Markov's inequality

by Marco Taboga, PhD

Markov's inequality is a probabilistic inequality. It provides an upper bound to the probability that the realization of a random variable exceeds a given threshold.

Statement

The proposition below formally states the inequality.

> **Proposition** Let X be an integrable random variable defined on a sample space Ω. Let $X(\omega) \geq 0$ for all $\omega \in \Omega$ (i.e., X is a positive random variable). Let $c \in \mathbb{R}_{++}$ (i.e., c is a strictly positive real number). Then, the following inequality, called Markov's inequality, holds:
>
> $$\mathrm{P}(X \geq c) \leq \frac{\mathrm{E}[X]}{c}$$

Reading and understanding the proof of Markov's inequality is highly recommended because it is an interesting application of many elementary properties of the expected value.

Proof

First note that

$$1_{\{X \geq c\}} + 1_{\{X < c\}} = 1$$

where $1_{\{X \geq c\}}$ is the indicator of the event $\{X \geq c\}$ and $1_{\{X < c\}}$ is the indicator of the event $\{X < c\}$. As a consequence, we can write

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[X \cdot 1] \\ &= \mathrm{E}[X \cdot (1_{\{X \geq c\}} + 1_{\{X < c\}})] \\ &= \mathrm{E}[X 1_{\{X \geq c\}}] + \mathrm{E}[X 1_{\{X < c\}}] \end{aligned}$$

Now, note that $X 1_{\{X < c\}}$ is a positive random variable and that the expected value of a positive random variable is positive:

$$\mathrm{E}[X 1_{\{X < c\}}] \geq 0$$

Therefore,

$$\mathrm{E}[X] = \mathrm{E}[X 1_{\{X \geq c\}}] + \mathrm{E}[X 1_{\{X < c\}}] \geq \mathrm{E}[X 1_{\{X \geq c\}}]$$

Now, note that the random variable $c \cdot 1_{\{X \geq c\}}$ is smaller than the random variable $X \cdot 1_{\{X \geq c\}}$ for any $\omega \in \Omega$:

$$c \cdot 1_{\{X \geq c\}} \leq X \cdot 1_{\{X \geq c\}}$$

because, trivially, c is always smaller than X when the indicator $1_{\{X \geq c\}}$ is not zero. Thus, by an elementary property of the expected value, we have that

$$c \cdot 1_{\{X \geq c\}} \leq X \cdot 1_{\{X \geq c\}} \Rightarrow \mathrm{E}[c \cdot 1_{\{X \geq c\}}] \leq \mathrm{E}[X \cdot 1_{\{X \geq c\}}]$$

Furthermore, by using the linearity of the expected value and the fact that the expected value of an indicator is equal to the probability of the event it indicates, we obtain

$$\mathrm{E}[c \cdot 1_{\{X \geq c\}}] = c\mathrm{E}[1_{\{X \geq c\}}] = c\mathrm{P}(X \geq c) \Rightarrow c\mathrm{P}(X \geq c) \leq \mathrm{E}[X 1_{\{X \geq c\}}]$$

The above inequalities can be put together:

$$\left.\begin{array}{l} \mathrm{E}[X] \geq \mathrm{E}[X 1_{\{X \geq c\}}] \\ \mathrm{E}[X 1_{\{X \geq c\}}] \geq c\mathrm{P}(X \geq c) \end{array}\right. \Rightarrow \mathrm{E}[X] \geq c\mathrm{P}(X \geq c)$$

Finally, since c is strictly positive we can divide both sides of the right-hand inequality to obtain Markov's inequality:

$$\mathrm{P}(X \geq c) \leq \frac{\mathrm{E}[X]}{c}$$

This property also holds when $X \geq 0$ almost surely (in other words, there exists a zero-probability event E such that $\{\omega \in \Omega : X(\omega) < 0\} \subseteq E$).

Example

Suppose that an individual is extracted at random from a population of individuals having an average yearly income of $40,000.

What is the probability that the extracted individual's income is greater than $200,000?

In the absence of more information about the distribution of income, we can use Markov's inequality to calculate an upper bound to this probability:

$$\mathrm{P}(X \geq 200{,}000) \leq \frac{40{,}000}{200{,}000} = \frac{1}{5}$$

Therefore, the probability of extracting an individual having an income greater than $200,000 is less than $1/5$.

Important applications

Markov's inequality has several applications in probability and statistics.

For example, it is used:

- to prove Chebyshev's inequality;
- in the proof that mean square convergence implies convergence in probability;
- to derive upper bounds on tail probabilities (Exercise 2 below).

Solved exercises

Below you can find some exercises with explained solutions.

Exercise 1

Let X be a positive random variable whose expected value is

$$\mathrm{E}[X] = 10$$

Find a lower bound to the probability

$$\mathrm{P}(X < 20)$$

Solution

First of all, we need to use the formula for the probability of a complement:

$$P(X < 20) = 1 - P(X \geq 20)$$

Now, we can use Markov's inequality:

$$P(X \geq 20) \leq \frac{E[X]}{20} = \frac{10}{20} = \frac{1}{2}$$

Multiplying both sides of the inequality by -1, we obtain

$$-P(X \geq 20) \geq -\frac{1}{2}$$

Adding 1 to both sides of the inequality, we obtain

$$1 - P(X \geq 20) \geq 1 - \frac{1}{2} = \frac{1}{2}$$

Thus, the lower bound is

$$P(X < 20) \geq \frac{1}{2}$$

Exercise 2

Let X be a random variable such that the expected value

$$E\left[|X|^4\right]$$

exists and is finite.

Use the latter expected value to derive an upper bound to the tail probability

$$P(|X| > c)$$

where c is a positive constant.

Solution

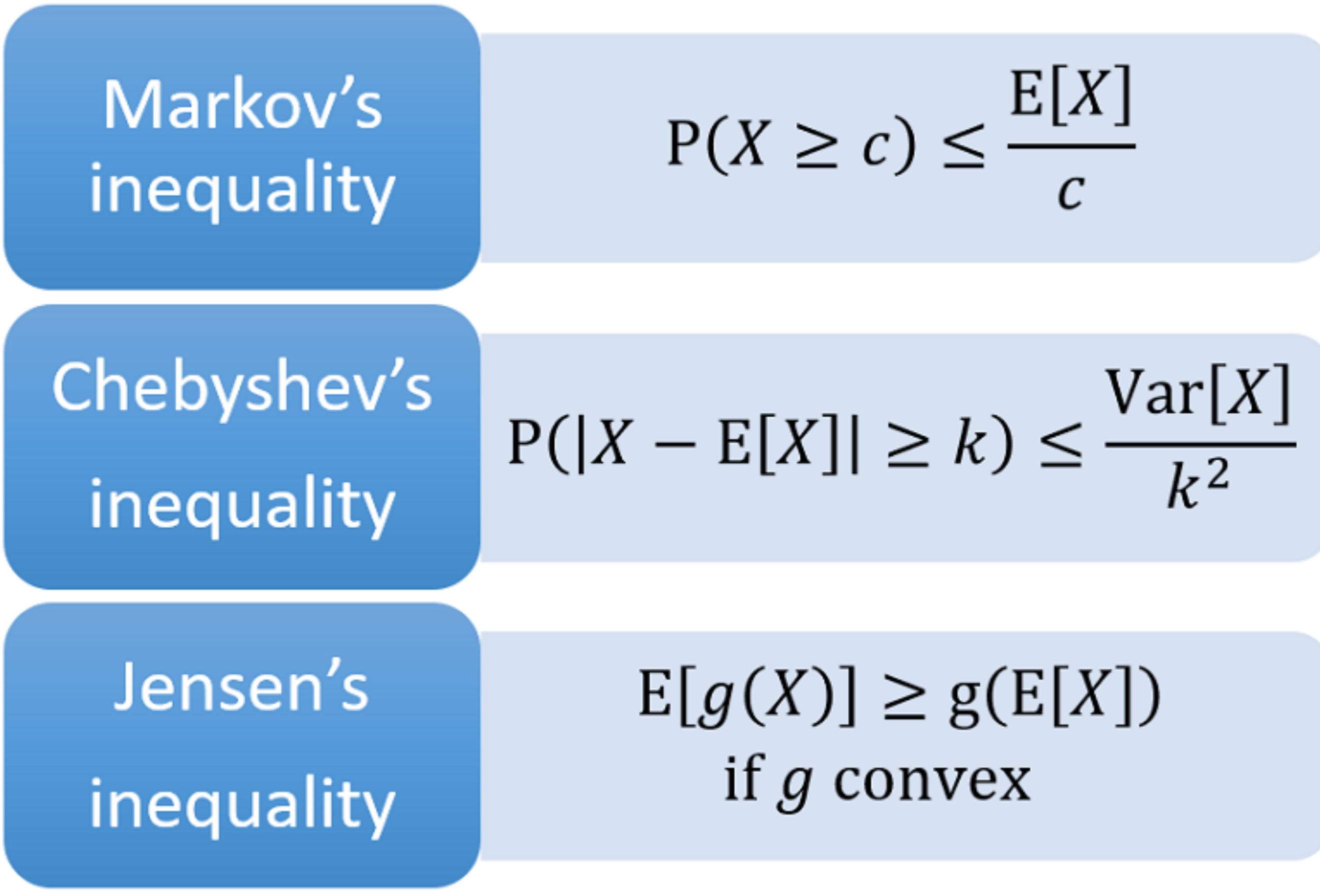
By Markov's inequality, we have

$$\begin{aligned} P(|X| > c) &= P\left(|X|^4 > c^4\right) \\ &\leq c^{-4} E\left[|X|^4\right] \end{aligned}$$

Other inequalities

If you like this page, StatLect has other pages on probabilistic inequalities:

- Chebyshev's inequality;
- Jensen's inequality.



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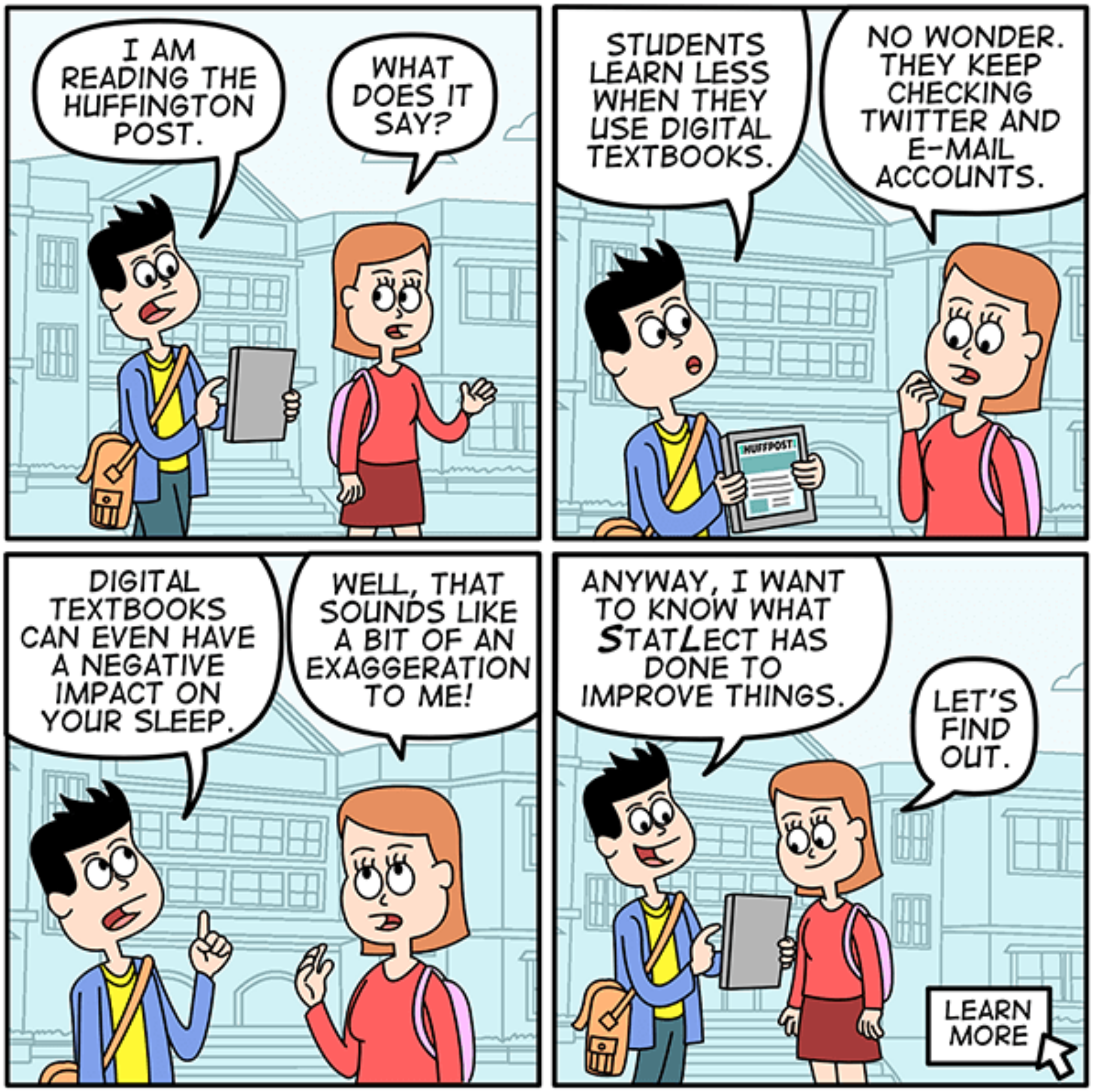
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