Nov 13, 2023

Risk & Uncertainty

I : indoor ceremony
S : outdoor ceremony, sun
R : outdoor ceremony, rain

$S \succ I \succ R$

Actual choice

Indoor vs. Outdoor
(I) (S or R)

: Choosing over lotteries

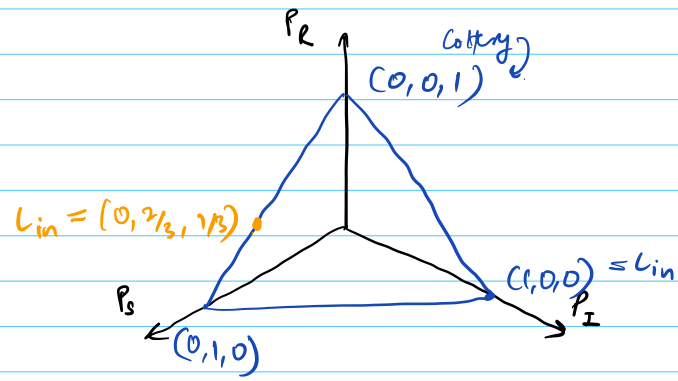
In general, $X = \{x_1, \ldots, x_n\}$ "consequences" or "prizes"
A lottery is a list
$L = (p_1, \ldots, p_n)$
$p_i \geq 0$
$\sum_{i=1}^{n} p_i = 1$

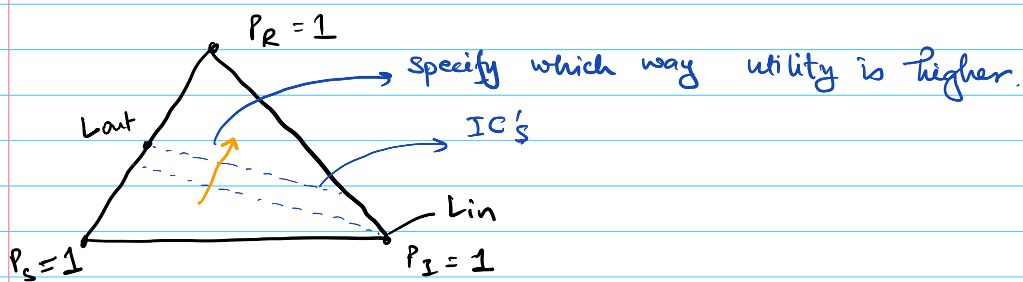
$X = \{I, S, R\}$ → outdoor
indoor ← $L_{in} = (1, 0, 0)$
p_I p_S p_R
Indoor

$L_{out} = (0, 2/3, 1/3)$

* Preferences :
$L_{in} \succ L_{out}$
$L_{out} \succ L_{in}$
$L_{out} \sim L_{in}$

$\mathcal{L} = \{(p_I, p_S, p_R) \in \mathbb{R}^3 : p_I, p_S, p_R > 0, \; p_I + p_S + p_R = 1\}$





Simplex

$\Delta(X) = \{(p_1, \ldots, p_n) : p_i \geq 0 \;\forall i, \; p_1 + \ldots + p_n = 1\}$

$(n-1)$ dimensional simplex.

If X is not finite, replace $(p_1, \ldots, p_n)$ with a pdf $f(x)$

$\Delta(x) = \{\text{pdfs } f(x)\}$

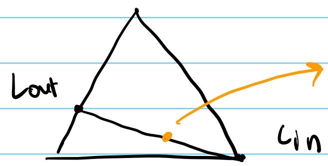
Compound lottery.

- Need a permit for an outdoor wedding
- Approval with prob 1/2
- Defines a new lottery

$L_{permit} = \frac{1}{2} L_{in} + \frac{1}{2} L_{out}$

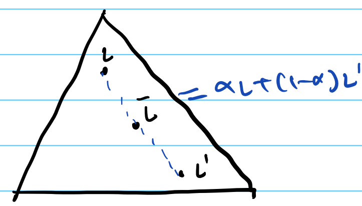
$$= \frac{1}{2}(1,0,0) + \frac{1}{2}\left(0, \frac{2}{3}, \frac{1}{3}\right)$$

$$= \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$$



L_{permit} : linear comb. of L_{out} & L_{in}

In general, $\mathcal{L}$ is <u>convex</u>.
$L, L' \in \mathcal{L}$, then $\alpha L + (1-\alpha)L' \in \mathcal{L} \quad \forall \alpha \in [0,1]$



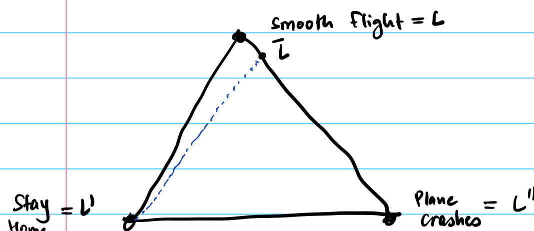
* $\mathcal{L}$ is (in principle) just like any other consumption space, X.

Can define a preference relation $\succsim$ on $\mathcal{L}$ just as before.

- Assume $\succsim$ is complete & transitive.

Two additional assumptions:-

① $\succsim$ is continous if for any $L, L', L'' \in \mathcal{L}$ s.t. $L \succsim L' \succsim L''$, $\exists$ some $\alpha \in [0,1]$ s.t. $\alpha L + (1-\alpha) L'' \sim L'$



Preferences cannot jump.
(Not too imp.)

If preferences are continuous, then we can represent them using a continuous utility f^n $u : \mathcal{L} \to \mathbb{R}$

$$L \succsim L' \iff u(L) \geq u(L')$$

Independence Axiom

$\succsim$ satisfies independence if $\forall\, L, L', L'' \in \mathcal{L}$ and all $\alpha \in (0,1)$,

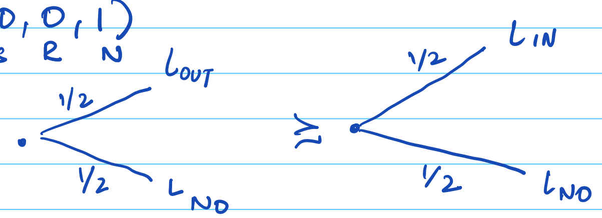
$$L \succsim L' \iff \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

Fourth Outcome: Groom gets cold feet

$$L_{NO} = (0, 0, 0, 1)$$

(I, S, R, N)

$L_{OUT} \succsim L_{IN} \iff$



Independence in "normal" consumer theory

$x = (2 \text{ apples}, 0 \text{ bananas})$

$x' = (0a, 2b)$

$x'' = (2a, 2b)$

$\alpha = 1/2$

$$x \succsim x' \Rightarrow \frac{1}{2}x + \frac{1}{2}x'' \succsim \frac{1}{2}x' + \frac{1}{2}x''$$

?

$$(2a, 1b) \succsim (1a, 2b)$$

$\Rightarrow$ This does not have to be true. Does not make sense to impose all the time in general consumer theory.

A utility function $u: \mathcal{L} \to \mathbb{R}$ has <u>expected utility form</u> if there are numbers $(u_1, \ldots, u_n)$ s.t.:

$\forall L \in \mathcal{L}$

$$u(L) = \sum_{i=1}^{n} p_i u_i$$

Theorem (von Neumann - Morgenstern, 1947)

A rational preference relation $\succsim$ on $\mathcal{L}$ is continuous and satisfies independence if & only if it admits an expected utility representation, that is, $\exists$ numbers $(u_1, \ldots u_n)$ s.t.

$$L \succsim L' \iff \sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} p_i' u_i$$

for any 2 $L = (p_1, \ldots, p_n)$, $L' = (p_1', \ldots, p_n')$

<u>Intuition</u>

EU $\Rightarrow$ independence is trivial

Independence (+ continuity) $\Rightarrow$ EU is harder

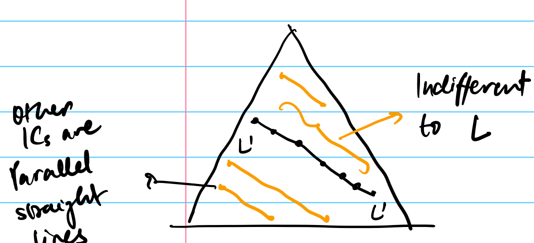
Take L, L' s.t. $L \sim L'$

Independence says:

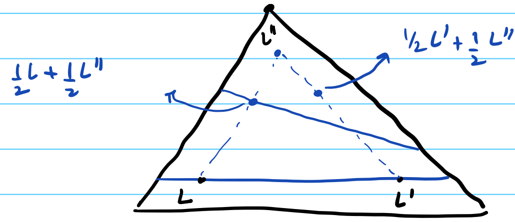
$$L \sim L' \iff \alpha L + (1-\alpha) L'' \sim \alpha L' + (1-\alpha) L''$$

Set $L'' = L$

$$L \sim L' \iff L \sim \alpha L' + (1-\alpha) L$$



What if not parallel?



Violates independence

$L \sim L'$ but

$\frac{1}{2}L + \frac{1}{2}L'' \nsim \frac{1}{2}L' + \frac{1}{2}L''$

$u(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$

- Straight line ICs
- MU $\frac{\partial U}{\partial x_1} = \alpha_1$

For lotteries, "goods" are units of probability:

$$U(p_1, \dots, p_n) = \sum_{i=1}^{n} p_i u_i$$

$$\frac{\partial U}{\partial p_i} = u_i$$

Further properties of EU:—

(1) A utility function has EU form iff it is linear.

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k) \quad \text{for} \quad \sum_{k=1}^{K} \alpha_k = 1$$

(2) EU is preserved only under <u>increasing</u> linear transformations.

Let $U: \mathcal{L} \to \mathbb{R}$ be an EU representation of $\succsim$.
Then, V is another EU representation if & only if
$V(L) = \alpha U(L) + \beta$ for some $\alpha > 0$.

EU is a cardinal property of utility

(3) Cardinality matters for $(u_1, \ldots, u_n)$

say $(u_1, u_2, u_3) = (1, x, 0)$

$L_1 = (1/2, 0, 1/2) \qquad L_2 = (0, 1, 0)$

$L_1 \gtrsim L_2$: $\frac{1}{2}(1) + \frac{1}{2}(0) \geq x$

$u_2 < \frac{1}{2}$: $L_1 \gtrsim L_2$

$u_2 > 1/2$: $L_2 \gtrsim L_1$

Side Note! 2 normalizations for "free" (corresponds to α, β)

wlog to set $u_1 = 1$, $u_n = 0$
But this pins down:
$u_2, \ldots, u_{n-1}$ exactly.

* If x is continuous, replace sums with integrals

$$u(F) = \int u(x)\, dF(x) = \int u(x)\, f(x)\, dx$$