ECON 7710 TA Session

Week 4

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Outline

Q1

i Event i is defined as we toss a die n times and **at least** one of the outcomes is equal to 6.

- The probability of no 6 through 1 toss, which is $\frac{5}{6}$.
- Repeat n times, we have $(\frac{5}{6})^n$.
- So we know the probability of event i.

$$P(i) = 1 - (\frac{5}{6})^n$$

ii Event ii is defined as we toss a die n times and an outcome equal to 6 is observed **exactly once**.

- There are 6^n ways of outcome through n times' toss. [Denominator]
- For the numerator, it is $n * 5^{n-1}$.
  - First, there are $\binom{n}{1}$ ways of choosing the one "6", which are n ways.
  - Once the "6" is chosen, it shows up exactly 1 time.
  - For the rest $n-1$ times, "1" to "5" can be freely chosen, so 5^{n-1} ways.

- So we know the probability of event ii.

$$P(ii) = \frac{n * 5^{n-1}}{6^n}$$

Q2

- **Setup:**
  - Mathematical modelling of this issue should be viewed as a multivariate uniform distribution consisting 2 or 3 random variables.
  - Correspondingly, when you do the math:
    - Lower bound: Earlist time one can arrive
    - Upper bound: Latest time one can arrive
  - $H, W, E \sim U[0, 1]$, PDFs:

$$h(x) = w(x) = e(x) = \begin{cases} 1, & \text{For } x \in [0, 1] \\ 0, & \text{Otherwise} \end{cases}$$

  - Normalize your time by taking hour as your unit:

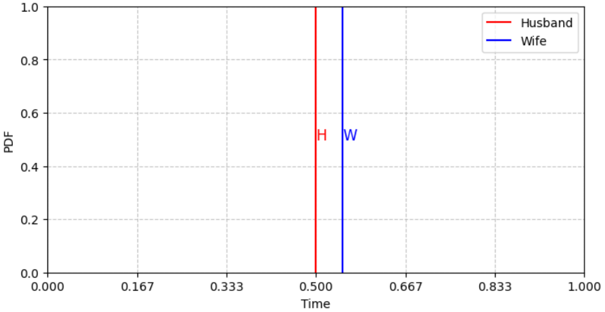
$$7:00 \;\; pm - 8:00 \;\; pm \Rightarrow [0, 1] \quad \begin{cases} 7:00 \;\; pm = "0" \\ 8:00 \;\; pm = "1" \\ 10 \text{ minutes} = \frac{1}{6} \end{cases}$$

- **Note:** Only when H,W meet each other, they vanish immediately. Discuss different cases, sequence of arrivals, who depends on whom.

Q2

a Event that date will occur is denoted by D. Then there are two subcases:
  - D_H: Husband arrives earlier and wife shows up within 10 minutes.
  - D_W: Wife arrives earlier and husband shows up within 10 minutes.
  - By symmetry: $P(D) = P(D_H) + P(D_W) = 2P(D_H)$

- D_H can be further divided into two cases:
  - D_{H1}, H shows up in $[0, \frac{5}{6}]$; W shows up within next 10 mins

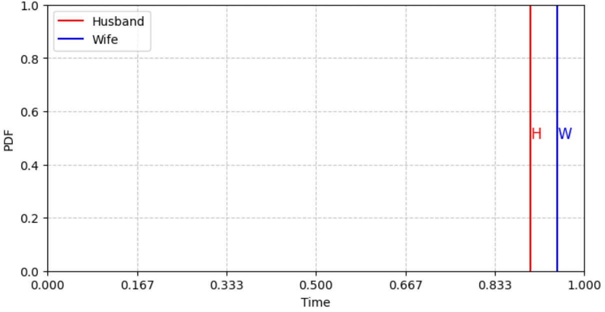


$$P(D_{H1}) = \int_0^{5/6} dH \int_H^{H+1/6} dW.$$

Q2

a Event that date will occur is denoted by D. Then there are two subcases:
  - D_H: Husband arrives earlier and wife shows up within 10 minutes.
  - D_W: Wife arrives earlier and husband shows up within 10 minutes.
  - By symmetry: $P(D) = P(D_H) + P(D_W) = 2P(D_H)$

- D_H can be further divided into two cases:
  - D_{H2}, H shows up in $[\frac{5}{6}, 1]$; W shows up later but before 8



$$P(D_{H2}) = \int_{5/6}^{1} dH \int_{H}^{1} dW.$$

Q2

a So probability of D_H is:

$$P(D_H) = \int_0^{5/6} dH \int_H^{H+1/6} dW + \int_{5/6}^{1} dH \int_H^{1} dW = \frac{5}{36} + \frac{1}{72} = \frac{11}{72}$$

Therefore,

$$P(D) = P(D_H) + P(D_W) = 2 * \frac{11}{72} = \frac{11}{36} \approx 0.306$$

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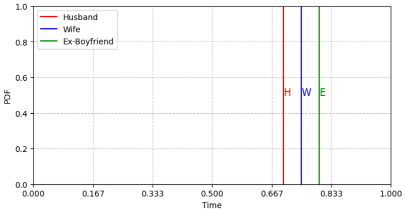
Q2

b.i We define M as the event all three meet together.
Note: E has to arrive before D happens. Otherwise, the couples will vanish immediately when they meet and E cannot get them caught.

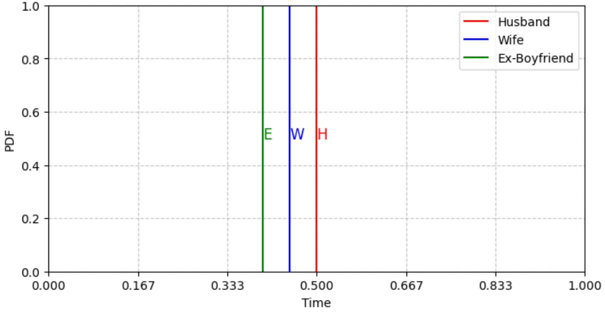


This doesn't work

- Since E is indifferent to whoever arrives the latest, by symmetry, there are two sub-events here with the same probability:
  - M_H where husband arrives latest.
  - M_W where wife arrives latest.
- Again, the probability of all three meet is
  $P(M) = P(M_H) + P(M_W) = 2P(M_H)$.
  WLOG, we analyze $P(M_H)$.

Q2

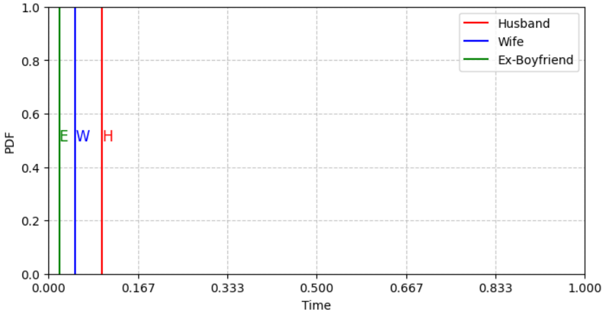
- Likewise, we need to consider 2 sub-cases
  - M_{H1} : H arrives between $[\frac{1}{6}, 1]$.
  - M_{H2} : H arrives between $[0, \frac{1}{6}]$.
- M_{H1} : H arrives between $[\frac{1}{6}, 1]$.



- $P(M_{H1}) = \int_{1/6}^{1} dH \int_{H-1/6}^{H} dW \int_{H-1/6}^{H} dE$

Q2

- Likewise, we need to consider 2 sub-cases
  - M_{H1} : H arrives between $[\frac{1}{6}, 1]$.
  - M_{H2} : H arrives between $[0, \frac{1}{6}]$.
- M_{H2} : H arrives between $[0, \frac{1}{6}]$.



- $P(M_{H2}) = \int_0^{1/6} dH \int_0^H dW \int_0^H dE$
- Then we know $P(M_H) = P(M_{H1}) + P(M_{H2}) = \frac{2}{81}$
- $P(M) = P(M_H) + P(M_W) = \frac{4}{81}$

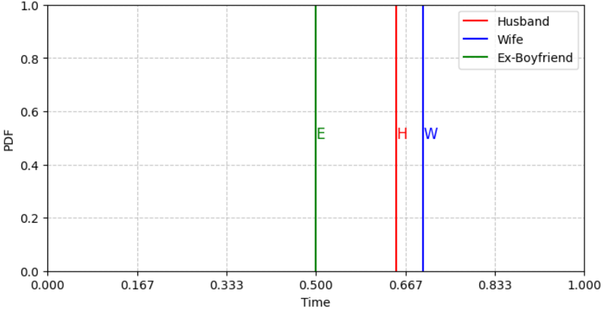
Q2

b.ii The event H and E meet with each other is named F, short for Fight. There are 3 sub-events.

- D happens and E meets both of W and H, which is M
  We already know $P(M) = \frac{4}{81}$.
- D happens and E meets H only. Denote this event by M'.
- D does not happen and E meets H. Denote this event by M".

Q2

- D happens and E meets H only. Denote this event by M'.
  Now E must arrives the earliest and E leaves before W comes. W has to arrive the latest.
  There are three sub-cases here:
  - M_1' If H arrives in $[\frac{1}{6}, \frac{5}{6}]$

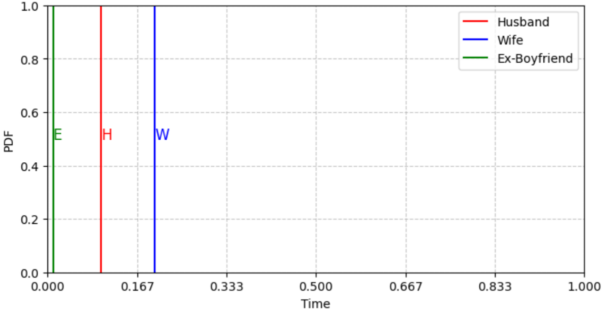


$$P(M_1') = \int_{1/6}^{5/6} dH \int_{H}^{H+1/6} dW \int_{H-1/6}^{W-1/6} dE$$

  - For E, he has to be earlist and not meeting W.
  - For W, she has to arrive latest and meet H

Q2

- D happens and E meets H only. Denote this event by M'.
  Now E must arrives the earliest and E leaves before W comes. W has to arrive the latest.
  There are three sub-cases here:
  - M_2' If H arrives in $[0, \frac{1}{6}]$

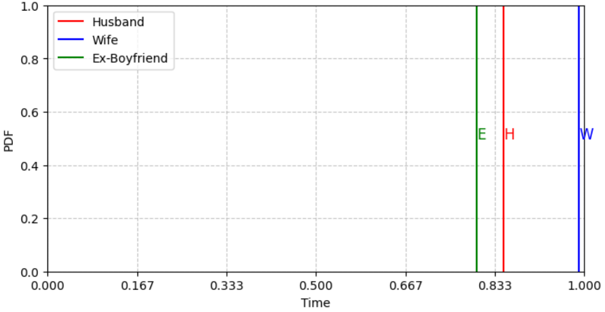


$$P(M_2') = \int_0^{1/6} dH \int_{1/6}^{H+1/6} dW \int_0^{W-1/6} dE$$

  - For E, he has to be earlist and not meeting W.
  - For W, she has to arrive latest and meet H , not E

Q2

- D happens and E meets H only. Denote this event by M'.
  Now E must arrives the earliest and E leaves before W comes. W has to arrive the latest.
  There are three sub-cases here:
  - M_3' If H arrives in $[\frac{5}{6}, 1]$



$$P(M_3') = \int_{5/6}^{1} dH \int_{H}^{1} dW \int_{H-1/6}^{W-1/6} dE$$

- For E, he has to be earlist and not meeting W.
- For W, she has to arrive latest and meet H and before 8pm.

Q2

b.ii The event H and E meet with each other is named F, short for Fight. There are 3 sub-events.

- D happens and E meets both of W and H, which is M
  We already know $P(M) = \frac{4}{81}$.
- D happens and E meets H only. Denote this event by M'. Therefore the probability of M' is

$$P(M') = P(M_1') + P(M_2') + P(M_3)' = \frac{1}{6^4} + \frac{1}{108} + \frac{1}{6^4}$$
$$= \frac{7}{648}$$

- D does not happen and E meets H. Denote this event by M''.

Q2

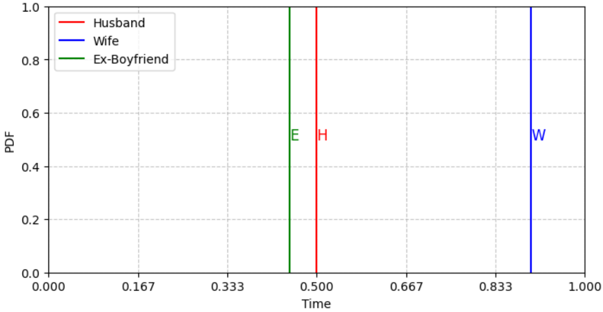
- D does not happen and E meets H. Denote this event by M".
- Wife meets with Ex-husband? Doesn't matter.
  There are two sub-events
  - Wife came too late, M''_W
  - Husband came too late, M''_H.

  Again, WLOG, we discuss M''_W

Q2

Again, WLOG, we discuss M''_W, consider these three sub-cases:

- M''_{W1}: If H arrives in $[\frac{1}{6}, \frac{5}{6}]$,

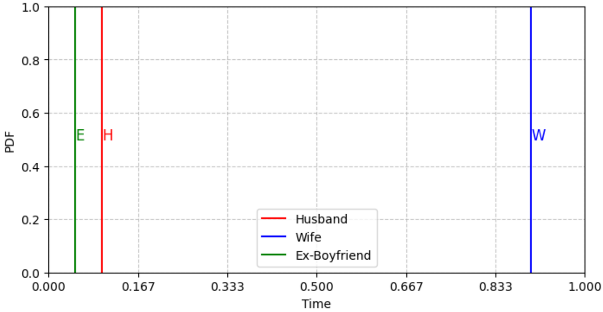


- $P(M''_{W1}) = \int_{1/6}^{5/6} dH \int_{H+1/6}^{1} dW \int_{H-1/6}^{H+1/6} dE$

Q2

Again, WLOG, we discuss M''_W, consider these three sub-cases:

- M''_{W2}: If H arrives in $[0, \frac{1}{6}]$,



- $P(M''_{W2}) = \int_0^{1/6} dH \int_{H+1/6}^{1} dW \int_0^{H+1/6} dE$

What if H arrives in $[\frac{5}{6}, 1]$? W and H will for sure meet.

Q2

b.ii The event H and E meet with each other is named F, short for Fight. There are 3 sub-events.

- D happens and E meets both of W and H, which is M
  We already know $P(M) = \frac{4}{81}$.
- D happens and E meets H only. Denote this event by M'.
  Therefore the probability of M' is

$$
\begin{aligned}
P(M') = P(M_1') + P(M_2') + P(M_3)' &= \frac{1}{6^4} + \frac{1}{108} + \frac{1}{6^4} \\
&= \frac{7}{648}
\end{aligned}
$$

- D does not happen and E meets H. Denote this event by M".
  $P(M_W'') = P(M_{W1}'') + P(M_{W2}'') = \frac{17}{162}$.
  $P(M'') = P(M_W'') + P(M_M'') = \frac{17}{81}$.

- So,
  $$P(F) = P(M) + P(M') + P(M'') = \frac{4}{81} + \frac{7}{648} + \frac{17}{81} = \frac{175}{648} \approx 0.270$$

Q3

This is false.
Think about a **mixed** random variable X, which is a Bernoulli distribution on point $x = 0$ and $x = 1$ with mass 0.5 and with the rest mass 0.5 being a uniform distribution on $[0, 1]$

$$X = \begin{cases} P(X) = \begin{cases} 0.5 & \text{If X=0} \\ 0.5 & \text{If X=1} \end{cases} & \text{(With mass 0.5)} \\ X \sim U[0,1] \quad \text{(With mass 0.5)} \end{cases}$$

Recall a function is pdf of a r.v. if and only if:

a $f_x(x) \geq 0 \forall x$
b $\int_{-\infty}^{\infty} f_x(x)dx = 1$(pdf)

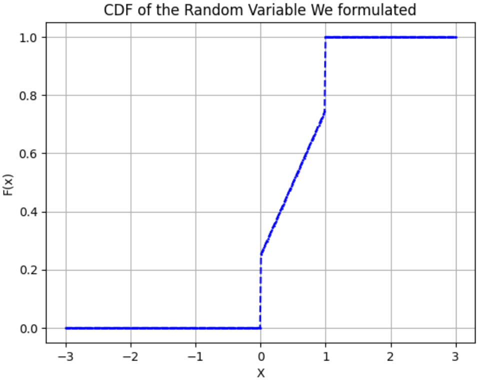
Then it has density on almost every point on $[0, 1]$
except for point $x = 0$ and $x = 1$.

Q3

- The corresponding CDF is:

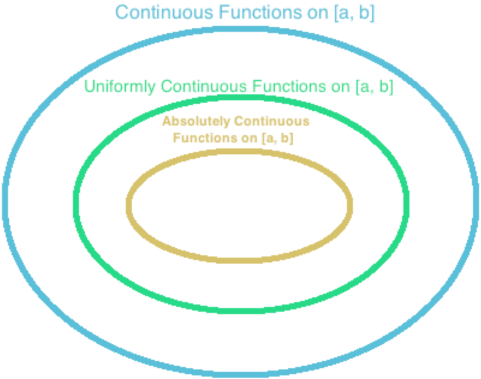
$$X = \begin{cases} P(X) = \begin{cases} 0.5 & \text{If X=0} \\ 0.5 & \text{If X=1} \end{cases} & \text{(With mass 0.5)} \\ X \sim U[0,1] \quad \text{(With mass 0.5)} \end{cases} \Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ 0.25 & x = 0 \\ 0.5x + 0.25 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

- We can plot the CDF as below:



Q3

- Clearly, this CDF is not continuous, let alone uniformly continuous.



- By Heine-Cantor theorem, a sufficient condition to make it true is that if random variable X, which distributed on a compact set $[a, b]$ has a continuous CDF, then we have a uniformly continuous CDF.

Q4

We define the event that no urn is empty as N. No limitation on urns' capacity and since no urn is empty, we need 1 urn contains 1 marble.

- Under the assumption of **throwing n marbles sequentially each time** [Ordered with Replacement].
  To calculate the probability, we know:
  - There are n^n ways of throwing the balls.
  - To make one in each urn, we have $n!$ ways.
    So the answer to $P(N) = \frac{n!}{n^n}$.

Note: Probability is frequency from repeating sampling many times.

Q4

We define the event that no urn is empty as N. No limitation on urns' capacity and since no urn is empty, we need 1 urn contains 1 marble.

* Under the assumption of **throwing n marbles simultaneously each time** [Unordered with Replacement].
  To calculate the probability, we know:
  - There are $\binom{2n-1}{n-1}$ ways of outcome.
  - But there's only one way of putting exactly one marble in every urn.

  So the answer to $P(N) = \frac{1}{\binom{2n-1}{n-1}}$

Note: Probability is frequency from repeating sampling many times.

Q5

Since $F(\cdot)$and $G(\cdot)$ are two distribution functions.
If $H(x) = F(G(x))$ is a distribution function, it has to satisfy three properties.

- **Monotonicity**
- $\mathbf{\lim_{x \to -\infty} H(x) = 0}$ **and** $\mathbf{\lim_{x \to +\infty} H(x) = 1}$
- **Right-continuity** $\mathbf{\lim_{x \downarrow x_0} H(x) = H(x_0)}$

Q5

Since $F(\cdot)$and $G(\cdot)$ are two distribution functions.
If $H(x) = F(G(x))$ is a distribution function, it has to satisfy three properties.

- **Monotonicity**
  - If $x_1 \leq x_2$, then $H(x_1) \leq H(x_2) \Rightarrow F(G(x_1)) \leq F(G(x_2))$.
  - Since if $x_1 \leq x_2$ then $G(x_1) \leq G(x_2)$ as $G(x)$ is a distribution function.
  - If $G(x_1) \leq G(x_2)$ then $F(G(x_1) \leq F(G(x_2))$ as $F(\cdot)$ is a distribution function. So if $x_1 \leq x_2$, then $H(x_1) \leq H(x_2)$.

  Monotonicity ✓

Q5

Since $F(\cdot)$and $G(\cdot)$ are two distribution functions.
If $H(x) = F(G(x))$ is a distribution function, it has to satisfy three properties.

- **Monotonicity** ✓
- $\lim_{\mathbf{x}\to-\infty} \mathbf{H}(\mathbf{x}) = \mathbf{0}$ **and** $\lim_{\mathbf{x}\to+\infty} \mathbf{H}(\mathbf{x}) = \mathbf{1}$
  - As we know $\lim_{x\to-\infty} G(x) = 0$ for $G(x)$ is a distribution function. We also know $G(x)$ satisfies monotonicity. So when $x \to -\infty$, $G(x)$ is non-increasing.
  - Hence, we will need to set $\lim_{x\downarrow 0} F(x) = 0$ to make H satisfies $\lim_{x\to-\infty} H(x) = 0$
  - Likewise, as we know $\lim_{x\to+\infty} G(x) = 1$ as $G(x)$ is a distribution function. Then we will need $\lim_{x\uparrow 1} F(x) = 1$.
  - Additionally, we want the support of $F(x)$ is $[0, 1]$

Q5

Since $F(\cdot)$and $G(\cdot)$ are two distribution functions.
If $H(x) = F(G(x))$ is a distribution function, it has to satisfy three properties.

- **Monotonicity** $\checkmark$
- $\lim\limits_{\mathbf{x}\to-\infty} \mathbf{H}(\mathbf{x}) = \mathbf{0}$ **and** $\lim\limits_{\mathbf{x}\to+\infty} \mathbf{H}(\mathbf{x}) = \mathbf{1}$ $\times$
- **Right-continuity** $\lim\limits_{\mathbf{x}\downarrow\mathbf{x_0}} \mathbf{H}(\mathbf{x}) = \mathbf{H}(\mathbf{x_0})$
    - As we know, $\lim\limits_{x\downarrow x_0} H(x) = H(x_0) \Leftrightarrow \lim\limits_{x\downarrow x_0} F(G(x)) = F(G(x_0))$
    - Since we know $\lim\limits_{x\downarrow x_0} G(x) = G(x_0)$ as $G(\cdot)$ is a distribution function.
    - Then we need $\lim\limits_{x\downarrow G(x_0)} F(G(x)) = F(G(x_0))$.
      Clearly this is satisfied as $F(\cdot)$ is a distribution function.

So **Right-continuity** $\checkmark$

Q5

Since $F(\cdot)$and $G(\cdot)$ are two distribution functions.
If $H(x) = F(G(x))$ is a distribution function, it has to satisfy three properties.

- **Monotonicity** $\checkmark$
- $\lim\limits_{\mathbf{x} \to -\infty} \mathbf{H}(\mathbf{x}) = \mathbf{0}$ **and** $\lim\limits_{\mathbf{x} \to +\infty} \mathbf{H}(\mathbf{x}) = \mathbf{1} \times$
  We need additional 3 conditions:

$$\lim_{x \downarrow 0} F(x) = 0$$
$$\lim_{x \uparrow 1} F(x) = 1$$
$$supp(F(x)) = [0, 1]$$

- **Right-continuity** $\lim\limits_{\mathbf{x} \downarrow \mathbf{x_0}} \mathbf{H}(\mathbf{x}) = \mathbf{H}(\mathbf{x_0}) \checkmark$

Practice Questions

2018 Midterm Q4 The cdf $F(x)$ of random variable X is continuous at 0. Find the distribution of random variable:

$$Y = \begin{cases} \frac{X}{|X|}, & \text{if X} \neq 0 \\ 1, \text{Otherwise} \end{cases}$$

Practice Questions

2018 Midterm Q4 The cdf $F(x)$ of random variable X is continuous at 0. Find the distribution of random variable:

$$Y = \begin{cases} \frac{X}{|X|}, \text{ if X} \neq 0 \\ 1, \text{Otherwise} \end{cases}$$

Y can be rewritten in this way:

$$Y = \begin{cases} -1 & \text{if } X < 0 \\ 1 & \text{if } X \geq 0 \end{cases}$$

Then we know Y is a Bernoulli random variable. and the distribution of Y is:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ p & \text{if } -1 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}, \quad p = P(X < 0) = F(0)$$