Nov 15, 2023.

Money lotteries & Risk Aversion

x: monetary outcome, $x \in \mathbb{R}$
$u(x)$: Bernoulli utility function

- A lottery is a CDF over $\mathbb{R}$, F.

$$F(x) = \text{Prob}(X \leq x)$$

$$U(F) = \int u(x)\, dF(x) = \int u(x) f(x)\,dx$$

$$U(L) = \sum_{i=1}^{n} p_i u_i$$

$E_F = \int x \,. dF(x)$ → expected value of the lottery.

Given a choice b/w $\underset{\text{lottery}}{F}$ & $\overset{\text{a constant no.}}{E_F}$, most people choose E_F – they are risk averse.

$$U(F) \leq U(E_F)$$

$$\int u(x).dF(x) \leq u\left(\int x.dF(x)\right) \quad \text{(Jensen's Inequality)}$$

↓ Need concavity

* let's consider lottery F
 x, prob p
 x', prob $1-p$

$$E_F = px + (1-p)x'$$

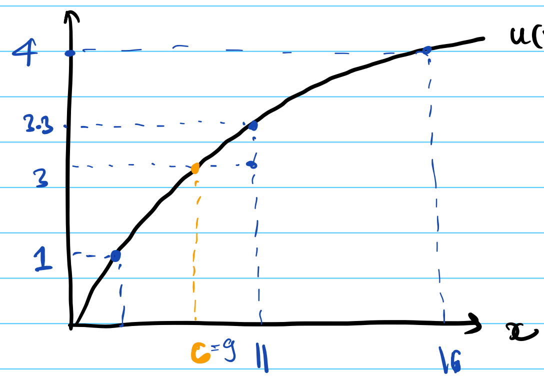
$$\int u(x)\,dF(x) = pu(x) + (1-p)u(x') \leq u(px + (1-p)x') = u(E_F)$$

Example : $u(x) = \sqrt{x}$, $x = 16$, $x' = 1$, $p = 2/3$

$$E_F = \frac{2}{3}(16) + \frac{1}{3}(1) = 11$$

$u(E_F) = \sqrt{11} \simeq 3.3$

$v(F) = \frac{2}{3}\sqrt{16} + \frac{1}{3}\sqrt{1} = 3$



A DM (decision maker) is risk averse iff
$\int u(x)\,dF(x) \leq u(E_F) \;\forall F$

Theorem : A DM is risk averse if & only if $u(\cdot)$ is concave.

Proof : Jensen's Inequality

- A DM is risk loving if & only if $u(\cdot)$ is convex
- A DM is risk neutral if & only if $u(\cdot)$ is linear

$u(x) = ax + b; \; a > 0 \;\&\; b$

Measuring Risk Aversion:

1. Certain equivalent / risk premium
2. Arrow - Pratt coefficient of risk aversion (curvature of $u(\cdot)$)

* c: Certain equivalent for lottery F.

$$U(F) = u(c)$$

$$\int u(x)\, dF(x) = u(c)$$

In example: find c s.t.

$$\int u(x) \cdot dF(x) = 3 = \sqrt{c}$$
$$c = 9$$

In general, c depends on both F, u, write $c(F, u)$

$$u(c(F,u)) = \int u(x)\, dF(x)$$

Risk Premium:

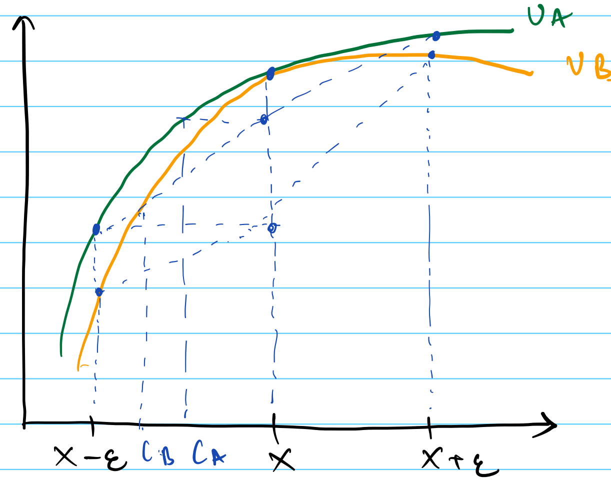
$$r(F,u) = E_F - c(F,u)$$

Given 2 utility functions, u_A, u_B, we say B is more risk averse than A if

$$r(F, u_B) \geq r(F, u_A) \text{ for all } F$$

or equivalently, $c(F, u_B) \leq c(F, u_A)$ for all F.

* Arrow - Pratt coefficient of risk aversion
(curvature of $u(\cdot)$)



$c_B < c_A$

Proposal: Measure risk aversion by $|u''(x)|$

B more risk averse than A $\iff$ $|u''_B(x)| \geq |u''_A(x)|$

$\tilde{u}(x) = k_1 u(x) + k_2$

$\tilde{u}$ & u represent the same prefs.

$\tilde{u}''(x) = k_1 u''(x) \neq u''(x)$

Define $r(x) = \dfrac{-u''(x)}{u'(x)}$ Arrow-Pratt coefficient of absolute risk aversion.

This is an example where A & B are two people.

$u_B(x) = u_1 u_A(x) + k_2$

$r_B(x) = \dfrac{-k_1 u''_A(x)}{k_1 u'_A(x)} = r_A(x)$ (risk measure at pt. x)

They have the same underlying prefs so their risk aversion should be same too

Theorem:– The following definitions of "more risk averse" are equivalent:

1) Whenever u_B prefers a lottery F to some certain outcome $\bar{x}$, then u_A does as well.

2) For all F, $c(F, u_B) \leq c(F, u_A)$

3) u_B is "more concave" than u_A: there is some increasing, concave g s.t.
$$u_B(x) = g(u_A(x))$$

4) $r_B(x) \geq r_A(x) \quad \forall x.$

Risk Aversion & Wealth

2 Pete's

1) Grad student Pete (poorer, wealth w_B)
2) Professor Pete (wealth $w_A > w_B$)

"Expect" risk aversion to be decreasing in wealth.
We say $u(\cdot)$ has decreasing absolute risk aversion (DARA) if $r(x) = -\dfrac{u''(x)}{u'(x)}$ is decreasing in x.

$$u_A(x) = u(w_A + x)$$
$$u_B(x) = u(w_B + x)$$

r is decreasing

$$r_A(x) = \frac{-u''(w_A+x)}{u'(w_A+x)} = r(w_A+x) < r(w_B+x)$$
$$= \frac{-u''(w_B+x)}{u'(w_B+x)}$$
$$= r_B(x)$$

Relative Risk Aversion

- 50% chance lose 1/2 wealth
- 50% chance wealth is doubled

Fix initial wealth w; utility $u(x)$

$$\tilde{u}(t) = u(tw)$$

Calculate Arrow-Pratt coefficient for $\tilde{u}$

$$\frac{-\tilde{u}''(t)}{\tilde{u}'(t)} = -\frac{w^2 u''(tw)}{w u'(tw)} = \frac{-w u''(tw)}{u'(tw)}$$

local measure, set $t = 1$

$$\rho(x) \quad -\frac{x u''(x)}{u'(x)}$$

coeff. of relative risk aversion

$$\rho(x) = x r(x)$$

$u(\cdot)$ has incr / decr / const. relative risk aversion if $\rho(\cdot)$ is incr / decr / const. respectively.

Common functional forms:

(1) Constant Absolute Risk Aversion (CARA)

$$u(x) = -e^{-\alpha x}, \quad r(x) = \alpha$$

(2) Constant Relative Risk Aversion (CRRA)

$$u(x) = \frac{x^{1-\sigma}}{1-\sigma}, \quad \rho(x) = \sigma \quad (\sigma = 1, \ u(x) = \log(x))$$

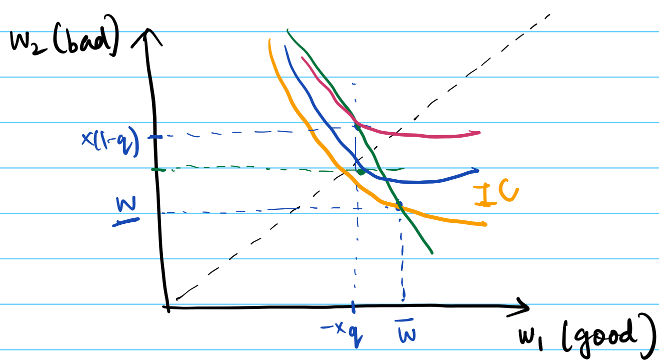
Insurance Example.

Farmer grows crops. Two states :

1) Prob $(1-p)$: high yield, worth $\bar{w}$

2) Prob p: low yield, worth $\underline{w}$

$u(\cdot)$ Bernoulli utility



$u_0 = pu(\underline{w}) + (1-p)u(\bar{w})$

IC : $\{(w_1, w_2) : pu(w_2) + (1-p)u(w_1) = u_0\}$

$\hat{U}(w_1, w_2) = pu(w_1) + (1-p)u(w_2)$

IC $= \quad U(w_1, w_2) = u_0$

u concave $\Rightarrow \hat{U}$ concave

Slope of IC on 45° line

$u_0 = (1-p)u(w_1) + pu(w_2(w_1))$ Implicit Function

Taking derivative wrt w_1

$0 = (1-p)u'(w_1) + pu'(w_2)\frac{dw_2}{dw_1}$

$$\frac{dw_2}{dw_1} = \frac{(1-p)u'(w_1)}{pu'(w_2)}$$

On 45° line, $w_1 = w_2$

$$\frac{dw_2}{dw_1} = -\frac{1-p}{p}$$

Insurance

Pay \$$q$ for 1 unit of a contract that pays out \$1 in bad state

Say farmer buys x units

State	Prob	Final Cons^m
Good	$1-p$	$\bar{w} - xq$
Bad	p	$\underline{w} - xq + x = \underline{w} + x(1-q)$

When does farmer choose on the 45° line $(w_1 = w_2)$?

slope of IC = slope of budget line

$$\frac{1-p}{p} = \frac{1-q}{q} \Rightarrow q = p \quad \text{(Actuarially Fair Insurance)}$$

Solve as a maximization problem:

$$\max_{x \geq 0} \; p\,u(\underline{w} + x(1-q)) + (1-p)\,u(\bar{w} - xq)$$

FOC $\quad p(1-q)\,u'(\underline{w} + x(1-q)) = q(1-p)\,u'(\bar{w} - xq)$

Suppose $p = q$: $\quad u'(\underline{w} + x(1-p)) = u'(\bar{w} - xp)$

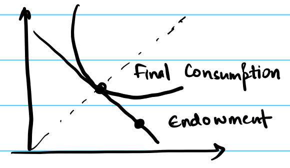
u strictly concave $\Rightarrow$ u' strictly monotonic

$\Rightarrow$ Arguments must be equal.

$$\underline{w} + x(1-p) = \bar{w} - xp$$

$$x^* = \bar{w} - \underline{w}$$

State	Prob	Final Cons^m	
Good	$1-p$	$\bar{w} - xq$	$\bar{w} - (\bar{w} - \underline{w})p$
Bad	p	$\underline{w} - xq + x = \underline{w} + x(1-q)$	$\underline{w} - (\bar{w} - \underline{w})(1-p) = \bar{w} - (\bar{w} - \underline{w})p$



What if $p < q$?

$p<q \Rightarrow p(1-q) < q(1-p)$

$\Rightarrow \dfrac{q(1-p)}{p(1-q)} > 1$

FOC:

$$\frac{u'(\underline{w} + x(1-q))}{u'(\bar{w} - xq)} = \frac{q(1-p)}{p(1-q)} > 1$$

$$u'(\overbrace{\underline{w} + x(1-q)}^{w_{bad}}) > u'(\overbrace{\bar{w} - xq}^{w_{good}})$$

u' strictly decreasing $\Rightarrow w_{bad} < w_{good}$

* What if u is risk neutral?

- Any choice of x is optimal.