact, its proof should go as follows: It is sufficient to prove that, for every $z \in \mathbb{R}^L \setminus Y$, there exist a $p >> 0$ such that $p \cdot z > \pi(p)$. Since Y is closed and convex, the separating hyperplane theorem implies the existence of such a nonzero vector p. The free disposal property implies that p must actually be nonnegative. If it is not strictly positive, then take the convex combination $(1 - \varepsilon)p + \varepsilon p^*$ with a sufficiently small $\varepsilon > 0$. Then it is strictly positive, and satisfies $((1 - \varepsilon)p + \varepsilon p^*) \cdot z > \pi((1 - \varepsilon)p + \varepsilon p^*)$ by the upper semicontinuity of $\pi(\cdot)$, which is implied by its convexity.] Since each Y_j is convex and satisfies the free disposal property, $\sum_j Y_j$ is also convex and satisfies the free disposal property. Since it is also assumed to be closed, Proposition 5.C.1(iii) implies that

$$\sum_j Y_j = \{y \in \mathbb{R}^L: p \cdot y \leq \pi^*(p) \text{ for all } p >> 0\}.$$

But here, by Proposition 5.E.1, $\pi^*(p) = \sum_j \pi_j(p)$ and hence

$$\sum_j Y_j = \{y \in \mathbb{R}^L: p \cdot y \leq \sum_j \pi_j(p) \text{ for all } p >> 0\}.$$

5.E.4 (a) Denote by $y_z(w) \subset \mathbb{R}^3$ the set of the supplies of the technology with characteristics $z = (z_1, z_2)$ at input prices $w = (w_1, w_2)$, then

$$y_z(w) = \begin{cases} \{(-z_1, -z_2, 1)\} & \text{if } w_1z_1 + w_2z_2 < 1, \\ \{(-\alpha z_1, -\alpha z_2, \alpha) : \alpha \in [0,1]\} & \text{if } w_1z_1 + w_2z_2 = 1, \\ \{0\} & \text{if } w_1z_1 + w_2z_2 > 1. \end{cases}$$

The area of the characteristics z for which the output may be one is depicted in the following picture.

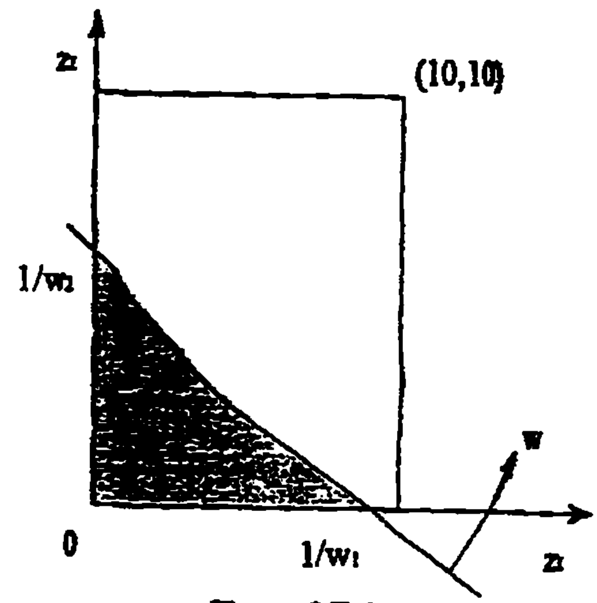


Figure 5.E.4

(b) [First printing errata: The phrase "More generally" should be deleted.] Denote the profit of the technology with characteristics $z = (z_1, z_2)$ at input prices $w = (w_1, w_2)$ by $\pi_z(w)$, then

$$\pi_z(w) = \begin{cases} 1 - w_1z_1 - w_2z_2 & \text{if } w_1z_1 + w_2z_2 \le 1, \\ 0 & \text{if } w_1z_1 + w_2z_2 > 1. \end{cases}$$

Thus, the aggregate (or, rigorously, average) profit is calculated by taking the integral of $1 - w_1z_1 - w_2z_2$ on the area $\{z \in [0,10] \times [0,10]: w_1z_1 + w_2z_2 \le 1\}$, which is depicted on the above figure. Thus the aggregate profit is

$$\pi(w) = \int_0^{1/w_1} \int_0^{(1-w_1z_1)/w_2} (1 - w_1z_1 - w_2z_2)\,dz_2\,dz_1 = 1/600w_1w_2.$$

(c) The aggregate input demand can also be obtained by integrating the input demands of the individual firms, but the following point is noteworthy: If a firm has characteristic $z = (z_1,z_2)$ with $w_1z_1 + w_2z_2 = 1$, then it has multiple input demands at input prices $w = (w_1,w_2)$. But those firms constitute only a negligible portion in the whole population. Hence it is harmless to assume that such a firm has input demand $z = (z_1,z_2)$ (in absolute values). Hence the aggregate demands are

$$\int_0^{1/w_1}\int_0^{(1-w_1z_1)/w_2} z_1 \frac{1}{100}\, dz_2 dz_1 = \frac{1}{600w_1^2w_2},$$

$$\int_0^{1/w_1}\int_0^{(1-w_1z_1)/w_2} z_2 \frac{1}{100}\, dz_2 dz_1 = \frac{1}{600w_1w_2^2}.$$

It is easy to check that these aggregate input demand functions can also be obtained by applying Hotelling's lemma to the aggregate profit function, which was obtained in (b).

(d) We need to find an aggregate production function whose input demand function is the same as the aggregate input demand function in (c). Denote such a production function by $f(\cdot)$, then the first order-conditions for profit maximization are $\partial f(z_1,z_2)/\partial z_1 = w_1$ and $\partial f(z_1,z_2)/\partial z_2 = w_2$. These expressions, when evaluated at the inputs demanded, must hold for all w. Thus

$$\frac{\partial f}{\partial z_1}\left(\frac{1}{600w_1^2w_2}, \frac{1}{600w_1w_2^2}\right) = w_1, \quad \frac{\partial f}{\partial z_2}\left(\frac{1}{600w_1^2w_2}, \frac{1}{600w_1w_2^2}\right) = w_2.$$

Let

$$z_1 = 1/600w_1^2w_2 \text{ and } z_2 = 1/600w_1w_2^2,$$

then

$$w_1 = (z_2/600z_1^2)^{1/3} \text{ and } w_2 = (z_1/600z_2^2)^{1/3}.$$

Thus

$$\frac{\partial f}{\partial z_1}(z_1,z_2) = (z_2/600z_1^2)^{1/3} \text{ and } \frac{\partial f}{\partial z_2}(z_1,z_2) = (z_1/600z_2^2)^{1/3}.$$

Therefore, $f(z_1,z_2) = 3(z_1z_2/600)^{1/3}$. The aggregate production function is a Cobb-Douglas one exhibiting decreasing returns to scale.

5.E.5 (a) Plant j's marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$. Since $\beta_j > 0$ for every j, the first-order necessary and sufficient conditions for cost minimization are that $\sum_j q_j = q$ and $MC_j(q_j) = MC_{j'}(q_{j'})$ for all j and j'. From these, we obtain $q_j = (q/\beta_j)/(\sum_h 1/\beta_h)$.

(b) (c) In both cases, it is cost-minimizing to concentrate on plants with the smallest $\beta_j < 0$, because the average cost is decreasing at the highest rate at such plants.

5.F.1 The production plan y in Figure 5.F.1(b) is not efficient but it maximizes profit for $p = (0,1)$.

5.G.1 Throughout the answer, we fix the price of the input at one and denote the price of the output by p. Suppose that there are I consumer-owners, indexed by $i = 1,\dots,I$. Denote their shares by $\theta_i > 0$. Of course, $\sum_i \theta_i = 1$. Since they have quasilinear utility functions, by Exercise 3.D.4(b), their indirect utility functions can be written as $v_i(p,w_i) = w_i + \phi_i(p)$. Note that the demand function $x_i(\cdot)$ for the output of consumer i does not depend on the wealth and satisfies $x_i(p) = -\phi_i'(p)$ by Roy's identity.

(a) When the input is z, the utility level of consumer i is

$$\theta_i(p(z)f(z) - z) + \phi_i(p(z)).$$

(Here we are assuming that the consumers have no source of wealth other than their shareholdings. But this does not affect our results below, because their demands for the output does not depend on their wealth levels. Thus, if an input level z maximizes his utility level, then it satisfies the following first-order condition:

$$\theta_i(p'(z)f(z) + p(z)f'(z) - 1) + \phi_i'(p(z))p'(z) = 0.$$

If an input level z is unanimously agreed, then this first-order condition must be satisfied at some z for all i. By taking the summation of the condition over i and using $x_i(p(z)) = -\phi_i'(p(z))$, we obtain

$$(p'(z)f(z) + p(z)f'(z) - 1) - \sum_i x_i(p(z))p'(z) = 0.$$

But, since $f(z) = \sum_i x_i(p(z))$, this implies $p(z)f'(z) - 1 = 0$. Plugging this into the first-order condition, we obtain

$$\theta_i p'(z)f(z) - x_i(p(z))p'(z) = 0.$$

Thus $\theta_i = x_i(p(z))/f(z)$.

(b) We know from (a) that, if ownership share are identical, then, in order for consumer-owners to unanimously agree on a production plan, it is necessary that they all consume the same amount of the output. But if their tastes are different for the output, then their consumption levels will be different. Hence they will instruct managers to carry out different output levels.

(c) If preferences and ownership shares are identical, then the first-order conditions are also identical and hence the consumer-owners unanimously agree on an input level. We showed in (a) that, a necessary condition for the unanimous agreement is that $p(z) = 1/f'(z)$. The right-hand side is the inverse of the marginal return, and hence equal to the marginal. This is nothing but profit maximization with respect to input, when the output price $p(z)$ is taken as given.

5.AA.1 From the unit isoquant,

$$z(w,1) = \begin{cases} (2,1) & \text{if } w_1 < w_2, \\ \{\lambda(2,1) + (1-\lambda)(1,2) \in \mathbb{R}: \lambda \in [0,1]\} & \text{if } w_1 = w_2, \\ (1,2) & \text{if } w_1 > w_2. \end{cases}$$

Thus

$$c(w,1) = \begin{cases} 2w_1 + w_2 & \text{if } w_1 \le w_2, \\ w_1 + 2w_2 & \text{if } w_1 > w_2. \end{cases}$$

This is differentiable at $w = (w_1, w_2)$ if and only if $w_1 \neq w_2$. Moreover, $\nabla c(w,1) = z(w,1)$ at $w = (w_1, w_2)$ with $w_1 \neq w_2$.

5.AA.2 (a) We shall first prove that if $\beta \in \mathbb{R}^{L-1}$, $(I - A)\beta \ge 0$, and $(I - A)\beta \neq 0$, then $b \cdot \beta > 0$, and that if $\beta \in \mathbb{R}^{L-1}$ and $(I - A)\beta = 0$, then $b \cdot \beta = 0$. In fact, in the proof of Proposition 5.AA.1, we showed that since A is productive, the inverse matrix $(I - A)^{-1}$ exists and all its entries are nonnegative. Thus, if $\beta \in \mathbb{R}^{L-1}$, $(I - A)\beta \ge 0$, and $(I - A)\beta \neq 0$, then $\beta = (I - A)^{-1}((I - A)\beta) \ge 0$ and $\beta \neq 0$. Since $b \gg 0$, this implies $b \cdot \beta > 0$. If $\beta \in \mathbb{R}^{L-1}$ and $(I - A)\beta = 0$, then $\beta = 0$ and hence $b \cdot \beta = 0$.

To derive efficiency from the above result, let $\alpha \in \mathbb{R}^{L-1}_+$ and $\alpha' \in \mathbb{R}^{L-1}_+$. If $(I - A)\alpha \ge (I - A)\alpha'$ and $(I - A)\alpha \neq (I - A)\alpha'$, then $(I - A)(\alpha - \alpha') \ge 0$ and $(I - A)(\alpha - \alpha') \neq 0$. Hence $b \cdot (\alpha - \alpha') > 0$, or $b \cdot \alpha > b \cdot \alpha'$. If $(I - A)\alpha = (I - A)\alpha'$, then $(I - A)(\alpha - \alpha') = 0$. Hence $b \cdot (\alpha - \alpha') = 0$, or $b \cdot \alpha = b \cdot \alpha'$. In any case, it is impossible that $\begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha \ge \begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha'$ and $\begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha \neq \begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha'$. Efficiency is thus established.

(b) By (a) and Proposition 5.F.2, any production plan with $\alpha \gg 0$ is profit-maximizing at some price vector. To establish its uniqueness, let $p \in \mathbb{R}^{L-1}_+$ be a supporting price vector. By $\alpha \gg 0$ and the zero-profit condition for activities being actually used, we must have $p \cdot (I - A) = b$, that is, $p = ((I - A)^{-1})^T b$ (where b is now a column vector). This implies the uniqueness and the strict positivity of p, because all entries of $(I - A)^{-1}$ are nonnegative, all its diagonal entries of are positive, and $b \gg 0$.

(c) For each ℓ, denote by e_ℓ the vector in $\mathbb{R}^{L-1}$ whose ℓth component is one and the other components are zero. As we saw in the remark following Proposition 5.AA.1, the total amounts of the producible goods necessary to realize a net output vector $e_\ell \in \mathbb{R}^{L-1}_+$ is equal to $(\sum_{n=0}^{\infty} A^n)e_\ell = (I - A)^{-1}e_\ell$. Hence the total amount of labor embodied in these necessary amounts of the producible goods equals $b\cdot(\sum_{n=0}^{\infty} A^n)e_\ell = b\cdot(I - A)^{-1}e_\ell = p_\ell$. Thus the price (row) vector $p = b\cdot(I - A)^{-1}$ can be interpreted as the amounts of primary factor directly or indirectly embodied in the production of one unit of each producible good.

(d) Let $\begin{bmatrix} I - A' \\ -b' \end{bmatrix} \in \mathbb{R}^{L\times(L-1)}$ be any alternative choice of activities that is productive. By the productivity, the inverse matrices $(I - A)^{-1}$ and $(I - A')^{-1}$ exist and are nonnegative. So denote them by $C = [c_1 \ldots c_{L-1}]$ and $C' = [c'_1 \ldots c'_{L-1}]$, where the c_ℓ and c'_ℓ $(\ell = 1,\ldots, L - 1)$ are $(L - 1)$-dimensional column vectors. Then $(I - A)c_\ell = e_\ell$ and $(I - A')c'_\ell = e_\ell$. Now, for each $\varepsilon > 0$, define $d_\ell(\varepsilon) = c_\ell + \varepsilon(\sum_{k\neq\ell} c_k)$ and $d'_\ell(\varepsilon) = c'_\ell + \varepsilon(\sum_{k\neq\ell} c'_k)$. Then $d_\ell(\varepsilon) \to c_\ell$ and $d'_\ell(\varepsilon) \to c'_\ell$ as $\varepsilon \to 0$. Moreover, $d_\ell(\varepsilon) \gg 0$, $d'_\ell(\varepsilon) \gg 0$, and

$$(I - A)d_\ell(\varepsilon) = (I - A')d'_\ell(\varepsilon) = e_\ell + \varepsilon(\sum_{k\neq\ell} e_k) \gg 0.$$

By (a), $\begin{bmatrix} I - A \\ -b \end{bmatrix} d_\ell(\varepsilon)$ is efficient. By this strict positivity and the assumption that the activities $\begin{bmatrix} I - A \\ -b \end{bmatrix}$ have been singled out by the nonsubstitution theorem, we must have $b\cdot d_\ell(\varepsilon) \le b'\cdot d'_\ell(\varepsilon)$. Taking the limit as $\varepsilon \to 0$, we obtain $b\cdot c_\ell \le b'\cdot c'_\ell$. Thus $b\cdot(I - A)^{-1} \le b'\cdot(I - A')^{-1}$. The assertion now follows from (c).

5.AA.3 (a) Denote the activity levels by α_1 and α_2. The resource constraint for labor is $\alpha_1 + 2\alpha_2 = 10$, or $\alpha_1/10 + \alpha_2/5 = 1$. Since the production level is given by $\begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} \alpha_1/10 \\ \alpha_2/5 \end{bmatrix}$, the production possibility frontier is as follows:

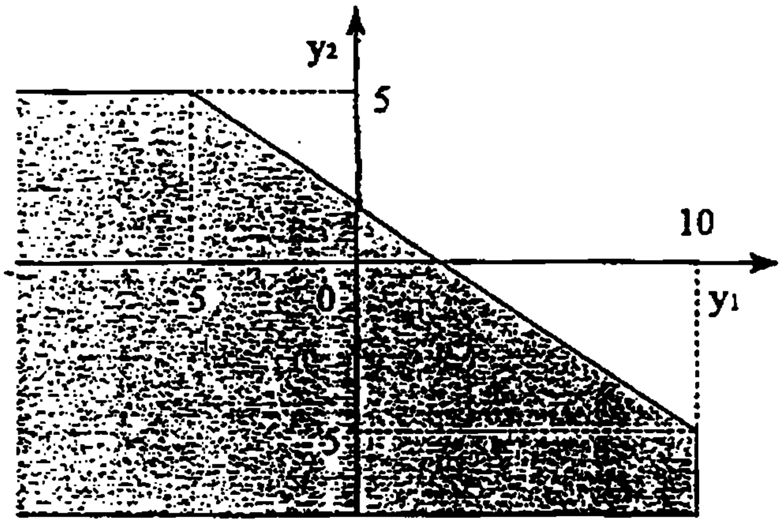


Figure 5.AA.3(a)

(b) By Exercise 5.AA.2(c), the equilibrium price vector is $b \cdot (I - A)^{-1}$ = (4,6).

(c) The amount of labor embodied in each commodity equals its price, as shown in Exercise 5.AA.2(c).

(d) The locus of amounts of good 1 and labor necessary to produce one unit of good 2 is equal to

$$\{\lambda(1,2) + (1 - \lambda)(1/2,\beta): \lambda \in [0,1]\} - \mathbb{R}^2_+,$$

assuming free disposal. It is represented in the following figure:

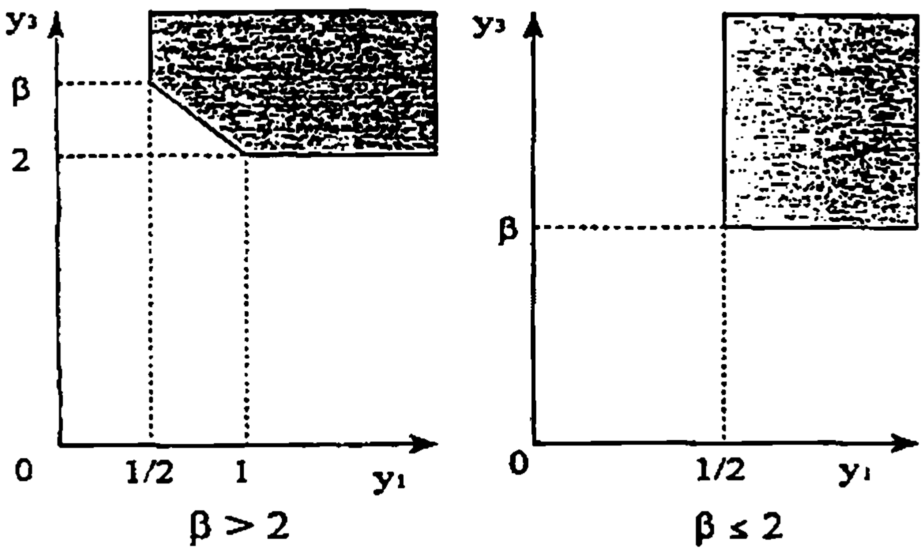


Figure 5.AA.3(d)

(e) In the context of (d), the nonsubstitution theorem says that it is possible to choose one of the two techniques to produce good 2 (or a combination of the two with a fixed proportion) in such a way that any efficient production plan with positive net outputs of the two producible goods can be attained by using the technique chosen for good 2.

We could determine which of the two techniques (or their mixtures) is efficient by actually plotting the frontier of the feasible output combinations from one unit of labor. In the following, however, we shall identify an efficient technique based on Exercise 5.AA.2(d). So let A' = $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and b' = $\begin{bmatrix} 1 \\ \beta \end{bmatrix}$. For each $\lambda \in [0,1]$, define $A(\lambda) = (1 - \lambda)A + \lambda A$, $b(\lambda) = (1 - \lambda)b + \lambda b'$, and $p(\lambda) = b(\lambda)\cdot(I - A(\lambda))^{-1}$ (where $p(\lambda)$ is a row vector). According to Exercise 5.AA.2(d), if $\lambda^* \in [0,1]$ and the convex combination of the first and the second technique with weight $1 - \lambda^*$ and λ^* is efficient, then $p(\lambda) \geq p(\lambda^*)$ for every $\lambda \in [0,1]$. Hence the switch of efficient techniques occurs precisely when the value of λ^* switches as β varies. We shall now find a value of β at which the value of λ^* switches.

Just as in (b), we can calculate

$$p(\lambda) = \frac{2}{\lambda + 2}\begin{bmatrix} (\beta - 2)\lambda + 4 \\ (2\beta - 5)\lambda + 6 \end{bmatrix} \text{ and } Dp(\lambda) = \frac{2\beta - 8}{(\lambda + 2)^2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence: if $\beta < 4$, then $\lambda^* = 0$; if $\beta > 4$, then $\lambda^* = 1$; and if $\beta = 4$, then λ^* can be any value in $[0,1]$. Thus the switching occurs at $\beta = 4$. More precisely: if $\beta < 4$, then it is efficient to continue using the first technique; if $\beta > 4$, then it is efficient to switch to the second technique; and if $\beta = 4$, every mixture of the two technique is efficient.

5.AA.4 (a) Since $y_2 = 3a_1 + a_4$, $y_3 = 3(a_2 + a_3 + a_4)$, and $y_4 = 4(a_2 + a_3)$, these three vectors are in the production set. But y_1 and y_5 are not. To see this, suppose that $y_1 \le \sum_j \alpha_j a_j$ with $\alpha_j \ge 0$. According to good 2, $\alpha_1 = \alpha_2 = 0$. By $\alpha_2 = 0$, according to good 3, $\alpha_3 = 0$. But there is no $\alpha_4 \ge 0$ for which $y_1 \le \alpha_4 a_4$. Suppose next that $y_5 \le \sum_j \alpha_j a_j$ with $\alpha_j \ge 0$. According to good 1, $\alpha_1 = \alpha_4 = 0$. By $\alpha_4 = 0$, according to good 4, $\alpha_3 = 0$. But there is no $\alpha_2 \ge 0$ for which $y_5 \le \alpha_2 a_2$.

(b) If $p = (1,3,3,2)$, then $p \cdot a_j \le 0$ for all j and $p \cdot y = 0$. Hence y maximizes profit at p. By Proposition 5.F.1, y is efficient.

(c) Since $y = a_1$, y is feasible. But, since $a_2 + a_3 + a_4 = (2, -1, 0, 0)$ is feasible, y cannot be efficient. (Note that $a_2 + a_3 + a_4$ represents a round-about production of good 1 out of good 2.)

5.AA.5 [First printing errata: The last elementary activity $a_8 = (-2, -4, 5, 2)$ should be $a_3 = (-2, -4, 5, -2)$.]

(a) The set Y is defined as $\{\sum_m \alpha_m a_m \in \mathbb{R}^4 : \alpha_m \ge 0 \text{ for each } m\}$. Let $\lambda \in [0,1]$, $y = \sum_m \alpha_m a_m \in Y$, and $y' = \sum_m \alpha'_m a_m \in Y$. Then

$$\lambda y + (1 - \lambda)y' = \sum_m (\lambda\alpha_m + (1 - \lambda)\alpha'_m)a_m.$$

Since $\lambda\alpha_m + (1 - \lambda)\alpha'_m \geq 0$ for every m, $\lambda y + (1 - \lambda)y' \in Y$. Thus Y is convex.

(b) Since all the activities use commodities 1 and 2 as an input, in order to produce any commodity in positive quantity, it is necessary to use commodities 1 and 2 as an inputs. The no-free-lunch property thus follows.

(c) Note that it is impossible to dispose of one of commodities 3 and 4 without increasing the output of the other, and that it is impossible to dispose of any one of commodities 1 and 2 without disposing the other. Hence Y does not satisfy the free-disposal property and it is necessary to add the four disposal activities to the given elementary activities in order for the free-disposal property to be satisfied.

(d) Note that $3a_1 \geq a_5$, $3a_1 \neq a_5$, $a_2 \geq a_4$, $a_2 \neq a_4$, $2a_3 \geq a_8$, $2a_3 \neq a_8$, $a_7 \geq 2a_6$, and $a_7 \neq 2a_6$. Hence a_5, a_4, a_8 and a_6 are not efficient.

(e) We can check that $(4/3)a_3 + (5/24)a_7 \geq a_1$, $(4/3)a_3 + (5/24)a_7 \neq a_1$, $a_3 + (1/2)a_7 \geq a_2$, and $a_3 + (1/2)a_7 \neq a_2$. Hence a_1 and a_2 are not efficient.

(f) We shall prove that the set of the efficient production vectors is equal to $\{\alpha_3 a_3 + \alpha_7 a_7 : \alpha_3 \geq 0, \alpha_7 \geq 0\}$. We have shown that every efficient production vector belongs to this set. Conversely, we can check that the production vectors in this set cannot be dominated by each other. Hence they are all efficient and the set of the efficient production vectors is equal to $\{\alpha_3 a_3 + \alpha_7 a_7 : \alpha_3 \geq 0, \alpha_7 \geq 0\}$.

(g) Since $\alpha_3 a_3 + \alpha_7 a_7 = (-\alpha_3 - 8\alpha_7, -2\alpha_3 - 5\alpha_7, 3\alpha_3, -\alpha_3 + 10\alpha_7)$, the problem of maximizing the net production of the third commodity is as follows:

$$\text{Max}_{(\alpha_3,\alpha_7)}\ 3\alpha_3$$

$$\begin{aligned}
\text{s.t.}\quad \alpha_3 + 8\alpha_7 &\le 480,\\
2\alpha_3 + 5\alpha_7 &\le 300,\\
\alpha_3 - 10\alpha_7 &\le 0,\\
\alpha_3 &\ge 0,\\
\alpha_7 &\ge 0.
\end{aligned}$$

(h) The feasible set is shaded in the (α_3,α_7)-space below.

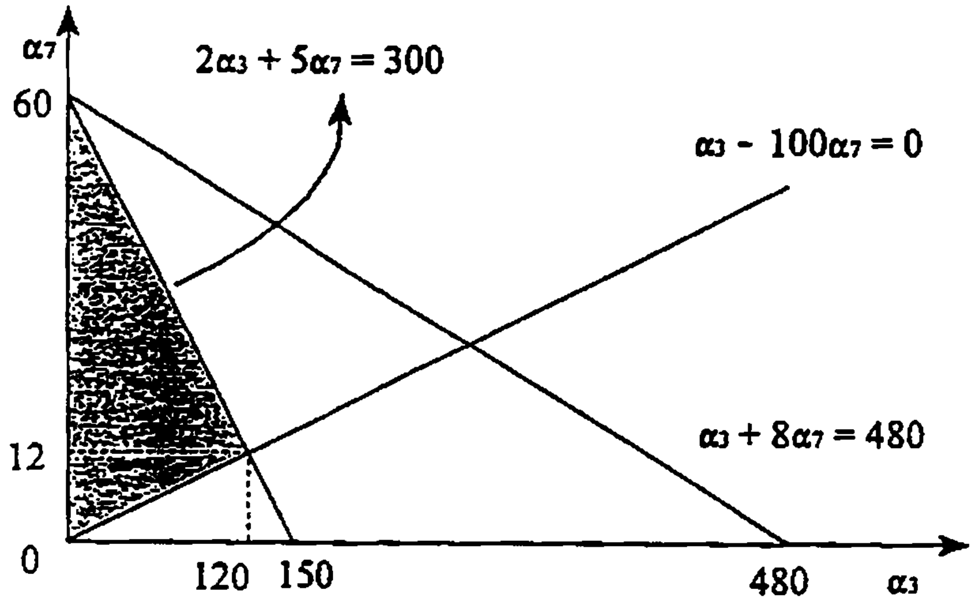


Figure 5.AA.5(h)

The solution to this problem satisfies $2\alpha_3 + 5\alpha_7 = 300$, $\alpha_3 = 10\alpha_7$. Thus $(\alpha_3,\alpha_7) = (120,12)$.

CHAPTER 6

6.B.1 Suppose first that $L \succ L'$. A first application of the independence axiom (in the "only-if" direction in Definition 6.B.4) yields

$$\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

If these two compound lotteries were indifferent, then a second application of the independence axiom (in the "if" direction) would yield $L' \succsim L$, which contradicts $L \succ L'$. We must thus have

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''.$$

Suppose conversely that $\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$, then, by the independence axiom, $L \succsim L'$. If these two simple lotteries were indifferent, then the independence axiom would imply

$$\alpha L' + (1 - \alpha)L'' \succsim \alpha L + (1 - \alpha)L'',$$

a contradiction. We must thus have $L \succ L'$.

Suppose next that $L \sim L'$, then $L \succsim L'$ and $L' \succsim L$. Hence by applying the independence axiom twice (in the "only if" direction), we obtain

$$\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''.$$

Conversely, we can show that if $\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''$, then $L \sim L'$.

For the last part of the exercise, suppose that $L \succ L'$ and $L'' \succ L'''$, then, by the independence axiom and the first assertion of this exercise,

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$$

and

$$\alpha L' + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

Thus, by the transitivity of $\succ$ (Proposition 1.B.1(i)),

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

6.B.2 Assume that the preference relation $\succsim$ is represented by an v.N-M expected utility function $U(L) = \sum_n u_n p_n$ for every $L = (p_1,\ldots,p_N) \in \mathcal{L}$. Let $L = (p_1,\ldots,p_N) \in \mathcal{L}$, $L' = (p'_1,\ldots,p'_N) \in \mathcal{L}$, $L'' = (p''_1,\ldots,p''_N) \in \mathcal{L}$, and $\alpha \in (0,1)$. Then $L \succsim L'$ if and only if $\sum_n u_n p_n \geq \sum_n u_n p'_n$. This inequality is equivalent to

$$\alpha(\textstyle\sum_n u_n p_n) + (1 - \alpha)(\sum_n u_n p''_n) \geq \alpha(\sum_n u_n p'_n) + (1 - \alpha)\ (\sum_n u_n p''_n).$$

This latter inequality holds if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Hence $L \succsim L'$ if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Thus the independence axiom holds.

6.B.3 Since the set C of outcomes is finite, there are best and worst outcomes in C. Let $\bar{L}$ be the lottery that yields a particular best outcome with probability one and $\underline{L}$ be the lottery that yields a particular worst outcome with probability one. We shall now prove that $\bar{L} \succsim L \succsim \underline{L}$ for every $L \in \mathcal{L}$ by applying the following lemma:

<u>Lemma</u>: Let $L_0, L_1, \ldots, L_K$ be $(1 + K)$ lotteries and $(\alpha_1,\ldots,\alpha_K) \geq 0$ be probabilities with $\sum_{k=1}^{K}\alpha_k = 1$. If $L_k \succsim L_0$ for every k, then $\sum_{k=1}^{K}\alpha_k L_k \succsim L_0$. If $L_0 \succsim L_k$ for every k, then $L_0 \succsim \sum_{k=1}^{K}\alpha_k L_k$.

<u>Proof of Lemma</u>: We shall prove this lemma by induction on K. If K = 1, there is nothing to prove. So let K > 1 and suppose that the lemma is true for K - 1. Assume that $L_k \succsim L_0$ for every k. By the definition of a compound lottery,

$$\sum_{k=1}^{K}\alpha_k L_k = (1 - \alpha_K)\sum_{k=1}^{K-1}\frac{\alpha_k}{1 - \alpha_K}L_k + \alpha_K L_K.$$

By the induction hypothesis, $\sum_{k=1}^{K-1}\frac{\alpha_k}{1 - \alpha_K}L_k \succsim L_0$. Hence, as our first application of the independence axiom, we obtain

$$(1 - \alpha_K)\sum_{k=1}^{K-1}\frac{\alpha_k}{1 - \alpha_K}L_k + \alpha_K L_K \succsim (1 - \alpha_K)L_0 + \alpha_K L_K$$

Applying the axiom once again, we obtain

$$(1 - \alpha_K)L_0 + \alpha_K L_K \succsim (1 - \alpha_K)L_0 + \alpha_K L_0 = L_0.$$

Hence, by the transitivity, $\sum_{k=1}^{K}\alpha_k L_k \succsim L_0$. The first statement is thus verified. The case of $L_0 \succsim L_k$ can similarly be verified.

Now, for each n, let L^n be the lottery that yields outcome n with probability one. Then $\bar{L} \succsim L^n$ because both of them can be identified with sure outcomes. Let $L = (p_1,\dots,p_N)$ be any lottery, then $L = \sum_n p_n L^n$. Thus, $\bar{L} \succsim L$ the above lemma. The same argument can be used to prove that $L \succsim \underline{L}$.

6.B.4 [First printing errata: On the the 11th and the 12th line of the exercise, the phrase "the lottery of B with probability q and D with probability 1 - q" should be "the lottery of A with probability q and D with probability 1 - q". Also, in the description of Criterion 2, the phrase "an unnecessary evacuation in 5%" should be "an unnecessary evacuation in 15%".]

(a) We can choose an assign utility levels (u_A, u_B, u_C, u_D) so that $u_A = 1$ and $u_D = 0$ as a normalization (Proposition 6.B.2). Then $u_B = p\cdot 1 + (1 - p)\cdot 0 = p$ and $u_C = q\cdot 1 + (1 - q)\cdot 0 = q$.

(b) The probability distribution under Criterion 1 is

$$(p_A, p_B, p_C, p_D) = (0.891,\ 0.099,\ 0.009,\ 0.001).$$

The probability distribution under Criterion 2 is

$$(p_A, p_B, p_C, p_D) = (0.8415,\ 0.1485,\ 0.0095,\ 0.0005).$$

The expected utility under Criterion 1 is thus $0.891 + 0.099p + 0.009q$. The expected utility under Criterion 2 is thus $0.8415 + 0.1485p + 0.0095q$. Hence the agency would prefer Criterion 1 if and only if $99 > 99p + q$, and it would prefer Criterion 2 if and only if $99 < 99p + q$.

6.B.5 (a) This follows from Exercise 6.B.1.

(b) The equivalence of the betweenness axiom and straight indifference curves can be established in the same way as in the part of Section 6.B on pp. 175-176 that explains how the independence axiom implies straight indifference curves. (Note that the argument there does not use the fully fledged independence axiom; as it is concerned with two indifferent lotteries, the betweenness axiom suffices.) Those straight lines need not be parallel, because the betweenness axiom imposes restrictions only on straight indifferent curves and nothing on the relative positions of different indifference lines. In fact, the argument for Figure 6.B.5(c) is not applicable to the betweenness axiom.

(c) Any preference represented by straight, but not parallel indifference curves satisfies the betweenness axiom but does not satisfy the independence axiom. Hence the former is weaker than the latter.

(d) Here is an example of a preference relation and its indifference map that satisfies the betweenness axiom and yields the choice of the Allais paradox.

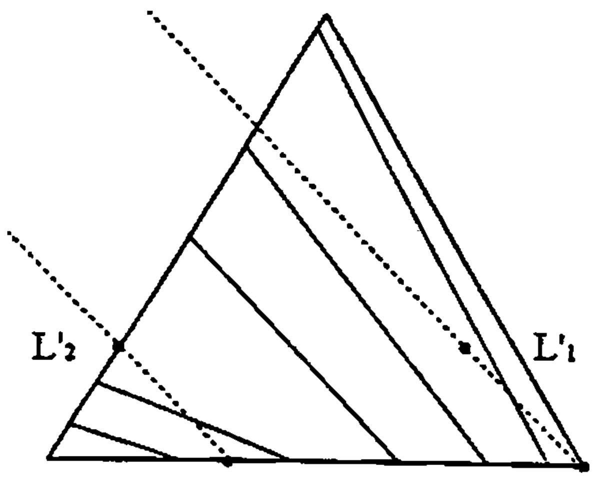


Figure 6.B.5(d)

6.B.6 Define $C = \{(u_1(a),\ldots,u_N(a)) \in \mathbb{R}^N\colon a \in A\}$, then

$$U(p) = \text{Max}\{p\cdot c \in \mathbb{R}\colon c \in C\} = -\text{Min}\{p\cdot c \in \mathbb{R}\colon c \in -C\}.$$

Hence $U(\cdot)$ is equal to $-\mu_{-C}(\cdot)$, the support function (Section 3.F) of $-C$ multiplied by -1, where the domain of the support function is restricted to the simplex $\{p \in \mathbb{R}^N_+\colon \sum_n p_n = 1\}$. Since any support function is concave, $U(\cdot)$ is convex. (A more direct proof is possible, which is essentially the same as the proof of concavity of support functions in Section 3.F.)

As an example of a nonlinear Bernoulli utility function, consider $A = B = \{1,2\}$ and define $u_1(1) = u_2(2) = 1$ and $u_2(1) = u_1(2) = 0$. Let $L = (p_1,p_2)$, then $U(L) = \text{Max}\ \{p_1,p_2\}$. (This is essentially the same as Example 6.B.4.)

6.B.7 Since the individual prefers L to L' and is indifferent between L and x_L and between L' and $x_{L'}$, by Proposition 1.B.1(iii), he prefers x_L to $x_{L'}$. By the monotonicity, this this is equivalent to $x_L > x_{L'}$.

6.C.1 If $\alpha = D > 0$ (complete insurance), then

$- q(1 - \pi)u'(w - \alpha q) + \pi(1 - q)u'(w - D + \alpha(1 - q))$

$= - q(1 - \pi)u'(w - Dq) + \pi(1 - q)u'(w - Dq)$

$= u'(w - Dq)(\pi(1 - q) - q(1 - \pi)) < 0$

$= u'(w - Dq)(\pi - q) < 0$

by $q > \pi$. Thus the first-order condition is not satisfied at $\alpha = D$. Hence the individual will not insure completely.

6.C.2 (a) Let $F(\cdot)$ be a distribution function, then

$$\int u(x)dF(x) = \int(\beta x^2 + \gamma x)dF(x) = \beta\int x^2 dF(x) + \gamma\int x dF(x)$$

$$= \beta(\text{mean of } F)^2 + \beta(\text{variance of } F) + \gamma(\text{mean of } F).$$

(b) We prove by contradiction that $U(\cdot)$ is not compatible with any Bernoulli utility function. So suppose that there is a Bernoulli utility function $u(\cdot)$ such that $U(F) = \int u(x)dF(x)$ for every distribution function $F(\cdot)$. Let x and y be two amounts of money, $G(\cdot)$ be the distribution that puts probability one at x, and $H(\cdot)$ be the distribution that puts probability one at y. Then

$u(x) = U(G) = (\text{mean of } G) - (\text{variance of } G) = x - 0 = x,$

$u(y) = U(H) = (\text{mean of } H) - (\text{variance of } H) = y - 0 = y.$

Thus, $x \geq y$ if and only if $u(x) \geq u(y)$. Hence $u(\cdot)$ is strictly monotone. Now let $F_0(\cdot)$ be the distribution that puts probability one on 0 and $F(\cdot)$ be the distribution that puts probability 1/2 on 0 and on $4/r > 0$. Since the mean and the variance of $F_0(\cdot)$ are zero, $U(F_0) = 0$. The strict monotonicity of $u(\cdot)$ thus implies that $U(F) > 0$. However, the mean of $F(\cdot)$ is $2/r$ and the variance is $4/r^2$. Hence $U(F) = 2/r - r(4/r^2) = - 2/r < 0$, which is a contradiction. Hence $U(\cdot)$ is not compatible with any Bernoulli utility function.

An example of two lotteries with the property requested in the exercise was given in the above proof of incompatibility. (Note that if all we need to show were the incompatibility of $U(\cdot)$ and any Bernoulli utility function, the equality $u(x) = x$ obtained above would be sufficient to complete the proof, because this implies the risk neutrality, which contradicts the fact that the variance of $F(\cdot)$ is subtracted in the definition of $U(\cdot)$.)

6.C.3 Suppose first that condition (i) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Let $F(\cdot)$ be the distribution that puts probability 1/2 on $x - \varepsilon$ and on $x + \varepsilon$, and $F_\varepsilon(\cdot)$ be the distribution that puts probability $1/2 - \pi(x,\varepsilon,u)$ on $x - \varepsilon$ and $1/2 + \pi(x,\varepsilon,u)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if} \quad z < x - \varepsilon, \\ 1/2 & \text{if } x - \varepsilon \le z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \ge z. \end{cases}$$

$$F_\varepsilon(z) = \begin{cases} 0 & \text{if} \quad z < x - \varepsilon, \\ 1/2 - \pi(x,\varepsilon,u) & \text{if } x - \varepsilon \le z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \ge z. \end{cases}$$

Then $\int z dF(z) = x$ and $\int u(z)dF(z) \le u(x) = \int u(z)dF_\varepsilon(z)$ by (i). But

$$\int u(z)dF(z) = (1/2)u(x - \varepsilon) + (1/2)u(x + \varepsilon),$$

$$\begin{aligned}\int u(z)dF_\varepsilon(z) &= (1/2 - \pi(x,\varepsilon,u))u(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u))u(x + \varepsilon). \\ &= (1/2)u(x - \varepsilon) + (1/2)u(x + \varepsilon) + \pi(x,\varepsilon,u))(u(x + \varepsilon) - u(x - \varepsilon)).\end{aligned}$$

Since $u(x + \varepsilon) - u(x - \varepsilon) > 0$, the above inequality is equivalent to $\pi(x,\varepsilon,u) \ge 0$. Thus (i) implies (iv).

Suppose conversely that condition (iv) holds. Let $y \in \mathbb{R}$, $z \in \mathbb{R}$, and $y > z$. Define $x = (y + z)/2$ and $\varepsilon = (y - z)/2$, then $y = x + \varepsilon$, $z = x - \varepsilon$, and

$$\begin{aligned}u(x) &= (1/2 + \pi(x,\varepsilon,u))u(y) + (1/2 - \pi(x,\varepsilon,u))u(z) \\ &= (1/2)u(y) + (1/2)u(z) + \pi(x,\varepsilon,u)(u(y) - u(z)).\end{aligned}$$

Since $\pi(x,\varepsilon,u) \ge 0$ and $u(y) \ge u(z)$, this implies

$$(1/2)u(y) + (1/2)u(z) \le u(x) = u((1/2)y + (1/2)z).$$

Although we omit the proof, this is sufficient for the concavity of $u(\cdot)$.

Hence (iv) implies (ii). Since the equivalence of (i), (ii), and (iii) have already been established, this completes the proof of the equivalence of all four conditions.

6.C.4 (a) Let $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\ldots,\alpha'_N) \in \mathbb{R}^N_+$, and $\alpha \geq \alpha'$, then $\sum_n \alpha_n z_n \geq \sum_n \alpha'_n z_n$ for almost every realization $(z_1,\ldots,z_N)$, because all the returns are nonnegative with probability one. Since $u(\cdot)$ is increasing, this implies that $u(\sum_n \alpha_n z_n) \geq u(\sum_n \alpha'_n z_n)$ with probability one. Hence

$$\int u(\textstyle\sum_n \alpha_n z_n) dF(z_1,\ldots,z_N) \geq \int u(\sum_n \alpha'_n z_n) dF(z_1,\ldots,z_N),$$

that is, $U(\alpha) \geq U(\alpha')$.

(b) Let $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\ldots,\alpha'_N) \in \mathbb{R}^N_+$, and $\lambda \in [0,1]$, then, by the concavity of $u(\cdot)$,

$$\begin{aligned} u(\textstyle\sum_n (\lambda\alpha_n + (1-\lambda)\alpha'_n) z_n) &= u(\lambda \textstyle\sum_n \alpha_n z_n + (1-\lambda)\sum_n \alpha'_n z_n) \\ &\geq \lambda u(\textstyle\sum_n \alpha_n z_n) + (1-\lambda) u(\sum_n \alpha'_n z_n) \end{aligned}$$

for almost every realization $(z_1,\ldots,z_n)$. Hence

$$\begin{aligned} &U(\lambda\alpha + (1-\lambda)\alpha') \\ &= \int u(\textstyle\sum_n (\lambda\alpha_n + (1-\lambda)\alpha'_n) z_n) dF(z_1,\ldots,z_N) \\ &\geq \int (\lambda u(\textstyle\sum_n \alpha_n z_n) + (1-\lambda) u(\sum_n \alpha'_n z_n)) dF(z_1,\ldots,z_N) \\ &= \lambda \int u(\textstyle\sum_n \alpha_n z_n) dF(z_1,\ldots,z_N) + (1-\lambda)\int u(\sum_n \alpha'_n z_n)) dF(z_1,\ldots,z_N) \\ &= \lambda U(\alpha) + (1-\lambda) U(\alpha'). \end{aligned}$$

(c) Let $(\alpha^m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N_+$ converging to $\alpha \in \mathbb{R}^N_+$, then there exists a positive number B such that $\alpha^m \leq (B,\ldots,B)$ for every m. Of course, $U(B,\ldots,B)$ is finite. But this is equivalent to saying that the (measurable) function $z \to u(\sum_n Bz_n)$ is integrable. Since $u(\cdot)$ is monotone and all the returns are nonnegative with probability one, $u(\sum_n \alpha^m_n z_n) \leq u(\sum_n Bz_n)$ for every m and for every realization $(z_1,\ldots,z_N)$. Moreover, since $u(\cdot)$ is continuous, $u(\sum_n \alpha^m_n z_n)$

converges to $u(\sum_n \alpha_n z_n)$ for almost every realization $(z_1,\ldots,z_N)$. Hence, by Lebesgue's dominated convergence theorem,

$$\int u(\textstyle\sum_n \alpha_n^m x_n)dF(x_1,\ldots,x_N) \to \int u(\textstyle\sum_n \alpha_n x_n)dF(x_1,\ldots,x_N).$$

That is, $U(\alpha^m) \to U(\alpha)$.

6.C.5 (a) Let $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$ and $\lambda \in [0,1]$. In analogy with expression (6.C.1) the value $\lambda u(x) + (1 - \lambda)u(y)$ can be considered as the expected utility from the lottery that yields x with probability λ and y with probability $1 - \lambda$. On the other hand, the value $u(\lambda x + (1 - \lambda)y)$ is the utility from consuming the mean $\lambda x + (1 - \lambda)y$ of the lottery with probability one. The concavity of $u(\cdot)$ would then imply that consuming the mean bundle of the L commodities with probability one is at least as good as entering into the lottery. But this is the defining property of risk aversion in Definition 6.C.1.

(b) [First printing errata: The Bernoulli utility function $u(\cdot)$ for wealth should be denoted by another symbol, say $\tilde{u}(\cdot)$, to avoid confusion with the original utility function $u(\cdot)$ defined on $\mathbb{R}^L_+$.] Let $p >> 0$ be a fixed price vector, w and w' be two wealth levels, and $\lambda \in [0,1]$. Denote the demand function by $x(\cdot)$ and let $x = x(p,w)$ and $x' = x(p,w')$, then $p\cdot(\lambda x + (1 - \lambda)x') \le \lambda w + (1 - \lambda)w'$. Thus $u(\lambda x + (1 - \lambda)x') \le \tilde{u}(\lambda w + (1 - \lambda)w')$. If $u(\cdot)$ is concave, then

$$u(\lambda x + (1 - \lambda)x') \ge \lambda u(x) + (1 - \lambda)u(x') = \lambda\tilde{u}(w) + (1 - \lambda)\tilde{u}(w').$$

Hence $\tilde{u}(\lambda w + (1 - \lambda)w') \ge \lambda\tilde{u}(w) + (1 - \lambda)\tilde{u}(w')$. Thus $\tilde{u}(\cdot)$ also exhibits risk aversion.

The following interpretation can be given to this result. Although, in the text, we are mainly concerned with the cases where outcomes are monetary amounts, in many cases in economic theory, utilities do not directly come from money, but from physical commodities. It is therefore desirable to *derive* risk aversion of Bernoulli utility functions for money from the properties of the underlying utility function for the commodities. The above result says that, if an individual has a risk-averse utility function for commodities, the his Bernoulli utility functions exhibits risk aversion.

(c) We shall give an example with the properties stated in the exercise. Let $L = 2$. Define $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ by $u(x) = \sqrt{\mathrm{Max}\{x_1, x_2\}}$, then $u(\cdot)$ is not concave. Now consider the price vector $p = (1,2)$, then, for each $w \geq 0$, $x(p,w) = (w,0)$. Hence $\tilde{u}(w) = \sqrt{w}$, which is concave and exhibits risk aversion. The lesson from this example is that, in order to obtain the risk aversion of $\tilde{u}(\cdot)$ for a fixed price vector, all that matters is the risk attitude along the wealth expansion path.

6.C.6 (a) Suppose that condition (ii) is true and let $F(\cdot)$ be any distribution function, then

$$\psi(u_1(c(F,u_2))) = u_2(c(F,u_2)) = \int u_2(x)dF(x).$$

Since $\psi(\cdot)$ is concave,

$$\int u_2(x)dF(x) = \int \psi(u_1(x))dF(x) \leq \psi(\int u_1(x)dF(x)).$$

Thus $\psi(u_1(c(F,u_2))) \leq \psi(\int u_1(x)dF(x))$. Since $\psi(\cdot)$ is increasing, this implies that $u_1(c(F,u_2)) \leq \int u_1(x)dF(x)$. Since $\int u_1(x)dF(x) = u_1(c(F,u_1))$, this implies that $u_1(c(F,u_2)) \leq u_1(c(F,u_1))$. Since $u_1(\cdot)$ is increasing, we obtain $c(F,u_2) \leq c(F,u_1)$. Condition (iii) is thus established.

Conversely, suppose that (iii) is true. Let $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $\lambda \in [0,1]$. We shall prove that

$$\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) \geq \lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)).$$

Let $F(\cdot)$ be the distribution function that puts probability λ on x and probability $1 - \lambda$ on y. Then, $\lambda u_1(x) + (1 - \lambda)u_1(y) = u_1(c(F,u_1))$ and hence $\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) = u_2(c(F,u_1))$. On the other hand, by the definition,

$$\lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)) = \lambda u_2(x) + (1 - \lambda)u_2(y) = u_2(c(F,u_2)).$$

By (iii) and the increasingness of $u_2(\cdot)$, we obtain

$$\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) \geq \lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)).$$

(b) Suppose first that condition (iii) holds. If $\int u_2(x)dF(x) \geq u_2(\bar{x})$, then $u_2(c(F,u_2)) \geq u_2(\bar{x})$. Thus $c(F,u_2) \geq \bar{x}$. By condition (iii), $c(F,u_1) \geq \bar{x}$. Hence $u_1(c(F,u_1)) \geq u_1(\bar{x})$, or $\int u_1(x)dF(x) \geq u_1(\bar{x})$. Thus condition (v) holds.

Suppose next that (v) holds, then $\int u_1(x)dF(x) \geq u_1(c(F,u_2))$. Since $\int u_1(x)dF(x) = u_1(c(F,u_1))$, we have $u_1(c(F,u_1)) \geq u_1(c(F,u_2))$ and hence $c(F,u_1) \geq c(F,u_2)$.

6.C.7 Suppose first that condition (iii) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Denote by $F(\cdot)$ the distribution function that puts probability $1/2 - \pi(x,\varepsilon,u_2)$ on $x - \varepsilon$ and $1/2 + \pi(x,\varepsilon,u_2)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if } z < x - \varepsilon, \\ 1/2 - \pi(x,\varepsilon,u_2) & \text{if } x - \varepsilon \leq z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \geq z. \end{cases}$$

Then $c(F,u_2) = x$. By (iii), $c(F,u_1) \geq x$. Thus $u_1(c(F,u_1)) \geq u_1(x)$. But here, we have

$$\begin{aligned} & u_1(c(F,u_1)) \\ &= (1/2 - \pi(x,\varepsilon,u_2))u_1(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u_2))u_1(x + \varepsilon) \\ &= (1/2)u_1(x - \varepsilon) + (1/2)u_1(x + \varepsilon) + \pi(x,\varepsilon,u_2)(u_1(x + \varepsilon) - u_1(x - \varepsilon)) \end{aligned}$$

and

$$\begin{aligned} & u_1(x) \\ &= (1/2 - \pi(x,\varepsilon,u_1))u_1(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u_1))u_1(x + \varepsilon) \end{aligned}$$

$= (1/2)u_1(x - \varepsilon) + (1/2)u_1(x + \varepsilon) + \pi(x,\varepsilon,u_1)(u_1(x + \varepsilon) - u_1(x - \varepsilon)).$

Thus the last inequality is equivalent to $\pi(x,\varepsilon,u_2) \geq \pi(x,\varepsilon,u_1)$. Hence condition (iv) holds.

Suppose now that condition (iv) holds. Since $\pi(x,0,u_1) = \pi(x,0,u_2) = 0$, (iv) implies that $\partial\pi(x,0,u_2)/\partial\varepsilon \geq \partial\pi(x,0,u_2)/\partial\varepsilon$. Since $r_A(x,u_1) = 4\partial\pi(x,0,u_1)/\partial\varepsilon$ and $r_A(x,u_2) = 4\partial\pi(x,0,u_2)/\partial\varepsilon$, (i) follows.

6.C.8 Let w_1 and w_2 be two wealth levels such that $w_1 > w_2$ and define $u_1(z) = u(w_1 + z)$ and $u_2(z) = u(w_2 + z)$, then $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$ by Proposition 6.C.3. It was shown in Example 6.C.2 continued that the demand for the risky asset of $u_1(\cdot)$ is greater than that of $u_2(\cdot)$. This means that the demand for the risky asset of $u(\cdot)$ is greater at wealth level w_1 than at w_2.

6.C.9 [First printing errata: The function $u(\cdot)$ on the left-hand side of the equality on the fifth line should be denoted by a different symbol, because, on the right-hand side, $u(\cdot)$ is used for the utility function on the first period.]

(a) The first-order condition for the first problem is $u'(w - x_0) = v'(x_0)$. For the second problem, let's first define a function $\phi(\cdot)$ by

$$\phi(x) = u(w - x) + E[v(x + y)].$$

Then $\phi'(x) = - u'(w - x) + E[v'(x + y)]$ and $\phi''(x) = u''(w - x) + E[v''(x + y)]$. Note also that $\phi'(x^*) = 0$ and $\phi''(x) \leq 0$ for every x, which implies that if $\phi'(x) > 0$, then $x^* > x$. Now, since $E[v'(x_0 + y)] > v'(x_0)$,

$$\phi'(x_0) = - u'(w - x_0) + E[v'(x_0 + y)] = - v'(x_0) + E[v'(x_0 + y)] > 0.$$

Hence $x^* > x_0$.

(b) Define two functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ by $\eta_1(x) = -v_1'(x)$ and $\eta_2(x) = -v_2'(x)$. Then $\eta_1(\cdot)$ and $\eta_2(\cdot)$ are increasing and the coefficients of absolute prudence of $v_1(\cdot)$ and of $v_2(\cdot)$ are equal to the coefficients of absolute risk aversion of $\eta_1(\cdot)$ and of $\eta_2(\cdot)$. Thus, if the coefficient of absolute prudence of $v_1(\cdot)$ is not larger than that of $v_2(\cdot)$, then the coefficient of absolute risk aversion of $\eta_1(\cdot)$ is not larger than that of $\eta_2(\cdot)$. Moreover, since $E[v_1'(x_0 + y)] > v_1'(x_0)$, we have $E[\eta_1(x_0 + y)] < \eta_1(x_0)$. Thus, by applying Proposition 6.C.2 to $\eta_1(\cdot)$ and $\eta_2(\cdot)$, we obtain $E[\eta_2(x_0 + y)] < \eta_2(x_0)$. Hence $E[v_2'(z_0 + y)] > v_2'(z_0)$.

The implication of this fact to part (a) is that, if the coefficient of absolute prudence of the first is not larger than that of the second, and if the risk y induces the first individual to save more, then it also induces the second to do so. Hence coefficients of absolute prudence measure how much individuals are willing to save when faced with a risk in the future.

(c) If $v'''(x) > 0$, then $\eta''(x) = -v'(x) < 0$ and hence $\eta(\cdot)$ exhibits risk aversion. Thus $E[\eta(x + y)] < \eta(x)$, that is, $E[v_1'(x + y)] > v_1'(x)$.

(d) Since

$$r_A'(x,v) = -\frac{v'''(x)v'(x) - v''(x)^2}{v'(x)^2} = \frac{v''(x)}{v'(x)}\left(-\frac{v'''(x)}{v''(x)} + \frac{v''(x)}{v'(x)}\right) < 0,$$

the assertion follows.

6.C.10 Throughout this answer, we let x_1 and x_2 be two fixed wealth levels such that $x_1 > x_2$ and define $u_1(z) = u(x_1 + z)$ and $u_2(z) = u(x_2 + z)$. It is sufficient to prove that each of the five conditions of Proposition 6.C.3 is equivalent to its counterpart of Proposition 6.C.2.

Since $r_A(z,u_1) = r_A(x_1 + z, u)$ and $r_A(z,u_2) = r_A(x_2 + z, u)$, property (i) of Proposition 6.C.3 is equivalent to (i) of Proposition 6.C.2.

Property (ii) of Proposition 6.C.3 is nothing but a restatement of (ii) of Proposition 6.C.2.

As for property (iii), note that

$$\int u_1(z)dF(z) = \int u(x_1 + z)dF(z) = u(c_{x_1}) = u((c_{x_1} - x_1) + x_1) = u_1(c_{x_1} - x_1)$$

and likewise for $u_2(\cdot)$. Thus the certainty equivalent for $u_1(\cdot)$ is smaller than that for $u_2(\cdot)$ if and only if $c_{x_1} - x_1 < c_{x_2} - x_2$. Thus property (iii) of Proposition 6.C.3 is equivalent to (iii) of Proposition 6.C.2.

As for property (iv), since

$$u(x_1) = (1/2 - \pi(x_1,\varepsilon,u))u(x_1 - \varepsilon) + (1/2 + \pi(x_1,\varepsilon,u))u(x_1 + \varepsilon),$$

we have

$$u_1(0) = (1/2 - \pi(x_1,\varepsilon,u))u_1(- \varepsilon) + (1/2 + \pi(x_1,\varepsilon,u))u_1(\varepsilon).$$

Hence $\pi(x_1,\varepsilon,u) = \pi(0,\varepsilon,u_1)$. Similarly, $\pi(x_2,\varepsilon,u) = \pi(0,\varepsilon,u_2)$. Hence (iv) of Proposition 6.C.3 is equivalent to (iv) of Proposition 6.C.2.

Note that $\int u(x_1 + z)dF(z) \geq u(x_1)$ if and only if $\int u_1(z)dF(z) \geq u_1(0)$, and likewise for $u_2(\cdot)$. Thus property (v) of Proposition 6.C.3 is equivalent to (v) of Proposition 6.C.2.

6.C.11 For any wealth level x, denote by $\gamma(x)$ the optimal proportion of x invested in the risky asset. We shall give a direct proof that if the coefficient of relative risk aversion is increasing, then $\gamma'(x) < 0$; along the same line of proof, we can show that if it is decreasing, then $\gamma'(x) > 0$. As shown in Exercise 6.C.2, $\gamma(x)$ is positive and satisfies the following first-order condition for every x:

$$\int u'((1 - \gamma(x) + \gamma(x)z)x)(z - 1)xdF(z) = 0.$$

Hence

$$\gamma'(x) = \frac{-\int u''((1 - \gamma(x) + \gamma(x)z)x)(1 - \gamma(x) + \gamma(x)z)(z - 1)x\,dF(z)}{\int u''((1 - \gamma(x) + \gamma(x)z)x)(z - 1)^2x^2 dF(z)}.$$

Since the denominator is negative, it is sufficient to show that the numerator is positive.

By the definition of the coefficient of relative risk aversion,

$$- u''((1 - \gamma(x) + \gamma(x)z)x)(1 - \gamma(x) + \gamma(x)z)x$$

$$= r_R((1 - \gamma(x) + \gamma(x)z)x)u'((1 - \gamma(x) + \gamma(x)z)x)$$

for every realization z. Note also that if $z > 1$, then $(1 - \gamma(x) + \gamma(x)z)x > x$ by $\gamma(x) > 0$. Since the coefficient of relative risk aversion is increasing, this implies that $r_R((1 - \gamma(x) + \gamma(x)z)x) > r_R(x)$. Hence

$$- u''((1 - \gamma(x) + \gamma(x)z)x)(1 - \gamma(x) + \gamma(x)z)x$$

$$> r_R(x)u'((1 - \gamma(x) + \gamma(x)z)x).$$

By $z - 1 > 0$,

$$- u''((1 - \gamma(x) + \gamma(x)z)x)(1 - \gamma(x) + \gamma(x)z)x(z - 1)$$

$$> r_R(x)u'((1 - \gamma(x) + \gamma(x)z)x)(z - 1).$$

We can similarly show that this last inequality also holds for every $z < 1$. Therefore,

$$- \int u''((1 - \gamma(x) + \gamma(x)z)x)(1 - \gamma(x) + \gamma(x)z)x(z - 1)dF(z)$$

$$> \int r_R(x)u'((1 - \gamma(x) + \gamma(x)z)x)(z - 1)dF(z)$$

$$= r_R(x)\int u'((1 - \gamma(x) + \gamma(x)z)x)(z - 1)dF(z) = 0$$

by the first-order condition.

6.C.12 (a) [<u>First printing errata</u>: The coefficient β should be positive if $\rho < 1$ and negative if $\rho > 1$. This makes $u(\cdot)$ increasing.] It is easy to check that, if $u(x) = \beta x^{1-\rho} + \gamma$ with $\rho \neq 1$ and $\gamma \in \mathbb{R}$, then $u(\cdot)$ exhibits constant relative risk aversion ρ. Suppose conversely that $u(\cdot)$ exhibits constant risk aversion ρ, then $u''(x)/u'(x) = -\rho/x$. Thus $\ln(u'(x)) = -\rho \ln x + c_1$ for some

$c_1 \in \mathbb{R}$. Thus $u'(x) = (\exp c_1)x^{-\rho}$. Hence $u(x) = (\exp c_1)x^{1-\rho}/(1 - \rho) + c_2$ for some $c_2 \in \mathbb{R}$. Letting $\beta = (\exp c_1)/(1 - \rho)$ and $\gamma = c_2$, we complete the proof.

(b) It is easy to check that, if $u(x) = \beta \ln x + \gamma$ with $\beta > 0$ and $\gamma \in \mathbb{R}$, then $u(\cdot)$ exhibits constant relative risk aversion one. The other direction can be shown in the same way as in (a).

(c) By L'hopital's rule,

$$\lim_{\rho \to 1} [(x^{1-\rho} - 1)/(1 - \rho)] = \lim_{\rho \to 1} (-\ln x)x^{1-\rho}/(-1) = \ln x.$$

6.C.13 Let $\pi(\cdot)$ be the profit function and $F(\cdot)$ be the distribution function of the random price. Since $\pi(\cdot)$ is convex, $\int \pi(p)dF(p) \geq \pi(\int p dF(p))$ by Jensen's inequality. But the left-hand side is the expected payoff from the uncertain prices and the right-hand side is the utility of the expected price vector. Thus the firm prefers the uncertain prices.

6.C.14 Define a function $g(\cdot)$ by $g(\alpha) = k\alpha + v(u^{-1}(\alpha))$, then $g(u(x)) = ku(x) + v(x) = u^*(x)$. It is thus sufficient to show that $g(\cdot)$ is concave. For this, in turn, it is sufficient to prove that $(v \circ u^{-1})(\cdot)$ is concave.

Let $\alpha, \beta \in \mathbb{R}$ and $\lambda \in [0,1]$. Since $u(\cdot)$ is increasing and concave, $u^{-1}(\cdot)$ is convex. Thus

$$u^{-1}(\lambda\alpha + (1 - \lambda)\beta) \leq \lambda u^{-1}(\alpha) + (1 - \lambda)u^{-1}(\beta).$$

Since $v(\cdot)$ is nonincreasing, this implies

$$v(u^{-1}(\lambda\alpha + (1 - \lambda)\beta)) \geq v(\lambda u^{-1}(\alpha) + (1 - \lambda)u^{-1}(\beta)).$$

Since $v(\cdot)$ is concave,

$$v(\lambda u^{-1}(\alpha) + (1 - \lambda)u^{-1}(\beta)) \geq \lambda v(u^{-1}(\alpha)) + (1 - \lambda)v(u^{-1}(\beta)).$$

Thus,

$$v(u^{-1}(\lambda\alpha + (1 - \lambda)\beta)) \geq \lambda v(u^{-1}(\alpha)) + (1 - \lambda)v(u^{-1}(\beta)).$$

or, equivalently,

$$(v \circ u^{-1})(\lambda\alpha + (1 - \lambda)\beta) \geq \lambda(v \circ u^{-1})(\alpha) + (1 - \lambda)(v \circ u^{-1})(\beta).$$

(b) [First printing errata: The entire interval $[0, +\infty]$ should be $[0, +\infty)$.] Suppose that we have $u^*(x) = ku(x) + v(x)$ for a non-constant $v(\cdot)$. Since $v(\cdot)$ is decreasing and concave, $v(x + 1) - v(x)$ is negative and decreasing with x. On the other hand, since $u(\cdot)$ is increasing, concave, and bounded above, $u(x + 1) - u(x)$ is positive and decreasing, and converges to zero. Since

$$u^*(x + 1) - u^*(x) = k(u(x + 1) - u(x)) + (v(x + 1) - v(x)),$$

$u^*(x + 1) - u^*(x)$ is negative for any sufficiently large x. That is, $u^*(\cdot)$ is not increasing around such x. But this is a contradiction to the assumption that $u^*(\cdot)$ is increasing. Thus, if $u(\cdot)$ is bounded, then there is no non-constant $v(\cdot)$ such that $u^*(x) = ku(x) + v(x)$ for all $x \in [0, +\infty)$.

(c) By (a) and (b), it is sufficient to find $u(\cdot)$ and $u^*(\cdot)$ such that $u^*(\cdot)$ is more risk averse (in the Arrow-Pratt sense) than $u(\cdot)$ and $u(\cdot)$ is bounded. Define $u(x) = -\exp(-\alpha x)$ and $u^*(x) = -\exp(-\beta x)$, where $0 < \alpha < \beta$. By Example 6.C.4, $u(\cdot)$ and $u^*(\cdot)$ exhibit constant absolute risk aversion with coefficients α and β. Hence, $u^*(\cdot)$ is more risk averse than $u(\cdot)$, but, since $u(x) < 0$ for all x, $u^*(\cdot)$ is not strongly more risk averse than $u(\cdot)$.

6.C.15 Throughout this answer, we assume that $a \neq b$, because, otherwise, there would be no uncertainty involved in the payment of the second asset.

(a) If Min $\{a,b\} \geq 1$, the risky asset pays at least as high a return as the riskless asset at both states, and a strictly higher return at one of them. Then all the wealth is invested to the risky asset. Thus, Min $\{a,b\} < 1$ is a necessary condition for the demand for the riskless asset to be strictly positive.

(b) If $\pi a + (1 - \pi)b \leq 1$, then the expected return does not exceed the payments of the riskless asset and hence the risk-averse decision maker does not demand the risky asset at all. Thus, $\pi a + (1 - \pi)b > 1$ is a necessary condition for the demand for the risky asset to be strictly positive.

In the following answers, we assume that the demands for both assets are always positive.

(c) Since the prices of the two assets are equal to one, their marginal utilities must be equal. Thus

$$\pi u'(x_1 + x_2 a) + (1 - \pi)u'(x_1 + x_2 b) = \pi a u'(x_1 + x_2 a) + (1 - \pi)b u'(x_1 + x_2 b).$$

That is,

$$\pi(1 - a)u'(x_1 + x_2 a) + (1 - \pi)(1 - b)u'(x_1 + x_2 b) = 0.$$

This and $x_1 + x_2 = 1$ constitute the first-order condition.

(d) Taking b as constant, define

$$\phi(a,\pi,x_1) = \pi(1 - a)u'(x_1 + (1 - x_1)a) + (1 - \pi)(1 - b)u'(x_1 + (1 - x_1)b),$$

then

$$\partial\phi/\partial a = - \pi u'(x_1 + (1 - x_1)a) + \pi(1 - a)(1 - x_1)u''(x_1 + (1 - x_1)a) < 0,$$

$$\partial\phi/\partial x_1 = \pi(1 - a)^2 u''(x_1 + (1 - x_1)a) + (1 - \pi)(1 - b)^2 u''(x_1 + (1 - x_1)b) < 0.$$

Thus, by the implicit function theorem (Theorem M.E.1),

$$dx_1/da = - \frac{\partial\phi/\partial a}{\partial\phi/\partial x_1} < 0.$$

(e) It follows from the condition of (b) that $b > 1$, that is, that a is the worse outcome of the risky asset. Thus, if the probability π of the worse outcome is increased, then it is anticipated that the demand for the riskless asset is increased.

(f) Since $b > 1$,

$$\partial\phi/\partial\pi = (1 - a)u'(x + (1 - x)a) - (1 - b)u'(x + (1 - x)b)$$
$$= (1 - a)u'(x + (1 - x)a) + (b - 1)u'(x + (1 - x)b) > 0,$$

because $a < 1 < b$. Thus $dx/d\pi = - \dfrac{\partial\phi/\partial\pi}{\partial\phi/\partial x} > 0$, as anticipated.

6.C.16 Throughout the answer, we assume that $u(\cdot)$ is continuous, so that the maximum and the minimum are attained.

(a) If the individual owns the lottery, his random wealth is $(w + G, w + B)$. Thus the minimal selling price R_s is defined by

$$pu(w + G) + (1 - p)u(w + B) = u(w + R_s).$$

(b) If he buys the lottery at price R, his random wealth is $(w - R + G, w - R + B)$. The maximal buying price R_b is defined by

$$pu(w - R_b + G) + (1 - p)u(w - R_b + B) = u(w).$$

(c) In general, these two prices are different. However, if $u(\cdot)$ exhibits constant absolute risk aversion, then they are the same. In fact, the above two equations can be restated as $c_w = w + R_s$ and $c_{w-R_b} = w$, where c_w and c_{w-R_b} are defined as in (iii) of Proposition 6.C.3. According to the proposition, the constant absolute risk aversion implies that

$$w - c_w = (w - R_b) - c_{w-R_b}.$$

This is equivalent to $R_s = R_b$.

(d) By a direct calculation,

$$R_s = 5[(7 - 4\sqrt{3})p^2 + (4\sqrt{3} - 6)p + 1],$$

and R_b is one of the solutions to the quadratic equation

$$(1 - 2p^2)R_b^2 - 10(2p^3 + 7p^2 - 8p + 1)R_b - 25(23p^2 - 54p + 29) = 0.$$

6.C.17 According to Exercise 6.C.12, if $u(\cdot)$ exhibits constant relative risk aversion ρ, then $u(x) = \beta x^{1-\rho} + \gamma$ or $u(x) = \beta\ln(x) + \gamma$. In this answer, we assume $u(x) = \beta x^{1-\rho}$. The case of $\beta\ln(x) + \gamma$ can be proven by the same argument. Let's first consider the portfolio problem of the individual in period $t = 1$, after a realization of the random return has generated wealth level w_1. Denoting the distribution function of the return by $F(\cdot)$, his problem is

$$\text{Max}_{0\le\alpha_1\le 1} \int u(((1 - \alpha_1)R + \alpha_1 x_2)w_1)dF(x_2).$$

As discussed in Example 6.C.2 continued (and also in Exercise 6.C.11), we can show that the solution does not depend on the value of w_1. Denote the solution by α^*. If he chooses portfolio α_0 at $t = 0$, then his random wealth at $t = 1$ is $w_1 = ((1 - \alpha_0)R + \alpha_0 x_1)w_0$. Given the solution α^* at $t = 1$, his problem in period $t = 0$ is

$$\text{Max}_{0\le\alpha_1\le 1} \int\int u(((1 - \alpha^*)R + \alpha^* x_2)((1 - \alpha_0)R + \alpha_0 x_1)w_0)dF(x_2)dF(x_1).$$

Since the distributions of x_1 and x_2 are independent and $u(x) = \beta x^{1-\rho}$, we can rewrite the objective function as

$$[\int((1 - \alpha^*)R + \alpha^* x_2)^{1-\rho}dF(x_2)][\int u(((1 - \alpha_0)R + \alpha_0 x_1)w_0)dF(x_1)].$$

Since the first integral does not depend on the choice of α_0, the solution is again $\alpha_0 = \alpha^*$. This completes the proof.

For the case of a utility function exhibits constant absolute risk aversions, the absolute amounts of wealth invested on the risky asset may vary over the two periods $t = 0,1$, but those in period $t = 1$ do not depend on the realization of x_1. To see this, let $u(x) = -\beta e^{-\rho}$. The individual's problem at $t = 1$ is

$$\text{Max}_{0\le\alpha_1\le w} \int u((w_1 - \alpha_1)R + \alpha_1 x_2)dF(x_2).$$

The solution turns out to be independent of the value of w_1, and hence of x_0. Denote the solution by α^*. If he chooses portfolio α_0 at $t = 0$, then his random wealth at $t = 1$ is $w_1 = (w_0 - \alpha_0)R + \alpha_0 x_1$. Given the solution α^* at $t = 1$, his problem in period $t = 0$ is

$$\text{Max}_{0\le\alpha_1\le 1} \int\int u(((w_0 - \alpha_0)R + \alpha_0 x_1 - \alpha^*)R + \alpha^* x_2)dF(x_2)dF(x_1).$$

Since the distributions of x_1 and x_2 are independent and $u(x) = -\beta e^{-\rho}$, we can rewrite the objective function as

$$[\int \exp(-\alpha^* R + \alpha^* x_2)dF(x_2)][\int -\beta\exp(-((w_0 - \alpha_0)R + \alpha_0 x_1)R\rho)dF(x_1)].$$

Since the first integral does not depend on the choice of α_0, the solution of this maximization problem is the same as the solution of the problem of maximizing

$$\int -\beta\exp(-((w_0 - \alpha_0)R + \alpha_0 x_1)R\rho)dF(x_1).$$

But the latter is the same as what the consumer would choose at $t = 1$ if his coefficient of absolute risk aversion is equal to $R\rho$.

Now, if $R = 1$, then the consumer invests a constant absolute amount of wealth over two periods. Thus, their proportions out of the total wealths are larger if the total wealths are smaller. So, the proportion α_1/w_1 now depends on w_1 and hence on the realization x_1. Hence the proportions can no longer be constant.

6.C.18 (a) A direct calculation shows that the coefficient of absolute risk aversion at $w = 5$ is 0.1. Exercise 6.C.12(a) shows that the coefficient of relative risk aversion is 0.5, which is constant over w.

(b) By a direct calculation, the certainty equivalent is 9 and the probability premium is $(\sqrt{10} - 3)/2$.

(c) By a direct calculation, the certainty equivalent is 25 and the probability premium is $(\sqrt{26} - 5)/2$.

For each of these two lotteries, the difference between the mean of the lottery and the certainty equivalent is equal to one. However, the probability premium for the first lottery is larger. This is because $u(\cdot)$ exhibits constant relative risk aversion and hence decreasing absolute risk aversion.

6.C.19 Foe each n, denote by β_n the wealth invested in risky asset n. The wealth invested in the riskless asset is then $w - \sum_n \beta_n$. If the individual takes portfolio $\beta = (\beta_1,\ldots,\beta_N) \in \mathbb{R}^N$, then his random consumption is $x = (w - \sum_n \beta_n)r + \sum_n \beta_n z_n$, where z_n denotes the random return of asset n. By linearity of normal distributions, x is a normal distribution with mean $(w - \sum_n \beta_n)r + \sum_n \beta_n \mu_n$ and variance $\beta \cdot V\beta$. The expected utility from x is $E[- \exp(- \alpha x)]$. But this is equal to the value, multiplied by $- 1$, at $- \alpha$ of the moment-generating function of the normal distribution with mean $(w - \sum_n \beta_n)r + \sum_n \beta_n \mu_n$ and variance $\beta \cdot V\beta$. Therefore,

$$E[- \exp(- \alpha x)] = - \exp[((w - \textstyle\sum_n \beta_n)r + \sum_n \beta_n \mu_n)(- \alpha) - (\beta \cdot V\beta)(- \alpha)^2/2].$$

By applying the monotone transformation $u \to (- 1/\alpha)\ln(- u)$ to this utility function, we obtain

$$((w - \textstyle\sum_n \beta_n)r + \sum_n \beta_n \mu_n) + (\beta \cdot V\beta)\alpha/2.$$

The first-order condition for a maximum of this objective function with respect to β gives the optimal portfolio $\beta^* = \alpha^{-1}V^{-1}(\mu - re)$, where e is the vector of $\mathbb{R}^N$ whose components are all equal to one.

6.C.20 For each $\varepsilon \geq 0$, let $F_\varepsilon(\cdot)$ be the distribution function of the lottery that pays $x + \varepsilon$ with probability 1/2 and $x - \varepsilon$ with probability 1/2. Then, $c(F_\varepsilon,u)$ is defined as the solution to the equation

$$(1/2)u(x + \varepsilon) + (1/2)u(x - \varepsilon) - u(c) = 0$$

with respect to c. Hence, by the implicit function theorem (Theorem M.E.1), $c(F_\varepsilon,u)$ is a differentiable function of ε and

$$(1/2)u'(x + \varepsilon) - (1/2)u'(x - \varepsilon) - u'(c(F_\varepsilon,u))(\partial c(F_\varepsilon,u)/\partial\varepsilon) = 0.$$

By putting $\varepsilon = 0$, we obtain $\partial c(F_0,u)/\partial\varepsilon = 0$. Also, by further differentiating the left-hand side of this equality with respect to ε, we obtain

$$(1/2)u''(x + \varepsilon) + (1/2)u''(x - \varepsilon)$$
$$- u''(c(F_\varepsilon,u))(\partial c(F_\varepsilon,u)/\partial\varepsilon)^2 - u'(c(F_\varepsilon,u))(\partial^2 c(F_\varepsilon,u)/\partial\varepsilon^2) = 0.$$

Thus, by putting $\varepsilon = 0$ and substituting $\partial c(F_0,u)/\partial\varepsilon = 0$, we obtain

$$u''(x) - u'(c(F_\varepsilon,u))(\partial^2 c(F_\varepsilon,u)/\partial\varepsilon^2) = 0.$$

Thus $\partial^2 c(F_\varepsilon,u)/\partial\varepsilon^2 = -r_A(x)$.

6.D.1 Let $L = (p_1,p_2,p_3)$ and $L' = (p_1',p_2',p_3')$ be two lotteries and $F(\cdot)$ and $G(\cdot)$ be their distribution functions.

(a) If a Bernoulli utility function is increasing, then there exists $p \in (0,1)$ such that the decision maker is indifferent between the sure outcome of \$2 and the lottery that pays \$1 with probability p and \$3 with probability 1 - p. Thus, the indifference line that goes through the \$2-vertex must hit some point on the (\$1,\$3)-face (excluding the vertices) and all indifference lines must be parallel to it. Conversely, this condition implies that the Bernoulli utility function increasing. By varying p vary from 0 to 1, we can identify the area of the lotteries that are above all indifference curves going through L. The area is shaded in the following figure:

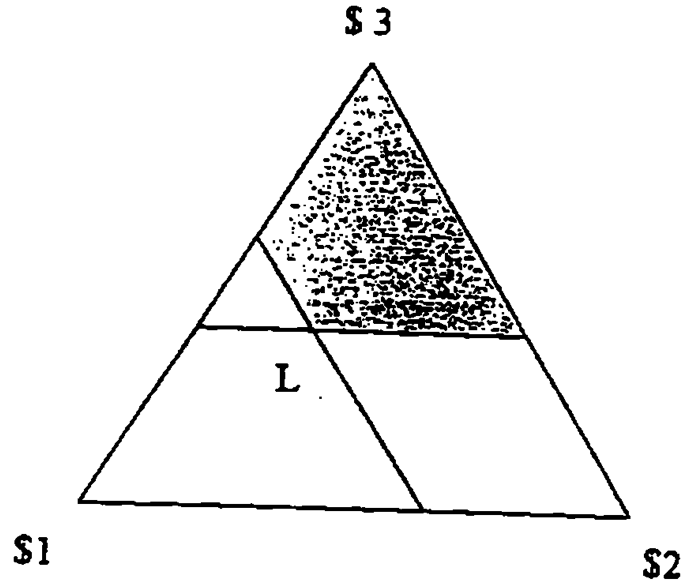


Figure 6.D.1(a)

Thus, $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if L' is located above the segment that goes through L and is parallel to the (\$1,\$2)-face and also above the segment that goes through L and parallel to the (\$2,\$3)-face.

(b) The distribution $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if $p_1 \geq p_1'$ and $p_1 + p_2 \geq p_1' + p_2'$. Since the second inequality is equivalent to $p_3 \leq p_3'$, $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if L' is located in the shaded area in the figure below:

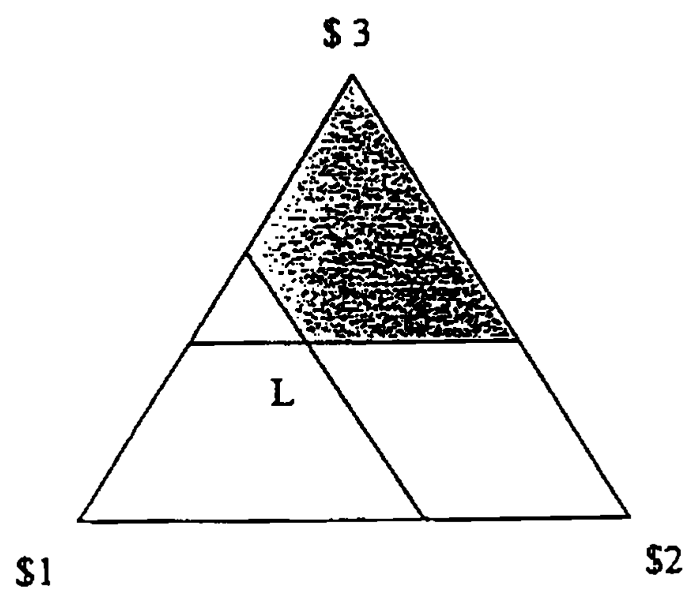


Figure 6.D.1(b)

6.D.2 [<u>First printing errata</u>: The phrase "the mean of x under $F(\cdot)$, $\int x dF(x)$, exceeds that under $G(\cdot)$, $\int x dG(x)$" should be "the mean of x under $G(\cdot)$, $\int x dG(x)$, cannot exceed that under $F(\cdot)$, $\int x dF(x)$". That is, the equality of the two means should be allowed.] For the first assertion, simply put $u(x) = x$ and apply Definition 6.D.1. As for the second, let $p \in (0,1/2)$ and consider the following two distributions:

$$F(z) = \begin{cases} 0 & \text{if } \quad z < 0, \\ p & \text{if } 0 \le z < 2, \\ 1 & \text{if } 2 \le z, \end{cases}$$

$$G(z) = \begin{cases} 0 & \text{if } z < 1, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Then $F(1/2) = p > 0 = G(1/2)$ and $\int x dF(x) = 2(1 - p) > 1 = \int x dG(x)$. Hence $F(\cdot)$ does not first-order stochastically dominate $G(\cdot)$, but the mean of $F(\cdot)$ is larger than that of $G(\cdot)$.

6.D.3 Any elementary increase in risk from a distribution $F(\cdot)$ is a mean-preserving spread of $F(\cdot)$. In Example 6.D.2, we saw that any mean-preserving spread of $F(\cdot)$ is second-order stochastically dominated by $F(\cdot)$. Hence the assertion follows.

6.D.4 Let $L = (p_1,p_2,p_3)$ and $L' = (p'_1,p'_2,p'_3)$ be two lotteries.

(a) By a direct calculation, the means of L and L' are $2 - p_1 + p_3$ and $2 - p'_1 + p'_3$. Thus the two lotteries have an equal mean if and only if $p_1 - p_3 = p'_1 - p'_3$. Hence they have an equal mean if and only if they are both on a segment that is parallel to the segment connecting the \$2-vertex and the middle point of the {\$1,\$3}-face, as depicted below:

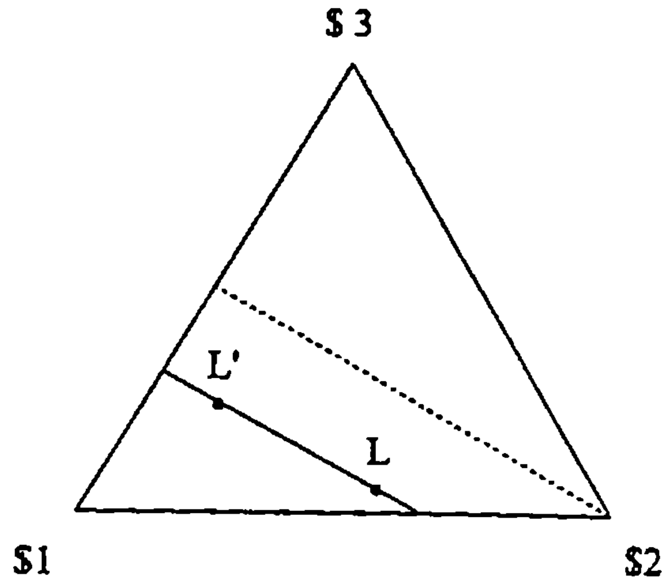


Figure 6.D.4(a)

(b) If the decision-maker exhibits risk aversion, then he prefers getting \$2 with probability one to the lottery yielding \$1 with probability 1/2 and \$3 with probability 1/2. Hence the indifference lines are steeper than the segment connecting the \$2-vertex and the middle point of the (\$1,\$3)-face. Hence, when L and L' have an equal mean, L is preferred to L' if and only if L is located on the right of L'. Therefore, L second-order stochastically dominates L' if and only if L is located on the right of L', as depicted in the figure below:

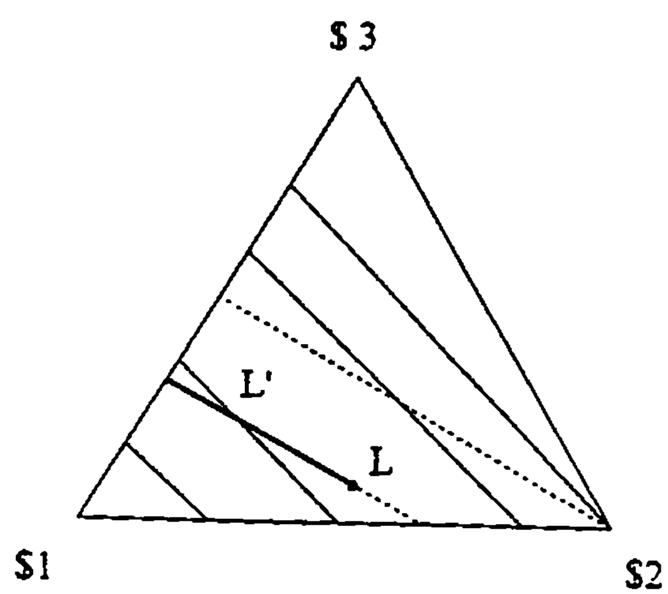


Figure 6.D.4(b)

(c) The distribution of L' is a mean preserving spread of that of L if and only if they are both on a segment that is parallel to the segment connecting the \$2-vertex and the middle point of the (\$1,\$3)-face, and L' is closer to the (\$1,\$3)-face than L. This is depicted below:

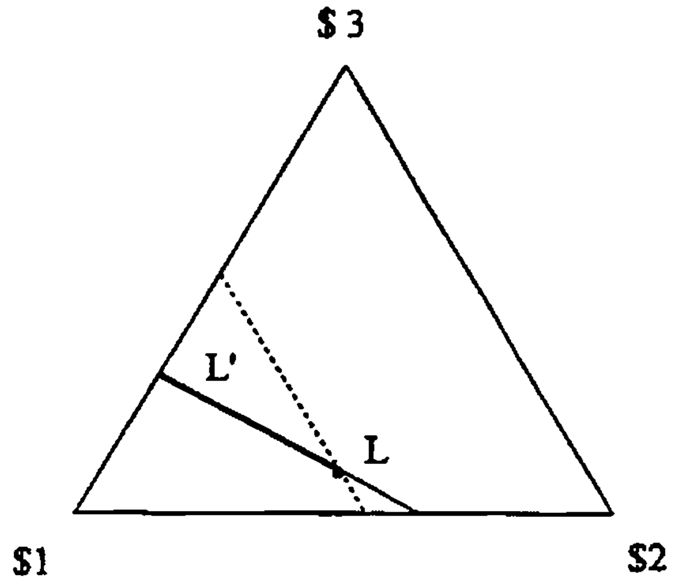


Figure 6.D.4(c)

(d) Inequality (6.D.1) holds if and only if $p_1' \geq p_1$ and $p_1' + (p_1' + p_2') \geq p_1 + (p_1 + p_2)$. But, since L and L' are assumed to have an equal mean, $p_1' - p_1 = p_3' - p_3$ and hence these two inequalities are equivalent to $p_1' \geq p_1$ alone. Thus, (6.D.1) holds if and only if L is located in the right of L', as depicted below:

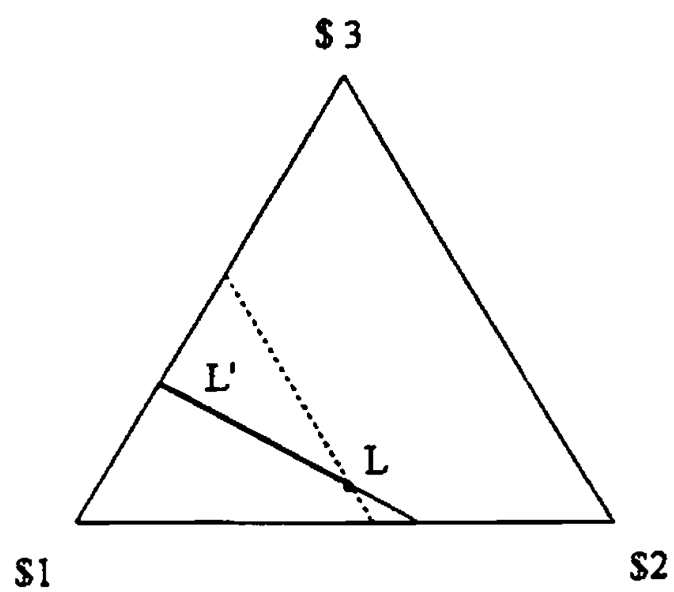


Figure 6.D.4(d)

6.E.1 Denote by $R(x,x')$ the expected regret associated with lottery x relative to x', and similarly for the other lotteries. A direct calculation yields:

$$R(x,x') = 2/3,\ R(x',x) = \sqrt{3}/3,$$

$$R(x',x'') = (\sqrt{2} + 1)/3,\ R(x'',x') = \sqrt{5}/3,$$

$$R(x'',x) = (\sqrt{2} + 1)/3,\ R(x,x'') = \sqrt{2}/3.$$

Thus, x' is preferred to x, x" is preferred to x', but x is preferred to x" .

6.E.2 (a) Denote the probability of state s by π_s and the expected utility from the contingent commodity vector (x_1,x_2) by $U(x_1,x_2)$, then $U(x_1,x_2) = \pi_1 u(x_1) + \pi_2(1 - \pi)u(x_2)$. Since $u(\cdot)$ is concave by the assumption of risk aversion, $U(\cdot)$ is also concave. Thus the preference ordering on (x_1,x_2) is convex.

(b) According to Exercise 6.C.5(a), the concavity of $U(\cdot)$ implies the risk aversion for the lotteries on (x_1,x_2).

(c) By the additive separability of $U(\cdot)$ and Exercise 3.G.4(c), both x_1 and x_2 are normal goods.

6.E.3 Since $g^*(s) = 1 + \alpha(g(s) - 1)$ for every s, we have

$$g^*(s) > g(s) \text{ if } g(s) < 1;$$

$$g^*(s) = g(s) \text{ if } g(s) = 1;$$

$$g^*(s) < g(s) \text{ if } g(s) > 1.$$

Thus $G^*(x) \le G(x)$ for every $x < 1$ and $G^*(x) \ge G(x)$ for every $x > 1$. Since $G(\cdot)$ and $G^*(\cdot)$ are continuous from the right, we have $G^*(1) \ge G(1)$. Hence property (6.D.2) holds and thus $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$ weakly. (If $g(s) \ne 1$ for some s, then $G^*(x) < G(x)$ for some $x < 1$ and

$G^*(x) > G(x)$ for some $x > 1$. Hence, in this case, $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$ strictly.)

6.F.1 We shall first prove the uniqueness of the utility function on money up to origin and scale. Suppose that two utility function $u(\cdot)$ and $\hat{u}(\cdot)$ satisfy the condition of the theorem. Since the state preferences $\succsim_s$ are represented by both $\int(\pi_s u(x_s) + \beta_s)dF_s(x_s)$ and $\int(\hat{\pi}_s \hat{u}(x_s) + \hat{\beta}_s)dF_s(x_s)$, by applying Proposition 6.B.2 to the set of all the lotteries in some state s, we know that $\pi_s u(\cdot) + \beta_s$ and $\hat{\pi}_s \hat{u}(\cdot) + \hat{\beta}_s$ are the same up to origin and scale. Hence so are $u(\cdot)$ and $\hat{u}(\cdot)$.

It remains to verify the uniqueness of subjective probability. Suppose that both $\sum_s \pi_s(\int u(x_s)dF_s(x_s))$ and $\sum_s \hat{\pi}_s(\int \hat{u}(x_s)dF_s(x_s))$ represents the same preference relation on $\mathcal{L}$. Now that we have shown that $u(\cdot)$ and $\hat{u}(\cdot)$ are the same up to origin and scale, without loss of generality, we can assume that $u(\cdot) = \hat{u}(\cdot)$. We can normalize $u(\cdot)$ so that $u(0) = 0$ and $u(1) = 1$. Note here that if a distribution function $F_s(\cdot)$ puts probability p_s on 1 and probability $1 - p_s$ on 0, then the expected utility is p_s. Thus, by choosing p_s suitably for each s, any point in $[0,1]^S$ can be represented in the form

$$(\int u(x_1)dF_1(x_1),\dots,\int u(x_S)dF_S(x_S)).$$

Hence, if $(\pi_1,\dots,\pi_S) \neq (\hat{\pi}_1,\dots,\hat{\pi}_S)$, then there would exist $(F_1,\dots,F_S) \in \mathcal{L}$ and $(F_1',\dots,F_S') \in \mathcal{L}$ such that

$$\sum_s \pi_s(\int u_s(x_s)dF_s(x_s)) > \sum_s \pi_s(\int u_s(x_s)dF_s'(x_s)),$$

$$\sum_s \hat{\pi}_s(\int u_s(x_s)dF_s(x_s)) < \sum_s \hat{\pi}_s(\int u_s(x_s)dF_s'(x_s)).$$

This contradicts the assumption that they represent the same preference. Thus $(\pi_1,\dots,\pi_S) = (\hat{\pi}_1,\dots,\hat{\pi}_S)$.

6.F.3 (a) If $P = \{\pi\}$, then $U_W(H) = \pi$ and $U_B(H) = 1 - \pi$. Hence they are determined from the expected utility $\pi u(1000) + (1 - \pi)u(0)$. Moreover, $U_W(R) > U_W(H)$ if and only if $0.49 > \pi$. But this is equivalent to $0.51 < 1 - \pi$, which is, in turn, equivalent to $U_B(R) < U_B(H)$.

(b) We have $U_W(R) > U_W(H)$ if and only if $0.49 > \text{Min } P$. We have $U_B(R) > U_B(H)$ if and only if $0.51 > \text{Min}\{1 - \pi : \pi \in P\}$, which is equivalent to $0.49 < \text{Max } P$. Hence $\text{Min } P < 0.49 < \text{Max } P$ if and only if $U_W(R) > U_W(H)$ and $U_B(R) > U_B(H)$.

CHAPTER 7

7.C.1

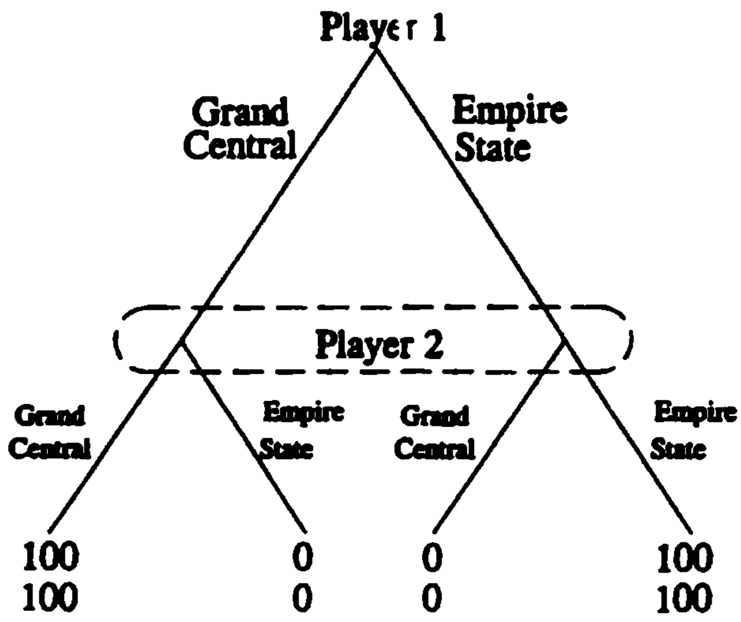


Figure 7.C.1

7.D.1 Player i has $M_1 \times M_2 \times M_3 \times \cdots \times M_N$ strategies in this game. □

7.D.2

		Player 2: H	Player 2: T
Player 1	H	-1 , 1	1 , -1
	T	1 , -1	-1 , 1

Figure 7.D.2

7.E.1 (a) In order to specify a strategy for player 1, we need to determine his moves in all of the three information sets in which he moves. Thus a typical strategy for player 1 can be written as a triple. The set of strategies for player 1 are:

S_1 = { (L, x, x), (L, x, y), (L, y, x), (L, y, y), (M, x, x), (M, x, y), (M, y, x), (M, y, y), (R, x, x), (R, x, y), (R, y, x), (R, y, y) }

If player 1 uses strategy (L, x, y), he plays L at the root of the game, x in his information set following his move M (we refer to this information set as "Information Set 2") and y in his information set following his move R We refer to this information set as "Information set 3").

Similarly player 2's strategy specify her move at her information set (we refer tho this information set as "Information Set 1"). Thus, $S_2 = \{(l),(r)\}$.

(b) A behavior strategy for player 1 consists of a randomization of his possible moves at each information set in which he has to move. Suppose that at the root, player 1 plays L, M, and R with probabilities of p_1, p_2 and p_3 respectively ($p_1+p_2+p_3=1$); at information set 2, player 1 plays x, y with probabilities of q_1 and q_2 respectively ($q_1+q_2=1$); at information set 3, player 1 plays x, y with probabilities of s_1 and s_2 respectively.

TYPO $s_1 = r_1$ $s_2 = r_2$

Assume that player 2 plays l and r with probabilities $\sigma(l)$ and $\sigma(r)$ respectively ($\sigma(l)+\sigma(r)=1$). Thus, if player 1 is using the above behavioral strategy and player 2 is using this mixed strategy, the probability that we reach each terminal node will be:

$\Pr(T_0) = p_1$; $\Pr(T_1) = p_2\ \sigma(l)\ q_1$; $\Pr(T_2) = p_2\ \sigma(l)\ q_2$; $\Pr(T_3) = p_2\ \sigma(r)\ q_1$;
$\Pr(T_4) = p_2\ \sigma(r)\ q_2$; $\Pr(T_6) = p_3\ \sigma(l)\ r_2$; $\Pr(T_7) = p_3\ \sigma(r)\ r_1$;
$\Pr(T_8) = p_3\ \sigma(r)\ r_2$.

Now the following mixed strategy for player 1 is realization equivalent to the above behavior strategy:

(L, x, x) with probability p_1, (M, x, x) with probability p_2q_1,
(M, y, x) with probability p_2q_2, (R, x, x) with probability p_3r_1,
(R, x, y) with probability p_3r_2. [Note: $p_1 + p_2q_1 + p_2q_2 + p_3r_1 + p_3r_2 = p_1 + p_2(q_1 + q_2) + p_3(r_1 + r_2) = p_1 + p_2\ 1 + p_3\ 1 = 1$]

If player 1 is using the above mixed strategy and player 2 is using the mixed strategy σ, the probability that we reach each terminal node will be the same as shown before for the behavior strategy. Therefore, the above mixed strategy is realization equivalent to the behavior strategy.

(c) Suppose that player 1 uses the following mixed strategy:

(L, x, x) with probability p_1 ; (L, x, y) with probability p_2,

(L, y, x) with probability p_3 ; (L, y, y) with probability p_4,

(M, x, x) with probability p_5 ; (M, x, y) with probability p_6,

(M, y, x) with probability p_7 ; (M, y, y) with probability p_8,

(R, x, x) with probability p_9 ; $(R, x, \underline{y})$ with probability p_{10}. TYPO Should be P_{11}

(R, y, x) with probability p_{11} ; $(R, y, \underline{y})$ with probability p_{12}.

$[p_i \geq 0$ for all i and $\sum p_i = 1]$ TYPO should be P_{10}

If Player 2 uses the mixed strategy σ, the probability that we reach each terminal node will be: $\Pr(T_0) = p_1 + p_2 + p_3 + p_4$, $\Pr(T_1) = (p_5 + p_6)\,\sigma(l)$, $\Pr(T_2) = (p_7 + p_8)\sigma(l)$, $\Pr(T_3) = (p_5 + p_6)\,\sigma(r)$, $\Pr(T_4) = (p_7 + p_8)\,\sigma(r)$, $\Pr(T_5) = (p_9 + p_{10})\,\sigma(l)$, $\Pr(T_6) = (p_{11} + p_{12})\,\sigma(l)$, $\Pr(T_7) = (p_9 + p_{10})\,\sigma(r)$, $\Pr(T_8) = (p_{11} + p_{12})\,\sigma(r)$.

The following behavioral strategy for player 1 is realization equivalent: At the root of the game, player 1 plays L, M, R with probabilities of $(p_1 + p_2 + p_3 + p_4)$, $(p_5 + p_6 + p_7 + p_8)$ and $(p_9 + p_{10} + p_{11} + p_{12})$ respectively; at information set 2, player 1 plays x, y with probabilities of $(p_5 + p_6)/(p_5 + p_6 + p_7 + p_8)$ and $(p_7 + p_8)/(p_5 + p_6 + p_7 + p_8)$ respectively; at information set 3, player 1 plays x, y with probabilities of $(p_9 + p_{10})/(p_9 + p_{10} + p_{11} + p_{12})$ and $(p_{11} + p_{12})/(p_9 + p_{10} + p_{11} + p_{12})$ respectively.

(d) Note that if player 1 reaches his (only) information set after player 2 moves, he will not remember whether he chose M or R. Thus, the game is not of perfect recall.

The result of part (b) still holds: there exists a mixed strategy for player 1 which is realization equivalent to any behavior strategy. Suppose player 1 uses the following behavior strategy:

At information set 1, player 1 plays L, M, R with probabilities of p_1, p_2 and p_3 respectively; at information set 2, player 1 plays x, y with probabilities of q_1 and q_2 respectively. If player 2 is using the mixed strategy σ, then the probability that we reach each terminal node will be:

$\Pr(T_0) = p_1$, $\Pr(T_1) = p_2\ \sigma(l)\ q_1$, $\Pr(T_2) = p_2\ \sigma(l)\ q_2$, $\Pr(T_3) = p_2\ \sigma(r)\ q_1$, $\Pr(T_4) = p_2\ \sigma(r)\ q_2$, $\Pr(T_5) = p_3\ \sigma(l)\ q_1$, $\Pr(T_6) = p_3\ \sigma(l)\ q_2$, $\Pr(T_7) = p_3\ \sigma(r)\ q_1$, $\Pr(T_8) = p_3\ \sigma(r)\ q_2$.

The following mixed strategy for player 1 is realization equivalent:

(L, x) with probability p_1, (M, x) with probability p_2q_1,

(M, y) with probability p_2q_2, (R, x) with probability p_3q_1, (M, y) with probability p_3q_2.

However, there does not always exist a behavior strategy that is realization equivalent to a mixed strategy. Consider the following example. Player 1 uses the mixed strategy playing (M, x) and (R, y) both with probability 1/2. Player 2 uses the pure strategy (l). Suppose there exist a behavior strategy for player 1 which is realization equivalent to the mixed strategy: at the root of the game, player 1 plays L, M, R with probabilities of p_1, p_2 and p_3 respectively; at his information set after player 2 moves, player 1 plays x, y with probabilities of q_1 and q_2 respectively. The mixed strategy generates the following distribution over the terminal nodes:

$\Pr(T_1) = \Pr(T_6) = 1/2$

$\Pr(T_0) = \Pr(T_2) = \Pr(T_3) = \Pr(T_4) = \Pr(T_5) = \Pr(T_7) = \Pr(T_8) = 0$ The behavior strategy generates:

$\Pr(T_3) = \Pr(T_4) = \Pr(T_7) = \Pr(T_8) = 0$

$\Pr(T_0) = p_1$, $\Pr(T_1) = p_2q_1$, $\Pr(T_2) = p_2q_2$, $\Pr(T_5) = p_3q_1$, $\Pr(T_6) = p_3q_2$.

in order for these distributions to be equivalent, we need: $\Pr(T_1) = p_2q_1 = 1/2 \Rightarrow p_2$ and $q_1 \neq 0$, $\Pr(T_2) = p_2q_2 = 0 \Rightarrow q_2 = 0$ since $p_2 \neq 0$, $\Pr(T_6) = p_3q_2 =$

1/2 which cannot hold since $q_2 = 0$, a contradiction. There exists no behavior strategy that is realization equivalent to the above mixed strategy.

In a game that is not of perfect recall the following holds:

- for any behavior strategy there exists a mixed strategy that is realization equivalent,
- not for all mixed strategies does there exist a realization equivalent behavior strategy. [Note: for a general proof of these results refer to Fudenberg/Tirole (1991) *Game Theory*. MIT press, p. 87]

CHAPTER 8

8.B.1 Firm i chooses h_i to maximize $\alpha\Sigma_j h_j + \beta(\Pi_j h_j) - w_i(h_i)^2$. The F.O.C. is: $\alpha + \beta(\Pi_{j\neq i} h_j) - 2w_i h_i = 0$. The best response function for firm i is therefore: $h_i = [\alpha + \beta\,(\Pi_{j\neq i} h_j)]\frac{1}{2w_i}$. Therefore firm i has a strictly dominant strategy iff $\beta = 0$, i.e., if the best response function of i is not dependent on the action of the other firms. If $\beta = 0$, firm i's strictly dominant strategy is $h_i = \frac{\alpha}{2w_i}$.

8.B.2 (a) Suppose $s_i^1 \in S_i$ and $s_i^2 \in S_i$ are two weakly dominant strategies for player i. This implies that $u_i(s_i^1, s_{-i}) \geq u_i(s_i^*, s_{-i})\ \forall\ s_i^* \in S_i$ and $\forall\ s_{-i} \in S_{-i}$, and $u_i(s_i^2, s_{-i}) \geq u_i(s_i^*, s_{-i})\ \forall\ s_i^* \in S_i$ and $\forall\ s_{-i} \in S_{-i}$. In particular, $u_i(s_i^1, s_{-i}) \geq u_i(s_i^2, s_{-i})$ and $u_i(s_i^2, s_{-i}) \geq u_i(s_i^1, s_{-i})\ \forall\ s_{-i} \in S_{-i}$. Therefore, $u_i(s_i^1, s_{-i}) = u_i(s_i^2, s_{-i})\ \forall\ s_{-i} \in S_{-i}$.

(b)

		Player 2: L	Player 2: R
Player 1	U	1 , 4	2 , 5
	D	1 , 2	2 , 3

Figure 8.B.2

Both of player 1's strategies (U) and (D) are weakly dominant. However, player 2 prefers that player 1 uses strategy (U).

8.B.3 Suppose not. Assume bidder i bids $b_i > v_i$. Then if some other bidder bids something larger than b_i, bidder i is just as well of as if he would have bid v_i. If all other players bid lower than v_i, then bidder i obtains the object and pays the amount of the second highest bid. If the second highest bid is $b_j < v_i$, this results in the same payoff for player i as if he bid v_i. However, suppose that the second highest bid of the other is $b_j > v_i$. Then, by bidding b_i bidder i will win the object and obtain a negative payoff. By bidding v_i he will not win the object and obtain a payoff of zero. Therefore, bidding $b_i > v_i$ is weakly dominated by bidding v_i.

Suppose bidder i bids $b_i < v_i$. Then if all other bidders bid something smaller than b_i, bidder i is just as well of as if he would have bid v_i. He will win the object and pay the the second highest bid. If some other player bids higher than v_i, then bidder i does not win the object regardless whether he bids b_i or v_i. However, suppose that nobody bids higher than v_i and the highest bid of the other players is b_j with $b_i < b_j < v_i$. Then by bidding b_i bidder i will not win the object, therefore getting a payoff of 0. By bidding v_i, he would win the object, pay $b_j < v_i$, and thus obtain a payoff of $v_i - b_j > 0$. Therefore, bidding $b_i < v_i$ is weakly dominated by bidding v_i. This argument implies that bidding v_i is a weakly dominant strategy.

8.B.4 Call the set of strategies for player i that remain after N rounds of deletion of strictly dominated strategies Σ_i^N. Suppose $s_i \in \Sigma_i^N$ is a strictly dominated strategy given the strategies Σ^N_{-i} of the other players Therefore, there exists a strategy $s_i^* \in \Sigma_i^N$, which is not a strictly dominated strategy given Σ^N_{-i}, which strictly dominates s_i. Suppose further that s_i will not be deleted in the N+1 round.

Since s_i was strictly dominated by s_i^* given Σ^N_{-i}, it will still be strictly dominated by s_i^* given $\Sigma^{N+1}_{-i} \subseteq \Sigma^N_{-i}$, and $s_i^* \in \Sigma_i^{N+1}$ (with the given assumptions). Thus the strategy s_i will be deleted in the next round. (Note: If s_i is only weakly dominated by s_i^* given Σ^N_{-i}, then s_i may no longer be weakly dominated given Σ^{N+1}_{-i} since Σ^{N+1}_{-i} may no longer include the strategies of the opponent relative to which some other strategy of player i will strictly be better. Thus the order of deletion does matter for the set of strategies surviving a process of iterated deletion of weakly dominated strategies).

8.B.5 (a) Suppose, player j produces q_j. Player i's best response can be calculated by maximizing (this is symmetric for both players):

$$\max\ [a - b(q_i \cdot\cdot\ q_j) - c]q_i$$

which yields the F.O.C: $[a - b(2q_i + q_j) - c] = 0$, so the best response is: $b_i(q_j) = \frac{a-c}{2b} - \frac{q_j}{2}$.

Now, since $q_1 \geq 0$, $q_2 \leq (a-c)/2b$ (all other strategies would be strictly dominated by $q_2 = (a-c) / 2b$). Therefore, since $q_2 \leq (a-c)/2b$, we have that $q_1 \geq (a-c)/2b - (a-c)/4b = (a-c)/4b$. Thus, since $q_1 \geq (a-c) / 4b$, then $q_2 \leq (a-c)/2b - (a-c)/8b = 3(a-c)/8b$. Continuing in this fashion we will obtain: $q = (a-c)/2b - q/2$. Thus, after successive elimination of strictly dominated strategies, $q_1 = q_2 = (a-c)/3b$.

(b) Suppose, player j produces q_j and player h produces q_h. Player i's best response can be calculated by maximizing $[a - b(q_i + q_j + q_h) - c]q_i$, which yields the F.O.C $[a - b(2q_i + q_j + q_h) - c] = 0$, implying the best response: $b_i(q_j,q_h) = (a-c)/2b - (q_j + q_h)/2$.

Now, since q_2, $q_3 \geq 0$, $q_1 \leq (a-c)/2b$ (all other strategies would strictly be dominated by $q_1 = (a-c)/2b$). Thus, since $q_1 \leq (a-c)/2b$ and similarly $q_3 \le (a-c)/2b$, we have $q_2 \ge (a-c)/2b - [2(a-c)/2b]/2 = 0$. Therefore, successive elimination of strictly dominated strategies, implies that $q_1, q_2, q_3 \ge 0$ and $q_1, q_2, q_3 \le (a-c)/2b$. However, a unique prediction cannot be obtained.

8.B.6 Suppose s_i^* is strictly dominated by the strategy σ_i^*. Suppose further that σ_i is a mixed strategy in which s_i^* is played with strictly positive probability $\sigma_i(s_i^*)>0$. We claim that σ_i is strictly dominated by the mixed strategy σ_i', which is equivalent to σ_i except that instead of playing s_i^* with probability $\sigma_i(s_i^*)$ it plays σ_i^* with probability $\sigma_i(s_i^*)$. This follows since:

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \Sigma\, \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}) \\ &= \sum_{s_i \ne s_i^*} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}) + \sigma_i(s_i^*)\, u_i(s_i^*, \sigma_{-i}) \\ &< \sum_{s_i \ne s_i^*} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}) + \sigma_i(s_i^*)\, u_i(\sigma_i^*, \sigma_{-i}) \\ &= u_i(\sigma_i', \sigma_{-i}) \quad \text{for all } \sigma_{-i} \in \Delta(S_{-i}). \end{aligned}$$

8.B.7 Suppose in negation that σ_i is a strictly dominant mixed strategy of player i, and suppose, $s_i^1, \ldots, s_i^N$ are the pure strategies that are played with positive probability in the mixed strategy σ_i. Since σ_i is a strictly dominant strategy: $u_i(\sigma_i, s_{-i}) > u_i(s_i^*, s_{-i})$ $\forall\, s_i^* \in S_i$ and $\forall\, s_{-i} \in S_{-i}$. In particular, $u_i(\sigma_i, s_{-i}) > u_i(s_i^j, s_{-i})$ $\forall\, j = 1,\ldots, N$. This implies that

$$u_i(\sigma_i, s_{-i}) > \sum_{j=1}^{N} [\sigma_i(s_i^j)\, u_i(s_i^j, s_{-i})] = u_i(\sigma_i, s_{-i})$$, a contradiction.

8.C.1 Notice that the elimination of strategies that are never a best-response is more demanding than strictly dominated strategy elimination. Thus, in every round of elimination, the deletion of never a best response deletes more strategies than the deletion of strictly dominated strategies.

Therefore, if the elimination of strictly dominated strategies yields a unique prediction in a game, then the elimination of strategies that are never a best-response cannot yield more than one strategy. Since a rationalizable strategy always exist, the elimination of strategies that are never a best-response will then also yield a unique prediction.

If the unique rationalizable strategy is not the unique prediction after elimination of strictly dominated strategies, then there exist a round of elimination in which this unique rationalizable strategy was strictly dominated. However, if this strategy was strictly dominated it was also never a best-response. This contradicts the assumption that the strategy is rationalizable. Therefore, both procedures must yield the same prediction.

8.C.2 Call the set of strategies for player i that remain after N rounds of deletion of never best-response strategies Σ_i^N. Suppose s_i is never a best-response to any strategy in Σ^N_{-i}. Suppose further that s_i will not be deleted in the N+1 round. Since s_i was never a best response to a strategy in Σ^N_{-i}, it will clearly not be a best-response to a strategy in $\Sigma^{N+1}_{-i} \subseteq \Sigma^N_{-i}$. Thus this strategy will be deleted in the next round.

8.C.3 Suppose that s_1 is a pure strategy of player 1 that is never a best response for any mixed strategy of player 2. Suppose in negation that s_1 is not strictly dominated. Construct the following correspondence for any $\sigma_i \in \Delta(S_i)$ for i = 1, 2:

$$(\sigma_1, \sigma_2) \to \{\hat{\sigma}_1 | \ \hat{\sigma}_1 \in \text{argmax } g_1(\sigma_1,\sigma_2)\} \times \{\hat{\sigma}_2 | \ g_1(s_1, \hat{\sigma}_2) \geq g_1(\sigma_1, \hat{\sigma}_2) \}.$$

The first part of this correspondence is the best response function for player 1 and therefore satisfies all the conditions of the Kakutani fixed point theorem. The second part of the correspondence is the set of mixed strategies of player 2, for which s_1 is not strictly dominated (it is a non-empty set since s_1 is not a strictly dominated strategy, i.e., it is not strictly dominated by σ_1). Therefore, the second part of the correspondence is convex valued and upper hemicontinuous due to the usual assumptions. Thus, by Kakutani's theorem there exists a fixed point (σ_1^*, σ_2^*) of this correspondence such that $g_1(s_1, \sigma_2^*) \geq g_1(\sigma_1^*, \sigma_2^*)$ from the second part of the correspondence, and $g_1(\sigma_1^*, \sigma_2^*) \geq g_1(\sigma_1, \sigma_2^*)$ for all $\sigma_1 \in \Delta(S_1)$. Therefore, $g_1(s_1, \sigma_2^*) \geq g_1(\sigma_1, \sigma_2^*)$ for all $\sigma_1 \in \Delta(S_1)$, which contradicts the assumption that s_1 is a pure strategy of player 1 that is never a best response for any mixed strategy of player 2. Therefore, if s_1 is a pure strategy of player 1 that is never a best response for any mixed strategy of player 2, then s_1 is strictly dominated by some mixed strategy of player 1.

8.C.4 [First Printing Errata: a typo appears in the lower left box of the payoff matrix. Player 1's payoff should be $\pi + 4\varepsilon$ and <u>not</u> $\eta + 4\varepsilon$.]
For the continuation of this answer, a strategy for player 2 is to play u with probability α and D with probability $1-\alpha$, and for player 3 is to play l with probability β and r with probability $1-\beta$. Denote by P_A the expected payoff of player 1 when action $A \in \{L,M,R\}$ is taken given α and β. Direct calculation and simple algebra yield:

$$P_M = \pi + \left(\frac{3\alpha + 3\beta}{2} - 3\alpha\beta - 1\right)\eta$$

$$P_L = \pi + (2\beta - 1)\varepsilon$$

$$P_R = \pi + (1 - 2\beta)\varepsilon$$

(a) To show that M is never a best response to any pair of strategies of players 2 and 3, (α,β), we have three cases:

<u>Case 1:</u> $\beta > 1/2$

Note that in this case $\dfrac{\partial P_M}{\partial \alpha} = \eta[3/2 - 3\beta] < 0$. Thus the highest payoff for player 1 if he plays M is obtained when $\alpha = 0$, and his payoff will be $P_M(\alpha=0) = \pi + \eta[\frac{3}{2}\beta - 1] < \pi + 4\varepsilon[\frac{3}{2}\beta - 1] < \pi + 4\varepsilon[2\beta - 1] = P_L$. Further note that P_L is independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response in this case.

Case 2: $\beta < 1/2$

Now, $\frac{\partial P_M}{\partial \alpha} > 0$, the highest payoff for player 1 if he plays M is obtained when $\alpha = 1$, and his payoff is $P_M(\alpha=1) = \pi + \eta[\frac{3}{2} + \frac{3}{2}\beta - 3\beta - 1] = \pi + \eta[\frac{1}{2} - \frac{3}{2}\beta] < \pi + \eta[\frac{1}{2} - \frac{3}{2}\beta + \frac{1}{2} - \beta] < \pi + 4\varepsilon[1 - 2\beta] = P_R$. Further note that P_R is independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response in this case.

Case 3: $\beta = 1/2$

In this case $P_M = \pi - \frac{\eta}{4} < \pi = P_R = P_L$. This concludes that M can never be a best response.

(b) Suppose in negation that there exists a mixed strategy, in which player 1 plays R with probability γ and L with probability $1-\gamma$, that strictly dominates M.

Case 1: $\gamma \leq 1/2$.

If $\beta = 0$ and $\alpha = 1$ then $P_M = \pi + \eta/2 > \pi$. The mixed strategy will give a payoff of $\pi - 4\varepsilon(1-2\gamma) \leq \pi$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case.

Case 2: $\gamma > 1/2$.

If $\beta = 1$ and $\alpha = 0$ then $P_M = \pi + \eta/2 > \pi$. The mixed strategy will give a payoff of $\pi - 4\varepsilon(2\gamma-1) \leq \pi$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case. This implies a contradiction, so that M cannot be strictly dominated.

(c) Suppose players correlate in the following way: Players 2 and 3 play

(U, r) with probability 1/2 and (D, l) with probability 1/2. Any mixed strategy for player 1 involving only L and R will give him a payoff of π. However, playing M will yield him a payoff of $\pi + \eta/2$. Thus M is a best-response to the above correlated strategy of player 2 and 3.

8.D.1 We know already from section 8.C that a_4 and b_4 are not rationalizable strategies. Thus, these strategies cannot be played with positive probability in a mixed strategy Nash equilibrium. Suppose that there exists a mixed strategy equilibrium in which a_1 and a_3 are both played with a strictly positive probability. Then the expected payoff from playing either one of them has to be equal (see exercise 8.D.2). This implies that the probability that player 2 plays b_1 has to be equal to the probability that he plays b_3. Now, suppose that player 2 plays b_1 and b_3 with probability α and b_2 with probability $1-2\alpha$. The expected payoff for player 1 obtained by playing either a_1 or a_3 equals: $7\alpha + (1-2\alpha)2$. The expected payoff for player 1 when playing a_2 equals: $5\alpha + 5\alpha + (1-2\alpha)3 = 10\alpha + (1-2\alpha)3 > 7\alpha + (1-2\alpha)2$. Therefore, in a mixed strategy equilibrium a_1 and a_3 cannot both be played with positive probability since playing a_2 would give the player a larger payoff.

Suppose, there exists a mixed strategy equilibrium in which player 1 plays a_1 and a_2 with strictly positive probability. Clearly, player 2's best response to this strategy of player 1 does not involve playing b_3 with strictly positive probability (given the strategy of player 1, playing b_2 is strictly better for player 2). Thus player 2 will play b_1 with probability β and b_2 with probability $1-\beta$. The payoff for player 1 from playing a_1 equals: $(1-\beta)2$, playing a_2 yields: $5\beta + (1-\beta)3 > (1-\beta)2$. Therefore, in a mixed strategy equilibrium a_1 and a_2 cannot both be played with strictly positive probability since playing a_2 is always better.

Similarly, it can be shown that there exists no mixed strategy equilibrium in which a_2 and a_3 are both played with strictly positive probability. Therefore, player 1 always plays a_2 in a Nash equilibrium. Player 2 will then play his best response b_2. Thus (a_2, b_2) being played with certainty is the unique mixed strategy equilibrium.

8.D.2 We will show that any Nash equilibrium (NE) must be in S^∞, the set of strategies which survive iterated strict dominance. Since it is assumed that this set contains one element, this will prove the required result.

Let $(s_1^*, s_2^*, \ldots, s_I^*)$ be a (mixed) NE and suppose in negation that it does not survive iterated strict dominance. Let i be the player whose strategy is first ruled out in the iterative process (say in the k^{th} round). Therefore, there exists σ_i and a_i such that $u_i(\sigma_i, s_{-i}) > u_i(a_i, s_{-i})\ \forall\ s_{-i} \in S_{-i}^{k-1}$, and a_i is played with positive probability $s_i^*(a_i)$. Since k is the first round at which any of the NE strategies, $(s_1^*, s_2^*, \ldots, s_I^*)$, are ruled out, we must have that $s_{-i}^* \in S_{-i}^{k-1}$. Hence, $u_i(\sigma_i, s_{-i}^*) > u_i(a_i, s_{-i}^*)$. Let the strategy s_i' be derived from s_i^* execpt that any probability of playing a_i is replaced by playing σ_i. We thus have that:

$$u_i(s_i', s_{-1}^*) = u_i(s_i^*, s_{-1}^*) + s_i^*(a_i)\cdot[u_i(\sigma_i, s_{-1}^*) - u_i(a_i, s_{-1}^*)] > u_i(s_i^*, s_{-1}^*)$$

which contradicts the assumption that $(s_1^*, s_2^*, \ldots, s_I^*)$ is a NE.

8.D.3 First of all, notice that the first auction bid is a simultaneous move game where a strategy for a player consists of a bid. Let b_1 be the bid of Player 1, and b_2 be the bid of Player 2.

(i) If $b_1 > b_2$, Player 1 gets the object and pays b_1 for it; Player 2 does not get the object. Thus, in this case:

$$\begin{cases} u_1(b_1,b_2)=v_1-b_1 & \text{[1's valuation of the object minus what he has to pay for it].} \\ u_2(b_1,b_2)=0 & \end{cases}$$

(ii) Similarly, if $b_2 > b_1$: $\begin{cases} u_1(b_1,b_2)=0 \\ u_2(b_1,b_2)=v_2-b_2 \end{cases}$

(iii) If $b_1 = b_2$, each player gets the object with probability $\frac{1}{2}$:

$u_1(b_1 + b_2) = \frac{1}{2}(v_1 - b_1) + \frac{1}{2} \cdot 0 = \frac{(v_1 - b_1)}{2}$.

Similarly: $u_2(b_1, b_2) = \frac{(v_2 - b_2)}{2}$

Therefore, we have for $i,j \in \{1,2\}$, $i \neq j$:

$$u_i(b_1, b_2) = \begin{cases} 0, & b_i < b_j \\ \frac{1}{2}(v_i - b_i), & b_1 = b_2 \\ (v_i - b_i), & b_i > b_j \end{cases}$$

(a) We claim that no strategy for player 1 is strictly dominated. Suppose in negation that b_1 is strictly dominated by b_1', i.e., for any b_2: $u_1(b_1', b_2) > u_1(b_1, b_2)$. Take $b_2^* = \max\{b_1, b_1'\} + 1$, then: $b_2^* > b_1$, and $b_2^* > b_1'$. Hence, $u_1(b_1', b_2^*) = u_1(b_1, b_2^*) = 0$, a contradiction. Therefore, no strategy for player 1 is strictly dominated. Similarly, one can prove that no strategy for player 2 is strictly dominated, and thus no strategies are strictly dominated.

(b) We now claim that any strategy b_1 for Player 1 such that $b_1 > v_1$, is weakly dominated by v_1. Note that, if $b_1 > v_1$:

(i) If $b_2 < v_1$: $u_1(v_1, b_2) = v_1 - v_1 = 0$

$u_1(b_1, b_2) = v_1 - b_1 < 0 = u(v_1, b_2)$

(ii) If $b_2 = v_1$: $u_1(v_1, b_2) = \frac{1}{2}(v_1 - v_1) = 0$

$u_1(b_1, b_2) = v_1 - b_1 < 0 = u_1(v_1, b_2)$

(iii) If $v_1 < b_2 < b_1$: $u_1(v_1, b_2) = 0$

$u_1(b_1, b_2) = v_1 - b_1 < 0 = u_1(v_1, b_2)$

(iv) If $v_1 < b_2 = b_1$: $u_1(v_1 , b_2) = 0$

$u_1(b_1,b_2) = 1/2(v_1 - b_1) < 0 = u_1(v_1,b_2)$

(v) If $b_2 > b_1$: $u_1(v_1, b_2) = 0$

$u_1(b_1, b_2) = 0 = u_1(v_1, b_2)$

Thus, in all cases, $u_1(v_1, b_2) \geq u_1(b_1, b_2)$, and in some cases strict inequality holds. Thus, $b_1 > v_1$ is weakly dominated by v_1. Similarly, any strategy b_2 for Player 2 such that $b_2 > v_2$ is weakly dominated by v_2.

Now suppose $v_1 > 2$. We claim that, in this case, $b_1 = 1$ weakly dominates $b_1' = 0$. Observe that:

(i) If $b_2 = 0$: $u_1(1, 0) = v_1 - 1$

$u_1(0, 0) = \frac{1}{2} v_1$

Since $v_1 > 2$, $u_1(1,0) - u_1(0,0) = (v_1 - 1) - \frac{v_1}{2} = \frac{v_1}{2} - 1 > 0$

(ii) If $b_2 = 1$: $u_1(1, 1) = \frac{1}{2} (v_1 - 1) > 0$

$u_1(0,1) = 0$

(iii) If $b_2 > 1$: $u_1(1, b_2) = u_1(0,b_2) = 0$.

Thus, in all cases $u_1(1, b_2) \geq u_1(0,b_2)$, with strict inequality in some cases. Finally, suppose that $v_1 \in (1,2)$. We claim that, in this case,

$b_1 = v_1 - 1$ weakly dominates $b_1 = v_1$:

(i) If $b_2 < v_1 - 1$: $u_1(v_1 - 1,b_2) = 1 > 0 = u_1(v_1,b_2)$

(ii) If $b_2 = v_1 - 1$: $u(v_1 - 1, b_2) = 1/2 > 0 = u_1(v_1,b_2)$

(iii) If $b_2 > v_1 - 1$: $u(v_1 - 1, b_2) = 0 = u_1(v_1,b_2)$

Thus, in all cases, $u(v_1 - 1, b_2) \geq u(v_1, b_2)$, with strict inequality in (i) and (ii). Similarly, it can be shown that:

-if $v_2 > 2$, $b_2 = 1$ weakly dominates $b_2' = 0$

-if $v_2 \geq 1$, $b_2 = v_2 - 1$ weakly dominates $b'_2 = v_2$

(c) Define the best response correspondence for Player 1 as the set of maximizers of 1's utility, given the strategy for 2. We already have the expression for $u_1(b_1, b_2)$; in order to find out the best response correspondence, all we have to do is maximize this function. Denoting this best response by $R_1(b_2)$, direct maximization of $u(b_1, b_2)$ yields:

$$R_1(b_2) = \begin{cases} \{b_2 + 1\}, & \text{if } b_2 < v_1 - 2 \\ \{b_2, b_2+1\}, & \text{if } b_2 = v_1 - 2 \\ \{b_2\}, & \text{if } b_2 = v_1 - 1 \\ \{0, 1, 2, \ldots v_1\}, & \text{if } b_2 = v_1 \\ \{0, 1, 2, \ldots, b_2 - 1\} & \text{if } b_2 > v_1 \end{cases}$$

Similarly, if $R_2(b_1)$ is the best response correspondence for Player 2:

$$R_2(b_1) = \begin{cases} \{b_1 + 1\}, & \text{if } b_1 < v_2 - 2 \\ \{b_1, b_1+1\}, & \text{if } b_1 = v_2 - 2 \\ \{b_1\}, & \text{if } b_1 = v_2 - 1 \\ \{0, 1, \ldots, v_2\}, & \text{if } b_1 = v_2 \\ \{0, 1, \ldots b_1 - 1\}, & \text{if } b_1 > v_2 \end{cases}$$

A Nash equilibrium is a pair (b_1^*, b_2^*) where $b_1^* \in R_1(b_2^*)$, and $b_2^* \in R_2(b_1^*)$. It can be verified that a NE always exists, and, for sufficiently large values of v_1 and v_2 Nash equilibria are not unique.

More explicitly, it can be shown that, for $v_1, v_2 \geq 2$ the following are Nash equilibria:

(i) If $v_1 = v_2$: (v_1, v_2), (v_1-1, v_2-1), (v_1-2, v_2-2)

(ii) If $v_1 = v_2+1$: (v_1-1, v_1-2), (v_1-2, v_1-2)

(iii) If $v_2 = v_1+1$: (v_2-2, v_2-1), (v_2-2, v_2-2)

(iv) If $v_1 > v_2+1$: (v_1-x, v_1-x-1), with $1 \leq x \leq v_1 - v_2$

Note: if $v_1 = v_2+2$: (v_1-2, v_1-2) is also a NE.

(v) If $v_2 > v_1+1$: (v_2-x-1, v_2-x), with $1 \leq x \leq v_2 - v_1$

Note: if $v_2 = v_1+2$: (v_2-2, v_2-2) is also a NE.

Thus, generally, uniqueness of NE does not hold, although existence does.

8.D.4 (a) If player i demands $y \geq 100$, then any strategy of player j with $x \geq 0$ is payoff equivalent. Therefore, there exists no strictly dominated strategy.

(b) Any strategy demanding more than \$100 is weakly dominated.

Case 1: player 2 demands $y \geq 100$. Then any strategy of player 1 with $x \geq 0$ is payoff equivalent. Case 2: player 2 demands $0 \leq y < 100$. Then, player 1 could demand $x = 100 - y$ and would obtain a payoff of $100 - y$. Demanding $x > 100$ will give player 1 a payoff of 0. Therefore, any strategy demanding more than \$100 is weakly dominated.

(c) Any pair $(x, 100 - x)$ with $100 \geq x \geq 0$ is a pure strategy Nash equilibrium of this game.

Proof: Suppose, player 1 demands x with $100 \geq x \geq 0$. If player 2 demands $y = 100 - x$, his payoff will equal $100 - x \geq 0$. If player 2 demands $y > 100 - x$, the demands sum to more than \$100 and both players get 0. If player 2 demands $0 \leq y < 100 - x$, he will obtain his demand and therefore be worse off than if he would have demanded $100 - x$. Thus, if player 1 demands x, player 2's best response is to demand $y = 100 - x$. Similarly if player 2 demands $100 - x$, player 1's best response is to demand x.

8.D.5 (a) Let x_1 be the location of Vendor 1 and x_2 be the location of Vendor 2. Thus, we can associate a strategy for Player i with $x_i \in [0,1]$. First, let us find out the payoff function for each of the vendors. Since the price of the ice cream is regulated, we can identify the profit of each vendor with the number of customers s/he gets. Suppose that $x_1 < x_2$. In this case, all consumers located to the left of (below) $\frac{x_1 + x_2}{2}$ will purchase from Vendor 1, while all customers located to the right of $\frac{x_1 + x_2}{2}$ will buy ice cream from Vendor 2. Thus:

$$u_1(x_1, x_2) = \frac{x_1 + x_2}{2} \qquad (= \text{length of } [0, \frac{x_1 + x_2}{2}])$$

$$u_2(x_1, x_2) = 1 - \frac{x_1 + x_2}{2} \qquad (= \text{length of } [\frac{x_1 + x_2}{2}, 1])$$

We can derive a similar result for $x_2 < x_1$:

$$u_1(x_1, x_2) = 1 - \frac{x_1 + x_2}{2}$$

$$u_2(x_1, x_2) = \frac{x_1 + x_2}{2}$$

Now, if $x_1 = x_2$, the vendors split the business so that $u_1(x_1,x_2) = u_2(x_1,x_2) = \frac{1}{2}$. Thus, summarizing:

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1 + x_2}{2}, & x_1 < x_2 \\ \frac{1}{2}, & x_1 = x_2 \\ 1 - \frac{x_1 + x_2}{2}, & x_1 > x_2 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} 1 - \frac{x_1 + x_2}{2}, & x_1 < x_2 \\ \frac{1}{2}, & x_1 = x_2 \\ \frac{x_1 + x_2}{2}, & x_1 > x_2 \end{cases}$$

It is straightforward to check that $x_1 = x_2 = 1/2$ constitutes a NE (no firm can do better by deviating). To show uniqueness, suppose first that $x_1 = x_2 < 1/2$. Then any firm can do better by moving by $\varepsilon > 0$ to the right, since it will sell almost $1 - x_1 > 1/2$ units rather than $1/2$ units. Similarly it can be shown that $x_1 = x_2 > 1/2$ does not constitute a NE. Suppose now that

$x_1 < x_2$. Then firm 1 can do better by moving to $x_2 - \varepsilon$, with $\varepsilon > 0$, therefore this could not have been a NE. Similarly it can be shown that $x_1 > x_2$ does not constitute a NE.

(b) Suppose that an equilibrium (x_1^*, x_2^*, x_3^*) exists. Suppose, first, that $x_1^* = x_2^* = x_3^*$. Then each firm will sell 1/3. But any firm can increase its sales by moving to the right (if $x_1^* = x_2^* = x_3^* < 1/2$) or the left (if $x_1^* = x_2^* = x_3^* \geq 1/2$), a contradiction. Suppose that two firms locate at the same point, let's say $x_1^* = x_2^*$. If $x_1^* = x_2^* < x_3^*$, then firm 3 can do better by moving to $x_1^* + \varepsilon$. If $x_1^* = x_2^* > x_3^*$, then firm 3 can do better by moving to $x_1^* - \varepsilon$, a contradiction. Finally, suppose that all 3 firms are located at different points. But then the firm that is located the farthest on the right will be able to increase its sales by moving to the left by $\varepsilon > 0$, a contradiction. Thus, there exists no pure strategy NE in this game.

8.D.6 <u>Case 1</u>: $u > w$ and $l > y$.

In this case player 1 always plays his dominant strategy a_1. Player 2 will play his best response to this strategy, i.e. if $v > m$ he will play b_1, if $v < m$ he will play b_2 and otherwise he will be indifferent.

<u>Case 2</u>: $u < w$ and $l < y$.

In this case player 1 always plays his dominant strategy a_2. Player 2 will play his best response to this strategy, i.e. if $x > z$ he will play b_1, if $x < z$ he will play b_2 and otherwise he will be indifferent.

<u>Case 3</u>: $v > m$ and $x > z$.

In this case player 2 always plays his dominant strategy b_1. Player 2 will play his best response to this strategy, i.e. if $u > w$ he will play a_1, if $u < w$ he will play a_2 and otherwise he will be indifferent.

<u>Case 4</u>: $v < m$ and $x < z$.

In this case player 2 always plays his dominant strategy b_2. Player 2 will play his best response to this strategy, i.e. if $l > y$ he will play a_1, if $l < y$ he will play a_2 and otherwise he will be indifferent.

Case 5: all other cases.

Suppose player 2 plays b_1 with probability α and b_2 with probability $1-\alpha$. Player 1's best response will be a mixed strategy if he obtains the same payoff from playing either of his strategies:

$$\alpha u + (1-\alpha)l = \alpha w + (1-\alpha)y, \quad \Rightarrow \quad \alpha = \frac{y - l}{u + y - l - w}.$$

Similarly suppose player 1 plays a_1 with probability β and a_2 with probability $1-\beta$. Player 2's best response will be a mixed strategy if he obtains the same payoff from playing either of his strategies:

$$\beta v + (1-\beta)x = \beta m + (1-\beta)z, \quad \Rightarrow \quad \beta = \frac{z - x}{v + z - x - m}.$$

Player 1 playing a_1 (a_2) with probability β ($1-\beta$) and player 2 playing b_1 (b_2) with probability α ($1-\alpha$) as defined above is a mixed strategy Nash equilibrium.

8.D.7 (a) Suppose $\underline{w}_i = u_i(\sigma_i^*, \sigma_{-i}^*)$. The following is true by definition:

$$\underline{v}_i = \min_{\sigma_{-i}} [\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})] \geq \min_{\sigma_{-i}} [u_i(\sigma_i^*, \sigma_{-i})] = u_i(\sigma_i^*, \sigma_{-i}^*) = \underline{w}_i.$$

(b) Suppose $(\sigma_i^*, \sigma_{-i}^*)$ is a mixed strategy Nash equilibrium. We have:

$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^*)$ and $u_{-i}(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_{-i}} u_{-i}(\sigma_i^*, \sigma_{-i})$. Since this is a zero-sum game we can rewrite the second equality:

$$-u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_{-i}} [-u_i(\sigma_i^*, \sigma_{-i})] \text{ or, } u_i(\sigma_i^*, \sigma_{-i}^*) = \min_{\sigma_{-i}} u_i(\sigma_i^*, \sigma_{-i}).$$

This yields: $\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^*) = u_i(\sigma_i^*, \sigma_{-i}^*) = \min_{\sigma_{-i}} u_i(\sigma_i^*, \sigma_{-i})$. Now,

$\min_{\sigma_{-i}} [\max_{\sigma_i} u_i(\sigma_i,\sigma_{-i})] \le \max_{\sigma_i} u_i(\sigma_i,\sigma^*_{-i}) = u_i(\sigma^*_i,\sigma^*_{-i})$, and also

$u_i(\sigma^*_i, \sigma^*_{-i}) = \min_{\sigma_{-i}} u_i(\sigma^*_i , \sigma_{-i}) \le \max_{\sigma_i} [\min_{\sigma_{-i}} u_i(\sigma_i , \sigma_{-i})]$. Taking these two inequalities together we get:

$$\underline{v}_i = \min_{\sigma_{-i}} [\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})] \le u_i(\sigma^*_i, \sigma^*_{-i}) \le \max_{\sigma_i} [\min_{\sigma_{-i}} u_i(\sigma_i , \sigma_{-i})] = \underline{w}_i.$$

But we know from (a) above that $\underline{v}_i \ge \underline{w}_i$. Therefore, we must have:

$\underline{v}_i = u_i(\sigma^*_i, \sigma^*_{-i}) = \underline{w}_i$.

(c) Suppose $(\sigma^*_i, \sigma^*_{-i})$ and $(\sigma'_i, \sigma'_{-i})$ are mixed strategy Nash equilibria. We must therefore have that:

(i) $u_i(\sigma^*_i,\sigma^*_{-i}) \ge u_i(\sigma'_i,\sigma^*_{-i}) = -u_{-i}(\sigma'_i,\sigma^*_{-i}) \ge -u_{-i}(\sigma'_i,\sigma'_{-i}) = u_i(\sigma'_i,\sigma'_{-i})$

(ii) $u_i(\sigma^*_i,\sigma^*_{-i}) = -u_{-i}(\sigma^*_i,\sigma^*_{-i}) \le -u_{-i}(\sigma^*_i,\sigma'_{-i}) = u_i(\sigma^*_i,\sigma'_{-i}) \le u_i(\sigma'_i,\sigma'_{-i})$

(these inequalities follow from the properties of NE and from the zero-sum property). From part (b) we know that $u_i(\sigma'_i,\sigma'_{-i}) = u_i(\sigma^*_i, \sigma^*_{-i}) = \underline{v}_i = \underline{w}_i$.
This, together with (i) and (ii) above yield:

$\underline{v}_i = u_i(\sigma'_i,\sigma'_{-i}) \le u_i(\sigma'_i,\sigma^*_{-i}) \le u_i(\sigma^*_i,\sigma^*_{-i}) \le u_i(\sigma^*_i,\sigma'_{-i}) \le u_i(\sigma'_i,\sigma'_{-i}) = \underline{v}_i$

Therefore, $u_i(\sigma'_i,\sigma^*_{-i}) = u_i(\sigma^*_i,\sigma^*_{-i}) = u_i(\sigma^*_i,\sigma'_{-i}) = u_i(\sigma'_i,\sigma'_{-i}) = \underline{v}_i$.

Since $(\sigma^*_i, \sigma^*_{-i})$ is an equilibrium we have: $u_i(\sigma^*_i,\sigma^*_{-i}) \ge u_i(\sigma_i,\sigma^*_{-i}) \quad \forall\ \sigma_i$, so $u_i(\sigma'_i, \sigma^*_{-i}) = u_i(\sigma^*_i, \sigma^*_{-i}) \ge u_i(\sigma_i, \sigma^*_{-i}) \quad \forall\ \sigma_i$. Similarly, $u_{-i}(\sigma'_i,\sigma'_{-i}) \ge u_{-i}(\sigma'_i,\sigma_{-i}) \quad \forall\ \sigma_{-i}$, so $u_{-i}(\sigma'_i,\sigma^*_{-i}) = u_{-i}(\sigma'_i,\sigma'_{-i}) \ge \ge u_{-i}(\sigma'_i,\sigma_{-i}) \quad \forall\ \sigma_{-i}$ (Note, that $u_{-i}(\sigma'_i, \sigma^*_{-i}) = u_{-i}(\sigma'_i, \sigma'_{-i})$ since $u_i(\sigma'_i,\sigma^*_{-i}) = u_i(\sigma'_i,\sigma'_{-i})$ and the game is zero-sum). This implies that $(\sigma'_i,\sigma^*_{-i})$ is a mixed strategy Nash equilibrium. Similarly it can be shown that $(\sigma^*_i, \sigma'_{-i})$ is a mixed strategy NE.

8.D.8 Let (σ_i,σ_{-i}) be a mixed strategy Nash equilibrium, and suppose in negation that σ_i assigns strictly positive probability to the pure strategies s^1_i and s^2_i, i.e. σ_i is not degenerate. This implies that s^1_i and s^2_i are each a best response to σ_{-i} and $u_i(s_i^1,\sigma_{-i}) = u_i(s_i^2, \sigma_{-i})$. By the convexity of S_i, $\alpha s_i^1 + (1-\alpha)s_i^2 \in S_i$, and since u_i is strictly quasiconvex we have that $u_i(\alpha s_i^1 + (1-\alpha)s_i^2, \sigma_{-i}) > u_i(s_i^1, \sigma_{-i}) = u_i(s_i^2,\sigma_{-i})$ for all $\alpha \in (0, 1)$. This contradicts the fact that s_i^1 and s_i^2 are each best response to σ_{-i}. Therefore, any mixed strategy NE of this game must be degenerate.

8.D.9 (a) Playing L or R is quite risky, since we do not know what player 1 will be playing. The risk of obtaining a payoff of -49 is very large compared to the payoff of 1 if player 2 played L, and the risk of obtaining a payoff of -100 is very large compared to the payoff of 2 if player 2 played LL or R. Therefore, it seems "reasonable" to play M.

(b) The two pure Nash equilibria of this game are (U,LL) and (D,R). To check for mixed strategy Nash equilibrium, player 1 must mix between U (with probability p) and D (with probability 1-p). Player 2 then has 11 possible mixing combinations: ⟨LL,L⟩, ⟨LL,M⟩, ⟨LL,R⟩, ⟨L,M⟩, ⟨L,R⟩, ⟨M,R⟩, ⟨M,L,R⟩, ⟨LL,M,R⟩, ⟨LL,L,R⟩, ⟨LL,L,M⟩, and ⟨LL,L,M,R⟩. We will show that only the first combination, ⟨LL,L⟩, is part of a mixed strategy NE.

For player 2 to mix between LL and L (with probabilities q and 1-q respectively), we must have that $p\cdot(2)+(1-p)\cdot(-100) = p\cdot(1)+(1-p)\cdot(-49)$ which yields $p = \frac{51}{52}$. The utility of player 2 from each strategy is then: $u_2(LL) = u_2(L) = \frac{1}{26}$, $u_2(M) = 0$, and $u_2(R) < 0$. Then, for player 1 to mix between U and D, we must have: $q\cdot(100)+(1-q)\cdot(-100) = q\cdot(-100)+(1-q)\cdot(100)$ which yields $q = \frac{1}{2}$, and $u_1(U) = u_1(D) = 0$. Therefore, $p = \frac{1}{26}$ and $q = \frac{1}{2}$ is a mixed strategy NE. For the rest of the answer we call this "NE*". We now show that no other mixing combination of player 2 can be part of a mixed strategy NE.

(i) If player 2 mixes with the combination ⟨LL,M⟩, we must have $p = \frac{50}{51}$, which gives utilities $u_2(LL) = u_2(M) = 0$, and $u_2(L) = \frac{1}{51} > 0$, so this cannot be part of a mixed strategy NE.

(ii) If player 2 mixes with the combination {LL,R}, we must have $p = \frac{1}{2}$, which gives utilities $u_2(LL) = u_2(R) = -49$, and $u_2(M) = 0$, so this cannot be part of a mixed strategy NE.

(iii) If player 2 mixes with one of the combinations {L,M}, {L,R}, {M,R}, {M,L,R}, then player 1 will have D as a strict best response, which in turn has R as player 2's strict best response.

(iv) If player 2 mixes with one of the combinations {LL,M,R}, {LL,L,R}, or {LL,L,M,R}, then the analysis of (ii) above implies that this cannot be part of a mixed strategy NE.

(v) If player 2 mixes with the combination {LL,L,M} then the analysis of (i) above implies that this cannot be part of a mixed strategy NE.

(c) The choice in part a) is not part of any NE described above. It is easy to see that strategy M is rationalizable: If plyer 1 plays $p = \frac{1}{2}$ then M is the unique best response of player 2.

(d) If preplay communication is possible, the players can agree to play one of the pure strategy NE, which are payoff equivalent and Pareto dominant for both players. Therefore, player 2 will play either LL or R depending on the agreed upon equilibrium.

8.E.1 There are four pure strategies contingent on the type of player:

AA: Attack if either weak or strong type,

AN: Attack if strong and Not Attack if weak,

NA: Not Attack if strong and Attack if weak,

NN: Never attack.

The expected payoff of each pair of strategies can be easily computed and are given in Figure 8.E.1:

Player 2

Player 1	AA	AN	NA	NN
AA	$\frac{M}{4}-\frac{s+w}{2}, \frac{M}{4}-\frac{s+w}{2}$	$\frac{M}{2}-\frac{s+w}{4}, \frac{M}{4}-\frac{s}{2}$	$\frac{3M}{4}-\frac{s+w}{4}, \frac{-w}{2}$	$M, 0$
AN	$\frac{M}{4}-\frac{s}{2}, \frac{M}{2}-\frac{s+w}{4}$	$\frac{M-s}{4}, \frac{M-s}{4}$	$\frac{M}{2}-\frac{s}{4}, \frac{M-w}{4}$	$\frac{M}{2}, 0$
NA	$\frac{-w}{2}, \frac{3M}{4}-\frac{s+w}{4}$	$\frac{M-w}{4}, \frac{M}{2}-\frac{s}{4}$	$\frac{M-w}{4}, \frac{M-w}{4}$	$\frac{M}{2}, 0$
NN	$0, M$	$0, \frac{M}{2}$	$0, \frac{M}{2}$	$0, 0$

Figure 8.E.1

Any NE of this normal form game is a Baysian NE of the original game.

<u>Case 1</u>: $M > w > s$, and $w > M/2 > s$

From the above payoffs we can see that (AA,AN) and (AN,AA) are both pure strategy Bayesian Nash equilibria.

<u>Case 2</u>: $M > w > s$, and $M/2 < s$

From the above payoffs we can see that (AA,NN) and (NN,AA) are both pure strategy Bayesian Nash equilibria.

<u>Case 3</u>: $w > M > s$, and $M/2 < s$

From the above payoffs we can see that (AN,AN), (AA,NN) and (NN,AA) are pure strategy Bayesian Nash equilibria.

<u>Case 4</u>: $w > M > s$, and $M/2 > s$

From the above payoffs we can see that (AA,AN), (AN,AA) and (AN,AN) are pure strategy Bayesian Nash equilibria.

8.E.2 (a) Suppose that all the bidders, use the bidding function $b(v)$, that is if their valuation is v they bid $b(v)$. The expected payoff for a bidder whose valuation is v_i is given by:

$(v_i - b(v_i))\cdot \Pr\{b(v_i) > b(v_j)\} + 0\cdot \Pr\{b(v_i) < b(v_j)\}$. (Note that we ignore a tie since it is a zero probability event given that $b(v_i)$ is a monotonic linear function.) Since both players use the same monotonic linear bidding function then $\Pr\{b(v_i) > b(v_j)\} = \Pr\{v_i > v_j\} = v_i / \bar{v}$ (since the valuations are uniformly distributed on $[0,\bar{v}]$. $b(v)$ will in fact be the equilibrium bidding function if it is not better for a player to pretend that his valuation is different. To check this let us solve a bidders problem whose valuation is v_i and who has to decide whether he wants to pretend to have a different valuation v'. The bidder maximizes: $(v_i - b(v))\cdot(v / \bar{v})$, and the FOC is: $(v_i - b(v))/\bar{v} - b'(v)\, v/\bar{v} = 0$. $b(v)$ is an equilibrium bidding function if it is optimal for the bidder not to pretend to have a different valuation, that is, if $v = v_i$ is the optimal solution to the above FOC, i.e., if $(v_i - b(v_i))/\bar{v} - b'(v_i)\, v_i/\bar{v} = 0$. This is a differential equation that has to be satisfied by the bidding function $b(v)$ in order to be an equilibrium bidding function. The solution to this differential equation is $b(v) = v/2$. Thus a bidder whose valuation is v will bid v/2 (a monotonic linear function).

(b) We can proceed as above by assuming that all bidders use the same bidding function $b(v)$. Now, $\Pr\{b(v_i) > b(v_j)\ \forall\ j \neq i\} = = \Pr\{v_i > v_j\ \forall\ j \neq i\} = (v_i)^{I-1}/\ \bar{v}$. Proceeding as in a) above, we get the following differential equation:

$(I-1)(v_i - b(v_i))(v_i)^{I-2}/\bar{v} - b'(v_i)\ (v_i)^{I-1}/\bar{v} = 0$. The solution to this differential equation is $b(v) = \frac{I-1}{I}\cdot v$. As $I \to \infty$, $b(v) = \frac{I-1}{I}\cdot v \to v$, i.e., as the number of players goes to infinity each player will bid his valuation.

8.E.3 A firm of type i = H or L, will maximize its expected profit, taken as given that the other firm will supply q_H or q_L depending whether this firm is of type H or L. A type $i \in \langle H,L \rangle$ firm 1 will maximize:

$$\underset{q_i^1}{\text{Max}}\ (1-\mu)[(a - b(q_i^1 + q_H^2) - c_i)q_i^1] + \mu[(a - b(q_i^1 + q_L^2) - c_i)q_i^1]$$

The FOC yields: $(1-\mu)[a - b(2q_i^1 + q_H^2) - c_i] + \mu[a - b(2q_i^1 + q_L^2) - c_i] = 0$. In a symmetric Bayesian Nash equilibrium: $q_H^1 = q_H^2 = q_H$ and $q_L^1 = q_L^2 = q_L$.

Plugging this into the F.O.C we get the following two eqations:

$$(1-\mu)[a - 3b\ q_H - c_H] + \mu[a - b(2q_H + q_L) - c_H] = 0,$$

$$(1-\mu)[a - b(q_H + 2q_L) - c_L] + \mu[a - 3b\ q_L - c_L] = 0.$$

Therefore, we obtain that

$$q_H = \left[a - c_H + \frac{\mu}{2}(c_L - c_H)\right]\cdot\frac{1}{3b},$$

$$q_L = \left[a - c_L + \frac{1-\mu}{2}(c_H - c_L)\right]\cdot\frac{1}{3b}.$$

8.F.1 For the proof of Proposition 8.F.1, we refer to:

Selten, R. (1975) "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*.

Another good source is Section 8.4 in Fudenberg & Tirole, (1991) *Game Theory*, MIT press.

8.F.2 For the solution of this question we refer to:

van Damme, E. (1983). *Refinements of the Nash Equilibrium Concept*. Berlin: Springer-Verlag (pp. 28-31).

8.F.3 For the proof of this statement, we refer to:

Selten, R. (1975) "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory.*

Another good source is Section 8.4 in Fudenberg & Tirole, (1991) *Game Theory*, MIT press.

CHAPTER 9

9.B.1 There are 5 subgames, each one beginning at a different node of the game (this includes the whole game itself).

9.B.2 **(a)** Clearly if σ° is a Nash equilibrium of Γ_E, and Γ_E is the only proper subgame of Γ_E, then σ° induces a NE in every proper subgame of the game Γ_E. Thus, by definition σ° is a subgame perfect NE of Γ_E.

(b) Assume in negation that σ° is a subgame perfect equilibrium of Γ_E, but it does not induce a subgame perfect Nash equilibrium in every proper subgame of Γ_E. Then there exists a proper subgame (say, Π_E) of Γ_E in which the restriction of σ° to Π_E is not a SPNE. This implies that there exists a proper subgame (say, Ω_E) of Π_E in which the restriction of σ° to Ω_E is not a NE. Since Ω_E is a proper subgame of Π_E and Π_E is a proper subgame of Γ_E, then Ω_E is also a proper subgame of Γ_E. Therefore, σ° does not induce a NE in a proper subgame of Γ_E - contradiction.

9.B.3 Let player 1's pure strategy be $s_1 \in \{L,R\}$, player 2's be $s_2 \in \{a,b\}$, and player 3's be $s_3 = (x,y,z)$ where $x,y,z \in \{l,r\}$, x is what 3 does after player 1 played L, y is what 3 does after 1 played R and 2 played a, and z is what 3 does after 1 played R and 2 played b. The pure strategy SPNE identified in the example is (R,a,(r,r,l)), which is easily seen to be a NE. Three other NE which are not SPNE but yield the same outcome are (R,a,(l,r,l)), (R,a,(l,r,r)) and (R,a,(r,r,r)). For each of these NE, player 3 is not choosing a rational move for some of his nodes. If player 3 is reached, he will always do the best thing for himself, therefore if 1 plays L, 3 will play r. To support this as a

NE we need strategies for 2 and 3 that give player 1 less than -1 in the subgame starting from player 2's node, but recall that these strategies need not be subgame perfect. Therefore, (L,b,(r,r,r)) would be another NE, in which player 3 is again not acting sequentially rational. There can be no NE with player 2 being reached and then choosing b, since after this move 3 will choose l, giving 2 a payoff of -1. Player 2 will then prefer to deviate and play a, resulting in a higher payoff no matter what 3 will do then.

9.B.4 Proposition 9.B.1 claims that in any game, in which no player has the same payoffs at any two terminal nodes, there exists a unique SPNE. Now suppose one of the players has the same payoff at one of the terminal nodes. This means that he will be indifferent between two actions that lead to either one of them. However, since the game is zero-sum, the other player will also be indifferent between the payoffs resulting from these two actions. Thus, in a finite zero-sum game of perfect information, there may exists many different (in terms of the strategies played) SPNE (because of potential indifference of the player between different terminal nodes), but all of them will yield the same payoffs for the players, i.e. there are unique SPNE payoffs.

9.B.5 Note: for parts **(a)** and (b), m_i denotes the number of strategies that player i has. For the remainder of the question, m_i denotes the move of player i. This is done to be consistent with the question.

(a) Since the game is simultaneous, each player i has m_i pure strategies. If we allow for mixed strategies then each player has a continuum of strategies: $\Sigma_i = \{ (p_1,\ldots p_{m_i}) \in \mathbb{R}_+^{m_i} : p_k \geq 0 \ \forall\ k,\ \sum_{k=1}^{m_i} p_k = 1 \}$.

(b) Since player 1 moves first, he cannot make his strategies contingent on any history, thus he still has m_1 (pure) strategies. Player 2 can, however, condition her play on player 1's move, thus allowing her to specify one of m_2 actions for each of player 1's m_1 plays. Therefore, she has $(m_2)^{m_1}$ (pure) strategies. There is, of course, a continuum of mixed strategies.

(c) Assume in negation that for all (m_1,m_2) and (m_1',m_2') where either $m_1 \neq m_1'$ or $m_2 \neq m_2'$ we have that $\phi_i(m_1,m_2) \neq \phi_i(m_1',m_2')$ for both i=1,2. Then, due to lack of indifference (the negation condition) player 2 will have a unique best response at each of her nodes. By backward induction, and by lack of indifference, player 1 will have a unique best response to the SPNE strategy of player 2, which contradicts multiple SPNE.

(d) Since player 2 has no indifference, she will have a unique best response after she is reached. Let (m_1^*,m_2^*) be the NE for the game in (a) yielding a payoff of π_1 to player 1. Clearly, player 1 playing m_1^* and player 2 playing m_2^* at each of her nodes is a NE in the extensive form game. However, this is not necessarily a SPNE since player 2 is not necessarily playing a best response at each of her nodes. Note that in any SPNE player 2 will play m_2^* after player 1 plays m_1^*, therefore player 1 can promise himself a payoff of π_1. Given player 2's unique SPNE strategy in the sequential game, player 1 can therefore do at least as well as the NE (m_1^*,m_2^*) in any SPNE. This conclusion would not necessarily hold for NE of the sequential game - low payoffs for player 1 could be sustained in a NE with incredible threats by player 2.

(e) (1) Consider the normal and extensive form versions of a game depicted in Figure 9.B.5(e.1):

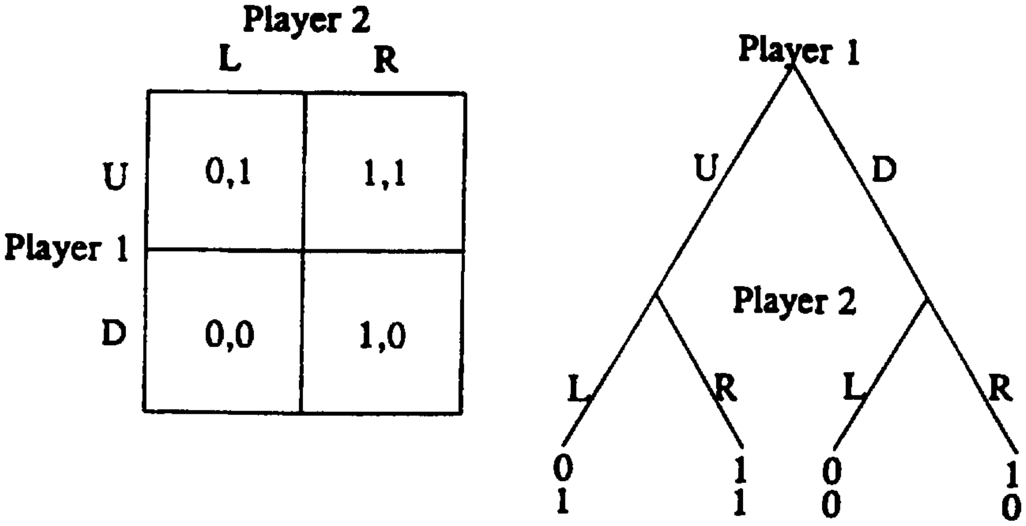


Figure 9.B.5(e.1)

In this game condition (ii) holds for some strategy pairs. It is easy to see (by backward induction) that any path in the extensive form game can be supported by a SPNE. Therefore, if we consider the NE (U,R) in the normal form version, then the conclusion of part (d) does not hold.

(2) Consider the normal and extensive form versions of the "matching pennies" game depicted in figure 9.B.5(e.2):

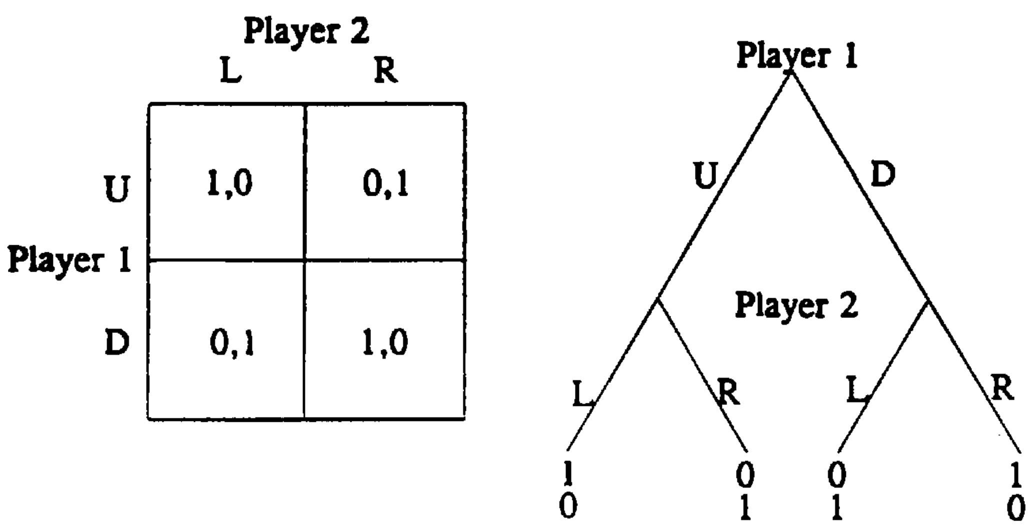


Figure 9.B.5(e.2)

The unique mixed strategy NE in the normal for game gives each player an expected payoff of 1/2. It is easily seen, however, that the only two SPNE in the extensive form game, (U,(R,L)) and (L,(R,L)), give player 1 a payoff of 0. Again, the conclusion of part (d) does not hold.

9.B.6 To find the mixed strategy equilibrium of the post-entry subgame, suppose that firm E plays Small with probability x and Large with probability 1-x; firm I plays Small with probability y and Large with probability 1-y. For firm I to be indifferent between playing Small and Large we need: $-6x - 1(1-x) = 1x - 3(1-x)$, or $x = 2/9$. For firm E to be indifferent between playing Small and Large we need: $-6y - 1(1-y) = 1y - 3(1-y)$, or $y = 2/9$. Thus in the mixed strategy equilibrium of the post-entry game firms E and I play Small with probability 2/9. This gives both firms a payoff of -19/9, which will cause firm E to choose not to enter. Therefore, the following strategies constitute a SPNE: firm E plays Out at the first node and randomizes between Small and Large in the second node (with probabilities 2/9 and 7/9 respectively). Firm I plays Small with probability 2/9 and Large with probability 7/9 given that firm E entered.

9.B.7 Consider the last period of the game. The offering player will offer (v, 0) and the offered player will accept. In the second to last period, the offering player will offer the other player a share that will make him indifferent between accepting now and rejecting and obtaining v in the next period. With the given cost c for making an offer, the offered player in the second to last period will accept any offer that gives him at least v - c. Thus, the offering player in the second to last period offers (c, v-c). Similarly, we can show that the offering player in the third to last period offers (v, 0).

If T is odd, then the player 1 will be the last player to make an offer. Thus, he will offer (v, 0) in every period in which he makes offers and accept an offer if he obtains at least v-c. Player 2 will offer (c, v-c) in every period in which she makes offers and accept if she obtains a non-negative payoff. The result is that in the first period player 1 will offer (v, 0) and player 2 will accept.

If T is even, then the player 2 will be the last player to make an offer. Thus she will offer (0, v) in every period in which she makes offers and accept an offer if she obtains at least v-c. Player 1 will offer (c, v-c) in every period in which he makes offers and accept if he obtains a non-negative payoff. The result is that in the first period player 1 will offer (c, v-c) and player 2 will accept.

The argument above holds for any finite game, but if $T = \infty$, then there can be many SPNE of this game. For an analysis of this case we refer to: Rubinstein, A. (1982) "Perfect Equilibria in a Bargaining Model," *Econometrica*, 50:97:109.

9.B.8 Take all the proper subgames that do not strictly contain another proper subgame. Since the whole game is finite, all proper subgames must be finite. Thus, by Proposition 8.D.2, there exists a mixed strategy NE in each of these subgames. Now construct a new finite game with terminal nodes at the root of each of the previous subgames, and associate the payoff for every player for each such terminal node by the payoff obtained from playing one of the mixed strategy NE in the subgame that followed in the original game. We can again look for all the proper subgames that do not strictly contain another proper subgame. Repeating the above process we can find the mixed strategy Nash equilibria in these subgames. Since the game is finite, repeating the above procedure will end after a finite number of rounds. We have therefore constructed a SPNE of the game. In every subgame players will play the strategies that constitute one of the mixed strategy Nash equilibria in this subgame.

9.B.9 The pure strategy NE of the one-shot game are (a_2, b_2) and (a_3, b_3). Thus any SPNE involves playing either of these in the second period. Thus, playing either of these strategies in both periods constitutes a SPNE. Additionally, the players could use them in any combination in the two periods. This results in the following two classes of SPNE (a total of four SPNE):

1) Player 1 plays a_i and player 2 plays b_i in both periods, $i \in \{2,3\}$.

2) Player 1 plays a_i in the first period and a_j in the second period; player 2 plays b_i in the first period and b_j in the second period, $i,j \in \{1,2\}$ and $i \neq j$.

However, there exist more SPNE in this game. The reason is that player 1 (or 2) can punish the other player by playing a_3 (b_3) in the second period, if the other player did not cooperate in the first period. [Note that this can only happen because there are more than one NE in the second stage]. This gives rise to two more classes of SPNE:

3) Player 1's strategy: Play a_i, $i \in \{1,2,3\}$ in period 1; Play a_2 in period 2 if player 2 played b_1 in period 1, otherwise play a_3.
Player 2's strategy: Play b_1 in period 1. Play b_2 in period 2 if player 1 played a_i in period 1, otherwise play b_3.

4) Player 2's strategy: Play b_i, $i \in \{1,2,3\}$ in period 1; Play b_2 in period 2 if player 1 played a_1 in period 1, otherwise play b_3.
Player 1's strategy: Play a_1 in period 1. Play a_2 in period 2 if player 2 played b_i in period 1, otherwise play a_3.

Too check that each of the 6 SPNE described by these classes in indeed a SPNE, note that by deviating a player looses 4 in the second period (no discounting) and no player can gain more than 3 in any of the described strategy profiles.

9.B.10 The simultaneous move game following the choice of both players to accommodate has the following Nash equilibria: both (U,l) and (D,r) if $x \geq 0$, and only (U,l) if $x < 0$. This implies that if $x < 0$, then the only continuation equilibrium payoffs after both firms accommodate is (3,1), and therefore the whole game will have the same SPNE as discussed in Example 9.B.2, except for the fact that the players will play (U, l) after they both accommodated.

If $x \geq 0$, the SPNE discussed in Example 9.B.3, with the addition that the players will play (U,l) after they both accommodated, will also constitute a SPNE in the modified game.

However, if $x \geq 0$, there will be additional SPNE. Now (D,r) is also a Nash equilibrium in the subgame after both players accommodated. In this case we can "replace" the simultaneous move game after the entry of firm E by the game in figure 9.B.10:

		Firm I: Acc.	Firm I: Fight
Firm E	Acc.	x,3	-2,-1
	Fight	1,-2	-3,-1

Figure 9.B.10

Suppose first that $x \geq 1$. Then player E playing In, both players accommodating, and then playing (D,r) constitutes another SPNE.

Now, suppose $0 \leq x < 1$. For the given value of x, this subgame no longer has a pure strategy equilibrium. The mixed strategy Nash equilibrium of this subgame is as follows: Firm E plays Accommodate with probability 1/5 and Fight with probability 4/5. Firm I plays Accommodate with probability $1/(2-x)$ and Fight with probability $(1-x)/(2-x)$. Firm E's payoff in the subgame after its entry is therefore $(3x-2)/(2-x)$, which for $0 \leq x < 2/3$ is negative. This analysis implies the following additional SPNE:

If $0 \leq x < 2/3$, firm E will play Out at the first node, given that when firm E plays In, the firms play the mixed strategy NE described above in the reduced continuation game.

If $2/3 < x < 1$, firm E will play In at the first node, given that when firm E plays In, the firms play the mixed strategy NE described above in the reduced continuation game.

If $x = 2/3$, firm E will randomize between playing In and Out at the first node, given that when firm E plays In, the firms play the mixed strategy NE described above in the reduced continuation game.

9.B.11 Direct calculation show that it is not worth for firm A to stay in the market for $t > 20$, even if it has a monopoly position. Similarly, firm B will not benefit by staying in the market if $t > 25$. Consider period 20. This is the last period in which A could still be in. If B continues to stay in, it will obtain monopoly profits during periods 21 through 25 of:

$$(51-2\cdot21) + (51-2\cdot22) + (51-2\cdot23) + (51-2\cdot24) + (51-2\cdot25) = 25.$$

If firm A were to stay in at period 20, then B will obtain its duopoly profits of $10.5 - 20 = -9.5$ in period. The net profit for firm B of staying in at period 19 is therefore $25 - 9.5 = 15.5 > 0$. Thus, at period 20 firm B will stay in no matter if firm A is in the market or not. This implies that if firm A stays in at period 20 it will make a loss of $(105 - 10\cdot 20) = -95$, which in turn implies that firm A will stay out from period 19 on.

The above argument can be applied backward and we will get the (unique) SPNE of this game. However, the following reasoning applies: The analysis above shows that with rational play firm 2 will exit after firm 1. Therefore, as far as firm 1 is concerned, it will never make monopoly profits. It follows that we only need calculate the last period in which firm 1 makes positive duopoly profits and this will be the last period in which both firms enter the market. In later periods, firm 2 will be in the market alone (until period 25). The unique SPNE is: For t = 1,2, ...,10 both firms stay in; in t = 11, 12, ..., 25 A is out an B stays in; for $t \geq 26$ both firms are out.

9.B.12 The unique SPNE strategies are as follows:

The offering player i offers $\left[\dfrac{(1-\delta_j)}{(1-\delta_i\delta_j)}\cdot v,\ \dfrac{(\delta_j-\delta_i\delta_j)}{(1-\delta_i\delta_j)}\cdot v\right]$. The offered player j accepts if and only if he is given a share of at least $\dfrac{(\delta_j-\delta_i\delta_j)}{(1-\delta_i\delta_j)}\cdot v$. To show that these strategies form a SPNE, suppose that the offered player j does not accept the offer. Then he will offer $\dfrac{(1-\delta_i)}{(1-\delta_i\delta_j)}\cdot v$ in the next period and the current offered player i will accept (if j offers less then i won't accept given the strategies). Player j's payoff will be $\delta_j\cdot\dfrac{(1-\delta_i)}{(1-\delta_i\delta_j)}\cdot v = \dfrac{(\delta_j-\delta_i\delta_j)}{(1-\delta_i\delta_j)}\cdot v$. Therefore he did not benefit by deviating. Suppose the offering player i deviates and uses a different strategy. He clearly does not benefit by offering more than $\dfrac{(\delta_j-\delta_i\delta_j)}{(1-\delta_i\delta_j)}\cdot v$ to player j. Suppose then that he offers less.

Player j will not accept, and offer $\frac{(\delta_i-\delta_i\delta_j)}{(1-\delta_i\delta_j)}\cdot v$ to player i in the next period. Player i will then accept and obtain a payoff of $\delta_i\cdot\frac{(\delta_i-\delta_i\delta_j)}{(1-\delta_i\delta_j)}\cdot v$ = $\delta_i^2\cdot\frac{(1-\delta_i)}{(1-\delta_i\delta_j)}\cdot v < \frac{(1-\delta_j)}{(1-\delta_i\delta_j)}\cdot v$. Thus he will choose not to deviating from the above strategy.

9.B.13 In the exercise with the cost of delay c being equal for both players, there will generally be many SPNE of this game. For an analysis of this case we refer to:

Rubinstein, A. (1982) "Perfect Equilibria in a Bargaining Model," *Econometrica*, 50:97-109 (pp. 107-108).

9.B.14 (a) The extensive form of the game is depicted in figure 9.B.14(a) below (the letter "d" is used instead of δ). Simple backward induction leads to the unique SPNE which is shown by arrows in the figure: Firm E enters at t=0, and always plays In thereafter. Firm I plays Accommodate for all t=1,2,3.

(b) The extensive form of the modified game is depicted in figure 9.B.14(b). Using Backward induction, firm I will always play Accommodate in period t=3, and therefore if t=3 is reached, firm E will play In. This causes firm I to choose Fight in t=2 since this causes firm E to exit the market, and due to condition A.2 we have that for firm I:

$z + \delta y + \delta^2 x = z + \delta(y + \delta x) > z + \delta(1 + \delta)z = z + \delta z + \delta^2 z$.

This causes firm E to choose Out in t=2. Working backward we get that at t=1, firm I chooses Accommodate and firm E chooses In. However, the choice of firm E at t=0 depends on the value of k. If k > 1 then firm E will choose not to enter, and if k < 1 then E will enter. For k = 1 both are part of the (unique) continuation SPNE, so there are two SPNE in this case.

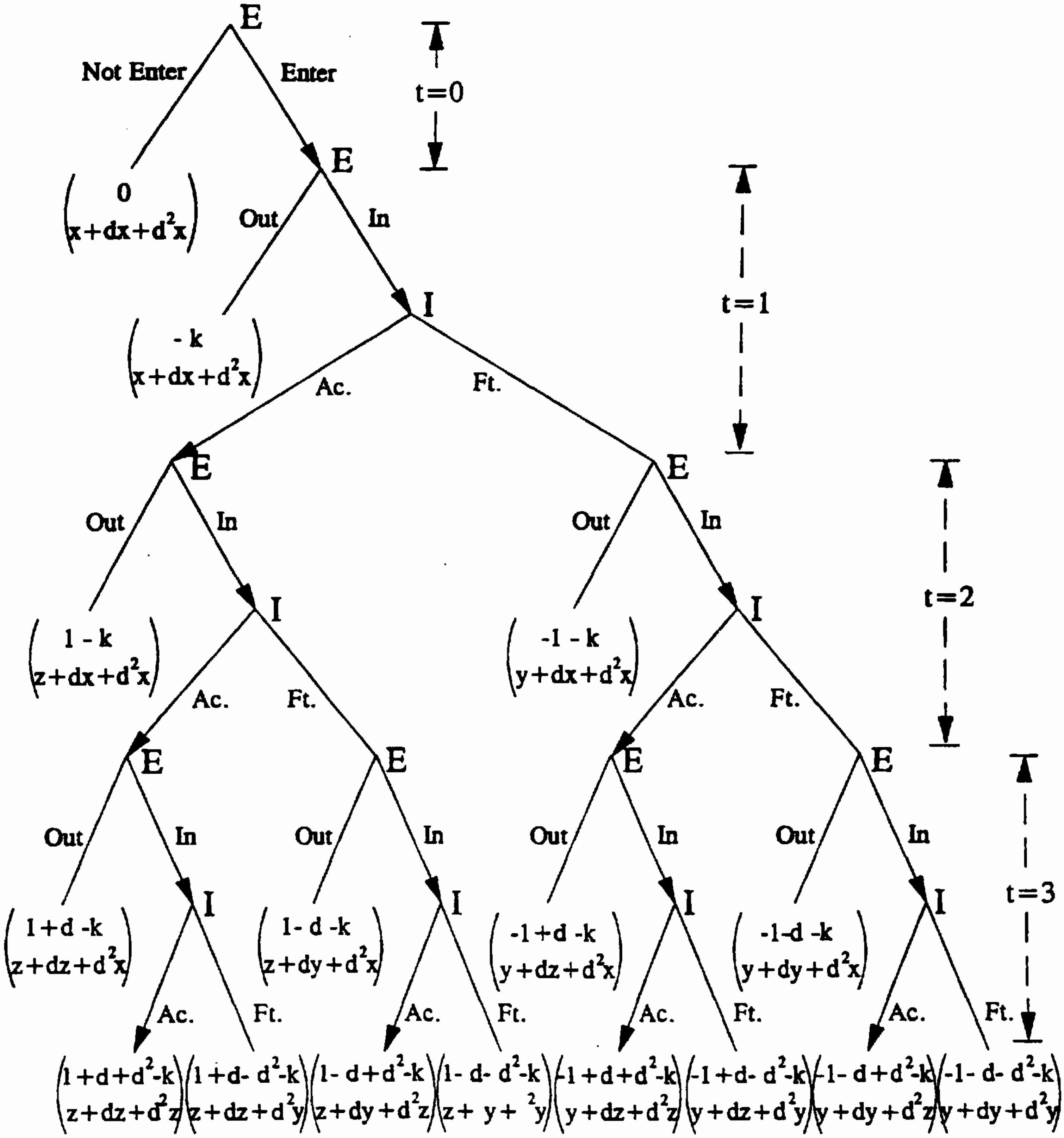


Figure 9.B.14(a)

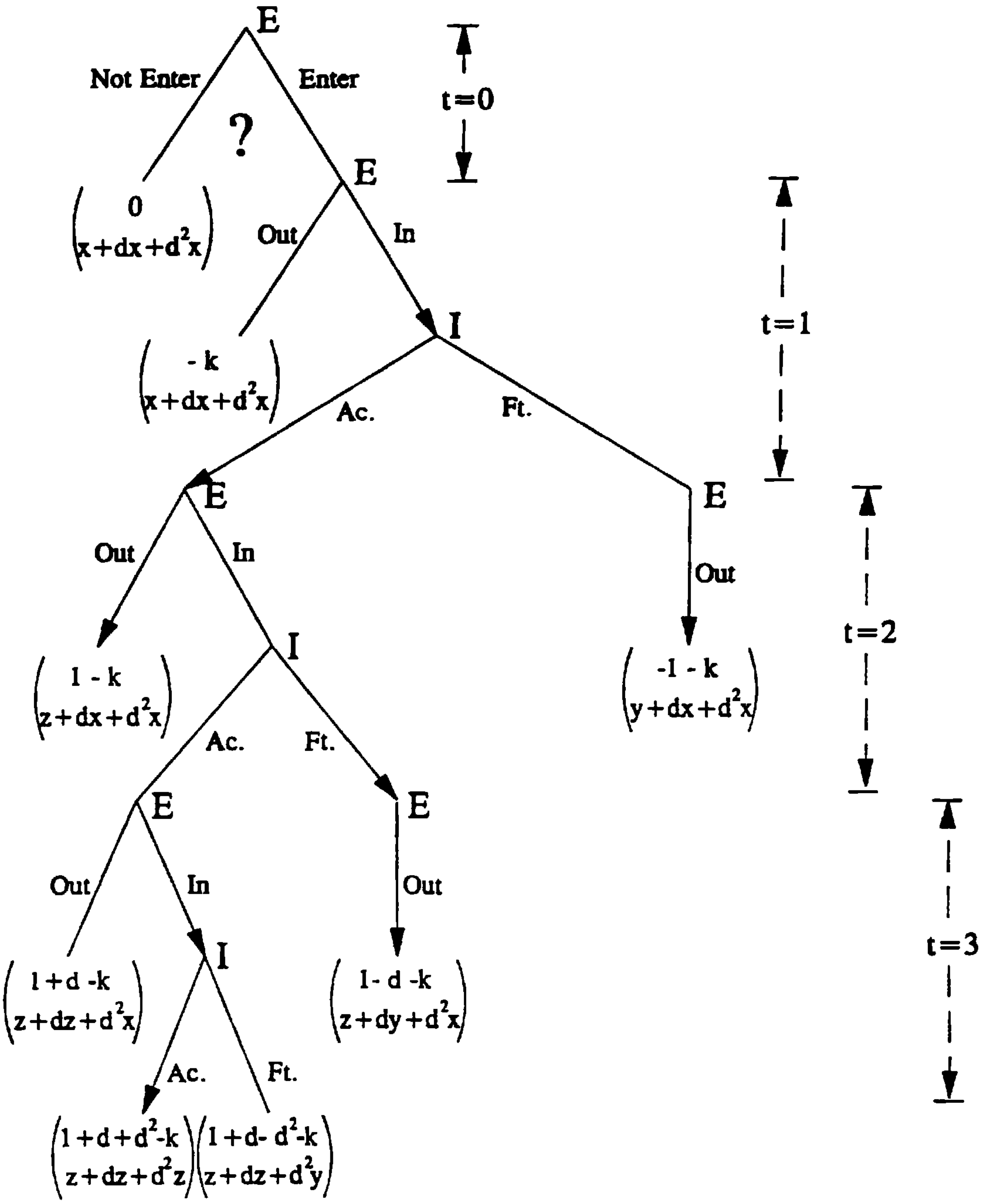


Figure 9.B.14(b)

9.C.1 First, if σ is a NE then (i) and (ii) must hold. If (i) weren't satisfied, then some player's information set is reached with positive probability in which he is not playing a best response to his opponents' strategies, contradicting the fact that σ is a NE. If (ii) were not satisfied then some player's information set is reached with positive probability in which his beliefs are not correct given his, and his opponents' strategies. Therefore, this cannot be an equilibrium.

Second, if (i) and (ii) are satisfied, then σ is clearly a NE since at each information set which is reached with positive probability, sequential rationality implies that each player is playing a best response to his opponents' strategies, with correct beliefs at each such information set.

9.C.2 Let σ_F be the probability that firm I fights after entry, let μ_1 be firm I's belief that In_1 was E's strategy if entry has occurred and let σ_0, σ_1, σ_2 denote the probabilities with which firm E actually chooses Out, In_1, and In_2 respectively.

As in example 9.C.3, we cannot have $\mu_1 < \frac{2}{3}$ (same analysis). We could, however, have $\mu_1 > \frac{2}{3}$: In this case firm I will play "Fight" with probability 1, and having $\gamma<0$ implies that firm E will choose "Out", which supports any beliefs in the information set of firm I. This concludes that one class of weak perfect Bayesian equilibria is with $\mu_1 \geq \frac{2}{3}$, firm E playing "Out" and firm I playing "Fight".

The second class of weak PBE is where $\mu_1 = \frac{2}{3}$, and the analysis is as in example 9.C.3 with $\sigma_F = 1/(\gamma+2)$, and $(\sigma_0, \sigma_1, \sigma_2) = (0, \frac{2}{3}, \frac{1}{3})$. This, however, is an equilibrium only if γ is "large enough" in the interval $(-1,0)$. When γ is close to -1, then σ_F is close to 1, and the expected payment for firm E of the strategy $(\sigma_0,\sigma_1,\sigma_2) = (0,\frac{2}{3},\frac{1}{3})$ is close to -1, therefore E would choose "Out" with probability 1. If, on the other hand, γ is close to 0, then σ_F is close to $\frac{1}{2}$, and the expected payment for firm E of the strategy $(\sigma_0,\sigma_1,\sigma_2) = (0,\frac{2}{3},\frac{1}{3})$ is close to 1, and E would be happy with the mixed strategy described. The set of γ's for which the equilibrium of example 9.C.3 will still be an equilibrium can be calculated.

9.C.3 (a) For given values of λ and δ, there exists the following weak perfect Bayesian equilibrium (WPBE):

<u>Case</u> <u>1</u>: $\lambda[(1-\delta)v_H + \delta v_L] + (1-\lambda)\delta v_L \geq v_L$.

The seller will demand a price of $(1-\delta)v_H + \delta v_L$ in the first period and v_L in the second period. A buyer of type L will buy the good if the price is at most v_L. A buyer of type H will buy the good in the first period if the price is at most $(1-\delta)v_H + \delta v_L$ and in the second period if the price is at most v_H.

To show that these strategies constitute an equilibrium suppose the high type buyer deviates and does not buy in the first period. Then he will buy the good in the second period and obtain a payoff of $\delta(v_H - v_L)$. Buying in the first period gives him a payoff of $v_H - (1-\delta)v_H - \delta v_L = \delta(v_H - v_L)$. Thus he does not benefit by deviating. A player of type L does not benefit by deviating and accepting in the first period since he will obtain a negative payoff. The seller will not benefit by demanding a higher price in the first period since nobody will buy at that price (both types will then buy in the second period). Suppose he offers a lower price. If this price is higher than v_L, then only the high type will buy, and the seller will obtain a lower price without increasing his sales volume. Suppose he offers the good for v_L, then both types will buy the good and the seller will make a profit of v_L. If he uses the above strategy he will make a profit of $\lambda[(1-\delta)v_H + \delta v_L] + (1-\lambda)\delta v_L \geq v_L$, therefore he will not deviate.

<u>Case 2</u>: $\lambda[(1-\delta)v_H + \delta v_L] + (1-\lambda)\delta v_L < v_L$.

The seller will offer the good for v_L in every period, and both types buy the good in period 1.

Clearly, both types of buyers will not benefit by deviating, since they obtain the good for the "lowest possible price" v_L, and the analysis of Case 1 shows that the seller will not deviate with the assumption of case 2.

(b) Suppose the buyer offers the good for the price x in the first period and y in the second period (clearly, $x,y \in [\underline{v},\bar{v}]$). Then there exists a unique type v^* who is indifferent between buying the good in the first period or in the second period, $v^* - x = \delta(v^* - y)$, or $v^* = \dfrac{(x - \delta y)}{(1-\delta)}$. Given that all types $v \geq v^*$ will buy the good in period 1, in equilibrium we must have that the price y was set to maximize the second period profits:

$$y \cdot \Pr\{v \geq y \mid v < v^*\} = y \cdot \left(1 - \frac{y - \underline{v}}{v^* - \underline{v}} \right) = \frac{y(v^* - y)}{v^*} .$$

The FOC implies that the optimum will be attained at $y = \frac{v^*}{2}$. Plugging this into the expression for v^*, we obtain: $v^* = \frac{x}{(1 - \delta/2)}$. The expressions for y and v^* obtained imply that if the seller charges x in the first period he will charge $\frac{x}{2 - \delta}$ in the second period, all buyers of type $v \geq v^* = \frac{x}{(1 - \delta/2)}$ will buy in the first period, and all types with $\frac{x}{(1 - \delta/2)} \geq v \geq \frac{x}{2 - \delta}$ will buy in the second period. The seller's profits will thus be:

$$\Pi = \left[\bar{v} - \frac{x}{(1 - \delta/2)} \right] \cdot x + \delta \cdot \frac{x}{2 - \delta} \cdot \left[\frac{x}{(1 - \delta/2)} - \frac{x}{2 - \delta} \right]$$

$$= \left[\bar{v} - \frac{x}{(1 - \delta/2)} \right] \cdot x + \delta \cdot \left[\frac{x}{2 - \delta} \right]^2$$

The optimal selling price for the first period can be obtained by maximizing

Summing up, the weak perfect Bayesian equilibrium of this game is as follows: The seller offers the good for $x = \dfrac{\bar{v}(2 - \delta)^2}{8 - 6\delta}$ in the first period and for $y = \dfrac{\bar{v}(2 - \delta)}{8 - 6\delta}$ in period 2. All buyers of type $v \in [\dfrac{\bar{v}(2 - \delta)}{4 - 3\delta}, \bar{v}]$ will buy in period 1, and all buyers of type $v \in [\dfrac{\bar{v}(2 - \delta)}{8 - 6\delta}, \dfrac{\bar{v}(2 - \delta)}{4 - 3\delta}]$ will buy in period 2.

9.C.4 [First Printing Errata: in part (c) it should read "Now allow Ms. P, after..."]

For an analysis of this problem see:

Nalebuff, B. (1987) "Credible Pretrial Negotiation," *Rand Journal of Economics*, 18(2):198-210.

9.C.5 For an analysis of this problem see:

Nalebuff, B. (1987) "Credible Pretrial Negotiation," *Rand Journal of Economics*, 18(2):198-210.

9.C.6 Example 9.C.3: Since the example restricts attention to $\gamma > 0$, there is a unique Weak Perfect Bayesian Equilibrium (WPBE). Since every Sequential Equilibrium (SE) is a WPBE, this is the only candidate for a SE. Let the sequences of strategies be: For firm I, $(\sigma_F^n, \sigma_A^n) = (\frac{1}{2+\gamma} - \frac{1}{n}, \frac{1}{n})$, and for firm E, $(\sigma_0^n, \sigma_1^n, \sigma_2^n) = (\frac{1}{n}, \frac{2}{3} - \frac{1}{2n}, \frac{1}{3} - \frac{1}{2n})$, where $n = 1,2,\ldots$. Clearly, these strategies converge to the WPBE strategies described in the textbook. The sequence of strategies for firm E generates a unique sequence of beliefs that firm E played In_1 as follows:

$$\mu_1^n = \frac{\frac{2}{3} - \frac{1}{2n}}{\frac{2}{3} - \frac{1}{2n} + \frac{1}{3} - \frac{1}{2n}} = \frac{4n - 3}{6n - 6} \xrightarrow[\infty]{n} \frac{2}{3}.$$

Thus, the WPBE described is a SE.

$$u_p(s,\lambda) = \begin{cases} \lambda\pi - c_p & \text{for } s \geq \lambda\pi - c_d \\ s & \text{for } s \leq \lambda\pi - c_d \end{cases} \qquad u_d(s,\lambda) = \begin{cases} -\lambda\pi - c_d & \text{for } s \geq \lambda\pi - c_d \\ -s & \text{for } s \leq \lambda\pi - c_d \end{cases}$$

Therefore, Ms. P's expected payoff for an offer s will be given by (recall that beliefs for Ms. P are determined by $f(\lambda)$):

$$E_\lambda u_p(s,\lambda) = \int_0^{(s-c_d)/\pi} (\lambda\pi - c_p) f(\lambda) d\lambda + \int_{(s-c_d)/\pi}^{1} s f(\lambda) d\lambda = \int_0^{(s-c_d)/\pi} (\lambda\pi - c_p) f(\lambda) d\lambda + s\left[1 - F\left(\frac{s-c_d}{\pi}\right)\right]$$

Ms. P will choose s to maximize this expression, and the FOC (assuming an interior solution) yields the pure strategy of Ms. P who solves:

$$\text{(i)} \quad 1 - F\left(\frac{s-c_d}{\pi}\right) = \frac{c_d+c_p}{\pi} \cdot f\left(\frac{s-c_d}{\pi}\right).$$

(b) Implicit differentiation of (i) above yields

$$\text{(ii)} \; - \left[\frac{c_d+c_p}{\pi^2} \cdot f'\left(\frac{s-c_d}{\pi}\right) + \frac{1}{\pi} \cdot f\left(\frac{s-c_d}{\pi}\right)\right] ds + \frac{1}{\pi} \cdot f\left(\frac{s-c_d}{\pi}\right) dc_p - \left[\frac{c_d+c_p}{\pi^2} \cdot f'\left(\frac{s-c_d}{\pi}\right)\right] dc_d$$

$$\left[\frac{s+c_p}{\pi^2} \cdot f\left(\frac{s-c_d}{\pi}\right) + \left(\frac{(c_d+c_p)(s-c_d)}{\pi^3}\right) \cdot f'\left(\frac{s-c_d}{\pi}\right)\right] d\pi = 0$$

which will determine the change in the optimal value of s given changes in any of the parameters. For example, if λ were uniformly distributed on the interval [0,1] then the expression in (ii) reduces to $s = \pi - c_p$ so that we have $\frac{ds}{d\pi} = 1$, $\frac{ds}{dc_p} = -1$, and $\frac{ds}{dc_d} = 0$.

(c) For an analysis of this part see:

Nalebuff, B. (1987) "Credible Pretrial Negotiation," *Rand Journal of Economics*, 18(2):198-210.

9.C.5 For an analysis of this problem first see exercise 9.C.4 above (for a guideline on how to solve such a problem), and then consult:

Nalebuff, B. (1987) "Credible Pretrial Negotiation," *Rand Journal* *of Economics*, 18(2):198-210.

9.C.6 Example 9.C.3: Since the example restricts attention to $\gamma > 0$, there is a unique Weak Perfect Bayesian Equilibrium (WPBE). Since every Sequential Equilibrium (SE) is a WPBE, this is the only candidate for a SE. Let the sequences of strategies be: For firm I, $(\sigma_F^n, \sigma_A^n) = (\frac{1}{2+\gamma} - \frac{1}{n}, \frac{1}{n})$, and for firm E, $(\sigma_O^n, \sigma_1^n, \sigma_2^n) = (\frac{1}{n}, \frac{2}{3} - \frac{1}{2n}, \frac{1}{3} - \frac{1}{2n})$, where n = 1,2,.... Clearly, these strategies converge to the WPBE strategies described in the textbook. The sequence of strategies for firm E generates a unique sequence of beliefs that firm E played In_1 as follows:

$$\mu_1^n = \frac{\frac{2}{3} - \frac{1}{2n}}{\frac{2}{3} - \frac{1}{2n} + \frac{1}{3} - \frac{1}{2n}} = \frac{4n-3}{6n-6} \xrightarrow[\infty]{n} \frac{2}{3}.$$

Thus, the WPBE described is a SE.

Example 9.C.4: The WPBE described in the textbook cannot be a SE since the only beliefs consistent with *any* mixed strategy of player 1 is $(\frac{1}{2},\frac{1}{2})$ in player 2's information set. Thus the unique SE must have player 2 believe that if his information set is reached then each node is as likely as the other. This causes him to play "r" if his information set is reached (since $\frac{1}{2}\cdot 10 + \frac{1}{2}\cdot 2 > 5$), which in turn induces player 1 to play "y". This is the unique SE (which is also a WPBE).

Example 9.C.5: The WPBE described in the text can be supported as a SE. Let a strategy of firm E be $(\sigma_O, \sigma_F, \sigma_A)$, the probabilities of playing (Out), (In,Fight), and (In,Accommodate) respectively. Let a strategy of firm I be (σ_f, σ_a), the probabilities of playing Fight, and Accommodate respectively (if reached). Consider the following sequences of strategies:For firm E, $(\sigma_O^n, \sigma_F^n, \sigma_A^n) = (1 - \frac{1}{n}, \frac{n-1}{n^2}, \frac{1}{n^2})$, and for firm I, $(\sigma_f^n, \sigma_a^n) = (1 - \frac{1}{n}, \frac{1}{n})$.

These sequences converge to the strategies described, and furthermore generate the belief sequence:

$$\mu_F^n = \frac{\frac{n-1}{n^2}}{\frac{n-1}{n^2}+\frac{1}{n^2}} \xrightarrow[\infty]{n} 1.$$

This supports the described strategies as a SE.

Another SE (and WPBE) is where firm E plays (In,Accommodate) and firm I plays Accommodate. This is supported by the sequences of strategies: For firm E, $(\sigma_O^n,\sigma_F^n,\sigma_A^n) = (\frac{1}{2n}, \frac{1}{2n}, 1-\frac{1}{n})$, and for firm I, $(\sigma_f^n,\sigma_a^n) = (\frac{1}{n}, 1-\frac{1}{n})$.

9.C.7 (a) The extensive form game is given in figure 9.C.7(a).

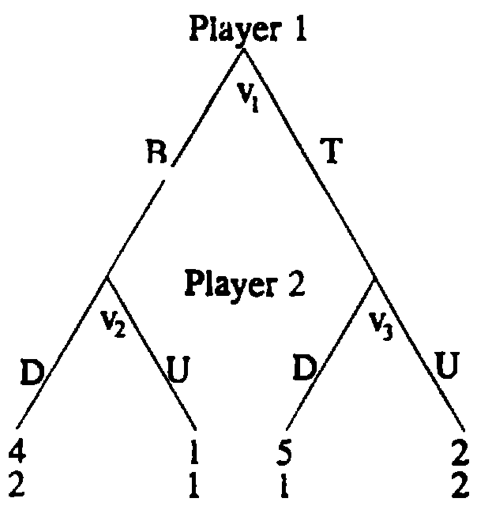


Figure 9.C.7(a)

The set of pure strategies for player 1 is S_1= {B,T}, and for player 2 is S_2= {DD,DU,UD,UU} where playing AB means playing A at node v_2 and B at node v_3. By backward induction it is easy to see that the unique SPNE is (B,DU). There are two more classes of NE: (i) Player 1 plays T, and player 2 plays UU with probability p and DU with probability 1-p, with $p \geq \frac{2}{3}$, and (ii) Player 1 plays B, and player 2 plays DU with probability p and DD with probability 1-p, with $p \geq \frac{1}{3}$.

(b) In this case the extensive form game, and its equivalent normal form game, are depicted in figure 9.C.7(b).

		Player 2	
		D	U
Player 1	T	5,1	2,2
	B	4,2	1,1

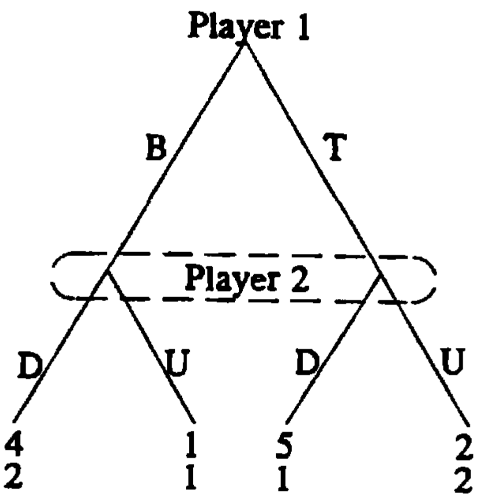


Figure 9.C.7(b)

Since playing T is a strictly dominant strategy for player 1, we have a unique NE: (T,U).

(c) The modified game is depicted in figure 9.C.7(c).

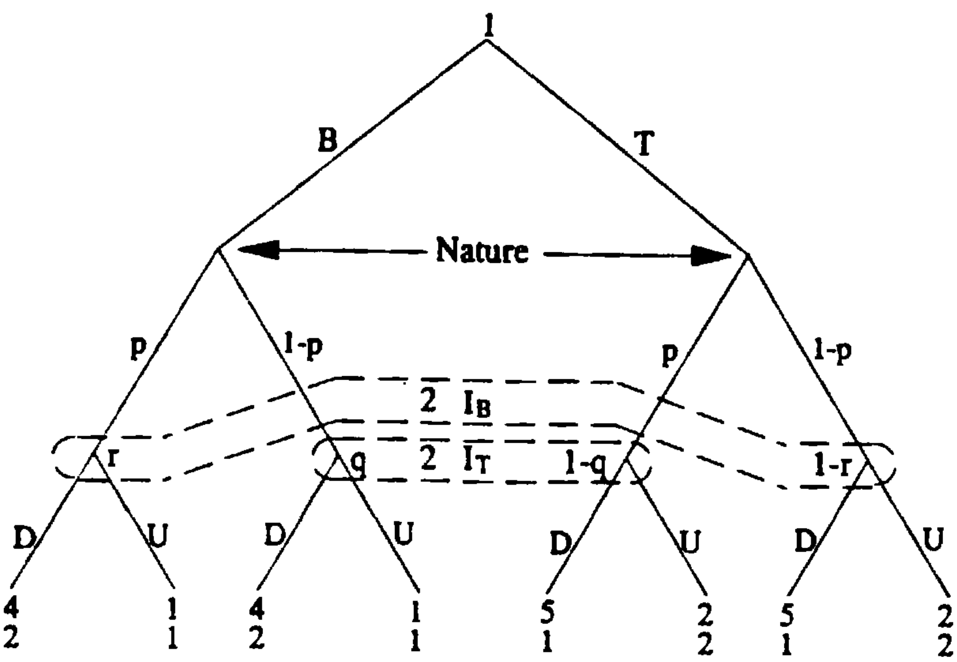


Figure 9.C.7(c)

I_k denotes player 2's information set after she observes $k \in \{B,T\}$, r is the probability she assigns to the event that player 1 played B after she finds herself in information set I_B, and similarly q is the probability she assigns to the event that player 1 played B after she finds herself in information set

I_T. Let $s \in [0,1]$ denote the probability that player 1 plays B. We can have three possible situations in a WPBE: First, player 1 playing $s = 1$; second, player 1 playing $s = 0$; and third, player 1 playing $s \in (0,1)$. Player 1 playing $s = 1$ cannot be part of a WPBE. Indeed, if this were the case we must have $q = r = 1$, which implies that player 2 will always play D. But given that 2 always plays D, player 1 will prefer to deviate and play T. Second, player 1 playing $s = 0$ is part of a WPBE. Indeed, if this is the case we must have $q = r = 0$, which implies that player 2 will always play U, and given that 2 always plays U, player 1 will prefer to to play T. Thus, player 1 playing T and player 2 playing U in each of her information sets is a WPBE.

To consider the possibility of a WPBE with $s \in (0,1)$, we first note that this will induce a unique pair of probability beliefs q and r derived by Bayes rule. In particular, in such an equilibrium we must have:

$$r = \frac{s \cdot p}{(1 - s)(1 - p) + s \cdot p}, \text{ and } q = \frac{s(1-p)}{s(1 - p) + p(1 - s)}.$$

Simple algebra shows that $s \gtreqless p$ if and only if $q \gtreqless \frac{1}{2}$, and that $s \gtreqless (1 - p)$ if and only if $r \gtreqless \frac{1}{2}$. This observation allows us to concentrate on 4 cases as follows:

(i) $\underline{s > p \text{ and } s > (1-p)}$: In this case we must have $q > \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will always play D, which in turn implies that player 1's best response is $s = 0$. Therefore there cannot be a WPBE in this case.

(ii) $\underline{s < p \text{ and } s < (1-p)}$: In this case we must have $q < \frac{1}{2}$ and $r < \frac{1}{2}$. This implies that player 2 will always play U, which in turn implies that player 1's best response is $s = 0$. This coincides with the pure strategy WPBE described earlier.

(iii) $\underline{(1-p) < s < p}$: (which implies $p > \frac{1}{2}$) In this case we must have $q < \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will play U in information set I_T and will play D in information set I_B. Player 1's best response will now depend on p.

Playing B will give 1 an expected payoff of $4p + 1(1-p)$, and playing T will give him $2p + 5(1-p)$. If $p \neq \frac{2}{3}$ then player 1 will have a unique best response which rules out such WPBE. However, if $p = \frac{2}{3}$ then we have a mixed strategy WPBE as follows: player 1 plays B with probability $s \in (\frac{1}{3}, \frac{2}{3})$, and player 2 will play U in information set I_T and will play D in information set I_B.

(iv) <u>$p < s < (1-p)$</u>: (which implies $p < \frac{1}{2}$) This case is symmetric to case (iii) above. If $p \neq \frac{1}{3}$ then player 1 will have a unique best response which rules out such WPBE. However, if $p = \frac{1}{3}$ the we have a mixed strategy WPBE as follows: player 1 plays B with probability $s \in (\frac{1}{3}, \frac{2}{3})$, and player 2 will play D in information set I_T and will play U in information set I_B.

To conclude, there exists a unique pure strategy WPBE as described earlier, and if p is randomly drawn from the interval (0,1) then the pure strategy WPBE is the unique WPBE with probability 1. However, if $p = \frac{1}{3}$ or $p = \frac{2}{3}$ then in addition there exists a mixed strategy WPBE as described in cases (iii) and (iv) above.

9.D.1 In a similar manner to the proof of proposition 9.B.1, let N be the maximal number of moves from a terminal node to the root of the tree. The statement is clearly true for N = 1 because only one player will have to play at this node, and since no two terminal nodes give him the same payoffs, he will have a unique optimal strategy that is not weakly dominated. Now suppose that the result is true for $N = n-1$, and consider the case where $N = n$. Let the set of final decision nodes (those that are before the terminal nodes) be V^0. Since for every player no two terminal nodes yield the same payoffs, then at each node $v_k \in V^0$ there is a unique optimal strategy for the player at that node. Let σ_k be this unique strategy. We can therefore associate with each node $v_k \in V^0$ a payoff to all players given σ_k. These payoffs induce a game with $N = n-1$, and by our hypothesis this game has a unique prediction obtained through iterated removal of weakly dominated strategies. Playing the optimal strategy in the n'th node given the unique prediction of the $N = n-1$ game will give us the unique prediction obtained through iterated removal of weakly dominated strategies.

CHAPTER 10

10.B.1. **(a)** Suppose that a feasible allocation (x,y) is strongly Pareto efficient, and take any allocation (x',y') for which $u_i(x_i') > u_i(x_i)$ for all i. Then, by strong Pareto efficiency of (x,y), (x',y') cannot be feasible. This, (x,y) must also be weakly Pareto efficient.

(b) [<u>First printing errata</u>: omit the assumption of inferiority, it plays no role given that $X_1 = X_2 = \mathbb{R}_+$ for all i.]

Suppose that a feasible allocation (x,y) is not strongly Pareto efficient, i.e. that there exists a feasible allocation (x',y') for which$u_i(x_i') \geq u_i(x_i)$ for all i, and $u_k(x_k') > u_k(x_k)$ for some k. Since $X_i = \mathbb{R}^L_+$ and preferences are strongly monotone, we must also have $u_k(x_k') > u_k(x_k) \geq u_k(0)$,and therefore $x_k' \neq 0$. Therefore, we must have $x_{ks}' > 0$ for at least one commodity s.

Let

$x_{il}'' = x_{il}'$ for $i \neq k$, $l \neq s$;

$x_{is}'' = x_{is}' + 1/(I-1)\ \varepsilon$ for $i \neq k$;

$x_{kl}'' = x_{kl}'$ for $l \neq s$;

$x_{ks}'' = x_{ks}' - \varepsilon$.

Intuitively, we are redistributing a small amount ε of commodity s from consumer k to all the other consumers, so that they become better off than under the original allocation (x,y). Observe that as long as $0 < \varepsilon < x_{ks}'$, (x'', y') will be a feasible allocation. By strong monotonicity of preferences, we must have $u_i(x_i'') > u_i(x_i') \geq u_i(x_i)$ for $i \neq k$, for any $\varepsilon > 0$. By continuity of $u_k(\cdot)$, we must have $u_k(x_k'') > u_k(x_k)$ for $\varepsilon > 0$ small enough. Therefore, we can find an $\varepsilon > 0$ small enough such that the corresponding allocation (x'', y') is feasible and makes every consumer strictly better off. Therefore, the original allocation (x,y) is not weakly Pareto efficient. Together with part (a) above, this implies that the notions of strong and weak Pareto efficiency are equivalent.

(c) [First printing errata: omit the assumption of interiority, it plays no role given that $X_1 = X_2 = \mathbb{R}_+$ for all i.]

Let $I = 2$, $L = 1$, $X_1 = X_2 = \mathbb{R}_+$, $u_1(x_1) \equiv 0$, $u_2(x_2) = x_2$, $\omega > 0$, $Y = -\mathbb{R}_+$ (no production). Then $(x_1, x_2) = (\omega/2, \omega/2)$ is a weakly Pareto efficient allocation, but it is not strongly Pareto efficient. Since consumer 1's preferences are not strongly monotone, we cannot reallocate utility from consumer 1 to consumer 2 to make every weak Pareto improvement a strong Pareto improvement, as we did in part (b).

10.B.2. Suppose that the allocation $(x_1^*, ..., x_I^*, y_1^*, ..., y_J^*)$ and price vector $p^* \in \mathbb{R}^L$ constitute a competitive equilibrium. Then the allocation $(x_1^*, ..., x_I^*, y_1^*, ..., y_J^*)$ and price vector αp^* $(\alpha > 0)$ also constitute a competitive equilibrium:

(i) Profit maximization:

$$y_j^* \text{ solves } \underset{y_j \in Y_j}{\text{Max}}\ p^* \cdot y_j \Longrightarrow y_j^* \text{ solves } \underset{y_j \in Y_j}{\text{Max}}\ \alpha p^* \cdot y_j = \alpha \underset{y_j \in Y_j}{\text{Max}}\ p^* \cdot y_j$$

(ii) Utility maximization:

Consumer i's new budget constraint is

$$\alpha p^* \cdot x_i \le \alpha p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} (\alpha p^* \cdot y_j^*) \Longleftrightarrow p^* \cdot x_i \le p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} (p^* \cdot y_j^*),$$

which is consumer i's old budget constraint. Therefore,

$$x_i \text{ solves } \underset{x_i \in X_i}{\text{Max}}\ u_i(x_i) \text{ s.t. the new budget constraint} \Longleftrightarrow$$

x_i solves $\underset{x_i \in X_i}{\text{Max}}\ u_i(x_i)$ s.t. the old budget constraint.

(iii) Market clearing: $\sum_{i=1}^{I} x_{li}^* = \omega_l + \sum_{j=1}^{J} y_{lj}^*$ for each $l = 1, ..., L$ - does not depend on prices at all.

10.C.1. (a) The consumer solves

$$\text{Max} \sum_{l=1}^{L} \log x_l \qquad \text{s.t.} \quad \sum_{l=1}^{L'} p_l x_l \le w.$$

The first-order condition for the Lagrangean of this program can be written as $x_l = \lambda/p_l$, $l=1,...,L$, where $\lambda > 0$. Substituting in the budget constraint, we find $\lambda = w/L$, therefore the demand function can be written as $x_l(p,w) = \frac{w}{Lp_l}$.

The wealth effect is $\partial x_l(p,w)/\partial w = \frac{1}{Lp_l}$.

(b) As $L \longrightarrow \infty$, the wealth effect $\partial x_l(p,w)/\partial w \longrightarrow 0$.

10.C.2. (a) The consumer solves

$$\underset{(x,m)}{\text{Max}}\ \alpha + \beta \ln x + m \qquad \text{s.t.} \quad p\,x + m \le \omega_m.$$ The

first-order condition (assuming interior solution) yields $x(p) = \beta/p$.

The firm solves $\underset{q \ge 0}{\text{Max}}\ pq - \sigma q$.

The firm's first-order condition (assuming interior solution) is $p = \sigma$.

(b) From the two first-order conditions and the consumer's budget constraint, the competitive equilibrium is $p^* = \sigma$, $x^* = \beta/\sigma$, $m^* = \omega_m - \beta$.

10.C.3. (a) Assuming interior solution, the first-order condition is $c_j'(q_j^*) = \lambda > 0$ for all j.

(b) $C'(q) = \sum_{j=1}^{J} c_j'(q_j^*)\, dq_j^*/dq = \sum_{j=1}^{J} \lambda\, dq_j^*/dq = \lambda\, d(\sum_{j=1}^{J} q_j^*)\,/dq = \lambda\, dq/dq = \lambda.$

Therefore, $C'(q) = c_j'(q_j^*)$ for all j.

(c) Each firm j solves $\underset{q_j}{\text{Max}}\ pq_j - c_j(q_j)$.

The first-order condition (assuming interior solution) is $c_j'(q_j) = p$.

If $p = C'(q)$, then we have $c_j'(q_j) = C'(q)$ for every j. If $c_j'(\cdot)$ is strictly increasing for all j, we must have $q_j = q_j^*(q)$ - the solution to the central authority's program in part (a) for total output q. Therefore, $\sum_{j=1}^{J} q_j = \sum_{j=1}^{J} q_j^*(q) = q$. In other words, if the market price is $C'(q)$, then the industry produces q. Therefore, $C'(\cdot)$ is the inverse of the industry supply function.

10.C.4. (a) The central authority's problem can be written as

$$\underset{(x_1,\ldots,x_I)\geq 0}{\text{Max}} \quad \sum_{i=1}^{I} \phi_i(x_i) \qquad \text{s.t.} \ \sum_{i=1}^{I} x_i \leq x.$$

Assuming interior solution, the first-order condition is

$\phi_i'(x_i^*) = \lambda > 0$ for all i.

(b) $\gamma'(x) = \sum_{i=1}^{I} \phi_i'(x_i^*)\, dx_i^*/dx = \sum_{i=1}^{I} \lambda\, dx_i^*/dx = \lambda\, d(\sum_{i=1}^{I} x_i^*)\,/dx = \lambda\, dx/dx = \lambda.$

Therefore, $\gamma'(x) = \phi_i'(x_i^*)$ for all i.

(c) Each consumer solves $\underset{x_i}{\text{Max}}\ \phi_i(x_i) - Px_i$.

The first-order condition (assuming interior solution) is $\phi_i'(x_i) = P$.

If $P = \gamma'(x)$, then we have $\phi_i'(x_i) = \gamma'(x)$ for every i. If $\phi_i'(\cdot)$ is strictly decreasing for all i, we must have $x_i = x_i^*(x)$ - the solution to the central authority's program in part (a) above for total consumption x. Therefore, $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} x_i^*(x) = x$. In other words, if the market price is $\gamma'(x)$, then the aggregate demand is x. Therefore, $\gamma'(\cdot)$ is the inverse of the aggregate demand function.

10.C.5. The system of equations (10.C.4)-(10.C.6) here takes the following form:

$$\phi_i'(x_i^*) = p^* + t, \qquad i = 1, \ldots, I,$$
$$c_j'(q_j^*) = p^*, \qquad j = 1, \ldots, J,$$
$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*.$$

These equations describe the equilibrium (x^*, q^*, p^*) as an implicit function of t. Differentiating with respect to t, we get

$$\phi_i''(x_i^*)\, x_i^{*\prime}(t) = p^{*\prime}(t) + 1, \qquad i = 1, \ldots, I,$$
$$c_j''(q_j^*)\, q_j^{*\prime}(t) = p^{*\prime}(t), \qquad j = 1, \ldots, J,$$
$$\sum_{i=1}^{I} x_i^{*\prime}(t) = \sum_{j=1}^{J} q_j^{*\prime}(t).$$

This system of linear equations should be solved for $(x_i^{*\prime}(t), q_j^{*\prime}(t), p^{*\prime}(t))$. This can be easily done, for e.:ample, by expressing dx_i^*/dt and dq_j^*/dt from the first two sets of equations and substituting into the third equation. We obtain

$$(p^{*\prime}(t) + 1) \sum_{i=1}^{I} [\phi_i''(x_i^*)]^{-1} = p^{*\prime}(t) \sum_{j=1}^{J} [c_j''(q_j^*)]^{-1}.$$

From here we can express $p^{*\prime}(t)$:

$$p^{*\prime}(t) = \frac{\sum_{i=1}^{I} [\phi_i''(x_i^*)]^{-1}}{\sum_{i=1}^{I} [\phi_i''(x_i^*)]^{-1} - \sum_{j=1}^{J} [c_j''(q_j^*)]^{-1}}.$$

Compare to the expression on page 324 of the textbook.

10.C.6. (a) If the specific tax t is levied on the consumer, then he pays $p+t$ for every unit of the good, and the demand at market price p becomes $x(p+t)$.

The equilibrium market price p^C is determined from equalizing demand and supply:

$$x(p^C + t) = q(p^C).$$

On the other hand, if the specific tax t is levied on the producer, then he collects $p-t$ from every unit of the good sold, and the supply at market price p becomes $q(p-t)$. The equilibrium market price p^P is determined from equalizing demand and supply:

$$x(p^P) = q(p^P - t).$$

It is easy to see that p solves the first equation if and only if $p+t$ solves the second one. Therefore, $p^P = p^C + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is $x(p^P) = x(p^C + t)$.

(b) If the ad valorem tax τ is levied on the consumer, then he pays $(1+\tau)p$ for every unit of the good, and the demand at market price p becomes $x((1+\tau)p)$. The equilibrium market price p^C is determined from equalizing demand and supply:

$$x((1+\tau)p^C) = q(p^C). \tag{1}$$

On the other hand, If the ad valorem tax τ is levied on the producer, collects $(1-\tau)p$ from then he pays $(1+\tau)p$ for every unit of the good sold, and the supply at market price p becomes $q((1-\tau)p)$. The equilibrium market price p^P is determined from equalizing demand and supply:

$$x(p^P) = q((1-\tau)p^P). \tag{2}$$

Consider the excess demand function for this case:

$$z(p) = x(p) - q((1-\tau)p).$$

Since $x(\cdot)$ is non-increasing and $q(\cdot)$ is non-decreasing, $z(p)$ must be non-increasing. From (1) we have

$$
\begin{aligned}
z((1+\tau)p^C) &= x((1+\tau)p^C) - q((1-\tau)[(1+\tau)p^C]) = \\
&= x((1+\tau)p^C) - q((1-\tau^2)p^C) \geq \\
&\geq x((1+\tau)p^C) - q(p^C) = 0,
\end{aligned}
$$

taking into account that $q(\cdot)$ is non-decreasing and using (1).

Therefore, $z((1+\tau)p^C) \geq 0$ and $z(p^P) = 0$. Since $z(\cdot)$ is non-increasing, this implies that $(1+\tau)p^C \leq p^P$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax to producers).

If $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1+\tau)p^C < p^P$. On the other hand, when the supply is perfectly inelastic, i.e. $q(p) = \bar{q} = const$, then (1) and (2) combined yield $x((1+\tau)p^C) = \bar{q} = x(p^P)$, and therefore $p^P = (1+\tau)p^C$. Here both taxes result in the same cost to consumers. However, the producers still bear a higher burden when the tax is levied directly on them: $(1-\tau)p^P = (1-\tau)(1+\tau)p^C < p^C$. Therefore, the two taxes are still not fully equivalent.

The intuition behind these results is simple: with a tax, there is always a wedge between the "consumer price" and the "producer price" Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax no the lower consumer price.

10.C.7. To compute the price received by producers, we can use equation (10.C.8) in the textbook:

$$
p_*'(0) = -\frac{x'(p_*)}{x'(p_*) - q'(p_*)} = -\frac{A\varepsilon p_*^{\varepsilon-1}}{A\varepsilon p_*^{\varepsilon-1} - \alpha\gamma p_*^{\gamma-1}} = -\frac{A\varepsilon p_*^{\varepsilon}}{A\varepsilon p_*^{\varepsilon} - \alpha\gamma p_*^{\gamma}} =
$$

$$= - \frac{\varepsilon x(p_*)}{\varepsilon x(p_*) - \gamma q(p_*)} = - \frac{\varepsilon}{\varepsilon - \gamma}.$$

(We have multiplied both the numerator and the denominator by p_* and used the fact that p_* is an equilibrium price, therefore $x(p_*) = q(p_*)$.) The price paid by consumers is $p_* + t$, and its derivative with respect to t at t=0 is

$$p'(0) + 1 = - \frac{\varepsilon}{\varepsilon - \gamma} + 1 = - \frac{\gamma}{\varepsilon - \gamma}.$$

From these expressions we can see that when $\gamma = 0$ (supply is perfectly inelastic) or $\varepsilon \longrightarrow -\infty$ (demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax. On the other hand, when $\varepsilon = 0$ (demand is perfectly inelastic) or $\gamma \longrightarrow \infty$ (supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax.

Computing the elasticity of the equilibrium price with respect to changes in k=1+τ is left as an exercise.

10.C.8. (a) Each firm's profit can be written as

$$\pi(q, \alpha) = p(\alpha)\, q - c(q, \alpha).$$

The first-order condition of the firm's profit-maximization problem can be written as

$$p(\alpha) = c_q(q, \alpha). \qquad \text{(FOC)}$$

Denoting the solution of the firms' problem by $q_*(\alpha)$, we can write the firms' reduced-form profits $\pi_*(\alpha) = \pi(q_*(\alpha), \alpha)$.

Using the Envelope Theorem (in simple words, taking into account that $\partial\pi(q, \alpha)/\partial q \,\big|_{q=q_*(\alpha)} = 0$), we can write

$$\pi_*'(\alpha) = \partial\pi(q_*(\alpha), \alpha)/\partial\alpha = p'(\alpha)\, q_*(\alpha) - c_\alpha(q_*(\alpha), \alpha). \qquad (*)$$

By assumption $c_\alpha(\cdot) \le 0$, and it is the first term,

$p'(\alpha)\, q_*(\alpha)$, which may present problems.

To determine how the market clearing price depends on α, consider the following system of two equations:

$$p'(\alpha) = c_{qq}(q_*(\alpha), \alpha)\; q_*'(\alpha) + c_{q\alpha}(q_*(\alpha), \alpha).$$

$$x'(p(\alpha))\; p'(\alpha) = Jq_*'(\alpha)$$

The first equation is obtained by differentiating (FOC), the second by differentiating the market clearing condition $x(p(\alpha)) = Jq_*(\alpha)$ (both with respect to α). Eliminating $q_*'(\alpha)$ from the system, we can solve for $p'(\alpha)$:

$$p'(\alpha) = c_{q\alpha}/(1 - J^{-1}c_{qq}\, x'(p(\alpha)))$$

(for convenience we have omitted arguments in the derivatives of $c(\cdot)$). Now we can rewrite (*) as

$$\pi_*'(\alpha) = c_{q\alpha}\; q_*(\alpha)/(1 - J^{-1}\; c_{qq} x'(p(\alpha))) - c_\alpha.$$

(b) The last expression can be rewritten as

$$\pi_*'(\alpha) = [c_{q\alpha}\; q_*(\alpha) - c_\alpha + c_\alpha\; J^{-1}\; c_{qq} x'(p(\alpha))]/[1 - J^{-1}\; c_{qq} x'(p(\alpha))].$$

Remember that by assumption $c_{qq} > 0$ (costs are strictly convex in q), and $x' \le 0$. Therefore, the denominator of the fraction is always positive, and

$$\text{sign } \pi_*'(\alpha) = \text{sign } [c_{q\alpha}\; q_*(\alpha) - c_\alpha + c_\alpha\; J^{-1}\; c_{qq} x'(p(\alpha))]. \qquad (**)$$

Since by assumption $c_\alpha > 0$, $c_{qq} > 0$, and $x' \le 0$, the last term must be non-positive, and

$$\text{sign } \pi_*'(\alpha) \le \text{sign } [c_{q\alpha}\; q_*(\alpha) - c_\alpha] =$$

$$= \text{sign } \partial/\partial q\; [c_\alpha(q,\alpha)/q] \text{ at } q = q_*(\alpha).$$

Thus, in order for profits to be increasing in α for any demand function, is sufficient to have $\partial/\partial q\; [c_\alpha(q,\alpha)/q] \le 0$ at $q = q_*(\alpha)$.

On the other hand, if this condition is not satisfied, we can take a demand function with $|x'(p(\alpha))|$ sufficiently small. For such a demand function, the last term in the right-hand side of (**) will be very small, and we will have

$$\text{sign } \pi_*'(\alpha) = \text{sign } [c_{q\alpha}\, q_*(\alpha) - c_\alpha] > 0,$$

i.e. profits increase in α.

(c) If α is the price of factor input k, then by Shepard's lemma (Proposition 5.C.2 (vi)) c_α is the conditional demand for factor k. Then the condition $c_{q\alpha} \le 0$ means that the conditional demand for k is non-increasing in output. i.e. that k is an inferior factor.

10.C.9. The first-order condition of the firms' profit-maximization problem can be written as $p(w) = c_q(w, q)$. The market clearing condition can be written as $x(p(w)) = Jq_*(w)$. Differentiating both equations with respect to w, we obtain the following system of two equations:

$$p'(w) = c_{qq}(w, q_*(w))\, q_*'(w) + c_{wq}(w, q_*(w)).$$

$$x'(p(w))\, p'(w) = Jq_*'(w)$$

Eliminating $q_*'(w)$ from the system, we can solve for $p'(w)$:

$$p'(w) = c_{wq}/(1 - J^{-1}c_{qq}\, x'(p(w)))$$

(for convenience we have omitted arguments in the derivatives of $c(\cdot)$).

Observe that the second-order condition for the firms' profit-maximization problem is $c_{qq} \ge 0$, and that in a partial equilibrium context we must have $x' \le 0$. It is then clear from the last expression for $p'(w)$ that the sign of $p'(w)$ coincides with the sign of c .

If w is the price of factor input k, then by Shepard's lemma (Proposition 5.C.2 (vi)) c_q is the conditional demand for factor k. Then the condition $c_{wq} \le 0$ means that the conditional demand for k is non-increasing in output. i.e. that k is an inferior factor. Therefore, $p'(w) \le 0$ if and only if k is an inferior factor.

10.C.10. The first-order condition for the firms' profit-maximization problem can be written as $p = c'(q)$. The market clearing condition can be written as $x(p) = Jq$. Substituting the functional forms for $c'(\cdot)$ and $x(\cdot)$, we obtain the following system of equations:

$$p = \beta q^{\eta},$$

$$\alpha p^{\varepsilon} = Jq.$$

Taking logs on both sides, we obtain a system of linear equations in log p and log q. The solution of this system is

$$\log p = (\log \beta + \eta \log \alpha - \eta \log J)/(1 - \varepsilon\eta),$$

$$\log q = (\varepsilon \log \beta + \log \alpha - \log J)/(1 - \varepsilon\eta).$$

From here we can compute the elasticities:

$$\partial \log p/\partial \log \alpha = \eta/(1 - \varepsilon\eta),$$

$$\partial \log p/\partial \log \beta = 1/(1 - \varepsilon\eta)$$

$$\partial \log q/\partial \log \alpha = 1/(1 - \varepsilon\eta),$$

$$\partial \log q/\partial \log \beta = \varepsilon/(1 - \varepsilon\eta).$$

We see that increasing the elasticity of demand $|\varepsilon|$ reduces the sensitivity of p and q to shifts in α, reduces the sensitivity of p to shifts in β, but increases the sensitivity of q to shifts in β. Similarly, increasing the inverse elasticity of supply η reduces the sensitivity of p and q to shifts in β, reduces the sensitivity of q to shifts in α, but increases the sensitivity of p to shifts in α.

10.C.11. Let us start with a situation where both firms' parameters are equal to α, and then increase α_1 and reduce α_2 marginally in such a way that the competitive equilibrium price does not change.

Firm's j's profit can be written as

$$\pi(q, \alpha_j) = pq - c(q, \alpha_j).$$

The first-order condition of the firm's profit-maximization problem can be written as

$$p(\alpha_j) = c_1(q, \alpha_j). \qquad \text{(FOC)}$$

Denoting the solution of the firms' problem by $q_*(\alpha)$, we can write the firms' reduced-form profits $\pi_*(\alpha_j) = \pi(q_*(\alpha_j), \alpha_j)$.

Using the Envelope Theorem (in simple words, taking into account $\partial\pi(q, \alpha_j)/\partial q \mid_{q=q_*(\alpha_j)} = 0$), we can write

$$\pi_*'(\alpha_j) = \partial\pi(q_*(\alpha_j), \alpha_j)/\partial\alpha_j = c_2(q_*(\alpha_j), \alpha_j).$$

If $\alpha_1 - \alpha_2 = \Delta\alpha$, then

$\pi_*(\alpha_1) - \pi_*(\alpha_2) \cong \pi_*'(\alpha)\,\Delta\alpha = c_2(q_*(\alpha), \alpha)\,\Delta\alpha.$

10.D.1. (i) Suppose in negation that

$(x_1, .., x_I, q_1, ..., q_J)$ and $(x_1', .., x_I', q_1', ..., q_J')$ are both solutions to (10.D.2), and that $x_k \neq x_k$ for some k. Take

$$x_i'' = 1/2\ x_i + 1/2\ x_i' \text{ for every } i = 1, ..., I,$$

$$q_j'' = 1/2\ q_j + 1/2\ q_j' \text{ for every } j = 1, ..., J.$$

Clearly, $(x_1'', .., x_I'', q_1'', ..., q_J'')$ satisfies the constraint in (10.D.2), since both $(x_1, .., x_I, q_1, ..., q_J)$ and $(x_1', .., x_I', q_1', ..., q_J')$ do. As cost functions are convex, we must have

$$c_j(q_j'') \le 1/2\ c_j(q_j) + 1/2\ c_j(q_j') \text{ for every } j = 1, ..., J.$$

As utility unctions ϕ_i's are strictly convex, we must have

$$\phi_i(x_i'') \ge 1/2\ \phi_i(x_i) + 1/2\ \phi_i(x_i') \text{ for every } i \neq k, \text{ and}$$

$$\phi_k(x_k'') > 1/2\ \phi_k(x_k) + 1/2\ \phi_k(x_k').$$

Adding up all the inequalities, we obtain

$$\sum_{i=1}^{I} \phi_i(x_i'') - \sum_{j=1}^{J} c_j(q_j'') > \quad 1/2 \left[\sum_{i=1}^{I} \phi_i(x_i') - \sum_{j=1}^{J} c_j(q_j') \right] + 1/2 \left[\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) \right],$$

which contradicts the assumption that $(x_1, .., x_I, q_1, ..., q_J)$ and $(x_1', .., x_I', q_1', ..., q_J')$ are both solutions to (10.D.2).

Therefore, individual consumption levels x_i's at a solution to (10.D.2) are uniquely determined.

(ii) The optimal aggregate production level is $\sum_{j=1}^{J} q_j = \sum_{i=1}^{I} x_i$, and it is uniquely determines since all the x_i's are uniquely determined.

(iii) Now suppose that the cost functions $c_j(\cdot)$ are strictly convex.

Suppose in negation that $(x_1, .., x_I, q_1, ..., q_J)$ and $(x_1', .., x_I', q_1', ..., q_J')$ are both solutions to (10.D.2), and that $q_k \neq q_k'$ for some k. Take

$x_i'' = 1/2\ x_i + 1/2\ x_i'$ for every $i = 1, ..., I$,

$q_j'' = 1/2\ q_j + 1/2\ q_j'$ for every $j = 1, ..., J$.

Clearly, $(x_1'', .., x_I'', q_1'', ..., q_J'')$ satisfies the constraint in (10.D.2), since both $(x_1, .., x_I, q_1, ..., q_J)$ and $(x_1', .., x_I', q_1', ..., q_J')$ do. As cost functions are strictly convex, we must have

$c_j(q_j'') \leq 1/2\ c_j(q_j) + 1/2\ c_j(q_j')$ for every $j \neq k$, and

$c_k(q_k'') < 1/2\ c_k(q_k) + 1/2\ c_k(q_k')$.

As utility unctions ϕ_i's are convex, we must have

$\phi_i(x_i'') \geq 1/2\ \phi_i(x_i) + 1/2\ \phi_i(x_i')$ for every $i = 1, ..., I$.

Adding up all the inequalities, we obtain

$$\sum_{i=1}^{I} \phi_i(x_i'') - \sum_{j=1}^{J} c_j(q_j'') > \quad 1/2 \left[\sum_{i=1}^{I} \phi_i(x_i') - \sum_{j=1}^{J} c_j(q_j') \right] + 1/2 \left[\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) \right],$$

which contradicts the assumption that $(x_1, .., x_I, q_1, ..., q_J)$ and $(x_1', ..,$

$x'_I, q'_1, \ldots, q'_J)$ are both solutions to (10.D.2).

Therefore, individual production levels q_j's at a solution to (10.D.2) are uniquely determined.

10.D.2. The program (10.D.2) for the economy in Exercise 10.C.2 can be written as follows:

$$\underset{x \geq 0}{\text{Max}} \quad \phi(x) - c(x) + \omega_m = \alpha + \beta \ln x - \sigma x + \omega_m$$

The first-order condition yields $x^* = \beta/\sigma$. This is the same consumption (and production) level as that obtained in Exercise 10.C.2.

10.D.3. (i) Suppose in negation that $\{x_i^*, m_i^*\}_{i=1}^I, \{z_j^*, q_j^*\}_{j=1}^J$ is a solution to (10.D.6), but not a Pareto optimal allocation. This means that there is an allocation $\{x'_i, m'_i\}_{i=1}^I, \{z'_j, q'_j\}_{j=1}^J$ which is feasible (i.e. satisfies (2l), (2m), (3)), such that

$$m'_i + \phi_i(x'_i) \geq m_i^* + \phi_i(x_i^*) \text{ for all } i = 1, \ldots, I, \text{ and}$$

$$m'_k + \phi_k(x'_k) > m_k^* + \phi_k(x_k^*) \text{ for some } k.$$

If $k = 1$, this means that the allocation $\{x'_i, m'_i\}_{i=1}^I, \{z'_j, q'_j\}_{j=1}^J$ satisfies all the constraints in (10.D.6) and yields a higher value of the objective function than $\{x_i^*, m_i^*\}_{i=1}^I, \{z_j^*, q_j^*\}_{j=1}^J$, which contradicts the assumption that the latter allocation is a solution to (10.D.6).

Suppose, therefore, that $k > 1$. Let $\Delta = [m'_k + \phi_k(x'_k)] - [m_k^* + \phi_k(x_k^*)]$. Take an allocation $\{x''_i, m''_i\}_{i=1}^I, \{z''_j, q''_j\}_{j=1}^J$ which has some numeraire redistributed from consumer k to consumer 1:

$$m''_1 = m'_1 + \Delta,$$

$$m''_k = m'_k - \Delta,$$

and which otherwise coincides with the allocation $\{x'_i, m'_i\}_{i=1}^I, \{z'_j, q'_j\}_{j=1}^J$.

It is easy to see that the new allocation $\{x''_i, m''_i\}_{i=1}^{I}, \{z''_j, q''_j\}_{j=1}^{J}$ satisfies all the constraints in (10.D.6) and yields a higher value of the objective function than $\{x^*_i, m^*_i\}_{i=1}^{I}, \{z^*_j, q^*_j\}_{j=1}^{J}$, which contradicts the assumption that the latter allocation is a solution to (10.D.6).

(ii) Let $\{x^*_i, m^*_i\}_{i=1}^{I}, \{z^*_j, q^*_j\}_{j=1}^{J}$ be a Pareto optimal allocation and let $\bar{u}_i = m^*_i + \phi_i(x^*_i)$ for $i = 2, ..., I$. We will prove that $\{x^*_i, m^*_i\}_{i=1}^{I}, \{z^*_j, q^*_j\}_{j=1}^{J}$ solves the program (10.D.6) with these utility levels.
First of all, it easy to see that $\{x^*_i, m^*_i\}_{i=1}^{I}, \{z^*_j, q^*_j\}_{j=1}^{J}$ satisfies all the constraints in (10.D.6). Suppose in negation that $\{x^*_i, m^*_i\}_{i=1}^{I}, \{z^*_j, q^*_j\}_{j=1}^{J}$ does not solve the program, i.e. that there is an allocation $\{x'_i, m'_i\}_{i=1}^{I}, \{z'_j, q'_j\}_{j=1}^{J}$ which satisfies all the constraint and yields a higher value of the objective function. Since $\{x'_i, m'_i\}_{i=1}^{I}, \{z'_j, q'_j\}_{j=1}^{J}$ satisfies (2l), (2m), (3), it is feasible. The constraint (1) implies that

$$m'_i + \phi_i(x'_i) \geq m^*_i + \phi_i(x^*_i) \text{ for every } i = 1, ..., I.$$

Since $\{x'_i, m'_i\}_{i=1}^{I}, \{z'_j, q'_j\}_{j=1}^{J}$ yields a higher value of the objective function, we have

$$m'_1 + \phi_1(x'_1) > m^*_1 + \phi_1(x^*_1).$$

Therefore, $\{x'_i, m'_i\}_{i=1}^{I}, \{z'_j, q'_j\}_{j=1}^{J}$ is a Pareto improvement over $\{x^*_i, m^*_i\}_{i=1}^{I}, \{z^*_j, q^*_j\}_{j=1}^{J}$, which contradicts the assumption that the latter allocation is Pareto optimal.

10.D.4. The Lagrangean for he program (10.D.6) can be written as

$$L = m_1 + \phi_1(x_1) + \sum_{i=1}^{I} \alpha_i x_i + \sum_{j=1}^{J} \beta_j q_j + \sum_{i=2}^{I} \gamma_i \,[m_i + \phi_i(x_i) - \bar{u}_i] - \mu\, [\sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j] -$$

$$- \nu[\sum_{i=1}^{I} m_i + \sum_{j=1}^{J} z_j - \omega_m] + \sum_{j=1}^{J} \kappa_j [z_j - c_j(q_j)],$$

where α_i, β_j, μ, ν, κ_j, γ_i are non-negative Lagrange multipliers.

The first-order conditions with respect to original variables are:

$\partial L/\partial m_1 = 1 - \nu = 0 \Rightarrow \nu = 1;$

$\partial L/\partial m_i = \gamma_i - \nu = 0$ for $i \geq 2 \Rightarrow \gamma_i = \nu$ for all $i \geq 2$.

$\partial L/\partial z_j = - \nu + \kappa_j = 0 \Rightarrow \kappa_j = \nu = 1$ for all j;

$\partial L/\partial x_i = \alpha_i + \phi_i'(x_i) - \mu = 0$ for all i; (*)

$\partial L/\partial q_j = \beta_j + \mu - \kappa_j c_j'(q_j) = \beta_j + \mu - c_j'(q_j) = 0$ for all j. (**)

Besides, there are first-order conditions with respect to dual variables and complementary slackness conditions. Complementary slackness conditions say that $\alpha_i x_i = 0$ for all i and $\beta_j q_j = 0$ for all j. Using this, (*) and (**) become equivalent to conditions (10.D.3) and (10.D.4) in the textbook respectively. The first-order condition with respect to μ yields condition (10.D.5) in the textbook. The remaining conditions serve to pin dow the variables m_i and z_j, which are not present in program (10.D.2).

10.E.1. (a) To determine how the domestic market price p depends on the tariff τ, observe that the demand $x(p)$, the domestic supply is $J_d q_d(p)$, and the foreign supply is $J_f q_f(p - \tau)$, where $q_d(\cdot)$ and $q_f(\cdot)$ are the supply functions of the domestic and foreign firms correspondingly. The market clearing condition can then be written as $x(p) = J_d q_d(p) + J_f q_f(p - \tau)$. Differentiating this condition with respect to τ and solving for $p'(\tau)$, we obtain

$$p'(\tau)\big|_{\tau=0} = \frac{J_f q_f'(p)}{J_d q_d'(p) + J_f q_f'(p) - x'(p)}. \qquad (*)$$

As $c_f(\cdot)$ are strictly convex, we have $c_f''(\cdot) > 0$, and therefore $q_f'(\cdot) < \infty$. The expression (*) then implies $p'(\tau) < 1$.

Domestic welfare is the sum of consumer surplus, domestic profits, and tax revenue (which can be, for example, distributed lump-sum to domestic consumers):

$$DW(\tau) = CS(p(\tau)) + J_d\pi_d(p(\tau)) + \tau J_f q_f(p(\tau)-\tau).$$

We will differentiate this expression at $\tau=0$, taking into account that $CS'(p) = -x(p)$ (Roy's identity) and $\pi_d'(p) = q_d(p)$ (Hotelling's lemma):

$$\begin{aligned} DW'(\tau)\big|_{\tau=0} &= -x(p)p'(0) + J_d q_d(p)p'(0) + J_f q_f(p) = \\ &= [-x(p) + J_d q_d(p)]p'(0) + J_f q_f(p) = \\ &= - J_f q_f(p)\ p'(0) + J_f q_f(p) = \\ &= J_f q_f(p)\ [1 - p'(0)]. \end{aligned}$$

Since we have established that $p'(0) < 1$, we have $DW'(0) > 0$. i.e. the imposition of a small tariff raises domestic welfare.

(b) Now we have $c_f''(\cdot) = 0$, and therefore $q_f'(\cdot) = \infty$. From (*) we see that $p'(\tau) = 1$, and from the result of part (a) above, $DW'(0) = J_f q_f(p)\ [1 - p'(0)] = 0$. Thus, there is no first-order effect on domestic welfare from a small tariff, and we need to look at the second derivative to determine how domestic welfare is affected by a small tariff. However, instead of computing the second derivative of $DW(\tau)$, one can answer the question with a simple observation. Observe that since foreign firms produce at constant returns to scale, they always have zero profits, and domestic welfare equals total welfare. Imposition of a tariff distorts the global economy away from the competitive equilibrium, and therefore lowers total welfare, which is equal to domestic welfare.

10.E.2. Substitute $s = x(p)$:

$$CS(\hat{p}) = \int_0^{x(\hat{p})} [P(s) - \hat{p}]\ ds = \int_{P(0)}^{P(x(\hat{p}))} [P(x(p)) - \hat{p}]\ dx(p) =$$

$$= \int_{\infty}^{\hat{p}} [p - \hat{p}]\, dx(p) = [p - \hat{p}]\, x(p)\Big|_{\infty}^{\hat{p}} - \int_{\infty}^{\hat{p}} x(p)dp = \int_{\hat{p}}^{\infty} x(p)dp.$$

(The second line carries out integration by parts.)

10.E.3. (a) [First printing errata: The first sentence should read: "Show that if $t'_1 = 0$ and $t'_l = t_1\ c_l + t_l$ for all $l \neq 1$, then tax vector t' raises the same amount of good 1 as does tax vector t."]

Given constant returns to scale, in equilibrium a firm should receive c_l units of numeraire for each unit of good l. Under the tax vector t, a consumer pays c_l units of numeraire directly to the firm for each unit of good l. For this transaction, the consumer has to pay the government t_l units of numeraire as a tax on good l, plus $c_l t_1$ units of numeraire as a tax on his transfer of numeraire to the firm. Thus, the total tax consumers pay on each unit of good l is $t_1\ c_l + t_l$. This is the same amount as under the tax vector t', and we know that $t'_1 = 0$. Therefore, for each tax scheme there is an equivalent tax scheme which leaves numeraire untaxed.

(b) The government's problem is

$$\begin{aligned} \underset{(t_2,\ \ldots,\ t_L)}{\text{Max}} \quad & \sum_{l=2}^{L} CS_l(c_l + t_l) \\ \text{s.t.} \quad & \sum_{l=2}^{L} t_l x_l(c_l + t_l) \geq R, \end{aligned}$$

where $x_l(p_l)$ is consumer demand in market l, and $CS_l(p_l)$ is consumer surplus in market l given price p_l. The Lagrangean of this program is

$$L = \sum_{l=2}^{L} CS_l(c_l + t_l) + \lambda[\sum_{l=2}^{L} t_l x_l(c_l + t_l) - R],$$

where $\lambda \geq 0$ is the Lagrange multiplier with the constraint. Differentiating with respect to t_l and using Roy's identity ($CS'_l(p_l) = -\ x_l(p_l)$), we obtain the following first-order condition:

$$-\ x_l(p_l) + \lambda\ [t_l x'_l(p_l) + x_l(p_l)] = 0$$

(for brevity we have substituted $p_l = c_l + t_l$). From here we can express t_l:

$$t_l = (1-\lambda)/\lambda \; x_l(p_l)/x_l'(p_l).$$

Denoting the elasticity of demand for good l by $\varepsilon_l = p_l x_l'(p_l)/x_l(p_l)$, the last expression can be rewritten as

$$t_l/p_l = (1-\lambda)/\lambda \; 1/\varepsilon_l.$$

This result is known as the *inverse elasticity rule*: optimal rates of taxation are inversely proportional to demand elasticities.

(c) From the inverse elasticity rule, optimal tax rates on two goods should be equal if demand elasticities for the two goods are equal; and markets with less elastic demands should be taxed more heavily than markets with more elastic demands.

Using the argument in part (a), if $t = (t_1, 0.., 0)$ is a tax scheme which taxes only numeraire, then $t' = (0, t_1 c_2 ..., t_1 c_L)$ is an equivalent tax scheme leaving numeraire untaxed. Such a scheme sets equal tax rates on all (non-numeraire) goods, which is only optimal when all markets have equally elastic demands.

10.F.1. (i) Suppose that $p > c'(0)$. Using Taylor expansion of $c(q)$ near zero, for a small $\varepsilon > 0$ we have $c(\varepsilon) = c(0) + c'(0)\varepsilon + o(\varepsilon) = c'(0)\varepsilon + o(\varepsilon)$. Therefore,

$$\pi(p) = \underset{q \geq 0}{\text{Max}}\; pq - c(q) \geq p\varepsilon - [c'(0)\varepsilon + o(\varepsilon)] = [p - c'(0)]\varepsilon + o(\varepsilon) > 0$$

for $\varepsilon > 0$ sufficiently small.

(ii) Suppose that $p \leq c'(0)$. Since $c(\cdot)$ is convex, we have

$$c(\varepsilon) \leq \varepsilon/q \; c(q) + (1-\varepsilon/q)\; c(0) = \varepsilon/q \; c(q)$$

for every $\varepsilon, q > 0$. Therefore,

$$c(q)/q \geq c(\varepsilon)/\varepsilon \longrightarrow c'(0) \quad \text{as } \varepsilon \longrightarrow 0,$$

and consequently $c(q) \geq c'(0)q$ for all $q \geq 0$.But then profits at any output q can be bounded:

$$pq - c(q) \leq pq - c'(0)q = [p-c'(0)]q \leq 0.$$

Therefore, $\pi(p) = \underset{q\geq 0}{\text{Max}}\ pq - c(q) \leq 0.$

10.F.2. (a) (i) Profit maximization $\Rightarrow p = c'(q) = \alpha + 2\beta q$.

(ii) Market clearing $\Rightarrow A - Bp = Jq$.

(iii) Free entry $\Rightarrow pq - c(q) = pq - K - \alpha q - \beta q^2 = 0$.

The solution to this system of equations is

$$q^* = \sqrt{K/\beta},$$

$$p^* = \alpha + 2\sqrt{K\beta},$$

$$J^* = (A - \alpha B)\sqrt{\beta/K} - 2\alpha\beta.$$

Aggregate output is $Q^* = J^*q^* = A - \alpha B - 2\alpha\beta\sqrt{K/\beta}$.

p^* and q^* are independent of A. This is not surprising, because p^* and q^* are entirely determined by technological parameters - q^* is the optimal scale of production, and p^* is the minimum average cost. J^* and Q^* are increasing in A.

(b) In the short run the number of firms J^* is fixed, p and q are determined by profit maximization (i) and market clearing (ii). Since we are interested in market responses to short-run changes in demand, we can differentiate (i) and (ii) treating p and q as functions of A:

$$p'(A) = 2\beta q'(A),$$

$$1 - Bp'(A) = J^*q'(A).$$

We can now solve for $p'(A)$ and $q'(A)$:

$$p'(A) = 1/(J^* + 2\beta),$$

$q'(A) = 2\beta/(J^* + 2\beta)$.

As A increases, $J^* = (A - \alpha B)\sqrt{\beta/K} - 2\,\alpha\beta$ increases, and $p'(A)$ and $q'(A)$ fall. When $A \longrightarrow \infty$, $J^* = (A - \alpha B)\sqrt{\beta/K} - 2\,\alpha\beta \longrightarrow \infty$, and $p'(A) \longrightarrow 0$.The intuition is simple: in a large market the equilibrium number of firms is large, each firm's production needs to change only slightly to accommodate a short-run shift in demand, and the market price becomes insensitive to short-run demand shifts.

10.F.3. [First printing errata: You need to assume that taxes are small, which is necessary for a definite comparison. Also, the condition $\phi''(\cdot) < 0$ in the third line of the exercise should instead be $\phi''(\cdot) > 0$.] Let p^*, q^*, and J^* denote respectively the market price, each firm's production, and the number of producing firms at the initial equilibrium. These variables should satisfy conditions (i)-(iii) on p.335 in the textbook.

Introduction of an *ad valorem* tax τ can be represented by replacing the demand function $x(p)$ with $x(p(1+\tau))$. This change leaves conditions (i) and (iii) intact. Therefore, the new long-run market price and firm output have to satisfy (i) and (iii), and thus have to coincide with p^* and q^*. The new long-run equilibrium number of firms $\hat{J}$ is determined from the modified (ii):

$$x(p^*(1+\tau)) = \hat{J}q^*$$

Differentiating this expression with respect to τ and substituting $\tau = 0$, we obtain

$$\hat{J}'(0) = p^* x'(p^*)/q^* \qquad (1)$$

The resulting tax revenue is $\hat{R}(\tau) = \tau p^* q^* \hat{J}$. The second-order Taylor expansion of this function around $\tau = 0$ can be computed as

$$\hat{R}(\tau) \approx \hat{R}'(0)\,\tau + \hat{R}''(0)\,\tau^2/2 =$$

$$= p^* q^* [(\tau\hat{J}'(\tau) + \hat{J}(\tau))|_{\tau=0}\,\tau + (\tau\hat{J}''(\tau) + 2\hat{J}'(\tau))|_{\tau=0}\,\tau^2/2\,] =$$

$$= p^* q^* J^* \tau + p^* q^* \hat{J}'(\tau)\tau^2 \qquad (2)$$

Introduction of a per firm tax T, on the other hand, modifies the free entry condition (iii), so that now it can be written as $p^\circ q^\circ - c(q^\circ) - T = 0$. The profit maximization condition (i) yields $p^\circ = c'(q^\circ)$. Combining with the previous equation and substituting our cost function we can write

$$\phi'(q^\circ) = (\phi(q^\circ) + K + T)/q^\circ.$$

This equation determines q°. Differentiating it with respect to T, we obtain

$$q^{\circ\prime}(T) = (\phi''(q^\circ)q^\circ)^{-1}.$$

The number of firms J° can be determined from (ii) and the profit maximization condition:

$$x(\phi'(q^\circ)) = J^\circ q^\circ.$$

Differentiating this equation with respect to T and evaluating at $T = 0$, we obtain

$$J^{\circ\prime}(0) = [x'(\phi'(q^\circ))\,\phi''(q^\circ) - J^\circ]\, q^{\circ\prime}(T)/q^\circ\big|_{T=0}$$
$$= [x'(p^*)\phi''(q^*) - J^*]/(\phi''(q^*)q^{*2}). \qquad (3)$$

The tax revenue from the per firm tax is $R^\circ(T) = TJ^\circ(T)$. The second-order Taylor expansion of this function around $T = 0$ can be computed as

$$R^\circ(T) \approx R^{\circ\prime}(0)\, T + R^{\circ\prime\prime}(0)\, T^2/2 =$$
$$= (TJ^{\circ\prime}(T) + J^\circ(T))\big|_{T=0}\, T + (TJ^{\circ\prime\prime}(T) + 2J^{\circ\prime}(T))\big|_{T=0}\, T^2/2 =$$
$$= J^* T + J^{\circ\prime}(0)T^2$$

Since at the initial equilibrium the two taxes raise the same revenue, we must have $T = \tau p^* q^*$. Substituting in the last Taylor expansion, we obtain

$$R^\circ(T) \approx p^* q^* J^* \tau + (p^* q^*)^2 J^{\circ\prime}(0)\tau^2 \qquad (4)$$

Comparing (2) and (4), we see that the first-order terms in τ coincide, and the second-order difference can be computed using (1) and (3):

$$R^\circ(T) - \hat{R}(\tau) \approx p^* q^* [p^* q^* J^{\circ\prime}(0) - \hat{J}'(0)]\tau^2 =$$
$$= p^* q^* [\, p^* x'(p^*)/q^* - p^* J^*/(\phi''(q^*)q^*) - p^* x'(q^*)/q^*]\tau^2 =$$

$= - p^{*2}\tau^2 J^*/\phi''(q^*) < 0.$

Therefore, a small *ad valorem* tax raises more revenue than the corresponding per firm tax.

10.F.4. [First printing errata: you should assume that $c(w, q)$ is a twice continuously differentiable function, and that most efficient scale is uniquely defined.]

(i) The long-run equilibrium price and output of each firm are given by the following conditions:

$$\left.\begin{array}{l} \text{Profit maximization} \Rightarrow p = c_q(w, q), \\ \text{Free entry} \Rightarrow pq - c(w, q) = 0. \end{array}\right\}$$

Let $f(w,q) = c(w,q)/q$ be a firm's average cost. These equations imply that $f_q(w,q) = c_q(w,q)/q - c(w,q)/q^2 = (p - p)/q = 0$. Intuitively, this is the first-order condition for a firm's operating at the minimum average cost in a long-run equilibrium. Now the above system can be rewritten as

$$\bar{p}(w) = f(w,\bar{q}(w)) \qquad (1)$$

$$f_q(w,\bar{q}(w)) = 0, \qquad (2)$$

where $\bar{p}(w)$ and $\bar{q}(w)$ are equilibrium price and firm output as a function of w. To find out how equilibrium price depends on w, differentiate (1) with respect to the input price vector w and use (2):

$$\nabla\bar{p}(w) = \nabla_w f(w,\bar{q}(w)) + f_q(w,\bar{q}(w))\,\nabla_w\bar{q}(w) = \nabla_w f(w,\bar{q}(w)) \qquad (*)$$

Now, using the definition of $f(w,q(w))$ and Shepard's Lemma (Proposition 5.C.2 in the textbook), we obtain

$$\nabla\bar{p}(w) = \nabla_w f(w,\bar{q}(w)) = \nabla_w c(w,\bar{q}(w))/\bar{q}(w) = z(w,\bar{q}(w))/\bar{q} \geq 0,$$

where $z(w, q)$ is a firm's input demand function. Therefore, the long-run equilibrium price is non-increasing in factor prices.

(ii) Take any $\alpha > 0$, and substitute αw instead of w in equations (1), (2):

$$\bar{p}(\alpha w) = f(\alpha w, \bar{q}(\alpha w)) \qquad (1')$$

$$f_q(\alpha w, \bar{q}(\alpha w)) = 0. \qquad (2')$$

Using the definition of $f(w,q)$ and homogeneity of $c(w,q)$ in w (Proposition 5.C.2 in the textbook), we obtain

$$f(\alpha w, q) = c(\alpha w, q)/q = \alpha\, c(w,q)/q = \alpha\, f(w,q),$$

$$f_q(\alpha w, q) = \alpha\, f_q(w,q).$$

Therefore, equation (2′) can be rewritten as $f_q(w, \bar{q}(\alpha w)) = 0$, which together with equation (2) implies that $\bar{q}(\alpha w) = \bar{q}(w)$. Using this fact, the above derivations, and (1), equation (1′) can be rewritten as

$$\bar{p}(\alpha w) = \alpha\, f(w, \bar{q}(\alpha w)) = \alpha\, f(w, \bar{q}(w)) = \alpha\, \bar{p}(w),$$

which means that $\bar{p}(w)$ is homogeneous of degree one in w.

(iii) Differentiating (*) with respect to the input price vector w and using (2), we obtain

$$D^2\bar{p}(w) = D_w^2 f(w, \bar{q}(w)) + \partial/\partial q\, [\nabla_w f(w, \bar{q}(w))]\, \nabla_w \bar{q}(w) =$$

$$= D_w^2 f(w, \bar{q}(w)) + \nabla_w f_q(w, \bar{q}(w))\, \nabla_w \bar{q}(w) = D_w^2 f(w, \bar{q}(w)).$$

(The order of differentiation has been changed, which we can do due to the twice continuous differentiability of $f(\cdot,\cdot)$.) Now, using the definition of $f(w,q(w))$, we obtain

$$D^2\bar{p}(w) = D_w^2 f(w, \bar{q}(w)) = D_w^2 c(w, \bar{q}(w))/\bar{q}(w).$$

According to Proposition 5.C.2 in the textbook, $c(w,q)$ is a concave function of w, and thus $D_w^2 c(w, \bar{q}(w))$ is a negative semidefinite matrix. Therefore, $D^2\bar{p}(w)$ is negative semidefinite, too, which implies that $\bar{p}(w)$ is a concave function of w.

10.F.5. (a) Let $q_L(p)$ be a firm's long-run choice of output as a function of its price, with p^* being the initial price and $q^* = q_L(p^*)$ the initial output. Let $z_k(p) = z_k(w,q_L(p))$ be the optimal choice of factor k when output price is p, with $z_k^* = z_k(p^*)$. Denote by $q_L(p|z_k)$ the firm's short-run choice of output as a function of p, the output price, and z_k the fixed input of factor k. Let $\pi_S(p|z_k)$ denote the short-run profits as a function of output price p and factor k's input z_k. Then the long-run profit function can be expressed as $\pi_L(p) = \pi_S(p|z_k(p))$. Since the firm can always keep its short-run demand of factor k in the long run, we must have $\pi_L(p) \geq \pi_S(p|z_k^*)$ for all p. Moreover, we know that the initial situation was a long-run equilibrium, which implies that $\pi_L(p^*) = \pi_S(p|z_k(p^*)) = \pi_S(p|z_k^*)$. Therefore, the function $h(p) = \pi_L(p) - \pi_S(p|z_k^*)$ achieves its minimum value of zero at $p = p^*$. For this to hold, the following second-order condition has to be satisfied:

$$h''(p^*) = \pi_L''(p^*) - \pi_S''(p^*|z_k^*) \geq 0.$$

By Hotelling's lemma, $q_L(p) = \pi_L'(p)$ and $q_S(p) = \pi_S'(p|z_k^*)$. Therefore, the last inequality can be rewritten as

$$q_L'(p^*) \geq q_S'(p^*).$$

(b) We will hold the number of firms J fixed both in the long run and in the short run. (If free entry occurs in the long run, then the equilibrium price will go back to its original level which is the minimum average cost, and the question becomes trivial.) If each firm's supply function is $q(\cdot)$, the market-clearing condition can be written as

$$x(p,\alpha) = Jq(p(\alpha)).$$

Differentiating with respect to α, we obtain

$$x_\alpha(p,\alpha) + x_p(p,\alpha)p'(\alpha) = Jq'(p(\alpha))p'(\alpha)$$

From here we can express $p'(\alpha)$:

$$p'(\alpha^*) = x_\alpha \,/\, [Jq'(p^*) - x_p], \qquad (*)$$

where α^* is the initial value of α, p^* $p(\alpha^*)$, $x_\alpha = x_\alpha(p^*, \alpha^*)$, $x_p = x_p(p^*, \alpha^*)$. From the exercise's assumptions, both the numerator and denominator of the fraction are positive. Denoting by $p_S(\alpha)$ and $p_L(\alpha)$ the short-run and long-run price responses as functions of α respectively and using (*), we can write

$$p_L'(\alpha^*) = x_\alpha \,/\, [Jq_L'(p^*) - x_p] \le x_\alpha \,/\, [Jq_S'(p^*) - x_p] = p_S'(\alpha^*),$$

i.e. the long-run response of the market price to a marginal increase in α is smaller than the short-run response.

The aggregate consumption in the market is $Q(\alpha) = x(p(\alpha),\alpha)$. Differentiating with respect to α and using the result from part (a), we obtain

$$Q'(\alpha^*) = x_\alpha + x_p\, p'(\alpha^*)$$

Denoting by $p_S(\alpha)$ and $p_L(\alpha)$ the short-run and long-run aggregate consumption as functions of α respectively and using (*) and $x'(p) < 0$, we obtain

$$Q_S'(\alpha^*) = x_\alpha + x_p\, p_S'(\alpha^*) \le x_\alpha + x_p\, p_L'(\alpha^*) = Q_L'(\alpha^*),$$

i.e. the long-run response of the aggregate consumption to a marginal increase in α is larger than the short-run response.

10.F.6. First we derive the long-run cost function for the Cobb-Douglas technology:

$$c(q) = \min_{(z_1,z_2)} \; w_1 z_1 + w_2 z_2$$
$$\text{s.t. } z_1^{\alpha} z_2^{1-\alpha} \ge q.$$

The first-order condition for this program can be written as

$$w_1/w_2 = \alpha z_2/(1-\alpha)z_1.$$

Together with the binding constraint in the program, this gives us factor demands:

$$z_1(q) = [\alpha w_2/(1-\alpha)w_1]^{1-\alpha}\ q,$$
$$z_2(q) = [(1-\alpha)w_1/\alpha w_2]^{\alpha}\ q.$$

Therefore, we can write $c(q) = w_1 z_1 + w_2 z_2 = Aq$, where

$$A = w_1[\alpha w_2/(1-\alpha)w_1]^{1-\alpha} + w_2[(1-\alpha)w_1/\alpha w_2]^{\alpha} = w_1^{\alpha}\ w_2^{(1-\alpha)}/\alpha^{\alpha}(1-\alpha)^{1-\alpha}.$$

The long-run equilibrium is given by

(i) Profit maximization $\Rightarrow p = c'(q) = A$.

(ii) Market clearing $\Rightarrow a - bp = Jq$.

(iii) Free entry $\Rightarrow pq - c(q) = pq - Aq = (p - A)q = 0$.

Observe that since technology exhibits constant return to scale, profit maximization (condition (i)) automatically implies zero profits (condition (iii). Also, the number of firms and the scale of every firm are indeterminate. Condition (i) gives the equilibrium price $p^* = A$, and condition (ii) gives the aggregate output $Q^* = J^* q^* = a - bp^* = a - bA$.

In the short run capital z_1 is fixed at z_1^*, and the number of firms J is fixed at J^*. From $f(z_1, z_2) = q$ we get the short-run demand for labor $z_2^S(q|z_1^*) = q^{1/(1-\alpha)}/z_1^{*\alpha/(1-\alpha)}$. The short-run cost function is

$$c(q|z_1^*) = w_1 z_1^* + w_2 z_2^S(q|z_1^*) = w_1 z^* + w_2 q^{1/(1-\alpha)}/z_1^{*\alpha/(1-\alpha)}.$$

The first-order condition for the firm's profit maximization is

$$p = c'(q|z^*) = w_2/(1-\alpha)\cdot(q/z_1^*)^{\alpha/(1-\alpha)}$$

Now, z_1^* can be computed as the long-run demand for capital:

$$z_1^* = z_1(q^*) = [\alpha w_2/(1-\alpha)w_1]^{1-\alpha}\ q^*,$$

and the first-order condition can be rewritten as

$$p = w_1^{\alpha}\ w_2^{(1-\alpha)}/\alpha^{\alpha}(1-\alpha)^{1-\alpha}\cdot(q/q^*)^{\alpha/(1-\alpha)} = A\ (q/q^*)^{\alpha/(1-\alpha)}.$$

From here we can solve for the short-run supply function of a firm:

$$q^S(p) = q^*(p/A)^{(1-\alpha)/\alpha}.$$

The short-run supply function of the industry is

$$Q^S(p) = J^*q^S(p) = J^*q^*(p/A)^{(1-\alpha)/\alpha} = Q^*(p/A)^{(1-\alpha)/\alpha}.$$

10.G.1. (i) Firm j solves

$$\underset{q_j \in \mathbb{R}^M_+}{\text{Max}} \sum_{m=1}^{M} p^*_m q_{mj} - c_j(q_{1j}, ..., q_{Mj}).$$

The first-order conditions for all firms are

$$p^*_m \le \partial c_j(q^*_{1j}, ..., q^*_{Mj})/\partial q_{mj}, \text{ with equality if } q^*_{mj} > 0, \qquad (1)$$
$$m = 1, .., M, \ j = 1, .., J.$$

The firms' first-order conditions give JM equations.

Consumer i solves

$$\underset{x_i \in \mathbb{R}^M_+,\ m_i \in \mathbb{R}}{\text{Max}} \quad m_i + \sum_{m=1}^{M} \phi_i(x_{1i}, ..., x_{Mi})$$
$$\text{s.t. } m_i + \sum_{m=1}^{M} p^*_m x_{mi} \le \omega_i + \sum_{j=1}^{J} \theta_{ij} \left(\sum_{m=1}^{M} p^*_m q^*_{mj} - c_j(q^*_{1j}, ..., q^*_{Mj})\right)$$

The first-order conditions for all consumers are

$$\partial \phi_i(x^*_{1i}, ..., x^*_{Mi})/\partial x_{mi} \le p^*_m, \text{ with equality if } x^*_{mi} > 0, \qquad (2)$$
$$m = 1, .., M, \ i = 1, .., I.$$

The consumers' first-order conditions give IM equations.

The market-clearing conditions are

$$\sum_{i=1}^{I} x^*_{mi} = \sum_{j=1}^{J} q^*_{mj}, \ m = 1, .., M. \qquad (3)$$

(By Lemma 10.B.1, we do not need to check that the market for numeraire clears.)

Thus, we have the total of $JM + IM + M = (J + I + 1)M$ equations and $(J + I + 1)M$ unknowns - prices, output vectors, and consumption vectors. Since the equations (1), (2), and (3) do not contain consumers' wealth levels

ω_i, the set of solutions to these equations is independent of consumer wealth.

(ii) Consider the following welfare-maximization problem:

$$\text{Max } \sum_{i=1}^{I} \phi_i(x_{1i}, \ldots, x_{Mi}) - \sum_{j=1}^{J} c_j(q_{1j}, \ldots, q_{Mj}) + \omega \qquad (*)$$

$$\text{s.t. } \sum_{i=1}^{I} x^*_{mi} - \sum_{j=1}^{J} q^*_{mj} = 0,\ m = 1, \ldots, M.$$

By examining the first-order conditions to (*), it is easy to see that an allocation $(x^*_1, \ldots, x^*_I, q^*_1, \ldots, q^*_I)$ solves (*) if and only $(p^*, x^*_1, \ldots, x^*_I, q^*_1, \ldots, q^*_I)$ solves (1)-(3) for some price vector p^*.

(For conciseness, every variable here

has M components corresponding to the non-numeraire goods.)

Suppose in negation that $(p, x_1, \ldots, x_I, q_1, \ldots, q_J)$ and $(p', x'_1, \ldots, x'_I, q'_1, \ldots, q'_J)$ are both solutions to (1) - (3), and therefore to (*), and that $x_k \neq x_k$ for some k. Take

$$x''_i = 1/2\ x_i + 1/2\ x'_i \text{ for every } i = 1, \ldots, I,$$

$$q''_j = 1/2\ q_j + 1/2\ q'_j \text{ for every } j = 1, \ldots, J.$$

Clearly, $(x''_1, \ldots, x''_I, q''_1, \ldots, q''_J)$ satisfies the constraint in (*), since both $(x_1, \ldots, x_I, q_1, \ldots, q_J)$ and $(x'_1, \ldots, x'_I, q'_1, \ldots, q'_J)$ do. As cost functions are convex, we must have

$$c_j(q''_j) \leq 1/2\ c_j(q_j) + 1/2\ c_j(q'_j) \text{ for every } j = 1, \ldots, J.$$

As utility unctions ϕ_i's are strictly convex, we must have

$$\phi_i(x''_i) \geq 1/2\ \phi_i(x_i) + 1/2\ \phi_i(x'_i) \text{ for every } i \neq k, \text{ and}$$

$$\phi_k(x_k'') > 1/2\ \phi_k(x_k) + 1/2\ \phi_k(x'_k).$$

Adding up all the inequalities, we obtain

$$\sum_{i=1}^{I} \phi_i(x''_i) - \sum_{j=1}^{J} c_j(q''_j) > \quad 1/2\ [\sum_{i=1}^{I} \phi_i(x'_i) - \sum_{j=1}^{J} c_j(q'_j)] + $$
$$+ 1/2\ [\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)],$$

which contradicts the assumption that $(x_1, \ldots, x_I, q_1, \ldots, q_J)$ and $(x'_1, \ldots,$

x'_I, q'_1, ..., q'_J) are both solutions to (*).

Therefore, individual consumption levels x_i's at a solution to (*), and thus at a solution to (1)-(3), are uniquely determined.

(iii) The optimal aggregate production levels are $\sum_{j=1}^{J} q_j = \sum_{i=1}^{I} x_i$, and they are uniquely determines since all the x_i's are uniquely determined.

(iv) Now suppose that the cost functions $c_j'(\cdot)$ are strictly convex.

Suppose in negation that (x_1, .., x_I, q_1, ..., q_J) and (x'_1, .., x'_I, q'_1, ..., q'_J) are both solutions to (1)-(3), and therefore to (*), and that $q_k \neq q'_k$ for some k. Take

$$x''_i = 1/2\ x_i + 1/2\ x'_i \text{ for every } i = 1, ..., I,$$

$$q''_j = 1/2\ q_j + 1/2\ q'_j \text{ for every } j = 1, ..., J.$$

Clearly, (x''_1, .., x''_I, q''_1, ..., q''_J) satisfies the constraint in (10.D.2), since both (x_1, .., x_I, q_1, ..., q_J) and (x'_1, .., x'_I, q'_1, ..., q'_J) do. As cost functions are strictly convex, we must have

$$c_j(q''_j) \le 1/2\ c_j(q_j) + 1/2\ c_j(q'_j) \text{ for every } j \neq k, \text{ and}$$

$$c_k(q''_k) < 1/2\ c_k(q_k) + 1/2\ c_k(q'_k).$$

As utility unctions ϕ_i's are convex, we must have

$$\phi_i(x''_i) \ge 1/2\ \phi_i(x_i) + 1/2\ \phi_i(x'_i) \text{ for every } i = 1, ..., I.$$

Adding up all the inequalities, we obtain

$$\sum_{i=1}^{I} \phi_i(x''_i) - \sum_{j=1}^{J} c_j(q''_j) > \quad 1/2\ [\sum_{i=1}^{I} \phi_i(x'_i) - \sum_{j=1}^{J} c_j(q'_j)] + $$
$$+ 1/2\ [\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)],$$

which contradicts the assumption that (x_1, .., x_I, q_1, ..., q_J) and (x'_1, .., x'_I, q'_1, ..., q'_J) are both solutions to (*).

Therefore, individual production levels q_i's at a solution to (*), and thus at a solution to (1)-(3), are uniquely determined.

10.G.2. The first-order conditions (1) in Exercise 10.G.1 give for output of good l:

$$p_l^* \le c'_{lj}(q_{lj}^*) \text{ , with equality if } q_{lj}^* > 0,\ j = 1, \ldots, J. \tag{1}$$

The first-order conditions (2) in Exercise 10.G.2 give for consumption of good l:

$$\phi'_{li}(x_{li}^*) \le p_l^*, \text{ with equality if } x_{li}^* > 0,\ i = 1, \ldots, I. \tag{2}$$

The market clearing condition for good l is

$$\sum_{i=1}^{I} x_{li}^* = \sum_{j=1}^{J} q_{lj}^*. \tag{3}$$

(1)-(3) is a system of $J+I+1$ equations with $J+I+1$ unknowns - $(p_l^*, x_{l1}^*, \ldots, x_{lI}^*, q_{l1}^*, \ldots, q_{lJ}^*)$. These equations do not contain prices and quantities in the other markets. From these equations, the price and quantities in market l can be determined independently of other markets.

Total welfare can be computed as

$$S(x_1, \ldots, x_I, q_1, \ldots, q_J) = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) =$$

$$= [\sum_{i=1}^{I} \phi_{li}(x_{li}) - \sum_{j=1}^{J} c_{lj}(q_{lj})] + [\sum_{i=1}^{I} \phi_{-l,i}(x_{-l,i}) - \sum_{j=1}^{J} c_{-l,j}(q_{-l,j})] =$$

$$= S_l(x_{l1}, \ldots, x_{lI}, q_{l1}, \ldots, q_{lJ}) + S_{-l}(x_{-l,1}, \ldots, x_{-l,I}, q_{-l,1}, \ldots, q_{-l,J}).$$

The first term describes the surplus in market l, and the second term describes the surplus in all the other markets. The first term takes the form

$$S_l(x_{l1}, \ldots, x_{lI}, q_{l1}, \ldots, q_{lJ}) = \sum_{i=1}^{I} \phi_{li}(x_{li}) - \sum_{j=1}^{J} c_{lj}(q_{lj}),$$

which is exactly equation (10.E.1) in the textbook. The welfare analysis of market l can proceed in the same way as in Section 10.E, without consideration of other markets.

If all goods are separable, every market l can be solved independently of others using equations (1)-(3). Moreover, total surplus can be written as

$$S(x_1, \ldots, x_I, q_1, \ldots, q_J) = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) =$$

$$= \sum_{l=1}^{L} \Big[\sum_{i=1}^{I} \phi_{li}(x_{li}) - \sum_{j=1}^{J} c_{lj}(q_{lj}) \Big] = \sum_{l=1}^{L} S_l(x_{l1}, \ldots, x_{lI}, q_{l1}, \ldots, q_{lJ}).$$

Therefore, welfare analysis of every market can be performed independently of other markets.

10.G.3. (a) Let p_2^*, p_3^* be the market prices of goods 2 and 3, let x_2^*, x_3^* be consumption of these goods by each consumer (since all consumers are identical, the will consume the same amounts), and let q_2^*, q_3^* be production of these goods by the firm. Assuming $(x_2^*, x_3^*) >> 0$, the market equilibrium with taxes t_1, t_2 on goods 1 and 2 is a solution to the following equations, which are a modification of conditions (1)-(3) of Exercise 10.G.1:

$$p_2^* = c_2 + t_2; \qquad (1.1)$$

$$p_3^* = c_3 + t_3; \qquad (1.2)$$

$$p_2^* = \partial\phi(x_2^*, x_3^*)/\partial x_2; \qquad (2.1)$$

$$p_3^* = \partial\phi(x_2^*, x_3^*)/\partial x_3; \qquad (2.2)$$

$$Ix_2^* = q_2^*; \qquad (3.1)$$

$$Ix_3^* = q_3^*. \qquad (3.2)$$

Equations (1.1) and (1.2) pin down equilibrium prices, and equations (2.1), (2.2) pin down consumption of goods 2 and 3. Equations (3.1) and (3.2) pin down aggregate production levels. Since production exhibits constant returns to scale, aggregate production can be allocated across firms in an arbitrary fashion.

Assuming that tax revenue is redistributed lump-sum across consumers, aggregate surplus can be computed as

$$S = I\phi(x_2^*, x_3^*) - (c_2 q_2^* + c_3 q_3^*) = I\phi(x_2^*, x_3^*) - c_2 I x_2^* - c_3 I x_3^*.$$

Let $x_2^*(p_2,p_3)$ and $x_3^*(p_2,p_3)$ be consumer demand functions, i.e. the solutions to (2.2), (2.3). Then aggregate surplus can be expressed as a function of taxes:

$$S(t_2,t_3) = I[\phi(x_2^*(c_2+t_2,\ c_3+t_3),x_3^*(c_2+t_2,\ c_3+t_3)) -$$
$$- c_2x_2^*(c_2+t_2,\ c_3+t_3)) - c_3x_3^*(c_2+t_2,\ c_3+t_3))] \qquad (*)$$

When we set $t_3 = 0$ and differentiate with respect to t_2, using (2.1); (2.2), (1.1), (1.2), we obtain

$$\partial S(t_2,0)/\partial t_2 = I[(p_2^* - c_2)\partial x_2^*(c_2+t_2,\ c_3)/\partial p_2 + (p_3^* - c_3)\partial x_3^*(c_2+t_2,\ c_3)/\partial t_2] =$$
$$= t_2I\partial x_2^*(c_2+t_2,\ c_3)/\partial p_2.$$

Thus, the marginal welfare change is the same as the marginal change in the area between supply and demand curves for good 2 (the derivative of expression (10.E.6) in the textbook), holding the price of good 3 constant at c_3.

(b) When we differentiate (*) with respect to t_2, using (2.1), (2.2), (1.1), (1.2), we obtain

$$\partial S(t_2,t_3)/\partial t_2 = I[t_2\partial x_2^*(c_2+t_2,\ c_3+t_3)/\partial p_2 + t_3\partial x_3^*(c_2+t_2,\ c_3 + t_3)/\partial p_2].$$

The first term in the expression is the change in the area between supply and demand curves for good 2 (the derivative of expression (10.E.6) in the textbook), holding the price of good 3 constant at c_3+t_3. The sign of the second term is the sign of $\partial x_3^*(p_2,p_3)/\partial p_2$. This derivative is positive when goods 2 and 3 are substitutes, then the area measure would overstate the welfare loss from taxation. When goods 2 and 3 are complements, this derivative is negative, and the area measure would understate the welfare loss from taxation.

10.G.4. Differentiating (*) in Exercise 10.G.3 with respect to t_2 and t_3, we obtain

$\partial S(t_2,t_3)/\partial t_2 = I[t_2\partial x_2^*(c_2+t_2,\ c_3+t_3)/\partial p_2 + t_3\partial x_3^*(c_2+t_2,\ c_3+t_3)/\partial p_2]$;

$\partial S(t_2,t_3)/\partial t_3 = I[t_2\partial x_2^*(c_2+t_2,\ c_3+t_3)/\partial p_3 + t_3\partial x_3^*(c_2+t_2,\ c_3+t_3)/\partial p_3]$.

Now, $S(t_2,t_3) - S(0,0)$ can be expressed, for example, as

$$S(t_2,t_3) - S(0,0) = S(t_2,0) - S(0,0) + S(t_2,t_3) - S(t_2,0) =$$

$$= \int_0^{t_2} \partial S(s,0)/\partial p_2\ ds + \int_0^{t_3} \partial S(t_2,s)/\partial p_3\ ds =$$

$$= \int_0^{t_2} Is\partial x_2^*(c_2+s,\ c_3)/\partial p_2\ ds +$$

$$+ \int_0^{t_3} I[t_2\partial x_2^*(c_2+t_2,\ c_3+s)/\partial p_3 + s\partial x_3^*(c_2+t_2,\ c_3+s)/\partial p_3]\ ds.$$

10.G.5. (a) Let p_2^*, p_3^* be the market prices of goods 2 and 3, let x_2^*, x_3^* be consumption of these goods by each consumer (since all consumers are identical, the will consume the same amounts), and let q_{2j}^*, q_{3j}^* be production of these goods by firm j. Assuming $(x_2^*,\ x_3^*) >> 0$, the market equilibrium with taxes t_1, t_2 on goods 1 and 2 is a solution to the following equations, which are a modification of conditions (1)-(3) of Exercise 10.G.1:

$$p_2^* = c_2'(q_2^*) + t_2; \qquad (1.1)$$

$$p_3^* = c_3'(q_3^*) + t_3; \qquad (1.2)$$

$$p_2^* = \partial\phi(x_2^*,x_3^*)/\partial x_2; \qquad (2.1)$$

$$p_3^* = \partial\phi(x_2^*,x_3^*)/\partial x_3; \qquad (2.2)$$

$$Ix_2^* = q_2^*; \qquad (3.1)$$

$$Ix_3^* = q_3^*. \qquad (3.2)$$

Eliminating prices and outputs from the equations, we obtain

$$\partial\phi(x_2^*,x_3^*)/\partial x_2 = c_2'(Ix_2^*) + t_2 \qquad (1)$$

$$\partial\phi(x_2^*,x_3^*)/\partial x_3 = c_3'(Ix_3^*) + t_3 \qquad (2)$$

Now, we differentiate these two equations with respect to t_2, treating x_2^* and x_3^* as functions of t_2:

$$\phi_{22}\, x_2' + \phi_{22}\, x_3' = I\, c_2''\, x_2' + 1$$

$$\phi_{23}\, x_2' + \phi_{33}\, x_3' = Ic_3''\, x_3'$$

(To avoid cumbersome expressions, we have defined $\phi_{ls} = \partial\phi(x_2^*,x_3^*)/\partial x_l \partial x_s$, $c_l'' = c_l''(Ix_l^*)$, $x_l' = dx_l^*/dt_2$).

From the above two equations we can express x_3':

$$x_3' = \phi_{23}/[\phi_{23}^{\,2} - \phi_{22}\phi_{33} + Ic_2''\phi_{33} + Ic_3''\phi_{22} - I^2c_2''c_3''].$$

Differentiating equation (1.2), we now obtain:

$$dp_3^*/dt_2 = Ic_3''x_3' = Ic_3''\phi_{23}/[\phi_{23}^{\,2} - \phi_{22}\phi_{33} + Ic_2''\phi_{33} + Ic_3''\phi_{22} - I^2c_2''c_3''].$$

Since $u(\cdot)$ is concave, $D^2u(\cdot)$ must be negative semidefinite, which implies, in particular, that ϕ_{22} it follows that $\phi_{22} \le 0$, $\phi_{33} \le 0$, $\phi_{23}^2 - \phi_{22}\phi_{33} \le 0$. Since the cost functions are strictly convex, we must have $c_2'' > 0$, $c_3'' > 0$. These inequalities together imply that the denominator of the above expression for dp_3^*/dt_2 is always negative. Therefore, the sign of dp_3^*/dt_2 is negative the sign of ϕ_{23}. When $\phi_{23} < 0$, goods 1 and 2 are substitutes, and an increase in t_2 raises p_3^*. Conversely, when $\phi_{23} > 0$, goods 1 and 2 are complements, and an increase in t_2 reduces p_3^*.

(b) [First printing errata: Assume that $c(q_2, q_3) = c_2q_2 + c_3(q_3)$.]

Assuming that tax revenue is redistributed lump-sum across consumers, aggregate surplus can be computed as

$$S = I\phi(x_2^*,x_3^*) - (c_2q_2^* + c_3(q_3^*)) = I\phi(x_2^*,x_3^*) - c_2Ix_2^* - c_3(Ix_3^*).$$

Let $x_2^*(p_2,p_3)$ and $x_3^*(p_2,p_3)$ be consumer demand functions, i.e. the solutions to (2.2), (2.3). Then aggregate surplus can be expressed as a function of taxes:

$$S(t_2,t_3) = I[\phi(x_2^*(c_2+t_2,\ p_3^*),x_3^*(c_2+t_2,\ p_3^*)) -$$
$$- c_2x_2^*(c_2+t_2,\ p_3^*))] - c_3(Ix_3^*(c_2+t_2,\ p_3^*))]$$

Differentiating the above expression with respect to t_2 and using (2.1), (2.2), (1.1), and (1.2), we obtain

$$\partial S(t_2,t_3)/\partial t_2 = It_2\ \partial x_2^*(c_2+t_2,\ p_3^*)/\partial p_2 + It_3\ \partial x_3^*(c_2+t_2,\ p_3^*)/\partial p_2 + \qquad (*)$$
$$+ [It_2\ \partial x_2^*(c_2+t_2,p_3^*)/\partial p_3 + It_3\ \partial x_3^*(c_2+t_2,p_3^*)/\partial p_3]\ dp_3^*/dt_2.$$

Now, in Exercise 10.G.3 we calculate this derivative assuming that p_3^* is fixed at $c_3 + t_3$, i.e. $dp_3^*/dt_2 = 0$. This is fine when technology exhibits constant returns to scale, because then the equilibrium price of good 3 is unaffected by t_2. But in the present case, an increase in t_2 raises p_3^* when goods are substitutes and lowers p_3 when goods are complements (as shown in part (a)). How would ignoring this effect bias are calculation of the welfare loss? We always know that $\partial x_3^*(\cdot,\cdot)/\partial p_3 < 0$. With complements, $\partial x_2^*(\cdot,\cdot)/\partial p_3 < 0$, and we know from part (a) that $dp_3^*/dt_2 < 0$. Thus, ignoring using the formula in Exercise 10.G.3 (b) ignoring the correction terms in (*) would result in an overstatement of the welfare loss. With substitutes, $\partial x_2^*(\cdot,\cdot)/\partial p_3 > 0$, and we know from part (a) that $dp_3^*/dt_2 > 0$. Thus, the first correction term is is negative and the second correction term is positive. The net effect is unclear. If $t_3 = 0$, the formula in Exercise 10.G.3 (b) would be overstating the welfare loss. But if t_3 is large and t_2 is close to zero, the formula in Exercise 10.G.3 (b) would be understating the welfare loss (which would actually be a gain).

CHAPTER 11

11.B.1 Note first that the technologies are CRS so that producers will either choose no production or will choose producing at full capacity. Assume that the prices of apples and honey in the market are such that producing both products is efficient. Since technologies are CRS, maximum profits are equivalent to minimum costs, and generically Jones will choose only one of the possible technologies. In particular, he will choose artificial production if and only if $H(p_m + w) < H(bp_b + kw)$, or, if and only if

$$p_m < bp_b + (k - 1)w .$$

Now, artificial production is socially optimal if and only if the total cost of producing both capacities is cheaper under artificial production of honey. i.e., if and only if

$$H(p_m + w) + Aw < H(bp_b + kw) + (A - \frac{Hb}{c})w ,$$

or, if and only if

$$p_m < bp_b + (k - \frac{b}{c})w .$$

So, if $\frac{b}{c} \leq 1$, Jones' choice of honey production will be socially efficient. If $\frac{b}{c} > 1$ then for $bp_b + (k - \frac{b}{c})w < p_m < bp_b + (k - 1)w$ Jones will choose artificial production but natural production is socially optimal (for other values of p_m Jones' choice will be socially optimal).

Clearly, k and p_b have no effect on the efficiency of Jones' choice. changes in the value of p_m will cause Jones' choice to be efficient or inefficient as described above. The values of b and c will determine the degree of the externality: as b increases, or as c decreases, the externality becomes stronger and thus the possibility of inefficiency is larger (the interval in which p_m will cause an inefficient choice becomes larger).

Smith's costs of producing A apples when Jones uses artificial honey production is Aw, while his costs when Jones uses natural honey production is $(A - \frac{Hb}{c})w$. Therefore, Smith will be willing to pay up to $\frac{Hb}{c}\cdot w$ in order to bribe Jones to switch to natural honey.

It is obvious that if both farms were owned by the same owner then full efficiency would be reached since the owner would minimize total costs. A government can achieve such efficiency by subsidizing bees so that when it is socially optimal to produce naturally, the subsidized price of bees will cause Jones to do so. This can be financed by a lump sum tax on Smith.

11.B.2 (a) The problem becomes:

$$\max_{h,T} \quad \phi_1(h) + w_1 - T$$

$$\text{s.t.} \quad \phi_2(h, w_2+T) \geq \bar{u}$$

Letting λ denote the Kuhn-Tucker multiplier, and assuming an interior solution, the FOCs are:

$$(1) \quad \phi_1'(h) + \lambda \cdot \frac{\partial \phi_2(h, w_2+T)}{\partial h} = 0$$

$$(2) \quad -1 + \lambda \cdot \frac{\partial \phi_2(h, w_2+T)}{\partial w} = 0\ ,$$

which together yield,

$$\phi_1'(h^o) = - \frac{\partial \phi_2(h^o, w_2+T^o)/\partial h}{\partial \phi_2(h^o, w_2+T^o)/\partial w}\ .$$

(b) Consider consumer 2's problem:

$$\max_{h} \quad \phi_2(h, w_2 - p_h \cdot h)$$

Letting $h(p_h, w_2)$ denote his demand for h, the FOC satisfies:

$$\frac{\partial \phi_2(h(p_h,w_2), w_2 - p_h \cdot h(p_h,w_2))}{\partial h} - p_h \cdot \frac{\partial \phi_2(h(p_h,w_2), w_2 - p_h \cdot h(p_h,w_2))}{\partial h} = 0$$

Differentiating this expression with respect to w_2 we get:

$$\frac{\partial^2\phi_2}{\partial h^2}\cdot\frac{\partial h(p_h,w_2)}{\partial w_2} + \frac{\partial^2\phi_2}{\partial w_2\partial h}\cdot\left(1 - p_h\cdot\frac{\partial h(p_h,w_2)}{\partial w_2}\right) - p_h\cdot\frac{\partial^2\phi_2}{\partial h\partial w_2}\cdot\frac{\partial h(p_h,w_2)}{\partial w_2}$$

$$- p_h\cdot\frac{\partial^2\phi_2}{\partial w_2^2}\cdot\left(1 - p_h\cdot\frac{\partial h(p_h,w_2)}{\partial w_2}\right) = 0 .$$

Under very general conditions (which we usually assume) we have that the cross partials are equivalent, i.e., that $\frac{\partial^2\phi_2}{\partial w_2\partial h} = \frac{\partial^2\phi_2}{\partial h\partial w_2}$, which changes the above to (we discard the subscript 2 since only consumer 2 is relevant):

$$(*) \quad \frac{\partial h(p_h,w)}{\partial w} = \frac{p_h\cdot\frac{\partial^2\phi}{\partial w_2^2} - \frac{\partial^2\phi}{\partial h\partial w}}{\frac{\partial^2\phi}{\partial h^2} - 2p_h\cdot\frac{\partial^2\phi}{\partial h\partial w} + p_h^2\cdot\frac{\partial^2\phi}{\partial w_2^2}} .$$

From the FOC we have that $p_h = \frac{\partial\phi/\partial h}{\partial\phi/\partial w}$, and substituting this into the expression (*) above will express the wealth effect in the required terms.

(c) From part (a) above we have that $\phi_1'(h^o) = -\frac{\partial\phi_2/\partial h}{\partial\phi_2/\partial w_2}$. Differentiating this with respect to w_2 and rearranging terms we get:

$$(**) \quad \frac{\partial h^o}{\partial w_2} = \frac{\frac{\partial\phi_2}{\partial h}\cdot\frac{\partial^2\phi_2}{\partial w_2^2} - \frac{\partial\phi_2}{\partial w_2}\cdot\frac{\partial^2\phi_2}{\partial h\partial w}}{\left(\frac{\partial\phi_2}{\partial w_2}\right)^2\cdot\phi_1''(h^o) + \frac{\partial\phi_2}{\partial w_2}\cdot\frac{\partial^2\phi_2}{\partial h^2} - \frac{\partial\phi_2}{\partial h}\cdot\frac{\partial^2\phi_2}{\partial h\partial w_2}} .$$

We need to show that (**) is positive (for a positive externality. To proceed,

we define consumer 2's indirect utility function in the usual way, denoted by:

$$\psi_2(p_h,w_2) \equiv \underset{h}{\text{Max}} \quad \phi_2(h,w_2 - p_h\cdot h) .$$

Now, consumer 2's demand for the externality, $h(p_h,w_2)$, can be obtained using

Roy's identity:

$$h(p_h,w_2) = -\frac{\partial\psi_2(p_h,w_2)/\partial p_h}{\partial\psi_2(p_h,w_2)/\partial w_2},$$

and differentiating this with respect to w_2 we get:

$$(3)\quad \frac{\partial h(p_h,w_2)}{\partial w_2} = -\frac{\partial^2\psi_2/\partial p_h\partial w_2}{\partial\psi_2/\partial w_2} + \frac{\partial\psi_2/\partial p_h}{(\partial\psi_2/\partial w_2)^2}\cdot\frac{\partial^2\psi_2}{\partial w_2^2}.$$

It is given that consumer 2's demand for the externality is normal, so that $\partial h(p_h,w_2)/\partial w_2 > 0$, which together with (3) above (after multiplying all terms in (3) by $(\partial\psi_2/\partial w_2)^2 > 0$) implies that:

$$(4)\quad \frac{\partial\psi_2}{\partial p_h}\cdot\frac{\partial^2\psi_2}{\partial w_2^2} > \frac{\partial^2\psi_2}{\partial p_h\partial w_2}\cdot\frac{\partial\psi_2}{\partial w_2}.$$

Since the externality is a normal good, it is not a Giffen good, which implies that $\frac{\partial h}{\partial p_h} < 0$, or in inverse terms, $\frac{\partial p_h}{\partial h} < 0$. Multiplying both sides of (4) by this we get:

$$(5)\quad \left(\frac{\partial\psi_2}{\partial p_h}\cdot\frac{\partial p_h}{\partial h}\right)\cdot\frac{\partial^2\psi_2}{\partial w_2^2} > \left(\frac{\partial^2\psi_2}{\partial p_h\partial w_2}\cdot\frac{\partial p_h}{\partial h}\right)\cdot\frac{\partial\psi_2}{\partial w_2}.$$

From consumer theory (chapter 3) we know that the following identities relate the indirect utility function with the utility function:

$$\left(\frac{\partial\psi_2}{\partial p_h}\cdot\frac{\partial p_h}{\partial h}\right) = \frac{\partial\phi_2}{\partial h} \quad ; \quad \left(\frac{\partial^2\psi_2}{\partial p_h\partial w_2}\cdot\frac{\partial p_h}{\partial h}\right) = \frac{\partial^2\phi_2}{\partial h\partial w} \quad ; \quad \frac{\partial\psi_2}{\partial w_2} = \frac{\partial\phi_2}{\partial w_2}.$$

Substituting these identities into (5) above we obtain:

$$\frac{\partial\phi_2}{\partial h}\cdot\frac{\partial^2\phi_2}{\partial w_2^2} < \frac{\partial^2\phi_2}{\partial h\partial w}\cdot\frac{\partial\phi_2}{\partial w_2},$$

which implies that the numerator of the expression (**) is negative. It is given in the model that $\phi_1'' < 0$, $\frac{\partial\phi_2}{\partial w_2} > 0$, and $\frac{\partial\phi_2}{\partial h} > 0$. Furthermore, strict quasiconcavity of $\phi_2(h,w_2)$ implies that $\frac{\partial^2\phi_2}{\partial w_2^2} < 0$, and $\frac{\partial^2\phi_2}{\partial h^2} < 0$. All these inequalities cause the denominator of (**) to be negative as well, thus we have shown that $\frac{\partial h^o}{\partial w_2} > 0$ for a positive externality. The reverse can be shown for the case of a negative externality using the same methods.

11.B.3 With a Pigovian tax, bargaining will change the level of the externality. To see this consider the situation where an optimal tax is chosen such that consumer 1 chooses the efficient level h^o, and assume (w.l.o.g.) that h is a negative externality. An infinitesimal reduction in h has a zero first-order effect on agent 1, and a positive first-order effect on agent 2. Therefore, there are gains from reduction of h below h^o, and the agents can bargain to obtain some $\hat{h} < h^o$ which will be the final choice of h given the Pigovian tax and the ability to bargain.

On the other hand, this argument will not hold for quotas. At the point h^o, an infinitesimal reduction in the level of h will reduce consumer 1's utility by $\phi_1'(h^o)$ and will increase 2's utility by $-\phi_2'(h^o)$, which cancel out given the

optimal choice of h^o. Therefore there are no gains from trade and the optimal level will not be bargained down (the opposite holds for a positive externality).

11.B.4 Assume that $\phi_2(h,e)$ is such that given any level of h chosen by consumer 1, there is a unique optimal level of e that is consumer 2's "best response". That is, there exists a function $e^*(h)$. Since e will be optimally chosen by 2, and it has no external effect on consumer 1, there is no reason to tax or subsidize this activity - it is not an externality. It may, however, still be optimal to impose some tax/subsidy on h, even though consumer 2 can choose an e in response.

11.B.5 (a) The firm solves $\underset{q,h}{\text{Max}}\ p\cdot q - c(q,h)$, and the FOCs are:

$$(1)\quad p \le \frac{\partial c(q^*,h^*)}{\partial q}\text{, with equality if } q^* > 0\text{ ,}$$

$$(2)\quad 0 \le \frac{\partial c(q^*,h^*)}{\partial h}\text{, with equality if } h^* > 0\text{ .}$$

(b) Since the consumer's utility function is quasilinear with respect to money, the utility posibility frontier is a linear line with slope -1, and therefore we can find the Pareto optimal level of h and q by maximizing the sum of the profit function and the utility function without wealth, i.e.,

$$\underset{q,h}{\text{Max}}\quad p\cdot q - c(q,h) + \phi(h)\text{ ,}$$

which yields the FOCs,

$$(3)\quad p \le \frac{\partial c(q^o,h^o)}{\partial q}\text{, with equality if } q^o > 0\text{ ,}$$

$$(4)\quad \phi'(h) \le \frac{\partial c(q^o,h^o)}{\partial h}\text{, with equality if } h^o > 0\text{ .}$$

(c) Let t denote the tax rate on output, the firm solves:

$$\underset{q,h}{\text{Max}}\quad p\cdot q - c(q,h) - t\cdot q\text{ ,}$$

which yields the FOCs,

$$(5)\quad p \le \frac{\partial c(q,h)}{\partial q} + t\text{ , with equality if } q > 0\text{ ,}$$

$$(6)\quad 0 \le \frac{\partial c(q,h)}{\partial h}\text{, with equality if } h > 0\text{ .}$$

This gives the same level of $h = h^*$ as in part (a) above. If, however, a tax is imposed on h, say τ, the firm solves:

$$\underset{q,h}{\text{Max}}\quad p\cdot q - c(q,h) - \tau\cdot h\text{ ,}$$

which yields the FOCs,

$$(7)\quad p \le \frac{\partial c(q,h)}{\partial q} \quad \text{, with equality if } q > 0 \text{ ,}$$

$$(8)\quad -\tau \le \frac{\partial c(q,h)}{\partial h} \text{, with equality if } h > 0 \text{ ,}$$

and if we set $\tau = -\phi_1'(h^o)$ then this gives the level $h = h^o$ as in part (b) above, and efficiency is restored.

(d) Let t denote the tax rate on output, the firm solves:

$$\underset{q}{\text{Max}} \quad p\cdot q - c(q,\alpha q) - t\cdot q \text{ ,}$$

which yields the FOC,

$$(9)\quad p \le \frac{\partial c(q,\alpha q)}{\partial q} + \frac{\partial c(q,\alpha q)}{\partial q} + t \quad \text{, with equality if } q > 0 \text{ ,}$$

and if we set $t = -\alpha\cdot\phi_1'(h^o)$ then the pair (q^o,h^o) will solve this FOC as in part (b) above, and efficiency is restored.

11.C.1 Following the analysis of he textbook, it can easily be shown that the subsidies which will lead to an efficient equilibrium are $s_i = \sum_{j\neq i}\phi_j'(q^o)$.

11.C.2 The analysis of the textbook regarding private provision of a public good yields two FOCs (numbered as in the textbook):

$$\phi_i'(x^*) \le p^* \quad \text{with equality if } x_i^* > 0 \qquad (11.C.3) \text{ ,}$$

$$p^* \le c'(q) \quad \text{with equality if } q > 0 \qquad (11.C.4) \text{ .}$$

If we have a subsidy s for the firm then (11.C.4) becomes:

$$p^* + s \le c'(q) \quad \text{with equality if } q > 0 \text{ .}$$

Let $k \equiv \underset{i}{\text{Argmax}}\ \{\phi_i'(x^o)\}$, i.e., k is the index of the consumer with the highest marginal utility at the optimal level x^o (there will generically be only one such consumer, if there are more then simple modifications will do). Set the subsidy in the following way: $s = \sum_{j\neq k}\phi_j'(x^o)$. The pair (x^o,p^o), where

$p^o = \phi_k'(x^o)$, will constitute a competitive equilibrium where consumer k will purchase exactly $x_i = x^o$, and all other consumers will have $\phi_j'(x^o) < p^o$ so they will not purchase any additional amount of the public good. Clearly, the firm will produce $q^o = x^o$ since it faces a gross price of $p^o + s = \sum_i \phi_i'(x^o)$.

11.C.3 First, we refer to the solution of exercise 10.E.3 for the basic setup and the results of the Ramsey taxation problem without a public good. Second, We modify the setup of this exercise so that the consumers have identical utility functions so that we can view the problem as if we had a representative consumer. The more general case is a straightforward extension, yet the algebra is messier. Furthermore, the qualitative results are not affected. Let each consumer's utility function be given by:

$$u(x_0, x_1, \ldots, x_L) = x_1 + \sum_{l=2}^{L} \phi_l(x_l, x_0).$$

The government budget constraint becomes (the cost is divided by the number of consumers since we are looking at a representative consumer):

$$\sum_{l=2}^{L} t_l x_l = \frac{c(x_0)}{I}.$$

The Lagrangian of the problem becomes:

$$L = x_1 + \sum_{l=2}^{L} \phi_l(x_l, x_0) + \lambda_1\left(\omega - x_1 - \sum_{l=2}^{L}(c_l + t_l)x_l\right) + \sum_{l=2}^{L} \lambda_l\left(c_l + t_l - \frac{\partial \phi_l(x_l, x_0)}{\partial x_l}\right)$$

$$+ \lambda_{L+1}\left(\frac{c(x_0)}{I} - \sum_{l=2}^{L} t_l x_l\right)$$

which yields the FOCs

(1) $$\frac{\partial L}{\partial x_1} = 1 - \lambda_1 = 0,$$

(2) $$\frac{\partial L}{\partial x_l} = \frac{\partial \phi_l(x_l, x_0)}{\partial x_l} - \lambda_1(c_l + t_l) - \lambda_l \frac{\partial^2 \phi_l(x_l, x_0)}{\partial x_l^2} - \lambda_{L+1} t_l = 0, \; l = 2, \ldots L,$$

(3) $$\frac{\partial L}{\partial x_0} = \sum_{l=2}^{L} \frac{\partial \phi_l(x_l, x_0)}{\partial x_0} - \sum_{l=2}^{L} \lambda_l \frac{\partial^2 \phi_l(x_l, x_0)}{\partial x_l \partial x_0} + \frac{c'(x_0)}{I} = 0,$$

(4) $$\frac{\partial L}{\partial t_l} = -\lambda_1 x_l + \lambda_l - \lambda_{L+1} x_l = 0, \ l=2,\dots,L.$$

Note that the first, second and fourth of these equations are the equivalent to those in the regular Ramsey taxation problem, and the conclusions of that model apply. The difference is that here the level of taxes to be generated is endogenous and is chosen to maximize the utility of the representative consumer. The condition for this is in (3) above, which states that the marginal social benefit from a marginal unit of the public good must equal the marginal cost of that marginal unit.

II.D.1 Suppose that $h_1 = h_2 = \cdots h_I \equiv h^*$ is a symmetric Nash equilibrium of the simultaneous move game. The it must be that for every i, h^* solves

$$\underset{h_i}{\text{Max}} \ \phi\left(\frac{h_i}{\frac{1}{I}h_i + \frac{I-1}{I}h^*}\right) - \frac{h_i^2}{2}$$

which yields the FOC

$$\phi'\left(\frac{h_i}{\frac{1}{I}h_i + \frac{I-1}{I}h^*}\right) \cdot \frac{\left(\frac{1}{I}h_i + \frac{I-1}{I}h^*\right) - h_i \cdot \frac{1}{I}}{\left(\frac{1}{I}h_i + \frac{I-1}{I}h^*\right)^2} - h_i = 0 \ .$$

For h^* to be a NE this should hold for $h_i = h^*$, i.e., $\phi'(1) \cdot \dfrac{\frac{I-1}{I} \cdot h^*}{(h^*)^2} - h^* = 0$, which yields $h^* = \left[(1 - \frac{1}{I}) \cdot \phi'(1)\right]$.

Now, let $h_1 = h_2 = \cdots h_I \equiv h^o$ be a symmetric Pareto optimum. The utility of each student i will be $u_i = \phi(1) - \dfrac{(h^o)^2}{2}$, which immediately implies that $h^o = 0$ is the Pareto optimal outcome. The intuition is simple; by studying, every student imposes a negative externality on others, and the competitive outcome has too much studying.

11.D.2 For the Pareto optimal outcome we solve $\underset{\{h_i\}}{\text{Max}} \sum_{i=1}^{I}\left(\phi_i(h_i,\sum_j h_j) + w_i\right)$,

which yields the FOCs $\dfrac{\partial\phi_i}{\partial h_i} + \dfrac{\partial\phi_i}{\partial\sum_j h_j} + \sum_{k\neq i}\left(\dfrac{\partial\phi_k}{\partial\sum_j h_j}\right) \leq 0$ with equality if $h_i^o > 0$

for all $i=1,\ldots I$. On the other hand, in a competitive equilibrium each individual solves $\underset{h_i}{\text{Max}} \; \phi_i(h_i,h_i+\sum_{j\neq i}h_j) + w_i$, which yields the FOC (for every i) $\dfrac{\partial\phi_i}{\partial h_i} + \dfrac{\partial\phi_i}{\partial\sum_j h_j} \leq 0$ with equality if $h_i^* > 0$. If, e.g., $\dfrac{\partial\phi_i}{\partial h_i} > 0$ and the externality is negative (i.e, $\dfrac{\partial\phi_k}{\partial\sum_j h_j}$ for all k), then if $h_i^* > 0$ we will necessarily have $h_i^* > h_i^o$ (due to the concavity of ϕ). To restore the Pareto optimal outcome in a competitive equilibrium, we must set an individual tax for each i of $t_i = \sum_{k\neq i}\dfrac{\partial\phi_k(h_k^o,\sum_j h_j^o)}{\partial\sum_j h_j}$.

11.D.3 In a competitive equilibrium a price-taking firm j solves

$$\underset{q_j}{\text{Max}} \quad \pi_j(q_j,q_{-j}) = p\cdot q_j - c(q_j,Q) ,$$

where $Q = q_j + \sum_{k\neq j}q_k$. The FOC for this program is

$$p - c_q(q_j,Q) - c_Q(q_j,Q) = 0 ,$$

and the SOC is satisfied by the assumptions in the exercise,

$$- c_{qq}(q_j,Q) - 2c_{qQ}(q_j,Q) - c_{QQ}(q_j,Q) < 0 .$$

Therefore, this program has a unique solution q^*, independent of j. Let $Q^* = Jq^*$, then a competitive equilibrium Q^* will be determined by

$$(1) \quad p = c_q(\tfrac{Q^*}{J},Q^*) + c_Q(\tfrac{Q^*}{J},Q^*) ,$$

which has a solution for $p > 0$ because $c_q > 0$, $c_Q < 0$, and $c_q + Jc_Q > 0$ together imply that $c_q(\frac{Q^*}{J},Q^*) + c_Q(\frac{Q^*}{J},Q^*) > 0$.

In contrast, the program for maximizing total surplus is

$$\underset{q_1,\ldots,q_J}{\text{Max}} \quad S \equiv \int_0^Q p(x)dx - \sum_{j=1}^{J} c(q_j,Q) ,$$

and the FOCs for this program are

$$\frac{\partial S}{\partial q_j} = p(Q) - c_q(q_j,Q) - c_Q(q_j,Q) - \sum_{k\neq j} c_Q(q_k,Q) = 0 ,$$

and we can check that the assumptions ensure that the SOC is satisfied. Let q^o denote the solution to this program (again, independent of j), and let $Q^o = Jq^o$. The optimal Q^o will then be determined by

$$(2) \quad p(Q^o) = c_q(\tfrac{Q^o}{J},Q^o) + c_Q(\tfrac{Q^o}{J},Q^o) + (J-1)c_Q(\tfrac{Q^o}{J},Q^o) .$$

(Again, by $c_q > 0$, $c_Q < 0$, and $c_q + Jc_Q > 0$, a solution will exist.) Since $c_Q < 0$, (2) implies that $p(Q^o) < c_q(\frac{Q^o}{J},Q^o) + c_Q(\frac{Q^o}{J},Q^o)$. Also, since $p'(Q) \leq 0$, and $\frac{d}{dQ}\left[c_q(\frac{Q}{J},Q) + c_Q(\frac{Q}{J},Q)\right] = \frac{1}{J}\left(c_{qq}(\frac{Q}{J},Q) + (J+1)c_{qQ}(\frac{Q}{J},Q) + Jc_{QQ}(\frac{Q}{J},Q)\right) > 0$ (by assumption), then we must have that $Q^o > Q^*$. This is intuitive since firms ignore the positive externality that they create, and we have an under-production competitive equilibrium. To restore efficiency the government can subsidize production with a subsidy of $s = -(J-1)c_Q(\frac{Q^o}{J},Q^o)$. Firm j's FOC will then be $p - (J-1)c_Q(\frac{Q^o}{J},Q^o) = c_q(q_j,Q) + c_Q(q_j,Q)$, and it is easy to see that $Q = Q^o$ and $p = p(Q^o)$ will cause $q_j = \frac{Q^o}{J}$ to solve this FOC.

11.D.4 For the Pareto optimal outcome we solve

$$\underset{\{h_i\}}{\text{Max}} \ \sum_{i=1}^{I}\phi_i(h_1,\ldots,h_J) + \sum_{j=1}^{J}\pi_j(h_j) ,$$

which yields the FOCs $\sum_{i=1}^{I}\left(\frac{\partial\phi_i(h_1^o,\ldots,h_J^o)}{\partial h_j}\right) \leq \pi_j'(h_j^o)$ with equality if $h_j^o > 0$ for all $j=1,\ldots J$. On the other hand, in a competitive equilibrium each firm maximizes profits individually, and we get the FOC shown in condition (11.D.1) in the textbook. To restore the Pareto optimal outcome in a competitive equilibrium, we must set an individual tax for each j of

$t_j = -\sum_{i=1}^{I}\left(\frac{\partial\phi_i(h_1^o,\ldots,h_J^o)}{\partial h_j}\right)$. Each firm will face the same tax rate if and only if we have $\sum_{i=1}^{I}\left(\frac{\partial\phi_i(h_1^o,\ldots,h_J^o)}{\partial h_j}\right) = \sum_{i=1}^{I}\left(\frac{\partial\phi_i(h_1^o,\ldots,h_J^o)}{\partial h_k}\right)$ for all j,k.

11.D.5 [First Printing Errata: the assumption that $f(0)=0$ should be added.]

(a) This is a model of free entry so fishermen will send out boats as long as there are positive profits from doing so. Therefore, the equilibrium number of boats, b^*, will be reached when $p\cdot\frac{f(b^*)}{b^*} - r = 0$, or, $\frac{f(b^*)}{b^*} = \frac{r}{p}$. This condition is that average revenue equals average cost. (We ignore integer problems, but if we are to give the integer equilibrium number then it is b^* such that $p\cdot\frac{f(b^*)}{b^*} - r \geq 0$ and $p\cdot\frac{f(b^*+1)}{b^*+1} - r < 0$.)

(b) To characterize the optimal number of boats we must solve for maximum total surplus, i.e., $\text{Max}_b\ p\cdot f(b) - r\cdot b$, the FOC is $p\cdot f'(b^o) - r \leq 0$, which is necessary and sufficient since the SOC, $p\cdot f''(b) < 0$, is satisfied. Therefore, the condition for the optimal number of boats is $f'(b^o) = \frac{r}{p}$, i.e., that marginal revenue equals marginal (and in this case average) cost. Assuming that $f(0) = 0$ ensures that $b^o \leq b^*$ (equality only at 0).

(c) To restore efficiency we need the equilibrium condition satisfied at b^o, i.e., we need the tax level to satisfy $\frac{f(b^*)}{b^*} = \frac{r+t}{p}$, or $t = p\cdot\frac{f(b^*)}{b^*} - r$.

(d) Clearly, if owned by a single individual, the problem to be solved is exactly that solved in part **(b)** above, which results in b^o.

11.D.6 (a) First, if the firm decides to go off and generate any level of the externality, absent of an agreement, it solves $\text{Max}_h\ p(h) = \alpha + \beta h - \mu h^2$, the (necessary and sufficient) FOC is $\beta - 2\mu h^* = 0$, or $h^* = \frac{\beta}{2\mu}$. This yields the firm profits of $\pi(h^*) = \alpha + \frac{\beta^2}{4\mu}$, which is the firm's reservation profits.

A coalition of θI consumers making a take-it-or-leave-it offer to the firm

firm, and we assume that the coalition splits T equally among its members. We can therefore maximize the utility of a representative member, subject to the firm accepting the offer

$$\underset{h,T}{\text{Max}} \quad \frac{1}{I}\cdot(\gamma - \eta h) - \frac{T}{\theta I}$$

$$\text{s.t. } \pi(h) + T \geq \pi(h^*) \ .$$

Clearly, at the optimum the constraint will bind, thus for any h that solves the program, $T = \pi(h^*) - \pi(h) = \alpha + \frac{\beta^2}{4\mu} - (\alpha + \beta h - \mu h^2) = \frac{\beta^2}{4\mu} - \beta h + \mu h^2$. Substituting this into the objective function we obtain the (necessary and sufficient) FOC, $-\frac{\eta}{I} + \frac{\beta}{\theta I} - \frac{2\mu h}{\theta I} = 0$, which yields $h^o(\theta) = \frac{\beta - \theta\eta}{2\mu}$, and $T^o(\theta) = \frac{\theta^2\eta^2}{4\mu}$.

We can now determine the subgame perfect Nash equilibrium level of θ by requiring that no (marginal) individual outside the coalition wishes to enter it, given that the values $h^o(\theta)$ and $T^o(\theta)$ will be offered, and the firm will accept the offer. Given θ, the utility of an outsider to the coalition is:

$$u_{out} = \frac{1}{I}\cdot(\gamma - \eta h^o(\theta)) \ .$$

If this individual joins the coalition then θ increases by $\frac{1}{I}$ which reduces the equilibrium externality level, but this individual now has to share the transfer T and pay $\frac{T^o(\theta + \frac{1}{I})}{(\theta + \frac{1}{I})\cdot I} = \frac{(\theta + \frac{1}{I})\cdot\eta^2}{4\mu I}$. Thus an outsider will be indifferent between staying out and entering the coalition if:

$$\frac{1}{I}\cdot(\gamma - \eta h^o(\theta^o)) = \frac{1}{I}\cdot(\gamma - \eta h^o(\theta^o + \frac{1}{I})) - \frac{(\theta^o + \frac{1}{I})\cdot\eta^2}{4\mu I} \ ,$$

and substituting for $h^o(\cdot)$ from above we get,

$$\frac{(\theta^o + \frac{1}{I})\cdot\eta^2}{4\mu I} = \frac{\eta^2}{4\mu I} \ ,$$

which gives us the equilibrium fraction $\theta^o = \frac{1}{I}$, i.e., the equilibrium coalition consists of only one individual (the second individual may be coalition consists of only one individual (the second individual may be indifferent, therefore may enter).

(b) Note that $h^o(1)$ (the externality generated when $\theta = 1$) is the socially optimal level of externality. From the analysis in part (a) above, when $I = 1$, we have $\theta^o = 1$. When $I = 2$, we have $\theta^o \in \{\frac{1}{2}, 1\}$. Generally, we have that for all $I \geq 2$, we have that $\theta^o \in \{\frac{1}{I}, \frac{2}{I}\}$, so that $\lim_{I\to\infty}\theta^O = 0$.

11.D.7 [First Printing Errata: (i) The question should begin with "A continuum of individuals can build..." (ii) Part (c) should continue with "...and allowing transfers between individuals."]

(a) Assume in negation that only one neighborhood is occupied. First assume it is B, and consider the most prestigious individual with $\theta = 1$. Since $\bar{\theta}_B = \frac{1}{2}$, then this individual's utility from staying in neighborhood B is $(1 + 1)(1 + \frac{1}{2}) - c_B = 3 - c_B \leq 3$. If he would move to neighborhood A his utility would be $(1 + 1)(1 + 1) - c_A = 4 - c_A > 3$, so all individuals in neighborhood B cannot be an equilibrium. Now assume that only A is occupied and again consider the most prestigious individual with $\theta = 1$. His utility from staying in neighborhood A is $(1 + 1)(1 + \frac{1}{2}) - c_A = 3 - c_A$, and his utility from moving to neighborhood B is $(1 + 1)(1 + 1) - c_B = 4 - c_B > 3 - c_A$ so all individuals in neighborhood A cannot be an equilibrium - contradiction.

(b) Let an equilibrium be a pair (Θ_A,Θ_B), where $\Theta_i \equiv \{\theta : \text{type } \theta \text{ locates in neighborhood } i\}$, and let $\bar{\Theta}_A,\bar{\Theta}_B$ be the average prestige levels associated with such an equilibrium.

Claim: $\bar{\Theta}_A$ must take on the form $[\hat{\theta},1]$, for some $\hat{\theta}$.

Proof: Assume θ' prefers A to B: $(1 + \theta')(1 + \bar{\Theta}_A) - c_A > (1 + \theta')(1 + \bar{\Theta}_B) - c_B$.

Rearranging gives us: $(1 + \theta') \geq \dfrac{c_A - c_B}{\bar{\Theta}_A - \bar{\Theta}_B}$, which implies that all types locates in which neighborhood, and it is calculated by solving:

$$(1 + \hat{\theta})(1 + \frac{1+\hat{\theta}}{2}) - c_A = (1 + \hat{\theta})(1 + \frac{\hat{\theta}}{2}) - c_B ,$$

which yields, $\hat{\theta} = 2(c_A - c_B) -1$.

(c) Starting at the equilibrium with $\hat{\theta}$ as given above, if a small group of individuals from the lower end of neighborhood A move to neighborhood B, then the average prestige in both neighborhoods will rise. In particular, if for some $\varepsilon > 0$ the segment $[\hat{\theta},\hat{\theta}+\varepsilon]$ moved from A to B, the average prestige in both neighborhoods would rise by $\frac{\varepsilon}{2}$. So, in <u>both</u> neighborhoods, an individual of type θ who did not move will have a positive change in utility of $(1+\theta)\frac{\varepsilon}{2}$. For a type θ individual who moved from A to B, there will be a negative change in utility equal to $(1 + \theta)(1 + \frac{\hat{\theta}}{2} + \frac{\varepsilon}{2}) - c_B - [(1 + \theta)(1 + \frac{1}{2} + \frac{\hat{\theta}}{2}) - c_A] =$ $(1 + \theta)(\frac{\varepsilon-1}{2}) + (c_A - c_B)$. We denote the total benefit from such a change as B, and the total cost as C, so that we have

$$B(\varepsilon) = \int_0^{\hat{\theta}} (1+\theta)\frac{\varepsilon}{2}\, d\theta + \int_{\hat{\theta}+\varepsilon}^{1} (1+\theta)\frac{\varepsilon}{2}\, d\theta ,$$

$$C(\varepsilon) = \int_{\hat{\theta}}^{\hat{\theta}+\varepsilon} [(1+\theta)(\frac{\varepsilon-1}{2}) + (c_A - c_B)]\, d\theta ,$$

and we can evaluate the effect of such a change when $\varepsilon = 0$:

$$\left.\frac{dB(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^{\hat{\theta}} (1+\theta)\frac{1}{2}\, d\theta + \int_{\hat{\theta}+\varepsilon}^{1} (1+\theta)\frac{1}{2}\, d\theta - (1+\hat{\theta}+\varepsilon)\frac{\varepsilon}{2}$$

$$= \frac{\hat{\theta}}{2} + \frac{\hat{\theta}^2}{4} + \frac{1}{2} + \frac{1}{4} - \frac{\hat{\theta}+\varepsilon}{2} - \frac{(\hat{\theta}+\varepsilon)^2}{4} - (1+\hat{\theta}+\varepsilon)\frac{\varepsilon}{2} = \frac{3}{4} ,$$

$$\left.\frac{dC(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_{\hat{\theta}}^{\hat{\theta}+\varepsilon} (1+\theta)\frac{1}{2}\, d\theta + [(1+\hat{\theta}+\varepsilon)(\frac{\varepsilon-1}{2}) + c_A - c_B]$$

$$= \frac{\hat{\theta}+\varepsilon}{2} - \frac{\hat{\theta}}{2} + \frac{(\hat{\theta}+\varepsilon)^2}{4} - \frac{\hat{\theta}^2}{4} + [2(c_A-c_B)+\varepsilon](\frac{\varepsilon-1}{2}) + c_A-c_B = 0.$$

[Note that the last equality is true since from the conclusion of part (b) above we have that $1 + \bar{\theta} = 2(c_A - c_B)]$. We conclude that a marginal increase in ε at $\varepsilon = 0$ will lead to higher benefits than costs, and therefore we can find an $\varepsilon > 0$ and transfers from the individuals who don't move to those that move that will make all individuals better off.

11.E.1 **(a)** The optimal quantity $\hat{h}$ is determined by solving

$$\underset{\hat{h}}{\text{Max}} \quad E_\eta[\phi(h,\eta)] + E_\theta[\pi(h,\theta)] \,,$$

which yields the FOC $E_\eta[\frac{\partial\phi(\hat{h},\eta)}{\partial h}] + E_\theta[\frac{\partial\pi(\hat{h},\theta)}{\partial h}] \le 0$, and substituting the functional form we have $\gamma - c\hat{h} + E[\eta] + \beta - b\hat{h} + E[\theta] \le 0$, from which we solve: $\hat{h} \ge \frac{\gamma + \beta}{c + b}$, with equality for $\hat{h} \ge 0$.

(b) Given a tax t, the firm will maximize profits and will choose h according to the FOC $\frac{\partial\pi(h,\theta)}{\partial h} - t = 0$, which yields the firm's "reaction" function to a tax t, given θ, as $h(t,\theta) = \frac{\theta + \beta - t}{b}$. The optimal tax $\hat{t}$ will be given by

$$\underset{\hat{t}}{\text{Max}} \quad E[\phi(h(t,\theta),\eta)] + E[\pi(h(t,\theta),\theta)] \,,$$

which yields the FOC $E[\frac{\partial\phi(h(t,\theta),\eta)}{\partial h}\cdot\frac{\partial h(t,\theta)}{\partial t}] + E[\frac{\partial\pi(h(t,\theta),\theta)}{\partial h}\cdot\frac{\partial h(t,\theta)}{\partial t}] = 0$.
Since $\frac{\partial h(t,\theta)}{\partial t} = -\frac{1}{b}$ is a constant, we can cancel out the $\frac{\partial h(t,\theta)}{\partial t}$ from the FOC, and substituting the functional form of our functions the FOC becomes:

$$\gamma - c\cdot\frac{E[\theta]+\beta-t}{b} + E[\eta] + \beta - b\cdot\frac{E[\theta]+\beta-t}{b} + E[\theta] = 0,$$

from which we solve: $\hat{t} = \frac{\beta c - \gamma b}{c + b}$.

(c) The choices of $\hat{h}$ and $\hat{t}$ are depicted in figure 11.E.1(a).
The intersection of the expected marginal profits and marginal utility curves solves for $\hat{h}$ and $\hat{t}$ as shown. Consider a realization of θ and η that gives rise to curves intersecting at the point x as shown. The optimal level of the externality is, therefore, h^*. If we use quantity regulation $\hat{h}$, the dead weight loss (or just "loss") is the triangle xuv, while if we use tax regulation $\hat{t}$, the firm will choose $h(\hat{t},\theta)$, and the loss is the triangle xyz.

We need to compare the expected difference in losses to determine which policy is better.

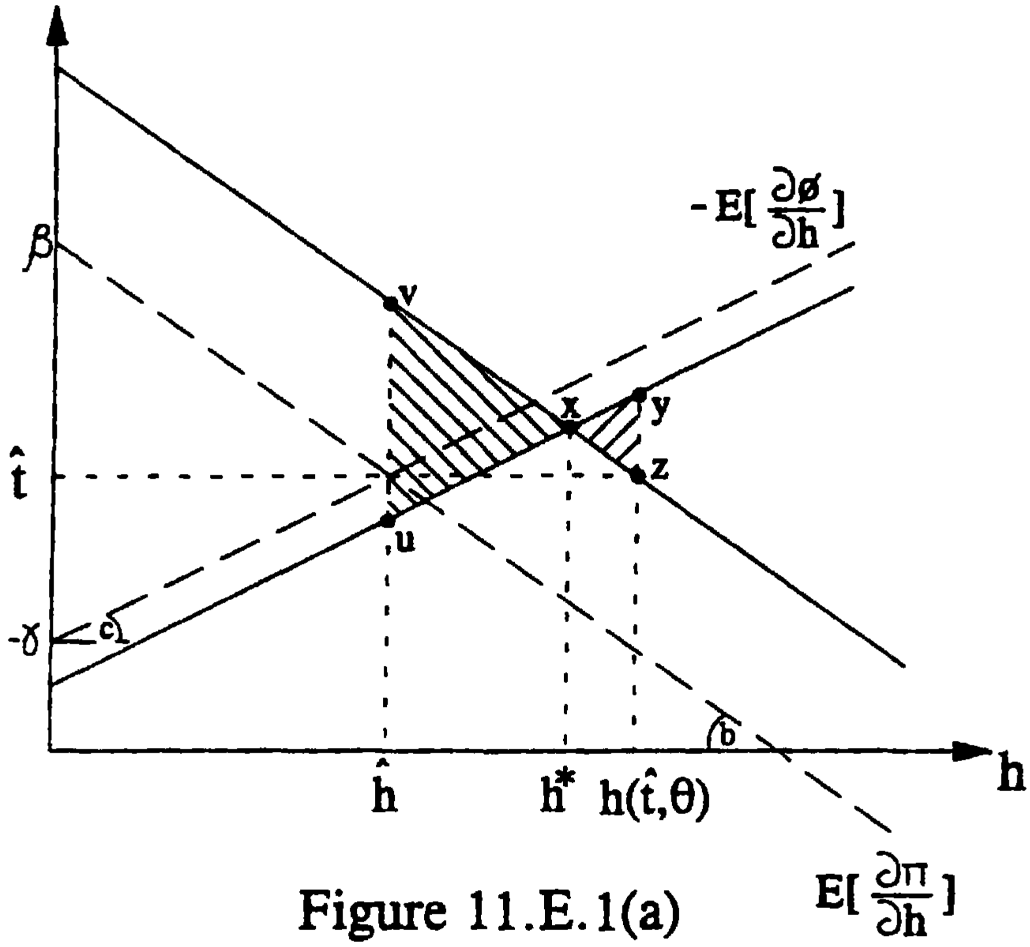


Figure 11.E.1(a)

Before we proceed, we will introduce a non standard, yet helpful way of calculating the area of a triangle. Consider the triangle in figure 11.E.1(b). The area is normally calculated using the formula $A = \frac{1}{2}ed$. We can divide the edge e into e_1 and e_2, and we can then write $d = \frac{e_1}{b}$, where b is the slope of the top edge of the triangle. We can also write $e_1 = \frac{b}{b+c} \cdot e$, which we can then substitute into the above to get: $A = \frac{1}{2} \cdot \frac{e^2}{b + c}$.

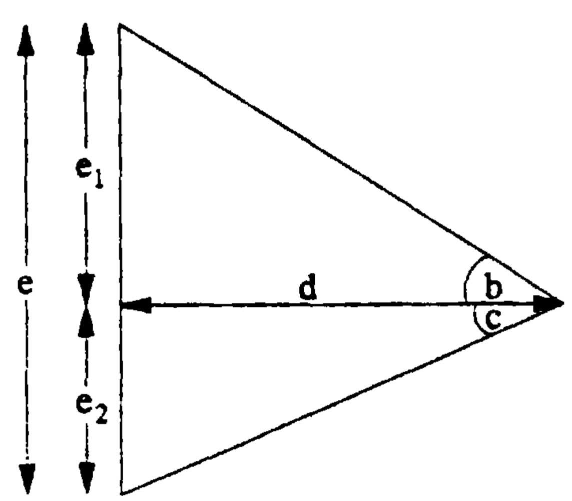


Figure 11.E.1(b)

Getting back to Figure 11.E.1(a), we can apply this to calculate the area of the triangle xuv: it is the edge uv squared, divided by twice the sum of the slopes of both marginal curves. The height of the edge uv is

$$\frac{\partial\pi(\hat{h},\theta)}{\partial h} - \left(-\frac{\partial\phi(\hat{h},\eta)}{\partial h}\right) = (\beta - b\cdot\frac{\gamma + \beta}{c + b} + \theta) + (\gamma - c\cdot\frac{\gamma + \beta}{c + b} + \eta) = \theta + \eta,$$

so that the loss from quantity regulation is,

$$L_h = \frac{(\theta + \eta)^2}{2(b + c)}.$$

To calculate the area of the loss from taxation, we need the height of the edge yz, which is calculated by

$$\frac{\partial\phi(h(\hat{t},\theta),\eta)}{\partial h} - \frac{\partial\pi(h(\hat{t},\theta),\theta)}{\partial h} = -(\gamma - c\cdot\frac{\theta + \beta - \hat{t}}{b} + \eta) - (\beta - b\cdot\frac{\theta + \beta - \hat{t}}{b} + \theta)$$

$$= -\gamma + \frac{c}{b}(\theta + \beta - \hat{t}) - \eta - \hat{t}$$

$$= c[h(\hat{t},\theta) - \hat{h}] - \eta,$$

where the last equality follows because we can rewrite $\hat{t} = c\cdot\hat{h} - \gamma$ (follows by simple algebra). We need to find $h(\hat{t},\theta) - \hat{h}$, and again using simple algebra we get that $h(\hat{t},\theta) - \hat{h} = \frac{\theta}{b}$, which gives us the loss from taxation,

$$L_t = \frac{(\frac{\theta c}{b} - \eta)^2}{2(b + c)}.$$

We can now calculate the expected difference in losses,

$$E[L_h - L_t] = \frac{1}{2(b + c)}\cdot E\left[(\theta + \eta)^2 - (\frac{\theta c}{b} - \eta)^2\right] = \frac{\sigma_\theta^2(b - c)}{2b^2},$$

which implies that the optimal choice between quantity and tax regulation depends on the sign of (b - c). When this term is positive, the economy is more sensitive to changes in the firm, and therefore taxation is better since it changes the level of the externality depending on the firm's realized marginal profits. The reverse is true when the term is negative.

11.E.2 The optimal quantities $(\hat{h}_1,\hat{h}_2)$ are determined by solving

$$\max_{\hat{h}_1,\hat{h}_2} \; E_\eta[\phi(h_1+h_2,\eta)] + E_\theta[\pi(h_1,\theta)] + E_\theta[\pi(h_2,\theta)],$$

which yields the FOCs

$$E_\eta[\frac{\partial\phi(\hat{h}_1+\hat{h}_2,\eta)}{\partial h}] + E_{\theta_1}[\frac{\partial\pi(\hat{h}_1,\theta_1)}{\partial h_1}] \le 0,$$

$$E_\eta[\frac{\partial\phi(\hat{h}_1+\hat{h}_2,\eta)}{\partial h}] + E_{\theta_2}[\frac{\partial\pi(\hat{h}_2,\theta_2)}{\partial h_2}] \le 0,$$

Substituting the functional form we get $\hat{h}_2 = \hat{h}_2 \ge \frac{\gamma + \beta}{2c + b}$, with equality for $\hat{h} \ge 0$.

Similarly, given a pair of taxes (t_1,t_2), each firm will maximize profits and will choose h_i according to the FOC $\frac{\partial\pi_i(h_i,\theta_i)}{\partial h_i} - t = 0$, which yields each firm's "reaction" function to a tax t, given θ_i, as $h_i(t,\theta_i) = \frac{\theta_i + \beta - t}{b}$.

The optimal taxes $\hat{t}_i$ will be given by

$$\underset{\hat{t}_1,\hat{t}_2}{\text{Max}}\ E[\phi(h_1(t_1,\theta_1)+h_2(t_2,\theta_2),\eta)] + E[\pi_1(h_1(t_1,\theta_1),\theta_1)] + E[\pi_2(h_2(t_2,\theta_2),\theta_2)],$$

Which can be calculated in a similar way as in exercise 11.E.1.

When comparing between the two instruments things become a little more cumbersome, and for an analysis we refer the reader to Section V in:

M. Weitzman (1974) "Prices vs. Quantities," *Review of Economic Studies* 41:477-91.

It is, however, worthwhile to convey the intuition for the effects of σ_{12}. If the marginal profits of each firm are perfectly correlated, then it is as if we have one producer and the analysis of exercise 11.E.1 follows through. As σ_{12} falls (they become less correlated) then it is more likely that taxes, which yield negatively correlated output decisions when the shocks are negatively correlated, will be more efficient than quantity quotas.

11.E.3 Assume that there exists another weakly dominant strategy that is not truth telling, i.e., there exists c′ such that the consumer announces $\hat{c}(c') = c'' \ne c'$. If $c'' > c'$ then when $c'' > \hat{b} > c'$, the consumer is not playing a best response to $\bar{b}$, and any $\bar{c} > \hat{b}$ is strictly better for him. Similarly, if $c'' < c'$ then when $c'' < \hat{b} < c'$, the consumer is not playing a best response to $\hat{b}$, and any $\hat{c} < \hat{b}$ is strictly better for him - a contradiction to there being a weakly dominant strategy which is not truth telling.

11.E.4 We can write the scheme for the consumer as $t(\hat{b},\hat{c}) + T(\hat{b})$, where $t(\hat{b},\hat{c})$ is the scheme described in the textbook. Given $\hat{b}$, the consumer will want to maximize his utility, $u(\hat{b},\hat{c}) = t(\hat{b},\hat{c}) + T(\hat{b}) - c$. But this is equivalent to maximizing $t(\hat{b},\hat{c}) - c$, the same as in the textbook. Therefore, adding $T(\hat{b})$ changes nothing (as does adding $T(\hat{c})$ to the firm).

11.E.5 Consider the scheme where every agent i announces a benefit $\hat{b}_i$. We call an agent i "pivotal to build" if $\sum_{i=1}^{I}\hat{b}_i \geq K$ and $\sum_{j\neq i}\hat{b}_j < K$. That is, if agent i's announcement causes the project to be built. Similarly, we call an agent i "pivotal to not build" if $\sum_{i=1}^{I}\hat{b}_i < K$ and $\sum_{j\neq i}\hat{b}_j \geq K$. (Note that if $\sum_{i=1}^{I}\hat{b}_i \geq K$ then there can be more than one pivotal agent to build but no pivotal agents not to build, and similarly, but reversed, for $\sum_{i=1}^{I}\hat{b}_i < K$.)
The scheme works as follows. If $\sum_{i=1}^{I}\hat{b}_i \geq K$ then the project is built, and each agent pays a tax t_i in the following way:

$$t_i = 0 \text{ if agent i is not pivotal to build,}$$

$$t_i = K - \sum_{j\neq i}\hat{b}_j > 0 \text{ if agent i is pivotal to build.}$$

If $\sum_{i=1}^{I}\hat{b}_i < K$ then the project is not built, and each agent pays a tax t_i in the following way:

$$t_i = 0 \text{ if agent i is not pivotal not to build,}$$

$$t_i = \sum_{j\neq i}\hat{b}_j - K > 0 \text{ if agent i is pivotal not to build.}$$

We need to show that this scheme chooses the choice of building in a Pareto optimal way, and that truth-telling is a weakly dominant strategy (we will deal with balancing the budget at the end). We show this for all cases:

<u>Case 1:</u> $\sum_{j\neq i}\hat{b}_j < K$ and $b_i + \sum_{j\neq i}\hat{b}_j \geq K$.

If agent i announces some $\hat{b}_i$ such that the project is not built, he is not pivotal in any way and his utility is zero. If, however, he announces $\hat{b}_i$ such that the project is built (in particular $\hat{b}_i = b_i$), then he is pivotal to build and his utility will be $u_i = b_i - (K - \sum_{j\neq i}\hat{b}_j) \geq 0$, so that in this case truth telling is a best response.

<u>Case 2:</u> $\sum_{j\neq i}\hat{b}_j < K$ and $b_i + \sum_{j\neq i}\hat{b}_j < K$.

If agent i announces some $\hat{b}_i$ such that the project is not built (in particular $\hat{b}_i$), he is not pivotal in any way and his utility is zero. If, however, he announces $\hat{b}_i$ such that the project is built then he is pivotal to build and his utility will be $u_i = b_i - (K - \sum_{j\neq i}\hat{b}_j) < 0$, so that in this case truth telling is a best response.

<u>Case 3:</u> $\sum_{j\neq i}\hat{b}_j > K$ and $b_i + \sum_{j\neq i}\hat{b}_j > K$.

If agent i announces some $\hat{b}_i$ such that the project is built (in particular $\hat{b}_i = b_i$), then he is not pivotal in any way and his utility is b_i. If, however, he announces $\hat{b}_i$ such that the project is not built, then he is pivotal to not build and his utility will be $u_i = 0 - (\sum_{j\neq i}\hat{b}_j - K) < b_i$, so that in this case truth telling is a best response.

<u>Case 4:</u> $\sum_{j\neq i}\hat{b}_j > K$ and $b_i + \sum_{j\neq i}\hat{b}_j < K$.

If agent i announces some $\hat{b}_i$ such that the project is built, then he is not pivotal in any way and his utility is b_i. If, however, he announces $\hat{b}_i$ such that the project is not built (in particular $\hat{b}_i = b_i$), then he is pivotal to not build and his utility will be $u_i = 0 - (\sum_{j\neq i}\hat{b}_j - K) > b_i$, so that in this case truth telling is a best response.

We conclude from the cases above that truth telling is a weakly dominant strategy, so that in equilibrium we will have all agents telling the truth.

This, together with the building decision rule build if and only if $\sum_{i=1}^{I}\hat{b}_i \geq K$, implies that the Pareto optimal decision will always be carried out. Finally, it is clear that if we add a constant to the tax paid by agents, then truth telling is still a weakly dominant strategy, and we can add such a constant that on average will balance the budget.

11.E.6 The analysis is almost identical to that of exercise 11.D.6, where each agent i announces a vector $(\hat{b}_i(1),\ldots,\hat{b}_i(N))$, project n is built if and only if $\sum_{i=1}^{I}\hat{b}_i(n) \geq K(n)$, and the taxes are similarly defined. It can then be shown that truth telling is a weakly dominant strategy for each project n, independent of the other projects, so that truth telling is a weakly dominant strategy for all projects. The balance budgeting will also be done by adding constants to the taxes.

11.E.7 We can describe the variable total tax as a function $T(h) \equiv -\phi(h,\bar{\eta})$, which promises that $T'(h) \equiv -\dfrac{\partial\phi(h,\bar{\eta})}{\partial h}$. Thus, when the firm maximizes its net profits, $\pi(h,\theta) - T(h)$, then for any realization of θ we will have h chosen according to the FOC $\dfrac{\partial\pi(h,\theta)}{\partial h} = -\dfrac{\partial\phi(h,\bar{\eta})}{\partial h}$, which is the condition for Pareto optimality.

CHAPTER 12

12.B.1 (a) The monopolists maximizes $x(p)p - c(x(p))$, which leads to the following first order condition: $x'(p^m)\cdot p^m + x(p^m) = c'(x(p^m))\cdot x'(p^m)$. Rearranging this we get $\dfrac{[p^m - c'(x(p^m))]}{p^m} = \dfrac{1}{\varepsilon(p^m)}$, where the elasticity of demand is $\varepsilon(p^m) = -x'(p^m)\cdot\dfrac{p^m}{x(p^m)}$.

(b) If $c'(x(p^m)) > 0$ always, then $\dfrac{[p^m - c'(x(p^m))]}{p^m} < \dfrac{[p^m - 0]}{p^m} = 1$. Therefore $1/\varepsilon(p^m) < 1$, or $\varepsilon(p^m) > 1$.

12.B.2 (a) The monopolist's problem is to maximize $\alpha p^{-\varepsilon}(p - c)$, which yields the FOC $\alpha(1-\varepsilon)p^{-\varepsilon} + c\alpha\varepsilon p^{-\varepsilon-1} = 0$, which gives us $p^m = \dfrac{-c\varepsilon}{1-\varepsilon} < 0$, so the problem is not well defined.

(b) Using the monopolists FOC, $p^m = \dfrac{-c\varepsilon}{1-\varepsilon}$, we get $q^m = \alpha\left(\dfrac{-c\varepsilon}{1-\varepsilon}\right)^{-\varepsilon}$.

Consumer surplus given price p: $S(p) = \int_p^{\infty} x(t)dt = \dfrac{\alpha p^{1-\varepsilon}}{(\varepsilon-1)}$. The deadweight welfare loss can be calculated by subtracting from the consumer surplus under competitive pricing, the consumer surplus plus profits under monopoly pricing:

$$DL = \frac{\alpha c^{1-\varepsilon}}{(\varepsilon-1)} - \frac{\alpha}{(\varepsilon-1)}\left(\frac{-c\varepsilon}{1-\varepsilon}\right)^{1-\varepsilon} - \left(\frac{-c\varepsilon}{1-\varepsilon} - c\right)\alpha\left(\frac{-c\varepsilon}{1-\varepsilon}\right)^{-\varepsilon} = \frac{\alpha}{1-\varepsilon}\left[\frac{1}{\varepsilon-1} - \left(\frac{-\varepsilon}{1-\varepsilon}\right)^{-\varepsilon}\right]c^{1-\varepsilon}$$

(c) After some messy algebra one can show that the deadweight welfare loss is decreasing in ε (i.e. $\partial DL(\varepsilon)/\partial\varepsilon < 0$). As ε increases the demand becomes more elastic without changing the size of the market, which forces the monopolist to price closer and closer to the competitive price. In the limit as $\varepsilon \to \infty$, the deadweight welfare loss goes to zero and the monopolist charges the competitive price.

12.B.3 Assume the partial derivatives satisfy $x_1 < 0$, $x_{11} < 0$, $c_1 > 0$, and $c_{11} > 0$ (these are standard assumptions). The monopolist's problem is Max $x(p,\theta)p - c(x(p,\theta),\phi)$. which yields the FOC,

$$x_1(p,\theta)\cdot p + x(p,\theta) - c_1(x(p,\theta),\phi)\cdot x_1(p,\theta) = 0.$$

Differentiating with respect to θ gives us:

$$\frac{\partial p}{\partial \theta} = \frac{-(p - c_1)x_{12} + x_2(x_1 c_{11} - 1)}{2x_1 + (p - c_1)x_{11} - x_1^2 c_{11}},$$

and under our assumptions above we will have $\partial p/\partial\theta > 0$ if $x_2 > 0$ and $x_{12} > 0$. Differentiating the FOC with respect to ϕ gives us:

$$\frac{\partial p}{\partial \theta} = \frac{x_1 c_{12}}{2x_1 + (p - c_1)x_{11} - x_1^2 c_{11}},$$

and under our assumptions above we will have $\partial p/\partial\phi > 0$ if $c_{12} > 0$.

12.B.4 Suppose $c_1 > c_2$, and p_1, p_2 are the monopoly prices given marginal costs c_1, c_2 respectively. By revealed preference of the monopolist we have $(p_1 - c_1)x(p_1) \geq (p_2 - c_1)x(p_2)$ and $(p_2 - c_2)x(p_2) \geq (p_1 - c_2)x(p_1)$. Adding these two inequalities and rearranging yields $(c_1 - c_2)(x(p_2) - x(p_1)) \geq 0$, and since $c_1 > c_2$ it must be that $x(p_2) \geq x(p_1)$. Therefore we must have that $p_2 \leq p_1$, or the monopoly price is increasing in c.

Suppose in general, that p_1, p_2 and q_1, q_2 are the monopoly prices and quantities given cost of $c(q_1,\phi_1)$, $c(q_2,\phi_2)$ respectively [that is, $q_i = x(p_i)$]. By revealed preference we must have that, $p_1q_1 - c(q_1,\phi_1) \geq p_2q_2 - c(q_2,\phi_1)$ and $p_2q_2 - c(q_2,\phi_2) \geq p_1q_1 - c(q_1,\phi_2)$. Adding these two inequalities and rearranging yields, $[c(q_1,\phi_2) - c(q_2,\phi_2)] - [c(q_1,\phi_1) - c(q_2,\phi_1)] \geq 0$, or

$$\int_{\phi_1}^{\phi_2}\int_{q_2}^{q_1} \frac{\partial^2 c(q,\phi)}{\partial q \partial \phi}\, dq d\phi \geq 0.$$

So, if $\dfrac{\partial^2 c(q,\phi)}{\partial q \partial \phi} \geq 0$, then $\phi_1 < \phi_2$ implies $q_2 < q_1$ and therefore $p_2 > p_1$. Thus, the monopoly price is increasing in ϕ. This is the condition we needed in exercise 12.B.3 to get $\partial p/\partial \phi > 0$.

12.B.5 (a) Since each consumer desires at most one unit, the monopolist will not gain by introducing a quantity discount. Since the monopolist is unable to discern any particular consumer's preferences he cannot price discriminate. Thus it is optimal for the monopolist to simply charge a price per unit.

(b) Since the monopolist is unable to discern any particular consumer's preferences he cannot price discriminate. However, suppose the monopolists charges different prices for different quantities bought. Denote the price for q units by $p(q)$. Since resale is costless and the resale market is competitive, the unit price of the good in the resale market must equals $\text{Min}_q(p(q)/q)$. This implies that consumers will only buy a quantity equal to $q^* = \text{Argmin}_q(p(q)/q)$ and then they resell each unit for $p(q^*)/q^*$. Thus, the monopolist is just as well off by charging a price per unit of $p(q^*)$.

12.B.6 Let t be the tax/subsidy per unit of output: The monopolist maximizes $p(q) - c(q) - tq$, and the FOC is $p'(q)q + p(q) - c'(q) - t = 0$. Since we want an efficient outcome, the monopolist must choose q so that $p(q) = c'(q)$, which implies that we need $t = p'(q)q < 0$, which in turn implies that the government subsidizes the monopoly.

12.B.7 (a) In a competitive equilibrium we must have $p^m = p^w = c$, and therefore $x_m = a - \theta_m c$, and $x_w = a - \theta_w c$.

(b) If the monopolist will serve both markets it solves,

$$\text{Max } (a - \theta_m p)(p - c) + (a - \theta_w p)(p - c) .$$

The FOC yields $p^* = \frac{a}{\theta_m + \theta_w} + \frac{c}{2}$. However, since the women are willing to pay higher prices than men for the same quantities $(\theta_m > \theta_w)$ then it may be better to charge a price above $\frac{a}{\theta_m}$. This cannot be captured in the program above since it causes negative quantities in the men's demand, so we have to compare the solution p^* above to the solution of $\text{Max}\ (a - \theta_w p)(p - c)$ which yields $\hat{p} = \frac{a}{2\theta_w} + \frac{c}{2}$. Figure 12.B.7 shows the two cases.

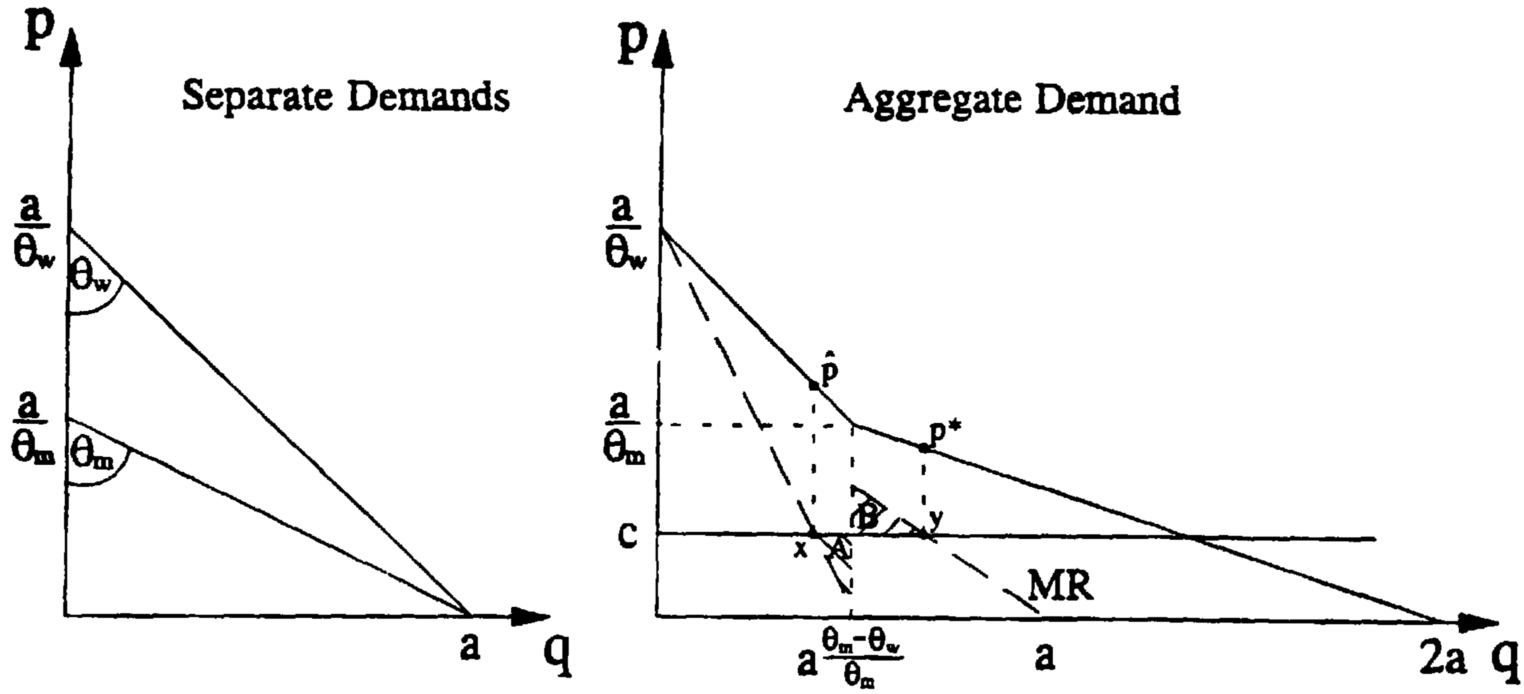


Figure 12.B.7

Due to the kink in the demand curve, the marginal revenue curve MR has a jump at that point (the dashed line). When equating the marginal cost, c, to MR, we could have one or two solutions. When we have one solution, it means that c cuts MR only to the left of the discontinuity, or only to the right. If c cuts MR only to the left, then the optimal price is above a/θ_m and we only serve the women. If c cuts MR only to the right, then the optimal price is below a/θ_m and we serve both markets. When we have two solutions, then c cuts MR both to the left and to the right of the discontinuity, and this is shown in the figure by points x and y. When the monopolist moves from point x to y (i.e., from $\hat{p}$ to p^*) it loses profits equal to the triangle A, and gains profits equal to the triangle B. The costs and benefits of such a move will determine if $\hat{p}$ is optimal (only the women are served) or if p^* is (both markets are served).

(c) Given quantity X, we maximize consumer surplus by solving,

$$\text{Max} \int_0^{q_m} p_m(x)dx + \int_0^{q_w} p_w(x)dx$$

$$\text{s.t. } q_m + q_w = X$$

and letting λ denote the Lagrange multiplier we get the FOCs $p_m(q_m) = \lambda$ and $p_w(q_w) = \lambda$, which yield $p_m(q_m) = p_w(q_m)$, or, $\dfrac{(a - q_m)}{\theta_m} = \dfrac{(a - q_w)}{\theta_w}$. Together with the constraint we solve for q_m and q_w.

(d) The discriminatory monopolist solves

$$\max_{p_m, p_w} \ (a - \theta_m p_m)(p_m - c) + (a - \theta_w p_w)(p_w - c) ,$$

which FOC's yield $p_m = \dfrac{a + c\theta_m}{2\theta_m}$, and $p_w = \dfrac{a + c\theta_w}{2\theta_w}$. These prices yield quantities of $q_m = (a - c\theta_m)/2$, and $q_w = (a - c\theta_w)/2$. Note that aggregate output under price discrimination is equal to the aggregate output <u>without</u> price discrimination if <u>both</u> markets were served ($Q = a - \dfrac{(\theta_m + \theta_w)c}{2}$). From part (c), we know that the welfare maximizing distribution for a given output is to set $p_m(q^m) = p_w(q^w)$, which is the case without price discrimination when both markets are served. Notice that under price discrimination $p_m(q^m) \neq p_w(q^w)$ if $\theta_m \neq \theta_w$, which implies that welfare is lower under price discrimination. If, however, without price discrimination only the women were served, then by allowing the monopolist to discriminate it will open the men's market (assuming that $c < \dfrac{a}{\theta_m}$) without changing the price in the women's market. This means that there is added surplus from the men's market and welfare under price discrimination is higher than welfare when price discrimination is forbidden.

12.B.8 (a) The monopolists intertemporal problem is

$$\text{Max}_{q_1, q_2} \ (a - bq_1 - c)q_1 + (a - bq_2 - (c - mq_1))q_2,$$

and the FOCs yield $q_1^m = q_2^m = \frac{a-c}{2b-m} > 0$ (by the exercise's assumptions).

(b) A benevolent social planner maximizes total surplus (assuming no discounting of the future, we just add up both periods' consumer surplus, to both periods' firm's profits, and subtract both periods' costs),

$$\text{Max}_{q_1, q_2} \ SW = \int_0^{q_1} p(x)dx + \int_0^{q_2} p(x)dx \ - cq_1 - (c-mq_1)q_2$$

$$= a(q_1+q_2) - \tfrac{1}{2}b(q_1^2+q_2^2) - cq_1 - (c-mq_1)q_2$$

and the FOCs are,

$$\text{(i)} \quad (a - bq_1) + mq_2 = c \ ,$$

$$\text{(ii)} \quad (a - bq_2) = c - mq_1 \ ,$$

which yield $q_1^* = q_2^* = \frac{a-c}{b-m} > 0$. Comparing this with the monopoly quantities we see that $q_i^m < q_i^*$. The way we wrote down the FOCs show that in fact there is a sense of "price equals marginal cost". Recall that price is marginal surplus, and the left hand side of both FOCs is exactly the effective marginal surplus from each period's good: In the first period, aside from marginal consumer surplus, given q_2, any additional unit of q_1 reduces marginal cost <u>next period</u> by mq_2. The right hand side is the effective marginal cost in each period.

(c) As we have seen, the social planner would want to produce more in every period. By increasing the output in the first period above q_i^m, welfare in the first period will be higher, and this will lead to a lower second period marginal cost. This lower second period marginal cost will induce the monopolist to produce more in the second period and will therefore further increase welfare.

12.B.9 (a) The monopolist will solve

$$\text{Max}_{q,I} \quad p(q)\cdot q - c(I)\cdot q - I,$$

which yields the FOCs are

$$\text{(i)} \quad p'(q^m)\cdot q^m + p(q^m) = c(I^m) \; ,$$

$$\text{(ii)} \quad -c'(I^m)q^m = 1 \; .$$

(b) The social planner will maximize total surplus,

$$\max_{q,I} \int_0^q p(x)dx - c(I)q - I \; ,$$

and the FOCs are,

$$\text{(iii)} \quad p(q^*) = c(I^*) \; ,$$

$$\text{(iv)} \quad -c'(I^*)q^* = 1 \; .$$

The monopolist produces less output than is socially optimal, $q^m < q^*$, and price is above marginal cost. Given this, equations (ii) and (iv) imply that $-c'(I^*) < -c'(I^m)$, which in turn implies that $I^* > I^m$. Note, however, that given the output level of the monopolist, he chooses the optimal level of investment.

(c) Given a level $\hat{I}$ set by the government, the monopolist will set q to maximize its profits, i.e., it will set q to equate MR = MC. Therefore, the governments problem is to maximize social surplus subject to the monopolist's behavior. That is,

$$\max_{q,I} \int_0^q p(x)\, dx - c(I)q - I$$

$$\text{s.t.} \; p'(q)\cdot q + p(q) = c(I)$$

The Lagrangian is $L = \int_0^q p(x)dx - c(I)q - I - \lambda[p'(q)q + p(q) - c(I)]$, which yields the FOCs ,

(v) $p(\hat{q}) - c(\hat{I}) - \lambda[p''(\hat{q})\hat{q} + 2p'(\hat{q})] = 0$,

(vi) $-c'(\hat{I})(\hat{q} - \lambda) = 1$.

Note that (vi) implies that the chosen investment level $\hat{I}$ is not optimal given the quantity $\hat{q}$, and since $\lambda > 0$ we will have $\hat{I}$ greater than optimal given $\hat{q}$. The intuition is that in this second-best environment, the social planer chooses an investment level larger than optimal for the given level of output in order to induce the monopolist to produce more.

12.B.10 Let $p(x,q)$ be the inverse demand function for a given quality level q. Since $x(p,q)$ is decreasing in p and increasing in q, then $p(x,q)$ is decreasing in x and increasing in q, and the monopolist solves,

$$\text{Max}_{x,q} \quad p(x,q)x - c(x,q),$$

and the FOCs are,

$$\text{(i)} \quad p(x^m,q^m) + p_1(x^m,q^m)x^m = c_1(x^m,q^m) \ ,$$

$$\text{(ii)} \quad p_2(x^m,q^m)x^m = c_2(x^m,q^m).$$

Given a quantity set by the monopolist, x^m, if the social planner could set the quality q it would maximizes social surplus,

$$\text{Max}_q \int_0^{x^m} p(s,q)ds - c(x^m,q) \ ,$$

and the FOC is,

$$\int_0^{x^m} p(s,q^*)ds = c_2(x^m,q^*) \ ,$$

so that generally we will have $q^* \neq q^m$, i.e., the monopolist will generally not choose quality optimally given his quantity. The intuition is that given the quantity picked by the monopolist, the social planner cares about the marginal change in quality over the <u>whole</u> market (i.e., the average marginal change $[\int_0^x p_2(x,s)\, dx]/x$) while the monopolist only cares about the marginal value of quality to the <u>marginal</u> consumer (i.e. $p_2(q,s)$). Note that for linear demand in quality, the two will coincide.

12.C.1 Following the arguments in the textbook, we cannot have a Nash Equilibrium (NE) with $p_i > c$ for all i. Note also that $p_1 = p_2 = \dots = p_J = c$ is a NE - none of the firms can gain by raising its price and it will lose money if it lowers its price. Extending the argument given in the textbook one can show that if at least two firms charge $p = c$ and all the other firms charge at least c we have a NE. Therefore there will be multiple Nash equilibrium, all yielding the competitive price.

12.C.2 Assume in negation that there is a mixed strategy Nash equilibrium (NE). It must be characterized by firm 1 mixing between prices p_1^1 and p_1^2 with probabilities q_1^1 and $q_1^2 = 1 - q_1^1$ respectively, and firm 2 mixing between prices p_2^1 and p_2^2 with probabilities q_2^1 and $q_2^2 = 1 - q_2^1$ respectively. In any NE both firms cannot lose money so we cannot have that for some i both $p_i^1 < c$ and $p_i^2 < c$. If for some i both $p_i^1 > c$ and $p_i^2 > c$, then $j \neq i$ can gain by setting $p_j^1 = p_j^2 = \text{Min}(p_i^1, p_i^2) - \varepsilon$. Therefore, we cannot have that for some i both $p_i^1 > c$ and $p_i^2 > c$. The only remaining case is that for both players we have (w.l.o.g.) $p_i^1 > c$ and $p_i^2 < c$. But then firm j can gain by setting $p_j^1 = p_j^2 = p_i^1 - \varepsilon$. This argument contradicts the existence of a mixed strategy NE, therefore only a pure strategy NE can exist, and Proposition 12.C.1 proves that $p_1 = p_2 = c$ is the unique pure strategy NE.

12.C.3 (a) Let $n \in \mathbb{N}$ satisfy $(n-1)\Delta \le c < n\Delta$. We have to show that both firms charging $n\Delta$ is a pure strategy Nash equilibrium (NE)of this game. Note first that both firms get a strictly positive profit by using this strategy. Neither firm wants to raise its price - if one of them did so, its sales would be zero and it would make zero profit. Neither firm wants to lower its price - if one of them did lower its price to at least $(n-1)\Delta$, then it would make negative or at most zero profit. Therefore, both firms charging $n\Delta$ is a pure strategy equilibrium. It also follows from the above argument that, given that the other firm charges $n\Delta$, charging $n\Delta$ is the <u>unique</u> best response. Hence it is not weakly dominated by any other strategy.

(b) This follows form (a) above and from the fact that and $n\Delta \to c$ as $\Delta \to 0$.

12.C.4 (a) First, there cannot be an equilibrium in which both p_1 and p_2 are strictly above c_2 (by the same argument as for the symmetric case). Second, firm 2 does not charge less than c_2, since it would otherwise make a negative profit. Third, firm 1 can guarantee itself a profit as close as possible to $(c_2-c_1)\ q(c_2)$, by charging $c_2 - \varepsilon$. Since the price charged cannot exceed c_2, this profit is also the highest that firm 1 can obtain. Unless one assumes that at common price c_2 firm 1 gets all the demand there is no equilibrium - firm 1 will want to choose ε as close as possible, but different from, 0. Such an ε does not exist. One can define the equilibrium as the limit, so $p_1 = c_2$ and firm 1's profit is $(c_2-c_1)\ q(c_2)$. When the monopoly price of firm 1 is lower than c_2, then it will charge this price without worrying about firm 2's threat (so in this case there exist an equilibrium)

(b) Let $n \in \mathbb{N}$ satisfy $(n-1)\Delta \le c_2 < n\Delta$. Firm 1 charging $(n-1)\Delta$ and firm 2 charging $n\Delta$ constitute a pure strategy NE of this game, which does not involve the play of weakly dominated strategies. As $\Delta \to 0$, $(n-1)\Delta \to c_2$, so the equilibrium approaches the NE of part (a) above.

Note that, firm 1 charging $(n-x)\Delta$ and firm 2 charging $(n-x+1)\Delta$, with x such that $c_1 \le (n-x-1)\Delta \le c_2$, also constitutes NE.

12.C.5 The first q agents bidding v_{q+1} and agent i with $i > q$ bidding v_{i+1} (the agent with the lowest valuation can bid anything strictly lower than his valuation) is a pure strategy Nash equilibrium not involving weakly dominated strategies. Clearly, the first q players cannot do better. Bidding higher will only reduce their payoff. If they bid lower, they will lose the object and therefore not make a positive payoff (if they bid v_{q+2} they will have 50% chance of loosing). Players with $i > q$ cannot do better by bidding lower, they will still not win the object and obtain a payoff of 0. By bidding higher, they will only be able to have a chance to win the object if they bid at least v_{q+1}. However, if they win the object they will have a negative payoff, since $v_i - v_{q+1} \leq 0$ for all $i > q$. Thus they do not gain by changing their bid. Clearly, this is a competitive equilibrium since the demand at this price is q, which equals the fixed supply.

Suppose there exists a pure strategy NE, in which one of the buyers 1 through q, assume player $j \leq q$, did not receive a unit. Then one of the other buyers, assume player $i > q$, must have received a unit. Since player i could have bid 0 and therefore obtain a payoff of 0, he has to get a positive payoff in the NE, which implies that his bid was at most v_i. Since player j did not get the object his payoff was 0. However, he could have bid slightly above v_i and obtain a strictly positive payoff of $v_j - v_i$ (since $j > i$), a Contradiction, since player j did not use his best-response. Thus in every pure strategy NE, buyers 1 through q receive a unit.

12.C.6 Each firm j maximizes $[a - b(q_j + q_k) - c] \cdot q_j$, and the FOC is $a - 2bq_j - bq_k - c = 0$ (when $q_j > 0$), which yields the best response function $b_j(q_k) = \text{Max}\{0, (a - c - bq_k)/(2b)\}$. To calculate the Nash equilibrium we have two equations (both BR functions) with two variables since we set $b_j(q_k) = q_j$ to get the equilibrium. Straightforward algebra yields the results.

12.C.7 Using equation 12.C.6 and plugging in the linear demand curve we obtain

$$-bQ^*/J + (a - bQ^*) = c,$$

which yields $Q^* = [J/(J+1)]\cdot[(a-c)/b]$, and $p = (a+Jc)/(J+1)$. For $J = 1$, we get the monopoly quantity $(a-c)/2b$ and price $(a+c)/2$. Clearly the price falls and the quantity increases as J increases. In addition, as $J \to \infty$, the price goes to the competitive price c and the quantity goes to the competitively supplied quantity $(a-c)/b$.

12.C.8 (a) Firm j maximizes

$$\max_{q_j \geq 0} \; p(q_j + Q_{-j})\, q_j - c(q_j).$$

The set of q_j's which solve this maximization problem clearly depends only on Q_{-j}, and we can denote this set by $b(Q_{-j})$.

(b) Consider the following example: there are 10 consumers of which 5 value the good at \$10 and the other 5 value the good at \$5, where each consumer only wishes to buy one unit of the good. There are two firms with zero marginal (and fixed) costs. If $q_2 = 0$ then both $q_1 = 5$ and $q_1 = 10$ are best response quantities for firm 1 which yield the (monopoly) profits of \$50.

(c) We will use a revealed preference argument. When the total output of other firms is $\hat{Q}$, firm j prefers to produce $\hat{q}$ rather than producing $Q + q - \hat{Q}$. (This quantity has been chosen to keep total output the same as when j responds optimally with q to other firms' producing Q.) Formally:

$$p(\hat{Q} + \hat{q})\hat{q} - c(\hat{q}) \geq p(\hat{Q} + (q + Q - \hat{Q}))\cdot(q + Q - \hat{Q}) - c(q + Q - \hat{Q}) =$$

$$= p(Q + q)\cdot(q + Q - \hat{Q}) - c(Q + q - \hat{Q}).$$

Similarly, when the total output of other firms is Q, firm j prefers to produce q rather than producing $\hat{Q} + \hat{q} - Q$. Formally:

$$p(Q + q)q - c(q) \geq p(Q + (\hat{Q} + \hat{q} - Q))\cdot(\hat{Q} + \hat{q} - Q) - c(\hat{Q} + \hat{q} - Q)$$

$$= p(\hat{Q} + \hat{q})\cdot(\hat{Q} + \hat{q} - Q) - c(\hat{Q} + \hat{q} - Q).$$

Adding up the two inequalities and rearranging terms, we obtain

$$(Q - \hat{Q})[p(\hat{Q} + \hat{q}) - p(Q + q)] \geq$$
$$[c(q) - c(q + Q - \hat{Q})] - [c(\hat{q} + \hat{Q} - Q) - c(\hat{q})]. \qquad (*)$$

Suppose in negation that $\hat{Q} + \hat{q} < Q + q$. Since $p(\cdot)$ is a decreasing function and $\hat{Q} > Q$, this implies that the left-hand side of (*) is negative. To analyze the right-hand side, observe that our negation assumption implies that $q + Q - \hat{Q} > \hat{q}$. Using this and the convexity of $c(\cdot)$, it is easy to see that the right-hand side of (*) is positive. This contradicts the inequality in (*). Therefore, we must have $\hat{Q} + \hat{q} \geq Q + q$.

Take a point (Q,q), with $q \in b(Q)$. Take an increasing sequence $Q_k \to Q$, and a corresponding sequence $q_k \in b(Q_k)$. By the result above, we have $Q_k + q_k \leq Q + q$. Therefore, $q_k \leq Q + q - Q_k$. But then

$$\liminf q_k \leq Q + q - \lim Q_k = q.$$

Mathematically, this implies that the infimum of $b(Q)$ is upper semicontinuous in Q. In words, $b(\cdot)$ can jump only upward.

When $b(\cdot)$ is a function, the above conclusion implies that $b(Q) + Q$ is increasing in Q. If $b(\cdot)$ is in addition differentiable at Q, we must have $b'(Q) + 1 \geq 0$, i.e. $b'(Q) \geq -1$.

(d) Since price is decreasing in total quantity and by convexity of $c(\cdot)$ average costs are increasing for each firm, there must exist a quantity $\bar{q}$ such that it is never a best response to produces $\bar{q}$, i.e., $b(Q) \in [0,\bar{q}]$ for all Q. Using this fact, and the upper semicontinuity of the infimum of $b(Q)$ established above, we can prove existence of a fixed point which will be a symmetric Cournot equilibrium i.e., that $q^* \in b((J-1)q^*)$. This is formally established in the Lemma presented in:

Roberts, J. and H. Sonnenschein (1976) "On the Existence of Cournot

Equilibrium without Concave Profit Functions," JET 13:112-117.

(e) Consider the following example: there are 10 consumers which all value the good at \$10 and each wishes to buy only one unit of the good. There are two firms with zero marginal (and fixed) costs. Any pair of quantities $(q_1, q_2) = (x, 10 - x)$ where $0 \le x \le 10$ will be a Cournot equilibrium.

(f) We will show that if the demand curve $p(q)$ is strictly concave, the symmetric Nash equilibrium is the unique Nash equilibrium in pure strategies. (Strict concavity $p(q)$ is equivalent to strict concavity of the demand function $x(\cdot) = p^{-1}(\cdot)$.) The proof utilizes the following claims, which follow from the strict concavity of $p(q)$:

Claim 1: The best-response correspondence $b(Q_{-j})$ is single-valued, i.e. $b(\cdot)$ is a function.

Proof: Firm j's profit maximization program formulated in part (a) is strictly convex, therefore, its solution is unique.

Claim 2: $b(Q)$ is non-increasing, i.e. best-response curves are downward-sloping.

Proof: Suppose that $Q \le \hat{Q}$, and let $q = b(Q)$, $\hat{q} = b(\hat{Q})$. Using a revealed preference argument, we obtain

$$p(q+Q)q - c(q) \ge p(\hat{q}+Q)\hat{q} - c(\hat{q}),$$
$$p(\hat{q}+\hat{Q})\hat{q} - c(\hat{q}) \ge p(q+\hat{Q})q - c(q).$$

Adding up the inequalities and rearranging terms, we have

$$[p(\hat{q}+Q) - p(\hat{q}+\hat{Q})]\hat{q} \le [p(q+Q) - p(q+\hat{Q})]q \qquad (*)$$

Suppose in negation that $q < \hat{q}$. As $p(\cdot)$ is strictly concave and downward-sloping, and $Q \le \hat{Q}$, we must then have

$$p(\hat{q}+Q) - p(\hat{q}+\hat{Q}) > p(q+Q) - p(q+\hat{Q}) \ge 0.$$

Multiplying by $\hat{q}$, we obtain

$$[p(\hat{q}+Q) - p(\hat{q}+\hat{Q})]\hat{q} > [p(q+Q) - p(q+\hat{Q})]\hat{q} \geq [p(q+Q) - p(q+\hat{Q})]q,$$

which contradicts (*). Therefore, we must have $q \geq \hat{q}$.

Claim 3: $Q < \hat{Q} \Rightarrow b(Q) + Q < b(\hat{Q}) + \hat{Q}$, i.e. the slope of the best-response function is *strictly* greater than -1.

Proof: We can use the same revealed preference argument as in part (b). The only modification is that due to uniqueness of a firm's best-response (Claim 1), the revealed preference inequalities can be written as strict. Continuing as in part (a), we obtain the claim.

Claim 4: The symmetric pure strategy Nash equilibrium is unique.

Proof: This follows from the fact that best-response curves are downward sloping (Claim 2). Suppose in negation that there are two symmetric Nash equilibria, q and q', and that $q > q'$. Since q is a Nash equilibrium, we must have $q = b((J-1)q)$. Using the fact that $b(\cdot)$ and $q > q'$, we obtain

$$q' < q = b((J-1)q) \leq b((J-1)q'),$$

which means that q' cannot be a Nash equilibrium - contradiction. Thus, the symmetric pure strategy Nash equilibrium is unique. (The existence of such an equilibrium has been shown in part (d)).

Using Claims 1-4, we can finally prove that no asymmetric pure strategy equilibrium exists. Suppose in negation that this is not true, i.e. that there exists a Nash equilibrium $(q_1^*, \dots, q_J^*)$ in which $q_l^* \neq q_k^*$ for a certain pair of firms l, k. Let $Q^* = \sum_{j \neq l,k} q_j^*$. Consider the two-firm Cournot game between firms j and k where the output of all other firms is fixed at Q^*. If $(q_1^*, \dots, q_J^*)$ was a Nash equilibrium in the original game, then (q_l^*, q_k^*) must constitute a Nash equilibrium in the two-firm game. The demand curve in this game is $p(Q^* + q)$, which is strictly concave in q as long as $p(\cdot)$ is strictly concave. Let $b(\cdot)$ be each firm's best-response function in this two-firm game. Let q^{o} be the symmetric Nash equilibrium of this game (which is unique by Claim 4), i.e. $q^{o} = b(q^{o})$. Suppose without loss of generality that $q_l^* \le q^{o}$ (If we had $q_l^* > q^{o}$, then by Claim 2, $q_k^* = b(q_l^*) \le b(q^{o}) = q^{o}$, and we would switch firms l and k). If we had $q_l^* = q^{o}$, then we would also have $q_k^* = b(q_l^*) = b(q^{o}) = q^{o}$, which would contradict our negation assumption that $q_l^* \ne q_k^*$. Therefore, we must have

$$q_l^* < q^{o}, \qquad (1)$$

and therefore

$$q_k^* = b(q_l^*) \ge q^{o}. \qquad (2)$$

Using Claim 3, (1) implies that

$$q_l^* + q_k^* = q_l^* + b(q_l^*) > q^{o} + b(q^{o}) = 2q^{o}.$$

But using the result of part (b), (2) implies that

$$q_l^* + q_k^* = q_k^* + b(q_k^*) \le q^{o} + b(q^{o}) = 2q^{o},$$

which contradicts the previous inequality. Therefore, in the Nash equilibrium $(q_1^*, \ldots, q_J^*)$ every two firms should produce the same outputs.

The last part of the proof is easily understood geometrically. Consider the space (q_l, q_k). Claim 3 implies that to the left of the point (q^{o}, q^{o}), firm l's best response curve lies above the (-45^{0}) line passing through (q^{o}, q^{o}), while firm k's best-response curve lies below that line. Therefore, they can never intersect to the left of (q^{o}, q^{o}).

12.C.9 (a) In a Nash equilibrium (q_j^*, q_{-j}^*), each firm j maximizes its profits:

$$\max_{q_j \ge 0} \pi_j(q_j, q_{-j}^*) = (a - b(q_j + q_{-j}^*))\, q_j - c_j q_j.$$

The objective function of this problem is concave in q_j. Therefore, $q_j = 0$ solves this problem if and only if $\partial\pi_j(q_j, q_{-j}^*)/\partial q_j\Big|_{q_j=0} = a - b\, q_{-j}^* - c_j \le 0.$

If this inequality holds and firm j does not produce in this equilibrium, the rival firm ($-j$) maximizes as a monopolist against the linear demand firm, and its optimal output is given by $q^*_{-j} = (a - c_{-j})/(2b)$. Substituting in the above inequality, we see that firm j may not produce in a Nash equilibrium if and only if $c_j \geq (a+c_{-j})/2$. Since we always have $a > c_{-j}$, the necessary condition for firm j not producing is $c_j > c_{-j}$, i.e. only firm 1 may not produce in a Nash equilibrium. Thus, if

$$c_1 \geq (a + c_2)/2, \qquad (*)$$

the output pair $(0, (a - c_2)/(2b))$ constitutes a Nash equilibrium .

Now, let us solve for an equilibrium where both firms produce. First-order conditions for the two firms' maximization problems can be written as

$$a - bq^*_2 - 2b\, q^*_1 - c_1 = 0,$$
$$a - bq^*_1 - 2b\, q^*_2 - c_2 = 0.$$

The solution to this system of equations is given by

$$\begin{aligned} q^*_1 &= (a + c_2 - 2c_1)/(3b), \\ q^*_2 &= (a + c_1 - 2c_2)/(3b). \end{aligned} \qquad (**)$$

Whenever inequality (*) is satisfied, the first expression in (**) produces $q^*_1 < 0$, which means that there is no equilibrium where both firms produce positive amounts. The only Nash equilibrium in this case has firm 1 produce zero. On the other hand, whenever (*) is violated, firm 1 has to produce a positive amount in equilibrium, and the equilibrium outputs are given by (**).

(b) As (**) shows, each firm's output increases as its own cost decreases, and as its rival's cost increases. Equilibrium profits are given by

$$\pi^*_1 = p(q^*_1 + q^*_2)\, q^*_1 - c_1 q^*_1 = (a - 2c_1 + c_2)/(9b),$$
$$\pi^*_2 = p(q^*_1 + q^*_2)\, q^*_2 - c_2 q^*_2 = (a - 2c_2 + c_1)/(9b).$$

Each firm's profit increases as its own cost decreases, and as its rival's cost increases.

(c) In Nash equilibrium, each firm solves

$$\max_{q_j \geq 0} \; p(q_j + Q^*_{-j})\, q_j - c_j q_j,$$

where $Q^*_{-j} = \sum_{k \neq j} q^*_k$. Without loss of generality, assume that all the J firms we consider produce positive quantities in equilibrium. (Otherwise, we would focus on the set of producing firms.) Differentiating the objective function with respect to q_j, we obtain the following first-order condition:

$$p - c_j = -\,p'(Q)\, q_j = (-\,p'(Q)Q/p)\; p \cdot (q_j/Q) = (1/\varepsilon)\; p\alpha_j,$$

where $Q = q_j + Q^*_{-j}$ is the total industry output, $p = p(Q)$ is the market price, $\alpha_j = q_j/Q$ is the market share of firm j, and ε is the elasticity of market demand curve. Industry profits can be written as the sum of individual firms' profits:

$$\Pi = \sum_{j=1}^{J} \pi_j = \sum_{j=1}^{J} (p - c_j)\, q_j = \sum_{j=1}^{J} (p\ \alpha_j q_j)/\varepsilon = \sum_{j=1}^{J} p\alpha_j^2\ (Q/\varepsilon) = H \cdot (pQ/\varepsilon),$$

where H is the Herfindahl index. Therefore, $\Pi/(pQ) = H/\varepsilon$.

12.C.10 (a) Each firm i chooses its output $q_i \geq 0$ to maximize its profits $p(Q_{-i} + q_i)q_i - \alpha_i \tilde{c}(q_i)$. Assuming a positive solution, the first-order condition for this maximization problem is

$$p(Q) = \alpha_i \tilde{c}'(q_i) - p'(Q)\, q_i \qquad \text{(FOC)},$$

where Q is the total output. Since $\tilde{c}(\cdot)$ is increasing and convex, the right-hand side of (FOC) is increasing in α_i and decreasing in q_i. Since (FOC) holds for every firm, we must have $q_i > q_j$ whenever $\alpha_j > \alpha_i$.

Subtracting (FOC) for firm i from (FOC) for firm j gives:

$$\alpha_j \tilde{c}'(q_j) - \alpha_i \tilde{c}'(q_i) = -p'(Q)\,(q_i - q_j).$$

If $\alpha_j > \alpha_i$ then $q_i > q_j$, and the last equality implies that $\alpha_j \tilde{c}'(q_j) > \alpha_i \tilde{c}'(q_i)$. Therefore, marginal costs across firms are not equalized, which implies that the aggregate output is not produced efficiently.

(b) The welfare loss in this case is equal to the loss of consumer surplus due to non-competitive pricing plus the higher production cost due to productive inefficiency. The latter loss has not been considered in section 10.E.

(c) The Cournot equilibrium output and price level obtained in Exercise 12.C.9 are

$$q_1 = (a-2c_1+c_2)/3b,$$
$$q_2 = (a-2c_2+c_1)/3b,$$
$$p = (a+c_1+c_2)/3.$$

Total profits of the two firms can now be computed as

$$(p-c_1)q_1 + (p-c_2)q_2 = (2a^2 + 5c_1^2 + 5c_2^2 - 2a(c_1+c_2) - 8c_1c_2)/9b.$$

Consumer surplus can be computed as

$$\int_0^{q_1+q_2} p(q)dq = (aq - bq^2/2)\Big|_0^{q_1+q_2} = (a-c_1-c_2)(5a+c_1+c_2)/18b.$$

Adding up total profits and consumer surplus, and differentiating with respect to c_1, we obtain

$$\partial S/\partial c_1 = (9c_1 - 9c_2 - 4a)/9b.$$

This derivative is positive when $c_1 > c_2 + (4/9)a$, i.e. when firm 1's costs are substantially higher when firm 2's costs. In this case a decrease in c_1 *reduces* social welfare. The reason is that when c_1 slightly falls, firm 1 steals more business from firm 2, which raises production inefficiency. When c_1 is substantially larger than c_2, this effect actually dominates the increase in consumer surplus due to a lower price.

12.C.11 (a) Suppose firm 2 charges $p_2^* = p(q_1 + q_2)$. If firm 1 charges $p_1 \le p(q_1 + q_2)$, it will sell at its full capacity, and its profit will be equal to $(p_1 - c)\, q_1$. Therefore, firm 1 will not find it optimal to charge a price lower than $p(q_1 + q_2)$.

Suppose instead that firm 1 charges a price above $p(q_1 + q_2)$. Given the described mode of rationing, firm 1's residual demand will be $x(p_1) - q_1$.

Therefore, firm 1 acts as a monopolist with the residual demand given by the inverse demand function $p(q_1 + q_2)$, and its profit maximization problem can be written as

$$\underset{x_1 \le q_1}{\text{Max}} \; [p(x_1 + q_2) - c]\, x_1.$$

Given that firm 2 sells q_2, the optimal output for firm 1 is $b(q_2)$. However, since by assumption $b(q_2)$ exceeds firm 1's capacity q_1, producing $b(q_2)$ is not possible. Given the assumptions of the exercise, firm 1's objective function is concave in x_1, therefore it is optimal for firm 1 to produce the closest feasible amount to $b(q_2)$, which is q_1. This is achieved by firm 1's charging $p(q_1 + q_2)$.

Therefore, it is optimal for firm 1 to charge $p(q_1 + q_2)$, given that firm 2 does so. By a symmetric argument, firm 2 also finds it optimal to charge $p(q_1 + q_2)$, given that firm 1 does so. Therefore, $p_1 = p_2 = p(q_1 + q_2)$ constitutes a Nash equilibrium of this game.

(b) [First printing errata: you need to assume that $p(q_1 + q_2) > c$.] Observe that given that $p(q_1 + q_2) > c$, each firm can make a positive profit by regardless of its rival's strategy by offering a very low price. Therefore, in a NE both firms must make positive sales.

We will look for a NE (p_1, p_2). Suppose first that the firms charge different prices, i.e. that $p_i < p_j$. Since firm j makes positive sales, firm i must sell at full capacity. Then firm i can slightly increase p_i while still selling at full capacity, which would raise firm i's profit.

Therefore, the firms cannot charge different prices in a pure-strategy NE, and we can restrict attention to equilibria where $p_1 = p_2$. There are three cases to consider:

(i) If $p_1 = p_2 > p(q_1 + q_2)$, then at least one firm sells below capacity, and this firm would be better off by slightly lowering its price and either selling at full capacity or stealing all customers from the rival.

(ii) If instead $p_1 = p_2 < p(q_1 + q_2)$, then both firms sell at full capacity. Then each firm could gain by increasing its price to $p(q_1 + q_2)$, which would still enable it to sell at full capacity.

(iii) Finally, suppose that $p_1 = p_2 = p(q_1 + q_2)$ and $q_i > b(q_j)$. Consider firm i's raising its price above $p(q_1 + q_2)$. As shown in part (a), firm i's best response would be to sell $q_i > b(q_j)$, which is now feasible (in contrast to (a)). For this purpose, firm i should set a price higher than $p(q_1 + q_2)$, therefore, this case does not describe an equilibrium.

As all possible cases have been exhausted, there is no pure strategy NE.

12.C.12 (a) Assume that $\Pi^i_{11}(q_i,q_j) < 0$, $i = 1, 2$.
The first-order condition for firm i's profit maximization problem is $\Pi^i_1(b_i(q_j),q_j) = 0$. Differentiating with respect to q_j gives: $\partial b_i(q_j)/\partial q_j = -\Pi^i_{12}(b_i(q_j),q_j) / \Pi^i_{11}(b_i(q_j),q_j)$. Therefore, the sign of $\partial b_i(q_j)/\partial q_j$ coincides with the sign of $\Pi^i_{12}(b_i(q_j),q_j)$. In words, firm i's best-response function is increasing (decreasing) provided that Π^i_{12} is positive (negative).

(b) In the Cournot model

$$\Pi^i(q_i,q_j) = p(q_i+q_j)q_i - c(q_i).$$

Then $\Pi^i_{12}(q_i,q_j) = p''(q_i+q_j)q_i + p'(q_i+q_j)$, which is negative if $p(\cdot)$ is downward sloping and not too convex. Therefore, the "normal" slope of best response functions in the Cournot model is negative.

12.C.13 For definiteness, suppose that it is firm that is playing its best response to p_2, and suppose in negation that some consumers strictly prefer not to purchase. This situation is depicted in Figure 12.C.6(a) in the textbook. Consider a small reduction in firm 1's price p_1. As long as firm 1 does not start to compete with firm 2, its sales are given by the consumer x who is indifferent between buying from firm 1 and not buying at all: $p_1 + tx = v$, which gives $x(p_1) = (v-p_1)/t$. Firm 1's profit can then be written as

$$\pi_1(p_1) = (p_1 - c)x(p_1) = (p_1 - c)(v - p_1)/t.$$

Differentiating with respect to p_1, we obtain

$$\pi_1'(p_1) = (v+c-2p_1)/t.$$

By assumption, some consumers strictly prefer not to buy rather than buying from firm 1, so at least this should be true for the consumer living at point $z = 1$, which means that $p_1 > v - t$. Substituting this inequality in the expression for $\pi_1'(p_1)$, we obtain $\pi_1'(p_1) < (c+2t-v)/t$, which is negative by assumption. Therefore, by raising p_1 slightly firm 1 could raise its profit, which contradicts the assumption that firm 1 is playing its best response.

12.C.14. There can be three types of equilibria (p_i^*, p_j^*) in the model:

Type 1: The consumer who is indifferent between buying from the two firms strictly prefers buying to not buying. The indifferent consumer is located at distance x from firm i, where $p_i^* + xt = p_j^* + (1-x)t$, and therefore $x = (t - p_i^* + p_j^*)/2t$. The utility of this consumer from buying from firm i is $v - p_i^* - xt = v - t/2 - (p_i^* + p_j^*)/2$. This consumer strictly prefers buying when

$$v > (p_i^* + p_j^*)/2 + t/2 \qquad (1)$$

In a type 1 equilibrium, the firms compete for a positive measure of consumers and each firm's demand is affected by the other firm's price.

Type 2: The consumer who is indifferent between buying from the two firms strictly prefers not buying to buying. Given the derivation above, this happens if and only if

$$v < (p_i^* + p_j^*)/2 + t/2 \qquad (2)$$

In a type 2 equilibrium, the sets of consumers buying from each firm are separate, and each firm's demand is unaffected by the other firm;s price.

Type 3: The consumer who is indifferent between buying from the two firms is also indifferent between buying and not buying.Given the derivation above, this happens if and only if

$$v = (p_i^* + p_j^*)/2 + t/2 \qquad (3)$$

In a type 3 equilibrium, if a firm lowers its price slightly, it starts stealing customers from the rival firm. On the other hand, if a firm raises its price slightly, the lost customers do not buy from the rival firm - they do not buy at all.

(a)-(b) In a type 1 equilibrium, if firm i changes its price slightly from p_i^* to p_i, its demand will be determined by the consumer x who is indifferent between buying from the two firms:

$$p_i + xt = p_j^* + (1-x)t.$$

Therefore, $x_i(p_i, p_j^*) = (t+p_j^*-p_i)M/2t$. Firm i chooses p_i to maximize profits:

$$\underset{p_i}{\text{Max}}\ (p_i - c)x_i(p_i, p_j^*) = (p_i - c)(t + p_j^* - p_i)M/2t.$$

The first-order condition for this program is

$$(t + p_j^* - 2p_i + c)M/2t = 0.$$

The first-order condition for firm j is symmetric. Combining the two first-order conditions, we obtain $p_i^* = p_j^* = c + t$.

Using (1), we see that this will indeed constitute a type 1 equilibrium if and only if $v > c + 3/2\ t$.

(c) In a type 2 equilibrium, if firm i changes its price slightly from p^* to

p_i, its demand will be determined by the consumer at the distance $x < 1/2$ from the firm who is indifferent between buying from firm i and not buying at all:

$$p_i + xt = v.$$

Therefore, $x_i(p_i, p_j^*) = (v-p_i)M/t$. Firm i chooses p_i to maximize profits:

$$\max_{p_i} (p_i - c)x_i(p_i, p^*) = (p_i - c)(v - p_i)M/t.$$

The first-order condition for this program is

$$(v - 2p_i + c)M/t = 0.$$

From here firm i's equilibrium price p_i^* can be determined:

$$p_i^* = (v + c)/2.$$

From a symmetric derivation for firm j,

$$p_j^* = (v + c)/2.$$

Using (2), we see that this will indeed constitute a type 2 equilibrium if and only if $v < c + t$.

(d) In a type 3 equilibrium, if firm i raises its price slightly from $p^* = v - t/2$ to p_i, its demand will be determined by the consumer who is indifferent between buying from firm i and not buying at all:

$$p_i + xt = v.$$

Therefore, $x_i(p_i, p_j^*) = x_i^+(p_i, p_j^*) = (v-p_i)M/t$ for $p_i \geq p_i^*$. Since firm i chooses p_i to maximize profits, a slight increase in p_i from p_i^* should not increase profits:

$$\partial/\partial p_i \left. (p_i-c)x_i^+(p_i, p_i^*)\right|_{p_i=p_i^*} = \partial/\partial p_i \left. (p_i - c)(v-p_i)M/t\right|_{p_i=p_i^*} =$$

$$= (v - 2p_i^* + c)M/t \leq 0,$$

i.e.,

$$v \leq 2p_i^* - c \qquad \text{(i.1)}$$

A symmetric derivation for firm j yields

$$v \leq 2p_j^* - c \qquad \text{(j.1)}$$

Also in a type 3 equilibrium, if firm i lowers its price slightly from p_i^* to

p_i, its demand will be determined by the consumer who is indifferent between buying from firm i and buying from firm j:

$$p_i + xt = p_j^* + (1-x)t.$$

Therefore, $x_i(p_i, p_j^*) = x_i^-(p_i, p_j^*) = (t+p_j^*-p_i)M/2t$ for $p_i \leq p_i^*$.

Since firm i chooses p_i to maximize profits, a slight decrease in p_i from p_i^* should not increase profits:

$$\partial/\partial p_i \ (p_i-c)x_i^-(p_i, p_j^*) \ \Big|_{p_i=p_i^*} = \partial/\partial p_i \ (p_i-c)(t+p_j^*-p_i)M/2t \ \Big|_{p_i=p_i^*}$$
$$= (t + p_j^* - 2p_i^* + c) \ M/2t \geq 0,$$

i.e.,

$$t + p_j^* - 2p_i^* + c \geq 0 \qquad \text{(i.2)}$$

A symmetric derivation for firm j yields

$$t + p_i^* - 2p_j^* + c \geq 0 \qquad \text{(j.2)}$$

Adding (i.1) and (j.1) and using (3), one obtains $v \geq c + t$. Adding (i.2) and (j.2) and using (3), one obtains $v \leq c + 3t/2$. Therefore, type 3 equilibria can only exist when

$$c + t \leq v \leq c + 3t/2. \qquad (*)$$

On the other hand, when (*) holds, $p_i^* = p_j^* = v - t/2$ satisfies (3), (i.1), (j.1), (i.2), (j.2), and is therefore a symmetric type 3 equilibrium. When the inequalities in (*) hold strictly, there also exists a continuum of asymmetric equilibria:

$$p_i^* = v - t/2 + \varepsilon,$$
$$p_j^* = v - t/2 - \varepsilon,$$

where ε is small enough in absolute value.

(e) In a type 1 equilibrium (cases (a)-(b)), a reduction in t makes competition more intense, prices and profits fall. At the same time, in a symmetric type 3 equilibrium, a small reduction in t increases prices and profits. However, when t falls substantially, a type 3 equilibrium is no longer possible, and the game switches to a type 1 equilibrium.

12.C.15. The consumer who is indifferent between buying from firm i and firm j is located at distance x from firm i, where

$$p_i + tx^2 = p_j + t(1-x)^2.$$

From here we obtain the demand for firm i's product:

$$x_i(p_i,p_j) = [1/2 + (p_j - p_i)/2t]\ M.$$

(Observe that the demand is the same as for linear transportation costs. The equilibrium is therefore also going to be the same.)

We are looking for an equilibrium (p_i^*,p_j^*). In equilibrium, firm i solves

$$\underset{p_i}{\text{Max}}\ (p_i - c)\ x_i(p_i,p_j^*) = (p_i - c)\ [1/2 + (p_j^* - p_i)/2t]\ M$$

The first-order condition for this maximization program is

$$[t + p_j^* - 2p_i^* + c]\ M/2t = 0.$$

A symmetric derivation for firm j yields

$$[t + p_i^* - 2p_j^* + c]\ M/2t = 0.$$

Solving these two equations, one obtains $p_i^* = p_j^* = c + t$.

12.C.16. If the circumference of the circle is 1, then the firms are located $1/J$ apart from each other. We will be looking for a symmetric equilibrium where all firms charge p^*. Suppose that firm i deviates and charges p_i. Then the consumer who is indifferent between buying from firm i and firm i+1 will be located at the distance x from firm i, where x solves

$$p_i + tx^2 = p^* + t[1/J - x]^2.$$

Observe that since consumers on both sides of firm i buy its product, the demand for the product is going to be $x_i(p_i,p^*) = 2xM$. Using the equation above, we find

$$x_i(p_1,p^*) = [1/J + (p^* - p_1)J/t]M.$$

In equilibrium, firm i solves

$$\underset{p_i}{\text{Max}}\ (p_i - c)\ x_i(p_i,p^*) = (p_i - c)\ [1/J + (p^* - p_i)J/t]M.$$

The first-order condition for this maximization program is

$$[1/J + (p^* - 2p_i + c)J/t]M = 0$$

Substituting the equilibrium value $p_i = p^*$, we can solve for p^*:

$$p^* = c + t/J^2.$$

As $t \longrightarrow 0$ or $J \longrightarrow 0$, we have $p^* \longrightarrow c$. Intuitively, as the transportation costs fall or the number of firms grows without bound, the industry becomes extremely competitive and price goes down to marginal cost.

12.C.17. For now, assume that prices are such that both firms sell positive amounts, i.e. that there exists a consumer who is indifferent between buying from the two firms. This consumer is located at the distance x from firm 1, where

$$p_1 + tx = p_2 + t(1-x).$$

From here we obtain the demands for the two firms' products:

$$x_1(p_1,p_2) = [1/2 + (p_2 - p_1)/2t]\ M,$$

$$x_2(p_1,p_2) = [1/2 + (p_1 - p_2)/2t]\ M.$$

We are looking for an equilibrium (p_1^*,p_2^*). In equilibrium, firm 1 solves

$$\underset{p_1}{\text{Max}}\ \pi_1 = (p_1 - c_1)\ x_1(p_1,p_2^*) = (p_1 - c_1)\ [1/2 + (p_2^* - p_1)/2t]\ M$$

The first-order condition for this maximization program is

$$[t + p_2^* - 2p_1^* + c_1]\ M/2t = 0.$$

A symmetric derivation for firm j yields

$$[t + p_1^* - 2p_2^* + c_2]\ M/2t = 0.$$

Solving these two equations, one obtains

$$p_1^* = (2c_1 + c_2)/3 + t,$$
$$p_2^* = (2c_2 + c_1)/3 + t.$$

After substituting these expressions into the expressions for profits, we obtain

$$\pi_1^* = \frac{t}{2}^2\left(1 + \frac{c_2 - c_1}{3t}\right)^2$$
$$\pi_2^* = \frac{t}{2}^2\left(1 + \frac{c_1 - c_2}{3t}\right)^2.$$

An increase in each firm's costs increases its equilibrium price and reduces its equilibrium profit.

Now we have to check when firm 2 (the higher-cost firm) makes no sales. The equilibrium demand for firm 2's product is

$$x_2(p_1^*,p_2^*) = [1/2 + (p_1^* - p_2^*)/2t]\ M = [1/2 + (c_1 - c_2)/6t]\ M.$$

When $c_2 - c_1 < 6t$, this demand is positive, both firms make positive sales, and our equilibrium analysis is correct. When $c_2 - c_1 \geq 6t$, in equilibrium firm 2 makes no sales. Since firm 2 can offer any price greater than c_2, in equilibrium firm 1 prices so that $p_1 + t = c_2$, i.e. the consumer living next to firm 2 is indifferent between buying from the two firms.

12.C.18. (a) Firm 1 chooses q_1 knowing that given its choice, firm 2 will produce $b_2(q_1)$, where $b_2(q_1)$ is firm 2's best-response function. Firm 1's program, therefore, is

$$\underset{q_1}{\text{Max}}\ \Pi^1(q_1,b_2(q_1)).$$

The first-order condition for this program can be written as

$$\Pi_1^1(q_1,b_2(q_1)) = -\ \Pi_2^1\ (q_1,b_2(q_1))\ b_2'(q_1) < 0$$

(since $\Pi_2^1\ (q_1,b_2(q_1)) < 0$ and $b_2'(q_1) < 0$). If the firms instead choose quantities simultaneously, the first-order condition becomes

$$\Pi_1^1\ (q_1,b_2(q_1)) = 0.$$

Since $\Pi^1_{11}(q_1, b_2(q_1)) < 0$, this implies that the Stackelberg leader picks a larger quantity in equilibrium than in the Cournot game. Since the best response function of firm 2 is downward-sloping with a slope larger than - (see Exercise 12.C.8(c)), this implies that the follower picks a smaller quantity and aggregate output increases (and therefore price decreases). Since the leader could have chosen the Cournot quantity, we know that his profits as a Stackelberg leader are higher. The follower produces less and obtains a lower price than in the Cournot outcome, which implies that his profits are lower.

(b) See Figure 12.C.18. *N* denotes the Nash equilibrium outcome, while *S* denotes the equilibrium of the Stackelberg game.

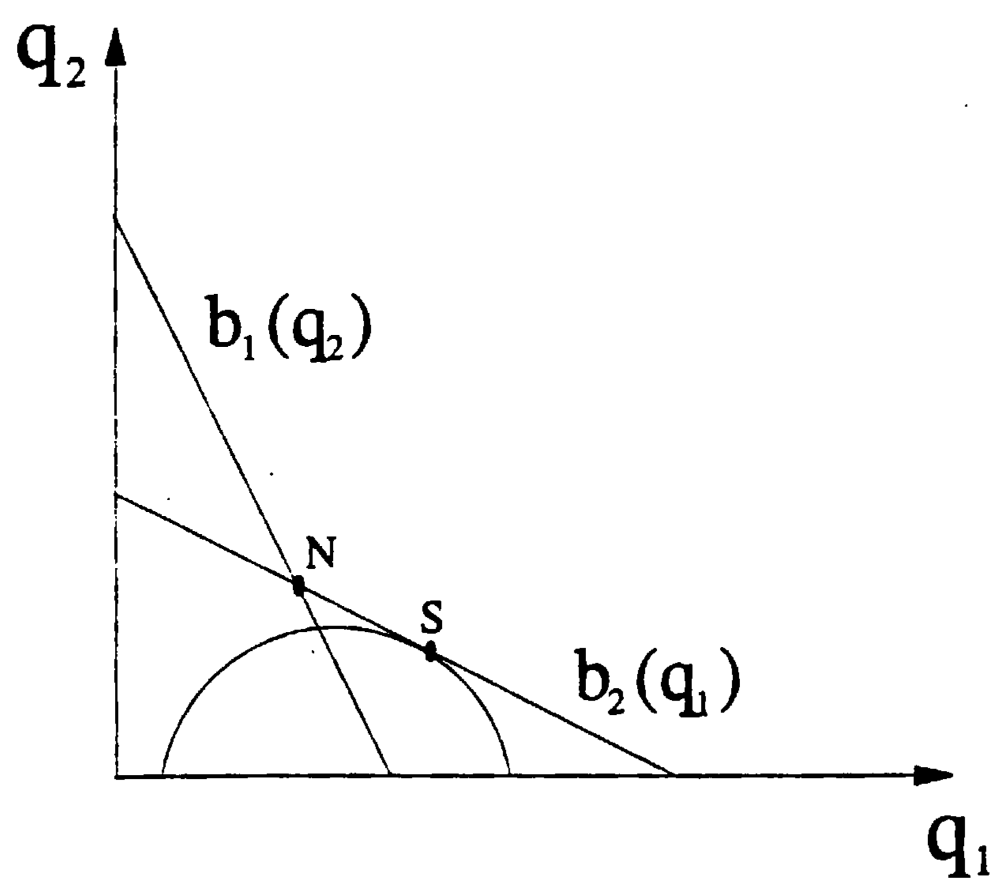


Figure 12.C.18

12.C.19. See Exercise 8.B.5 .

12.C.20. In a Nash equilibrium (q_1^*, q_2^*), firm j solves

$$\underset{q_j \geq 0}{\text{Max}}\ p(q_j + q_k^*)q_j - c(q_j).$$

Assuming positive production levels, the first-order conditions for the two firms can be written as

$$p'(q_1^* + q_2^*)\ q_1^* + p(q_1^* + q_2^*) = c'(q_1^*),$$
$$p'(q_1^* + q_2^*)\ q_2^* + p(q_1^* + q_2^*) = c'(q_2^*).$$

Adding these two equalities yields

$$p'(q_1^*+q_2^*)(q_1^*+q_2^*)/2 + p(q_1^*+q_2^*) = (c'(q_1^*) + c'(q_2^*))/2.$$

Observe that since $c(\cdot)$ is convex, $c'(\cdot)$ is increasing, and therefore

$$(c'(q_1^*) + c'(q_2^*))/2 \leq (c'(q_1^*+q_2^*) + c'(q_1^*+q_2^*))/2 = c'(q_1^*+q_2^*).$$

Combining this with the previous equality, we obtain

$$p'(q_1^*+q_2^*)(q_1^*+q_2^*)/2 + p(q_1^*+q_2^*) \leq c'(q_1^*+q_2^*). \qquad (*)$$

Since $p'(\cdot) < 0$, (*) implies that

$$p(q_1^*+q_2^*) > c'(q_1^*+q_2^*),$$

i.e. the Cournot equilibrium price is higher than the competitive price at the same aggregate output level.

Next we show that $q_1^* + q_2^* > q^m$, i.e., that the equilibrium duopoly price $p(q_1^*+q_2^*)$ is strictly less than the monopoly price $p(q^m)$. The argument is in two parts.

First, we argue that $q_1^* + q_2^* \geq q^m$. To see this, suppose that $q^m > q_1^* + q_2^*$. By increasing its quantity to $\hat{q}_j = q^m - q_k^*$, firm j would (weakly) increase the joint profit of the two firms (the firms' joint profit then equals the monopoly profit level, its largest possible level). In addition, because aggregate quantity increases, price must fall, and so firm k is strictly worse off. This implies that firm j is strictly better off, and so firm j would have a profitable deviation if $q^m > q_1^* + q_2^*$. We conclude that we must have $q_1^* + q_2^* \geq q^m$.

Second, condition (*) implies that we cannot have $q_1^* + q_2^* = q^m$ because then

$$p'(q^m)q^m/2 + p(q^m) \leq c'(q^m),$$

which is incompatible with the monopoly first-order condition (12.B.4) in the textbook. Thus, we must have $q_1^* + q_2^* > q^m$.

12.D.1. (a) Monopoly profit in period t is

$$\max_p \gamma^t x(p)\,(p-c) = \gamma^t \max_p x(p)\,(p-c) = \gamma^t \Pi^m.$$

If a firm deviates at $t = \tau$, it can obtain $\gamma^\tau \Pi^m$ in that period, and it will get zero forever after. If it does not deviate, its payoff is

$$\gamma^\tau \sum_{t=0}^{\infty} (\gamma\delta)^t\ \Pi^m/2 = 1/(1-\gamma\delta)\ \gamma^\tau\ \Pi^m/2.$$

Therefore, deviation is not profitable if and only if $1/(1-\gamma\delta)\ \gamma^\tau\ \Pi^m/2 \geq \gamma^\tau\ \Pi^m$, or $\delta \geq 1/2\gamma$.

(b) If a firm deviates, it can obtain Π^m in that period, and it will get zero forever after. If it does not deviate, its payoff is

$$\sum_{t=0}^{\infty} (\gamma\delta)^t\ \Pi^m/2 = 1/(1-\gamma\delta)\ \Pi^m/2.$$

Therefore, deviation is not profitable if and only if

$1/(1-\gamma\delta)\ \ \Pi^m/2 \geq \Pi^m$, or $\delta \geq 1/2\gamma$.

(c) If a firm deviates, it can obtain $\sum_{t=0}^{K-1} \delta^t\ \Pi^m = (1-\delta^K)/(1-\delta)\ \Pi^m$ in the next K periods, and it will get zero forever after. If it does not deviate, its payoff is

$$\sum_{t=0}^{\infty} \delta^t\ \Pi^m/2 = 1/(1-\delta)\ \Pi^m/2.$$

Therefore, deviation is not profitable if and only if

$1/(1-\delta)\ \Pi^m/2 \geq (1-\delta^K)/(1-\delta)\Pi^m$, or $\delta \geq (1/2)^{1/K}$.

12.D.2. Let $\Pi^* > 0$ be the firms' equilibrium joint profits, which in equilibrium are split equally among firms. The best deviation for a firm is to undercut the rivals by a small ε, in which case it can steal all the demand and obtain almost as much as Π^* in that period. In the following punishment phase, the deviator will get zero forever after. If the firm does not deviate, its payoff is $\sum_{t=0}^{\infty} \delta^t\ \Pi^*/J$.

Therefore, deviation is not profitable if and only if $1/(1-\delta)\Pi^*/J \geq \Pi^m$, or $\delta \geq (J-1)/J$.

Since $(J-1)/J$ increases in J, as J increases, δ has to increase in order for collusion to be still sustainable. Therefore,as the number of firms increases, it is harder to sustain a collusive outcome.

12.D.3. (a) Example 12.C.1 in the textbook solves for the static Nash equilibrium in the game. The equilibrium yields the Cournot outcome in which each firm makes a profit of $(a-c)^2/9b$. The maximum gain from deviation can be obtained by playing the best response to $q^m/2 = (a-c)/4b$, which is $3(a-c)/8b$. This maximum gain is $9/64\ (a-c)^2/b$.Monopoly profit can be calculated to be $(a-c)^2/4b$. The payoff from deviating is

$$9(a-c)^2/64b + \sum_{t=1}^{\infty} \delta^t\ (a-c)^2/9b = 9(a-c)^2/64b + \delta/(1-\delta)\ (a-c)^2/9b.$$

The payoff from not deviating is

$$\sum_{t=0}^{\infty} \delta^t\ (a-c)^2/8b = 1/(1-\delta)\ (a-c)^2/8b.$$

Therefore, deviation is not profitable if and only if

$1/(1-\delta)\ (a-c)^2/8b \geq 9(a-c)^2/64b + \delta/(1-\delta)\ (a-c)^2/9b$, or $\delta \geq 9/17$.

(b) The maximum gain from deviation can be obtained by playing the best response to q, which is $(a-c)/2b - q/2$. This maximum gain is $b[(a-c)/2b - q/2]^2$.

The payoff from deviating is

$$b[(a-c)/2b - q/2]^2 + \sum_{t=0}^{\infty} \delta^t (a-c)^2/9b =$$

$$= b[(a-c)/2b - q/2]^2 + \delta/(1-\delta)(a-c)^2/9b.$$

The payoff from not deviating is

$$\sum_{t=0}^{\infty} \delta^t \quad (a-c)^2/4b \quad = \quad 1/(1-\delta)\ (a-c)^2/4b.$$

Therefore, deviation is not profitable if and only if

$$1/(1-\delta) \quad (a-c)^2/4b = b[(a-c)/2b - q/2]^2 + \quad \delta/(1-\delta) \quad (a-c)^2/9b.$$

Thus,

$$\delta(q) = [(a-c)^2/4b - b[(a-c)/2b - q/2]^2]/[(a-c)^2/9b - b[(a-c)/2b - q/2]^2],$$

which is a decreasing, differentiable function of q.

12.D.4. (a) The most profitable price that can be sustained for $\delta \in [1/2, 1)$ is the monopoly price. As Exercise 12.B.4 shows, the monopoly price is increasing in the cost of production c.

(b) Let $p_m(c)$ be the optimal monopoly price as a function of cost, and $\pi_m(c)$ be the resulting monopoly profit. Let c^* be the marginal cost in period one and $c \geq c^*$ be the marginal cost starting from period two. When $c = c^*$, the monopoly price and profits can be sustained in period 1 (see Proposition 12.D.1). When c is raised above c^*, the gain from deviating in period one stays the same, but the future payoff from complying falls. Therefore, only lower profits can be supported in period one.

Formally, the highest profits in period 1 can be supported by the strategies which result in the best collusive equilibrium after compliance and revert to the Bertrand punishment after a deviation. The best collusive equilibrium starting from period 2 brings the firms $\pi_m(c)$ in every period. The highest supportable joint profits π^* in period 1 therefore have to satisfy

$$\pi^*/2 \le \pi^* + \delta/(1-\delta)\ \pi_m(c)/2, \text{ i.e. } \pi^* \le \delta/(1-\delta)\ \pi_m(c).$$

Observe that by assumption $\delta/(1-\delta) \ge 1/2$. We need to consider two cases:

Case (i): $\delta/(1-\delta)\ \pi_m(c) \ge \pi_m(c^*)$ - a small cost increase. Then monopoly profits can still be sustained in period 1, and the most profitable price will still be $p_m(c^*)$. The *highest* sustainable price may be higher than $p_m(c^*)$, but an increase in c reduces it.

Case (ii): $\delta/(1-\delta)\ \pi_m(c) < \pi_m(c^*)$ - a large cost increase. Then monopoly profits in period 1 can no longer be sustained. (As an extreme case, imagine $c=\infty$, in which case no profits can be sustained in period 1.) Therefore, the monopoly price can no longer be sustained in period 1, nor any higher price. The most profitable price will now be the highest sustainable price, and it will be lower than $p_m(c^*)$.

12.D.5. Let π_H and π_L be the profits the firms collect in high and low states respectively:

$$\pi_H = (p_H - c)\ x(p_H),$$
$$\pi_L = (p_L - c)\ \alpha x(p_L).$$

In order for the strategies described to constitute an equilibrium, a firm should not find it profitable to deviate in either high or low states. If a firm deviates in a high state, it can undercut the rival by a small ε and collect π_H. The deviator will be punished by a reversion to Bertrand competition and will collect zero profits thereafter. If the firm does not deviate, it collects $\pi_H/2$ in the current period, and its expected present value of future profits is

$$1/2\ (\delta + \delta^2 + \ldots)(\lambda\pi_H(p_H) + (1-\lambda)\pi_L(p_L)) =$$

$$= 1/2\ \delta/(1-\delta)\ (\lambda\pi_H(p_H) + (1-\lambda)\pi_L(p_L)).$$

Therefore, the incentive constraint in the high state can be written as

$$\pi_H/2 \le 1/2\ \delta/(1-\delta)\ (\lambda\pi_H + (1-\lambda)\pi_L). \qquad (IC_H)$$

Similarly, if a firm deviates in a low state, it can collect π_H in this period and will collect zero profits thereafter. If the firm does not deviate, it collects $\pi_L/2$ in the current period, and its expected present value of future profits is $\delta/(1-\delta)\ (\lambda\pi_H + (1-\lambda)\pi_L)$. Therefore, the incentive constraint in the low state can be written as

$$\pi_L/2 \le 1/2\ \delta/(1-\delta)\ (\lambda\pi_H + (1-\lambda)\pi_L). \qquad (IC_L)$$

(a) If $p_H = p_L = p^m$, then $\pi_L = (p^m - c)\ \alpha x(p^m) = \alpha\pi_H$. Substituting in the incentive constraints (IC_H) and (IC_L), we obtain

$$1 \le \delta/(1-\delta)\ (\lambda + (1-\lambda)\alpha),$$

$$\alpha \le \delta/(1-\delta)\ (\lambda + (1-\lambda)\alpha).$$

Since $\alpha < 1$, the first constraint clearly implies the second one.

Expressing δ from the first constraint, we obtain

$$\delta \ge \underline{\delta} = 1/(1+\lambda + (1-\lambda)\alpha).$$

It is easy to see that $\underline{\delta} \in (1/2, 1)$. When $\delta \ge \underline{\delta}$, there is a SPNE in which both firms set $p_H = p_L = p^m$.

(b) From the above derivation it is clear that when $\delta < \underline{\delta}$, charging p^m in all periods in both states would violate (IC_H).

To find out which prices are sustainable, we rewrite (IC_H) as

$$\pi_H \le \frac{\delta(1-\lambda)\ \pi_L}{1 - \delta - \lambda\delta}.$$

Observe that $\pi_L \le \pi_L^m = (p^m - c)\ \alpha x(p^m)$. Therefore, in order for a price p_H to be sustainable in high demand periods, it should satisfy $(p_H - c)\ x(p_H) \le$

$$\frac{\delta(1-\lambda)}{1 - \delta - \lambda\delta}\ (p^m - c)\ \alpha x(p^m).$$

Observe that a price higher than p^m is not sustainable in the high periods, because the deviator would have the option of undercutting by charging p^m. The maximum price sustainable in the high periods is therefore the smallest root of the following equation:

$$(p_H - c)\, x(p_H) = \frac{\delta(1-\lambda)}{1 - \delta - \lambda\delta} (p^m - c)\, \alpha x(p^m). \qquad (*)$$

Suppose that this root is p_H^*, and the corresponding profit in the high periods is $\pi_H^* = (p_H^* - c)\, x(p_H^*)$. Since the right-hand side of (*) is increasing in δ, and we know that $\delta < \underline{\delta}$, we can write

$$\pi_H^* \geq \frac{\underline{\delta}(1-\lambda)\, \pi_L^m}{1 - \underline{\delta} - \lambda\underline{\delta}} = \pi_L^m/\alpha > \pi_L^m$$

Since the profit pair (π_H^*, π_L^m) is known to satisfy (IC_H) and $\pi_H^* > \pi_L^m$, it is easy to see that (IC_L) is also satisfied. In words, it is possible to sustain the highest possible profit π_H^* in the high demand periods and at the same time the monopoly profit π_L^m in the low demand periods. This is equilibrium outcome is optimal for the firms. Observe that in this equilibrium $p_H = p_H^* < p^m = p_L$, i.e. the price in high demand periods is lower than the price in low demand periods. Intuitively, collusion is harder to sustain in high demand periods, because the gains from deviation are higher, and the future expected benefit of cooperation is the same as in the low demand periods.

(c) Now $\delta < 1/2$. Suppose in negation that either $\pi_H > 0$ or $\pi_L > 0$. Then we must have $\lambda\pi_H + (1-\lambda)\pi_L > 0$. ($IC_H$) yields

$$\pi_H < \lambda\pi_H + (1-\lambda)\pi_L.$$

Similarly, (IC_L) yields

$$\pi_L < \lambda\pi_H + (1-\lambda)\pi_L.$$

Multiplying the first inequality by λ and the second one by $1-\lambda$ and adding up, we obtain

$$\lambda\pi_H + (1-\lambda)\pi_L < \lambda\pi_H + (1-\lambda)\pi_L,$$

which is false. Therefore, we have $\pi_H = \pi_L = 0$, and consequently $p_H = p_L = c$.

12.E.1. In a two-stage model the most efficient firm is not necessarily active. Take the following example: $p(q) = 1 - q$, $c_1 = 1/2$, $c_2 = 2/5$, entry cost $K = 1/17$. Firm one entering and charging $p = 3/4$ and firm 2 not entering is a SPNE of the two stage game (As shown in Exercise 12.C.4, if both firms enter, the second stage equilibrium price is 1/2, and firm 2 will make a profit of 1/20 which is not high enough to compensate for the fixed cost.) However, firm 2 is more efficient than firm 1.

12.E.2.

$$\pi_j = p(Jq_j)\, q_j - c(q_j).$$

$d\pi_j/dJ = [p(Jq_j) - c'(q_j)]\, dq_j/dJ + p'(Jq_j)\, q_j\, d(Jq_j)/dJ < 0$, since $[p(Jq_j) - c'(q_j)] > 0$ by (A3), $dq_j/dJ < 0$ by (A2), $p'(\cdot) < 0$, and $d(Jq_j)/dJ > 0$ by (A1).

12.E.3. The socially optimal number of firms J^o solves

$$\max_J W(J) = \int_0^{Jq_j} p(s)\, ds - J\, c(q_j) - J\, K,$$

<u>Example 12.E.1</u>: $q_j = (a-c)/b\ 1/(J+1)$, $p(Q) = a - bQ$, $c(Q) = cQ$.

The first-order condition to the welfare maximization problem yields

$$J^o = [(a-c)^2/bK]^{1/3} - 1.$$

The equilibrium number of entrants in Example 12.E.1 is

$$J^* = [(a-c)^2/bK]^{1/2} - 1.$$

As $K \rightarrow 0$, both J^o and J^* tend to infinity, but J^* at a higher rate. As for the welfare loss from excess entry, it is

$$W(J^o) - W(J^*) = \int_{Q_*}^{Q_o} p(s)\, ds - c(Q_o - Q_*) - (J^o - J^*)\, K,$$

where $Q_o = J^o q_{j^o}$, $Q_* = J^* q_{j^*}$.

Observe that this expression in bounded by $K(J^* - J^o)$, since the social loss of excess entry arises from replicating the fixed cost K. It is easy to see that $\lim_{K \to 0} K(J^* - J^\circ) = 0$, which implies that the welfare loss goes to zero as K goes to zero.

Example 12.E.2: Since two firms in this Example already bring price down to marginal cost, the socially optimal number of firms J° is either one or two. If $J^\circ = 1$, free entry results in the socially optimal number of firms. If $J^\circ = 2$, free entry yields one firm less than socially optimal. The welfare loss in this case is

$$W(J^\circ) - W(J^*) = \int_{Q_*}^{Q_\circ} p(s)\, ds \; - c(Q^\circ - Q^*) - (J^\circ - J^*)\, K,$$

where $Q_\circ = J^0 q_{J^\circ} = (a-c)/b$, $Q_* = J^* q_{J^*} = q^m = (a-c)/2b$. However, as $K \longrightarrow 0$, we must have $J^\circ = 2$, and therefore, there is no welfare loss from free entry.

12.E.4. Since the firms form a cartel no matter how many firms enter the market, industry output and price will be unchanged by the number of firms in the industry. Only the aggregate fixed costs increase as the number of firms increase. Therefore, the optimal number of firms in this industry is one.

If the planner cannot control entry, the equilibrium number of firms will be $J^* = \pi^m/K$. In terms of welfare this means that free entry leads to a complete dissipation of monopoly profits, without any benefit to consumers.

12.E.5. We assume that entering firms do not choose their locations. No matter how many firms enter, they spread evenly around the circle. According to the solution of Exercise 12.C.16, if J firms have entered, the second-stage equilibrium price is

$$p^* = c + t/J^2.$$

Since each firm in equilibrium serves I/J customers, its profits are

$$(p^* - c)\; I/J = tI/J^3.$$

The equilibrium number of firms in the two-stage entry game is determined by equalizing these second-stage profits to the fixed cost of entry K. We obtain

$$J^{*} = (tI/K)^{1/3}.$$

By assumption of Exercise 12.C.16, consumers' valuation for the product is high enough so that they always buy. In this situation the social planner does not care about the market price. Aggregate production/consumption is I, regardless of the price and of the number of firms producing. The number of firms affects only the aggregate fixed costs and the aggregate transportation costs. The social planner's decision will be determined by the tradeoff between the two.

To compute the transportation costs, observe that if a consumer lives at a distance $x \leq 1/(2J)$ to the right from firm j, his transportation costs are tx^2. The density of consumers on the circle is I. Therefore, the aggregate transportation costs of consumers with $x \in [0,1/(2J)]$ is

$$\int_0^{1/2J} tx^2 \, I \, dx = tI/(24J^3).$$

The aggregate transportation costs are obtained by multiplying this result by $2J$, which gives $It/(12J^2)$. The social planner's program can now be written as

$$\operatorname*{Min}_{J} \; tI/(12J^2) + JK.$$

The first-order condition is $tI/(6J^3) = K$, from where we obtain the socially optimal number of firms

$$J^{0} = (tI/6K)^{1/3}.$$

Observe that the socially optimal number of firms here is smaller than the free-entry equilibrium number. The "business stealing" effect here dominates the tendency to insufficient entry resulting from non-appropriation of social surplus when products are differentiated. (See also Mankiw and Whinston, 1986.)

12.E.6. Suppose there is a SPNE of the two-stage entry game where J firms enter, and each active firm produces q_J. We want to show that this will also be a Nash equilibrium outcome of the one-stage entry game. To show this, we need to consider all possible deviations in the one-stage entry game. There are three kinds of deviations: an active firm may change its output, an active firm may choose to stay inactive, and an inactive firm may choose to become active. We will show that all these deviations are unprofitable:

(i) If an active firm could profitably change its output, then this would not be a Nash equilibrium outcome of the second stage of the two-stage game. Therefore, this would not be a SPNE outcome of the two-stage game, which contradicts our assumption.

(ii) If an active firm preferred to stay inactive in the one-stage game, this would mean that the firm's equilibrium profits are negative. But then this would not be a SPNE of the two-stage game.

(iii) Suppose that an inactive firm prefers to enter and produce some quantity q. Then entering and producing q would also be a profitable deviation in the two-stage game. Indeed, the J rival firms would respond to the entry by choosing their outputs to be q_{J+1}. By condition (A2) in Section 12.E of the textbook, $q_{J+1} \geq q_J$. In words, in the two-stage game the active firms would respond to another entrant by reducing their outputs. But this means that when the deviator produces q, the market price and consequently the deviator's profits are higher in the two stage game than in the one-stage game. Therefore, the original outcome would not be a SPNE of the two-stage game. (Exercise: show that Cournot competition satisfies condition (A2).)

The reverse statement is not true, which can be seen from the following example: $p(q) = 1 - q$, $K = 1/16$ and $c = 1/4$. As follows from the analysis in

Example 12.E.1 in the textbook, the equilibrium of the two-stage game has two firms active, each firm producing 1/4. However, in the one-stage game on firm active and producing 3/8 is a NE (The best any other firm can do given that one firm produces 3/8 is to produce 3/16 which gives this firm a profit of 9/256 < 1/16, i.e. not high enough to cover the entry cost.)

Suppose that there is an equilibrium of the one-stage game where J firms are active. Using the notation of Section 12.E, this implies that $\pi_J \geq 0$ (otherwise a firm would prefer to stay inactive). But then the two-stage equilibrium number of firms is at least J (adopting the convention that an indifferent firm enters). In other words, we cannot have less firms active in the two-stage game then in the one-stage game.

12.E.7. Suppose in negation there is an equilibrium in which $J^\circ q^\circ \neq x(c^*)$. If $J^* q^\circ > x(c^*)$, then the industry price is below c^* and all firms make negative profits, which cannot constitute a NE. Suppose that instead $J^\circ q^\circ < x(c^*)$, i.e. the industry price p is higher than c^*. Then a firm could enter the market charging $c^*+\varepsilon$, producing q^*, and making a positive profit of εq^*. Therefore any equilibrium has $J^\circ q^\circ = x(c^*)$.Since there exist an integer J° such that $J^\circ q^\circ = x(c^*)$, such an equilibrium exists.

12.F.1. [<u>First printing errata</u>: Assume that $p(\cdot)$ and $c(\cdot)$ are twice continuously differentiable.]

Assume for simplicity that $c(0) = 0$, and restrict attention to $\alpha \geq 1$. If the demand function for the market of size α is $X(p) = \alpha x(p)$, then the inverse demand function for this market is $P(Q) = p(Q/\alpha)$. Firm j's best response is obtained by solving

$$\underset{q_j \geq 0}{\text{Max}} \; p((Q_{-j} + q_j)/\alpha)q_j - c(q_j). \qquad (*)$$

Since setting $q_j = 0$ is always an option, any solution to the problem satisfies $p((Q_{-j}+ q_j)/\alpha)q_j - c(q_j) \geq 0$. Therefore, at any solution q_j we must have $c(q_j)/q_j \leq p((Q_{-j}+ q_j)/\alpha) \leq p(0)$. Note that $c(0) = 0$ and $c''(\cdot) > 0$ imply that $c(q)/q$ is strictly increasing in q. Therefore, we must have $q_j \leq \bar{q}_j$, where $\bar{q}_j$ solves $c(\bar{q}_j)/\bar{q}_j = p(0)$.

The Kuhn-Tucker conditions for (*) can be written as

$$p'((q_j+Q_{-j})/\alpha)\, q_j + p((q_j+Q_{-j})/\alpha) - c'(q_j) \leq 0,$$
$$= 0 \text{ if } q_j > 0.$$

Denote the solution to (*) as $b(Q_{-j}|\alpha)$. Define $\bar{Q}_{-j}$ so that $p(\bar{Q}_{-j}) = c'(0)$. Then from the Kuhn-Tucker conditions we know that

$$b(Q_{-j}|\alpha) = 0 \text{ if } Q_{-j} \geq \alpha\bar{Q}_{-j}$$
$$> 0 \text{ if } Q_{-j} < \alpha\bar{Q}_{-j}.$$

Now, let us focus on the case where $Q_{-j} < \alpha\bar{Q}_{-j}$. Substituting $q_j = b(Q_{-j}|\alpha)$ in the Kuhn-Tucker condition and differentiating with respect to Q_{-j}, we obtain

$$\partial b(Q_{-j}|\alpha)/\partial Q_{-j} = -\frac{\phi(Q_{-j}|\alpha)}{\phi(Q_{-j}|\alpha) + p'((b(Q_{-j}|\alpha)+Q_{-j})/\alpha)/\alpha - c''(b(Q_{-j}|\alpha)/\alpha)},$$

where $\phi(Q_{-j}|\alpha) = p''((b(Q_{-j}|\alpha)+Q_{-j})/\alpha)\, b(Q_{-j}|\alpha)/\alpha^2 + p'((b(Q_{-j}|\alpha)+Q_{-j})/\alpha)/\alpha$.

We know that $p'(\cdot) \leq 0$ and $c''(\cdot) \leq 0$. Hence, the result will be proved once we show that $\phi(Q_{-j}|\alpha) < 0$ for α large enough.

Note that Kuhn-Tucker conditions imply that

$$p((b(Q_{-j}|\alpha)+Q_{-j})/\alpha) \geq c'(b(Q_{-j}|\alpha)/\alpha) \geq c'(0) = c'(0) = p(\bar{Q}_{-j})$$

Hence, we must have $(b(Q_{-j}|\alpha)+Q_{-j})/\alpha \leq \bar{Q}_{-j}$. Define

$$\bar{p}' = \text{Max } \{p'(Q) \mid Q \in [0, \bar{Q}_{-j}]\} < 0,$$
$$\bar{p}'' = \text{Max } \{p''(Q) \mid Q \in [0, \bar{Q}_{-j}]\}.$$

Then we have $\phi(Q_{-j}|\alpha) \leq |\bar{p}''|\, \bar{q}_j/\alpha^2 + \bar{p}'/\alpha \sim \bar{p}'/\alpha < 0$ for α large enough, which proves the result.

12.F.2. (a) Aggregate demand is $X(p) = Ix(p) = I(a - bp)$. Inverting this function, we obtain the inverse demand function: $P(x) = A - Bx$, where $A = a/b$ and $B = 1/bI$.

(b) Two-Stage Entry Model:

Example 12.E.1 in the textbook solves for the equilibrium in the two-stage entry model with Cournot competition and linear demand $P(x) = A - Bx$. The equilibrium number of firms J^* (ignoring the integer problem) is given by

$$J^* + 1 = (A-c)/(BK)^{1/2} = (a - bc)(I/bK)^{1/2}.$$

The equilibrium output per firm is then determined in Example 12.E.1 (Exercise 12.C.7):

$$q_{J^*} = (A-c)/B(J^*+1) = (IbK)^{1/2}.$$

The equilibrium aggregate output is $J^* q_{J^*}$, and the equilibrium price is equal to

$$P(J^* q_{J^*}) = A - BJ^* q_{J^*} = A - B(J^*+1)q_{J^*} + Bq_{J^*} = A - B(A-c)/B + Bq_{J^*} =$$

$$= c + Bq_{J^*} = c + (K/Ib)^{1/2} \longrightarrow c \text{ as } I \longrightarrow \infty.$$

Therefore, consumer (per capita) surplus increases to the competitive level as $I \longrightarrow \infty$.

One-Stage Entry Model: As shown in Exercise 12.E.6, the number of active firms in an equilibrium of the one-stage game cannot exceed J^*, the equilibrium number of firms in the two-stage game obtained above. What is the minimum number of active firms in the one-stage game? Suppose that J firms are active in an equilibrium of the game, each firm producing q_J. If an inactive firm decides to deviate and enter, its optimal output choice q will solve

$$\underset{q}{\text{Max}}\ (A - B(Jq_J + q) - c)q$$

The first-order condition yields $q^* = (A - BJq_J - c)/2B$. Substituting q_J from Example 12.E.1, we obtain $q^* = (A - c)/2B(J+1)$. The deviation will not be profitable if

$$K \geq (A - B(Jq_J + q^*) - c)q^* = (A-c)^2/4B(J+1)^2.$$

This gives a lower bound on the number of active firms in equilibrium of the one-stage game:

$$J + 1 \geq \quad J^0 + 1 = (A-c)/2(BK)^{1/2} = \quad (a - bc)(I/bK)^{1/2}/2$$

(we have substituted A and B from part (a)). When J^0 firms are active in the market, each firm will produce

$$q_{J^0} = (A-c)/B(J^0+1) = I(a-bc)/(J^0+1) = 2(IbK)^{1/2}.$$

The corresponding upper bound on the equilibrium market price is

$$P(Jq_{J^0}) = A - BJ^0 q_{J^0} = A - B(J^0+1)q_{J^0} + Bq_{J^0} = A - B(A-c)/B + Bq_{J^0} =$$

$$= c + Bq_{J^0} = c + 2(K/Ib)^{1/2} \longrightarrow c \text{ as } I \longrightarrow \infty.$$

Therefore, consumer (per capita) surplus in any equilibrium of the one-stage entry model converges to the competitive level as $I \longrightarrow \infty$.

12.F.3. For a fixed α, we can use the result of exercise 12.E.3.

12.F.4. [<u>First printing errata</u>: The demand function for firm j as a function of the price vector is given by $x_j(p) = \alpha(\gamma - \beta p_j/\bar{p}^{-1+\varepsilon})$, where $\bar{p} = \sum_k p_k/J$, and $\varepsilon > 0$.]

First we solve for the second-stage Nash equilibrium. Each active firm j solves

$$\operatorname*{Max}_{p_j} \; p_j \, x_j(p) = p_j \, \alpha[\gamma - \beta p_j/\bar{p}^{-1+\varepsilon}]$$

Differentiating with respect to p_j and taking into account that $\bar{p}$ depends on p_j, we obtain the following first-order condition:

$$\gamma - 2\,\beta\, p_j/\bar{p}^{-1+\varepsilon} + \beta(1+\varepsilon)\, p_j^2/(J\bar{p}^{-2+\varepsilon}) = 0.$$

To find a symmetric equilibrium, substitute $p_j = \bar{p}$ and solve for $\bar{p}$. This gives

$$\bar{p} = [\frac{\beta}{\gamma}\,(2 - \frac{1+\varepsilon}{J})]^{1/\varepsilon}.$$

The equilibrium second-period profit for each firm, with J active firms, is therefore

$$\pi_j = \bar{p}\,\alpha[\gamma - \beta\bar{p}/\bar{p}^{-1+\varepsilon}] = \alpha\gamma\,[\frac{\beta}{\gamma}\,(2 - \frac{1+\varepsilon}{J})]^{1/\varepsilon}\,(1 - \frac{1+\varepsilon}{J})/(2 - \frac{1+\varepsilon}{J}).$$

Observe that as $J \longrightarrow \infty$, $\pi_j \longrightarrow \pi^* = \frac{1}{2}\,\alpha\gamma\,[\frac{2\beta}{\gamma}\,]^{1/\varepsilon}$. When α (the size of the market) or β (the substitution parameter) are large enough, we have $\pi^* > K$. Therefore, in equilibrium of the two-stage game infinitely many firms will enter. However, the competitive limit (Proposition 12.F.1) will not be achieved: the price will converge to $[\frac{2\beta}{\gamma}\,]^{1/\varepsilon} > 0$, while the marginal cost is zero.

12.G.1.

The Cournot Duopoly

The equilibrium for this model (q_1^*,q_2^*) has been obtained in Exercise 12.C.9(a). Assuming that both firms produce, the equilibrium is

$$q_1^* = (a + c_2 - 2c_1)/3b,$$
$$q_2^* = (a + c_1 - 2c_2)/3b.$$

From here we see that $\partial q_2^*/\partial c_1 = 2/3b > 0$. When firm 1's cost goes up, firm 2's equilibrium quantity goes up. Firm 1's equilibrium profit is

$$\pi_1(q_1^*,q_2^*|\ c_1) = (a - b(q_1^* + q_2^*))q_1^* - cq_1^*.$$

The strategic effect on firm 1 is

$$\partial\pi_1(q_1^*,q_2^*|\ c_1)/\partial q_2\ \partial q_2^*/\partial c_1 = -bq_1^*\ \partial q_2^*/\partial c_1 < 0.$$

Intuitively, an increase in firm 1's cost makes firm 2 more aggressive, which strategically hurts firm 1.

The Linear City Model

Assuming that prices are such that both firms produce, the equilibrium (p_1^*,p_2^*) has been obtained in Exercise 12.C.17:

$$p_1^* = (2c_1 + c_2)/3 + t,$$
$$p_2^* = (2c_2 + c_1)/3 + t.$$

From here we see that $\partial p_2^*/\partial c_1 = 1/3 > 0$. When firm 1's cost goes up, firm 2's equilibrium price goes up.

Firm 1's equilibrium profit is

$$\pi_1(p_1^*, p_2^* | \, c_1) = (p_1^* - c_1)\, [1/2 + (p_2^* - p_1^*)/2t]\, M$$

The strategic effect on firm 1 is

$$\partial\pi_1(p_1^*, p_2^* | \, c_1)/\partial p_2 \;\; \partial p_2^*/\partial c_1 = 1/2\; (p_1^* - c_1)\; \partial q_2^*/\partial c_1 > 0.$$

Intuitively, an increase in firm 1's cost makes firm 2 less aggressive, which is strategically beneficial for firm 1.

12.AA.1. [First printing errata: The statement in the exercise is incorrect. Here is a counterexample:

	L	R
U	-1, 9	-1,10
D	-10,-1	0, 0

The only NE of this game is (D,R). However, for some δ we can sustain (U,L) ir a SPNE using Nash reversion strategies. If player 1 deviates and plays D, he will lose 9 immediately and gain 1 forever after; he will prefer not do this provided that $9 \geq \delta/(1-\delta)$, i.e. $\delta \leq 0.9$. If player 2 deviates and plays R, she will gain 1 immediately and lose 9 forever after; she will prefer not to do this provided that $1 \leq 9\delta/(1-\delta)$, i.e. $\delta \geq 0.1$. Using Lemma 12.AA.1, we see that (U,L) can be sustained in a SPNE using Nash reversion strategies if and only if $0.1 \leq \delta \leq 0.9$.

Intuitively, (U,L) can be sustained in a SPNE using Nash reversion, because player 1, whose payoff is below its Nash equilibrium level, is playing his short-run best response. Thus, any deviation brings short-run losses to player 1, and if he is sufficiently impatient, he will choose not to deviate despite a permanent future gain. On the other hand, player 2 who receives a higher than NE payoff, will not deviate if he is sufficiently patient.

The correct statement which is the closest to the statement in the textbook is as follows: No pair of actions q such that $\pi_i(q_1, q_2) < \pi_i(q_1^*, q_2^*)$ *for all* $i = 1,2$ can be sustained as a stationary SPNE outcome path using Nash reversion.]

To prove this statement, observe that by assumption (q_1^*, q_2^*) is the unique NE, and therefore (q_1, q_2) is not a NE. Thus, one of the players has a profitable short-run deviation. Suppose for definiteness that it is player 1, and that his profitable short-run deviation is q_1', i.e. $\pi_1(q_1', q_2) > \pi_1(q_1, q_2)$. If player 1 deviates in this way, the game will revert to the Nash equilibrium, and player 1 will receive $\pi_1(q_1^*, q_2^*)$ forever after. Since we know that $\pi_1(q_1^*, q_2^*) > \pi_1(q_1, q_2)$, by deviating player 1 can increase his payoff in very period of the game. Therefore, (q_1, q_2) cannot be sustained in a SPNE using Nash reversion.

12.AA.2. The first-order Taylor expansion of $\pi_i(q_i,q_j)$ in the neighborhood of (q_i^*,q_j^*) is

$$\pi_i(q_i,q_j) \simeq \pi_i(q_i^*,q_j^*) +$$
$$+ \partial\pi_i(q_i^*,q_j^*)/\partial q_i \ (q_i-q_i^*) + \partial\pi_i(q_i^*,q_j^*)/\partial q_j \ (q_j-q_j^*).$$

Since (q_i^*,q_j^*) is a Nash equilibrium, we must have $\partial\pi_i(q_i^*,q_j^*)/\partial q_i = 0$. Therefore,

$$\pi_i(q_i,q_j) \simeq \pi_i(q_i^*,q_j^*) + \partial\pi_i(q_i^*,q_j^*)/\partial q_j \ (q_j - q_j^*), \ i = 1,2. \qquad (*)$$

Since by assumption $\partial\pi_i(q_i^*,q_j^*)/\partial q_j \neq 0$ for $i = 1,2$, there exists a point (q_1,q_2) close enough to (q_1^*,q_2^*) such that $\pi_i(q_1,q_2) < \pi_i(q_1^*,q_2^*)$ for $i = 1, 2$. Consider a strategy profile where the players are to play an outcome path involving (q_1,q_2) in period one and (q_1^*,q_2^*) in every period thereafter, and where the outcome path is restarted after a deviation. This strategy profile gives player i the payoff $v_i' = \pi_i(q_1,q_2) + \delta/(1-\delta)\ \pi_i(q_1^*,q_2^*)$. We have

$$(1-\delta)\ v_i' = (1-\delta)\ \pi_i(q_1,q_2) + \delta\ \pi_i(q_1^*,q_2^*) < \pi_i(q_1^*,q_2^*),$$

as required by the proposition. It only remains to show that the described strategy profile constitutes a SPNE.

To do this, it will be useful to estimate $\hat{\pi}_i(q_j)$ (using the notation of Appendix A in the textbook). For this purpose, we can write $\hat{\pi}_i(q_j) = \pi_i(b_i(q_j), q_j)$, and differentiate with respect to q_j at the point q_j^*:

$$\hat{\pi}_i'(q_j^*) = \partial\pi_i(q_i^*, q_j^*)/\partial q_i \; b_i'(q_j^*) + \partial\pi_i(q_i^*, q_j^*)/\partial q_j.$$

Since q_i^* is a best-response to q_j^*, we must have $\partial\pi_i(q_i^*, q_j^*)/\partial q_i = 0$. Therefore, $\hat{\pi}_i'(q_j^*) = \partial\pi_i(q_i^*,q_j^*)/\partial q_j$. Since (q_1, q_2) can be chosen to be as close as necessary to (q_1^*,q_2^*), we can use the Taylor expansion of $\hat{\pi}_i(q_j)$ around q_j^*:

$$\hat{\pi}_i(q_j) \approx \pi_i(q_i^*, q_j^*) + \partial\pi_i(q_i^*,q_j^*)/\partial q_j \; (q_j - q_j^*) \approx \pi_i(q_i,q_j) <$$
$$< (1-\delta)\; \pi_i(q_1,q_2) + \delta\; \pi_i(q_1^*,q_2^*) = (1-\delta)\; v_i'. \qquad (**)$$

(We have made use of (*).) Now we can finally check that the strategies above constitute a SPNE by checking all possible deviations after two types of histories: those which prescribe playing (q_1^*,q_2^*) forever, and those which prescribe playing (q_1,q_2) in the next period (this occurs in period one or immediately after a deviation).

(i) As has been discussed in Appendix A, the strategies prescribing playing (q_1^*,q_2^*) forever constitute a SPNE, and therefore no player can profitably deviate.

(ii) Suppose that player i deviates after a history which prescribes playing (q_1,q_2) in the next period. If this deviation continues for T periods (we also allow for $T = \infty$), player j will keep playing q_j for $T+1$ periods, and player i will get at most $(1 + \ldots + \delta^{T-1})\; \hat{\pi}_i(q_j) + \delta^T\; v_i' < (1-\delta^T)v_i' + \delta^T v_i' = v_i'$, which is what player i would get if he did not deviate. (We have made use of (**).)

Therefore, both kinds of deviations are unprofitable, and the strategies constitute a SPNE.

12.BB.1. Figure 12.BB.1 describes the case where entry deterrence is possible but not inevitable, and where point S lies to the right of point Z. This figure is a small modification of Figure 12.BB.8 in the textbook. Geometrically it is clear that $k_\pi \geq q_1(S) > q_1(Z) = k_z$. Since entry is not blockaded, point Z is to the right of the incumbent's optimal monopoly point. In words, k_z is higher than the monopoly output, but lower than k_π. Therefore, by the argument on p.427 of the book, the incumbent prefers to deter entry.

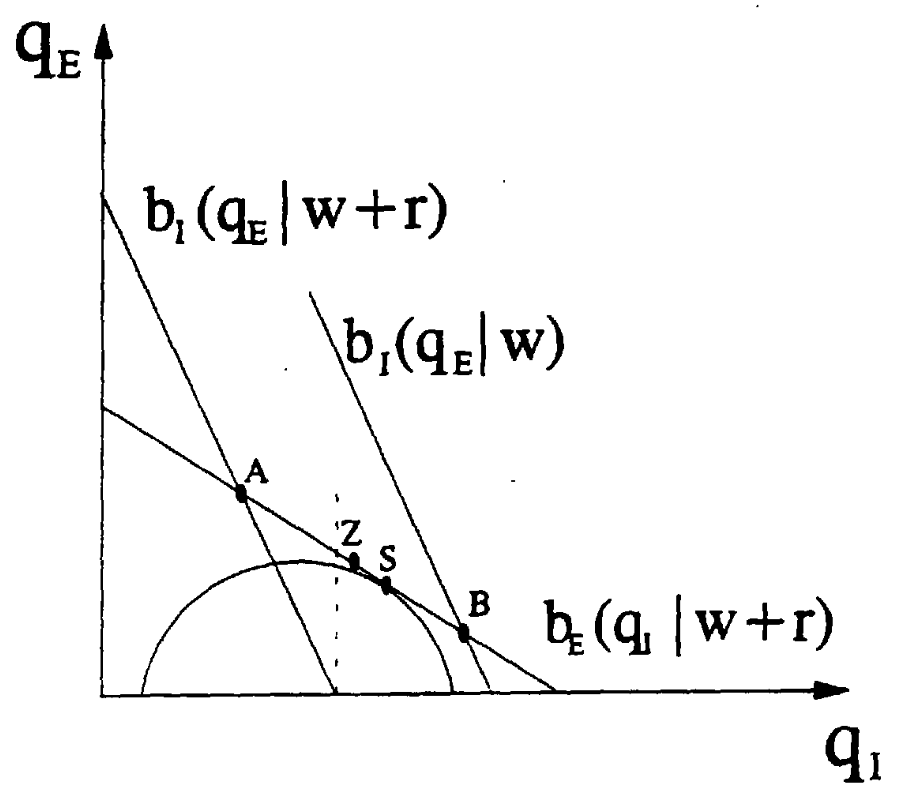


Figure 12.BB.1

12.BB.2. Observe that when accommodating, the incumbent ends up producing less then when deterring entry. In order for the incumbent to be indifferent between these two outcomes, the equilibrium price in case of accommodation must be higher than in case of deterrence. Also, in case of accommodation the fixed cost of production is duplicated. Therefore social welfare is higher if the incumbent chooses deterrence.

As we have seen in Section 12.E of the textbook, an entrant may decide to enter even when it lowers social welfare. This may happen because of the "business stealing" effect - by entering, the entrant incurs a negative externality on other firms in the market. It is thus not surprising that entry deterrence may sometimes raise social welfare.

12.BB.3. If firm I enters at both ends of the city, then firm E will not enter. Indeed, if firm E entered in this situation at one of the ends, it would engage in Bertrand competition with firm I's plant existing at this location, it would collect zero profits and would not be able to cover the fixed cost of entry.

Now consider the situation where firm I has entered at one end of the city. If firm E enters at the other end, the analysis of Example 12.C.2 suggests that it will collect $tM/2 - F$ in net profits. Entering on top of firm I is clearly dominated by not entering at all, in which case firm E gets zero. Therefore, firm E will enter at the opposite end if and only if $tM/2 > F$. Now let us go back one stage and examine firm I's decision. Firm I has three choices:

(i) Firm I can enter at both ends and become a monopoly. Then firm I has a choice: it can either charge a price at which the midpoint consumer buys or charge a price at which the consumer does not buy. If firm firm I chooses the first option, it will charge the maximum possible price $v-t/2$. Firm I may consider to switch to the second option by raising its price and losing some customers. Suppose that it charges some $p > v-t/2$. Then the demand at each end will be $M(v-p)/t$. Firm I's profits will then be $\pi(p) = 2\ (p - c)\ M(v-p)/t$. Differentiating with respect to p, we obtain

$$\pi'(p) = 2M(v + c - 2p) < 2M(v + c - 2v + t) = 2M(c + t - v) < 0,$$

since by assumption $v > c + 3t$. Therefore, firm I will choose to charge $v-t/2$, and its net profit will be $M(v - t/2 - c) - 2F$.

(ii) Firm I can enter at one end. Then if $tM/2 > F$, firm E will enter at the opposite end. The resulting competition will bring the net profit of $tM/2 - F$ to firm E. On the other hand, if $tM/2 > F$, firm E will not enter at all. Then firm I again a choice: it can either charge a price at which the most distant consumer buys or charge a price at which the consumer does not buy. If firm firm I chooses the first option, it will charge the maximum possible price $v-t$. Firm I may consider to switch to the second option by raising its price and losing some customers. Suppose that it charges some $p > v-t$. Then the demand will be $M(v-p)/t$. Firm I's profits will then be $\pi(p) = (p - c)$ $M(v-p)/t$. Differentiating with respect to p, we obtain

$$\pi'(p) = M(v + c - 2p) < M(v + c - 2v + 2t) = M(c + 2t - v) < 0,$$

since by assumption $v > c + 3t$. Therefore, firm I will choose to charge $v-t$, and its net profit will be $M(v - t - c) - F$.

(iii) If firm one does not enter at all, its net profit is zero.

All these cases can be summarized as follows:

If $F < Mt/2$, firm I will never choose not to enter, since it always gets positive profits from entering. If it enters at both ends, it gets $M(v - t/2 - c) - F$, while if it enters at one end, it gets $Mt/2 - F$. Since by assumption $v > c + 3t$, firm I will choose to enter at both ends and deter firm E's entry.

If $F > Mt/2$, firm E's entry is not a threat. Firm I decides whether it wants to be a monopolist with two plants, a monopolist with one plant, or not to enter at all. With two plants firm I will get $M(v - t/2 - c) - 2F$, while with one plant firm I will get $M(v - t - c) - F$. The difference in profits is $Mt/2 - F < 0$, therefore firm I would rather enter with one plant than with two. On the other hand, if $F > M(v - t - c)$, firm I will not enter at all.

We can now summarize how the equilibrium outcome depends on F:

If $F < Mt/2$ then firm I enters at both ends, firm E does not enter.

If $Mt/2 < F < M(v - t - c)$ then firm I enters at one end, firm E does not enter.

If $M(v - t - c) < F$ then neither firm enters.

There is a simple intuitive explanation for why firm E never enters in this game. The reason is that whenever firm E could enter and collect a positive profit, firm I could preempt by entering at the same site and collecting at least as high a profit. In general, we can expect firm I to collect strictly higher profits by preemption because then it does not have to compete with itself. This is called the *efficiency effect*. For a more detailed discussion, see Jean Tirole's "The Theory of Industrial Organization", 1988, section 8.6.2.

This effect does not work, however, when firm I can only enter with one plant. Now when $F < Mt/2$, firm I will only able to enter at one end of the city, and firm E will enter at the other end. Preemption becomes infeasible.

13.B.1 The three functions are graphed in figure 13.B.1(a). This graph also depicts the situation in figure 13.B.1 in the text.

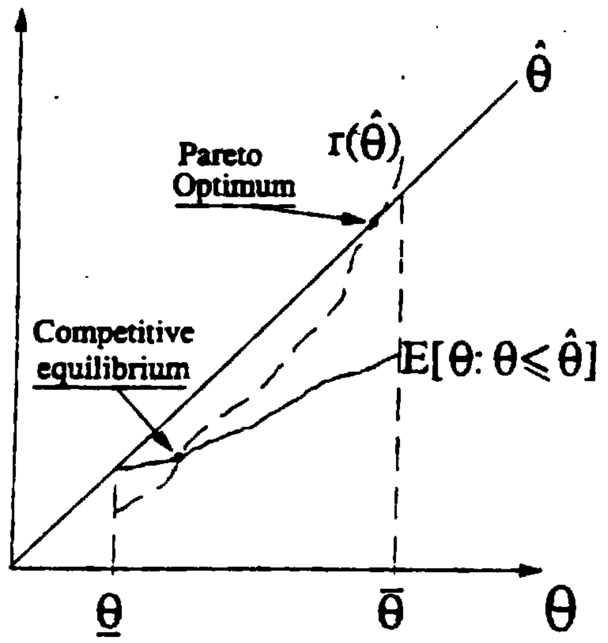


Figure 13.B.1(a)

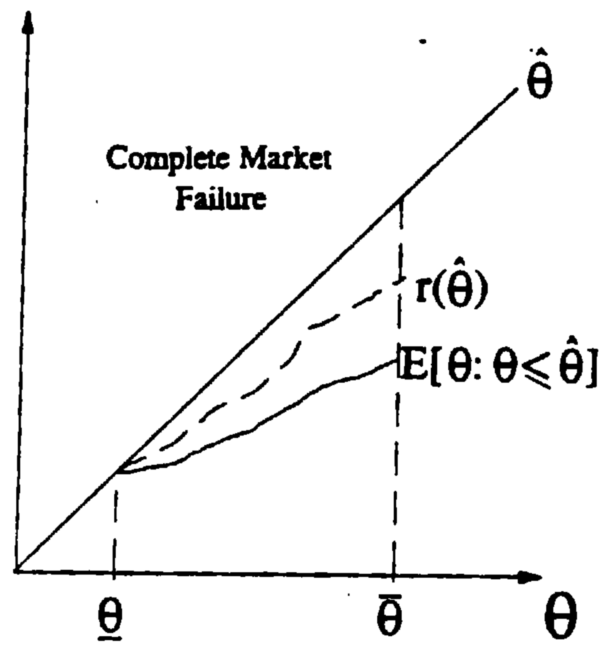


Figure 13.B.1(b)

Figure 13.B.1(b) depicts the situation in figure 13.B.2 in the text.

Figure 13.B.1(c) depicts the situation in figure 13.B.3 in the text.

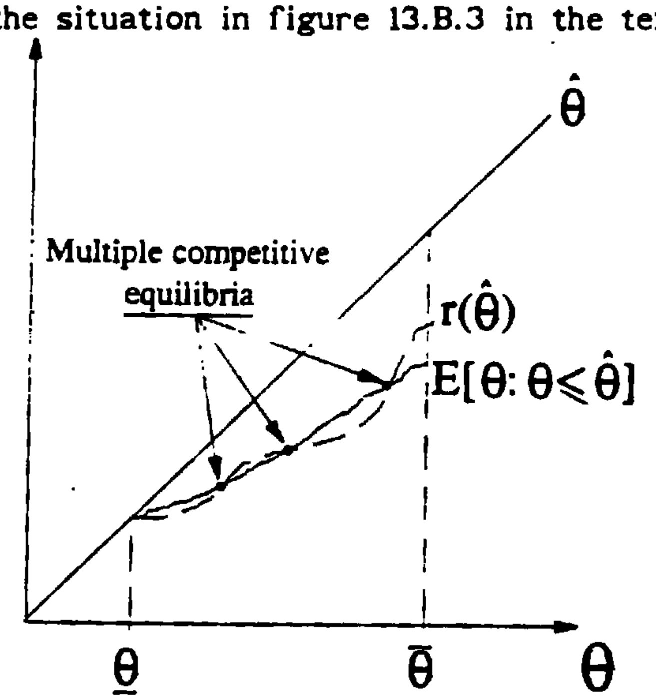


Figure 13.B.1(c)

13.B.2 If $w = \hat{\theta}$, only workers of type $\theta \leq \hat{\theta}$ will accept the wage w and work. But $E[\theta \mid \theta \leq \hat{\theta}] < \hat{\theta} = w$, which implies that firms demand less worker than there are in supply. This implies that the market does not clear. If $w > \hat{\theta}$, only workers of type $\theta \le \theta^* > \hat{\theta}$, with $r(\theta^*) = w$, will accept the wage w and work (since $r()$ is an increasing function). But $E[\theta \mid \theta \le \theta^*] < \theta^* = w$, which implies that firms demand less worker than there are in supply. This implies that the market does not clear.

Thus, to obtain market clearing firms have to offer a wage $w < \hat{\theta}$, which implies that some workers of type $\theta < \hat{\theta}$ will not work, and there will be underemployment in the competitive equilibrium (in an equilibrium with perfect information all workers of type $\theta \le \hat{\theta}$ will work).

13.B.3 (a) Suppose firms offer a wage of w. All workers of type θ, with $r(\theta) \le w$, will accept the wage and work. Suppose there exists a θ^* with $r(\theta^*) = w$. Then all workers of type $\theta \ge \theta^*$ will work, since $r(\theta) \le r(\theta^*) = w$ and $r()$ is decreasing. Thus, the more capable workers are the ones who will work at any given work.

(b) Firms can offer the wage $w = \bar{\theta}$, and since $r(\bar{\theta}) > \bar{\theta}$ no workers of type $\bar{\theta}$ will work. From part (a), no worker of any type will work. Therefore, the competitive equilibrium is Pareto efficient, i.e. nobody will work.

(c) If $w = \hat{\theta}$, only workers of type $\theta \ge \hat{\theta}$ will accept the wage w and work. But $E[\theta \mid \theta \ge \hat{\theta}] > \hat{\theta} = w$, which implies that firms demand more workers than there are in supply, and the market will not clear. If $w < \hat{\theta}$, only workers of type $\theta \ge \theta^* > \hat{\theta}$, with $r(\theta^*) = w$, will accept the wage w and work (since $r()$ is a decreasing function). But $E[\theta \mid \theta \ge \theta^*] > \theta^* = w$, which implies that firms demand more workers than there are in supply, and the market will not clear.

Thus, to obtain market clearing, firms have to offer a wage $w > \hat{\theta}$, which implies that some workers of type $\theta < \hat{\theta}$ will accept the job, and there is over employment in the competitive equilibrium (in an equilibrium with perfect information only workers of type $\theta \ge \hat{\theta}$ will work).

13.B.4 We can think of the true valuation of the good, y, as a state of nature $s \in S$ where S is the set of all states of nature, and both agents 1 (say the seller) and 2 (say the buyer) have a common prior that is common knowledge. Let $H_i(s)$ be the set of states that agent i believes is possible given the true state s (this will depend on the signal observed). We can define the following two events:

$$T_1 \equiv \left\{ s \in S \,\middle|\, E[\, y \mid H_1(s) \cap T_2 \,] \leq p \right\}$$

$$T_2 \equiv \left\{ s \in S \,\middle|\, E[\, y \mid H_2(s) \cap T_1 \,] \geq p \right\}$$

That is, T_i is the event that agent i will say "trade" given that he knows that event $H_i(s)$ has occurred, and that he believes agent j will say "trade". Assume that there exists an equilibrium where both agents say "trade". Then, each agent i *knows* that event T_i occurred, and since this is an equilibrium then each agent i *believes* with probability 1 that agent j knows that event T_j occurred. Therefore, each agent believes with probability 1 that event $T = T_1 \cap T_2$ occurred. The seller (1) then prefers to trade if and only if $E[y|T] \leq p$, and the buyer (2) prefers to trade if and only if $E[y|T] \geq p$. Assuming a generic distribution of values both can hold with probability zero, therefore the set T must occur with probability zero.

13.B.5 (a) When $r(\theta) = r$ for all θ, and $E[\theta] \geq r > \underline{\theta}$, then the firms will make zero profits (a necessary condition for a competitive equilibrium) in two cases: Either $w^* = E[\theta]$ or $w^* = \underline{\theta}$. In this case, if $w^* > E[\theta]$ then firms are losing money, while if $w^* \in [r, E(\theta))$ then $\Theta^* = [\underline{\theta}, \bar{\theta}]$ and firms make positive profits. Finally if $w^* \in (\underline{\theta}, r)$ then no one will accept employment, and together with the assumption that in this case firms believe that any worker who might accept employment is a $\underline{\theta}$ type worker, we must have the offered wage to be $\underline{\theta}$.

If, however, $\underline{\theta}\geq r$, then when $w^*=\underline{\theta}$ firms will make positive profits since all workers will work, and $E(\theta)>w^*$, so the only competitive equilibrium is $w^*=E(\theta)$. On the other hand, if $r>E(\theta)$ then when $w^*=E(\theta)$ no worker will wish to be employed, and due to the assumption on firm beliefs, we must have $w^*=\underline{\theta}$.

(b) In the equilibrium with $w^*=E(\theta)$, both firms have zero profit, there is full employment, and all workers have a utility of $E(\theta)>r$. If, however, $w^*=\underline{\theta}$ then again both firms have zero profits but workers have a utility of only r, this is Pareto dominated by the full employment equilibrium.

(c) The analysis of SPNE is as in proposition 13.B.1 which applies (with straightforward modifications due to a constant r() function). When $E(\theta)=r$, then both competitive equilibria are SPNE. This follows because no firm can offer a wage $w\in[\underline{\theta},r]$ and make positive profits (because if $w<E(\theta)=r$ no worker will accept employment) and a wage $w>r$ will cause losses.

(d) Clearly, when $E(\theta)\geq r>\underline{\theta}$ and when $\underline{\theta}\geq r$ then the highest wage competitive equilibrium has full employment, and is therefore Pareto optimal. For the case where $r>E(\theta)$ a simpler version of proposition 13.B.2 can be applied.

13.B.6 For a similar analysis we refer to:

Wilson, C. (1980) "The Nature of Equilibrium in Markets with Adverse Selection," *The Bell Journal of Economics*, 11:108-30.

13.B.7 First assume there is a Pareto improving market intervention $(\tilde{w}_e,\tilde{w}_u)$ that reduces employment with respect to a competitive equilibrium with wage w^*. Clearly, we cannot have $\tilde{w}_e< w^*$ since then those workers who are employed are worse off. Similarly we cannot have $\tilde{w}_u< 0$. Now assume that $\tilde{w}_e> w^*$ and $\tilde{w}_u> 0$. We can then reduce both $\tilde{w}_e$ and $\tilde{w}_u$ by $\varepsilon > 0$ such that $\tilde{w}_e- \varepsilon > w^*$ and $\tilde{w}_u- \varepsilon > 0$ will still hold, the same groups of agents will be employed/unemployed, and the government now has a surplus (it broke even with $(\tilde{w}_e, \tilde{w}_u)$ since it must have a balanced budget). The government can then distribute this surplus by raising only $\tilde{w}_u$, and this would still be a Pareto improvement relative to the competitive equilibrium. This can be done as long as $\tilde{w}_e > w^*$, and since it was employment reducing we must have $\tilde{w}_e - w^* < \tilde{w}_u - 0$, implying that we can reduce both $\tilde{w}_e$ and $\tilde{w}_u$ until $\tilde{w}_e = w^*$. The result is a Pareto improving intervention of the form $(\hat{w}_e, \hat{w}_u)$ with $\hat{w}_e = w^*$ and $\hat{w}_u > 0$. The converse is trivial: if $(\hat{w}_e, \hat{w}_u)$ is a Pareto improving market intervention of the form $\hat{w}_e = w^*$ and $\hat{w}_u > 0$, then we must have less employment since the marginal type θ^* who decided to work had $w^* = r(\theta^*)$ and now can be unemployed and receive $r(\theta^*) + \hat{w}_u > w^*$ so he will decide to be unemployed, and by continuity a positive mass of types will decide so as well.

Second, assume there is a Pareto improving market intervention $(\tilde{w}_e, \tilde{w}_u)$ that increases employment with respect to a competitive equilibrium with wage w^*. Clearly, we cannot have $\tilde{w}_e < w^*$ since then those workers who are employed are worse off. Similarly we cannot have $\tilde{w}_u < 0$. Now assume that $\tilde{w}_e - \varepsilon > w^*$ and $\tilde{w}_u - \varepsilon > 0$. We can then reduce both $\tilde{w}_e$ and $\tilde{w}_u$ by $\varepsilon > 0$ such that $\tilde{w}_e > w^*$ and $\tilde{w}_u > 0$ will still hold, the same groups of agents will be employed/unemployed, and as before, the government can distribute the generated surplus by raising only $\tilde{w}_e$, and this would still be a Pareto improvement relative to the competitive equilibrium. This can be done as long as $\tilde{w}_u > 0$, and since it was employment increasing we must have $\tilde{w}_e - w^* > \tilde{w}_u - 0$, implying that we can reduce both $\tilde{w}_e$ and $\tilde{w}_u$ until $\tilde{w}_u = 0$. The result is a Pareto improving intervention of the form $(\hat{w}_e, \hat{w}_u)$ with $\hat{w}_e > w^*$ and $\hat{w}_u = 0$. As before, the converse is trivial.

Finally, these facts give a simple proof of proposition 13.B.2. by contradiction: If there were a Pareto improving intervention, it must be either employment increasing or reducing. Therefore, it must be of one of the two forms: (i) $(\hat{w}_e, \hat{w}_u) = (w^* + \delta, 0)$, or (ii) $(\hat{w}_e, \hat{w}_u) = (w^*, \varepsilon)$. If it is of form (i), then a firm could have deviated, proposed $w^* + \frac{\delta}{2}$, and by the conditions given would have made positive profits - a contradiction to w^* being a SPNE. If it is of form (ii) then the government cannot balance its budget since $r(\cdot)$ is strictly increasing, therefore the best of the formerly employed types will choose to be unemployed, and by paying w^* to the remaining employed workers the government must lose money, in addition to the unemployment benefit it is paying out, a contradiction to (w^*, ε) being a feasible intervention.

13.B.8 Assume that we have a model similar to the one studied in section 13.B as displayed in figure 13.B.8 (let $r(\theta^*) = w^*$).

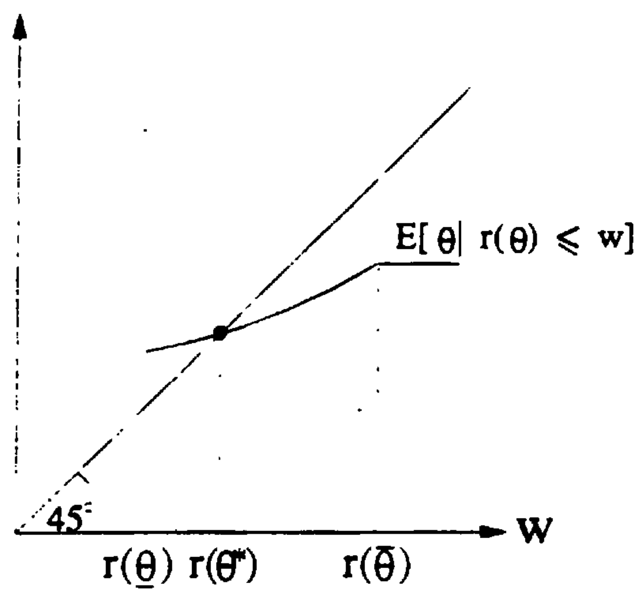


Figure 13.B.8

In our new model, since all workers of type $\theta \in [\theta^*, \theta]$ don't work, they get to consume the product x. Since the government observes the level of consumption of x, and prior to intervention the economy is in a competitive equilibrium, the government is able to identify the type of all individuals for which $\theta > \theta^*$. (Because their consumption is a function of θ, $x(\theta)$, which is known to the government, and $x(\theta)$ is invertible because it is increasing.) In the competitive equilibrium of the model of section 13.B., firms break even, and workers' payoffs are:

$$- \text{ if } \theta \leq \theta^*, \quad u(\theta) = w^*$$

$$- \text{ if } \theta > \theta^*, \quad u(\theta) = r(\theta)$$

Consider the following scheme: suppose the government offers a worker of type $\theta > \theta^*$ [which has been identified] wage of $r(\theta) + \varepsilon$, where ε is small enough so that $r(\theta) + \varepsilon < \theta$ for all $\theta \in (\theta^*, \theta]$. All of these workers will accept this offer, and improve their payoff by ε. At the same time, $r(\theta) + \varepsilon < \theta$ for all $\theta > \theta^*$, and the firm makes positive profits out of hiring these highly skilled workers. These profits are equal to:

$$\Pi^* = \int_{\theta^*}^{\bar{\theta}} [\theta - r(\theta) - \varepsilon] \, f(\theta) d\theta$$

Since the firm was breaking even with wage w^* for workers with $\theta < \theta^*$, we must have:

$$\int_{\underline{\theta}}^{\theta^*} \theta f(\theta) \, d\theta \;=\; w^* \Pr(\theta \ll \theta^*).$$

If the firm raises the wages of all workers with $\theta < \theta^*$ by d (to $w^* + d$), it will incur a loss of:

$$\int_{\underline{\theta}}^{\theta^*} d \, f(\theta) \, d\theta = d \Pr(\theta \leq \theta^*)$$

If $d < \dfrac{\Pi^*}{2 \cdot \Pr(\theta \leq \theta^*)}$, this loss will be smaller than the gain from the highly skilled workers because:

$$d \Pr(\theta \leq \theta^*) < \frac{\Pi^*}{2 \cdot \Pr(\theta \leq \theta^*)} \cdot \Pr(\theta \leq \theta^*) = \Pi^*/2 < \Pi^*$$

Thus, the firm is now making positive profits, thereby having raised its utility level. All workers of type $\theta \leq \theta^*$ have also raised their utility level (from w^* to $w^* + d$). Additionally, all workers of type $\theta \geq \theta^*$ have raised their utility level from $r(\theta)$ to $r(\theta) + \varepsilon$. Thus, this change is Pareto improving.

13.B.9 In the model described there are three candidates for a highest wage competitive equilibrium (recall firms should get zero profits):

<u>Case 1:</u> $w^* = \theta_H$ and only high types are employed. This implies that we must have $\theta_H < r(\theta_L)$ or else low types would ask for employment and firms would lose money. (This case is fully Pareto optimal.)

<u>Case 2:</u> $w^* < \text{Min}(\ r(\theta_H)\ ,\ r(\theta_L)\ )$ and no one is employed. This implies that $r(\theta_H) \geq r(\theta_L)$, since if $r(\theta_H) < r(\theta_L)$ were the case, a firm who would deviate to $w = r(\theta_H)$ would make positive profits and this would not be an equilibrium.

<u>Case 3:</u> $w^* = \lambda\theta_H + (1-\lambda)\theta_L$ and both types are employed. This implies that both $r(\theta_H) \leq w^*$ and $r(\theta_L) \leq w^*$.

Consider case 3 which is not fully Pareto optimal (only the high type should be employed) and assume that $r(\theta_H) < r(\theta_L) < w^*$. Let $w_u = \varepsilon$ be the unemployment benefit where $\varepsilon < \text{Min}(\ r(\theta_L) - r(\theta_H),\ \theta_H - r(\theta_H)\ )$, and let w^* remain the employment wage. Only low types will choose not to work, the government will lose $(1-\lambda)\varepsilon$ and gain $(1-\lambda)(w^* - \theta_L)$ due to the low types relocation, and for small enough ε this is beneficial and yield a Pareto improvement.

Turning back to Exercise 13.B.7, we can say that a competitive equilibrium involving full employment is constrained Pareto optimal, if there is no employment reducing, Pareto improving intervention of the form (w^*,ε). □

13.B.10 As in proposition 13.B.2, we still have that for all (w_e,w_u), the set of workers accepting employment is of the form $[\hat{\theta},\hat{\theta}]$. This follows since $r(\cdot)$

is still an increasing function. Therefore, the details of the proof in the textbook go through.

13.C.1 Suppose some workers, whose types do not equal to $\underline{\theta}$, do not submit to the test. Call the worker with the highest type that did not submit to the test $\theta^* > \underline{\theta}$ Firms will now offer all the workers that did not submit to the test a wage justified for the mean worker that did not submit to the test, i.e. they will offer a wage equal to the expected value of θ for workers that did not submit to the test. This expected value will be lower than the type of worker θ^*, while if worker θ^* submits to the test he will receive a wage of θ^* which is higher than the wage he would receive if he would not submit to the test. Therefore all workers (except for the lowest type worker) submit to the test in the unique SPNE of this game. It follows that firms offer no more than $\underline{\theta}$ to workers who do not submit to the test (they loose money otherwise).

13.C.2 For simplicity, assume μ=1 (this does not change the qualitative results). The competitive equilibrium with perfect information is given in the following Claim:

Claim 1: At the competitive equilibrium with perfect information, type θ_H (resp. θ_L) gets education level e_H^* (resp. e_L^*), where $\frac{dc}{de}(e_H^*, \theta_H) = \theta_H$. (resp. $\frac{dc}{de}(e_L^*, \theta_L) = \theta_L$). The wage for θ_H (resp. θ_L) is given by $w_H^* = \theta_H + \theta_H e_H^*$ (resp. $w_L^* = \theta_L + \theta_L e_L^*$).

Proof of Claim 1: If a worker of quality θ gets education level e, then his marginal productivity is $\theta + \theta e$, and his wage will be equal to $\theta + \theta e$, since firms are competitive. Workers of type θ will thus choose their level of education to maximize their utility, given this wage level:

$$\max_{e} \; w - c(e, \theta) = \theta + \theta e - c(e, \theta)$$

and the FOC is: $\frac{dc}{de}(e^*, \theta) = \theta$. Q.E.D.

Remark: The competitive equilibrium with perfect information is Pareto efficient.

As you may expect, both the separating equilibrium and the pooling equilibrium look similar as in the original model in which education does not affect productivity. That is:

- the equilibrium contracts provide the firms with zero profits.
- the low productivity type will obtain the optimal level of education in a separating equilibrium.

The good news is that the high productivity type may also obtain the optimal level of education, i.e the above competitive equilibrium with perfect information may emerge as a separating equilibrium. In Figure 13.C.2(a) the competitive equilibrium with perfect information is sustained as a separating equilibrium:

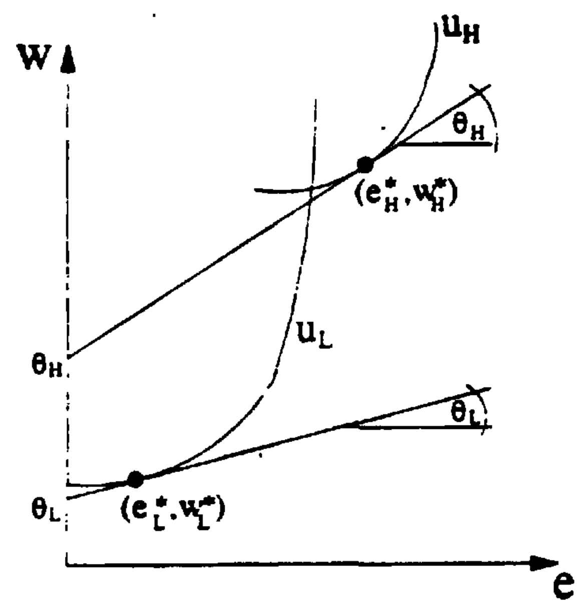


Figure 13.C.2(a)

In Figure 13.C.2(b) below, the outcome of the competitive equilibrium cannot be attained as an equilibrium with imperfect information. Type θ_L would like to pretend to be of type θ_H, if he is offered $(e_L^*, \theta_L + \theta_L e_L^*)$ and $(e_H^*, \theta_H + \theta_H e_H^*)$. Thus, the incentive constraint for the low productivity type is not satisfied.

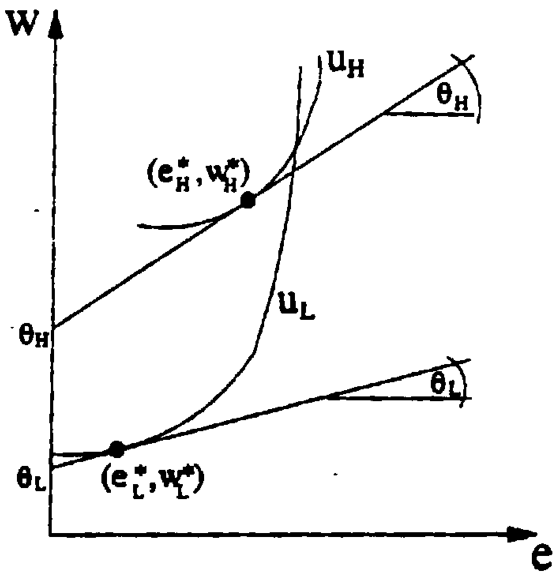


Figure 13.C.2(b)

With a similar argument as in the text we can show:

Claim 2: The separating equilibrium of this model is as follows: $(e_L^*, \theta_L + \theta_L e_L^*)$ and $(e_H^*, \theta_H + \theta_H e_H^*)$, with $e_H^* \in [e', e'']$. This equilibrium is depicted in Figure 13.C.2(c).

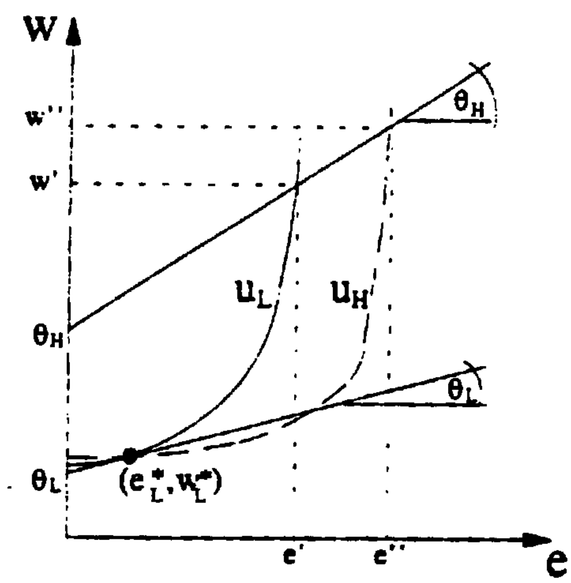


Figure 13.C.2(c)

Claim 3: The pooling equilibrium of this model is as follows: $(e^*, (1-\lambda)(\theta_L + \theta_L e^*) + \lambda(\theta_H + \theta_H e^*))$, with $e^* \in [0, e']$. This equilibr depicted in Figure 13.C.2(c).

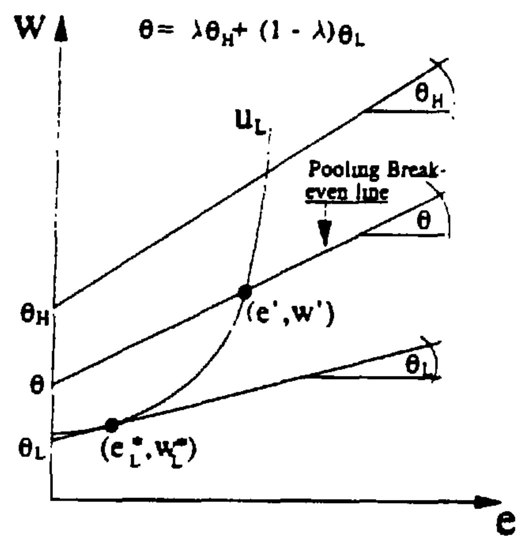


Figure 13.C.2(d)

13.C.3 We basically need to find an example that demonstrates the situation in Figure 13.C.11 in the textbook. Let the probability of a high type be $\lambda = \frac{5}{6}$, let $\theta_H = 4$ and $\theta_L = 0$, and let the utility functions of both types be:

$$u_L(w_L,e_L) = w_L - e_L \,, \quad \text{and} \quad u_H(w_H,e_H) = \begin{cases} w_E - \frac{e}{10} & \text{for } e \le \frac{1}{2} \\ w_E - e + \frac{9}{20} & \text{for } e \ge \frac{1}{2} \end{cases}$$

The separating equilibrium is $(w_L,e_L) = (0,0)$, and $(w_H,e_H) = (4,1)$, and the utilities of both types are $u_L(w_L,e_L) = 0$, and $u_H(w_H,e_H) = 3.45$. This equilibrium is depicted in Figure 13.C.3(a) on the next page.

Notice the linear line between the points $(w,e)=(3\frac{1}{3},0)$ and $(w.e)=(4,1)$. As we move the wage of the L type up from 0, his indifference curve will shift up. In order to make zero profits we must give the H type a wage equal to the wage determined by the intersection between the L type's new indifference curve and the linear line described above (this is easily shown algebraically and depends on the values of λ, θ_H and θ_L). Therefore, to give an example as required we must have the H type's indifference curve through the separating equilibrium point first go down below this linear line, and then cut it and reach the w-axis above this linear line. This is how such an example can be generated. A central authority who will offer $(w_L,e_L) = (0.5,0)$, and $(w_H,e_H) = (3.9,0.88)$, will maintain zero profits since $\frac{5}{6}\cdot 3.9 + \frac{1}{6}\cdot 0.5 = 3\frac{1}{3} = E(\theta)$, and the utilities of both types are $u_L(w_L, e_L) = 0.5$, and $u_H(w_H, e_H) = 3.47$. Note that since $E(\theta) = 3\frac{1}{3} < 3.45 = u_H(4,1)$, we would not get a Pareto improvement by banning the signal since the H type would be worse off.

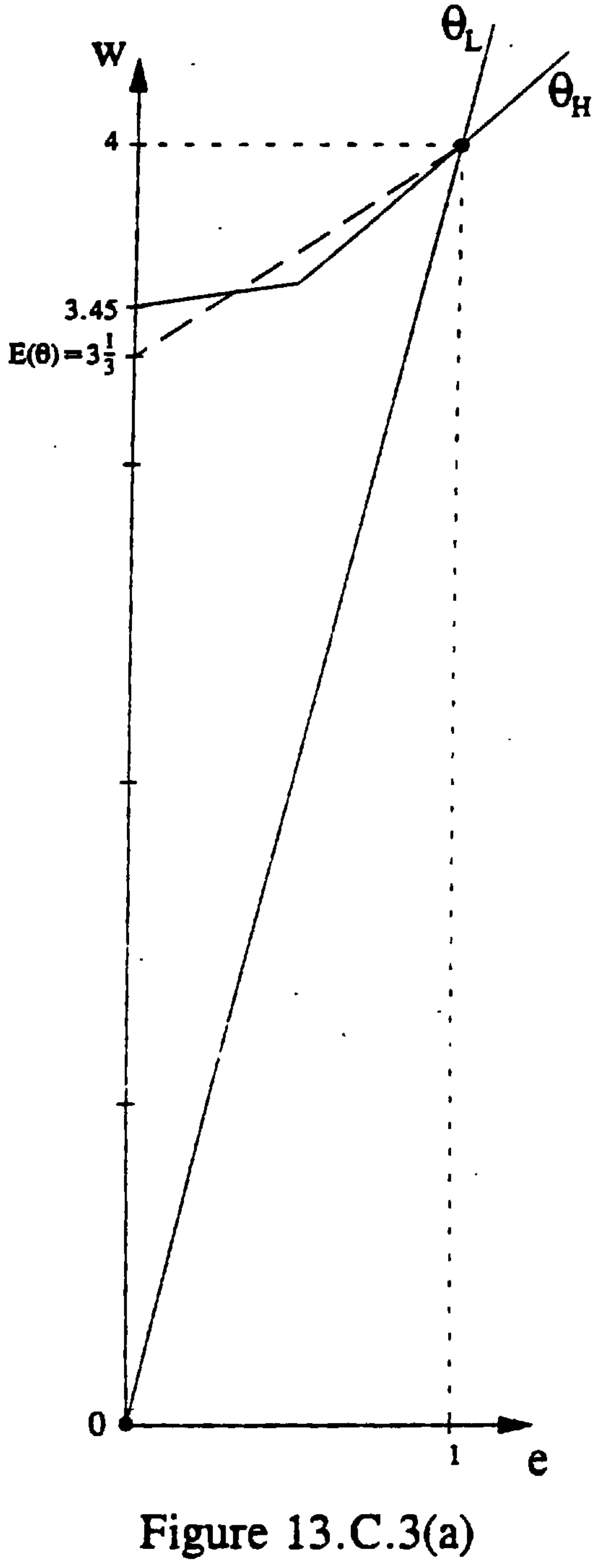


Figure 13.C.3(a)

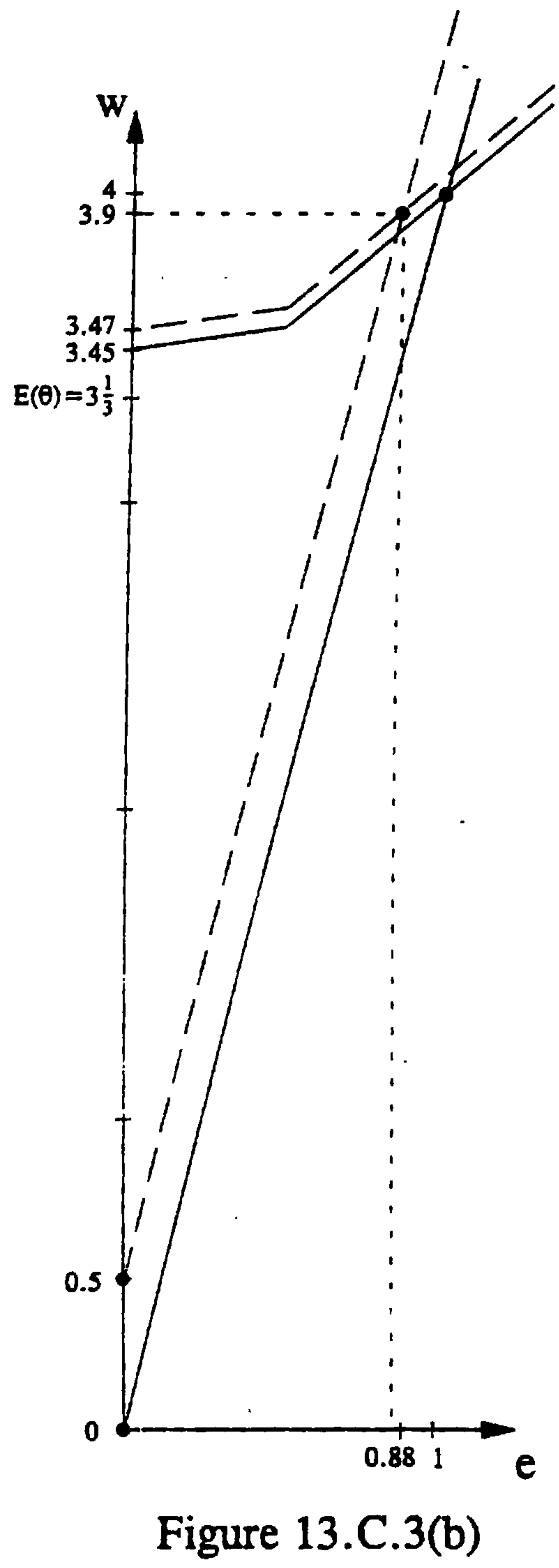


Figure 13.C.3(b)

13.C.4 [First Printing Errata: the question should end with "Derive the (nuique) separating perfect Bayesian equilibrium."]

The firms will offer a wage $w(e)$ as a function of the observed level of education. Given a wage schedule $w(\cdot)$, an agent of type θ^* chooses e to maximize his utility $w(e) - e^2/\theta^*$, and the FOC is: $w'(e^*) = \frac{2e^*}{\theta^*}$. Since the firms are competitive we must have zero profits, $w(e^*) = \theta^*$, in equilibrium. Combining this with the FOC gives us the differential equation: $w(e^*)\cdot w'(e^*) = 2e^*$, which implies that $w(e) = \sqrt{2}\cdot e$. This wage function, together with each type choosing e that satisfies the FOC above, that is, $e(\theta) = \theta / \sqrt{2}$, is the unique separating PBE.

There also exist many (in fact, a continuum of) pooling equilibria (w^*,e^*) of the form: $w^* = E(\theta)$, and $e \in [0,\hat{e}]$, where $\hat{e}$ is calculated by equating: $u(w^*,\hat{e}\,|\,\underline{\theta}) = u(\underline{\theta},0\,|\,\underline{\theta})$.

13.C.5 (a) The consumer will buy the product if the expected value of the product is higher than the price, i.e., if $\lambda\, v_H + (1-\lambda)\, v_L \geq p$.

(b) Suppose there exists a separating equilibrium in which the high quality producers spends A on advertising and only the high quality product will be bought (in a separating equilibrium consumers know the quality of a product, so low quality products will not be bought since $p > v_L$). This implies that the low quality producer makes no profit and the high quality producer makes a non-negative profit, $\Pi_H = p - c_H - A \geq 0$.

However, a low quality producer can make a positive profit buy spending A on advertising, since the consumer will then mistake him for a high quality producer and buy the good from him. The low quality producer's profit will equal $\Pi_L = p - c_L - A > p - c_H - A \geq 0$. Therefore, no separating equilibrium can exist.

(Note, that the banks will not be compensated for the risk that they assume since they are risk-neutral): $\lambda(p_G R + (1-p_G)0) + (1-\lambda)(p_B R + (1-p_B)0) = 1 + r$, or, $R = (1+r) / (\lambda p_G + (1-\lambda)p_B)$.

An entrepreneur of type i will pursue a project if: $(p_i(\Pi-R) + (1-p_i)0) \geq 0$ for $i \in \langle G,B\rangle$, or if $\Pi \geq (1+r)/[\lambda p_G + (1-\lambda)p_B]$. Since, by assumption $\Pi \geq (1+r)/p_G$ and $\Pi \leq (1+r)/p_B$, the entrepreneurs will pursue a project if λ is large enough. In other words, if the fraction of good projects (λ) is large enough, then the banks will set a lower interest rate (R) and thus, the entrepreneurs will undertake the project, i.e. $\Pi \geq R = (1+r)/[\lambda p_G + (1-\lambda)p_B]$.

(b) **(i)** The entrepreneur's expected payoff from a project of type $i \in \langle G,B\rangle$ is: $p_i[\Pi-(1-x)R] + (1-p_i)0 - (1+\rho)x = p_i(\Pi-R) - x[(1+\rho) - p_i R]$.

(ii) In a separating equilibrium the banks will know the type of the project. Thus, the banks will offer an entrepreneur that pursues a good project an interest rate of $R = (1+r)/p_G \leq \Pi$ and an entrepreneur that pursues a bad project an interest rate of $R = (1+r)/p_B \geq \Pi$. Therefore, in a separating equilibrium no bad projects will be pursued.

Therefore, the minimum level of x, that allows the entrepreneur with the good project to signal his type, has to give an entrepreneur with a bad project a negative expected payoff if he contributes this level x of internal funds and obtains financing from the bank at an interest rate of $R = (1+r)/p_G$. That is, $p_B(\Pi-R) - x((1+\rho) - p_B R) = 0$. Substituting in the equilibrium level of R we get: $p_B[\Pi-(1+r)/p_G] - x[(1+\rho) - p_B(1+r)/p_G] = 0$, or, $x = [(\Pi-(1+r)/p_G)]/[(1+\rho)/p_B - (1+r)/p_G] < 1$.

It follows that as p_G increases and r decreases, x increases. As Π or p_B increase, x increases. Since an entrepreneur with a bad project will obtain a zero expected payoff by pursuing the project, an entrepreneur with a good project will obtain a positive payoff.

The separating equilibrium is then: Entrepreneurs with bad projects will contribute $x = 0$ and accept a bank's offer if $R \leq \Pi$. Entrepreneurs with good projects will contribute $x = [(\Pi-(1+r)/p_G)]/[(1+\rho)/p_B - (1+r)/p_G]$ and accept a bank's offer if $R \leq (1+r)/p_G$. The banks will offer an interest rate of $R = (1+r)/p_B$ if the entrepreneur contribute $x = 0$ and $R = (1+r)/p_G$ if he contributed $x = [(\Pi-(1+r)/p_G)]/[(1+\rho)/p_B - (1+r)/p_G]$.

(iii) The entrepreneurs with the bad project will not be better off, and may be worse off in the separating equilibrium of part (ii). For large enough λ, all projects will be financed in the equilibrium of part (i) and the entrepreneurs with the bad project will make a strictly positive expected profit. In the separating equilibrium of part (ii), however, entrepreneurs with bad projects will always obtain a payoff of zero. For small λ, entrepreneurs with bad projects will obtain a payoff of zero in both equilibria.

For small λ, entrepreneurs with good projects will be better off in the separating equilibrium, since they obtain a positive payoff in the separating equilibrium and a zero payoff in the equilibrium of part (i) (since the projects will not be financed).

As λ becomes larger, projects will also be funded in the equilibrium of part (i). Now, the entrepreneur will have to pay a higher interest rate R on the bank loan in the equilibrium of part (i). In the separating equilibrium, however, the entrepreneur has to contribute his own funds, which is costly to him since he is liquidity constraint. Thus as λ becomes large enough, the entrepreneur will be better of in the equilibrium of part (i).

13.D.1 This is the screening analog of the signaling model of Exercise 13.C.2. The competitive equilibrium with perfect information is given in the following Claim:

<u>Claim 1</u>: At the competitive equilibrium with perfect information, type θ_H

(resp. θ_L) has a task level of t_H^* (resp. t_L^*), where $\frac{dc}{dt}(t_H^*, \theta_H) = \theta_H$. (resp. $\frac{dc}{dt}(t_L^*, \theta_L) = \theta_L$. The wage for θ_H (resp. θ_L) is given by $w_H{}^* = \theta_H + \theta_H t_H^*$ (resp. $w_L{}^* = \theta_L + \theta_L t_L^*$) .

Proof of Claim 1: A worker of quality θ performing a task level of t, produces $\theta + \theta t$, and his wage will be equal to $\theta + \theta t$, since firms are competitive. Workers of type θ will thus choose their task level to maximize their utility, given this wage level: max $w - c(t, \theta) = \theta + \theta t - c(t, \theta)$, and the FOC is: $\frac{dc}{dt}(t^*, \theta) = \theta$. Q.E.D.

Remark: The competitive equilibrium with perfect information is Pareto efficient.

As one may expect, the equilibrium looks similar as in the original model in which the task level is unproductive. That is:

- the equilibrium contracts provide the firms with zero profits.
- there exists no pooling equilibrium
- the low productivity type will provide the optima' task level in a separating equilibrium.

The good news is that the high productivity type may also obtain the optimal task level, i.e the above competitive equilibrium with perfect information may emerge as a separating equilibrium. In Figure 13.D.1(a), the competitive equilibrium with perfect information is sustained as a separating equilibrium:

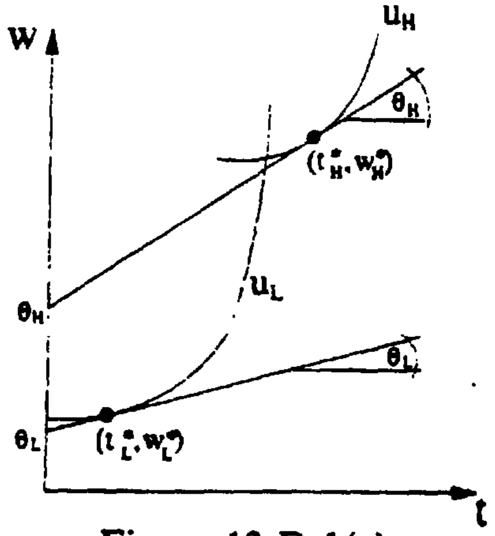


Figure 13.D.1(a)

In Figure 13.D.1(b), the outcome of the competitive equilibrium cannot be attained as an equilibrium with imperfect information:

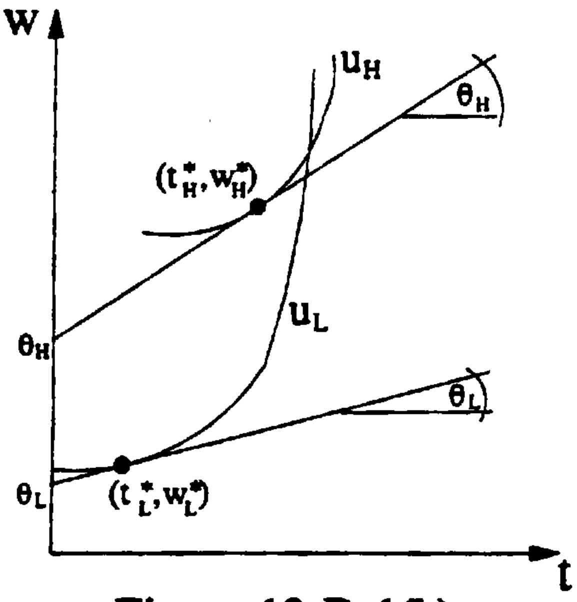


Figure 13.D.1(b)

Type θ_L would like to pretend to be of type θ_H, if he is offered $(t_L^*, \theta_L + \theta_L t_L^*)$ and $(t_H^*, \theta_H + \theta_H t_H^*)$. Thus, the incentive constraint for the low productivity type is not satisfied.

By a similar argument as in the text we can show:

<u>Claim 2</u>: The separating equilibrium of this model exists and is as follows:

- $(e_L^*, \theta_L + \theta_L e_L^*)$ and $(e_H^*, \theta_H + \theta_H e_H^*)$,
- the indifference curve of the high productivity type is above the pooling break-even line.
- no cross-subsidizing contract, that will be accepted by both types, exists that gives a firm a positive profit.

This equilibrium is shown in Figure 13.D.1(c):

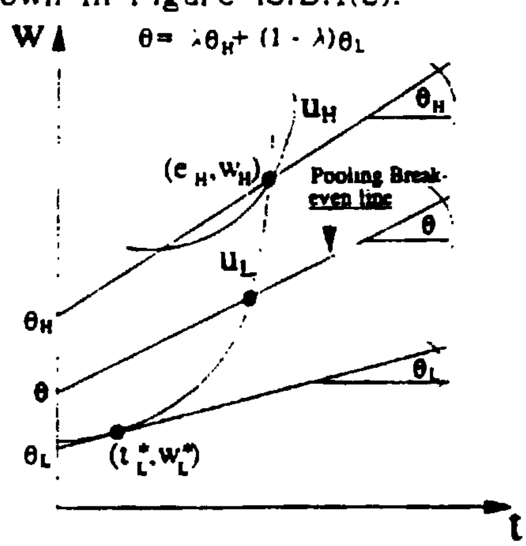


Figure 13.D.1(c)

<u>Claim 4</u>: No equilibrium exists in the following case: The indifference

curve of the high productivity type is at some points below the pooling break-even line. This equilibrium is shown in Figure 13.D.1(d):

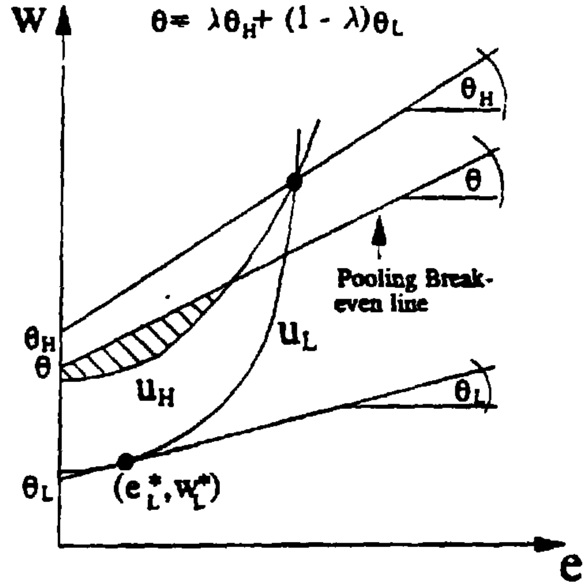


Figure 13.D.1(d)

13.D.2 (a) Once M and R are given, the wealth levels of an insured individual in the two states, denoted by (w_1, w_2), are given by:

$$(w_1, w_2) = (W - M, W - L - M + R).$$

Therefore, we can think of the insurance contract as specifying the wealth levels (w_1, w_2) in the two states. The premium M and the repayment R can be obtained by the following equations: $M = W - w_1$ and $R = w_2 + L - w_1$.

(b) This game is analogous to the screening game studied in this chapter. Therefore there exists no pooling equilibrium, and the existence of a separating equilibrium is not always assured. If there exists a separating equilibrium, then the high risk types are completely insured, i.e. $w_1^H = w_2^H$. The low risk types will not be completely insured, in fact $w_1^L > w_2^L$.

For a proof of these results, we refer to the original paper by Rothschild, M. and Stiglitz, J.(1977) "Equilibrium in Competitive Insurance Markets with Adverse Selection," QJE, p. 629 - 649 and to Laffont (1989) *The Economics of Uncertainty and Information*, MIT press, Chapter 8.

13.D.3 (a) Assume for simplicity that μ=1, which does not change the qualitative results. The high (respectively, low) type workers output is

$\theta_H(1+T)$ (respectively, $\theta_L(1+T)$). A firm can now deduce from the observed output level the worker's type. Therefore, any firm will propose the following contract:

- $\theta_H(1+T)$ if observed output is $\theta_H(1+T)$
- $\theta_L(1+T)$ if observed output is $\theta_L(1+T)$.

If $\theta_L(1+T) \geq c$, both types will accept the offer and work.

If $\theta_H(1+T) \geq c > \theta_L(1+T)$, then only high type worker will accept the contract.

If $c > \theta_H(1+T)$, none of the workers will accept the offer.

(b) Denote by (w_G, w_B) the contract offered by a firm, in which it pays w_G (w_B) if the output realization is good (bad). The firm chooses its contract,

(c) The problem only differs in respect to which workers will accept the offer given in part b):

A worker of high type accepts, if $p_H\, u(w_G) + (1-p_H)\, u(w_B) \geq u(c)$.

A worker of low type accepts, if $p_L u(w_G) + (1-p_L)\, u(w_B) \geq u(c)$.

13.D.4 It can be shown that under conditions (i) and (ii), Proposition 13.D.2 still remains true. However, with these added conditions The model differs from the model of section 13.D. with respect to the existence of equilibria. In this model the existence of an equilibrium is guaranteed if no firm can offer a pooling contract, that attracts both types of workers, and makes a positive profit. Thus, the existence of an equilibrium is guaranteed in Figure 13.D.7(a) in the textbook. In Figure 13.D.7(b), however, there exists no equilibrium, as is the case for the original model. The difference for this model is for the situation displayed in Figure 13.D.8 of the textbook. Contrary to the model discussed in the text, firms cannot offer multiple contracts and thus cannot cross-subsidize between the workers, and an equilibrium will exist.

For a proof of these results, we refer to the original paper by Rothschild, M. and Stiglitz, J.(1977) "Equilibrium in Competitive Insurance Markets with Adverse Selection," QJE, p. 629 - 649 and to Laffont (1989) *The Economics of Uncertainty and Information*, MIT press, Chapter 8.

13.AA.1 An example appears in Section V of:

Cho, I-K., and D.M. Kreps (1987) "Signaling Games and Stable Equilibria," QJE, 102:179-221.

In their paper, Cho and Kreps use the notion of "Riley outcome" to refer to the textbook notion of "best separating equilibrium". The example itself appears under the sub-title "Case B: More than two types," on pages 212-13.

CHAPTER 14

14.B.1 The answer is yes, and the argument is supplied in footnote 8, immediately after Lemma 14.B.1 in the textbook.

14.B.2 Now, the program cannot be split into a minimization program and afterward a maximization program since the principal is risk averse over π - w(π). Therefore, letting $u(\cdot)$ denote the principal's utility function, the program becomes:

$$\underset{w(\pi)}{\text{Max}} \int u(\pi - w(\pi))f(\pi|e_H)d\pi$$

$$\text{s.t.} \quad \text{(i)} \quad \bar{u} - \int v(w(\pi))f(\pi|e_H)d\pi + g(e_H) \le 0$$

$$\text{(ii)} \int v(w(\pi))f(\pi|e_L)d\pi - \int v(w(\pi))f(\pi|e_H)d\pi + g(e_H) + g(e_L) \le 0$$

where constraint (i) is the participation constraint and (ii) is the incentive constraint (assuming that e_H is the desirable action). Letting γ and μ be the Kuhn-Tucker multipliers for (i), and (ii) respectively, the Kuhn-Tucker FOC is:

$$-u'(\pi - w(\pi))f(\pi|e_H) + \gamma v'(w(\pi))f(\pi|e_H) - \mu[f(\pi|e_L) - f(\pi|e_H)]v'(w(\pi)) = 0$$

which in turn yields:

$$\frac{u'(\pi - w(\pi))}{v'(w(\pi))} = \gamma + \mu\left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right].$$

Note that in this case the incentive constraint may not bind, i.e., we may have μ = 0. The reason is that due to optimal risk sharing it may be optimal for the agent to have enough risk such that (ii) does not bind.

14.B.3 [First Printing Errata: part (c) should end with "What effect do changes in ϕ and σ^2 have?"]

(a) Direct calculation gives:

$$\begin{aligned} Eu(w(\pi),e) &= E[\alpha + \beta\pi|e] - \phi VAR[\alpha + \beta\pi|e] - g(e) \\ &= \alpha + \beta E[\pi|e] - \phi\beta^2 VAR[\pi|e] - g(e) \\ &= \alpha + \beta e - \phi\beta^2\sigma^2 - g(e) . \end{aligned}$$

(b) Optimal risk sharing will result in a fixed wage for the agent, and maximizing the principal's profits ensures that this wage will exactly compensate the agent for his effort. Therefore, the first-best contract with observable (and verifiable) effort is the solution to:

$$\underset{e}{\text{Max}} \quad E[\pi|e] - g(e) \ ,$$

and the FOC (which is necessary and sufficient) gives $g'(e^*) = 1$, which gives us the optimal effort level e^*, and $w = g(e^*)$ is the wage.

(c) As in section 14.B of the textbook, the principal's problem can be divided into two steps. First, for a given effort level e', the optimal individually rational and incentive compatible compensation scheme (using the result from part (a) above) is given by:

$$\begin{aligned} &\underset{\alpha,\beta}{\text{Min}} \quad \alpha + \beta e' \\ &\text{s.t. (i)} \quad \alpha + \beta e' - \phi\beta^2\sigma^2 - g(e') \geq 0 \\ &\qquad \text{(ii)} \ \alpha + \beta e' - \phi\beta^2\sigma^2 - g(e') \geq \alpha + \beta e - \phi\beta^2\sigma^2 - g(e) \quad \forall\ e \neq e' \end{aligned}$$

The incentive constraint (ii) implies that given e', $\beta e - g(e)$ should reach a maximum at e'. The condition of the question allow us to replace (ii) with the FOC: $g'(e') = \beta$, which uniquely determines β given e'. We also know that (i) will bind, so $\beta = g'(e')$ implies: $\alpha = g(e') + \phi g'(e')^2\sigma^2 - g'(e')e'$. We can now find the optimal compensation scheme by solving:

$$\underset{e}{\text{Max}} \cdot E[\pi|e] - E[g(e) + \phi g'(e)^2\sigma^2 - g'(e)e + g'(e)\pi|e] ,$$

which reduces to:

$$\underset{e}{\text{Max}} \quad e - g(e) - \phi g'(e)^2\sigma^2.$$

Given the conditions of the question, this is a concave program, so the FOC which is necessary and sufficient yields: $1 - g'(e) - 2\phi\sigma^2 g'(e)g''(e) = 0$, which gives us: $g'(e) = \dfrac{1}{1 + 2\phi\sigma^2 g'(e)}$. This implies that $0 < \beta < 1$, which is "economically" reasonable because the incentives are not fully aligned with profits due to optimal risk sharing. As ϕ increases, the agent is more averse to risk (through variance) and therefore β will be lower, i.e., lower incentives. The same happens as σ^2 increases.

14.B.4 [First Printing Errata: in the hint of part (b) it should read "v_L and v_H" and <u>not</u> "v_1 and v_2"]

(a) Let $p_i \equiv f(\pi_H|e_i)$ so that $p_1 = \frac{2}{3}$, $p_2 = \frac{1}{2}$, and $p_3 = \frac{1}{3}$. When effort is observable, the principal will pay exactly $g(e)$ for effort level e, so that:

$$\pi(e_1) = \frac{2}{3}\cdot 10 + \frac{1}{3}\cdot 0 - \frac{25}{9} > 3 ,$$

$$\pi(e_2) = \frac{1}{2}\cdot 10 + \frac{1}{2}\cdot 0 - \frac{64}{25} < 3 ,$$

$$\pi(e_3) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 - \frac{16}{9} < 2 ,$$

and therefore e_1 is optimal with a wage of $w = \frac{25}{9}$.

(b) Let a contract specify a pair (v_H, v_L) where $v_k \equiv v(w_k)$. For e_2 to be implementable, three conditions must hold:

$$\text{(i)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq 0 ,$$

$$\text{(ii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3} ,$$

calculations show that (ii) implies $v_H \le \frac{2}{5} + v_L$, and (iii) implies $v_H \ge \frac{8}{5} + v_L$, and clearly both cannot be satisfied simultaneously. For e_2 to be implementable we must have both (ii) and (ii) satisfied. Rewriting both:

$$\text{(ii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - g(e_2) \ge \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3} ,$$

$$\text{(iii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - g(e_2) \ge \frac{1}{3}\cdot v_H + \frac{2}{3}\cdot v_L - \frac{4}{3} ,$$

or,

$$\text{(ii)} \quad 10 - 6\cdot g(e_2) + v_L \ge v_H ,$$

$$\text{(iii)} \quad 6\cdot g(e_2) - 8 + v_L \le v_H .$$

Both can be satisfied if and only if $g(e_2) \le \frac{3}{2}$.

(c) ~~[First printing errata: the end of this part of the question should read: "What effects do changes in ϕ and σ^2 have?".]~~ no it shouldn't

We established in (b) above that e_2 cannot be implemented. To implement e_3 we get the first best contract by paying w ~~[illegible]~~ $= \frac{16}{9}$, and the principal's expected profit is:

$$\pi(e_3) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 - \frac{16}{9} = \frac{14}{9} .$$

To implement e_1 we must have both individual rationality and incentive constraints satisfied. Consider the individual rationality constraint, and the incentive constraint with respect to e_3:

$$\text{(i)} \quad \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3} = 0 ,$$

$$\text{(ii)} \quad \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3} = \frac{1}{3}\cdot v_H + \frac{2}{3}\cdot v_L - \frac{4}{3} ,$$

which together give: $(v_H, v_L) = (2,1)$, or in terms of wages: $(w_H, w_L) = (4,1)$. It is easy to check that incentives with respect to e_2 are satisfied. The principal's expected profit is:

$$\pi(e_1) = \frac{2}{3}\cdot 10 + \frac{1}{3}\cdot 0 - \frac{2}{3}\cdot 4 - \frac{1}{3}\cdot 1 = \frac{33}{9} ,$$

and therefore e_1 is optimal with the compensation scheme $(w_H, w_L) = (4,1)$.

(d) When effort is observable, the principal will pay exactly $g(e)$ for effort level e, so that:

$$\pi(e_1) = x\cdot 10 + (1-x)\cdot 0 - 8 \longrightarrow 2 ,$$

$$\pi(e_2) = \frac{1}{2}\cdot 10 + \frac{1}{2}\cdot 0 - \frac{64}{25} = \frac{61}{25} > 2 ,$$

$$\pi(e_3) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 - \frac{16}{9} = \frac{14}{9} < 2 ,$$

e_2 is optimal with a wage of $\frac{64}{25}$.

and therefore as $x \longrightarrow 1$, ~~e_1 is optimal with a wage of $w = 8$.~~

When effort is not observable, the principal can still implement e_3 as in **(c)** above with $\pi(e_3) = \frac{14}{9}$. To implement e_2 we must have (similar to **(b)** above):

$$\text{(i)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq 0 ,$$

$$\text{(ii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq x\cdot v_H + (1-x)\cdot v_L - \sqrt{8} ,$$

$$\text{(iii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq \frac{1}{3}\cdot v_H + \frac{2}{3}\cdot v_L - \frac{4}{3} .$$

Supposing that (i) and (iii) bind, we compute that $(v_H,v_L) = (\frac{12}{5},\frac{4}{5})$, or in terms of wages: $(w_H,w_L) = (\frac{144}{25},\frac{16}{25})$, and it is easy to verify that (ii) is satisfied. We therefore get:

$$\pi(e_2) = \frac{1}{2}\cdot[10 - \frac{144}{25}] + \frac{1}{2}\cdot[0 - \frac{16}{25}] = \frac{49}{25} < 2 .$$

To implement e_1 the principal can use a scheme so that as x is arbitrarily close to 1, the cost of implementing e_1 will be arbitrarily close to 8, the wage which will exactly compensate the agent for his efforts. To see this, take some number $\delta > 0$, and let $x = 1-\varepsilon$ where $\varepsilon \longrightarrow 0$. Set the utility compensation values $(v_H,v_L) = ([1 + \delta]\cdot\sqrt{8},[1 + \delta - \frac{\delta}{\varepsilon}]\cdot\sqrt{8})$. It is easy to check that for all ε, given e_1 this scheme gives the agent an expected utility of 0. Furthermore, as $\varepsilon \longrightarrow 0$ we have that $v_L \longrightarrow -\infty$, so that the incentive constraints will easily be satisfied, and as $\varepsilon \longrightarrow 0$ we can make δ arbitrarily small so that both the individual rationality and incentive constraints are satisfied, and the expected wage that the principal will pay will approach 8, resulting in $\pi(e_1) \longrightarrow 2$. therefore, as $\varepsilon \longrightarrow 0$ the principal will prefer to implement e_1, a higher level than the first best level of effort.

14.B.5 Assume this is not true, i.e., that the optimal incentive scheme is as shown in Figure 14.B.5(a) below:

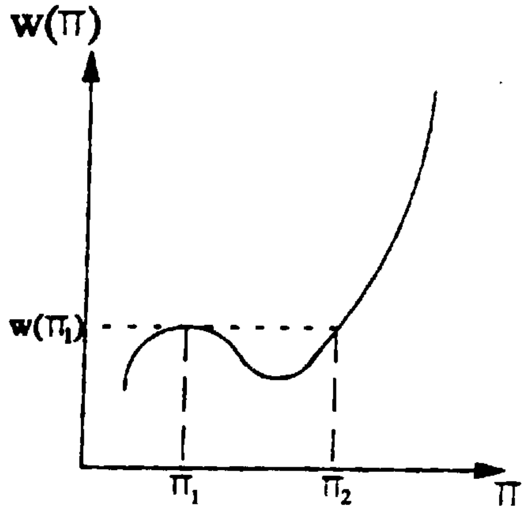


Figure 14.B.5(a)

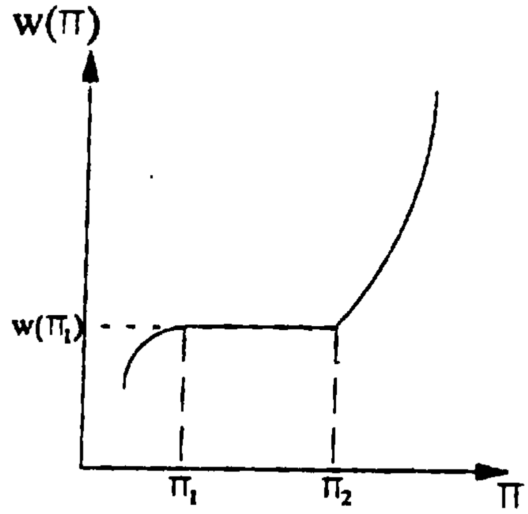


Figure 14.B.5(b)

Then, if any profit $\pi \in (\pi_1, \pi_2)$ was realized, the agent will dispose of any excess profits above π_1, and receive $w(\pi_1) > w(\pi)$. Thus, for all realizations $\pi \in (\pi_1, \pi_2)$ the principal will end up paying $w(\pi_1)$. But this can be achieved by the scheme depicted in Figure 14.B.5(b) above - a contradiction.

14.B.6 **(a)** The principal's problem becomes:

$$\underset{w(R,C)}{\text{Min}} \int_{\underline{R}}^{\bar{R}} \int_{\underline{C}}^{\bar{C}} w(R,C) f_R(R|e_C) f_C(C|e_C)\, dC\, dR$$

$$\text{s.t.} \quad \text{(i)} \quad \int_{\underline{R}}^{\bar{R}} \int_{\underline{C}}^{\bar{C}} w(R,C) f_R(R|e_C) f_C(C|e_C)\, dC\, dR - g(e_C) \geq 0$$

$$\text{(ii)} \quad \int_{\underline{R}}^{\bar{R}} \int_{\underline{C}}^{\bar{C}} w(R,C) f_R(R|e_C) f_C(C|e_C)\, dC\, dR - g(e_C)$$

$$\geq \int_{\underline{R}}^{\bar{R}} \int_{\underline{C}}^{\bar{C}} w(R,C) f_R(R|e_R) f_C(C|e_R)\, dC\, dR - g(e_R) ,$$

where constraint (i) is the participation constraint and (ii) is the incentive constraint (assuming that e_C is the desirable action). Letting γ and μ be the Kuhn-Tucker multipliers for (i) and (ii) respectively, the FOC can be worked out as is done in section 14.B and further algebra yields:

$$\frac{1}{v'(w(R,C))} = \gamma + \mu \cdot \left[1 - \left(\frac{f_R(R|e_R)}{f_R(R|e_C)} \right) \left(\frac{f_C(C|e_R)}{f_C(C|e_C)} \right) \right]$$

As R increases, $\dfrac{f_R(R|e_R)}{f_R(R|e_C)}$ increases, and the concavity of $v(\cdot)$ implies that $w(\cdot,\cdot)$ should decrease. This makes sense because we want to suppress the incentives of the agent to choose the revenue enhancing effort. Similarly, as C increases, $\dfrac{f_C(C|e_R)}{f_C(C|e_C)}$ increases, and the concavity of $v(\cdot)$ implies that $w(\cdot,\cdot)$ should decrease. This makes sense because we want to strengthen the incentives of the agent to choose the cost reducing effort.

(b) In this case the principal can no longer use R as a variable in the compensation scheme. The intuition is straightforward: no compensation scheme that induces the agent to choose e_C will have $\dfrac{\partial w}{\partial R} > 0$, and if $\dfrac{\partial w}{\partial R} < 0$ for some values of R, the manager will dispose of some revenues. Therefore we must have $\dfrac{\partial w}{\partial R} = 0$ for all values of R. Thus, the optimal scheme will be $w(C)$ with $\dfrac{\partial w}{\partial C} < 0$. (The FOC will be as in condition 14.B.10 of the textbook.)

(c) In this case no compensation scheme can induce the manager to exert effort level e_C. That is, only e_R is implementable.

14.B.7 For an analysis of this problem we refer to:

Rogerson, W. (1985) "Repeated Moral Hazard," *Econometrica*, 53:69-76.

It is worth mentioning that the conclusion of this model is counter-intuitive: The optimal compensation scheme is history dependent. The reason is that in addition to supplying the agent with correct incentives, the compensation scheme also serves as a consumption smoothing device for the agent.

14.B.8 For an analysis of this problem we refer to:

Dye, R. (1986) "Optimal Monitoring policies in agencies," *The Rand Journal of Economics*, 17:339-50.

14.C.1 Let $\theta \in \{\theta_1, \theta_2, \ldots, \theta_N\}$, where state θ_i occurs with probability $\lambda_i > 0$, and $\sum_{i=1}^{N} \lambda_i = 1$. The revelation principle still holds, so we can restrict ourselves to a menu of contracts of the form $\{(w_i, e_i)\}_{i=1}^{N}$. Program 14.C.8 in the textbook now becomes:

$$\underset{\{(w_i, e_i)\}}{\text{Max}} \quad \sum_{i=1}^{N} \lambda_i [\pi(e_i) - w_i]$$

$$\text{s.t. (i)} \quad w_i - g(e_i, \theta_i) \geq v^{-1}(\bar{u}) \qquad \forall\, i$$

$$\text{(ii)} \quad w_i - g(e_i, \theta_i) \geq w_j - g(e_j, \theta_j) \ \forall\, i \text{ and } \forall\, j \neq i$$

There are N individual rationality (IR) constraints given by (i) above, and (N-1)N incentive compatibility (IC) constraints given by (ii) above. However, an analog of Lemma 14.C.1 implies that only one IR constraint, that of type θ_1, binds. To see this note that the IC constraint ensuring that type θ_2 will not choose (w_1, e_1), the IR constraint of type θ_1, and the assumption that $g_{e\theta}(\cdot,\cdot) < 0$, together imply that:

$$w_2 - g(e_2, \theta_2) \geq w_1 - g(e_1, \theta_2) > w_1 - g(e_1, \theta_1) \geq v^{-1}(\bar{u}) \ ,$$

and this can be repeated inductively. Now that we have only one IR constraint, Lemma 14.C.2 holds as before, and we have: $w_1 - g(e_1, \theta_1) = v^{-1}(\bar{u})$.

The analog to Lemma 14.C.3 is that two type of inequalities hold:

$$\text{(I)} \quad e_i \le e_i^* \ , \quad \text{and,}$$

$$\text{(II)} \quad e_N = e_N^* \ .$$

The analog to Lemma 14.C.4 is that $e_i \le e_i^*$ for all $i < N$. The proof of this can be seen with a graphical argument as in the textbook extended to N types, or in a similar (yet more cumbersome) way to that of Appendix B in the textbook. Moreover, we will only have the "downward" IC constraints binding, i.e., (ii) above can be replaced with:

$$\text{(ii}'\text{)} \ w_i - g(e_i,\theta_i) \ge w_{i-1} - g(e_{i-1},\theta_i) \ \forall \ i > 1 \ .$$

To see this, note first that for all $i > 2$ wee can drop the IC constraints with respect to all $j < i - 1$ because:

$$\text{(1)} \qquad w_{i-1} - g(e_{i-1},\theta_{i-1}) \ge w_{i-2} - g(e_{i-2},\theta_{i-1}) \ ,$$

$$\text{(2)} \qquad w_i - g(e_i,\theta_i) \ge w_{i-1} - g(e_{i-1},\theta_i) \ ,$$

summing (1) and (2) we get that:

$$w_i - g(e_{i-1},\theta_{i-1}) - g(e_i,\theta_i) \ge w_{i-2} - g(e_{i-2},\theta_{i-1}) - g(e_{i-1},\theta_i).$$

Rewriting this, and using the fact that $g_{e\theta}(\cdot,\cdot) < 0$ we get:

$$\begin{aligned} w_i - g(e_i,\theta_i) &\ge w_{i-2} - g(e_{i-2},\theta_{i-1}) + [g(e_{i-1},\theta_{i-1}) - g(e_{i-1},\theta_i)] \\ &> w_{i-2} - g(e_{i-2},\theta_{i-1}) \\ &> w_{i-2} - g(e_{i-2},\theta_i) \ . \end{aligned}$$

To show that all the "upward binding IC constraints are not binding we can follow a similar manner as suggested at the end of Appendix B in the textbook.

14.C.2 The manager's utility function becomes $u(w,e,\theta) = w - g(e,\theta)$. It is easy to show (similar to the textbook analysis where the state θ is observable) that the first best effort level must satisfy $\pi'(e_i^*) = g_e(e_i^*,\theta_i)$

for $i \in \{H,L\}$. Furthermore, when θ is observable the manager has his individual rationality constraint binding, that is:

$$\lambda[w_H - g(e^*_H,\theta_H)] + (1-\lambda)[w_L - g(e^*_L,\theta_L)] = \bar{u} ,$$

or,

$$\lambda w_H + (1-\lambda)w_L = \bar{u} + \lambda g(e^*_H,\theta_H) + (1-\lambda)g(e^*_L,\theta_L) .$$

This, in turn, gives the owner expected profits equal to:

$$\begin{aligned} E\pi &= \lambda[\pi(e^*_H) - w_H] + (1-\lambda)[\pi(e^*_L) - w_L)] \\ &= \lambda\pi(e^*_H) + (1-\lambda)\pi(e^*_L) - [\bar{u} + \lambda g(e^*_H,\theta_H) + (1-\lambda)g(e^*_L,\theta_L)]. \end{aligned}$$

(Note that due to risk uncertainty for both the manager and the owner, any pair (w_H,w_L) that satisfies the manager's individual rationality constraint above with equality, will give the owner the same expected profits as above.) Now suppose that θ is not observable, and the owner offers the manager the following compensation scheme: $w(\pi) = \pi - \alpha$, where

$$\alpha = \lambda\pi(e^*_H) + (1-\lambda)\pi(e^*_L)- w_H] - [\bar{u} + \lambda g(e^*_H,\theta_H) + (1-\lambda)g(e^*_L,\theta_L)].$$

A manager of type i who faces this scheme will choose a level of effort that maximizes $w(\pi(e)) - g(e,\theta_i)$, which is (recall that α is a constant):

$$\max_e \quad \pi(e) - \alpha - g(e,\theta_i)$$

and the FOC is just the first best condition: $\pi'(e^*_i) = g_e(e^*_i,\theta_i)$. We only need to check that the owner makes the same profit as in the first best scenario, and that the manager will choose to participate. First, since the owner will pay $w(\pi) = \pi - \alpha$ for any realization π, he is left with a profit of α, which is exactly his expected utility in the first best scenario. Second, the manager's expected utility if he accepts the contract is:

$$Eu = \lambda\pi(e^*_H) + (1-\lambda)\pi(e^*_L) - \alpha - \lambda g(e^*_H,\theta_H) - (1-\lambda)g(e^*_L,\theta_L) = \bar{u}.$$

This is shown in the (w,π) space in Figure 14.C.2(a), and in the (w,e) space in Figure 14.C.2(b).

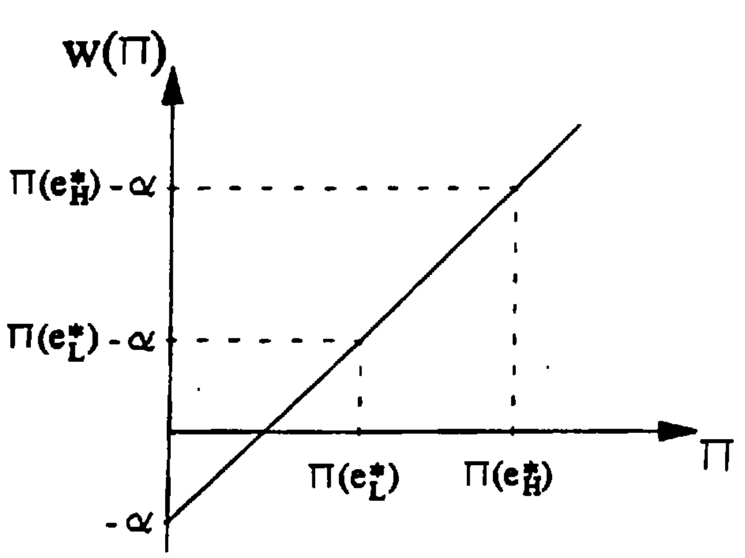


Figure 14.C.2(a)

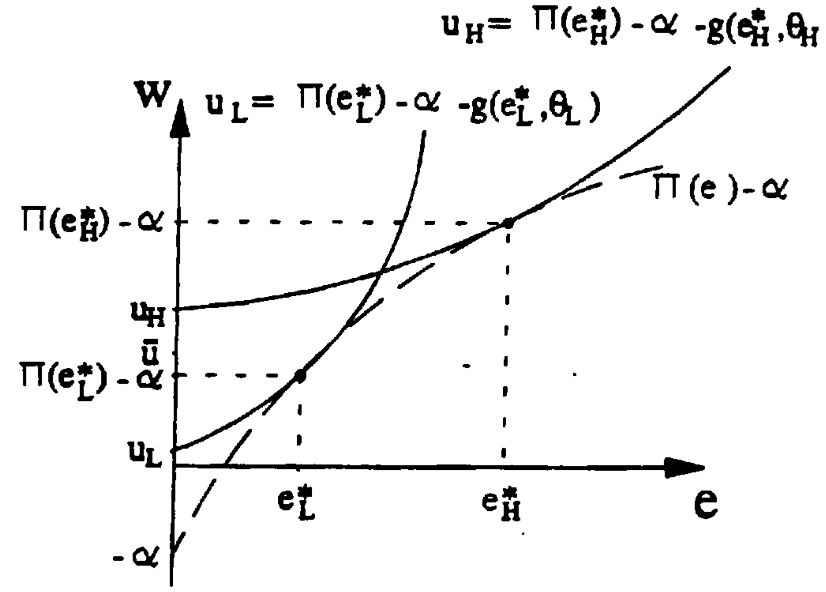


Figure 14.C.2(b)

The revelation mechanism which yields the same outcome is just the two points which result from the compensation scheme above. That is, given α as above, $(w_H, e_H) = (\pi(e_H^*) - \alpha, e_H^*)$, and $(w_L, e_L) = (\pi(e_L^*) - \alpha, e_L^*)$.

14.C.3 (a) Assuming the same conditions on $v(\cdot)$ and $g(\cdot,\cdot)$ the program for the optimal contract under full observability is:

$$\text{Max} \quad \lambda[\pi(e_H) - w_H] + (1 - \lambda)[\pi(e_L) - w_L]$$

$$\text{s.t.} \quad \lambda[v(w_H) - g(e_H,\theta_H)] + (1 - \lambda)[v(w_L) - g(e_L,\theta_L)] \geq \bar{u} .$$

Letting γ denote the Kuhn-Tucker multiplier, and noting that as in the textbook analysis the assumptions imply that the FOCs must hold with equalities, we get the following four FOCs:

$$\text{(i)} \quad -\lambda + \gamma[\lambda v'(w_H^*)] = 0 ,$$

$$\text{(ii)} \quad -(1 - \lambda) + \gamma[(1 - \lambda)v'(w_L^*)] = 0 ,$$

$$\text{(iii)} \quad \lambda\pi'(e_H^*) - \gamma\lambda g_e(e_H^*,\theta_H) = 0 ,$$

$$\text{(iv)} \quad (1 - \lambda)\pi'(e_L^*) - \gamma(1 - \lambda)g_e(e_L^*,\theta_L) = 0 .$$

From (i) and (ii) we get that $w_L^* = w_H^*$. This makes sense because wages and

(i) $-\lambda + \gamma[\lambda v'(w_H^*)] = 0$,

(ii) $-(1 - \lambda) + \gamma[(1 - \lambda)v'(w_L^*)] = 0$,

(iii) $\lambda\pi'(e_H^*) - \gamma\lambda g_e(e_H^*,\theta_H) = 0$,

(iv) $(1 - \lambda)\pi'(e_L^*) - \gamma(1 - \lambda)g_e(e_L^*,\theta_L) = 0$.

From (i) and (ii) we get that $w_L^* = w_H^*$. This makes sense because wages and disutility of effort are separable now, and the risk aversion is only on monetary income so that an optimal contract will have both wage levels equal. Using this we can rewrite (iii) and (iv), for states H and L respectively, as:

$$\pi'(e_i^*) = \frac{1}{v'(w_i^*)}\cdot g_e(e_i^*,\theta_i) \ .$$

This again makes sense because having $w_L^* = w_H^*$ implies that $e_L^* < e_H^*$.

(b) This contract is clearly not feasible when θ is unobservable. The reason is that $w_L^* = w_H^* = w^*$, and $e_L^* <= e_H^*$, will cause the manager to choose the L pair if the H state occurs, i.e., the H type will misrepresent.

14.C.4 For an analysis of this problem we refer to:

Hart, O. (1983) "Optimal Labor Contracts under Asymmetric Information: An Introduction," *The Review of Economic Studies*, 50:3-35.

14.C.5 [First Printing Errata: the first line of the question should read "... in Section 14.C would not change if..."]

The objective function in (14.C.1) and (14.C.8) in the textbook changes to:

$$\lambda[\pi_H(e_H) - w_H] + (1 - \lambda)[\pi_L(e_L) - w_L]$$

and the first best effort levels are determined by: $\pi_i'(e_i^*) = g_e(e_i^*,\theta_i)$ for $i \in \{H,L\}$, and the condition $\pi_H'(e) \geq \pi_L'(e) > 0$ ensures that $e_H^* > e_L^*$, and this promises that the analysis in section 13.C would not change. If, however, we have $0 < \pi_H'(e) < \pi_L'(e)$, it may be that $e_H^* < e_L^*$, and in this case the analysis of section 13.C would no longer apply. (Note that the

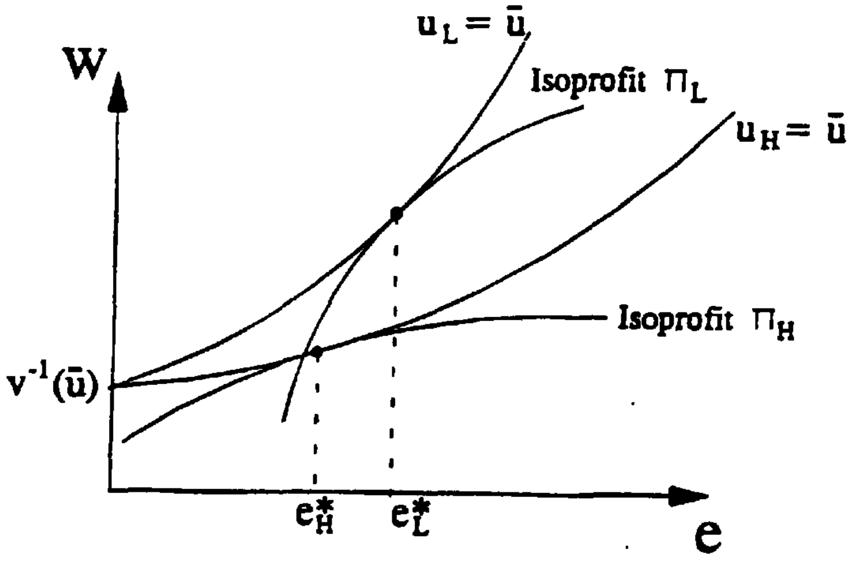


Figure 14.C.5(a)

We present a brief graphical presentation of two possibilities for optimal contracts when $e_H^* < e_L^*$. If the menu $\{(w_H,e_H),(w_L,e_L)\}$ is incentive compatible we must have $e_H > e_L$ (see Figure 14.C.4 in the textbook). Now start from the situation $(w_L,e_L)=(w_L^*,e_L^*)$ where w_L^* is chosen such that $u_L = \bar{u}$. Because$_L$ > e_H^*, the best contract for the H type will have $e_H = e_L = e_L^*$ since this has the least distortion of the H type's effort level. This means we will have a pooling contract with $(w,e)=(w_L^*,e_L^*)$. This, however, cannot be optimal because a first order reduction of w and of e along the indifference curve of the L type, $u_L = \bar{u}$, will cause no first order change for the L type yet will cause a first order benefit for the H type both with respect to a lower wage, and a better effort choice (closer to e_H^*) . Therefore, if the probability of θ_H is not too large, we will end up at a pooling contract of the type $(\hat{w},\hat{e})$, where $e_H^* \le \hat{e} \le e_L^*$, and $\hat{w}$ is chosen so that the L type's individual rationality constraint is bonding. An illustration of this situation is given in Figure 14.C.5(b). If, on the other hand, λ_H is large enough, then it will be worthwhile to further distort the L type contract so that we get $e_H = e_H^*$, and $e_L < e_H^*$. An illustration of this is given in Figure 14.C.5(c).

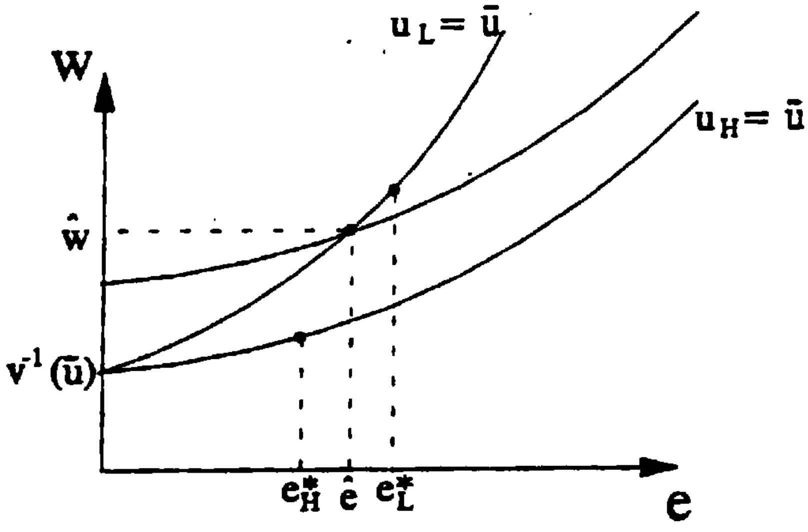


Figure 14.C.5(b)

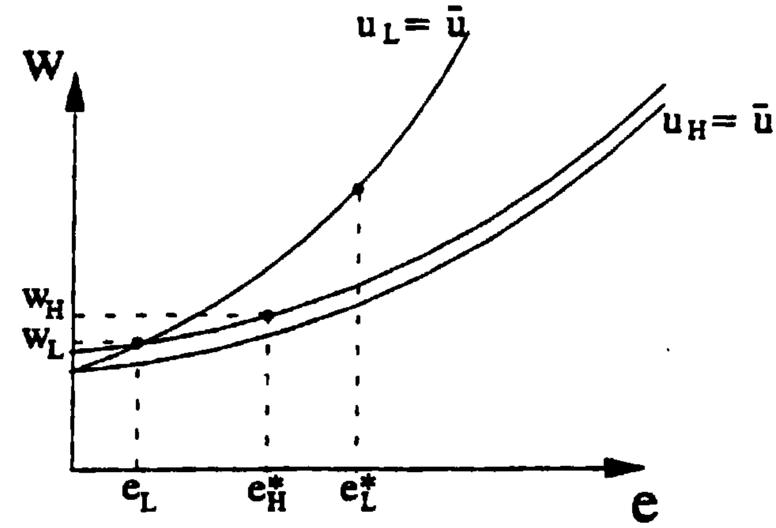


Figure 14.C.5(c)

14.C.6 This is a problem of monopolistic screening, and the firm will make positive profits in contrast to the situation in competitive screening described in exercise 13.D.1. Assuming that a worker of type θ_i has a marginal product of $\theta_i(1 + t)$ (recall that we took $\mu = 1$ which does not change the qualitative results), then the monopolist's problem is:

$$\underset{((w_H,t_H),(w_L,t_L))}{\text{Max}} \quad \lambda[\theta_H(1 + t_H) - w_H] + (1 - \lambda)[\theta_L(1 + t_L) - w_L]$$

$$\begin{aligned} \text{s.t.} \quad &\text{(i)} \quad w_L - c(t_L,\theta_L) \geq 0 \\ &\text{(ii)} \quad w_H - c(t_H,\theta_H) \geq 0 \\ &\text{(iii)} \quad w_H - c(t_H,\theta_H) \geq w_L - c(t_L,\theta_H) \\ &\text{(iv)} \quad w_L - c(t_L,\theta_L) \geq w_H - c(t_H,\theta_L) \end{aligned}$$

As we can expect, (i) together with (iii) and with the assumption that $c_{t\theta}(t,\theta) < 0$ imply that (ii) is redundant. All the conclusions of section 14.C will apply directly. That is, we will have the optimal contract satisfying $t_H = t_H^*$ (the first best value), $t_L < t_L^*$, $u(w_L,t_L,\theta_L) = 0$ and $u(w_H,t_H,\theta_H) > 0$. In summary, a monopolistic screening model will always have an equilibrium, it is never first best in the sense that $t_H = t_H^*$ and $t_L < t_L^*$, and the firm makes positive profits. In contrast, the competitive screening model of exercise 13.D.1 may not always have an equilibrium, we may have a first best separating equilibrium, if we have a separating equilibrium that is not first best then $t_H > t_H^*$ and $t_L = t_L^*$, and the firm makes no profits.

14.C.7 For an analysis of this problem we refer to:

Tirole, J. (1988) *The Theory of Industrial Organization*, MIT Press, Section 3.3 (pp 142-152) and section 3.5.1 (pp 153-154).

14.C.8 (a) The indifference curves of the two types and the firm's isoprofit curve are depicted in Figure 14.C.8.

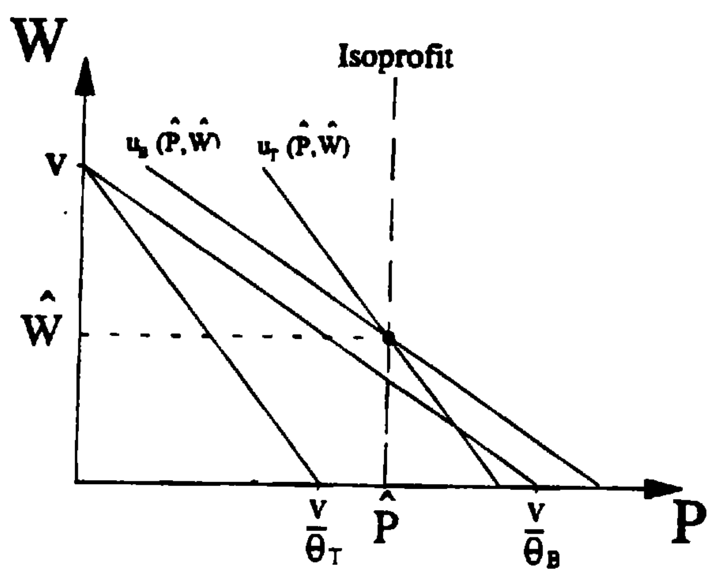


Figure 14.C.8

The formal problem that Air Shangri-La would want to solve is:

$$\underset{\{P_T, W_T, P_B, W_B\}}{\text{Max}} \quad \lambda P_T + (1 - \lambda) P_B$$

$$\begin{aligned}
\text{s.t.} \quad & \text{(i)} \quad \theta_T P_T + W_T \le v \\
& \text{(ii)} \quad \theta_B P_B + W_B \le v \\
& \text{(iii)} \quad \theta_T P_T + W_T \le \theta_T P_B + W_B \\
& \text{(iv)} \quad \theta_B P_B + W_B \le \theta_B P_T + W_T \\
& \text{(v)} \quad P_B, W_B, P_T, W_T \ge 0
\end{aligned}$$

(b) In the program above, constraint (i) and (iv), together with $\theta_B < \theta_T$, imply that constraint (ii) is redundant, so it is never binding. Constraint (i) must therefore bind for if it would not, we can reduce P_T and P_B by $\varepsilon > 0$, and all the remaining constraints will still be satisfied. This implies that tourists will be indifferent between buying and not buying a ticket.

(c) Assume that $\langle(P_T,W_T),(P_B,W_B)\rangle$ is an optimal, incentive compatible contract, and assume in negation that $W_B > 0$. Now reduce W_B by $\varepsilon > 0$, and increase P_B by $\frac{\varepsilon}{\theta_B}$ so that the B type's utility does not change, and the firm earns higher profits from the B type. We need to check that the T type will not choose this new compensation package. Indeed,

$$\theta_T P_T + W_T \leq \theta_T P_B + W_B = \theta_T(P_B + \frac{\varepsilon}{\theta_T}) + (W_B - \varepsilon) < \theta_T(P_B + \frac{\varepsilon}{\theta_B}) + (W_B -$$

contradicting that $\langle(P_T,W_T),(P_B,W_B)\rangle$ is an optimal, incentive compatible contract. Therefore, we must have $W_B = 0$. If, in an optimal contract, the business travelers were not indifferent between (P_T,W_T) and (P_B,W_B), we could slightly raise P_B and all the constraints would remain satisfied (recall that (ii) is redundant), and the firm would earn higher profits from the business types. Therefore, in an optimal contract we must have the business types indifferent between (P_T,W_T) and (P_B,W_B).

(d) The trade of that the firm faces is: By lowering P_T and increasing W_T so as to keep the tourists indifferent between buying a ticket or not, the firm can increase P_B (recall that $W_B = 0$). From parts (b) and (c) above, we can conclude that if the firm raises W_T by ε and lowers P_T by $\frac{\varepsilon}{\theta_T}$ so that the tourists remain indifferent between buying or not, then to keep the business types indifferent between their package and the new tourist package, it can increase P_B by $\frac{\varepsilon(\theta_T-\theta_B)}{\theta_T\theta_B}$. Since this trade off is linear, it is true no matter where we are in the (P,W) space, and therefore it will be profitable if and only if:

$$\lambda \cdot \frac{\varepsilon}{\theta_T} < (1 - \lambda) \cdot \frac{\varepsilon(\theta_T - \theta_B)}{\theta_T \theta_B} ,$$

or,

$$\frac{\lambda}{1 - \lambda} < \frac{\theta_T - \theta_B}{\theta_B} .$$

Note that this is independent of the cost c, because this is a revenue trade off (the costs are the same for both types). Therefore, the price discrimination scheme can take on two forms:

(1) If $\frac{\lambda}{1 - \lambda} < \frac{\theta_T - \theta_B}{\theta_B}$, then only the high types will be served and the scheme will be (this assumes that $c < \frac{v}{\theta_B}$:

$$\{(P_T, W_T), (P_B, W_B)\} = \{(0, v), (\frac{v}{\theta_B}, 0)\}$$

(2) If $\frac{\lambda}{1 - \lambda} > \frac{\theta_T - \theta_B}{\theta_B}$, then both types will be served and the scheme will be a pooling scheme with (this assumes that $c < \frac{v}{\theta_T}$):

$$(P_T, W_T) = (P_B, W_B) = (\frac{v}{\theta_T}, 0) .$$

Since the direction of the inequality in the condition above determines the type of scheme, it is easy to see how changes in λ, θ_T, and θ_B will affect the scheme: If the proportion of B types is large enough (λ small enough) the firm will choose to serve only the B types. If the B types suffer less from prices (θ_B is smaller) then the firm is more likely to serve only them. If the T types suffer less from prices (θ_T is smaller) then the firm is more likely to serve them as well as the B types. Changes in the cost c are discussed in part (e) below.

(e) As long as $c < \frac{v}{\theta_B}$, and we are in case (1) as described in part **(d)** above, the firm will decide to serve only the business types. If, however, we are in case (2) above, and $\frac{v}{\theta_T} < c < \frac{v}{\theta_B}$, then the scheme described in (d) above cannot be optimal because the firm is losing money. In such a case, the firm will choose the scheme described in case (1) of part (d), and serve only the business types. If $c > \frac{v}{\theta_B}$ the firm will choose not to operate at all.

14.C.9 (a) The monopolist will offer the individual a policy that fully insures him (optimal risk sharing) and keeps the individual at the same level of expected utility. That is, if we define $\bar{u} \equiv \theta u(W - L) + (1 - \theta)u(W)$, then the optimal insurance policy has $c_1 = c_2 = u^{-1}(\bar{u})$.

(b) The monopolist will offer an optimal screening contract of the form $\{(c_1^L, c_2^L), (c_1^H, c_2^H)\}$ that solves:

$$\text{Max} \quad \lambda[(1 - \theta_H)c_1^H + \theta_H c_2^H] + (1 - \lambda)[(1 - \theta_L)c_1^L + \theta_L c_2^L]$$

$$\text{s.t.} \quad \text{(i)} \quad (1 - \theta_H)u(c_1^H) + \theta_H u(c_2^H) \geq \bar{u}$$

$$\text{(ii)} \quad (1 - \theta_L)u(c_1^L) + \theta_L u(c_2^L) \geq \bar{u}$$

$$\text{(iii)} \quad (1 - \theta_H)u(c_1^H) + \theta_H u(c_2^H) \geq (1 - \theta_H)u(c_1^L) + \theta_H u(c_2^L)$$

$$\text{(iv)} \quad (1 - \theta_L)u(c_1^L) + \theta_L u(c_2^L) \geq (1 - \theta_L)u(c_1^H) + \theta_L u(c_2^H)$$

This is again a standard monopolistic screening problem and the standard analysis applies. Solving this program (again, (ii) and (iii) will be redundant) will have the H type fully insured, and the L type not fully insured. A graphical analysis is given using Figure 14.C.9 below. Both types start at the point A, with utility levels $\bar{u}_H$ and $\bar{u}_L$ respectively. If the monopolist would try to insure both types by offering the points B and C to H and L respectively, the (unobservable) H type would choose policy C instead of B, that is, the points B and C cannot be part of an incentive compatible contract.

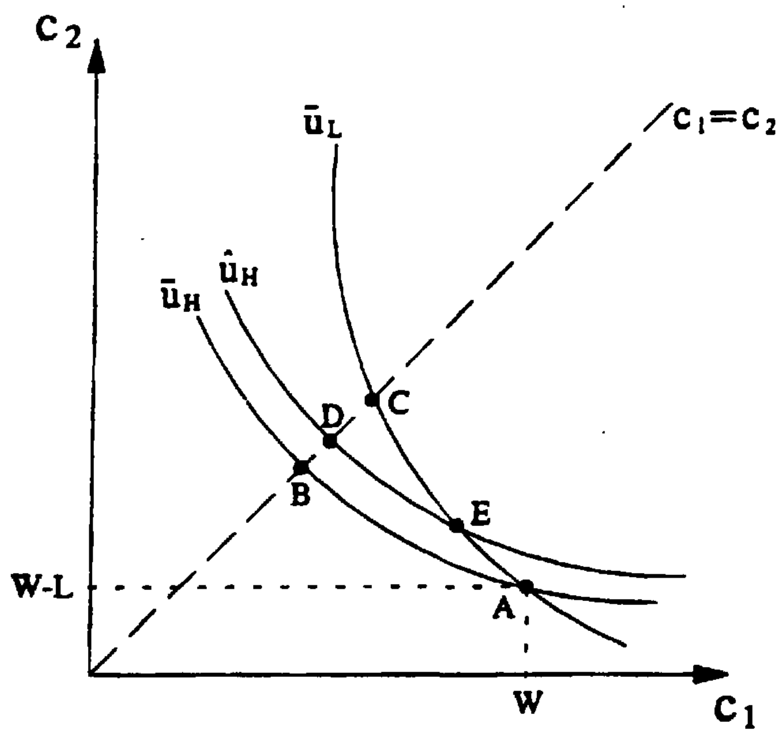


Figure 14.C.9

If the monopolist offers points A and B, then the H types would choose B (since they are indifferent), they would be fully insured, and the risk neutral monopolist would make profits from their choice. The L types, however, would prefer the point A to B and no profits would be made from them. If the proportion of High types is large enough, then the monopolist will find it profitable to slightly insure the L type at the cost of raising the utility of the H type. This means that the optimal contract will look like the two points D and E for the H and L types respectively. The mathematical analysis is straightforward, and results in a situation common to monopolistic screening (or hidden information): The H type will be at the first best insurance level and his participation constraint is not binding. The L type will be under insured (second best distortion so that screening is possible and profitable), his participation constraint will bind and his incentive compatibility constraint will not. (Note, that the pair of points A,B may be optimal if the proportion of L types is small. This is parallel to the hidden information case where the H type is at the first-best observable point with no surplus, and the L type has $e_L = 0$).

(c) The difference is who gets the surplus. In chapter 13 we discussed competitive markets, so that the insurer was left with zero profits and the individuals had utility levels above their reservation utility. Here, the monopolist makes positive profits and at least the L type has no surplus.

14.AA.1 For an analysis of this problem we refer to Proposition 5 in:

Milgrom, P. (1981) "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12:380-91.

14.AA.2 Sufficient conditions for the first order approach to be valid will promise that the agent's optimization program yields a unique solution. Given a compensation scheme (w_H, w_L), where w_i is the compensation when profits π_i are observed, the agent maximizes:

$$\underset{e}{\text{Max}} \quad f(\pi_H|e)\cdot v(w_H) + [1 - f(\pi_H|e)]\cdot v(w_L) - g(e) \ .$$

The FOC will be sufficient if the SOC is satisfied, i.e., if

$$f_{ee}(\pi_H|e)\cdot[v(w_H) - v(w_L)] - g''(e) < 0$$

So, if $v(w_H) - v(w_L) > 0$, and $f_{ee}(\pi_H|e) < 0$, then the first order approach will be valid. The first inequality is implies by MLRP, and the second is guaranteed by concavity of the density function. For more on this we refer to:

Rogerson, W. (1985) "The first-order Approach to Principal-Agent Problems," *Econometrica*, 53:1357-69.

14.AA.3 The program to be solved is:

$$\underset{((w_H, e_H),(w_L, e_L))}{\text{Max}} \quad \lambda[\pi(e_H) - w_H] + (1 - \lambda)[\pi(e_L) - w_L]$$

s.t. (i) $w_L - g(e_L, \theta_L) \geq 0$

(ii) $w_H - g(e_H, \theta_H) \geq 0$

(iii) $w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H)$

(iv) $w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L)$

and we have already established in the textbook that (ii) is redundant (Lemma 14.C.1). We proceed with two straightforward claims for the program without constraint (ii):

Claim 1: Constraint (i) must bind at a solution.

Proof: Assume not. Then reduce both w_L and w_H by $\varepsilon > 0$ such that (i) is still satisfied. This will not change the remaining two constraints and it will increase the objective function, a contradiction to being at an optimum. □

Claim 2: Constraint (iii) must bind at a solution.

Proof: Assume not. Then reduce w_H by $\varepsilon > 0$ such that (iii) is still satisfied. This will not affect constraint (i), constraint (iv) will only be further relaxed, and it will increase the objective function, a contradiction to being at an optimum. □

We now proceed to solve the "modified program" which is the original program without constraints (ii) and (iv), and with constraints (i) and (iii) binding. We will then proceed to show that at the solution to the modified program, constraint (iv) will be satisfied. The modified program is therefore:

$$\max_{\{(w_H, e_H), (w_L, e_L)\}} \lambda[\pi(e_H) - w_H] + (1 - \lambda)[\pi(e_L) - w_L]$$

s.t. (i) $w_L - g(e_L, \theta_L) = 0$

(iii) $w_H - g(e_H, \theta_H) = w_L - g(e_L, \theta_H)$

Letting γ and μ be the Lagrange multipliers for constraints (i) and (iii) respectively. Then, assuming an interior solution, we get the FOCs:

$$(1)\quad \lambda\pi'(e_H) - \mu g_e(e_H,\theta_H) = 0$$

$$(2)\quad (1-\lambda)\pi'(e_L) - \gamma g_e(e_L,\theta_L) + \mu g_e(e_L,\theta_H) = 0$$

$$(3)\quad -\lambda + \mu = 0$$

$$(4)\quad -(1-\lambda) + \gamma - \mu = 0$$

From (3) we have that $\mu = \lambda$, and plugging this into (4) we get that $\gamma = 1$. Substituting for μ and λ in (1) and (2) we get the well known conditions:

$$\pi'(e_H) = g_e(e_H,\theta_H)\ ,$$

$$\pi'(e_L) = g_e(e_L,\theta_L) + \frac{\lambda}{1-\lambda}\cdot[g_e(e_L,\theta_H) - g_e(e_L,\theta_H)]\ .$$

w_L and w_H are then computed using the two binding constraints. Denote the solution to the modified program by $\{(\hat{w}_H,\hat{e}_H),(\hat{w}_L,\hat{e}_L)\}$. We are left to show that this solution satisfies constraint (iv). We proceed with tow claims regarding the modified program.

Claim 3: At the solution to the modified program: $\pi(\hat{e}_H) - \hat{w}_H \geq \pi(\hat{e}_L) - \hat{w}_L$.

Proof: Assume not. Then the firm can offer the pair $(\hat{w}_L,\hat{e}_L)$ to both types. The L type will clearly accept it, and since we have shown that (iii) is binding then the H type will be indifferent between this pair and his original pair $(\hat{w}_H,\hat{e}_H)$. This, however, will raise the profits earned from the H type while leaving the profits earned from the L type unchanged, a contradiction to $\{(\hat{w}_H,\hat{e}_H),(\hat{w}_L,\hat{e}_L)\}$ being a solution to the modified program. □

Claim 4: Constraint (iv) must be satisfied at a solution to the modified program.

Proof: Assume not. That is, assume that $\hat{w}_L - g(\hat{e}_L,\theta_L) < \hat{w}_H - g(\hat{e}_H,\theta_L)$. Then, the firm can offer $(\hat{w}_H,\hat{e}_H)$ to both types, the H type will clearly accept as he did before, and the low type will prefer this to $(\bar{w}_L, \bar{e}_L)$ given our negation assumption. Furthermore, $(\hat{w}_H, \hat{e}_H)$ must satisfy (i) because $(\hat{w}_L, \hat{e}_L)$ did and the L type prefers $(\hat{w}_H, \hat{e}_H)$ to $(\hat{w}_L, \hat{e}_L)$. But now the firm is earning profits of $\pi(\hat{e}_H) - \hat{w}_H$ from both types, which is (from Claim 3) at least as good as the profits earned from the L type through $(\hat{w}_L, \hat{e}_L)$. The assumptions on $\pi(\cdot)$ and $g(\cdot,\cdot)$ guarantee a unique solution to the modified program, a contradiction. Therefore, constraint (iv) must be satisfied at a solution to the modified program.

CHAPTER 15

15.B.1 (a) The budget constraint implies that

$$p_1(x_{1i}(p) - \omega_{1i}) + p_2(x_{2i}(p) - \omega_{2i}) \le 0$$

for each $i = 1,2$. Suppose that the above weak inequality $\le$ held with strict inequality $<$, then there would be $(x_{1i},x_{2i}) \in B_i(p)$ such that $(x_{1i},x_{2i}) \succ_i (x_{1i}(p),x_{2i}(p))$, because the preference of each consumer is locally nonsatiated. This would contradict the fact that $(x_{1i}(p),x_{2i}(p))$ is the demand at p. We must thus have

$$p_1(x_{1i}(p) - \omega_{1i}) + p_2(x_{2i}(p) - \omega_{2i}) = 0$$

for each $i = 1,2$. Summing over $i = 1,2$, we obtain

$$p_1(\textstyle\sum_i x_{1i}(p) - \bar{\omega}_1) + p_2(\textstyle\sum_i x_{2i}(p) - \bar{\omega}_2) = 0.$$

(b) If the market for good 1 clears at p^*, then $\sum_i x_{1i}(p^*) - \bar{\omega}_1 = 0$ and hence $p_2(\sum_i x_{2i}(p^*) - \bar{\omega}_2) = 0$ by the equality of (a). By $p_2^* > 0$, this implies that $\sum_i x_{2i}(p^*) - \bar{\omega}_2 = 0$. Hence the market for good 2 clears at p^* as well and p^* is a Walrasian equilibrium price vector.

15.B.2 As we saw in Example 15.B.1, the offer curves of the two consumers are given by

$$OC_1(p) = (\alpha p\cdot\omega_1/p_1,\ (1 - \alpha)p\cdot\omega_1/p_2),$$

$$OC_2(p) = (\beta p\cdot\omega_2/p_1,\ (1 - \beta)p\cdot\omega_2/p_2).$$

The total demand for good 2 at p is thus

$$(p_1/p_2)((1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}) + ((1 - \alpha)\omega_{21} + (1 - \beta)\omega_{22}).$$

By (b) of Exercise 15.B.1, equating this to $\bar{\omega}_2 = \omega_{21} + \omega_{22}$ gives the equilibrium price ratio $p_1^*/p_2^* = \dfrac{\alpha\omega_{21} + \beta\omega_{22}}{(1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}}$. By substituting this into the offer curves, we obtain the equilibrium allocations: $OC_i(p^*)$ is equal to

$$(\omega_{11}\omega_{21} + \beta\omega_{11}\omega_{22} + (1 - \beta)\omega_{21}\omega_{12})\left(\frac{\alpha}{\alpha\omega_{21} + \beta\omega_{22}}, \frac{1 - \alpha}{(1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}}\right),$$

and $OC_2(p^*)$ is equal to

$$(\omega_{12}\omega_{22} + (1 - \alpha)\omega_{11}\omega_{22} + \alpha\omega_{21}\omega_{12})\left(\frac{\beta}{\alpha\omega_{21} + \beta\omega_{22}}, \frac{1 - \beta}{(1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}}\right).$$

It is easy to check that

$$\partial(p_1^*/p_2^*)/\partial\omega_{11} < 0,$$

$$\partial OC_{11}(p^*)/\partial\omega_{11} > 0,$$

$$\partial OC_{21}(p^*)/\partial\omega_{11} = \frac{(1 - \alpha)(1 - \beta)\omega_{12}}{\alpha\omega_{21} + \beta\omega_{22}} > 0,$$

$$\partial OC_{12}(p^*)/\partial\omega_{11} > 0,$$

and, since $\partial OC_{21}(p^*)/\partial\omega_{11} > 0$ and $\bar{\omega}_2$ is constant,

$$\partial OC_{22}(p^*)/\partial\omega_{11} < 0.$$

15.B.3 Let p^* be a Walrasian equilibrium price vector and x_i^* be the demand of consumer i (i = 1,2). Since the preference of consumer 1 is locally non-satiated, the upper contour sets $\{x_1 \in \mathbb{R}^2_+ : x_1 \succsim_1 x_1^*\}$ lies on or above the budget line and the strict upper contour sets $\{x_1 \in \mathbb{R}^2_+ : x_1 \succ_1 x_1^*\}$ lies strictly above the budget line. Symmetrically, the upper contour sets $\{x_2 \in \mathbb{R}^2_+ : x_2 \succsim_2 x_2^*\}$ lies on or below the budget line and the strict upper contour sets $\{x_2 \in \mathbb{R}^2_+ : x_2 \succ_2 x_2^*\}$ lies strictly above the budget line. Hence the two sets $\{x_1 \in \mathbb{R}^2_+ : x_1 \succsim_1 x_1^*\}$ and $\{x_2 \in \mathbb{R}^2_+ : x_2 \succ_2 x_2^*\}$ do not intersect; the other two, $\{x_1 \in \mathbb{R}^2_+ : x_1 \succ_1 x_1^*\}$ and $\{x_2 \in \mathbb{R}^2_+ : x_2 \succsim_2 x_2^*\}$, do not either. Hence the Walrasian equilibrium allocation $x^* = (x_1^*, x_2^*)$ is Pareto optimal. See the figure below.

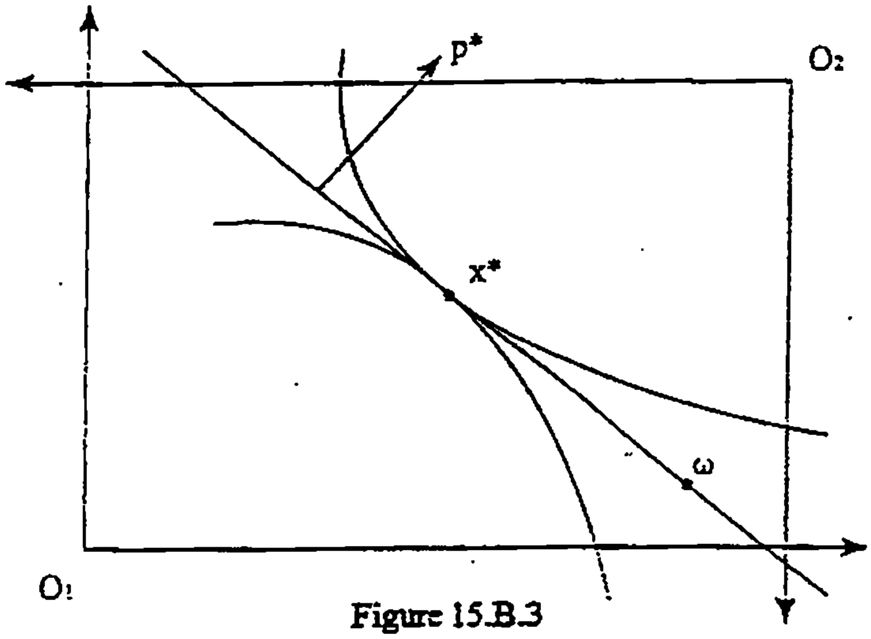


Figure 15.B.3

15.B.4 (a) Here is an example of an offer curve (of consumer 1) with the gross substitute property. The dotted curve is the indifference curve of consumer 1 that goes through ω.

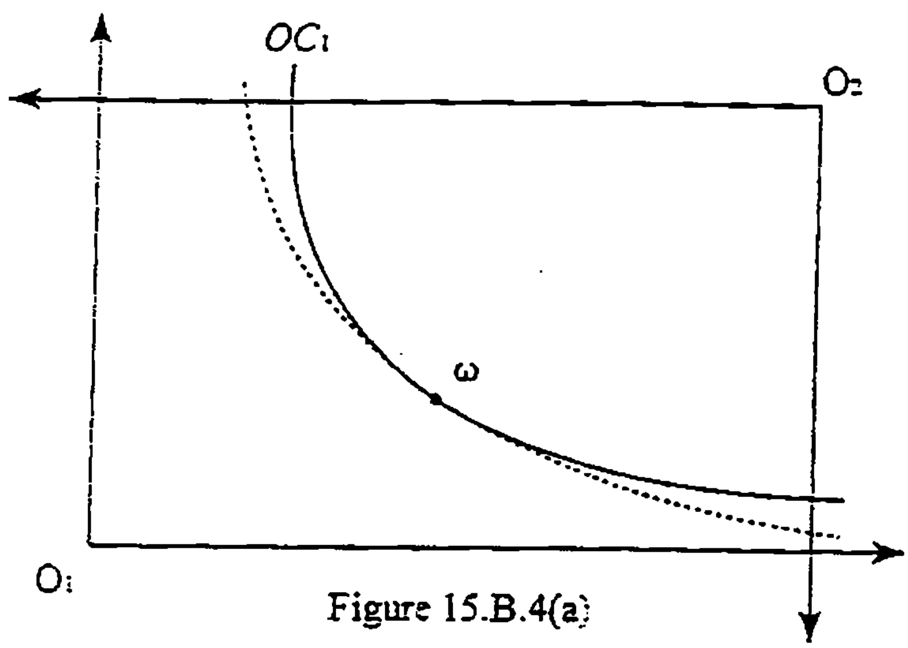
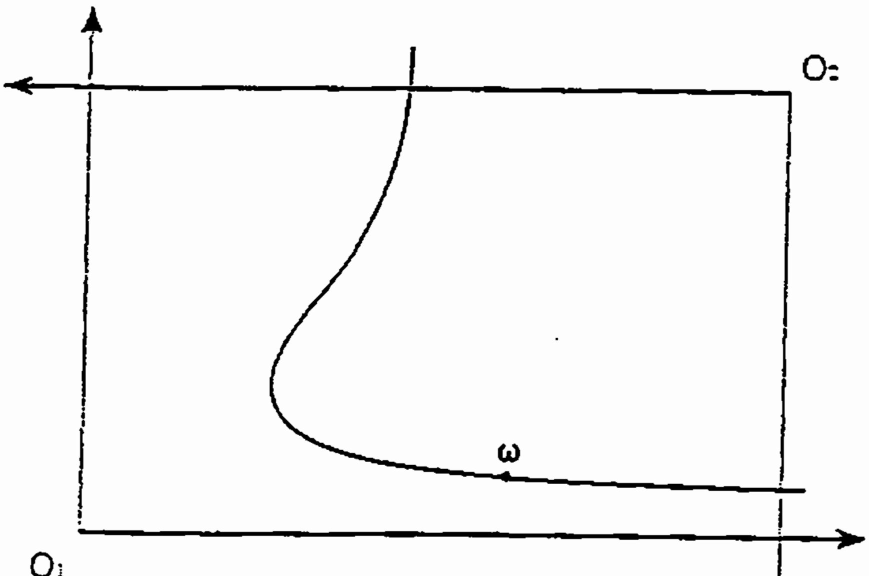


Figure 15.B.4(a)

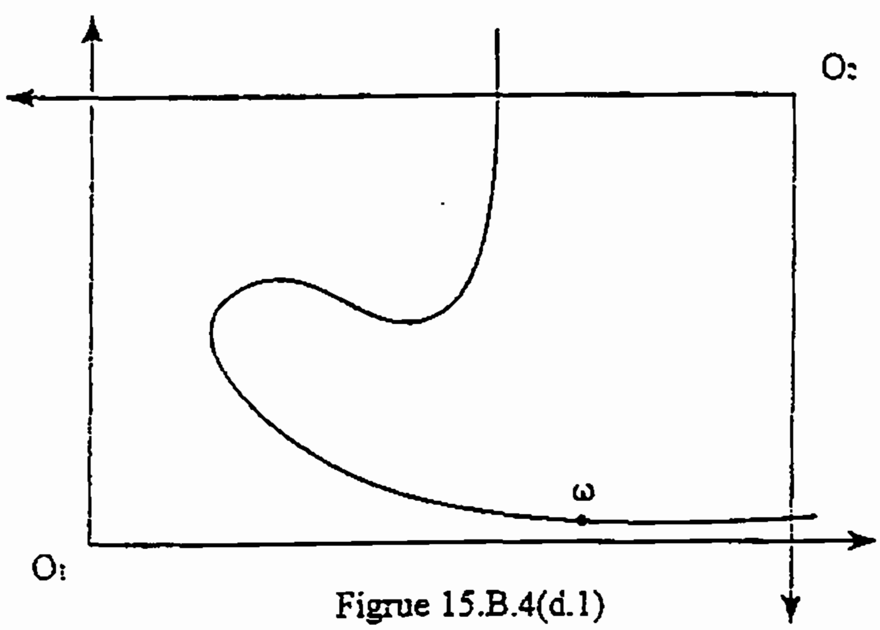
(b) As we discussed in the text, at any equilibrium, the offer curves of the two consumers intersect and, conversely, any intersection of the offer curves at an allocation different from ω corresponds an equilibrium. In order to verify that the offer curves intersect only once (not counting the intersection at the initial endowments), it is therefore sufficient to show that there is only one equilibrium price ratio p_1^*/p_2^*. So, let $p^* = (p_1^*,1)$ be an equilibrium price vector. If $p_1 > p_1^*$, then $OC_{11}(p_1,1) < OC_{11}(p_1^*,1)$ and $OC_{12}(p_1,1) < OC_{12}(p_1^*,1)$ by the gross substitute property. Hence $OC_{11}(p_1,1) + OC_{12}(p_1,1) < OC_{11}(p_1^*,1) + OC_{12}(p_1^*,1) = \bar{\omega}_1$. Hence $(p_1,1)$ is not an equilibrium price vector. Symmetrically, if $p_1 < p_1^*$, then there $(P_1,1)$ is not an equilibrium price vector either. Thus the offer curves intersect only once.

(c) Here is an example of a normal offer curve that does not satisfy the gross substitute property.



Figrue 15.B.4(c)

(d) Here is an example of a preference, an initial endowment, and the corresponding offer curve that is not normal.



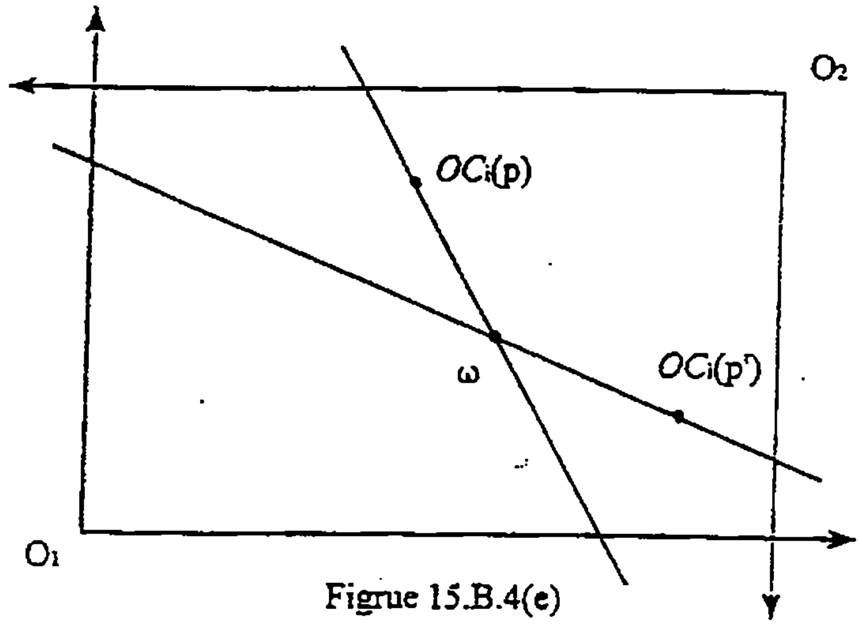
Figrue 15.B.4(d.1)

As for the second statement, we consider the case in which the price of commodity 1 increases. (The case in which the price of commodity 1 increases can be symmetrically proved.) Let (p_1,p_2) be the initial price vector and $p_1' > p_1$. Assume that $OC_{1i}(p_1',p_2) > OC_{1i}(p_1,p_2)$ and $OC_{2i}(p_1',p_2) < OC_{2i}(p_1,p_2)$. It is sufficient to prove that if commodity 1 is not inferior, then commodity 2 must be inferior. Suppose so. Then, by $p_1' > p_1$ and $OC_{1i}(p_1',p_2) > OC_{1i}(p_1,p_2)$, the real wealth must have increased from p_1 to p_1'. (That is, $p_1'OC_{1i}(p_1,p_2) + p_2OC_{2i}(p_1,p_2) < p_1'\omega_{1i} + p_2\omega_{2i}$). Since the relative price of commodity 2 has decreased, this and $OC_{2i}(p_1',p_2) < OC_{2i}(p_1,p_2)$ together imply that commodity 2 must be inferior. This completes the proof.

(e) Assume that the offer curve of consumer 1 is normal and that of consumer 2 satisfies the gross substitute property. If the initial endowments (ω_1,ω_2) constitute a Walrasian equilibrium (and preferences are strictly convex), then the two offer curves intersect only at the initial endowments, because they are contained in the upper contour sets of ω_1 and ω_2. So suppose that the initial endowments do not constitute a Walrasian equilibrium. Then we first need to establish the following assertion:

If both p and p' are equilibrium price vectors, then, for every $i = 1,2$ and every $\ell = 1,2$, we have $(OC_{\ell i}(p) - \omega_{\ell i})(OC_{\ell i}(p') - \omega_{\ell i}) > 0$.

In fact, suppose that we have $(OC_{\ell i}(p) - \omega_{\ell i})(OC_{\ell i}(p') - \omega_{\ell i}) \le 0$. Since the initial endowments do not constitute an equilibrium, the above weak inequality is satisfied with strict inequality. Thus one of $OC_{\ell i}(p) - \omega_{\ell i}$ and $OC_{\ell i}(p') - \omega_{\ell i}$ must be positive and the other must be negative. By the weak axiom of revealed preference, $OC_i(p)$ and $OC_i(p')$ must be outside the other budget constraints, as shown in the figure below:



Figrue 15.B.4(e)

In the Edgeworth box, however, this implies that the offer curve of the other consumer does not satisfy the weak axiom. This is a contradiction. Hence, for every ℓ and i, we must have

$$(OC_{\ell i}(p) - \omega_{\ell i})(OC_{\ell i}(p') - \omega_{\ell i}) > 0.$$

Therefore, by relabelling the indexes of the commodities if necessary, we can assume that $OC_{11}(p) - \omega_{11} > 0$ for every equilibrium price vector p. Now, let $p^* = (p_1^*,1)$ be an equilibrium price vector such that if $p = (p_1,1)$ is any other equilibrium price vector, then $p_1^* < p_1$. For any $p = (p_1,1)$ with $p_1^* < p_1$, by normality, we have either

$$OC_{11}(p^*) > OC_{11}(p)$$

or

$$OC_{11}(p^*) < OC_{11}(p) \text{ and } OC_{21}(p^*) < OC_{21}(p).$$

Since $OC_{11}(p^*) - \omega_{11} > 0$ and $p_1^* < p_1$, if the second case applies, then $p_1(OC_{11}(p) - \omega_{11}) + OC_{21}(p) - \omega_{2i} > 0$, which is a contradiction to the budget constraint. Thus the second case is actually impossible and the first case applies. On the other hand, by the gross substitute property, we have $OC_{12}(p^*) > OC_{12}(p)$. Hence $\omega_1 = OC_{11}(p^*) + OC_{12}(p^*) > OC_{11}(p) + OC_{12}(p)$. Thus p is not an equilibrium price vector. Thus the Walrasian equilibrium price vector p^* is unique. Therefore, the two offer curves intersect only once, except for the initial endowments.

(f) Here is an example in which two normal offer curves intersect several times.

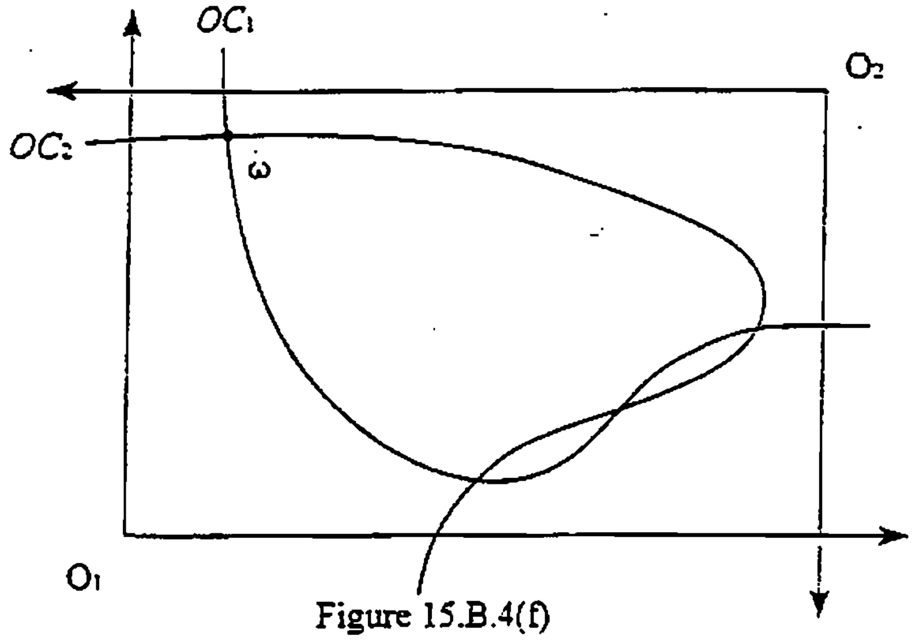


Figure 15.B.4(f)

15.B.5 First, we can derive from the quasilinearity that $OC_{21}(p_1,p_2)^{-9} = p_2/p_1$, that is, $OC_{21}(p_1,p_2) = (p_2/p_1)^{-1/9}$. The budget constraint then implies that $OC_{11}(p_1,p_2) = 2 + r(p_2/p_1) - (p_2/p_1)^{8/9}$. Hence the offer curve of consumer 1 is as given in Example 15.B.2. We can similarly show that the offer curve of consumer 2 is also as given in the example. Rearranging the equality of the total demand of the second good and its total supply and substituting $r = 2^{8/9} - 2^{1/9}$, we obtain

$$(p_1/p_2)^{8/9} - (p_1/p_2)^{1/9} = (2^{8/9} - 2^{1/9})(p_1/p_2 - 1).$$

Then $p_1/p_2 = 1/2, 1, 2$ are solutions of this equation, and hence equilibrium price ratios.

15.B.6 We can obtain the offer curves from the first-order conditions of the utility maximization problem:

$$OC_1(p) = \frac{p_1}{p_1^{2/3} + (12/37)p_2^{2/3}}(p_1^{-1/3}, (12/37)p_2^{-1/3}),$$

$$OC_2(p) = \frac{p_2}{(12/37)p_1^{2/3} + p_2^{2/3}}((12/37)p_1^{-1/3}, p_2^{-1/3}).$$

Set $p_2 = 1$ and write $q = p_1^{1/3}$, then

$$OC_{11}(p) + OC_{12}(p) = \frac{q^2}{q^2 + 12/37} + \frac{(12/37)q^{-1}}{(12/37)q^2 + 1}$$

$$= \frac{q^5 + (37/12)q^3 + q^2 + 12/37}{q^5 + (12/37 + 37/12)q^3 + q}.$$

We can check that $OC_{11}(p) + OC_{12}(p) = 1$ if and only if

$$12q^3 - 37q^2 + 37q - 12 = (q - 1)(4q - 3)(3q - 4) = 0.$$

Thus $q = 1, 3/4, 4/3$. Hence the equilibrium price ratios are given by

$$p_1^*/p_2^* = 1, (3/4)^{1/3}, (4/3)^{1/3}.$$

15.B.7 We shall prove that the set of Pareto optimal allocation looks like a curve in the Edgeworth box. More precisely, we show that there is a one-to-one, continuous mapping from a (non-degenerated) bounded closed interval of $\mathbb{R}$ into the Edgeworth whose image is equal to the Pareto set. This is sufficient for the first assertion of the exercise. For each $i = 1,2$, let $u_i(\cdot)$ be a utility function of consumer i. It is continuous, strongly monotone, and strictly quasiconcave.

Note first that, since $u_2(\cdot)$ is continuous, the set $\{u_2(x_2) \in \mathbb{R}: 0 \le x_2 \le \bar{\omega}\}$ is a (non-degenerated) closed bounded interval. Denote it by $[\delta_0,\delta_1]$. For each $\delta \in [\delta_0,\delta_1]$, consider maximizing $u_1(x_1)$ under the constraints $0 \le x_1 \le \bar{\omega}$ and $u_2(\bar{\omega} - x_1) \ge \delta$. This maximization problem is feasible and, by the compactness of $\{x_1: 0 \le x_1 \le \bar{\omega}, u_2(\bar{\omega} - x_1) \ge \delta\}$, there is at least one solution. We shall now prove that the strict quasiconcavity of the $u_i(\cdot)$ implies that such a solution is unique. Let x_1 and x_1' be distinct solutions, then $u_1((1/2)x_1 + (1/2)x_1') > u_1(x_1) = u_1(x_1')$. We can assume without loss of generality that $u_2(\bar{\omega} - x_1) \ge u_2(\bar{\omega} - x_1') \ge \delta$. Then, by continuity and strong monotonicity, there exists a unique $\lambda \in [0,1]$ such that $u_2(\lambda(\bar{\omega} - x_1)) = u_2(\bar{\omega} - x_1')$. By $x_1 \neq x_1'$ and $u_1(x_1) = u_1(x_1')$, we have $\lambda(\bar{\omega} - x_1) \neq \bar{\omega} - x_1'$. Hence

$$u_2((1/2)\lambda(\bar{\omega} - x_1) + (1/2)(\bar{\omega} - x_1')) > u_2(\lambda(\bar{\omega} - x_1)) = u_2(\bar{\omega} - x_1').$$

Thus $u_2(\bar{\omega} - ((1/2)x_1 + (1/2)x_1')) > \delta$. Therefore $(1/2)x_1 + (1/2)x_1'$ is feasible and attains a higher utility than x_1 and x_1', a contradiction. Thus there must be a unique solution, which we denote by $\varphi_1(\delta) \in \mathbb{R}^2_+$. Define the mapping φ: $[\delta_0,\delta_1] \to \{(x_1,x_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ : x_1 + x_2 = \bar{\omega}\}$ by

$$\varphi(\delta) = (\varphi_1(\delta),\ \bar{\omega} - \varphi_1(\delta)).$$

It is now sufficient to prove that $\varphi(\cdot)$ is one-to-one and continuous, and its image is equal to the Pareto set. The equality to the Pareto set follows from its construction. As for being one-to-one, note that $u_2(\bar{\omega} - \varphi_1(\delta)) = \delta$; otherwise, by strong monotonicity, a small transfer collinear with $\bar{\omega} - \varphi_1(\delta)$ from consumer 2 to 1 would increase the utility level of consumer 1, a contradiction. It thus remains to verify the continuity. For this, it suffices to prove that $\varphi_1(\cdot)$ is continuous. Let $\{\delta^n\}$ be a sequence in $[\delta_0,\delta_1]$ converging to δ. We shall prove that if $\varphi_1(\delta^n) \to x_1$, then $x_1 = \varphi_1(\delta)$. Note first that, by continuity, $u_2(\bar{\omega} - x_1) \geq \delta$ and hence $u_1(x_1) \leq u_1(\varphi_1(\delta))$. By strong monotonicity, for any sufficiently large n, we can find x_1^n such that $u_2(\bar{\omega} - x_1^n) \geq \delta^n$ and $x_1^n \to \varphi_1(\delta)$. Thus $u_1(x_1^n) \leq u_1(\varphi_1(\delta^n))$. Hence $u_1(\varphi_1(\delta)) \leq u_1(x_1)$. Thus $u_1(\varphi_1(\delta)) = u_1(x_1)$. By $u_2(\bar{\omega} - x_1) \geq \delta$, we obtain $x_1 = \varphi_1(\delta)$.

For the second assertion of the exercise, it is sufficient to prove that if the preferences of the consumers are homothetic and the Pareto set ever cuts the diagonal in the interior of the Edgeworth box, then the Pareto set must coincide with the diagonal. Let $(\bar{\delta}\bar{\omega}, (1 - \bar{\delta})\bar{\omega})$ $(\bar{\delta} \in (0,1))$ a Pareto optimal allocation on the diagonal. By the definition of a homothetic preference (Definition 3.B.6), $(\delta\bar{\omega}, (1 - \delta)\bar{\omega})$ is Pareto optimal for every $\delta \in$

(0,1). By strong monotonicity, $(0,\bar{\omega})$ and $(\bar{\omega},0)$ are also Pareto optimal. Hence every allocation on the diagonal is Pareto optimal. By our previous result (the existence of the one-to-one, continuous mapping $\varphi(\cdot)$), the diagonal exhausts all Pareto optimal allocations. Hence the Pareto set is equal to the diagonal.

15.B.8 Suppose that the preference of consumer i (i = 1,2) is represented by a utility function u_i: $\mathbb{R}^2_+ \to \mathbb{R}$ of the quasi-linear form $u_i(x_i) = x_{1i} + \phi_i(x_{2i})$, where ϕ_i: $\mathbb{R}_+ \to \mathbb{R}$ is continuous. We shall first prove that $\phi_i(\cdot)$ is strictly monotone and strictly concave. In fact, its strict monotonicity is an immediate consequence of that of the preference. As for the strict concavity, let $x_{2i} \neq x'_{2i}$ and $\lambda \in (0,1)$. By definition,

$$u_i(\phi_i(x'_{2i}) - \phi_i(0),\ x_{2i}) = u_i(\phi_i(x_{2i}) - \phi_i(0),\ x'_{2i}).$$

Hence, by the strict convexity of the preference,

$$u_i(\lambda(\phi_i(x'_{2i}) - \phi_i(0)) + (1 - \lambda)(\phi_i(x_{2i}) - \phi_i(0)),\ \lambda x_{2i} + (1 - \lambda)x'_{2i})$$
$$> \lambda u_i(\phi_i(x'_{2i}) - \phi_i(0),\ x_{2i}) + (1 - \lambda)u_i(\phi_i(x_{2i}) - \phi_i(0),\ x'_{2i}).$$

This is equivalent to

$$\phi_i(\lambda x_{2i} + (1 - \lambda)x'_{2i}) > \lambda\phi_i(x_{2i}) + (1 - \lambda)\phi_i(x'_{2i}).$$

Thus $\phi_i(\cdot)$ is strictly concave.

Now define φ: $[0,\bar{\omega}_2] \to \mathbb{R}$ by $\varphi(x_{21}) = \phi_1(x_{21}) + \phi_2(\bar{\omega}_2 - x_{21})$, then $\varphi(\cdot)$ is continuous and strictly concave. Hence there exists a unique maximum $x^*_{21} \in [0,\bar{\omega}_2]$.

In order to verify the assertion of the exercise, it is now sufficient to prove that if $x = (x_1,x_2) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ is a nonwasteful feasible allocation in the interior of the Edgeworth box and $x_{21} \neq x^*_{21}$, then x is not Pareto optimal. In fact, let x be as such. Then, by the strict concavity of $\varphi(\cdot)$, for every $\lambda \in (0,1)$, we have

$$\varphi((1 - \lambda)x_{21} + \lambda x^*_{21}) > (1 - \lambda)\varphi(x_{21}) + \lambda\varphi(x^*_{21}) > \varphi(x_{21}).$$

For each $i = 1,2$, define $\delta_{\lambda i} \in \mathbb{R}$ as

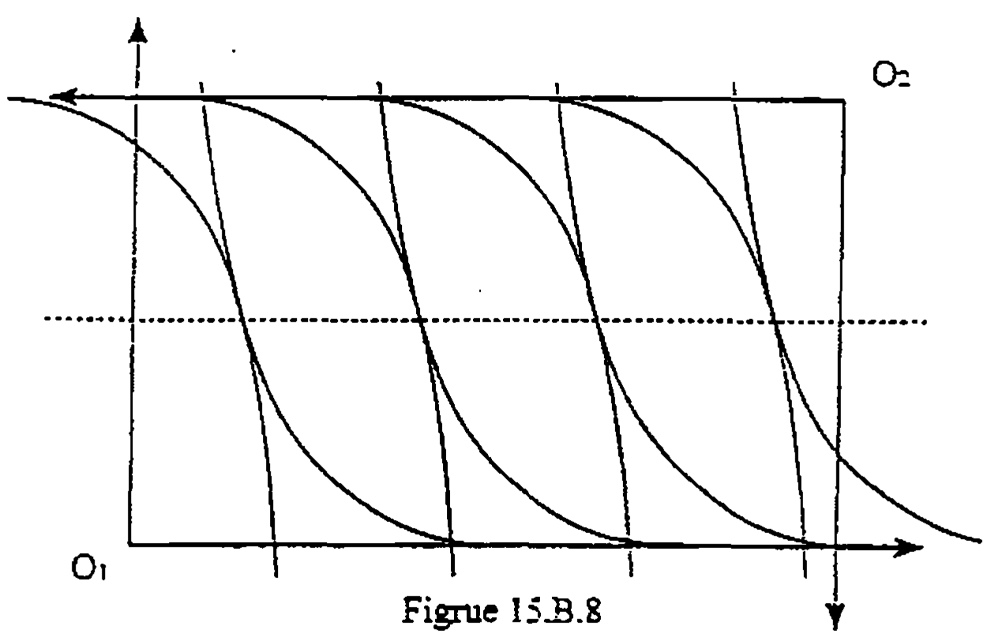
$$- \phi_i((1 - \lambda)x_{2i} + \lambda x^*_{2i}) + \phi_i(x_{2i}) + (1/2)(\varphi((1 - \lambda)x_{21} + \lambda x^*_{21}) - \varphi(x_{21})),$$

then $\delta_{\lambda 1} + \delta_{\lambda 2} = 0$ by the definition of $\varphi(\cdot)$, and $\delta_{\lambda i} \to 0$ as $\lambda \to 0$ by continuity. Since $x_{1i} > 0$, we have $x_{1i} + \delta_{\lambda 1} > 0$ for each i for any sufficiently small $\lambda \in (0,1)$. Moreover,

$$u_i(x_{1i} + \delta_{\lambda i}, (1 - \lambda)x_{2i} + \lambda x^*_{2i}) = x_{1i} + \delta_{\lambda i} + \phi_i((1 - \lambda)x_{2i} + \lambda x^*_{2i})$$
$$= x_{1i} + \phi_i(x_{2i}) + (1/2)(\varphi((1 - \lambda)x_{21} + \lambda x^*_{21}) - \varphi(x_{21}))$$
$$> x_{1i} + \phi_i(x_{2i}) = u_i(x_{1i}, x_{2i}).$$

Thus x is not a Pareto optimal allocation. The figure below depicts the set of the Pareto optimal allocations in the interior of the Edgeworth box.

[It is worthwhile to note that if the consumptions sets were taken to be $(-\infty, \infty) \times \mathbb{R}$ as in Definition 3.B.7, then the Edgeworth "box" would have an infinite length in the horizontal direction and the assertion of this exercise could more easily be proved, without the interiority assumption or an explicit use of the utility functions. The reason is that, then, for any allocation x with $x_{21} \neq x^*_{21}$, there would always exist a feasible allocation x' with $x'_{21} = x^*_{21}$ that is Pareto superior to x. This is not always true when we have the nonnegativity constraints.]



Figrue 15.B.8

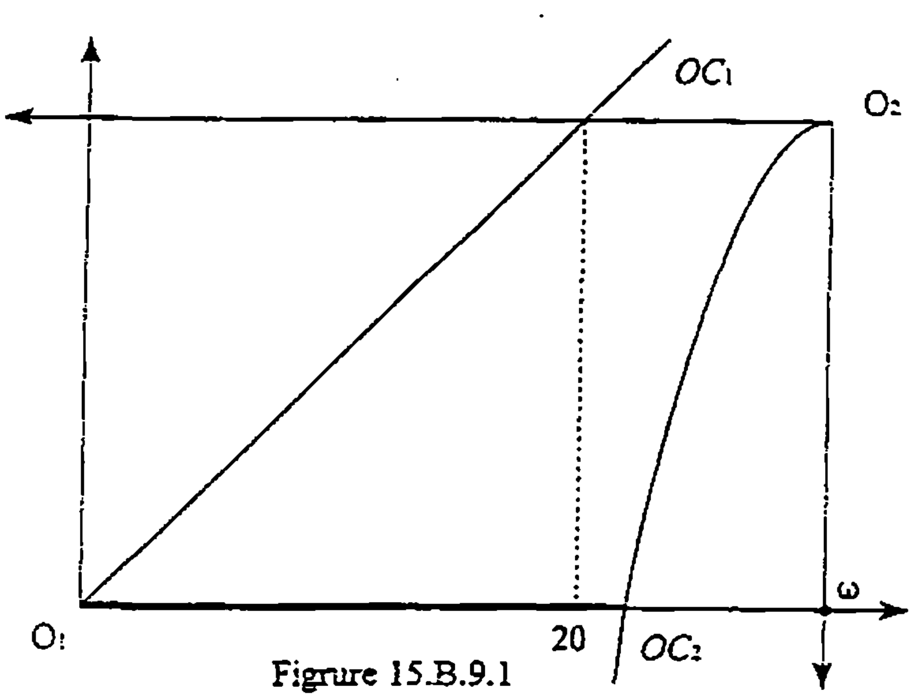
In connection with the discussion of Chapter 10, this fact implies that in the absence of wealth effects, a Pareto optimal allocation (in the interior of the Edgeworth box) of the non-numeraire commodities are uniquely determined. The differences in the consumers' utility levels at different Pareto optimal allocations (this time including the numeraire) can all be generated simply by redistributing the numeraire.

15.B.9 The offer curves of the two consumers are rather trivial when the prices of the two commodities are both positive. Their graphs are $\{(x_{p\alpha},x_{b\alpha}) \in \mathbb{R}^2_{++}: x_{p\alpha} = x_{b\alpha}\}$ and $\{(x_{p\beta},x_{b\beta}) \in \mathbb{R}^2_{++}: x_{p\beta} = (x_{b\beta})^{1/2}\}$. When one of the two commodities has zero price, the offer curves are given by

$$OC_\alpha(p_p,p_b) = \begin{cases} \{(x_{p\alpha},0): x_{p\alpha} \geq 0\} & \text{if } p_p = 0 \text{ and } p_b > 0, \\ \{(\omega_{p\alpha},x_{b\alpha}): x_{b\alpha} \geq \omega_{p\alpha}\} & \text{if } p_p > 0 \text{ and } p_b = 0. \end{cases}$$

$$OC_\beta(p_p,p_b) = \begin{cases} \{(x_{p\beta},20): x_{p\beta} \geq 20^{1/2}\} & \text{if } p_p = 0 \text{ and } p_b > 0, \\ \{(0,x_{b\alpha}): x_{b\alpha} \geq 0\} & \text{if } p_p > 0 \text{ and } p_b = 0. \end{cases}$$

These are depicted in the following figures. (Note that $4 < 20^{1/2} < 5$.)



Figrure 15.B.9.1

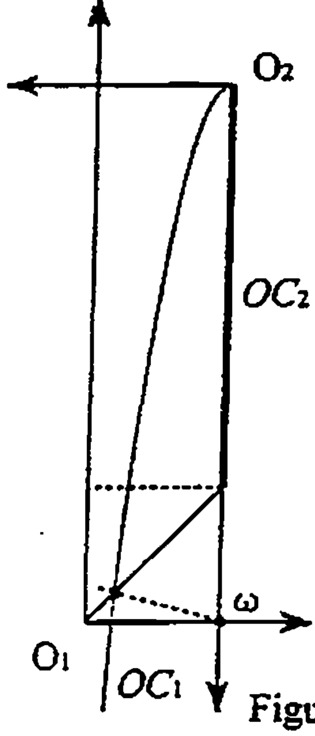


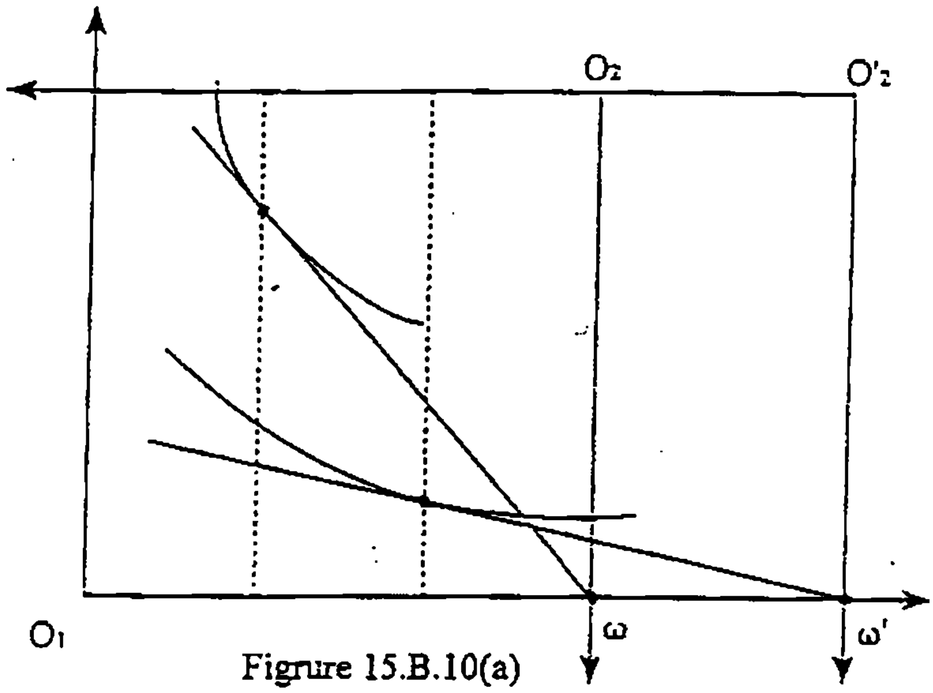
Figure 15.B.9.2

Thus, when $\omega_{p\alpha} = 30$, all equilibria are on the boundary of the Edgeworth box and the equilibrium price ratio and allocations are given by $p_p/p_b = 0$ and $\{((30 - x_{p\beta}, 0), (x_{p\beta}, 20)) : 20^{1/2} \le x_{p\beta} \le 30\}$. When $\omega_{p\alpha} = 5$, the interior equilibrium is the intersection on the above figure . The boundary equilibria have the price ratio $p_b/p_p = 0$ and allocations $\{((5, x_{b\alpha}), (0, 20 - x_{b\alpha})) : 5 \le x_{b\alpha} \le 20\}$.

Note that, when $\omega_{p\alpha}$ decreases from 30 to 5, Alphanse's utility level increases and Betatrix's utility level decreases, regardless of the choice of equilibria to be compared. This is because, when $\omega_{p\alpha} = 30$, Perrier is too abundant relative to Brie and its equilibrium price is zero, implying that Betatrix essentially consumes the total endowment of the economy. When $\omega_{p\alpha} = 5$, Perrier is scarce enough to have positive price and Alphanse can afford positive consumptions of both goods. The price of Brie can even be driven down to zero, in which case he essentially consumes the total endowment of the economy.

15.B.10 (a) Suppose that the preferences of the two consumers are quasilinear with respect to commodity 2 and that $\omega_{21} = 0$. Suppose for a moment that $\omega'_{21} = 0$, so that there is no increase in the endowment of consumer 1 for commodity

2 Here is an example in which an increase in the endowment of consumer 1 for commodity 1 may lead a decrease in his utility.



Figrure 15.B.10(a)

Since equilibrium allocations depend continuously on the initial endowments, we can still have a decrease in the utility of of consumer 1 when ω'_{21} is positive, but sufficiently small. We can thus take $\omega'_2 >> \omega_2$ as asked for in the exercise.

In this example, the small increment in the initial endowment leads to a substantial decrease in the relative price of commodity 1 . Since the wealth of consumer 1 comes exclusively from commodity 1, his real wealth then decreases, despite the increase in his endowment. Hence his utility decreases. This fact is often discussed in the theory of a quantity-setting monopoly: It is not in the monopolist's best interest to supply all it could potentially do, because an increase in supply leads a decrease in price.

(b) Let (p,x) be an equilibrium of the original endowments (ω_1,ω_2) and (p',x') be an equilibrium of the new endowments (ω'_1,ω'_2). In order to apply the result of Exercise 15.B.8, assume that both x and x' belong to the interior of the Edgeworth box. By the first fundamental theorem of welfare economics, both x and x' are Pareto optimal. Thus by the result of Exercise 15.B.8, we have x_{11}

$= x'_{11}$ and $x'_{12} = x'_{12}$. Hence, by the definition of quasilinearity (Definition 3.B.7), we can assume without loss of generality that $p = p'$. By the strong monotonicity of the preferences, $p \gg 0$. Hence, by $\omega'_1 \geq \omega_1$ and $\omega'_1 \neq \omega_1$, we have $p \cdot \omega'_1 > p \cdot \omega_1$. Since $p \cdot x'_1 = p \cdot \omega'_1 > p \cdot \omega_1 = p \cdot x_1$, we obtain $x'_1 \succ_1 x_1$.

(c) Here is an example of the transfer paradox:

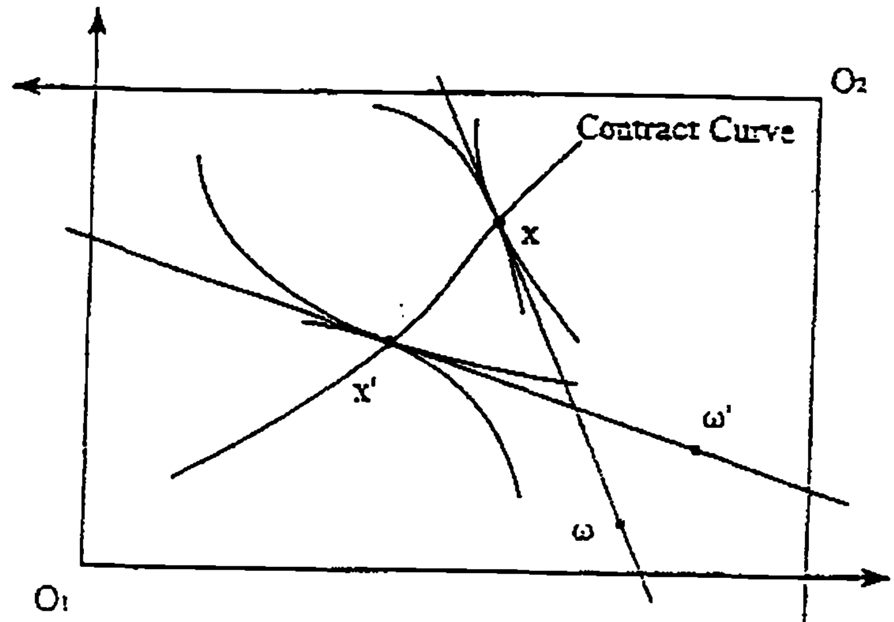


Figure 15.B.10(c)

In this example, a positive transfer is made from consumer 2 to consumer 1. If the price ratio were kept to be at the original level p_1/p_2, then there would be an excess demand for commodity 1. Thus the price ratio needs to change to recover an equilibrium. In this example, p_1/p_2 decreases, but this decrease induces a negative wealth effect on consumer 1 because he is the net supplier of commodity 1. Hence his equilibrium consumptions goes down from x_1 to x'_1.

(d) Following the hint, we shall prove that there are other equilibria at the original endowment ω. In the figure of (c), draw the budget line that goes through the original endowment ω and the new equilibrium x'. This budget line must be steeper than that of p' because $\omega_1 \ll \omega'_1$. Hence the demand of consumer 1 on this budget line must be in the north-west of x'. Of course, his offer curve with ω_1 must go through this demand, as well as the original equilibrium allocation x. Hence, as the relative price of commodity 1 increases, his offer curve must cut the contract curve from above at x. Symmetrically, as the relative price of commodity 1 increases, the offer curve of consumer 2 must cut the contract curve from below at x. This is illustrated in the following figure:

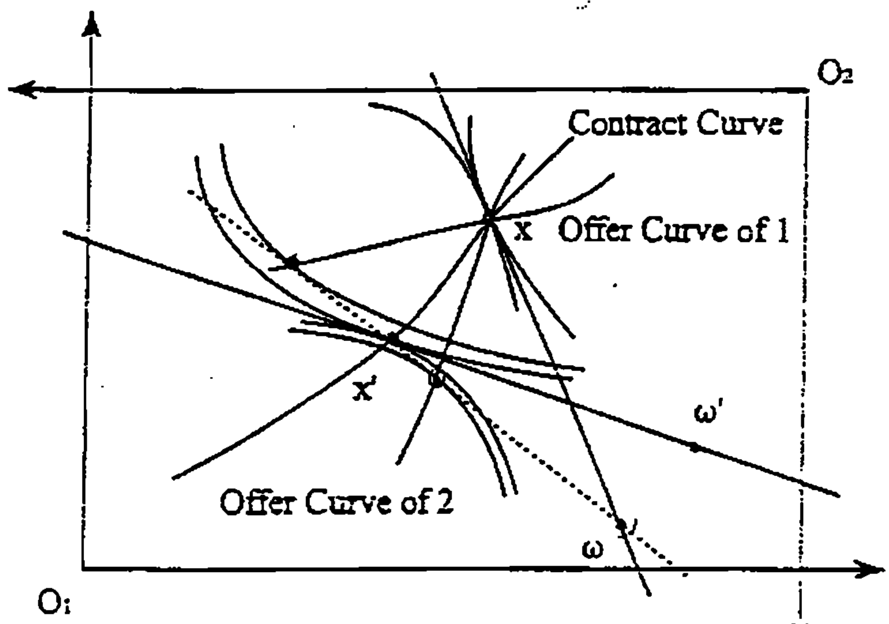


Figure 15.B.10(d)

Now recall that these two offer curves must go through the initial endowments. It will then be easy to convince yourself that, in whatever way you will extend the offer curves, they must intersect at at least two other points. Hence there are at least three equilibria with the original endowments.

15.C.1 (a) This is a simple consequence of two simple facts, both of which are already mentioned in the text: First, a Walrasian equilibrium is Pareto optimal. Second, under the strict convexity assumptions, there is a unique Pareto optimal allocation.

(b) Here is an example in which the slope of the excess demand function may change its sign. Here, given p = 1, the equilibrium wage level is w^* and, at wage levels w_1 and w_2 with $w_1 < w_2 < w^*$, we have $0 < z_1(w_1) < z_1(w_2)$. Hence

$z_1(\cdot)$ must have a positive slope somewhere between w_1 and w_2, and a negative slope somewhere between w_2 and w^*.

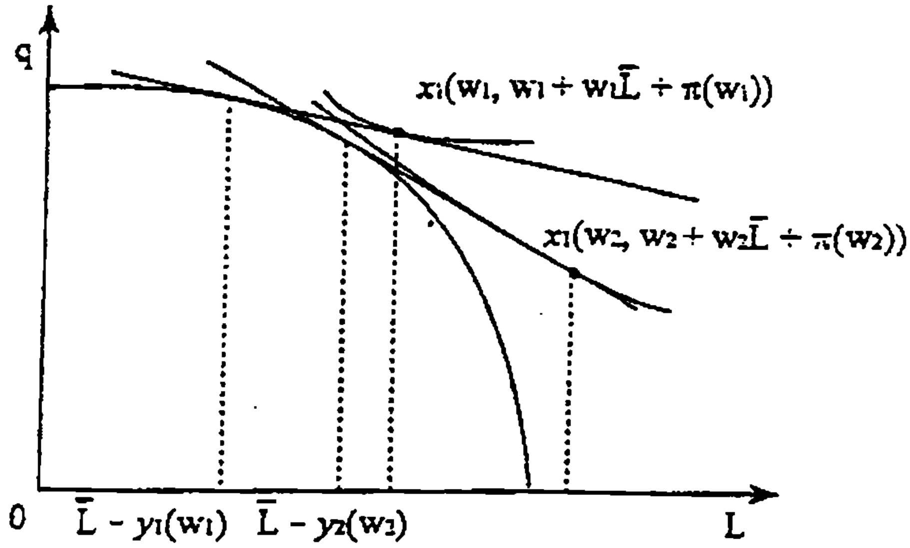


Figure 15.C.1(b)

To prove that the slope of the excess demand function is negative in a neighborhood of the equilibrium wage level w^*, assume its differentiability and denote the wealth level by $v = w\bar{L} + \pi(w)$, then

$$z_1'(w^*) = (\partial x_1/\partial w)(w^*, w^*\bar{L} + \pi(w^*)) + (\partial x_1/\partial v)(w^*, w^*\bar{L} + \pi(w^*))(\bar{L} + \pi'(w^*)) + y_1'(w^*).$$

Here, by (vi) of Proposition 5.C.1, $\pi'(w^*) = -y_1(w^*)$ and hence $\bar{L} + \pi'(w^*) = \bar{L} - y_1(w^*) = x_1(w^*, w^*\bar{L} + \pi(w^*))$. Thus the sum of the first two terms is equal to the diagonal element corresponding to labor of the Slutsky matrix of this consumer. Hence it is negative by Propositions 3.G.2 and 3.G.3. By (vii) of Proposition 5.C.1, $y_1'(w^*) < 0$. Hence $z_1'(w^*) < 0$.

(c) Suppose that the two consumers are endowed with the same amount of labor. Then, at any wage level, the total wealth of the economy is split equally between then. Here is an example of multiple equilibria. It is a modification of Example 4.C.1.

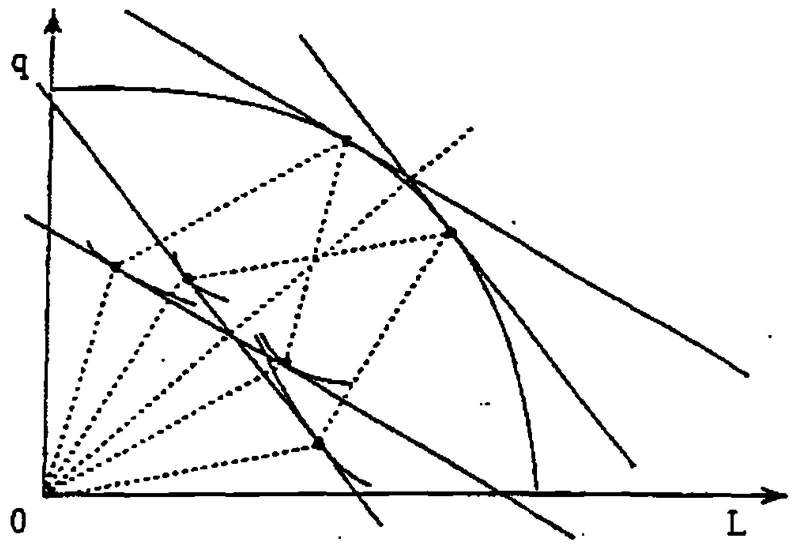


Figure 15.C.1(c.1)

If the firm operates under constant returns to scale, then there is a unique equilibrium allocation. To prove this, note first that the profit of the firm must be zero at any equilibrium. Thus, to find an equilibrium, we can assume that $\pi(w) = 0$ for every w. Since, in addition, the individuals are endowed with labor alone, they are always net suppliers of labor. Thus their offer curves look as follows.

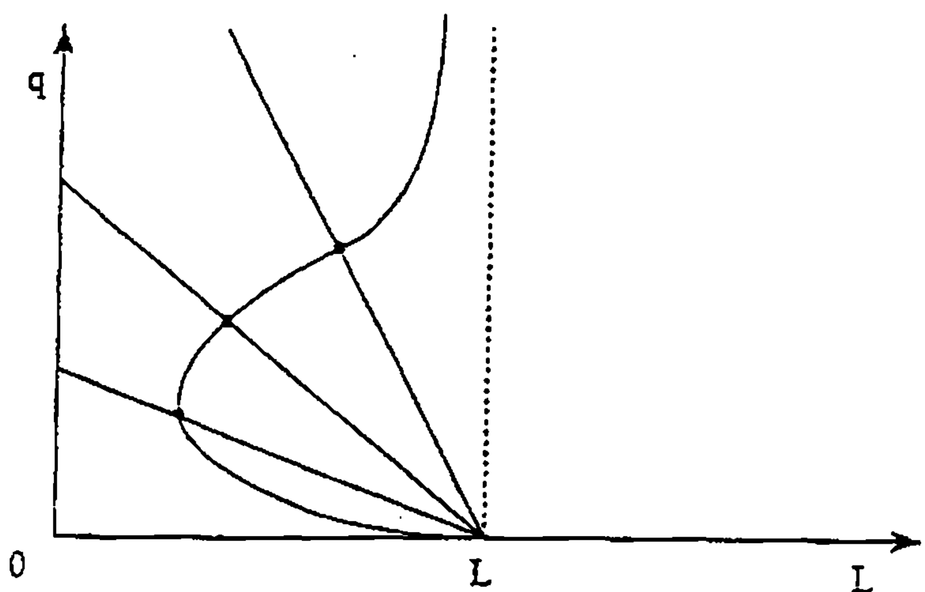


Figure 15.C.1(c.2)

Hence the total offer curve, that is, the sum of the two offer curves, also looks as above, and the equilibrium is described as an intersection of the total offer curves and the boundary of the production set, according to the profit maximization requirement of an equilibrium. Since the total offer curve can cross the boundary only once (otherwise, the labor demand function would be multi-valued), an equilibrium must be unique.

15.C.2 It is sufficient to calculate the Pareto optimal production levels, which is a solution to the maximization problem

$$u(z^{1/2}, 1 - z) = \ln z^{1/2} + \ln (1 - z).$$

By the first-order condition, $z = 1/3$. If we fix the output price to be one, then the equilibrium wage is $3^{1/2}/2$. The equilibrium profit is $1/(2\cdot 3^{1/2})$. The equilibrium consumption is $(1/3^{1/2}, 2/3)$.

15.D.1 (a) For any allocation $z = (z_1, z_2)$ in the Edgeworth box, we have $z_{21}/z_{11} > \bar{z}_2/\bar{z}_1 > z_{22}/z_{12}$ if and only if z lies above the diagonal; $z_{21}/z_{1i} <$ $\bar{z}_2/\bar{z}_1 < z_{22}/z_{12}$ if and only if z lies below the diagonal. Hence the assertion follows.

(b) For this question, first recall that the differentiability of the cost function of $c_j(\cdot)$ (or equivalently, the uniqueness of $a_j(w)$ at every w) is equivalent to saying that $f_j(\cdot)$ is strictly quasiconcave. This implies that the marginal rates of substitution changes strictly monotonically along the unit isoquant curve, and thus (together with the homogeneity of degree zero) that, for any z_j and z'_j, if $z_{2j}/z_{1j} \neq z'_{2j}/z'_{1j}$, then the marginal rates of substitution at z_j and z'_j are different.

Now suppose that a ray from the origin of firm 1 and the Pareto set of factor allocations intersect at $z = (z_1, z_2)$ (which is not the origin) and let $z' = (z'_1, z'_2)$ be another point on the ray. (The case in which a ray starts from the origin of firm 2 can be similarly proved.) It is sufficient to prove that z' is not Pareto optimal. By definition, $z_{21}/z_{11} = z'_{21}/z'_{11} \neq \bar{z}_2/\bar{z}_1$.

Thus $z_{22}/z_{12} \neq z'_{22}/z'_{12}$. The equality $z_{21}/z_{11} = z'_{21}/z'_{11}$ implies that the marginal rates of substitution of firm 1 at z_1 and z'_1 are the same. The inequality $z_{22}/z_{12} \neq z'_{22}/z'_{12}$ implies that the marginal rates of substitution of firm 2 at z_2 and z'_2 are different. By Pareto optimality, the marginal rate of substitution of firm 1 at z_1 and that of firm 2 at z_2 are the same. Therefore, the marginal rate of substitution of firm 1 at z'_1 and that of firm 2 at z'_2 are different. Hence z' is not Pareto optimal. Note that this result is equivalent to saying that the factor intensities at different Pareto optimal allocations are different.

Let's now show the (strict) monotonicity of the factor intensities and of the supporting relative factor prices along the Pareto set. It is sufficient to prove that of the former. By Exercise 15.B.7, there exist a bounded, closed (non-degenerated) interval $[\delta_0,\delta_1]$ and a continuous, one-to-one map $\varphi(\cdot)$ from $[\delta_0,\delta_1]$ into the Edgeworth box whose image coincides with the Pareto set of factor allocations. Define a map $\psi(\cdot)$ from $[\delta_0,\delta_1]$ into $\mathbb{R}$ by letting $\psi(\delta)$ be the factor intensity at $\delta \in [\delta_0,\delta_1]$, then $\psi(\cdot)$ is continuous. We want to prove that it is (strictly) monotone. In fact, if not, then the intermediate value theorem would imply that it is not one-to-one. That is, two different Pareto optimal allocations would have the same factor intensity. But this contradicts the result in the preceding paragraph. Hence $\psi(\cdot)$ is strictly monotone, implying the (strict) monotonicity of the factor intensity along the Pareto set.

15.D.2 Let $z = (z_1,z_2)$ and $z' = (z'_1,z'_2)$ be two feasible factor allocations. Let $\lambda \in [0,1]$. We want to prove that the consumption vector

$$\lambda(f_1(z_1),f_2(z_2)) + (1-\lambda)(f_1(z'_1),f_2(z'_2))$$
$$= (\lambda f_1(z_1) + (1-\lambda)f_1(z'_1),\ \lambda f_2(z_2) + (1-\lambda)f_2(z'_2))$$

is in the production possibility set. Clearly,

$$(\lambda z_1 + (1-\lambda)z_1') + (\lambda z_2 + (1-\lambda)z_2')) \le \bar{\omega}$$

and, by concavity,

$$f_1(\lambda z_1 + (1-\lambda)z_1') \ge \lambda f_1(z_1) + (1-\lambda)f_1(z_1'),$$

$$f_2(\lambda z_2 + (1-\lambda)z_2') \ge \lambda f_2(z_2) + (1-\lambda)f_2(z_2').$$

Hence the proof is completed.

15.D.3 We shall give two proofs. The first one uses Figure 15.D.6(a). The second one is more formal along the line of the Proof of Proposition 15.D.1. In both proofs, we consider the case in which the price of good 1 increases. The case in which the price of good 2 increases can be treated similarly.

For the first proof, suppose that the price of good 1 increases from p_1 to λp_1, where $\lambda > 1$. Let $w^* = (w_1^*, w_2^*)$ be the equilibrium factor price vector of (p_1, p_2) and $w^{**} = (w_1^{**}, w_2^{**})$ be that of $(\lambda p_1, p_2)$. We want to show that $w_1^{**} > \lambda w_1^*$. Of course, both w^* and λw^* are on the same ray. Since p_2 does not change, both w^* and w^{**} are on the same unit-cost curve $\{w: c_2(w) = p_2\}$ of firm 2, which is downward-sloping. By the equilibrium condition and the homogeneous of degree one of $c_1(\cdot)$, both w^{**} and λw^* are on the same unit-cost curve $\{w: c_1(w) = \lambda p_1\}$. The positions of w^*, λw^*, and w^{**} are depicted in the figure below:

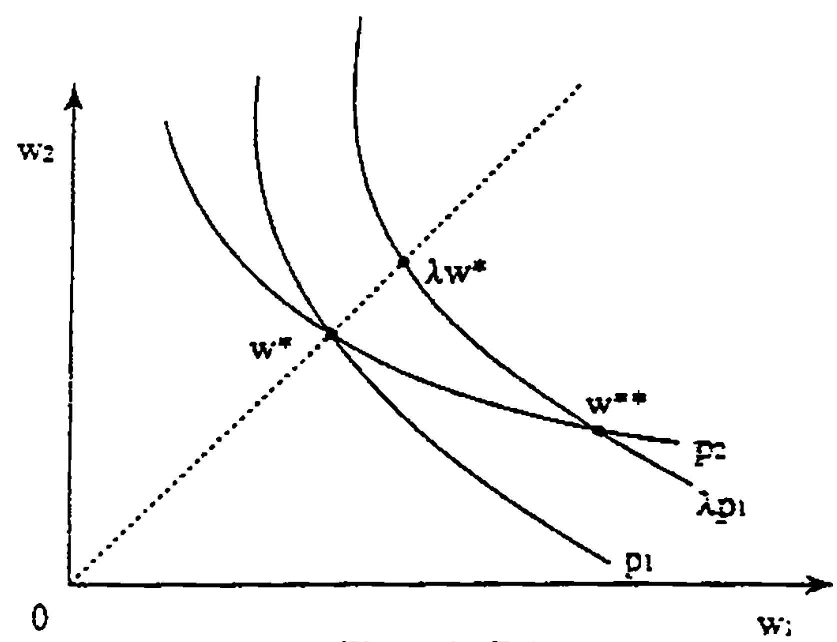


Figure 15.D.3

Hence we must have $w_1^{**} > \lambda w_1^*$.

For the second proof, note from the Proof of Proposition 15.D.1 that $dw_1 = (a_{22}(w^*)/|A|)dp_1$. Thus, by $p_1 = a_{11}(w^*)w_1^* + a_{21}(w_2^*)w_2^*$,

$$
\begin{aligned}
& dw_1/w_1^* - dp_1/p_1 \\
&= ((a_{22}(w^*)/|A|w_1^* - 1/p_1)dp_1 \\
&= \frac{a_{22}(w^*)}{w_1^*}\left(\frac{1}{|A|} - \frac{1}{a_{11}(w^*)a_{22}(w^*) + a_{21}(w^*)a_{22}(w^*)(w_2^*/w_1^*)}\right)dp_1 > 0.
\end{aligned}
$$

15.D.4 (a) The utility function of consumer i (i = 1,2) is denoted by $u_i(\cdot)$. The production function of firm j (j = 1,2) is denoted by $f_j(\cdot)$, which is assumed to be homogeneous of degree one. A price vector of the two consumption goods is denoted by $p = (p_1,p_2) \in \mathbb{R}^2_{++}$, a price vector of the two inputs by $w = (w_1,w_2) \in \mathbb{R}^2_{++}$, the consumption vector of consumer i by $x_i \in \mathbb{R}^2_+$, and the input demand of firm j by $z_j \in \mathbb{R}^2_+$. Write $x = (x_1,x_2)$ and $z = (z_1,z_2)$ for short. Then an equilibrium is defined as a vector (p^*,w^*,x^*,z^*) such that

1. (Utility Maximization) For each i, x_i^* solves the constraint maximization problem

$$\text{Max } u_i(x_i) \text{ s.t. } p^*\cdot x_i \le w_i^*.$$

2. (Profit Maximization) For each j, z_j^* solves the constraint maximization problem

$$\text{Max } p_j^* f_j(z_j) - w^*\cdot z_j \text{ s.t. } z_j \in \mathbb{R}^2_+.$$

3. (Market Clearing) $\sum_i x_i^* = (f_1(z_1^*), f_2(z_2^*))$ and $\sum_j z_j^* = (1,1)$.

Of course, Condition 2 can be replaced by the first-order condition $p_j^* = c_j(w_1^*,w_2^*)$ and $z_{1j}^*/z_{2j}^* = a_{1j}(w^*)/a_{2j}(w^*)$.

(b) Suppose now that we have two equilibria (p^*,w^*,x^*,z^*) and $(p^{**},w^{**},x^{**},z^{**})$. Assume without loss of generality that $p_1^* = p_1^{**} = 1$ and $f_1(z_1^*) \le f_1(z_1^{**})$ and $f_2(z_2^*) \ge f_2(z_2^{**})$. Note that $w_1^*/p_2^* = f_2(z_2^*)$, $w_2^* = f_1(z_1^*)$, $w_1^{**}/p_2^{**} = f_2(z_2^{**})$, $w_2^{**} = f_1(z_1^{**})$ by the utility maximization and the market clearing. According to Exercise 15.D.1(b), the input allocations z^* and z^{**} can be depicted as follows:

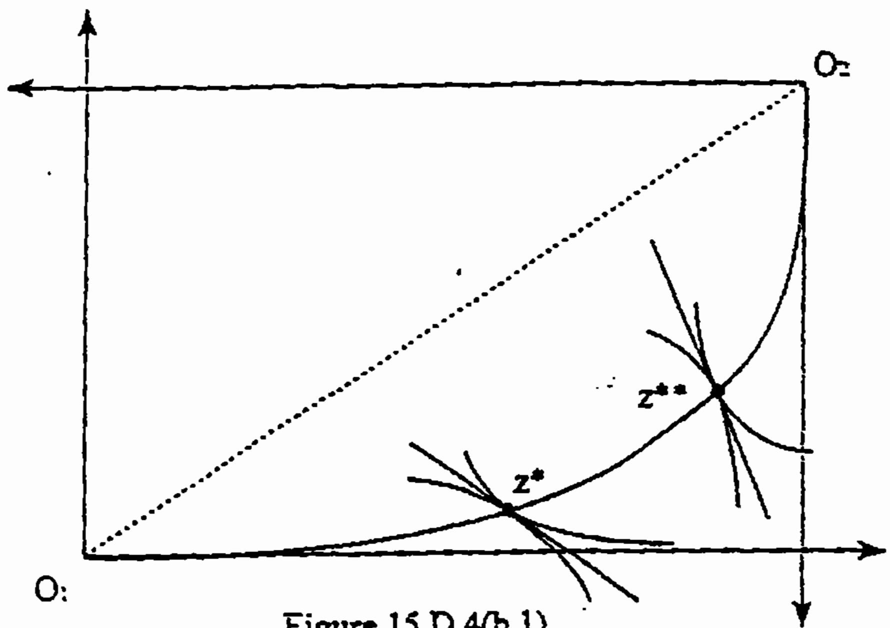


Figure 15.D.4(b.1)

Hence $w_2^*/w_1^* \geq w_2^{**}/w_1^{**}$. Thus, as we can see with the unit cost curves (and because of $p_1^* = p_1^{**} = 1$), we must have $w_2^* \geq w_2^{**}$, that is, the price of input 2 cannot increase even in the absolute value:

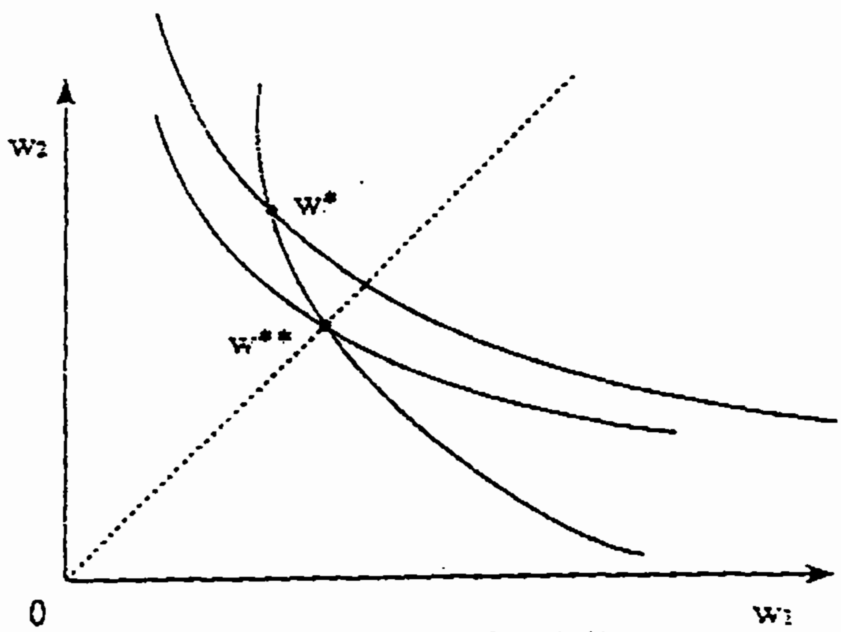


Figure 15.D.4(b.2)

Hence $f_1(z_1^*) \geq f_1(z_1^{**})$. Thus $f_1(z_1^*) = f_1(z_1^{**})$ and $f_2(z_2^*) = f_2(z_2^{**})$. Since the $f_j(\cdot)$ are strictly quasiconcave, this implies that $z^* = z^{**}$. Hence, by $p_1^* = p_1^{**} = 1$, $w^* = w^{**}$ and $p_2^* = p_2^{**}$.

(c) Suppose now that we have two equilibria (p^*,w^*,x^*,z^*) and $(p^{**},w^{**},x^{**},z^{**})$. Assume without loss of generality that $p_1^* = p_1^{**} = 1$. We only show that it is not inconsistent to have $f_1(z_1^*) < f_1(z_1^{**})$, $f_2(z_2^*) > f_2(z_2^{**})$, $w_1^* = f_1(z_1^*)$, $w_1^{**} = f_1(z_1^{**})$, $w_1^*/p_2^* = f_2(z_2^*)$, and $w_2^{**}/p_2^{**} = f_2(z_2^{**})$ at the same time. In fact, again, we know from the unit cost curves that $w_1^* < w_1^{**}$ and $w_2^* > w_2^{**}$. But, as we saw in the proof of Exercise 15.D.3, the profit maximization and the factor market clearing implies that $w_2^{**}/w_2^* < p_2^{**}/p_2^*$, that is, $w_2^{**}/p_2^{**} < w_2^*/p_2^*$. This last inequality is nothing but $f_2(z_2^*) > f_2(z_2^{**})$.

15.D.5 Denote the initial factor allocation by $z = (z_1,z_2)$ and the new factor allocation by $z' = (z_1',z_2')$, after the endowment of input 1 increases from $\bar{z}_1$ to $\bar{z}_1'$. Note on Figure 15.D.7 that z_j and z_j' are proportional for each $j = 1,2$, and that $z_1' >> z_1$. In particular, $z_{21}' > z_{21}$. Since the endowment of input is fixed at the level of $\bar{\omega}_2$, this implies that $z_{22}' < z_{22}$. By proportionality, $z_{12}' < z_{12}$. Thus, $\bar{z}_1' - z_{11}' < \bar{z}_1 - z_{11}$, that is, $z_{11}' - z_{11} > \bar{z}_1' - \bar{z}_1$. Hence, by dividing the left hand side by z_{11} and the right hand side by $\bar{z}_1$ (and because of $z_{11} < \bar{z}_1$), we obtain $z_{11}'/z_{11} > \bar{z}_1'/\bar{z}_1$. By the homogeneity of degree one and the proportionality, we have $f_1(z')/f_1(z) > \bar{z}_1'/\bar{z}_1$.

15.D.6 (a) Writing $w^* = (w_1^*,w_2^*)$, the equilibrium conditions for w^* and (q_1^*,q_2^*) are that

$$c_1(w^*) \geq p_1, \text{ and } c_1(w^*) = p_1 \text{ whenever } q_1^* > 0;$$

$$c_2(w^*) \geq p_2, \text{ and } c_2(w^*) = p_2 \text{ whenever } q_2^* > 0;$$

$$a_{11}(w^*)q_1^* + a_{12}(w^*)q_2^* = \bar{z}_1;$$

$$a_{21}(w^*)q_1^* + a_{22}(w^*)q_2^* = \bar{z}_2.$$

(b) Suppose first that an equilibrium $\hat{w}$ and (q_1^*,q_2^*) satisfies

$$c_1(\hat{w}) = p_1 \text{ and } q_1^* > 0; \; c_2(\hat{w}) = p_2 \text{ and } q_2^* > 0.$$

By the factor intensity condition, (the inverse of) the slope $a_{11}(\hat{w})/a_{21}(\hat{w})$

of the optimal input vector of firm 1 must be greater the slope $a_{12}(\hat{w})/a_{22}(\hat{w})$ of the optimal input vector of firm 2. By the market clearing condition and $q_1^* > 0$, $q_2^* > 0$, the slope $\bar{z}_1/\bar{z}_2$ of the endowment vector must be between these two slopes. This is equivalent to saying that $\bar{z}$ belongs to the diversification cone.

Suppose conversely that $\bar{z}$ belongs to the diversification cone. If $q_1^* = 0$, then the market clearing condition implies that $q_2^*(a_{12}(\hat{w}), a_{22}(\hat{w})) = \bar{z}$. Thus $a_{12}(\hat{w})/a_{22}(\hat{w}) = \bar{z}_1/\bar{z}_2$ and hence $\bar{z}$ does not belong to the diversification cone. Similarly, if $q_2^* = 0$, then z does not belong to the diversification cone either. We must thus have $(q_1^*, q_2^*) \gg 0$.

(c) We shall prove the assertion of this question by showing that if the unit-dollar isoquants intersect more than once, then the factor intensity condition is not satisfied. The next paragraph is devoted to a proof of the statement in the hint that if they intersect more than once, then there are two points (one in each isoquant) proportional to each other and such that the slopes of the isoquants at these points are identical. As the proof is technical (and perhaps unnecessarily long), it can be skipped. The slope of the unit-dollar isoquant of firm j at point (z_{1j}, z_{2j}) is denoted by $s_j(z_{1j})$. (Since, for each z_{1j}, there is only one z_{2j} such that (z_{1j}, z_{2j}) lies on the isoquant, we can suppress z_{2j} from the augment of the slope function $s_j(\cdot)$.)

Let $v \in \mathbb{R}^2_{++}$ and $v' \in \mathbb{R}^2_{++}$ be two different intersections of the two isoquants. Denote by C the region of the unit-dollar isoquant of firm 1 bounded by v and v' inclusive, then C is a compact set. If there are infinitely many intersections of the two isoquants on C, let $\{v^n\}$ be a sequence of different intersections in C. We can assume without loss of generality that it converges to a point $\bar{v} \in C$. Then $\bar{v}$ is also an intersection. Since $s_j(\bar{v}_1) = \lim \dfrac{v_2^n - \bar{v}_2}{v_1^n - \bar{v}_1}$, we have $s_1(\bar{v}_1) = s_2(\bar{v}_1)$. Hence the hint is verified for this case of infinitely many intersections on C. If there are only finitely many intersections on C, pick up two consecutive intersections. To simplify notation, let v and v' be as such and $v_1 < v'_1$. Since one of the two isoquants is above the other everywhere between v and v', we have $(s_1(v_1) - s_2(v_1))(s_1(v'_1) - s_2(v'_1)) \leq 0$. If we have either $s_1(v) = s_2(v)$ or $s_1(v') = s_2(v')$, then there is nothing to prove. So suppose not, then $(s_1(v_1) - s_2(v_1))(s_1(v'_1) - s_2(v'_1)) < 0$. Suppose also that the isoquant of firm 2 is above that of firm 1 everywhere between v and v'. (The other case can be proved similarly.) Then $s_1(v_1) < s_2(v_1) < 0$ and $s_2(v'_1) < s_1(v'_1) < 0$. For each $z_{11} \in [v_1,v'_1]$, let $\lambda(z_{11}) \geq 1$ be such that, if (z_{11},z_{21}) lies on the isoquant of firm 1, then $\lambda(z_{11})(z_{11},z_{21})$ lies on that of firm 2. Note that $\lambda(v_1) = \lambda(v'_1) = 1$ and $\lambda(z_{11})z_{11} \in [v_1,v'_1]$ for every $z_{11} \in [v_1,v'_1]$. Now define $g\colon [v_1,v'_1] \to \mathbb{R}$ by $g(z_{11}) = s_1(z_{11}) - s_2(\lambda(z_{11})z_{11})$, then $g(v_1) < 0$ and $g(v'_1) > 0$. By the continuity of $g(\cdot)$ (which is implied by the continuous differentiability of the $f_j(\cdot)$) and the intermediate value theorem, there must exist $z_{11} \in (v_1,v'_1)$ such that $g(z_{11}) = 0$. But this implies that $s_1(z_{11}) = s_2(\lambda(z_{11})z_{11})$, that is, if (z_{11},z_{21}) lies on the isoquant of firm 1, then $\lambda(z_{11})(z_{11},z_{21})$ lies on that of firm 2 and the slopes of the two isoquants are the same on those two points. The Hint is thus proved.

Now suppose that there are more than one intersections. By the hint, there are z^*_1 and $\lambda > 0$ such that z^*_1 lies on the unit-dollar isoquant of firm 1, λz^*_1 lies on the unit-dollar isoquant of firm 2, and the slopes at those points are the same. Hence if $w = (w_1,w_2) \in \mathbb{R}^2_{++}$ is such that w_2/w_1 is equal to the slope, then z^*_1 attains the minimum of $w \cdot z_1$ on $\{z_1\colon f_1(z_1) \geq 1/p_1\}$ and λz^*_1 is the minimum of $w \cdot z_2$ on $\{z_2\colon f_2(z_2) \geq 1/p_2\}$. By homogeneity of degree one, this implies that $p_1 z^*_1$ attains the minimum of $w \cdot z_1$ on $\{z_1\colon f_1(z_1) \geq 1\}$ and $p_2 \lambda z^*_1$ attains the minimum of $w \cdot z_2$ on $\{z_2\colon f_2(z_2) \geq 1\}$. But since $p_1 z^*_1$ and $p_2 \lambda z^*_1$ are proportional, the factor intensity condition is not satisfied.

As for the graphical construction of the diversification cone, if $\hat{w}$ is an equilibrium input price vector, then, for each j, $\hat{w}$ supports $\{z_j: f_j(z_j) \geq 1/p_j\}$ at $(1/p_j)(a_{1j}(\hat{w}),a_{2j}(\hat{w}))$. Moreover $\hat{w}\cdot\{(1/p_j)(a_{1j}(\hat{w}),a_{2j}(\hat{w}))\} = 1$. Hence we obtain the following figure.

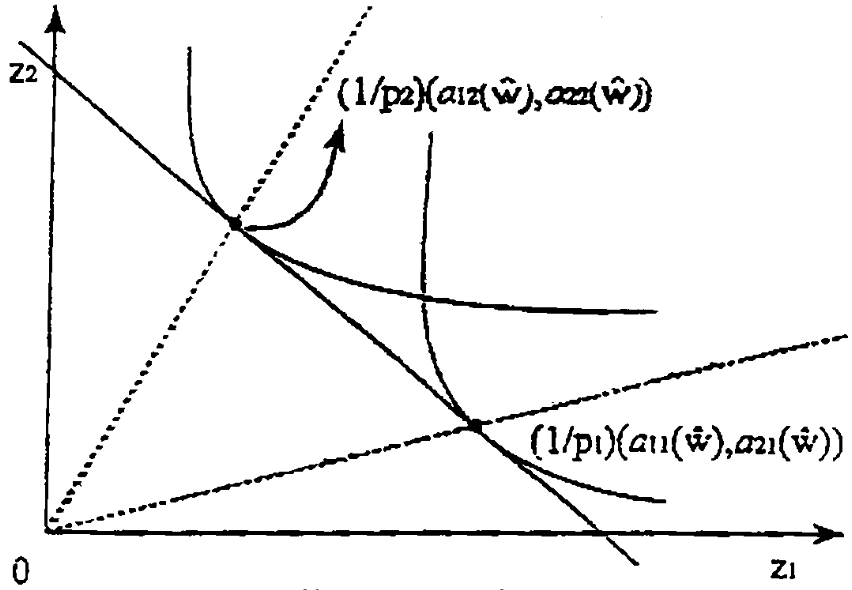


Figure 15 D.6(c)

(d) As we saw in the discussion preceding Proposition 15.D.2 (Rybcszynski Theorem), the factor intensity condition implies that there exists exactly one factor price vector $\hat{w} = (\hat{w}_1,\hat{w}_2)$ such that, for any total initial endowments, the factor price vector of any equilibrium involving positive production of both goods is equal to $\hat{w}$. By (b), the total initial endowment vector $\bar{z}$ gives rise to an (unique) equilibrium that involves positive production of both goods if and only if $\bar{z}$ belongs to the diversification cone of $\hat{w}$. If $\bar{z}$ lies below the cone, that is, $\bar{z}_1/\bar{z}_2 \geq a_{11}(\hat{w})/a_{21}(\hat{w})$, then the economy specializes in production of good 1 and the equilibrium factor price vector w^* is determined so that $a_1(w^*) = (1/f_1(\bar{z}))\bar{z}$ and $c_1(w^*) = p_1$. If, on the other hand, $\bar{z}$ lies above the cone, that is, $\bar{z}_1/\bar{z}_2 \leq a_{12}(\hat{w})/a_{22}(\hat{w})$, then the economy specializes in production of good 2 and the equilibrium factor price vector w^{**} is determined so that $a_2(w^{**}) = (1/f_2(\bar{z}))\bar{z}$ and $c_2(w^{**}) = p_2$. These are illustrated in the following picture.

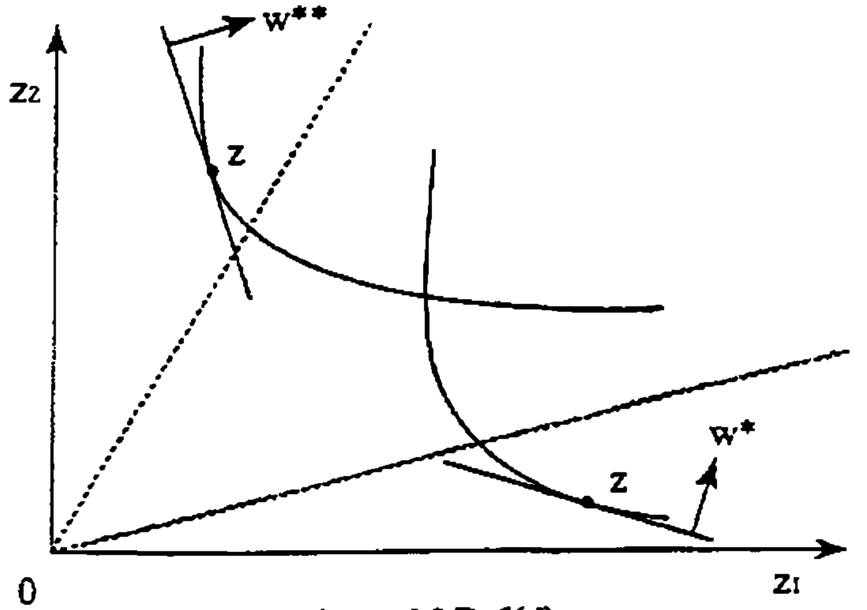


Figure 15.D.6(d)

15.D.7 It is straightforward either from (15.D.1) and (15.D.2) or from (15.D.5) that $z^*_{11} = 4\bar{z}_1/5$, $z^*_{21} = \bar{z}_2/5$, $z^*_{12} = \bar{z}_1/5$, $z^*_{22} = 4\bar{z}_2/5$ $w^*_1 = 5^{1/2}/2\bar{z}_1^{-1/2}$, and $w^*_2 = 5^{1/2}/2\bar{z}_2^{-1/2}$.

15.D.8 It is easy to check that the production of good 1 is relatively more intensive in factor 1 than in the production of good 2. We can thus apply the graphical apparatus obtained in Exercise 15.D.6 to answer this question. By some straightforward calculations, we can show that the unique equilibrium factor price vector $\hat{w} = (\hat{w}_1, \hat{w}_2)$ that involves positive production of both goods is equal to $(2^{2/3}/3,\ 2^{2/3}/3)$ and is independent of the total factor endowments; and that $a_{11}(\hat{w}) = 2^{1/3}$, $a_{21}(\hat{w}) = 2^{-2/3}$, $a_{12}(\hat{w}) = 2^{-2/3}$, $a_{22}(\hat{w}) = 2^{1/3}$. Since the unit-dollar isoquants are equal to the (standard) isoquants $\{z_j \in \mathbb{R}^2_+ : f_j(z_j) = 1\}$ by p = (1,1), these results yield the following figure:

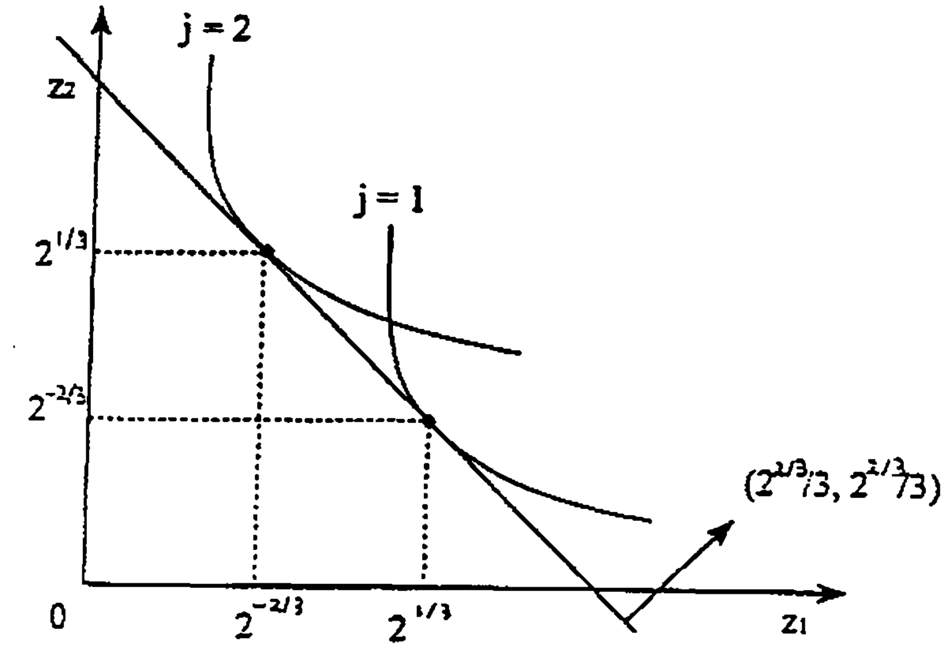


Figure 15.D.8

Hence the total factor endowments $\bar{z}$ gives rise to positive production of both goods if and only if $2^{-2/3}/2^{1/3} < \bar{z}_1/\bar{z}_2 < 2^{1/3}/2^{-2/3}$, that is, $1/2 < \bar{z}_1/\bar{z}_2 < 2$. The equilibrium production level $q^* = (q_1^*, q_2^*)$ must then satisfy $\bar{\omega} = A^T q^*$, where A is the 2×2 matrix as defined in the Proof of Proposition 15.D.1 and A^T is its transpose. Solving this, we obtain the equilibrium factor allocation:

$$z_{11}^* = q_1^* a_{11}(\hat{w}) = (2/3)(2\bar{z}_1 - \bar{z}_2),\ z_{21}^* = q_1^* a_{21}(\hat{w}) = (1/3)(2\bar{z}_1 - \bar{z}_2),$$
$$z_{12}^* = q_1^* a_{12}(\hat{w}) = (1/3)(2\bar{z}_2 - \bar{z}_1),\ z_{22}^* = q_2^* a_{22}(\hat{w}) = (2/3)(2\bar{z}_2 - \bar{z}_1).$$

If $\bar{z}$ lies below the diversification cone, that is, $\bar{z}_1/\bar{z}_2 \geq 2$, then the economy specializes in production of good 1. The equilibrium factor price vector w^* is determined so that $a_1(w^*) = (1/f_1(\bar{z}))\bar{z}$ and $c_1(w^*) = p_1$. Thus

$$w^* = \frac{1}{2^{-2/3}\bar{z}_1 + 2^{4/3}\bar{z}_2}(2\bar{z}_2, \bar{z}_1).$$

Symmetrically, if $\bar{z}_1/\bar{z}_2 \leq 1/2$, then the economy specializes in production of good 2. The equilibrium factor price vector w^* is given by

$$w^* = \frac{1}{2^{4/3}\bar{z}_1 + 2^{-2/3}\bar{z}_2}(\bar{z}_2, 2\bar{z}_1).$$

15.D.9 Let $(p^*, w_A^*, w_B^*, z_A^*, z_B^*, q_A^*, q_B^*, x_A^*, x_B^*)$ be an equilibrium of this two-country model, where $p^* \in \mathbb{R}^2_{++}$ is the international price vector of the two consumption goods, $w^*_A \in \mathbb{R}^2_{++}$ is the factor price vector in country A, $z^*_A \in \mathbb{R}^4_{++}$ is the factor allocation in country A, $q^*_A \in \mathbb{R}^2_{++}$ is the output levels of the two consumption goods in country A, and $x^*_A \in \mathbb{R}^2_{++}$ is the aggregate demand for the two consumption goods of country A, and similarly for country B. By the assumptions on the utility functions, x^*_A and x^*_B are proportional. By the market clearing condition, $x^*_A + x^*_B = q^*_A + q^*_B$ and this sum is proportional to x^*_A and x^*_B. By the budget constraints, $p^* \cdot (x^*_A - q^*_A) = p^* \cdot (x^*_B - q^*_B) = 0$. Hence, in order to verify this theorem, it is sufficient to prove that $q^*_{1A}/q^*_{2A} > q^*_{1B}/q^*_{2B}$.

We shall now prove this inequality. Let $\lambda = \bar{z}_{21}/\bar{z}_{22} > 0$ and consider an auxiliary country C, which is endowed with the total factor allocation $\lambda \bar{z}_2 \in \mathbb{R}^2_{++}$. It is easy to see that, faced with the international price vector p^* of the two consumption goods, the factor price vector w^*_B would clear the input markets in country C; the corresponding factor allocation would be equal to λz^*_B and the output levels to λq^*_B. Country A is endowed with the same amount of factor 2 as country C and a larger amount of factor 1 than country C. Hence (as neither of country A nor C specializes) the Rybcszynski Theorem is applicable to comparison between countries A and C, implying that $q^*_{1A} > \lambda q^*_{1B}$ and $q^*_{2A} < \lambda q^*_{2B}$. Thus $q^*_{1A}/q^*_{2A} > \lambda q^*_{1B}/\lambda q^*_{2B} = q^*_{1B}/q^*_{2B}$.

CHAPTER 16

16.C.1 By Proposition 3.C.1, there exists a continuous utility function u_i: $X_i \to \mathbb{R}$ that represents $\succsim_i$. Since X_i is nonempty, closed, and bounded, Theorem M.F.2 implies that the function $u_i(\cdot)$ has a maximizer, denoted by $x_i^* \in X_i$. Hence $u_i(x_i^*) \geq u_i(x_i)$, that is, $x_i^* \succsim_i x_i$ for every $x_i \in X_i$. Thus $\succsim_i$ is globally (and hence locally) satiated.

16.C.2 We shall prove by contradiction the property "If $x_i \succsim_i x_i^*$, then $p \cdot x_i \geq w_i$." Suppose that there is $x_i \in X_i$ such that $x_i \succsim_i x_i^*$ and $p \cdot x_i < w_i$. Then, any $x_i' \in X_i$ sufficiently close to x_i satisfies $p \cdot x_i' < w_i$ and, by the local nonsatiation, there must exist at least one such x_i' which moreover satisfies $x_i' \succ_i x_i$. By the transitivity, $x_i' \succ_i x_i^*$. But this contradicts the assumption that x_i^* is maximal for $\succsim_i$ in set $\{x_i \in X_i: p \cdot x_i \leq w_i\}$.

16.C.3 (a) For the first assertion, if we had two different two global satiation points x_i and x_i', then the convexity of X_i would imply that $(1/2)x_i + (1/2)x_i' \in X_i$ and the strict convexity of $\succsim_i$ would imply that $(1/2)x_i + (1/2)x_i' \succ_i x_i \sim_i x_i'$. This would contradict the hypothesis that x_i and x_i' are global satiation points. Hence every i can have at most one global satiation point.

For the second assertion, let $x_i \in X_i$ be different from the global satiation point. Then there exists $x_i' \in X_i$ such that $x_i' \succ_i x_i$. Then, for any $\alpha \in (0,1]$, $\alpha x_i' + (1 - \alpha)x_i \succ_i x_i$. Hence x_i is locally nonsatiated.

(b) Let (x^*,y^*,p) be a price equilibrium with transfers $(w_1,\ldots,w_I)$. We shall show that if an allocation (x,y) Pareto dominates (x^*,y^*), then $\sum_i p \cdot x_i > \sum_i w_i$, which, as we saw in the Proof of Proposition 16.C.1, implies that (x,y) is not feasible. In fact, if x_i^* is a global satiation point for $\succsim_i$, then it is unique by (a) and hence $x_i = x_i^*$, implying that $p\cdot x_i = p\cdot x_i^* = w_i$. If x_i^* is not a global satiation point, then $\succsim_i$ is locally nonsatiated at x_i^* by (a). Thus $p\cdot x_i \geq p\cdot x_i^* = w_i$. In addition, if $x_i \succ_i x_i^*$, then $p\cdot x_i > p\cdot x_i^* = w_i$. Since there is at least one such i, the summation over i yields $\sum_i p\cdot x_i > \sum_i w_i$.

16.C.4 For the first assertion, it is sufficient to prove that if x is not Pareto optimal relative to the $u_i(\cdot)$'s, then it is not Pareto optimal relative to the $U_i(\cdot)$'s either. So suppose that x' Pareto dominates x relative to the $u_i(\cdot)$'s, then $u_i(x_i') \geq u_i(x_i)$ for every i and $u_i(x_i') > u_i(x_i)$ for some i. Thus $U_i(x') > U_i(x)$ for every i. Thus x' Pareto dominates x relative to the $U_i(\cdot)$'s too.

This result does not mean that a community of altruists can use competitive markets to attain Pareto optima. In fact, suppose that, in an Edgeworth box pure exchange economy, consumer 1 owns most of the total endowments of the economy and his utility function $U_1(\cdot)$ puts much more emphasis on the pleasure function $u_2(\cdot)$ of consumer 2. (Consumer 2, on the other hand, does not so much care about $u_1(\cdot)$.) Then a Walrasian equilibrium allocation, at which consumer 1 consumes most of the total endowments, is not Pareto optimal relative to $U_1(\cdot)$ and $U_2(\cdot)$. Both of them prefer a transfer from consumer 1 to 2 to attain a Pareto optimal allocation (relative to the $u_i(\cdot)$'s) at which consumer 2 consumes most of the total endowments.

The above argument does not depend on the concavity.

16.D.1 Suppose that if $x_i \succ_i x_i^*$, then $p\cdot x_i \geq p\cdot x_i^*$. Let $x_i \succsim_i x_i^*$. By the local nonsatiation, there exists a sequence $\{x_i^n\}_n$ in X_i converging to x_i such that $x_i^n \succ_i x_i$ for every n. By the transitivity, $x_i^n \succ_i x_i^*$. Hence $p\cdot x_i^n \geq p\cdot x_i^*$. Thus, by $n \to \infty$, we obtain $p\cdot x_i \geq p\cdot x_i^*$. Hence x_i^* is expenditure minimizing for p in $\{x_i \in X_i: x_i \succsim_i x_i^*\}$. The implication of the other direction of obvious.

16.D.2 Here is an example of the failure of Proposition 16.D.1, in which the preference is not locally nonsatiated and a "thick" indifference curve of the preference is drawn. Any point in the intersection of the interior of the production set and the thick indifference curve is Pareto optimal, but can be supported neither as a price equilibrium with transfers nor as a price quasiequilibrium with transfers.

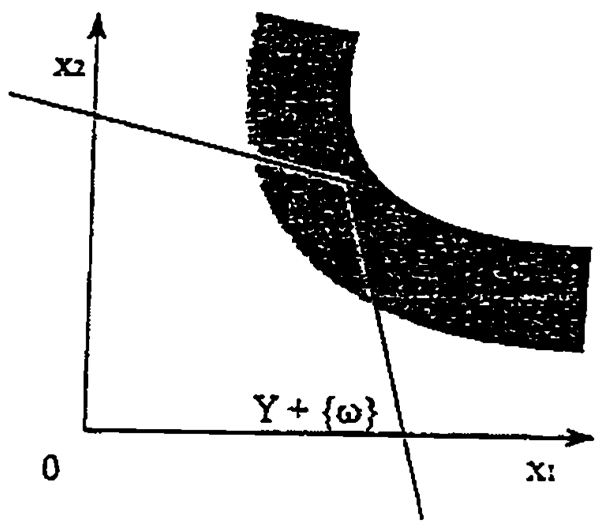


Figure 16.D.2

16.D.3 Let (x^*,y^*,p) be a quasiequilibrium with transfers $(w_1,\ldots,w_I)$. Suppose that $y = (y_1,\ldots,y_J) \in Y_1 \times \ldots \times Y_J$ satisfies $\sum_j y_j + \bar{\omega} >> 0$, then $p\cdot(\sum_j y_j + \bar{\omega}) > 0$. Since $p\cdot(\sum_j y_j^*) \geq p\cdot(\sum_j y_j)$ by profit maximization, we have $p\cdot(\sum_j y_j^* + \bar{\omega}) = \sum_i w_i > 0$. Hence $w_i > 0$ for some i. By Proposition 16.D.2 (and the continuity) this consumer must then be maximizing her preferences in $\{x_i \in \mathbb{R}^L_+: p\cdot x_i \leq w_i\}$. By the strong monotonicity, this implies that $p >> 0$. As we saw in the small-letter paragraph on page 556, this implies that (x^*,y^*,p) be an equilibrium with transfers.

16.D.4 Denote the identical preference by $\succsim$ and let $V = \{x \in \mathbb{R}^2_+ : x \succsim \omega\}$. Write $e = (1,1) \in \mathbb{R}^2_{++}$.

We shall first show that if $\omega - \lambda e \notin \text{Co } V$ for any $\lambda > 0$, then the symmetric allocation in which every consumer gets ω is a Walrasian equilibrium. In fact, if $\omega - \lambda e \notin \text{Co } V$ for any $\lambda > 0$, then, by the separating hyperplane theorem (Theorem M.G.3), there exists $p_\lambda \in \mathbb{R}^2$ with $p_\lambda \neq 0$ such that $p_\lambda \cdot (\omega - \lambda e) \leq p_\lambda \cdot x$ for every $x \in \text{Co } V$. By the strong monotonicity, $p_\lambda \in \mathbb{R}^2_{++}$ and we can thus assume that $p_\lambda \cdot e = 1$. Moreover, we can assume that there exists $p \in \mathbb{R}^2_+$ such that $p \cdot e = 1$ and $p_\lambda \to p \in \mathbb{R}^2_+$. Then $p \cdot \omega \leq p \cdot x$ for every $x \in \text{Co } V$. Hence ω is cost minimizing at p. Since $\omega \in \mathbb{R}^2_{++}$, $p \cdot \omega > 0$ and hence ω is utility maximizing. Thus the symmetric allocation in which every consumer gets ω is a Walrasian equilibrium.

Therefore, if the symmetric allocation in which every consumer gets ω is not a Walrasian equilibrium, then there exists $\lambda > 0$ such that $\omega - \lambda e \in \text{Co } V$. Then there exist a positive integer N, $\mu_n > 0$, and $x_n \in V$ $(n = 1,\ldots,N)$ such that $\sum_n \mu_n = 1$ and $\omega - \lambda e = \sum_n \mu_n x_n$. Write $x'_n = x_n + \lambda e$, then $\sum_n \mu_n x'_n = \omega$ and $x'_n \succ \omega$ by the transitivity and the strong monotonicity. Now, if r is large enough, then there exist nonnegative integers r_n such that $\sum_n r_n = r$ and $(\mu_n r/r_n) x'_n \succ \omega$, by the continuity of $\succ$. (That is, for each n, the fraction r_n/r approximates μ_n.) The allocation in which, for each n, r_n consumers (out of r) get the consumption bundle $(\mu_n r/r_n) x'_n$ is feasible, because $\sum_n r_n (\mu_n r/r_n) x'_n = r(\sum_n \mu_n x'_n) = r\omega$. Hence the symmetric allocation in which every consumer gets ω is not Pareto optimal.

16.E.1 (a) Suppose that consumer 1 cares only about commodity 1 and consumer 2 cares only about consumer 2. Then

$$U = \{u \in \mathbb{R}^2 : u_1 \leq u_1(\bar{\omega}_1) \text{ and } u_2 \leq u_2(\bar{\omega}_2)\},$$

$$U' = \{u \in U : u_1 \geq u_1(0) \text{ and } u_2 \geq u_2(0)\}.$$

(Recall that the disposal technologies are assumed to be available). These sets are illustrated in the following figure.

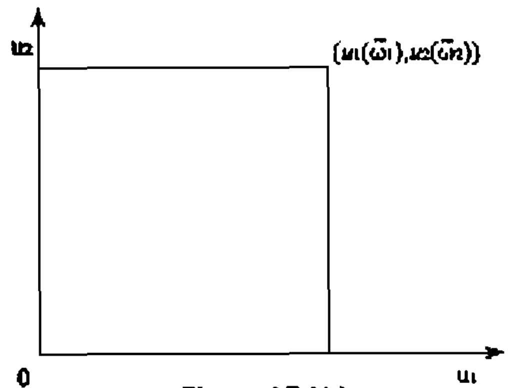


Figure 16.E.1(a)

It is easy to see that the unique Pareto optimum is $(u_1(\bar{\omega}_1), u_2(\bar{\omega}_2)) \in U'$.

(b) We can assume without loss of generality that $u_i(0) = 0$ for every i. The strong monotonicity implies that $u_i(x_i) \geq 0$ for every $x_i \in R^L_+$, and hence $U' \subset U \cap R^L_+$.

We shall prove that if $u \in U'$ and u is not a Pareto optimum, then u belongs to the interior of U. So let u be as such. Let x be a feasible consumption allocation such that $u_i = u_i(x_i)$ for every i. Since u is not a Pareto optimum, there exists $u' \in U'$ such that $u' \geq u$ and $u' \neq u$. Let x' be a feasible consumption allocation such that $u'_i = u_i(x'_i)$ for every i. Pick up i for which $u'_i > u_i$. Since $u \in U'$, $u \in R^L_+$ and hence $u_i(x'_i) > 0$. Thus $x'_i \neq 0$. Pick up ℓ for which $x'_{\ell i} > 0$. Denote by $e_\ell \in R^L_+$ the vector whose ℓth component is one and the other components are zero. By the continuity, $u_i(x'_i - \varepsilon e_\ell) > u_i$ for any sufficiently small $\varepsilon > 0$. By the strong monotonicity, $u_k(x'_k + \frac{\varepsilon}{I - 1} e_\ell) > u'_k \geq u_k$ for any $k \neq i$. The consumption allocation in which consumer i gets $x'_i - \varepsilon e_\ell$ and any other $k \neq i$ gets $x'_k + \frac{\varepsilon}{I - 1} e_\ell$ is simply a redistribution of x', and hence feasible. Since all consumers' utility levels are increased by this redistribution from u, u belongs to the interior of U.

Therefore, if u belongs to U' and the boundary of U, then u is a Pareto optimum.

(c) Let (x^*,p) be a quasiequilibrium with transfers $(w_1,\ldots,w_I)$. Let x be an allocation such that $u_i(x_i) > u_i(x_i^*)$ for every i. It is sufficient to show that x is not feasible.

By condition (ii) of a quasiequilibrium (Definition 16.D.1), $p\cdot x_i \geq w_i$ for every i. Since $p \geq 0$ (by the profit maximization of the free disposal technologies) and $\bar{\omega} \in \mathbb{R}^L_{++}$, $p\cdot\bar{\omega} = \sum_i w_i > 0$. Hence $w_i > 0$ for some i. Since the consumption sets are $\mathbb{R}^L_+$, Proposition 16.D.2 implies that $p\cdot x_i > w_i$ for such i. Summing over i, we obtain $p\cdot\sum_i x_i > p\cdot\bar{\omega}$. Hence x is not feasible.

16.E.2 Let (x,y) and (x',y') be two feasible allocations and $\lambda \in [0,1]$. We want to show that there exists another feasible allocation (x",y") such that $u_i(x_i'') \geq \lambda u_i(x_i) + (1-\lambda)u_i(x_i')$ for each i. In fact, let

$$x'' = \lambda x + (1-\lambda)x' \text{ and } y'' = \lambda y + (1-\lambda)y',$$

where, of course, the summation is taken component-wise. By the convexity of the production sets and the consumption sets, (x",y") is a feasible allocation. Moreover, the concavity of the utility functions implies that $u_i(x_i'') \geq \lambda u_i(x_i) + (1-\lambda)u_i(x_i')$ for each i.

16.F.1 Suppose first that an allocation (x,y) is a solution to problem (16.F.1). If it is not Pareto optimal, then there exists another feasible allocation (x',y') that Pareto dominates (x,y). Because of the feasibility, (x',y') satisfies constraints (2) and (3). The Pareto dominance implies that it also satisfies constraint (1). If $u_1(x_1') > u_1(x_1)$, then this contradicts the hypothesis that (x,y) is a solution. If $u_1(x_1') = u_1(x_1)$, then, by Pareto dominance, there is a consumer $i \neq 1$ such that $u_i(x_i') > u_i(x_i) \geq \bar{u}_i$. By the strong monotonicity, a small transfer from consumer i to consumer 1, such as $\varepsilon x_i'$, increases the utility level of consumer 1 and, by the continuity of $u_i(\cdot)$, still satisfies the constraints. But, again, this yields a contradiction to the hypothesis that (x,y) is a solution. Hence (x,y) must be Pareto optimal.

Suppose conversely that (x,y) is Pareto optimal. If we put $u_i = u_i(x_i)$ for each $i \neq 1$, then (x,y) is a solution to problem (16.F.1) by the definition of Pareto optimality.

16.F.2 Denote by $(\nu_{11},\ldots,\nu_{L1},\ldots,\nu_{LI},\ldots,\nu_{LI}) \geq 0$ the multipliers associated with the nonnegativity constraints $x_{\ell i} \geq 0$. By augmenting the domains of all the $u_i(\cdot)$ and $F_j(\cdot)$ in the obvious way to include all the variables of (x,y), and then by applying Theorem M.K.2, we obtain the first-order necessary conditions:

$$\partial u_1/\partial x_{\ell 1} = \mu_\ell - \nu_{\ell 1} \text{ for every } \ell;$$

$$0 = -\delta_i(\partial u_i/\partial x_{\ell i}) + \mu_\ell - \nu_{\ell i} \text{ for every } \ell \text{ and } i \neq 1;$$

$$0 = -\mu_\ell + \gamma_j(\partial F_j/\partial y_{\ell j}) \text{ for every } \ell \text{ and } j.$$

Rearranging these terms and by eliminating the $\nu_{\ell i}$ by the complementary slackness condition $\nu_{\ell i}x_{\ell i} = 0$, we obtain (16.F.2) and (16.F.3).

16.F.3 Since $\delta_1 = 1$, $\nabla u_i(x_i) >> 0$, and $\nabla F_j(y_j) >> 0$, we have $\delta_i > 0$ and $\gamma_j > 0$ for all i and j. Thus, by (16.F.2),

$$\frac{\partial u_i/\partial x_{\ell i}}{\partial u_i/\partial x_{\ell' i}} = \frac{\mu_\ell/\delta_i}{\mu_{\ell'}/\delta_i} = \mu_\ell/\mu_{\ell'} \text{ and } \frac{\partial u_{i'}/\partial x_{\ell i'}}{\partial u_{i'}/\partial x_{\ell' i'}} = \frac{\mu_\ell/\delta_{i'}}{\mu_{\ell'}/\delta_{i'}} = \mu_\ell/\mu_{\ell'}.$$

Hence (16.F.4) is established. By (16.F.3),

$$\frac{\partial F_j/\partial y_{\ell j}}{\partial F_j/\partial y_{\ell' j}} = \frac{\mu_\ell/\gamma_j}{\mu_{\ell'}/\gamma_j} = \mu_\ell/\mu_{\ell'} \text{ and } \frac{\partial F_{j'}/\partial y_{\ell j'}}{\partial F_{j'}/\partial y_{\ell' j'}} = \frac{\mu_\ell/\gamma_{j'}}{\mu_{\ell'}/\gamma_{j'}} = \mu_\ell/\mu_{\ell'}.$$

Hence (16.F.5) is established. Condition (16.F.6) is implied by the above equations as well.

16.F.4 For (16.F.7), denote by $(\delta_2,\dots,\delta_I) \geq 0$ and $(\mu_1,\dots,\mu_L) \geq 0$ the multipliers associated with constraints (1) and (2), respectively, and define $\delta_1 = 1$. Then the first order conditions of this problem is the same as (16.F.2), which implies (16.F.4), as we saw in the answer to Exercise 16.F.3.

For (16.F.8), denote by $(\sigma_2,\dots,\sigma_L) \geq 0$ and $(\gamma_1,\dots,\gamma_J) \geq 0$ the multipliers associated with constraints (1) and (2), respectively . Then the first-order conditions of this problem are

$$1 = \gamma_j(\partial F_j/\partial y_{1j}) \text{ and } 0 = -\sigma_\ell + \gamma_j(\partial F_j/\partial y_{\ell j}).$$

Hence, for every j and $\ell \neq 1$, we have $\dfrac{\partial F_j/\partial y_{\ell j}}{\partial F_j/\partial y_{1j}} = \dfrac{\partial F_{j'}/\partial y_{\ell j'}}{\partial F_{j'}/\partial y_{1j'}}$, which implies (16.F.5).

Finally, for (16.F.9), denote by $\rho \geq 0$ the multiplier associated with the constraint. Then the first-order conditions of the problem are

$$\partial u/\partial \bar{x}_1 = \rho \text{ and } \partial u/\partial \bar{x}_\ell = -\rho(\partial f/\partial \bar{y}_\ell).$$

Applying the Envelop Theorem (Theorem M.L.1) to (16.F.7) and (16.F.8), we obtain $\partial u/\partial \bar{x}_\ell = \mu_\ell$ for every $\ell \geq 1$ and $\partial f/\partial \bar{y}_\ell = -\sigma_\ell$ for every $\ell \geq 2$. Thus the first-order conditions for (16.F.9) can be written as $\mu_\ell/\mu_1 = \sigma_\ell$. But here $\mu_\ell/\mu_1 = \dfrac{\partial u_i/\partial x_{\ell i}}{\partial u_i/\partial x_{1i}}$ for every i and and $\sigma_\ell = \dfrac{\partial F_j/\partial y_{\ell j}}{\partial F_j/\partial y_{1j}}$ for every j. This implies (16.F.6).

16.G.1 The Pareto optimality and the strict monotonicity of the $u_i(\cdot)$ imply condition (iii). The Pareto optimality also implies conditions (16.F.2) and (16.F.3) and, since $\delta_1 = 1$, $\nabla u_i(x_i^*) \gg 0$, and $\nabla F_j(y_j^*) \gg 0$, we have $\mu_\ell > 0$, $\delta_i > 0$, and $\gamma_j > 0$ for all ℓ, i, and j. Hence, by putting $p = (\mu_1,\dots,\mu_L)$, we get condition (i), and the first-order condition $\nabla u_i(x_i^*) = (1/\delta_i)p$. For each i, define $w_i = p\cdot x_i^*$, then $\sum_i w_i = p\cdot\bar{\omega} + \sum_j p\cdot y_j$ by condition (iii), and the quasiconcavity of $u_i(\cdot)$ implies that condition (ii) is also satisfied.

16.G.2 Let the input be good 1 and the output be good 2. By applying the implicit function theorem (Theorem M.E.1) to the equation $F_j(y_j) = 0$, we know that the marginal productivity of the input is equal to $\frac{\partial F_j/\partial y_{1j}}{\partial F_j/\partial y_{2j}}$. By (16.G.1), this is equal to p_1/p_2. Hence $p_1 = \frac{\partial F_j/\partial y_{1j}}{\partial F_j/\partial y_{2j}} p_2$.

16.G.3 Let goods 1 and 2 be the inputs and good 3 be the output. The production set Y of the firm is given by $Y = \{y \in \mathbb{R}^3: y_1 \le 0, y_2 \le 0, \text{ and } F(y) \le 0\}$, where $F: (-\mathbb{R}) \times (-\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(y) = y_3 - 2^{1-1/\rho}((-y_1)^{\rho} + (-y_2)^{\rho})^{1/\rho}$$

and $\rho > 1$ is close to 1. The domain of $F(\cdot)$ is not the whole $\mathbb{R}^3$, but this will turn out to be irrelevant. Note that this production sets exhibits constant returns to scale, but, for any $q > 0$, the set $\{y \in Y: y_3 \ge q\}$ is not convex, as illustrated below.

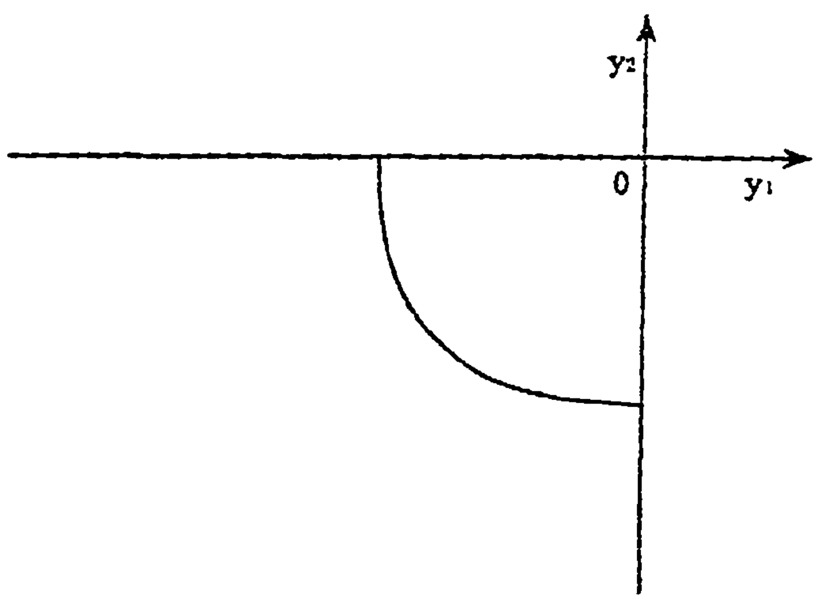


Figure 16.G.3.1

Note also that, as $\rho \to 0$, the boundary $\{y \in Y: y_3 = q\}$ becomes flatter and flatter so as to coincide with $\{(y_1,y_2,q) \in \mathbb{R}^3: y_1 \le 0, y_2 \le 0, y_1 + y_2 = -q\}$.

The utility function of the consumer is $u(x) = x_1^{1/3}x_2^{1/3}x_3^{1/3}$ and the initial endowment is $\bar{\omega} = (3,3,0)$. It is easy to show via the first-order conditions that, for every $\rho > 1$, the production and consumption plans and the price vector

$$x^* = (2,2,2),\ y^* = (-1, -1, 2), \text{ and } p = (1,1,1)$$

constitute a marginal cost pricing equilibrium. Moreover, if ρ is sufficiently close to 1, then it is the unique one, which can be shown as follows: Let (x,y) be any marginal cost pricing equilibrium allocation. Note first that $x \in \mathbb{R}^3_{++}$. Hence $y_1 > -3$, $y_2 > -3$, and $y_3 > 0$. If, furthermore, $y_1 < 0$ and $y_2 < 0$, then, by the equality of the consumer's marginal rate of substitution at x and the firm's marginal rate of technological substitution at y, we must have $x_1 = x_2$ and $y_1 = y_2$. Although one of y_1 and y_2 may well be zero, by letting $\rho \to 1$, the marginal rate of technological substitution can be made as much close to 1 as needed and thus we can have neither $y_1 = 0$ nor $y_2 = 0$. This can be illustrated by an Edgeworth box of input allocations:

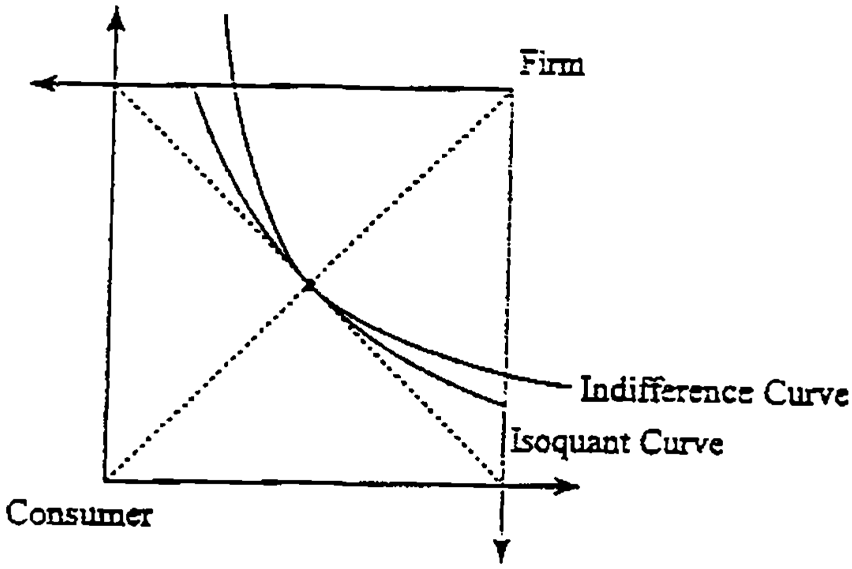


Figure 16.G.3.2

Hence at any marginal cost pricing equilibrium, we must have $y_1 < 0$ and $y_2 < 0$. Then, as we discussed, $y_1 = y_2$ and $x_1 = x_2$. Then, given this equality, the equality of the consumer's marginal rate of substitution between the output and the inputs and the marginal productivity implies that $y_3 = x_3 = 2$

and $y_1 = y_2 = -1$. Hence the above marginal cost pricing equilibrium is the only one.

It is now clear from the first figure that, given $p = (1,1,1)$, in order to produce $y_3 = 2$, it is not cost minimizing (and in fact, cost maximizing) to use the input combination $(y_1, y_2) = (-1, -1)$. The cost can be decreased by moving towards concentrating on only one of the two inputs.

16.G.4 Suppose that the set of all feasible allocations is nonempty and compact. Then there exists an allocation (x,y) at which the utility of the single consumer is maximized. Hence it is Pareto optimal and satisfies (16.F.2) and (16.F.3). Hence, by Proposition 16.G.1 (as proved in Exercise 16.G.1), there exists a price vector p such that (x,y,p) is a marginal cost pricing equilibrium.

16.G.5 (a) Let (L_e, T_e) and (L_f, T_f) be an optimal allocation of the endowments and let $e^* = (\min\{L_e, T_e\})^2$ and $f^* = (L_f T_f)^{1/2}$. Then $e^* > 0$ and $f^* > 0$. Hence $L_e = T_e$ and $L_e + L_f = T_e + T_f = 1$. Hence $e^* = L_e^2$ and $f^* = 1 - L_e$. Thus $u(e^*, f^*) = L_e(1 - L_e)^{1/2}$. By the first-order conditions, $L_e = 2/3$. Hence $T_e = 2/3$, $e^* = 4/9$, and $f^* = 1/3$.

(b) Let (p_L, p_T, p_e, p_f) be a supporting price vector of the Pareto optimal allocation identified in (a). As a normalization, put $p_L = 1$. The zero profit condition and the cost minimization condition of the food production implies that $p_T = 1$ and $p_f = 2$. Under $(p_L, p_T) = (1,1)$, the marginal cost of education at the input levels (L,T) with $L = T$ is equal to $1/L$. Thus the marginal cost at the optimum is $3/2$ and hence $p_e = 3/2$. Thus the deficit of the education sector is $(2/3 + 2/3) - (3/2)(4/9) = 2/3$, which is to be subtracted from the consumer's budget as a lump-sum tax. Hence his net wealth is $4/3$ and his utility maximization problem is to maximize $e^{1/2} f^{1/2}$

subject to $(3/2)e + 2f \leq 4/3$. The solution to this problem is in fact $e^* = 4/9$ and $f^* = 1/3$.

(c) As calculated in the answer to (b), the lump sum tax is 2/3.

(d) Let $e \in [0,1]$ be a planned level of education and suppose that it is produced by the input combination $L_e = T_e = e^{1/2}$. (It can be shown that this equality always holds at all Pareto optimal allocations.) Let $p_L = 1$, then, as we saw in the answer to (b), the zero profit condition and the cost minimization condition of the food production implies that $p_T = 1$ and $p_f = 2$. Hence, if the government follows the marginal cost pricing rule, then it sets $p_e = 1/e^{1/2}$ and the deficit is $2e^{1/2} - e/e^{1/2} = e^{1/2}$. This is the transfer from the landowner when the planned level of education is e.

(e) As we saw in the answer to (d), at any marginal cost pricing equilibrium with a planned level e of education, the equilibrium price vector must be (a scalar multiple of) $(p_L,p_T,p_e,p_f) = (1,1,1/e^{1/2},2)$. Faced with this price vector, the demand of the laborowner is $(e^{1/2}/2,1/4)$ and the demand of the landowner is $\left(\frac{e^{1/2}(1 - e^{1/2})}{2}, \frac{1 - e^{1/2}}{4}\right)$. Hence the aggregate demand for food is $\frac{2 - e^{1/2}}{4}$. The inputs used for education are $(L_e,T_e) = (e^{1/2},e^{1/2})$ and thus the food supply is $1 - e^{1/2}$. Hence we must have $e = 4/9$. This is the planned level of education compatible with a marginal cost pricing equilibrium.

We shall now prove that the marginal cost pricing equilibrium is Pareto optimal by showing that the production possibility frontier is below the Scitovsky contour (page 120), as illustrated in the figure below.

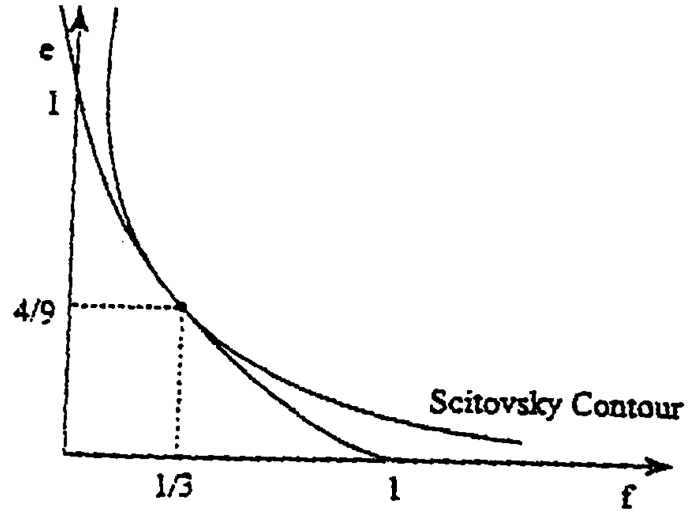


Figure 16.G.5(e.1)

Note first that the production possibility frontier is given by the equation $f = 1 - e^{1/2}$, or equivalently, $e = (1 - f)^2$. On the other hand, since the aggregate demand at the marginal cost pricing equilibrium is $(f^*,e^*) = (1/3,4/9)$ and both the laborowner and the landowner have the identical, homothetic utility function $u(e,f) = e^{1/2}f^{1/2}$, the Scitovsky contour is given by $e^{1/2}f^{1/2} = (4/9)^{1/2}(1/3)^{1/2}$, or equivalently, $e = 4/27f$. It is sufficient to show that the Scitovsky contour touches on the production possibility frontier only at $(f^*,e^*) = (1/3,4/9)$ and the former is above the latter everywhere else. For this, it suffices to prove that $(1 - f)^2 < 4/27f$ for every $f \in [0,1]$ with $f \neq 1/3$. But this follows from $(1 - f)^2 f < 4/27$, as depicted in the figure below.

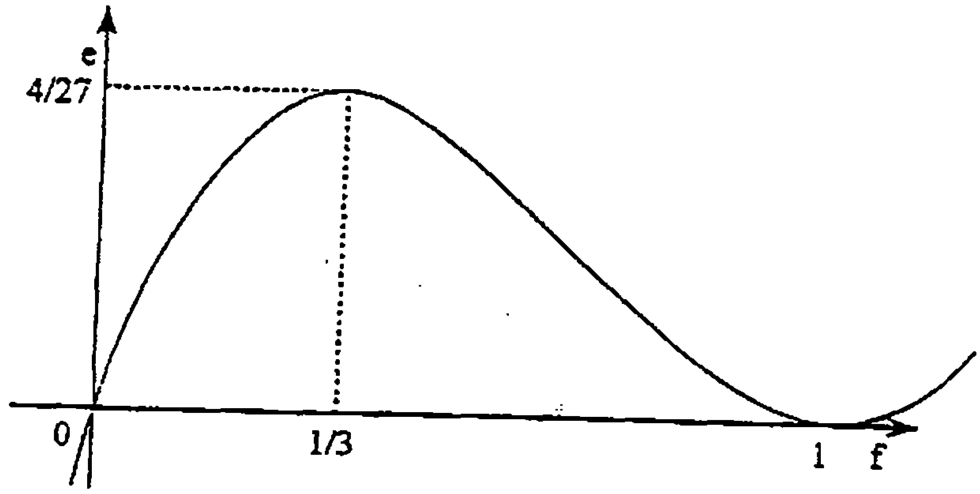


Figure 16.G.5(e.2)

16.AA.1 Let $((x^n,y^n))_n$ be a sequence in A converging in $\mathbb{R}^{LI} \times \mathbb{R}^{LJ}$ to (x,y). By definition, then, $x_i^n \to x_i$ and $y_j^n \to y_j$. Since every X_i and every Y_j is closed, and $x_i^n \in X_i$ and $y_j^n \in Y_j$ for every n , we have $x_i \in X_i$ and $y_j \in Y_j$. Since $\sum_i x_i^n = \bar{\omega} + \sum_j y_j^n$, we have $\sum_i x_i = \bar{\omega} + \sum_j y_j$. Hence $(x,y) \in A$.

16.AA.2 We showed in Exercise 16.AA.1 that the set A is closed. It is thus sufficient to prove that it is bounded. First observe that if $((x^n,y^n))_n$ be an arbitrary sequence in A such that the sequence (y^n) is bounded, then the whole sequence $((x^n,y^n))_n$ is also bounded. To show this, let $s > 0$ satisfy $|y_\ell^n| < s$ for every ℓ and n. Then, by $\sum_i x_{\ell i}^n = y_\ell^n + \bar{\omega}_\ell$, we have $-r < x_{\ell i}^n < (I-1)r + s + \bar{\omega}_\ell$ for every ℓ and i, where $r > 0$ provides a lower bound for the X_i. Thus the sequence $(x_{\ell i}^n)_n$ is bounded, and hence so is the whole $((x^n,y^n))_n$.

Now suppose that A is not bounded, then there exists an unbounded sequence $((x^n,y^n))_n$ in A. According to the observation of the previous paragraph, by taking a subsequence if necessary, we can assume that $\|\sum_i x_i^n - \bar{\omega}\| = \|y^n\| \to \infty$ and the sequence $((1/\|y^n\|)y^n)_n$ converges to $y \in \mathbb{R}^L$. Since Y is convex and $0 \in Y$, $(1/\|y^n\|)y^n = (1 - 1/\|y^n\|)0 + (1/\|y^n\|)y^n \in Y$. Since Y is closed, $y \in Y$.

By the equality $\sum_i x_i^n = y^n + \bar{\omega}$, we also have $\frac{1}{\|\sum_i x_i^n - \bar{\omega}\|}(\sum_i x_i^n - \bar{\omega}) \to y$.

But here, we have $\sum_i x_{i\ell}^n - \bar{\omega}_\ell > - rI - \bar{\omega}_\ell$ and hence

$$\frac{1}{\|\sum_i x_i^n - \bar{\omega}\|}(\sum_i x_{\ell i}^n - \bar{\omega}_\ell) > \frac{- rI - \bar{\omega}_\ell}{\|\sum_i x_i^n - \bar{\omega}\|}.$$

Since the right-hand side converges to zero as $n \to \infty$, we obtain $y \in \mathbb{R}_+^L$. But since $y \in Y$ and $\|y\| = 1$, this is a contradiction.

Here are four examples to show that each of the four assumptions is indispensable to guarantee the boundedness. Note that each example violates only one of the four conditions.

The first example shows that assumption (i) is indispensable:

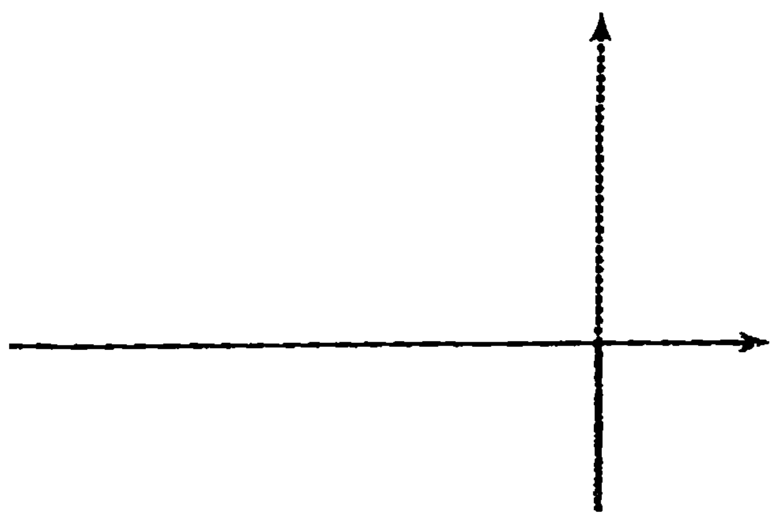


Figure 16.AA.2.1

The second example shows that (ii) is indispensable:

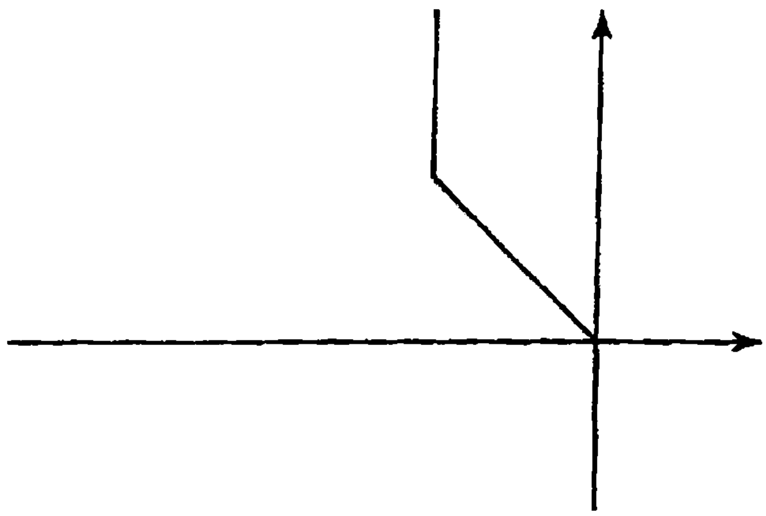


Figure 16.AA.2.2

The third examples shows that (iii) is indispensable.

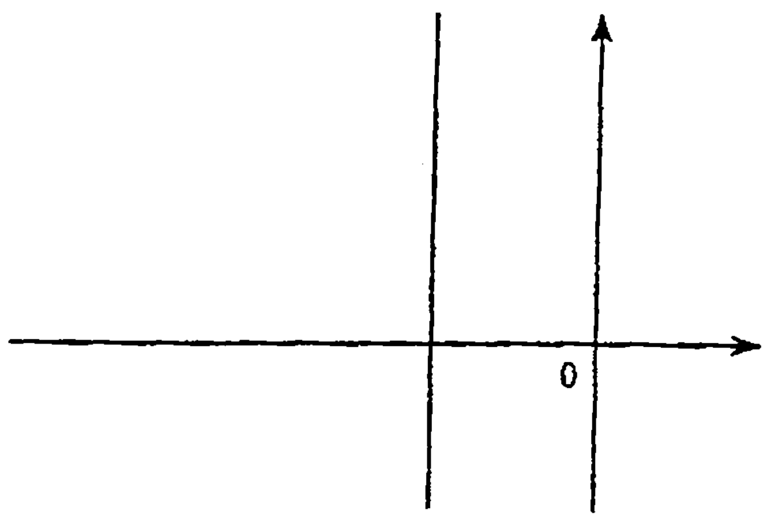


Figure 16.AA.2.3

The last example shows that (iv) is indispensable.

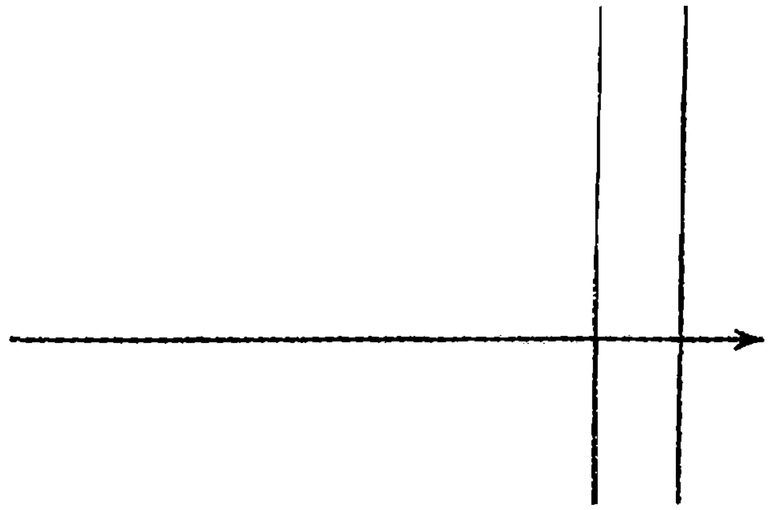


Figure 16.AA.2.4

16.AA.3 We shall first establish the following lemma

Suppose that, for each $j = 1,2$, $(v_j^n)_n$ is a sequence in $\mathbb{R}^L$ and v_j is a vector of $\mathbb{R}^L$ such that $\|v_j^n\| \to \infty$, $(1/\|v_j^n\|)v_j^n \to v_j$, and $v_1 + v_2 \neq 0$. Then $\|v_1^n + v_2^n\| \to \infty$ as $n \to \infty$.

To prove this, note first that, since $(v_1 + v_2)\cdot v_j > 0$ for each j, there exist a positive integer N_0 such that $((1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n)\cdot v_j^n > 0$ for every $n > N_0$. Then, for any $s > 0$, there exists a positive integer $N > N_0$ such that, for every $n > N$,

$$\|(1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n\| > (1/2)\|v_1 + v_2\|;$$

$$\|v_j^n\| > 2s/\|v_1 + v_2\| \text{ for each j.}$$

Now pick any $n > N$. Assume for a moment that $\|v_1^n\| \leq \|v_2^n\|$. Then

$$\|v_1^n + v_2^n\| = \|v_1^n\|\cdot\|(1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n + (1/\|v_1^n\| - 1/\|v_2^n\|)v_2^n\|.$$

But here

$$\|(1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n + (1/\|v_1^n\| - 1/\|v_2^n\|)v_2^n\|^2$$
$$= \|(1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n\|^2$$
$$+ 2(1/\|v_1^n\| - 1/\|v_2^n\|)((1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n)\cdot v_2^n$$
$$+ (1/\|v_1^n\| - 1/\|v_2^n\|)^2\|v_2^n\|^2$$

$\geq \|(1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n\|^2.$

Hence

$$\|v_1^n + v_2^n\| \geq \|v_1^n\| \cdot \|(1/\|v_1^n\|)v_1^n + (1/\|v_2^n\|)v_2^n\| > s.$$

We can establish this inequality in the same way for the case of $\|v_1^n\| \geq \|v_2^n\|$. Thus $\|v_1^n + v_2^n\| \to \infty$ as $n \to \infty$.

We shall now prove the assertion of this exercise by contradiction. Suppose that the assertion is false. Then there exist sequences $\{y_1^n\}_n$ in Y_1 and $\{y_2^n\}_n$ in Y_2 and vectors y_1 and y_2 in $\mathbb{R}^L$ such that

$\|y_1^n\| \to \infty$ as $n \to \infty$;

$(1/\|y_1^n\|)y_1^n \to y_1$ and $(1/\|y_2^n\|)y_2^n \to y_2$;

$y_1^n + y_2^n + \bar{\omega} \geq 0.$

By Exercise 16.AA.2, the sequence $\{y_1^n + y_2^n\}_n$ in $Y = Y_1 + Y_2$ must be bounded. Thus we also have $\|y_2^n\| \to \infty$ as $n \to \infty$, and, by the lemma in the preceding paragraph, we must have $y_1 + y_2 = 0$. Since $\|y_1\| = \|y_2\| = 1$ and the irreversibility assumption holds for Y, in order to derive a contradiction, it remains to show that $y_1 \in Y$ and $y_2 \in Y$. Since Y is convex and $0 \in Y$, we have $(1/\|y_1^n + y_2^1\|)(y_1^n + y_2^1) \in Y$ for every n. Moreover, by taking a subsequence if necessary, we can assume that the sequence $\{(1/\|y_1^n + y_2^1\|)(y_1^n + y_2^1)\}$ in Y is convergent. Our second application of the lemma (let $v_1^n = y_1^n$ and $v_2^n = -(y_1^n + y_2^1)$) yields $(1/\|y_1^n + y_2^1\|)(y_1^n + y_2^1) \to y_1$. Since Y is closed, $y_1 \in Y$. We can similarly show that $y_2 \in Y$.

CHAPTER 17

17.B.1 Suppose first that $y_1^* \le 0$, $p \cdot y_1^* = 0$, and $p \ge 0$. Since $Y_1 = -\mathbb{R}_+^L$, $y_1^* \in Y_1$. By $p \ge 0$, $p \cdot y_1 \le 0$ for every $y_1 \in Y_1$ and hence $p \cdot y_1^* \ge p \cdot y_1$ for every $y_1 \in Y_1$.

Suppose conversely that $y_1^* \in Y_1$ and $p \cdot y_1^* \ge p \cdot y_1$ for every $y_1 \in Y_1$. Since $Y_1 = -\mathbb{R}_+^L$, $y_1^* \le 0$. If $p_\ell < 0$ for some ℓ, then we could make $p \cdot y_1$ as big as needed by taking $y_{\ell 1}$ very small. But this is a contradiction to $p \cdot y_1^* \ge p \cdot y_1$ for every $y_1 \in Y_1$. Thus $p \ge 0$. Hence $p \cdot y_1 \le 0$ for all $y_1 \in Y_1$. Finally, since $0 \in Y_i$, $p \cdot y_1^* = 0$.

17.B.2. We shall prove property (v) by contradiction. Suppose that there exists a sequence $(p^n)_n$ such that $p^n >> 0$, $p^n \to p$, $p \ne 0$, $p_\ell = 0$, and the sequence $(\mathrm{Max}\{z_1(p^n),\ldots,z_L(p^n)\})_n$ in $\mathbb{R}$ does not diverge to infinity. Suppose that consumer i satisfies $p \cdot \omega_i > 0$. We first observe that, by the strong monotonicity, the demand of consumer i at p is not well defined, that is, there is no $x_i^* \in \mathbb{R}_+^L$ such that $p \cdot x_i^* = p \cdot \omega_i$ and $x_i^* \succsim_i x_i$ for every $x_i \in \mathbb{R}_+^L$ with $p \cdot x_i \le p \cdot \omega_i$. This is because, by simply adding a positive amount of good ℓ, whose price is zero, consumer i can get better off.

Since the consumption sets $\mathbb{R}_+^L$ are bounded below and the sequence $(\mathrm{Max}\{z_1(p^n),\ldots,z_L(p^n)\})_n$ in $\mathbb{R}$ does not diverge to infinity, the sequence $(\mathrm{Max}\{z_{1i}(p^n),\ldots,z_{Li}(p^n)\})_n$ in $\mathbb{R}$, which consists of the excess demands of consumer i along $(p^n)_n$, does not diverge to infinity either. Hence, by taking a subsequence if necessary, we can assume that the sequence $(z_i(p^n))_n$ is convergent. Denote its limit by z_i^* and define $x_i^* = z_i^* + \omega_i$, then $x_i^* \in \mathbb{R}_+^L$ and $p \cdot x_i^* = p \cdot \omega_i$. We shall now derive $x_i^* \succsim_i x_i$ for every $x_i \in \mathbb{R}_+^L$ with $p \cdot x_i \le p \cdot \omega_i$, which is a contradiction to the observation in the first paragraph. In fact, let $x_i \in \mathbb{R}^L_+$ and $p \cdot x_i \le p \cdot \omega_i$. For each n, let $\lambda^n = p^n \cdot \omega_i / p \cdot \omega_i \ge 0$; this is well defined because $p \cdot \omega_i > 0$. Then $\lambda^n x_i \in \mathbb{R}^L_+$ and $p^n \cdot (\lambda^n x_i) \le p^n \cdot \omega_i$. Thus $z_i(p^n) + \omega_i \succsim_i \lambda^n x_i$ by the definition of the excess demand function. Since $z_i(p^n) \to z_i^*$ and $\lambda^n \to 1$, $z_i(p^n) + \omega_i \to x_i^*$ and $\lambda^n x_i \to x_i$. By the continuity of $\succsim_i$, we obtain $x_i^* \succsim_i x_i$.

17.B.3 (a) Let $\{p^n\}_n$ be a sequence such that $p^n \gg 0$, $p^n \to p$ and $p_\ell > 0$. By property (v) of Proposition 17.B.2, it is sufficient to prove that the sequence $\{z_\ell(p^n)\}_n$ of aggregate excess demands for commodity ℓ is bounded. For this, it is sufficient to show that, for each i, the sequence $\{x_{\ell i}(p^n, p^n \cdot \omega_i)\}_n$ is bounded. In fact, by $p_\ell > 0$, $p \cdot \omega_i / p_\ell \ge 0$ is well defined and $p^n \cdot \omega_i / p^n_\ell \to p \cdot \omega_i / p_\ell$ as $n \to \infty$. Hence $p^n \cdot \omega_i / p^n_\ell < p \cdot \omega_i / p_\ell + 1$ for any sufficiently large n. Since $0 \le x_{\ell i}(p^n, p^n \cdot \omega_i) \le p^n \cdot \omega_i / p^n_\ell$, this completes the proof. [Step 4 of the proof of Proposition 17.C.1 contains a similar proof of this fact.]

(b) We consider the case of $I = 1$, $u_1(x_1) = x_{11} + x_{21}^{1/2} + x_{31}^{1/2}$, and $\omega_1 = (1,2,1)$. For each n, define $p^n = (1/n, 1/2n, 1)$. Then

$$x_1(p^n, p^n \cdot \omega_1) = (n + 3/2 - 1/4n,\ 1,\ 1/4n^2)$$

and hence

$$z(p^n) = (n + 1/2 - 1/4n,\ -1,\ 1/4n^2 - 1).$$

Hence, although $p^n_2 \to 0$, commodity 2 remains in net supply.

17.B.4 We shall first prove that the profit functions $\pi_j(\cdot)$ and the supply functions $y_j(\cdot)$ are well defined. Let $r > 0$ provide an upper bound for the Y_j (that is, $y_{\ell j} < r$ for every $y_j \in Y_j$ and ℓ). Let $p \gg 0$ and $\bar{y}_j \in Y_j$, and consider the subset $\{y_j \in Y_j: p\cdot y_j \geq p\cdot \bar{y}_j\}$ of Y_j. Then this subset is bounded because it is included in

$$\{y_j \in Y_j: p\cdot \bar{y}_j/p_\ell - (L-1)\textstyle\sum_{k\neq\ell}(p_k/p_\ell)r \leq y_{\ell j} \leq r \text{ for every } \ell\}.$$

Thus it is compact. Hence there is a $y_j^* \in Y_j$ such that $p\cdot y_j^* \geq p\cdot \bar{y}_j$ and $p\cdot y_j^* \geq p\cdot y_j$ for all $y_j \in Y_j$ with $p\cdot y_j \geq p\cdot \bar{y}_j$. Then $p\cdot y_j^* \geq p\cdot y_j$ for all $y_j \in Y_j$ and hence y_j^* is a profit-maximizing production vector. Its uniqueness is shown in Proposition 5.C.1. Hence both $\pi_j(\cdot)$ and $y_j(\cdot)$ are well defined functions.

The homogeneity of the $\pi_j(\cdot)$ and the $y_j(\cdot)$ is also shown in Proposition 5.C.1.

Let $p^* \gg 0$ and let C be a compact subset of $\{p \in \mathbb{R}^L: p \gg 0\}$ such that $p^* \in \mathrm{Int}_{\mathbb{R}^L} C$. Let $\bar{y}_j \in Y_j$ and consider the subset $\{y_j \in Y_j: p\cdot y_j \geq p\cdot \bar{y}_j$ for some $p \in C\}$ of Y_j. We now show that this is bounded. In fact, let $b \in \mathbb{R}$ satisfy $p\cdot \bar{y}_j \geq b$ for all $p \in C$, and $c \in \mathbb{R}$ satisfy $b/p_\ell \geq c$ for all $p \in C$. Let $d > 0$ satisfy $p_k/p_\ell \geq d$ for all $p \in C$. Then, for every $y_j \in Y_j$ and $p \in C$, if $p\cdot y_j \geq p\cdot \bar{y}_j$, then $p\cdot y_j \geq b$ and hence

$$y_{\ell j} \geq b/p_\ell - \textstyle\sum_{k\neq\ell}(p_k/p_\ell)y_{kj} \geq c - \textstyle\sum_{k\neq\ell}(p_k/p_\ell)r \geq c - (L-1)dr.$$

Therefore, for every $p \in C$,

$$\{y_j \in Y_j: p\cdot y_j \geq p\cdot \bar{y}_j\} \subset \{y_j \in Y_j: c - (L-1)dr \leq y_{\ell j} \leq r \text{ for every } \ell\}.$$

Given this boundedness, we can now prove the continuity of the $y_j(\cdot)$ and the $\pi_j(\cdot)$. The subset

$$Y_j(C) = \{y_j \in Y_j: c - (L-1)rd \leq y_{\ell j} \leq r \text{ for every } \ell\},$$

is compact. Let $\hat{y}_j: C \to Y_j(C)$ be the correspondence that associates p with the maximizers of $p\cdot y_j$ over $Y_j(C)$ and $\hat{\pi}_j: C \to \mathbb{R}$ be the corresponding profit function. By the theorem of the maximum (Theorem M.K.6), $\hat{y}_j(\cdot)$ is an upper hemicontinuous correspondence and $\hat{\pi}_j(\cdot)$ is a continuous function. But since $Y_j(C) \supset \{y_j \in Y_j: p\cdot y_j \geq p\cdot \bar{y}_j\}$ for every $p \in C$, $\hat{y}_j(p) = y_j(p)$ and $\hat{\pi}_j(p) =$

$\pi_j(p)$ for every $p \in C$. Thus both $y_j(\cdot)$ and $\pi_j(\cdot)$ are continuous functions at p^*. Since the choice $p^* \gg 0$ has been arbitrarily chosen, this shows that both $y_j(\cdot)$ and $\pi_j(\cdot)$ are continuous functions everywhere on $\{p \in \mathbb{R}^L: p \gg 0\}$.

Properties (i) and (ii) of Proposition 17.B.2 for the production inclusive aggregate excess demand function are immediate consequences of the homogeneity and the continuity we established in the preceding paragraphs.

For property (iii), simply note that

$$\begin{aligned}&\textstyle\sum_i(p\cdot\omega_i + \sum_j\theta_{ij}\pi_j(p)) - \sum_i p\cdot\omega_i - \sum_j p\cdot y_j(p)\\ &= \textstyle\sum_i p\cdot\omega_i + \sum_j\pi_j(p) - \sum_i p\cdot\omega_i - \sum_j\pi_j(p) = 0.\end{aligned}$$

As for (iv), let $r > 0$ provide an upper bound for the Y_j and a lower bound for the X_i, then, for every ℓ and p, $\tilde{z}_\ell(p) > - rI - \sum_i\omega_{\ell i} - rJ$.

Finally, we shall establish (v). Let $(p^n)_n$ be a sequence such that $p^n \gg 0$, $p^n \to p$, $p \neq 0$, and $p_\ell = 0$ for some ℓ. Since there exists a $\bar{y} \in Y$ such that $\sum_i\omega_i + \bar{y} \in \mathbb{R}^L_{++}$, we have $\sum_i p\cdot\omega_i + \sum_j\pi_j(p) \geq p\cdot(\sum_i\omega_i + \bar{y}) > 0$. (Although we do not have $p \gg 0$, $\pi_j(p)$ can still be defined as the supremum of the profits $p\cdot y_j$ for $y_j \in Y_j$, because Y_j is bounded above.) Since

$$\textstyle\sum_i p\cdot\omega_i + \sum_j\pi_j(p) = \sum_i(p\cdot\omega_i + \sum_j\theta_{ij}\pi_j(p)),$$

there is a consumer i such that $p\cdot\omega_i + \sum_j\theta_{ij}\pi_j(p) > 0$. For such i, we can show that $\mathrm{Max}\{z_{1i}(p^n),\ldots,z_{Li}(p^n)\} \to \infty$, as in Exercise 17.B.2. Since

$$\tilde{z}_\ell(p^n) > z_{\ell i}(p^n) - r(I - 1) - \textstyle\sum_i\omega_{\ell i} - rJ$$

for every ℓ, we also have $\mathrm{Max}\{\tilde{z}_1(p^n),\ldots,\tilde{z}_L(p^n)\} \to \infty$.

17.B.5 By the Shephard's Lemma (Proposition 5.C.2(vi)), for each j and p, $\nabla c_j(p)$ represents the input demand of firm j at p for unit output. Define a function a_j: $\{p \in \mathbb{R}^L: p \gg 0\} \to \mathbb{R}^L$ by

$$a_{\ell j}(\hat{p}) = \begin{cases} 1 & \text{if good } \ell \text{ is the output of firm } j; \\ -\nabla_\ell c_j(p) & \text{if good } \ell \text{ belongs to the domain of } c_j(\cdot); \\ 0 & \text{otherwise.} \end{cases}$$

Then a pair (p,α) formed by a price vector $p >> 0$ and a vector $\alpha \in R^J_+$ of activity levels constitute an equilibrium if and only if they solve

$$z(p) - \sum_j \alpha_j a_j(p) = 0$$

and

$$p \cdot a_j(p) \le 0,\ \alpha_j(p \cdot a_j(p)) = 0 \text{ for all } j.$$

17.B.6 Let (v^*,p) be an equilibrium in the sense of this exercise. For each i, let $y_i^* \in Y$ solve the maximization problem of consumer i together with v_i^*. Define $y^* = \sum_i y_i^*$, then $y^* \in Y$ because Y is a convex cone. For each i, define $x_i^* = v_i^* + y_i^* \in X_i$. Write $x^* = (x_1^*,\ldots,x_I^*)$. We shall show that (x^*,y^*,p) is a Walrasian equilibrium.

First of all, since $\sum_i v_i^* = \sum_i \omega_i$, we have $\sum_i x_i^* = \sum_i \omega_i + y^*$.

Denote $e = (1,\ldots,1) \in R^L$. Note that, by the monotonicity of preferences, $p \cdot e > 0$; otherwise, the consumers can always get better off simply by adding e to v_i^* (without changing the production plans y_i^*), whose price is nonpositive.

We shall prove that $p \cdot y_i^* \ge 0$ for every i. Define $v_i = -(p \cdot y_i^* / p \cdot e)e + y_i^*$, then $p \cdot v_i = 0$. Thus $v_i^* + v_i$ satisfies the budget constraint. Hence if this vector is chosen and no production is carried out (that is, $y_j = 0$), then the resulting consumption is

$$v_i^* + v_i = (v_i^* + y_i^*) - (p \cdot y_i^*/p \cdot e)e = x_i^* - (p \cdot y_i^*/p \cdot e)e.$$

Since (v_i^*,y_i^*) is maximal for $\succsim_i$, we must have $x_i^* \succsim_i x_i^* - (p \cdot y_i^*/p \cdot e)e$. By the monotonicity, $p \cdot y_i^* \ge 0$. Thus $p \cdot y^* \ge 0$.

We shall next prove that $p \cdot y \le 0$ for every $y \in Y$. Let $y \in Y$. Define $v =$

$(p\cdot y/p\cdot e)e - y$, then $p\cdot v = 0$. Thus, $v_i^* + v$ satisfies the budget constraint. Also, we have $y_i^* + y \in Y$ because Y is a convex cone. Hence if the choice $v_i^* + v$ and the production plan $y_i^* + y$ are combined, then the resulting consumption is

$$(v_i^* + v) + (y_i^* + y) = x_i^* + (p\cdot y/p\cdot e)e.$$

Since (v_i^*, y_i^*) is maximal for $\succsim_i$, we must have $x_i^* \succsim_i x_i^* + (p\cdot y/p\cdot e)e$. By the monotonicity, $p\cdot y \le 0$.

The profit maximization condition of a Walrasian equilibrium is therefore established. To establish the utility maximization condition, note that, since $p\cdot y \le 0$ for every $y \in Y$,

$$\{v_i + y_i \in \mathbb{R}^L: p\cdot v_i \le p\cdot\omega_i \text{ and } y_i \in Y\} = \{x_i \in \mathbb{R}^L: p\cdot x_i \le p\cdot\omega_i\}.$$

Hence $p\cdot x_i^* \le p\cdot\omega_i$ and x_i^* is maximal for $\succsim_i$ in $\{x_i \in \mathbb{R}^L: p\cdot x_i \le p\cdot\omega_i\}$.

A possible interpretation of this implication is that if a single convex, constant returns to scale (in fact, additivity suffices) production set Y is accessible to every consumer, then the profit maximization is a consequence of their utility maximization. This result thus gives a justification for the assumption of profit maximization (to the extent that utility maximization is justified).

17.C.1 Let $p \in \Delta$, $q' \in f(p)$, $q'' \in f(p)$, and $\lambda \in [0,1]$. We consider two cases, $p \in$ Interior Δ and $p \in$ Boundary Δ, separately.

Let $p \in$ Interior Δ, then, for every $q \in \Delta$,

$$z(p)\cdot((1-\lambda)q' + \lambda q'') = (1-\lambda)z(p)\cdot q' + \lambda z(p)\cdot q''$$

$$\ge (1-\lambda)z(p)\cdot q + \lambda z(p)\cdot q = z(p)\cdot q.$$

Hence $(1-\lambda)q' + \lambda q'' \in f(p)$.

Let $p \in$ Boundary Δ. If $p_\ell > 0$, then $q'_\ell = q''_\ell = 0$ and hence $(1-\lambda)q'_\ell +$

$\lambda q''_\ell = 0$. Hence $(1 - \lambda)q' + \lambda q'' \in f(p)$.

17.C.2 For each positive integer n, we let $\Delta^n = \{p \in \Delta: p_\ell \geq 1/n\}$.

Step 1: For each n, there exists $r^n > 0$ such that $z(p) \subset [-r^n, r^n]^L$ for every $p \in \Delta^n$. This is an immediate consequence of property (i) and the compactness of Δ^n.

Step 2: Construction of the fixed-point correspondences. For each n, define $f^n: \Delta^n \times [-r^n, r^n]^L \to \Delta^n \times [-r^n, r^n]^L$ by

$$f^n(p,z) = \{q \in \Delta^n: z \cdot q \geq z \cdot q' \text{ for every } q' \in \Delta^n\} \times z(p).$$

Step 3: The fixed-point correspondences are convex-valued and upper hemicontinuous. This can be proved as in Step 4 of the proof of Proposition 17.C.1, by property (i), and the convex-valuedness of $z(\cdot)$.

Step 4: For each n, $f^n(\cdot)$ has a fixed point. This follows from Step 3 and Kakutani's fixed-point theorem. Denote it by (p^n, z^n), then $p^n \cdot z^n \geq p \cdot z^n$ for every $p \in \Delta^n$ and $z^n \in z(p^n)$. By (iii), $p^n \cdot z^n = 0$ and hence $p \cdot z^n \leq 0$ for every $p \in \Delta^n$. Define

$$\Pi^n = \{z \in \mathbb{R}^L: p \cdot z \leq 0 \text{ for every } p \in \Delta^n \text{ and } z_\ell \geq -s \text{ for every } \ell\},$$

then $\Pi^n \supset \Pi^{n+1}$, $z^n \in \Pi^n$, and hence $z^n \in \Pi^N$ for every N and $n \geq N$. Moreover, each Π^n is bounded; to show this, let $z \in \Pi^n$ and, for every ℓ, $q^\ell = (q_1^\ell, \ldots, q_L^\ell)$ be the vector in Δ^n whose ℓth coordinate is equal to $1 - (L - 1)/n$ and the other coordinates are equal to $1/n$. Then

$$(1 - (L - 1)/n)z_\ell = q_\ell^\ell z_\ell = \textstyle\sum_{k \neq \ell} q_k^\ell(-z_k) \leq (\sum_{k \neq \ell} q_k^\ell)s = (L - 1)s/n.$$

Hence

$$-s \leq z_\ell \leq \frac{(L - 1)s/n}{1 - (L - 1)/n} = \frac{L - 1}{n - L + 1}s.$$

Thus the sequence $\{z^n\}$ is bounded.

Step 5: The sequence $\{p^n\}_n$ in Δ *has a subsequence converging to a price vector in Interior* Δ. Since Δ is compact, it has a subsequence converging to a price vector $p^* \in \Delta$. By (v) and the boundedness of $\{z^n\}$, p^* must belong to Interior Δ. Therefore, by taking a subsequence if necessary, we can assume that $p^n \to p^* \in$ Interior Δ and $z^n \to z^* \in \mathbb{R}^L$.

Step 6: The limit p^* *is a Walrasian equilibrium price vector.* Since $z^* \in \Pi^n$ for all n, the upper bounds $\frac{L - 1}{n - L + 1}$s for the Π^n obtained in Step 4 imply that $z^* \le 0$. By (i), $z^* \in z(p^*)$. By (iv), $z^* = 0$.

17.C.3 [<u>First</u> <u>Printing</u> <u>Errata</u>: In the definition of an equilibrium with taxes, the budget set $B_i(p, p\cdot\omega_i + (1/I)\sum_{\ell,i} t_{\ell i} p_\ell x_{\ell i})$ should be written as $B_i(p, p\cdot\omega_i + (1/I)\sum_{\ell j} t_\ell p_\ell x^*_{\ell j})$. That is, first, the relevant consumptions are the equilibrium ones; second, the indexes for the consumers in the sum of the tax revenues should be something other than i, because it is already being used. Also, the difficulty level should be C.]

(a) For example, if we take $t_{11} = t_{21} = t_{12} = 0$ and $t_{22} > 0$, then an equilibrium in an Edgeworth box looks like the following figure.

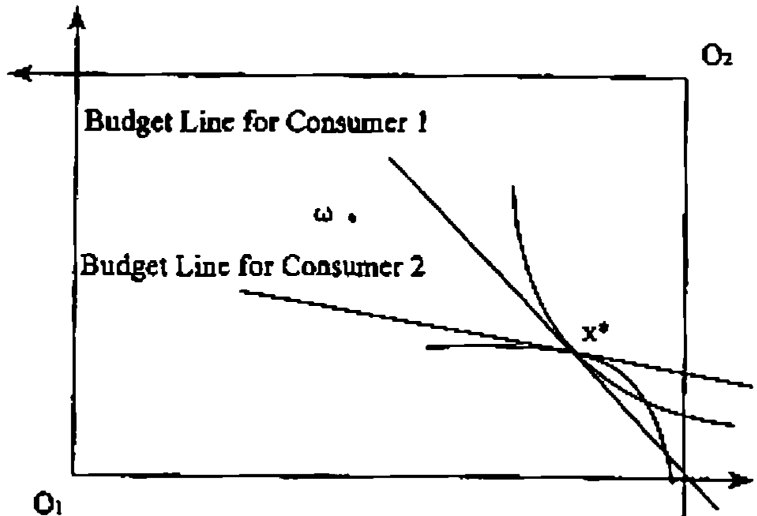


Figure 17.C.3(a)

As we can see in this figure, the relative prices the consumers are faced with can be different, and hence an equilibrium with the taxes need not be a Pareto optimum.

(b) [First printing errata: The phrase "Apply Proposition 17.C.1" should really be "Apply the idea of Proposition 17.C.1."] For each i, define τ_i: $\mathbb{R}^L_+ \to \mathbb{R}^L_+$ by

$$\tau_i(x_i) = (x_{1i}/(1 + t_{1i}), \dots, x_{Li}/(1 + t_{Li}))$$

for every $x_i \in \mathbb{R}^L_+$. For each i, define another preference $\succsim_i^*$ on $\mathbb{R}^L_+$ by the following rule: For each $x_i \in \mathbb{R}^L_+$ and $y_i \in \mathbb{R}^L_+$, $x_i \succsim_i^* y_i$ if and only if $\tau_i(x_i) \succsim_i \tau_i(y_i)$. Denote by $x_i^*(p,w_i)$ the Walrasian demand function of $\succsim_i^*$ and by $x_i^t(p,w_i)$ the demand function of consumer i under the budget constraint $B_i(p,w_i)$. Then

$$x_i^t(p,w_i) = \tau_i(x_i^*(p,w_i))$$

for every (p,w_i), because, for every $x_i \in \mathbb{R}^L_+$, $\tau_i(x_i) \in B_i(p,w_i)$ if and only if $p \cdot x_i \le w_i$. For each $p \gg 0$ and $r \ge 0$, define

$$z_i^t(p,r) = x_i^t(p,\ p \cdot \omega_i + r) - \omega_i.$$

This is the excess demand function of consumer i under the budget constraint with the tax rates $t_{\ell i}$ and rebate r. Define $z^t(p,r) = \sum_i z_i^t(p,r)$. This is the aggregate excess demand function. Note that if p is an equilibrium price vector and r is the associated tax rebate for each consumer, then $z^t(p,r) = 0$. We shall now show that, conversely, if a price vector p and the associate tax rebate r satisfies $z^t(p,r) = 0$, then they constitute an equilibrium with taxes. In fact, then, $\sum_i x_i^t(p, p\cdot\omega_i + r) = \sum_i \omega_i$. Moreover,

$$\sum_{\ell i} t_{\ell i} p_\ell x_{\ell i}^t(p, p\cdot\omega_i + r)$$
$$= \sum_{\ell i}(1 + t_{\ell i})p_\ell x_{\ell i}^t(p, p\cdot\omega_i + r) - \sum_{\ell i} p_\ell x_{\ell i}^t(p, p\cdot\omega_i + r).$$

By the budget constraints, the first term is equal to $p\cdot(\sum_i \omega_i) + Ir$. The second term is equal to

$$\sum_{\ell i} p_\ell (z_{\ell i}^t(p,r) + \omega_{\ell i}) = p\cdot z^t(p,r) + p\cdot(\sum_i \omega_i) = p\cdot(\sum_i \omega_i).$$

Hence $\sum_{\ell i} t_{\ell i} p_\ell x_{\ell i}^t(p, p\cdot\omega_i + r) = Ir$. Thus p is an equilibrium price vector and r is the associated rebate for each consumer if and only if $z^t(p,r) = 0$.

~~To~~ To prove the existence of an equilibrium along the line of Proposition 17.C.1, note first that the aggregate excess demand function $z^t(\cdot)$ with taxes does not satisfy Walras' law. (That is, $p\cdot z^t(p,r) \neq 0$.) So we modify $z^t(\cdot)$ to satisfy Walras' law: Define

$$z^w(p,r) = z^t(p,r) - (p\cdot z^t(p,r)/p\cdot p)p \in \mathbb{R}^L,$$

then $p\cdot z^w(p,r) = 0$ for every (p,r). Just like the original excess demand function $z^t(\cdot)$, this modified excess demand function $z^w(\cdot)$ is continuous (in both p and r). Moreover, for each fixed $r \geq \bar{0}$, $z^w(\cdot,r)$ satisfies properties (iv) and (v) of Proposition 17.B.2. For this, since $z^t(\cdot,r)$ satisfies these properties, it is sufficient to prove that $p\cdot z^t(p,r)$ is bounded above. But this follows from

$$p\cdot z^t(p,r) = p\cdot(\sum_i x_i^t(p, p\cdot\omega_i + r) - \sum_i \omega_i)$$

$$\leq \textstyle\sum_i p\cdot x_i^t(p,\ p\cdot\omega_i + r) = p\cdot(\sum_i\omega_i) + Ir$$
$$\leq (\mathrm{Max}\{\textstyle\sum_i\omega_{\ell i}:\ \ell = 1,\ldots,L\}) + Ir.$$

Note also that if a price vector p and a rebate r satisfies $z^w(p,r) = 0$ and $p\cdot z^t(p,r) = 0$, then they constitute an equilibrium with taxes. In fact, then $z^w(p,r) = z^t(p,r)$ and hence $z^t(p,r) = 0$.

To construct the correspondence to which the fixed-point theorem is to be applied, define $\Delta = \{p \in \mathbb{R}^L_+:\ \sum_\ell p_\ell = 1\}$. Let

$$\bar{t} = \mathrm{Max}\{t_{\ell i}:\ \ell = 1,\ldots,L \text{ and } i = 1,\ldots,I\}$$

and let $\bar{r} > 0$ satisfy

$$(I/(1 + \bar{t}))\bar{r} - [\bar{t}/(1 + \bar{t})]p\cdot(\textstyle\sum_i\omega_i) > 0.$$

We shall now prove that $p\cdot z^t(p,r) > 0$ for every $r \geq \bar{r}$. In fact,

$$\begin{aligned}
p\cdot z^t(p,r) &= \textstyle\sum_{\ell i}p_\ell(x^t_{\ell i}(p,\ p\cdot\omega_i + r) - \omega_{\ell i})\\
&= \textstyle\sum_{\ell i}p_\ell x^t_{\ell i}(p,\ p\cdot\omega_i + r) - \sum_{\ell i}p_\ell\omega_{\ell i}\\
&\geq (1 + \bar{t})^{-1}\textstyle\sum_{\ell i}(1 + t_{\ell i})p_\ell x^t_{\ell i}(p,\ p\cdot\omega_i + r) - p\cdot(\sum_i\omega_i)\\
&= (1 + \bar{t})^{-1}(p\cdot(\textstyle\sum_i\omega_i) + Ir) - p\cdot(\sum_i\omega_i)\\
&= (I/(1 + \bar{t}))r - (\bar{t}/(1 + \bar{t}))p\cdot(\textstyle\sum_i\omega_i) > 0.
\end{aligned}$$

Now define a correspondences f: $\Delta \times [0,\ \bar{r}] \to \Delta$ by

$$f(p,r) = \{q \in \Delta:\ z^w(p,r)\cdot q \geq z^w(p,r)\cdot q' \text{ for all } q' \in \Delta\} \text{ if } p \in \text{Interior } \Delta,$$

$$f(p,r) = \{q \in \Delta:\ p\cdot q = 0\} \text{ if } p \in \text{Boundary } \Delta.$$

Note that we cannot have $p \in f(p,r)$ for $p \in$ Boundary Δ. Also, since $z^w(\cdot)$ is continuous (in both p and r) and $z^w(\cdot,r)$ satisfies properties (iii), (iv) and (v) of Proposition 17.B.2, we can prove just as in the proof of Proposition 17.C.1 that $f(\cdot)$ is convex-valued and upper hemicontinuous. (The proof for upper hemicontinuity can be made easier by establishing this property with respect to p and r separately.) Define then another correspondence g: $\Delta \times [0,\bar{r}] \to [0,\bar{r}]$ by

$$g(p,r) = \{s \in [0,\bar{r}]: p\cdot z^t(p,r)s \le p\cdot z^t(p,r)s' \text{ for all } s' \in [0,\bar{r}]\}.$$

Note that we cannot have $\bar{r} \in g(p,\bar{r})$ because $p\cdot z^t(p,\bar{r}) > 0$. We cannot have $0 \in g(p,0)$ either, because $p\cdot z^t(p,0) = -\sum_{\ell i} t_{\ell i} p_\ell x^t_{\ell i}(p, p\cdot\omega_i) < 0$. It is easy to check that $g(\cdot)$ is convex-valued and upper hemicontinuous.

We shall now prove that the a fixed point of the product correspondence

$$(f \times g): \Delta \times [0,\bar{r}] \to \Delta \times [0,\bar{r}],$$

which maps (p,r) to $f(p,r) \times g(p,r)$, constitutes an equilibrium with taxes. In fact, if $(p^*,r^*) \in f(p^*,r^*) \times g(p^*,r^*)$, then $p^* \in f(p^*,r^*)$ and $r^* \in g(p^*,r^*)$. Hence $p^* \gg 0$ and $r^* \in (0,\bar{r})$. Thus $z^w(p^*,r^*) = 0$ and $p^*\cdot z^t(p^*,r^*) = 0$. Hence, as we argued before, $z^t(p^*,r^*) = 0$ and p^* is an equilibrium price vector and r^* is the associated rebate.

Finally, the product correspondence $(f \times g)(\cdot)$ is convex-valued and upper hemicontinuous because so are $f(\cdot)$ and $g(\cdot)$. Thus the existence of a fixed point is guaranteed. This completes the proof of the existence of an equilibrium.

(c) [<u>First Printing Errata</u>: On the right-hand side of the equation defining $B_i(p,T_i)$, $p_{\ell i}$ should be replaced by p_ℓ.] In this case, a budget set has a kink at a consumption $x_i \in \mathbb{R}^L_+$ with $x_{\ell i} = \omega_{\ell i}$ if $t_\ell > 0$. Here is an example in which $0 < t_{1i} < 1$, $0 < t_{2i} < 1$, and $T_i > 0$. (If $t_\ell \ge 1$, then the consumers will not increase his disposal wealth by selling their initial endowments of commodity ℓ. In the case of $L = 2$, therefore, the budget line will then have nonnegative slope at x_i with $x_{\ell i} > \omega_{\ell i}$.)

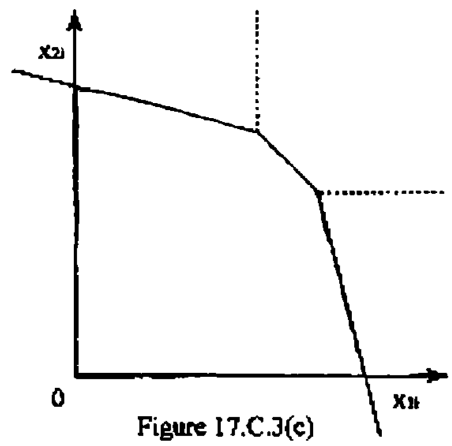


Figure 17.C.3(c)

(d) Denote the buying tax rate for good ℓ by $t_\ell^b \geq 0$ and the selling tax rate for good ℓ by $t_\ell^s \geq 0$. The budget set $B_i(p, T_i)$ is then equal to

$$\{x_i \in R_+^L: p\cdot(x_i - \omega_i) + \sum_\ell t_\ell^b p_\ell \text{Max}\{x_{\ell i} - \omega_{\ell i}, 0\}$$
$$+ \sum_\ell t_\ell^s p_\ell \text{Max}\{\omega_{\ell i} - x_{\ell i}, 0\} \leq T_i\}.$$

(e) The existence of an equilibrium for the modification in (c) can be guaranteed by applying the proof of Proposition 17.BB.2. The key fact in this application is that the game-theoretic approach to the existence of an equilibrium allows the budget sets to be different among consumers and to have kinks. The details, such as checking convex-valuedness and upper hemicontinuity, are almost identical as the proof, so we omit them here.

17.C.4 (a) According to the rule, the consumers' after-tax wealths are

$$p\cdot\omega_1 - (1/2)(p\cdot\omega_1 - (1/2)p\cdot(\omega_1 + \omega_2)) = p\cdot((3/4)\omega_1 + (1/4)\omega_2),$$
$$p\cdot\omega_2 - (1/2)(p\cdot\omega_2 - (1/2)p\cdot(\omega_1 + \omega_2)) = p\cdot((1/4)\omega_1 + (3/4)\omega_2).$$

Hence, by $\omega_1 = (1,2)$ and $\omega_2 = (2,1)$, the after-tax wealths are $(5/4)p_1$ +

$(7/4)p_2$ and $(7/4)p_1 + (5/4)p_2$.

(b) Because of the after-taxes wealths in (a), the after-tax aggregate excess demand function is equal to the standard aggregate excess demand function of the economy in which the initial endowment of consumer 1 is equal to $(3/4)\omega_1 + (1/4)\omega_2$ and that of consumer 2 is $(1/4)\omega_1 + (3/4)\omega_2$. It thus satisfies the conditions in Proposition 17.C.1.

17.C.5 (a) Let $x_i \in \mathbb{R}^L_+$ and $x'_i \in \mathbb{R}^L_+$. Suppose that $x_i \succsim^*_i x'_i$ does not hold. Then, by the definition of $\succsim^*_i$, there exists a $y'_i \in Y_i$ with $x'_i + y'_i \in \mathbb{R}^L_+$ such that for every $y_i \in Y_i$ with $x_i + y_i \in \mathbb{R}^L_+$, we do not have $x_i + y_i \succsim_i x'_i + y'_i$. By the completeness of the original preference $\succsim_i$, we then have $x'_i + y'_i \succsim_i x_i + y_i$. By the definition of $\succsim^*_i$, we then have $x'_i \succsim^*_i x_i$. Thus $\succsim^*_i$ is complete.

Let $x_i \in \mathbb{R}^L_+$, $x'_i \in \mathbb{R}^L_+$, and $x''_i \in \mathbb{R}^L_+$. Suppose that $x_i \succsim^*_i x'_i$ and $x'_i \succsim^*_i x''_i$. Let $y''_i \in Y_i$. Then, by the definition of $\succsim^*_i$, there exists a $y'_i \in Y_i$ with $x'_i + y'_i \in \mathbb{R}^L_+$ such that $x'_i + y'_i \succsim_i x''_i + y''_i$, and there also exists a $y_i \in Y_i$ such that $x_i + y_i \succsim_i x'_i + y'_i$. By the transitivity of the original preference $\succsim_i$, this means that $x_i + y_i \succsim_i x''_i + y''_i$. By the definition of $\succsim^*_i$, we have $x_i \succsim^*_i x''_i$. Hence $\succsim^*_i$ is transitive.

(b) Let $x_i \in \mathbb{R}^L_+$, $x'_i \in \mathbb{R}^L_+$, and $\lambda \in [0,1]$. Suppose that $x'_i \succsim^*_i x_i$ and $x''_i \succsim^*_i x_i$. Let $y_i \in Y_i$. Then, by the definition of $\succsim^*_i$, there exists a $y'_i \in Y_i$ such that $x'_i + y'_i \in \mathbb{R}^L_+$ and $x'_i + y'_i \succsim_i x_i + y_i$, and there also exists a $y''_i \in Y_i$ such that $x''_i + y''_i \in \mathbb{R}^L_+$ and $x''_i + y''_i \succsim_i x_i + y_i$. By the convexity of the original preference $\succsim_i$, this means that

$$\begin{aligned}&\lambda(x'_i + y'_i) + (1-\lambda)(x''_i + y''_i)\\ &= (\lambda x'_i + (1-\lambda)x''_i) + (\lambda y'_i + (1-\lambda)y''_i) \succsim_i x_i + y_i.\end{aligned}$$

Since and $\lambda y_i' + (1 - \lambda)y_i'' \in Y_i$ by the convexity of Y_i, the definition of $\succsim_i^*$ implies that $\lambda x_i' + (1 - \lambda)x_i'' \succsim_i^* x_i$. Hence $\succsim_i^*$ is convex.

(c) Assume that, among the L goods, the first M goods are the marketed goods and the last L - M goods are the non-marketed household goods. We say that a price vector $p \in \mathbb{R}^M$ and an allocation $(x_1^*,\ldots,x_I^*)$ for the marketed goods constitute an equilibrium if they constitute a standard Walrasian equilibrium of a pure exchange economy $(\{\mathbb{R}_+^M,\succsim_i^*\}_{i=1}^I,\{\omega_i\}_{i=1}^I)$ with M goods. By the definition of the $\succsim_i^*$, this means that the consumers maximize preferences by using the household technology Y_i and the marketed goods x_i^* as inputs. This is an instance in which a non-Walrasian model can be reduced to a Walrasian one of Sections B and C after some modification, so that the results in these sections are also applicable to non-Walrasian settings.

17.C.6 We are going to draw the images of the aggregate excess demand functions $z(\cdot)$ just like the offer curve of a consumer, except that the initial endowment has been displaced to the origin. Any equilibrium then corresponds to a price vector at which the graph intersects the origin. (Note however that, unlike the offer curves of consumers, the image of an aggregate excess demand function may intersect the origin more than once.)

Here is an example to show that condition (i) is indispensable:

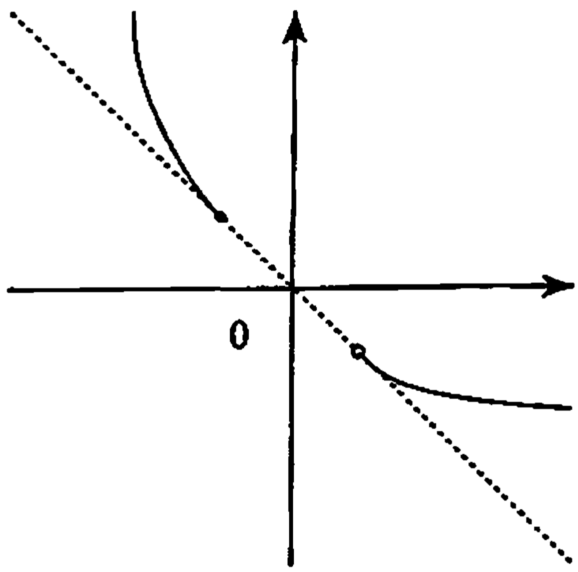


Figure 17.C.6.1

Here is an example to show that condition (iii) is indispensable:

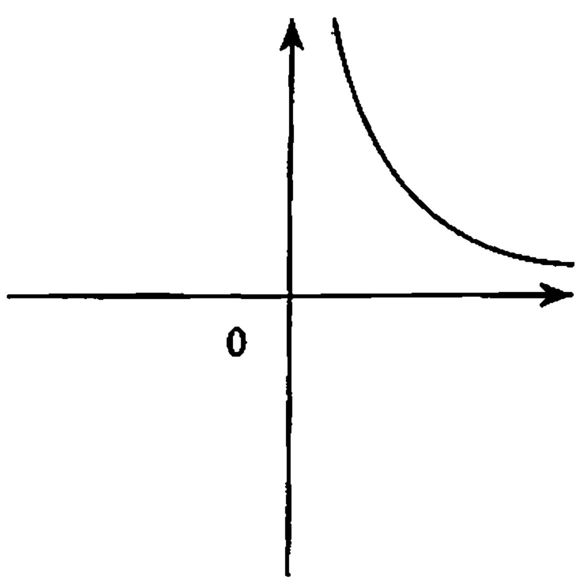


Figure 17.C.6.2

Here is an example to show that condition (iv) is indispensable. Note that we have taken the example so that as $p^n \to (0,1)$, $z_2(p^n) \to \infty$.

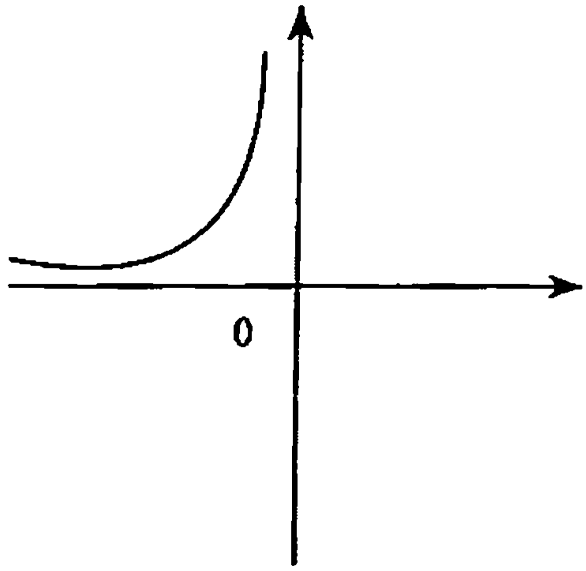


Figure 17.C.6.3

Here is an example to show that condition (v) is indispensable:

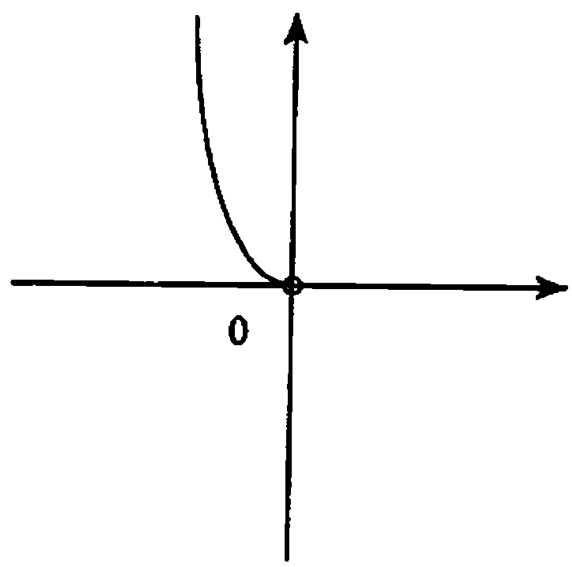


Figure 17.C.6.4

Condition (ii) is not included in the list because it is dispensable. That is, even if (ii) is not satisfied, the existence of an equilibrium price vector is guaranteed as long as the given function z: $\mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies the other four conditions on the restricted domain $\mathbb{R}^L_{++} \cap \Delta$ of normalized price vectors.

17.D.1 From the first-order conditions, for any normalized price vector $p = (p_1, 1)$ with $p_1 > 0$,

$$z_{11}(p_1, 1) = \frac{p_1}{p_1 + (p_1/2)^{1/(1-\rho)}} - 1 \text{ and } z_{12}(p_1, 1) = \frac{1}{p_1 + (2p_1)^{1/(1-\rho)}}.$$

It is then easy to check that $z_{11}(1,1) + z_{12}(1,1) = 0$. Hence $p = (1,1)$ is an equilibrium price vector. Also, the derivative $\partial z_1(1,1)/\partial p_1$ is equal to

$$\frac{-\rho - 1}{(1 + 2^{-1/(1-\rho)})^2(1 + 2^{1/(1-\rho)})^2(1 - \rho)}(2 + 2^{1/(1-\rho)} + 2^{-1/(1-\rho)}) > 0.$$

Hence, by the index theorem (Proposition 17.D.2) (or the intermediate value theorem, as L = 2), there must exist at least two other equilibria.

17.D.2 Since rank $Df(\bar{v}) = M$, we can assume, by relabeling the indexes of the unknowns, that the Jacobian matrix with respect to the first M unknowns $(v_1,\ldots,v_M)$,

$$\begin{bmatrix} \partial f_1(\bar{v})/\partial v_1 & \cdots & \partial f_1(v)/\partial v_M \\ & \ddots & \\ \partial f_M(v)/\partial v_1 & \cdots & \partial f_M(v)/\partial v_M \end{bmatrix},$$

is non-singular. Thus, by the implicit function theorem, there are open neighborhoods A' and B' of $(\bar{v}_1,\ldots,\bar{v}_M)$ and $(\bar{v}_{M+1},\ldots,\bar{v}_N)$, and a continuously differentiable function η: B' $\to$ A' such that for any $(v_1,\ldots,v_M) \in A'$ and $(v_{M+1},\ldots,v_N) \in B'$, we have $(v_1,\ldots,v_M) = \eta(v_{M+1},\ldots,v_N)$ if and only if $f(v_1,\ldots,v_M,v_{M+1},\ldots,v_N) = 0$. Hence, in the neighborhood A' $\times$ B' of $\bar{v}$ the solution set can be parameterized by means of (N − M) parameters $(v_{M+1},\ldots,v_N)$.

17.D.3 Denote by $\hat{z}_1(p;\omega_1) \in \mathbb{R}^{L-1}$ the excess demand for the first L - 1 commodities of consumer 1 with initial endowment vector ω_1. Then $D_{\omega_1}\hat{z}(p;\omega) = D_{\omega_1}\hat{z}_1(p;\omega_1) \in \mathbb{R}^{(L-1)\times L}$. Denote by $\partial\hat{x}_1(p,p\cdot\omega_1)/\partial w_1 \in \mathbb{R}^{L-1}$ (a column vector) the partial derivative of his demand function for the first L - 1 commodities with respect to wealth, and by $\hat{I}_L$ the (L - 1) × L matrix

$$\begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}.$$

Then

$$D_{\omega_1}\hat{z}_1(p;\omega_1) = [\partial\hat{x}_1(p,p\cdot\omega_1)/\partial w_1]p^T - \hat{I}_L.$$

Thus, for every $\hat{v} \in \mathbb{R}^{L-1}$, let $v = \begin{bmatrix} \hat{v} \\ -\sum_{\ell<L} p_\ell \hat{v}_\ell \end{bmatrix} \in \mathbb{R}^L$, then $p\cdot v = 0$ and hence $D_{\omega_1}\hat{z}(p;\omega)v = -\hat{v}$. Thus the range of matrix $D_{\omega_1}\hat{z}(p;\omega)$ is equal to $\mathbb{R}^{L-1}$. Thus its rank is L - 1.

17.D.4 Denote the upper-left (L - 1) × (L - 1) submatrix of the Slutsky matrix $S_i(p,w_i) \in \mathbb{R}^{L\times L}$ (p. 33) of consumer i by $\hat{S}_i(p,w_i)$. We shall first prove that if $\hat{z}(p;\omega) = 0$ and $\hat{v} \in \mathbb{R}^{L-1}$, then there exists $(t_1,\ldots,t_I) \in \mathbb{R}^L \times \ldots \times \mathbb{R}^L$ such that $\sum_i t_i = 0$ and $D\hat{z}(p;\omega)(\hat{v},t_1,\ldots,t_I) = (\sum_i \hat{S}_i(p,p\cdot\omega_i))\hat{v}$.

Let p, ω, and $\hat{v}$ be as such. Write $e = (0,\ldots,0,1) \in \mathbb{R}^L_+$. For each i, define $t_i = (\hat{v}\cdot\hat{z}_i(p;\omega_i))e \in \mathbb{R}^L$. Since $\sum_i \hat{z}_i(p;\omega_i) = 0$, $\sum_i t_i = 0$. Moreover,

$$D\hat{z}_i(p;\omega_i)(\hat{v},t_i) = D_p\hat{z}_i(p;\omega_i)\hat{v} + D_{\omega_i}\hat{z}_i(p;\omega_i)t_i$$

$$= \hat{S}_i(p,p\cdot\omega_i)\hat{v} - (\partial\hat{x}_i(p,p\cdot\omega_i)/\partial w_i)\hat{z}_i(p;\omega_i)^T\hat{v} + (\hat{v}\cdot\hat{z}_i(p;\omega_i))(\partial\hat{x}_i(p,p\cdot\omega_i)/\partial w_i)$$

$$= \hat{S}_i(p,p\cdot\omega_i)\hat{v}.$$

Hence $D\hat{z}(p;\omega)(\hat{v},t_1,\ldots,t_I) = \sum_i D\hat{z}_i(p,p\cdot\omega_i)(\hat{v},t_i) = (\sum_i \hat{S}_i(p,p\cdot\omega_i))\hat{v}$.

The appropriate differentiability condition (Appendix A of Chapter 3, p.

95) says that $S_i(p,p\cdot\omega_i)$ has rank L - 1 and negative semidefinite (Propositions 3.G.2(ii) and 3.G.3). Since $S_i(p,p\cdot\omega_i)p = 0$ and $p\cdot S_i(p,p\cdot\omega_i) = 0$, $S_i(p,p\cdot\omega_i)$ is negative definite on $T_p = \{v \in \mathbb{R}^L: p\cdot v = 0\}$. Hence so is the sum $\sum_i S_i(p,p\cdot\omega_i)$. By Theorem M.D.4(iii), the $(L - 1) \times (L - 1)$ matrix $\sum_i \hat{S}_i(p,p\cdot\omega_i)$ is negative definite (on the whole $\mathbb{R}^{L-1}$). So it has rank L - 1. Therefore, the result in the previous paragraph,

$$D\hat{z}(p;\omega)(\hat{v},t_1,\ldots,t_I) = \sum_i D\hat{z}_i(p;\omega_i)(\hat{v},t_i) = (\sum_i \hat{S}_i(p,p\cdot\omega_i))\hat{v},$$

implies that the matrix $D\hat{z}(p;\omega)$ has rank L - 1 (while the total endowment is fixed).

Hence, by the transversality theorem, for almost all $\omega \in \mathbb{R}^{LI}_{++}$ with $\sum_i \omega_i = \bar{\omega}$, the mapping $z(\cdot;\omega)$: $\mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L-1}$ has the property that rank $D_p z(p;\omega) = L - 1$ for every p with $z(p,\omega) = 0$. Hence, each equilibrium is locally isolated. By property (v) of Proposition Proposition 17.B.2, we know that there is $\varepsilon \in (0,1)$ such that if p is an equilibrium price vector, then $p \in [\varepsilon,1/\varepsilon]^L$. Since the set $[\varepsilon,1/\varepsilon]^L$ is compact and each equilibrium price vector is locally isolated, Theorem M.F.3(ii) implies that there are only finitely many equilibrium price vectors. Since each equilibrium price vector p uniquely determines an equilibrium allocation, the number of equilibria is finite.

17.D.5 For given initial endowment vectors $(\omega_1,\omega_2) \in \mathbb{R}^4_+$, a price vector $p = (p_1,1) \in \mathbb{R}^2_{++}$ and an allocation $(x_1,x_2) \in \mathbb{R}^4_{++}$ constitute an equilibrium if and only if there exists $(\lambda_1,\lambda_2) \in \mathbb{R}^2_{++}$ such that

$\lambda_i \nabla u_i(x_i) - (p_1,1) = 0$ for each $i = 1,2$;

$x_{11} + x_{12} - \omega_{11} - \omega_{12} = 0$;

$p_1(x_{1i} - \omega_{1i}) + (x_{2i} - \omega_{2i}) = 0$ for each $i = 1,2$.

The left hand sides of this system of equations define a continuously differentiable mapping from $\mathbb{R}^4_{++} \times \mathbb{R}^2_{++} \times \mathbb{R}_{++} \times \mathbb{R}^4_{+}$ into $\mathbb{R}^4 \times \mathbb{R} \times \mathbb{R}^2$, where the unknowns are $(x_1,x_2,\lambda_1,\lambda_2,p_1,\omega_1,\omega_2)$. Denote the mapping by $\psi(\cdot)$. Then the above equilibrium condition can be rephrased as the equation $\psi(x_1,x_2,\lambda_1,\lambda_2,p_1,\omega_1,\omega_2) = 0$. We want to show that if $\psi(x_1,x_2,\lambda_1,\lambda_2,p_1,\omega_1,\omega_2) = 0$, then rank $D\psi(x_1,x_2,\lambda_1,\lambda_2,p_1,\omega_1,\omega_2) = 7$. It is then sufficient to prove that the row rank of this Jacobian is 7, that is, for any $(v_1,v_2,s,t_1,t_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, if

$$(v_1^T,v_2^T,s,t_1,t_2)D\psi(x_1,x_2,\lambda_1,\lambda_2,p_1,\omega_1,\omega_2) = 0,$$

then $(v_1,v_2,s,t_1,t_2) = 0$. In fact, suppose that the above equation holds. Then, by the partial derivative of $\psi(\cdot)$ with respect to ω_i, we know that $s(-1, 0) - t_ip^T = 0$. Thus $s = t_i = 0$. By the partial derivative of $\psi(\cdot)$ with respect to λ_i, we know that $\nabla u_i(x_i)v_i = 0$. By the partial derivative of $\psi(\cdot)$ with respect to x_i and by $s = t_i = 0$, we know that $v_i \cdot \lambda_i D^2u_i(x_i) = 0$ and hence $v_i \cdot \lambda_i D^2u_i(x_i)v_i = 0$. By the differentiable strict quasiconcavity (that is, if $\nabla u_1(x_1,x_2)v_1 = 0$ and $v_1 \neq 0$, then $v_1 \cdot D^2u_1(x_1)v_1 < 0$) and $\nabla u_i(x_i)v_i = 0$, we have $v_i = 0$.

17.D.6 The system of equations that defines an equilibrium in this context is the same as in the answer to Exercise 17.D.5, except that the first equation

$$\lambda_1\nabla u_1(x_1) - (p_1,1) = 0$$

is now replaced by

$$\lambda_1\nabla_{x_1}u_1(x_1,x_2) - (p_1,1) = 0.$$

With this replacement, we can continue to use the same symbol $\psi(\cdot)$ to denote this new system of equations that defines an equilibrium in this context. In the same way as before, we can prove that for any $(v_1,v_2,s,t_1,t_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \times$

$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, if

$$(v_1^T, v_2^T, s, t_1, t_2)D\psi(x_1, x_2, \lambda_1, \lambda_2, p_1, \omega_1, \omega_2) = 0,$$

then $(v_1, v_2, s, t_1, t_2) = 0$. (We need to assume the differentiable strict quasiconcavity with respect to the own consumption, that is, if $\nabla_{x_1} u_1(x_1, x_2)v_1 = 0$ and $v_1 \neq 0$, then $v_1 \cdot D^2_{x_1} u_1(x_1, x_2)v_1 < 0$.) The assertion that, generically, there are only finitely many equilibria follows from this, just as we saw in Exercise 17.D.5.

17.D.7 (a) Assume that, for each island n ($n = 1,\dots,N$), there are L_n commodities and I_n consumers. Since there is no communication among the N islands, an array $(x_1,\dots,x_N)$ of the consumption allocations for the overall economy, where $x_n \in \mathbb{R}^{L_n I_n}$ is an allocation of island n, is an equilibrium of the overall economy if and only if, for each n, x_n is an equilibrium allocation of island n. Since each island has three equilibrium allocations, the number of the equilibrium allocations in the overall economy is 3^N.

Note that this result does not imply that the number of equilibrium price vectors is 3^N. In fact, there is a continuum of equilibrium price vectors for the overall economy, because, if $q_n \in \mathbb{R}^{L_n}$ is an equilibrium price vector of island n, then, for any positive numbers $\lambda_1 > 0, \dots, \lambda_N > 0$, the array $(\lambda_1 q_1,\dots,\lambda_N q_N)$ is an equilibrium price vector of the overall economy.

(b) [First printing errata: The consumers' preferences should be assumed to be strictly convex. This implies that there is no equilibrium of the overall economy generated by the replication, which is discussed in Section I.] Let $L_n = L$ and $I_n = I$ for each n. Let $\bar{\omega} \in \mathbb{R}^L_{++}$ be the initial endowments of the L commodities in each island. Let $q_1, q_2, q_3 \in \mathbb{R}^L$ be the equilibrium price vectors of each identical island such that $q_{11} = q_{12} = q_{13} = 1$. (That is, the price of commodity 1 is normalized to be one.) We shall prove that the equilibrium price vectors of the overall economy are (scalar multiples of) $(q_1,\ldots,q_1)$, $(q_2,\ldots,q_2)$, $(q_3,\ldots,q_3) \in \mathbb{R}^{LN}$. It is clear that they are equilibrium price vectors. Suppose that $(p_1,\ldots,p_N) \in \mathbb{R}^{LN}$ is an equilibrium price vector of the overall economy and $p_{11} = 1$. (That is, the price of commodity 1 in island 1 is normalized to be one.) If there are $\ell = 1,\ldots,L$, $n = 1,\ldots,N$, and $n' = 1,\ldots,N$ such that $p_{\ell n} < p_{\ell n'}$, then the demand for commodity ℓ endowed in island n' is zero, which cannot happen at equilibrium. Hence, for any ℓ, n, and n', we have $p_{\ell n} = p_{\ell n'}$, that is, $p_1 = \ldots = p_N$. Denote them by $p \in \mathbb{R}^L$. Since the consumers' preferences are strictly convex, the aggregate demand vector of each island for the L commodities at p, which is denoted by $x(p) \in \mathbb{R}^L$. Then the aggregate demand for commodity ℓ of the overall economy is equal to $Nx_\ell(p)$ and the aggregate supply from the N islands is $N\bar{\omega}_\ell$. Hence we must have $Nx_\ell(p) = N\bar{\omega}_\ell$ for every ℓ, or equivalently, $x(p) = \bar{\omega}$. Thus, by $p_{11} = 1$, $p \in \{q_1,q_2,q_3\}$ and the number of the normalized equilibrium price vectors is three.

17.D.8 Let the consumer's utility function be $u(x) = \Pi_\ell x_\ell^{\alpha_\ell}$ with $\alpha_\ell > 0$ and $\sum_\ell \alpha_\ell = 1$, and his initial endowment vector be $\omega \in \mathbb{R}^L_{++}$. Then his excess demand function is given by

$$z(p) = x(p,p\cdot\omega) - \omega = (\alpha_1 p\cdot\omega/p_1,\ldots,\alpha_L p\cdot\omega/p_L) - \omega.$$

By the chain rule, for every $p \in \mathbb{R}^L_{++}$, $\ell < L$, and $k < L$,

$$\partial\hat{z}_\ell(p)/\partial p_k = \partial x_\ell(p,w)/\partial p_k + (\partial x_\ell(p,w)/\partial w)\omega_k,$$

where $w = p\cdot\omega$, the wealth of the consumer at price vector p. Moreover,

$$\partial x_\ell(p,w)/\partial p_k = \begin{cases} -\alpha_\ell w/p_\ell^2 & \text{if } \ell = k, \\ 0 & \text{otherwise,} \end{cases}$$

and $\partial x_\ell(p,w)/\partial w = \alpha_\ell/p_\ell$. Hence the $(L-1) \times (L-1)$ matrix $D\hat{z}(p)$ is the sum of the following two matrices: the first one is the diagonal matrix whose ℓth diagonal entry is $-\alpha_\ell w/p_\ell^2$; the second one is equal to

$$\begin{bmatrix} \alpha_1/p_1 \\ \vdots \\ \alpha_{L-1}/p_{L-1} \end{bmatrix} [\omega_1 \;\; \cdots \;\; \omega_{L-1}].$$

By (17.E.1) and $\partial z_\ell(p)/\partial p_L = (\alpha_\ell/p_\ell)\omega_L$ for every $\ell < L$,

$$D\hat{z}(p)p = -\omega_L \begin{bmatrix} \alpha_1/p_1 \\ \vdots \\ \alpha_{L-1}/p_{L-1} \end{bmatrix}$$

and hence

$$D\hat{z}(p)((1/p_1)p) = -(\omega_L/p_1) \begin{bmatrix} \alpha_1/p_1 \\ \vdots \\ \alpha_{L-1}/p_{L-1} \end{bmatrix}.$$

This equality means that if we add to the first column of $D\hat{z}(p)$ the ℓth column of $D\hat{z}(p)$ multiplied by p_ℓ/p_1 for all $\ell < L$, then we obtain the vector on the right hand side, which we denote by v. Denote by Z the matrix obtained from $D\hat{z}(p)$ by replacing its first column by v, then $\det Z = \det D\hat{z}(p)$. We now obtain another new matrix Z' by adding, for each $\ell = 2,\ldots,L-1$, the vector $(\omega_\ell p_1/\omega_L)v$ to the ℓth column vector of Z (and hence of $D\hat{z}(p)$). Since

$$(\omega_\ell p_1/\omega_L)v = -\omega_\ell \begin{bmatrix} \alpha_1/p_1 \\ \vdots \\ \alpha_{L-1}/p_{L-1} \end{bmatrix},$$

$\det Z' = \det Z$, and Z' is the sum of the following two matrices: the first one is the diagonal matrix, whose first diagonal is zero and the ℓth diagonal is equal to $-\alpha_\ell w/p_\ell^2$ for each $\ell = 2,\ldots, L-1$; the second one has the property that its first column is equal to v and all other columns are equal to zero.

Thus

$$\det Z' = \left(\prod_{\ell=2}^{L-1} (-\alpha_\ell w/p_\ell^2)\right)\cdot(-\alpha_1\omega_L/p_1^2) = (-1)^{L-1}\left(\prod_{\ell=2}^{L-1} (\alpha_\ell w/p_\ell^2)\right)\cdot(\alpha_i\omega_L/p_1^2).$$

The (normalized) equilibrium price vector is

$$p = (\alpha_1\omega_L/\alpha_L\omega_1, \ldots, \alpha_{L-1}\omega_L/\alpha_L\omega_{L-1}, 1).$$

Plugging this into the above equality, we obtain

$$\det Z' = (-1)^{L-1}\left(\prod_{\ell=1}^{L} \omega_\ell^2/\alpha_\ell\right)\cdot(\alpha_L/\omega_L)^{L+1}.$$

By $\det D\hat{z}(p) = \det Z'$, the index of the equilibrium is $+1$.

17.E.1 By differentiating both sides of $z(\lambda p) = z(p)$ with respect to $\lambda > 0$ and then evaluating at $\lambda = 1$, we obtain (17.E.1).

By differentiating both sides of $p\cdot z(p) = 0$ with respect to p, we obtain (17.E.2).

17.E.2 By $z_i(p) = x_i(p, p\cdot\omega_i)$ for all p and the chain rule,

$$Dz_i(\bar{p}) = D_p x_i(\bar{p},\bar{p}\cdot\omega_i) + D_{w_i} x_i(\bar{p},\bar{p}\cdot\omega_i)\omega_i^T$$

for every $\bar{p}$. Since $D_p x_i(\bar{p},\bar{p}\cdot\omega_i) = S_i(\bar{p},\bar{p}\cdot\omega_i) - D_{w_i} x_i(\bar{p},\bar{p}\cdot\omega_i)x_i(\bar{p},\bar{p}\cdot\omega_i)^T$, we have

$$\begin{aligned} Dz_i(\bar{p}) &= S_i(\bar{p},\bar{p}\cdot\omega_i) - D_{w_i} x_i(\bar{p},\bar{p}\cdot\omega_i)x_i(\bar{p},\bar{p}\cdot\omega_i)^T + D_{w_i} x_i(\bar{p},\bar{p}\cdot\omega_i)\omega_i^T \\ &= S_i(\bar{p},\bar{p}\cdot\omega_i) - D_{w_i} x_i(\bar{p},\bar{p}\cdot\omega_i)z_i(\bar{p},\bar{p}\cdot\omega_i)^T. \end{aligned}$$

Hence, by summing over i and replacing $\bar{p}$ by p, we obtain (17.E.3). (We used $\bar{p}$ in place of p to emphasize that $D_p x_i(\bar{p},\bar{p}\cdot\omega_i)$ is the partial derivative with respect to prices evaluated at $(\bar{p},\bar{p}\cdot\omega_i)$; that is, D_p does not operate for $\bar{p}$ in $\bar{p}\cdot\omega_i$.)

17.E.3 Let p, A, a^ℓ, e^ℓ, and x_i be as defined in the proof of Proposition

17.E.2. For each ℓ, define $q^{\ell} = (q_{\ell 1},\ldots,q_{\ell L}) \in \mathbb{R}^L_{++}$ by $q_{\ell\ell} = p_\ell/2$ and $q_{\ell k} = p_k$ for any $k \neq \ell$. For each ℓ and i, define $b_i^{\ell} = (1/q_{\ell i})q^{\ell} \in \mathbb{R}^L_{++}$. Note that, for each i, the family $(b_i^1,\ldots,b_i^L)$ is linearly independent. So, for each i, define $u_i\colon \mathbb{R}^L_+ \to \mathbb{R}$ by $u_i(y_i) = \min\{b_i^{\ell}\cdot y_i - b_i^{\ell}\cdot x_i\colon \ell = 1,\ldots,L\}$. Then $u_i(\cdot)$ is continuous, concave, and strictly monotone. Note also that

$$u_i(x_i) = \min\{b_i^{\ell}\cdot x_i - b_i^{\ell}\cdot x_i\colon \ell = 1,\ldots,L\} = 0,$$
$$p = \textstyle\sum_\ell (L - 1/2)^{-1} q_{\ell i} b_i^{\ell}.$$

The second equality means that the price vector p can be spanned by the "gradient vectors" b_i^{ℓ} of $u_i(\cdot)$ with strictly positive coefficients $(L - 1/2)^{-1}q_{1i}, \ldots, (L - 1/2)^{-1}q_{Li}$. Hence the upper contour set $\{y_i \in \mathbb{R}^L_+\colon u_i(y_i) \geq u_i(x_i)\}$ is strictly supported by p at x_i. The utility maximization condition is thus satisfied at x_i under p, implying that $z_i(p) = x_i - \omega_i = p_i(a^i)^T$. Moreover, since $b_i^{\ell}\cdot(x_i + e^i) - b_i^{\ell}\cdot x_i = b_i^{\ell}\cdot e^i = 1$ for all ℓ, the wealth expansion path of consumer i under p must be a straight line parallel to e^i (locally around x_i). Hence $D_{w_i}x_i(p,p\cdot\omega_i) = (1/p_i)e^i$. Since the upper contour set $\{y_i \in \mathbb{R}^L_+\colon u_i(y_i) \geq u_i(x_i)\}$ is strictly supported by p at x_i, $S_i(p,p\cdot\omega_i) = 0$. Hence $Dz_i(p) = -D_{w_i}x_i(p,p\cdot\omega_i)z_i(p) = e^i a^i$.

17.E.4 Here is the offer curve of a function $z(\cdot)$ that is continuous, homogeneous of degree zero, satisfies Walras' law, and cannot be generated from a rational preference. The last property follows from the fact that the weak axiom of revealed preference is not satisfied at p and p'. On the other hand, it is easy to check that if the offer curve does not go through the initial endowment point as in Figure 17.E.2, then the weak axiom always holds.

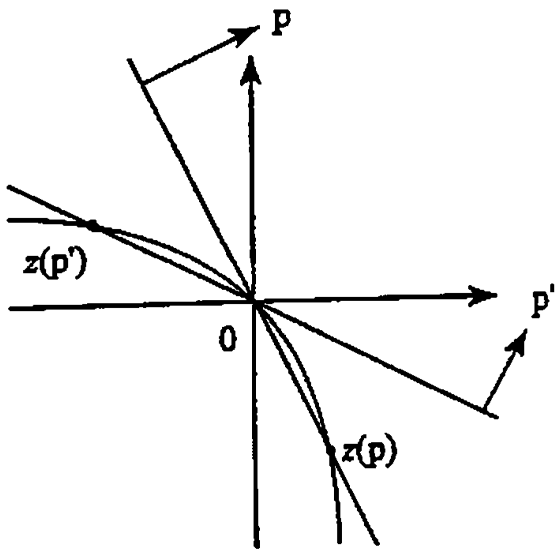


Figure 17.E.4

17.E.5 We shall prove the assertion by contradiction. Suppose that there is an economy that generates the given function $z(\cdot)$, whose initial endowment $\bar{\omega}$ satisfies $0 \le \bar{\omega}_1 \le 1$ and $0 \le \bar{\omega}_2 \le 1$, and whose consumers have the consumption sets $\mathbb{R}^2_+$. Let $x_i(\cdot)$ be the demand function of consumer i, $z_i(\cdot)$ be his excess demand function, and $\omega_i \in \mathbb{R}^2_+$ be his initial endowment. Then $z_{1i}(p) \ge -\omega_{1i}$ for every i. According to Figure 17.E.3, $\sum_i z_{1i}(p) = -1 \le -\bar{\omega}_1 = -\sum_i \omega_{1i}$. We must thus have $z_{1i}(p) = -\omega_{1i}$, or $x_{1i}(p, p\cdot\omega_i) = 0$ for every i. We can similarly show that $x_{2i}(p', p'\cdot\omega_i) = 0$ for every i. These demands can be depicted as follows.

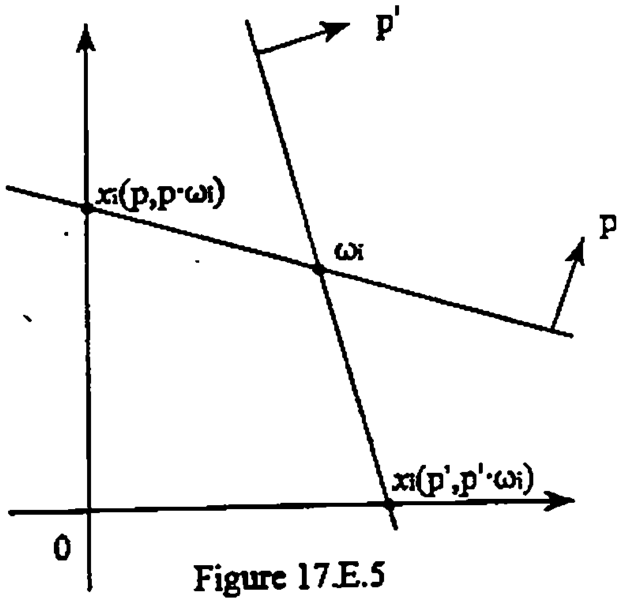


Figure 17.E.5

This is a violation to the weak axiom of revealed preference. Hence there is no economy that generates $z(\cdot)$.

17.E.6 Let $p \in \mathbb{R}^L_{++}$, $p' \in \mathbb{R}^L_{++}$, $\|p\| = \|p'\| = 1$, and $p \neq p'$. Since $p\cdot z(p) = p'\cdot z(p') = 0$, it is sufficient to prove that we have either $p\cdot z(p') > 0$ or $p'\cdot z(p) > 0$. So suppose that $p_i \geq p'_i$. Since $p\cdot p' < \|p\|\|p'\| = 1$, $p\cdot z(p') = p_i - p'_i(p\cdot p') > p_i - p'_i \geq 0$. If $p_i \leq p'_i$, we can similarly show that $p'\cdot z(p) > 0$.

17.E.7 We first note the following observation: If $\|p\| = \|p'\| = 1$ and p is directly revealed preferred to p' by consumer i, then $p_i < p'_i$. This follows from the fact that $p\cdot z(p') = p_i - p'_i(p\cdot p') \leq 0$ and $0 < p\cdot p' < \|p\|\|p'\| = 1$.

Now suppose that p is indirectly revealed preferred to p'. Then there exists a finite chain $p^1,\ldots,p^N$ such that $p = p^1$, $p^N = p'$, and p^n is directly revealed preferred to p^{n+1} for all $n \leq N - 1$. Hence, by the above observation, $p^n_i < p^{n+1}_i$ for all $n \leq N - 1$. Thus $p_i < p'_i$. Hence, again by the above observation, p' cannot be directly revealed preferred to p. The strong axiom is therefore satisfied.

17.F.1 Let S_i, x_i, $\bar{x}$, and $\bar{\omega}_i$ be as defined just before (17.F.2) and define

$$C = \sum_i \frac{1}{p\cdot\omega_i}[x_i - \frac{p\cdot\omega_i}{p\cdot\bar{\omega}}\bar{x}][x_i - \frac{p\cdot\omega_i}{p\cdot\bar{\omega}}\bar{x}]^T,$$

$$D = \sum_i \frac{1}{p\cdot\omega_i}[x_i - \frac{p\cdot\omega_i}{p\cdot\bar{\omega}}\bar{x}][\omega_i - \frac{p\cdot\omega_i}{p\cdot\bar{\omega}}\bar{\omega}]^T.$$

Then, by (17.F.2), $Dz(p) = \sum_i S_i - \frac{1}{p\cdot\bar{\omega}}\bar{x}z(p)^T - (C - D)$. It is thus sufficient to prove that $dp\cdot(C - D)dp \geq 0$ for any dp with $dp\cdot z(p) = 0$.

If the initial endowments ω_i are proportional among themselves, then $\omega_i - \frac{p\cdot\omega_i}{p\cdot\bar{\omega}}\bar{\omega} = 0$ and hence $D = 0$. Since C is always positive semidefinite, our assertion follows.

If the demands x_i are proportional among themselves, then $x_i - \frac{p\cdot\omega_i}{p\cdot\bar{\omega}}\bar{x} = 0$ and hence $C = D = 0$. The proof is thus completed.

17.F.2 By construction,

$$\begin{aligned} q\cdot z(q') &= (p_1' z_{11}(\alpha\hat{p}_1,\alpha\hat{p}_2) + p_2' z_{21}(\alpha\hat{p}_1,\alpha\hat{p}_2)) + \alpha(\hat{p}_3 z_{32}(p_3',p_4') + \hat{p}_4 z_{42}(p_3',p_4')) \\ &= (p_1' z_{11}(\hat{p}_1,\hat{p}_2) + p_2' z_{21}(\hat{p}_1,\hat{p}_2)) + \alpha(\hat{p}_3 z_{32}(p_3',p_4') + \hat{p}_4 z_{42}(p_3',p_4')) \end{aligned}$$

Note that $p_1' z_{11}(\hat{p}_1,\hat{p}_2) + p_2' z_{21}(\hat{p}_1,\hat{p}_2)$ does not depend on $\alpha > 0$ and $\hat{p}_3 z_{32}(p_3',p_4') + \hat{p}_4 z_{42}(p_3',p_4') < 0$. Hence if $\alpha > 0$ is sufficiently large, then $q\cdot z(q')$ is negative. Similarly,

$$\begin{aligned} q'\cdot z(q) &= \alpha(\hat{p}_1 z_{11}(p_1',p_2') + \hat{p}_2 z_{21}(p_1',p_2')) + (p_3' z_{32}(\alpha\hat{p}_3,\alpha\hat{p}_4) + p_4' z_{42}(\alpha\hat{p}_3,\alpha\hat{p}_4)) \\ &= \alpha(\hat{p}_1 z_{11}(p_1',p_2') + \hat{p}_2 z_{21}(p_1',p_2')) + (p_3' z_{32}(\hat{p}_3,\hat{p}_4) + p_4' z_{42}(\hat{p}_3,\hat{p}_4)) \end{aligned}$$

Note that $\hat{p}_1 z_{11}(p_1',p_2') + \hat{p}_2 z_{21}(p_1',p_2') < 0$ and $p_3' z_{32}(\hat{p}_3,\hat{p}_4) + p_4' z_{42}(\hat{p}_3,\hat{p}_4)$ does not depend on $\alpha > 0$. Hence if $\alpha > 0$ is sufficiently large, then $q\cdot z(q')$ is negative.

17.F.3 Define

$$\omega_1 = (\omega_{11}, \omega_{21}, 0, 0) = (1,0,0,0) \text{ and } \omega_2 = (\omega_{12}, \omega_{22}, 0, 0) = (0,1,0,0).$$

The aggregate excess demand function is then

$$z(p) = (-1, -1, p_1/p_3, p_2/p_4).$$

Hence, if we let $p = (4,1,8,4)$ and $p' = (1,4,4,8)$, then

$$z(p) = (-1, -1, 1/2, 1/4) \text{ and } z(p') = (-1, -1, 1/4, 1/2)$$

and hence $p' \cdot z(p) = p \cdot z(p') = -1$. Thus the weak axiom of revealed preference is not satisfied.

17.F.4 If the individual excess demand functions satisfies the (weak) gross substitution property, then so does the aggregate excess demand function (p. 612). It is thus sufficient to show that the excess demand function of every consumer has this property. So suppose that consumer i has initial endowment $\omega_i \in \mathbb{R}^L_+$ with $\omega_i \neq 0$ and spends all his wealth on good ℓ. Denote by e_ℓ the vector in $\mathbb{R}^L_+$ whose ℓth coordinate is one and all the other coordinates are zero. Then his excess demand function $z_i(\cdot)$ is given by

$$z_i(p) = (p \cdot \omega_i / p_\ell) e_\ell - \omega_i.$$

Thus

$$\partial z_{ki}(p)/\partial p_m = 0 \text{ for every } k \neq \ell \text{ and } m;$$

$$\partial z_{\ell i}(p)/\partial p_k = \omega_{ki}/p_\ell \geq 0 \text{ for every } k \neq \ell;$$

Hence it has the weak gross substitution property.

17.F.5 For the gross substitute property with any initial endowments, recall from Exercise 3.G.4(c) and (d) that if $-[x_{\ell i} u''_{\ell i}(x_{\ell i})/u'_{\ell i}(x_{\ell i})] < 1$, then

$\partial x_{ki}(\bar{p},\bar{p}\cdot\omega_i)/\partial w_i \geq 0$ and $\partial x_{ki}(\bar{p},\bar{p}\cdot\omega_i)/\partial p_\ell > 0$ for every k and ℓ with $k \neq \ell$. (The assumption in (d) that all $u_\ell(\cdot)$ are identical is not necessary for this result.) Hence

$$\partial z_{ki}(\bar{p})/\partial p_\ell = \partial x_{ki}(\bar{p},\bar{p}\cdot\omega_i)/\partial p_\ell + (\partial x_{ki}(\bar{p},\bar{p}\cdot\omega_i)/\partial w_i)\omega_{\ell i} > 0.$$

We shall next prove that if $u_i(x_i) = (\sum_\ell \alpha_{\ell i} x_{\ell i}^\rho)^{1/\rho}$ with $0 < \rho < 1$, then $-[x_{\ell i}u''_{\ell i}(x_{\ell i})/u'_{\ell i}(x_{\ell i})] < 1$. In fact, if $u_{\ell i}(x_{\ell i}) = \alpha_{\ell i}x_{\ell i}^\rho$, then

$$u'_{\ell i}(x_{\ell i}) = \alpha_{\ell i}\rho x_{\ell i}^{\rho-1} \text{ and}$$

$$u''_{\ell i}(x_{\ell i}) = \alpha_{\ell i}\rho(\rho - 1)x_{\ell i}^{\rho-2}.$$

Thus $-[x_{\ell i}u''_{\ell i}(x_{\ell i})/u'_{\ell i}(x_{\ell i})] = 1 - \rho < 1$.

17.F.6 [First printing errata: The hypothesis made do not seem to be sufficient for the desired property. We proceed to give a proof under the additional hypothesis that the utility functions of the consumers for the two consumption goods are Cobb-Douglas.] For each $i = 1,\dots,I$, let $u_i\colon \mathbb{R}^2_+ \to \mathbb{R}$ be the utility function of consumer i. Denote the initial endowment of consumer i by $\omega_i \in \mathbb{R}^2_+$. For each $\ell = 1,2$, let $f_\ell\colon \mathbb{R}^2_+ \to \mathbb{R}$ be the production function of consumption good ℓ. By the Cobb-Douglas assumption, we have

$$u_i(x_i) = u_i(x_{1i},x_{2i}) = x_{1i}^{\alpha_i}x_{2i}^{1-\alpha_i},$$

$$f_\ell(v_\ell) = f_\ell(v_{1\ell},v_{2\ell}) = v_{1\ell}^{\beta_\ell}v_{2\ell}^{1-\beta_\ell},$$

where $\alpha_i \in (0,1)$ and $\beta_\ell \in (0,1)$ for every i and ℓ. Let $p = (p_1,p_2) \in \mathbb{R}^2_+$ be a vector of consumption good prices and $q = (q_1,q_2) \in \mathbb{R}^2_+$ be a vector of input prices. Denote the demand function of consumer i by $x_i(p,w_i) = (x_{1i}(p,w_i),x_{2i}(p,w_i)) \in \mathbb{R}^2_+$, and the input demand function and the cost function for unit output by $z_\ell(q) = (z_{1\ell}(q),z_{2\ell}(q)) \in \mathbb{R}^2_+$ and $c_\ell(q) \geq 0$. Then, by Example 3.D.1,

$$x_i(p,w_i) = (\alpha_i w_i/p_1, (1 - \alpha_i)w_i/p_2).$$

By Example 5.C.1,

$$z_\ell(q) = ((\beta_\ell/(1 - \beta_\ell))^{1-\beta_\ell}(q_2/q_1)^{1-\beta_\ell}, ((1 - \beta_\ell)/\beta_\ell)^{\beta_\ell}(q_1/q_2)^{\beta_\ell}),$$

$$c_\ell(q) = ((\beta_\ell/(1 - \beta_\ell))^{1-\beta_\ell} + ((1 - \beta_\ell)/\beta_\ell)^{\beta_\ell})q_1^{\beta_\ell}q_2^{1-\beta_\ell}.$$

Let's now propose an equilibrium concept for the induced economy, which will later shown to be equivalent to the standard equilibrium concept for the original production economy. We say that an allocation $(v_1^*,\ldots,v_I^*) \in \mathbb{R}_+^2 \times \ldots \times \mathbb{R}_+^2$ of input vectors and a price vector $q \in \mathbb{R}_+^2$ for inputs constitute an induced-economy equilibrium if

(i) For every i, $v_i^* = \sum_\ell x_{\ell i}(c_1(q),c_2(q),q\cdot\omega_i)z_\ell(q)$.

(ii) $\sum_i v_i^* \le \sum_i \omega_i$.

Condition (i) can be motivated from the standard equilibrium concept in the following way: If the input price vector is equal to to q, then the wealth of consumer i is $q\cdot\omega_i$, and the zero profit condition implies that the output price vector must be equal to $(c_1(q),c_2(q))$. Then his consumption demand vector is $x_i(c_1(q),c_2(q),q\cdot\omega_i)$. The vector of inputs necessary to produce this consumption demand is $\sum_\ell x_{\ell i}(c_1(q),c_2(q),q\cdot\omega_i)z_\ell(q)$. By condition (ii), an induced-economy equilibrium is defined to be a point where the sum of these input vectors is less than or equal to the aggregate resource vector $\sum_i \omega_i$.

Suppose now that the consumption vectors $(x_1^*,\ldots,x_I^*) \in \mathbb{R}_+^2 \times \ldots \times \mathbb{R}_+^2$, input demands $(z_1^*,z_2^*) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$, and price vectors p and q constitute a standard Walrasian equilibrium. For each i, define $v_i^* = \sum_\ell x_{\ell i}^* z_\ell(q)$. We shall now prove that $(v_1^*,\ldots,v_I^*) \in \mathbb{R}_+^2 \times \ldots \times \mathbb{R}_+^2$ and $q \in \mathbb{R}_+^2$ for inputs constitute an induced-economy equilibrium. Since $p = (c_1(q),c_2(q))$, condition (i) is satisfied. By the market-clearing condition for inputs of the original equilibrium,

$$\sum_i v_i^* = \sum_i \sum_\ell x_{\ell i}^* z_\ell(q) = \sum_\ell (\sum_i x_{\ell i}^*) z_\ell(q) \le \sum_i \omega_i.$$

Hence condition (ii) is satisfied.

Suppose conversely that $(v_1^*, \ldots, v_I^*) \in \mathbb{R}_+^2 \times \ldots \times \mathbb{R}_+^2$ and $q \in \mathbb{R}_+^2$ for inputs constitute an induced-economy equilibrium. For each i, define $x_i^* = x_i(c_1(q), c_2(q), q\cdot\omega_i)$. For each ℓ, define $z_\ell^* = (\sum_i x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i)) z_\ell(q)$. Let $p = (c_1(q), c_2(q))$. We shall now prove that $(x_1^*, \ldots, x_I^*)$, input demands $(z_1^*, z_2^*) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$, and price vectors p and q constitute a standard Walrasian equilibrium. First, the profit maximization conditions are satisfied because $p = (c_1(q), c_2(q))$ and z_ℓ^* is proportional to $z_\ell(q)$. The utility maximization condition follows from $p = (c_1(q), c_2(q))$ and $x_i^* = x_i(c_1(q), c_2(q), q\cdot\omega_i)$. As for the market-clearing condition for the inputs,

$$\begin{aligned}\sum_\ell z_\ell^* &= \sum_\ell (\sum_i x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i)) z_\ell(q) \\ &= \sum_i (\sum_\ell x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i) z_\ell(q)) = \sum_i v_i^* \le \sum_i \omega_i.\end{aligned}$$

For the outputs, since $z_\ell^* = (\sum_i x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i)) z_\ell(q)$, we have $f_\ell(z_\ell^*) = (\sum_i x_i^*) z_\ell(q)$.

By this equivalence, it is sufficient to prove that there is a unique reduced-economy equilibrium. Since it is given by the inequality

$$\sum_i \sum_\ell x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i) z_\ell(q) - \sum_i \omega_i \le 0,$$

it is sufficient to prove that for each i, the input demand function

$$\sum_\ell x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i) z_\ell(q)$$

has the gross substitute property. In fact, by the explicit expressions for $x_i(p, w_i)$, $z_\ell(q)$, and $c_\ell(q)$ given above, we can obtain

$$\begin{aligned}&\sum_\ell x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i) z_{1\ell}(q) \\ &= ((\alpha_i/\gamma_1)(\beta_1/(1-\beta_1))^{1-\beta_1} + ((1-\alpha_i)/\gamma_2)(\beta_2/(1-\beta_2))^{1-\beta_2})(q\cdot\omega_i/q_1) - \omega_{1i},\end{aligned}$$

$$\sum_\ell x_{\ell i}(c_1(q), c_2(q), q\cdot\omega_i) z_{2\ell}(q)$$

$= ((\alpha_i/\gamma_1)((1 - \beta_1)/\beta_1)^{\beta_1} + ((1 - \alpha_i)/\gamma_2)((1 - \beta_2)/\beta_2)^{\beta_2})(q \cdot \omega_i/q_2) - \omega_{2i}$,

where $\gamma_\ell = (\beta_\ell/(1 - \beta_\ell))^{1-\beta_\ell} + ((1 - \beta_\ell)/\beta_\ell)^{\beta_\ell} > 0$. It is then easy to show by a straightforward calculation that the gross substitute property holds. In fact, if we define

$$\delta_{1i} = ((\alpha_i/\gamma_1)(\beta_1/(1 - \beta_1))^{1-\beta_1} + ((1 - \alpha_i)/\gamma_2)(\beta_2/(1 - \beta_2))^{1-\beta_2}),$$

$$\delta_{2i} = ((\alpha_i/\gamma_1)((1 - \beta_1)/\beta_1)^{\beta_1} + ((1 - \alpha_i)/\gamma_2)((1 - \beta_2)/\beta_2)^{\beta_2}),$$

then $\delta_{1i} + \delta_{2i} = 1$ and the input demand function is equal to the standard demand function derived from the Cobb-Douglas "utility" function $v_{1i}^{\delta_{1i}} v_{2i}^{\delta_{2i}}$ for the inputs. Thus the gross substitution property follows from this fact and Example 17.F.2.

Note finally that the above proof shows that the uniqueness of equilibrium for this setup can also be obtained from Exercise 17.F.9. To see this, suppose that the consumption vectors $(x_1^*,\dots,x_I^*) \in \mathbb{R}_+^2 \times \dots \times \mathbb{R}_+^2$ and an input price vector q constitute a standard Walrasian equilibrium. Just as above, if we define $v_i^* = \sum_\ell x_{\ell i}^* z_\ell(q)$ for each i, then $q \cdot v_i^* \le q \cdot \omega_i$ and $f_\ell(x_{\ell i}^* z_\ell(q)) = x_{\ell i}^*$. Thus no trade is required to attain the allocation if the technologies of the $f_\ell(z_\ell)$ are freely available.

17.F.7 Suppose that $z(p) =$ and $z(p') \neq 0$. By the homogeneity of degree zero, we can assume that $p_2 = p_2'$. Then $p \cdot z(p') = p_1 z_1(p') + p_2' z_2(p')$. By Walras' law, $p_1' z_1(p') + p_2' z_2(p') = 0$. The gross substitute property implies that if $p_1 > p_1'$, then $0 = z_1(p) < z_1(p')$ and hence $p_1 z_1(p') > p_1' z_1(p')$. Also, if $p_1 < p_1'$, then $0 = z_1(p) > z_1(p')$ and hence $p_1 z_1(p') > p_1' z_1(p')$. Therefore, in either case, we have

$$p \cdot z(p') = p_1 z_1(p') + p_2' z_2(p') > p_1' z_1(p') + p_2' z_2(p') = 0.$$

17.F.8 Let p and p' be equilibrium price vectors and $\lambda \in [0,1]$. Define $p'' = \lambda p + (1 - \lambda)p'$. We shall prove that $z(p'') = 0$ by contradiction. Suppose not, then, by (17.F.3), $p \cdot z(p'') > 0$ and $p' \cdot z(p'') > 0$. Hence $p'' \cdot z(p'') = \lambda p \cdot z(p'') + (1 - \lambda)p' \cdot z(p'') > 0$, a contradiction to Walras' law.

17.F.9 [First printing errata: The production set Y should be convex. It is not necessary to assume that preferences are continuous.] Let $(x_1,\ldots,x_I)$ be an equilibrium allocation such that $x_i - \omega_i \in Y$. Let $(p',x'_1,\ldots,x'_I)$ also be an equilibrium. We shall prove that $x_i = x'_i$ for every i. By $x_i - \omega_i \in Y$, the assumption of constant returns to scale, and the profit maximization condition of an equilibrium, we have $p' \cdot x_i \le p' \cdot \omega_i$ for every i. Thus, by the utility maximization condition of an equilibrium, $x'_i \succsim_i x_i$ for every i. Since Y is convex, the consumption allocation

$$((1/2)x_1 + (1/2)x'_1, \ldots, (1/2)x_I + (1/2)x'_I)$$

is feasible. Moreover, by the strict convexity of the $\succsim_i$, if $x_i \ne x'_i$, then $(1/2)x_i + (1/2)x'_i \succ_i x_i$. On the other hand, since preferences are strongly monotone, the first welfare theorem holds, implying that there is no i for whom $(1/2)x_i + (1/2)x'_i \succ_i x_i$. We must thus have $x_i = x'_i$ for every i.

17.F.10 Let p be an equilibrium price vector and $dp \in \mathbb{R}^L$. By the definition,

$$dp \cdot Dz(p)dp = \lim_{\varepsilon\to 0}(1/\varepsilon^2)((p + \varepsilon dp) - p)\cdot(z(p + \varepsilon dp) - z(p)).$$

But by (17.F.3),

$$((p + \varepsilon dp) - p)\cdot(z(p + \varepsilon dp) - z(p)) = (-p)\cdot z(p + \varepsilon dp) \le 0.$$

Hence

$$\lim_{\varepsilon\to 0}(1/\varepsilon^2)((p + \varepsilon dp) - p)\cdot(z(p + \varepsilon dp) - z(p)) \le 0$$

and thus $dp \cdot Dz(p)dp \le 0$.

17.F.11 Let p be as in the exercise, fix an ℓ, and denote by $\hat{Z}$ the $(L-1) \times (L-1)$ matrix obtained from $Dz(p)$ by deleting the ℓth row and column. Then $\hat{Z}$ is negative semidefinite and, by Theorem M.D.4(i), $\det \hat{Z} \neq 0$. Denote by $\hat{I}$ the $(L-1) \times (L-1)$ identity matrix. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(\alpha) = |\alpha\hat{I} - \hat{Z}|$, then $f(0) = (-1)^{L-1}|\hat{Z}|$ and $f(\alpha) \to \infty$ as $\alpha \to \infty$ (because $f(\cdot)$ is a polynomial of degree $L-1$ and the coefficient for α^{L-1} is one). It is sufficient to show that $f(0) > 0$. For this, it is sufficient to show that, for any $\alpha \in \mathbb{R}$, if $f(\alpha) = 0$, then $\alpha < 0$. So let $f(\alpha) = 0$, then there exists $v \in \mathbb{R}^{L-1}$ such that $\|v\| = 1$ and $(\alpha\hat{I} - \hat{Z})v = 0$. Hence $v \cdot \hat{Z}v = \alpha$. Since $\hat{Z}$ is negative semidefinite, we must have $\alpha \le 0$. Since $\det \hat{Z} \neq 0$, we actually have $\alpha < 0$.

17.F.12 Let p be as in the exercise, fix an ℓ, and denote by $\hat{Z}$ the $(L-1) \times (L-1)$ matrix obtained from $Dz(p)$ by deleting the ℓth row and column. For each $k \neq \ell$, let $\hat{Z}_k$ be the row vector of $\hat{Z}$ that corresponds to the kth row $Dz_k(p)$ of $Dz(p)$. Denote by $\hat{p} \in \mathbb{R}^{L-1}_{++}$ the vector obtained from p by deleting the ℓth entry. We shall prove that $\hat{p}$ is has the property in the definition of a dominant diagonal (Definition M.D.2), that is,

$$|p_k \partial z_k(p)/\partial p_k| > \textstyle\sum_{m \neq \ell, k} |p_m \partial z_k(p)/\partial p_m|.$$

By $Dz(p)p = 0$, $\hat{Z}_k\hat{p} = -p_\ell \partial z_k(p)/\partial p_\ell$ for any $k \neq \ell$. By the gross substitution sign pattern $\hat{Z}_k\hat{p} < 0$. But, by the definition and again the gross substitution sign pattern,

$$\begin{aligned}\hat{Z}_k\hat{p} &= p_k \partial z_k(p)/\partial p_k + \textstyle\sum_{m \neq \ell, k} p_m \partial z_k(p)/\partial p_m \\ &= -|p_k \partial z_k(p)/\partial p_k| + \textstyle\sum_{m \neq \ell, k} |p_m \partial z_k(p)/\partial p_m|.\end{aligned}$$

Thus $|p_k \partial z_k(p)/\partial p_k| > \sum_{m \neq \ell, k} |p_m \partial z_k(p)/\partial p_m|$.

17.F.13 [First printing errata: The coefficient ρ should be less than -1, that is, $\rho < -1$.] Let $\ell = 1$ be the home good of country 1 and $\ell = 2$ be the home good of country 2. Then, by the first-order condition of utility maximization, if their demands at a price vector p are in the interior $\mathbb{R}^2_{++}$, then their individual excess demand functions for $\ell = 1$ are given by

$$z_{11}(p) = -(-\rho)^{1/(1-\rho)}(p_1/p_2)^{\rho/(1-\rho)},$$
$$z_{12}(p) = (-\rho)^{1/(1-\rho)}(p_1/p_2)^{-1/(1-\rho)}.$$

Hence the aggregate excess demand function for $\ell = 1$ at $p = (p_1,1)$ is

$$z_1(p_1,1) = (-\rho)^{1/(1-\rho)}(p_1{}^{-1/(1-\rho)} - p_1{}^{\rho/(1-\rho)}).$$

Hence

$$\partial z_1(p_1,1)/\partial p_1 = (-\rho)^{1/(1-\rho)}(p_1{}^{-1/(1-\rho)} - p_1{}^{\rho/(1-\rho)}).$$

Thus, by $\rho < -1$,

$$\partial z_1(1,1)/\partial p_1 = (-\rho)^{1/(1-\rho)}\left(\frac{-1}{1-\rho} - \frac{\rho}{1-\rho}\right) > 0.$$

Thus the index at $p = (1,1)$ is negative.

17.F.14 (a) For any two different ℓ and ℓ', and for any v, $\partial^2 f(v)/\partial v_\ell \partial v_{\ell'} = \alpha_\ell \alpha_{\ell'} f(v)/v_\ell v_{\ell'} > 0$. Hence any two inputs are complementary at any v.

(b) [First printing errata: The statement is not exactly true. Just as in the case of perfectly substitutable inputs (Exercise 5.C.10), we can find only weak complements, that is, $\partial^2 f(v)/\partial v_\ell \partial v_{\ell'} \geq 0$, if we do not impose any further assumption. To obtain a strict complement, we should assume that rank $D^2 f(v) = L - 2$. We should also assume that $v \gg 0$.] By the constant returns to scale, $Df(v)v = f(v)$. By differentiating both sides with respect to v and rearranging the derivatives, we obtain $D^2 f(v)v = 0$. Thus, for every ℓ, $\sum_{\ell'}(\partial^2 f(v)/\partial v_\ell \partial v_{\ell'})v_{\ell'} = 0$. By the concavity, $\partial^2 f(v)/\partial v_\ell^2 \leq 0$. Thus there must exist an $\ell' \neq \ell$ such that $(\partial^2 f(v)/\partial v_\ell \partial v_{\ell'})v_{\ell'} \geq 0$, that is, $\partial^2 f(v)/\partial v_\ell \partial v_{\ell'} \geq 0$. We have thus obtained a weak complement. To obtain a strict complement from rank $D^2 f(v) = L - 2$, suppose that there is no strict complement. Then, by $\sum_{\ell'}(\partial^2 f(v)/\partial v_\ell \partial v_{\ell'})v_{\ell'} = 0$, we must have $\partial^2 f(v)/\partial v_\ell \partial v_{\ell'} = 0$ for every ℓ'. So, denote by e the vector of $\mathbb{R}^L$ whose ℓth coordinate is one and the other coordinates are zero. Then $e \cdot D^2 f(v) = 0$. By the symmetry, $D^2 f(v)e = 0$. Since e and v are not proportional, this implies that rank $D^2 f(v) \leq L - 3$, violating our assumption. Hence there must exist at least one strict complement.

(c) This is the same as Exercise 5.C.7.

(d) The above result says that if goods ℓ and ℓ' are complements, then the input demand for ℓ will decrease as the price of good ℓ' increases. This direction of change is opposite to that required by the gross substitute property. Hence the presence of complements makes the gross substitute property less likely to hold for production inclusive excess demand functions (17.B.3).

17.F.15 Denote the consumer's preference by $\succsim$, his initial endowment vector by ω, and the (aggregate) production set by Y. An allocation (x^*, y^*) and a producer price vector q constitute a distorted equilibrium if

(i) $q \cdot y \leq q \cdot y^*$ for every $y \in Y$.

(ii) x^* is maximal for $\succsim$ in $\{x \in \mathbb{R}^L_+ : p \cdot x \leq q \cdot \omega + r \cdot x^*\}$, where

$$p_\ell = (1 + t_\ell)q_\ell \text{ and } r_\ell = t_\ell q_\ell$$

(iii) $x^* = \omega + y^*$.

Note that we are assuming that when the consumer sells his initial endowments, he receives the producer prices.

The uniqueness of of the equilibrium can be proved as follows. First, since the production sector is of the Leontief type, by the Nonsubstitution Theorem (Proposition 5.AA.2), we can assume that there is only one elementary activity for each producible good. Then, by Exercise 5.AA.2(b), the equilibrium producer price vector is unique up to scalar multiplication (assuming that every producible good is actually produced by means of Y at equilibrium). Let q be an equilibrium price vector. Suppose that there are two different equilibrium allocations (x,y) and (x',y'). Then $x \neq x'$. Since $\succsim$ is strictly convex, this means that the two corresponding budget sets are different. Thus $r\cdot x \neq r\cdot x'$. Assume without loss of generality that $r\cdot x > r\cdot x'$. Then, by the normality condition, $x > x'$. Hence $y > y'$ by the feasibility condition (iii). Thus $q\cdot y > q\cdot y'$. But this contradicts the profit maximization condition (i) for y'. The equilibrium allocation must therefore be unique.

Here is an example which shows that the normality condition is indispensable. In the example, $L = 2$, good 1 is the primary factor, $t_1 = 0$, $t_2 > 0$, and $\omega_2 = 0$.

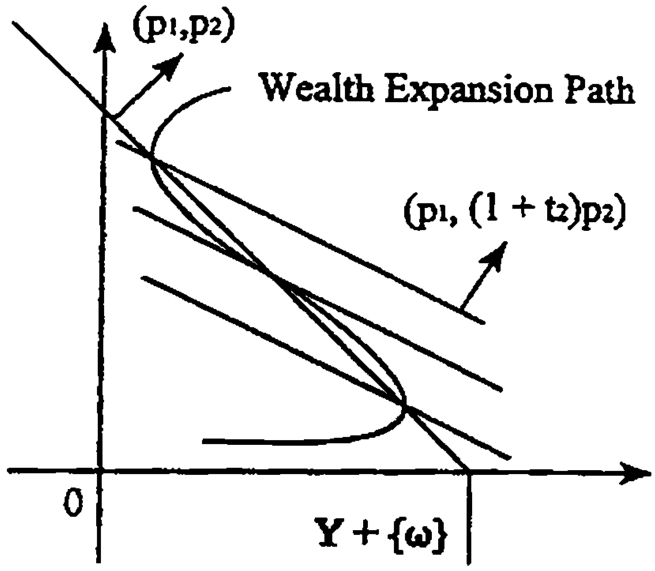


Figure 17.F.15

Since good 2 is not normal, the wealth expansion path for p can intersect the boundary of Y more than once.

17.F.16. (a) Let $p \in [0,r]^N$, $p' \in [0,r]^N$, $n \in \{1,\ldots,N\}$, $p'_n > p_n$, and $p'_m = p_m$ for any $m \neq n$. Then, by the SGS property, $\alpha g_m(p') + p_m > \alpha g_m(p) + p_m$, that is, $g_m(p') > g_m(p)$ for any $m \neq n$. Hence $g(\cdot)$ satisfies the GS property.

(b) An example that satisfies the GS property but not the SGS property can be constructed as follows. Define h: $[-1, 1] \to \mathbb{R}$ by $h(p) = p^2\sin(\pi/2p^2)$ if $p \neq 0$ and $h(0) = 0$. Define g: $[0,2] \to \mathbb{R}$ by $g(p) = h(p - 1) - p$. Note that, as required at the beginning of this exercise, we have $g(0) = 1 > 0$ and $g(2) = -1 < 0$. The function $g(\cdot)$ trivially satisfies the GS property, as it does not impose any constraint for functions defined on a subset of $\mathbb{R}$. We shall prove that it does not satisfy the SGS property. For this, we now prove that it is differentiable. It is sufficient to prove that $h(\cdot)$ is differentiable.
If $p \neq 0$, then, by the definition, $h(\cdot)$ is differentiable at p and

$$h'(p) = 2p\cdot\sin(\pi/2p^2) + p^2\cos(\pi/2p^2)\cdot(-\pi/p^3)$$

$$= 2p\cdot\sin(\pi/2p^2) - (\pi/p)\cos(\pi/2p^2).$$

The differentiability at p = 0 follows from

$$\lim_{p\to0}|h(p) - h(0)|/|p - 0| \le \lim_{p\to0}|p| = 0.$$

Thus $g(\cdot)$ is also differentiable and $g'(p) = h'(p - 1) - 1$ for every $p \in [0,2]$.

We can now show that, for any $\alpha > 0$, the function $\alpha g(p) + p$ is not increasing. In fact, for every $p \in (0,1)$, $h'(p) \le 2p - (\pi/p)\cos(\pi/2p^2)$. Hence, for every $\beta > 0$, there exists $p \in (0,1)$ such that $h'(p) < -\beta$. Thus, for every $\alpha > 0$, there exists $p \in (1,2)$ such that

$$\alpha g'(p) + 1 = \alpha(h'(p - 1) - 1) - 1 < 0.$$

Thus the function $\alpha g(p) + p$ is not increasing. Hence the SGS property does not hold. [Note however that there exists an $\alpha > 0$ such that $\alpha g(p) + p \in [0,2]$ for all $p \in [0,2]$. To establish this, define $D = \{p \in [0,2]: h(p - 1) \le 0\}$, then $0 \in D$ and D is compact. Consider the function $p/(p - h(p - 1))$ defined on D. This is continuous and $p/(p - h(p - 1)) > 0$ for every $p \in D$. So the minimum is attained, which is positive. Let $\alpha > 0$ be smaller than the minimum, then we can show that $\alpha g(p) + p \in [0,2]$ for all $p \in [0,2]$.]

As for the continuously differentiable case, write $q = (r,\ldots,r) \in \mathbb{R}^N_{++}$. Let $g: [0,r]^N \to \mathbb{R}^N$ be continuously differentiable and satisfy $g(0) \gg 0$ and $g(q) \ll 0$. We remark the following two facts:

For every $p \in [0,r]^N$ and every n, if $p_n = 0$, then $g_n(p) > 0$.

For every $p \in [0,r]^N$ and every n, if $p_n = r$, then $g_n(p) < 0$.

If $p \in \{0,q\}$, then these facts follow directly from the assumption. If not, they are immediate consequences of the GS property.

For each n, define $C_n = \{p \in [0,r]^N: g_n(p) \ge 0\}$ and $D_n = \{p \in [0,r]^N: g_n(p) \le 0\}$. Then, by the above facts, $C_n \subset \{p \in [0,r]^N: p_n < r\}$ and $D_n \subset \{p \in$

$[0,r]^N$: $p_n > 0\}$. Thus, by the continuous differentiability (or, actually, just by the continuity), the following two minima, $\text{Min}\{(r - p_n)/g_n(p)$: $p \in C_n\}$ and $\text{Min}\{- p_n/g_n(p)$: $p \in D_n\}$, exist and are positive. Let $K_n > 0$ be smaller than those two minima. Then, for every $\alpha \in (0,K_n)$ and every $p \in [0,r]^N$, we have $0 \le \alpha g_n(p) + p_n \le r$.

For each n, define $L_n = \text{Max}\{|\partial g_n(p)/\partial p_n|$: $p \in [0,r]^I\}$. This is well defined because each $\partial g_n(p)/\partial p_n$ is a continuous function defined on the compact set $[0,r]^N$. Then, for each $\alpha \in (0,1/L_n)$,

$$\partial[\alpha g_n(p) + p_n]/\partial p_n = \alpha(\partial g_n(p)/\partial p_n) + 1 \ge - \alpha K + 1 > 0.$$

Note also that $\partial[\alpha g_m(p) + p_m]/\partial p_n = \alpha(\partial g_m(p)/\partial p_n) > 0$ for any $m \ne n$ by the GS property.

Now let $K = \text{Min}\{K_1,\ldots,K_N,1/L_1,\ldots,1/L_N\}$ and $\alpha \in (0,K)$, then $\alpha g(p) + p \in [0,r]^N$ and $\partial[\alpha g_m(p) + p_m]/\partial p_n > 0$ for every $p \in [0,r]^N$, n, and m. Hence $g(\cdot)$ has the SGS property.

(c) Pick an $\alpha > 0$ for which the SGS property is met. Then the function h: $[0,r]^N \to [0,r]^N$ defined by $h(p) = \alpha g(p) + p$ satisfies the conditions of the Tarski fixed point theorem (Theorem M.I.3). Hence there exists $p \in [0,r]^N$ such that $p = h(p)$. By the construction of $h(\cdot)$, this implies that p is an equilibrium. A graphical illustration is given below:

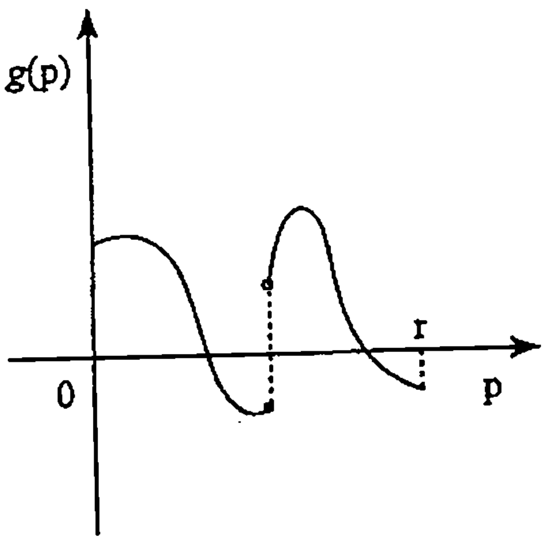


Figure 17.F.16(c)

(d) Define g: $[0,4]^2 \to [0,4]^2$ by

$$g_1(p) = -(p_1 - 1)(p_1 - 2)(p_1 - 3) - p_1 + p_2,$$
$$g_2(p) = -(p_2 - 1)(p_2 - 2)(p_2 - 3) + p_1 - p_2.$$

then $g(0,0) = (6,6)$, $g(4,4) = (-6, -6)$, $g(\cdot)$ is continuously differentiable and satisfies the GS property. Hence, by (b), it satisfies the SGS property. Moreover, $g^{-1}(0) \supset \{(1,1),(2,2),(3,3)\}$.

(e) Suppose that $g(p) = g(p') = 0$. We shall now prove that there exists p" such that $p" \geq p$, $p" \geq p'$, and $g(p") = 0$. (The existence of p" for which $p" \leq p$, $p" \leq p'$, and $g(p") = 0$ can be symmetrically proved.) Define $q \in [0,r]^N$ by $q_n = \text{Max}\{p_n, p'_n\}$ for each n. Assume that $g(\cdot)$ satisfies the SGS property with $\alpha > 0$. Then, for any $s \in [q_1,r] \times \ldots \times [q_N,r]$, $\alpha g(s) + s \geq \alpha g(q) + q \geq \alpha g(p) + p = p$ and $\alpha g(s) + s \geq \alpha g(q) + q \geq \alpha g(p') + p' = p'$. Hence $\alpha g(s) + s \geq q$ and the function h: $[q_1,r] \times \ldots \times [q_N,r] \to [q_1,r] \times \ldots \times [q_N,r]$ defined by $h(p) = \alpha g(p) + p$ is in fact well defined. By the Tarski fixed point theorem, there exists $p" \in [q_1,r] \times \ldots \times [q_N,r]$ such that $h(p") = p"$, that is, $p" \geq q$ and $g(p") = 0$. The proof is thus completed.

(f) We shall prove the existence of p^{max}. (The existence of p^{min} can be symmetrically proved.) For each n, let $q_n = \text{Sup}\{p_n : p \in [0,r]^N \text{ and } g(p) = 0\}$. Define $q = (q_1,\ldots,q_N) \in [0,r]^N$. Clearly, if $g(p) = 0$, then $q \geq p$. It is thus sufficient to show that $g(q) = 0$. For this, we need the following lemma:

> Suppose that a sequence $\{p^m\}_m$ in $[0,r]^N$ converges to a $p \in [0,r]^N$ with $p^m \leq p$ for all m, and the sequence $\{g(p^m)\}_m$ converges to $z \in [0,r]^N$, then $z \leq g(p)$.

In fact, by the SGS property, $g_n(p^m) - g_n(p) \leq (p_n - p_n^m)/\alpha$ for every n and m. Thus $z_n - g_n(p) \leq 0$, implying that $z \leq g(p)$.

For each m, by applying (e) (N - 1)-times, there exists $p^m \in [0,r]^N$ such that $g(p^m) = 0$ and $q_n - 1/m \leq p^m_n \leq q_n$ for each n. Thus $p^m \to q$. Thus, by the above lemma, $g(q) \geq 0$ and hence $\alpha g(q) + q \geq q$. Thus, by using the same fixed point argument as in (e), we can show that there exists $s \geq q$ such that $g(s) = 0$. By the definition of q, $s \leq q$ and hence $s = q$. Therefore $g(q) = 0$.

(g) Analogously to the index theorem, it is sufficient to prove that, for every $p \in [0,r]^n$, if $g(p) = 0$, then $(-1)^N \text{sgn}|Dg(p)| = 1$. (In fact, the boundary behavior of $g(\cdot)$ should require some attention to establish this sufficiency. But we shall neglect this. Showing the above equality on the sign will turn out to be sufficient for the application in (g).) Define $f: \mathbb{R} \to \mathbb{R}$ by $f(\lambda) = |\lambda I - Dg(p)|$, where I is the N x N identity matrix. Then $f(0) = (-1)^N|Dg(p)|$ and $f(\lambda) \to \infty$ as $\lambda \to \infty$ (because $f(\cdot)$ is a polynomial of degree N and the coefficient for λ^N is one). It is sufficient to show that $f(0) > 0$. For this, it is sufficient to show that, for any $\lambda \in \mathbb{R}$, if $\lambda \geq 0$, then $f(\lambda) \neq 0$. So let $\lambda \geq 0$, then $(\lambda I - Dg(p))v = \lambda v - Dg(p)v \gg 0$ since $Dg(p)v \ll 0$. By the GS sign pattern, this means that $\lambda I - Dg(p)$ has a positive dominant diagonal. By Theorem M.D.5(i), $f(\lambda) \neq 0$.

(h) It is sufficient to show that, under the given assumptions, $Dg(p)$ has a negative dominant diagonal. In fact, let $\partial g(p)/\partial p_{N+1} \in \mathbb{R}^N$ be the partial derivative of the demands for the first N + 1 goods with respect to p_{N+1}. Then, by applying (17.E.1) to the usual excess demand system for the N + 1 good, we obtain $Dg(p)p = -\partial g(p)/\partial p_{N+1}$. By the gross substitution property, $-\partial g(p)/\partial p_{N+1} \ll 0$. Hence $Dg(p)$ has the negative dominant diagonal. Hence the equilibrium is unique by (g).

17.F.17 Let $\omega \in \mathbb{R}^2_{++}$ be the identical initial endowment vector of the consumers and let $p \in \mathbb{R}^2_{++}$ be a price vector. The demands for good 1 constitute a uniform distribution on $[0, p\cdot\omega/p_1]$ and hence the average demand is $p\cdot\omega/2p_1$. Thus the average excess demand for good 1 is $p\cdot\omega/2p_1 - \omega_1$. Similarly, the average excess demand for good 2 is $p\cdot\omega/2p_2 - \omega_2$. Thus

$$z(p) = (p\cdot\omega/2p_1 - \omega_1,\ p\cdot\omega/2p_2 - \omega_2).$$

This can be generated by a consumer whose utility function is $u(x) = x_1^{1/2}x_2^{1/2}$ (Cobb Douglas) and initial endowment vector is ω.

17.G.1 Let $\hat{x}_1(\cdot)$ be the demand function of the first consumer for the first L - 1 good, with the price of the last commodity normalized to be one. Then $\hat{z}_1(\bar{p};\hat{\omega}_1) = \hat{x}_1(\bar{p},\bar{p}\cdot\omega_1) - \hat{\omega}_1$. Thus

$$D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1) = D_{w_1}\hat{x}_1(\bar{p},\bar{p}\cdot\bar{\omega}_1)\hat{\bar{p}}^T - I_{L-1},$$

where $\hat{\bar{p}} = (\bar{p}_1,\ldots,\bar{p}_{L-1})$ and I_{L-1} is the $(L - 1) \times (L - 1)$ identity matrix.

Thus, for every $v \in \mathbb{R}^{L-1}$ with $\hat{\bar{p}}\cdot v = 0$, we have $D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1)v = -v$. Thus

$\{D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1)v \in \mathbb{R}^{L-1}: v \in \mathbb{R}^{L-1}\} \supset \{v \in \mathbb{R}^{L-1}: \hat{\bar{p}}\cdot v = 0\}$. Moreover, writing $e = (1,0,\ldots,0) \in \mathbb{R}^{L-1}_+$, we have

$$\hat{\bar{p}}\cdot D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1)e = \hat{\bar{p}}\cdot(D_{w_1}\hat{x}_1(\bar{p},\ \bar{p}\cdot\bar{\omega}_1)\bar{p}_1 - e) = \bar{p}_1(\hat{\bar{p}}\cdot D_{w_1}\hat{x}_1(\bar{p},\ \bar{p}\cdot\bar{\omega}_1) - 1)$$

$$= -\ \bar{p}_1 D_{w_1}x_{L1}(\bar{p},\ \bar{p}\cdot\bar{\omega}_1) < 0,$$

by the strict normality assumption. Thus $D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1)e \notin \{v \in \mathbb{R}^{L-1}: \hat{\bar{p}}\cdot v = 0\}$.

Hence $\{D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1)v \in \mathbb{R}^{L-1}: v \in \mathbb{R}^{L-1}\} = \mathbb{R}^{L-1}$ and rank $D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\hat{\bar{\omega}}_1) = L - 1$.

Since rank $D_p\hat{z}(\bar{p};\bar{q}) = L - 1$ (which is implied by the differentiability of $p(\cdot)$), this and (17.G.1) imply that rank $Dp(\bar{q}) = L - 1$.

17.G.2 Let $\hat{A}$ be any $(L - 1) \times (L - 1)$ negative definite matrix. Write $e = (1,0,\ldots,0) \in \mathbb{R}^{L-1}_{+}$. Since $e\cdot\hat{A}e < 0$, there exists $\hat{v} \in \mathbb{R}^{L-1}_{++}$ such that $e\cdot\hat{A}\hat{v} < 0$ and $\hat{\bar{p}}\cdot\hat{v} < 1$, where $\hat{\bar{p}} = (\bar{p}_1,\ldots,\bar{p}_{L-1})$. Define $v = (\hat{v}, 1 - \hat{\bar{p}}\cdot\hat{v}) \in \mathbb{R}^{L}_{++}$, then $\bar{p}\cdot v = 1$. Now let the first consumer's preference be represented by $u_1(x_1) = \min\{x_{11}/v_1,\ldots,x_{L1}/v_L\}$, then $D_{w_1}\hat{x}_1(p,w_1) = \hat{v}$.

Consider an L-consumer economy such that the aggregate excess demand at $\bar{p}$ is equal to $- z_1(\bar{p};\bar{\omega}_1)$ and the $(L - 1) \times (L - 1)$ submatrix of the Jacobian of the aggregate excess demand function is equal to $\hat{A}^{-1} - D_p\hat{z}_1(\bar{p};\bar{\omega}_1)$; the existence of such an economy is guaranteed by Proposition 17.E.2, as applied in the proof of Proposition 17.G.1. Then $\bar{p}$ is an equilibrium price vector of the $(L + 1)$-consumer economy, which now includes the first consumer as well. For this $(L + 1)$-consumer economy, we have $D_p\hat{z}(\bar{p};\hat{\bar{\omega}}_1) = \hat{A}^{-1}$, which is negative definite. By (17.G.1),

$$Dp(\hat{\bar{\omega}}_1) = - [D_p\hat{z}(\bar{p};\bar{\omega}_1)]^{-1}D_{\hat{\omega}_1}\hat{z}_1(\bar{p};\bar{\omega}_1) = - \hat{A}D_{w_1}\hat{x}_1(p,\hat{\bar{p}}\cdot\hat{\bar{\omega}}_1 + \omega_{L1})p^T = - \hat{A}\hat{v}p^T.$$

Thus $\partial p_1(\hat{\bar{\omega}}_1)/\partial\omega_{11} = e\cdot(- \hat{A}\hat{v}p^T)e = - p_1e\cdot\hat{A}\hat{v}$. By our choice of $\hat{v}$, $- p_1e\cdot\hat{A}\hat{v} > 0$. Hence $\partial p(\hat{\bar{\omega}}_1)/\partial\omega_{11} > 0$.

17.G.3 [<u>First printing errata</u>: On the third line of this exercise, "the SDS property" should be "the SGS property".] We shall prove that $\hat{p}^{max} \geq p^{max}$. (The other inequality $\hat{p}^{min} \geq p^{min}$ can be symmetrically proved.) For this, it is sufficient to show that there exists $p \geq p^{max}$ such that $\hat{g}(p) = 0$. But since $\hat{g}(p^{max}) \geq g(p^{max}) = 0$, we can apply the fixed point argument to the function $\alpha\hat{g}(p) + p$ on the domain $[p_1^{max}, r] \times \ldots \times [p_N^{max}, r]$, just as we did in (e) of Exercise 17.F.16. Thus there exists $p \geq p^{max}$ such that $\hat{g}(p) = 0$. This result is illustrated in the following picture for the case of $N = 1$.

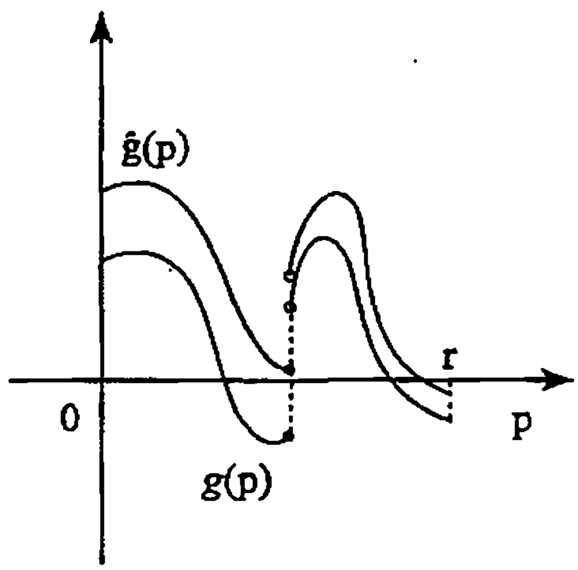


Figure 17.G.3

17.H.1 Let $t \in \mathbb{R}_+$. Define $M = \{\ell = 1,\ldots,L: \psi(p(t)) = z_\ell(p(t))/p_\ell(t)\}$ and $N = \{1,\ldots,L\}\setminus M$. Then $\psi(p(t))p(t) - z(p(t)) \in \mathbb{R}^L_+$ and $\psi(p(t))p_\ell(t) - z_\ell(p(t)) > 0$ if and only if $\ell \in N$. Then, by $Dz(p(t))p(t) = 0$,

$$dz(p(t))/dt = Dz(p(t))(dp(t)/dt) = Dz(p(t))z(p(t))$$

$$= - Dz(p(t))(\psi(p(t))p(t) - z(p(t))).$$

Hence, by the differential version of the gross substitution property, $dz_\ell(p(t))/dt < 0$ for every $\ell \in M$. Therefore, for every $\ell \in M$,

$$d[z_\ell(p(t))/p_\ell(t)]/dt = \frac{(dz_\ell(p(t))/dt)p_\ell(t) - z_\ell(p(t))^2}{p_\ell(t)^2} < 0.$$

Thus $\psi(p(t))$ is decreasing through time.

(b) The assertion was already established as Proposition 17.H.1. Here we shall give an alternative proof based on the fact that the function $\psi(\cdot)$ is a Lyapunov function.

Let $\varepsilon = \inf\ \{\psi(p(t)): t \geq 0\}$, then $\psi(p(t)) \to \varepsilon$ because $\psi(\cdot)$ is decreasing. By property (v) of Proposition 17.B.2 and the fact that $\sum_\ell p_\ell(t)^2$ is constant over t, we can suppose, by taking a subsequence if necessary, that $p(t) \to p^*$ for some $p^* \in \mathbb{R}^L_{++}$. Then $\psi(p^*) = \varepsilon$ and hence $\psi(\cdot)$ is not decreasing at p^*. Hence p^* must be an equilibrium price vector. (So, in fact, $\varepsilon = 0$.)

17.H.2 (a) A Walrasian price dynamics $\pi(\cdot)$ is given by

$$d\pi(t)/dt = z(\pi(t)) - p^{-1}(\pi(t)),$$

for every t. Or, more generally, it has the property that, for every $t \geq 0$, the sign of $d\pi(t)/dt$ is equal to that of $z(\pi(t)) - p^{-1}(\pi(t))$. It is interpreted as the trajectory of prices (perhaps in fictional time) adjusted by the auctioneer, who increases the price whenever there is an excess demand.

A Marshallian quantity dynamics $\zeta(\cdot)$ is given by

$$d\zeta(t)/dt = z^{-1}(\zeta(t)) - p(\zeta(t))$$

for every t. Or, more generally, it has the property that, for every $t \geq 0$, the sign of $d\zeta(t)/dt$ is equal to that of $z^{-1}(\zeta(t)) - p(\zeta(t))$. It may be interpreted as the trajectory of quantities adjusted by the producer, who increases the production level whenever the consumers' willingness to pay (demand price) exceeds the marginal cost of production (supply price).

(b) Let $f(p) = z(p) - p^{-1}(p)$, then $d\pi(t)/dt = f(\pi(t))$ and the system is Walrasian stable if and only if $f'(1) < 0$. Also, let $g(z) = z^{-1}(z) - p(z)$ and

$d\zeta(t)/dt = g(\zeta(t))$ and the system if Walrasian if and only if $g'(1) < 0$. By $(p^{-1})'(1) = 1/p'(1)$ and $(z^{-1})'(1) = 1/z'(1)$, we have

$$f'(1) = z'(1) - 1/p'(1),$$

$$g'(1) = 1/z'(1) - p'(1).$$

Since the technology is nearly of the constant return type, $p'(1)$ is very close to zero. Thus $f'(1) < 0$. On the other hand, we have $g'(1) < 0$ if and only if $1/z'(1) < p'(1)$. (In particular, if $z'(1) < 0$, that is, the slope of the excess demand function is negative, then this condition is satisfied.) This is depicted in the figures below:

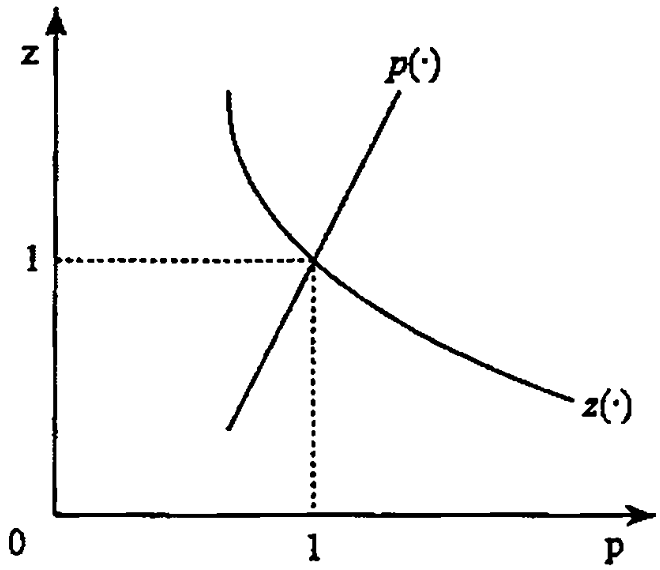


Figure 17.H.2(b.1)

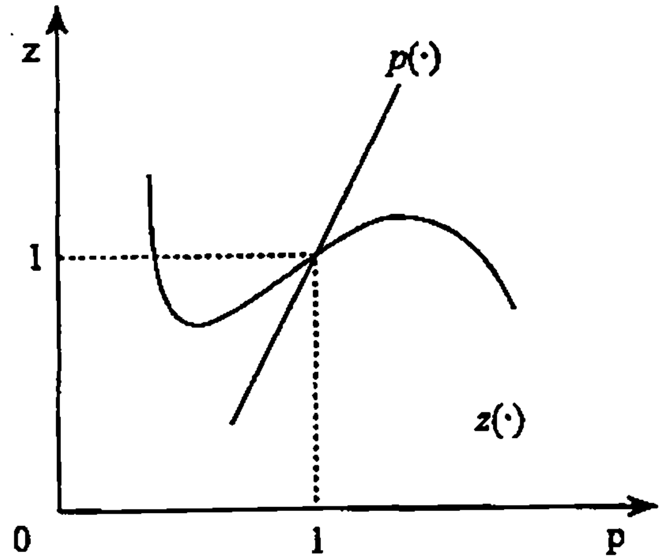


Figure 17.H.2(b.2)

(c) In the following general price and quantity dynamics, the price $\pi(t)$ is considered as the price that the consumers have to pay and the quantity $\zeta(t)$ is considered as the quantity that is produced (by the producers). Then the dynamics are given by

$$d\pi(t)/dt = z(\pi(t)) - \zeta(t),$$

$$d\zeta(t)/dt = \pi(t) - p(\zeta(t)).$$

That is, prices are adjusted according to the difference between the consumers' demand and the output level; quantities produced are adjusted according to the difference between the price and the marginal cost of production.

To draw a phase diagram and argue about dynamic trajectories, let's define $f(p,z) = z(p) - z$ and $g(p,z) = p - p(z)$. Then the Jacobian of this system at $(p,z) = (1,1)$ is given by

$$\begin{bmatrix} df(1,1)/dp & df(1,1)/dz \\ dg(1,1)/dp & dg(1,1)/dz \end{bmatrix} = \begin{bmatrix} z'(1) & -1 \\ 1 & -p'(1) \end{bmatrix}.$$

Hence the trace of this Jacobian is equal to $z'(1) - p'(1)$ and the determinant is equal to $1 - z'(1)p'(1)$. In the typical case of $z'(1) < 0$, then it is quite possible that $z'(1) - p'(1) < 0$ and $1 - z'(1)p'(1) > 0$. This is equivalent to saying that the characteristic roots of the Jacobian are either two negative real numbers or two conjugate complex numbers with negative real parts. But this is a necessary and sufficient condition for the system to be locally stable. If, furthermore,

$$(z'(1) - p'(1))^2 - 4(1 - z'(1)p'(1)) = (z'(1) + p'(1))^2 - 4 < 0,$$

then the characteristic roots are in fact two complex numbers. Hence the dynamic trajectories spiral around equilibrium. This is illustrated in the following figure.

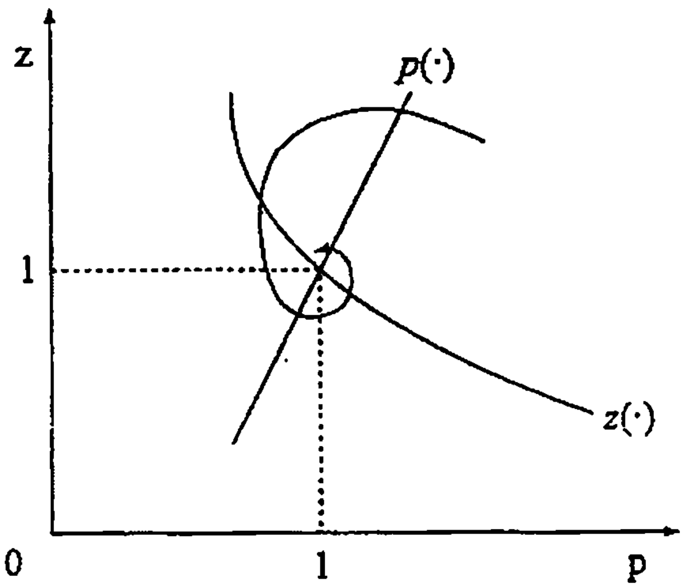


Figure 17.H.2(c)

(d) The system in (c) is locally stable if and only if $z'(1) - p'(1) < 0$ and $1 - z'(1)p'(1) > 0$. As we saw in (b), the Marshallian stability obtains if and only if $1/z'(1) < p'(1)$. If we take $p'(1)$ to be 0, then these two conditions are equivalent to $z'(1) < 0$. Hence the system in (c) is locally stable if and only if the equilibrium is Marshallian stable.

(e) The simplest price and quantity dynamics in this limit case is

$$d\pi(t)/dt = 1 - \zeta(t),$$

$$d\zeta(t)/dt = \pi(t) - 1.$$

Hence

$$\begin{bmatrix} df(1,1)/dp & df(1,1)/dz \\ dg(1,1)/dp & dg(1,1)/dz \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Hence the characteristic roots of the Jacobian is ± i. Thus the dynamic trajectories are spirals that keep the same distance from the equilibrium (1,1), as depicted below.

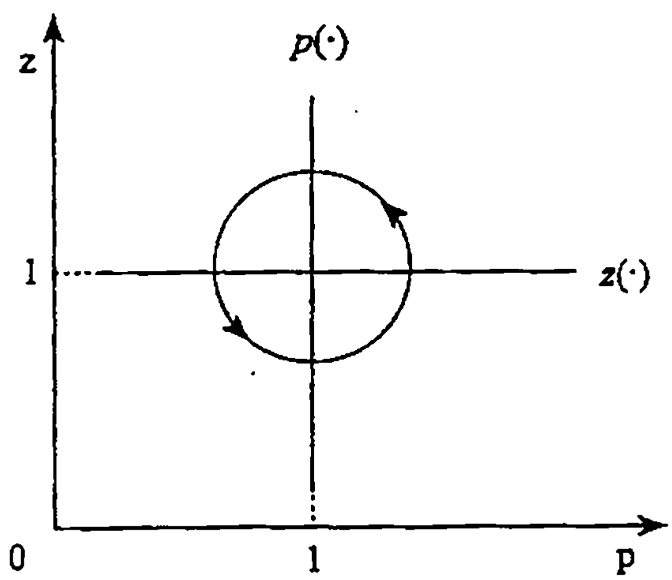


Figure 17.H.2(e)

A modified quantity dynamics is given by

$$d\zeta(t)/dt = \pi(t) - 1 + \theta(1 - \zeta(t)),$$

where $\theta > 0$. If we define $h(p,z) = p - 1 + \theta(1 - z)$, then $d\zeta(t)/dt = h(\pi(t),\zeta(t))$ and

$$\begin{bmatrix} df(1,1)/dp & df(1,1)/dz \\ dh(1,1)/dp & dh(1,1)/dz \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -\theta \end{bmatrix}.$$

Hence the characteristic roots of this Jacobian is $-\theta/2 \pm (1 - \theta^2/4)i$. Hence the modified dynamics is stable and the dynamic trajectories are spirals converging to $(1,1)$.

17.H.3 Here is an example of tatonnement trajectories for $L = 3$ in which there is a single equilibrium that is locally totally unstable.

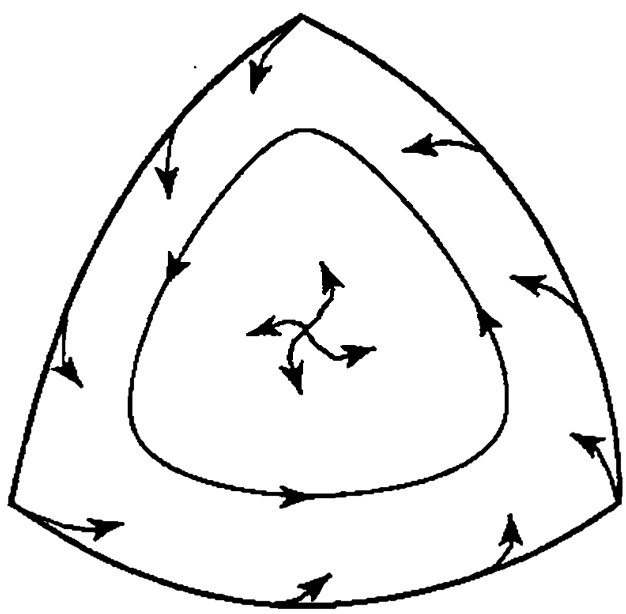


Figure 17.H.3

It is impossible to make such a single equilibrium a saddle. The reason is that a saddle has index -1 (as pointed out in Figure 17.H.2) and, according to the index theorem, if an equilibrium is a saddle, then there must be at least two others.

17.I.1 Suppose that we are initially given I types of consumers $(\succsim_i, \omega_i)$ $(i = 1,\dots,I)$ and a positive, rational distribution $(\pi_1,\dots,\pi_I)$ for the I types, with $\sum_i \pi_i = 1$. For each i, let m_i and n_i be positive integers such that $\pi_i =$

n_i/m_i. Let K be a common multiplier of the m_i. We shall now show that the initially given distribution $(\pi_1,\ldots,\pi_I)$ can be generated by (any replication of) an economy with K consumers. Note first that $K\pi_i$ is a positive integer and $\sum_i K\pi_i = K$. We can thus define a K-consumer economy $(\succsim^*_k,\omega^*_k)$ $(k = 1,\ldots,K)$ by $(\succsim^*_k,\omega^*_k) = (\succsim_i,\omega_i)$ if $\sum_{m<i}K\pi_m + 1 \le k \le \sum_{m\le i}K\pi_m$. Then the proportion of the consumers k for which $(\succsim^*_k,\omega^*_k) = (\succsim_i,\omega_i)$ is $K\pi_i/K = \pi_i$. Any replication of this K-consumer economy thus results in the same distribution as the initially given one.

17.I.2 Suppose the the input is measured by the horizontal axis and the output is measured by the vertical axis. Then the production set Y of the production function $q = v^2$ is depicted in the following figure:

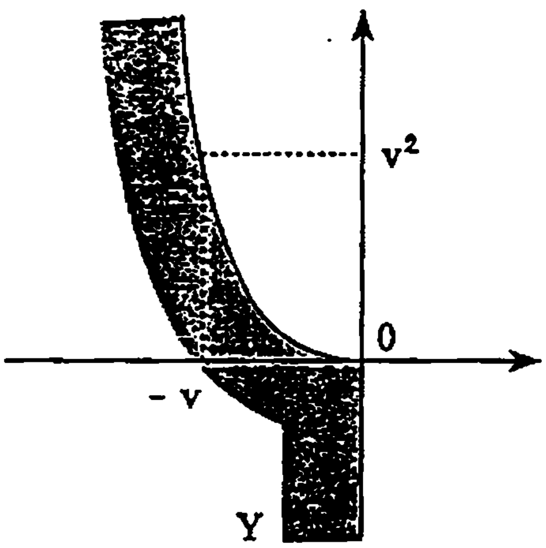


Figure 17.I.2

As we can see in the picture (and also in the equality $q/v = v$), Y is additive and $Y^* = \{y = (y_1,y_2) \in \mathbb{R}^2$: if $y_2 > 0$, then $y_1 < 0\}$.

The nonconvexity of Y is large in the sense that marginal productivity of the input is increasing on any part of the boundary of Y, however far away it is from the origin.

Hence, however big the number of the consumers is (and whatever the positive price of the output is), the firm gets an arbitrary large profit by supplying an arbitrarily large amount. Hence there is no profit-maximizing production plan and there is no useful sense in which an equilibrium, even nearly, exists.

17.I.3 (a) The convexity assumptions on the consumption sets and the preferences are not satisfied.

(b) Let (p_1, p_2, w) and (v_1, v_2) be the price of the high-quality good, the price of the the low-quality good, the wage level, the input demand of the production of the high-quality good, and the input demand of the production of the low-quality good at an equilibrium (assuming that it exists). We want to prove that $v_1 > 0$ and $v_2 > 0$.

We shall first show that $p_1 > 0$, $p_2 > 0$, and $w > 0$. The positivity of p_1 and p_2 follows from the utility maximization condition of an equilibrium. Suppose now that $w \le 0$. Then the profit maximization condition implies that the supply of the low-quality good must be one. However, the wealth of the poor does not exceed zero, implying that the demand for the low-quality good is zero. The feasibility condition of an equilibrium (together with $p_2 > 0$) is thus violated. Hence $w > 0$.

Since $w > 0$, the rich, as well as the poor, get positive wealth. Hence the demand for the high-quality good must be positive. Hence $v_1 > 0$. As for the low-quality good, since $f_2'(v_2) \to \infty$ as $v_2 \to 0$, the positivity of p_2 implies that of v_2.

(c) We assume that each agent has one unit of labor and normalize the total population of the economy to be one, so that the populations of the rich and of the poor are both 1/2. (This means that consumptions are measured in per capita term.) Let (p_1, p_2, w) and (v_1, v_2) be the equilibrium values as in (b). We can assume that $w = 1$. Since $0 < v_1 < 1$ and $0 < v_2 < 1$, $p_1 = w = 1$ and $p_2 f_2'(v_2) = 1$. Hence $v_2 = (\beta p_2)^{1/(1-\beta)}$, $f_2(v_2) = (\beta p_2)^{\beta/(1-\beta)}$, and the profit from the production of the low-quality good is $(\beta^{\beta/(1-\beta)} - \beta^{1/(1-\beta)})p_2^{1/(1-\beta)}$. Hence the (per capita) wealth of the rich is

$$1/2 + (\beta^{\beta/(1-\beta)} - \beta^{1/(1-\beta)})p_2^{1/(1-\beta)}.$$

We shall now prove by contradiction that $p_2 = 1/2$. If $p_2 > 1/2$, then the poor do not demand the low-quality good at all, contradicting $v_2 > 0$. We must thus have $p_2 \leq 1/2$. Suppose that $p_2 < 1/2$, then the poor concentrate on the low-quality good. Since their wealth comes exclusively from labor, it is equal to 1/2 (in per capita term). Thus the demand for the low-quality good is $1/2p_2$ and $v_2 = (1/2p_2)^{1/\beta}$. On the other hand, we also have $v_2 = (\beta p_2)^{1/(1-\beta)}$ (which followed from the profit maximization). Hence $p_2 = 2^{-(1-\beta)}\beta^{-\beta}$, $v_2 = 2^{(2/\beta-1)}\beta$, and the profit at the equilibrium is $(1 - \beta)/2$ and the rich's (per capita) wealth is $1/2 + (1 - \beta)/2 = 1 - \beta/2$. By $p_1 = 1$, this is the (per capita) demand for the high-quality good; by $f_1(v_1) = v_1$, this also implies that $v_1 = 1 - \beta/2$. Hence $v_1 + v_2 = 1 - \beta/2 + 2^{(2/\beta-1)}\beta = 1 + \beta(2^{2/\beta} - 1)/2 > 1$, a contradiction to the feasibility condition.

We have shown that $p_2 = 1/2$ if an equilibrium exists. We shall now prove that this in fact constitutes an equilibrium. Note first that $v_2 = (\beta/2)^{1/(1-\beta)}$ and $f_2(v_2) = (\beta/2)^{\beta/(1-\beta)} < 1$. This and the feasibility condition of labor implies that $v_1 = f_1(v_1) = 1 - (\beta/2)^{1/(1-\beta)}$. Also, the profit from the production of the low-quality good is

$$(\beta^{\beta/(1-\beta)} - \beta^{1/(1-\beta)})(1/2)^{1/(1-\beta)}$$

and the (per capita) wealth of the rich is

$$1/2 + (\beta^{\beta/(1-\beta)} - \beta^{1/(1-\beta)})(1/2)^{1/(1-\beta)}.$$

This is also equal to their (per capita) demand for the high quality good. The most important part of this exercise lies in the consumption of the poor, who has the role of clearing the output markets because of the multiplicity of their demands. More specifically, since $p_1 = 1$, $p_2 = 1/2$, and each agent is endowed with one unit of labor, the demand of each of the poor consists of one unit of the high-quality good and two units of the low-quality good. We can thus prescribe the fractions of the poor who choose one of these two demand vectors to clear the markets in the following way:

The fraction $(\beta/2)^{\beta/(1-\beta)}$ of the poor consumes two units of the low-quality good. The (per capita) consumption for it is thus $2\cdot(\beta/2)^{\beta/(1-\beta)}\cdot(1/2) = (\beta/2)^{\beta/(1-\beta)}$

The fraction $1 - (\beta/2)^{\beta/(1-\beta)}$ of the poor consumes one unit of the high-quality good. The (per capita) consumption for it is thus $1\cdot(1 - (\beta/2)^{\beta/(1-\beta)})\cdot(1/2) = 1/2 - 2^{-1/(1-\beta)}\beta^{\beta/(1-\beta)}$.

It is easy to check that the feasibility condition is satisfied for labor and the two goods.

To summarize, there exists a unique equilibrium of this economy, in which:

$(p_1,p_2,w) = (1,1/2,1)$;

$v_1 = 1 - (\beta/2)^{1/(1-\beta)}$ and $v_2 = (\beta/2)^{1/(1-\beta)}$;

each of the rich consumes $1 + (\beta^{\beta/(1-\beta)} - \beta^{1/(1-\beta)})(1/2)^{\beta/(1-\beta)}$ units of the high-quality good;

the fraction $(\beta/2)^{\beta/(1-\beta)}$ of the poor consumes two units of the low-quality good; and

the fraction $1 - (\beta/2)^{\beta/(1-\beta)}$ of the poor consumes one unit of the high-quality good.

17.AA.1 Let x and x' be Pareto optimal allocations such that $(u_1(x_1),\ldots,u_I(x_I))$ and $(u_1(x_1'),\ldots,u_I(x_I'))$ are proportional to $(s_1,\ldots,s_I) \gg 0$. By Pareto optimality, we must have $(u_1(x_1),\ldots,u_I(x_I)) = (u_1(x_1'),\ldots,u_I(x_I'))$. Define another allocation x" by $x_i'' = (1/2)x_i + (1/2)x_i'$ for each i. This is feasible and, by the strict convexity of the preferences, if $x_i \neq x_i'$, then $u_i(x_i'') > u_i(x_i) = u_i(x_i')$. Thus, by Pareto optimality, we must have $x_i = x_i'$ for every i, that is, x = x'. The proof is thus completed.

17.AA.2 (a) It is sufficient to prove that the mappings x: $\Delta \to \mathbb{R}_+^{LI}$ and p: $\Delta \to \mathbb{R}_+^L$ are well defined and continuous. By assuming that the $u_i(\cdot)$ are continuously differentiable and noticing that for every s, there exists an i for which $p(s) = \nabla u_i(x_i(s))$ (if $x(\cdot)$ is well defined), it is then sufficient to prove that $x(\cdot)$ is well defined and continuous. Write $u(x) = (u_1(x_1),\ldots,u_I(x_I))$.

We shall first show the well-definedness. By $\sum_i \omega_i \gg 0$ and $u_i(0) = 0$, there exists a feasible allocation $\bar{x}$ such that $u(\bar{x}) \gg 0$. Hence (by free disposal), for each s $\in \Delta$, there exists a feasible allocation x such that $u(x)$ is proportional to s. Since the set of all feasible allocations is compact (Proposition 16.AA.1), for each s $\in \Delta$, there exists a feasible allocation x such that $u(x) = \lambda s$ and, if x' is another feasible allocation for which $u(x') = \lambda' s$, then $\lambda \geq \lambda'$. The strong monotonicity then implies that x is Pareto optimal (which can be proved by the argument in the answer to Exercise

16.E.1(b)). Moreover, by the strict convexity of the preferences, there is no other feasible allocation x' for which $u(x) = u(x')$. So, by taking $x(s)$ to be this x, we know that $x(\cdot)$ is well defined.

We shall next prove that $x(\cdot)$ is continuous. Let $\{s^n\}_n$ be a sequence in Δ converging to an $s \in \Delta$. By the compactness of the set of feasible allocations, we can assume that the sequence $\{x(s^n)\}_n$ converges to a feasible allocation x. Also, for each n, $u(x(s^n))$ belongs to the boundary of U and to U', where U was defined in Section 16.E and U' in Appendix A to Chapter 16. By Proposition 16.AA.2, U is closed and U' is bounded. Hence we can assume that the sequence $\{u(x(s^n))\}_n$ converges to a point u on the boundary of U, which is included in U itself. By the continuity of the $u_i(\cdot)$, $u = u(x)$. Hence $u \in U'$. Therefore, by Exercise 16.E.1(b), x is Pareto optimal. Note that u is proportional to s, as $u(x(s^n)) \geq u(\bar{x}) >> 0$, where the feasible allocation $\bar{x}$ was introduced in the preceding paragraph. Thus $x = x(s)$, establishing the continuity of $x(\cdot)$.

(b) For each s, $\sum_i g_i(s) = p(s)\cdot(\sum_i \omega_i - \sum_i x_i(s)) = 0$ because $\sum_i x_i(s) = \sum_i \omega_i$.

(c) If $s_i = 0$, then $u_i(x_i(s)) = 0$. Thus, by the strong monotonicity and Pareto optimality, $x_i(s) = 0$. Hence $p(s)\cdot x_i(s) = 0$. On the other hand, by $\omega_i >> 0$, $p(s)\cdot\omega_i > 0$. Hence $g_i(s) = p(s)\cdot(\omega_i - x_i(s)) > 0$.

(d) Our existence proof here will proceed in the same way as the proof of Proposition 17.C.1. Namely, we shall first construct a mapping to which the fixed point theorem is applied and then show that, by the construction, a fixed point corresponds to an equilibrium.

Define a mapping $f\colon \Delta \to \Delta$ by $f_i(s) = \dfrac{s_i + \max(g_i(s),0)}{1 + \sum_h \max(g_h(s),0)}$ for each i and each $s \in \Delta$, then this is a well defined continuous mapping. Note that if $s \in \Delta$ and $s_i = 0$ for some i, then $f(s) \neq s$. In fact, then, $g_i(s) > 0$ by (c) and hence $f_i(s) = g_i(s) > 0 = s_i$. Note also that if $s \in \Delta$, $s >> 0$, and $g(s) \neq 0$, then $f(s) \neq s$. In fact, then, there exist i and k such that $g_i(s) < 0$ and $g_k(s) > 0$ by (b), and hence $1 + \sum_h \max(g_h(s),0) > 1$ and $s_i + \max(g_i(s),0) = s_i$, implying that $f_i(s) < s_i$. Therefore if s^* is a fixed point of $f(\cdot)$, then $g(s^*) = 0$. Brouwer's Fixed Point Theorem (Theorem M.I.1) guarantees the existence of such an s^*.

17.AA.3 For each $\lambda \in \Delta$, consider a maximization problem:

$$\operatorname{Max}_{x \in \mathbb{R}_+^{LI}} \sum_i \lambda_i u_i(x_i) \text{ s.t. } \sum_i x_i \leq \bar{\omega}.$$

By the compactness of the set of feasible allocations (Proposition 16.AA.1), a solution always exists. Suppose that whenever x is a solution to this problem, we have $x_i \in \mathbb{R}_{++}^L$ for every i with $\lambda_i > 0$. Then $x_i = 0$ for every i with $\lambda_i = 0$ (by the strong monotonicity) and there exists a unique $p \in \mathbb{R}_+^L$ such that $\lambda_i \nabla u_i(x_i) = p$ for every i with $\lambda_i > 0$.

We can thus associate with every λ a solution $x(\lambda)$ to the above maximization problem and the unique $p(\lambda) \in \mathbb{R}_+^L$ such that $\lambda_i \nabla u_i(x_i(\lambda)) = p(\lambda)$ for every i with $\lambda_i > 0$. Then define a mapping $h: \Delta \to \mathbb{R}^I$ by

$$h(\lambda) = (p(\lambda)\cdot(\omega_1 - x_1(\lambda)), \ldots, p(\lambda)\cdot(\omega_I - x_I(\lambda))).$$

Then $(p(\lambda),x(\lambda))$ is an equilibrium if and only if $h(\lambda) = 0$.

17.BB.1 Here is a graphical example in which a Walrasian quasiequilibrium with strictly positive prices is not an equilibrium, and the given four conditions are satisfied. Here $I = 1$, $X_1 = \{x_1 \in \mathbb{R}_+^2 : x_{11} + x_{21} \geq 2\}$, $\omega_1 =$

(1,1), and the preference is depicted in the figure by means of the indifference curves. Note that the indifference curve at ω_1 is tangent to the boundary of X_1.

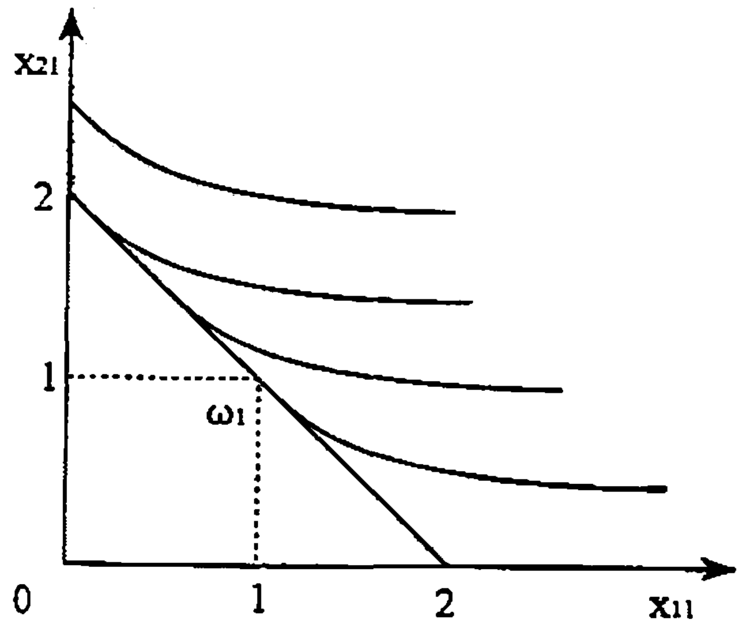


Figure 17.BB.1

The initial endowment ω_1 and its supporting price vector $p = (1,1)$ constitute the unique Walrasian quasiequilibrium. But it is not a Walrasian equilibrium because, in the budget set of $p = (1,1)$, he could afford $(0,2) \in X_1$, which is strictly better than ω_1. This example does not contradict any result given in the text, because, in this example, there is no $\hat{x}_i \in X_i$ such that $\omega_i \geq \hat{x}_i$ and $\omega_i \neq \hat{x}_i$.

17.BB.2 We shall prove the assertion by contradiction. Suppose that (p,x) with $p \in \mathbb{R}^L_+$ and $p \neq 0$ is a Walrasian quasiequilibrium, but it is not a Walrasian equilibrium. Then, by Proposition 17.BB.1 and the assumption that the consumption sets are nonnegative orthants, there exists a consumer h such that $p \cdot \omega_h = 0$. Also, by $\sum_i \omega_i \gg 0$, $p \cdot \omega_k > 0$ for some k. So define $H = \{h: p \cdot \omega_h = 0\}$ and $K = \{k: p \cdot \omega_k > 0\}$, then H and K constitute a nontrivial partition of the consumers. An application of the indecomposability condition implies that the consumers in K desires some of the commodities owned by the consumers in J, say, commodity ℓ. Again, by Proposition 17.BB.1, x_k is preference-maximizing for every $k \in K$. Since (some of) the consumers in K has strongly monotone preferences for ℓ, we must then have $p_\ell > 0$. Hence $p \cdot \omega_h > 0$ for some $h \in H$, a contradiction.

17.BB.3 In order to argue the existence of an equilibrium, first of all, we assume that $\omega_i >> 0$ for both i; without this, only the existence of a Walrasian quasiequilibrium is guaranteed (Figure 15.B.10(a)). We also assume that the initial endowments (ω_1, ω_2) do not constitute an equilibrium; if they do, there is nothing further to be proved. A final additional assumption is that the preferences are representable by differentiable utility functions. This assumption enables us to discuss tangency of offer curves and indifference curves, although it is not strictly necessary; an analogous argument is available for continuous, quasiconcave utility function as well, but it requires some knowledge on nonsmooth analysis, which we omit here.

Given these assumptions, the indifference curves of the two consumers that go though the initial endowment point must intersect. Since the offer curve are tangent to those indifference curves (p.518), they also intersect. This fact is illustrated in the figure below:

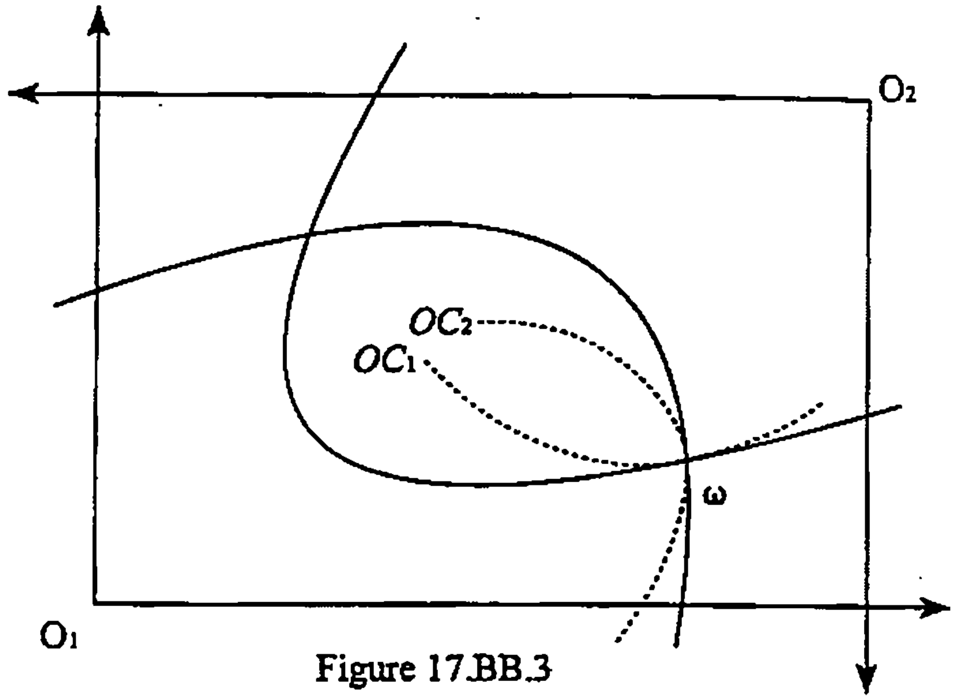


Figure 17.BB.3

It is thus sufficient to prove that the offer curves intersect once again at a point not equal to the initial endowments (p. 519). In other words, we need to show that they do not get stuck within the Edgeworth Box.

Since the Edgeworth Box is bounded, in order to show that the offer curves cannot get stuck within it, it is sufficient to show that offer curves extend indefinitely. Formally, this can be established by proving the following three assertions for both consumers i. Here, by saying that $OC_i(p)$ is well defined, we mean that there is the unique utility maximizer under p.

1. The set $\{p \in \mathbb{R}^2: OC_i(p)$ is well defined$\}$ is open in $\mathbb{R}^2$.

2. Suppose that a sequence $\{p^n\}$ in $\mathbb{R}^2$ converges to a $p \in \mathbb{R}^2$ with $p \neq 0$, $OC_i(p^n)$ is well defined for every n, and $OC_i(p)$ is not. Then

$$\text{Max } \{OC_{1i}(p^n), OC_{2i}(p^n)\} \to \infty.$$

3. If $p \in -\mathbb{R}^2_+$, then $OC_i(p)$ is not well defined.

In fact, take the set of normalized price vectors to be $S = \{p \in \mathbb{R}^2: \|p\| = 1\}$. Denote by D_i the intersection of S and the set $\{p \in \mathbb{R}^2: OC_i(p)$ is well defined$\}$. Starting from any strictly positive price vector on S, move a price vector p along S towards $(-1, 0)$ or $(0, -1)$ until it gets out of D_i. By Assertion 3, D_i does not intersect with $-\mathbb{R}^2_+$ and hence p will eventually be outside D_i. By Assertion 1, $S\backslash D_i$ is closed and hence, for each direction towards $(-1, 0)$ or $(0, -1)$, there is the "first" vector p^* to be outside D_i. Then, by Assertion 2, $\mathrm{Max}(OC_{1i}(p), OC_{2i}(p)) \to \infty$ as $p \to p^*$. Hence the offer curve extends indefinitely.

We shall prove Assertion 1 by contradiction. Suppose that the set $\{p \in \mathbb{R}^2 : OC_i(p)$ is well defined$\}$ is not open. Then there exist a sequence $\{p^n\}$ in $S\backslash D_i$ converging to a $p \in D_i$. By $\omega_i >> 0$, the correspondence that associates each p to the budget set $\{x_i \in \mathbb{R}^2_+ : p \cdot x_i \leq p \cdot \omega_i\}$ is continuous. Hence there exists a sequence $(x_i^n)_n$ in $\mathbb{R}^2_+$ such that $p^n \cdot x_i^n \leq p^n \cdot \omega_i$ and $x_i^n \to OC_i(p)$. We shall now construct another sequence $(y_i^n)_n$ in $\mathbb{R}^2_+$ such that $p^n \cdot y_i^n \leq p^n \cdot \omega_i$, $y_i^n \succsim_i x_i^n$, and $\|y_i^n - x_i^n\| = 1$. In fact, since $OC_i(p^n)$ is not well defined, there must exist a $\hat{y}_i^n \in \mathbb{R}^2_+$ such that $p^n \cdot \hat{y}_i^n \leq p^n \cdot \omega_i$, $\hat{y}_i^n \succ_i x_i^n$, and $\|\hat{y}_i^n - x_i^n\| > 1$; otherwise, the utility maximizer on the restricted budget set $\{x_i \in \mathbb{R}^2_+ : p^n \cdot x_i \leq p^n \cdot \omega_i$ and $\|x_i - x_i^n\| \leq 1\}$, which always exists by the compactness, is the utility maximizer on the overall budget set $\{x_i \in \mathbb{R}^2_+ : p \cdot x_i \leq p \cdot \omega_i\}$. Define

$$y_i^n = \frac{1}{\|\hat{y}_i^n - x_i^n\|}\hat{y}_i^n + \left(1 - \frac{1}{\|\hat{y}_i^n - x_i^n\|}\right)x_i^n,$$

then $p^n \cdot y_i^n \leq p^n \cdot \omega_i$, $y_i^n \succ_i x_i^n$ by the strict convexity, and $\|y_i^n - x_i^n\| = 1$. Thus y_i^n is as desired. Hence the sequence $(y_i^n)_n$ is bounded. By taking a subsequence if necessary, we can assume that it is convergent. Denote the limit by y_i, then it is in the budget set under p and satisfies $\|y_i - OC_i(p)\| = 1$ and $y_i \succsim_i OC_i(p)$ by the continuity. By the strict convexity, we obtain $(1/2)y_i + (1/2)OC_i(p) \succ_i OC_i(p)$, which is a contradiction.

Assertion 2 can also be proved by contradiction. Suppose that there is a sequence (p^n) in D_i converging to a $p \in S\setminus D_i$ such that the sequence $\{\mathrm{Max}(OC_{1i}(p^n), OC_{2i}(p^n))\}_n$ does not diverge to infinity. Then, by taking a subsequence if necessary, we can assume that $OC_i(p^n)$ converges to a $x_i \in \mathbb{R}^2_+$. Then $p\cdot x_i \le p\cdot\omega_i$ and, by the continuity of the budget correspondence and of the preference, $x_i \succsim_i y_i$ for all $y_i \in \mathbb{R}^2_+$ with $p\cdot y_i \le p\cdot\omega_i$, implying that $x_i = OC_i(p^n)$. This is a contradiction.

In order to prove Assertion 3, it is sufficient to prove that if $p \in -\mathbb{R}^2_{++}$, then $OC_i(p)$ is not well defined, because of Assertion 1 and the fact that the closure of $-\mathbb{R}^2_{++}$ is equal to $-\mathbb{R}^2_+$. So suppose that $p \in -\mathbb{R}^2_{++}$ and $OC_i(p)$ is well defined. By the utility maximization,

$$\{x_i \in \mathbb{R}^2_+ : x_i \succsim_i OC_i(p)\} \subset \{x_i \in \mathbb{R}^2_+ : p\cdot x_i \ge p\cdot\omega_i\}.$$

By $p \in -\mathbb{R}^2_{++}$, the set $\{x_i \in \mathbb{R}^2_+ : p\cdot x_i \ge p\cdot\omega_i\}$ is compact, and hence so is $\{x_i \in \mathbb{R}^2_+ : x_i \succsim_i OC_i(p)\}$. By the continuity, then, there exists a maximum on this set, which is in fact a global maximum on the whole $\mathbb{R}^2_+$. But this contradicts the local nonsatiation. Assertion 3 thus follows.

It follows directly from Assertion 3 that at equilibrium at least one price must be positive.

17.BB.4 Define $y_1'^* = y_1^* + (\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*) \in \mathbb{R}^L$. Then $\sum_i x_i^* = \sum_i \omega_i + y_1'^* + \sum_{j\ge 2} y_j^*$. Since $\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^* \in -\mathbb{R}^L_+$ by (iii') and Y_1 satisfies free disposal, $y_1'^* \in Y_1$. Since $p\cdot(\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*) = 0$ again by (iii'), $p\cdot y_1'^* = p\cdot y_1^*$. Thus, by (i), $p\cdot y_1'^* \ge p\cdot y_1$ for all $y_1 \in Y_1$. Hence $(x^*, y_1'^*, y_2^*, \dots, y_J^*, p)$ is a Walrasian quasiequilibrium.

17.BB.5 Suppose that $x_{i\alpha} = \alpha x_i + (1 - \alpha)x_i'$ and we have both $x_i \succ_i x_{i\alpha}$ and $x_i' \succ_i x_{i\alpha}$. By the continuity and the local nonsatiation, there exists an $x_i'' \in$

X_i such that $x_i \succ_i x_i'' \succ_i x_{i\alpha}$ and $x_i' \succ_i x_i'' \succ_i x_{i\alpha}$. By the convexity,

$$x_{i\alpha} = \alpha x_i + (1 - \alpha)x_i' \succsim_i x_i'',$$

which is a contradiction.

17.BB.6 Let $z_j \in \tilde{y}_j(x,y,p)$, $z_j' \in \tilde{y}_j(x,y,p)$, $\alpha \in [0,1]$, $y_{j\alpha} = \alpha z_j + (1 - \alpha)z_j'$, and $y_j'' \in \hat{Y}_j$. Then $p\cdot y_{j\alpha} = \alpha p\cdot z_j + (1 - \alpha)p\cdot z_j' \geq \alpha p\cdot y_j'' + (1 - \alpha)p\cdot y_j'' = p\cdot y_j''$. Hence $y_{j\alpha} \in \tilde{y}_j(x,y,p)$. Thus $\tilde{y}_j(x,y,p)$ is convex.

Let $q \in \tilde{p}(x,y,p)$, $q' \in \tilde{p}(x,y,p)$, $\alpha \in [0,1]$, $p_\alpha = \alpha q + (1 - \alpha)q'$, and $p'' \in \Delta$. Then

$$\begin{aligned}
&(\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot p_\alpha \\
&= \alpha(\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot q + (1 - \alpha)(\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot q' \\
&\geq \alpha(\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot p'' + (1 - \alpha)(\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot p'' \\
&= (\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot p''.
\end{aligned}$$

Hence $p_\alpha \in \tilde{p}(x,y,p)$. Thus $\tilde{p}(x,y,p)$ is convex.

17.BB.7 Let $p^n \to p$, $y^n \to y$, and $x^n \to x$. To prove that $\tilde{y}_j(\cdot)$ is upper hemicontinuous, let $z_j^n \to z_j$ and $z_j^n \in \tilde{y}_j(x^n,y^n,p^n)$. We need to show that $z_j \in \tilde{y}_j(x,y,p)$. Let $y_j' \in \hat{Y}_j$, then $p^n\cdot z_j^n \geq p^n\cdot y_j'$. As $n \to \infty$, we obtain $p\cdot z_j \geq p\cdot y_j'$. Hence $z_j \in \tilde{y}_j(x,y,p)$.

To prove that $\tilde{p}(\cdot)$ is upper hemicontinuous, let $q^n \to q$ and $q^n \in \tilde{p}(x^n,y^n,p^n)$. We need to show that $q \in \tilde{p}(x,y,p)$. Let $p' \in \Delta$, then

$$(\textstyle\sum_i x_i^n - \sum_i \omega_i - \sum_j y_j^n)\cdot q^n \geq (\sum_i x_i^n - \sum_i \omega_i - \sum_j y_j^n)\cdot p'.$$

As $n \to \infty$, we obtain

$$(\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot q \geq (\sum_i x_i - \sum_i \omega_i - \sum_j y_j)\cdot p'.$$

Hence $q \in \tilde{p}(x,y,p)$.

17.BB.8 We assume that there are I consumers, indexed by $i = 1,\ldots,I$, with preferences $\succsim_i$ on the consumption set $\mathbb{R}^L_+$ and initial endowments $\omega_i \in \mathbb{R}_+$ of labor.

(a) We say that a vector (x^*,v^*,p) in $\mathbb{R}^{LI}_+ \times \mathbb{R}^{J_1}_+ \times \ldots \times \mathbb{R}^{J_{L-1}}_+ \times \mathbb{R}^L$ is an equilibrium if

(i) For every firm j in sector ℓ,

$$p_\ell \alpha_\ell (v^*_{\ell j})^{\beta_\ell} - p_L v^*_{\ell j} \ge p_\ell \alpha_\ell (v_{\ell j})^{\beta_\ell} - p_L v_{\ell j}$$

for every $v_{\ell j} \ge 0$, where $\alpha_\ell = \gamma_\ell(\sum_j v^*_{\ell j})^{\rho_\ell}$.

(ii) For ever i, $p\cdot x^*_i \le p^*_L\omega_i + (1/I)\sum_{\ell,j}(p_\ell\alpha_\ell(v^*_{\ell j})^{\beta_\ell} - p_L v^*_{\ell j})$ and if $x_i \succ_i$

x^*_i, then $p\cdot x_i > p^*_L\omega_i + (1/I)\sum_{\ell,j}(p_\ell\alpha_\ell(v^*_{\ell j})^{\beta_\ell} - p_L v^*_{\ell j})$.

(iii) For every $\ell < L$, $\sum_i x^*_{\ell i} = \sum_j \alpha_\ell (v^*_{\ell j})^{\beta_\ell}$, and $\sum_{\ell,j} v^*_{\ell j} = \sum_i \omega_i$.

(b) Condition (i) of Proposition 17.BB.2 is clearly satisfied. We assume that $\omega_i > 0$ for all i, which guarantees that every Walrasian quasiequilibrium is actually an equilibrium. We can show that condition (iii) is met as follows: Let $\bar{\omega} = \sum_i \omega_i > 0$ be the total endowment of labor. Then the maximum producible amount of good ℓ is attained by using input $\bar{\omega}/J_\ell$ for each of the J_ℓ firms (by $0 < \beta_\ell \le 1$) in sector ℓ, which yields the production level

$$J_\ell \cdot \gamma_\ell \cdot \bar{\omega}^{\rho_\ell} \cdot (\bar{\omega}/J_\ell)^{\beta_\ell} = \gamma_\ell \cdot J_\ell^{1-\beta_\ell} \cdot \bar{\omega}^{\rho_\ell + \beta_\ell}.$$

Hence the set of feasible allocations is bounded, and thus compact. Let $r > \gamma_\ell \cdot \bar{\omega}^{\rho_\ell+\beta_\ell} \cdot J_\ell^{1-\beta_\ell}$ for every ℓ, and define $\hat{X}_i = \{x_i \in \mathbb{R}^L_+ : x_{\ell i} \le r\}$ and

$$\hat{Y}_{\ell j} = \{y_{\ell j} \in \mathbb{R}^L : -\bar{\omega} - 1 \le y_{L\ell j} \le 0,\ 0 \le y_{\ell\ell j} \le r,$$
$$\text{and } z_{k\ell j} = 0 \text{ for any } k \ne \ell, L.\}$$

(Note that $\hat{Y}_{\ell j}$ is not a subset of $Y_{\ell j}$, unlike the $\hat{Y}_j$ in the proof of

Proposition 17.BB.2. But we will construct a fixed point correspondence so that, at a fixed point, the production plans are in the $Y_{\ell j}$.)

Define the wealth functions $w_i(\cdot)$ and the best responses correspondences $\tilde{x}_i(\cdot)$ and $\tilde{p}(\cdot)$ just as in the proof of Proposition 17.BB.2. The best response correspondences are convex valued and upper hemicontinuous.

As for the production sector, condition (ii) is not immediately satisfied by the current model because of the externalities. However, for each industry ℓ, once we fix the value of the coefficient α_ℓ by determining the labor inputs of the firms in the sector, then the production set of each firm satisfies (ii). Thus we can define the best response correspondence $\tilde{y}_{\ell j}(\cdot)$ of firm j in sector ℓ by taking $\tilde{y}_{\ell j}(x,y,p)$ to be the set of profit maximizers on

$$\{z_{\ell j} \in \hat{Y}_{\ell j}: z_{\ell\ell j} \le \gamma_\ell(-\textstyle\sum_j y_{L\ell j})^{\rho_\ell}(-z_{L\ell j})^{\beta_\ell}\}.$$

Then, $\bar{y}_{\ell j}(\cdot)$ is convex valued and upper hemicontinuous. Thus there exists a fixed point (x^*,y^*,p^*) with $y^*_{\ell j} \in Y_{\ell j}$, implying that it corresponds to a Walrasian equilibrium.

(c) Given an input level v_ℓ, the maximum output level of good ℓ is attained by using v_ℓ/J_ℓ units for each of the J_ℓ firms (by $0 < \beta_\ell \le 1$), yielding the aggregate production level

$$\hat{f}_\ell(v_\ell) = J_\ell \gamma_\ell v_\ell^{\rho_\ell}(v_\ell/J_\ell)^{\beta_\ell} = \gamma_\ell J_\ell^{1-\beta_\ell} v_\ell^{\rho_\ell+\beta_\ell}.$$

Hence the aggregate production set of sector ℓ is given by this aggregate production function $\hat{f}_\ell(\cdot)$. It thus exhibits increasing, constant, and decreasing returns to scale, respectively if and only if $\rho_\ell + \beta_\ell > 1$, $\rho_\ell + \beta_\ell = 1$, and $\rho_\ell + \beta_\ell < 1$. While there exists an equilibrium in the cases of constant and decreasing returns to scale, there is no equilibrium in the case of increasing returns to scale, because unboundedly large profits can be made at any (positive) output prices.

(d) If the externally of sector ℓ is internalized and the aggregate production set exhibits increasing returns to scale (that is, $\rho_\ell + \beta_\ell > 1$), then the sector can attain an arbitrarily large profit. Hence a Walrasian equilibrium does not exist.

(e) We assume that $\rho_1 > 0$. Then, by $\beta_1 = 1$, the individual production function exhibits constant returns to scale and the aggregate production function exhibits increasing returns to scale. Because of the quasilinearity, there exists a normative representative consumer (Example 4.D.2) and his (direct) utility function (Exercise 4.D.4) is also quasilinear. Let it be $u(x_1) + x_2$. We assume that $u(\cdot)$ is differentiable and strictly concave. Let $v^* > 0$ be the equilibrium level of labor input and $v^{**} > 0$ be the socially optimal level of labor input. They can be graphically illustrated as follows. Note that $v^* < v^{**}$.

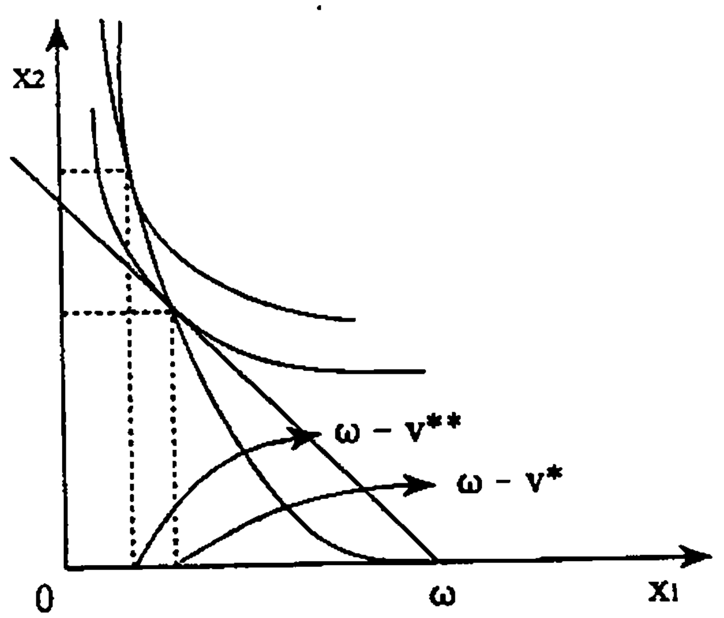


Figure 17.BB.8

We shall now analytically prove that $v^* < v^{**}$. Note first that the equilibrium price of good 1 (with labor being the numeraire) is equal to $\alpha_1^{-1} = \gamma_1^{-1} v_1^{*\,-\rho_1}$. Thus the budget constraint of the representative consumer at the equilibrium is given by

$$\gamma_1^{-1} v^{*\,-\rho_1} x_1 + x_2 \le \bar{\omega}.$$

Hence, for every $v \le v^*$, the consumption vector $(\gamma_1 v^{\rho_1+1}, \bar{\omega} - v)$, which is made available by using the labor input v, satisfies this constraint, because

$$\gamma_1^{-1} v^{*\,-\rho_1}(\gamma_1 v^{\rho_1+1}) + \bar{\omega} - v = \bar{\omega} + v((v/v^*)^{\rho_1} - 1) \le \bar{\omega}.$$

Thus

$$u(\gamma_1 v^{\rho_1+1}) + (\bar{\omega} - v) \le u(\gamma_1 v^{*\,\rho_1+1}) + (\bar{\omega} - v^*).$$

By the strict concavity of $u(\cdot)$, the demand is unique and hence, for any $v < v^*$,

$$u(\gamma_1 v^{\rho_1+1}) + (\bar{\omega} - v) < u(\gamma_1 v^{*\,\rho_1+1}) + (\bar{\omega} - v^*).$$

We must thus have $v^{**} \ge v^*$. To show that this weak inequality actually holds with equality, note that the first-order necessary condition for the utility maximization at the equilibrium is given by

$$u'(\gamma_1 v^{*\,\rho_1+1}) = \alpha_1^{-1} = \gamma_1^{-1} v^{*\,-\rho_1}.$$

On the other hand, the first-order necessary condition for the social optimum is given by

$$u'(\gamma_1 v^{**\,\rho_1+1}) = \gamma_1^{-1} v^{**\,-\rho_1}(\rho_1 + 1)^{-1}.$$

By $\rho_1 > 0$, these two first-order necessary conditions are not satisfied at the same labor input. Hence $v^* < v^{**}$.

17.BB.9 We shall only outline how the existence argument for the two-player-game approach should proceed. The details of the proofs, such as checking convexity and upper hemicontinuity, are exactly the same as in the proof of Proposition 17.BB.2, so we do not provide them here.

First, for each j, define a correspondence $\tilde{y}_j\colon \Delta \to \hat{Y}_j$ by taking $\tilde{y}_j(p)$ to be the set of the profit maximizers over $\hat{Y}_j$ under p. This is a convex valued, upper hemicontinuous correspondence. Associated with $\tilde{y}_j(\cdot)$, consider the profit function $\pi_j\colon \Delta \to \mathbb{R}_+$ by letting $\pi_j(p)$ be the maximized profit. This is a continuous function.

For each i, define the wealth function $w_i\colon \Delta \to \mathbb{R}$ by $w_i(p) = p\cdot\omega_i + \sum_j \theta_{ij}\pi_j(p)$. This is continuous (as so are the $\pi_j(\cdot)$) and satisfies $w_i(p) \geq p\cdot\hat{x}_i$ for every $p \in \Delta$. Moreover, $\sum_i w_i(p) = \sum_i p\cdot\omega_i + \sum_j \pi_j(p)$. Define then the individual demand correspondence $\tilde{x}_i\colon \Delta \to \hat{X}_i$ by letting $x_i \in \tilde{x}_i(p)$ if and only if $p\cdot x_i \leq w_i(p)$ and $x_i \succsim_i x_i'$ for all $x_i' \in \hat{X}_i$ satisfying $p\cdot x_i' < w_i(p)$. Then it is convex valued and upper hemicontinuous.

Define $Z = \sum_i \hat{X}_i - \{\sum_i \omega_i\} - \sum_j \hat{Y}_j$, then it is a convex, compact subset of $\mathbb{R}^L$. Define a correspondence $\tilde{z}\colon \Delta \to Z$ by $z(p) = \sum_i \tilde{x}_i(p) - \{\sum_i \omega_i\} - \sum_j \tilde{y}_j(p)$, then it is convex valued and upper hemicontinuous. Define a correspondence $\tilde{p}\colon Z \to \Delta$ by letting $\tilde{p} \in p(z)$ if and only if $p\cdot z \geq q\cdot z$ for every $q \in \Delta$, then it is convex valued and upper hemicontinuous. Finally, define a correspondence $\Psi\colon \Delta \times Z \to \Delta \times Z$ by $\Psi(p,z) = \tilde{p}(z) \times \tilde{z}(p)$, then it is convex valued and upper hemicontinuous.

By Kakutani's fixed point theorem, $\Psi(\cdot)$ has a fixed point (p^*,z^*). Then $p^* \in p(z^*)$, $z^* \in z(p^*)$, $z^* \leq 0$ and $p^*\cdot z^* = 0$. By construction, there exists $(x_1^*,\ldots,x_I^*,y_1^*,\ldots,y_J^*)$ such that $x_i^* \in \tilde{x}_i(p^*)$, $y_j^* \in \tilde{y}_j(p^*)$, and $z^* = \sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*$. Then $(x_1^*,\ldots,x_I^*,y_1^*,\ldots,y_J^*,p^*)$ is a free disposal quasiequilibrium for the truncated economy. It is thus a true quasiequilibrium of the original, untruncated economy.

18.B.1 Let (x^*, y^*, p) be an equilibrium. Suppose that there is a coalition S that can improve upon x^*. Then, for each $i \in S$, there exist $x_i \in \mathbb{R}^L_+$ and $y \in Y$ such that $x_i \succ_i x_i^*$ and $\sum_{i \in S} x_i = y + \sum_{i \in S} \omega_i$. By the preference maximization condition of the equilibrium, $p \cdot x_i > p \cdot \omega_i$. Hence $p \cdot y = \sum_{i \in S}(p \cdot x_i - p \cdot \omega_i) > 0$. Since Y is a convex cone, $p \cdot y^* = 0$ and hence $p \cdot y > p \cdot y^*$. This contradicts the profit maximization condition. Hence x^* must have the core property.

18.B.2 Define a three-consumer exchange economy by

$$u_i(x_i) = x_{1i} + x_{2i}^{1/2} - 2, \text{ for } i = 1,2,3$$

$$\omega_1 = \omega_2 = (1,1), \ \omega_3 = (0,4).$$

Note that consumers 1 and 2 have the same type. We shall show that an allocation $x^* = (x_1^*, x_2^*, x_3^*)$ defined by

$$x_1^* = (2 - 2^{1/2}, 2),$$

$$x_2^* = (8^{1/2} + 1 - 10^{1/2}, 2),$$

$$x_3^* = (10^{1/2} - 1 - 2^{1/2}, 2).$$

is a core allocation and does not have the equal treatment property. In fact, the quasilinearity implies that, for each feasible allocation x and for each coalition S, if

$$\sum_{i \in S} u_i(x_i) \geq \sum_{i \in S} \omega_{1i} + \#S(\sum_{i \in S} \omega_{2i}/\#S)^{1/2} - 2(\#S),$$

then S cannot improve upon x; this can be shown by applying the argument in the answer to Exercise 15.B.8 to coalition S. Denote the value on the right-hand side by $v(S)$. (This is the same notation as in Appendix A; by the quasilinearity, the exchange economy give rise to a TU-game v.) Then

$$v(\{i\}) = v(\{1,2\}) = 0 \text{ for every } i,$$

$$v(\{1,3\}) = v(\{2,3\}) = 10^{1/2} - 3,$$

$$v(\{1,2,3\}) = 18^{1/2} - 4.$$

On the other hand,

$$u_1(x_1^*) = 0,\ u_2(x_2^*) = 18^{1/2} - 10^{1/2} - 1,\ u_3(x_3^*) = 10^{1/2} - 3.$$

Hence it is now easy to check that x^* has the core property and $x_{11}^* < x_{12}^*$, that is, x^* does not have the equal treatment property.

It is impossible to construct an example of this effect only with two consumers. In fact, the initial endowment allocation (ω_1,ω_1) is the only allocation in the core. To see this, let (x_1,x_2) be an allocation in the core and $(x_1,x_2) \neq (\omega_1,\omega_1)$, then $x_1 \succsim_1 \omega_1$ and $x_2 \succsim_1 \omega_1$. Hence, by the strict convexity, $(1/2)x_1 + (1/2)x_2 \succ_1 \omega_1$. But since $(1/2)x_1 + (1/2)x_2 = \omega_1$, this is a contradiction.

18.B.3 Suppose that there exists a Walrasian equilibrium $(x^*,p) \in \mathbb{R}_+^{LHN} \times \mathbb{R}^L$ such that the nth and the mth individuals of type h consume different bundles, that is, $x_{hm}^* \neq x_{hn}^*$. Then

$$\begin{aligned} &p\cdot((1/2)x_{hm}^* + (1/2)\ x_{hn}^*) \\ &= (1/2)p\cdot x_{hm}^* + (1/2)p\cdot x_{hn}^* \\ &\le (1/2)p\cdot\omega_h + (1/2)p\cdot\omega_h = p\cdot\omega_h \end{aligned}$$

and, by the strict convexity, we must have either

$$(1/2)x_{hm}^* + (1/2)x_{hn}^* \succ_h x_{hm}^* \text{ or } (1/2)x_{hm}^* + (1/2)x_{hn}^* \succ_h x_{hn}^*.$$

This is a contradiction to the utility maximization condition of an equilibrium. Thus x^* must have the equal treatment property.

18.B.4 Write $v = x_1 - \omega_1$. By Taylor's formula (or simply the definition of the directional partial derivative), there exists a function $\varphi(\cdot)$ such that

$$u_h(x_h + \alpha v) - u_h(x_h) = \alpha \nabla u_h(x_h)\cdot v + \varphi(\alpha)$$

and $|\varphi(\alpha)|/\alpha \to 0$ as $\alpha \to 0$. Since $\nabla u_h(x_h)\cdot v$ is a positive constant (independent of α), $u_h(x_h + \alpha v) - u_h(x_h) > 0$ for any sufficiently small $\alpha > 0$.

18.B.5 (a) [First printing errata: To be rigorous, the phrase in the parentheses "i.e., every individual consumes the same number of shoes of each kind" should be "i.e., every individual consumes the same number of shoes of each kind, possibly except for one individual, who consumes one more left shoes than right shoes". This is because there is one more left shoes endowed with the economy and extra left shoes do not decrease utility.] It is easy to see that if all pairs are matched in a feasible allocation, then it is Pareto optimal. It is also easy to see that if some pair is not matched in a feasible allocation, then we can find a Pareto superior allocation by combining the unmatched pair. The statement of this question is simply the combination of these two facts.

(b) We claim that an feasible allocation has the core property if and only if, at the allocation, each owner of right shoes receives a pair of shoes.

To prove this, first let x be a feasible allocation at which each owner of right shoes receives a pair of shoes. Suppose that there is a blocking coalition S, which contains J owners of right shoes. This means that the whole coalition S owns J right shoes and, at x, it receives J pairs of shoes. By the definition of a blocking coalition, every member of S must receive at least as many as pairs as before, and some must receive more. The whole S must thus receives more than J pairs at the blocking allocation. However, this is not feasible because the whole S owns only J right shoes. Thus there is no coalition that blocks x and x is in the core.

Next, let x be a Pareto optimal allocation in which some owner of right shoes receives no pair. Since there are $I + 1$ owners of the left shoe and only I right shoes are available in the economy, some owner of the left shoe receives none of the right shoes. Thus these two consumers' utility levels at x are both zero. But if they form a coalition, then at least one of them can get a pair (and the other nothing). Hence the coalition can block x, implying that x does not have the core property. Thus, if x is in the core, then each owner of the right shoe receives a pair of the shoes.

(c) We claim that (x,p) is an equilibrium if and only if p is a positive multiple of $(p_R, p_L) = (1,0)$ and, at x, each owner of right shoes receives a pair of shoes.

Let's first suppose that (x,p) is an equilibrium. By the utility maximization condition, $p \geq 0$ and $p \neq 0$. However, we cannot have $p \gg 0$. The reason is as follows. If $p \gg 0$, then each consumer can afford only a shoe. Thus the aggregate demand consists of at most $2I$ shoes (including both the right and the left ones). Hence at least a shoe is left unsold in the market, which contradicts the market clearing condition of an equilibrium and the profit maximization for the free disposal technology, together with $p \gg 0$. We must thus have either $p_R = 0$ or $p_L = 0$. If $p_R = 0$ (and hence $p_L > 0$), every owner of the left shoe demands a pair, which, again, contradicts the feasibility condition because there are $I + 1$ owners of left shoes. We must thus have $p_L = 0$ and $p_R > 0$. Then each owner of the right shoe receives a pair.

It is easy to see that if a price vector p is a positive multiple of $(p_R, p_L) = (1,0)$ and, at a feasible allocation x, each owner of the right shoe receives a pair of the shoes, then (x,p) is an equilibrium. (The remaining left shoe can be assigned to any consumer.) The proof is thus completed.

(d) The core and the set of equilibrium allocations coincide, regardless of the value of I.

18.C.1 We shall prove the stated properties of an effective budget set by showing that it can be expressed as a parallel shift of the epigraph of some strictly concave function. (Let D be a subset of $\mathbb{R}^{L-1}$ and f: $D \to \mathbb{R}$, then the epigraph of $f(\cdot)$ is defined to be the subset $\{(z,z_L) \in D \times \mathbb{R}: z_L \le f(z)\}$ of $\mathbb{R}^L$, which we denote by epi f. It follows immediately from the definition of strict concavity that if D is convex and $f(\cdot)$ is strictly concave, then its epigraph has an upper boundary containing no straight line.) Our strategy of proof can be more specifically explained as follows. Fix a consumer i. Let $a_{-i} = (a'_{\ell k},a''_{\ell k})_{\ell<L,k\ne i}$ be the nonnegative bids of the other consumers with $\sum_{k\ne i}a'_{\ell k} > 0$ and $\sum_{k\ne i}a''_{\ell k} > 0$ for every $\ell < L$. For each $\ell < L$, define h_ℓ: $(-\infty, \sum_{k\ne i}a'_{\ell k}) \to \mathbb{R}$ by

$$h_\ell(z_\ell) = \frac{(\sum_{k\ne i}a''_{\ell k})z_\ell}{(\sum_{k\ne i}a'_{\ell k}) - z_\ell},$$

then $h_\ell(\cdot)$ is strictly increasing and strictly convex. Write

$$U = (-\infty, \sum_{k\ne i}a'_{1k}) \times \dots \times (-\infty, \sum_{k\ne i}a'_{L-1,k}) \subset \mathbb{R}^{L-1}.$$

Define f: $U \to \mathbb{R}$ by $f(z) = -\sum_{\ell<L}h_\ell(z_\ell)$, then $f(\cdot)$ is strictly decreasing and strictly concave. Define

$$E = \{z \in U: \sum_{\ell<L}\text{Max}\{h_\ell(z_\ell),0\} \le \omega_{Li}\},$$

then E is convex and closed, because the function $\sum_{\ell<L}\text{Max}\{h_\ell(z_\ell),0\}$ is convex on U and

$$E \subset \left(-\infty, \frac{(\sum_{k\ne i}a'_{1k})\omega_{Li}}{(\sum_{k\ne i}a''_{1k}) + \omega_{Li}}\right] \times \dots \times \left(-\infty, \frac{(\sum_{k\ne i}a'_{L-1,k})\omega_{Li}}{(\sum_{k\ne i}a''_{L-1,k}) + \omega_{Li}}\right].$$

Define then $D = \{z \in E: z_\ell \ge -\omega_{\ell i}$ for every $\ell < L\}$, then D is convex and closed. We shall now prove that:

The effective budget set $\{x_i \in \mathbb{R}^L_+ : x_i - \omega_i \le g(a_i; p(a_i, a_{-i}))$ for some $a_i \in A_i\}$ is equal to $(\text{epi}\ (f|D) + \{\omega_i\}) \cap (\mathbb{R}^{L-1} \times \mathbb{R}_+)$, where $g(\cdot)$, $p(\cdot)$, and A_i are as explained in Example 18.C.3, and $f|D$ denotes the restriction of the function $f(\cdot)$ onto $D \subset U$.

To ease the notation, for each $\ell \le L$, define $\hat{p}_\ell : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$\hat{p}_\ell(b_\ell) = \frac{(\sum_{k \ne i} a''_{\ell k}) + b''_\ell}{(\sum_{k \ne i} a'_{\ell k}) + b'_\ell},$$

where $b_\ell = (b'_\ell, b''_\ell) \in \mathbb{R}^2_+$. This function assigns to each bid b_ℓ of consumer i at trading post ℓ the market clearing price. Define $\hat{g}: \mathbb{R}^{2(L-1)}_+ \to \mathbb{R}^L$ by $\hat{g}_\ell(b) = \hat{p}_\ell(b_\ell)^{-1} b''_\ell - b'_\ell$ for each $\ell < L$, and $\hat{g}_L(b) = \sum_{\ell < L}(\hat{p}_\ell(b)b'_\ell - b''_\ell)$, where $b = (b_1, \dots, b_{L-1})$. This function assigns to bids b of consumer i the transaction bundle that he ends up with. (In fact, for each $\ell < L$, $\hat{g}_\ell(b)$ is uniquely determined by b_ℓ.) With this notation, we have

$$\begin{aligned}&\{x_i \in \mathbb{R}^L_+ : x_i - \omega_i \le g(a_i; p(a_i, a_{-i})) \text{ for some } a_i \in A_i\} \\ &= \{z \in \mathbb{R}^L : -\omega_i \le z \le \hat{g}(b) \text{ for some } b \in A_i\} + \{\omega_i\}.\end{aligned}$$

Although our formulation allows the consumers to put both the good (on the offer side) and the money (on the bid side) simultaneously at a trading post, it turns out that they can enjoy exactly the same excess demands by concentrating on one side at each post. Formally, for each $b \in A_i$, there exists $c \in A_i$ such that $c'_\ell c''_\ell = 0$ for every $\ell < L$ and $\hat{g}(b) = \hat{g}(c)$. To see this, for any given $b \in A_i$ and any $\ell < L$, define $c \in \mathbb{R}^{2(L-1)}_+$ as follows:

If $\hat{g}(b) = 0$, the $c_\ell = 0$;

If $\hat{g}_\ell(b) > 0$, then $c_\ell = (0, \hat{p}_\ell(b)\hat{g}_\ell(b))$;

If $\hat{g}_\ell(b) < 0$, then $c_\ell = (-\hat{g}_\ell(b), 0)$.

It follows immediately from the definition that $c'_\ell \le b'_\ell$ and $c''_\ell \le b''_\ell$, implying that $c \in A_i$. With a bit of calculation, we can show that $\hat{g}(b) = \hat{g}(c)$.

We shall next show that $D = \{(\hat{g}_1(b),\ldots,\hat{g}_{L-1}(b)) \in \mathbb{R}^{L-1}: b \in A_i\}$. First let $b \in A_i$ with $b'_\ell b''_\ell = 0$. Then

$$\hat{g}_\ell(b) = 0 \text{ if } b_\ell = 0;$$

$$\hat{g}_\ell(b) = - b'_\ell < 0 \text{ if } b'_\ell > 0;$$

$$\hat{g}_\ell(b) = \frac{\sum_{k\neq i} a'_{\ell k}}{(\sum_{k\neq i} a''_{\ell k}) + b''_\ell} b''_\ell > 0 \text{ if } b''_\ell > 0;$$

Hence $- \omega_{\ell i} \leq \hat{g}_\ell(b) < \sum_{k\neq i} a'_{\ell k}$. Moreover

$$h_\ell(\hat{g}_\ell(b)) = 0 \text{ if } b_\ell = 0;$$

$$h_\ell(\hat{g}_\ell(b)) = - \frac{(\sum_{k\neq i} a''_{\ell k})b'_\ell}{(\sum_{k\neq i} a'_{\ell k}) - b'_\ell} < 0 \text{ if } b'_\ell > 0;$$

$$h_\ell(\hat{g}_\ell(b)) = b''_\ell > 0 \text{ if } b''_\ell > 0;$$

Thus $\sum_{\ell<L} \mathrm{Max}\langle h_\ell(z_\ell),0\rangle = \sum_{\ell<L} b''_\ell \leq \omega_{Li}$. Hence $(\hat{g}_1(b),\ldots,\hat{g}_{L-1}(b)) \in D$. Conversely, let $z \in D$, then define $b \in \mathbb{R}^{2(L-1)}_+$ with $b'_\ell b''_\ell = 0$ as follows:

If $z_\ell = 0$, then $b_\ell = 0$;

If $z_\ell > 0$, then $b_\ell = (0, h_\ell(z_\ell))$;

If $z_\ell < 0$, then $b_\ell = (- z_\ell, 0)$.

Then $b'_\ell = - z_\ell \leq \omega_{\ell i}$ and $\sum_\ell b''_\ell = \sum_\ell h_\ell(z_\ell) \leq \omega_{Li}$. Hence $b \in A_i$. We have thus proved that $D = \{(\hat{g}_1(b),\ldots,\hat{g}_{L-1}(b)) \in \mathbb{R}^{L-1}: b \in A_i\}$.

It is easy to show that $\hat{g}_L(b) = f(\hat{g}_1(b),\ldots,\hat{g}_{L-1}(b))$ for every $b \in A_i$. Hence

$$\{(z,z_L) \in D \times \mathbb{R}: z_L = f(z)\} = \{\hat{g}(b) \in \mathbb{R}^L: b \in A_i\}.$$

By taking the free disposal (comprehensive) hull in $D \times \mathbb{R}$, we obtain

$$\{(z,z_L) \in D \times \mathbb{R}: (z_L,z) \leq (z',f(z')) \text{ for some } z' \in D\}$$

$$= \{(z,z_L) \in D \times \mathbb{R}: (z_L,z) \leq \hat{g}(b) \text{ for some } b \in A_i\}.$$

Since $f(\cdot)$ is strictly decreasing, the left-hand side is equal to epi $(f|D)$. By the definition of D, the right-hand side is equal to

$\{(z,z_L) \in D \times \mathbb{R}$: there exists $b \in A_i$ such that $z_L \le \hat{g}_L(b)$

and $-\omega_{\ell i} \le z_\ell \le \hat{g}_\ell(b)$ for every $\ell < L\}$.

By adding the initial endowment ω_i to both sides, we know that epi $(f \mid D)$ + $\{\omega_i\}$ is equal to

$\{(z,z_L) \in D \times \mathbb{R}$: there exists $b \in A_i$ such that $z_L \le \hat{g}_L(b)$

and $-\omega_{\ell i} \le z_\ell \le \hat{g}_\ell(b)$ for every $\ell < L\} + \{\omega_i\}$.

Finally, by taking the intersection with $\mathbb{R}^{L-1} \times \mathbb{R}_+$, we obtain

$$(\text{epi } (f \mid D) + \{\omega_i\}) \cap (\mathbb{R}^{L-1} \times \mathbb{R}_+)$$

$$= \{(z_L,z) \in \mathbb{R}^L: -\omega_i \le z \le \hat{g}(b) \text{ for some } b \in A_i\} + \{\omega_i\}.$$

The proof is thus completed.

When L = 3 and $\omega_i >> 0$, the effective budget set can be depicted as follows.

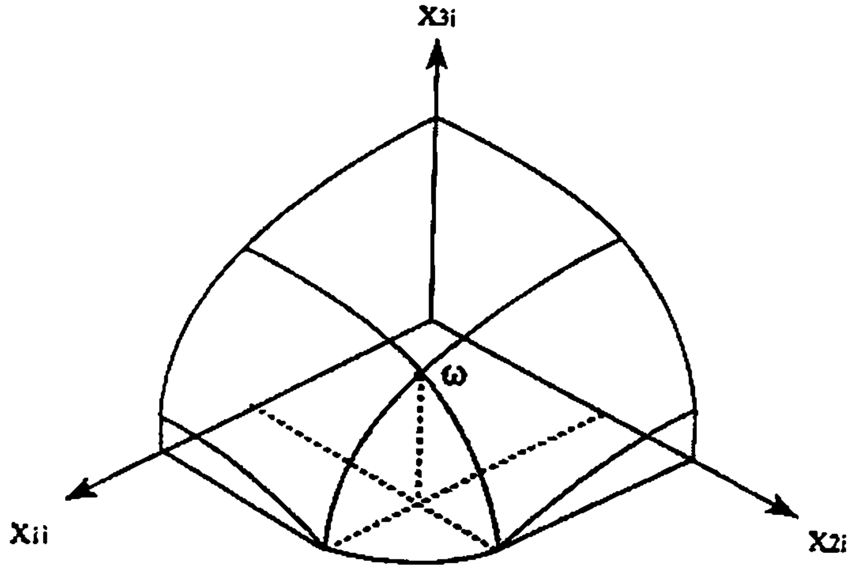


Figure 18.C.1

18.D.1 Let (x_1^*, x_2^*) be a feasible allocation such that $x_i^* \succsim_i \omega_i$ for both i. Write $z_i = x_i^* - \omega_i$. It is sufficient to prove that $\omega_i + z_i \succsim_i \omega_i - z_i$ for both i. Suppose not, then $\omega_i - z_i \succ_i \omega_i + z_i$ for some i. Then, by the strict convexity,

$$(1/2)(\omega_i - z_i) + (1/2)(\omega_i + z_i) = \omega_i \succ_i \omega_i + z_i = x_i^*.$$

This contradicts the assumption that $x_i^* \succsim_i \omega_i$. We must thus have $\omega_i + z_i \succsim_i \omega_i - z_i$ for both i.

18.E.1 The homogeneity of degree zero can be proved by noticing that, for every $\lambda > 0$, any allocation $(x_1^*,\ldots,x_H^*)$ for the H types solves the maximization problem (18.E.2) with a profile $(\mu_1,\ldots,\mu_H)$ if and only if it does so with the profile $(\lambda\mu_1,\ldots,\lambda\mu_H)$.

As for the concavity, let $\mu \in \mathbb{R}_+^H$, $\mu' \in \mathbb{R}_+^H$, and $t \in [0,1]$. Let $x = (x_1,\ldots,x_H) \in \mathbb{R}^{LH}$ be a solution with μ and $x' = (x_1',\ldots,x_H') \in \mathbb{R}^{LH}$ be a solution with μ'. Define $\mu'' = t\mu + (1-t)\mu' \in \mathbb{R}_+^H$. Define $x'' = (x_1'',\ldots,x_H'') \in \mathbb{R}^{LH}$ by $x_h'' = (t\mu_h/\mu_h'')x_h + ((1-t)\mu_h'/\mu_h'')x_h'$ for each h, then x" is feasible with μ''. Moreover, since $u_h(\cdot)$ is concave,

$$u_h(x_h'') \geq (t\mu_h/\mu_h'')u_h(x_h) + ((1-t)\mu_h'/\mu_h'')u_h(x_h').$$

Hence, by multiplying μ_h'' and taking the summation over h, we obtain

$$\textstyle\sum_h \mu_h'' u_h(x_h'') \geq t v(\mu) + (1-t)v(\mu').$$

By the definition of $v(\cdot)$, $v(\mu'') \geq \sum_h \mu_h'' u_h(x_h'')$. Thus $v(\mu'') \geq t v(\mu) + (1-t)v(\mu')$. Hence $v(\cdot)$ is concave.

18.E.2 Because of the strong monotonicity of the $u_h(\cdot)$ and the interior solution, we can replace the inequality constraints (i) of (18.E.2) by the equalities and neglect (ii). Since $\partial[\sum_k \bar{\mu}_k u_k(x_k^*)]/\partial\mu_h = u_h(x_h^*)$ and $\partial[\sum_k \bar{\mu}_k(x_k^* - \omega_k)]/\partial\mu_h = -(\omega_h - x_h^*)$, the envelop theorem (Theorem M.L.1) implies (18.E.5).

18.E.3 [First printing errata: In order to obtain an example of the desired property, we should not require the value function $v(\cdot)$ to be differentiable. If $v(\cdot)$ is not differentiable, we should define the marginal contribution of type h at $\bar{\mu}$ is defined as

$$\lim_{\mu_h < \bar{\mu}_h, \mu_h \to \bar{\mu}_h} \frac{v(\bar{\mu}_1, \ldots, \bar{\mu}_{h-1}, \mu_h, \bar{\mu}_{h+1}, \ldots, \bar{\mu}_H) - v(\bar{\mu})}{\mu_h - \bar{\mu}_h}.$$

That is, it is defined as the partial derivative of $v(\cdot)$ with respect to μ_h from the left.] We consider an example with two types, $h = 1,2$. Let $\omega_1 = (2,0,0)$ and $\omega_2 = (0,2,0)$. For both h, define $\psi_h\colon \mathbb{R}^2_+ \to \mathbb{R}$ by $\psi_h(x_{1h},x_{2h}) = \sqrt{\min\{x_{1h},x_{2h}\}}$, then $\psi_h(\cdot)$ is concave and not differentiable at any (x_{1h},x_{2h}) with $x_{1h} = x_{2h}$. Define then $u_h(x_h) = \psi_h(x_{1h},x_{2h}) + x_{3h}$, then $u_h(\cdot)$ is concave and not differentiable at any x_h with $x_{1h} = x_{2h}$.

Let $\bar{\mu} = (1,1)$. By the symmetry and the L-shaped preferences, the following constitute the class of the equilibria:

$p = (p_1,p_2,p_3) \in \mathbb{R}^3_+$ with $p_1 + p_2 = 1/2$ and $p_3 = 1$;

$x_1^* = (1, 1, p_1 - p_2)$ and $x_2^* = (1, 1, p_2 - p_1)$.

Note that the kinks of the identical L-shaped preferences admit multiple supporting price vectors at $(x^*_{1h},x^*_{2h}) = (1,1)$ $(h = 1,2)$, and the consumption allocations for the numeraire depend on the choice of the equilibrium price vectors. The utility levels at the equilibrium allocations are given by $u_1(x_1^*) = 1 + p_1 - p_2$ and $u_2(x_2^*) = 1 - p_1 + p_2$.

Let's now calculate the values $v(\mu)$ for μ near $\bar{\mu} = (1,1)$. If $0 \le \mu_1 \le 1$, then the L-shaped preferences and the strict concavity of the square root imply that the following constitute the class of the solution to problem (18.E.2):

$x_1 = (2\mu_1/(\mu_1 + 1), x_{21}, x_{31})$, $x_2 = (2\mu_1/(\mu_1 + 1), x_{22}, x_{32})$,

$x_{21} \geq 2\mu_1/(\mu_1 + 1)$, $x_{22} \geq 2\mu_1/(\mu_1 + 1)$, $x_{21} + x_{22} \leq 2$,

$x_{31} + x_{32} = 0$.

Hence $v(\mu_1,1) = (\mu_1 + 1)\sqrt{2\mu_1/(\mu_1 + 1)} = \sqrt{2\mu_1/(\mu_1 + 1)}$. Likewise, if $0 \leq \mu_2 < 1$, then $v(1,\mu_2) = \sqrt{2\mu_2/(\mu_2 + 1)}$.

The marginal contribution of each type at $\bar{\mu} = (1,1)$ can be calculated as follows:

$$\lim_{\mu_1<1,\mu_1\to 1} \frac{v(\mu_1,1) - v(1,1)}{\mu_1 - 1} = \lim_{\mu_2<1,\mu_2\to 1} \frac{v(\mu_2,1) - v(1,1)}{\mu_2 - 1} = 3/2.$$

Hence, for example, if $p_1 = p_2 = 1/2$, then both types get less than their marginal contribution at the equilibrium.

18.AA.1 Any partition $S_1,\ldots,S_N$ (in the standard sense) is a generalized partition with $\delta_n = 1$ for every n. Here is another kind of example: $I = \{1,2,3,4,5\}$, $N = 4$, $S_1 = \{1,2,3\}$, $S_2 = \{3,4,5\}$, $S_3 = \{1,2\}$, $S_4 = \{4,5\}$, and $\delta_n = 1/2$ for all n.

Suppose first that a TU game (I,v) has a nonempty core. Let a utility outcome $u \in \mathbb{R}^I$ be in its core. Let $S_1,\ldots,S_N$ be a generalized partition and the δ_n $(1 \leq n \leq N)$ be the corresponding weights. Then

$$\sum_{n=1}^{N} \delta_n v(S_n) \leq \sum_{n=1}^{N} \delta_n (\textstyle\sum_{i\in S_n} x_i) = \displaystyle\sum_{\{(i,n):i\in S_n\}} \delta_n x_i = \sum_{i=1}^{I} x_i (\sum_{\{n:S_n \ni i\}} \delta_n)$$
$$= \sum_{i=1}^{I} x_i = v(I).$$

Hence (I,v) is balanced.

Suppose conversely that (I,v) is balanced. To show the nonemptiness of its core, we consider a maximization problem whose solution is guaranteed to exist by the balancedness, then apply the duality theorem of linear programming (Theorem M.M.1) to derive the existence of a solution to the dual minimization problem, and show that this existence implies the nonemptiness of the core.

Denote by $\mathscr{S}$ the set of all coalitions (that is, the set of all nonempty subsets of I). Consider the following maximization problem:

$$\mathrm{Max}_{((\delta_S)_{S\in\mathscr{S}},\gamma)\geq 0} \textstyle\sum_{S\in\mathscr{S}} v(S)\delta_S - v(I)\gamma$$

$$\text{s.t.} \qquad \textstyle\sum_{S\ni i}\delta_S - \gamma \leq \quad 1 \text{ for all } i,$$

$$- \textstyle\sum_{S\ni i}\delta_S + \gamma \leq - 1 \text{ for all } i.$$

Note that the objective function is linear with coefficients $v(S)$ $(S \in \mathscr{S})$ and $- v(I)$, and the left-hand side of the constraints can be written by means of the $2I \times 2^I$ matrix such that, after assigning a number m $(1 \leq m \leq 2^I - 1)$ to each coalition:

The (ℓ,m) entry of the matrix is 1 if $\ell \leq I$, $m < 2^I$, and player ℓ belongs to coalition m;

The (ℓ,m) entry of the matrix is $- 1$ if $\ell > I$, $m < 2^I$, and player $\ell - I$ belongs to coalition m;

The (ℓ,m) entry of the matrix is 0 if $\ell \leq I$, $m < 2^I$, and player ℓ does not belong to coalition m;

The (ℓ,m) entry of the matrix is 0 if $\ell > I$, $m < 2^I$, and player $\ell - I$ does not belong to coalition m;

The $(\ell,2^I)$ entry of the matrix is $- 1$ if $\ell \leq I$;

The $(\ell,2^I)$ entry of the matrix is 1 if $\ell > I$.

Define $((\delta^*_S)_{S\in\mathscr{S}},\gamma^*)$ by $\delta^*_S = 0$ for all $S \neq I$, $\delta^*_I = 1$, and $\gamma^* = 0$. We now show that $((\delta^*_S)_{S\in\mathscr{S}},\gamma^*)$ is a solution. It is easy to see that it satisfies that constraints and $\sum_{S\in\mathscr{S}} v(S)\delta^*_S - v(I)\gamma^* = v(I)$. Let $((\delta_S)_{S\in\mathscr{S}},\gamma) \geq 0$ satisfy the constraints, then $\sum_{S\ni i}\delta_S - \gamma = 1$ for all i and hence the $(1/(1 + \gamma))\delta_S$ constitute balancing weights for the generalized partition $\mathscr{S}$. Thus, by the balancedness, $\sum_S v(S)(1/(1+\gamma))\delta_S - v(I) \le 0$, that is, $\sum_S v(S)\delta_S - v(I)\gamma \le v(I)$. Hence $((\delta^*_S)_{S\in\mathcal{S}}, \gamma^*)$ is a solution. Thus, by the Duality Theorem, the minimization problem:

$$\text{Min}_{((y_i),(z_i))\ge 0} \sum_{i\in I} y_i - \sum_{i\in I} z_i$$

$$\text{s.t.} \qquad \sum_{i\in S} y_i - \sum_{i\in S} z_i \ge \quad v(S) \text{ for all } S \in \mathcal{S},$$

$$- \sum_{i\in I} y_i + \sum_{i\in I} z_i \ge - v(I),$$

has a solution $((y^*_i),(z^*_i)) \ge 0$. By the constraints, $(y^*_i - z^*_i)_{i\in I}$ belongs to the core of (I,v). So the core is nonempty.

18.AA.2 This can be proved by Exercise 18.AA.1 and the fact that S and T constitute a generalized partition with coefficients $\delta_S = \delta_T = 1$. But a more direct proof is also available. Namely, suppose that $v(S) + v(T) > v(I)$. Let u be a utility outcome such that $\sum_{i\in S} u_i \ge v(S)$ and $\sum_{i\in T} u_i \ge v(T)$, then $\sum_{i\in I} u_i = \sum_{i\in S} u_i + \sum_{i\in T} u_i \ge v(S) + v(T) > v(I)$. So u is not feasible for the grand coalition and hence does not belong to the core. Hence the core must be empty if $v(S) + v(T) > v(I)$. The contraposition of this is the statement of the exercise.

18.AA.3 It was shown in the text that the proportional allocation belongs to the core. To prove that it is the only one, let $f(z) = cz$ with $c > 0$, then any feasible utility outcome $u = (u_1,\ldots,u_I)$ satisfies $\sum_i u_i = c(\sum_i \omega_i)$ and the proportion allocation $u^* = (u^*_1,\ldots,u^*_I)$ satisfies $u^*_i = c\omega_i = v(\{i\})$. Thus if a feasible utility outcome u is not equal to u^*, then there must exist $i \in I$ such that $u_i < v(\{i\})$. Hence u does not belong to the core. Thus the proportional allocation is the only allocation that belongs to the core.

18.AA.4 The following proof is a bit lengthy, but it is direct in the sense that it uses neither the four axioms nor the linear combination of Exercises 18.AA.5 and 18.AA.6.

If $\#S = 1$, then (18.AA.2) trivially holds. So suppose $\#S \geq 2$ and $\{i,h\} \subset S$. For any $T \subset S$, let $\Pi(T)$ be the set of all one-to-one functions from T onto itself. Then

$$\begin{aligned}\textstyle\sum_{i\in S} Sh_i(S,v) &= \textstyle\sum_{\pi\in\Pi(S)}(1/(\#S)!)m(S(\pi,i),i)\\ &= (1/(\#S)!)\textstyle\sum_{\pi\in\Pi(S)} m(S(\pi,i),i) = (1/(\#S)!)(\#S)!v(S) = v(S).\end{aligned}$$

To show the first equality of (18.AA.2), note that

$$Sh_i(S\setminus\{h\},v) = (1/(\#S - 1)!)\textstyle\sum_{\pi\in\Pi(S\setminus\{h\})} m(\pi,i).$$

Here, for each $\pi \in \Pi(S\setminus\{h\})$, there are exactly $\#S$ elements of $\Pi(S)$ whose restriction onto $S\setminus\{h\}$ is equal to π. Hence

$$\begin{aligned}Sh_i(S\setminus\{h\},v) &= (1/(\#S - 1)!)\textstyle\sum_{\pi\in\Pi(S)}(1/\#S)m(\pi|(S\setminus\{h\}),i)\\ &= (1/(\#S)!)\textstyle\sum_{\pi\in\Pi(S)} m(\pi|(S\setminus\{h\}),i),\end{aligned}$$

where $\pi|(S\setminus\{h\})$ denotes the restriction of $\pi \in \Pi(S)$ onto $S\setminus\{h\}$. Hence

$$Sh_i(S,v) - Sh_i(S\setminus\{h\},v) = (1/(\#S)!)\textstyle\sum_{\pi\in\Pi(S)}(m(\pi,i) - m(\pi|(S\setminus\{h\}),i)).$$

Now define

$$\Pi_1 = \{\pi \in \Pi(S)\colon \pi(i) < \pi(h)\},$$

$$\Pi_2 = \{\pi \in \Pi(S)\colon \pi(i) > \pi(h)\},$$

then Π_1 and Π_2 constitute a nontrivial partition of Π. Define $P\colon \Pi(S) \to \Pi(S)$ so that, for each $\pi \in \Pi(S)$, $P(\pi)$ is obtained from π by permuting i and h. Then the restrictions $P|\Pi_1$ and $P|\Pi_2$ are one-to-one and onto mappings between Π_1 and Π_2 (and, in fact, each one of them is the inverse of the other).

If $\pi \in \Pi_1$, then $S(\pi,i) = S(\pi|(S\setminus\{h\},i)$ and hence

$$m(\pi,i) - m(\pi|(S\setminus\{h\}),i) = 0.$$

If $\pi \in \Pi_2$, then

$$m(\pi,i) - m(\pi|(S\setminus\{h\}),i)$$

$$= v(S(\pi,i) \cup \{i\}) - v(S(\pi,i)) - v((S(\pi,i)\setminus\{h\}) \cup \{i\}) + v(S(\pi,i)\setminus\{h\}).$$

Therefore,

$$Sh_i(S,v) - Sh_i(S\setminus\{h\},v)$$

$$= (1/(\#S)!)(\textstyle\sum_{\pi\in\Pi_2} v(S(\pi,i) \cup \{i\}) - \sum_{\pi\in\Pi_2} v(S(\pi,i))$$

$$- \textstyle\sum_{\pi\in\Pi_2} v((S(\pi,i)\setminus\{h\}) \cup \{i\}) + \sum_{\pi\in\Pi_2} v(S(\pi,i)\setminus\{h\})).$$

Similarly,

$$Sh_h(S,v) - Sh_h(S\setminus\{i\},v)$$

$$= (1/(\#S)!)(\textstyle\sum_{\pi\in\Pi_1} v(S(\pi,h) \cup \{h\}) - \sum_{\pi\in\Pi_1} v(S(\pi,h))$$

$$- \textstyle\sum_{\pi\in\Pi_1} v((S(\pi,h)\setminus\{i\}) \cup \{h\}) + \sum_{\pi\in\Pi_1} v(S(\pi,h)\setminus\{i\})).$$

Hence, if $\pi_i \in \Pi_1$ and $\pi_2 = P(\pi_1)$ (and hence $\pi_2 \in \Pi_2$ and $\pi_1 = P(\pi_2)$), then

$$S(\pi_2,i) \cup \{i\} = S(\pi_1,h) \cup \{h\},$$

$$S(\pi_2,i) = (S(\pi_1,h)\setminus\{i\}) \cup \{h\},$$

$$(S(\pi_2,i)\setminus\{h\}) \cup \{i\} = S(\pi_1,h),$$

$$S(\pi_2,i)\setminus\{h\} = S(\pi_1,h)\setminus\{i\}.$$

We thus obtain $Sh_i(S,v) - Sh_i(S\setminus\{h\},v) = Sh_h(S,v) - Sh_h(S\setminus\{i\},v)$.

18.AA.5 Suppose that a utility outcome u of the unanimity game (I,v) of S satisfies the efficiency, symmetry, and dummy axioms. By the efficiency axiom, $\sum_{i\in I} u_i = v(S)$. By the definition of a unanimity game, if $i \notin S$, then $v(T \cup \{i\}) - v(T) = 0$ for every T. Hence, by the dummy axiom, $u_i = 0$. Thus $\sum_{i\in S} u_i = v(S)$. To apply the symmetry axiom, note that if another game (I,v') is identical to (I,v) except that the roles of the two players $i \in S$ and $h \in S$ are permuted, then, in fact, (I,v') is identical to (I,v). Thus the symmetry axiom requires $u_i = u_h$. Hence, by $\sum_{i\in S} u_i = v(S)$, we obtain $u_i = v(S)/\#S$ for every $i \in S$.

18.AA.6 Let $\mathcal{S}$ be the set of all coalitions and, for each $S \in \mathcal{S}$, denote by (I, v_S) the unanimity game such that, for every $T \in \mathcal{S}$, $v(T) = 1$ if $S \subset T$ and $v(T) = 0$ otherwise. Denote by G the vector space of all TU-games on I. By $\#\mathcal{S} = 2^I - 1$, $\dim G = 2^I - 1$. Define f: $\mathbb{R}^{2^I-1} \to G$ by $f((\alpha_S)_{S\in\mathcal{S}}) = \sum_{S\in\mathcal{S}}\alpha_S v_S$, then this is a linear transformation. For the first statement, therefore, it is sufficient to prove that $f(\cdot)$ is onto. For this, by $\dim G = 2^I - 1$, it is sufficient to show that if $f((\alpha_S)_{S\in\mathcal{S}}) = 0$, then $\alpha_S = 0$ for all $S \in \mathcal{S}$. We shall now establish this by induction on $\#S$.

Suppose that $f((\alpha_S)_{S\in\mathcal{S}}) = 0$. Let $S \in \mathcal{S}$ and $\#S = 1$. Then, for every $T \in \mathcal{S}$, $v_T(S) = 1$ if $T = S$ and $v_T(S) = 0$ otherwise. Hence $f((\alpha_T)_{T\in\mathcal{S}})(S) = \sum_{T\in\mathcal{S}}\alpha_T v_T(S) = \alpha_S$. Thus $\alpha_S = 0$. Suppose now that $S \in \mathcal{S}$ and, for every $T \in \mathcal{S}$, if $\#T < \#S$, then $\alpha_T = 0$. Then

$$f((\alpha_T)_{T\in\mathcal{S}})(S) = \sum_{\{T\in\mathcal{S}: T\subset S\}}\alpha_T v_T(S) = \sum_{\{T\in\mathcal{S}: T\subset S, \#T\geq\#S\}}\alpha_T v_T(S).$$

But here, $\{T \in \mathcal{S}: T \subset S, \#T \geq \#S\} = \{S\}$. Hence $f((\alpha_T)_{T\in\mathcal{S}})(S) = \alpha_S$. Thus $\alpha_S = 0$. This completes the proof of the first part.

To show the second part, for each $S \in \mathcal{S}$, let u^S be the utility outcome for v^S identified in Exercise 18.AA.5. (That is, $u_i^S = 1/\#S$ for every $i \in S$ and $u_i^S = 0$ for any $i \notin S$.) Let $v = \sum_{S\in\mathcal{S}}\alpha_S v_S$ be a TU-game on I. We shall prove that the utility outcome $\sum_{S\in\mathcal{S}}\alpha_S u^S$ is the only one that satisfies the four axioms. Since each u^S satisfies the efficiency, symmetry, and dummy axioms, the linearity axiom implies that if there exists a utility outcome that satisfies the four axioms, then it must be $\sum_{S\in\mathcal{S}}\alpha_S u^S$. It is therefore sufficient to prove that $\sum_{S\in\mathcal{S}}\alpha_S u^S$ actually satisfies the four axioms. Our of the four, the linearity axiom is satisfied by construction. The efficiency axiom can be easily checked. It thus remains to verify the dummy and symmetry axioms.

Let $i \in I$ satisfy $v(S) - v(S\setminus\{i\}) = 0$ for every $S \in \mathcal{S}$. We shall prove by induction on $\#S$ that, for every $S \in \mathcal{S}$, if $i \in S$, then $\alpha_S = 0$. Let $S \in \mathcal{S}$ be as such. Let $\#S = 1$, then $S = \{i\}$. Hence $v(S) = \alpha_S$. On the other hand, $v(S) - v(\varnothing) = v(S) = 0$. Thus $\alpha_S = 0$. Suppose now that $S \in \mathcal{S}$, $i \in S$, and, for every $T \in \mathcal{S}$, if $\#T < \#S$ and $i \in T$, then $\alpha_T = 0$. Then

$$v(S) - v(S\setminus\{i\}) = \sum_{\{T\in\mathcal{S}: i\in T, T\subset S\}}\alpha_T v_T(S) = \sum_{\{T\in\mathcal{S}: i\in T, T\subset S, \#T\geq\#S\}}\alpha_T v_T(S).$$

But here, $\{T \in \mathcal{S}: i \in T, T \subset S, \#T \geq \#S\} = \{S\}$. Thus $\alpha_S = 0$. This completes the proof of the dummy axiom.

As for the symmetry condition, assume that another game (I,v') is identical to (I,v) except that the roles of the two players i and h are permuted. Define $\tau: \mathcal{S} \to \mathcal{S}$ by $\tau(S) = S$ if $\{i,h\} \subset S$ or $\{i,h\} \cap S = \varnothing$, $\tau(S) = (S\setminus\{i\}) \cup \{h\}$ if $i \in S$ and $h \notin S$, and $\tau(S) = (S\setminus\{h\}) \cup \{i\}$ if $i \notin S$ and $h \in S$. Then $\tau(\cdot)$ is one-to-one and onto, $\tau = \tau^{-1}$, $\{S \in \mathcal{S}: i \in S\} = \{S \in \mathcal{S}: h \in \tau(S)\}$, and $\#S = \#\tau(S)$. Moreover, for each $S \in \mathcal{S}$ and $T \in \mathcal{S}$, $S \subset \tau(T)$ if and only if $\tau(S) \subset T$. Thus $v' = \sum_S \alpha_S v_{\tau(S)} = \sum_S \alpha_{\tau(S)} v_S$. Hence $u_i' = \sum_{S\ni i}\alpha_{\tau(S)}/\#S = \sum_{S\ni i}\alpha_{\tau(S)}/\#\tau(S) = \sum_{S\ni h}\alpha_S/\#S = u_h$. Similarly, $u_h' = u_i$. The symmetry axiom is thus verified.

18.AA.7 [<u>First printing errata</u>: The difficulty level should perhaps be B.] We shall first establish two lemmas. Denote by the set of all coalitions by $\mathcal{S}$. For each $S \in \mathcal{S}$ and $i \notin S$, define $\varphi_{S,i}: [0,1] \to \mathbb{R}^n_+$ by $\varphi_{S,i}(t) = \sum_{h\in S}\omega_h + t\omega_i$. Note that if $S \subset T$ and $i \notin T$, then $\varphi_{S,i}(t) \leq \varphi_{T,i}(t)$ for all t.

Lemma 1: Let $S \in \mathcal{S}$ and $i \notin S$, then

$$v(S \cup \{i\}) - v(S) = \int_0^1 \nabla f(\varphi_{S,i}(t))\omega_i dt.$$

In fact,

$$v(S \cup \{i\}) - v(S) = f(\varphi(1)) - f(\varphi(0)) = \int_0^1 \nabla f(\varphi_{S,i}(t))(d\varphi_{S,i}(t)/dt)dt$$
$$= \int_0^1 \nabla f(\varphi_{S,i}(t))\omega_i dt.$$

Lemma 2: Let $z \in \mathbb{R}^n_+$, $z' \in \mathbb{R}^n_+$, and $z' \geq z$, then $\nabla f(z') \geq \nabla f(z)$.

In fact, define ψ: $[0,1] \to \mathbb{R}^n_+$ by $\psi(t) = z + t(z' - z)$, then

$$\nabla f(z') - \nabla f(z) = \nabla f(\psi(1)) - \nabla f(\psi(0)) = \int_0^1 D^2 f(\psi(t))(d\psi(t)/dt)dt$$
$$= \int_0^1 D^2 f(\varphi_{S,i}(t))(z' - z)dt \geq 0,$$

where the integral is taken component-wise.

Now let $S \subset T$ and $i \notin T$, then, by Lemma 1,

$$v(S \cup \{i\}) - v(S) = \int_0^1 \nabla f(\varphi_{S,i}(t))\omega_i dt \text{ and}$$
$$v(T \cup \{i\}) - v(T) = \int_0^1 \nabla f(\varphi_{T,i}(t))\omega_i dt.$$

But, by Lemma 2,

$$\int_0^1 \nabla f(\varphi_{S,i}(t))\omega_i dt \leq \int_0^1 \nabla f(\varphi_{T,i}(t))\omega_i dt.$$

Hence

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

The convexity is thus proved.

18.AA.8 Let $n = 2$ and define f: $\mathbb{R}^2_+ \to \mathbb{R}_+$ by $f(z) = (z_1^2 + z_2^2)^{1/2}$, then $f(\cdot)$ is convex. Let $I = 2$ and $\omega_1 = (1,0)$ and $\omega_2 = (0,1)$. Then $v(\{1\}) = v(\{2\}) = 1$ and $v(\{1,2\}) = 2^{1/2}$. Hence, by Exercise 18.AA.2, the core is empty.

18.AA.9 (a) It is easy to check that $v(S) = f(\sum_{i \in S}\omega_i)$, where $f(z) = \text{Min}(z_1,z_2)$.

(b) A utility outcome u is in the core if and only if $u_i \geq 0$, $\sum_i u_i = 2$, $u_1 + u_3 \geq 1$, $u_1 + u_4 \geq 1$, $u_2 + u_3 \geq 1$, and $u_2 + u_4 \geq 1$. This is equivalent to saying that $u_i \geq 0$, $\sum_i u_i = 2$, and $u_1 + u_3 = u_1 + u_4 = u_2 + u_3 = u_2 + u_4 = 1$.

This is, in turn, equivalent to $u = (\alpha, \alpha, 1 - \alpha, 1 - \alpha)$ for some $\alpha \in [0,1]$.

(c) After the modification, if a utility outcome u is in the core, then $u_1 + u_3 + u_4 \geq 2$. Hence $u_1 + u_3 + u_4 = 2$ and $u_2 = 0$. The other inequalities defining the core are now $u_1 + u_3 \geq 1$, $u_1 + u_4 \geq 1$, $u_3 \geq 1$, and $u_4 \geq 1$. Thus $u_1 = 0$ and $u_3 = u_4 = 1$. Hence, the welfare of consumer 1 is worse off.

(d) By the symmetry axiom, $Sh_1(v) = Sh_2(v)$ and $Sh_3(v) = Sh_4(v)$. By the efficiency axiom, $Sh_1(v) + Sh_3(v) = 1$. By (18.AA.3), we can obtain $Sh_1(v) = 1/4$. Hence $Sh(v) = (1/4,1/4,1/4,1/4)$.

(e) Denote by v' the TU-game obtained after the modification of (c). Define another TU-game v'' by $v(S) = 1$ if $S = \{1,3,4\}$ and $v(S) = 0$ otherwise. Then $v' = v + v''$. Moreover, $Sh_2(v'') = -1/4$ by (18.AA.3) and, thus, $Sh_i(v'') = 1/12$ for any $i \neq 2$, by the efficiency and symmetry axioms. Thus, by the linearity axiom, $Sh(v') = Sh(v) + Sh(v'') = (1/3,0,1/3,1/3)$. Hence the Shapley value for player 1 increases by the modification, which is in the opposite direction to that of the core outcome.

18.AA.10 (a) Let $I = 2$ and the spaces (x_1,x_2) are given. Define a TU-game $(I,v_{(x_1,x_2)})$ by $v_{(x_1,x_2)}(S) = -C(\sum_{i\in S} x_i)$. The proposed cost allocation is then $(-Sh_1(I,v_{(x_1,x_2)}), -Sh_2(I,v_{(x_1,x_2)}))$. This can be explicitly calculated as:

$$((1/2)((x_1 + x_2)^{\gamma} + x_1^{\gamma} - x_2^{\gamma}), (1/2)((x_1 + x_2)^{\gamma} - x_1^{\gamma} + x_2^{\gamma})).$$

(b) The marginal cost on division one is $(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_1^{\gamma-1})$. The marginal cost on division two is $(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_2^{\gamma-1})$.

(c) [First printing errata: In (c) and (d), we should assume that $\gamma > 1$.] By re-numbering the two divisions if necessary, we can assume that $\alpha_1 \geq \alpha_2$. The answer depends on the values of α_1 and α_2. That is, the cost allocation rule proposed in (b) leads to a profit-maximizing choice of overhead if and only if $\alpha_1 \geq 2\alpha_2$. To prove this, suppose first that $\alpha_1 \geq 2\alpha_2$. We shall show that the allocation $(x_1, x_2) = ((\alpha_1/\gamma)^{1/(\gamma-1)}, 0)$ is the unique profit-maximizing allocation, and led by the cost allocation rule in (b). In fact, the former follows from $\alpha_1 > \alpha_2$ and and $\partial C(x_1 + x_2)/\partial x_1 = \gamma x_1^{\gamma-1} = \alpha_1$. The latter follows from

$$(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_1^{\gamma-1}) = \gamma x_1^{\gamma-1} = \alpha_1,$$
$$(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_2^{\gamma-1}) = (\gamma/2)x_1^{\gamma-1} = \alpha_1/2 \geq \alpha_2.$$

Suppose next that $2\alpha_2 > \alpha_1 > \alpha_2$. Then the allocation $(x_1, x_2) = ((\alpha_1/\gamma)^{1/(\gamma-1)}, 0)$ is the unique profit-maximizing allocation, but it cannot be led by the proposed cost allocation rule because

$$(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_2^{\gamma-1}) = (\gamma/2)x_1^{\gamma-1} = \alpha_1/2 < \alpha_2.$$

Suppose finally that $\alpha_1 = \alpha_2$. Then an allocation (x_1, x_2) is profit-maximizing if and only if $x_1 + x_2 = (\alpha_1/\gamma)^{1/(\gamma-1)}$. So assume that (x_1, x_2) is as such. If $x_1 \neq x_2$, then

$$(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_1^{\gamma-1}) \neq (\gamma/2)((x_1 + x_2)^{\gamma-1} + x_2^{\gamma-1})$$

and hence such an allocation cannot be attained by the proposed rule. If $x_1 = x_2$, then

$$(\gamma/2)((x_1 + x_2)^{\gamma-1} + x_1^{\gamma-1}) < \gamma(x_1 + x_2)^{\gamma-1} = \alpha_1.$$

Hence the allocation cannot be attained either.

(d) Rigorously speaking, the answer depends on how to interpret efficient decentralization choice. In fact, we shall establish the following two statements.

(i) There is no allocation rule $\psi_1(\cdot)$ and $\psi_2(\cdot)$ with the following two properties: First, $\psi_1(x_1,x_2) + \psi_2(x_1,x_2) = C(x_1 + x_2)$ for all (x_1,x_2); second, for every (α_1,α_2) and for *every* combination (x_1,x_2) that is profit-maximizing under (α_1,α_2), the two divisions are led to choose (x_1,x_2) when faced with $\psi_1(\cdot)$ and $\psi_2(\cdot)$ under (α_1,α_2) in the sense of (c).

(ii) There exists a cost allocation rule $\psi_1(\cdot)$ and $\psi_2(\cdot)$ with the following two properties: First, $\psi_1(x_1,x_2) + \psi_2(x_1,x_2) = C(x_1 + x_2)$ for all (x_1,x_2); second, for every (α_1,α_2) and for *some* combination (x_1,x_2) that is profit-maximizing under (α_1,α_2), the two divisions are led to choose (x_1,x_2) when faced with $\psi_1(\cdot)$ and $\psi_2(\cdot)$ under (α_1,α_2) in the sense of (c).

Thus the requirement for "efficient decentralized choice" is stronger in (i), because, for every (α_1,α_2), (i) requires all profit-maximizing combinations (x_1,x_2) under (α_1,α_2) to be attained as a result of decentralized choice; on the other hand, (ii) requires some profit-maximizing combination (x_1,x_2) to be attained. As we saw in (c), there are multiple profit-maximizing combinations (x_1,x_2) if and only if $\alpha_1 = \alpha_2$. The difference in the notion of decentralized choice between (i) and (ii) therefore arises only when $\alpha_1 = \alpha_2$. But, when it comes to the existence of a cost allocation rule of the desired properties, this seemingly subtle difference brings about two different conclusions.

Let's prove (i) first. Suppose that an allocation rule $\psi_1(\cdot)$ and $\psi_2(\cdot)$ satisfies $\psi_1(x_1,x_2) + \psi_2(x_1,x_2) = C(x_1 + x_2)$ for all (x_1,x_2). We shall now show that there exists a combination $(\bar{x}_1,\bar{x}_2)$ such that either $\partial\psi_1(\bar{x}_1,\bar{x}_2)/\partial x_1 \neq C'(\bar{x}_1 + \bar{x}_2)$ or $\partial\psi_2(\bar{x}_1,\bar{x}_2)/\partial x_2 \neq C'(\bar{x}_1 + \bar{x}_2)$. In fact, suppose that $\partial\psi_1(x_1,x_2)/\partial x_1 = C'(x_1 + x_2)$ and $\partial\psi_2(x_1,x_2)/\partial x_2 = C'(x_1 + x_2)$ for every

(x_1,x_2). Then $\partial\psi_1(x_1,x_2)/\partial x_2 = 0$ and $\partial\psi_2(x_1,x_2)/\partial x_1 = 0$ for every (x_1,x_2). We can thus write $\psi_1(x_1,x_2) = \psi_1(x_1)$ and $\psi_2(x_1,x_2) = \psi_2(x_2)$. That is, the cost on each division depends only on its own usage. Hence $\psi_1(x_1) + \psi_2(x_2) = C(x_1 + x_2)$ for every (x_1,x_2). Now differentiate both sides twice, first with respect to x_1 and then with respect to x_2. Then we obtain

$$0 = \partial^2 C(x_1 + x_2)/\partial x_1 \partial x_2 = \gamma(\gamma - 1)(x_1 + x_2)^{\gamma-2} > 0,$$

which is a contradiction. Hence there exists a combination $(\bar{x}_1,\bar{x}_2)$ such that either $\partial\psi_1(\bar{x}_1,\bar{x}_2)/\partial x_1 \neq C'(\bar{x}_1 + \bar{x}_2)$ or $\partial\psi_2(\bar{x}_1,\bar{x}_2)/\partial x_2 \neq C'(\bar{x}_1 + \bar{x}_2)$.

By re-numbering the two divisions if necessary, we can assume that $\partial\psi_1(\bar{x}_1,\bar{x}_2)/\partial x_1 \neq C'(\bar{x}_1 + \bar{x}_2)$. Let $\alpha_1 = \alpha_2 = C'(\bar{x}_1 + \bar{x}_2)$, then $(\bar{x}_1,\bar{x}_2)$ is a profit-maximizing combination under (α_1,α_2). But division one does not choose $\bar{x}_1$ because $\partial\psi_1(\bar{x}_1,\bar{x}_2)/\partial x_1 \neq \alpha_1$. Statement (i) is thus proved.

Let's next prove (ii). Define

$$\psi_1(x_1,x_2) = \begin{cases} C(x_1) & \text{if } x_1 \geq x_2, \\ C(x_1 + x_2) - C(x_2) & \text{if } x_1 < x_2, \end{cases}$$

$$\psi_2(x_1,x_2) = \begin{cases} C(x_1 + x_2) - C(x_1) & \text{if } x_1 \geq x_2, \\ C(x_2) & \text{if } x_1 < x_2. \end{cases}$$

It immediately follows from this construction that $\psi_1(x_1,x_2) + \psi_2(x_1,x_2) = C(x_1 + x_2)$ for all (x_1,x_2). To verify efficient decentralized choice, let $\alpha_1 \geq \alpha_2$. Then $(x_1,x_2) = ((\alpha_1/\gamma)^{1/(\gamma-1)},0)$ is a profit-maximizing allocation (and the unique one if $\alpha_1 > \alpha_2$), and

$$\partial\psi_1((\alpha_1/\gamma)^{1/(\gamma-1)},0)/\partial x_1 = \alpha_1,$$

$$\partial\psi_2((\alpha_1/\gamma)^{1/(\gamma-1)},0)/\partial x_2 = \alpha_1 \geq \alpha_2.$$

Thus $((\alpha_1/\gamma)^{1/(\gamma-1)},0)$ is chosen by the two divisions in the sense of (c). Likewise, we can show that if $\alpha_1 < \alpha_2$, then $(x_1,x_2) = (0,(\alpha_2/\gamma)^{1/(\gamma-1)})$ is the unique profit-maximizing allocation and chosen by the two divisions.

Statement (ii) is thus proved.

CHAPTER 19

19.C.1 Let $(x_1,\ldots,x_S) \in \mathbb{R}^{LS}_+$, $(x'_1,\ldots,x'_S) \in \mathbb{R}^{LS}_+$, and $\lambda \in [0,1]$. For each s, by the concavity of $u_s(\cdot)$, $u_s(\lambda x_s + (1-\lambda)x'_s) \geq \lambda u_s(x_s) + (1-\lambda)u_s(x'_s)$. Multiplying $\pi_s \geq 0$ and summing over s, we obtain $U(\lambda x + (1-\lambda)x') \geq \lambda U(x) + (1-\lambda)U(x')$. Thus $U(\cdot)$ is concave.

19.C.2 Let (x_1,x_2) be a (interior) Pareto optimal allocation, then

$$\pi_1 u'_1(x_{11})/\pi_2 u'_1(x_{21}) = \pi_1 u'_2(x_{12})/\pi_2 u'_2(x_{22}).$$

Hence $u'_1(x_{11})/u'_1(x_{21}) = u'_2(x_{12})/u'_2(x_{22})$. Denote this value by $k > 0$. Then $k \geq 1$ if and only if $u'_1(x_{11}) \geq u'_1(x_{21})$ and $u'_2(x_{12}) \geq u'_2(x_{22})$. But this is equivalent to $x_{11} \leq x_{21}$ and $x_{12} \leq x_{22}$. But this implies that

$$2 = \omega_{11} + \omega_{21} = x_{11} + x_{21} \leq x_{12} + x_{22} = \omega_{21} + \omega_{22} = 1,$$

a contradiction. We must thus have $k < 1$. Hence, if $p = (p_1,p_2)$ is a supporting price vector, then

$$p_1/p_2 = \pi_1 u'_1(x_{11})/\pi_2 u'_1(x_{21}) = (\pi_1/\pi_2)k < \pi_1/\pi_2.$$

19.C.3 Write $x_i = (x_{1i},\ldots,x_{Si}) \in \mathbb{R}^S$ and define

$$\bar{x}_i = (\textstyle\sum_s \pi_{si}x_{si},\ldots,\sum_s \pi_{si}x_{si}) \in \mathbb{R}^S.$$

Then, by the concavity of the Bernoulli utility function, $\bar{x}_i \succsim_i x_i$. Hence (by the strong monotonicity, which implies the local nonsatiation), $p\cdot\bar{x}_i \geq p\cdot x_i$. But here

$$\begin{aligned} p\cdot\bar{x}_i - p\cdot x_i &= (\textstyle\sum_s p_s)(\sum_s \pi_{si}x_{si}) - \sum_s p_s x_{si} = \sum_s \pi_{si}\bar{p}x_{si} - \sum_s p_s x_{si} \\ &= \textstyle\sum_s (\pi_{si}\bar{p} - p_s)x_{si}. \end{aligned}$$

Hence $\sum_s (\pi_{si}\bar{p} - p_s)x_{si} \geq 0$.

By dividing both sides of this inequality by $\bar{p}$, we obtain

$\sum_s(\pi_{si} - p_s/\bar{p})x_{si} \geq 0$. A possible interpretation is thus that the equilibrium consumption x_i is biased towards the states s whose equilibrium price ratio $p_s/\bar{p}$ is lower than the subjective probability π_i that consumer assigns to states s.

19.C.4 [First printing errata: Every $u_i(\cdot)$ should be just concave, not strictly concave. This is because risk neutrality needs to be allowed for.]

(a) Let (x,p) be an interior Arrow-Debreu equilibrium. Then

$$p_1/p_2 = \pi_{11}/\pi_{21} = \pi_{12}u_2'(x_{12})/\pi_{22}u_2'(x_{22}).$$

By the equality of the subjective probabilities, $u_2'(x_{12})/u_2'(x_{22}) = 1$. Thus $x_{12} = x_{22}$ by the strict concavity $u_1(\cdot)$.

(b) We now have $u_2'(x_{12})/u_2'(x_{22}) < 1$ if and only if $\pi_{11}/\pi_{21} < \pi_{12}/\pi_{22}$. But $u_2'(x_{12})/u_2'(x_{22}) < 1$ if and only if $x_{12} > x_{22}$. Hence consumer 2 consumes more in state 1 when his probability assessment for the state is larger than that of consumer 1. Symmetrically, consumer 2 consumes more in state 2 when his probability assessment for the state is larger than that of consumer 1.

Consumer 1 will not gain from trade because, at equilibrium, $p_1/p_2 = \pi_{11}/\pi_{21}$ and hence his indifference "line" going through his initial endowment coincides with his budget line.

19.C.5 An allocation $(x_1^*,\dots,x_I^*) \in X_1 \times \dots \times X_I$ and a price vector $p = (p_1,\dots,p_S) \in \mathbb{R}^S$ constitute an Arrow-Debreu equilibrium if

(i) For every i, x_i^* is maximal for $\succsim_i$ in $\{x_i \in X_i: p\cdot x_i \leq p\cdot\omega_i\}$.

(ii) $\sum_i x_i^* = \sum_i \omega_i$.

Denote the common probability by $\pi = (\pi_1,\dots,\pi_S) \in \mathbb{R}^S_+$. For each i, define x_i^*

$= (\pi\cdot\omega_i,\ldots,\pi\cdot\omega_i) \in \mathbb{R}^S$. We shall prove that the allocation $(x_1^*,\ldots,x_I^*) \in \mathbb{R}^{SI}$ and a price vector $p = \pi$ constitute an Arrow-Debreu equilibrium. First, for each i, $\pi\cdot x_i^* = \sum_s \pi_s \pi\cdot\omega_i = \pi\cdot\omega_i$ and hence x_i^* is in the budget set. Let $x_i \in X_i$ and $\pi\cdot x_i \le \pi\cdot\omega_i$. By the risk aversion, $(\pi\cdot x_i,\ldots,\pi\cdot x_i) \succsim_i x_i$. By the monotonicity, $x_i^* = (\pi\cdot\omega_i,\ldots,\pi\cdot\omega_i) \succsim_i (\pi\cdot x_i,\ldots,\pi\cdot x_i)$. Hence $x_i^* \succsim_i x_i$ by the transitivity, and condition (i) is met. As for condition (ii), let $k = \sum_i \omega_{si}$ for all s. Then $\sum_i x_{si}^* = \pi\cdot(\sum_i \omega_i) = \sum_s \pi_s k = k$. Hence (ii) is met.

19.D.1 Out of the two inequalities (i) and (ii) that constitute the budget constraint of (19.D.1), (i) remains true. Hence we can normalize the security price vector q (so that, for example, $q_1 = 1$ or $\sum_s q_s = 1$). The second inequality (ii) must be replaced by $p_s\cdot x_{si} \le p_s\cdot\omega_{si} + z_{si}$ for every s. This family of the inequalities is not homogeneous of degree one with respect to p_s because the z_{si}-term does not involve p_s. Thus we can no longer normalize the price of one physical commodity equal to 1.

19.D.2 Let $\bar{\omega} = (\bar{\omega}_1,\bar{\omega}_2) \in \mathbb{R}^2_{++}$ be the aggregate endowment vector in each state, then $\omega_1 = (\bar{\omega}_1,\bar{\omega}_2,0,0)$ and $\omega_2 = (0,0,\bar{\omega}_1,\bar{\omega}_2)$. Define $\hat{p} = \nabla u((1/2)\bar{\omega}) \in \mathbb{R}^2_{++}$ and $p = (\hat{p},\hat{p}) \in \mathbb{R}^4_{++}$. Define $x_1^* = x_2^* = (1/2)(\bar{\omega},\bar{\omega}) \in \mathbb{R}^4_{++}$. Then it is easy to check that (p,x^*) is an Arrow-Debreu equilibrium. Let's now figure out the corresponding Radner equilibrium with a price vector (q_1,q_2) for the contingent first-good commodities and consumption plan (z_{1i}^*,z_{2i}^*). Just as in the proof of Proposition 19.D.1, let $q_1 = q_2 = \hat{p}_1 = \partial u((1/2)\bar{\omega})/\partial x_{1si}$. By $\hat{p}\cdot((1/2)\bar{\omega} - \bar{\omega}) = -(\hat{p}\cdot\bar{\omega})/2$, we must have

$$z_{1i}^* = -(\hat{p}\cdot\bar{\omega})/2\hat{p}_1 = \frac{-\nabla u((1/2)\bar{\omega})\cdot\bar{\omega}}{2\partial u((1/2)\omega)/\partial x_{1si}}.$$

By the budge constraint for the contingent commodities, $z_{2i}^* = -z_{1i}^*$. By the feasibility constraint, $z_{12}^* = -z_{22}^* = -z_{11}^*$. Hence

$$\alpha = \beta = -z_{11}^* = -z_{22}^* = \frac{\nabla u((1/2)\bar{\omega})\cdot\bar{\omega}}{2\partial u((1/2)\omega)/\partial x_{1si}}.$$

19.D.3 By including the consumption at $t = 0$, we have $X_i = \mathbb{R}_+^{L(1+S)}$, $\omega_i \in \mathbb{R}^{L(1+S)}$, and a spot price vector $p_0 = (p_{10},\ldots,p_{L0}) \in \mathbb{R}^L$ at $t = 0$ in addition to the spot price vectors $p_s = (p_{1s},\ldots,p_{Ls}) \in \mathbb{R}^L$ in each state s at $t = 1$. Write $p = (p_0,p_1,\ldots,p_S) \in \mathbb{R}^{L(1+S)}$. A contingent commodity price vector $q = (q_1,\ldots,q_S) \in \mathbb{R}^S$ is as before, and the L physical goods at $t = 0$ and the S contingent commodities are traded in the markets opened at $t = 0$. The utility maximization problem (19.D.1) is now

$$\operatorname*{Max}_{\substack{(x_{0i},x_{1i},\ldots,x_{Si})\in\mathbb{R}_+^{L(1+S)} \\ (z_{1i},\ldots,z_{Si})\in\mathbb{R}^S}} U_i(x_{0i},x_{1i},\ldots,x_{Si})$$

s.t. (i) $p_0\cdot x_{0i} + \sum_s q_s z_{si} \le p_0\cdot\omega_{0i}$,

(ii) $p_s\cdot x_{si} \le p_s\cdot\omega_{si} + p_{1s}z_{si}$ for every $s = 1,\ldots,S$.

A collection formed by a price vector $q = (q_1,\ldots,q_S) \in \mathbb{R}^S$ for contingent first-good commodities, a spot price vector p_s for each period and state $s = 0,1,\ldots,S$, and, for every consumer i, consumption plan $z_i^* = (z_{1i}^*,\ldots,z_{Si}^*)$ and $x_i^* = (x_{0i}^*,x_{1i}^*,\ldots,x_{Si}^*) \in \mathbb{R}^{L(1+S)}$ at each period and state, constitute a *Radner equilibrium* if

(i) For every i, the consumption plan (z_i^*,x_i^*) solves the above maximization problem.

(ii) $\sum_i z_{si}^* \le 0$ for every $s = 1,\ldots,S$, and $\sum_i x_{si}^* \le \sum_i \omega_{si}$ for every $s = 0,1,\ldots,S$.

The statement of Proposition 19.D.1 remains the same, except that the commodity bundles and price vectors now lie in $\mathbb{R}^{L(1+S)}$ and, in part (ii), the "multiplied" contingent commodity vector is now given as $(p_0, \mu_1 p_1, \ldots, \mu_S p_S)$. In its proof, the definitions of the two budget sets, B_i^{AD} and B_i^R, should be easily modified to incorporate the consumption at t = 0 (as was done for B_i^R in the above formulation of the utility maximization problem). The equivalence between the two modified budget sets can be proved just as before.

19.D.4 (a) The given event-tree structure implies that S = 4, T + 1 = 3, and $\mathscr{F}_0 = \{\{1,2,3,4\}\}$, $\mathscr{F}_1 = \{\{1,2\},\{3,4\}\}$, and $\mathscr{F}_2 = \{\{1\},\{2\},\{3\},\{4\}\}$. The consumption sets $X_i \subset \mathbb{R}^{H\cdot 3\cdot 4}$ and the initial endowments $\omega_i \in \mathbb{R}^{H\cdot 3\cdot 4}$ must satisfy the measurability constraint, which was explained in the small-type discussion at the end of Section 19.B. The admissible date-event pairs tE (defined in the small type discussion at the end of Section 19.D) are thus 0{1,2,3,4}, 1{1,2}, 1{3,4}, 2{1}, 2{2}, 2{3}, and 2{4}. Denote the set of the admissible date-event pairs by $\mathscr{A}$.

An allocation $(x_1^*, \ldots, x_I^*) \in X_1 \times \ldots \times X_I$ and a price vector $p = (p_{tE})_{tE\in\mathscr{A}}$ with $p_{tE} \in \mathbb{R}_+^H$ for every $tE \in \mathscr{A}$ constitute an Arrow-Debreu equilibrium if

(i) For every i, x_i^* solves the maximization problem

$$\mathrm{Max}_{x_i \in X_i} \ U_i(x_i)$$

$$\text{s.t.} \quad \textstyle\sum_{tE\in\mathscr{A}} p_{tE} x_{tEi} \le \sum_{tE\in\mathscr{A}} p_{tE}\omega_{tEi}.$$

(Recall that x_{tEi} and ω_{tEi} are well defined by the measurability constraint.)

(ii) $\sum_i x_i^* \le \sum_i \omega_i$.

(b) A collection formed by a price vector

$$q = ((q_{tE}(t + 1, E'))_{E'\in\mathscr{F}_{t+1}, E'\subset E})_{tE\in\mathscr{A}}$$

for contingent first-good commodities, a spot price vector $p = (p_{tE})_{tE\in\mathscr{A}}$ with $p_{tE} \in \mathbb{R}^H_+$ for every $tE \in \mathscr{A}$, and, for every consumer i, consumption plan

$$z_i^* = ((z_{tEi}^*(t+1, E'))_{E'\in\mathscr{S}_{t+1}, E'\subset E})_{tE\in\mathscr{A}}$$

and $x_i^* \in X_i$ constitute a *Radner equilibrium* if

(i) For every i, the consumption plans (z_i^*, x_i^*) maximizes $U_i(x_i)$ with respect to $x_i \in X_i$ and $z_i = ((z_{tEi}(t+1, E'))_{E'\in\mathscr{S}_{t+1}, E'\subset E})_{tE\in\mathscr{A}}$ subject to the budget constraint in the small-type discussion at the end of Section 19.D.

(ii) $\sum_i z_{tEi}^*(t+1, E') \le 0$ for every $tE \in \mathscr{A}$ and $E' \in \mathscr{S}_{t+1}$ with $E' \subset E$, and $\sum_i x_{tEi}^* \le \sum_i \omega_{tEi}$ for every $tE \in \mathscr{A}$.

Note that, since there is no predecessor at $tE = 0\{1,2,3,4\}$, the budget constraint on this admissible pairs becomes

$$p_{0E}\cdot x_{0Ei} + \sum_{E'\in\{\{1,2\},\{3,4\}\}} q_{0E}(1,E')z_{0Ei}(1,E') \le p_{0E}\cdot\omega_{0Ei}.$$

Also, since there is no successor at $tE \in \{2\{1\},2\{2\},2\{3\},2\{4\}\}$, the budget constraint on these pair becomes

$$p_{2E}\cdot x_{2Ei} \le p_{2E}\cdot\omega_{2Ei} + p_{12E}z_{1\{1,2\}i}(2,E) \text{ for } E \in \{\{1\},\{2\}\}, \text{ and}$$
$$p_{2E}\cdot x_{2Ei} \le p_{2E}\cdot\omega_{2Ei} + p_{12E}z_{1\{3,4\}i}(2,E) \text{ for } E \in \{\{3\},\{4\}\}.$$

(c) Just as in the proof of Proposition 19.D.1, it is essentially sufficient to show that the Arrow-Debreu budget set and the Radner budget set are the same, after appropriately choosing equilibrium price vectors.

For an Arrow-Debreu equilibrium price vector $p = (p_{tE})_{tE\in\mathscr{A}}$ with $p_{tE} \in \mathbb{R}^H_+$ for every $tE \in \mathscr{A}$, define a price vector

$$q = ((q_{tE}(t+1, E'))_{E'\in\mathscr{S}_{t+1}, E'\subset E})_{tE\in\mathscr{A}}$$

for the contingent first good commodities by $q_{tE}(t+1, E') = p_{1,t+1,E'}$. For $x_i \in X_i$ in the Arrow-Debreu budget set under p, define the transaction of the contingent first-good commodities,

$$z_i = ((z_{tEi}(t+1, E'))_{E' \in \mathcal{P}_{t+1}, E' \subset E})_{tE \in \mathcal{A}}$$

by the following rule: If $t = 1$, then

$$z_{tEi}(t+1, E') = (p_{2E'} \cdot (x_{2E'i} - \omega_{2E'i}))/p_{12E'}$$

If $t = 0$ and $E' = \{1,2\}$, then

$$z_{tEi}(t+1, E') = (p_{1\{1,2\}} \cdot (x_{1\{1,2\}i} - \omega_{1\{1,2\}i}))/p_{11\{1,2\}}$$
$$+ \sum_{E'' \in \{\{1\},\{2\}\}} q_{1\{1,2\}}(2,E'') z_{1\{1,2\}i}(2,E''))/p_{11\{1,2\}};$$

If $t = 0$ and $E' = \{3,4\}$, then

$$z_{tEi}(t+1, E') = (p_{1\{3,4\}} \cdot (x_{1\{3,4\}i} - \omega_{1\{3,4\}i}))/p_{11\{3,4\}}$$
$$+ \sum_{E'' \in \{\{3\},\{4\}\}} q_{1\{3,4\}}(2,E'') z_{1\{3,4\}i}(2,E''))/p_{11\{3,4\}}.$$

It is routine to show that $x_i \in X_i$ is also in the Radner budget set attained by the transactions z_i of the contingent first-good commodities. It is also routine to show that, conversely, if $x_i \in X_i$ is in the Radner budget set attained by z_i under price vectors (p,q), then x_i is also in the Arrow-Debreu budget set.

For a Radner equilibrium price vector

$$q = ((q_{tE}(t+1, E'))_{E' \in \mathcal{P}_{t+1}, E' \subset E})_{tE \in \mathcal{A}}$$

for the contingent first-good commodities and a spot price vector $p = (p_{tE})_{tE \in \mathcal{A}}$ with $p_{tE} \in \mathbb{R}^H_+$ for every $tE \in \mathcal{A}$, define $\mu = (\mu_{tE})_{tE \in \mathcal{A}}$ by:

$\mu_{tE} = 1$ if $t = 0$,

$\mu_{tE} = q_{0\{1,2,3,4\}}(1,E)/p_{1,0\{1,2,3,4\}}$ if $t = 1$,

$\mu_{tE} = (q_{1\{1,2\}}(2,E)/p_{11\{1,2\}})\mu_{1\{1,2\}}$ if $t = 2$ and $E \in \{\{1\},\{2\}\}$,

$\mu_{tE} = (q_{1\{3,4\}}(2,E)/p_{11\{3,4\}})\mu_{1\{3,4\}}$ if $t = 2$ and $E \in \{\{3\},\{4\}\}$.

It is routine to show that $x_i \in X_i$ is also in the Arrow-Debreu budget set under the price vector $(\mu_{tE} p_{tE})_{tE \in \mathcal{A}}$. It is also routine to show that, if $x_i \in X_i$ is in the Arrow-Debreu budget set under the price vector $(\mu_{tE} p_{tE})_{tE \in \mathcal{A}}$, then it is also in the Radner budget set under price vectors (p,q), attained by the transaction z_i of the contingent first-good commodities defined as in the previous paragraph.

19.E.1 Let $v_{0i}(p_0,w_{0i})$ and $v_{1i}(p_s,w_{si})$ be the indirect utility functions derived from $u_{0i}(x_{0i})$ and $u_{1i}(x_{1i})$. Suppose that, at a Radner equilibrium, we have asset prices $q = (q_1,\ldots,q_K)$ and spot prices $p = (p_0,p_1,\ldots,p_S) \in \mathbb{R}^{L(1+S)}$. Denote by $z_i^* \in \mathbb{R}^K$ and $x_i^* \in \mathbb{R}^{L(1+S)}$ the optimal choice of consumer i. Write $w_{0i}^* = p_0 \cdot x_{0i}^*$ and $w_{si}^* = p \cdot x_{si}^*$ for each $s = 1,\ldots,S$. Assume that $w_{0i}^* > 0$ and $w_{si}^* > 0$ for every s, so that the solution is in the interior. The first-order condition is now that there exists $\alpha_i > 0$ such that

$$\alpha_i = \partial v_{0i}(p_0,w_{0i}^*)/\partial w_{0i},$$

$$\alpha_i q_k = \textstyle\sum_s \pi_{si}(\partial v_{1i}(p_s,w_{si}^*)/\partial w_{si}) r_{sk} \text{ for every } k = 1,\ldots,K.$$

Hence, by defining $\mu = (\mu_1,\ldots,\mu_S)$ by

$$\mu_s = \pi_{si}\frac{\partial v_{1i}(p_s,w_{si}^*)/\partial w_{si}}{\partial v_{0i}(p_0,w_{0i}^*)/\partial w_{0i}} \text{ for every } s = 1,\ldots,S,$$

we obtain $q_k = \sum_s \mu_s r_{sk}$.

(b) In the single good case, the first-order conditions in terms of the direct utility function is that there exists $\alpha_i > 0$ such that

$$\alpha_i = u_{0i}'(x_{0i}^*),$$

$$\alpha_i q_k = \textstyle\sum_s \pi_{si} u_{1i}'(x_{si}^*) r_{sk} \text{ for every } k = 1,\ldots,K.$$

Hence, by defining $\mu = (\mu_1,\ldots,\mu_S)$ by

$$\mu_s = \pi_{si} u_{1i}'(x_{si}^*)/u_{0i}'(x_{0i}^*) \text{ for every } s = 1,\ldots,S,$$

we obtain $q_k = \sum_s \mu_s r_{sk}$.

19.E.2 By relabeling the S states if necessary, we can assume that $r_1 > r_2 >$

$\ldots > r_S > 0$. For each $s \in \{1,\ldots,S-1\}$, let $c_s \in (r_{s+1}, r_s)$ and let asset s be the option with strike price c_s. Let asset S be the primary asset. Then the return matrix is

$$\begin{bmatrix} r_1 - c_1 & r_1 - c_2 & \cdots & r_1 - c_{S-1} & r_1 \\ 0 & r_2 - c_2 & \cdots & r_2 - c_{S-1} & r_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{S-1} - c_{S-1} & r_{S-1} \\ 0 & 0 & \cdots & 0 & r_S \end{bmatrix}.$$

Since $r_s - c_s > 0$ for every $s < S$ and $r_S > 0$, this matrix is nonsingular. Hence the asset structure is complete.

19.E.3 It is sufficient to prove that if (x_i, z_i) is in the budget set of the constructed Radner equilibrium, then x_i is also in the budget set of the original Arrow-Debreu equilibrium. To this effect, let (x_i, z_i) be as such, then $p_s \cdot (x_{si} - \omega_{si}) \le \sum_k p_{1s} z_{ki} r_{sk}$. Summing over s, we obtain $p \cdot (x_i - \omega_i) \le \sum_s \sum_k p_{1s} z_{ki} r_{sk}$. But here

$$\sum_s \sum_k p_{1s} z_{ki} r_{sk} = \sum_k (\sum_s p_{1s} r_{sk}) z_{ki} = \sum_k q_k z_{ki} \le 0.$$

Hence $p \cdot (x_i - \omega_i) \le 0$. Thus x_i is in the budget set of the original Arrow-Debreu equilibrium.

19.E.4 Suppose first that $q_3 > \alpha_1 q_1 + \alpha_2 q_2$ and consider a portfolio $\bar{z} = (\alpha_1 q_3, \alpha_2 q_3, -(\alpha_1 q_1 + \alpha_2 q_2))$. Then $q \cdot \bar{z} = 0$. Since unlimited short sales are possible, this means that the consumers can always add $\bar{z}$ to any portfolio, while satisfying the budget constraint. Moreover, since $r_3 = \alpha_1 r_1 + \alpha_2 r_2$,

$$\begin{aligned} \sum_k p_{1s} r_{sk} \bar{z}_k &= p_{1s}(r_{s1}\alpha_1 q_3 + r_{s2}\alpha_2 q_3 - r_{s3}(\alpha_1 q_1 + \alpha_2 q_2)) \\ &= p_{1s} r_{s3}(q_3 - (\alpha_1 q_1 + \alpha_2 q_2)). \end{aligned}$$

for each s. Since r_3 is nonnegative and nonzero, $\sum_k p_{1s} r_{sk} \bar{z}_k \ge 0$ for every s

and $\sum_k p_{1s} r_{sk} \bar{z}_k > 0$ for some s. By the strict monotonicity of preferences, the consumers can always increase their utility levels by adding $\bar{z}$ to their portfolio. However, this cannot happen at equilibrium. We must thus have $q_3 \le \alpha_1 q_1 + \alpha_2 q_2$.

Suppose next that $q_3 < \alpha_1 q_1 + \alpha_2 q_2$. We can then show that, by subtracting $\bar{z}$, the consumers can always increase their utility levels, which again cannot happen at equilibrium. Hence we conclude that $q_3 = \alpha_1 q_1 + \alpha_2 q_2$.

19.E.5 It is sufficient to prove that if a terminal-node consumption $x_i \in \mathbb{R}^4_+$ is in the Arrow-Debreu budget set with price $p = (\lambda_b\mu_1, \lambda_b\mu_2, \lambda_c\mu_3, \lambda_c\mu_4)$, then there exists a dynamic asset trading $(z_{1ai}, z_{2ai}, z_{1bi}, z_{2bi}, z_{1ci}, z_{2ci})$ such that

(1) $z_{1ai} + q_a z_{2ai} \le 0$

(2) $z_{1bi} + q_b z_{2bi} = z_{1ai} + q_b z_{2ai}$,

(3) $z_{1ci} + q_c z_{2ci} = z_{1ai} + q_c z_{2ai}$,

(4) $x_{1i} = \omega_{1i} + z_{1bi}$,

(5) $x_{2i} = \omega_{2i} + z_{1bi} + z_{2bi}$,

(6) $x_{3i} = \omega_{3i} + z_{1ci}$,

(7) $x_{4i} = \omega_{4i} + z_{1ci} + z_{2ci}$.

In fact, given such an x_i, we can uniquely determine z_{1bi} and z_{1ci} so as to satisfy (4) and (6). We can then apply (5) and (7) to uniquely determine z_{2bi} and z_{2ci}. These values uniquely determine the left-hand sides of (2) and (3). Under the condition $q_b \ne q_c$, there exists a unique pair (z_{1ai}, z_{2ai}) that satisfies both (2) and (3). It thus remains to prove that the pair (z_{1ai}, z_{2ai}) so determined satisfies (1). From the Arrow-Debreu budget constraint and (4) through (7), we obtain

$$\lambda_b\mu_1 z_{1bi} + \lambda_b\mu_2(z_{1bi} + z_{2bi}) + \lambda_c\mu_3 z_{1ci} + \lambda_c\mu_4(z_{1ci} + z_{2ci}) \le 0.$$

Since $\mu_1 + \mu_2 = \mu_3 + \mu_4 = 1$, the left-hand side is equal to

$$\lambda_b(z_{1bi} + \mu_2 z_{2bi}) + \lambda_c(z_{1ci} + \mu_4 z_{2ci}).$$

By (2), (3), $\mu_2 = q_b$, and $\mu_4 = q_c$, this is equal to

$$\lambda_b(z_{1ai} + \mu_2 z_{2ai}) + \lambda_c(z_{1ai} + \mu_4 z_{2ai}).$$

By $\lambda_b + \lambda_c = 1$ and $\lambda_b\mu_2 + \lambda_c\mu_4 = q_a$, this is equal to $z_{1ai} + q_a z_{2ai}$. Hence (1) holds.

19.E.6 (a) The return vector is $r = (1 - 0.75)r_1 = (16,4,1)$. The price is $q = (1 - 0.75)q_1 = 8$.

(b) The return vector is $r = ((1 - 0.75)\cdot 64,\ (1 - 0.75)\cdot 16,\ 0) = (16,4,0)$. Hence $r = (1/4)r_1 - r_2$. Hence the price is $q = (1/4)q_1 - q_2 = 7$.

(c) The return vector is $r = ((1 - 0.75)\cdot 64,\ 0,\ 0) = (16,0,0)$. It is easy to check that there is no $(\alpha_1,\alpha_2) \in \mathbb{R}^2$ such that $r = \alpha_1 r_1 + \alpha_2 r_2$. That's why this derivative asset cannot be priced by arbitrage.

(d) Write the riskless asset $r_3 = (1,1,1)$ and its price $q_3 = 1$, then $r_3 - r_1 = (1,1,0)$ and $q_3 - q_1 = 0$. Thus there is an arbitrage opportunity. Hence the derivative asset r in (c) cannot be priced by arbitrage.

(e) The return vector is $r = (\text{Max}\{16,1\}, \text{Max}\{4,1\}, \text{Max}\{1,1\}) = (16,4,1)$, which is the same as in (a). So its price is $q = 8$.

(f) The return vector is $r = (\text{Max}\{16,1\}, \text{Max}\{4,1\}, \text{Max}\{0,1\}) = (16,4,1)$, which is the same as in (e). So its price is $q = 8$.

19.F.1 Let $(\hat{p},\hat{x}_1^*,\ldots,\hat{x}_I^*) \in \mathbb{R}^L \times \mathbb{R}^L \times \ldots \times \mathbb{R}^L$ be a Walrasian equilibrium of the spot economy, which exists under the standard assumptions. Define $p = (\hat{p},\ldots,\hat{p}) \in \mathbb{R}^{LS}$ and $x_i^* = (\hat{x}_i^*,\ldots,\hat{x}_i^*) \in \mathbb{R}^{LS}$ for each i. This is a sunspot-free consumption. Assuming that the asset structure is the return matrix $R = (r_{sk}) \in \mathbb{R}^{S\times K}$, define $q^T = \pi\cdot R \in \mathbb{R}^K$. We claim that $(p,q,x_1^*,\ldots,x_I^*)$, combined with no asset transaction at all, is a Radner equilibrium. It is sufficient to prove that if (x_i,z_i) is in the budget set of this Radner equilibrium, then $\sum_s\pi_s u_i(x_{si}) \le u_i(x_i^*) = u_i(\hat{x}_i^*)$. In fact, let (x_i,z_i) be as such. Then the average spot consumption $\sum_s\pi_s x_{si} \in \mathbb{R}^L$ is affordable under $\hat{p}$, because

$$\hat{p}\cdot(\sum_s\pi_s x_{si} - \omega_i) = \sum_s\pi_s(\hat{p}\cdot(x_{si} - \omega_i)) \le \sum_s\pi_s(\sum_k r_{sk}z_{ki}) = \sum_k(\sum_s\pi_s r_{sk})z_{ki}$$
$$= \sum_k q_k z_{ki} \le 0.$$

Thus, by the utility maximization condition of the equilibrium, we have $u_i(\hat{x}_i^*) \ge u_i(\sum_s\pi_s x_{si})$. By the concavity, $u_i(\sum_s\pi_s x_{si}) \ge \sum_s\pi_s u_i(x_{si})$. Hence $u_i(\hat{x}_i^*) \ge \sum_s\pi_s u_i(x_{si})$.

19.F.2 The utility function $U_i^*(\cdot)$ is continuous and concave if so is the original utility function $U_i(\cdot)$. If the returns are strictly positive, then the monotonicity of $U_i(\cdot)$ implies that $U_i^*(\cdot)$ is strictly monotone. These can be proven by the same way as the proof of Exercise 6.C.4, except that the integral sign $\int$ is replaced by the summation sigh $\sum$, because, here, the number S of the states is finite.

19.F.3 (a) Assume that there are two goods in each state. For each $i \in \{1,2\}$, let $u_i: \mathbb{R}^2_+ \to \mathbb{R}$ be the Bernoulli utility function of consumer i for each of the two states. For each $i \in \{1,2\}$ and $s \in \{1,2\}$, let $\omega_{si} \in \mathbb{R}^2_{++}$ be the initial endowment of consumer i at state s. We assume that $u_i(\cdot)$ is twice continuously differentiable on $\mathbb{R}^2_{++}$, $\nabla u_i(x_i) \in \mathbb{R}^2_{++}$ and $D^2u_i(x_i)$ is negative definite for every $x_i \in \mathbb{R}^2_{++}$. We assume that $\omega_{111} = \omega_{122}$, $\omega_{121} = \omega_{112}$, $\omega_{211} = \omega_{221}$, and $\omega_{212} = \omega_{222}$. That is, the uncertainty on the endowments for good 1 is simply a swapping between the two consumers acoss the two states, and there is no uncertainty on the endowments for good 2. These assumptions imply that there is no aggregate uncertainty. So denote the total endowment vector by $\bar{\omega}$. We can use the Edgeworth box of size $\bar{\omega}$ for both states.

We shall now consider the equilibria of economies with the $u_i(\cdot)$ and three different ownership assignments of the total endowment vector $\bar{\omega}$. Let $(x^*_{11}, x^*_{12}, p^*)$ be the unique equilibrium for state 1 (with initial endowment vectors ω_{11} and ω_{12}), where $x^*_{11} \in \mathbb{R}^2_{++}$ is the consumption vector of consumer 1, $x^*_{12} \in \mathbb{R}^2_{++}$ is the consumption vector of consumer 2, and $p^* \in \mathbb{R}^2_{++}$ is a price vector. Let $(x^*_{21}, x^*_{22}, p^*)$ be the unique equilibrium for state 2 (with initial endowment vectors ω_{21} and ω_{22}), where $x^*_{21} \in \mathbb{R}^2_{++}$ is the consumption vector of consumer 1 and $x^*_{22} \in \mathbb{R}^2_{++}$ is the consumption vector of consumer 2. Note that equilibrium price vector is equal to $p^* \in \mathbb{R}^2_{++}$, which is the same as in state 1. Finally, consider the "mean" initial endowment vectors $(1/2)\omega_{11} + (1/2)\omega_{21}$ and $(1/2)\omega_{12} + (1/2)\omega_{22}$, where the mean is taken over the two states. Let $(x^{**}_{11}, x^{**}_{12}, p^{**})$ be the unique equilibrium with these mean initial endowment vectors, where $x^{**}_{11} \in \mathbb{R}^2_{++}$ is the consumption vector of consumer 1, $x^{**}_{12} \in \mathbb{R}^2_{++}$ is the consumption vector of consumer 2, and $p^{**} \in \mathbb{R}^2_{++}$ is a price vector. These three equilibria, corresponding to the three different initial endowment vectors, are depicted in the following figure:

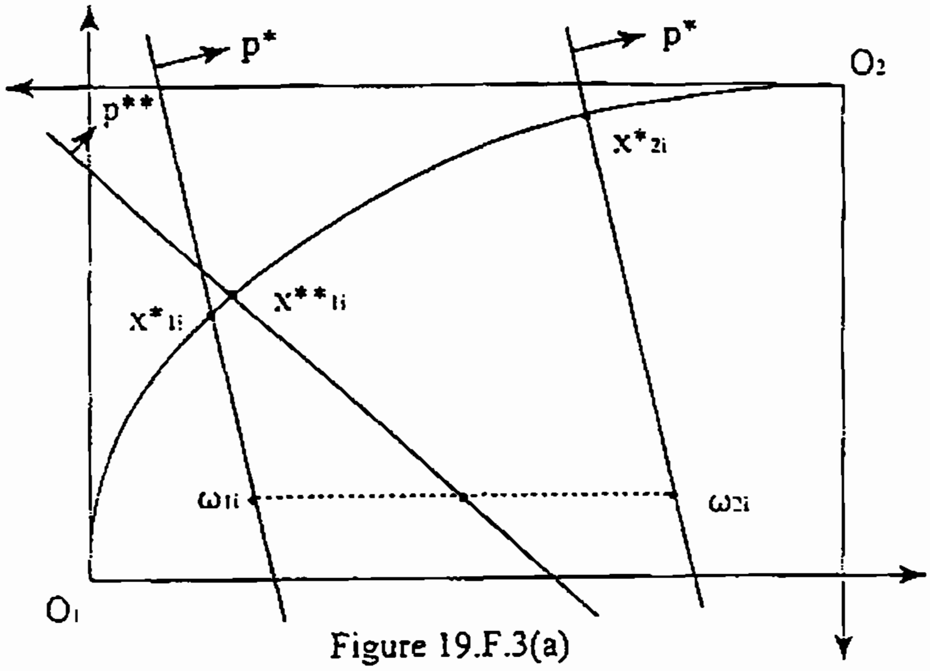


Figure 19.F.3(a)

Note that $p_1^*/p_2^* > p_1^{**}/p_2^{**}$, that is, the relative price of good 1 is higher for the first two equilibria, and thus that the allocation $(x_{11}^{**}, x_{12}^{**})$ of the third equilibrium is close to the allocation (x_{11}^*, x_{12}^*) of the first equilibrium. This implies that

$$(1/2)u_1(x_{11}^*) + (1/2)u_1(x_{21}^*) > u_1(x_{11}^{**}),$$

that is, the expected utility that consumer 1 obtains from consuming x_{11}^* with probability 1/2 and x_{21}^* with probability 1/2 is higher than the utility that he obtains from consuming x_{11}^{**} for sure.

By multiplying a sufficiently small positive number to $u_2(\cdot)$ if necessary, we can assume that

$$((1/2)u_1(x_{11}^*) + (1/2)u_1(x_{21}^*)) + ((1/2)u_2(x_{12}^*) + (1/2)u_2(x_{22}^*))$$
$$> u_1(x_{11}^{**}) + u_2(x_{12}^{**}),$$

that is, the sum of the expected utilities of the two consumers for the first two equilibria is higher than the sum of the utilities that they obtain from the third equilibrium.

We shall now show that these three equilibria (all for the same Edgeworth box) correspond to two equilibria for the two states with and without assets.

First of all, an equilibrium without assets is simply a combination of an equilibrium for the first state and another for the second. Hence the consumption vector $(x^*_{11}, x^*_{21}) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ of consumer 1, the consumption vector $(x^*_{12}, x^*_{22}) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ of consumer 2, and the price vector $(p^*, p^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ constitute a unique equilibrium for the no-asset case. As for the case with two Arrow securities (contingent first good commodities), define

$$z^{**}_{si} = (1/2)(\omega_{11i} + \omega_{12i}) - \omega_{1si}.$$

This is simply the deviation of the initial endowment of consumer i for good 1 in state s from the mean. Then $z^{**}_{1i} + z^{**}_{2i} = 0$ for $i \in \{1,2\}$. Moreover, since $\omega_{111} = \omega_{122}$ and $\omega_{121} = \omega_{112}$, we have $z^{**}_{s1} + z^{**}_{s2} = 0$ for each $s \in \{1,2\}$. Write $e = (1,0) \in \mathbb{R}^2$, then

$$\omega_{si} + z^{**}_{si} e = (1/2)(\omega_{11i} + \omega_{12i}).$$

Thus the transactions z^{**}_{si} of the first good contingent commodities result in the mean endowment vector. We claim that the consumption plan $(x^{**}_{11}, x^{**}_{11}) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ and $(z^{**}_{11}, z^{**}_{21})$ of consumer 1, the consumption plan $(x^{**}_{12}, x^{**}_{12}) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ and $(z^{**}_{12}, z^{**}_{22})$ of consumer 2, the spot price vector $(p^{**}, p^{**}) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$, and the asset price vector $q^{**} = (1,1) \in \mathbb{R}^2$ constitute a complete-market Radner equilibrium; note that both consumers are fully insured. The market-clearing condition (condition (ii) of Definition 19.E.2) follows immediately from the construction. Since the market is complete, by Proposition 19.E.2(i), in order to verify the utility maximization condition (condition (i) of Definition 19.E.2), it is sufficient to show that the above allocation constitutes an Arrow-Debreu equilibrium. But this follows from the complete insurance and $q^{**} = (1,1)$.

At the first, incomplete-market equilibrium, the utility level of consumer 1 is equal to $(1/2)u_1(x^*_{11}) + (1/2)u_1(x^*_{21})$ and the utility level of consumer 2 is equal to $(1/2)u_2(x^*_{12}) + (1/2)u_2(x^*_{22})$. At the second, complete-market equilibrium, the utility level of consumer 1 is equal to $u_1(x^{**}_{11})$ and the utility level of consumer 2 is equal to $u_2(x^{**}_{12})$. Hence, by our construction, the sum of the utilities at the complete-market equilibrium is smaller than the sum of the utilities at the incomplete-market equilibrium.

(b) This is just a matter of fixing up the necessary notation for all four states. For each $s \in \{1,2\}$, define $u_{s1}(\cdot) = u_1(\cdot)$ and $u_{s2}(\cdot) = u_2(\cdot)$. For each $s \in \{3,4\}$, define $u_{s1}(\cdot) = u_2(\cdot)$ and $u_{s2}(\cdot) = u_1(\cdot)$. Also define $\omega_{31} = \omega_{12}$, $\omega_{41} = \omega_{22}$, $\omega_{32} = \omega_{11}$, and $\omega_{42} = \omega_{21}$. For the purpose of presenting equilibria in (c), let's also define $x^*_{31} = x^*_{12}$, $x^*_{41} = x^*_{22}$, $x^*_{32} = x^*_{11}$, $x^*_{42} = x^*_{21}$, $x^{**}_{21} = x^{**}_{11}$, $x^{**}_{22} = x^{**}_{12}$, $x^{**}_{31} = x^{**}_{41} = x^{**}_{12}$, and $x^{**}_{32} = x^{**}_{42} = x^{**}_{11}$. Define $z^{**}_{31} = z^{**}_{12}$, $z^{**}_{41} = z^{**}_{22}$, $z^{**}_{32} = z^{**}_{11}$, $z^{**}_{42} = z^{**}_{21}$. Finally, for each i, define $x^*_i = (x^*_{1i}, x^*_{2i}, x^*_{3i}, x^*_{4i})$ and likewise for x^{**}_i and z^{**}_i.

(c) [<u>First printing errata</u>: We should assume explicitly that this economy has three dates $t = 0,1,2$. At $t = 0$, all four states are indistinguishable. At $t = 1$, the first two states $s = 1,2$ and the second two states $s = 3,4$ become distinguishable. At $t = 2$, the uncertainty is completely resolved. (According to the notation in the small-type discussion in Section B, $\mathscr{F}_0 = \{\{1,2,3,4\}\}$, $\mathscr{F}_1 = \{\{1,2\},\{3,4\}\}$, and $\mathscr{F}_2 = \{\{1\},\{2\},\{3\},\{4\}\}$.) We should also specify the asset structure to be introduced as follows: At $t = 0$, there is no asset transaction allowed. At $t = 1$ and event $\{1,2\}$, the Arrow securities for the first two states $s = 1,2$ are traded. At $t = 1$ and event $\{3,4\}$, the Arrow securities for the second two states $s = 3,4$ are traded.] If there are no assets traded, then the consumption vectors x^*_i and the spot price vector p^* (for each of the four states) constitute the unique equilibrium. At this equilibrium, the utility level of consumer 1 (at t = 0) is equal to

$$(1/4)u_{11}(x^*_{11}) + (1/4)u_{21}(x^*_{21}) + (1/4)u_{31}(x^*_{31}) + (1/4)u_{41}(x^*_{41})$$
$$= (1/4)u_1(x^*_{11}) + (1/4)u_1(x^*_{21}) + (1/4)u_2(x^*_{12}) + (1/4)u_2(x^*_{22}),$$

and the utility level of consumer 2 (at t = 0) is equal to

$$(1/4)u_{12}(x^*_{12}) + (1/4)u_{22}(x^*_{22}) + (1/4)u_{32}(x^*_{32}) + (1/4)u_{42}(x^*_{42})$$
$$= (1/4)u_1(x^*_{11}) + (1/4)u_1(x^*_{21}) + (1/4)u_2(x^*_{12}) + (1/4)u_2(x^*_{22}).$$

If the four Arrow securities are available for trade at t = 1, then the consumption vectors x^{**}_i, the portfolio plans z^{**}_i, the spot price vector p^{**} (for each of the four states), and the asset price vector $q^{**} = (1,1)$ (for each of the two events (1,2) and (3,4) at t = 1) constitute a Radner equilibrium. At this equilibrium, the utility level of consumer 1 (at t = 0) is equal to

$$(1/4)u_{11}(x^{**}_{11}) + (1/4)u_{21}(x^{**}_{21}) + (1/4)u_{31}(x^{**}_{31}) + (1/4)u_{41}(x^{**}_{41})$$
$$= (1/2)u_1(x^{**}_{11}) + (1/2)u_2(x^{**}_{12}),$$

and the utility level of consumer 2 (at t = 0) is equal to

$$(1/4)u_{12}(x^{**}_{12}) + (1/4)u_{22}(x^{**}_{22}) + (1/4)u_{32}(x^{**}_{32}) + (1/4)u_{42}(x^{**}_{42})$$
$$= (1/2)u_1(x^{**}_{11}) + (1/2)u_2(x^{**}_{12}).$$

We saw in (a) that

$$(1/2)u_1(x^*_{11}) + (1/2)u_1(x^*_{21}) + (1/2)u_2(x^*_{12}) + (1/2)u_2(x^*_{22})$$
$$> u_1(x^{**}_{11}) + u_2(x^{**}_{12}).$$

Thus both consumers get worse off after the introduction of the four Arrow securities at t = 1.

19.F.4 We consider an economy with I = 2, L = 2, T = 1, S = 2, and K = 2. We assume that there is no consumption at t = 0. The utility functions and the initial endowments are as follows:

$$U_1(x_1) = (1/4)\ln x_{111} + (3/4)\ln x_{211} + x_{121} + x_{221}.$$

$$U_2(x_2) = (3/4)\ln x_{112} + (1/4)\ln x_{212} + x_{122} + x_{222}.$$

$$\omega_1 = (\omega_{111}, \omega_{211}, \omega_{121}, \omega_{221}) = (1/2, 3/4, 1, 1).$$

$$\omega_2 = (\omega_{112}, \omega_{212}, \omega_{122}, \omega_{222}) = (1/2, 1/4, 1, 1).$$

Note that if state 2 is realized, then the two physical goods are perfect substitutes. Moreover, the $U_i(\cdot)$ are quasilinear with these goods.

The return structure of the two assets are as follows: The first asset pays one unit of good 1 in both states. The second asset pays one unit of good 2 in both states. Thus, when the spot prices are $p = (p_{11}, p_{21}, p_{12}, p_{22})$, the return matrix is given by

$$V(p) = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix}.$$

Because of the perfect substitutability, we must have $p_{12} = p_{22}$ at equilibrium. So normalize $p_{12} = p_{22} = 1$. Then

$$V(p) = \begin{bmatrix} p_{11} & p_{21} \\ 1 & 1 \end{bmatrix}.$$

Hence rank $V(p) = 2$, that is, the asset structure is complete, if and only if $p_{11} = p_{21}$. On the other hand, $V(p) = 1$, that is, (since the returns are strictly positive) it is impossible to transfer wealth across the two state, if and only if $p_{11} = p_{21}$. Note that, unlike in the case in which the returns of assets are only in amounts of a single physical good, the completeness of the asset structure depends on spot prices.

We shall now prove that there is no Radner equilibrium in this economy, by considering, separately, two possibilities of equilibrium prices, $p_{11} \neq p_{21}$ and $p_{11} = p_{21}$. So suppose first that $p_{11} \neq p_{21}$. Then the market is complete and, together with the quasilinearity, we can calculate the consumers' demands as follows:

$$x_{111} = 1/4p_{11},\ x_{211} = 3/4p_{21},\ x_{112} = 3/4p_{11},\ x_{212} = 1/4p_{21}.$$

Hence, from the market clearing conditions $x_{111} + x_{112} = 1/2 + 1/2$ and $x_{211} + x_{212} = 3/4 + 1/4$, we obtain $p_{11} = p_{12} = 1$, which is a contradiction. Thus we cannot have $p_{11} \neq p_{12}$ at equilibrium. So suppose next that $p_{11} = p_{21}$. Then no wealth transfer across the two states is possible, and hence $p = (p_{11},p_{21},1,1)$ is a Radner equilibrium price vector if and only if (p_{11},p_{21}) is an equilibrium of the spot economy of state 1. It is a standard Cobb-Douglas economy, so it is easy to check that we cannot have $p_{11} = p_{21}$. (In fact, $p_{11} = 3/4$ and $p_{21} = 1$.) This is another contradiction. Hence there is no Radner equilibrium.

19.F.5 (a) Let $(c_a^*,c_d^*) \in \mathbb{R}_+^2$ satisfy $c_a^*/2 + c_d^*/2 \le k/2$ and

$$(1/2)U_a(c_a^*) + (1/2)U_d(c_d^*) \ge (1/2)U_a(c_a) + (1/2)U_d(c_d)$$

for every $(c_a,c_d) \in \mathbb{R}_+^2$ with $c_a/2 + c_d/2 \le k/2$. We claim that if there is a full set of Arrow-Debreu markets, then an Arrow-Debreu equilibrium allocation is such that every consumer consumes c_a^* units of the single consumption good when he is able and c_d^* units when disabled. In fact, suppose that the contingent commodities (each of which involves a complete specification on which consumers are able and which are not) are priced so that the delivery contingent on any particular consumer's being able (that is, the portfolio so formed as to deliver one unit of the single consumption good if and only if he is able) costs 1/2, and the delivery contingent on any particular consumer's being disable also costs 1/2. Since a consumer can get k units of the consumption good when he is able and nothing when disabled, his utility maximization problem is equivalent to the following one:

$$\operatorname*{Max}_{(c_a,c_d)\in\mathbb{R}_+^2} (1/2)U_a(c_a) + (1/2)U_d(c_d),$$

s.t. $\quad c_a/2 + c_d/2 \le k/2.$

Hence (c_a^*, c_d^*) is a solution. Since there are sufficiently many worker, it is always true that a half of all workers are able and the other half disabled. The budget constraint $c_a^*/2 + c_d^*/2 \le k/2$ thus implies that the feasibility (market-clearing) condition is met.

(b) An insurance is the contingent commodity that is a title to receive some positive payment t of the consumption good if and only if (after the realization of a state of the world) the holder claim that he is disabled. Suppose that the riskless asset (which promises an uncontingent payment of a unit of the consumption good and makes it possible for the consumers to pay the insurance fee) is also available in the market and its price is normalized to be one. An insurance payment $t^* \ge 0$ and an insurance fee $q^* \ge 0$ constitute a competitive insurance market equilibrium if the following two condition are satisfied:

(i) $k \ge t^*$.

(ii) For every $(t,q) \in \mathbb{R}_+^2$ with $q \ge t/2$ and $k \ge t$, we have

$$(1/2)U_a(k - q^*) + (1/2)U_d(t^* - q^*) \ge (1/2)U_a(k - q) + (1/2)U_d(t - q).$$

Condition (i) is the incentive compatibility condition. A consumer cannot get a higher consumption level by claiming that he is disabled and stop working, when he can actually work.

Condition (ii) embodies the idea of perfect competition in the insurance market industry. Namely, if this condition is met, then it is impossible to introduce a new insurance (t,q), in the presence of (t^*,q^*), to obtain a positive profit. To see this, we shall now prove that if it is possible to do so, then condition (ii) must fail to hold. So suppose that (t,q) generates a positive profit. Then it must be preferred to (t^*,q^*) and hence

$$(1/2)U_a(k - q^*) + (1/2)U_d(t^* - q^*) < (1/2)U_a(k - q) + (1/2)U_d(t - q).$$

Since $(c_a, c_d) = (k, 0)$ (no insurance at all) can be attained by $(t, q) = (0, 0)$,

$$(1/2)U_a(k) + (1/2)U_d(0) < (1/2)U_a(k - q) + (1/2)U_d(t - q).$$

Hence $t - q > 0$. Suppose now that $k < t$. Then a buyer of this insurance can get a higher consumption level by claiming that he is disabled. Hence he would always do so and get $t - q$ units, regardless of whether he is actually disabled or not. The firm's profit is then $q - t < 0$, meaning that it gets a negative profit, a contradiction. We must thus have $k \geq t$. Moreover, a buyer of the insurance works whenever he is able to do so and gets $k - q$ units when he is able and $t - q$ when disabled. The firm's expected profit is then $q - t/2$, which is assumed to be positive. That is, $q > t/2$, implying that condition (ii) is not met.

We shall now prove that the equilibrium allocation (c_a^*, c_d^*) of the Arrow-Debreu markets can be attained at a competitive insurance market equilibrium. In fact, define $t^* = k - c_a^* + c_d^*$ and $q^* = k - c_a^*$. We shall first show that (t^*, q^*) satisfies condition (ii). Let $(t, q) \in \mathbb{R}_-^2$ with $q \geq t/2$. Define $c_a = k - q$ and $c_d = t - q$, then $c_a/2 + c_d/2 = k/2 - t/2 - q \leq k/2$. Hence, by the definition of (c_a^*, c_d^*), $(1/2)U_a(c_a^*) + (1/2)U_d(c_c^*) \geq (1/2)U_a(c_a) + (1/2)U_c(c_c)$. This implies condition (ii). As for condition (i), note that the first-order conditions of the utility maximization in (a) implies that $U_a'(c_a^*) = U_d'(c_d^*)$. The given condition on the derivative then implies that $c_a^* > c_c^*$, that is, $k - q^* > t^* - q^*$. This implies condition (i). The consumption bundle (c_a^*, c_c^*) is depicted in the following figure

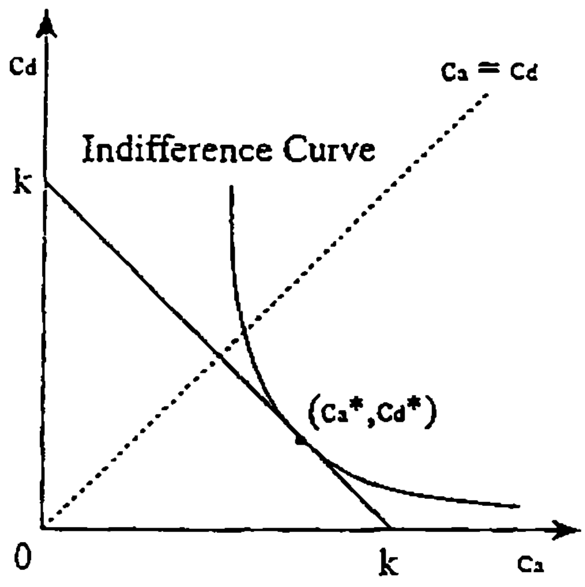


Figure 19.F.5(b)

(c) Under the new assumption, we have $c_a^* < c_d^*$, that is, $k < t^*$. Hence condition (i) is not met. Hence (c_a^*, c_d^*) is not reachable at any insurance market equilibrium. (Every worker would claim disability and stop working.) Since

$$\{(k - q,\ t - q) \in \mathbb{R}^2_+ : (t,q) \in \mathbb{R}^2_+,\ q \geq t/2,\ k \geq t\}$$
$$= \{(c_a, c_d) \in \mathbb{R}^2_+ : c_a/2 + c_d/2 \leq k/2,\ c_a \geq c_d\},$$

the insurance market equilibrium under the new assumption is $(t^{**}, q^{**}) = (k, k/2)$ and the resulting consumption bundle is $(c_a^{**}, c_d^{**}) = (k/2, k/2)$. This is depicted in the following figure.

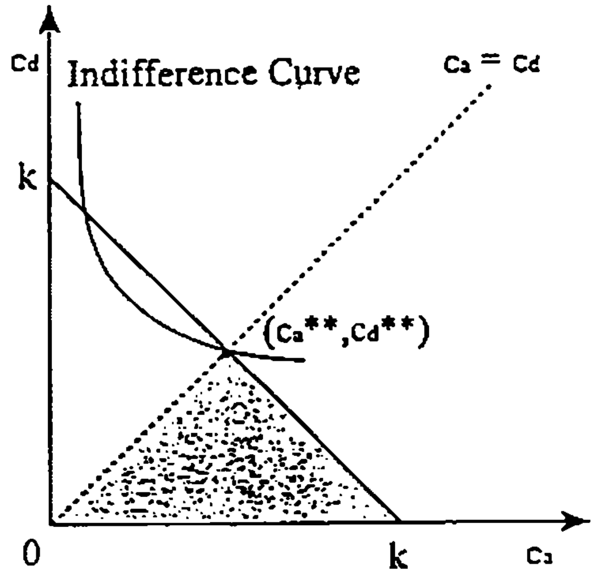


Figure 19.F.5(c)

The equilibrium allocation (c_a^{**},c_d^{**}) is optimal relative to the allocations the government can achieve. The reason is that, since it also cannot observe disability, the consumption allocations it can choose must belong to the set $\{(c_a,c_d) \in \mathbb{R}^2_+: c_a/2 + c_d/2 \le k/2,\ c_a \ge c_d\}$ and (c_a^{**},c_d^{**}) is optimal in this set.

19.G.1 Let $x_i \in \mathbb{R}^{LS}_+$ for which there exists $(z_i,z_{ai}) \in \mathbb{R}^K \times \mathbb{R}$ such that

$$p\cdot(x_{si} - \omega_{si}) \le \textstyle\sum_k p_{1s} r_{sk} z_{ki} + p_{1s} a_s z_{ai}$$

for every s, and

$$q\cdot z_i + v(a,q)z_{ai} \le \theta_i v(a,q).$$

Since $a = \sum_k \alpha_k r_k$, the right-hand side of the first inequality is equal to $\sum_k p_{1s} r_{sk}(z_{ki} + z_{ai}\alpha_k)$. Since $v(a,p) = \sum_k \alpha_k q_k$, the second inequality is equivalent to $q\cdot(z_i + z_{ai}\alpha) \le \theta_i v(a,q)$, where $\alpha = (\alpha_1,\ldots,\alpha_K) \in \mathbb{R}^K$. Hence x_i belongs to the budget set of (19.G.1) by using the portfolio $z_i + a_{ai}\alpha \in \mathbb{R}^K$.

19.G.2 [<u>First printing errata</u>: The last phrase "one consumer is significantly better off and the other is significantly worse off" should be replaced by "both consumers are significantly better off".] Suppose that the two consumers have the same utility function $u_i(x_{1i}, x_{2i}) = x_{1i}x_{2i}$. Their initial endowments are $\omega_1 = (1,0)$ and $\omega_2 = (0,1)$. The return vector of the initial asset is $(1,0)$. Define $a = (0,1)$.

If the share of the firm is not traded, then the consumers consumption their initial endowments and their share-holdings, which are $x_1 = (1, \varepsilon/2)$ and $x_2 = (0, 1 + \varepsilon/2)$. Hence their utility levels are $u_1(x_1) = \varepsilon/2$ and $u_2(x_2) = 0$. Note that $u_1(x_1) \to 0$ as $\varepsilon \to 0$.

If the share is traded, then the asset market is complete. It is thus sufficient to find the Arrow-Debreu equilibrium allocation, taking $(1, \varepsilon/2)$ and $(0, 1 + \varepsilon/2)$ as the consumers' initial endowments. The equilibrium allocation is then given by

$$x_1^* = ((1 + \varepsilon)/(2 + 3\varepsilon),\ (1 + \varepsilon)^2/(2 + 3\varepsilon)),$$

$$x_2^* = ((1 + 2\varepsilon)/(2 + 3\varepsilon),\ (1 + \varepsilon)(1 + 2\varepsilon)/(2 + 3\varepsilon)).$$

Note that $x_i^* \to (1/2, 1/2)$ as $\varepsilon \to 0$ for both $i = 1,2$. Hence $u_i(x_i^*) \to 1/4$. Hence, however small ε may be, at the new equilibrium, both consumers are significantly better off. The reason is that if the share of the firm is traded, then the asset market becomes complete and the consumers can insure each other by trading the share and the initial asset. This enhanced insurance opportunity, by itself, increases their utility levels significantly, however small the associated increment in the total endowment of the economy may be.

19.C.3 Assume that there are K assets, with return matrix $R \in \mathbb{R}^{S \times K}$. By the spanning condition, there exists an $\alpha = (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K$ such that $a = R\alpha$.

that is, $a_s = \sum_s \alpha_k r_{sk}$ for each s.

Suppose first that $((q,q_a),p,x_1^*,\ldots,x_I^*,(z_1^*,z_{a1}^*),\ldots,(z_I^*,z_{aI}^*))$ be an equilibrium of the first scenario. By the no-arbitrage condition, $q_a = \sum_s \alpha_k q_k$. We shall now prove that

$$(q,\ p,\ x_i^*,\ \ldots,\ x_I^*,\ z_1^* + (z_{a1}^* - \theta_1)\alpha,\ \ldots,\ z_I^* + (z_{aI}^* - \theta_I)\alpha)$$

is an equilibrium in the second scenario. It is easy to check that the feasibility condition is satisfied. Define

$$B_i^1 = \{x_i \in \mathbb{R}_+^S\text{: there exists } (z_i,z_{ai}) \in \mathbb{R}^K \times \mathbb{R} \text{ such that } p\cdot(x_{si} - \omega_{si}) \le \textstyle\sum_k r_{sk}z_{ki} + a_s z_{ai} \text{ for every } s \text{ and } q\cdot z_i + q_a z_{ai} \le q_a\theta_i\},$$

$$B_i^2 = \{x_i \in \mathbb{R}_+^S\text{: there exists } z_i \in \mathbb{R}^K \text{ such that } p\cdot(x_{si} - \omega_{si} - \theta_i a) \le \textstyle\sum_k r_{sk}z_{ki} \text{ for every } s \text{ and } q\cdot z_i \le 0\}.$$

Let $x_i \in B_i^1$ and suppose that it is implemented by a portfolio $(z_i,z_{ai}) \in \mathbb{R}^K \times \mathbb{R}$ in the first scenario, then it is routine to check that $x_i \in B_i^2$ and it is implemented by a portfolio $z_i + (z_{ai} - \theta_i)\alpha \in \mathbb{R}^K$. This shows that $B_i^1 \subset B_i^2$ and x_i^* is implemented by $z_i^* + (z_{ai}^* - \theta_i)\alpha$ in the second scenario. It thus remains to show that $B_i^1 \supset B_i^2$. But if $x_i \in B_i^2$ and it is implemented by a portfolio $z_i \in \mathbb{R}^K$, then it is routine to check that $x_i \in B_i^1$ and it is implemented by a portfolio (z_i,θ_i).

Suppose next that $(q,p,x_1^*,\ldots,x_I^*,z_1^*,\ldots,z_I^*)$ be an equilibrium of the second scenario. Let $q_a = \sum_s \alpha_k q_k$. We shall now prove that

$$((q,q_a),p,x_1^*,\ldots,x_I^*,(z_1^*,\theta_1),\ldots,(z_I^*,\theta_I))$$

is an equilibrium in the first scenario. It is easy to check that the feasibility condition is satisfied. Define B_i^1 and B_i^2 as just before. Let $x_i \in B_i^2$ and suppose that it is implemented by a portfolio $z_i \in \mathbb{R}^K$ in the second scenario, then it is routine to check that $x_i \in B_i^1$ and it is implemented by a portfolio $(z_i,\theta_i) \in \mathbb{R}^K \times \mathbb{R}$ in the first scenario. This shows that $B_i^1 \supset B_i^2$ and

x_i^* is implemented by (z_i^*, θ_i) in the first scenario. It thus remains to show that $B_i^1 \subset B_i^2$. But if $x_i \in B_i^1$ and it is implemented by a portfolio $(z_i, z_{ai}) \in \mathbb{R}^K \times \mathbb{R}$, then it is routine to check that $x_i \in B_i^2$ and it is implemented by a portfolio $z_i + (z_{ai} - \theta_i)\alpha \in \mathbb{R}^K$.

19.G.4 Let S = 2 and, for both i, define $U_i: \mathbb{R}_+^2 \to \mathbb{R}$ by

$$U_i(x_i) = (1/2)x_{1i}^{1/2} + (1/2)x_{2i}^{1/2}$$

and let $\omega_1 = (1,0)$ and $\omega_2 = (0,1)$. Let $a^1 = (0,2)$ and $a^2 = (2,0)$. Then

$$U_1(\omega_1 + (1/2)a^1) = U_2(\omega_2 + (1/2)a^2) = 1,$$

$$U_1(\omega_1 + (1/2)a^2) = U_2(\omega_2 + (1/2)a^1) = (1/2)2^{1/2} = 2^{-1/2}.$$

Hence, for both i, consumer i prefers return vector a^i and they are not unanimous in their preference for a^1 or a^2.

19.H.1 (a) For the first assertion, if $\sigma(\cdot)$ is completely revealing, then

$$B_i^{\sigma(\cdot)} = \{x_i \in \mathbb{R}^{2S}: px_{1si} + x_{2si} \le w_i \text{ for every } s\}.$$

Thus, for every s, if $px_{1si} + x_{2si} \le w_i$, then $u_{si}(x_i^{\sigma(\cdot)}) \ge u_{si}(x_{si})$. Hence $x_i^{\sigma(\cdot)}$ is ex post optimal.

As for the second assertion, consider the following example: Let S = 2, $\pi_{1i} = \pi_{2i} = 1/2$, $p = 1$, $w_i = 4$, and

$$u_{1i}(x_{11i}, x_{21i}) = 2\ln x_{11i} + x_{21i},$$

$$u_{2i}(x_{12i}, x_{22i}) = 4\ln x_{12i} + x_{22i}.$$

Assume that $\sigma(1) = \sigma(2)$ (no information provided by $\sigma(\cdot)$). Then, for each s and i, $u_i(x_i | \sigma(s)) = 3\ln x_{1si} + x_{2si}$. Hence $x_i^{\sigma(\cdot)} = (3,1;3,1)$. Thus $u_{1i}(x_i^{\sigma(\cdot)}) = 2\ln 3 + 1$ and $u_{2i}(x_i^{\sigma(\cdot)}) = 4\ln 3 + 1$. But consider $x_i = (2,2;4,0)$, then $u_{1i}(x_{1i}) = 2\ln 2 + 2$ and $u_{2i}(x_{2i}) = 4\ln 4$. Thus $u_{1i}(x_i^{\sigma(\cdot)}) < u_{1i}(x_{1i})$ and $u_{2i}(x_i^{\sigma(\cdot)}) < u_{2i}(x_{2i})$. Hence $x_i^{\sigma(\cdot)}$ is not ex post optimal.

(b) This property is nothing but the definition of $x_i^{\sigma(\cdot)}$.

(c) The allocation $x_i^{\sigma(\cdot)}$ is measurable with respect to $\sigma'(\cdot)$. Thus the assertion follows from the definition of $x_i^{\sigma'(\cdot)}$.

(d) By the definition,

$$u_i(x_i^{\sigma'(\cdot)}|\sigma(s)) = \sum_{s'}(\pi_{s'i}|\sigma(s))u_{s'i}(x_{s'i}^{\sigma'(s')}).$$

Denote by $((\pi_{s'i}|\sigma'(s))|\sigma(s))$ the posterior updated by $\sigma(\cdot)$ of another posterior $(\pi_{s'i}|\sigma'(s))$ at state s. Then, by the law of iterated conditional expectation,

$$\sum_{s'}(\pi_{s'i}|\sigma(s))u_{s'i}(x_{s'i}^{\sigma'(s')}) = \sum_{s'}((\pi_{s'i}|\sigma'(s))|\sigma(s))u_{s'i}(x_{s'i}^{\sigma'(s')}).$$

Since $\sigma'(\cdot)$ is at least as informative as $\sigma(\cdot)$,

$$\sum_{s'}((\pi_{s'i}|\sigma'(s))|\sigma(s))u_{s'i}(x_{s'i}^{\sigma'(s')}) \geq \sum_{s'}((\pi_{s'i}|\sigma'(s))|\sigma(s))u_{s'i}(x_{s'i}^{\sigma(s')}).$$

By the definition, if $\sigma(s') = \sigma(s'') = \sigma(s)$, then $x_{s'i}^{\sigma(s')} = x_{s''i}^{\sigma(s'')}$. Hence

$$\sum_{s'}((\pi_{s'i}|\sigma'(s))|\sigma(s))u_{s'i}(x_{s'i}^{\sigma(s')}) = \sum_{s'}(\pi_{s'i}|\sigma(s))u_{s'i}(x_{s'i}^{\sigma(s')}).$$

By the definition,

$$\sum_{s'}(\pi_{s'i}|\sigma(s))u_{s'i}(x_{s'i}^{\sigma(s')}) = u_i(x_i^{\sigma(\cdot)}|\sigma(s)).$$

Thus $u_i(x_i^{\sigma'(\cdot)}|\sigma(s)) \geq u_i(x_i^{\sigma(\cdot)}|\sigma(s))$.

19.H.2 As in the proof of Proposition 19.H.1, for each signal $\sigma(\cdot)$, define

$$B_i^{\sigma(\cdot)} = \{x_i \in \mathbb{R}^{2S}: p(s)\cdot x_{1si} + x_{2si} \leq w_i(s) \text{ for each s and}$$
$$x_{si} = x_{s'i} \text{ whenever } \sigma(s) = \sigma(s')\}.$$

If $\sigma'(\cdot)$ is at least as informative as $\sigma(\cdot)$, and $p(\cdot)$ and $w_i(\cdot)$ remain unaltered, then $B_i^{\sigma(\cdot)} \subset B_i^{\sigma'(\cdot)}$. Hence Proposition 19.H.1 is still valid under this set of assumptions.

19.H.3 For the first observation, let $x_i \in B_i^{\sigma(\cdot)}$. Then, by the definition of

$x_i^{\sigma(\cdot)}$,

$$u_i(x_{si}^{\sigma(\cdot)}|\sigma(s)) \geq u_i(x_{si}|\sigma(s)),$$

or equivalently, if $\sigma(s) = c \in \mathbb{R}$, then we can write

$$u_i(x_{ci}^{\sigma(\cdot)}|c) \geq u_i(x_{ci}|c)$$

by the measurability constraint. Hence, by taking the expectation over $c \in \mathbb{R}$, we obtain

$$\sum_{c\in\sigma(\{1,\ldots,S\})}(\sum_{s\in\sigma^{-1}(c)}\pi_{si})u_i(x_{ci}^{\sigma(\cdot)}|c)$$

$$\geq \sum_{c\in\sigma(\{1,\ldots,S\})}(\sum_{s\in\sigma^{-1}(c)}\pi_{si})u_i(x_{ci}|c).$$

Here, by the definition of $u_i(\cdot|\sigma(s)) = u_i(\cdot|c)$,

$$\sum_{c\in\sigma(\{1,\ldots,S\})}(\sum_{s\in\sigma^{-1}(c)}\pi_{si})u_i(x_{ci}^{\sigma(\cdot)}|c) = \sum_s\pi_{si}u_{si}(x_{si}^{\sigma(\cdot)}) = U_i(x_i^{\sigma(\cdot)}),$$

$$\sum_{c\in\sigma(\{1,\ldots,S\})}(\sum_{s\in\sigma^{-1}(c)}\pi_{si})u_i(x_{ci}|c) = \sum_s\pi_{si}u_{si}(x_{si}) = U_i(x_i).$$

Hence $U_i(x_i^{\sigma(\cdot)}) \geq U_i(x_i)$.

For the second observation let $x_i \in B_i^{\sigma(\cdot)}$, then $x_{si} = x_{s'i}$ whenever $\sigma(s) = \sigma(s')$. Since $\sigma(s) = \sigma(s')$ whenever $\sigma'(s) = \sigma'(s')$, $x_{si} = x_{s'i}$ whenever $\sigma'(s) = \sigma'(s')$. Thus $x_i \in B_i^{\sigma'(\cdot)}$. Hence $B_i^{\sigma(\cdot)} \subset B_i^{\sigma'(\cdot)}$.

19.H.4 For each i, if $\sigma_i = 2$, then there are two possibilities: The first one is that $\beta = 1$ and $\varepsilon_i = 1$; the second one is that $\beta = 2$ and $\varepsilon_i = -2$. By the assumption of independent distributions, both possibilities have probability 1/50. Hence the conditional probability of $\beta = 1$ given $\sigma_i = 2$ is 1/2 and that of $\beta = 2$ given $\sigma_i = 2$ is also 1/2. The same argument can be applied to the case of $\sigma_i = 3$ to know that the conditional probability of $\beta = 1$ given $\sigma_i = 3$ is 1/2 and that of $\beta = 2$ given $\sigma_i = 3$ is also 1/2. Hence, for both $n = 2,3$, and for both $i = 1,2$, the expected utility function remains to be

$$u_i(x_i|\sigma_i = n) = (3/2)\ln x_{1i} + x_{2i}.$$

Thus the equilibrium price of good 1 is 3/2.

19.H.5 We shall consider two cases, separately, on the value of β (which is revealed by the pooled information equilibrium price function $\tilde{p}(\cdot)$) and show that, in both cases, consumer i with nontrivial ε_i has incentive to pay a sufficiently small $\delta > 0$ to know the realization of ε_i. We assume that consumer i has no endowment of the second good. (Because of the quasilinearity, any specification of the endowment of the second good does not affect the conclusion.)

<u>Case 1</u>: $\beta = 1$.

Suppose first that consumer i does not know ε_i, then, by $E[\varepsilon_i] = 0$, his expected utility function is $u_i(x_i|\beta) = \ln x_{1i} + x_{2i}$. Hence his demand is $(1/p, p - 1)$ and thus his expected utility level is $-\ln p + p - 1$.

Suppose second that consumer i knows ε_i, then his utility function (further updated by ε_i) is $u_i(x_i|\beta,\varepsilon_i) = (1 + \varepsilon_i)\ln x_{1i} + x_{2i}$. Hence his demand is $((1 + \varepsilon_i)/p,\ p - (1 + \varepsilon_i))$ and thus his utility level is

$$(1 + \varepsilon_i)\ln(1 + \varepsilon_i) - (1 + \varepsilon_i)\ln p + p - 1 - \varepsilon_i.$$

Hence his ex ante utility level (before observing ε_i) is the expected value of this:

$$E[(1 + \varepsilon_i)\ln(1 + \varepsilon_i)] - \ln p + p - 1.$$

Hence the increment in the utility levels by receiving ε_i is $E[(1 + \varepsilon_i)\ln(1 + \varepsilon_i)]$. It is easy to check that $(1 + \varepsilon_i)\ln(1 + \varepsilon_i)$ is a strictly convex function of ε_i. Thus, by Jensen's Inequality

$$E[(1 + \varepsilon_i)\ln(1 + \varepsilon_i)] > 0.$$

Note that this does not depend on the value of p.

<u>Case 2</u>: $\beta = 2$

Suppose first that consumer i does not know ε_i, then, by $E[\varepsilon_i] = 0$, his expected utility function is $u_i(x_i|\beta) = 2\ln x_{1i} + x_{2i}$. Hence his demand is $(2/p,\ p - 2)$ and thus his expected utility level is $2\ln 2 - 2\ln p + p - 2$.

Suppose second that consumer i knows ε_i, then his utility function (further updated by ε_i) is $u_i(x_i|\beta,\varepsilon_i) = (2 + \varepsilon_i)\ln x_{1i} + x_{2i}$. Hence his demand is $((2 + \varepsilon_i)/p,\ p - (2 + \varepsilon_i))$ and thus his utility level is

$$(2 + \varepsilon_i)\ln(2 + \varepsilon_i) - (2 + \varepsilon_i)\ln p + p - 2 - \varepsilon_i.$$

Hence his ex ante utility level (before observing ε_i) is the expected value of this:

$$E[(2 + \varepsilon_i)\ln(2 + \varepsilon_i)] - 2\ln p + p - 2.$$

Hence the increment in the utility levels by receiving ε_i is $E[(2 + \varepsilon_i)\ln(2 + \varepsilon_i)] - 2\ln 2$. It is easy to check that $(2 + \varepsilon_i)\ln(2 + \varepsilon_i)$ is a strictly convex function of ε_i. Thus, by Jensen's Inequality,

$$E[(2 + \varepsilon_i)\ln(2 + \varepsilon_i)] - 2\ln 2 > 0$$

Note that this does not depend on the value of p.

Therefore, if

$$\delta < \mathrm{Min}\ \{E[(1 + \varepsilon_i)\ln(1 + \varepsilon_i)],\ E[(2 + \varepsilon_i)\ln(2 + \varepsilon_i)] - 2\ln 2\},$$

then every consumer i with nontrivial ε_i is willing to pay δ (in terms of the numeraire) to know ε_i.

19.H.6 (a) Let (p,x) be an Arrow-Debreu equilibrium. Since any of the six contingent commodities are perfect substitute for consumer 1 and any Arrow-Debreu equilibrium is in the interior, $p_{11} = p_{21} = p_{12} = p_{22} = p_{13} = p_{23}$. Thus $(1/2)x_{1s2}^{-1/2} = 1$ for every s. Thus $x_{1s2} = 1/4$ for every s. His budget constraint then implies that $x_{2s2} = \omega_{1s2} - 1/4$, and the market-clearing condition implies that $x_{1s1} = \omega_{1s2} - 1/4$. Thus $x_{2s1} = \omega_{21} - \omega_{1s2} + 1/4$.

(b) Denote the price of the noncontingent delivery of (one unit of) good ℓ by $q_\ell \geq 0$. Taking (q_1, q_2) as given, consumer i solves the following maximization problem:

$$\underset{(z_{1i}, z_{2i}) \in \mathbb{R}^2}{\text{Max}} \quad (1/3)\textstyle\sum_s u_i(\omega_{1si} + z_{1i}, \omega_{2si} + z_{2i}),$$

$$\text{s.t.} \qquad q_1 z_{1i} + q_2 z_{2i} \leq 0,$$

where $\omega_{1s1} = 0$, $\omega_{2s1} = \omega_{21}$, and $\omega_{2s2} = 0$ for every s. (We are assuming that there is no spot market open after a state occurs.)

An asset (noncontingent delivery) price vector (q_1, q_2) and an asset allocation $(z^*_{11}, z^*_{21}, z^*_{12}, z^*_{22})$ constitute an equilibrium if

(i) For each i, (z^*_{1i}, z^*_{2i}) solves the above maximization problem.

(ii) $z^*_{11} + z^*_{12} \leq 0$ and $z^*_{21} + z^*_{22} \leq 0$.

We claim that no equilibrium allocation is Pareto optimal. To show this, note first that, by the quasilinearity, if an allocation x is Pareto optimal, then $x_{112} = x_{122} = x_{132} = 1/4$. Note second that, if an allocation x is attained at an equilibrium, then

$$x_{112} = \omega_{112} + z^*_1, \; x_{122} = \omega_{122} + z^*_1, \; x_{132} = \omega_{132} + z^*_1.$$

Hence we cannot have $x_{112} = x_{122} = x_{132}$ (unless $\omega_{112} = \omega_{122} = \omega_{132}$). Thus no equilibrium allocation is Pareto optimal.

(c) If state 1 has occurred and the asset (contingent commodity) prices are (q_{11}, q_{21}), then the utility maximization problem is

$$\underset{(z_{11i}, z_{21i}) \in \mathbb{R}^2}{\text{Max}} \quad u_i(\omega_{11i} + z_{11i}, \omega_{21i} + z_{21i}),$$

$$\text{s.t.} \qquad q_{11} z_{11i} + q_{21} z_{21i} \leq 0.$$

If state 1 has not occurred and the asset (noncontingent delivery) prices are (q_{12}, q_{22}), then it is

$$\operatorname{Max}_{(z_{12i}, z_{22i}) \in \mathbb{R}^2} (1/2)\sum_{s=2,3} u_i(\omega_{1si} + z_{12i}, \omega_{2si} + z_{22i}).$$

$$\text{s.t.} \qquad q_{12}z_{12i} + q_{22}z_{22i} \le 0.$$

An asset price vector $(q_{11}, q_{21}; q_{12}, q_{22})$ and an asset allocation $(z^*_{111}, z^*_{211}, z^*_{112}, z^*_{212}; z^*_{121}, z^*_{221}, z^*_{122}, z^*_{222})$ constitute an equilibrium if

(i) For each i, (z^*_{11i}, z^*_{21i}) and (z^*_{12i}, z^*_{22i}) solve the above maximization problems.

(ii) $\sum_i z^*_{\ell ji} \le 0$ for every ℓ and j.

(d) If state 1 has occurred, then the utility maximization problem is the same as in (c). If state 1 has not occurred, then the utility maximization problem is that of the Arrow-Debreu contingent commodity markets for states 2 and 3.

(e) Since all equilibria are in the interior, the marginal rate of substitution of consumer 1 between any two of the six contingent commodities is equal to one. Hence the budget sets of consumer i must be as follows:

In (a), $B_i^a = \{x_i \in \mathbb{R}^6_+ : \sum_{\ell,s} x_{\ell si} \le \sum_{\ell,s}\omega_{\ell si}\}$.

In (b), $B_i^b = \{x_i \in \mathbb{R}^6_+ : \sum_{\ell,s} x_{\ell si} \le \sum_{\ell,s}\omega_{\ell si}$ and

$x_{\ell 1i} - \omega_{\ell 1i} = x_{\ell 2i} - \omega_{\ell 2i} = x_{\ell 3i} - \omega_{\ell 3i}$ for both $\ell\}$.

In (c), $B_i^c = \{x_i \in \mathbb{R}^6_+ : \sum_\ell x_{\ell 1i} \le \sum_\ell \omega_{\ell 1i}, \sum_{\ell,s>1} x_{\ell si} \le \sum_{\ell,s>1}\omega_{\ell si}$, and

$x_{\ell 2i} - \omega_{\ell 2i} = x_{\ell 3i} - \omega_{\ell 3i}$ for both $\ell\}$.

In (d), $B_i^d = \{x_i \in \mathbb{R}^6_+ : \sum_\ell x_{\ell 1i} \le \sum_\ell \omega_{\ell 1i}$ and $\sum_{\ell,s>1} x_{\ell si} \le \sum_{\ell,s>1}\omega_{\ell si}\}$.

Thus $B_i^a \supset B_i^d \supset B_i^c \supset B_i^b$ and we can conclude:

(i) At all of the equilibrium allocations of (a), (b), (c), and (d), consumer 1 attains utility level ω_{21}.

(ii) The equilibrium allocations of (a) and (d) are Pareto optimal. (For (d), note that there is an $x_i \in B_i^d$ such that $x_{112} = x_{122} = x_{132} = 1/4$.)

(iii) The equilibrium allocation of (b) is Pareto optimal if and only if ω_{112}

$= \omega_{122} = \omega_{132}$.

(iv) The equilibrium allocation of (c) is Pareto optimal if and only if $\omega_{122} = \omega_{132}$.

Thus the information in (c) and (d) are socially valuable. [The important fact here is that the equilibrium prices are not changed by the introduction of information or contingent commodities, and consumer 1 attains the same utility level at every state. Nothing as in Example 19.H.1 happens in this example.]

20.H.7 [First printing errata: The utility function of consumer 2 in state 2 should be $2\ln x_{12} + x_{22}$.]

(a) As usual, the quasilinearity makes the necessary calculations easier. We can obtain $(\hat{p}_1(\varepsilon), \hat{p}_2(\varepsilon)) = (1, 6/(6 + \varepsilon))$.

(b) While the utility functions of consumer 2 are as in (a), consumer 1 maximizes his expected utility function $3\ln x_{11} + x_{21}$. We can thus obtain $(\bar{p}_1(\varepsilon), \bar{p}_2(\varepsilon)) = (7/6, 5/(6 + \varepsilon))$.

(c) If $\varepsilon = 0$, then $\hat{p}_1(\varepsilon) = \hat{p}_2(\varepsilon)$ and hence consumer 1 can actually be informed by observing the spot prices. Thus $(p_1^*(\varepsilon), p_2^*(\varepsilon)) = (\hat{p}_1(\varepsilon), \hat{p}_2(\varepsilon)) = (1, 6/(6 - \varepsilon))$.

(d) If $\varepsilon = 0$, then there is no rational expectations equilibrium at which consumer 1 is informed by observing the spot prices, because $\hat{p}_1(0) = \hat{p}_2(0) = 1$. However, there is also no rational expectations equilibrium at which he is not informed, because $\bar{p}_1(\varepsilon) > 1 > \bar{p}_2(\varepsilon)$.

This example has the same nature as Example 19.H.3, in that both of them illustrates the possibility of the non-existence of a rational expectations equilibrium. A difference between the two is that, here, there is uncertainty in endowments, while, in Example 19.H.3, there is uncertainty in utility functions. In both cases, however, there is uncertainty in utility functions for excess demands. This is the uncertainty that is essential in models of rational expectations equilibrium.

20.B.1 Let c be a nonzero consumption stream, then $u(c_t) \geq u(0)$ for all t and $u(c_t) > u(0)$ for some t, because $u(\cdot)$ is strictly increasing. Thus

$$V(c) > \sum_t \delta^t u(0) = (1 - \delta)^{-1}u(0).$$

Hence

$$u(0) + \delta V(c) < V(c).$$

But, if $c' = (0, c_0, c_1, \ldots, c_{t-1}, \ldots)$, then

$$V(c') = u(0) + \sum_{t=1}^{t=\infty}\delta^t u(c_{t-1}) = u(0) + \delta\sum_{t=0}^{t=\infty}\delta^t u(c_t) = u(0) + \delta V(c).$$

Hence $V(c') < V(c)$. Hence $V(\cdot)$ exhibits time impatience.

20.B.2 For the first statement, let c and c' be two consumption streams such that $c_t = c'_t$ for every $t \leq T - 1$. Then

$$\begin{aligned} V(c) - V(c') &= \sum_{t\geq 0}\delta^t u(c_t) - \sum_{t\geq 0}\delta^t u(c'_t) \\ &= \sum_{t\geq T}\delta^t u(c_t) - \sum_{t\geq T}\delta^t u(c'_t) \\ &= \delta^T(\sum_{t\geq 0}\delta^t u(c_{t+T}) - \sum_{t\geq 0}\delta^t u(c'_{t+T})) \\ &= \delta^T(V(c^T) - V(c'^T)). \end{aligned}$$

Hence $V(c) - V(c') \geq 0$ if and only if $V(c^T) - V(c'^T) \geq 0$. Hence stationarity holds.

As for the second statement, let L = 1 and define $u: \mathbb{R}_+ \to \mathbb{R}$ by $u(c_t) = c_t^{1/2}$. Let $\delta_1 = \delta_2 = 3/4$ and $\delta_3 = 1/2$. Compare two consumption streams c = (0,1,0,0,0,...) and c' = (0,0,1,1,0,...). Then $V(c) = 3/4$, $V(c') = (3/4)^2 + (1/2)^3 = 11/16$, and hence $V(c) > V(c')$. But $V(c^1) = 1$, $V(c'^1) = 3/4 + (3/4)^2 = 21/16$, and hence $V(c^1) < V(c'^1)$. Hence stationarity is violated.

20.B.3 Let $\bar{c}'$, c", and $\hat{c}"$ satisfy $V(\bar{c}', c") \geq V(\bar{c}', \hat{c}")$, then

$$\sum_{\tau\le t}u_\tau(\bar{c}_\tau) + \sum_{\tau>t}u_\tau(c_\tau) \ge \sum_{\tau\le t}u_\tau(\bar{c}_\tau) + \sum_{\tau>t}u_\tau(\hat{c}_\tau).$$

Hence $\sum_{\tau>t}u_\tau(c_\tau) \ge \sum_{\tau>t}u_\tau(\hat{c}_\tau)$. Thus, for any $c' = (c_0,\ldots,c_t)$,

$$\sum_{\tau\le t}u_\tau(c_\tau) + \sum_{\tau>t}u_\tau(c_\tau) \ge \sum_{\tau\le t}u_\tau(c_\tau) + \sum_{\tau>t}u_\tau(\hat{c}_\tau).$$

Hence $V(c',c'') \ge V(c',\hat{c}'')$. The first property is thus proved. The second property can be similarly proved. These two properties can be interpreted as the property that the preference between two consumption streams does not depend on the consumptions on the periods where they assign the same consumptions.

20.B.4. Let u: $\mathbb{R}^L_+ \to \mathbb{R}_+$ be a one-period utility function and G: $\mathbb{R}^L_+ \to \mathbb{R}_+$ be the aggregator function defined in the exercise. Define M to be the set of all bounded sequences in $\mathbb{R}^L_+$. We shall prove that there exists V: $M \to \mathbb{R}_+$ such that $V(c) = G(u(c_0),V(c^1))$ for every $c \in M$, where c^T is the T-period backward shift of c for every T = 0,1,.... Define

$$M_0 = \{c \in M: c_t = c_{t+1} \text{ for all } t = 0,1,\ldots\},$$
$$M_1 = \{c \in M: c^T \in M_0 \text{ for some } T = 0,1,\ldots\}.$$

We shall establish the existence of such a $V(\cdot)$ by first defining it on M_0, then extending it to M_1, and finally to the whole M.

To define a $V(\cdot)$ on M_0, define f: $[\delta^{1/(1-\alpha)},\infty) \to \mathbb{R}_+$ by

$$f(V) = V - \delta V^\alpha,$$

then it is easy to check that $f(\delta^{1/(1-\alpha)}) = 0$ and

$$f'(V) = 1 - \delta\alpha V^{\alpha-1} \ge 1 - \delta\alpha(\delta^{1/(1-\alpha)})^{\alpha-1} = 1 - \alpha > 0.$$

Hence $f(\cdot)$ is strictly increasing and onto. Thus so is the function $V \to f(V)^{1/\alpha}$. Hence we can define V: $M_0 \to \mathbb{R}$ by letting, for each $x \in \mathbb{R}^L_+$, $V(x,x,\ldots)$ satisfy $f(V(x,x,\ldots))^{1/\alpha} = u(x)$. Then, by the definition of $f(\cdot)$, we have $V(x,x,\ldots) = G(u(x),V(x,x,\ldots))$ and, by the strict monotonicity of

$f(\cdot)$, for any $x \in \mathbb{R}^L_+$ and $x' \in \mathbb{R}^L_+$, if $u(x) \geq u'(x)$, then $V(x,x,\ldots) \geq V(x',x',\ldots)$.

We now extend $V(\cdot)$ to M_1 as follows: Let $c \in M_1$. If $c^T \in M_0$, then define

$$V(c) = G(u(c_0),G(u(c_1),\ldots,G(u(c_{T-1}),V(c^T))\ldots)).$$

This definition does not depend on the choice of T. In fact, if $c^T \in M_0$ and $c^{T+1} \in M_0$, then $V(c^T) = G(u(c_T),V(c^{T+1}))$ and hence

$$\begin{aligned} &G(u(c_0),G(u(c_1),\ldots,G(u(c_{T-1}),V(c^T))\ldots)) \\ &= G(u(c_0),G(u(c_1),\ldots,G(u(c_{T-1}),G(u(c_T),V(c^{T+1})))\ldots)). \end{aligned}$$

By the construction, $V(c) = G(u(c_0),G(c^1))$ for every $c \in M_1$. Moreover, by the monotonicity of $u(\cdot)$ and $V(\cdot)$ on M_0, if $c \in M_1$, $c' \in M_1$, and $c \geq c'$ (where the inequality $\geq$ is taken coordinatewise), then $V(c) \geq V(c')$.

To finally extend $V(\cdot)$ to the whole M, for each $T = 0,1,\ldots$, define h_T: $M \to M_1$ by letting $h_T(c) = (c_0,c_1,\ldots,c_T,0,0,\ldots)$ for each $c \in M$. Then, for any $c \in M$ and T, we have $h_{T+1}(c) \geq h_T(c)$ and hence $V(h_{T+1}(c)) \geq V(h_T(c))$. Moreover, by letting $\bar{c} \in \sup\{c_t: t = 0,1,\ldots\} \in \mathbb{R}^L_+$, we have $(\bar{c},\bar{c},\ldots) \in M_0$ and $V(\bar{c},\bar{c},\ldots) \geq V(h_T(c))$ for every T. Thus the sequence $(V(h_T(c)))_T$ in $\mathbb{R}_+$ is nondecreasing and bounded above: Hence it is convergent. We define $V(c)$ to be the limit of the sequence.

It thus remains to prove that $V(c) = G(u(c_0),V(c^1))$ for every $c \in M$. So let $c \in M$. By the continuity of $G(\cdot)$,

$$G(u(c_0),V(c^1)) = G(u(c_0),\lim_T V(h_T(c^1))) = \lim_T G(u(c_0),V(h_T(c^1))).$$

Since $h_T(c^1) = h_{T+1}(c)^1$ and $c_0 = h_{T+1}(c)_0$,

$$G(u(c_0),V(h_T(c^1))) = G(u(h_{T+1}(c)_0),V(h_{T+1}(c)^1)) = V(h_{T+1}(c)),$$

and hence $\lim_T G(u(c_0),V(h_T(c^1))) = \lim_T V(h_{T+1}(c)) = V(c)$. Thus $G(u(c_0),V(c^1)) = V(c)$.

20.B.5 Let c and c' be two bounded consumption streams and let $\lambda \in [0,1]$, then the convex combination $\lambda c + (1 - \lambda)c'$ is also a bounded consumption stream and hence $V(\lambda c + (1 - \lambda)c')$ is well defined. Since $u(\cdot)$ is concave, $u(\lambda c_t + (1 - \lambda)c'_t) \geq \lambda u(c_t) + (1 - \lambda)u(c'_t)$ for every t. Hence

$$
\begin{aligned}
V(\lambda c + (1 - \lambda)c') &= \textstyle\sum_t \delta^t u(\lambda c_t + (1 - \lambda)c'_t) \\
&\geq \textstyle\sum_t \delta^t (\lambda u(c_t) + (1 - \lambda)u(c'_t)) = \lambda \sum_t \delta^t u(c_t) + (1 - \lambda)\sum_t \delta^t u(c'_t) \\
&= \lambda V(c) + (1 - \lambda)V(c').
\end{aligned}
$$

Thus $V(\cdot)$ is concave.

For the cardinality of the additively separable form, consider a monotone transformation $f(V) = -\exp(-V)$. Then the function $-\exp(-\sum_t \delta^t u(c_t))$ represents the same preference over the bounded consumption steams, but it is not additively separable. [Or we could apply Exercise 3.G.4(a) to see that only linear (affine) transformations can preserve separability; although it dealt with finite sums, its proof can be easily modified to incorporate infinite sums.]

20.C.1 The own rate of interest of commodity ℓ at t is defined as $(p_{\ell,t+1} - p_{\ell t})/p_{\ell t}$.

20.C.2 Let $(y_0, y_1, \ldots y_t, \ldots)$ be a myopically profit-maximizing production path and $(y'_0, y'_1, \ldots, y'_t, \ldots)$ be any other production path. Then, for every t,

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \geq p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}.$$

Thus, by summing over $t = 0,1,\ldots,T$ for any T, we obtain

$$\textstyle\sum_{t=0}^{t=T} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \geq \sum_{t=0}^{t=T} (p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}).$$

Hence $(y_0, y_1, \ldots y_t, \ldots)$ is also profit-maximizing over any finite horizon.

20.C.3 A production path $(y_0, y_1, \ldots y_t, \ldots)$ is said to be *weakly efficient* if there is no other production path $(y'_0, y'_1, \ldots, y'_t, \ldots)$ such that

$$y_{a,t-1} + y_{bt} \ll y'_{a,t-1} + y'_{bt}$$

for every t.

For a weakly efficient production plan, Proposition 20.C.1 can be stated as follows: Suppose that a production path $(y_0, y_1, \ldots y_t, \ldots)$ is myopically profit-maximizing with respect to the nonzero, nonnegative price sequence $(p_0, p_1, \ldots, p_t, \ldots)$. Suppose also that the production path and the price sequence satisfy the transversality condition $p_{t+1} \cdot y_{at} \to 0$. Then the path $(y_0, y_1, \ldots y_t, \ldots)$ is weakly efficient. We now prove this by contradiction. Suppose that a production path $(y_0, y_1, \ldots y_t, \ldots)$ is myopically profit-maximizing with respect to the nonzero, nonnegative price sequence $(p_0, p_1, \ldots, p_t, \ldots)$. Suppose also that the production path and the price sequence satisfy the transversality condition $p_{t+1} \cdot y_{at} \to 0$. Let $(y'_0, y'_1, \ldots, y'_t, \ldots)$ be a production path such that $y_{a,t-1} + y_{bt} \ll y'_{a,t-1} + y'_{bt}$ for every t. Since $p_t \geq 0$,

$$p_t \cdot (y_{a,t-1} + y_{bt}) \leq p_t \cdot (y'_{a,t-1} + y'_{bt})$$

for every t. Moreover there exists a t such that $p_t \neq 0$ and hence,

$$p_t \cdot (y_{a,t-1} + y_{bt}) < p_t \cdot (y'_{a,t-1} + y'_{bt}).$$

So define a sequence $(\pi_T)_T$ in $\mathbb{R}_+$ by

$$\pi_T = \sum_{t=0}^{t=T} (p_t \cdot (y'_{a,t-1} + y'_{bt}) - p_t \cdot (y_{a,t-1} + y_{bt}))$$

for every T, then it is nondecreasing and there exists a T such that $\pi_T > 0$. Thus, by the transversality condition, there exists a T such that $\pi_T > p_{T+1} \cdot y_{aT}$. Fix such a T, then we have

$$\sum_{t=0}^{t=T} p_t \cdot (y'_{a,t-1} + y'_{bt}) > \sum_{t=0}^{t=T} p_t \cdot (y_{a,t-1} + y_{bt}) + p_{T+1} \cdot y_{aT}.$$

By rearranging terms and recalling the convention $y_{a,-1} = y'_{a,-1} = 0$, we obtain

$$\sum_{t=0}^{t=T-1}(p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}) + p_T \cdot y'_{bT} > \sum_{t=0}^{t=T}(p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}).$$

Hence we must have either $p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at} > p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}$ for some $t \le T - 1$, or $p_T \cdot y'_{bT} > p_T \cdot y_{bT} + p_{T+1} \cdot y_{aT}$. The first possibility is a direct violation to the assumption that $(y_0, y_1, \ldots y_t, \ldots)$ is myopically profit-maximizing with respect to $(p_0, p_1, \ldots, p_t, \ldots)$. As for the second possibility, note that $(y'_{bT}, 0) \in Y$ by the possibility of truncation. Thus it is also a violation to the myopic profit maximization. Hence no such $(y'_0, y'_1, \ldots, y'_t, \ldots)$ can exist. Thus $(y_0, y_1, \ldots y_t, \ldots)$ is weakly efficient.

20.C.4 (a) Let a production path $(y_0, y_1, \ldots, y_t, \ldots)$ be overall profit-maximizing with respect to $(p_0, p_1, \ldots, p_t, \ldots) \gg 0$. Then it is easy to check that $(y_0, y_1, \ldots, y_t, \ldots)$ be myopically profit-maximizing with respect to $(p_0, p_1, \ldots, p_t, \ldots) \gg 0$. Since $p_t \to 0$, we have $\alpha\|p_t\| \to 0$ and hence $p_{t+1} \cdot y_{at} \to 0$. Thus the transversality condition is satisfied. Hence, by Proposition 20.C.1, $(y_0, y_1, \ldots, y_t, \ldots)$ is efficient.

(b) Let a (bounded) production path $(y_0, y_1, \ldots, y_t, \ldots)$ be myopically profit-maximizing with respect to $(p_0, p_1, \ldots, p_t, \ldots) \gg 0$ (with $\sum_t p_t < \infty$) and $(y'_0, y'_1, \ldots, y'_t, \ldots)$ be any other (bounded) production path. Then, by Exercise 20.C.2,

$$\sum_{t=0}^{t=T}(p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \ge \sum_{t=0}^{t=T}(p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}).$$

By the boundedness and $\sum_t p_t < \infty$, the limits of both sides as $T \to \infty$ exist and they satisfy

$$\sum_t(p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \ge \sum_t(p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}).$$

Hence $(y_0, y_1, \ldots, y_t, \ldots)$ is overall profit maximizing.

20.C.5 [First printing errata: The overall difficulty level of this exercise should perhaps be B, although part (c) is harder, as already indicated.]

(a) Suppose that there is a production path $(y_0, y_1, \ldots, y_t, \ldots)$ that is myopically profit-maximizing with respect to $(p_0, p_1, \ldots, p_t, \ldots) \gg 0$ and is not T-efficient for some T. Then there exists another production path $(y'_0, y'_1, \ldots, y'_t, \ldots)$ such that $\#\{t: y_t \neq y'_t\} \leq T$, $y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt}$ for every t, and $y_{a,t-1} + y_{bt} \neq y'_{a,t-1} + y'_{bt}$ for some t. Take a T' so large that if $y_t \neq y'_t$, then $t \leq T'$. Then, by $p_t \gg 0$,

$$\textstyle\sum_{t=0}^{t=T'+1} p_t \cdot (y_{a,t-1} + y_{bt}) < \sum_{t=0}^{t=T'+1} p_t \cdot (y'_{a,t-1} + y'_{bt}).$$

Here, the left-hand side is equal to

$$\textstyle\sum_{t=0}^{t=T'} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) + p_{T'+1} \cdot y_{b,T'+1},$$

and the right-hand side is equal to

$$\textstyle\sum_{t=0}^{t=T'} (p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}) + p_{T'+1} \cdot y'_{b,T'+1}.$$

Since $p_{T'+1} \cdot y_{b,T'+1} = p_{T'+1} \cdot y'_{b,T'+1}$ by the definition of T',

$$\textstyle\sum_{t=0}^{t=T'} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) < \sum_{t=0}^{t=T'} (p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}).$$

Hence there must exist a $t \leq T'$ such that

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} < p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}.$$

This contradicts the myopic profit maximization. (We could also apply Exercise 20.C.2 to derive this contradiction.) Hence $(y_0, y_1, \ldots, y_t, \ldots)$ must be T-efficient for all T.

(b) Let $(y_0, y_1, \ldots, y_t, \ldots)$ be a 2-efficient production plan, then, by the smoothness assumption, for each t, there exists a unique $q_t = (q_{bt}, q_{at}) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{++}$ such that $\|q_t\| = 1$ and q_t is an outward normal vector of Y at y_t. By the 2-efficiency, for every t, there exists a $\beta_t > 0$ such that $q_{a,t-1} = \beta_t q_{bt}$. Hence, just in the small-type discussion at the end of Section 20.C, there exists a price sequence $(p_0,p_1,\ldots,p_t,\ldots)$ such that $(y_0,y_1,\ldots,y_t,\ldots)$ be myopically profit-maximizing with respect to $(p_0,p_1,\ldots,p_t,\ldots)$ and (p_t,p_{t+1}) is proportional to $q_t = (q_{bt},q_{at})$ for every t. Hence $(p_0,p_1,\ldots,p_t,\ldots) \gg 0$. Therefore, by (a), $(y_0,y_1,\ldots,y_t,\ldots)$ is T-efficient for all T.

(c) Here is an example of general linear activity technologies for which the conclusion of (b) fails. Let L = 4 and define the production set Y to be the set of the $y = (y_b,y_a) \in \mathbb{R}^4 \times \mathbb{R}^4$ that satisfies the following weak inequalities:

$$y_{\ell b} \le 0 \text{ for every } \ell;$$

$$y_{1b} + \max\{y_{1a},0\} + \max\{y_{2a},0\} + 2\max\{y_{4a},0\} \le 0;$$

$$y_{2b} + y_{3a} \le 0;$$

$$y_{3b} + y_{4a} \le 0.$$

Then Y satisfies assumptions (i) through (iv) on page 737. It also exhibits a general linear activity model. In fact, the elementary activities are

$$v_1 = (v_1',v_1'') = (-1, 0, 0, 0; 1, 0, 0, 0),$$

$$v_2 = (v_2',v_2'') = (-1, 0, 0, 0; 0, 1, 0, 0),$$

$$v_3 = (v_3',v_3'') = (0, -1, 0, 0; 0, 0, 1, 0),$$

$$v_4 = (v_4',v_4'') = (0, 0, -1, 0; 0, 0, 0, 1),$$

$$v_5 = (v_5',v_5'') = (-1, 0, 0, 0; 0, 0, 0, 1/2),$$

and the free disposal technologies.

This technology can be understood as follows: Good 1 is the primary factor, goods 2 and 3 are intermediate goods, and good 4 is the final output. The first elementary activity v_1 is a simple storage of the primary factor. The second one v_2 transforms one unit of the primary input into one unit of the first intermediate good. The third one v_3 transforms one unit of the first intermediate good into one unit of the second intermediate good. The fourth one v_4 transforms one unit of the second intermediate good into one unit of the final output. The last one v_5 transforms one unit of the primary input directly into a half unit of the final output.

The crucial feature of this technology is that there is two ways to produce the final output from the primary factor. The first way is to use v_5, which yields, at the next period, a half unit of the final output out of one unit of the primary factor. The second way is a round-about production: Use v_2, v_3, and then v_4. Although this takes three periods, one unit of the final output can be produced out of one unit of the primary factor.

Now consider the following two production paths $(y_0, y_1, \ldots, y_t, \ldots)$ and $(y'_0, y'_1, \ldots, y'_t, \ldots)$:

$$y_0 = v_1,\ y_1 = v_1,\ y_2 = v_5, \text{ and } y_t = 0 \text{ for all } t \geq 3.$$

$$y'_0 = v_2,\ y'_1 = v_3,\ y'_2 = v_4, \text{ and } y'_t = 0 \text{ for all } t \geq 3.$$

Then

$$y_{b0} = (-1, 0, 0, 0),$$

$$y_{a0} + y_{b1} = y_{a1} + y_{b2} = 0,$$

$$y_{a2} + y_{b3} = (0, 0, 0, 1/2),$$

$$y_{a,t-1} + y_{bt} = 0 \text{ for all } t \geq 4.$$

$$y'_{b0} = (-1, 0, 0, 0),$$

$$y'_{a0} + y'_{b1} = y'_{a1} + y'_{b2} = 0,$$

$$y_{a2} + y_{b3} = (0, 0, 0, 1),$$

$$y'_{a,t-1} + y'_{bt} = 0 \text{ for all } t \geq 4.$$

Hence $(y_0, y_1, \ldots, y_t, \ldots)$ is inferior to $(y'_0, y'_1, \ldots, y'_t, \ldots)$. Thus $(y_0, y_1, \ldots, y_t, \ldots)$ is not 3-efficient. We shall show that $(y_0, y_1, \ldots, y_t, \ldots)$ is nevertheless 2-efficient. So suppose that a production path $(y''_0, y''_1, \ldots, y''_t, \ldots)$ satisfies $\#\{t\colon y''_t \neq y_t\} \leq 2$ and

$$y''_{a,t-1} + y''_{bt} \geq y_{a,t-1} + y_{bt}$$

for all t. We need to show that, in fact,

$$y''_{a,t-1} + y''_{bt} = y_{a,t-1} + y_{bt}$$

for all t.

Suppose first that if $y''_t \neq y_t$ and $y''_{t'} \neq y_{t'}$ then $|t - t'| \geq 2$. Then we must have $y''_t \geq y_t$ and $y''_{t'} \geq y_{t'}$. Hence $y''_t = y_t$ and $y''_{t'} = y_{t'}$. Thus

$$y''_{a,t-1} + y''_{bt} = y_{a,t-1} + y_{bt}$$

for all t.

Suppose now that there exists a t such that $y''_s = y_s$ for any $s \notin \{t, t + 1\}$, then

$$y''_{bt} \geq y_{bt},$$

$$y''_{at} + y''_{b,t+1} \geq y_{at} + y_{b,t+1},$$

$$y''_{a,t+1} \geq y_{a,t+1}.$$

If $t \geq 3$, then $y''_{bt} \geq 0$ and hence $y''_t = 0 = y_t$. Thus $y''_{b,t+1} \geq 0$ and hence $y''_{t+1} = 0 = y_{t+1}$. If $t = 2$, then $y''_{b2} \geq (-1,0,0,0)$ and $y''_{a2} \geq y''_{b3} + y''_{a2} \geq (0,0,0,1/2)$. Thus $y''_2 = y_2$ and hence $y''_3 = y_3$. If $t = 1$, then $y''_{2b1} \geq 0$ and hence $y''_{2b1} = 0$ and $y''_{3a1} = 0$. By $y''_{4a2} \geq 1/2$, we must have $y''_{1b2} \leq -1$. Thus, by $y''_{1a1} + y''_{1b2} \geq 0$, we must have $y''_{1a1} \geq 1$. Hence $y''_{1b1} \leq -1$. Since $y''_{1b1} \geq y_{1b1} = -1$, we actually have $y''_{1b1} = -1$. Thus, by $y''_{1a1} \geq 1$, $y''_1 = y_1$. Hence $y''_2 = y_2$. Finally, if $t = 0$, then $y''_{1a1} \geq 1$ and hence $y''_{1b1} \leq -1$. By $y''_{1a0} + y''_{1b1} \geq 0$, $y''_{1a0} \geq 1$. Hence $y''_{1b0} \leq -1$. Since $y''_{1b0} \geq y_{1b0} = -1$, we actually have $y''_{1b0} = -1$. Thus, by $y''_{1a0} \geq 1$, $y''_0 = y_0$. Hence $y''_1 = y_1$.

[It will be instructive to see how the smoothness assumption of (b) is violated and the argument in the above answer to (b) does not go through. Define $Z = \{(z_0,z_1,z_2) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$: there exist $y''_0 \in Y$ and $y''_1 \in Y$ such that $z_0 = y''_{b0}$, $z_1 = y''_{a0} + y''_{b1}$, and $z_2 = y''_{a1}\}$, then Z is the set of three-period production paths (made up by two production plans) and it is closed and convex. Since the production path $(y_0, y_1, \ldots, y_t, \ldots)$ is 2-efficient, the vector $(y_{bt}, y_{at} + y_{b,t+1}, y_{a,t+1})$ belong to the boundary of Z for every t. By the supporting hyperplane theorem (Theorem M.G.3), there is a nonzero supporting vector $q_0 = (q_{00}, q_{10}, q_{20}) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$ of Z at $(y_{b0}, y_{a0} + y_{b1}, y_{a1})$ and there is another nonzero supporting vector $q_1 = (q_{11}, q_{21}, q_{31}) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$ of Z at $(y_{b1}, y_{a1} + y_{b2}, y_{a2})$. We shall now show by contradiction that, unlike the case of smooth production sets, it is impossible that $(q_{10}, q_{20}) = (q_{11}, q_{21})$. Suppose so. By multiplying positive scalars if necessary, we can assume that $q_{110} = q_{111} = 1$. Then, since $q_{00} \cdot y_{b0} + q_{10} \cdot y_{a0} = q_{00} \cdot v_1' + q_{10} \cdot v_1'' = 0$, $q_{100} = q_{110} = 1$. Since $q_{00} \cdot v_2' + q_{10} \cdot v_2'' \le 0$, $q_{211} \le q_{110} = 1$. On the other hand, since $q_{11} \cdot y_{b1} + q_{21} \cdot y_{a1} = q_{11} \cdot v_1' + q_{21} \cdot v_1'' = 0$, $q_{121} = q_{111} = 1$. Since $q_{21} \cdot y_{b2} + q_{31} \cdot y_{a2} = q_{21} \cdot v_5' + q_{31} \cdot v_5'' = 0$, $q_{431} = 2q_{121} = 2$. Since $q_{21} \cdot v_4' + q_{31} \cdot v_4'' \le 0$, $q_{321} \ge q_{431} = 2$. Since $q_{11} \cdot v_3' + q_{21} \cdot v_3'' \le 0$, $q_{211} \ge q_{321} \ge 2$. We have thus got $2 \le q_{211} \le 1$, a contradiction.]

20.C.6 By the first-order conditions for myopic profit maximization in Example 20.C.6, if we define $w_t = q_{t+1} \nabla F_2(k_t, \ell_t)$ for each t, then $(k_0, k_1, \ldots, k_t, \ldots)$ is myopic profit maximizing for $((q_0, w_0), (q_1, w_1), \ldots, (q_t, w_t), \ldots)$.

20.D.1 [First printing errata: The budget constraint of problem (20.C.3) should be (20.C.4).] We denote a sequence of money borrowed and lent at each period by $(m_0, m_1, \ldots, m_t, \ldots)$. Then the sequential budget constraints are

$$p_t \cdot c_t \le p_t \cdot \omega_t + m_t \text{ for each } t,$$
$$\textstyle\sum_t m_t \le 0.$$

We shall now prove that these sequential budget constraints are equivalent to the single budget constraint of (20.D.4). We assume that the infinite sum $\lim_{T\to\infty}\sum_{t=0}^{t=T}p_t\cdot\omega_t = \sum_t p_t\cdot\omega_t$ exists and is finite. (Otherwise, the single budget constraint would be strictly larger than the sequential budget constraint, because the former would impose no constraint but the latter would do.) Suppose first that a consumption stream $(c_0,c_1,\ldots,c_t,\ldots)$ satisfies the single budget constraint. Define a sequence $(m_0,m_1,\ldots,m_t,\ldots)$ by $m_t = p_t\cdot c_t - p_t\cdot\omega_t$ for each t, then the budget constraint at each period is satisfied. Moreover, since the infinite sum $\lim_{T\to\infty}\sum_{t=0}^{t=T}p_t\cdot c_t = \sum_t p_t\cdot c_t$ exists, the infinite sum $\sum_t m_t = \sum_t(p_t\cdot c_t - p_t\cdot\omega_t)$ also exists and equals $\sum_t p_t\cdot c_t - \sum_t p_t\cdot\omega_t \le 0$. Thus $\sum_t m_t \le 0$. Suppose conversely that

$$((c_0,m_0),(c_1,m_1),\ldots,(c_t,m_t),\ldots)$$

satisfies the sequential budget constraints. Since $p_t\cdot c_t \ge 0$, the sequence $\{\sum_{t=0}^{t=T}p_t\cdot c_t\}_T$ is nonnegative and nondecreasing. Moreover, for every $\varepsilon > 0$ and for every sufficiently large T, $\sum_{t=0}^{t=T}p_t\cdot c_t \le \sum_{t=0}^{t=T}p_t\cdot\omega_t + \sum_{t=0}^{t=T}m_t < \sum_t p_t\cdot\omega_t + \varepsilon$. Thus the infinite sum $\lim_{T\to\infty}\sum_{t=0}^{t=T}p_t\cdot c_t = \sum_t p_t\cdot c_t$ exists and is not greater than $\sum_t p_t\cdot\omega_t$.

20.D.2 To show that (ii') implies (ii), let $(y_b,y_a) \in Y$, then the infinite sum $\sum_{s\ne t}(p_s\cdot y_{bs}^* + p_{s+1}\cdot y_{as}^*) + (p_t\cdot y_b + p_{t+1}\cdot y_a)$ exists and, by (ii'),

$$\sum_s(p_s\cdot y_{bs}^* + p_{s+1}\cdot y_{as}^*) \ge \sum_{s\ne t}(p_s\cdot y_{bs}^* + p_{s+1}\cdot y_{as}^*) + (p_t\cdot y_b + p_{t+1}\cdot y_a).$$

Thus

$$p_t\cdot y_{bt}^* + p_{t+1}\cdot y_{at}^* \ge p_t\cdot y_b + p_{t+1}\cdot y_a.$$

Hence (ii) holds.

To show the converse, let $(y_0,y_1,\ldots,y_t,\ldots)$ be a feasible production path. The myopic profit maximization implies that, for every T,

$$\sum_{t=0}^{t=T}(p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \le \sum_{t=0}^{t=T}(p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*).$$

Since $0 \in Y$, $p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \ge 0$ for every t. Together with $w < \infty$, the limit $\lim_T \sum_{t=0}^{t=T}(p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*) = \sum_t (p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*)$ exists (and it is finite). Hence

$$\sum_{t=0}^{t=T}(p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*) \le \sum_t (p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*).$$

Thus $\sum_{t=0}^{t=T}(p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \le \sum_t (p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*)$ for every T.

20.D.3 By the definition, a consumption stream $(c_0, c_1, \ldots, c_t, \ldots)$ is myopically utility-maximizing if and only if, for every t, (c_t, c_{t+1}) solves the following maximization problem:

$$\text{Max}_{(c_t', c_{t+1}')} \; u(c_t') + \delta u(c_{t+1}')$$

$$\text{s.t.} \qquad p_t \cdot c_t' + p_{t+1} \cdot c_{t+1}' \le p_t \cdot c_t + p_{t+1} \cdot c_{t+1}.$$

Thus, by the interiority assumption $(c_0, c_1, \ldots, c_t, \ldots) \gg 0$, the necessary and sufficient first-order condition for the above maximization problem is that there exists $\lambda_t > 0$ for which (20.D.5) holds. Hence (20.D.5) is equivalent to the myopic utility maximization.

20.D.4 We shall prove that $p_T c_T / (\sum_{t \ge T} p_t c_t) = 1 - \delta$ for all T. As shown in Example 20.D.1, $p_T c_T = \delta^T (1 - \delta) w$. Also,

$$\sum_{t \ge T} p_t c_t = (1 - \delta) w \sum_{t \ge T} \delta^t = (1 - \delta) w \delta^T / (1 - \delta) = \delta^T w.$$

Hence $p_T c_T / (\sum_{t \ge T} p_t c_t) = \delta^T (1 - \delta) w / \delta^T w = 1 - \delta$.

20.D.5 Define $k^* = (\alpha\delta)^{1/(1-\alpha)} > 0$, then $1 - \delta F'(k^*) = 0$, $1 - \delta F'(k_t) > 0$ if $k_t > k^*$, and $1 - \delta F'(k_t) < 0$ if $k_t < k^*$. Let $(k_0, k_1, \ldots, k_t, \ldots)$ be an optimal investment policy, then the Euler equation is

$$1 - \delta F'(k_t) + g'(k_t - k_{t-1}) = \delta g'(k_{t+1} - k_t)$$

for every $t \geq 1$. We shall prove that:

<u>Claim 1</u>. If $\bar{k}_0 \geq k^*$, then $(k_0, k_1, \ldots, k_t, \ldots) = (\bar{k}_0, k^*, \ldots, k^*, \ldots)$.

<u>Claim 2</u>. If $\bar{k}_0 < k^*$, then $k_t > k_{t-1}$ for every $t \geq 1$ and $k_t \to k^*$ as $t \to \infty$.

To this end, it is convenient to first establish the following lemmas:

<u>Lemma 1</u>. For every $t \geq 1$, $1 - \delta F'(k_t) + g'(k_t - k_{t-1}) \geq 0$.

<u>Lemma 2</u>. There is no $t \geq 1$ such that $k_t \geq k^*$ and

$$1 - \delta F'(k_t) + g'(k_t - k_{t-1}) > 0.$$

Lemma 1 follows directly from the Euler equation and $g'(k_{t+1} - k_t) \geq 0$. We shall prove Lemma 2 by contradiction. Suppose that there is a $t \geq 1$ such that $k_t \geq k^*$ and $1 - \delta F'(k_t) + g'(k_t - k_{t-1}) > 0$. Then $1 - \delta F'(k_t) \geq 0$ and hence

$$\delta g'(k_{t+1} - k_t) = 1 - \delta F'(k_t) + g'(k_t - k_{t-1}) > 0.$$

Thus, by $\delta \in (0,1)$,

$$g'(k_{t+1} - k_t) > 1 - \delta F'(k_t) + g'(k_t - k_{t-1})$$

and

$$k_{t+1} - k_t > k_t - k_{t-1} \geq 0.$$

Write $\xi = 1 - \delta F'(k_t) + g'(k_t - k_{t-1}) > 0$. We shall now prove that $g'(k_{\tau+1} - k_\tau) > \xi$ for every $\tau \geq t$. We have already shown this for $\tau = t$. Let $\tau > t$ and assume that

$$g'(k_{\tau'+1} - k_{\tau'}) > \xi$$

for every $\tau' \in \{t, t+1, \ldots, \tau - 1\}$. Then, $k_\tau > k^*$ and hence $1 - \delta F'(k_\tau) > 0$. Thus, by the Euler equation, $\delta g'(k_{\tau+1} - k_\tau) - g'(k_\tau - k_{\tau-1}) > 0$. Hence $g'(k_{\tau+1} - k_\tau) > g'(k_\tau - k_{\tau-1}) > \xi$. Thus $\beta(k_{\tau+1} - k_\tau)^{\beta-1} > \xi$, that is, $k_{\tau+1} - k_\tau > (\xi/\beta)^{1/(\beta-1)}$.

Therefore, the $(t-1)$-period backward shift investment policy $(k_{t-1}, k_t, \ldots)$ is strictly increasing and unbounded. Hence, for any sufficiently large $\tau \geq t$, we have $k_\tau > 1$ and hence

$$F(k_\tau) - k_{\tau+1} < F(k_\tau) - k_\tau = k_\tau^\alpha - k_\tau < 0.$$

This contradicts the feasibility of the optimal investment policy. The proof is thus completed.

We now prove Claim 1. Suppose first that there is a $t \geq 1$ such that $k_t < k^*$. By taking t smaller if necessary, we can assume that if $k_\tau < k^*$, then $\tau \geq t$. Then $k_t < k^* \leq k_{t-1}$ and hence $1 - \delta F'(k_t) + g'(k_t - k_{t-1}) < 0$, a contradiction to Lemma 1. Thus $k_t \geq k^*$ for every $t \geq 1$. Suppose next that there is a $t \geq 1$ such that $k_t > k^*$. By taking t smaller if necessary, we can assume that if $k_\tau > k^*$, then $\tau \geq t$. Then $k_t > k^*$ and hence

$$1 - \delta F'(k_t) + g'(k_t - k_{t-1}) > 0,$$

a contradiction to Lemma 2. Thus $k_t \leq k^*$ for every $t \geq 1$. We have thus proved Claim 1.

We next prove Claim 2. We shall first prove by contradiction that $k_t < k^*$ for every $t \geq 1$. Suppose that there is a $t \geq 1$ such that $k_t \geq k^*$. By taking t smaller if necessary, we can assume that if $k_\tau \geq k^*$, then $\tau \geq t$. Then $k_t \geq k^* > k_{t-1}$ and hence $1 - \delta F'(k_t) + g'(k_t - k_{t-1}) > 0$, a contradiction to Lemma 2. Thus $k_t < k^*$ for every $t \geq 1$. We shall next prove by contradiction that $k_t > k_{t-1}$ for every $t \geq 1$. Suppose that there is a $t \geq 1$ such that $k_t \leq k_{t-1}$, then $g'(k_t - k_{t-1}) = 0$ and thus, by Lemma 1, $1 - \delta F'(k_t) \geq 0$. But this contradicts $k_t < k^*$. Hence $k_t > k_{t-1}$ for every $t \geq 1$. We shall finally prove that $k_t \to k^*$. Since $F(k_t) - k_{t+1} \geq F(k_t) - k_t = k_t^\alpha - k_t \geq 0$, we must have $k_t \leq 1$ for every $t \geq 1$. Thus the investment policy $(k_0, k_1, \ldots, k_t, \ldots)$ is bounded. We have shown that it is also increasing. Hence it has a limit, which we denote by k^{**}. By taking $t \to \infty$ in the Euler equation, we obtain $1 - \delta F'(k^{**}) = 0$, that is, $k^{**} = k^*$. The proof is thus completed.

20.D.6 We shall prove that an interior investment policy $(k_0^*, k_1^*, \ldots, k_t^*, \ldots)$ satisfies the Euler equations (20.D.9) if and only if it solves the following maximization problem:

$$\text{Max} \quad \textstyle\sum_t \delta^t u(k_{t-1}, k_t)$$

$$\text{s.t. } (k_{t-1}, k_t) \in A \text{ for every } t,\ k_0 = \bar{k}_0, \text{ and } \#\{t: k_t \neq k_t^*\} < \infty.$$

But $(k_0^*, k_1^*, \ldots, k_t^*, \ldots)$ solves the above maximization problem if and only if it solves the following maximization problem for every T:

$$\text{Max} \quad \textstyle\sum_{t=0}^{t=T+1} \delta^t u(k_{t-1}, k_t)$$

$$\text{s.t. } (k_{t-1}, k_t) \in A \text{ for every } t \le T,\ k_0 = \bar{k}_0, \text{ and } k_{T+1} = k_{T+1}^*.$$

But the necessary and sufficient first-order condition for a (interior) maximum of the above problem with terminal period T + 1 is that the Euler equation (20.D.9) holds for $t \in \{1, \ldots, T\}$. Thus the investment policy $(k_0^*, k_1^*, \ldots, k_t^*, \ldots)$ satisfies the Euler equations (20.D.9) for every t if and only if it solves the first maximization problem.

20.D.7 By the functional forms given,

$$\nabla_2 u(k_{t-1}, k_t) + (1/2)\nabla_1 u(k_t, k_{t+1}) = -1/(2k_{t-1} - k_t) + 1/(2k_t - k_{t+1}) = 0.$$

Hence, by multiplying $(2k_{t-1} - k_t)(2k_t - k_{t+1})$ to both sides, we obtain

$$-(2k_t - k_{t+1}) + (2k_{t-1} - k_t) = k_{t+1} - 3k_t + 2k_{t-1} = 0.$$

Hence $k_{t+1} = 3k_t - 2k_{t-1}$. By substituting the given solution

$$k_t = k_0 + (k_1 - k_0)(2^t - 1)$$

into this equality, we obtain

$$3(k_0 + (k_1 - k_0)(2^t - 1)) - 2(k_0 + (k_1 - k_0)(2^{t-1} - 1))$$
$$= k_0 + (k_1 - k_0)(3\cdot 2^t - 2^t - 1)$$
$$= k_0 + (k_1 - k_0)(2^{t+1} - 1) = k_{t+1}.$$

[By the standard technique from linear difference equations, we can show that the above solution is in fact the unique one.]

20.D.8 Let k and k' be two initial capital investment, and $\lambda \in [0,1]$. Let $(k_0,k_1,\ldots,k_t,\ldots)$ and $(k'_0,k'_1,\ldots,k'_t,\ldots)$ satisfy $(k_{t-1},k_t) \in A$ and $(k'_{t-1},k'_t) \in A$ for every t, $k_0 = k$, $k'_0 = k'$, $V(k) = \sum_t \delta^t u(k_{t-1},k_t)$, and $V(k') = \sum_t \delta^t u(k'_{t-1},k'_t)$. Then the trajectory

$$(\lambda k_0 + (1 - \lambda)k'_0,\ \lambda k_1 + (1 - \lambda)k'_1,\ldots,\ \lambda k_t + (1 - \lambda)k'_t,\ldots)$$

satisfies $\lambda k_0 + (1 - \lambda)k'_0 = \lambda k + (1 - \lambda)k'$ and

$$(\lambda k_{t-1} + (1 - \lambda)k'_{t-1},\ \lambda k_t + (1 - \lambda)k'_t) \in A$$

for every t, because A is convex. By the (strict) concavity of $u(\cdot)$

$$\sum_t \delta^t u(\lambda k_{t-1} + (1 - \lambda)k'_{t-1},\ \lambda k_t + (1 - \lambda)k'_t)$$
$$\geq \lambda \sum_t \delta^t u(k'_{t-1},k'_t) + (1 - \lambda)\sum_t \delta^t u(k'_{t-1},k'_t).$$

By the definition,

$$V(\lambda k + (1 - \lambda)k') \geq \sum_t \delta^t u(\lambda k_{t-1} + (1 - \lambda)k'_{t-1},\ \lambda k_t + (1 - \lambda)k'_t).$$

Hence

$$V(\lambda k + (1 - \lambda)k') \geq \lambda V(k) + (1 - \lambda)V(k').$$

We have thus established property (i). Property (ii) follows directly from (20.D.11).

20.D.9 For any given k, define a function $f(\cdot)$ by

$$f(z) = V(k + z) - (u(k + z,\ \psi(k)) + \delta V(\psi(k))),$$

then, by property (ii), $f(z) \geq 0$ for every z and $f(0) = 0$. Thus $f'(0) = 0$ and $f''(0) \geq 0$. But here

$$f'(0) = V'(k) - \nabla_1 u(k,\psi(k)),$$
$$f''(0) = V''(k) - \nabla^2_{11} u(k,\psi(k)).$$

Thus $V'(k) = \nabla_1 u(k,\psi(k))$ and $V''(k) \geq \nabla^2_{11} u(k,\psi(k))$.

20.E.1 By giving up one unit of good ℓ at period t, a consumer can save $p_{\ell t}$ units of money. With this amount of money, he can buy $p_{\ell t}/p_{\ell,t+1}$ units of good ℓ at period t + 1. Since $p_{\ell t}/p_{\ell,t+1} = 1 + r$, r can be interpreted as the rate of interest.

20.E.2 (a) Let $(y_0,y_1,\ldots,y_t,\ldots)$ be a proportional, efficient production path and assume that $y_t = (1 + n)y_{t-1}$ for every t. Let $\bar{q} = (\bar{q}_0,\bar{q}_1) \in \mathbb{R}^L \times \mathbb{R}^L$ be a vector that supports Y at y_0. Then, by the constant returns to scale, $\bar{q}$ supports Y at y_t for every t. By the small-type discussion at the end of Section 20.C, there exists a price path $(p_0,p_1,\ldots,p_t,\ldots)$ with respect to which the production path $(y_0,y_1,\ldots,y_t,\ldots)$ is myopically profit-maximizing. But this implies that for every t, there exists $\lambda_t > 0$ such that $(p_t,p_{t+1}) = \lambda_t(\bar{q}_0,\bar{q}_1)$. This, in particular, implies that $p_t = \lambda_t\bar{q}_0$, $p_{t+1} = \lambda_t\bar{q}_1$, $p_t = \lambda_{t-1}\bar{q}_1$, and $p_{t+1} = \lambda_{t+1}\bar{q}_0$. Thus $p_{t+1} = (\lambda_t/\lambda_{t-1})p_t = (\lambda_{t+1}/\lambda_t)p_t$. Thus $\lambda_t/\lambda_{t-1} = \lambda_{t+1}/\lambda_t$. Denote this ratio by α, then we have $p_t = \alpha^t p_0$.

(b) We shall prove the following assertion: Let $(y_0,y_1,\ldots,y_t,\ldots)$ be a proportional production path such that $y_t = (1 + n)y_{t-1}$ for every t, and assume that it can be myopically supported by a price sequence $(p_0,p_1,\ldots,p_t,\ldots)$ such that $p_t = \alpha^t p_0$ for every t. Define $r = (1 - \alpha)/\alpha$. Then, the production path is efficient if $r > n$; it is not efficient if $r < n$.

The proof of this assertion is as follows: Suppose first that $r > n$. Then $p_{t+1}\cdot y_{at} = ((1 + r)^{-t}p_0)\cdot((1 + n)^t y_{a0}) = ((1 + n)/(1 + r))^t p_0\cdot y_{a0}$. Since $(1 + n)/(1 + r) < 1$, $p_{t+1}\cdot y_{at} \to 0$ as $t \to \infty$. Thus the transversality condition (Proposition 20.C.1) is satisfied. Thus the proportional path is efficient.

Suppose next that $r < n$. Consider a vector $((1+n)^{-1}e, -e)$, where $e = (1,\ldots,1) \in \mathbb{R}^L$. Then

$$(p_0, (1+r)^{-1}p_0)\cdot((1+n)^{-1}e, -e) = ((1+n)^{-1} - (1+r)^{-1})p_0\cdot e < 0.$$

Thus, for any sufficiently small $\varepsilon > 0$,

$$y_0' = y_0 + \varepsilon((1+n)^{-1}e, -e) \in Y.$$

Hence, if we define

$$y_t' = (1+n)^t y_0' = y_t + \varepsilon((1+n)^{t-1}e, -(1+n)^t e),$$

then $y_t' \in Y$ by the constant returns to scale. But here,

$$y_{b0}' = y_{b0} + \varepsilon(1+n)^{-1}e$$

and

$$y_{a,t-1}' + y_{bt}' = y_{a,t-1} + y_{bt}$$

for every $t \geq 0$. Thus the alternative production path $(y_0', y_1', \ldots, y_t', \ldots)$ Pareto dominates $(y_0, y_1, \ldots, y_t, \ldots)$. Hence the latter is not efficient.

20.E.3 Let a production path $(k_0, k_1, \ldots, k_t, \ldots)$ be as in the statement of this exercise, then $F(k_t) \leq F(\bar{k} - \varepsilon)$. Define a output price sequence $(q_0, q_1, \ldots, q_t, \ldots)$ by $q_0 = 1$ and $q_t = q_{t-1}/\nabla_1 F(k_{t-1}, 1)$ for every $t \geq 1$. Then $\nabla_1 F(k_t, 1) \geq \nabla_1 F(\bar{k} - \varepsilon, 1) > \nabla_1 F(\bar{k}, 1) = 1$. Thus $q_t \leq q_0/(\nabla_1 F(\bar{k} - \varepsilon, 1))^t$. Since $q_0/(\nabla_1 F(\bar{k} - \varepsilon, 1))^t \to 0$, $q_t \to 0$. Hence (20.C.2) holds. Thus, as explained in Example 20.C.6, $(k_0, k_1, \ldots, k_t, \ldots)$ is efficient.

20.E.4 Property (i) follows from $c'(k) = \nabla_1 F(k,1) - 1$, $c''(k) = \nabla_1^2 F(k,1) < 0$, and $\nabla_1 F(\bar{k}, 1) = 1$. Property (ii) follows from $r'(k) = \nabla_1^2 F(k,1) < 0$. For property (iii), note first that $\nabla_1 F(k,1)k + \nabla_2 F(k,1) = F(k,1)$ by the homogeneity and Euler's formula (Theorem M.B.2). Thus $w(k) = F(k,1)/\nabla_1 F(k,1) - k$. By differentiating both sides with respect to k, we obtain

$$w'(k) = -\frac{\nabla_1^2 F(k,1)}{(\nabla_1 F(k,1))^2} F(k,1).$$

Since $\nabla_1^2 F(k,1) < 0$ by the concavity, $w'(k) > 0$.

20.E.5 For a path $(k_0, k_1, \ldots, k_t, \ldots)$ of the N investment goods (and labor inputs $\ell_t = 1$ for every t), according to the notation in Example 20.C.4, the input at period t - 1 is equal to $(-k_{t-1}, 0, -1)$ and output at period t is equal to $(k_t, G(k_{t-1}, k_t), 0)$. Hence the profit at period t - 1 is equal to

$$q_t \cdot k_t + s_t G(k_{t-1}, k_t) - q_{t-1} \cdot k_{t-1} - w_{t-1}.$$

If a steady state k is myopically supported by price sequence (s_t, q_t), then this profit is maximized at $(k_{t-1}, k_t) = (k,k)$. The first-order condition for the maximum with respect to k_{t-1} is $s_t \nabla_1 G(k,k) - q_{t-1} = 0$. This proves the first equality of (20.E.1). The second equality is the first-order condition for with respect to k_t, which is $q_t + s_t \nabla_2 G(k,k) = 0$.

20.E.6 If $N = 1$, then (20.E.1) implies that $q_t/q_{t+1} = -\nabla_1 G(k,k)/\nabla_2 G(k,k)$. But the right-hand side is nothing but the slope of the level curve at (k,k).

20.E.7 [First printing errata: On the last line of this exercise, the inequality $\alpha < F(k,k')$ should be replaced by $\alpha < G(k,k')$.] By differentiating both sides of $G(k, k' + \alpha) = G(k,k') - \alpha$ with respect to α and evaluating at $\alpha = 0$, we obtain $\nabla_2 G(k,k') = -1$. Thus, by the second equality of (20.E.1), $q_t = s_t$ for every t. By the first equality, $\nabla_1 G(k,k) = q_{t-1}/q_t$. Hence $r(k) = \nabla_1 G(k,k) - 1$. Thus $r'(k) = \nabla_1^2 G(k,k) + \nabla_2 \nabla_1 G(k,k)$. But here $\nabla_2 \nabla_1 G(k,k) = \nabla_1 \nabla_2 G(k,k) = 0$ since $\nabla_2 G(k,k') = -1$ for every (k,k'). Hence $r'(k) = \nabla_1^2 G(k,k) < 0$. The assertion thus follows.

20.E.8 Assuming that $\ell_t = (1 + n)^t$ for every t, the per-capita surplus maximization problem among the proportional production paths with rate n of growth can be written as

$$\mathrm{Max}_{k_0 \geq 0}\ F(k_0, 1 + n)/(1 + n) - k_0.$$

The first-order condition for a maximum is $\nabla_1 F(k_0, 1 + n)/(1 + n) - 1 = 0$. Thus the interest rate $\nabla_1 F(k,1) - 1$ is equal to n.

20.E.9 Suppose that the golden rule path $(\bar{y},\bar{y},\ldots,\bar{y},\ldots)$ constitutes an equilibrium with a price sequence $(p_0,p_1,\ldots,p_t,\ldots)$. Then the golden rule path must be myopically supported by this price sequence. Under the smoothness assumption of Y, it is a unique supporting price sequence (up to a scalar multiplication). Thus, by Definition 20.E.2, it must be constant. So let $\bar{p} = p_t$ for every t

By the equilibrium condition (20.D.2), $c_t = \bar{y}_a + \bar{y}_b + \omega_t$. Assuming that the sequence $(\omega_0,\omega_1,\ldots,\omega_t,\ldots)$ of initial endowments is constant, the sequence $(c_0,c_1,\ldots,c_t,\ldots)$ of consumptions is also constant. So let $\bar{c} = c_t$. Since this consumption sequence is myopically utility-maximizing (Definition 20.D.2), Exercise 20.D.2 implies that for every t there exists $\lambda_t > 0$ such that $\lambda_t\bar{p} = \nabla u(\bar{c})$ and $\lambda_t\bar{p} = \delta\nabla u(\bar{c})$. But since $\delta < 1$, this is a contradiction.

20.F.1 Define $X = \{x = (x_1,x_2) \in \mathbb{R}^2_+: x_1 + x_2 \leq \alpha\}$. Write $\gamma(w) = (\gamma_1(w),\gamma_2(w)) \in \mathbb{R}^2_+$. Then $\gamma(x_1 + x_2) \in X$ for every $x = (x_1,x_2) \in X$. Define ξ: $X \to \mathbb{R}$ by

$$\xi(x) = (x_1 + x_2)^{1/2}$$

and ζ: $X \to \mathbb{R}$ by

$$\zeta(x) = \|x - \gamma(x_1 + x_2)\|^2 = ((x_1 - \gamma_1(x_1 + x_2))^2 + ((x_2 - \gamma_2(x_1 + x_2))^2.$$

Then $\xi(\cdot)$ is twice continuously differentiable on $\{x \in X\colon x \neq 0\}$ and $\zeta(\cdot)$ is twice continuously differentiable on X.

For the gradients of these functions, note that

$$\nabla\xi(x) = (1/2)(x_1 + x_2)^{-1/2}\begin{bmatrix}1\\1\end{bmatrix} \gg (1/2)\alpha^{-1/2}\begin{bmatrix}1\\1\end{bmatrix}.$$

Since $\nabla\zeta(\cdot)\colon X \to \mathbb{R}^2$ is continuous, there exists a $\beta_1 > 0$ such that $\nabla\zeta(x) \ll \beta_1\begin{bmatrix}1\\1\end{bmatrix}$ for every $x \in X$. Define $\varepsilon_1 = (1/2)\alpha^{-1/2}\beta_1^{-1} > 0$, then $\varepsilon\nabla\zeta(x) \ll (1/2)\alpha^{-1/2}\begin{bmatrix}1\\1\end{bmatrix}$ for every $\varepsilon \in (0,\varepsilon_1)$ and $x \in X$. Hence $\nabla\xi(x) - \varepsilon\nabla\zeta(x) \gg 0$ for every $\varepsilon \in (0,\varepsilon_1)$ and $x \in X$.

As for the Hessians, note that

$$D^2\xi(x) = (-1/4)(x_1 + x_2)^{-3/2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}.$$

Thus $D^2\xi(x)$ is negative semidefinite on $\mathbb{R}^2$ and negative definite on $\{v \in \mathbb{R}^2\colon v_1 + v_2 \neq 0\}$. Note also that, for every $x \in X$, $x' \in X$, and $v \in \mathbb{R}^2$, if $0 < x_1 + x_2 < x_1' + x_2'$ and $v_1 + v_2 \neq 0$, then $v\cdot D^2\xi(x)v < v\cdot D^2\xi(x')v < 0$. On the other hand, for every $v \in \mathbb{R}^2$ with $v_1 + v_2 = 0$ and $c \in \mathbb{R}$, $\gamma((x_1 + x_2) + c(v_1 + v_2)) = \gamma(x_1 + x_2)$ and hence, by denoting the 2×2 identity matrix by I, we have

$$\begin{aligned}
&d[\zeta(x + cv)]/dc \\
&= \nabla\zeta(x + cv)\cdot v \\
&= 2(x + cv - \gamma(x_1 + x_2))\cdot(I - \begin{bmatrix}\gamma_1''(x_1 + x_2) & \gamma_1''(x_1 + x_2)\\ \gamma_2''(x_1 + x_2) & \gamma_2''(x_1 + x_2)\end{bmatrix})v \\
&= 2(x - \gamma(x_1 + x_2))\cdot(I - \begin{bmatrix}\gamma_1''(x_1 + x_2) & \gamma_1''(x_1 + x_2)\\ \gamma_2''(x_1 + x_2) & \gamma_2''(x_1 + x_2)\end{bmatrix})v \\
&\qquad + (2c)v\cdot(I - \begin{bmatrix}\gamma_1''(x_1 + x_2) & \gamma_1''(x_1 + x_2)\\ \gamma_2''(x_1 + x_2) & \gamma_2''(x_1 + x_2)\end{bmatrix})v \\
&= 2(x - \gamma(x_1 + x_2))\cdot v + (2c)\|v\|^2.
\end{aligned}$$

Thus $d^2[\zeta(x + cv)]/dc^2 = 2\|v\|^2 > 0$. On the other hand, by the definition, $d^2[\zeta(x + cv)]/dc^2|_{c=0} = v\cdot D^2\zeta(x)v$. Hence $v\cdot D^2\zeta(x)v > 0$.

Now define $S = \{v \in \mathbb{R}^2: \|v\| = 1\}$. Consider the function $(x,v) \to v \cdot D^2\zeta(x)v$ defined on $X \times S$. Since this function is continuous and $X \times S$ is compact, there exists $\beta_1 < 0$ such that $v \cdot D^2\zeta(x)v \geq \beta_1$ for every $(x,v) \in X \times S$. We shall now prove that there exists a $c > 0$ such that, for every $x \in X$ and $v \in S$, if $\left\| v - \begin{bmatrix} 2^{-1/2} \\ -2^{-1/2} \end{bmatrix} \right\| < c$ or $\left\| v - \begin{bmatrix} -2^{-1/2} \\ 2^{-1/2} \end{bmatrix} \right\| < c$, then $v \cdot D^2\zeta(x)v > 0$. In fact, suppose that this claim is false, then, for each positive integer n, there exists a $(x^n,v^n) \in X \times S$ such that $\left\| v^n - \begin{bmatrix} 2^{-1/2} \\ -2^{-1/2} \end{bmatrix} \right\| < 1/n$ or $\left\| v^n - \begin{bmatrix} -2^{-1/2} \\ 2^{-1/2} \end{bmatrix} \right\| < 1/n$, and $v^n \cdot D^2\zeta(x^n)v^n \leq 0$. By the compactness, and by taking a subsequence (and by multiplying -1 to v^n), we can assume that there exists an $x \in X$ such that $x^n \to x$ and $v^n \to v = \begin{bmatrix} 2^{-1/2} \\ -2^{-1/2} \end{bmatrix}$. Then, by the continuity, $v \cdot D^2\zeta(x)v \leq 0$. This is a contradiction. Hence such a $c > 0$ actually exists. So define

$$S' = \left\{v \in S: \left\| v^n - \begin{bmatrix} 2^{-1/2} \\ -2^{-1/2} \end{bmatrix} \right\| \geq c \text{ and } \left\| v^n - \begin{bmatrix} -2^{-1/2} \\ 2^{-1/2} \end{bmatrix} \right\| \geq c\right\},$$

then S' is compact, $v \cdot D^2\zeta(x)v > 0$ for every $x \in X$ and $v \notin S'$, $\begin{bmatrix} 2^{-1/2} \\ -2^{-1/2} \end{bmatrix} \notin S'$, and $\begin{bmatrix} -2^{-1/2} \\ 2^{-1/2} \end{bmatrix} \notin S'$. Thus $v \cdot D^2\xi(x)v < 0$ for every $x \in X$ with $x \neq 0$ and $v \in S'$. So let

$$\beta_2 = \max\{v \cdot D^2\xi(x)v: x \in X, x \neq 0, v \in S'\};$$

the maximum exists because for every $x \in X$, $x' \in X$, and $v \in \mathbb{R}^2$, if $0 < x_1 + x_2 < x_1' + x_2'$ and $v_1 + v_2 \neq 0$, then $v \cdot D^2\xi(x)v < v \cdot D^2\xi(x')v$. Now define $\varepsilon_2 = \beta_2/\beta_1 > 0$. Then let $\varepsilon \in (0,\varepsilon_2)$, $x \in X$, $x \neq 0$, $v \in \mathbb{R}^2$, and $v \neq 0$. If $v \notin S'$, then

$$v \cdot (D^2\xi(x) - \varepsilon D^2\zeta(x))v < v \cdot D^2\xi(x)v \leq 0.$$

If $v \in S'$, then

$$v \cdot (D^2\xi(x) - \varepsilon D^2\zeta(x))v \leq \beta_2 - \varepsilon\beta_1 < \beta_2 - (\beta_2/\beta_1)\beta_1 = 0.$$

Thus $D^2\xi(x) - \varepsilon D^2\zeta(x)$ is negative definite on the whole $\mathbb{R}^2$. Hence, if $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$ and we define u: $X \to \mathbb{R}$ by $u(x) = \xi(x) - \varepsilon\zeta(x)$, then $u(\cdot)$ is strongly monotone and strictly concave.

20.F.2 A couple of possible graphs for a policy function and the graphical dynamic analysis are given below. The uniqueness of a steady state can also be proved as follows: Since $|\psi'(k)| < 1$, $d[\psi(k) - k]/dk = \psi'(k) - 1 < 0$ and hence the function $k \to \psi(k) - k$ is strongly decreasing. Since a steady state k is characterized as $\psi(k) - k = 0$, this implies that there is only one steady state.

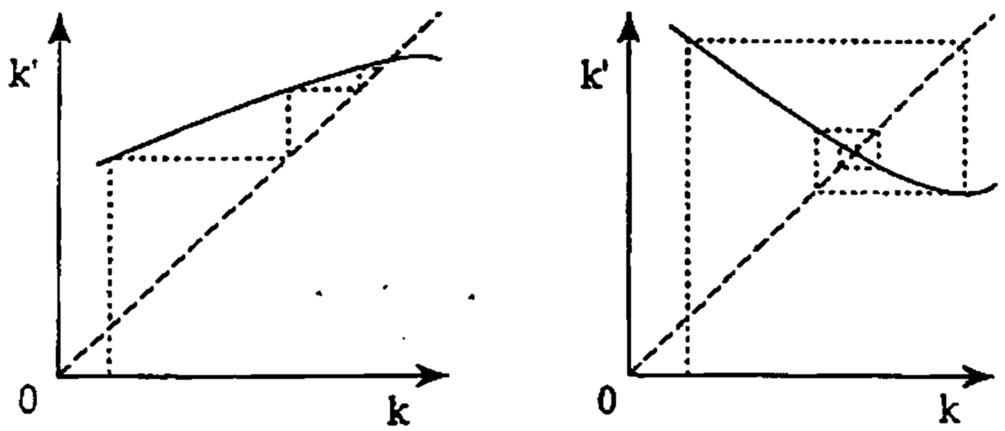


Figure 20.F.2

20.F.3 For the Ramsey-Solow technology, we have

$$\nabla_1 u(k,k') = u'(F(k) - k')F'(k)$$

and hence

$$\nabla_{12} u(k,k') = - u''(F(k) - k')F'(k) > 0.$$

As for the cost-of-adjustment model, we have

$$\nabla_1 u(k,k') = u'(F(k) - k' - \gamma(k' - k))(F'(k) + \gamma'(k' - k))$$

and hence

$$\nabla_{12} u(k,k') = - u''(F(k) - k' - \gamma(k' - k))(1 + \gamma'(k' - k))(F'(k) + \gamma'(k' - k))$$
$$+ u'(F(k) - k' - \gamma(k' - k))\gamma''(k' - k) > 0,$$

because $\gamma''(k' - k) \geq 0$.

20.F.4 While a diagrammatic proof is given in Figure 20.F.5, a proof based on the first-order conditions is also possible. Denote by $k_1^{tr}(\theta)$ the solution of the optimum problem when the shock is transitory and takes value θ. Then $k_1^{tr}(0) = 0$. By (20.F.3) and $u(k,k') = g(k) + h(k')$,

$$\partial h(k_1^{tr}(\theta),\theta)/\partial k' + \delta \partial V(k_1^{tr}(\theta),0)/\partial k' = 0.$$

Since $\partial V(k_1^{tr}(\theta),0)/\partial k' = \nabla_1 u(k_1^{tr}(\theta),\bar{k},0) = \partial g(k_1^{tr}(\theta),0)/\partial k$, we have

$$\partial h(k_1^{tr}(\theta),\theta)/\partial k' + \delta \partial g(k_1^{tr}(\theta),0)/\partial k = 0.$$

By differentiating both sides with respect to θ, we obtain

$$(\partial^2 h(k_1^{tr}(\theta),\theta)/\partial k'^2)(dk_1^{tr}(\theta)/d\theta) + \partial^2 h(k_1^{tr}(\theta),\theta)/\partial k'\partial\theta$$
$$+ \delta(\partial^2 g(k_1^{tr}(\theta),0)/\partial k^2)(dk_1^{tr}(\theta)/d\theta) = 0.$$

Hence

$$dk_1^{tr}(\theta)/d\theta = - \frac{\partial^2 h(k_1^{tr}(\theta),\theta)/\partial k'\partial\theta}{\partial^2 h(k_1^{tr}(\theta),\theta)/\partial k'^2 + \delta\partial^2 g(k_1^{tr}(\theta),0)/\partial k^2}$$

Since $\partial^2 h(\bar{k},0)/\partial k'^2 + \delta\partial^2 g(\bar{k},0)/\partial k^2 < 0$, $dk_1^{tr}(0)/d\theta \gtrless 0$ if and only if $\partial^2 h(\bar{k},0)/\partial k'\partial\theta \gtrless 0$.

20.F.5 A diagrammatic proof goes as follows. Since $\partial^2 h(\bar{k},0)/\partial k'\partial\theta > 0$ and $\partial^2 g(\bar{k},0)/\partial k'\partial\theta > 0$, as θ goes up, the indifference curve going through $(\bar{k},V(\bar{k},0))$ becomes flatter and the graph of $V(\cdot,\theta)$ becomes steeper. Hence the tangency on the graph of $V(\cdot,\theta)$ is attained in the right of $(\bar{k},V(\bar{k}))$, that is, $k_1^p > \bar{k}$. Moreover, since the indifference curves are of quasilinear form (and the graph of $V(\cdot,\theta)$ becomes steeper), the tangency is attained in the right of $(k_1^{tr},V(k_1^{tr}))$.

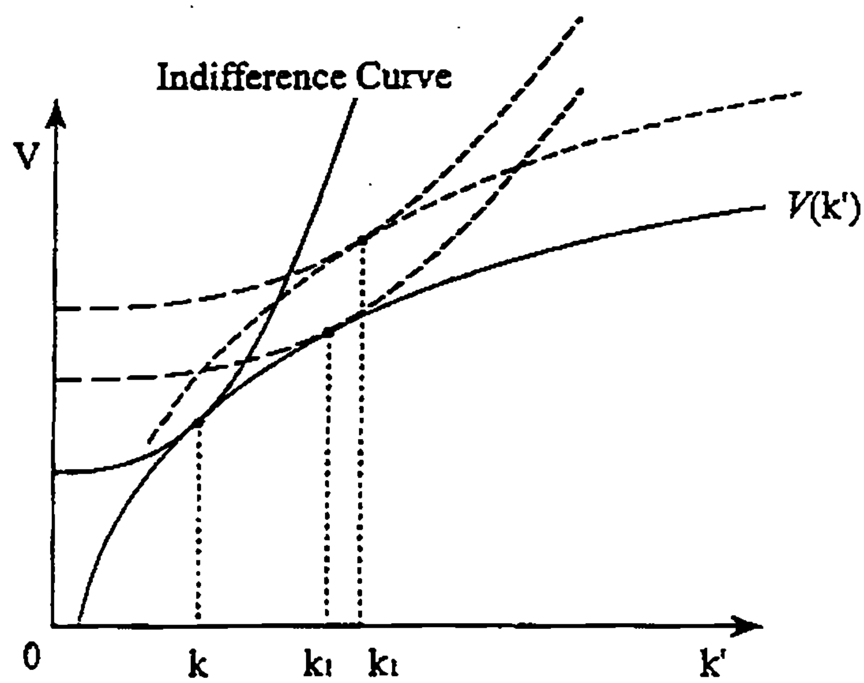


Figure 20.F.5

A proof based on the first-order conditions goes as follows. Denote by $k_1^p(\theta)$ the solution of the optimum problem when the shock is permanent and takes value θ. Then $k_1^p(0) = 0$. By (20.F.3) and $u(k,k') = g(k) + h(k')$,

$$\partial h(k_1^p(\theta),\theta)/\partial k' + \delta \partial V(k_1^p(\theta),\theta)/\partial k' = 0.$$

Since $\partial V(k_1^p(\theta),\theta)/\partial k' = \nabla_1 u(k_1^p(\theta),k_1^p(\theta),\theta) = \partial g(k_1^p(\theta),\theta)/\partial k$, we have

$$\partial h(k_1^p(\theta),\theta)/\partial k' + \delta \partial g(k_1^p(\theta),\theta)/\partial k = 0.$$

By differentiating both sides with respect to θ, we obtain

$$(\partial^2 h(k_1^p(\theta),\theta)/\partial k'^2)(dk_1^p(\theta)/d\theta) + \partial^2 h(k_1^p(\theta),\theta)/\partial k'\partial\theta$$
$$+ \delta(\partial^2 g(k_1^p(\theta),0)/\partial k^2)(dk_1^p(\theta)/d\theta) + \delta\partial^2 g(k_1^p(\theta),\theta)/\partial k'\partial\theta = 0.$$

Hence

$$dk_1^p(\theta)/d\theta = - \frac{\partial^2 h(k_1^p(\theta),\theta)/\partial k'\partial\theta + \delta\partial^2 g(k_1^p(\theta),\theta)/\partial k\partial\theta}{\partial^2 h(k_1^p(\theta),\theta)/\partial k'^2 + \delta\partial^2 g(k_1^p(\theta),0)/\partial k^2}.$$

Since $\partial^2 h(\bar{k},0)/\partial k'^2 + \delta\partial^2 g(\bar{k},0)/\partial k^2 < 0$, $\partial^2 h(\bar{k},0)/\partial k'\partial\theta > 0$, and $\partial^2 g(\bar{k},0)/\partial k'\partial\theta > 0$, we conclude that $dk_1^p(0)/d\theta > dk_1^{tr}(0)/d\theta > 0$.

20.G.1 Suppose that a price sequence $(p_0,p_1,\ldots,p_t,\ldots)$ and the consumption streams $(c_{0i},c_{1i},\ldots,c_{ti},\ldots)$ $(i = 1,2)$ constitute a Walrasian equilibrium such that $c_{ti} > 0$ for every t and i. Then $p_{t+1}/p_t = \delta u_i(c_{t+1,i})/u_i(c_{ti})$ for every t and i. Hence $\delta u_1(c_{t+1,1})/u_1(c_{t1}) = \delta u_2(c_{t+1,2})/u_2(c_{t2})$ for every t. Thus, for every t, $c_{t+1,1} \gtrless c_{t1}$ if and only if $c_{t+1,2} \gtrless c_{t2}$. Since $c_{t1} + c_{t2} = c_{t+1,1} + c_{t+1,2} = 1$ for every t, we must have $c_{t+1,1} = c_{t1}$ and $c_{t+1,2} = c_{t2}$. Thus the equilibrium consumption streams is stationary and the sequence of equilibrium prices is proportional at rate δ.

20.G.2 The first statement is an immediate consequence of the single budget constraint, which is represented in (20.G.3) and mentioned in footnote 28. (And the in-period marginal utilities of wealth are equal to λ in (20.D.6).) Its implication on transfers of wealth is that as the timing of transfers of wealth does not matter, it can take the form of a transfer of commodities on any period.

20.G.3 (a) For each $\eta_1 \in (0,1)$, let

$$c_i(\eta_1) = (c_{0i}(\eta_1),c_{1i}(\eta_1),\ldots,c_{ti}(\eta_1),\ldots) \ (i = 1,2)$$

be the unique Pareto optimal (interior) consumption allocation such that $\eta_1\nabla\delta^t u_1(c_{t1}) = (1 - \eta_i)\nabla\delta^t u_2(c_{t2})$. (Such an allocation can be characterized as the unique solution to the problem of maximizing the weighted sum

$\eta_1\sum_t\delta^t u_1(c_{t1}) + (1-\eta_1)\sum_t\delta^t u_2(c_{t2})$ subject to $c_{t1} + c_{t2} = \omega_{t1} + \omega_{t2}$ for every t.) The Pareto frontier of the utility possibility set can be parameterized by

$$\eta_1 \to (\sum_t\delta^t u_1(c_{t1}(\eta_1)), \sum_t\delta^t u_2(c_{t2}(\eta_1))).$$

(b) Let $p(\eta_1) = (p_0(\eta_1), p_1(\eta_1), \ldots, p_t(\eta_1), \ldots)$ be a supporting price sequence. A Walrasian equilibrium $\eta_1 \in (0,1)$ can be characterized by

$$\sum_t p_t(\eta_1)\cdot c_{t1}(\eta_1) - \sum_t p_t(\eta_1)\cdot\omega_{t1} = 0.$$

In fact, by the constraint $c_{t1}(\eta_1) + c_{t2}(\eta_1) = \omega_{t1} + \omega_{t2}$, we obtain

$$\sum_t p_t(\eta_1)\cdot c_{t2}(\eta_1) - \sum_t p_t(\eta_1)\cdot\omega_{t2} = 0.$$

Hence the budget constraints for both consumers are met and thus the price sequence $p(\eta_1)$ and the consumption streams $c_i(\eta_1)$ constitute a Walrasian equilibrium.

(c) Assume that the function $\eta_1 \to \sum_t p_t(\eta_1)\cdot c_{t1}(\eta_1) - \sum_t p_t(\eta_1)\cdot\omega_{t1}$ is as many times differentiable as necessary for the transversality theorem (Proposition 17.D.3) to be applicable. We shall take ω_{101} (the initial endowment for good 1 at period 0 of consumer 1) as the exogenous parameter (and the other variables are fixed). Define f: $[0,1] \times (0, +\infty) \to \mathbb{R}$ by $f(\eta_1,\omega_{101}) = \sum_t p_t(\eta_1)\cdot\omega_{t1} - \sum_t p_t(\eta_1)\cdot c_{t1}(\eta_1)$, then $\partial f(\eta_1,\omega_{101})/\partial\omega_{101} = p_{10}(\eta_1) > 0$ and hence $\nabla f(\eta_1,\omega_{101}) \neq 0$, that is, $\nabla f(\eta_1,\omega_{101})$ is of full rank. Hence, by the transversality theorem, for almost all $\omega_{101} \in (0, +\infty)$, if $\eta_1 \in (0,1)$ and $f(\eta_1,\omega_{101}) = 0$, then $\partial f(\eta_1,\omega_{101})/\partial\eta_1 \neq 0$. Thus, for almost all $\omega_{101} \in (0, +\infty)$, the equilibria are discrete. Together with an appropriate boundary condition on $f(\cdot)$ (such as $f(0,\omega_{101}) > 0$ and $f(1,\omega_{101}) < 0$, which are analogous to Exercise 17.AA.2(c)), this implies that generically the equilibria are finite.

20.G.4 Suppose that a price sequence $(p_0, p_1, \ldots, p_t, \ldots)$ and the consumption streams $c_i = (c_{0i}, c_{1i}, \ldots, c_{ti}, \ldots)$ $(i = 1, \ldots, I)$ constitute an intertemporal (overall) Walrasian equilibrium. To prove footnote 32, we shall first prove by contradiction that $(c_{t1}, \ldots, c_{tI}) = (c_{t'1}, \ldots, c_{t'I})$ for every t and t', that is, the consumption streams are constant over time. So suppose that there exist two periods t and t' such that $(c_{t1}, \ldots, c_{tI}) \neq (c_{t'1}, \ldots, c_{t'I})$. Then consider the consumption streams that are the same as $(c_1, \ldots, c_I)$ except that $(c_{t1}, \ldots, c_{tI})$ and $(c_{t'1}, \ldots, c_{t'I})$ are both replaced by

$$\left(\frac{\delta^t}{\delta^t + \delta^{t'}} c_{t1} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} c_{t'1}, \ldots, \frac{\delta^t}{\delta^t + \delta^{t'}} c_{tI} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} c_{t'I}\right).$$

Let $y_t \in Y$ and $y_{t'} \in Y$ be the production plans at period t and t'. Since there is no possibility of intertemporal production, we can take $y_t \in \mathbb{R}^L$ and $y_{t'} \in \mathbb{R}^L$. Then $\sum_i c_{ti} = \sum_i \omega_i + y_t$ and $\sum_i c_{t'i} = \sum_i \omega_i + y_{t'}$. Hence

$$\begin{aligned}
&\textstyle\sum_i\left(\frac{\delta^t}{\delta^t + \delta^{t'}} c_{ti} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} c_{t'i}\right) \\
&= \frac{\delta^t}{\delta^t + \delta^{t'}} \textstyle\sum_i c_{ti} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} \sum_i c_{t'i} \\
&= \frac{\delta^t}{\delta^t + \delta^{t'}} (\textstyle\sum_i \omega_i + y_t) + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} (\sum_i \omega_i + y_{t'}) \\
&= \textstyle\sum_i \omega_i + \left(\frac{\delta^t}{\delta^t + \delta^{t'}} y_t + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} y_{t'}\right)
\end{aligned}$$

Since Y is convex, this proposed allocation is feasible. Moreover, by the strict concavity,

$$\begin{aligned}
&u_i\left(\frac{\delta^t}{\delta^t + \delta^{t'}} c_{ti} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} c_{t'i}\right) \\
&\geq \frac{\delta^t}{\delta^t + \delta^{t'}} u_i(c_{ti}) + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} u_i(c_{t'i})
\end{aligned}$$

and the strict inequality holds if $c_{ti} \neq c_{t'i}$. By multiplying $\delta^t + \delta^{t'}$ to both sides, we obtain

$$\delta^t u_i\left(\frac{\delta^t}{\delta^t + \delta^{t'}} c_{ti} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} c_{t'i}\right) + \delta^{t'} u_i\left(\frac{\delta^t}{\delta^t + \delta^{t'}} c_{ti} + \frac{\delta^{t'}}{\delta^t + \delta^{t'}} c_{t'i}\right)$$

$$\geq \delta^t u_i(c_{ti}) + \delta^{t'} u_i(c_{t'i})$$

and the strict inequality holds if $c_{ti} \neq c_{t'i}$. Hence the proposed allocation is Pareto superior to $(c_1,\ldots,c_I)$. By Proposition 20.G.1, this contradicts the fact that the consumption streams $(c_1,\ldots,c_I)$ constitutes an equilibrium. Hence they must be constant.

Given this constancy, it is routine to check that the given intertemporal equilibrium is an infinite, constant repetition of a one-period Walrasian equilibrium and that $p_t = \delta^t p_0$ for every t.

Conversely, it is also easy to check that an infinite, constant repetition of any one-period Walrasian equilibrium constitute an intertemporal equilibrium, and that $p_t = \delta^t p_0$ for every t.

20.G.5 By the concavity, $\nabla^2_{11}u(\cdot) \leq 0$ and $\nabla^2_{22}u(\cdot) \leq 0$. Thus

$$\left|\frac{dk_{t+1}}{dk}\right| = \frac{-(\nabla^2_{22}u(\cdot) + \delta\nabla^2_{11}u(\cdot))}{\delta|\nabla^2_{12}u(\cdot)|}.$$

Hence

$$\left|\frac{dk_{t+1}}{dk}\right|^2 = \frac{(\nabla^2_{22}u(\cdot) + \delta\nabla^2_{11}u(\cdot))^2}{\delta^2(\nabla^2_{12}u(\cdot))^2}.$$

But here, by the concavity, $(\nabla^2_{12}u(\cdot))^2 \leq \nabla^2_{11}u(\cdot)\nabla^2_{22}u(\cdot)$ and hence

$$\begin{aligned}
&(\nabla^2_{22}u(\cdot) + \delta\nabla^2_{11}u(\cdot))^2 \\
&= (\nabla^2_{22}u(\cdot))^2 + 2\delta\nabla^2_{22}u(\cdot)\nabla^2_{11}u(\cdot) + \delta^2(\nabla^2_{11}u(\cdot))^2 \\
&\geq 2\delta\nabla^2_{22}u(\cdot)\nabla^2_{11}u(\cdot) \\
&\geq 2\delta(\nabla^2_{12}u(\cdot))^2.
\end{aligned}$$

By $\delta \in (0,1)$ and $\nabla^2_{12}u(\cdot) \neq 0$, $2\delta(\nabla^2_{12}u(\cdot))^2 > \delta^2(\nabla^2_{12}u(\cdot))^2$. Hence $\left|\frac{dk_{t+1}}{dk}\right| >$

20.H.1 If we allow for the possibility for bubbles at an equilibrium consisting of $(y_0, y_1, \ldots, y_t, \ldots)$, $(p_0, p_1, \ldots, p_t, \ldots)$, and $(c_{0i}, c_{1i}, \ldots, c_{ti}, \ldots)$ $(i = 1, \ldots, I)$, then the budget constraint for each consumer i is that, for some $M_i \geq 0$,

$$\Sigma_t p_t c_{ti} = \Sigma_t \theta_{ti} \pi_t + \Sigma_t p_t \omega_{ti} + M_i.$$

With the finite number of consumers, we can sum both sides over i to obtain

$$\Sigma_i \Sigma_t p_t c_{ti} = \Sigma_t \pi_t + \Sigma_i \Sigma_t p_t \omega_{ti} + \Sigma_i M_i.$$

On the other hand, the feasibility constraints remain the same: For every t,

$$\Sigma_i c_{ti} = y_{a,t-1} + y_{bt} + \Sigma_i \omega_{ti}.$$

Multiply p_t to both sides, we obtain

$$\Sigma_i p_t c_{ti} = p_t(y_{a,t-1} + y_{bt}) + \Sigma_i p_t \omega_{ti}.$$

Summing both sides over $t = 0, 1, \ldots, T$, we obtain

$$\Sigma_{t=0}^{T} \Sigma_i p_t c_{ti} = \Sigma_{t=0}^{T} \pi_t + \Sigma_{t=0}^{T} \Sigma_i p_t \omega_{ti} - p_{T+1} y_{aT},$$

or

$$\Sigma_{t=0}^{T} (\pi_t + \Sigma_i p_t \omega_{ti}) - \Sigma_{t=0}^{T} \Sigma_i p_t c_{ti} = p_{T+1} y_{aT}.$$

Taking the limit $T \to \infty$, we obtain

$$- \Sigma_i M_i = \lim_{T\to\infty} p_{T+1} y_{aT}.$$

Now, by the possibility of truncation, $(y_{bT}, 0) \in Y$ and hence the myopic profit maximization implies that $p_T y_{bT} + p_{T+1} y_{aT} \geq p_T y_{bT}$, or $p_{T+1} y_{aT} \geq 0$. Hence $\lim_{T\to\infty} p_{T+1} y_{aT} \geq 0$. On the other hand, $- \Sigma_i M_i \leq 0$. Thus, in fact, we must have $\Sigma_i M_i = \lim_{T\to\infty} p_{T+1} y_{aT} = 0$. Hence $M_i = 0$ for every i. Hence no bubbles can arise at equilibrium.

20.H.2 (a) Note first that he slope of the offer curve at price (p_0, p_a) is equal to $\nabla_1 z_a(p_b,p_a)/\nabla_1 z_b(p_b,p_a)$. If $\nabla_1 z_b(p_b,p_a) \le 0$, then, by differentiating both sides of the budget constraint

$$p_b z_b(p_b,p_a) + p_a z_a(p_b,p_a) = (1 - \varepsilon)p_b$$

with respect to p_b, we obtain

$$z_b(p_b,p_a) + p_b \nabla_1 z_b(p_b,p_a) + p_a \nabla_1 z_a(p_b,p_a) = 1 - \varepsilon.$$

Since $\nabla_1 z_b(p_b,p_a) \le 0$ and $z_b(p_b,p_a) \le 0$, this implies that $\nabla_1 z_a(p_b,p_a) > 0$. Hence condition (20.H.3) is satisfied and the slope of the offer curve is negative (or the curve is vertical if $\nabla_1 z_b(p_b,p_a) = 0$). If $\nabla_1 z_b(p_b,p_a) > 0$, then condition (20.H.3) is satisfied if and only if the slope is greater than one. Therefore, the offer curve must look as in the following figures (or a mixture of the two, in which case the curve has both positive and negative slopes):

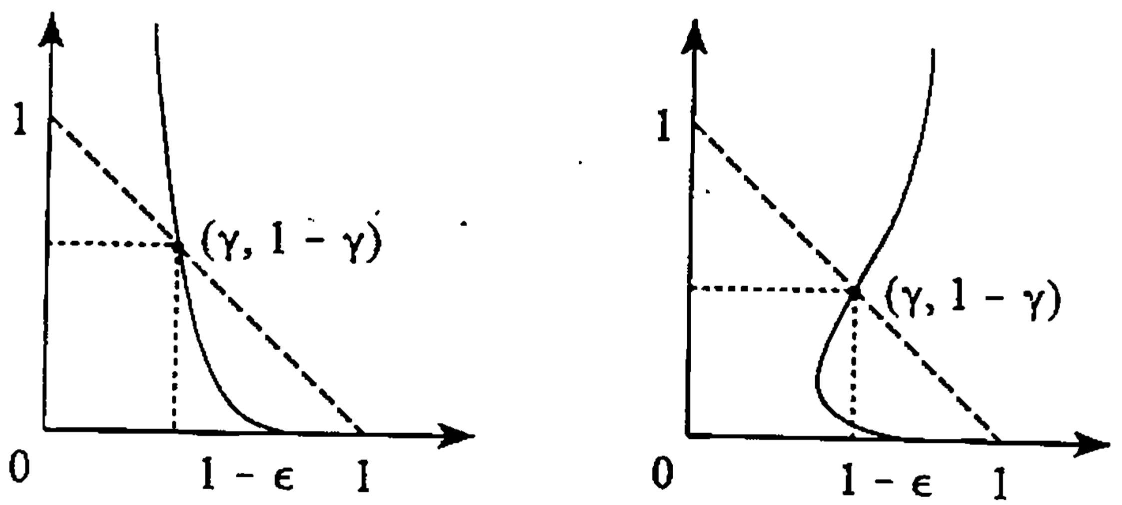


Figure 20.H.2(a)

In either case, if we plot iterates on the offer curve from any $c_{a0} \neq 1 - \gamma$ in the same way as in Figures 20.H.1 and 20.H.4, then we can see that they must unavoidably leave the picture. Hence we can conclude that the steady state is the only equilibrium.

(b) Let $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ be equilibrium consumption streams that constitute a Pareto optimum. By the same logic as in (a), we can show under condition (20.H.3) that if $c_{at} > 1 - \gamma$ for some $t \geq 0$, then the sequence of those iterates must unavoidably leave the picture. Thus, if $c_{at} \leq 1 - \gamma$ for all $t \geq 0$. As we saw in (a), condition (20.H.3) also implies that $\nabla_1 z_a(p_b, p_a) > 0$. Hence, by $z(1,1) = (\gamma, 1 - \gamma)$ and $c_{at} \leq 1 - \gamma$, $p_{t+1} \geq p_t$ for all $t \geq 1$. Thus $u(c_{bt}, c_{at}) \leq u(\gamma, 1 - \gamma)$ for all $t \geq 1$. As for $t = 0$, we have $c_{b0} = 1$ and $c_{a0} \leq 1 - \gamma$, implying that $u(c_{b0}, c_{a0}) \leq u(1, 1 - \gamma)$. Since $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ is Pareto optimal, we must have $u(c_{b0}, c_{a0}) = u(1, 1 - \gamma)$ and $u(c_{bt}, c_{at}) = u(\gamma, 1 - \gamma)$ for all $t \geq 1$. Thus $(c_{b0}, c_{a0}) = (1, 1 - \gamma)$ and $(c_{ot}, c_{at}) = (\gamma, 1 - \gamma)$ for all $t \geq 1$. The proof is thus completed.

(c) We shall prove that any such equilibrium consumption stream as in Figure 20.H.4 (as well as the monetary steady state) is Pareto optimal. So let $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$ be such a stream and $M \geq 0$ be the associated possibility for a bubble. Then $(c^*_{bt}, c^*_{at}) \to (\gamma, 1 - \gamma)$, and, as seen on page 774, $p_t c^*_{a,t-1} = M$ for every $t \geq 1$. Hence $M/p_t = c^*_{a,t-1} \to 1 - \gamma > 0$. Therefore, according to the italicized claim on page 774, the stream $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$ is Pareto optimal.

(d) According to Example 17.F.2, if $-v''(c)c/v'(c) < 1$ for all $c \geq 0$, then $\nabla_1 z_o(p_o, p_a) < 0$ and $\nabla_1 z_a(c_b, c_a) > 0$. Hence condition (20.H.3) is satisfied.

The following is a more general condition: If

$$-\frac{z_a(p_b, p_a)v''(z_a(p_b, p_a))}{v(z_a(p_b, p_a))} - 1 < p_b/p_a$$

for all (p_b, p_a), then (20.H.3) holds. Note that, since there is no initial endowment in the second period of life, the first term of the left-hand side is nothing but the same measure $-v''(c)c/v'(c)$, evaluated at the demand under (p_b, p_a). This condition thus allows the measure to be greater than one, but requires the difference to be bounded by the relative price p_b/p_a.

Let's now prove that the above condition is in fact sufficient. By the first-order condition,

$$p_a v'(z_b(p_b,p_a) + 1) = p_b \delta v'(z_b(p_b,p_a)).$$

Differentiate both sides with respect to p_b, then we obtain

$$p_a v''(z_b(p_b,p_a) + 1)(\partial z_b(p_b,p_a)/\partial p_b)$$
$$= \delta v'(z_b(p_b,p_a)) + p_b \delta v''(z_b(p_b,p_a))(\partial z_a(p_b,p_a)/\partial p_b) .$$

Differentiate Walras' law $p_b z_b(p_b,p_a) + p_a z_a(p_b,p_a) = 0$, then we obtain

$$z_b(p_b,p_a) + p_b(\partial z_b(p_b,p_a)/\partial p_b) + p_a(\partial z_a(p_b,p_a)/\partial p_b) = 0.$$

Solve those two equation with respect to $\partial z_b(p_b,p_a)/\partial p_b$ and $\partial z_a(p_b,p_a)/\partial p_b$, then we obtain

$$\partial z_b(p_b,p_a)/\partial p_b = (- \delta v'(z_a(p_b,p_a))p_a + \delta v''(z_a(p_b,p_a))p_b z_b(p_b,p_a))/\Delta,$$
$$\partial z_a(p_b,p_a)/\partial p_b = (\delta v'(z_a(p_b,p_a))p_b + v''(z_b(p_b,p_a) + 1)p_a z_b(p_b,p_a))/\Delta,$$

where

$$\Delta = - (v''(z_b(p_b,p_a) + 1)p_a^2 + \delta v''(z_a(p_b,p_a))p_b^2) > 0.$$

Hence (20.H.3) is satisfied if and only if

$$- \delta v'(z_a(p_b,p_a))p_a + \delta v''(z_a(p_b,p_a))p_b z_b(p_b,p_a)$$
$$< \delta v'(z_a(p_b,p_a))p_b + v''(z_b(p_b,p_a) + 1)p_a z_b(p_b,p_a).$$

By Walras' law, this is equivalent to

$$- \delta v'(z_a(p_b,p_a))p_a - \delta v''(z_a(p_b,p_a))p_a z_a(p_b,p_a)$$
$$< \delta v'(z_a(p_b,p_a))p_b + v''(z_b(p_b,p_a) + 1)p_a z_b(p_b,p_a).$$

Divide both sides by $\delta v'(z_a(p_b,p_a))p_a$, then we obtain

$$- 1 - \frac{z_a(p_b,p_a)v''(z_a(p_b,p_a))}{v(z_a(p_b,p_a))} < p_b/p_a + \frac{v''(z_b(p_b,p_a) + 1)z_b(p_b,p_a)}{\delta v'(z_a(p_b,p_a))p_a}.$$

Here, the second term of the right-hand side is positive by $z_b(p_b,p_a) < 0$.

Hence this is implied by

$$- 1 - \frac{z_a(p_b,p_a)v''(z_a(p_b,p_a))}{v(z_a(p_b,p_a))} < p_b/p_a.$$

But this is the sufficient condition we proposed before. The proof is thus completed.

Note that, by the first-order condition and Walras' law,

$$p_b/p_a = v'(z_b(p_b,p_a) + 1)/\delta v'(z_a(p_b,p_a)) = - z_a(p_b,p_a)/z_b(p_b,p_a).$$

Hence the righ-hand side of the above inequality can be replaced by these values, which are given in terms of excess demands and marginal utilities.

Let $\varepsilon > 0$ and suppose that the above inequality is satisfied for all (p_b,p_a) with $p_b/p_a > \varepsilon$ (but not so for some (p_b,p_a) with $p_b/p_a \le \varepsilon$). If there are some equilibria with non-steady states, then these equilibria must involve the recurrence of generations facing (p_b,p_a) with $p_b/p_a \le \varepsilon$. This is because expression (20.H.3) holds at all (p_b,p_a) with $p_b/p_a > \varepsilon$ and hence there is no equilibrium with non-steady states and $p_t/p_{t+1} > \varepsilon$ for all t.

Note also that if the inequality is satisfied for all (p_b,p_a), then the supremum of the measure $- v''(c)c/v'(c)$ must be at most one. Hence, if, for example, the measure is nondecreasing with c, then $- v''(c)c/v'(c) \le 1$ for all c and hence the offer curves looks like that in Figure 20.H.2.

Further elaborations on conditions for (20.H.3) can be found in Jean-Michel Grandmont's "On Endogenous Competitive Business Cycle" in Econometrica, Vol. 53, No. 5 (September, 1985), pp. 995 - 1045.

20.I.1 Since $z(\cdot,\cdot)$ is homogeneous of degree zero, Theorem M.B.1 implies that $\nabla z_a(\cdot,\cdot)$ and $\nabla z_o(\cdot,\cdot)$ are homogeneous of degree - 1. Hence $\nabla_2 z_a(1/\rho,1) = \rho\nabla_2 z_a(1,\rho)$ and the first equality follows. The homogeneity also implies that $\nabla_1 z_a(1,\rho) + \nabla_2 z_a(1,\rho)\rho = 0$. The second equality follows from this.

CHAPTER 21

21.B.1 (i) Symmetry: Let $\pi:\{1,2,\ldots,I\} \longrightarrow \{1,2,\ldots,I\}$ be any permutation (onto), then $\sum_{i=1}^{I}\alpha_i = \sum_{i=1}^{I}\alpha_{\pi(i)}$, which implies $\text{Sign}\left(\sum_{i=1}^{I}\alpha_i\right) = \text{Sign}\left(\sum_{i=1}^{I}\alpha_{\pi(i)}\right)$ which in turn implies that $F(\alpha_1,\alpha_2,\ldots,\alpha_I) = F(\alpha_{\pi(1)},\alpha_{\pi(2)},\ldots,\alpha_{\pi(I)})$.

(ii) Neutrality: $F(\alpha_1,\alpha_2,\ldots,\alpha_I) = \text{Sign}\left(\sum_{i=1}^{I}\alpha_i\right) = \text{Sign}\left(-\sum_{i=1}^{I}(-\alpha_i)\right) = -\text{Sign}\left(\sum_{i=1}^{I}(-\alpha_i)\right) = -F(-\alpha_1,-\alpha_2,\ldots,-\alpha_I)$.

(iii) Positive Responsiveness: Assume that $F(\alpha_1,\ldots,\alpha_I) \geq 0$, then $\text{Sign}\left(\sum_{i=1}^{I}\alpha_i\right) \geq 0$, which implies that $\sum_{i=1}^{I}\alpha_i \geq 0$. Take $(\alpha'_1,\ldots,\alpha'_I) \geq (\alpha_1,\ldots,\alpha_I)$ such that $(\alpha'_1,\ldots,\alpha'_I) \neq (\alpha_1,\ldots,\alpha_I)$. Then, $\sum_{i=1}^{I}\alpha'_i > 0$, which implies that $\text{Sign}\left(\sum_{i=1}^{I}\alpha'_i\right) > 0$, which in turn implies that $F(\alpha'_1,\ldots,\alpha'_I) = 1$.

21.B.2 (i) Let I=2. A Dictatorship functional $F(\alpha_1,\alpha_2) = \alpha_1$ is both neutral between alternatives and positively responsive, but is not symmetric.

(ii) A constant functional $F(\alpha_1,\ldots,\alpha_I) = 1$ for all $(\alpha_1,\ldots,\alpha_I)$ is both symmetric and positive responsive, but is not neutral between alternatives.

(iii) A constant functional $F(\alpha_1,\ldots,\alpha_I) = 0$ for all $(\alpha_1,\ldots,\alpha_I)$ is both symmetric and neutral between alternatives, but is not positive responsive.

21.B.3 Assume majority voting implements k = 1. Then, there exists i^* such that $v_{i^*} > 0$ (if not, either all $v_i = 0$ and k = 1 is Pareto optimal and this shows the result, or $v_i \leq 0$ for all i and $v_i < 0$ for some i which implies that k = 0 should have been the case). Now, if the decision is reversed to k = 0, then agent i^* is made worse off, which implies that k = 1 was indeed Pareto optimal. A symmetric argument hods for majority voting implementing k = 0.

Now consider transfers between agents. An example can easily be constructed to show that majority voting is not Pareto optimal. Let $I = 3$, $v_1 = v_2 = -1$ and $v_3 = 5$. Majority voting will implement $k = 0$, but if the agents were allowed to choose $k = 1$ and agent 3 would pay $1 + \varepsilon$ to each of the other two agents, then all agents would be strictly better off.

This contrast is due to the nature of quasilinear preferences. In such a case we can perform interpersonal comparisons of utility (use the "mean"), which is a cardinal property, while majority voting is only considered with ordinality (which is the "median").

21.C.1 <u>Claim:</u> If for some $\{x,y\} \subset X$, $S \subset I$ is decisive for x over y, then for ant third alternative z, S is decisive for z over y.

<u>Proof:</u> Consider a preference profile $(\succsim_1,\ldots,\succsim_I)$ satisfying $z \succ_i x \succ_i y$ for all $i \in S$, and $y \succ_i z \succ_i x$ for all $i \in I \backslash S$. Since S is decisive for x over y we must have $x\ F_p(\succsim_1,\ldots,\succsim_I)\ y$. Since $z \succ_i x$ for every i, by the Pareto property we must have $z\ F_p(\succsim_1,\ldots,\succsim_I)\ x$, therefore, by transitivity we must have $z\ F_p(\succsim_1,\ldots,\succsim_I)\ y$. Moreover, by pairwise independence this relation should hold regardless of individual ranking of x, which implies that whenever $z \succ_i y$ for all $i \in S$, and $y \succ_i z$ for all $i \in I \backslash S$, we should have $z\ F_p(\succsim_1,\ldots,\succsim_I)\ y$, i.e., S is decisive for z over y.

21.C.2 (a) Majority voting with two alternatives.

(b) Majority voting when the domain is single peaked (see section 21.D for more on this).

(c) Majority voting with more than two alternatives: The Condorcet paradox illustrates how the resulting social welfare functional can have cycles when the domain is unrestricted (see example 21.C.2 in the textbook).

(d) The Borda count does not satisfy pairwise independence (see the continuation of example 21.C.1 in the textbook).

(e) A constant social welfare functional (i.e., which is independent of the preference profile).

(f) A dictatorial social welfare functional, i.e., choose agent 1 as the dictator, and $x \ F_p(\succsim_1,\ldots,\succsim_I) \ y$ if and only if $x \succ_1 y$.

21.C.3 Let X be the finite set of alternatives, and let there be I individuals. Consider the following social welfare functional :

$x \ F_p(\succsim_1,\ldots,\succsim_I) \ y$ if one of the following hold:

(1) $x \succ_1 y$

(2) $x \sim_1 y$ and $x \succ_2 y$

(3) $x \sim_1 y$, $x \sim_2 y$ and $x \succ_3 y$

$\vdots$

(I) $x \sim_i y$ for all $i = 1,\ldots,I-1$ and $x \succ_I y$

This is a *lexical dictatorship* social welfare functional in the sense that agent 1 gets to determine the strict preference, but if he is indifferent then agent 2 decides, and so on, and if all are indifferent then the two alternatives are socially indifferent. It is easy to see that for all profiles of preferences where there is some indifference for all agents, and the preferences of the agents do not coincide then the social choice functional will not be identical to any of the individual preferences.

21.D.1 Since X is finite, let N be the total number of elements in X, and we can therefore order its elements as $\{x_1,x_2,\ldots,x_n,\ldots x_N\}$. Define the preference relation $\succsim$ as follows:

(i) For any $X' \subset X$, $x_i \in X'$ implies $x_i \sim x_i$.

(ii) For any $X' \subset X$ such that $x_1 \notin X'$ or $x_N \notin X'$ then for all $\{x_i, x_j\} \subset X'$,

$x_i \succ x_j$ if and only if $j > i$.

(iii) For any $X' \subset X$ such that $x_1 \in X'$ and $x_N \in X'$ then $x_N \succ x_1$, and for all

$\{x_i, x_j\} \subset X'$ such that $N \notin \{i, j\}$, $x_i \succ x_j$ if and only if $j > i$.

Clearly, this preference relation is complete and reflexive. Furthermore, for any strict subset of X it has a maximal element: For subsets not including both x_1 and x_N then x_i with the smallest i is the maximal element, while for subsets including both x_1 and x_N then x_N is the maximal element. This preference is clearly complete, and it is reflexive. Furthermore, it has a maximal element on every strict subset of X. Note, however, that it is not acyclic.

21.D.2 The preferences generated from oligarchy are clearly quasitransitive: If $x\ F_p(\succsim_1,\ldots,\succsim_I)\ y$ and $y\ F_p(\succsim_1,\ldots,\succsim_I)\ z$, this means that for all $h \in S$ (the members of the oligarchy) we have $x \succ_h y$ and $y \succ_h z$. Since the preferences of each individual are taken from $\mathcal{R}$, transitivity implies that $x \succ_h z$ for all $h \in S$, which in turn implies that $x\ F_p(\succsim_1,\ldots,\succsim_I)\ z$. To see that social indifference may not be transitive, let $I = \{1,2,\ldots,I\}$ be the set of individuals and let $S = \{1,2\}$ be the oligarchy. Further assume that the current preferences are $x \succ_1 z \succ_1 y$ and $y \succ_2 x \succ_2 z$. From the definition of the oligarchic social preferences we have $x\ F(\succsim_1,\ldots,\succsim_I)\ y$ and $y\ F(\succsim_1,\ldots,\succsim_I)\ x$, so that x and y are socially indifferent, and we also have that $y\ F(\succsim_1,\ldots,\succsim_I)\ z$ and $z\ F(\succsim_1,\ldots,\succsim_I)\ y$, so that y and z are socially indifferent. However, since both $x \succ_1 z$ and $x \succ_2 z$, we have that $x\ F_p(\succsim_1,\ldots,\succsim_I)\ z$, which shows that social indifference is not transitive. This can happen as long as there are more than one member in the oligarchy. They can all agree that one alternative is strictly better than some other, but when comparing each of these two alternatives to all other alternatives there can be disagreement.

21.D.3 Clearly, the generated social preferences are acyclic: for all possibilities of individual preferences, agent 1's most preferred alternative will be at least as good (but not necessarily <u>better</u>) than any other alternative. To see that quasitransitivity may not hold, let the preferences be $x \succ_1 z \succ_1 y$ and $y \succ_2 x \succ_2 z$. From the definition of social preferences we have that $x\ F_p(\succsim_1,\ldots,\succsim_I)\ z$ and $z\ F_p(\succsim_1,\ldots,\succsim_I)\ y$, but we do not have $x\ F_p(\succsim_1,\ldots,\succsim_I)\ y$ because individual 2 can veto this. Note also, that in spite of the veto power of 2, x is the only maximal element in this case: we have that $x\ F_p(\succsim_1,\ldots,\succsim_I)\ z$ and $x\ F(\succsim_1,\ldots,\succsim_I)\ y$. However, y is not a maximal element since $z\ F_p(\succsim_1,\ldots,\succsim_I)\ y$, which happens because individual 2 cannot veto z preferred to y.

21.D.4 Assume $\succsim$ is single peaked with $x^* \in [0,1]$ as the peak alternative. Assume that y and z are two distinct alternatives, neither of them equals x^*, such that $y \sim z$. We need to show that for all $\lambda \in (0,1)$, $\lambda y + (1-\lambda)z \succ y$. First, we must have either $y < x^* < z$ or $z < x^* < y$ (or else we could not have $y \sim z$). W.l.o.g. assume that $y < x < z$, and choose some $\lambda \in (0,1)$. Letting $x' = \lambda y + (1-\lambda)z$, we either have that $x' \in (y,x^*]$, in which case $x' \succ y$, or we have that $x' \in (x^*,z)$, in which case $x' \succ z$, which implies that $x' \succ y$.

21.D.5 The six possible orderings are: (1) (x,y,z); (2) (x,z,y); (3) (y,x,z); (4) (y,z,x); (5) (z,x,y); and (6) (z,y,x). Recall that the preferences of the Condorcet paradox are $x \succ_1 y \succ_1 z$, $z \succ_2 x \succ_2 y$, and $y \succ_3 z \succ_3 x$. Agent 1's preferences are not single peaked for orderings (2) and (4), agent 2's preferences are not single peaked for orderings (1) and (6), and agent 3's preferences are not single peaked for orderings (3) and (5).

21.D.6 Let $X = \{x,y,z\}$, $I = \{1,2,3,4\}$, and consider the following profile of preferences: $x \succ_1 y \succ_1 z$, $z \succ_2 y \succ_2 x$, $x \succ_3 z \succ_3 y$, and $y \succ_4 x \succ_4 z$. We thus have $\#\{i : x \succ_i y\} = \#\{i : y \succ_i x\} = \#\{i : z \succ_i y\} = \#\{i : y \succ_i z\} = 2$, which implies that x is socially indifferent to y, and y is socially indifferent to z, so we can write $z\ \hat{F}(\succsim_1,\ldots,\succsim_6)\ y$ and $y\ \hat{F}(\succsim_1,\ldots,\succsim_6)\ x$. It is not true, however, that $z\ \hat{F}(\succsim_1,\ldots,\succsim_6)\ x$ since $\#\{i : x \succ_i z\} = 3$ and $\#\{i : z \succ_i x\} = 1$ which implies $x\ \hat{F}_p(\succsim_1,\ldots,\succsim_6)\ z$. Therefore, in this case majority voting fails to generate a fully transitive social welfare functional.

21.D.7 (a) Since the cone spanned by $\{\nabla u_1(0),\nabla u_2(0),\nabla u_3(0)\}$ is the entire space $\mathbb{R}^2$, we must have no linear dependence between the $\nabla u_i(0)$'s, and the directions of the gradients must be as drawn in Figure 21.D.7(a1) (they cannot all point in directions that can be "bounded" by some straight line going through the origin). The dashed lines p_1, p_2, and p_3 are the perpendicular lines to the direction of each gradient respectively, and the indifference curve of agent i that passes through the origin is tangent to p_i. These indifference curves are shown in Figure 21.D.7(a2).

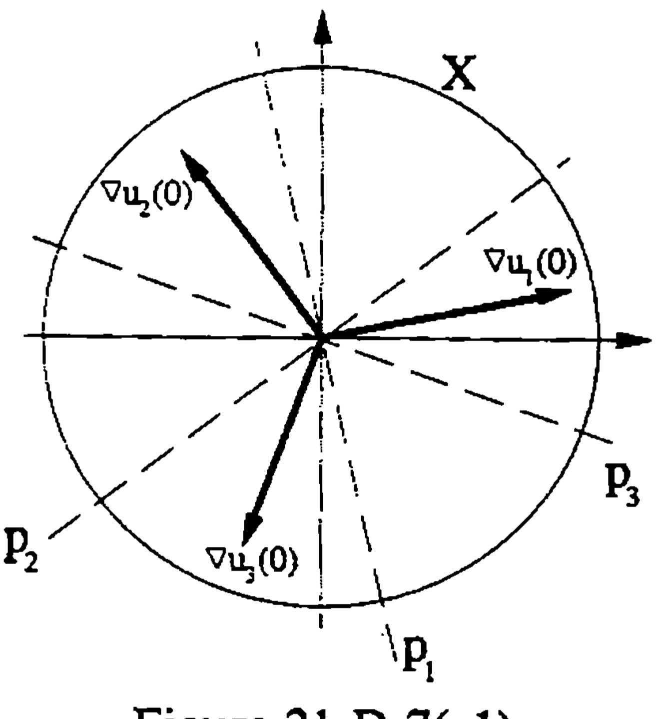


Figure 21.D.7(a1)

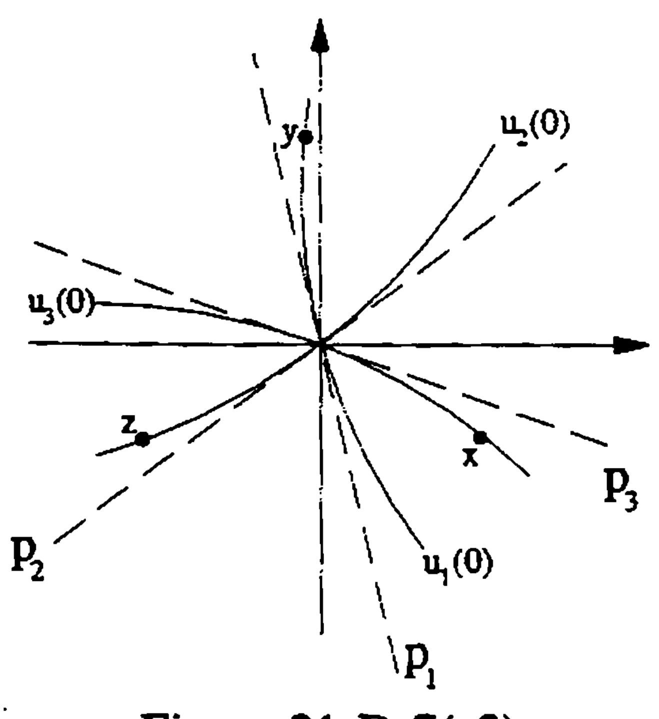


Figure 21.D.7(a2)

(We show convex preferences though they can be linear, as used for part **(b)** of this exercise.) Given the directions of the gradients, we can find three alternatives x, y, and z as shown in the figure for which $x \succ_1 y \succ_1 z$, $y \succ_2 z \succ_2 x$, and $z \succ_3 x \succ_3 y$, and a Condorcet cycle exists.

(b) (We restrict attention to linear preferences which is not too restrictive since close enough to the origin, any indifference curve going through the origin will approach a linear line.)

Claim: Given a point $x \in \mathbb{R}^2$, either one or two gradients $\nabla u_i(0)$ pass through the circle centered at x, with radius $\|x\|$.

proof: Assume in negation that none, or all three gradients would pass through the circle. Then they would all be one the same side of the dashed line in Figure 21.D.7(b1), and they would not span the entire $\mathbb{R}^2$ by their cones, a contradiction to our assumption. □

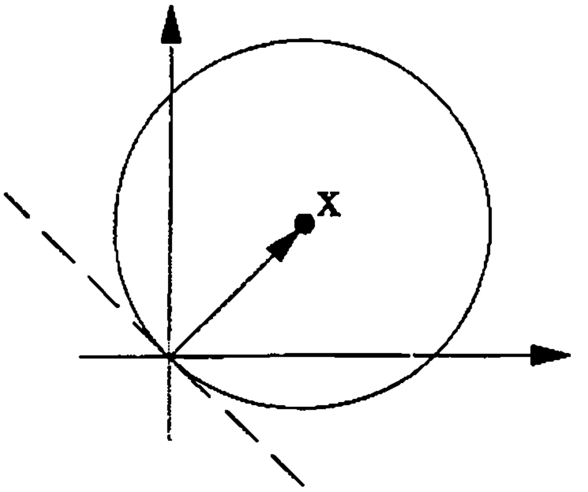


Figure 21.D.7(b1)

Thus, we divide the proof into two cases:

Case 1: Two gradients pass through. Assume w.l.o.g. that $\nabla u_1(0)$ and $\nabla u_2(0)$ pass through the circle as shown in Figure 21.D.7(b2).

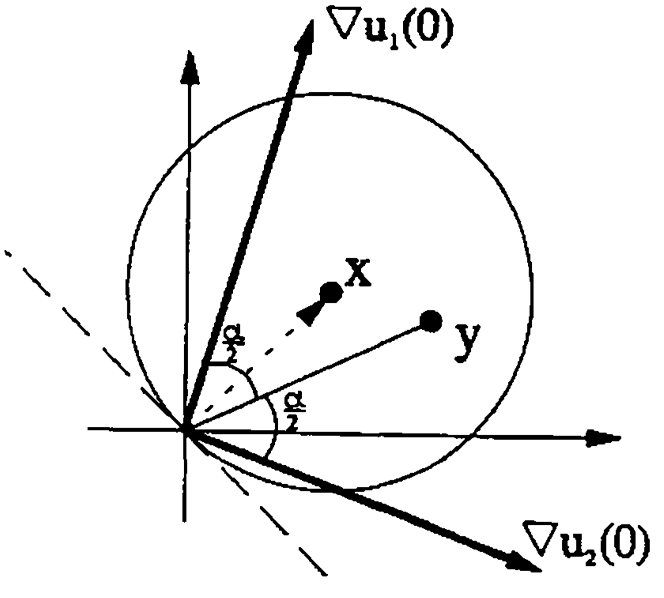


Figure 21.D.7(b2)

Then the angle between both gradients must be some $\alpha < 180^\circ$. Now bisect α and let y be on this bisecting line so that $\|x-y\| < \|x\|$ (i.e., y is within the circle). Since $\frac{\alpha}{2} < 90^\circ$ we must have that both $\nabla u_1(0)\cdot y > 0$ and $\nabla u_2(0)\cdot y > 0$, which implies that $u_1(y) > u_1(0)$ and $u_2(y) > u_2(0)$. This concludes case 1.

Case 2: One gradient passes through. Assume w.l.o.g. that $\nabla u_1(0)$ passes through the circle as shown in Figure 21.D.7(b3).

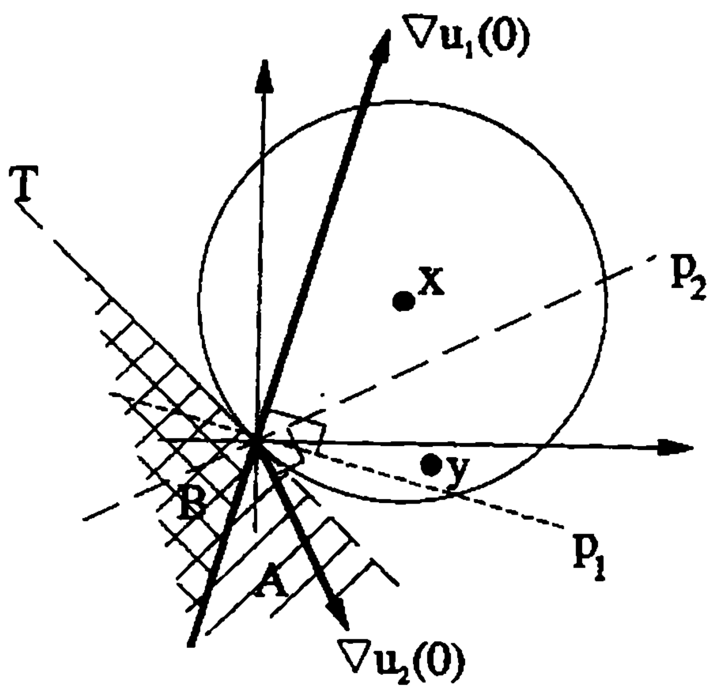


Figure 21.D.7(b3)

The dashed line T is tangent to the circle, so that $\nabla u_2(0)$ and $\nabla u_3(0)$ must pass below the line T. By the spanning condition, we must have that either $\nabla u_2(0)$ or $\nabla u_3(0)$ will be strictly inside the region A, and the remaining gradient will be strictly inside the region B. Assuming w.l.o.g that $\nabla u_2(0)$ is in the region A, let p_1 and p_2 be the perpendicular lines to $\nabla u_1(0)$ and $\nabla u_2(0)$ respectively, and there must be (by the spanning condition) an "area" between these lines so that we can choose a point y in this area such that $\nabla u_1(0)\cdot y > 0$ and $\nabla u_2(0)\cdot y > 0$, which implies that $u_1(y) > u_1(0)$ and $u_2(y) > u_2(0)$. This concludes case 1.

Remark: The idea in both cases is finding two gradients with an angle less than 180° between them.

21.D.8 (a) The situation is depicted in Figure 21.D.8(a) below:

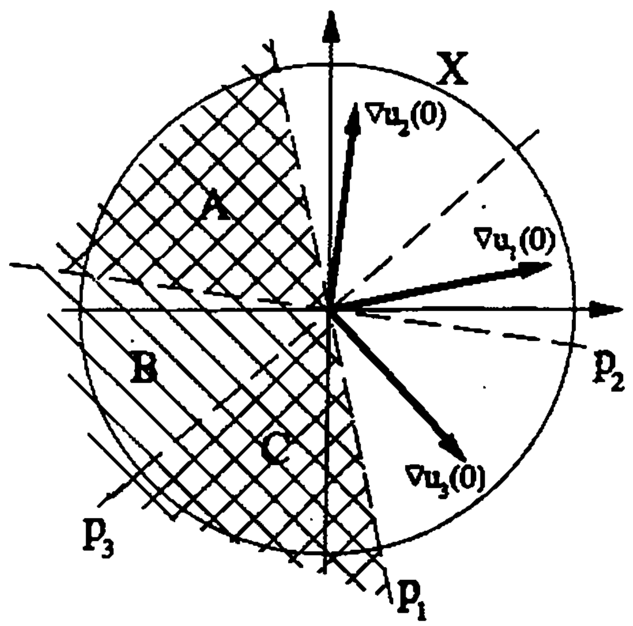


Figure 21.D.8(a)

Since the gradients form a pointed cone, assume w.l.o.g. that agent 1's gradient is in the "middle" as shown in the figure. Letting P_i denote the perpendicular line to the gradient of agent i at the origin, we show that no alternative to the left of P_1 can be preferred by a majority to the origin. Any such alternative must be either in region A, or in B, or in C. The origin is preferred to any alternative in region A by both agents 1 and 3. Similarly, the origin is preferred to any alternative in region C by both agents 1 and 2, and finally, the origin is preferred to any alternative in region A by all agents. Therefore, any alternative preferred to the origin will be preferred by agent 1 who is a directional median. (Note that alternatives in the "slice" opposite from A will be preferred to the origin by agents 1 and 3, and similarly for the other slices.)

(b) In Figure 21.D.8(b) we show the desired situation. Agent 1 has linear indifference curves with the (fixed) gradient $\nabla u_1(\cdot)$ (the line with long dashes), agent 2 has linear indifference curves with the (fixed) gradient $\nabla u_2(\cdot)$ (the line with short dashes), and agent 3 has convex indifference curves with the (variable) gradient $\nabla u_3(\cdot)$ (the line with long dashes). We have three alternatives, A, B, and C, such that $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, and $C \succ_3 A \succ_3 B$.

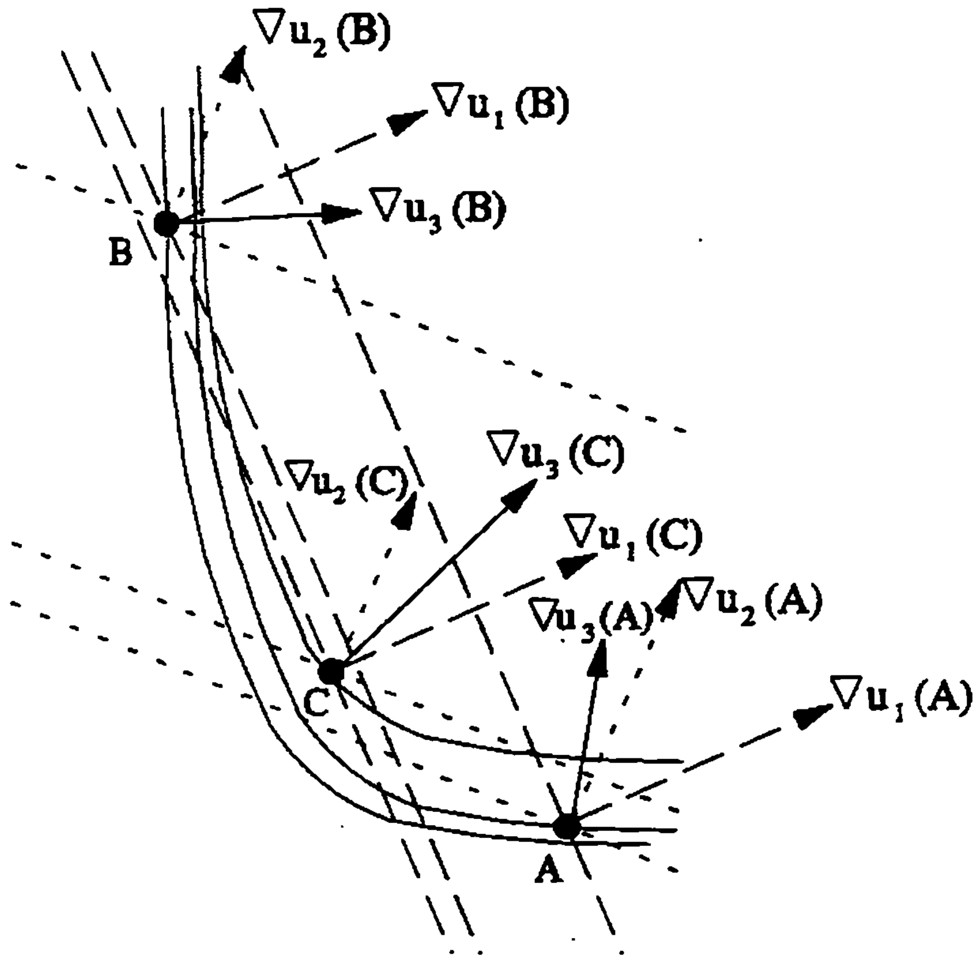


Figure 21.D.8(b)

At each point we have a different median agent, and we clearly have a Condorcet cycle.

(c) The result follows immediately from (a) above and from the transitivity of individual preferences: Assume w.l.o.g. that agent 1 is a median agent for all alternatives. If $x\ F_p\ y$ and $y\ F_p\ x$ then we must have from part (a) that $x \succ_1 y$ and $y \succ_1 z$. Assume in negation that $z\ F_p\ x$. Then (a) above implies that $z \succ_1 x$, but this contradicts the transitivity of $\succ_1$, therefore we must have $x\ F_p\ z$.

21.D.9 (a) Assume that $u(x) \geq u(y)$ and $u'(x) \geq u'(y)$. Then, since γ and ψ

are positive we have that $\psi u(x) + \gamma u'(x) \geq \psi u(y) + \gamma u'(y)$, i.e., the preference relation represented by $\psi u(\cdot) + \gamma u'(\cdot)$ is intermediate between the preferences represented by $u(\cdot)$ and by $u'(\cdot)$.

(b) For every pair (x,y) define the half space,

$$A(x,y) \equiv \{\beta \in \mathbb{R}^N: \beta\cdot(a_x - a_y) > 0\}$$

where $a_x = (h_1(x),\ldots,h_N(x))$ and $a_y = (h_1(y),\ldots,h_N(y))$. We can thus write $B(x,y) = \mathbb{R}^N_{++} \cap A(x,y)$.

(c) <u>Claim 1:</u> The set $\bar{B}(x,y) \equiv \{\beta \in \mathbb{R}^N: u_\beta(x) \geq u_\beta(y)\}$ is convex.

<u>Proof:</u> Assume that β and β' are both in the set $\bar{B}(x,y)$, that is, $u_\beta(x) \geq u_\beta(y)$ and $u_{\beta'}(x) \geq u_{\beta'}(y)$. Because the preferences represented by β'' are intermediate between those represented by β and β', we have $u_{\beta''}(x) \geq u_{\beta''}(y)$, so $\beta'' \in \bar{B}(x,y)$.

<u>Claim 2:</u> The set $\bar{L}(x,y) \equiv \{\beta \in \mathbb{R}^N: u_\beta(x) \leq u_\beta(y)\}$ is convex.

<u>Proof:</u> Symmetric to the proof of claim 1.

Assuming that $u_\beta(\cdot)$ is continuous in β, then the conclusion of part (b) above is still correct as follows: From Claims 1 and 2 above, it must be that $\bar{B}(x,y) \cap \bar{L}(x,y)$ is convex. However, $\bar{B}(x,y) \cap \bar{L}(x,y)$ can be convex only if $\bar{B}(x,y)$ and $\bar{L}(x,y)$ are translated half spaces. Of course, if β is restricted to $\mathbb{R}^N_{++}$ then $B(x,y)$ must also be restricted to $\mathbb{R}^N_{++}$. (See Figure 21.D.9(c))

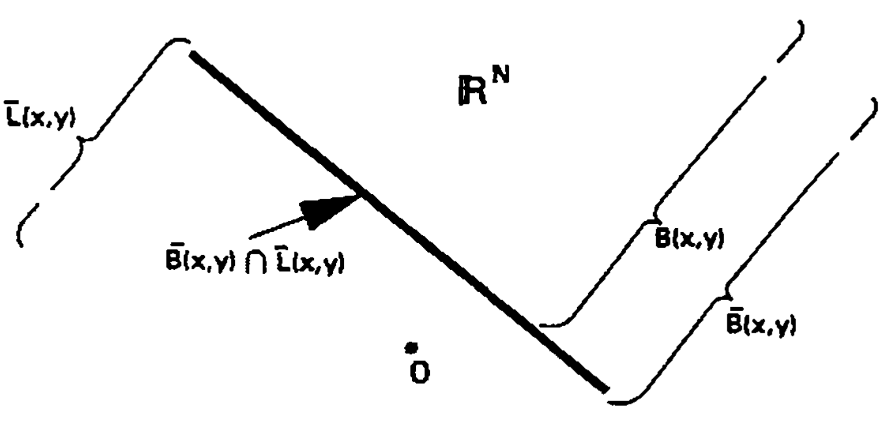


Figure 21.D.9(c)

(d) Existence of a median agent: Take N=2 and consider a density $g(\cdot)$ that is uniform over a circle $K \subset \mathbb{R}^2_{++}$ with β^* being the center of the circle (see Figure 21.D.9(d1)). It is straightforward to see that β^* is a median agent.

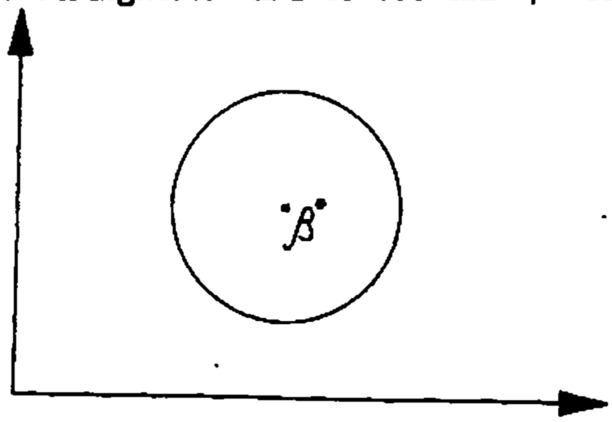


Figure 21.D.9(d1)

Non-existence of a median agent: Take N=2 and consider a density $g(\cdot)$ that is uniform over an equi-triangle $A \subset \mathbb{R}^2_{++}$ where the length of each of the triangle's edges is a, as depicted in Figure 21.D.9(d2)

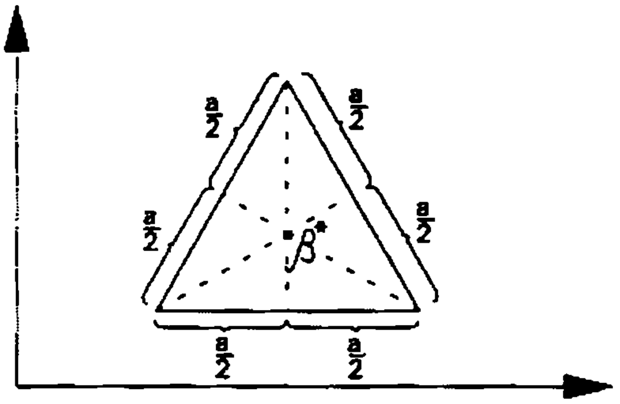


Figure 21.D.9(d2)

The only candidate for a median agent would be β^* shown in Figure 21.D.9(d2). However, if we draw a horizontal line through β^* as shown in figure 21.D.9(d3) then it splits the triangle into two unequal parts, which implies that β^* cannot be a median agent.

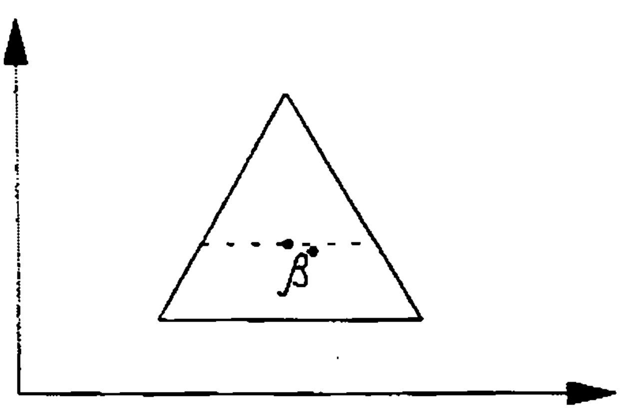


Figure 21.D.9(d3)

(e) Consider the situation depicted in Figure 21.D.9(e) below:

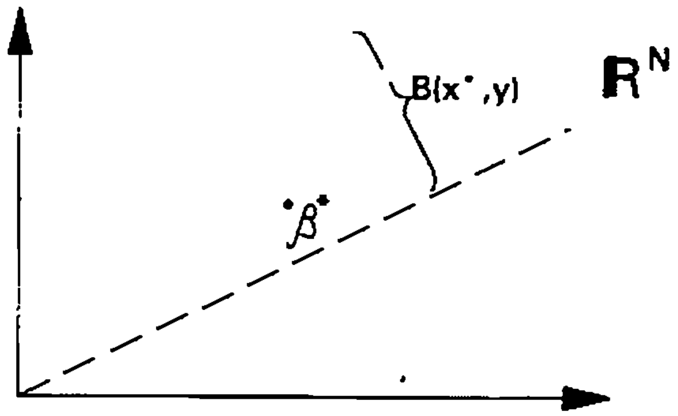


Figure 21.D.9(e)

From part (b), for any alternative y, $B(x^*,y)$ is a half-space. Furthermore, because x^* is the single most preferred alternative of the median agent we must have that $\beta^* \in B(x,y)$. Since β^* is a median agent, and $g(\beta^*) > 0$, then the mass of agents in the region $B(x^*,y)$ is greater than $\frac{1}{2}$, hence, x^* defeats y.

(f) Let $X = \mathbb{R}^N$ be the set of alternatives. For any $x \in X$ consider the N + 1 functions given by,

$$h_0(x) = -\sum_{n=1}^{N} x_n^2 ,$$

$$h_1(x) = 2x_1 ,$$

$$h_2(x) = 2x_2 ,$$

$$\vdots$$

$$h_N(x) = 2x_N .$$

For $\beta \in \mathbb{R}^N_{++}$ define,

$$u_\beta(x) \equiv h_0(x) + \beta_1 h_1(x) + \beta_2 h_2(x) + \cdots + \beta_N h_N(x)$$

This function is as in part (b) above, except that β is not restricted to be non-negative, and that given x, there is a constant term $h_0(x)$. This does not change any of the conclusions of parts (b) and (c) above. Computing $u_\beta(x)$ explicitly,

$$u_\beta(x) = -\sum_{n=1}^{N} x_n^2 + 2\beta_1 x_1 + 2\beta_2 x_2 + \cdots + 2\beta_N x_N$$

$$= -\sum_{n=1}^{N} (x_n - \beta_n)^2 + \sum_{n=1}^{N} \beta_n^2$$

$$= -\| x - \beta \|^2 + \sum_{n=1}^{N} \beta_n^2$$

But notice that $= -\| x - \beta \|^2 + \sum_{n=1}^{N} \beta_n^2$ is an increasing transformation of $-\| x - \beta \|^2$, that is, both functions represent the same preferences over X. Therefore, this is exactly the same model as in Example 21.D.6.

21.D.10 [First Printing Errata: The sentence starting in the fourth line of part (c) should read "Show that preferences over *X* are..."]

(a) As w^* and $\bar{w}$ become further apart, this means that the median is moving away from the mean, or in other words we are moving away from equality.

(b) Each agent's "peak", t_i, will be chosen to maximize $(1-t)w_i + t\bar{w}$. Clearly, if $w_i > \bar{w}$ then $t_i = 0$, and if $\bar{w} > w_i$ then $t_i = 1$. If $w^* > \bar{w}$ then more than half the agents prefer $t = 0$ to any other alternative, and if $w^* < \bar{w}$ then the opposite is true.

(c) Now, to find t_i we maximize $(1-t)^2 w_i + t(1-t)\bar{w}$. The FOC is: $-2w_i + 2tw_i + \bar{w} - 2t\bar{w} = 0$, and the SOC is satisfied if $2w_i - 2\bar{w} < 0$, i.e., if $w_i < \bar{w}$. Clearly, for $w_i > \bar{w}$ we will have $t_i = 0$. To solve for t_i for all agents such that $w_i < \bar{w}$, the FOC is necessary and sufficient and yields $t_i = \dfrac{w_i - \frac{1}{2}\bar{w}}{w_i - \bar{w}}$, which gives us $t_i \in [0, \frac{1}{2}]$ for $w_i \in [0, \frac{1}{2}\bar{w}]$, and due to the non-negativity of t_i we have $t_i = 0$ for $w_i > \frac{1}{2}\bar{w}$. So, the above implies that $t_c \in [0, \frac{1}{2}]$. Furthermore, if $w^* \in [0, \frac{1}{2}\bar{w})$, then $t_c \in (0, \frac{1}{2}]$, and if $w^* > \frac{1}{2}\bar{w}$ then $t_c = 0$. The difference with part (b) is that here the dead weight loss causes agents to be more averse to taxes. As a tax is levied, it both decreases one's own initial wealth, and it decreases the median initial wealth.

(d) To find t_i we maximize $(1-t)(1-t^2)w_i + t(1-t^2)\bar{w}$, which yields the FOC: $3(w_i - \bar{w})t^2 - 2tw_i - (w_i - \bar{w}) = 0$. Clearly, this FOC can have multiple solutions so that more than "peak" can exist for each individual.

21.D.11 Consider the following example: Let I = 5, and consider a mechanism of pairwise majority voting with the modification that agents 1 and 2 have a weight of two votes each, while the other agents (3,4, and 5) count once (as in regular voting). That is, $x\ F_p(\succsim_1,\ldots,\succsim_5)\ y$ if and only if $\sum_i \alpha_i \beta_i > 0$ where,

$$\alpha_i = \begin{cases} +1 & \text{if } x \succ_i y \\ -1 & \text{if } y \succ_i x \end{cases}, \text{ and } \quad \beta_i = \begin{cases} 1 & \text{if } i \in \{3,4,5\} \\ 2 & \text{if } i \in \{1,2\} \end{cases}.$$

This is a social welfare functional because it is equivalent to a majority voting mechanism in a society with 7 agents, which consists of our 5 agents with one more agent of types 1 and 2, and by Proposition 21.D.2 this indeed gives a social ordering. Furthermore, the social welfare functional is clearly pairwise independent, and Paretian, but it is not equivalent to pairwise majority voting on our society of the 5 agents.

21.D.12 (a) Consider the alternative $(t_1^*, t_2^*, t_3^*) \gg 0$, $t_1^* + t_2^* + t_3^* = c$. The following alternative is preferred by agents 1 and 2: $(t_1', t_2', t_3') = (0, t_2^* - \varepsilon, t_3^* + t_1^* + \varepsilon)$. Similarly we can find other alternatives preferred by 1 and 3, or by 2 and 3.

(b) With $(t_1^*, t_2^*, t_3^*) \geq 0$ such that $t_i^* = 0$ for <u>one</u> i only, we have the same problem as in (a) above: reduce the tax paid by each of the two agents by ε, and increase the tax of agent i by 2ε; this is preferred by both agents with positive initial payments. If, however, two agents have $t_i^* = 0$, then this is a "weak" Condorcet winner. this alternative will be preferred by the two agents who pay nothing to any alternative in which they pay something. For alternatives in which one of the two still pay nothing, and the third agent pays less while one of the two pays out the difference, we have social indifference - the agent who still pays nothing will be indifferent and the agent who pays less will prefer the new alternative. If we ignore indifference in the sense that a status quo is changed only if a majority strictly prefers the new alternative, then any (t_1^*, t_2^*, t_3^*) with two agents paying 0 will be a Condorcet winner.

21.D.13 (a) Suppose in negation that $y \notin \{x_1, \ldots, x_J\}$. Since J is finite, we can trace a line through the point y which does not contain any of the points $\{x_1, \ldots, x_J\}$ (see Figure 21.D.13(a1))

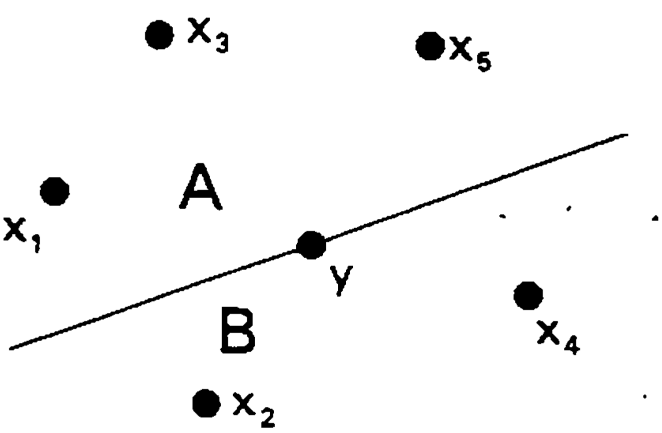


Figure 21.D.13(a1)

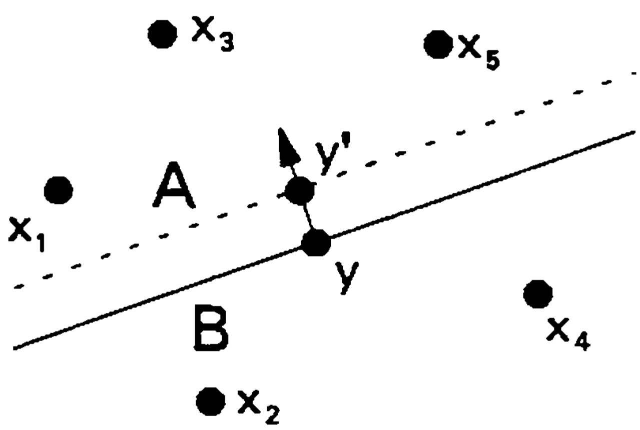


Figure 21.D.13(a2)

Denote by A and B the two (strict) half planes that lie on both sides of the line traced through y. Because J is odd, we cannot have the same number of most preferred points, and say (w.l.o.g.) that A contains more. Since we have $\alpha_1 = \alpha_2 = \cdots = \alpha_J$, then the mass of agents in A is larger than that in B. Therefore, any alternative y′ that is a small displacement of y into the region A, will defeat y (see Figure 21.D.13(a2)), hence, y could not have been a Condorcet winner.

Note that if J is even this does not hold. Figure 21.D.13(a3) demonstrates this for I = 4 and $\alpha_i = 1/4$ for i = 1,2,3,4.

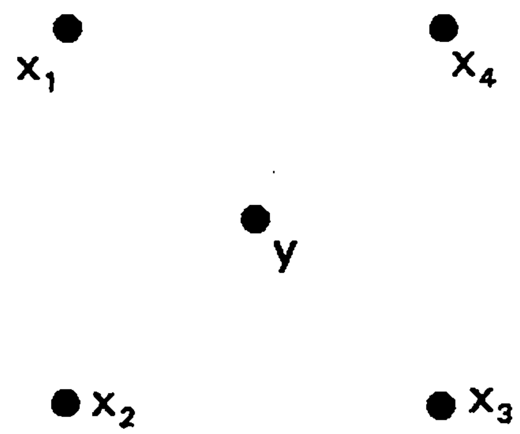


Figure 21.D.13(a3)

(b) Consider any point $y \notin x_h$ and trace the straight line that connects y with x_h. By the "general position" assumption, there can be at most one other preferred point on this line. Denote by A and B the two half spaces created by

this line and by C the closed half-segment of the line created by the point y, and that does not contain x_h. (See Figure 21.D.13(b1))

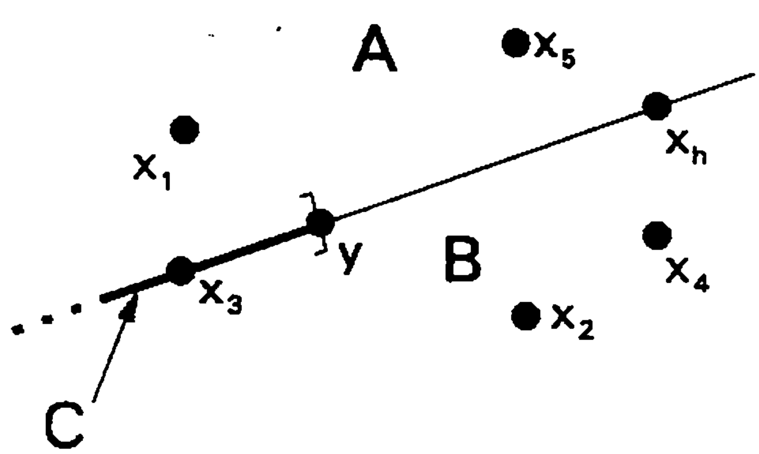


Figure 21.D.13(b1)

We must have that the mass of C is less than α_h (because h is dominant and C contains at most one other type). Also, we either have mass(A) ≥ mass(B), or the reverse. The following argument is similar to the one in part (a) above.

Assume w.l.o.g. that mass(A) ≥ mass(B) and consider a point y' that is in region A, close to the point y, in such a way that $(y' - y)$ creates an acute (but almost straight) angle with $(x_h - y)$ as depicted in Figure 21.D.13(b2).

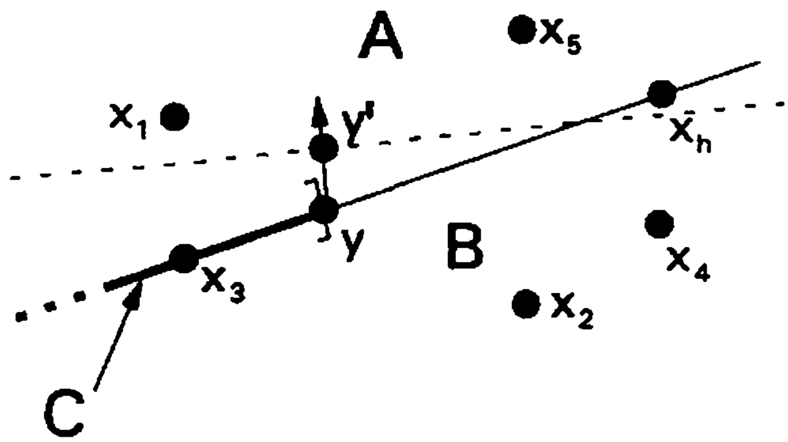


Figure 21.D.13(b2)

Then, mass(A) + α_h > mass(B) + mass(C), which implies that y' defeats y. Therefore y could not have been a Condorcet winner.

21.D.14 (a) Due to indifference we can view majority voting in one of two ways: if agent i is indifferent between alternatives x and y, he will either vote for both "x preferred to y" and for "y preferred to x", or he will abstain from voting at all. Considering the first voting procedure, and given the preferences of the five agents, x is defeated by w (2 vs. 3), y is defeated by w (3 vs. 4), z is defeated by x (3 vs. 4), v is defeated by z (3 vs. 4), and w is defeated by v (3 vs. 4).

(b) The linear order $x < y < z < v < w$ will satisfies this problem. It is easy to verify this by drawing the preferences of each agent given this order.

(c) The following utility functions shown in Figure 21.D.14(c) for each agent will give rise to the preferences over the alternatives given in the exercise (the utility functions are actually weakly concave due to the "weak" single peakedness):

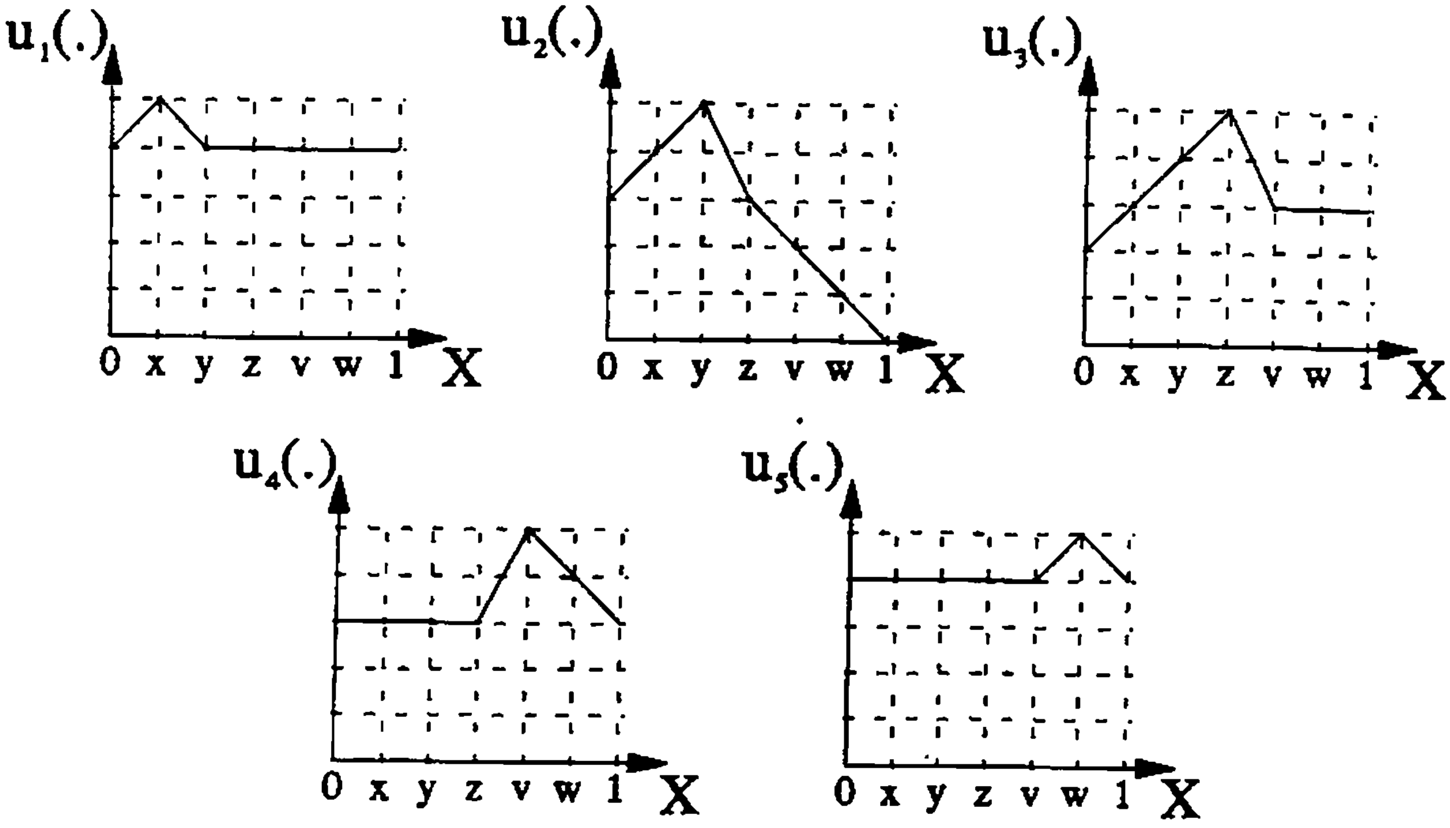


Figure 21.D.14(c)

(d) We construct an example which is an extension of the preferences given to a continuum of alternatives, where the "gaps" are filled with indifference. In particular let $X = [0,1]$, and let the agents' utility functions be given by the graphs in Figure 21.D.14(d). For example, the utility function for 3 is:

$$u_3(x) = \begin{cases} 3 & \text{for } 0 \le x < 0.2 \\ 4 & \text{for } 0.2 \le x < 0.4 \text{ and for } 0.4 < x \le 1 \\ 5 & \text{for } x = 0.4 \end{cases}$$

Similarly for the other agents. Note that these utility functions are quasiconcave.

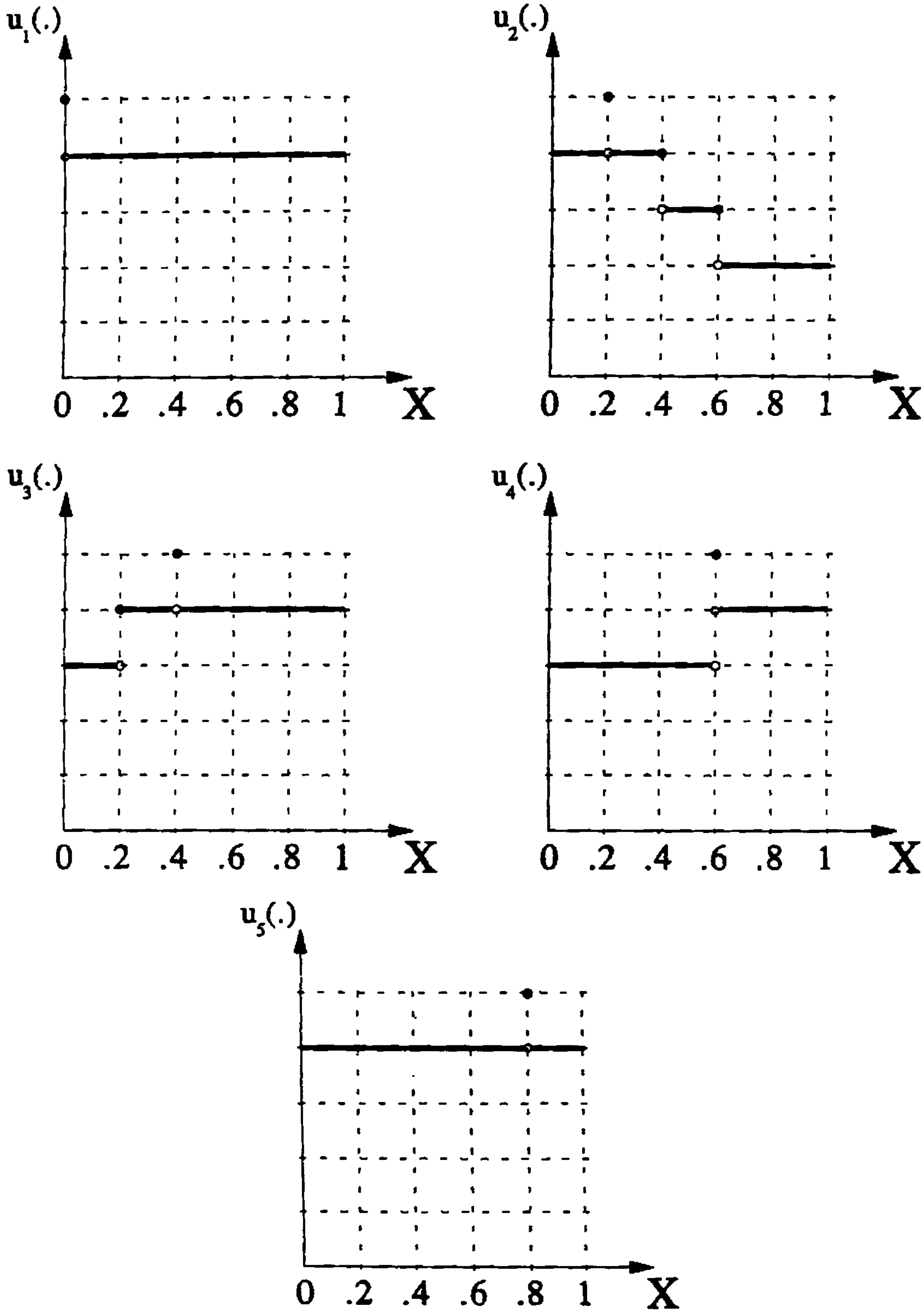


Figure 21.D.14(d)

Using majority voting as described in part (a) above, we can see that 0.2 beats any alternative in the open interval (0,0.2). This follows because all agents but agent 2 and 3 are indifferent between any two alternatives in (o,0.2], and agents 2 and 3 strictly prefer 0.2 to any alternative in (0,0.2). Similarly, 0.4 beats any alternative in the open interval (0.2,0.4), 0.6 beats any alternative in the open interval (0.4,0.6), and 0.8 beats any alternative in the interval (0.6,1]. So, we are left with the alternatives {0,0.2,0.4,0.6,0.8}. Now observe that if we label these alternatives as x = 0, y = 0.2, z = 0.4, v = 0.6, and w = 0.8 then the agents' preferences are identical to that in the previous parts of the exercise and we know that there exists no Condorcet winner in this case.

21.E.1 First, we show that a unique Condorcet winner exists. From Proposition 21.D.1 we know that there exists a Condorcet winner associated with the median voter (there may generally be more tan one). In our case preferences are strict and there is an odd number of individuals, so we cannot have two Condorcet winners. (Because when voting between any two alternatives x and x' we would have a strict majority for one or the other).

Second, we show that $f(\succsim_1,\ldots,\succsim_I)$ that assigns the Condorcet winner to every preference profile satisfies weak Pareto. Suppose in negation that there exists a profile $(\succsim_1',\ldots,\succsim_I')$ and $x = f(\succsim_1',\ldots,\succsim_I')$, and there exists $y \in X$ such that $y \succ_i' x$ for all $i = 1,\ldots I$. But then with preferences $(\succsim_1',\ldots,\succsim_I')$, y defeats x in majority voting, contradicting that x is the Condorcet winner. Therefore, $f(\cdot,\ldots,\cdot)$ must satisfy weak Pareto.

Finally, we show that $f(\succsim_1,\ldots,\succsim_I)$ is monotonic. Let $x = f(\succsim_1',\ldots,\succsim_I')$. Since I is odd and preferences are strict we must have that for all $y \in X$, $y \neq x$, $\#\{i : x \succ_i' y\} > I/2$. Take some $(\succsim_1'',\ldots,\succsim_I'') \in A$ such that x maintains its position from $(\succsim_1',\ldots,\succsim_I')$ to $(\succsim_1'',\ldots,\succsim_I'')$. Then, for all $y \in X$, $y \neq x$,

$\#\{i : x \succ_i'' y\} > \#\{i : x \succ_i' y\} > I/2$, which implies that $x = f(\succsim_1'', \ldots, \succsim_I'')$.

21.E.2 First, the first agent is a dictator by definition 21.E.5, so that the defined social choice function (SCF) is dictatorial. Second, the SCF is weakly Paretian because that dictator is always getting one of his most preferred alternatives, and we can therefore not make <u>all</u> agents strictly better off. Finally, the SCF is monotonic since if a chosen x maintains its position, it will maintain its position for the dictator. Therefore, for the dictator, x's lower contour set (weakly) increases, implying that it must still be the smallest indexed alternative among the dictator's best alternatives, and it must be again chosen.

The argument for strict preferences is even simpler, and along the same lines, since then agent one has a unique best alternative which is always chosen.

21.E.3 Suppose that we had $z = f(\succsim_1'', \ldots, \succsim_I'')$. Consider a profile $(\succsim_1', \ldots, \succsim_I') \in A$ that takes $\{y,z\}$ to the top from $(\succsim_1'', \ldots, \succsim_I'')$. Since z maintains its position from $(\succsim_1'', \ldots, \succsim_I'')$ to $(\succsim_1', \ldots, \succsim_I')$, it follows from monotonicity that $f(\succsim_1', \ldots, \succsim_I') = z$. But $(\succsim_1', \ldots, \succsim_I')$ also takes $\{y,z\}$ to the top from $(\succsim_1, \ldots, \succsim_I)$. Therefore we conclude that $z\ F(\succsim_1, \ldots, \succsim_I)\ y$, a contradiction to the assumption that $y\ F(\succsim_1, \ldots, \succsim_I)\ z$. Hence, $z \neq f(\succsim_1'', \ldots, \succsim_I'')$

21.E.4 (i) Let $I \geq 3$, $X = \{x,y\}$, and define $F: \mathcal{P}^I \to \mathcal{P}$ such that:

(1) if $x \succ_i y$ for all i then $x\ F_p(\succsim_1, \ldots, \succsim_I)\ y$,

(2) if (1) does not hold then $y\ F_p(\succsim_1, \ldots, \succsim_I)\ x$ if and only if

$\#\{i : x \succ_i y\} \geq I/2$.

That is, x is socially preferred to y if it the <u>unanimous</u> Condorcet winner, and otherwise the Condorcet loser is socially preferred between the two alternatives. The induced $f(\cdots)$ is vacuously pairwise independent, and clearly satisfies the Pareto condition. Now consider $(\succsim_1',\ldots,\succsim_I')$ and $(\succsim_1'',\ldots,\succsim_I'')$ in $\mathcal{P}^I$ such that $f(\succsim_1',\ldots,\succsim_I') = y$, and

$$\#\{i : x \succ_i' y\} < \frac{I}{2} < \#\{i : x \succ_i'' y\} < I.$$

By definition, $x\ F(\succsim_1',\ldots,\succsim_I')\ y$ and $y\ F(\succsim_1'',\ldots,\succsim_I'')\ x$, which implies that $f(\succsim_1',\ldots,\succsim_I') = x$ and $f(\succsim_1'',\ldots,\succsim_I'') = y$. But x maintained its position from $(\succsim_1',\ldots,\succsim_I')$ to $(\succsim_1'',\ldots,\succsim_I'')$, implying that $f(\cdots)$ is not monotonic.

(ii) Let $I = 3$, $X = \{x,y,z\}$, and restrict the domain of preferences to $\mathcal{A} = \{(\succsim_1',\ldots,\succsim_I'),(\succsim_1'',\ldots,\succsim_I'')\}$ such that preferences $(\succsim_1',\ldots,\succsim_I')$ are given by $x \succ_1' y \succ_1' z$, $z \succ_2' y \succ_2' x$, and $z \succ_3' y \succ_3' x$, and preferences $(\succsim_1'',\ldots,\succsim_I'')$ are given by $x \succ_1'' y \succ_1'' z$, $y \succ_2'' z \succ_2'' x$, and $z \succ_3'' y \succ_3'' x$. Define the social welfare functional $F(\cdots)$ such that $y\ F_p(\succsim_1',\ldots,\succsim_I')\ z$, $z\ F_p(\succsim_1',\ldots,\succsim_I')\ x$, $z\ F_p(\succsim_1'',\ldots,\succsim_I'')\ y$, and $y\ F_p(\succsim_1'',\ldots,\succsim_I'')\ x$. $F(\cdots)$ is clearly Pareto, and looking at the ranking between x and z, and between x and y, it is also pairwise independent. However, even though y maintains its position from $(\succsim_1',\ldots,\succsim_I')$ to $(\succsim_1'',\ldots,\succsim_I'')$, we have $f(\succsim_1',\ldots,\succsim_I') = y$ and $f(\succsim_1'',\ldots,\succsim_I'') = z$, so that $f(\cdots)$ is not monotonic.

22.B.1. The set of alternatives X in an exchange economy (Example 22.B.1) is given by $X = \{ \sum_{i=1}^{I} x_i \le \sum_{i=1}^{I} \omega_i \}$, where $(\omega_i)_{i=1}^{I}$ is the vector of initial endowments. Observe that this set is convex. A sufficient condition for the UPS in this economy to be convex is that $u_i(\cdot)$ be concave for every i. Indeed, suppose that $u' = (u_i(x_i'))_{i=1}^{I} \in U$ and $u'' = (u_i(x_i''))_{i=1}^{I} \in U$, where $x', x'' \in X$. Then by convexity of X for any $\lambda \in [0, 1]$ we must have $\lambda x' + (1-\lambda)x'' \in X$. Concavity of $u_i(\cdot)$ implies that $\lambda u_i(x_i') + (1-\lambda)\ u_i(x_i'') \ge u_i(\lambda x_i' + (1-\lambda)x_i'')$ for any i. Then by definition of a UPS we have $\lambda u' + (1-\lambda)\ u'' = (\lambda u_i(x_i') + (1-\lambda)\ u_i(x_i''))_{i=1}^{I} \in U$.

22.B.2. The Lagrangean for the constrained maximization problem can be written as

$$L = v(p_1, p_2) + \lambda\ [(p_1-1)\ x_1(p_1) + (p_2-2)\ x_2(p_2)].$$

The first-order conditions are obtained by differentiating the Lagrangean with respect to p_1, p_2 and using Roy's identity for the quasilinear case:

$$\frac{\partial}{\partial p_1} v(p_1, p_2) = -\ x_1(p_1);\ \frac{\partial}{\partial p_2} v(p_1, p_2) = -\ x_2(p_2).$$

22.B.3. The second-best problem can be written as

$$\max_{p_2}\ v(\hat{p}_1, p_2) + (\hat{p}_1-1)\ x_1(\hat{p}_1, p_2) + (p_2-1)\ x_2(\hat{p}_1, p_2),$$

where the last two terms represent the tax revenue which can be, for example, distributed among the population lump-sum. The first-order conditions (22.B.2) are then obtained by differentiating the objective function with respect to p_2 and using Roy's identity for quasilinear utility:

$\frac{\partial}{\partial p_2} v(\hat{p}_1, p_2) = - x_2(\hat{p}_1, p_2).$

22.B.4. The one-dimensional "boundary" of the UPS in Example 22.B.4 is given parametrically:

$$u(p_2) = (v_1(p_2) + (p_2-1) \sum_{i=1}^{I} x_i(p_2), v_2(p_2), \ldots, v_I(p_2)).$$

We will examine the "curvature" of this boundary around $p_2 = 1$. For this purpose, consider the following derivatives:

$$u_1'(p_2) = \sum_{i=2}^{I} x_i(p_2) + (p_2-1) \sum_{i=1}^{I} x_i'(p_2);$$

$$u_k'(p_2) = - x_k(p_2) \text{ for } k > 1.$$

(We have used Roy's identity for quasilinear utility: $\frac{\partial}{\partial p} v(p) = - x_i(p)$.)

Furthermore,

$$u_1''(p_2) = x_1'(p_2) + 2 \sum_{i=2}^{I} x_i'(p_2) + (p_2-1) \sum_{i=1}^{I} x_i''(p_2);$$

$$u_k''(p_2) = - x_k'(p_2) \text{ for } k > 1.$$

Substituting $p_2 = 0$ and observing that $x_i'(p_2) < 0$ (by the Law of Demand which holds for the quasilinear case), we can see that

$$u_1'(0) > 0,\ u_1''(0) < 0,\ u_1''(0) < 0,\ u_k''(0) > 0. \qquad (*)$$

Now, take any $k > 1$, and consider the boundary of the UPS's projection on the plane (u_1, u_k). The slope of this boundary at $p_2 = 0$ is $\frac{du_k}{du_1} = \frac{u_k'(0)}{u_1'(0)} < 0$, which is not surprising. The curvature of this boundary at this point is equal to

$$\frac{du_k^2}{du_1^2} = \frac{d}{dp_1}\left(\frac{du_k}{du_1}\right) \cdot \frac{1}{u_1'(0)} = \frac{u_k''(0)\, u_1'(0) - u_k'(0)\, u_1''(0)}{(u_1'(0))^3} > 0,$$

using the signs in (*).

This, the boundary is convex at $p_2 = 0$, and the UPS is not concave.

22.C.1. (i) $U \subset \mathbb{R}^I$ is symmetric, convex; W: $\mathbb{R}^I \longrightarrow \mathbb{R}^I$ is increasing, symmetric, strictly concave.

Suppose that $W(u^*) = \max\limits_{u \in U} W(u)$, and $u_j^* \neq u_k^*$. Let

$$u_i^{**} = \begin{cases} u_k^*, & i \quad j \\ u_j^*, & i = k \\ u_i^*, & i \quad j,\ i \neq k \end{cases}$$

By symmetry of U, $u^{**} \in U$. By symmetry of $W(\cdot)$, $W(u^{**}) = W(u^*)$. By convexity of U, $1/2\ u^* + 1/2\ u^{**} \in U$. By strict concavity of $W(\cdot)$, $W(1/2\ u^* + 1/2\ u^{**}) > W(u^*)$, which contradicts the assumption that u^* was the optimal utility profile in U.

(ii) $U \subset \mathbb{R}^I$ is symmetric, strictly convex; W: $\mathbb{R}^I \longrightarrow \mathbb{R}^I$ is increasing, symmetric concave. Suppose that $W(u^*) = \max\limits_{u \in U} W(u)$, and $u_j^* \neq u_k^*$. Let

$$u_i^{**} = \begin{cases} u_k^*, & i \quad j \\ u_j^*, & i = k \\ u_i^*, & i \quad j,\ i \neq k \end{cases}$$

By symmetry of U, $u^{**} \in U$. By symmetry of $W(\cdot)$, $W(u^{**}) = W(u^*)$. By strict convexity of U, $1/2\ u^* + 1/2\ u^{**} \in Interior(U)$. Therefore, $1/2\ u^* + 1/2\ u^{**} + \varepsilon e \in U$ for $\varepsilon > 0$ sufficiently small, where $e = (1,\ldots,1)$. Since $W(\cdot)$ is increasing and concave,

$$W(1/2\ u^* + 1/2\ u^{**} + \varepsilon e) > W(1/2\ u^* + 1/2\ u^{**}) \geq W(u^*),$$

which contradicts the assumption that u^* was the optimal utility profile in U.

22.C.2. (a) According to Proposition 6.E.1 (Extended Expected Utility Theorem), if the planner's preferences over lotteries satisfy the continuity and extended independence axioms, they can be described by the utility function

$$U(x_1, ..., x_I) = u_1(x_1) + ... + u_I(x_I).$$

Suppose that all components of the outcome except x_i are fixed. If the planner's preferences are non-paternalistic, then the planner's preferences over lotteries over x_I should coincide with individual i's preferences. But then $u_i(\cdot)$ must be individual i's von Neumann-Morgenstern (vNM) utility function. Since vNM utility functions are defined up to a liner rescaling, we can conclude that the planner's preferences can be represented by a (possibly) nonsymmetric utilitarian SWF:

$$W(v_1, ..., v_I) = \beta_1 v_1 + ... + \beta_I v_I,$$

where v_i is the utility level of individual i, and $\beta_i > 0$ for every i.

(b) If in addition the planner's preference relation is symmetric across individuals, then it can be represented by

$$U(x_1, ..., x_I) = u(x_1) + ... + u(x_I).$$

Therefore, the planner's preferences can be described by a utilitarian social welfare function where all individuals are ascribed identical utility functions.

22.C.3. [First printing errata: The definition of a Paretian social utility function should be strengthened. It should say that, in addition, $U(p) \geq U(p')$ whenever $U_i(p) \geq U_i(p')$ for every i.]

(a) Let final outcome 1 give the object to individual 1, and final outcome 2 give the object to individual 2. If p is the probability of implementing outcome 1, then $U(p) = u_1 p + u_2(1-p)$. If we are indifferent between outcome 1 for sure and outcome 2 for sure, we must have $U(1) = u_1 = U(0) = u_2$. But then $U(p) = u_1 p + u_1(1-p) = u_1$ regardless of p, i.e. all the lotteries are socially indifferent. This means that the society does not care how expected utility is distributed between the two individuals. This conclusion follows from the linearity imposed by the independence axiom. For example, if $U(p)$ were a strictly concave function, we would have $U(1/2) > U(0) = U(1)$. In this case, society would prefer to allocate the object using a fair randomization, rather than giving it to either individual with probability one.

(b) See Figure 22.C.3(a). The vectors U_1 and U_2 represent the normals to the two agents' indifference curves. The vector U represents the normal to the social indifference curves. The social utility function is Paretian when U lies "between" U_1 and U_2, i.e. the slope of social indifference curves is between the slopes of individual indifference curves. If U does not lie between U_1 and U_2, we can find two points A and B such that both individuals prefer B to A, while society prefers A to B, i.e. the social utility function is not **Paretian (see Figure 22.C.3(b).**

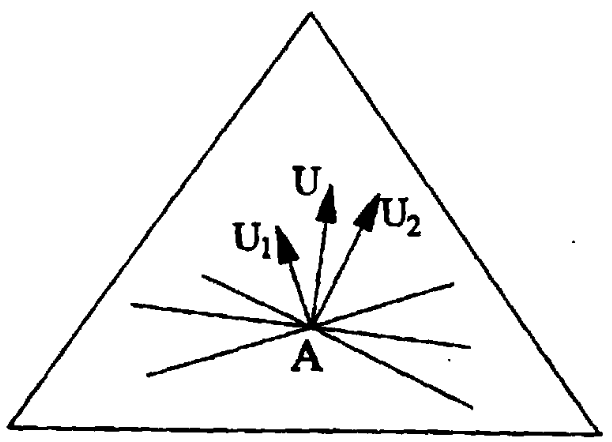


Figure 22.C.3(a)

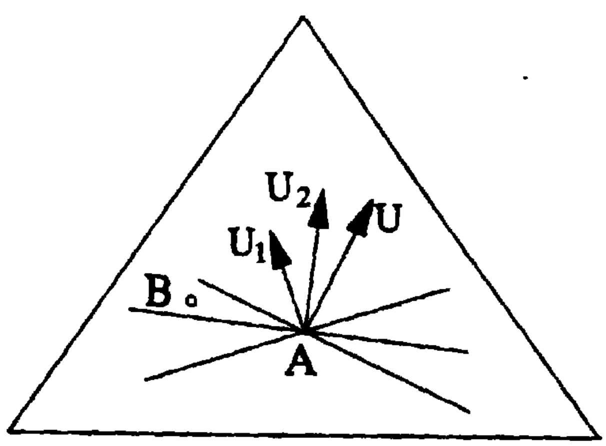


Figure 22.C.3(b)

(c) In this part we will restrict attention to the case where $N = 3$ and $I = 2$. When the two individuals have the same preferences over lotteries, then the only Paretian social preference over lotteries is the one which agrees with both individuals. Indeed, if a social preference relation does not agree with the individual preferences, there will exist two points A and B such that each individual preferes B to A, while society prefers A to B (see Figure 22.C.3(c)). Therefore, in this case the Paretian condition determines the social preference uniquely.

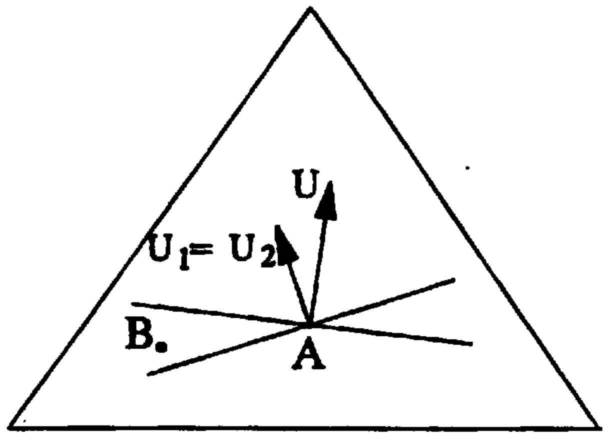


Figure 22.C.3(c)

When the two individuals have different preferences, any social preference relation whose indifference curves have the slope between the slopes of the two individuals' indifference curves, is Paretian (see Figure 22.C.3(a)).

When the two individuals have strictly opposing preferences, any social utility function satisfies the Paretian property vacuously.

(d) The social utility function can be written as $U(p) = U\cdot p$, where $U = (u^1, ..., u^N)$. Similarly, every individual i's social utility function can be written as $U_i(p) = U_i\cdot p$, where $U_i = (u_i^1, ..., u_i^N)$. In the case where $N = 3$ and $I = 2$, the geometrical analysis in parts (b)-(c) suggests that the social preference function $U(\cdot)$ is Paretian if and only if the vector U lies in the convex cone of U_1 and U_2, i.e. $U = \beta_1 U_1 + \beta_2 U_2$, with $\beta_1,\beta_2 \geq 0$, and $\beta_i > 0$ for some i. We will show that this result holds generally:

Sufficiency: Suppose $U(p) = \sum_{i=1}^{I} \beta_i U_i(p)$, where $\beta_i \neq 0$ for every i, and $\beta_i > 0$ for some i. Suppose that $U_i(p) - U_i(p') \geq 0$ for every i. Then we must have

$$U(p) - U(p') = \sum_{i=1}^{I} \beta_i [U_i(p) - U_i(p')] \geq 0.$$

Suppose furthermore that $U_i(p) - U_i(p') > 0$ for every i. Then we must have

$$U(p) - U(p') = \sum_{i=1}^{I} \beta_i [U_i(p) - U_i(p')] \geq \beta_k [U_k(p) - U_k(p')] > 0.$$

Therefore, $U(\cdot)$ is a Paretian social function.

Necessity: Let C be the convex cone of $(U_1, ..., U_I)$, i.e.
$C = \{\sum_{i=1}^{I} \beta_i U_i \mid \beta_i \geq 0 \text{ for all } i\}$. If $U(\cdot)$ cannot be written as $\sum_{i=1}^{I} \beta_i U_i(p)$, where $\beta_i \neq 0$ for every i, and $\beta_i > 0$ for some i, then either (i) U lies outside C, or (ii) $U = 0$, i.e. the social functional exhibits complete indifference.

In case (i), the Separating Hyperplane Theorem (Theorem M.G.2 in the Mathematical Appendix) says that there exist vectors $c, d \in \mathbb{R}^N$ such that

$$U \cdot d < c \qquad (1.1)$$

$$c < u \cdot d \text{ for every } u \in C. \qquad (1.2)$$

Substituting $o \in C$ in (1.2), we get $c < 0$, which together with (1.1) implies

$$U \cdot d < 0. \qquad (2.1)$$

Also, inequality (1.2) implies that

$$u \cdot d \geq 0 \text{ for every } u \in C. \qquad (2.2)$$

Indeed, if we had $u \cdot d$ for some $u \in C$, then by definition of C we would have $\beta u \in C$ for any $\beta \geq 0$, and $(\beta u) \cdot d = \beta(u \cdot d) < c$ for β large enough, which contradicts (1.2).

Now, take any interior probability distribution $p^* \in \mathrm{Int}(\Delta)$, and $\varepsilon > 0$

small enough so that $p^* + \varepsilon d \in \Delta$. On the one hand, inequality (2.1) implies that $U(p^* + \varepsilon d) - U(p^*) = U \cdot (\varepsilon d) < 0$. On the other hand, inequality (2.2) implies that for every individual i, $U_i(p^* + \varepsilon d) - U_i(p^*) = U_i \cdot (\varepsilon d) \geq 0$, since $U_i \in C$ by construction of C. These inequalities show that the social utility function $U(\cdot)$ is not Paretian.

In case (ii), observe that by assumption
$\sum_{i=1}^{I} \beta_i U_i \neq U = 0$ as long as $\beta_i \geq 0$ for all i and $\beta_i > 0$ for some i.
We can formulate the following useful lemma, which is a strengthening of the Supporting Hyperplane Theorem (Theorem M.G.3 in the Mathematical Appendix) in application to convex cones:

Lemma: Let $C = \{\sum_{i=1}^{I} \beta_i U_i \mid \beta_i \geq 0 \text{ for } i = 1,\ldots,I\} \subseteq \mathbb{R}^N$, where
$\sum_{i=1}^{I} \beta_i U_i \neq 0$ as long as $\beta_i \geq 0$ for all i, and $\beta_i > 0$ for some i. (*)
Then there exists a vector $d^* \in \mathbb{R}^N$ such that $U_i \cdot d^* < 0$ for all i.

Proof: We will use induction on N, the dimensionality of the space. The statement is obvious for $N = 1$. Suppose it is true for N-1 - dimensional spaces, and look at an N - dimensional space. Observe that (*) implies that that $0 \notin \text{Int}(C)$. This allows us to use the Supporting Hyperplane Theorem, which implies that there exists a vector $d \in \mathbb{R}^N$ such that $u \cdot d \leq 0$ for all $u \in C$. Now, restrict attention to the hyperplane $H_d = \{u \in \mathbb{R}^N \mid u \cdot d = 0\}$, which is an N-1 -dimensional space. The induction statement implies that there exists a vector $d' \in H_d$ such that $U_i \cdot d < 0$ for any $U_i \in H_d$. Now, consider a vector $d^* = d + \varepsilon d'$, where $\varepsilon > 0$ is small enough. First, for any $U_i \notin H_d$ we have $U_i \cdot d > 0$, and therefore $U_i \cdot d^* = U_i \cdot d + \varepsilon U_i \cdot d' > 0$ for ε small enough. Second, for any $U_i \in H_d$ we have $U_i \cdot d^* = U_i \cdot d + \varepsilon U_i \cdot d' = \varepsilon U_i \cdot d' > 0$. Therefore, the statement is true for N-dimensional spaces. By induction, it must be true for any N.■

Applying the Lemma, we see that $U_i \cdot d < 0$ for all i. Take any interior probability distribution $p^* \in \mathrm{Int}(\Delta)$, and $\varepsilon > 0$ small enough so that $p^* + \varepsilon d \in \Delta$. On the one hand, we have $U_i(p^* + \varepsilon d) - U_i(p^*) = U_i \cdot (\varepsilon d) < 0$ for all i. On the other hand, we know that $U(p^* + \varepsilon d) = U \cdot (p^* + \varepsilon d) = 0 = U(p^*)$. Therefore, the social utility function $U(\cdot)$ is not Paretian. This completes the proof.

The result says that any social utility function which is Paretian and satisfies the independence axiom corresponds to a weighted utilitarian SWF. The coeeficients β_i are the weights which the SWF places on the utility of individual i. Different individuals can have different weights.

22.C.4. (a) Let $\succ_R$ represent the Rawlsian social preference, while $\succ_{LM}$ represent the leximin social preference. Observe that

$$u' \succ_R u \Rightarrow \min_i u'_i > \min_i u_i \Rightarrow u'^r_1 > u^r_1 \Rightarrow u' \succ_{LM} u.$$

On the other hand, we can have $u' \sim_R u$ but $u' \succ_{LM} u$, for example when when $u'^r_1 = u^r_1$ and $u'^r_2 = u^r_2$. Thus, leximin is a refinement of the Rawlsian preference.

(b) Observe that the leximin ordering coincides with the lexicographic ordering above the 45^0 line. As shown in Example 3.C.1, lexicographic preferences cannot be represented by a utility function.

(c) Suppose u is a LM optimum, but not a Pareto optimum. Then there exists u' such that $u' \geq u$, $u' \neq u$. Let $k = \min\{i \mid u'^r_i > u^r_i\}$. (Observe that the set is non-empty.) We must have

$$u'^r_i = u^r_i \text{ for any } i < k,$$
$$u'^r_k > u^r_k.$$

Thus, $u' \succ_{LM} u$ – contradiction to u being a LM optimum.

22.C.5. Let $\underline{u} = \min_i u_i$. Then we can write

$W_\rho(u) = [\sum_i u_i^{1-\rho}]^{1/(1-\rho)} = \underline{u}\,[\sum_i (\frac{u_i}{\underline{u}})^{1-\rho}]^{1/(1-\rho)}$. Observe that when $\rho > 1$,

at least one term in the sum is exactly one, and others may be between zero and one. Thus, the sum in the square brackets is between 1 and I for $\rho > 1$. When $\rho \to \infty$, $1/(1-\rho) \to 0$, and both $1^{1/(1-\rho)}$ and $I^{1/(1-\rho)}$ go to one. Therefore, $W_\rho(u) \to \underline{u}$ as $\rho \to \infty$.

22.C.6. (a) See Figure 22.C.6(a).

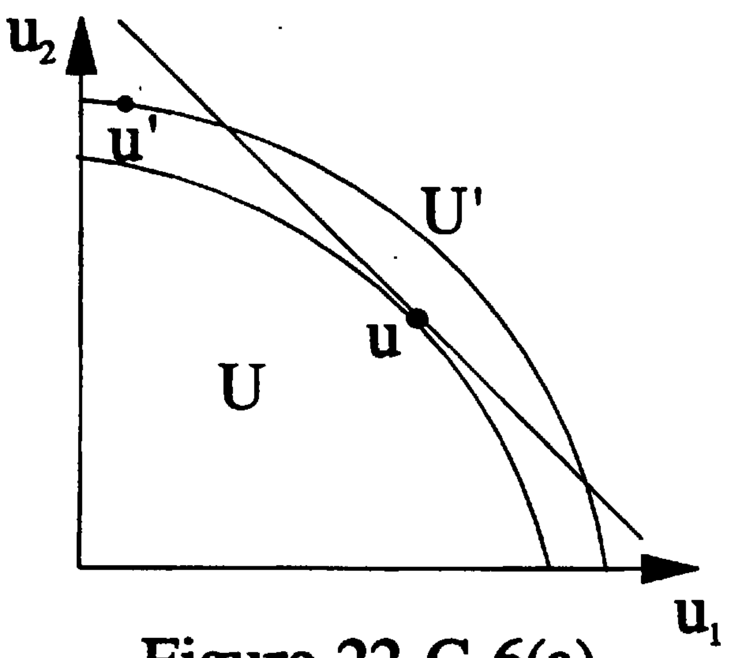


Figure 22.C.6(a)

(b) The utility possibility sets for quasilinear economies are of the form $U = \{u \in \mathbb{R}^I \mid \sum_{i=1}^{I} u_i \le \sum_{i=1}^{I} \bar{u}_i\}$, $U' = \{u \in \mathbb{R}^I \mid \sum_{i=1}^{I} u_i \le \sum_{i=1}^{I} \bar{u}'_i\}$. If U' passes the weak compensation test over (U, u), then by definition the exists a $u' \in U'$ such that $u'_i \ge \bar{u}_i$ for every i. Adding up, we obtain $\sum_{i=1}^{I} u'_i \ge \sum_{i=1}^{I} \bar{u}_i$. On the other hand, $u' \in U'$ implies $\sum_{i=1}^{I} u'_i \le \sum_{i=1}^{I} \bar{u}'_i$. Combining, we see that $\sum_{i=1}^{I} \bar{u}_i \le \sum_{i=1}^{I} \bar{u}'_i$, i.e. $\bar{u}'$ is a utilitarian improvement over $\bar{u}$. Also, the last intequality implies that $U \subset U'$, i.e. U' passes a strong compensation test over U. This argument is graphically illustrated in Figure 22.C.6(b). If we use a nonutilitarian SWF, $\bar{u}'$ may not be an improvement over $\bar{u}$.

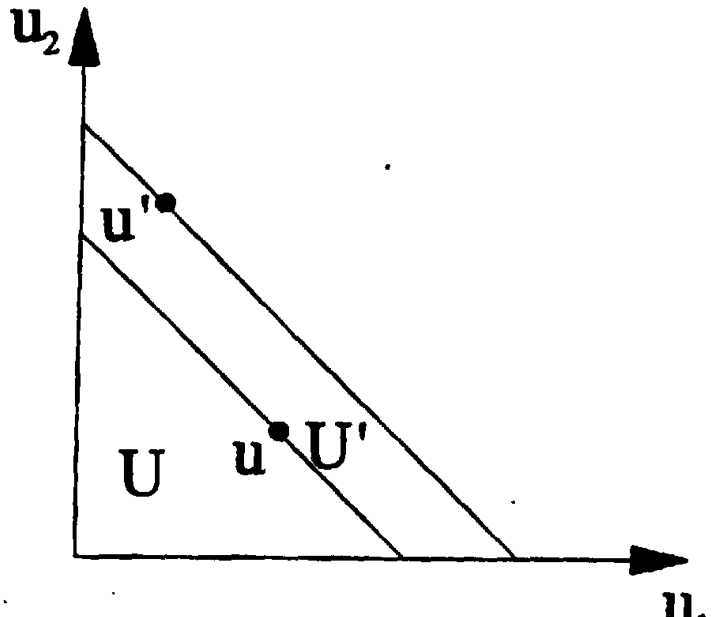


Figure 22.C.6(b)

22.C.7. [First printing errata: "differing only in distribution of the initial endowments" should be interpreted to allow for different initial endowments.] Take two consumers with utility functions

$$u_1(x_{11}, x_{21}) = x_{11}^{1/3}\, x_{21}^{2/3},$$
$$u_1(x_{12}, x_{22}) = x_{12}^{2/3}\, x_{22}^{1/3}.$$

Take the total endowments for U to be $(2,1)$, and the total endowments for U' to be $(1,2)$. Choose then individual initial endowments to be

$\omega_1 = (2-\varepsilon, 1-\varepsilon)$, $\omega_2 = (\varepsilon, \varepsilon)$ for U,

$\omega_1 = (\varepsilon, \varepsilon)$, $\omega_2 = (1-\varepsilon, 2-\varepsilon)$ for U'.

For $\varepsilon > 0$ small enough, the competitive equilibria in the two economies will yield utility vectors u and u' located as in Figure 22.C.7 in the textbook.

22.C.8. (a) See Figure 22.C.8(a) for a situation where Kaldor comparability is possible. Kaldor comparability is not possible in Figure 22.C.7. in the textbook.

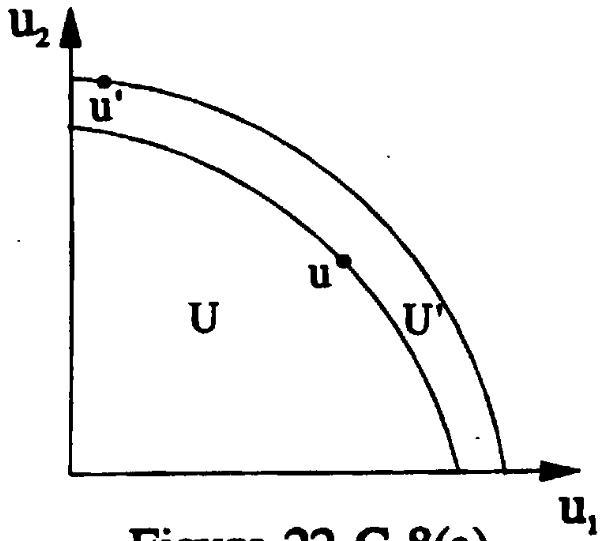


Figure 22.C.8(a)

(b) A relation $\succ$ is asymmetric if $a \succ b$ and $b \succ a$ cannot hold at once.

If (U',u') passes the Kaldor compensation test over (U,u), then by definition U' passes the weak compensation test over (U,u), therefore, again by definition, (U,u) does not pass the Kaldor compensation test over (U', u'). Therefore, Kaldor comparability is asymmetric.

(c) See Figure 22.C.8(b). (U',u') passes the Kaldor compensation test over (U,u), and (U'',u'') passes the Kaldor compensation test over (U',u'). But U'' does not pass the weak compensation test over (U,u), and, therefore, (U'',u'') does not pass the Kaldor compensation test over (U,u).

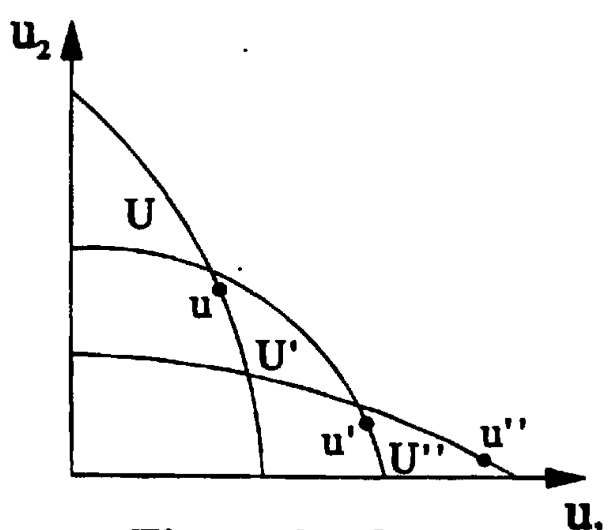


Figure 22.C.8(b)

22.D.1. (a) $F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is always complete, since we always have either x $F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ y or y $F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ x (or both).

F is Paretian:

$\tilde{u}_i(x) \geq \tilde{u}_i(y)$ for every i $\Rightarrow$ $x\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ y$.

$\tilde{u}_i(x) > \tilde{u}_i(y)$ for every i $\Rightarrow$ $x\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ y$ and not $y\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ x \Rightarrow$

$\Rightarrow$ $x\ F_p(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ y$.

$\tilde{u}_i(y) \geq \tilde{u}_i(x)$ for every i $\Rightarrow$ $y\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ x$.

$\tilde{u}_i(y) > \tilde{u}_i(x)$ for every i $\Rightarrow$ $y\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ x$ and not $x\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ y \Rightarrow$

$\Rightarrow$ $y\ F_p(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ x$.

F satisfies PI, since whenever

$\tilde{u}_i(x) = \tilde{u}'_i(x)$ and $\tilde{u}_i(y) = \tilde{u}'_i(y)$, we have $\tilde{u}_i(\cdot) \equiv \tilde{u}'_i(\cdot)$ (there are no other alternatives).

Suppose in negation that F can be represented by a social welfare function $W(\cdot)$. Choose a utility profile $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ so that

$$\tilde{u}_1(x) = \tilde{u}_2(x) = \tilde{u}_3(x) = 1,\ \tilde{u}_1(y) = \tilde{u}_2(y) = 1,\ \tilde{u}_3(y) = 2.$$

Then by definition of F we have $y\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ x$, but not $x\ F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ y$, i.e. $y\ F_p(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\ x$. This implies that $W(1,1,2) > W(1,1,1)$. Now choose another utility profile, $(\tilde{u}'_1, \tilde{u}'_2, \tilde{u}'_3)$, so that

$$\tilde{u}'_1(y) = \tilde{u}'_2(y) = \tilde{u}'_3(y) = 1,\ \tilde{u}'_1(x) = \tilde{u}'_2(x) = 1,\ \tilde{u}'_3(x) = 2.$$

Then by definition of F we have $x\ F(\tilde{u}'_1, \tilde{u}'_2, \tilde{u}'_3)\ y$. This implies that $W(1,1,1) \leq W(1,1,2)$ – contradiction to the previous paragraph.

(b) The Paretian condition fails to be satisfied, since we can take $\tilde{u} \in \mathcal{U}^I$ such that $\tilde{u}_i(x) < \tilde{u}_i(y)$ for all i, and we still have $x\ F_p(\tilde{u}_1, \dots, \tilde{u}_I)\ y$. All the other conditions of Proposition 22.D.1 are satisfied. However, F cannot be represented by means of a social welfare function. Indeed, suppose that $\succsim$ is a social preference relation on utility vectors which represents F. Choose a utility profile $\tilde{u}$ so that

$$\tilde{u}_1(x) = \tilde{u}_2(x) = \tilde{u}_3(x) = 0, \text{ and } \tilde{u}_1(y) = \tilde{u}_2(y) = \tilde{u}_3(y) = 1.$$

Since $x\ F_p(\tilde{u}_1, ..., \tilde{u}_I)\ y$, we must have $(0,0,0) \succ (1,1,1)$. On the other hand, consider a utility profile $\tilde{u}'$ such that

$$\tilde{u}_1(x) = \tilde{u}_2(x) = \tilde{u}_3(x) = 1, \text{ and } \tilde{u}_1(y) = \tilde{u}_2(y) = \tilde{u}_3(y) = 0.$$

Now we must again have $x\ F_p(\tilde{u}_1, ..., \tilde{u}_I)\ y$, which implies $(1,1,1) \succ (0,0,0)$. Contradiction.

(c) Consider the Borda count.

22.D.2. Take a profile of utility functions $\tilde{u} \in \mathcal{U}^I$ with $u_i(x) = u_i$, $\tilde{u}_i(y) = u_i'$, and $\tilde{u}_i(z) = u_i$ for every i. Since $u \succsim u'$ using the pair x,y, we must have $x\ F(\tilde{u}_1, ..., \tilde{u}_I)\ y$. By the Pareto property, we also have $z\ F(\tilde{u}_1, ..., \tilde{u}_I)\ x$. Hence, by the transitivity of $F(\tilde{u}_1, ..., \tilde{u}_I)$, we obtain $z\ F(\tilde{u}_1, ..., \tilde{u}_I)\ y$.

22.D.3. (i) Let $u' \geq u$, where $u,u' \in \mathbb{R}^I$. Choose two alternatives $x,y \in X$ and a utility profile $\tilde{u} \in \mathcal{U}^I$ such that $\tilde{u}_i(x) = u_i'$, $\tilde{u}_i(y) = u_i$ for all i. Since the functional F is Paretian, $\tilde{u}_i(x) = u_i' \geq \tilde{u}(y_i) = u_i$ for all i implies that $x\ F_p(\tilde{u})\ y$. Since the functional is represented by the social preference relation $\succsim$, we must have $\tilde{u}(x) = u' \succsim \tilde{u}(y) = u$.

(ii) Let $u' \gg u$, where $u,u' \in \mathbb{R}^I$. Choose two alternatives $x,y \in X$ and a utility profile $\tilde{u} \in \mathcal{U}^I$ such that $\tilde{u}_i(x) = u_i'$, $\tilde{u}_i(y) = u_i$ for all i. Since the functional F is Paretian, $\tilde{u}_i(x) = u_i' > \tilde{u}(y_i) = u_i$ for all i implies that $x\ F_p(\tilde{u})\ y$. Since the functional is represented by the social preference relation $\succsim$, we must have $\tilde{u}(x) = u' \succ \tilde{u}(y) = u$.

22.D.4. The social welfare functional is defined by

$x\ F(\tilde{u}_1, \ldots, \tilde{u}_I)\ y \Leftrightarrow\ \tilde{u}_{h_j}(x) \geq \tilde{u}_{h_j}(y)$ for $j = 1, .., k$ and

$(\tilde{u}_{h_k}(x) > \tilde{u}_{h_k}(y)$ or $k = n)$ for some $k \leq n$.

(a) Pareto: Suppose $\tilde{u}_i(x) \geq \tilde{u}_i(x)$ for every i. Then $\tilde{u}_{h_j}(x) \geq \tilde{u}_{h_j}(x)$

for every $k = 1, \ldots, n$, which implies $x\ F(\tilde{u}_1, \ldots, \tilde{u}_I)\ y$. If, moreover,

$\tilde{u}_i(x) > \tilde{u}_i(y)$, then we know that $\tilde{u}_{h_1}(x) > \tilde{u}_{h_1}(y)$, which implies

$x\ F(\tilde{u}_1, \ldots, \tilde{u}_I)\ y$.

Pairwise Independence: Let $x,y \in X$ and $(\tilde{u}_1, \ldots, \tilde{u}_I) \in \mathcal{U}^I$,

$(\tilde{u}'_1, \ldots, \tilde{u}'_I) \in \mathcal{U}^I$, with $\tilde{u}_i(x) = \tilde{u}'_i(x)$ and $\tilde{u}_i(y) = \tilde{u}'_i(y)$ for all i. Then

$$x\ F(\tilde{u}_1, \ldots, \tilde{u}_I)\ y \Leftrightarrow$$

$$\Leftrightarrow \tilde{u}_{h_j}(x) \geq \tilde{u}_{h_j}(y) \text{ for } j = 1, .., k \text{ and } (\tilde{u}_{h_k}(x) > \tilde{u}_{h_k}(y) \text{ or } k = n) \Leftrightarrow$$

$$\Leftrightarrow \tilde{u}'_{h_j}(x) \geq \tilde{u}'_{h_j}(y) \text{ for } j = 1, .., k \text{ and } (\tilde{u}'_{h_k}(x) > \tilde{u}'_{h_k}(y) \text{ or } k = n) \Leftrightarrow$$

$$\Leftrightarrow x\ F(\tilde{u}'_1, \ldots, \tilde{u}'_I)\ y.$$

Does not allow interpersonal comparisons: Let $\tilde{u}'_i(x) = \beta_i\ \tilde{u}_i(x) + \alpha_i$ for all i

and x, with $\beta_i > 0$ for all i. Then for all $x,y \in X$

$$x\ F(\tilde{u}_1, \ldots, \tilde{u}_I)\ y \Leftrightarrow$$

$$\Leftrightarrow \tilde{u}_{h_j}(x) \geq \tilde{u}_{h_j}(y) \text{ for } j = 1, .., k \text{ and } (\tilde{u}_{h_k}(x) > \tilde{u}_{h_k}(y) \text{ or } k = n) \Leftrightarrow$$

$$\Leftrightarrow \tilde{u}'_{h_j}(x) \geq \tilde{u}'_{h_j}(y) \text{ for } j = 1, .., k \text{ and } (\tilde{u}'_{h_k}(x) > \tilde{u}'_{h_k}(y) \text{ or } k = n) \Leftrightarrow$$

$$\Leftrightarrow x\ F(\tilde{u}'_1, \ldots, \tilde{u}'_I)\ y.$$

Thus, $F(\tilde{u}_1, \ldots, \tilde{u}_I) = F(\tilde{u}'_1, \ldots, \tilde{u}'_I)$.

(b) When n=1, F can be generated from a social welfare function

$W(u_1, \ldots, u_I) = u_{h_1}$. When n>1, F cannot be generated from a social welfare function, since it induces lexicographic social preferences on utility profiles (u_{h_1}, u_{h_2}) (for simplicity we keep utilities of all other agent constants), and as we know from Example 3.C.1, such preferences cannot be represented by a utility function.

(c) Let k be the dictator. Suppose in negation that $b_j \neq 0$ for some j. Then take any $x,y \in X$ and $(\tilde{u}_1, ..., \tilde{u}_I) \in \mathcal{U}^I$ such that $\tilde{u}_i(x) = \tilde{u}_i(y)$ for $i \neq j,k$, $\tilde{u}_k(x) = \tilde{u}_k(y) + b_j/2$, and $\tilde{u}_j(x) = \tilde{u}_j(y) - b_k$. Then agent k prefers x to y, while $W(\tilde{u}_1(x), ..., \tilde{u}_I(x)) - W(\tilde{u}_1(y), ..., \tilde{u}_I(y)) = b_jb_k/2 - b_jb_k < 0$, i.e. socially y is preferred to x. This contradicts agent k being a dictator.

22.D.5. We will prove Arrow's impossibility theorem (Proposition 21.C.1) only for the case where preference domain allows no indifference: $\mathcal{A} = \mathcal{P}^I$. (Remark: To extend the proof to the domain $\mathcal{A} = \mathcal{R}^I$, one needs to bring the definition of of a Paretian social welfare functional defined on preferences, Definition 21.C.2, in correspondence with the corresponding definition for a functional defined on utility profiles, Definition 22.D.2.)

For every social welfare functional $F()$ defined on preference profiles from $\mathcal{P}^I$ we can construct a social welfare functional $F'()$ on utility functions by letting $F'(\tilde{u}_1, .., \tilde{u}_I) = F(\succsim_1, .., \succsim_I)$, where $\succsim_i$ is the preference relation induced by the utility function $\tilde{u}_i(\cdot)$.

<u>$F'()$ inherits the Pareto property from $F()$</u>:

(i) If $\tilde{u}_i(x) \geq \tilde{u}_i(y)$ for all i, then either $x \succ_i y$ $\forall i$ or $x = y$. In the first case, we must have $x\ F_p(\succsim_1, .., \succsim_I)\ y$ since $F()$ is Paretian (Definition 21.C.2). In both cases, therefore, we have $x\ F(\succsim_1, .., \succsim_I)\ y$, and correspondingly $x\ F'(\tilde{u}_1, .., \tilde{u}_I)\ y$.

(ii) If $\tilde{u}_i(x) \geq \tilde{u}_i(y)$ for all i, then $x \succ_i y$ $\forall i$. Since $F()$ is Paretian

(Definition 21.C.2), we must have $x \; F_p(\succsim_1, \ldots, \succsim_I) \; y$, and correspondingly

$x \; F_p'(\tilde{u}_1, \ldots, \tilde{u}_I) \; y$.

Thus, both parts of definition 22.D.2 are satisfied.

$F'()$ the Pairwise Independence property from $F()$:

Suppose that $\tilde{u}_i(x) = \tilde{u}_i'(x)$ and $\tilde{u}_i(y) = \tilde{u}_i'(y)$ for all i. This implies that

$x \succsim_i y \Leftrightarrow \tilde{u}_i(x) \geq \tilde{u}_i(y) \Leftrightarrow \tilde{u}_i'(x) \leq \tilde{u}_i'(y) \Leftrightarrow x \succsim_i' y$, and that

$y \succsim_i x \Leftrightarrow \tilde{u}_i(y) \geq \tilde{u}_i(x) \Leftrightarrow \tilde{u}_i'(y) \leq \tilde{u}_i'(x) \Leftrightarrow y \succsim_i' x$.

Since $F()$ satisfies PI (Definition 21.C.3), we therefore have

$$x \; F'(\tilde{u}_1, \ldots, \tilde{u}_I) \; y \Leftrightarrow x \; F(\succsim_1, \ldots, \succsim_I) \; y \Leftrightarrow x \; F(\succsim_1', \ldots, \succsim_I') \; y \Leftrightarrow$$
$$\Leftrightarrow x \; F'(\tilde{u}_1', \ldots, \tilde{u}_I') \; y.$$

Therefore, $F'()$ satisfies PI (Definition 22.D.3).

$F'()$ does not allow interpersonal comparisons of utility:

Suppose that for all i and x we have $\tilde{u}_i(x) = \beta_i \; \tilde{u}_i'(x) + \alpha_i$, with $\beta_i > 0$. Then

for all x, y we have

$$x \succsim_i y \Leftrightarrow \tilde{u}_i(x) \geq \tilde{u}_i(y) \Leftrightarrow \beta_i \; \tilde{u}_i'(x) + \alpha_i \geq \beta_i \; \tilde{u}_i'(y) + \alpha_i \Leftrightarrow$$
$$\Leftrightarrow \tilde{u}_i'(x) \geq \tilde{u}_i'(y) \Leftrightarrow x \succsim_i' y.$$

Therefore, $\succsim_i$ and $\succsim_i'$ represent the same preferences, and

$$x \; F'(\tilde{u}_1', \ldots, \tilde{u}_I') \; y \Leftrightarrow x \; F(\succsim_1', \ldots, \succsim_I') \; y \Leftrightarrow x \; F(\succsim_1, \ldots, \succsim_I) \; y \Leftrightarrow$$
$$\Leftrightarrow x \; F'(\tilde{u}_1, \ldots, \tilde{u}_I) \; y.$$

Therefore, $F'()$ does not allow interpersonal comparisons of utility

(Definition 2.D.5).

A dictator for $F'()$ is a dictator for $F()$:

Suppose that h is a dictator for $F'()$, i.e., that for every pair $x,y \in X$

$\tilde{u}_h(x) > \tilde{u}_h(y)$ implies that $x \; F_p'(\tilde{u}_1, \ldots, \tilde{u}_I) \; y$. Then for every $x,y \in X$ we have

$$x \succ_h y \Rightarrow \tilde{u}_h(x) > \tilde{u}_h(y) \Rightarrow x \; F_p'(\tilde{u}_1, \ldots, \tilde{u}_I) \; y \Rightarrow x \; F_p(\succsim_1, \ldots, \succsim_I) \; y.$$

Therefore, h is a dictator for $F()$.

Therefore, for every Paretian, PI social welfare functional $F()$ defined on preference profiles we can construct a Paretian, PI social welfare functional $F'()$ on utility functions which does not allow interpersonal comparisons of utility, so that a dictator for $F'()$ is a dictator for $F()$. Proposition 22.D.3 establishes that $F'()$ admits a dictator, and therefore, so does $F()$. This establishes Arrow's impossibility theorem.

22.D.6. (a) Since SWF is defined up to a monotonic transformation, we can always normalize the nonsymmetric utilitarian SWF $W(u) = \sum_i b_i u_i$ so that $\sum_i b_i = 1$. Then we can write

$W(u) = \sum_i b_i u_i = \bar{u} + \sum_i b_i(u_i - \bar{u})$.

(b) [First printing errata: we need to assume concavity of $W(\cdot)$.] Suppose in negation that $g(u_1 - \bar{u},...u_I - \bar{u}) < 0$, where $\bar{u} = \frac{1}{I}\sum_j u_j$. Observe that for every i we have $\frac{1}{n!}\sum_\pi u_{\pi(i)} = \bar{u}$, where the sum is taken over all permutations of agents. Since $g(\cdot)$ is symmetric, we must have

$$g(u_{\pi(1)} - \bar{u},...u_{\pi(I)} - \bar{u}) = g(u_1 - \bar{u},...u_I - \bar{u}) < 0$$

for every permutation π of agents. Let $u_\pi = (u_{\pi(1)}, ..., u_{\pi(I)})$ for every permutation π. On one hand, we have $W(\frac{1}{n!}\sum_\pi u_\pi) = W(\bar{u}, ..., \bar{u}) = \bar{u} - g(0) = \bar{u}$.

On the other hand, we have

$$\frac{1}{n!}\sum_\pi W(u_\pi) = \frac{1}{n!}\sum_\pi (\bar{u} - g(u_{\pi(1)} - \bar{u},...u_{\pi(I)} - \bar{u})) =$$

$$= \frac{1}{n!}\sum_\pi (\bar{u} - g(u_1 - \bar{u},...u_I - \bar{u})) =$$

$$= \bar{u} - g(u_1 - \bar{u},...u_I - \bar{u}) > \bar{u} = W(\frac{1}{n!}\sum_\pi u_\pi),$$

which contradicts the concavity of $W(\cdot)$. Intuitively, concavity of $W\cdot()$ means that the SWF favors equality. In order for inequality to be penalized, we need to have $g(u_1 - \bar{u},...u_I - \bar{u}) \geq g(0) = 0$.

(c) The symmetric Rawlsian welfare function can be written as $W(u) = \min\{u_1, .., u_I\} = \bar{u} + \min\{u_1 - \bar{u}, .., u_I - \bar{u}\}$. As for the asymmetric Rawlsian social welfare function, it is invariant to identical changes of origins and therefore cannot be expressed in the form (22.D.1). For example, take $W(u_1,u_2) = \min\{b_1u_1, b_2u_2\}$ with $b_2 > b_1 > 0$. Take two utility vectors $u' = (1,0)$, $u'' = (0,1)$. Then $W(u') = W(u'') = 0$, and at the same time $W(u'_1 - 1, u'_2 - 1) = -a_2 < -a_1 = W(u''_1 - 1, u''_2 - 1)$.

(d) Take, for example, $W(u) = \bar{u} - \sum_i k\ (u_i - \bar{u})^2$ with $k \geq 0$. This SWF punishes for a variance of utilities; it is more sensitive to inequality than the utilitarian SWF, but less sensitive than the Rawlsian SWF.

(e) If $g(\cdot)$ is homogeneous of degree one, $g(\alpha z) = \alpha\ g(z)$ for all $\alpha \geq 0$. Differentiation with respect to α yields $z\cdot\nabla g(\alpha z) = g(z)$. Substituting $\alpha = 0$, we obtain $g(z) = z\cdot\nabla g(0)$, i.e. $g(\cdot)$ is linear.

22.D.7. (a) $W_\rho(\lambda u) = [\sum_i (\lambda u_i)^{1-\rho}]^{1/(1-\rho)} = \lambda\ [\sum_i u_i^{1-\rho}]^{1/(1-\rho)} = \lambda\ W_\rho(u)$, therefore, the social preferences are invariant to common changes in units.

(b) Let $W'_\rho(u) = W_\rho(u+a)$. When does $W'_\rho(u)$ induce the same social preferences as $W_\rho(u+a)$? To answer the question, we can look at marginal rates of substitution:

$$MRS_{i,j} = \frac{\partial W_\rho(u)/\partial u_j}{\partial W_\rho(u)/\partial u_i} = \left(\frac{u_j}{u_i}\right)^{-\rho};\ MRS'_{i,j} = \frac{\partial W'_\rho(u)/\partial u_j}{\partial W'_\rho(u)/\partial u_i} = \left(\frac{u_j + a}{u_i + a}\right)^{-\rho}.$$

For social preferences to be invariant to a, we need to have $MRS_{i,j} = MRS'_{i,j}$, which implies $\left(\frac{u_j}{u_i}\right)^{-\rho} = \left(\frac{u_j + a}{u_i + a}\right)^{-\rho}$. This holds for $\rho = 0$

(both sides are equal to one) and $\rho = \infty$ (both sides are equal ∞ when $u_j < u_i$, 0 when $u_j > u_i$, one when $u_j = u_i$). For $0 < \rho < \infty$, marginal rates of substitution can only be equalized if $\frac{u_j}{u_i} = \frac{u_j + a}{u_i + a}$, which cannot hold unless $u_i = u_j$ or $a = 0$.

22.D.8. (a) If $f(\cdot)$ is an increasing function, we have

$$W(f(u_1), \ldots, f(u_I)) = \min\{f(u_1), \ldots, f(u_I)\} =$$
$$= f(\min\{u_1, \ldots, u_I\}) = f(W(u_1, \ldots, u_I)).$$

Therefore, $W(f(u_1), \ldots, f(u_I))$ induces the same social ordering as $W(u_1, \ldots, u_I)$.

(b) If $f(\cdot)$ is an increasing function, we have

$$W(f(u_1), \ldots, f(u_I)) = \max\{f(u_1), \ldots, f(u_I)\} =$$
$$= f(\max\{u_1, \ldots, u_I\}) = f(W(u_1, \ldots, u_I)).$$

Therefore, $W(f(u_1), \ldots, f(u_I))$ induces the same social ordering as $W(u_1, \ldots, u_I)$.

(c) If k the dictator, we can use $W(u) = u_k$. If $f(\cdot)$ is an increasing function, we have

$$W(f(u_1), \ldots, f(u_I)) = f(u_k) = f(W(u_1, \ldots, u_I)).$$

Therefore, $W(f(u_1), \ldots, f(u_I))$ induces the same social ordering as $W(u_1, \ldots, u_I)$.

(d) We can find a monotone increasing transformation of the utility axis which increases u'_1, u'_2, while keeping u_1, u_2 unchanged. For example, pick $a \in (0,1)$ and $b > 1$ and consider the following ordinal transformation:

$$f(u) = \begin{cases} u,\ u_1 & u \le u_2 \\ u_1 + a(u-u_1), & u < u_1 \\ u_2 + b\, u-u_2\,, & u > u_2 \end{cases}$$

Suppose that $W(u_1', u_2') = W(u_1, u_2)$. Since W is invariant to f, we must also have $W(f(u_1'), f(u_2')) = W(f(u_1), f(u_2))$. And from the way f was constructed, $W(u_1, u_2) = W(f(u_1), f(u_2))$. All this together implies $W(u_1', u_2') = W(f(u_1'), f(u_2'))$. But this contradicts W being increasing, since $u_1' > f(u_1')$, $u_2' > f(u_2')$. Therefore, we cannot have $W(u_1', u_2') = W(u_1, u_2)$. Geometrically this implies that social indifference curves above the 45^0 line cannot have a slope between zero and infinity. The same argument, of course, works below the 45^0 line (by switching u_1 and u_2). Thus, we are left with four possibilities regarding slopes of indifference curves above and below the 45^0 line:

- 0 above the line, 0 below the line: 2 is a dictator;
- 0 above the line, ∞ below the line: anti-Rawlsian SWF;
- ∞ above the line, 0 below the line: Rawlsian SWF;
- ∞ above the line, ∞ below the line: 1 is a dictator.

22.E.1. (a) Egalitarian solution $f_e(U)$(Example 22.E.1) solves $\max_U \min_i u_i$.

- IUO is satisfied by construction (see footnote 26 in text).
- Paretian: Suppose $u^* = f_e(U)$. Suppose in negation that $u' \in U$ and $u' \gg u$. We then have $\max_U \min_i u_i' > \max_U \min_i u_i$, which contradicts u^* being the egalitarian solution.
- Symmetry: follows from Exercise 22.C.1.
- Individual rationality: Suppose $u^* = f_e(U)$. Since u^* solves $\max_U \min_i u_i$ and $0 \in U$, we must have $\min_i u_i^* \ge 0$.

(b) Utilitarian solution $f_u(U)$ (Example 22.E.2) solves $\max_{U \cap \mathbb{R}^I_+} \sum_i u_i$.

- IUO is satisfied by construction (see footnote 26 in text).

- Paretian: Suppose $u^* = f_u(U)$. Suppose in negation that $u' \in U$ and $u' \gg u$. We then have $\sum_i u'_i > \sum_i u_i$, which contradicts u^* being the utilitarian solution.

- Symmetry: follows from Exercise 22.C.1.

- Individual rationality: $f_u(U) \geq 0$ by definition, since the maximization domain is restricted to the positive orthant.

(c) Nash solution (Example 2.E.3) solves $\max_{U \cap \mathbb{R}^I_+} \sum_i \ln u_i$

- IUO is satisfied by construction (see footnote 26 in text).

- Paretian: Suppose $u^* = f_n(U)$. Suppose in negation that $u' \in U$ and $u' \gg u$. We then have $\sum_i \ln u'_i > \sum_i \ln u_i$, which contradicts u^* being the Nash solution.

- Symmetry: follows from Exercise 22.C.1.

- Individual rationality: $f_n(U) \geq 0$ by definition, since the maximization domain is restricted to the positive orthant.

(d) Kalai-Smorodinsky solution (Example 2.E.4) is $f_k(U) = t^*(u_1(U),u_2(U))$, where $t^* = \max \{t \mid t(u_1(U),u_2(U)) \in U\}$.

- IUO is satisfied by construction (see footnote 26 in text).

- Paretian: Suppose $u^* = f_k(U)$. Suppose in negation that u' and $u' \gg u$. Let $t = \min_i (u'_i/u_i(U)) > t^*$. We then have $t(u_1(U),u_2(U)) \leq \iota$ and therefore $t(u_1(U),u_2(U))$, which contradicts u^* being the Kalai-Smorodinsky solution.

- Symmetry: If U is symmetric, then $u_1(U) = u_2(U)$, and therefore

$f_{k1}(U) = tu_1(U) = tu_2(U) = f_{k2}(U)$.

- Individual rationality: since $t = 0$ is feasible, we must have $t^* \geq 0$.

Therefore, $f_k(U) \geq 0$.

22.E.2. Take a vector of weights $(\lambda_1, ..., \lambda_I) >> 0$. Then define

(a) Non-symmetric egalitarian solution: solves $\max_U \min_i \lambda_i u_i$.

(b) Non-symmetric utilitarian solution: solves $\max_U \sum_i \lambda_i u_i$.

(c) Non-symmetric Nash solution: solves $\max_U \sum_i \lambda_i \ln u_i$.

(d) Non-symmetric Kalai-Smorodinsky solution: $f_k(U) = t^*(\lambda_1 u_1(U), \lambda_2 u_2(U))$, where $t^* = \max \{t|\ t(\lambda_1 u_1(U), \lambda_2 u_2(U)) \in U\}$.

These solutions satisfy all the axioms which the "standard" solutions satisfy, except symmetry. Weights (λ_i) can be interpreted as the relative importance of individuals in bargaining.

22.E.3. (a) The Nash solution $f = (f_1, f_2) \in U$ is given by $f = \operatorname{argmax}_U W(u) = \sum_i \ln u_i$. Take the line which separates U and $V = \{u|\ W(u) > W(f)\}$. (The existence of such a line follows from the separating hyperplane theorem, Theorem M.G.2 in the Mathematical Appendix.) If the boundary of U is smooth, then this line is tangent to it. We know that the boundary of V is smooth, therefore this line is tangent to its boundary. From the equation of the boundary of V, a normal to the line can be written as $(\partial W/\partial u_1, \partial W/\partial u_2) = (1/f_1, 1/f_2)$. Since this line passes through $N = (f_1, f_2)$, its equation can be written as $u_1/f_1 + u_2/f_2 = f_1/f_1 + f_2/f_2 = 2$.

From this we can see that this line crosses the vertical axis at $A = (0, 2f_2)$, and the horizontal axis at $B = (2f_1, 0)$. Using similarity of triangles (see Figure 22.E.3(a)), one can deduce that N is the midpoint of AB.

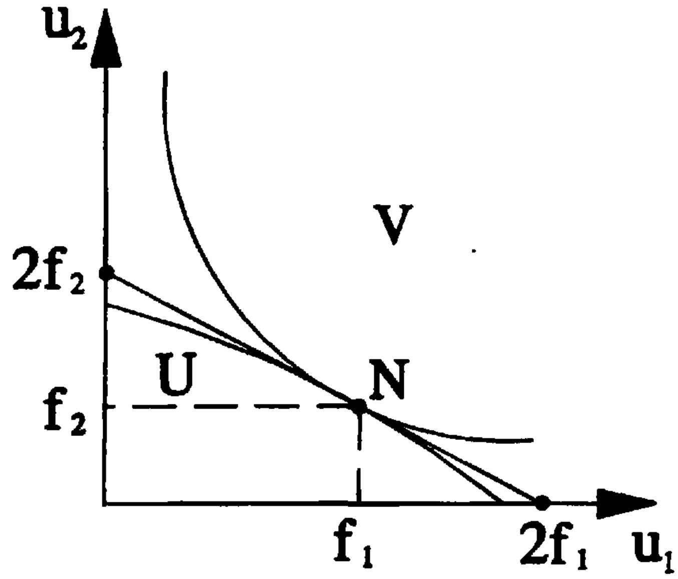


Figure 22.E.3(a)

(b) The idea here is that the rescaling should transform the tangent line AB into a line with slope -1. Take the rescaling $u_1' = \beta_1 u_1$. Then A is mapped into $A' = (0, 2\beta_2 f_2)$, and B is mapped into $B' = (2\beta_1 f_1, 0)$. For $A'B'$ to have slope -1, we can take, for example, $\beta_1 = 1/f_1$, for this gives us $A' = (0, 1)$ and $B' = (1, 0)$. The Nash point in the new utilities will then be $N' = (\beta_1 f_1, \beta_2 f_2) = (1, 1)$. (See Figure 22.E.3(b).)

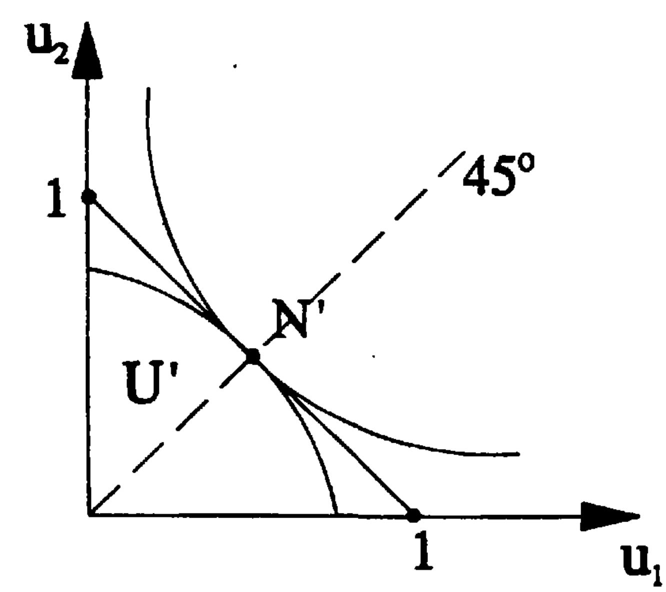


Figure 22.E.3(b)

Now it is clear that in the new utilities the Nash point N' is both the utilitarian solution (since U is supported at N' by a line AB with slope -1) and the egalitarian solution (since N' is at the intersection of the 45^0 line and on the boundary of the convex set U).

22.E.4. (a) The Kalai-Smorodinsky solution (Example 2.E.4) can be written as $f_k(U) = t^*(u_1(U),u_2(U))$, where $t^* = \max \{t \mid t(u_1(U),u_2(U)) \in U\}$. Take a linear rescaling $u_1' = \beta_1 u_1$, and let this rescaling transform U into U'. We need to figure out how $f_k(U') = t'^*(u_1(U'),u_2(U'))$ relates to $f_k(U)$. First, it is clear that $u_1(U') = \beta_1 u_1(U)$. Then

$t'^* = \max \{t' \mid t'(u_1(U'), u_2(U')) \in U'\} =$

$= \max \{t' \mid t'(\beta_1 u_1(U), \beta_2 u_2(U)) \in U'\} = \max \{t' \mid t'(u_1(U), u_2(U)) \in U\} = t^*$.

Thus, $f_k(U') = t^*(u_1(U'), u_2(U')) = t^*(\beta_1 u_1(U), \beta_2 u_2(U))$. This means that $f_{k1}(U') = \beta_1 f_{k1}(U)$, i.e. f_k is independent of utility units.

(b) Take $U = \{u \mid u_1 + u_2 \le 2\}$, $U' = \{u \mid u_1 + u_2 \le 2, u_2 \le 1\}$. We have $U' \subset U$, and by symmetry $f_k(U) = (1,1)$. At the same time, $u_1(U') = 2$ and $u_2(U') = 1$, which implies that $f_{k2}(U')/f_{k1}(U') = 1/2$. Therefore, $f_k(U') \neq (1, 1) = f_k(U)$. Thus, the Kalai-Smorodinsky solution does not satisfy IIA.

22.E.5. (a) The egalitarian solution can be written as $f_e(U) = t^* e$, where $e = (1,\dots,1)$, and $t^* = \max \{t \mid te \in U\}$.

(i) First we show that the egalitarian solution satisfies the four properties. The properties of IUO, Pareto, and symmetry are shown in Exercise 22.E.1. Monotonicity: let $U \subset U'$. Then $t'^* = \max \{t \mid te \in U'\} \ge t^* = \max \{t \mid te \in U\}$, and therefore $f_e(U') = t'^* e \ge f_e(U) = t^* e$.

(ii) Now we show that a solution that satisfies the four properties is egalitarian. We will show this only for utility possibility sets for which the weak and strong Pareto properties coincide. (The general proof is more difficult.)

Independence of utility origins allows us to normalize the threat point to be the origin, and to write our solution as $f(U)$. Now, consider an arbitrary UPS U and let its egalitarian solution be $f_e(U) = (a,\ldots,a)$. Define $U' = \{u \in \mathbb{R}^I_+ | \sum_i u_i \leq Ia\}$. The properties of Pareto and symmetry of $f(\cdot)$ imply that $f(U') = (a,\ldots,a)$. Since $U \cup U' \subset U$, by monotonicity of $f(\cdot)$ we must have $f(U \cup U') \leq f(U') = (a,\ldots,a)$. But we know that $(a,\ldots,a) \in U \cup U'$. By (strict) Pareto optimality of $f(U')$, we must therefore have $f(U \cup U') = (a,\ldots,a)$. Now, since $U \cup U' \subset U$, by monotonicity of $f(\cdot)$ we must have $f(U) \geq f(U \cup U') = (a,\ldots,a) = f_e(U)$. But we know from (i) that $f_e(U)$ is a (weak) Pareto optimum in U. Therefore, it isalso a strict Pareto optimum, and we must have $f(U) = f_e(U)$.(b) Independence of utility origins allows us to normalize the threat point to be the origin, and to write our solution as $f(U)$. Let us start with looking at the family of symmetric UPS with linear boundaries:

$U(a) = \{u \in \mathbb{R}^I_+ | \sum_i u_i \leq a\}$, and let us define $v(a) = f(U(a))$. First we establish that $v(a)$ is continuous in $a \in [0,+\infty)$. By Pareto optimality $\sum_i v_i(a) = a$ for all a. Using monotonicity,

$v(a+\varepsilon) \in \{u \in \mathbb{R}^I_+ | \sum_i u_i = \sum_i v_i(a) + \varepsilon,\ u \geq v(a)\}$. This implies that $|v(a+\varepsilon) - v(a)| \leq \varepsilon$. Therefore, $v(\cdot)$ describes a continuous curve. By definition, $v(0) = 0$. Also, since $\sum_i v_i(a) = a$, we must have $|v(a)| \geq [I(a/I)^2]^{1/2} = aI^{-1/2}$, therefore the curve $v(\cdot)$ is unbounded.

Now, take an arbitrary UPS U. On one hand, we have $v(0) = 0 \in U$. On the othe hand, since $v(\cdot)$ is unbounded and $U \cap \mathbb{R}^I_+$ is bounded, we must have $v(a^\circ) \notin U$ for some $a^\circ \in (0,+\infty)$. Since $v(\cdot)$ is continuous and U is closed, there exists an $a' \in (0,a^\circ)$ such that $v(a')$ lies on the boundary of U. Denote $U' = U(a')$, and apply the argument of part (a)(ii) of this exercise to show that $f(U) = f(U') = v(a')$. Thus, for every UPS U, $f(U)$ is the intersection of the boundary of U with the curve $v(\cdot)$.

22.E.6. The Kalai-Smorodinsky (K-S) solution (Example 2.E.4) can be written as $f_k(U) = t^*(u_1(U),u_2(U))$, where $t^* = \max\,\{t \mid t(u_1(U),u_2(U)) \in U\}$.

(i) First we show that the K-S solution satisfies the five properties. The properties of IUO, Pareto, and symmetry are shown in Exercise 22.E.1. The property of IUU is shown in Exercise 22.E.4. Partial monotonicity:
let U' expand U in the direction of agent $j \neq i$, i.e.
$U \subset U'$ and $u_i(U) = u_i(U')$. Observe that $U \subset U'$ implies $u_j(U) \leq u_j(U')$. Let $\hat{t} = f_j(U)/u_j(U') \leq f_j(U)/u_j(U) = t^*$. Then

$$\hat{t}(u_1(U'),\ u_j(U')) \leq (tu_1(U),\ f_j(U)) = f(U) \qquad \text{(see Figure 22.E.6(a))}.$$

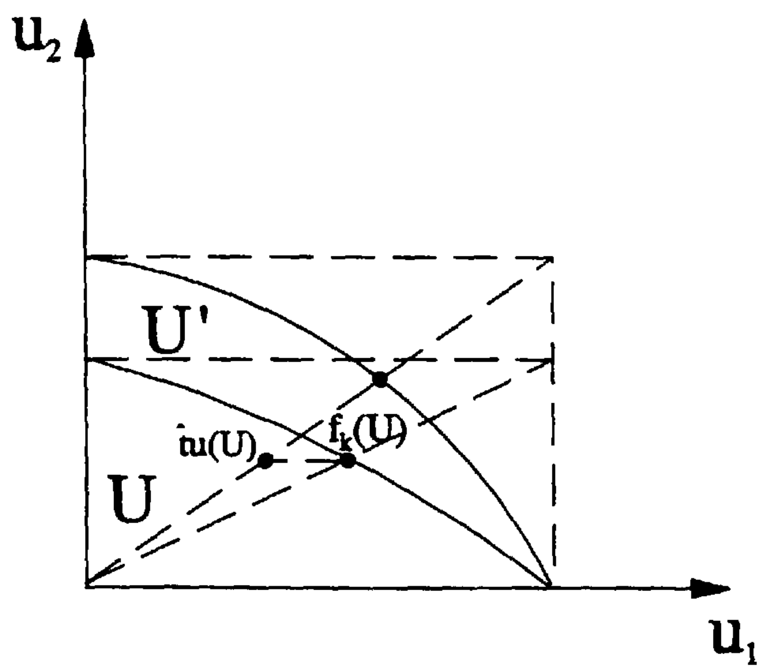


Figure 22.E.6(a)

Therefore, $\hat{t}(u_1(U'), u_j(U')) \in U \subset U'$. This implies that

$$t'^{*} = \max \{t' \mid t'(u_1(U'), u_j(U')) \in U'\} \geq \hat{t}.$$

Therefore, $f_{kj}(U') = t'^{*} u_j(U') \geq \hat{t}\, u_j(U') \geq \hat{t}\, u_j(U) = f_j(U)$.

(ii) Now we show that a solution that satisfies the five properties has to coincide with the K-S solution. Independence of utility origins allows us to normalize the threat point to be the origin, and to write our solution as $f(U)$. Independence of utility units allows us to rescale utilities to normalize $u_1(U) = u_2(U) = 1$. Let $f_k(U)$ be the K-S solution for U. Once we show that $f(U) = f_k(U)$, independence of utility units and origins of both $f(U)$ and $f_k(U)$ will allow us to extend this equality to all bargaining problems.To show that $f(U) = f_k(U)$, define

$$U' = \{u \mid (1-f_{k2}(U))\, u_1 + f_{k1}(U)\, u_2 \leq f_{k1}(U),$$
$$f_{k2}(U))\, u_1 + (1-f_{k1}(U))\, u_2 \leq f_{k2}(U)\}.$$

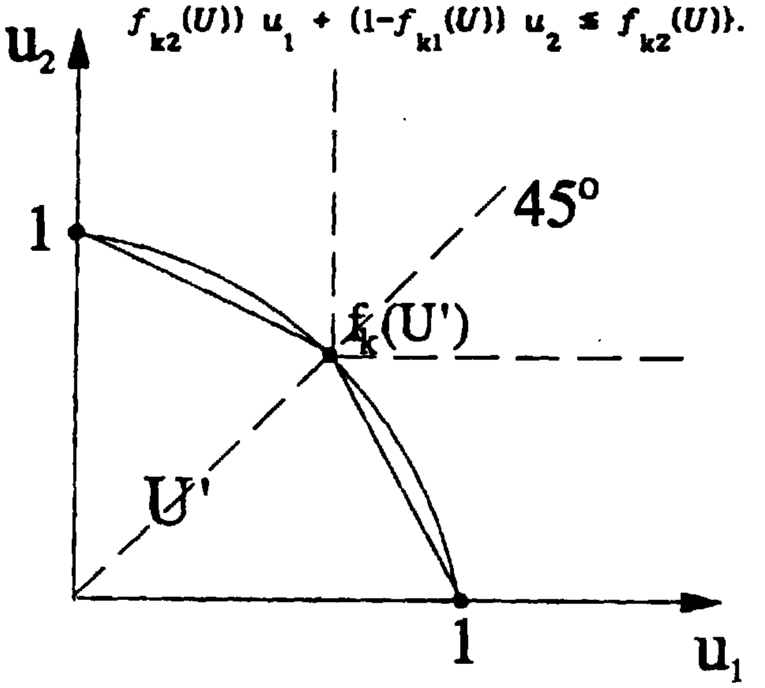


Figure 22.E.6(b)

Geometrically, U' is bounded by two lines: one connecting $f_k(U)$ and (0,1), and the other connecting $f_k(U)$ to (1,0). (See Figure 22.E.6(b).) The properties of symmetry and Pareto imply that $f(U') = f_k(U)$. By convexity of U we have $U' \subset U$. We also have $u_1(U') = u_1(U) = 1$, $u_2(U') = u_2(U) = 1$. Now we can use partial monotonicity in both directions: U expands U' in the direction of agent 1 $\Rightarrow$

$f_1(U) \geq f_1(U') = f_{k1}(U)$. Also, U expands U' in the direction of agent 2 $\Rightarrow$

$f_2(U) \geq f_2(U') = f_{k2}(U)$. To summarize, $f(U) \geq f_k(U)$. As the Kalai-Smorodinsky solution for $I = 2$ is always strictly Pareto optimal (check), this implies

$f(U) = f_k(U)$.

22.E.7. Let $S \subset I$. For any $u_{-S} \in \mathbb{R}^{I\setminus S}$, define

$$U^S(u_{-S}) = \{u_{-S} \in \mathbb{R}^S \mid (u_S, u_{-S}) \in U\}.$$

In words, $U^S(u_{-S})$ is the bargaining set left for coalition S when utilities of the other players have been fixed at u_{-S}. Consistency then means that for any $S \subset I$, $f^I_S(U) = f^S(U^S(f^I_{-S}(U)))$.

Our solution for any I is given by $f^I(U) = \underset{U}{\operatorname{argmax}} \sum_I g(u_i)$, and uniqueness of argmax is guaranteed by strict convexity of $g(\cdot)$ and convexity of U.

Suppose in negation that $f^I(U)$ is not consistent, i.e. that for some U and some S we have $u'_S = f^S(U^S(f^I_{-S}(U))) \neq f^I_S(U)$. This means that

$u' = (u'_S, f^I_{-S}(U)) \in U$ and that $\sum_S g(u'_{Si}) = \max_{U^S(f^I_{-S}(U))} \sum_S g(u_i) > \sum_S g(f^I_i(U))$.

But then we have

$$\sum_I g(u'_i) = \sum_S g(u'_{Si}) + \sum_{I\setminus S} g(f^I_i(U)) > \sum_S g(f^I_i(U)) + \sum_{I\setminus S} g(f^I_i(U)) = \sum_I g(f^I_i(U)),$$

which contradicts the definition of $f^I(U)$.

22.E.8. Take $I = 3$, and let $U = \{u \in \mathbb{R}^3 \mid u_1 \leq 1, u_1/2 + u_2 + u_3 \leq 1\}$.

It is easy to see that $u^1(U) = u^2(U) = u^3(U) = 1$. To find the Kalai-Smorodinsky solution, therefore, we solve Max $\{t \mid (t,t,t) \in U\}$.

This gives $t = 0.4$, and therefore $f_k(U) = (0.4, 0.4, 0.4)$.

Now, define

$U' = \{(u_1, u_2) | (u_1, u_2, 0.4) \in U\} = \{(u_1, u_2) | u_1 \le 1, u_1/2 + u_2 \le 0.6\}$.

It is easy to see that $u^1(U') = 1$, $u^2(U') = 0.6$. To find the Kalai-Smorodinsky solution for this two-dimensional UPS, we solve Max $\{t | (t, 0.6t) \in U'\}$.

This gives $t = 6/11$. Therefore,

$f_k^{\{1,2\}}(U') = (t, 0.6t) = (6/11, 3.6/11) \neq (0.4, 0.4)$.

Therefore, the Kalai-Smorodinsky solution is not consistent.

22.E.9. (i) The utilitarian solution satisfies IUO, Pareto, and symmetry (Exercise 22.E.1). Since it is given by a maximization problem it satisfies IIA. The utilitarian solution requires comparing utilities across agents, so it clearly does not satisfy IUU.

(ii) The solution given by $f(U) = 0\ \forall U$ clearly satisfies all the properties except Pareto.

(iii) The solution $f(U) = (u_1(U), 0)$ satisfies all properties except symmetry.

(iv) The Kalai-Smorodinsky solution satisfies IUO, Pareto, symmetry (Exercise 22.E.1), and IUU (Exercise 22.E.4), but does not satisfy IIA (Exercise 22.E.4).

(v) Take $f(U, u^*) = f_n(U)$ - the Nash solution where the threat point is taken to be the origin regardless of u^*. Just as the Nash solution, this solution satisfies Pareto, symmetry, IUU and IIA, but it does not satisfy IUO, since now utility origins matter.

22.E.10. Take $U = \{u \in \mathbb{R}^2 | u_1 + u_2 \le 2\}$, $U' = \{u \in \mathbb{R}^2 | u_1 \le 1, u_2 \le 1\}$. Observe that $U' \in U$, but and that we can take, for example, $f(U) = (0,2)$, while we must have $f(U') = (1,1)$.

22.E.11. (a) Take a SPE in stationary strategies, and let v_i be player i's continuation payoff in this equilibrium at a stage when he makes an offer. Denote the players' inverse utility functions by $\phi_i(\cdot) = u_i^{-1}(\cdot)$, $i = 1,2$. (Observe that $\phi_i(\cdot)$ is increasing and convex as the inverse of the increasing concave function $u_i(\cdot)$.) In a stationary SPE, player i will demand $\phi_i(v_i)$ every time, and his offer will be accepted. Indeed, if he demanded something other than $\phi_i(v_i)$ and his offer were accepted, then his continuation utility would be different from v_i. On the other hand, if his offer were rejected, then in equilibrium it would always be rejected, and we would have $v_i = 0$, which can be easily ruled out as a SPE outcome (show this). This argument implies, in particular, that player j's strategy ($j \neq i$) should involve rejecting any offer giving him less than $m - \phi_i(v_i)$.

Now, in order for player j to accept player i's offer in equilibrium, he must be at least as well off doing this as he would be by rejecting i's offer and demanding $\phi_j(v_j)$ at the next stage, which would be accepted. On the other hand, if he was strictly better off accepting player i's offer , player i could raise his demand a little bit and player j would still accepted it. Therefore, player j should be as well off accepting player i's offer as he would be rejecting it and making an offer at the next stage. This can be written as $u_j(m - \phi_i(v_i)) = \delta v_j$, or equivalently $m - \phi_i(v_i) = \phi_j(\delta v_j)$. Substituting in turn i=1 and i=2, we obtain a system of equations:

$$\begin{cases} \phi_1(v_1) + \phi_2(\delta v_2) = m \\ \phi_1(\delta v_1) + \phi_2(v_2) = m \end{cases} \qquad (*)$$

To show uniqueness of the solution to (*), we need the following lemma:

Lemma: $\phi_1(x) - \phi_1(\delta x)$ is increasing in x for $x > 0$.

Proof: Take $x' > x > 0$. We can have the following two cases:

Case (i): $\delta x' \geq x$. Then by the Mean Value Theorem,

$$\phi_1(x') - \phi_1(\delta x') = (1-\delta)\, x'\, \phi_1'(y'),$$
$$\phi_1(x) - \phi_1(\delta x) = (1-\delta)\, x\, \phi_1'(y),$$

where $\delta x \leq y \leq x \leq \delta x' \leq y' \leq x'$. From convexity of $\phi_1(\cdot)$ we have $\phi_1'(y') \geq \phi_1'(y)$. Therefore,

$$\phi_1(x') - \phi_1(\delta x') = (1-\delta)\, x'\, \phi_1'(y') > (1-\delta)\, x\, \phi_1'(y') \geq$$
$$\geq (1-\delta)\, x\, \phi_1'(y) = \phi_1(x) - \phi_1(\delta x).$$

Case (ii): $\delta x' < x$. Then by the Mean Value Theorem,

$$\phi_1(x') - \phi_1(x) = (x' - x)\, \phi_1'(y'),$$
$$\phi_1(\delta x') - \phi_1(\delta x) = \delta(x' - x)\, \phi_1'(y),$$

where $\delta x \leq y \leq \delta x' < x \leq y' \leq x'$. From convexity of $\phi_1(\cdot)$ we have $\phi_1'(y') \geq \phi_1'(y)$. Therefore,

$$\phi_1(x') - \phi_1(x) = (x' - x)\, \phi_1'(y') > \delta\, (x' - x)\, \phi_1'(y') \geq$$
$$\geq \delta\, (x' - x)\, \phi_1'(y) = \phi_1(\delta x') - \phi_1(\delta x).$$

This can be rewritten as $\phi_1(x') - \phi_1(\delta x') > \phi_1(x) - \phi_1(\delta x)$. □

Now, suppose in negation that (v_1', v_2') and (v_1'', v_2'') are two different solutions to (*), and without loss of generality suppose that $v_1'' > v_1'$. Then each of the two equations implies that $v_2'' < v_2'$. Subtracting the second equation in (*) from the first, we obtain $\phi_1(v_1) - \phi_1(\delta v_1) = \phi_2(v_2) - \phi_2(\delta v_2)$, which should hold for every solution, in particular (v_1', v_2'). But then the Lemma implies

$$\phi_1(v_1'') - \phi_1(\delta v_1'') > \phi_1(v_1') - \phi_1(\delta v_1') = \phi_2(v_2') - \phi_2(\delta v_2') > \phi_2(v_2'') - \phi_2(\delta v_2''),$$

which means that (v_1'', v_2'') cannot be a solution to (*) -contradiction.

(b) The utility possibility set can be written as

$$U = \{(v_1,v_2) \in \mathbb{R}^2 \mid \phi_1(v_1) + \phi_2(v_2) \le m\}.$$

The Nash bargaining solution solves $\max_U [\ln u_1 + \ln u_2]$. The first-order condition for this maximization problem can be written as $u_1\, \phi_1'(v_1) = u_2\, \phi_2'(v_2)$. Since the Nash bargaining solution has to lie on the boundary of U, it can be described by the following system of equations:

$$\begin{cases} u_1\, \phi_1'(v_1) = u_2\, \phi_2'(v_2) \\ \phi_1(v_1) + \phi_2(v_2) = m. \end{cases} \qquad (**)$$

Now, we will show that the solution to (*) goes to the solution to (**) as $\delta \longrightarrow 1$. First, both equations in (*) converge to the second equation in (**). Second, let subtracting the second equation in (*) from the first results in $\phi_1(v_1) - \phi_1(\delta v_1) = \phi_2(v_2) - \phi_2(\delta v_2)$. Taking the first-order Taylor expansion of both sides around $\delta = 1$ yields: $(1-\delta)\, v_1\, \phi_1'(v_1) \approx (1-\delta)\, v_2\, \phi_2'(v_2)$, which means that asymptotically as $\delta \longrightarrow 1$ the first equation in (*) becomes satisfied. Therefore, as $\delta \longrightarrow 1$ the solution to (*) converges to the solution to (**).

(c) Let $\bar{v}_i$ and $\underline{v}_i$ be respectively the maximum and the minimum continuation payoffs achieved by player i when he makes an offer in a SPE, $i = 1,2$. Observe that

(i) Player $j \ne i$ will always reject an offer leaving him less than $\delta \underline{v}_j$, since in any SPE starting next period he will get at least $\underline{v}_j$. Thus, player i will not demand more than $m - \phi_j(\delta \underline{v}_j)$, and his continuation payoff in any SPE cannot exceed $u_i(m - \phi_j(\delta \underline{v}_j))$. This implies that $\bar{v}_i \le u_i(m - \phi_j(\delta \underline{v}_j))$, or equivalently $\phi_i(\bar{v}_i) + \phi_j(\delta \underline{v}_j) \le m$.

(ii) Player $j \ne i$ will always accept an offer giving him more than $\delta \bar{v}_j$, since in any SPE starting next period he will get at most $\bar{v}_j$. Thus, player i can always demand almost as much as $m - \phi_j(\delta\bar{v}_j)$, and his continuation payoff in any SPE cannot fall short of $u_i(m - \phi_j(\delta\bar{v}_j))$. This implies that $\underline{v}_i \geq u_i(m - \phi_j(\delta\bar{v}_j))$, or equivalently $\phi_i(\underline{v}_i) + \phi_j(\delta\bar{v}_j) \geq m$.

Substituting in turn $i = 1$ and $i = 2$, we obtain a system of four inequalities:

$$\begin{cases} \phi_1(\bar{v}_1) + \phi_2(\delta\underline{v}_2) \leq m, \\ \phi_1(\underline{v}_1) + \phi_2(\delta\bar{v}_2) \geq m, \\ \phi_2(\bar{v}_2) + \phi_1(\delta\underline{v}_1) \leq m, \\ \phi_2(\underline{v}_2) + \phi_1(\delta\bar{v}_1) \geq m. \end{cases} \qquad (***)$$

We will show that these inequalities imply $\bar{v}_1 = \underline{v}_1$, $\bar{v}_2 = \underline{v}_2$. Suppose in negation that it is not true, and without lose of generality $\bar{v}_1 > \underline{v}_1$.

Then subtracting the first inequality from the second, we obtain

$\phi_2(\delta\bar{v}_2) - \phi_2(\delta\underline{v}_2) \geq \phi_1(\bar{v}_1) - \phi_1(\underline{v}_1) > 0$, which implies $\bar{v}_2 > \underline{v}_2$. Then adding up the first and the third inequalities and subtracting the second and the fourth inequalities results in

$$[(\phi_1(\bar{v}_1) - \phi_1(\delta\bar{v}_1)) - (\phi_1(\underline{v}_1) - \phi_1(\delta\underline{v}_1))] +$$
$$+ [(\phi_2(\bar{v}_2) - \phi_2(\delta\bar{v}_2)) - (\phi_2(\underline{v}_2) - \phi_1(\delta\underline{v}_2))] \leq 0.$$

But the Lemma proven in (a) implies that whenever $\bar{v}_1 > \underline{v}_1$ and $\bar{v}_2 > \underline{v}_2$, the expressions in both square brackets are positive. Thus, we obtain a contradiction.

Now, when we substitute $\bar{v}_1 = \underline{v}_1 = v_1$, $\bar{v}_2 = \underline{v}_2 = v_2$ in the system (***), the system is reduced to (*). Therefore, the payoffs achieved in a SPE are unique.

22.F.1. IUO allows us to normalize the threat point to be the origin. The UPS in the TU case can be described as $U = \{u \in \mathbb{R}^I \mid \sum_i u_i \leq \hat{u}\}$.

Pareto property implies that $\sum_i f_i(U) = \hat{u}$. Symmetry implies that $f_1(U) = \ldots = f_I(U)$. Therefore, we must have $f_1(U) = \ldots = f_I(U) = \hat{u}/I$.

22.F.2. $f_{Si}(v) = \frac{1}{I!}\sum_{\pi}[v(\{h|\ \pi(h) \le \pi(i)\}) - v(\{h|\ \pi(h) < \pi(i)\})]$ (*)

Invariance to changes in utility origins:

Take $u_i' = u_i + \alpha_i$ for every i. Then in new utilities

$v'(S) = v(S) + \sum_{j\in S}\alpha_j$ for every $S \subset I$. Substituting in (*), we obtain

$$f_{Si}(v') = \frac{1}{I!}\sum_{\pi}[v(\{h|\ \pi(h) \le \pi(i)\}) + \sum_{j\in\{h|\pi(h)\le\pi(i)\}}\alpha_j - v(\{h|\ \pi(h) < \pi(i)\}) - \sum_{j\in\{h|\pi(h)<(i)\}}\alpha_j] =$$

$$= \frac{1}{I!}\sum_{\pi}[v(\{h|\ \pi(h) \le \pi(i)\}) - v(\{h|\ \pi(h) < \pi(i)\}) + \alpha_i] =$$

$$= f_{Si}(v) + \frac{1}{I!}\sum_{\pi}\alpha_i = f_{Si}(v) + \alpha_i.$$

Invariance to common changes in utility units:

Take $u_i' = \beta\, u_i$ for every i. Then in new utilities $v'(S) = \beta\, v(S)$

for every $S \subset I$. Substituting in (*), we obtain

$$f_{Si}(v') = \frac{1}{I!}\sum_{\pi}[\beta\ v(\{h|\ \pi(h) \le \pi(i)\}) - \beta\ v(\{h|\ \pi(h) < \pi(i)\})] = \beta\ f_{Si}(v).$$

Pareto:

$$\sum_i f_{Si}(v) = \frac{1}{I!}\sum_{\pi}\sum_i[v(\{h|\ \pi(h) \le \pi(i)\}) - v(\{h|\ \pi(h) < \pi(i)\})] =$$

$$= \frac{1}{I!}\sum_{\pi}[v(S) - v(\emptyset)] = v(S).$$

Symmetry:

Suppose $v(S) = v'(\mu(S))$ for all $S \subset I$, where $\mu: I \to I$ is a permutation of agents. Then from (*) we can write

$$f_{S\mu(i)}(v') = \frac{1}{I!}\sum_{\pi}[v'(\{h|\ \pi(h) \le \pi\mu(i)\}) - v'(\{\mu(h)|\ \pi(h) < \pi\mu(i)\})] =$$

(replace h with $\mu(h)$)

$$= \frac{1}{I!}\sum_{\pi}[v'(\{\mu(h)|\ \pi\mu(h) \le \pi\mu(i)\}) - v'(\{\mu(h)|\ \pi\mu(h) < \pi\mu(i)\})] =$$

$$= \frac{1}{I!}\sum_{\pi}[v'(\mu(\{h|\ \pi\mu(h) \le \pi\mu(i)\})) - v'(\mu(\{h|\ \pi\mu(h) < \pi\mu(i)\}))] =$$

(let $\pi' = \pi\mu$ and use $v'(\mu(S)) = v(S)$)

$$= \frac{1}{I!}\sum_{\pi'}[v(\{h|\ \pi'(h) \le \pi'(i)\}) - v(\{h|\ \pi'(h) < \pi'(i)\})] =$$

$$= f_{Si}(v).$$

The dummy axiom:

If $v(S\cup\{i\}) = v(S)$ for all $S \subset I$, then from (*) we obtain

$$f_{Si}(v) = \frac{1}{I!}\sum_{\pi} 0 = 0.$$

22.F.3. (a)

$$f_{Si}(v+v') = \frac{1}{I!}\sum_{\pi}[v(\{h|\ \pi(h) \le \pi(i)\}) + v'(\{h|\ \pi(h) \le \pi(i)\}) -$$

$$- v(\{h|\ \pi(h) < \pi(i)\}) - v'(\{h|\ \pi(h) < \pi(i)\})]$$

$$= \frac{1}{I!}\sum_{\pi}[v(\{h|\ \pi(h) \le \pi(i)\}) - v(\{h|\ \pi(h) < \pi(i)\})] +$$

$$+ \frac{1}{I!}\sum_{\pi}[v'(\{h|\ \pi(h) \le \pi(i)\}) - v'(\{h|\ \pi(h) < \pi(i)\})] =$$

$$= f_{Si}(v') + f_{Si}(v').$$

(b) Suppose that the agents do not know which of the two bargaining situations v and v' will occur, and they expect each situation to occur with probability 1/2. Linearity of a bargaining solution then implies that the agents are indifferent as to whether the solution is applied before or after uncertainty is resolved:

$$E_{\tilde{v}}\, f_i(\tilde{v}) = 1/2\ f_i(v) + 1/2\ f_i(v') =$$

(by IUU)

$$= f_i(1/2v) + f_i(1/2\ v') = = f_i(1/2v + 1/2\ v') = f_i(E\tilde{v}).$$

22.F.4. (a) Let v be a T-unanimity game, where $T\subset I$. Then for each $i \in I\setminus T$ and each $S\subset I$ we have

$$v(S\cup\{i\}) = \begin{Bmatrix} v(I), \ T\subset S \\ 0 \text{ otherwise} \end{Bmatrix} = v(S).$$

Therefore, by the dummy axiom, $f_i(v) = 0$ for each $i \in I\setminus T$. By the Pareto property,

$$\sum_{i\in T} f_i(v) = \sum_{i\in I\setminus T} f_i(v) = v(I).$$

Sincy by summetry we must have $f_i(v) = f_j(v)$ for each $i,j \in T$, we must have $f_i(v) = 1/|T| \ v(I)$ for each $i \in T$.

(b) Let v, v' be any two TU characteristic form games on the set I of agents, and α be a real number. Then we can define two new TU characteristic function games on I, $v+v'$ and αv, so that for any $S\subset I$,

$$(v+v')(S) = v(S) + v'(S),$$

$$(\alpha v)(S) = \alpha v(S).$$

It is easy to check that with addition and multiplication by a real number defined as above, the set of TU characteristic function games is a linear space, which we will denote as V. Since a TU characteristic function game v is described by a vector $(v(S))_{S\subset I}$ of $2^{|I|}$ numbers, the dimensionality of V is $2^{|I|}$.

For any $T\subset I$, define the *normal T-unanimity game* v_T by

$$v_T(S) = \begin{cases} 1, \ T\subset S \\ 0 \text{ otherwise.} \end{cases}$$

We are going to show that normal unanimity games constitute a basis in V. First, there are $2^{|I|}$ different normal unanimity games. Second, we will show that normal unanimity games are linearly independent, i.e. that for any $\alpha \in \mathbb{R}^{2^I}$ such that $\sum_{T\subset I} \alpha_T \, v_T(S) = \sum_{T\subset S} \alpha_T = 0 \ \forall S$, we must have $\alpha_T = 0 \ \forall T\subset I$.

We will show this by induction on the number of elements in T. Setting $S=\varnothing$, we find $\sum_{T\subset\emptyset} \alpha_T = \alpha_\emptyset = 0$. Further, suppose that $\alpha_T = 0$ for all $T\subset I$ such that $|T| \le k < |I|$. Then, for any $S\subset I$ such that $|S| = k + 1$, we must have

$$\sum_{T\subset S} \alpha_T = \alpha_S + \sum_{T\subset S,\, |T|\le k+1} \alpha_T = \alpha_S + 0 = \alpha_S = 0,$$

i.e. the statement is true for all subsets of I containing $k+1$ elements.

By induction, $\alpha_T = 0\ \forall T\subset I$, which proves linear independence of normal unanimity games.

Now it is easy to prove that weak linearity of a cooperative solution $f(\cdot)$ implies its linearity. Weak linearity of $f(\cdot)$ means that for any unanimity game v and any $v' \in V$, $f(v) + f(v') = f(v+v')$. Now, take instead any two characterestic function games v, v' $v' \in V$. Let $\alpha \in \mathbb{R}^{2^I}$ be the coordinates of v in the basis of normal unanimity games, i.e. let

$v = \sum_{T\subset I} \alpha_T v_T$. Since $\alpha_T v_T$ is a unanimity game for all $T\subset I$, we can use weak linearity iteratively to obtain

$$f(v+v') = f\Big(\sum_{T\subset I} \alpha_T v_T + v'\Big) = \sum_{T\subset I} f(\alpha_T v_T) + f(v') =$$

$$= f\Big(\sum_{T\subset I} \alpha_T v_T\Big) + f(v') = f(v) + f(v'),$$

i.e. $f(\cdot)$ is linear.

(c) It is straightforward to check that the Shapley value satisfies all the mentioned properties. Here we will show that any cooperative solution $f(\cdot)$ which satisfies all the mentioned properties coincides with the Shapley value. Take any characteristic function game v, and let $\alpha \in \mathbb{R}^{2^I}$ be the coordinates of v in the basis of normal unanimity games, i.e.

$v = \sum_{T\subset I} \alpha_T v_T$. Using linearity, invariance to common changes of utility units, and the result of part (a), we can write

$$f_i(v) = \sum_{T\subset I} f_i(\alpha_T v_T) = \sum_{T\subset I} \alpha_T f_i(v_T) = \sum_{T\subset I | i\in T} \alpha_T f_i(v_T) = \sum_{T\subset I | i\in T} \alpha_T/|T|.$$

Now we can show that $f(\cdot)$ satisfies Definition 18.AA.7 of the Shapley value in Appendix A to Chapter 18 of the textbook (p.681). For this purpose, we define

$f_i(S,v) = \sum_{T\subset S | i\in T} \alpha_T/|T|$. Clearly, $f_i(v) = f_i(I,v)$, and it thus remains to show that $f_i(S,v)$ satisfies both conditions of (18.AA.2). To see that the first condition (preservation of utility differences) is satisfied, take any $S \subset I$ and any $i,h \in S$, and observe that

$$f_i(S,v) - f_i(S\setminus\{h\},v) = \sum_{T\subset S | i\in T} \alpha_T/|T| - \sum_{T\subset S\setminus\{h\} | i\in T} \alpha_T/|T| = \sum_{T\subset S | \{i,h\}\subset T} \alpha_T/|T|.$$

Since the last expression is symmetric in i and h, it is clear that $f_i(S,v) - f_i(S\setminus\{h\},v) = f_h(S,v) - f_h(S\setminus\{i\},v)$, i.e. utility differences are preserved. To see that the second (adding up) condition is satisfied, take any $S \subset I$ and observe that

$$\sum_{i\in S} f_i(S,v) = \sum_{i\in S} \sum_{T\subset S | i\in T} \alpha_T/|T| = \sum_{T\subset S} |T| \, \alpha_T/|T| = \sum_{T\subset S} \alpha_T = \sum_{T\subset I} \alpha_T \, v_T(S) = v(S).$$

(We have reduced the double summation to a single sum, noting that each set $T\subset S$ is encountered $|T|$ times in the double summation. The last equality follows from the defintion of α as the vector of coordinates of v in the basis of normal unanimity games.) Therefore, $f_i(S,v)$ satisfies both parts of (18.AA.2), and $f(\cdot)$ coincides with the Shapley value as defined in Definition 18.AA.7. For the relation between this and other definitions of Shapley value, see e.g. A. Mas-Colell and S. Hart, "Potential, Value, and Consistency", *Econometrica* 1989, 57(3), p.589(26).

22.F.5. (a) Suppose in negation that the two lowest excesses are not equal. Without loss of generality, suppose that

$$e(u,\{1,2\}) < e(u,\{1,3\}) = m_2(u) \le e(u,\{2,3\}) = m_1(u). \qquad (1)$$

(The individuals could always be renumbered to obtain this.) First, we show that $u_1 + u_2 > 0$. Indeed, suppose in negation that $u_1 + u_2 = 0$, then by exact feasibility of the utility profile we must have $u_3 = v(I)$. Then using the definition of excesses, by (1) we must have

$$e(u,\{1,2\}) = v(1,2) < e(u,\{1,3\}) = v(1,3) - v(I).$$

This implies that $v(I) < v(1,3) - v(1,2) \le v(1,3)$, which contradicts the assumption that $v(S) \le v(I)$ for any coalition S. Therefore, we must have either $u_1 > 0$, or $u_2 > 0$, or both.

When $u_1 > 0$, take a new exactly feasible utility point $u' = (u_1 - \varepsilon, u_2, u_3 + \varepsilon)$, where $\varepsilon > 0$ is small enough. Then we know that $e(u',\{2,3\}) = m_1(u) - \varepsilon$, $e(u',\{1,3\}) = m_2(u) \le m_1(u)$, and that $e(u',\{1,2\}) = e(u,\{1,2\}) + \varepsilon < \min\{e(u',\{2,3\}), e(u',\{1,3\})\}$. If $m_1(u) - \varepsilon \ge m_2(u)$, the above implies that

$$m_1(u') = e(u',\{2,3\}) = m_1(u) - \varepsilon < m_1(u),$$

which contradicts the assumption that u' is in the nucleolus. If, instead, $m_1(u) - \varepsilon < m_2(u)$, the above implies that

$$m_1(u') = e(u',\{1,3\}) = m_2(u) \le m_1(u),$$

$$m_2(u') = e(u',\{2,3\}) = m_1(u) - \varepsilon < m_2(u),$$

which again contradicts the assumption that u' is in the nucleolus.

When $u_2 > 0$, take a new exactly feasible utility point $u' = (u_1, u_2 - \varepsilon, u_3 + \varepsilon)$, where $\varepsilon > 0$ is small enough. Then we know that $e(u',\{2,3\}) = m_1(u)$, $e(u',\{1,3\}) = m_2(u) - \varepsilon < m_1(u)$, and that $e(u',\{1,2\}) = e(u,\{1,2\}) + \varepsilon < \min\{e(u',\{2,3\}), e(u',\{1,3\})\}$. This implies that $m_1(u') = e(u',\{2,3\}) = m_1(u)$, and $m_2(u') = e(u',\{1,3\}) = m_2(u) - \varepsilon < m_2(u)$. This contradicts the assumption that u' is in the nucleolus.

Therefore, we have shown that in all cases (1) contradicts the assumption that u' is in the nucleolus.

(b) Suppose that the nucleolus contains two utility profiles, u and u'. If u is in the nucleolus, by definition of the nucleolus we must have $m_1(u) \le m_1(u')$. Symmetrically, if u' is in the nucleolus, we must have $m_1(u') \le m_1(u)$. Therefore, if both utility profiles are in the nucleolus, we must have

$$m_1(u) = m_1(u'). \tag{2.1}$$

Using (2.1) fact and the fact that u is in the nucleolus, we must have $m_2(u) \le m_2(u')$. Symmetrically, using (2.1) and the fact that u' is in the nucelolus, we must have $m_2(u') \le m_2(u)$. Therefore, if both utility profiles are in the nucleolus, we must also have

$$m_2(u) = m_2(u'). \tag{2.2}$$

Now we are going to show that there exists a two-agent coalition S such that $m_1(u) = e(u, S) = m_1(u') = e(u', S)$. Suppose in negation that this is not true. For definiteness, suppose that

$$m_1(u) = e(u, \{1,2\}) = m_1(u') > e(u', \{1,2\}) = e(u', \{2,3\}),$$

$$m_1(u') = e(u', \{1,3\}) = m_1(u) > e(u, \{1,3\}) = e(u, \{2,3\}),$$

where the last equality in each line follows from the result of part (a). Now, take $u'' = 1/2\, u + 1/2\, u'$. Using linearity of the excesses in utility profiles, and the above relations, we can write

$$e(u'', \{1,2\}) = 1/2\, e(u, \{1,2\}) + 1/2\, e(u', \{1,2\}) <$$
$$< 1/2\, m_1(u) + 1/2\, m_1(u') = m_1(u),$$

$$e(u'', \{1,3\}) = 1/2\ e(u, \{1,3\}) + 1/2\ e(u', \{1,3\}) <$$
$$< 1/2\ m_1(u) + 1/2\ m_1(u') = m_1(u),$$
$$e(u'', \{2,3\}) = 1/2\ e(u, \{2,3\}) + 1/2\ e(u', \{2,3\}) <$$
$$< 1/2\ m_1(u) + 1/2\ m_1(u') = m_1(u).$$

This implies that

$$m_1(u'') = \max\{e(u'', \{1,2\}),\ e(u'', \{1,3\}),\ e(u'', \{2,3\})\} < m_1(u),$$

which contradicts the assumption that u is in the nucleolus. Therefore, we there must exist a two-agent coalition S such that

$$m_1(u) = e(u, S) = m_1(u') = e(u', S).$$

For definiteness, suppose that $S = \{1,2\}$, i.e.

$$m_1(u) = m_1(u') = e(u, \{1,2\}) = e(u', \{1,2\}).$$

Using the definition of excesses, this can be rewritten as

$$u_1 + u_2 = u'_1 + u'_2. \qquad (3.1)$$

Equation (2.2), together with the result of part (a), implies that

$$m_2(u) = m_2(u') = e(u, \{1,3\}) = e(u, \{2,3\}) =$$
$$= e(u', \{1,3\}) = e(u', \{2,3\}).$$

Using the definition of excesses, these equalities imply that

$$u_1 + u_3 = u'_1 + u'_3. \qquad (3.2)$$

$$u_2 + u_3 = u'_2 + u'_3. \qquad (3.3)$$

Equations (3.1), (3.2), and (3.3) together imply that $u = u'$. Therefore, the nucleolus cannot contain two different utility profiles.

(c) Suppose for definiteness that $v(1,2) = v(1,3)$. We are going to show that in that case $u_2 = u_3$ at the nucleolus solution. Suppose in negation that $u_2 \neq u_3$. Define $u' = (u_1, u_3, u_2) \neq u$. We then have

$$e(u, \{1,2\}) = v(1,2) - u_1 - u_2 = v(1,3) - u'_1 - u'_3 = e(u', \{1,3\}),$$
$$e(u, \{1,3\}) = v(1,3) - u_1 - u_3 = v(1,2) - u'_1 - u'_2 = e(u', \{1,2\}),$$

$$e(u, \{2,3\}) = v(2,3) - u_2 - u_3 = v(2,3) - u'_2 - u'_3 = e(u', \{2,3\}).$$

Therefore, we must have $m_1(u') = m_1(u)$ and $m_2(u') = m_2(u)$. This implies, by definition, that u' is also in the nucleolus, which contradicts uniqueness of the nucleolus solution proven in part (b).

(d) Agent 1 being a dummy means that $v(I) - v(2,3) = v(1,2) = v(1,3) = 0$. We have two cases to consider: (i) $m_1(u) = e(u, \{2,3\})$ at the nucleolus solution, and (ii) $m_1(u) \neq e(u, \{2,3\})$ at the nucleolus solution.

Case (i): $m_1(u) = e(u, \{2,3\})$ at the nucleolus solution. Using the result of part (a), we must then have

$e(u, \{1,2\}) = - u_1 - u_2 = e(u, \{1,3\}) = - u_1 - u_3$, which implies that $u_2 = u_3$. Using the fact that $u_1 + u_2 + u_3 = v(I)$, we can express u_2 and u_3 through u_1:

$$u_2 = u_3 = [v(I) - u_1]/2 \geq 0 \text{ as long as } u_1 \leq v(I).$$

Substituting these expressions into the formula for excesses, we obtain

$$m_1(u) = e(u, \{2,3\}) = v(I) - u_2 - u_3 = u_1 \geq 0,$$

$$e(u, \{1,2\}) = e(u, \{1,3\}) = - u_1 - u_2 = - [v(I) + u_1]/2 < 0 \leq m_1(u),$$

where the inequalities hold as long as $u_1 \geq 0$. Now, if we had $u_1 > 0$, we could define $u' = (0, v(I)/2, v(I)/2)$. At this new point, we have

$$m_1(u') = e(u', \{2,3\})) = 0 < u_1 = m_1(u),$$

which contradicts u being the nucleolus solution.

Case (ii): $m_1(u) \neq e(u, \{2,3\})$ at the nucleolus solution. For definiteness, suppose that $m_1(u) = e(u, \{1,3\})$ 3\}). Using the result of part (a), we must then have $e(u, \{2,3\}) = v(I) - u_2 - u_3 = e(u, \{1,2\}) = - u_1 - u_2$. At the same time, we know that $u_1 + u_2 + u_3 = v(I)$. Using these two equations, we can express u_2 and u_3 through u_1:

$$u_2 = v(I) + u_1; \; u_3 = -2u_1.$$

We see that we can have $u = (u_1, u_2, u_3) \geq 0$ only if $u_1 = 0$.

(e) First, let us make sure that there exists an exactly feasible utility point which euqalizes the three excesses. This point should satisfy the following conditions:

$$v(1,2) - u_1 - u_2 = v(1,3) - u_1 - u_3 = v(2,3) - u_2 - u_3,$$

$$u_1 + u_2 + u_3 = v(I),$$

$$u_1, u_2, u_3 \geq 0.$$

Let u^* denote the solution to the linear equations above. Solving for u^*, we obtain:

$$u_1^* = 1/3\ [v(I) - 2v(2,3) + v(1,2) + v(1,3)],$$

$$u_2^* = 1/3\ [v(I) - 2v(1,3) + v(1,2) + v(2,3)],$$

$$u_3^* = 1/3\ [v(I) - 2v(1,2) + v(1,3) + v(2,3)].$$

Using the assumption that $v(S) \geq v(I)/2$ for any coalition S of two agents, it is easy to see that all components of u^* are nonnegative. For example,

$$u_1^* = 1/3\ [v(I) - 2v(2,3) + v(1,2) + v(1,3)] \geq$$

$$\geq 1/3\ [v(I) - 2v(2,3) + v(I)/2 + v(I)/2] =$$

$$= 2/3\ [v(I) - v(2,3)] \geq 0.$$

Substituting u^* into the formula for excesses, we obtain that each excess at u^* is equal to $1/3\ [\sum_{|S|=2} v(S) - 2v(I)]$. Therefore,

$$m_1(u^*) = 1/3\ [\sum_{|S|=2} v(S) - 2v(I)].$$

In further analysis, we will use the following formula, which is easily derived from the definition of the three excesses:

$$m_1(u) + m_2(u) + m_3(u) = \sum_{|S|=2} v(S) - 2v(I), \tag{4}$$

where $m_3(u)$ stands for the smallest excess at the utility point u.

Now, suppose in negation that the nucleolus solution u is different from u^*, i.e. it does not equalize the three excesses. Using the result of part (a), we must then have $m_1(u) > m_2(u) = m_3(u)$. Using (4), this implies that

$$m_1(u) > m_2(u) = m_3(u) = [\sum_{|S|=2} v(S) - 2v(I)]/2 - m_1(u)/2.$$

Using this inequality and the expression for $m_1(u^*)$ obtained above, we see that

$$m_1(u) > 1/3\ [\sum_{|S|=2} v(S) - 2v(I)] \ = m_1(u^*),$$

which contradicts the assumption that u is the nucleolus solution.

(f) The characteristic function described satisfies $v(S) \geq v(I)/2 = 3$ for all two-agent coalitions S. By the result of part (e), this implies that the nucleolus solution of this game equalizes the three excesses, and it is given by the point u^* computed above. Substituting the numbers, we obtain

$$u_1^* = 4/3,\ u_2^* = u_3^* = 7/3.$$

To compute the Shapley value, observe that the three agents can enter a room in six different orderings. For example, let us focus on agent 1. There are two orderings in which he comes first, and in those orderings he contributes $v(1) - 0 = 0$. There are two orderings in which he comes last, in which case he contributes $v(I) - v(2,3) = 1$. There are two orderings in which he comes second. When he comes after agent 2, he contributes $v(1,2) - v(2) = 4$. When he comes after agent 3, he contributes $v(1,3) - v(3) = 4$. Weighting each ordering with the probability one-sixth, we obtain the Shapley value of agent 1 as his average contribution over orderings:

$$u_1^S = (2/6)\cdot 0 + (2/6)\cdot 1 + (1/6)\cdot 4 + (1/6)\cdot 4 = 5/3.$$

In the same manner, we can compute the Shapley values of the two other agents, which are $u_2^S = u_3^S = 13/6$.

Comparing to the nucleolus solution, we see that the Shapley value is more equitable: $u_1^S = 5/3 > u_1^* = 4/3$, which means that agent 1, whose contributions are smaller than those of agents 2 and 3, is punished less in the Shapley value than he is in the nucleolus.

(g) By definition of the core, an exactly feasible utility profile u is in the core if and only if

$$\begin{aligned} u_1 + u_2 &\geq v(1,2), \\ u_1 + u_3 &\geq v(1,3), \\ u_2 + u_3 &\geq v(2,3). \end{aligned} \tag{5}$$

If the core is non-empty, we can add up those inequalities to obtain

$$2v(I) \geq \sum_{|S|=2} v(S). \tag{6}$$

Suppose in negation that u, the nucleolus solution, is *not* in the core, i.e. at least one of the inequalities (5) is not satisfied. For definiteness, suppose that $u_1 + u_2 < v(1,2)$. This implies that

$$m_1(u) \geq e(u,\{1,2\}) = v(1,2) - u_1 + u_2 > 0.$$

We are now going to construct an exactly feasible utility profile u' such that $m_1(u') \leq 0 < m_1(u)$, which will contradict the assumption that u is the nucleolus solution. First, let us see if the point u^* which equalizes the three excesses is fit for this role. This point has been computed in part (e). We only need to check under what conditions the point gives non-negative utility levels to all agents. Using the expressions for u^* from part (e) and the inequality (6), we see that

$$u_1^* = 1/3\ [v(I) + \sum_{|S|=2} v(S) - 3v(2,3)] \geq 1/3\ [3/2 \sum_{|S|=2} v(S) - 3v(2,3)] =$$

$$= 1/2 \sum_{|S|=2} v(S) - v(2,3).$$

Similarly, we obtain that

$$u_2^* \geq 1/2 \sum_{|S|=2} v(S) - v(1,3),$$

$$u_3^* \geq 1/2 \sum_{|S|=2} v(S) - v(1,2).$$

Now, as long as $v(T) \leq 1/2 \sum_{|S|=2} v(S)$ for every two-agent coalition T, the utility profile u^*, which equalizes the three excesses, is non-negative. To obtain $m_1(u^*)$, we can use (4):

$$m_1(u^*) = 1/3\, [m_1(u^*) + m_2(u^*) + m_3(u^*)] = 1/3\, [\sum_{|S|=2} v(S) - 2v(I)] \leq 0$$

by the inequality (6). Therefore, $m_1(u^*) \leq 0 < m_1(u)$, which contradicts the assumption that u is the nucleolus solution.

We are left to consider the case where $v(T) > 1/2 \sum_{|S|=2} v(S)$ for some two-agent coalition T. Suppose for definiteness that $v(1,2) > 1/2 \sum_{|S|=2} v(S)$. Let us now find the exactly feasible utility profile u' such that $u'_3 = 0$ and $e(u', \{1,3\}) = e(u', \{2,3\})$. Solving a system of two linear equations with two unknowns (u'_1, u'_2), we obtain

$$u'_1 = 1/2\, [v(I) + v(1,3) - v(2,3)],$$

$$u'_2 = 1/2\, [v(I) + v(2,3) - v(1,3)].$$

Now we can compute the three excesses at u'. We obtain:

$$e(u',\{1,3\}) = e(u',\{2,3\}) = 1/2\, [\sum_{|S|=2} v(S) - v(1,2) - v(I)] <$$

$$< 1/2\, [1/2 \sum_{|S|=2} v(S) - v(I)] \leq 0$$

(we have used the assumption that $v(1,2) > 1/2 \sum_{|S|=2} v(S)$, and the inequality (6)). Also, $e(u',\{1,2\}) = v(1,2) - v(I) \leq 0$. Putting the above inequalities together, we see that $m_1(u') = \text{Max}_{|S|=2}\, e(u', S) \leq 0 < m_1(u)$, which contradicts the assumption that u is the nucleolus solution.

22.F.6. (a) The first-best price equals to marginal cost: $p^* = c'(q^*)$.

But since $c(\cdot)$ is concave, average cost exceeds marginal cost: $c(q^*)/q^* > c'(q^*)$. Therefore, the firm's profits at the first-best price are negative:

$$p^* q^* - c(q^*) = [c'(q^*) - c(q^*)/q^*]\, q^* < 0.$$

(See Figure 22.F.6(a) for an illustration. The average cost (AC) at a point on the cost curve is the slope of the line connecting the point with the origin. The marginal cost (MC) at this point is the slope of the tangent to the curve at this point. The average cost always exceeds the marginal cost for a convex cost curve.)

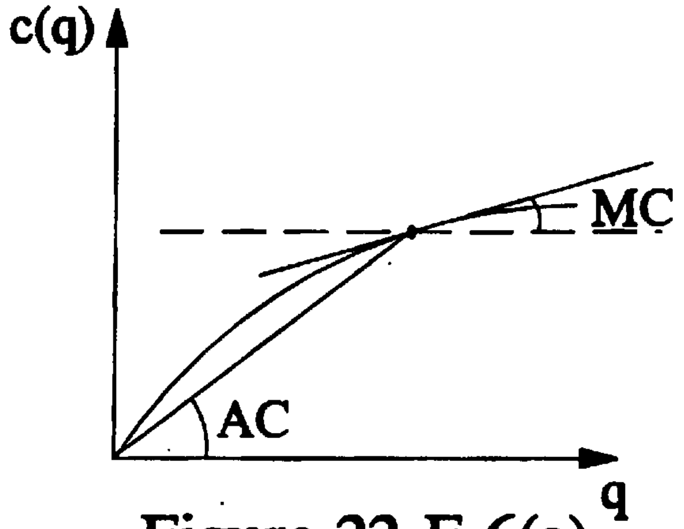


Figure 22.F.6(a)

On the other hand, If costs are covered and $p^0 = c(q^0)/q^0$, then we must have $S'(q^0) = p^0 > c'(q^0)$, which implies that production is socially suboptimal. (See Figure 22.F.6(b)).

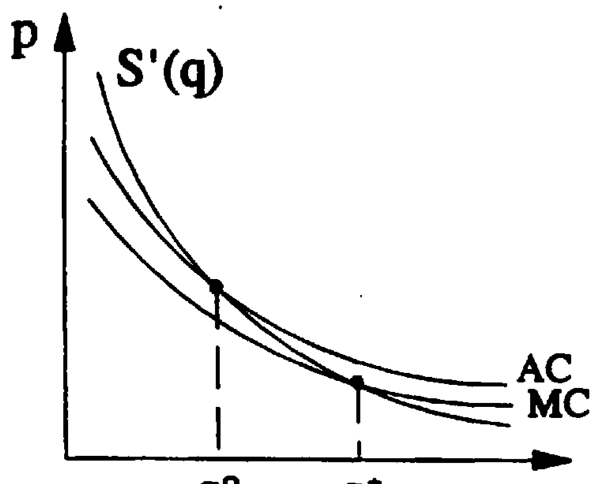


Figure 22.F.6(b)

(b) The second-best welfare problem can be written as

$$\max\ S(q) - c(q)$$

$$\text{s.t. } S'(q)q \geq c(q).$$

If the constraint were not binding, we would obtain the first-best solution, but from part (a) we know that then the costs would not be covered - contradiction. Therefore, the constraint is binding, and the second-best quantity is given by $S'(q)q = c(q)$, i.e. price equals average cost.

(c) We can interpret every unit of output as a "project", with $c(q)$ describing how much it costs to run q projects at once. The Pareto property of the Shapley value in this context means that the costs have to be fully covered, while the symmetry property implies that the price for every "project" should be the same. Thus, the Shapley value suggests that every unit of output should be priced at the average cost.

22.F.7. (a) The second-best problem can be written as

$$\max_{q_1, q_2}\ S_1(q_1) + S_2(q_2) - c(q_1) - c(q_2)$$

$$\text{s.t. } S_1'(q_1)\, q_1 + S_2'(q_2)\, q_2 - c(q_1) - c(q_2) \geq 0.$$

If we denote the Lagrange multiplier with the constraint by λ, the first-order conditions obtained by differentiation can be written as

$(1+\lambda)\ S_l'(q_l) - (1+\lambda)\ c'(q_l) + \lambda S_l''(q_l)\ q_l = 0,\ l = 1,2.$

If we denote the elasticities of demand by $\varepsilon_l(q_l) = -\dfrac{S_l''(q_l)\ q_l}{S'(q_l)}$,

the first-order conditions can be rewritten in the form

$$(S_l'(q_l) - c'(q_l))/S_l'(q_l) = \alpha/\varepsilon_l(q_l) \text{ for some } \alpha > 0.$$

Thie coincides with the Ramsey taxation formula (22.B.1).

(b) By symmetry of the problem, the Shapley value cost allocation should allocate the same value (c_1) to all projects of type 1, and the same value (c_2) to all projects of type 2. Take a unit project of type 1, and take a random ordering of the large number of small unit projects. Suppose that the proportion t of all the projects precede the given project. Since by the Law of Large Numbers with a very high probability our project will be preceded by an almost perfect sample of all projects, therefore the output of all preceding projects will be (tq_1, tq_2). The contribution of our project to the costs will therefore be $c'(tq_1)$. Since every t is equally likely, the Shapley value for type 1 projects is obtained by averaging:

$$c_1 = \int_0^1 c'(tq_1)\, dt = (1/q_1)\int_0^1 c'(tq_1)\, d(tq_1) = c(q_1)/q_1.$$

Similarly, the Shapley value for type 2 projects is

$$c_2 = c(q_1)/q_1.$$

(For a rigorous derivation of Shapley value with a continuum of agents, see R. Aumann, "Value of Markets with a Continuum of Traders", *Econometrica* 1975, 43, pp. 611-646)

(c) If outputs are priced according to their Shapley values obtained in part (b), this results in average cost pricing for each output. The resulting production levels $(\bar{q}_1, \bar{q}_2)$ will be determined by the intersection of demand curves and average cost curves:

$$S'(\bar{q}_1) = c(\bar{q}_1)/\bar{q}_1;\ S'(\bar{q}_2) = c(\bar{q}_2)/\bar{q}_2.$$

Clearly, production is lower than the first-best levels, as long as costs are strictlty concave and average costs exceed marginal costs. In general, this is also different from the second-best allocation obtained in part (a). Indeed, under average cost pricing, production level for each output is entirely determined by demand and cost functions for this output, and there is no cross-subsidization. In contrast, under Ramsey pricing in obtained in (a), production level for each output is in general determined by demand elasticities for both outputs (which enter from the break-even condition defining α), and in general we should expectcross-subsidization. For example, if demand for good 1 is perfectly inelastic, under the Ramsey pricing good 2 is priced at marginal cost and subsidized from the non-distortionary mark-up on good 1.

CHAPTER 23

23.B.1. In (a) and (d) agent 2 is indifferent between truth telling or lying, and therefore is willing to tell the truth. In (b) and (c) agent 2 strictly prefers truth telling. However, in (e) the agent *always* prefers to lie because $z \succsim_2(\theta')y$, and $y \succsim_2(\theta')z$.

23.B.2. Agent 2's problem is: $\max_{\hat{\theta}_2} E_{\theta_1}\left[\theta_2 - \frac{\theta_1 + \hat{\theta}_2}{2} \mid \theta_1 < \hat{\theta}_2\right]$

which is equivalent to: $\max_{\hat{\theta}_2} \left[\theta_2 - \frac{\hat{\theta}_2/2 + \hat{\theta}_2}{2}\right]\hat{\theta}_2$

Taking the FOC (note that the SOC is satisfied) we get $\hat{\theta}_2 = \frac{2}{3}\theta_2$. This result is similar to the understatement in the first-price auction in example 23.b.4. (see footnote 6).

23.B.3. Showing this is basically following the argument at the end of example 23.B.4. Agent i can choose $b_i(\theta_i) = \alpha_i\theta_i$ where $\alpha_i \in [0,1]$. Assume all other agents are announcing $\alpha_j\theta_j < \theta_i$ and let $\alpha_j\theta_j^* = \max_j\{\alpha_j\theta_j\}$. Agent i is then indifferent between $\alpha_i \in \left(\frac{\alpha_j\theta_j^*}{\theta_i}, 1\right]$ and his utility is $\theta_i - \alpha_j\theta_j^*$ (lower values of α_i will give him zero utility). If there exists j s.t. $\alpha_j\theta_j > \theta_i$ then agent i is indifferent between $\alpha_i \in [0,1]$. Therefore, $\alpha_i = 1$ is a weakly dominant strategy.

23.B.4. (a) Normalize the seller's (agent 1) and buyer's (agent 2) no-trade utility levels to zero. The parties' payoffs as a function of their strategies (b_1,b_2) are both zero if $b_1 \geq b_2$, and if $b_1 < b_2$ then $u_1 = \frac{b_1+b_2}{2} - \theta_1$, and $u_2 = \theta_2 - \frac{b_1+b_2}{2}$. To find a Bayesian equilibrium with $b_i(\theta_i) = \alpha_i + \beta_i\theta_i$, first note that a seller of type θ_1 will bid b_1 to maximize his expected payoff:

$$\max_{b_1} \; Eu_1 = E_{\theta_2}\left(\frac{b_1+\alpha_2+\beta_2\theta_2}{2} - \theta_1 \;\middle|\; \alpha_2+\beta_2\theta_2 > b_1 \right) = \int_{(b_1-\alpha_2)/\beta_2}^{1} \left(\frac{b_1+\alpha_2+\beta_2\theta_2}{2} - \theta_1 \right) d\theta_2$$

the FOC yields (the SOC is satisfied):

$$b_1 = \tfrac{1}{3}(\alpha_2+\beta_2) + \tfrac{2}{3}\theta_1 \qquad \text{(i)}$$

Similarly, the buyer maximizes her expected payoff:

$$\max_{b_2} \; Eu_2 = E_{\theta_1}\left(\theta_2 - \frac{b_2+\alpha_1+\beta_1\theta_1}{2} \;\middle|\; \alpha_1+\beta_1\theta_1 < b_2 \right) = \int_{0}^{(b_2-\alpha_1)/\beta_1} \left(\theta_2 - \frac{b_2+\alpha_1+\beta_1\theta_1}{2} \right) d\theta_1$$

the FOC yields (the SOC is satisfied):

$$b_2 = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\theta_2 \qquad \text{(ii)}$$

The two equations (i) and (ii) above yield: $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{12}$, $\beta_1 = \beta_2 = \frac{2}{3}$. Trade occurs if and only if $b_1(\theta_1) < b_2(\theta_2)$, and given the equilibrium strategies, trade occurs if and only if $\theta_1 + \frac{1}{4} < \theta_2$. The transfer in case of trade is $t(\theta_1,\theta_2) = \frac{b_1(\theta_1)+b_2(\theta_2)}{2} = \frac{1}{3}(\theta_1+\theta_2) + \frac{1}{6}$. This social choice function is not Pareto optimal because $\theta_1 < \theta_2$ does not guarantee trade, and efficient trade may not always occur.

(b) The social choice rule above has $x(\theta_1,\theta_2)=1$ and $t(\theta_1,\theta_2)=\frac{1}{3}(\theta_1+\theta_2)+\frac{1}{6}$ when $\theta_1+\frac{1}{4}<\theta_2$, and $x(\theta_1,\theta_2)=t(\theta_1,\theta_2)=0$ otherwise. We will check if it is truthfully implementable in a direct revelation mechanism, i.e., that truth-telling is a Bayesian Nash equilibrium in the direct mechanism. The seller, assuming that the buyer reveals herself truthfully, solves:

$$\max_{\hat{\theta}_1} \; Eu_1 = E_{\theta_2}[t(\hat{\theta}_1,\theta_2)-\theta_1 x(\hat{\theta}_1,\theta_2)] = \int_{\frac{1}{4}-\hat{\theta}}^{1}[\tfrac{1}{3}(\hat{\theta}_1+\theta_2)+\tfrac{1}{6}]d\theta_2$$

the FOC yields (the SOC is satisfied):

$$\frac{1}{3}(1-\frac{1}{4}-\hat{\theta}_1)-\frac{1}{3}(\hat{\theta}_1+\frac{1}{4}+\hat{\theta}_1)-\frac{1}{6}-\theta_1=0$$

which implies that $\hat{\theta}_1=\theta_1$. Similarly, the buyer solves:

$$\max_{\hat{\theta}_2} \; Eu_2 = E_{\theta_1}[\theta_2 x(\theta_1,\hat{\theta}_2)-t(\theta_1,\hat{\theta}_2)] = \int_{0}^{\hat{\theta}_2-\frac{1}{4}}[\theta_2-\tfrac{1}{3}(\theta_1+\hat{\theta}_2)-\tfrac{1}{6}]d\theta_1$$

the FOC yields (the SOC is satisfied):

$$\theta_2-\frac{1}{3}(\hat{\theta}_2-\frac{1}{4}+\hat{\theta}_2)-\frac{1}{6}-\frac{1}{3}(\hat{\theta}_2-\frac{1}{4})=0$$

which implies that $\hat{\theta}_2=\theta_2$. Therefore truth telling is an equilibrium.

23.C.1. Assume the preference reversal property is satisfied for all i, θ_i', θ_i'', and θ_{-i}, and assume in negation that $f(\cdot)$ is not truthfully implementable in dominant strategies. That is, there exists i, θ_i', θ_i'', and θ_{-i}, such that

$$u_i(f(\theta_i'',\theta_{-i}),\theta_i') > u_i(f(\theta_i',\theta_{-i}),\theta_i')$$

but this contradicts the preference reversal property, therefore, $f(\cdot)$ must be truthfully implementable in dominant strategies.

23.C.2. Each agent i can have at most three types: θ_i^1, for which $x_1 \succ x_2$, θ_i^2, for which $x_2 \succ x_1$, and θ_i^0, for which $x_1 \sim x_2$. The majority voting social choice function is defined by (let x_1 be chosen as a tie-breaker):

$$f(\theta) = \begin{cases} x_1 \text{ if } \#(\theta_i^1) \geq \#(\theta_i^2) \\ x_2 \text{ if } \#(\theta_i^1) < \#(\theta_i^2) \end{cases}$$

It is easy to see that if agents are asked to vote one way or another (i.e., a "direct mechanism") then no agent has an incentive to lie no matter what his preferences, or the other agent's announcements are. In cases where an agent is pivotal then he will strictly prefer to tell the truth (telling the truth will cause his preferred outcome to be chosen while lying will cause the other outcome to be chosen). In all other cases (no influence) then he is indifferent. Therefore, $f(\cdot)$ is implementable in dominant strategies.

23.C.3. Let $\mathcal{R}_i = \mathcal{P}$ for all i, $f(\cdot)$ be *ex post* efficient, and assume in negation that there exists $\bar{x} \in X$ such that for all $\theta \in \Theta$, $f(\theta) \neq \bar{x}$. Since $\mathcal{R}_i = \mathcal{P}$ for all i, then there exists $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_I) \in \Theta$ such that $\bar{x} \succ_i (\bar{\theta}_i)\ x$ for all $x \in X$, and for all i. This, however, contradicts $f(\cdot)$ being *ex post* efficient, therefore $f(\Theta) = X$.

23.C.4. Assume that $f: \Theta \to X$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i for $i = 1, \ldots I$. This implies (by definition) that for all i and for all $\theta_i \in \Theta_i$, $u_i(f(\theta_i, \theta_{-i}), \theta_i) >$ $u_i(f(\theta_i', \theta_{-i}), \theta_i)$ for all $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. This implies that for all i, and for all $\theta_i \in \hat{\Theta}_i \subset \Theta_i$, $u_i(f(\theta_i, \theta_{-i}), \theta_i) > u_i(f(\theta_i', \theta_{-i}), \theta_i)$ for all $\theta_i' \in \hat{\Theta}_i$ and all $\theta_{-i} \in \hat{\Theta}_{-i}$, which in turn implies the desired result.

23.C.5. Define $F(\cdot)$ to be the pairwise majority voting social welfare functional (note that single-peaked domain ensures that this functional is rational). Observe that $F(\cdot)$ satisfies non-negative responsiveness. Therefore, if we define $f(\cdot)$ to be the social choice function that selects a Condorcet winner, i.e., the maximizer of $F(\cdot)$, we can apply Exercise 23.C.7 below to see that $f(\cdot)$ is truthfully implementable in dominant strategies.

23.C.6. (a) Let $X=\{x,y\}$ and let there be two type profiles $\theta'=(\theta'_1,\ldots,\theta'_I)$ and $\theta''=(\theta''_1,\ldots,\theta''_I)$ where all agents are indifferent between x and y no matter which type profile occurs. Let $f(\theta')=x$ and $f(\theta'')=y$. This is a trivial example for $f(\cdot)$ satisfying IPM and not being monotonic.

(b) Suppose that $I=2$, $X=\{w,x,y,z\}$, $\Theta_1=\{\theta'_1,\theta''_1\}$, $\Theta_2=\{\theta'_2,\theta''_2\}$, and the possible preferences are:

$\succsim_1(\theta'_1)$	$\succsim_1(\theta''_1)$	$\succsim_2(\theta'_2)$	$\succsim_2(\theta''_2)$
z	w	x	x
y	x	y	y
x	y	w	w
w	z	z	z

Let $f(\theta'_1,\theta'_2)=f(\theta'_1,\theta''_2)=x$, and $f(\theta''_1,\theta'_2)=f(\theta''_1,\theta''_2)=y$. The only two preference

changes for which the "if" part of the monotonicity definition is satisfied are the change from (θ'_1,θ''_2) to (θ'_1,θ'_2), and from (θ''_1,θ'_2) to (θ''_1,θ''_2). Indeed, the "then" part of the definition is also satisfied for these two changes. For all other preference changes, the "if" part of the definition is not satisfied, therefore the definition is vacuously satisfied. We conclude that $f(\cdot)$ is monotonic. Now observe that $f(\theta'_1,\theta'_2) \notin L_1(f(\theta''_1,\theta'_2),\theta''_1)$. Therefore, IPM is not satisfied.

(c) From Proposition 23.C.2 we know that $f(\cdot)$ satisfies IPM if and only if it is truthfully implementable in dominant strategies. Now consider two profiles of types, θ and θ', such that $L_i(f(\theta),\theta_i) \subset L_i(f(\theta),\theta'_i)$ for all i (if no such two profiles exist in Θ then $f(\cdot)$ is vacuously monotonic). We need to show that $f(\theta')=f(\theta)$, and this can be done by following the proof of Claim 1 in the proof of Proposition 23.C.3.

(d) Suppose in negation that $f(\cdot)$ does not satisfy IPM, i.e., there exists θ_i, θ'_i, and θ_{-i} such that $f(\theta) \in L_i(f(\theta'_i, \theta_{-i}), \theta'_i)$. Since $\mathcal{R}_i = \mathcal{P}_i$ this implies that $f(\theta) \neq f(\theta'_i, \theta_{-i})$ and that

$$u_i(f(\theta), \theta'_i) > u_i(f(\theta'_i, \theta_{-i}), \theta'_i).$$

Consider the strict preference θ''_i such that $f(\theta)$ is at the top, $f(\theta'_i, \theta_{-i})$ is second, and all other alternatives are below these two in some arbitrary order. This construction implies that $L_i(f(\theta), \theta_i) \subset L_i(f(\theta), \theta''_i)$, therefore monotonicity implies that $f(\theta''_i, \theta_{-i}) = f(\theta)$. Note, however, that this construction also implies that $L_i(f(\theta'_i, \theta_{-i}), \theta'_i) \subset L_i(f(\theta'_i, \theta_{-i}), \theta''_i)$, therefore monotonicity implies that $f(\theta''_i, \theta_{-i}) = f(\theta'_i, \theta_{-i})$, a contradiction.

23.C.7. We assume that $f(\theta_1, \ldots, \theta_I) = \{x \in X: x\ F(\succsim_1(\theta_1), \ldots, \succsim_I(\theta_I))\ y\ \ \forall\ y \in X\}$ is single-valued. We want to show that if $F(\cdot)$ satisfies non-negative responsiveness, then $f(\cdot)$ is implementable in dominant strategies. By the Revelation Principle, it suffices to check the direct revelation mechanism, i.e., to show that for all i:

$$f(\theta_i, \theta_{-i}) \succsim_i(\theta_i)\ f(\hat{\theta}_i, \theta_{-i}) \text{ for all } \theta_i, \hat{\theta}_i, \theta_{-i}$$

Suppose in negation that there exists an agent who can gain by misrepresenting, that is, there exists $i \in I$, $\theta = (\theta_i, \theta_{-i})$, and $\hat{\theta}_i$, such that $f(\theta) = x$, $f(\hat{\theta}_i, \theta_{-i}) = y$, and $y \succ_i(\theta_i)\ x$. By definition, $f(\cdot)$ maximizes $F(\cdot)$, and is single valued, so we have:

$$f(\theta) = x \text{ implies } x\ F_p(\succsim_i(\theta_i), \succsim_{-i}(\theta_{-i}))\ y \quad \text{(i)}$$

$$f(\hat{\theta}_i, \theta_{-i}) = y \text{ implies } y\ F_p(\succsim_i(\hat{\theta}_i), \succsim_{-i}(\theta_{-i}))\ x \quad \text{(ii)}$$

Now we apply non-negative responsiveness of $F(\cdot)$ to (ii), by noticing that when we move from $(\hat{\theta}_i, \theta_{-i})$ to θ, y can only rise relative to x for agent i's preferences, and does not move for the other agents. Thus we must have:

$$y \; F_p(\succsim_i(\theta_i), \succsim_{-i}(\theta_{-i})) \; x$$

which contradicts (i) above. Therefore, no agent can gain by misrepresenting his preferences.

23.C.8. (a) It is easy to check that we cannot improve upon $f(\cdot)$ for one agent without hurting the other, therefore $f(\cdot)$ is *ex post* efficient.

(b) It is again easy to check that given the preferences, $f(\cdot)$ satisfies the property identified in Proposition 23.C.2.

(c) Truth-telling is not the unique (weakly) dominant strategy - another one is for each agent i to *always* announce $\hat{\theta}_i = \theta_i''$, which causes outcome a to be implemented. The agents are indifferent between this strategy and truth-telling because with preferences (θ_1', θ_2') the agents are indifferent between a and b, and for all other preferences, alternative a would have been implemented anyway.

23.C.9 We begin by showing that if $f(\cdot)$ is truthfully implementable in dominant strategies then both conditions are satisfied. Actually, the second condition was shown to be satisfied in section 23.C, and is exactly the condition in equation (23.C.13). Therefore, we only need to show that $\frac{\partial k(\theta)}{\partial \theta_i} \geq 0$. Let $\theta = (\theta_1, \ldots, \theta_I)$ and take $\theta_i' > \theta_i$. Because truth-telling is always optimal, we must have:

$$v_i(k(\theta_i', \theta_{-i}), \theta_i') + t_i(\theta_i', \theta_{-i}) \geq v_i(k(\theta), \theta_i') + t_i(\theta) \qquad \text{(i)}$$

$$v_i(k(\theta), \theta_i) + t_i(\theta) \geq v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i}) \qquad \text{(ii)}$$

(i) and (ii) imply:

$$v_i(k(\theta_i',\theta_{-i}),\theta_i') - v_i(k(\theta),\theta_i') \geq v_i(k(\theta_i',\theta_{-i}),\theta_i) - v_i(k(\theta),\theta_i) \qquad \text{(iii)}$$

Now let $\theta_i'=\theta_i+d\theta_i$, which for small $d\theta_i$ implies that (ignoring higher order terms) $dk = \frac{\partial k(\theta)}{\partial \theta_i}\cdot d\theta_i$, and we can rewrite (iii) as

$$\frac{\partial v_i(k(\theta),\theta_i')}{\partial k}\cdot\frac{\partial k(\theta)}{\partial \theta_i}\cdot d\theta_i \geq \frac{\partial v_i(k(\theta),\theta_i)}{\partial k}\cdot\frac{\partial k(\theta)}{\partial \theta_i}\cdot d\theta_i$$

which is equivalent to:

$$\left(\frac{\partial v_i(k(\theta),\theta_i')}{\partial k} - \frac{\partial v_i(k(\theta),\theta_i)}{\partial k}\right)\cdot\frac{\partial k(\theta)}{\partial \theta_i}\cdot d\theta_i \geq 0 \qquad \text{(iv)}$$

but since $\theta_i'>\theta_i$, and we assumed that $\partial^2 v_i(k,\theta_i)/\partial k\partial\theta_i>0$, then the term in brackets in (iv) is strictly positive. Therefore, for (iv) to be satisfied (recall that $d\theta_i>0$) we must have $\frac{\partial k(\theta)}{\partial \theta_i} \geq 0$.

We are left to show the converse, i.e., that both conditions imply that $f(\cdot)$ is truthfully implementable in dominant strategies. Assume in negation that there exists an agent i, such that given types $\theta=(\theta_i,\theta_{-i})$, agent i strictly prefers lying and announcing that he is of type $\hat{\theta}_i$. Define agent i's utility, given types are (θ_i,θ_{-i}) but he announces $\hat{\theta}_i$ as:

$$u(\theta_i,\hat{\theta}_i) = v_i(k(\hat{\theta}_i,\theta_{-i}),\theta_i) + t(\hat{\theta}_i,\theta_{-i})$$

Our negation assumption implies that $u(\theta_i,\hat{\theta}_i) - u(\theta_i,\theta_i) > 0$, or

$$\int_{\theta}^{\hat{\theta}} \frac{\partial u(\theta_i,\tau)}{\partial \tau}d\tau = \int_{\theta}^{\hat{\theta}}\left(\frac{\partial v(k(\tau,\theta_{-i}),\theta_i)}{\partial k}\cdot\frac{\partial k(\tau,\theta_{-i})}{\partial \tau} + \frac{\partial t(\tau,\theta_{-i})}{\partial \tau}\right)d\tau > 0 \qquad \text{(v)}$$

<u>Case 1:</u> Suppose $\hat{\theta}_i>\theta_i$. From the first condition, i.e., $k(\theta)$ is increasing in θ_i, we know that $\frac{\partial k(\tau,\theta_{-i})}{\partial \tau} \geq 0$, and from the assumption $\partial^2 v_i(k,\theta_i)/\partial k\partial\theta_i>0$

we know that:

$$\frac{\partial v(k(\tau,\theta_{-i}),\tau)}{\partial k} \geq \frac{\partial v(k(\tau,\theta_{-i}),\theta_i)}{\partial k} \quad \text{for all} \quad \tau \geq \theta_i. \qquad \text{(vi)}$$

using (v) and (vi) we get:

$$u(\theta_i,\hat{\theta}_i) - u(\theta_i,\theta_i) \leq \int_{\theta}^{\hat{\theta}} \left(\frac{\partial v(k(\tau,\theta_{-i}),\tau)}{\partial k} \cdot \frac{\partial k(\tau,\theta_{-i})}{\partial \tau} + \frac{\partial t(\tau,\theta_{-i})}{\partial \tau} \right) d\tau = 0$$

because the bracketed term equals zero for all τ (see equation (23.C.12)). This, however, contradicts our negation assumption that $u(\theta_i,\hat{\theta}_i)-u(\theta_i,\theta_i)>0$ so $f(\cdot)$ must be truthfully implementable.

<u>Case 2:</u> Suppose $\hat{\theta}_i<\theta_i$. We can proceed as before, however the inequality in (vi) above will be reversed, and we will have a minus sign before the integral, so we will get the same contradiction.

23.C.10 [<u>First Printing Errata:</u> At the end of the first paragraph insert: "Assume throughout that conditions are such that (23.C.8) holding is a necessary condition for $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ to be truthfully implementable in dominant strategies." Also, in the second line of part c) insert the word "implementable" before "ex post efficient social choice function".]

a) <u>Sufficiency:</u> Suppose that we can write $V^*(\theta) = \sum_i V_i(\theta_{-i})$. Consider the transfer functions of the form

$$t_i(\theta) = \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + h_i(\theta_{-i}) \ ,$$

where for all i,

$$h_i(\theta_{-i}) = -(I - 1)V_i(\theta_{-i}) \quad \text{for all } \theta_{-i} \ .$$

By proposition 23.C.4, $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is truthfully implementable in dominant strategies. Moreover, for all θ we have,

$$\sum_i t_i(\theta) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= (I - 1)V^*(\theta) - (I - 1)\sum_i V_i(\theta_{-i}) = 0$$

<u>Necessity:</u> Suppose $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is ex post efficient and is truthfully implementable in dominant strategies. Since (23.C.8) is necessary (by assumption) for truthful implementation, this means that there exist functions $\{h_i(\theta_{-i})\}_{i=1}^{I}$ such that

$$(I - 1)V^*(\theta) + \sum_i h_i(\theta_{-i}) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= \sum_i t_i(\theta) = 0$$

But this implies that by defining

$$V_i(\theta_{-i}) = \left(\frac{-1}{I - 1}\right)h_i(\theta_{-i}) \ ,$$

we can then write $V^*(\theta) = \sum_i V_i(\theta_{-i})$.

b) If $v_i(k,\theta_i) = \theta_i k - \frac{1}{2} k^2$ for all i, then, $k^*(\theta) = \text{Argmax}_k(\sum_i\theta_i)k - \frac{3}{2} k^2$ for all θ, and so the FOC implies that $k^*(\theta) = \frac{\sum_i\theta_i}{3}$. Hence,

$$V^*(\theta) = \sum_{i=1}^{3}\left[\theta_i\left(\frac{\sum_i\theta_i}{3} \right) - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3} \right)^2 \right]$$

$$= \left(\frac{\sum_i\theta_i}{3} \right)\sum_i\left[\theta_i - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3} \right) \right]$$

$$= (\theta_1+ \theta_2+ \theta_3)[\theta_1+ \theta_2+ \theta_3 - \tfrac{1}{2}(\theta_1+ \theta_2+ \theta_3)]$$

$$= \tfrac{1}{2}(\textstyle\sum_i\theta_i)^2$$

$$= (\theta_1^2 + \theta_2^2 + \theta_3^2 + 2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3) \ .$$

We now define,

$$V_1(\theta_2,\theta_3) = \frac{\theta_2^2 + \theta_3^2}{2} + 2\theta_2\theta_3 \ ,$$

$$V_2(\theta_1,\theta_3) = \frac{\theta_1^2 + \theta_3^2}{2} + 2\theta_1\theta_3 \ ,$$

$$V_3(\theta_1,\theta_2) = \frac{\theta_1^2 + \theta_2^2}{2} + 2\theta_1\theta_2 \ ,$$

and the result then follows from part a) above since

$$V^*(\theta) = V_1(\theta_2,\theta_3) + V_2(\theta_1,\theta_3) + V_3(\theta_1,\theta_2) \ .$$

c) If $V^*(\theta) = \sum_i V_i(\theta_{-i})$ then clearly $\dfrac{\partial^I V^*(\theta)}{\partial\theta_1\cdots\partial\theta_I} = 0$.

d) In this case, $V^*(\theta_1,\theta_2) = v_1(k^*(\theta),\theta_1) + v_2(k^*(\theta),\theta_2)$, therefore,

$$\frac{\partial V^*}{\partial\theta_1} = \left(\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k}\right)\frac{\partial k}{\partial\theta_1} + \frac{\partial v_1}{\partial\theta_1} \ ,$$

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = \left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) + \frac{\partial^2 v_2}{\partial k\partial\theta_2}\cdot\frac{\partial k}{\partial\theta_1} + \frac{\partial^2 v_1}{\partial k\partial\theta_1}\cdot\frac{\partial k}{\partial\theta_2},$$

Since,

$$\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} = 0 \ ,$$

we have,

$$\frac{\partial^2 v_i}{\partial k\partial\theta_i} = -\frac{\partial k}{\partial\theta_i}\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right),$$

which in turn implies that

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = -\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) \neq 0 \ ,$$

thus proving the statement.

23.C.11 Let agent i's Bernoulli utility function be $u_i(v_i(k,\theta_i) + \bar{m}_i + t_i$ and assume in negation that Proposition 23.C.4 no longer holds. That is, there exists i, θ_i, $\hat{\theta}_i$, and θ_{-i} such that:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i) + \bar{m}_i + t_i(\hat{\theta}_i,\theta_{-i}) > u_i(v_i(k^*(\theta),\theta_i) + \bar{m}_i + t_i(\theta)$$

Substituting from (23.C.8) we get:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i)+\bar{m}_i+\sum_{j\neq i} v_j(k^*(\hat{\theta}_i,\theta_{-i}),\theta_j) + h_i(\theta_{-i})) >$$

$$> u_i(v_i(k^*(\theta),\theta_i) + \bar{m}_i + \sum_{j\neq i} v_j(k^*(\theta),\theta_j) + h_i(\theta_{-i}))$$

Given $u_i'(\cdot) > 0$ this implies that:

$$\sum_{j=1}^{I} v_j(k^*(\hat{\theta}_i,\theta_{-i}),\theta_j) > \sum_{j=1}^{I} v_j(k^*(\theta),\theta_j)$$

a contradiction to $k^*(\cdot)$ satisfying (23.C.7). Thus, $f(\cdot)$ must be truthfully implementable in dominant strategies.

23.D.1 (a) The set of alternatives X are all pairs (k,y) where

k=1,2,... is

the period in which trade occurs, and $y \in \mathbb{R}$ is the transfer from the buyer to the seller (the price).

(b) It is clear that when $\theta=9$, the seller will not want to announce that $\theta=0$ since then he will receive a net payment of (-4) immediately. The relevant incentive constraint is therefore $\delta^k(9.5 - 0) \leq 5$, i.e., that a $\theta=0$ type will not misrepresent given that trade will occur after k periods. Solving this yields the minimum k required: $k \geq [\ln(10/19)]/\ln(\delta)$.

23.D.2 (a) Letting the seller be agent 1, it is optimal to trade if and only if $\theta_2>\theta_1$, so we must have:

$$k(\theta_1,\theta_2) = \begin{cases} 0 & \text{if} \quad \theta_2\leq\theta_1 \\ 1 & \text{if} \quad \theta_1>\theta_2 \end{cases}$$

and of course, $v_1(k)=\theta_1$ if k=0 (and zero otherwise) whereas $v_2(k)=\theta_2$ if k=1 (and zero otherwise). From equations (23.D.8) and (23.D.9)we can derive the transfer functions for each agent. For the seller:

$$E_{\theta_2}[v_2(k(\theta_1,\theta_2),\theta_2)] = E_{\theta_2}[\ \theta_2 \mid \theta_2>\theta_1] = \int_{\theta_1}^{1} \theta_2 d\theta_2 = \frac{1 - \theta_1^2}{2}$$

$$\tau_1(\theta_2) = -E_{\theta_1}[v_1(k(\theta_1,\theta_2),\theta_1)] = -E_{\theta_1}[\ \theta_1 \mid \theta_1 \geq \theta_2] = -\int_{\theta_2}^{1} \theta_1 d\theta_1 = \frac{\theta_2^2 - 1}{2}$$

and since the optimal transfer for agent 1 is the sum of these two terms we have:

$$t_1(\theta_1,\theta_2) = \frac{\theta_2^2 - \theta_1^2}{2}$$

By symmetry, it is clear that:

$$t_2(\theta_1,\theta_2) = \frac{\theta_1^2 - \theta_2^2}{2}$$

(b) We will show that for each agent i, truth-telling is optimal in expectations given the belief that agent $j \neq i$ is always truth-telling. The problem for the seller is:

$$\underset{\hat{\theta}_1}{\text{Max}} \quad E_{\theta_2}\left(\theta_1 \cdot \text{Prob}(\theta_1 \geq \theta_2) + \frac{\theta_2^2 - \theta_1^2}{2} \right)$$

which reduces to:

$$\underset{\hat{\theta}_1}{\text{Max}} \quad \theta_1 \int_0^{\hat{\theta}_1} 1 d\theta_2 - \frac{\hat{\theta}_1^2}{2} + E_{\theta_2}\left(\frac{\theta_2^2}{2} \right) \qquad (1)$$

which yields the FOC (the SOC is satisfied): $\theta_1 - \hat{\theta}_1 = 0$

For the buyer, the problem is:

$$\underset{\hat{\theta}_2}{\text{Max}} \quad E_{\theta_1}\left(\theta_2 \cdot \text{Prob}(\theta_1 < \theta_2) + \frac{\theta_1^2 - \theta_2^2}{2} \right)$$

which symmetrically gives us the FOC: $\theta_2 - \hat{\theta}_2 = 0$.

23.D.3. It is easily seen that the conditions for the Revenue Equivalence Theorem are satisfied for the two auction settings. We will now verify that both examples 23.B.5 and 23.B.6 yield the same expected revenue to the seller. In the sealed-bid first-price auction we have:

$$E[-t_1(\theta)] = E\left(\frac{\theta_1}{2}\cdot \text{Prob}(\theta_2 \le \theta_1) \right) = \int_0^1\int_0^1 \frac{\theta_1}{2}\cdot \text{Prob}(\theta_2 \le \theta_1)\, d\theta_2 d\theta_1 =$$

$$= \int_0^1 \frac{\theta_1}{2} \int_0^{\theta_1} 1 d\theta_2 d\theta_1 = \int_0^1 \frac{\theta_1^2}{2}\, d\theta_1 = \frac{1}{6}$$

Since the problem is symmetric for player 2 we get that $E[-t_2(\theta)] = 1/6$ as well. Now consider the sealed-bid second-price auction:

$$E[-t_1(\theta)] = E\left(\theta_2\cdot \text{Prob}(\theta_2 \le \theta_1) \right) = \int_0^1\int_0^1 \theta_2\cdot \text{Prob}(\theta_2 \le \theta_1)\, d\theta_2 d\theta_1 =$$

$$= \int_0^1 \theta_1 \int_0^{\theta_1} \theta_2 d\theta_2 d\theta_1 = \int_0^1 \frac{\theta_1^2}{2}\, d\theta_1 = \frac{1}{6}$$

Since the problem is symmetric for player 2 we get that $E[-t_2(\theta)] = 1/6$ as well. Therefore, both the first and second price auctions yield an expected revenue of 1/3 for the seller (for a general formula with any I and with valuations in [0,1] see exercise 23.D.6).

23.D.4. (a) Let $\theta_i' < \theta_i''$, let b′ and b″ be agent i's bids associated with his two types, and assume in negation that $b' > b''$. Denote by π' and π'' the probabilities of winning the auction given bids b_i' and b_i'' respectively. By optimality (using a revealed-preference argument) we have that $(\theta_i' - b_i')\pi' \ge (\theta_i' - b_i'')\pi''$ which implies:

$$\frac{\theta_i' - b_i''}{\theta_i' - b_i'} \le \frac{\pi'}{\pi''} \qquad \text{(i)}$$

and similarly, $(\theta_i'' - b_i'')\pi'' \ge (\theta_i'' - b_i')\pi'$ which implies:

$$\frac{\theta_i'' - b_i''}{\theta_i'' - b_i'} \geq \frac{\pi'}{\pi''} \qquad \text{(ii)}$$

(i) and (ii) imply (after some simple algebra) that $(\theta_i'' - \theta_i')(b_i'' - b_i') \geq 0$ which is a contradiction to our assumption that $\theta_i'' - \theta_i' > 0$ and $b_i'' - b_i' < 0$.

(b) Assume in negation that there is a symmetric equilibrium with (θ', θ''), such that $b^*(\theta) = \bar{b}$ for all $\theta \in (\theta', \theta'')$. Let $\pi > 0$ denote the (strictly positive) probability that *all* agents' types are in this interval (it must be that in this case all agents get the good with equal probability because the equilibrium is symmetric). Let $p_i(\theta_i)$ be the probability that agent i gets the good if he bids $b^*(\theta_i)$. Now assume that for all $\theta_i \in (\theta', \theta'')$ agent i deviates and bids $\bar{b} + \epsilon > \bar{b}$. For ϵ small, the loss is $p_i(\theta_i) \cdot \epsilon$, while the gain is that he will get the good with probability 1 in the event that all types are in (θ', θ''), and will increase the probability of getting the good in all other events.Ignoring the second gain, the total change is then at least:

$$\pi[1 - p_i(\theta_i)](\theta_i - b_i - \epsilon) - p_i(\theta_i) \cdot \epsilon \xrightarrow[0]{\epsilon} \pi[1 - p_i(\theta_i)](\theta_i - b_i) > 0$$

So for small enough ϵ this deviation is beneficial. Therefore, we must have $b^*(\theta_i)$ strictly increasing.

(c) We need to verify that the conditions for the Revenue Equivalence Theorem are satisfied. First, all valuations are drawn from an interval with positive density. Second, for the lowest valuation each agent gets the good with probability zero so the expected utility is zero. Finally, since $b^*(\theta_i)$ is strictly increasing, then the highest valuation agent gets the good with probability one and all others get nothing. These conditions are equivalent to those that would be satisfied for a sealed-bid second-price auction, and therefore the Revenue Equivalence Theorem holds.

23.D.5. Let $\theta_i' < \theta_i''$, let b_i' and b_i'' be agent i's bids associated with his two types, and assume in negation that $b_i' > b_i''$. Denote by π' and π'' the probabilities of winning the auction given bids b_i' and b_i'' respectively. Note that $b' > b''$ implies that $\pi' \geq \pi''$. By optimality (using a revealed-preference argument) we have that $\theta_i'\pi' - b_i' \geq \theta_i'\pi'' - b_i''$ which implies:

$$b_i'' - b_i' \geq \theta_i'(\pi'' - \pi') \qquad \text{(i)}$$

Similarly, $\theta_i''\pi'' - b_i'' \geq \theta_i''\pi' - b_i'$ which implies:

$$\theta_i''(\pi'' - \pi') \geq b_i'' - b_i' \qquad \text{(ii)}$$

(i) and (ii) imply that $(\theta_i'' - \theta_i')(\pi'' - \pi') \geq 0$. If this is strict, we have a contradiction to our assumption that $\theta_i'' - \theta_i' > 0$ and $\pi'' - \pi' \leq 0$. If this holds with equality this implies that $b_i'' - b_i' = 0$, a contradiction to our assumption that $b_i'' - b_i' > 0$.

Now to show that the bid must be strictly increasing in the type's valuation, an almost identical argument to that in part (b) of exercise 23.D.4 can be used (see above). Therefore, the argument of part (c) of exercise 23.D.4 applies here as well and this concludes the answer.

23.D.6. Going to stand in line t hours before 9:00am is equivalent to biding a monetary amount of βt dollars (ignoring the cost of the ticket which is assumed to be captured in θ). Since this is equivalent to a sealed-bid first-price auction, we know from the Revenue Equivalence Theorem that the expected revenue from the auction (i.e., the expected highest bid) is equal to that of a sealed-bid second-price auction. Therefore, we need to calculate the expected second-highest valuation of the agents. It is straightforward to see that the density function of the second highest valuation is $I\cdot(I-1)\cdot F(t)^{I-2}\cdot f(t)\cdot[1-F(t)]$, and therefore the expected second highest valuation is:

$$E[b] = I(I-1)\int_0^1 t\cdot t^{I-2}\cdot 1\cdot(1-t)\,dt = I(I-1)\left[\frac{t^I}{I} - \frac{t^{I+1}}{I+1}\right|_0^1 = \frac{I-1}{I+1}$$

Given a monetary bid of b, the respective waiting time is $t=b/\beta$, so that the expected waiting time of the first individual is:

$$E[t] = \frac{I-1}{\beta(I+1)}$$

Therefore, if β doubles, the waiting time goes down by 50%, and if I doubles, the waiting time goes up but this increase diminishes to zero as I goes to infinity.

23.E.1. From the solution to Exercise 23.D.2, we know that the seller will tell the truth, and his expected utility will be given by the function (this is (i) from the solution to 23.D.2, and taking $\theta_1=\hat{\theta}_1$):

$$E_{\theta_2}u_1 = \theta_1^2 - \frac{\theta_1^2}{2} + \frac{1}{2}\int_0^1 \theta_2^2\,d\theta_2 = \frac{\theta_1^2}{2} + \frac{1}{6}$$

Therefore, the seller will gain from not participating if $\theta_1 > \frac{\theta_1^2}{2} + \frac{1}{6}$, that is, if $\theta_1 > 1 - \sqrt{2/3}$. Note that since the buyer's reservation utility is 0, he will always be better off participating under the assumption that the seller is always participating.

23.E.2 (a) The Myerson-Satterthwaite Theorem tells us that there is no Bayesian incentive compatible (BIC) and interim individually rational (IIR) social choice function (SCF). We know that if a SCF is dominant strategy incentive compatible (DSIC), then it is BIC. Therefore, a SCF which is not BIC, is not DSIC. This concludes that there is no SCF that is DSIC and IIR.

(b) We know that if a SCF is *ex post* individually rational, then it is IIR. Therefore, a SCF which is not IIR is not *ex post* individually rational. This, with the Myerson-Satterthwaite Theorem concludes that there is no SCF which is BIC and *ex post* individually rational.

23.E.3 Let $\theta_i \in \{\underline{\theta}, \bar{\theta}\}$ where $\underline{\theta}_1 = 1$, $\bar{\theta}_1 = 3$, $\underline{\theta}_2 = 2$, and $\bar{\theta}_2 = 4$. *ex post* efficiency implies that $k(1,\theta_2) = k(3,4) = 1$ and $k(3,2) = 0$ (where k=1 denotes trade and k=0 no trade). Furthermore we must have $t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2) = 0$ for *ex post* efficiency. Now for individual rationality we must have the following:

$$t_1(1,\theta_2) \geq 1 \qquad\qquad t_2(\theta_1,2) \geq -2$$

$$t_1(3,\theta_2) \geq 3 \qquad\qquad t_2(\theta_1,4) \geq -4$$

Now set $t_1(1,2) = t_1(1,4) = 2$, $t_1(3,2) = 0$, and $t_1(3,4) = 3.5$. Symmetrically set $t_2(\theta_1,\theta_2) = -t_1(\theta_1,\theta_2)$. This example gives us the required result, which concludes that without a strictly positive density function (i.e., with discrete values) the Myerson-Satterthwaite Theorem may not hold.

23.E.4. Since both agents are risk neutral, we can interpret this game in the following way: If trade occurs at date t, this is equivalent to trade occurring with probability δ^{t-1} due to the discounting. Then, trade occurring in the first period is equivalent, in the interpretation model, to trade occurring with probability one. But the Myerson-Satterthwaite theorem exactly tells us that in this setting we cannot have trade ocuuring with probability one.

23.E.5. (a) The expost efficient trading rule will have all the highest valuation agents owning a unit of the good. i.e., we may have only buyers, only sellers or a mixture of both ending up with the good, as long as there is some θ^* such that all agents with $\theta \geq \theta^*$ have one unit of the good and all others do not. The total amount of good is given by the continuum $[\underline{\theta}_1, \bar{\theta}_1]$.

(b) Define the following "competitive" social choice function as follows: Let q_S and q_D denote market supply and demand, and be defined as:

$$q_D = \begin{cases} \bar{\theta}_2 - \underline{\theta}_2 & \text{for } p \leq \underline{\theta}_2 \\ \bar{\theta}_2 - p & \text{for } \underline{\theta}_2 \leq p \leq \bar{\theta}_2 \\ 0 & \text{for } p \geq \bar{\theta}_2 \end{cases} \qquad q_S = \begin{cases} 0 & \text{for } p \leq \underline{\theta}_1 \\ p - \underline{\theta}_1 & \text{for } \underline{\theta}_1 \leq p \leq \bar{\theta}_1 \\ \bar{\theta}_1 - \underline{\theta}_1 & \text{for } p \geq \bar{\theta}_1 \end{cases}$$

The market equilibrium price p* will cause efficient trade, and will lead to the efficient outcome described in (a) above. There will be no need for announcements, so that is is incentive compatible (vacuously) and it is individually rational since a buyer will buy if and only if $p \leq \theta_2$, and a seller will sell if and only if $p \geq \theta_1$. This example concludes that for a continuum of buyers and sellers the Myerson-Satterthwaite Theorem no longer holds.

23.E.6. (a) The efficient trading rule is such that if $\theta_i > \theta_j$ then j should sell the good to i for a price $p \in [\theta_i, \theta_j]$.

(b) Let b_i denote agent i's bid. The utilities of the agents, conditional on the relative values of b_1 and b_2 are:

	$b_1 \geq b_2$	$b_1 < b_2$
$u_1(b_1, b_2) =$	$2\theta_1 - b_1$	$b_2 - \theta_1$
$u_2(b_1, b_2) =$	$b_1 - \theta_2$	$2\theta_2 - b_2$

Restricting the bids to linear bids, $b_i = \alpha_i + \beta_i \theta_i$, agent 1 maximizes his expected utility:

$$Eu_1 = E_{\theta_2}[2\theta_1 - b_1 \mid b_1 \geq b_2] + E_{\theta_2}[b_2 - \theta_1 \mid b_1 < b_2] =$$

$$= \int_0^{\frac{b_1-\alpha_2}{\beta_2}} (2\theta_1 - b_1)d\theta_2 + \int_{\frac{b_1-\alpha_2}{\beta_2}}^{1} (\alpha_2+\beta_2\theta_2-\theta_1)d\theta 2$$

Using Leibnitz's rule we obtain the FOC:

$$(2\theta_1-b_1)\frac{1}{\beta_2} - (2\theta_1-b_1)\cdot 0 + \int_0^{\frac{b_1-\alpha_2}{\beta_2}} (-1)d\theta_2 + (\alpha_2+\beta_2\theta_2-\theta_1)\cdot 0 + (\alpha_2+\beta_2\theta_2-\theta_1)\frac{1}{\beta_2} + \int_{\frac{b_1-\alpha_2}{\beta_2}}^{1} 0 d\theta 2 = 0$$

which yields (after some simple algebra):

$$b_1 = \frac{\alpha_2}{3} + \theta_1 \qquad \text{(i)}$$

The symmetric problem for agent 2 yields:

$$b_2 = \frac{\alpha_1}{3} + \theta_2 \qquad \text{(ii)}$$

(i) and (ii) imply that $\alpha_1=\alpha_2=0$ and therefore $b_i= \theta_i$ for $i=1,2$.

(c) The social choice function that is implemented is: $\theta_i>\theta_j$ implies that i buys the good from j at price θ_i. The analysis in (b) above shows that it is Bayesian incentive compatible, and it is clearly *ex post* efficient and individually rational. The result differs from the Myerson-Satterthwaite Theorem because of the symmetry of the agents - each agent is a buyer and a seller so there is *always* an efficient trade.

23.E.7 (a) Let $y(\theta_1,\theta_2)$ be the probability of trade and $t(\theta_1,\theta_2)$ be the payment from the buyer to the seller given that they announced (θ_1,θ_2). ex *post* efficiency entails that $y(\theta_1,\theta_2)=1$ if $\theta_1<\theta_2$, and $y(\theta_1,\theta_2)=0$ otherwise. Denote the expected payment that the seller anticipates to get given that he announces θ_1' by:

$$\bar{t}_1(\theta_1') = \int_0^1 t(\theta_1',\theta_2)d\theta_2 \ .$$

For incentive compatability on behalf of the seller we must have that truth telling solves :

$$\underset{\theta_1'}{\text{Max}} \ \ \theta_1\cdot\text{Prob}\{\theta_1'>\theta_2\} + \bar{t}_1(\theta_1')$$

Given that $y(\theta_1,\theta_2)$ is *ex post* efficient, the objective function is just $\theta_1\cdot\theta_1' + \bar{t}_1(\theta_1')$, and for $\theta_1'=\theta_1$ to be a solution we must have the following FOC:

$$\theta_1 + \frac{d\bar{t}_1(\theta_1)}{d\theta_1} = 0 \quad \text{for all } \theta_1$$

Integrating this we get:

$$\int_{\theta_1}^{1} \theta_1 d\theta_1 + \int_{\theta_1}^{1} \frac{d\bar{t}_1(\theta_1)}{d\theta_1} = 0$$

Note that from interim IR, we must have $\bar{t}_1(1)\geq 0$, otherwise the seller can leave and consume the good, getting a utility of 1. Using this, the integral above yields: $\bar{t}_1(\theta_1) \geq \frac{1}{2} - \frac{1}{2}\theta_1^2$. Therefore, the interim expected utility of the seller, given that truth telling is optimal (IC), is given by:

$$E_{\theta_2}u_1(\theta_1) = \theta_1^2 + \bar{t}_1(\theta_1) \geq \frac{1}{2} + \frac{1}{2}\theta_1^2$$

Similarly, denote the expected payment that the buyer anticipates to pay given that he announces θ_2' by:

$$\bar{t}_2(\theta_2') = \int_0^1 t(\theta_1,\theta_2')d\theta_2 \ .$$

For incentive compatability on behalf of the seller we must have that truth telling solves :

$$\underset{\theta_2'}{\text{Max}} \ \ \theta_2\cdot\text{Prob}\{\theta_1<\theta_2'\} - \bar{t}_2(\theta_2')$$

Given that $y(\theta_1,\theta_2)$ is *ex post* efficient, the objective function is just

$\theta_2 \cdot \theta_2' - \bar{t}_2(\theta_1')$, and for $\theta_2' = \theta_2$ to be a solution we must have the following FOC:

$$\theta_2 - \frac{d\bar{t}_2(\theta_2)}{d\theta_2} = 0 \quad \text{for all } \theta_2$$

Note that from interim IR, we must have $\bar{t}_2(0) \leq 0$, otherwise the buyer can leave and get a utility of 0. Integrating the FOC from 0 to θ_2, and using this interim IR condition yields: $\bar{t}_2(\theta_2) \leq \frac{1}{2}\theta_2^2$. Therefore, the interim expected utility of the buyer, given that truth telling is optimal (IC), is given by:

$$E_{\theta_1} u_2(\theta_2) = \theta_2^2 - \bar{t}_2(\theta_2) \geq \frac{1}{2}\theta_2^2$$

Adding the interim expected utilities, we get that the *ex ante* sum of expected utilities is:

$$E(u_1+u_2) \geq \frac{1}{2} + \int_0^1 \frac{1}{2}\theta_1^2 d\theta_1 + \int_0^1 \frac{1}{2}\theta_2^2 d\theta_2 = \frac{5}{6}$$

Remark: The result above can be derived using inequality (23.E.4). The minimum total expected utility for the two agents should be the expected utility of the seller *without* trade, plus the minimum expected gains from trade. The latter can be calculated by rearranging (23.E.4) as follows:

$$\int_0^1\int_0^1 y(\theta_1,\theta_2)[\theta_1-\theta_2]\phi_1(\theta_1)\phi_2(\theta_2)d\theta_2 d\theta_1 \geq$$

$$\geq \int_0^1\int_0^1 y(\theta_1,\theta_2)\left(\frac{1-\Phi_2(\theta_2)}{\phi_2(\theta_2)} + \frac{\Phi_1(\theta_1)}{\phi_1(\theta_1)}\right)\phi_1(\theta_1)\phi_2(\theta_2)d\theta_1 d\theta_2$$

The right hand side of this inequality is the expected gains from trade, and solving the left hand side with our parameters gives $\frac{1}{3}$.

Adding the seller's expected utility without trade, which is $\frac{1}{2}$, gives us $\frac{5}{6}$ as calculated above.

(b) If the true observations are (θ_1, θ_2), the total utility that can be produced is at most $\text{Max}\{\theta_1, \theta_2\}$, even if all informational asymmetries are overcome. Thus, the *ex ante* total expected surplus can be at most:

$$\int_0^1 \text{Max}\{\theta_1, \theta_2\}\, d\theta_1 d\theta_2 = \frac{2}{3}$$

Therefore, no SCF can have the sum of expected utilities exceeding $\frac{2}{3}$.

23.F.1 We prove the statement in three steps:

<u>Claim 1:</u> If $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is ex post classically efficient then it is ex-ante classically efficient.

<u>Proof:</u> Suppose not, i.e., there exists a feasible social choice function (SCF) $\hat{f}(\cdot) = (\hat{k}(\cdot), \hat{t}_1(\cdot), \ldots, \hat{t}_I(\cdot))$ such that ,

$$E_\theta[v_i(\hat{k}(\theta), \theta_i) + \hat{t}_i(\theta)] \geq E_\theta[v_i(k(\theta), \theta_i) + t_i(\theta)], \qquad \text{(i)}$$

with strict inequality for at least one i. This implies that

$$E_\theta[\textstyle\sum_i \hat{t}_i(\theta)] - E_\theta[\sum_i t_i(\theta)] \geq E_\theta[v_i(k(\theta), \theta_i)] - E_\theta[v_i(\hat{k}(\theta), \theta_i)]. \qquad \text{(ii)}$$

From ex post efficiency of $f(\cdot)$ we must have that the right hand side of (ii) is non-negative, and that $E_\theta[\sum_i t_i(\theta)] = 0$. However, feasibility of $\hat{f}(\cdot)$ implies that $E_\theta[\sum_i \hat{t}_i(\theta)] \leq 0$, a contradiction.

<u>Claim 2:</u> If $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is ex ante classically efficient and $f(\cdot) \in F^*$ then it is ex-ante incentive efficient.

<u>Proof:</u> By definition (see Definition 23.F.1 and the preceding paragraph).

<u>Claim 3:</u> If $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is ex ante incentive efficient then it is interim incentive efficient.

<u>Proof:</u> his is just restating Proposition 23.F.1.

23.F.2 Think of the buyer as the agent, and the monopolist as the principal, in the setting of example 23.F.1. The agent's utility is given by $u_1(\theta,x,t) = \theta v(x) - t$ (so that a higher θ results in a higher utility level and a higher marginal utility - the latter is the "single crossing" condition). The sellers utility is given by $u_0(\theta,x,t) = t - c\cdot x$. Using the revelation principal we can concentrate on a direct mechanism $(x(\theta),t(\theta))$ which solves the sellers problem:

$$\max_{x(\theta),t(\theta)} \quad E[t(\theta) - c\cdot x(\theta)]$$

$$\text{s.t.} \quad (x(\theta),t(\theta)) \text{ is Bayesian incentive compatible and individually rational.}$$

We now follow the analysis of linear utility in section 23.D [with k=x, and $\tilde{v}(\theta)=v(x(\theta))$]. Letting $U_1(\theta) = \theta v(x(\theta)) - t(\theta)$ denote the utility of type θ from truth-telling, proposition 23.D.2 implies that the principal's problem is equivalent to (recall that $v'(\cdot)>0$ so that constraint (i) can be written as shown below):

$$\max_{x(\theta),U(\theta)} \quad E[\theta v(x(\theta)) - U(\theta) - cx(\theta)]$$

s.t. (i) $x(\cdot)$ is non-decreasing

$$\text{(ii) } U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(x(s))ds \quad \text{for all } \theta\in[\underline{\theta},\bar{\theta}]$$

$$\text{(iii) } U(\theta) \geq 0 \quad \text{for all } \theta\in[\underline{\theta},\bar{\theta}]$$

Following the same steps as in example 23.F.1 the problem becomes:

$$\max_{x(\theta),U(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} [\theta v(x(\theta)) - cx(\theta)]\phi(\theta)d\theta - U(\underline{\theta})$$

s.t. (i) $x(\cdot)$ is non-decreasing

(ii) $U(\underline{\theta}) \geq 0$

Clearly, (ii) will bind at a solution, and using integration by parts the problem becomes:

$$\underset{x(\theta)}{\text{Max}} \int_{\underline{\theta}}^{\bar{\theta}} \left(v(x(\theta)) \left[\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)} \right] - cx(\theta) \right) \phi(\theta) d\theta$$

s.t. $x(\cdot)$ is non-decreasing

Ignoring the constraint, and assuming an interior solution (see footnote 22 in chapter 23) we get that the function $x(\theta)$ which solves the problem must satisfy the FOC:

$$v'(x(\theta)) \left[\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)} \right] - c = 0 \quad \text{for all } \theta \in [\underline{\theta}, \bar{\theta}]$$

To see that the solution $x(\cdot)$ is indeed non-decreasing, the same argument as in example 23.F.1 applies here as well. The SOC is satisfied since $v''(\cdot)<0$. Note that the highest valuation consumer is set at the first-best level since we get $v'(x(\bar{\theta}))\bar{\theta} - c = 0$, but all other consumer types are distorted (this is a common result in the screening literature).

23.F.3. We will follow the analysis of example 23.F.2, and in particular the result of equation (23.F.9). From symmetry and the fact that $\underline{\theta}=0$, there exists $\hat{\theta}$ such that $J_i(\theta_i)<0$ for all $\theta_i \in [\underline{\theta}, \hat{\theta})$. Therefore, we can implement the result of (23.F.9) using a second-price sealed-bid auction with a reserve price of $\hat{\theta}$.

In the general case, for each bidder i there exists $\hat{\theta}_i$ such that $J_i(\theta_i)<0$ for all $\theta_i \in [\underline{\theta}_i, \hat{\theta}_i)$ (it could be that for some i we will have $\hat{\theta}_i = \underline{\theta}_i$). The seller can then offer an auction where each agent i is restricted to submit bids greater than $\hat{\theta}_i$, and the agent which generates the highest $J_i(\theta_i)$ wins the auction. If agent i won the auction, we calculate the value $\tilde{\theta}_i$ such that

$J_i(\bar{\theta}_i) = \text{Max}_{h\neq i}\{J_h(\theta_h)\}$ (i.e., the value for which agent i's $J(\cdot)$ function generates the second highest $J(\cdot)$ generated), and agent i pays $\text{Max}\{\hat{\theta}_i, \bar{\theta}_i\}$. (For more on this see Bullow and Roberts [1989]).

23.F.4. The analysis of example 23.F.2. applies here as well, yet the objective function in (23.F.8) changes and becomes:

$$\int_{\underline{\theta}_1}^{\bar{\theta}_1}\cdots\int_{\underline{\theta}_I}^{\bar{\theta}_I}\left(\sum_{i=1}^{I} y_i(\theta)\left[\theta_i - \frac{1-\Phi(\theta_i)}{\phi(\theta_i)}\right] + \theta_0\left[1 - \sum_{i=1}^{I} y_i(\theta)\right]\right)\left[\prod_{i=1}^{I}\phi_i(\theta_i)\right]d\theta_1\cdots d\theta_I$$

and therefore, (23.F.9) becomes:

$$y_i(\theta) = 1 \quad \text{if} \quad J_i(\theta_i) > \text{Max}\{\theta_0, \text{Max}_{h\neq i}\{J_h(\theta_h)\}\}$$

and

$$y_i(\theta) = 0 \quad \text{if} \quad J_i(\theta_i) < \text{Max}\{\theta_0, \text{Max}_{h\neq i}\{J_h(\theta_h)\}\}$$

23.F.5. (a) Letting $p_i(x_i)$ denote agent i's inverse demand function, the monopolist's problem is:

$$\underset{x_1, x_2}{\text{Max}} \quad p_1(x_1)\cdot x_1 + p_2(x_2)\cdot x_2$$

$$\text{s.t.} \quad x_1 + x_2 \leq 1$$

Using Kuhn-Tucker, the FOCs are:

$$p_i'(x_i)\cdot x_i + p_i(x_i) - \lambda \geq 0 \quad \text{for } i=1,2$$

$$[p_i'(x_i)\cdot x_i + p_i(x_i) - \lambda]\cdot x_i = 0 \quad \text{for } i=1,2$$

Therefore, depending on the demand functions we will have one, both, or none of the x_i positive, and there sum being less than one.

(b) The results here resemble those of example 23.F.2. The monopolist sets prices (quantities) to maximize his payoff. Interpreting quantities as probabilities, and prices as valuations, the results here are similar to the optimal auction program. For more on this see Bulow and Roberts (1989).

23.F.6. Let $x = (p,T)$ where p is the price and T is the total revenue given to the firm, $T = p\cdot x(p) + t$. Then, $u_1(x,\theta) = \theta[-x(p)] + T$, and using the notation on page 887 of the textbook we obtain that $u_1(x,\theta) = v_1(k) + t_1$, which in turn is equal to $\tilde{v}_1(k)$.

a) Consider a mechanism as $[p(\theta),T(\theta)]$. Then Proposition 23.D.2 tells us that $[p(\theta),T(\theta)]$ is implementable if and only if,

(i) $-x(p(\theta))$ is non-decreasing, i.e., $p(\theta)$ is non-decreasing,

$$\text{(ii)}\ u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \tilde{v}_1(s)ds$$

$$= u(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} x(p(s))ds$$

$$= u(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} x(p(s))ds$$

b) The regulator wants to choose $[p(\theta),T(\theta)]$, or equivalently, $[p(\theta),u(\theta)]$, to maximize,

$$\int_{\underline{\theta}}^{\bar{\theta}}\left(\left[\int_{p(\theta)}^{\infty} x(s)ds - t(\theta)\right] + \alpha u(\theta)\right)\phi(\theta)d\theta\ .$$

subject to $u(\theta) \geq 0$ for all θ, and the two conditions given in part a) above. Since $u(\theta) = T(\theta) - \theta x(p(\theta))$ the objective function can be rewritten as,

$$\int_{\underline{\theta}}^{\bar{\theta}}\left(\left[\int_{p(\theta)}^{\infty} x(s)ds - \{u(\theta) + \theta x(p(\theta)) - p(\theta)x(p(\theta))\}\right] + \alpha u(\theta)\right)\phi(\theta)d\theta$$

$$\int_{\underline{\theta}}^{\bar{\theta}} u(\theta)\phi(\theta)d\theta = \Big[u(\theta)\Phi(\theta)\Big]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} u'(\theta)\Phi(\theta)d\theta$$

$$= u(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} x(p(\theta))\Phi(\theta)d\theta \ .$$

Substituting, we want to choose $[p(\theta), u(\bar{\theta})]$ to solve,

$$\text{Max} \ \int_{\underline{\theta}}^{\bar{\theta}} \left(\left[\int_{p(\theta)}^{\infty} x(s)ds \right] - [p(\theta)-\theta]x(p(\theta)) - (1-\alpha)x(p(\theta))\frac{\Phi(\theta)}{\phi(\theta)} \right) \phi(\theta)d\theta - (1-\alpha)u(\bar{\theta})$$

s.t. (i) $u(\bar{\theta}) \geq 0$ (which implies $u(\theta) \geq 0$ for all θ)

(ii) $p(\theta)$ is non-decreasing.

Start by setting $u(\bar{\theta}) = 0$, and let's ignore constraint (ii). Taking the FOC w.r.t. $p(\theta)$, we get,

$$-x(p(\theta)) + x(p(\theta)) + (p(\theta) - \theta)x'(p(\theta)) - (1-\alpha)x'(p(\theta))\frac{\Phi(\theta)}{\phi(\theta)} = 0$$

rearranging and dividing by $x'(p(\theta))$ we get,

$$p(\theta) = \theta + (1-\alpha)\frac{\Phi(\theta)}{\phi(\theta)} \ . \qquad (*)$$

So, $\frac{\Phi(\theta)}{\phi(\theta)}$ nondecreasing and $(1-\alpha) > 0$ implies that constraint (ii) is satisfied, and (*) is the regulator's optimal price rule. If $\alpha = 1$ then we want $p(\theta) = \theta$. If, however, $\alpha > 1$, then we have no solution: for any transfer function $t(\cdot)$, we can increase "welfare" by raising it to $t(\cdot) + \epsilon$ for $\epsilon > 0$.

23.F.7. This question is analyzed in Dana-Spier (1994). Following the analysis in section 2 of the paper, one can derive the following "virtual welfare function" corresponding to a monopoly awarded to firm i (see the paper for this definition) :

$$W^i(\theta) = \frac{(1-c)^2}{8} + \lambda\left(\frac{(1-c)^2}{8} - \theta\right) - (\lambda-1)\frac{F(\theta_i)}{f(\theta_i)} \ , \qquad \text{for } i=1,2$$

and the virtual welfare function associated with a duopoly is:

$$W^d(\theta) = \frac{2(1-c)^2}{9} + \lambda\left(\frac{2(1-c)^2}{9} - \theta_1 - \theta_2\right) - (\lambda-1)\left(\frac{F(\theta_1)}{f(\theta_1)} - \frac{F(\theta_2)}{f(\theta_2)}\right), \quad \text{for } i=1,2$$

The government awards a monopoly to firm i if $W^i(\theta) = \text{Max}(W^1(\theta), W^2(\theta), W^d(\theta))$, and a duopoly right if $W^d(\theta) = \text{Max}(W^1(\theta), W^2(\theta), W^d(\theta))$.

23.F.8. For both θ_L and θ_H, the buyer values the good more than the seller does. If $y(\theta)<1$ for some θ, then increasing $y(\theta)$ to 1 gives the buyer more utility than the seller looses, and we can therefore increase the transfer $t(\theta)$ to (more than) compensate the seller for his loss so both agents are better off. Therefore, any *ex post* classically efficient social choice function must have $y_L=y_H=1$.

23.F.9. (a) We can rewrite (23.F.15) and (23.F.16) as follows:

$$t_H - t_L \geq 40(y_H - y_L) \qquad (23.F.15)$$

$$t_H - t_L \leq 20(y_H - y_L) \qquad (23.F.16)$$

Both the above yield $20(y_H - y_L) \geq (t_H - t_L) \geq 40(y_H - y_L)$, which implies that $y_H \leq y_L$, and $t_H \leq t_L$.

(b) *ex post* efficiency implies that $y_L=y_H=1$. This, with (23.F.13) imply that $t_H \geq 40$. This in turn, together with $t_H \leq t_L$ (from (i) above) imply that $t_L \geq 40$. Using these conclusions, we could evaluate the expected utility of the buyer:

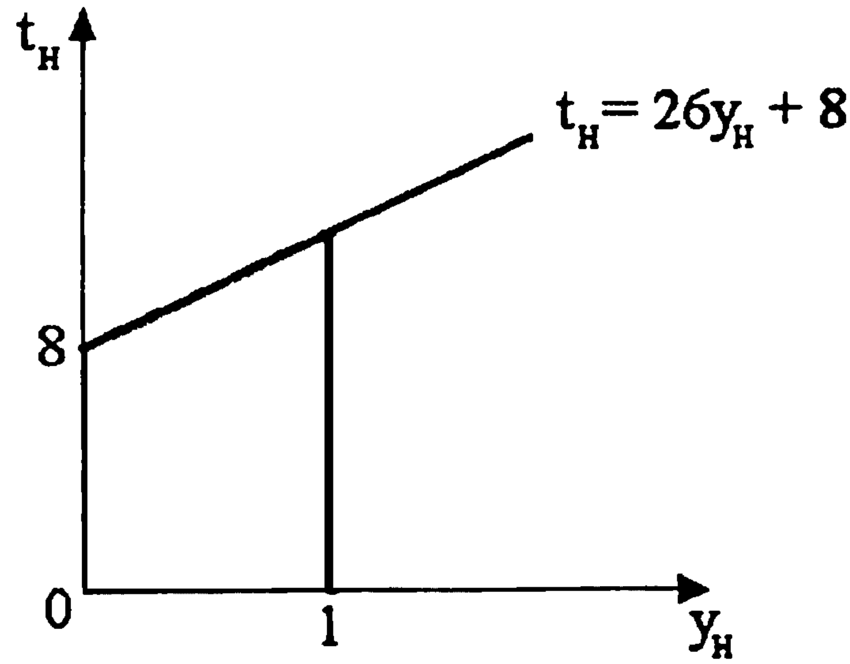
$$EU_2 = .2(50 - t_H) + .8(30 - t_L) \leq .2(50 - 40) + .8(30 - 40) \leq -6$$

so that (23.F.14) is violated. Therefore, no feasible social choice function can be *ex post* efficient.

(c) (23.F.16) together with $y_H \geq 0$ implies that:

$$t_L - 20y_L \geq t_H - 20y_h \geq t_H - 40y_H.$$

23.F.10. The set of interim efficient social choice functions are those that solve (23.F.17) for some $\bar{u}_2 \geq 0$ and $\bar{u}_{1H}$ (the change is that we no longer require $\bar{u}_{1H} \geq 0$). The logic leading to 23.F.19 and Figure 23.F.1 in the textbook is unchanged from the logic in the textbook. Now, however, because $\bar{u}_{1H}$ can take on negative values the set of (y_H, t_H) pairs in interim incentive efficient social choice functions is the standard set in Figure 23.F.10 below:



The boundary at $y_H = 1$ comes from the feasibility requirement that $y_H \leq 1$. Following, again, the same sort of logic as in the textbook (and of Figure 23.F.3 there), the set of (y_H, t_H) in ex ante incentive efficient social choice functions is the heavily traced boundary at $y_h = 1$ in figure 23.F.10 above. Thus, when trade is not voluntary for the seller, all ex ante incentive efficient social choice functions have trade occurring with probability one.

23.AA.1. Consider as a mechanism the following game where player 1 chooses rows, 2 chooses columns, and the mechanism chooses the resulting outcome in the matrix:

		player 2		
		l	m	r
	U	b	a	c
player 1	C	a	a	d
	B	c	d	e

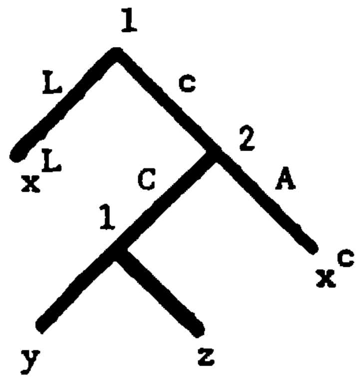
It is easy to check that if player 1's type is θ_1' then U is his unique (weakly) dominant strategy, and when he is type θ_1'' then C is his unique (weakly) dominant strategy. Similarly, if player 2's type is θ_1' then l is her unique (weakly) dominant strategy, and when she is type θ_1'' then m is her unique (weakly) dominant strategy.

23.AA.2. The answer to the first part is no. This follows from the fact that <u>every</u> possible valuation function arises from Θ_i for all i, so that for all $\alpha\in\mathbb{R}$, and every θ_i, there exists a θ_i' such that $v_i(k_n,\theta_i') = v_i(k_n,\theta_i) + \alpha$. Therefore, these two valuation functions, $v_i(k_n,\theta_i')$ and $v_i(k_n,\theta_i)$ not only rank the projects ordinally the same, but measure the differences in utility between projects identically. This implies that $k^*(\theta_i,\theta_{-i}) = k^*(\theta_i',\theta_{-i})$, which in turn implies that $t_i(\theta_i,\theta_{-i}) = t_i(\theta_i',\theta_{-i})$ so agent i is indifferent between announcing θ_i or the shifted θ_i'.

This is the only type of modification that would keep the agent indifferent, implying that the answer to the second art of the question is no. If the agent chooses to misrepresent, since the value of k_0 is "anchored" to zero, then any misrepresentation must involve changing the monetary values of the differences between some alternatives, which might change the decision of the optimal project, which in turn will change the utility of the deviant agent. therefore, truth-telling will be the unique (weakly) dominant strategy.

23.BB.1. (a) In the normal form game associated with the extensive form game described, agent 1 has three strategies, {Ly,Lz,cy,cz} (where "$\alpha\beta$" means "announce α, and if challenged choose β") and agent 2 has two strategies: {A,C} for "Accept" or "Reject". The normal and extensive form games can be depicted as shown in the following figure:

		2	
		A	C
	Ly	x^L	x^L
1	Lz	x^L	x^L
	cy	x^c	y
	cz	x^c	z



It is easy to check that (cz,A) and (Lα,C), $\alpha\in\{y,z\}$ are both Nash equilibria outcomes in the normal form game, and therefore, any strategy profile which supports these outcomes will be a Nash equilibrium.

(b) Looking at the extensive form of the game and using backward induction for each type profile, it is easy to check that the only strategy profile which is a subgame perfect equilibrium is:

$$s_1(\theta_1,\theta_2) = \begin{cases} \text{Ly} & \text{if } \theta_1 = \text{L} \\ \text{cz} & \text{if } \theta_1 = \text{c} \end{cases} \qquad s_2(\theta_1,\theta_2) = \begin{cases} \text{C} & \text{if } \theta_1 = \text{L} \\ \text{A} & \text{if } \theta_1 = \text{c} \end{cases}$$

□

23.BB.2. Clearly, any equilibrium in dominant strategies is a Nash equilibrium, so the answer to the first part is "yes". If the dominant strategy equilibrium is strict (i.e., not weakly dominant) then clearly we would have strong implementation. However, this is not the case if the dominant equilibrium is weak. Consider the following example:

Two agents, 1 and 2, with the following preferences:

$\succsim_1(\theta_1')$	$\succsim_1(\theta_1'')$	$\succsim_2(\theta_2')$	$\succsim_2(\theta_2'')$
a-b	b-c	a-b	b-c
c	a	c	a

and the social choice function (SCF) is $f(\theta_1',\theta_2')=a$, $f(\theta_1'',\theta_2'')=c$, and $f(\theta_1',\theta_2'')=f(\theta_1'',\theta_2')=b$. It is easy to check that this SCF is implementable in (weakly) dominant strategies. However, to see that it is not strongly Nash implementable, the following is a Nash equilibrium of the direct mechanism: agent 1 always announces himself to be a θ_1' type, and agent 2 always announces herself to be a θ_1'' type. This is a NE that always implements outcome b, no matter the types, and therefore it does not implement $f(\cdot)$.

23.BB.3. <u>Claim 1:</u> Property (i) must be satisfied.

<u>Proof:</u> By Proposition 23.BB.1, if $f(\cdot)$ is Nash implementable the $f(\cdot)$ is monotonic. Let $\theta=(\theta_i,\theta_{-i})$ be such that $k^*(\theta)=1$, and let $\theta'=(\theta_i',\theta_{-i})$ satisfy $\theta_i' > \theta_i$, so monotonicity implies that $k^*(\theta')=1$. Similarly, if $k^*(\theta)=0$, and $\theta'=(\theta_i',\theta_{-i})$ satisfies $\theta_i' \le \theta_i$, then $k^*(\theta')=0$. Now assume in negation that property (i) didn't hold, e.g., w.l.o.g. assume that $t_i(\theta_i',\theta_{-i}) > t_i(\theta_i,\theta_{-i})$. Then when agent i's type is θ_i', and all others' are θ_{-i}, then agent i would prefer to misrepresent and announce θ_i instead, contradicting that $f(\cdot)$ is Nash implementable. Therefore, we must have $t_i(\theta_i',\theta_{-i}) = t_i(\theta_i,\theta_{-i})$.

<u>Claim 2:</u> Property (ii) need not be satisfied.

<u>Proof:</u> Consider a SCF that is implementable in Strong Nash equilibrium. By proposition 23.BB.1 such a SCF must satisfy monotonicity. This implies that if $k^*(\theta_i',\theta_{-i}) = 1$ and $k^*(\theta_i,\theta_{-i}) = 0$ then we must have that $\theta_i' > \theta_i$. However, this imposes no restrictions on the change in the vector of transfers from $(t_1(\theta_i,\theta_{-i}),\ldots,t_I(\theta_i,\theta_{-i}))$ to $(t_1(\theta_i',\theta_{-i}),\ldots,t_I(\theta_i',\theta_{-i}))$. Therefore, any change will be compatible with monotonicity, and we need only to satisfy incentive compatibility which can be satisfied in many ways.

$u_1(\theta,x,t) = \theta v(x) - t$ (so that a higher θ results in a higher utility leve and a higher marginal utility - the latter is the "single crossing" condition). The sellers utility is given by $u_0(\theta,x,t) = t - c\cdot x$. Using the revelation principal we can concentrate on a direct mechanism $(x(\theta),t(\theta))$ which solves the sellers problem:

$$\max_{x(\theta),t(\theta)} \quad E[t(\theta) - c\cdot x(\theta)]$$

s.t. $(x(\theta),t(\theta))$ is Bayesian incentive compatible and individually rational.

We now follow the analysis of linear utility in section 23.D [with $k=x$, and $\bar{v}(\theta)=v(x(\theta))$]. Letting $U_1(\theta) = \theta v(x(\theta)) - t(\theta)$ denote the utility of type θ from truth-telling, proposition 23.D.2 implies that the principal's problem is equivalent to (recall that $v'(\cdot)>0$ so that constraint (i) can be written as shown below):

$$\max_{x(\theta),U(\theta)} \quad E[\theta v(x(\theta)) - U(\theta) - cx(\theta)]$$

s.t. (i) $x(\cdot)$ is non-decreasing

(ii) $U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(x(s))ds$ for all $\theta\in[\underline{\theta},\bar{\theta}]$

(iii) $U(\theta) \geq 0$ for all $\theta\in[\theta,\bar{\theta}]$

Following the same steps as in example 23.F.1 the problem becomes:

$$\max_{x(\theta),U(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} [\theta v(x(\theta)) - cx(\theta)]\phi(\theta)d\theta - U(\underline{\theta})$$

s.t. (i) $x(\cdot)$ is non-decreasing

(ii) $U(\underline{\theta}) \geq 0$

23.F.6. Let $x = (p,T)$ where p is the price and T is the total revenue given to the firm, $T = p\cdot x(p) + t$. Then, $u_i(x,\theta) = \theta[-x(p)] + T$, and using the notation on page 887 of the textbook we obtain that $u_i(x,\theta) = v_i(k) + t_i$, which in turn is equal to $\bar{v}_i(k)$.

a) Consider a mechanism as $[p(\theta),T(\theta)]$. Then Proposition 23.D.2 tells us that $[p(\theta),T(\theta)]$ is implementable if and only if,

(i) $-x(p(\theta))$ is non-decreasing, i.e., $p(\theta)$ is non-decreasing,

(ii)
$$\begin{aligned} u(\theta) &= u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \bar{v}_i(s)ds \\ &= u(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} x(p(s))ds \\ &= u(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} x(p(s))ds \end{aligned}$$

b) The regulator wants to choose $[p(\theta),T(\theta)]$, or equivalently, $[p(\theta),u(\theta)]$, to maximize,

$$\int_{\underline{\theta}}^{\bar{\theta}}\left(\left[\int_{p(\theta)}^{\infty} x(s)ds - t(\theta)\right] + \alpha u(\theta)\right)\phi(\theta)d\theta \ ,$$

subject to $u(\theta) \geq 0$ for all θ, and the two conditions given in part a) above. Since $u(\theta) = T(\theta) - \theta x(p(\theta))$ the objective function can be rewritten as,

$$\int_{\underline{\theta}}^{\bar{\theta}}\left(\left[\int_{p(\theta)}^{\infty} x(s)ds - \{u(\theta) + \theta x(p(\theta)) - p(\theta)x(p(\theta))\}\right] + \alpha u(\theta)\right)\phi(\theta)d\theta$$

$$= \int_{\underline{\theta}}^{\bar{\theta}}\left(\left[\int_{p(\theta)}^{\infty} x(s)ds\right] - [p(\theta) - \theta]x(p(\theta)) - (1 - \alpha)u(\theta)\right)\phi(\theta)d\theta \ .$$

Using integration by parts we have,