11 Oct 2023.

Defn : X_n is a Cauchy sequence in probability (a.s., in L_r) if $\lim_{n,m\to\infty} P(|X_n - X_m| > \varepsilon) = 0$

For almost surely:-

$$\lim_{n,m\to\infty} P\left(\sup_{k\geq n} |X_k - X_m| > \varepsilon\right) = 0$$

For almost surely in L_r:-

$$\lim_{n,m\to\infty} E\left[|X_n - X_m|^r\right] = 0$$

Theorem : X_n converges in probability (a.s, or in L_r) iff X_n is a Cauchy sequence in probability (a.s, in L_r)

Defn : Weak Convergence

A sequence of distn functions F_n converges weakly to a limit F if $\forall$ bounded & continuous functions g

$$\int_{-\infty}^{+\infty} \underbrace{g(x)\, dF_n(x)}_{\text{What does this mean?}} \longrightarrow \int_{-\infty}^{+\infty} g(x) . dF(x) \qquad \underbrace{(F_n \Rightarrow F)}_{\text{notation.}}$$

Theorem: $F_n \Rightarrow F$ iff $F_n(x) \to F(x)$ for all x where where $F(\cdot)$ is continuous.

Lemma: If $F(\cdot)$ is a continuous function, then $F_n \Rightarrow F$ implies $\sup_x |F_n(x) - F(x)| \to 0$, $n \to \infty$

Def^n: Suppose that sequence of r.v. X_n with dist^n functions F_n is s.t. $F_n \Rightarrow F$, where F is the distribution function of r.v. X, then X_n converges to X in distribution

$(X_n \xrightarrow{d} X)$

Example:

$$F_n(x) = \begin{cases} 0, & x < -n \\ 1/2, & -n \leq x < n \\ 1, & x \geq n \end{cases}$$

- Monotone, left continuity, lim at $-\infty, +\infty$ → satisfies dist^n function
- $\forall x$, $\exists N$, s.t. $-N \leq x \leq N$

$F_n(x) \to \frac{1}{2}$, $n \to \infty$

$F_n \Rightarrow \frac{1}{2}$

$1/2$ is not a dist^n f^n as $\lim_{-\infty} \neq 0$ $\lim_{+\infty} \neq 1$. one of the dist^n f^n requirement.

Def^n: Class $\mathcal{G}$ of functions G is s.t.

1. $G(\cdot)$ is monotone increasing
2. $G(\cdot)$ is continuous from the right & has left limit at all pts x in $\mathbb{R}$ (cadlag)
3. $\lim_{x \to +\infty} G(x) \leq 1$, $\lim_{x \to -\infty} G(x) \geq 0$

(Third cond^n is more flexible than the third cond^n of dist^n functions).

Theorem: Class $\mathcal{G}$ is compact wrt weak convergence

the limits of all sequence is within the class

Denote by $\mathcal{F}$ the class of all distribution functions

Def^n: Sequence of dist^n functions F_n is asymtotically tight if $\forall \varepsilon > 0$, $\exists C$, s.t. :-

(Asymptotic Textbook describes it v. well)

$$\underbrace{\inf_n | F_n(C) - F_n(-C)| > 1 - \varepsilon}_{\text{How much is contained}}$$

The above example is not asymtotically tight as $\exists$ $n > C$ with probability mass 1/2. Inf here would be 0.

(why?)

Def^n : A class of bounded continuous functions $\mathcal{L}$ defines distributions if $\int_{-\infty}^{+\infty} g(x)\, dF(x) = \int_{-\infty}^{+\infty} g(x)\, dG(x)$ for all $g(\cdot) \in \mathcal{L}$ implies that
$G(\cdot) = F(\cdot)$

$(G, F \in \mathcal{G})$

Theorem : Suppose that $\mathcal{L}$ defines distributions and $F_n \Rightarrow F$ with $F_n, F \in \mathcal{F}$ iff

1. Sequence F_n is asymptotically tight
2. $\int_{-\infty}^{+\infty} g(x)\, dF_n(x)$ has a limit $\forall\, g \in \mathcal{L}$

Example : Class $\mathcal{L} = \{e^{itx},\ t \in \mathbb{R}\}$ defines distribution

Theorem : $F_n \Rightarrow F$ iff for $\phi_n(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF_n(x)$ and
$\phi(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x)$: $\phi_n(t) \longrightarrow \phi(t)\ \forall\, t \in \mathbb{R}$