Econ 7010 - Microeconomics I
University of Virginia

Problem Set 4 Solutions

1. *The UVA Economics Department is looking to hire a new professor next year. For simplicity, assume there are only 3 new Ph.D. graduates on the market, named $\mathcal{X} = \{Ann, Bob, Charlie\}$. It may be that only some (and not all) of the candidates apply. For any potential set of applicants $Y \subset \mathcal{X}$, the department has a choice rule $C(Y)$ that determines who will be given an offer. The following condition, known as Sen's α, is a (partial) alternative to WARP. Let Y, Z be two sets of potential applicants.*

   - Sen's α: *If $x \in Y$, $Y \subseteq Z$, and $x \in C(Z)$, then $x \in C(Y)$.*

   (a) *Translate this condition into words. (It may be helpful, though not necessary, to draw a Venn diagram.)*

   Consider the following two alternative choice procedures:

   - Second-best: *The department has the following preference relation: $Ann \succ Bob \succ Charlie$. However, rather than choose the best alternative, they choose the second-best applicant from any applicant pool Y (perhaps because they think the best applicant is unlikely to accept).*
   - Satisficing: *The department assigns a value $v(A)$, $v(B)$, and $v(C)$ to each possibility, a threshold v^*, and a fixed ordering of the candidates O. For any Y, the candidates are considered in in this order O, and, the first candidate with a value exceeding v^* is chosen; if no candidate exceeds the threshold, the choice is the last candidate in the list. For example, assume that $v(A) = 5$, $v(B) = 3$, and $v(C) = 10$, and the threshold value is $v^* = 4$. If $Y = \{Ann, Bob, Charlie\}$ and the candidates are ordered Bob, followed by Ann, followed by Charlie, $C(Y) = Ann$, because she is the first one to exceed the threshold. If the order were Bob, Charlie, Ann, then $C(Y) = Charlie$.*

   (b) *Show that the second-best procedure does* not *satisfy Sen's α. Does it satisfy WARP?*

(c) *Show that satisficing* does *satisfy Sen's α. (A formal "proof" is not necessary, but make sure you provide a clear, logical argument.)*

(d) *Show that in general, WARP implies Sen's α.*[1] *(Hint: Assume that there exists an x such that $x \in Y$, $Y \subseteq Z$, and $x \in C(Z)$, but $x \notin C(Y)$, and apply WARP.)*

Answer

Part (a)

The condition says if x is chosen from a larger set of alternatives, then it must be chosen from any smaller set that contains it; that is, eliminating some elements from Z should not affect whether you pick Y. This is also known as *independence of irrelevant alternatives.*

Part (b)

We have $C(Ann, Bob, Charlie) = Bob$, but $C(Bob, Charlie) = Charlie$. That is, eliminating Ann, who was not chosen in the first place, alters the choice between Bob and Charlie. This also violates WARP: Charlie was chosen when Bob was (from $\{Bob, Charlie\}$), but when offered $\{Ann, Bob, Charlie\}$, Bob and Charlie are both available, while Bob is chosen and Charlie is not.

Part (c)

If *Ann*, say, was chosen from some larger set Z, then she is the first to reach the threshold for the candidates in Y. If we shrink the choice set to Y and consider fewer candidates in the same order, then she clearly will still be the first to reach the threshold. If no candidate reached the threshold, then she was chosen because she was the last in the order under Z. She will also be the last in the order, and hence chosen, under Y.

Part (d)

Consider some $x \in Y \subseteq Z$ and $x \in C(Z)$. We want to show that $x \in C(Y)$. Assume not. Then, there is some other $y \in C(Y)$. Since $y \in C(Y)$, it is obvious that $y \in Y$ and also $y \in Z$. Thus, both x and y are in Z, and so x was chosen when y was available. Since $x \in Y$ and $y \in C(Y)$, WARP implies $x \in C(Y)$.

[1] The converse, that Sen's α implies WARP, is false, but you are not required to show this.

2. *Prove the following proposition: A preference relation can be represented by a utility function only if it is rational*

   Answer

   We need to show that if $u(\cdot)$ represents $\succeq$, then $\succeq$ is complete and transitive.

   Completeness: Consider two elements $x, y \in X$, and their corresponding utilities $u(x)$ and $u(y)$. Since $u(x)$ and $u(y)$ are real numbers, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ (or both). Since $u(\cdot)$ represents $\succeq$, this means that either $x \succeq y$ or $y \succeq x$ (or both). But this is simply the definition of completeness.

   Transitivity: Let $x \succeq y$ and $y \succeq z$. We must show that $x \succeq z$ holds. Since $u(\cdot)$ represents $\succeq$, $x \succeq y$ and $y \succeq z$ imply $u(x) \geq u(y)$ and $u(y) \geq u(z)$. By transitivity of the real numbers, $u(x) \geq u(z)$. Since u represents $\succeq$, this implies $x \succeq z$.

3. *Let $x(p, m)$ be an agent's Marshallian demand function. Prove that if the agent's preferences are convex, then $x(p, m)$ is convex, and further, if the agent's preferences are strictly convex, then $x(p, m)$ is a singleton. (Note: you may assume a continuous utility representation $u(x)$ exists, but do not assume it is differentiable. Let $x', x' \in x(p, m)$, and define $\bar{x} = \alpha x + (1-\alpha)x'$. To prove the first part, it is sufficient to show the following: (i) $\bar{x}$ is feasible at (p, m) and (ii) $\bar{x}$ gives at least as much utility as x, x'. Use the definition of quasiconcavity that says $u(\bar{x}) \geq \min\{u(x), u(x')\}$.)*

   Answer

   Since preferences are convex, there exists a quasiconcave utility function u.

   First, we show that $\bar{x}$ is feasible. Since x and x' are both feasible, we have $p \cdot x \leq m$ and $p \cdot x' \leq m$. Multiply the former inequality by α, the latter by $(1-\alpha)$, and add the two to get:

$$\begin{aligned} \alpha p \cdot x + (1-\alpha) p \cdot x' &\leq \alpha m + (1-\alpha) m \\ p \cdot (\alpha x + (1-\alpha) x') &\leq m \\ p \cdot \bar{x} &\leq m \end{aligned}$$

   Therefore, $\bar{x}$ is feasible. To show that it is optimal, let $u(x) = u(x') = u^*$. By quasiconcavity of u, we hae

$$u(\bar{x}) \geq \min\{u(x), u(x')\} = u^*$$

Therefore, $\bar{x}$ gives at least as much utility as x or x', and therefore, must also be optimal.

For the strict convexity implies uniqueness part, assume not, and let $x, x' \in x(p, m)$ be two distinct maximizers. The same reasoning as above delivers that $\bar{x}$ is feasiable. Strict quasiconcavity of u implies $u(\bar{x}) > \min\{u(x), u(x')\} = u^*$, which contradicts that x and x' are optimal.

4. *MWG 3.D.4*

   Answer

   (a) Normalize $p_1 = 1$, let $x_{-1} = (x_2, \ldots, x_n)$, and let $u(x) = x_1 + v(x_{-1})$ be the agent's quasilinear utility function. Let $x^* = x(p, w)$ be the Marshallian demand at (p, w), and $(p, w') = (p, w + \alpha)$; that is, the prices are the same, but the agent has additional wealth α.

   We show that $x' = (x_1^* + \alpha, x_{-1}^*)$ is optimal at (p, w'). First, note that because $p_1 = 1$, the bundle x' is affordable at (p, w'). What remains to show is that x' is optimal. Take any other $\hat{x}$ that is affordable at (p, w'). Since $\hat{x}$ is affordable, we have $p \cdot \hat{x} \leq w + \alpha$, and so $(\hat{x}_1 - \alpha) + p_2\hat{x}_2 + \cdots p_n\hat{x}_n \leq w$, i.e., the bundle $(\hat{x}_1 - \alpha, \hat{x}_{-1})$ is affordable at wealth w, which implies that x^* (which was optimal at w) must give at least as much utility:

   $$(\hat{x}_1 - \alpha) + v(\hat{x}_{-1}) \leq u(x^*) = x_1^* + v(x_{-1}^*)$$

   Rearranging, we have

   $$u(\hat{x}_1, \hat{x}_{-1}) \leq \hat{x}_1 + v(\hat{x}_{-1}) \leq (x_1^* + \alpha) + v(x_{-1}^*) = u(x_1^* + \alpha, x_{-1}^*)$$

   In other words, the bundle $(x_1^* + \alpha, x_{-1}^*)$ gives higher utility than the bundle $(\hat{x}_1, \hat{x}_{-1})$. Since $\hat{x}$ was any arbitrary bundle affordable at wealth w', this implies that $(x_1^* + \alpha, x_{-1}^*)$ is optimal at wealth $w' = w + \alpha$.

   Therefore, at w', all additional wealth is spent on good 1, and so $x_k(p, w) = x_k(p, w')$ for all w, w'.

   (b) Preferences are quasilinear, so we can write them as $u(x) = x_1 + \tilde{u}(x_2, \ldots, x_n)$. Thus, we can write
   $v(p, w) = x_1(p, w) + \tilde{u}(x_2(p), \ldots, x_n(p))$

where the demands of goods 2 through n do not depend on wealth by part (a). The budget constraint gives us that $x_1(p, w) = w - \sum p_k x_k(p)$. Plugging this in, we see

$$v(p, w) = w + \left[\tilde{u}(x_2(p), \ldots, x_n(p)) - \sum p_k x_k(p)\right]$$

Letting the term in brackets be $\phi(p)$, we are done.

(c) In the quaslinear case, the marginal rate of substitution between goods 1 and 2 is simply $1/\eta'(x_2)$, and the price ratio is $1/p_2$, so, *if we ignore* the non negativity constraint on x_1, the optimal x_2 is given by the solution to the first order condition

$$\eta'(x_2) = p_2$$

Since marginal utility is (strictly) increasing, this function is invertible, and we can write $x_2 = (\eta')^{-1}(p_2)$. When is it justified to ignore the non-negativity constraint? Exactly when the level of consumption of x_1 implied by the above solution is positive. That is, if $w - p_2[(\eta')^{-1}(p_2)] \geq 0$, then the solution ignoring the non negativity constraint also implicitly satisfies it, and so must indeed be optimal. Another way to say this is that the non negativity constraint **binds** when:

$$(\eta')^{-1}(p_2) \geq \frac{w}{p_2}$$

If we fix prices, when w is small, the consumer spends all of her money on x_2 and consumes no x_1. As w increases and the inequality above no longer holds, she then fixes the amount of x_2 she purchases at the level determined by the first order condition and proceeds to spend all of her remaining wealth on good 1.

5. *MWG 2.D.1, 2.D.2*

Answers

2.D.1 $B_{p,w} = \left\{\mathbf{x} \in \Re^2_+ : p_1 x_1 + p_2 x_2 \leq w\right\}$ where x_i, p_i is the consumption level and price in period $i = 1, 2$ and w is lifetime wealth.

2.D.2 $B_p = \left\{(x, h) \in \Re^2_+ : px + h \leq L\right\}$ where L is the consumer's allocation of time.

6. *Consider the utility function*

$$u(x_1, x_2) = (x_1 - a_1)^{b_1}(x_2 - a_2)^{b_2},$$

where $b_1, b_2 > 0, b_1 + b_2 < 1,$ *and* $a_1, a_2 > 0$. *Find:*

(a) *the Marshallian Demands,*

(b) *the expenditure function,*

(c) *the indirect utility function, and*

(d) *the money metric indirect utility function.*

(e) *Confirm the identities* $e(p, v(p, m)) = m$ *and* $v(p, e(p, \bar{u})) = \bar{u}$.

(f) *Assume you are given the expenditure function and indirect utility function from parts (b) and (c), but have no other knowledge of the consumer (in particular, you do not know the direct utility function* $u(x_1, x_2)$*). Show how to calculate the consumer's Hicksian demand using only this information. Then, show how to calculate the consumer's Marshallian demand without using Roy's identity (Roy's identity is of course a valid way to calculate the Marshallian demand given knowledge of* $v(p, m)$*, but this question is asking you to do this in another way).*

Answer Note that this is a Stone-Geary utlity function, which we'll see with some frequency. This is messy stuff, but straightforward.

(a)

$$x_i\left(p, m\right) = a_i + \frac{b_i}{b_1 + b_2}\left(\frac{m - a_1p_1 - a_2p_2}{p_i}\right), i = 1, 2$$

(b)

$$e\left(p, \bar{u}\right) = a_1p_1 + a_2p_2 + \left(b_1 + b_2\right)\left[u\left(\frac{p_1}{b_1}\right)^{b_1}\left(\frac{p_2}{b_2}\right)^{b_2}\right]^{\frac{1}{b_1+b_2}}$$

(c)

$$v\left(p, m\right) = \left(\frac{b_1}{p_1}\right)^{b_1}\left(\frac{b_2}{p_2}\right)^{b_2}\left(\frac{m - a_1p_1 - a_2p_2}{b_1 + b_2}\right)^{b_1+b_2}$$

(d)

$$\mu\left(p; q, m\right) = a_1p_1 + a_2p_2 + \left(m - a_1p_1 - a_2p_2\right)\left[\left(\frac{p_1}{q_1}\right)^{b_1}\left(\frac{p_2}{q_2}\right)^{b_2}\right]^{\frac{1}{b_1+b_2}}$$

(e) Confirming these identities is simply algebra. After plugging in the expression you found for $e(p, \bar{u})$ in part (b) into the equation for $v(p, m)$ from part (c) and simplifying, you can show that $v(p, e(p, \bar{u})) = \bar{u}$. Similarly, after substituting the expression for $v(p, m)$ from (c) into the equation you found in (b), you should find that $e(p, v(p, m)) = m$. I will leave the algebraic details to you.

(f) To solve this problem, we use Shepard's lemma and the duality identity $x_i(p, m) = h_i(p, v(p, m))$. Shephard's lemma gives (for x_1, say):

$$h_1(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_1} = a_1 + \bar{u}^{1/(b_1+b_2)} \left[\left(\frac{p_1}{b_1}\right)^{b_1} \left(\frac{p_2}{b_2}\right)^{b_2} \right]^{\frac{1}{b_1+b_2}-1} \left[\left(\frac{p_1}{b_1}\right)^{b_1-1} \left(\frac{p_2}{b_2}\right)^{b_2} \right]$$

Plugging in the expression for $v(p, m)$ found in part (c), we get

$$h_1(p, v(p, m)) = a_1 + \Gamma \left(\left(\frac{b_1}{p_1}\right)^{b_1} \left(\frac{b_2}{p_2}\right)^{b_2} \right)^{1/(b_1+b_2)} \left[\left(\frac{p_1}{b_1}\right)^{b_1} \left(\frac{p_2}{b_2}\right)^{b_2} \right]^{\frac{1}{b_1+b_2}-1} \left[\left(\frac{p_1}{b_1}\right)^{b_1-1} \left(\frac{p_2}{b_2}\right)^{b_2} \right]$$

where $\Gamma = \frac{m - a_1p_1 - a_2p_2}{b_1+b_2}$. Using the duality identity and simplifying the exponents gives

$$x_1(p, m) = h_1(p, v(p, m)) = a_1 + \frac{b_1}{p_1}\Gamma = a_1 + \frac{b_1}{b_1 + b_2} \left(\frac{m - a_1p_1 - a_2p_2}{p_1} \right)$$

which is the same expression for Marshallian demand we found in part (a). The result for good 2 is symmetric.

7. *Consider the indirect utility function,*

$$v(p, m) = mp_1^{-1/2} p_2^{-1/2}.$$

Find expressions for the expenditure function, the Hicksian and Marshallian demands, and the (direct) utility function.

Answer

To find the expenditure function, use the identity $v(p, e(p, u)) = u$ to solve for $e(p, u)$. Hicksian demands then fall from the expenditure function via Shepard's Lemma. To get Marshallin demands we can use Roy's identity and the indirect utility function. Finally, find the utility function using the Hicksian demands to solve for u as a function of demanded bundles x_1, x_2 (i.e. substitute to eliminate prices in the Hicksian demands and rearrange).

$$\begin{gathered}
e(p, \bar{u}) = up_1^{1/2} p_2^{1/2} \\
h_1(p, \bar{u}) = \tfrac{1}{2}\left(\tfrac{p_2}{p_1}\right)^{1/2} u, \; h_2(p, \bar{u}) = \tfrac{1}{2}\left(\tfrac{p_1}{p_2}\right)^{1/2} u \\
x_1(p, m) = \tfrac{m}{2p_1}, \; x_2(p, m) = \tfrac{m}{2p_2} \\
u(x_1, x_2) = 2x_1^{1/2} x_2^{1/2}.
\end{gathered}$$

8. *Show that Roy's identity is a direct consequence of the duality identities (those that relate the expenditure function with the indirect utility function and the Marshallian demand function with the Hicksian demand function) and Shepard's lemma. (Hint: Start with the identity $v(p, e(p, \bar{u})) = \bar{u}$ and differentiate this equation with respect to p_i.)*

   Answer

   Differentiating $v(p, e(p, \bar{u})) = \bar{u}$ with respect to p_i and evaluating at $e(p, \bar{u}) = m$, we get

$$\frac{\partial v(p,m)}{\partial p_i} + \frac{\partial v(p,m)}{\partial m}\frac{\partial e(p,\bar{u})}{\partial p_i} = 0$$

   Rearranging, we get

$$\frac{\partial e(p,\bar{u})}{\partial p_i} = -\frac{\partial v(p,m)/\partial p_i}{\partial v(p,m)/\partial m}$$

   The RHS looks very similar to Roy's identity, but we somehow need to relate this equation to the Marshallian demand function $x(p, m)$. This is where Shepard's lemma comes in: by Shepard's lemma, $\partial e(p, \bar{u})/\partial p_i = h_i(p, \bar{u}) = x_i(p, m)$, where again we evaluate at $e(p, \bar{u}) = m$. Equating the two expressions we have found for $\partial e(p, \bar{u})/\partial p_i$, we find

$$x_i(p,m) = -\frac{\partial v(p,m)/\partial p_i}{\partial v(p,m)/\partial m},$$

   which is simply Roy's identity.

9. *MWG 3.E.1*

   Answer

   The Lagrangean for the expenditure minimization problem is

   $$\mathcal{L} = p \cdot x + \lambda(\bar{u} - u(x)) - \mu_1 x_1 - \cdots - \mu_n x_n$$

   The first order conditions are

   $$p_i = \lambda \frac{\partial u(x)}{\partial x_i} + \mu_i \text{ for all } i$$

   the complementary slackness conditions are are

   $$\lambda(\bar{u} - u(x)) = 0 \text{ and } \mu_i x_i = 0 \text{ for all } i$$

   and the multiplier positivity conditions are $\lambda, \mu_i \geq 0$.

   A more succinct way of writing the FOCs is

   $$\lambda \frac{\partial u(x)}{\partial x_i} \leq p_i, \text{ with equality if } x_i > 0.$$

   You should notice that these are the exact same as the first order conditions of the utility maximization problem.

10. *MWG 3.G.4, parts (a)-(c). The wording of this problem is a bit unclear. For part (a), you need to show that if u is additively separable, and $g(x)$ is a linear monotonic transformation (i.e., $g(x) = ax + b$ for some constants $a > 0$ and b), then, the transformed utility $v(x) = g(u(x))$ has an additively separable representation (i.e., we can write $v(x) = \sum_{i=1}^{N} g_i(x_i)$ for some single-variable functions $g_i(\cdot)$). For part (b), partition the consumption space $X = Y \times Z$, where $Y = \mathbb{R}^m$ and $Z = \mathbb{R}^{n-m}$. Then, consider consumption bundles $(a, c), (a, d), (b, c)$ and (b, d), where $a, b \in Y$ and $c, d \in Z$. The question is asking you to prove that $(a, c) \succeq (b, c)$ if and only if $(a, d) \succeq (b, d)$.[2] For part (c), you just need to show that there are no inferior goods (i.e., the wealth effects for all goods are nonnegative).*

    Answer

[2] That is, the ordering between bundles a and b in Y is independent of the bundle you have from Z.

(a) Let $g(x) = ax + b$. Then,

$$g(u(x)) = \sum_{i=1}^{N} au_i(x_i) + b$$

Define $g_1(x_1) = au_1(x_1) + b$ and $g_i(x_i) = au_i(x_i)$. (these are slightly different, since we need to stick the b term somewhere; here, without loss of generality, we stuck it in the function g_1. This is not the only choice; we also could have used, for example, $g_i(x_i) = au_i(x_i) + \frac{b}{n}$ for each i.) Using these definitions, we write

$$g(u(x)) = \sum_{i=1}^{N} g_i(x_i)$$

The converse also holds (i.e., if $g(u(x))$ is additively separable, then g is affine) but we aren't going to prove it.

(b) Let $u(x) = u(y, z)$ where $y \in Y$ and $z \in Z$. Fix $z = c$, and notice that $u(a, c) \geq u(b, c)$ reduces to $\sum_{i=1}^{m} u_i(a_i) \geq \sum_{i=1}^{m} u_i(b_i)$. Now consider fixing some other $z = d$. We want to show $u(a, d) \geq u(b, d)$. This once again reduces to $\sum_{i=1}^{m} u_i(a_i) \geq \sum_{i=1}^{m} u_i(b_i)$, which we know holds from the above. Thus, the result is proved. Since this holds for any arbitrary decomposition of the consumption space, $X = Y \times Z$, we are done.

(c) An inferior good is one that has negative wealth effects. Since we are told to assume interiority, we can use the FOC (MRS=price ratio) to write

$$u_j'(x_j(p, m)) = \frac{p_j}{p_k} u_k'(x_k(p, m))$$

for all j, k.

If we differentiate this with respect to m, we get

$$u_j''(x_j(p, m)) \frac{\partial x_j(p, m)}{\partial m} = \frac{p_j}{p_k} u_k''(x_k(p, m)) \frac{\partial x_k(p, m)}{\partial m}$$

By strict concavity, u_j'' and u_k'' are both strictly negative. Thus, the signs of the wealth effects for any two goods must be the same, i.e., either all wealth effects are positive, or all wealth effects are negative. So, if one good is inferior, then *all* goods are inferior. However, if all goods are inferior, then as wealth rises, demand for all goods decreases, which is a violation of local non satiation. Therefore, all goods are normal.