Micro theory Aug 23, 2023

Grading: 40% exam1
40% exam2
20% Problem sets

Content:
I) Firm theory (Rec: Varian)
II) Consumer theory (Rec:Mas-collel)
III) Choice under uncertainty (Rec: Mas-collel)

Firm Theory: Assumptions
- Firms are price takers (no market power)
- Technology is exogenous
- Firms maximize profits

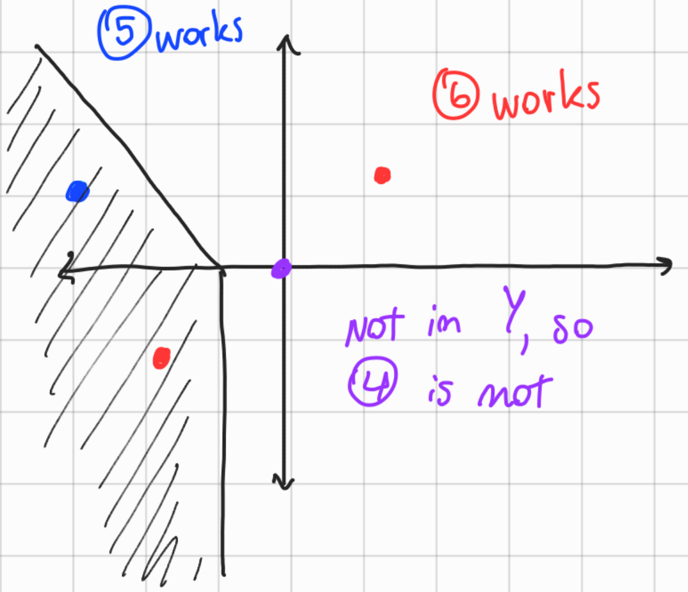
Technology:
n commodities y =(y_1 , y_2, ... , y_n)
y_i < 0 : Input
y_i > 0 : Output
Production set: Y subset R^n

Properties of production sets

1. $Y \neq 0$
2. Y is closed (includ boundry)
3. if $y \in Y$ and $y \geq 0$, then $y = 0$

always assumed

4. Shutdown is possible: $0 \in Y$
5. Free disposal: $y \in Y$ and $y' \in y_i$ then $y' \in Y$
6. Irreversibility: $y \in Y$, $y \neq 0$, then $y \notin Y$. $y(-1,1)$ in out not the opposite $-y(1,-1)$



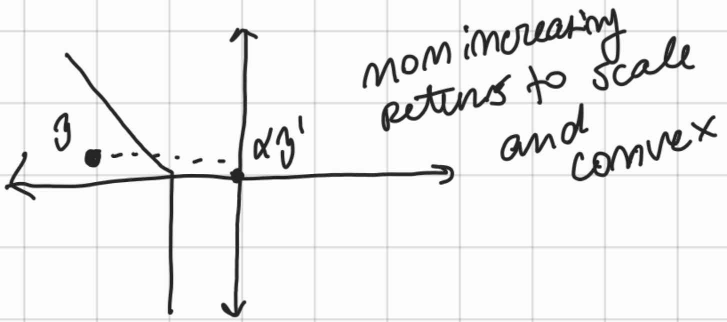
Returns to scale and convexity

i) Y has nonincreasing returns to scale if $y \in Y$
$\Rightarrow \alpha y \in Y \;\forall\, \alpha \in [0,1]$

ii) Nondecreasing: $y \in Y$
$\Rightarrow \alpha y \in Y \;\forall\, \alpha \geq 1.$

iii) Constant Returns to scale: $y \in Y \rightarrow \alpha y \in Y \;\;\forall \alpha \geq 0$

Y is convex if $y, y' \in Y \Rightarrow \alpha y + (1-\alpha) y' \in Y \;\;\forall\, \alpha \in [0,1]$



<u>Theorem</u>: For any convex Y with $0 \in Y$, there is a constant returns production set $Y' \subseteq \mathbb{R}^{n+1}$ s.t
$Y = \{ y \in \mathbb{R}^n : (y,-1) \in Y' \}$

Idea: The farmer controls a subset of possible inputs

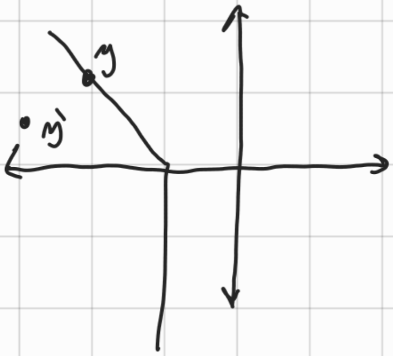
<u>proof</u>: Define $Y' = \{ y' \in \mathbb{R}^{n+1} : y' = \alpha(y-1) \text{ for some } y \in Y \wedge \alpha \geq 0 \}$

Check: Y' is constant returns

Technological efficiency:

Production plan $y \in Y$ is tech efficient if there doesn't exist $y' \in Y$ s.t $y' > y$.

$\hookrightarrow y_i' \geq y_i \; \forall i$ and $y_i' \gneq y$ for some i



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lecture #2: Aug 28, 2023

* Homework (on Canvas).

Transformation frontier (Function):

→ Firm is a production set Y

→ Transformation function $T: \mathbb{R}^n \to \mathbb{R}$

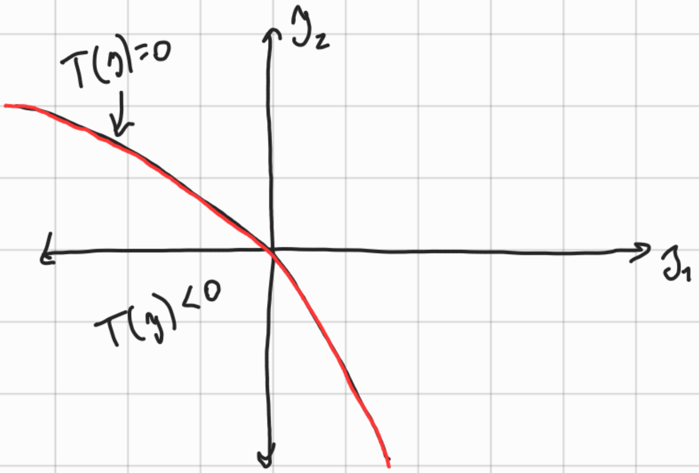
$T(y) < 0 \longleftrightarrow y$ is inefficient

$T(y) = 0 \longleftrightarrow y$ is efficient

$T(y) > 0 \longleftrightarrow y$ is infeasible.

$\Rightarrow Y = \{y \in \mathbb{R}^n : T(y) \leq 0\}$ production set.

$\Rightarrow \{y \in \mathbb{R}^n : T(y) = 0\}$ production frontier.



Lets fix goods $\bar{y}_3, \dots, \bar{y}_n$. Consider y_1, y_2

Define a function $y_2(y_1)$ by

$$T(y_1, \underbrace{y_2(y_1)}_{\text{implicit function of } y_1}, \bar{y}_3, \dots, \bar{y}_n) = 0$$

Differenciate both sides wrt. y_1

$$\underbrace{\frac{\partial T}{\partial y_1} + \frac{\partial T}{\partial y_2} \cdot \frac{dy_2}{dy_1}}_{\text{Chain Rule.}} = 0 \quad \Rightarrow \quad \frac{dy_2}{dy_1} = -\frac{\partial T/\partial y_1}{\partial T/\partial y_2}$$

* for any good : $\frac{dy_k}{dy_j} = -\frac{\partial T/\partial y_j}{\partial T/\partial y_k} = \underbrace{-MRT_{j,k}(\bar{y})}_{\text{marginal rate of Transformation.}}$

Special case: many inputs, one output

Inputs : $(x_1, \dots, x_n) \geq 0$
Output : $y \geq 0$
production function : $f(x)$
production set : $Y = \{\underbrace{-x_1, \dots, -x_n, y}_{n+1.}\} : x_i \geq 0 \ \forall i, \ y \leq f(x_1, \dots, x_n)$

$T(z) = y - f(x)$, $z = (-x, y)$
eff. frontier is
$T(z) = 0$
$y - f(x) = 0$
$y = f(x)$

Example: Cobb-Douglas

→ 2 inputs, capital x_k and labor x_L

→ Output y.

→ $f(x_k, x_L) = x_k^{\alpha} x_L^{\beta}$, $\alpha + \beta = 1$.

→ $Y = \{(-x_k, -x_L, y) : x_k, x_L \geq 0, y \leq x_k^{\alpha} x_L^{\beta}\}$

→ $T(-x_k, -x_L, y) = y - x_k^{\alpha} x_L^{\beta}$.

$$MRT_{(-x_k, y)} = -\frac{\partial T / \partial(-x_k)}{\partial T / \partial y} = -\frac{-\alpha x_k^{\alpha-1} x_L^{\beta}}{1} = \alpha x_k^{\alpha-1} x_L^{\beta} = \underbrace{\alpha\left(\frac{x_L}{x_k}\right)^{\beta}}_{\substack{\text{mg prod of } k \\ (MRK)}}$$

MRT for capital and Labor

→ output remains constant.

$$MRT_{(-x_k),(-x_L)} = -\frac{\partial T / \partial(-x_k)}{\partial / \partial(-x_L)} = \frac{\alpha}{\beta}\left(\frac{x_L}{x_k}\right)$$

$$\frac{d(-x_L)}{d(-x_k)} = -MRT_{(-x_k)(-x_L)} = \frac{dx_L}{dx_k} = \frac{-\alpha}{\beta}\left(\frac{x_L}{x_k}\right)$$

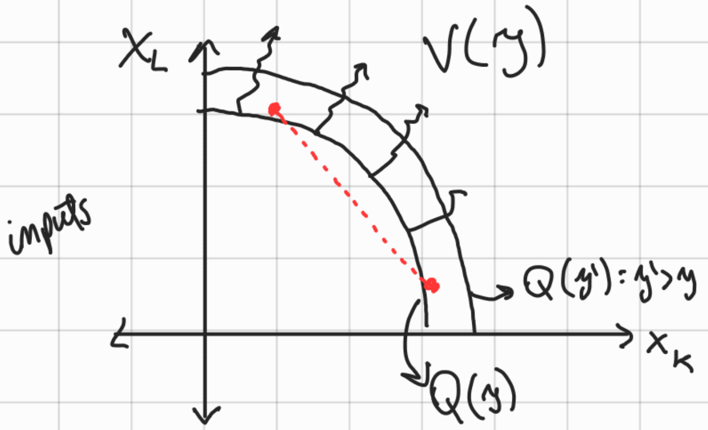
slightly Different interpretation (?).

Single output Fix y

$V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$ → Input Requirement set

$Q(y) = \{x \in \mathbb{R}^n : f(x) = y\}$ → Isoquant

Example: $f(x_1, x_2) = x_K^2 + x_L^2 = y$ (constant)



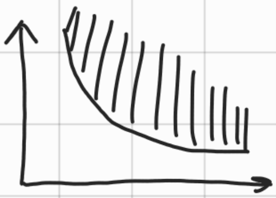
with a single output, we can define the mg rate of technical substitution between 2 inputs

$$\boxed{MRTS_{x_K, x_L} = \frac{\partial f / \partial x_K}{\partial f / \partial x_L}}$$

$$\frac{\partial x_L}{\partial x_K} = -MRTS_{x_K, x_L} = -\frac{f_{x_K}}{f_{x_L}}$$

In our example, the production function is increasing in K (not usual) because V it is not convex.

$\Rightarrow$



Theorem: If Y is convex, then $V(y)$ is also convex
$(P \Rightarrow Q \iff \neg Q \Rightarrow \neg P)$.

proof (Direct). : $(-x, y) \in Y$, $(-x', y) \in Y$ (two differents inputs that yield the same output)

Y is convex implies: $t(-x, y) + (1-t)(-x', y) \in Y$ (most of proofs will be by contradiction)

$$(-tx - (1-t)x', ty + (1-t)y) \in Y$$

$$(-tx - (1-t)x', y) \in Y$$

$$(-x'', y) \in Y. \quad (-x'' \text{ produces } y)$$

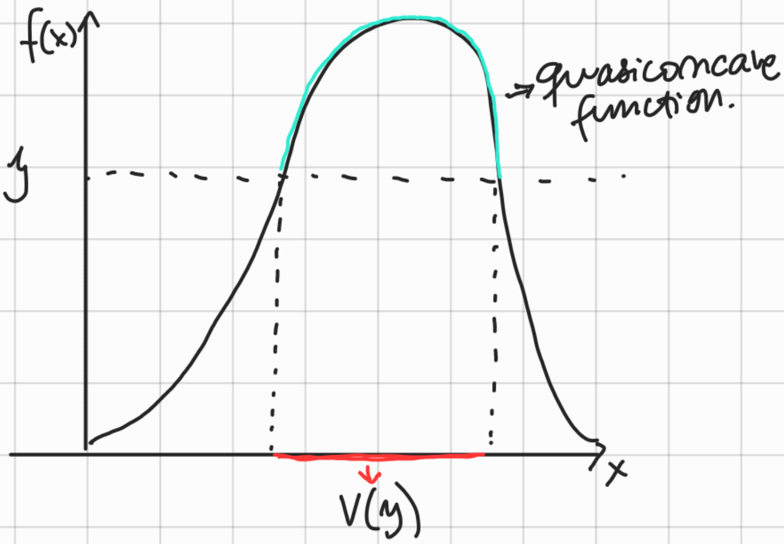
$\Rightarrow x'' \in V(y)$. meaning $V(y)$ is convex

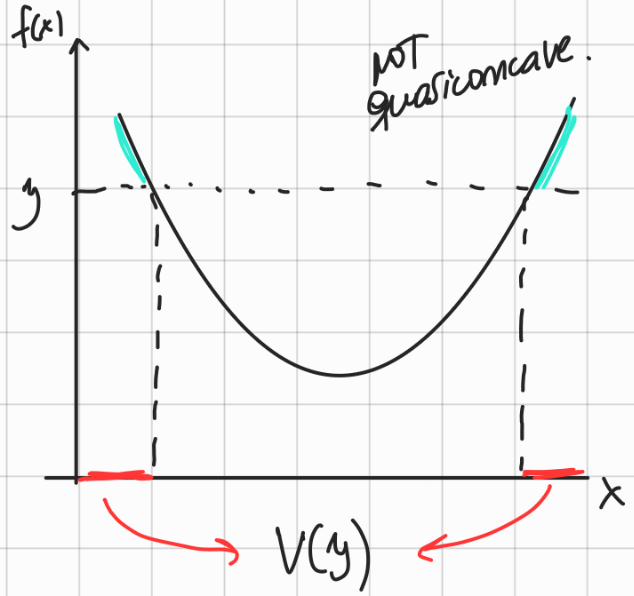
Theorem: $V(y)$ is convex IFF $f(x)$ is quasiconcave.

"**proof**": $V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$

↳ upper contour set of f

Def: Quasiconcave function is one that has convex upper contour sets.





Homogeneity and homotheticity.

A function $f: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree K if $f(tx) = t^k f(x)$ $\forall t > 0$

$k=0$: $f(tx) = f(x)$

$k=1$: $f(tx) = t f(x)$.

Example: Cobb Douglas $f(x_K, x_L) = x_K^{\alpha} x_L^{\beta}$, $\alpha + \beta = 1$

$$f(tK, tL) = (tK)^{\alpha} (t \cdot L)^{\beta}$$
$$= t^{\alpha} (K)^{\alpha} t^{\beta} \cdot (L)^{\beta}$$
$$= t \, K^{\alpha} L^{\beta}$$
$$\underbrace{f(tK, tL) = t \cdot f(K, L)}_{\text{constant Returns to scale}}$$

(if I double the input, I get double output).

$$MRTS = \frac{\alpha}{\beta} \frac{L}{K}$$

$$\frac{\alpha}{\beta} \frac{t \cdot L}{t \cdot K} = \frac{\alpha L}{\beta K}$$

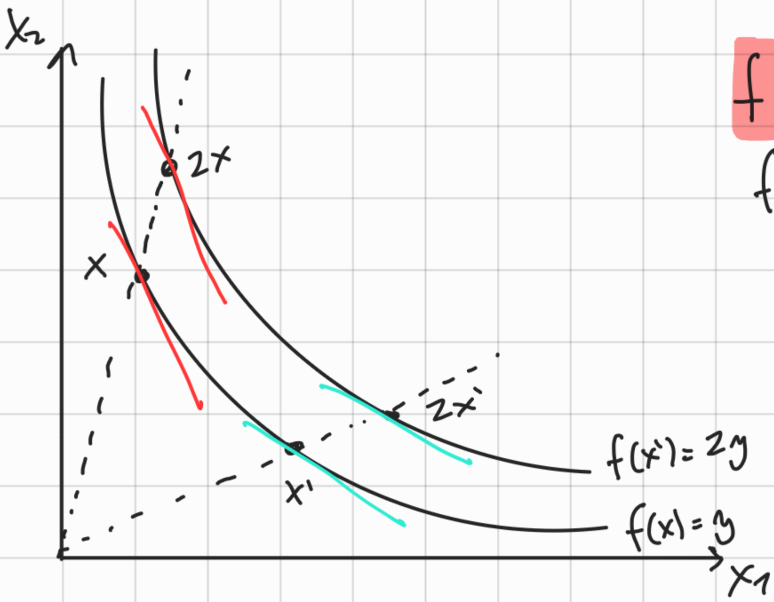
↓

MRTS doesn't change

$\Rightarrow$ MRTS is homogeneous of degree 0.

Theorem: If is hd-k, then the partial derivative f_{x_j} is hd-(k-1).

Proof:
$$f(t\cdot x) = t^k f(x)$$
$$t\cdot f_{x_i}(t\cdot x) = t^k f_{x_i}(x)$$
$$f_{x_i}(t\cdot x) = t^{k-1} f_{x_i}(x). \; //.$$



f is hd-1
$f(2x) = 2f(x)$.

→ partial derivatives does not change.

Theorem: If a production function $f(x)$ is hd-1, then the associated MRTS is hd-0.

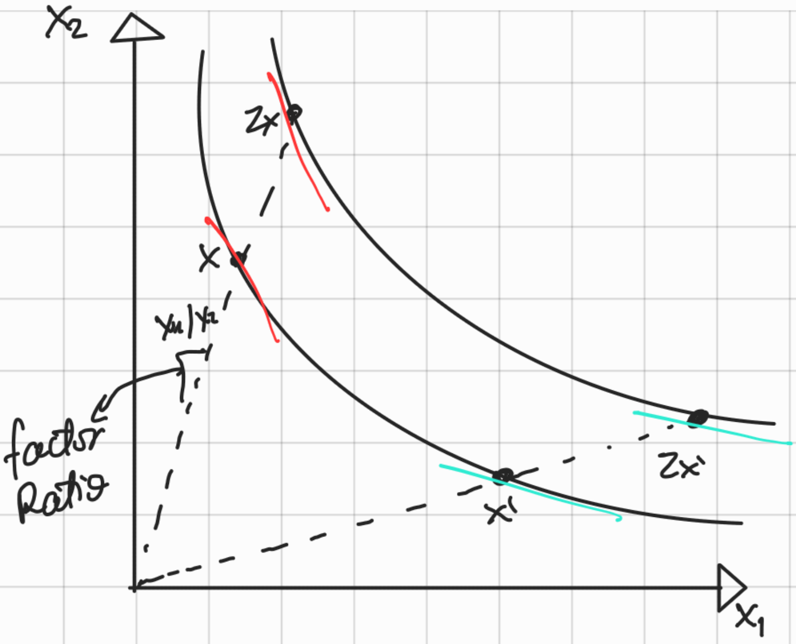
{Aug 30, 2023}

*Hd-1: $f(tx) = t\cdot f(x) \quad \forall t>0$

Proof:
$$MRTS_{ij}(\bar{x}) = \frac{f_{x_i}(\bar{x})}{f_{x_j}(\bar{x})}$$

$$MRTS_{ij}(t\cdot\bar{x}) = \frac{f_{x_i}(t\cdot\bar{x})}{f_{x_j}(t\cdot\bar{x})} = \frac{t\cdot f_{x_i}(\bar{x})}{t\cdot f_{x_j}(\bar{x})} = MRTS_{ij}(\bar{x})$$

Euler's Theorem.



$f(x) = f(x') = 10$
$f(2x) = f(2x') = 20$

<u>A homothetic function</u>: Is a positive monotonic transformation of an hd-1 function i.e., $f: \mathbb{R}^n \to \mathbb{R}$ is homothetic if $\exists$ a $g: \mathbb{R} \to \mathbb{R}$ strictly increasing and and $h: \mathbb{R}^n \to \mathbb{R}$ that is hd-1 s.t $f(x) = g(h(x))$.

$$MRTS \text{ of } f = \frac{f_{x_j}}{f_{x_i}} = \frac{g'(h(x))h_{x_i}}{g'(h(x))h_{x_j}} = \frac{h_{x_i}}{h_{x_j}} \to hd\text{-}0$$

<u>Returns to scale with a single output.</u>

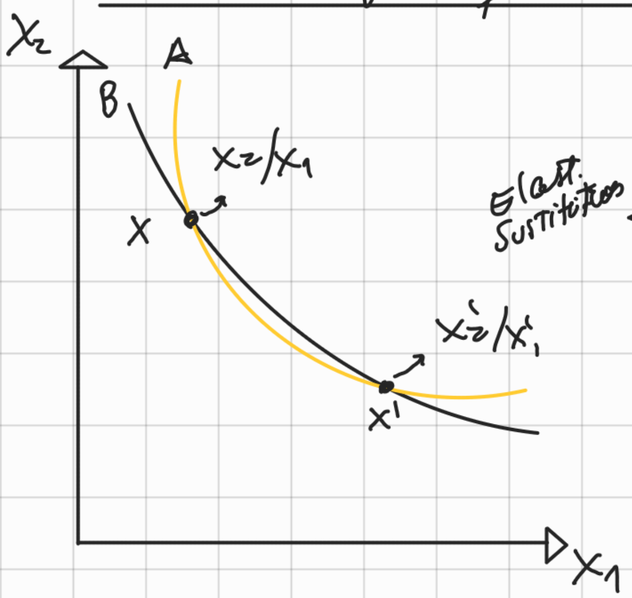
Production function f exhibits:

- Decreasing Returns to scale
  if $f(tx) \leq t f(x) \quad \forall t \geq 1$

- Increasing returns to scale
  if $f(tx) \geq t \cdot f(x) \quad \forall t \geq 1$

- Constant returns to scale
  if $f(tx) = t \cdot f(x) \quad \forall t \geq 0$

Curvature of isoquants:



Elasticity of complementarity (for "large" jumps).

$$\underset{\text{Elast. Sustitution}}{\frac{1}{\sigma}} = \frac{\frac{\Delta MRTS}{MRTS}}{\frac{\Delta(x_2/x_1)}{x_2/x_1}}$$

* Denominators are the same for A & B in example.

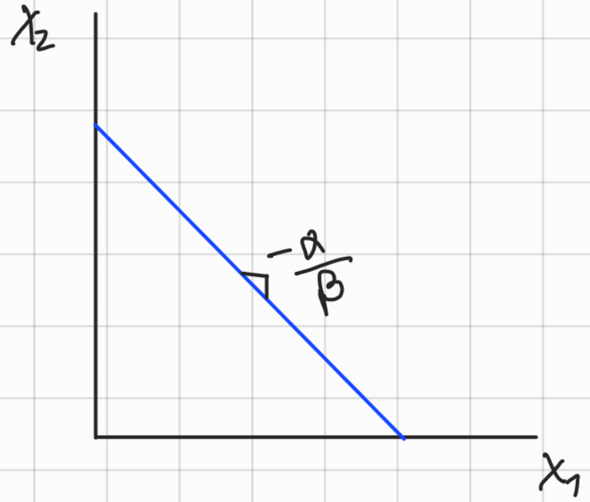
** Numerator is much higher for A

$$\frac{1}{\sigma_A} > \frac{1}{\sigma_B}$$

In the limit:

$$\frac{1}{\sigma} = \frac{x_2/x_1}{MRTS} \cdot \frac{d\,MRTS}{d(x_2/x_1)}.$$

Perfect substitution: $f(x_1, x_2) = \alpha x_1 + \beta x_2$



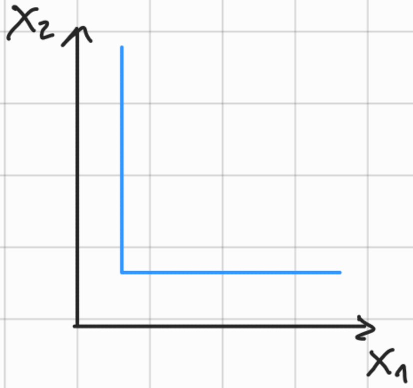
$\frac{1}{\sigma} = 0$, because MRTS is a fixed numers.

Elasticity of substitution: $\sigma = \frac{MRTS}{x_2/x_1} \cdot \frac{d(x_2/x_1)}{d(MRTS)}$

↳ for perfect sustitutes: $\sigma = \infty$

↳ Less curvature → bigger σ.

Perfect Complements: $f(x_1,x_2) = \min\{x_1, x_2\}$



$\sigma = 0$

For any two quantities, y, x, the elasticity of y with respect to x: $\varepsilon_{yx} = \dfrac{d\log(y)}{d\log(x)}$

* way to see this: $d\log(y) = \frac{1}{y}\,dy$ (Total derivative).

$$d\log(x) = \frac{1}{x}\cdot dx.$$

$\hookrightarrow$ $\varepsilon_{yx} = \dfrac{x}{y}\cdot\dfrac{dy}{dx}$ but $\dfrac{d\log(y)}{d\log(x)}$ is easier to remember.

<u>Example</u>: σ for Cobb-Douglas

MRTS: $\dfrac{\alpha}{\beta}\cdot\dfrac{x_L}{x_K}$ $\Rightarrow$ $\sigma = \dfrac{MRTS}{x_L/x_K}\;\dfrac{d\left(x_L/x_K\right)}{d\,MRTS}$.

a change variable to readability.

$\theta = \dfrac{\alpha}{\beta}\dfrac{x_L}{x_K} = MRTS$

$$\sigma = \frac{\theta}{\frac{\beta}{\alpha}\theta}\cdot\frac{d\left(\frac{\beta}{\alpha}\theta\right)}{d\theta} = \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1$$

Property of Cobb-Douglas.

(NOT the same as slope)

CES : Constant elasticity of sust. production.

$$f(x) = \left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{\frac{1}{\rho}} \quad , \quad \rho = \underbrace{0, 1, \infty}_{\text{subcases.}}$$

The Profit maximization function (PMP)

Recall assumptions
- Price takers
- max profits

$$\pi(p) = \max_{\text{s.t } y \in Y} p \cdot y, \quad \text{where } p \cdot y = \sum_{i=1}^{n} p_j y_j$$

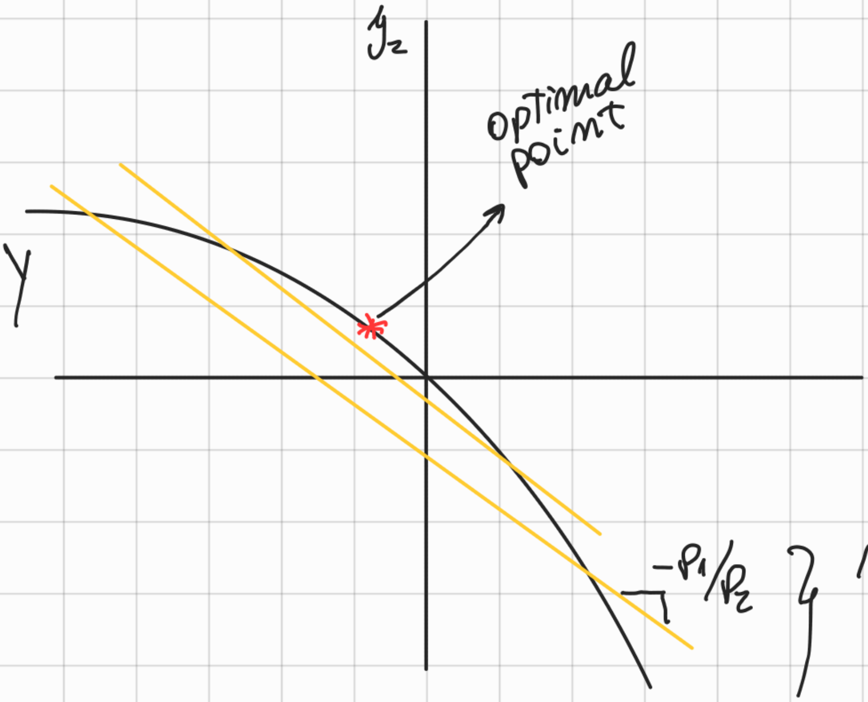
positive $y \Rightarrow$ output
negative $y \Rightarrow$ inputs

Equivalently, $\pi(p) = \max p \cdot y$
s.t $T(y) \leq 0$

$\pi : \mathbb{R}^n \to \mathbb{R}$ is also called profit function.
Example of value function in constrained optimization
$p \cdot y$ is called the objective function.

$y^*(p) = \{ y \in Y : p \cdot y = \pi(p) \}$ is called the supply correspondence
set valued mapping.

$y(p)$ could be multivalued (or $y(p) = \emptyset$).

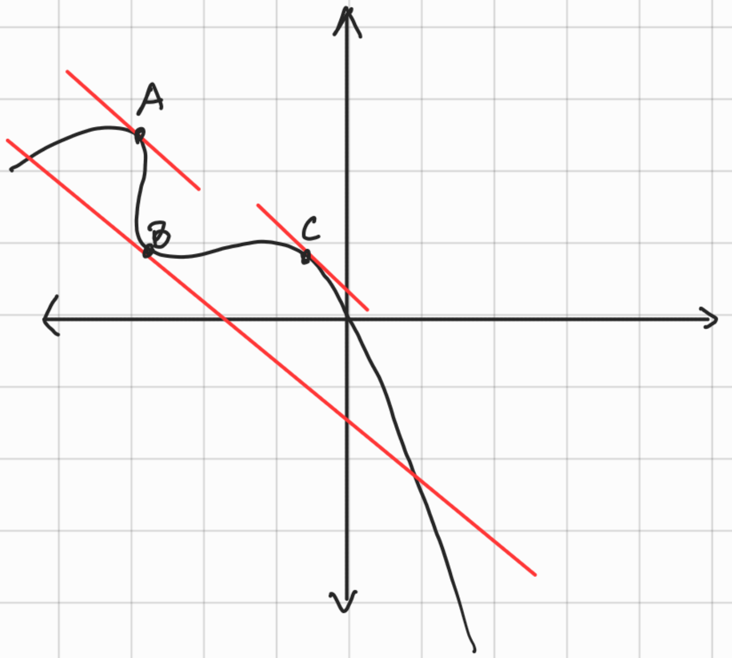


} isoprofit lines

Objective: $p_1 y_1 + p_2 y_2$
Level sets: Fix an $m \in \mathbb{R}$
set of (y_1, y_2) s.t $p_1 y_1 + p_2 y_2 = m$

$$y_2 = \frac{m}{p_2} - \frac{p_1}{p_2} y_1$$

In more dimensions: Solution is hyperplane.



Sept 4, 2023

max $p \cdot y$

s.t $T(y) \leq 0$

$$\mathcal{L} = p \cdot y - \lambda T(y)$$

K.T. Conditions:

$$\frac{\partial \mathcal{L}}{\partial y_i} = 0 \Rightarrow P_i = \frac{\partial T}{\partial y_i} \cdot \lambda$$

c.s.c. $\lambda \cdot T(y) = 0$

$\lambda \geq 0$

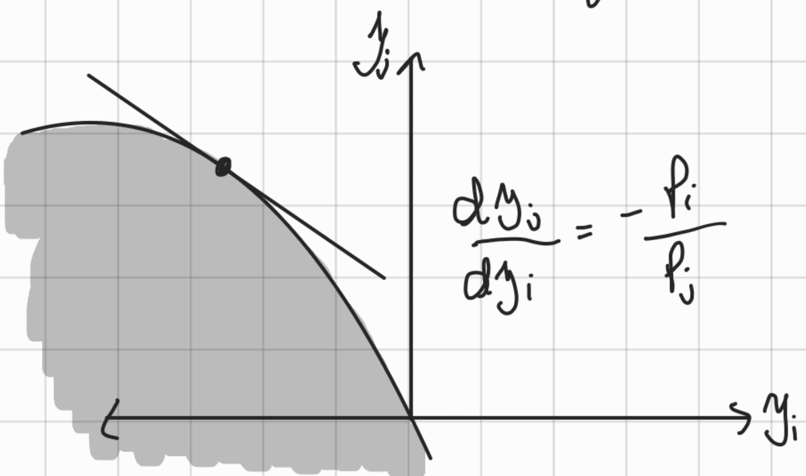
Implication of the f.o.c

$$\frac{P_i}{P_j} = \underbrace{\frac{\partial T / \partial y_i}{\partial T / \partial y_j}}_{MRT_{ij}} \quad \forall\, i,j$$

* Recall $\frac{dy_j}{dy_i} = -MRT$

$\Rightarrow$ Optimum at

$$\frac{P_i}{P_j} = -\frac{dy_j}{dy_i}$$



* Economic interpretation in: $\dfrac{\partial T/\partial y_i}{P_i} \gtreqless \dfrac{\partial T/\partial y_j}{P_j}$

Single output case: Production function $f(y)$

$$\max \; p \cdot f(x) - w \cdot x$$
s.t. $x_j \geq 0$

w: Vector of input prices
p: output price
μ: multiplier.

$$\mathcal{L} = p \cdot f(x) - wx + \mu \cdot x$$

alternatively.
s.t $-x_j \leq 0$
$\ldots \; -\mu(-x)$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \Rightarrow p \cdot \frac{\partial \mathcal{L}}{\partial x_i} - w_i + \mu_i = 0 \quad (\star)$$

csc. $\mu_i x_i = 0 \quad \forall i$
$\mu_i \geq 0 \quad \forall i$

Another way for $(\star)$: $\underbrace{p \cdot \dfrac{\partial f}{\partial x_i}}_{\text{Mg. Revenue of add. } x_i} \leq \underbrace{w_i}_{\text{Mg. cost of add. } x_i} \; \forall i$, with equality if $x_i^* > 0$

Divide i by j: $\underbrace{\dfrac{\partial f/\partial x_i}{\partial f/\partial x_j}}_{\text{MRTS}} = \dfrac{w_i}{w_j} \quad \forall \; x_i^*, x_j^* > 0$

$$\frac{\partial f/\partial x_i}{w_i} = \frac{\partial f/\partial x_j}{w_j}$$

* NOTE
$f(x) \leq 0$
$-\mu f(x)$
$f(x) \geq 0$
$\mu f(x)$

Notation for simple output problem:

$x^*(p,w)$: factor demand correspondence (or function).

$y^*(p,w)$: Supply correspondence

$\pi(p,w) = p \cdot f(x^*(p,w)) - w \cdot x^*(p,w)$

* K.T. cond. are necessary but not sufficient (in general).

** S.O.C are informative but locally.

Back: $\max\ p \cdot y$
$\text{s.t}\ T(y) \leq 0$

Theorem: Assume $T(y)$ is convex. Then, if a point y^* satisfies the KT conditions for some $\lambda \geq 0$, then y^* is a solution to maximization.

$$\max\ \underbrace{g(x)}^{\text{quasiconcave}}$$
$$\text{st.}\ \underbrace{h(x) \geq 0}_{\text{quasiconvex}}$$

Then KT are necessary and sufficient.

Single output: $\max\ p \cdot f(x) - wx$
$\text{s.t}\ x_i \geq 0$

Theorem: Assume f is concave. Then, if x^* satisfies

$$p \cdot \frac{\partial f}{\partial x_i} \leq w_i \quad \forall i, \text{ with equality if } x_j^* > 0,$$

Then x^* is a solution to the problem.

Caveats: KT/FOC are useful but be careful with
- nondifferentiabilities
- nonconvexities
- Existence / uniqueness
- Corner solution (some variable can be optimal at 0).

Example: single-output, single-input $f(\cdot)$

$$y = f(x) = \frac{1}{2}x$$

$$\max_{x \geq 0} \; p f(x) - wx$$

$$\mathcal{L} = p\left(\frac{1}{2}x\right) - wx + \mu x$$

$$\mathcal{L}_x = \frac{1}{2}p - w + \mu = 0 \qquad \text{csc: } \mu x = 0, \; \mu \geq 0.$$

Solve by cases:

I) $x = 0$, $\mu > 0$.

$\rightarrow \mu = w - \frac{1}{2}p > 0$?

what if $w - \frac{1}{2}p < 0$?

Economic interpretation:

$2w < p$ \
Mg cost < mg I } Firm should produce inf. so, no solution.

II) $x > 0$, $\mu = 0$.

$\underbrace{\frac{1}{2}p = \mu}$

Economic sense.

Solution:

$$x(p,w) = \begin{cases} 0 & , p < 2w \\ [0,\infty) & , p = 2w \\ \emptyset & , p > w \end{cases}$$

Example 2: $f(x_1,x_2) = \log(1 + x_1 + x_2)$ $\qquad x_1 \geq 0$, $x_2 \geq 0$

$$\mathcal{L} = p \cdot \log(1 + x_1 + x_2) - w_1 x_1 + \mu_1 x_1 + \mu_2 x_2$$

$$\partial\mathcal{L}/\partial x_1 = 0 \longrightarrow \frac{p}{1 + x_1 + x_2} - w_1 + \mu_1 = 0$$

$$\partial\mathcal{L}/\partial x_2 = 0 \longrightarrow \frac{p}{1 + x_1 + x_2} - w_2 + \mu_2 = 0$$

csc

$\mu_1 x_1 = 0$

$\mu_2 x_2 = 0$

$\mu_1 \geq 0$, $\mu_2 \geq 0$

<u>Case I:</u> $\mu_1 = 0, \ \mu_2 = 0, \ x_1 > 0, \ x_2 > 0$

$$w_1 = \frac{p}{1+x_1+x_2} \ ; \ w_2 = \frac{p}{1+x_1+x_2} \rightarrow w_1 = w_2 = \bar{w}.$$

$$\rightarrow x_1 + x_2 = \frac{p}{\bar{w}} - 1.$$

implicitly. $\frac{p}{\bar{w}} - 1 > 0$

$p > \bar{w}$ condition

<u>Case II:</u> $\mu_1 = 0, \ \mu_2 \geq 0, \ x_1 \geq 0, \ x_2 = 0$

$$\frac{p}{1+x_1} = w_1 \Rightarrow x_1^* = \frac{p}{w_1} - 1$$

$w_2 \geq w_1$ condition

$$\mu_2 = w_2 - \frac{p}{1+x_1} = w_2 - w_1 \geq 0$$

<u>Case III:</u> $\mu_1 \geq, \ \mu_1 = 0, \ x_1 = 0, \ x_2 \geq 0$

$w_1 \geq w_2$ condition

$$\rightarrow \mu_1 = w_1 - \frac{p}{1+x_2} = w_1 - w_2 \geq 0$$

wrapping all together: Factor demands

assume $p \geq \min\{w_1, w_2\}$

$$x_1(p,w) = \begin{cases} \frac{p}{w_1} - 1, & w_1 < w_2 \\ \alpha\left(\frac{p}{\bar{w}} - 1\right), & w_1 = w_2 = \bar{w}, \ \alpha \in [0,1] \\ 0, & w_1 > w_2 \end{cases}$$

$$x_2(p,w) = \begin{cases} \frac{p}{w_2} - 1, & w_2 > w_1 \\ (1-\alpha)\left(\frac{p}{\bar{w}} - 1\right), & w_1 = w_2 = \bar{w}, \ \alpha \in [0,1] \\ 0, & w_2 > w_1 \end{cases}$$

$$y(p,w) = \begin{cases} 0, & p \le \hat{\omega} \\ \log\left(\frac{p}{\hat{\omega}}\right), & p > \hat{\omega} \end{cases}, \quad \hat{\omega} = \min\{w_1, w_2\}$$

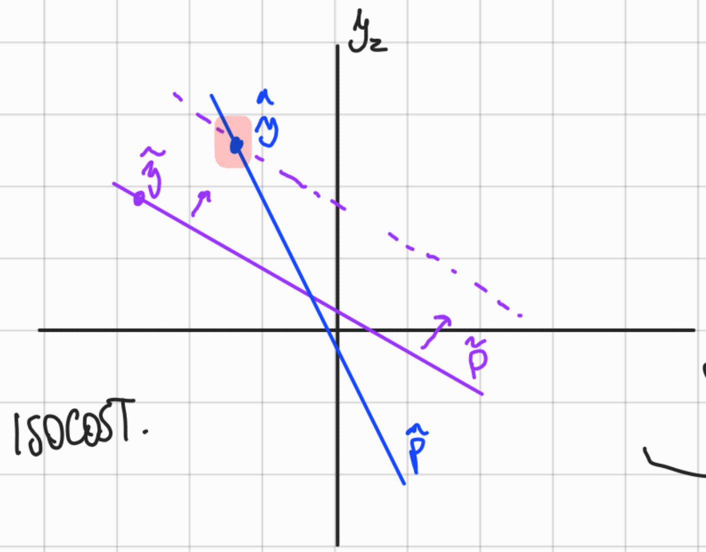
$$\pi(p,w) = \begin{cases} 0, & p \le \hat{\omega} \\ p \cdot \log\left(\frac{p}{\hat{\omega}}\right) - p + \hat{\omega}, & p > \hat{\omega} \end{cases}$$

Rationalizability:

Sept 06, 2023

Q: Given some data $\{(p^1, y^1), \dots, (p^T, y^T)\}$ vectors of data

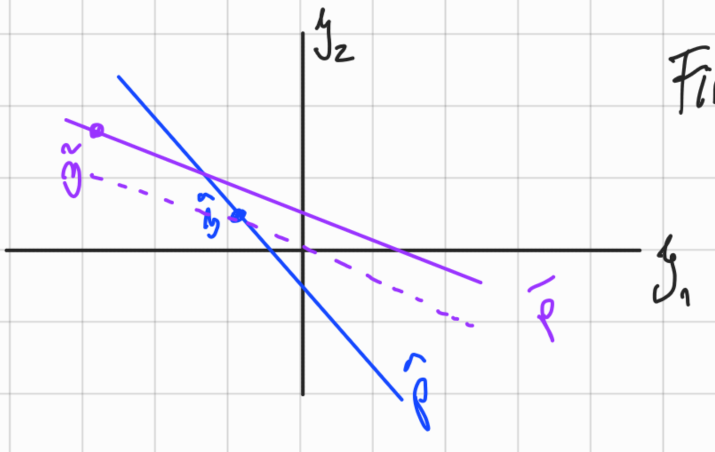
Can we "learn" about the firm's technology?



Firm is irrational: Violate the weak axiom of profit maximization: (WAPM)

$$p^t y^t \ge p^t y^s, \quad \forall s,t = 1,\dots,T$$

$\longrightarrow \tilde{p}\hat{y} \ge \tilde{p}\tilde{y}$ violates.



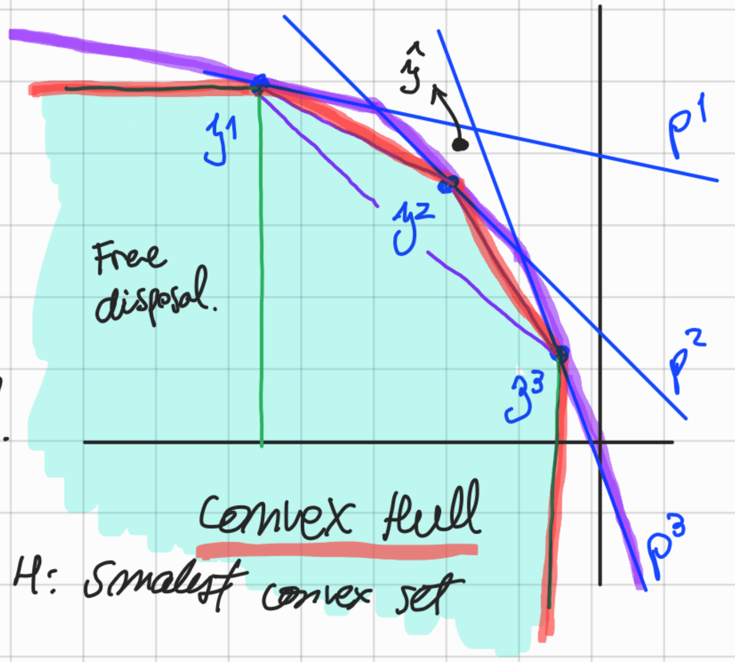
Firm could've chosen $\hat{y}$, but at a price.

What other production sets could "rationalize" the data?

a production set Y rationalizes the data if

$$y^t \in y^*(p^t) \quad \forall\, t = 1, \dots, T$$

$$y^*(p) = \underset{y \in Y}{\arg\max}\; p \cdot y$$



make some additional assumptions.
- Data satisfies WAMP
- Y is convex & free disposal.

$$Y^F = \mathrm{conv}_{fd}(\{y^1, \dots, y^T\})$$

*C.H: smallest convex set

Does Y^F rationalize the data?

Upper bound.

claim: yes.

Proof: Contradiction: Then $\exists$ a point $y \in Y^F$ s.t $p^t \cdot y > p^t y^t$ for some t.

By convexity: $y = \alpha^1 y^1 + \alpha^2 y^2 + \dots + \alpha^T y^T$, $\alpha^1 + \dots + \alpha^T = 1$.

① $p^t(\alpha^1 y^1 + \dots + \alpha^T y^T) > p^t \cdot y^t$

② $\alpha^1 p^t y^1 + \dots + \alpha p^t y^T \leq \alpha^1 p^t \cdot y^t + \dots + \alpha^T p^t \cdot y^t$

$$= (\alpha^1 + \dots + \alpha^T)\, p^t \cdot y^t$$

$$= p^t \cdot y^t$$

contradiction.

$\rightarrow$ Y^F is one production set that rationalizes the data. By def. for any other convex Y that rationalizes the data $Y^F \subseteq Y$

$Y^o = \{ y \in \mathbb{R}^n : p^t \cdot y \leq \pi(p^t),\ \forall t = 1, \dots, T \}$

$\hat{y} \notin Y^o$ because $p^2 \cdot \hat{y} > p^2 \cdot y^2 = \pi(p^2)$

$\mathbb{R}^n \setminus Y^o$ bundles that yield higher profits than some observed choice $= \{ y \in \mathbb{R}^n : p^t y > \pi(p^t) \text{ for some } t \}$.

Does Y^o rationalize the data?
claim. yes.

Proof: Assume not. Then, $\exists$ some $y \in Y^o$ s.t $p^t \cdot y > p^t \cdot y^t$
Directly contradicts definition of Y^o.

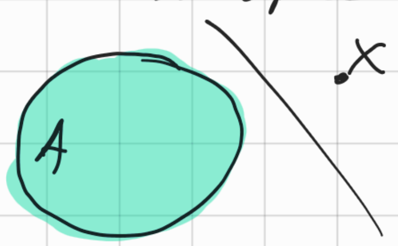
Theorem: For any production set Y that rationalizes the data, we have $Y^F \subseteq Y \subseteq Y^o$

What if we had infinite data? we would pin down the exact production function.

Suppose we observe $y(p)\ \forall p$. Let $\pi(p) = p \cdot y(p)$ be the observed profit function.

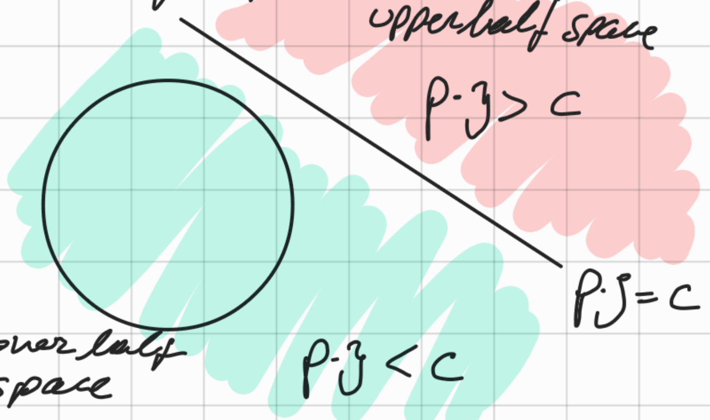
Theorem: Suppose Y is convex, closed & satisfies free disposal.
Then $Y = Y^o = \{ y \in \mathbb{R}^n : p y(p) \leq \pi(p)\ \forall p \in \mathbb{R}^n_+ \}$

Detour: Separating hyperplane theorem. In $\mathbb{R}^n$, a hyperplane is a subspace of dim $(n-1)$.



Given $p \in \mathbb{R}^n$, $c \in \mathbb{R}$, the $(p-c)$ hyperplane is

$$H_{p,c} = \{ y \in \mathbb{R}^n : p \cdot y = c \}$$



Theorem: Suppose $A \subseteq \mathbb{R}^n$ is convex and closed, and $x \notin A$. Then, there is a $p \in \mathbb{R}^n$, $p \neq 0$, and $c \in \mathbb{R}$ s.t. $p \cdot x < c$ & $p \cdot y < c \;\; \forall y \in A$.

* Convexity is important:

Proof: $Y = Y^o$

NTS ① $Y \subseteq Y^o$ & ② $Y^o \subseteq Y$

For ② : Take some $\hat{y} \in \mathbb{R}^n \setminus Y$

By S.H.T., $\exists$ some $p \in \mathbb{R}^n$ & c s.t.

$$p \cdot \hat{y} > c > p \cdot y \quad \forall y \in Y$$

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y \qquad (*)$$

By free disposal, $p \geq 0$ if $p_i < 0$ for some i, then $\max p \cdot y = \infty$ which contradicts $(*)$

Therefore, there exists $p \geq 0$ s.t. $p \cdot \hat{y} \; \max_{y \in Y} p \cdot y = \pi(p)$
i.e., $\hat{y} \notin Y^o$, or $Y^o \subseteq Y$.

Sept 18, 2023

Properties of profit & supply functions

$$\pi(p) = \max_{y \in Y} p \cdot y \qquad y(p) = \underset{y \in Y}{\operatorname{argmax}}\ p \cdot y$$

$$\pi : \underbrace{\mathbb{R}^n}_{\text{space of prices}} \to \underbrace{\mathbb{R}}_{\text{profit}} \qquad y : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$$

Note: Results do not rely on regularity assumptions (convexity, differentiability, etc..) unless stated otherwise

1. $\pi(p)$ is nonincreasing in input prices, nondecreasing in output prices.

Proof: Take a price vectors p, p' and choose $y \in y^*(p)$ and $y' \in y^*(p')$. Let $p_i' \geq p_i$ $\forall$ outputs, $p_i' \leq p_i$ for all inputs

By WAPM, $p' \cdot y' \geq p' \cdot y$

consider $(p'-p) \cdot y = (p_1'-p_1)y_1 + \cdots + (p_n'-p_n)y_n$

$y_i \geq 0 : p_i' \geq p_i \qquad y_i \leq 0 : p_i' \leq p_i \Rightarrow (p'-p)y \geq 0$

$\Rightarrow p' y \geq p \cdot y$

$\pi(p') = p' \cdot y' \geq p' \cdot y \geq p \cdot y = \pi(p)$

$\Rightarrow \quad \pi(p') \geq \pi(p)$

② $\pi(p)$ is hd-1. WTS: $\pi(tp) = t\pi(p)$

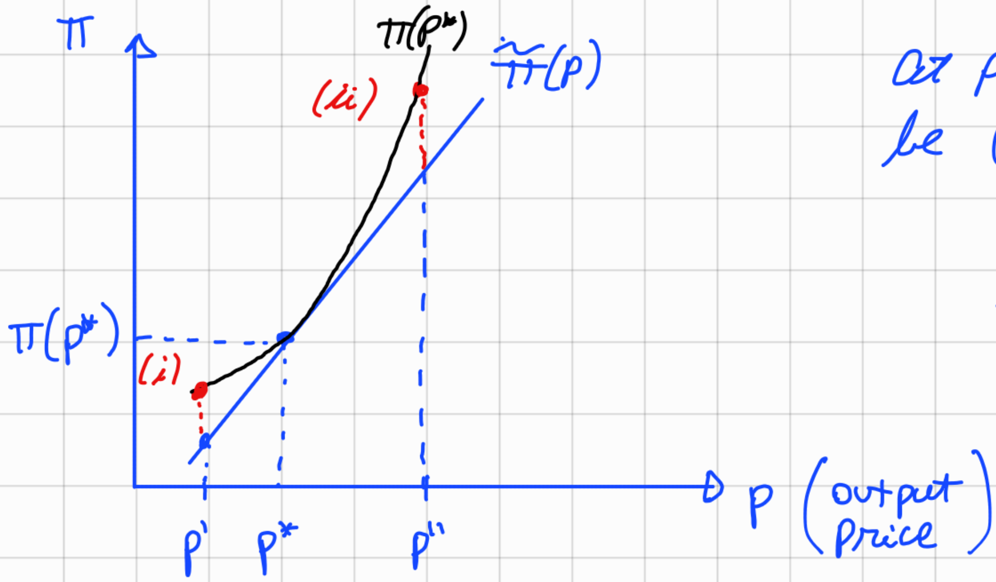
<u>Proof</u>: $\pi(tp) = \max\limits_{y \in Y} (tp)\cdot y$

$$= t\cdot \left[\max_{y\in Y} p\cdot y \right] = t\cdot \pi(p)$$

③ $\pi(p)$ is convex in p, i.e.,

$\pi(\alpha p + (1-\alpha)p') \leq \alpha\pi(p) + (1-\alpha)\pi(p')$, $\forall \alpha \in [0,1]$.

Graphical intuition: [Do the proof...].



at p^*, let optimal production be $(-x^*, y^*)$.

$\pi(p^*) = p^* y^* - w^* x^*$

$\tilde{\pi}(p) = p\, y^* - w^* x^*$

(i) and (ii) are reoptimized decisions of the firm, that will be above the first optimized point.

If we do assume π is differentiable

$$H(p) = D^2\pi(p) = \begin{bmatrix} \partial^2\pi/\partial p_1^2 & \cdots & \partial^2\pi/\partial p_1\partial p_n \\ \vdots & \ddots & \vdots \\ \partial^2\pi/\partial p_n\partial p_1 & \cdots & \partial^2\pi/\partial p_n^2 \end{bmatrix}$$

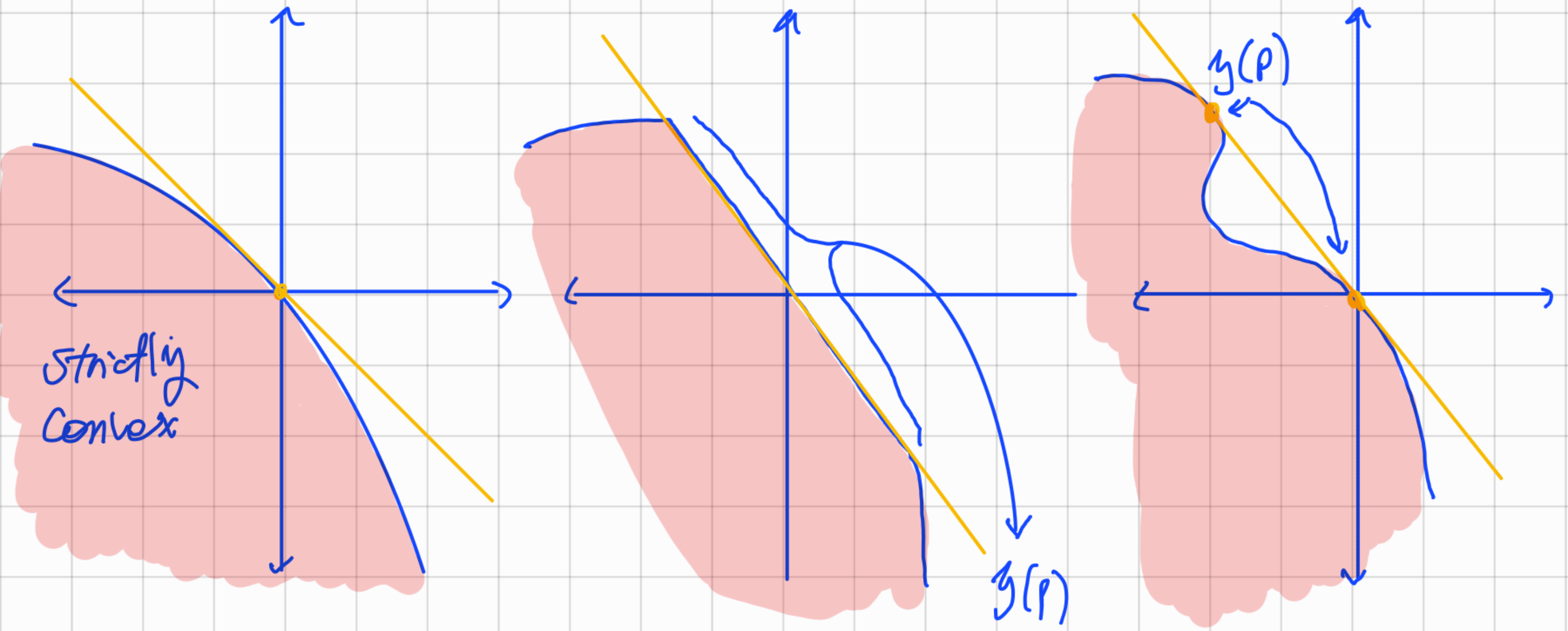
④ positive semidefinite.

Properties of $y(p)$

1. $y(p)$ is homogeneous of degree 0. (change in prices don't affect decisions)

Proof: $\max_{y \in Y} (\alpha p)\cdot y = \max_{y \in Y} p\cdot y$ $\quad y(\alpha p) = y(p)$

2. a) If Y is convex, then $y(p)$ is in convex set
   b) If Y is strictly convex then $y(p)$ is a singleton



Detour: Envelope theorem.

Q: How does the optimal value of a maximization problem change as a parameter changes?

$\max_x f(x,\theta)$

x: choice variable
θ: parameter

Define the value function

$$V(\theta) = \max_x f(x,\theta)$$

$$V(\theta) = f(x^*(\theta), \theta)$$

A change in θ has 2 effects:

- Direct effect: θ changes f directly through and argument
- Indirect effect: θ changes x^*, which changes f.

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x^*(\theta)}}_{0} + \frac{\partial f}{\partial \theta}\Big|_{x=x^*(\theta)}$$

$$\Rightarrow \boxed{\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta}\Big|_{x=x^*(\theta)}}$$

Found all over economics:

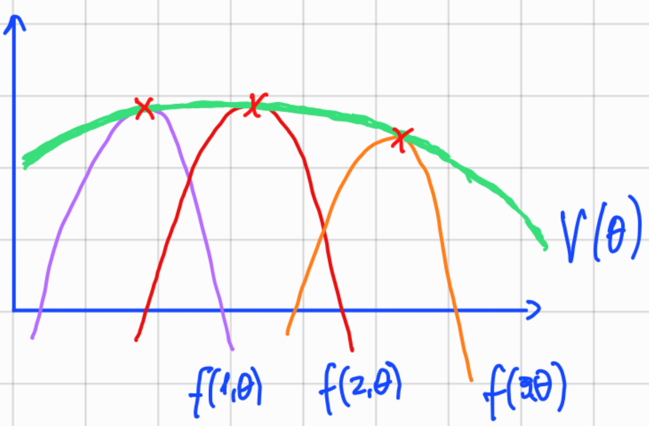
- Hotelling' Lemma
- Shepard's Lemma
- Myerson's Lemma
- Benveniste-Scheinkman formula
- ...

When there are constraints

$$\max_x f(x,\theta) \quad \text{s.t} \quad g_k(x,\theta) \leq c_k, \quad k=1,\dots,K.$$

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} - \lambda_1 \frac{\partial g}{\partial \theta} - \dots - \lambda_k \frac{\partial g}{\partial \theta}$$

$$\frac{dV}{d\theta} = \frac{\partial \mathcal{L}}{\partial \theta}\Big|_{x=x^*(\theta)}$$



Hotelling's Lemma: If $\pi(p)$ is differentiable at $\bar{p}$, and $\bar{p}_i > 0$, then
$$\frac{\partial \pi}{\partial p_i} = y_i^*(\bar{p})$$

Proof: For single output case

$$\pi(p, w) = \max_{x \geq 0} \underbrace{p \cdot f(x) - w \cdot x}_{h(x)}$$

By envelope theorem,

$$\frac{d\pi}{dp} = \left.\frac{\partial h}{\partial p}\right|_{x = x^*(p,w)} = f(x^*(p,w)) = y^*(p,w)$$

$$\frac{d\pi}{dw_i} = \left.\frac{\partial h}{\partial w_i}\right|_{x = x^*(p,w)} = -x_i^*(p,w)$$

Example: $f(x_1, x_2) = 30p\, x_1^{2/5} x_2^{2/5}$

$$\max\ 30p x_1^{2/5} x_2^{2/5} - w_1 x_1 - w_2 x_2$$

f.o.c.:
$$\left.\begin{aligned} 12p\, x_1^{-3/5} x_2^{2/5} &= w_1 \\ 12p\, x_1^{2/5} x_2^{-3/5} &= w_2 \end{aligned}\right\} \Rightarrow \boxed{\boxed{x_1^* = \frac{12^5 p^5}{w_1^3 w_2^2}};\ x_2^* = \frac{12^5 p^5}{w_1^2 w_2^3}}$$

$$\pi(p, w) = 30\, p\, \frac{12^4 p^4}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} = \frac{p}{2}\left[\frac{12^5 p^5}{w_1^2 w_2^2}\right]$$

$$\boxed{\frac{\partial \pi}{\partial w_1} = -\frac{2}{2}\left[\frac{12^5 p^5}{w_1^3 w_2^2}\right]}$$

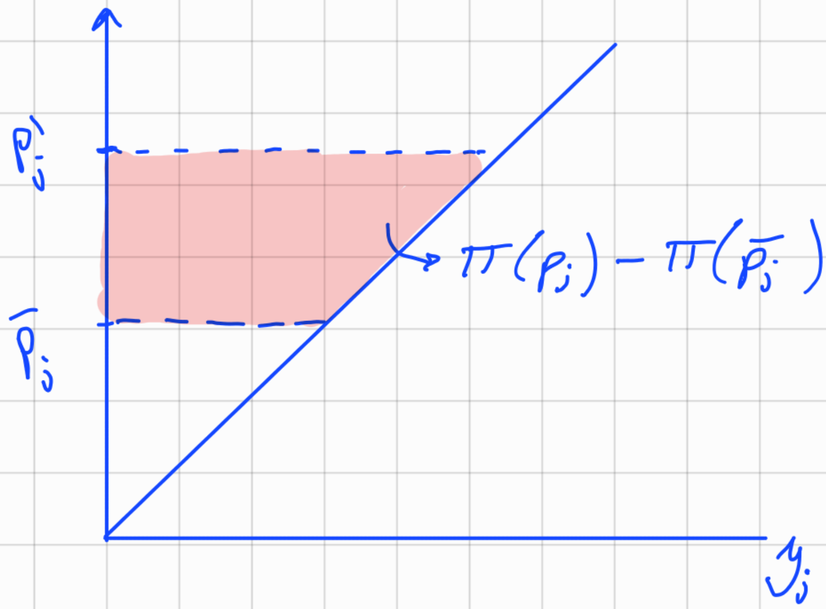
Integral form for Hotelling's Lemma

$-j$: everything else.

Hold all prices other than p_j fixed at $\bar{p}_{-j}$.
Then

$$\pi(p_j, \bar{p}_{-j}) - \pi(\bar{p}_j, \bar{p}_{-j}) = \int_{\bar{p}_j}^{p_j} y_j(s, \bar{p}_{-j})\, ds$$

Proof: $$\int_{\bar{p}_j}^{p_j} y_j(s, \bar{p}_{-j})\, ds = \int_{\bar{p}_j}^{p_j} \frac{d\pi(s, \bar{p}_{-j})}{dp_j}\, ds = \underbrace{\pi(p_j, \bar{p}_{-j}) - \pi(\bar{p}_j, \bar{p}_{-j})}_{\text{Surplus.}}$$



Lecture

Sept 20, 2023

Hotelling's Lemma

$$\frac{\partial \pi}{\partial p_i} = y_i(p) \longrightarrow \frac{\partial y_i(p)}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_i \partial p_j}$$

$D y(p) =$ Jacobian or substitution matrix.

$$D y(p) = \begin{bmatrix} \partial y_1/\partial p_1 & \cdots & \partial y_n/\partial p_1 \\ \vdots & \ddots & \vdots \\ \partial y_1/\partial p_n & \cdots & \partial y_n/\partial p_n \end{bmatrix} = \begin{bmatrix} \partial^2 \pi/\partial p_1^2 & \cdots & \frac{\partial^2 \pi}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \partial^2 \pi/\partial p_n \partial p_1 & \cdots & \partial^2 \pi/\partial p_n^2 \end{bmatrix}$$

π is convex, pos semidef

This implies:

① $\frac{\partial y_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2} \geq 0$ (Law of supply)

② $\frac{\partial y_i}{\partial p_j} = \underbrace{\frac{\partial^2 \pi}{\partial p_j \partial p_i} = \frac{\partial^2 \pi}{\partial p_j \partial p_i}}_{\text{Young's Law}} = \frac{\partial y_j}{\partial p_i}$

Lechatelier Principle: "Firms should respond more to price changes in the long-Run than in the short-Run."

Suppose $(y^*, x^*) = (y(p^*, w^*), x(p^*, w^*))$ at price (p^*, w^*)

Fix one input z_i^* in the short run, variable in the long-Run.
z_i^* is optimal at (p^*, w^*)

Q: How does x_j depend on w_j in the SR vs L-R?

Proof: $\longrightarrow$

$$\pi(p,w) = \max \; p \cdot f(x) - w \cdot x \quad ; \quad \pi^s(p,w) = \max \; p \cdot f(x) - w \cdot x$$
$$\text{s.t } x_j \geq 0 \qquad\qquad \text{s.t } x_j \geq 0, \; x_i = z_i^*$$

Compare

$$\left.\frac{\partial x_j(p,w)}{\partial w_j}\right|_{(p,w)=(p^*,w^*)} \quad \text{vs} \quad \left.\frac{\partial x_j^s(p,w)}{\partial w_j}\right|_{(p,w)=(p^*,w^*)}$$

$$h(p,w) = \pi(p,w) - \underbrace{\pi^s(p,w)}$$

this is lower because it is a more restricted problem

By definition, $h(p,w) \geq 0$ and $h(p^*,w^*) = 0$, (p^*,w^*) is a local minimum.

$$\frac{\partial^2 h(p^*,w^*)}{\partial w_j^2} \geq 0$$

$$\|$$

$$\frac{\partial^2 \pi(p^*,w^*)}{\partial w_j^2} - \frac{\partial^2 \pi^s(p^*,w^*)}{\partial w_j^2} \geq 0$$

Hotelling : $\dfrac{\partial \pi(p,w)}{\partial w_j} = -x_j(p,w)$

$$-\frac{\partial x_j(p^*,w^*)}{\partial w_j} + \frac{\partial x_j^s(p^*,w^*)}{\partial w_j} \geq 0$$

$$\left|\frac{\partial x_j(p^*,w^*)}{\partial w_j}\right| \geq \left|\frac{\partial x_j^s(p^*,w^*)}{\partial w_j}\right|$$

because $\dfrac{\partial x_j}{\partial w_j}$ are negative values $\left(\dfrac{\Delta \text{Demand}}{\Delta \text{price input}}\right)$

Cost minimization:

- Single output good
- pricing power in output markets, but not input markets
- Previous analysis doesn't work, but we can:

① Find the cheapest way to make any target output y (cost minimization problem).

② Use this "cost function" $c(y)$ to choose optimal p/y combination.

CMP:

- Fix a target level of output, y.
- Cost: $\underbrace{C(w, y)}_{\text{parameters.}} = \min w \cdot x$ s.t $x_i \geq 0$, $f(x) \geq y$

*Ignore this for now...

$$\mathcal{L} = wx - \lambda(f(x) - y)$$

F.O.C: $w_i - \lambda \dfrac{\partial f}{\partial x_i} = 0 \quad \forall i$

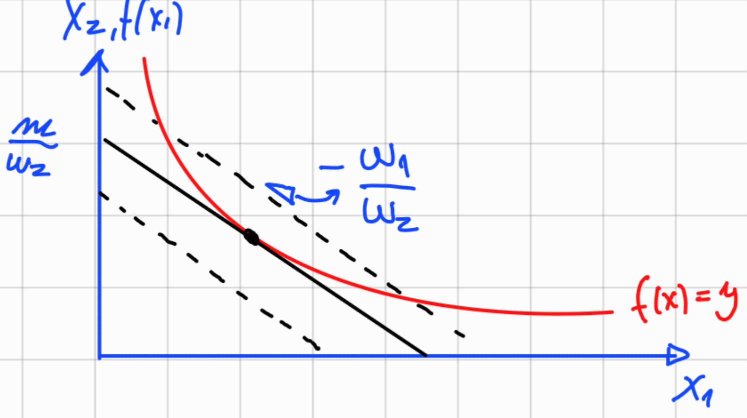
Divide i by j:

$$\underbrace{\frac{w_i}{w_j}}_{\text{Economic Rate of Substitution}} = \underbrace{\frac{\partial f / \partial x_i}{\partial f / \partial x_j}}_{\text{MRTS}}, \text{ for } (x_i^*, x_j^* > 0)$$

Constraint: Isoquant

objective: Isocost $w_1 x_1 + w_2 x_2 = m$

$$x_2 = \frac{m}{w_2} - \frac{w_1 x_1}{w_2}$$



$x^*(w,y)$ conditional factor demand correspondence.

$$c(w,y) = w \cdot x^*(w,y) = \min w \cdot x \quad , \quad \begin{array}{l} x_i \geq 0 \\ f(x) \geq y. \end{array}$$

Apply Envelope theorem.

$$\boxed{\frac{\partial c}{\partial y} = \frac{\partial \mathcal{L}}{\partial y} = \lambda}$$

→ λ is the mg cost of increasing production by 1 unit

Relationship with PMP

$$\max_{y \geq 0} \; p \cdot y - c(w,y)$$

F.O.C) $p = \frac{\partial c}{\partial y}$ $(= \lambda$ from CMP$)$

Things to keep in mind:

- We have assumed, differentiable f, interior solutions
- General F.O.C: $\lambda \cdot \frac{\partial f(x^*)}{\partial x_i} - w_i \leq 0 \quad \forall i$

  with equality if $x_i^* > 0$
- KT conditions are necessary, but not sufficient in general
- will be sufficient if $f(\cdot)$ is concave
- Existence / uniqueness issues

Weak Axiom of Cost minimization (WACM)

$$\{(w^1, x^1, y^1), \dots, (w^T, x^T, y^T)\}$$

WACM: $\underbrace{w^t x^t}_{\text{cost}} \leq \underbrace{w^t x^s}_{\text{other choices are more expensive.}} \quad \forall \; y^s \geq y^t$

Any firm that violates WACM is not Rational.

Implication of WACM: Downward sloping Demand

→ Take $y^s = y^t$ WACM gives 2 inequalities

1. $w^t x^t \leq w^t x^s$
2. $w^s \cdot x^s \leq w^s x^t$

1 + 2 : $w^t \cdot x^t + w^s x^s \leq w^t x^s + w^s x^t$

$$(w^t - w^s)(x^t - x^s) \leq 0$$

$$\Delta w \cdot \Delta x \leq 0$$

$$C(w, y) = \min_x x \cdot y \quad \text{s.t } f(x) \geq y$$

1. $C(w, y)$ is nondecreasing in w. Proof:

Take $w' \geq w$ and $x \in x^*(w, y)$, $x' \in x^*(w', y)$

$$c(w, y) = w \cdot x \underset{\text{WACM}}{\leq} w \cdot x' \underset{w' \geq w}{\leq} w' \cdot x' = c(w', y)$$

$$C(w, y) \leq C(w', y)$$

② $C(w,y)$ is nondecreasing in y.

Let $y'' \geq y'$, $x' \in x^*(w,y')$, $x'' \in x^*(w,y'')$

$$C(w,y') = \min_{f(x)\geq y'} w\cdot x = w\cdot x' \underset{\text{WACM}}{\leq} w\cdot x'' = \min_{f(x)\geq y''} w\cdot x = C(w,y'')$$

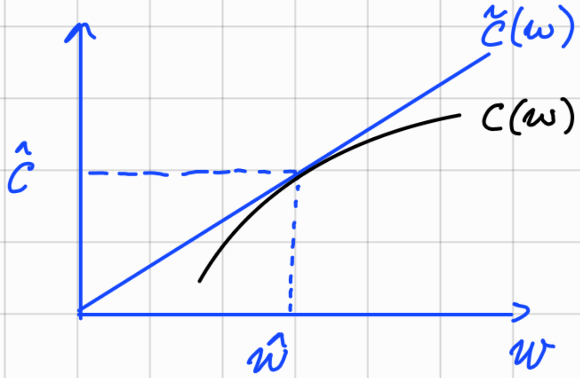
Inequality works because $f(x'') \geq y'' \geq y'$

③ $C(w,y)$ is hd-1 in w

$$C(tw,y) = t\,C(w,y)$$

④ $C(w,y)$ is concave in w (NOT TRUE for y)

(Same argument as in PMP problem)



⑤ If $V(y)$ is convex, then $x^*(w,y)$ is convex-valued.
If $V(y)$ is strictly convex, then $x^*(w,y)$ is single valued

⑥ $x^*(w,y)$ is hd-0 in w: $x^*(t\cdot w,y) = x^*(w,y)$

→ Implications:

$$x_i^*(tw,y) = x_i^*(w,y)$$

differentiate w.r.t. t, eval at $t=1$

$$\nabla w \cdot x_i^*(tw,y)\cdot w = 0$$

$$\sum_{j=1}^{\infty} \frac{\partial x_i^*(tw,y)}{\partial w_j}\cdot w_j = 0$$

We know $\frac{\partial x_i^*}{\partial w_j} \leq 0$

Say $\boxed{n=2}$ $\quad \frac{\partial x_i^*}{\partial w_i}\cdot w_i + \frac{\partial x_1^*}{\partial w_2}\cdot w_2 = 0$

$\frac{\partial x_1^*}{\partial w_2} \geq 0$: goods are substitutes

If $n>2$, then $\frac{\partial x_i^*}{\partial w_j} \geq 0$ for some good j

<u>Shepard's Lemma</u>: If $c(w,y)$ is C^1, and $\bar{w}>0$, then

$$x_i(\bar{w},y) = \frac{\partial c(\bar{w},y)}{\partial w_i}$$

<u>Proof 1</u>: Envelope Theorem.

$$\mathcal{L} = w\cdot x - \lambda\,(f(x)-y)$$

$$\frac{\partial c}{\partial w_j} = \frac{\partial \mathcal{L}}{\partial w_i}\Big|_{x=x^*(w,y)} \longrightarrow \boxed{\frac{\partial c}{\partial w_i} = x_i^*(w,y)}$$

Proof 2: $C(w,y) = w \cdot x^*(w,y)$

$$\frac{\partial C}{\partial w_i} = x_i^*(w,y) + \sum^n w_j \cdot \frac{\partial x_i^*(w,y)}{\partial w_i} \quad (*)$$

Recall the f.o.c of CMP

① $\lambda \dfrac{\partial f}{\partial x_i} = w_i$ ② $f(x^*(w,y)) = y$

Differentiate ② wrt w_i:

$$\sum_{j=1}^n \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j^*}{\partial w_i} = 0$$

plug (1) into (*)

$$\frac{\partial C}{\partial w_i} = x_i^*(w,y) + \overbrace{\sum^n \lambda \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j^*}{\partial w_i}}^{0}$$

$$\boxed{\frac{\partial C}{\partial w_i} = x_i^*(w,y)}$$

Substitution matrix.

$$\begin{bmatrix} \partial x_1^*/\partial w_1 & \cdots & \partial x_1^*/\partial w_n \\ \vdots & \ddots & \vdots \\ \partial x_n^*/\partial w_1 & \cdots & \partial x_n^*/\partial w_n \end{bmatrix} = \begin{bmatrix} \partial^2 C/\partial w_1^2 & \cdots & \partial^2 C/\partial w_n \partial w_1 \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 C}{\partial w_1 \partial w_n} & \cdots & \dfrac{\partial^2 C}{\partial w_n^2} \end{bmatrix}$$

Shepard's Lemma

$\partial^2 C$, which is concave.
(Hessian of C)

- Law of demand: $\frac{\partial x_i^*}{\partial w_i} \leq 0$, $\forall i$

- Symmetric cross-price effect: $\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$

<u>Theorem</u>: If f is constant returns to scale, then
$$c(w,y) = y \cdot \underbrace{c(w,y)}_{\text{unit cost function}}$$

<u>Proof</u>: Assume f C^1, int solution.
F.O.C at $y=1$:

$$\frac{w_i}{w_j} = \frac{f_i(x^*)}{f_j(x^*)} \quad \text{and} \quad f(x^*) = 1$$

For general $\hat{y}$: $\frac{w_i}{w_j} = \frac{f_i(\hat{x})}{f_j(\hat{x})}$, $f(\hat{x}) = y$

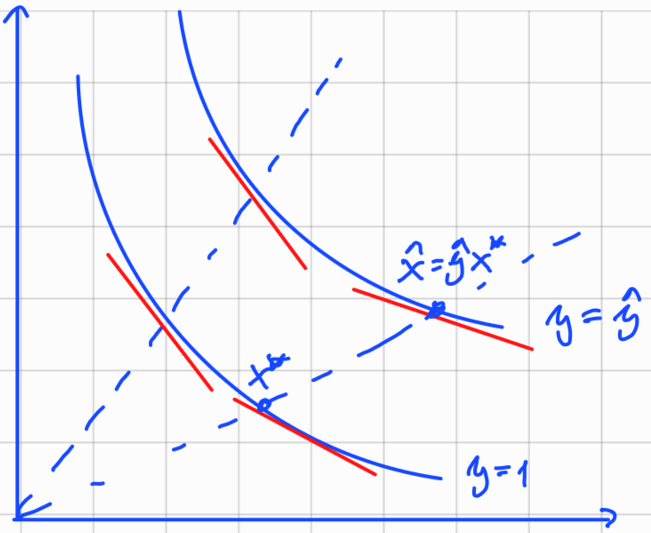
Consider input $\hat{x} = \hat{y}x^*$
claim: $\hat{x}$ solves the f.o.c. at $\hat{y}$

$$\frac{f_i(\hat{x})}{f_j(\hat{x})} = \frac{f_i(\hat{y}, x^*)}{f_j(\hat{y}, x^*)} \underset{\text{C.R.S.}}{=} \frac{f_i(x^*)}{f_j(x^*)} = \frac{w_i}{w_j}$$

$$f(\hat{y}x^*) \underset{\text{by C.R.S.}}{=} \hat{y} f(x^*) = \hat{y}$$

$\Rightarrow$ so $\hat{x} = \hat{y}x^*$ solves the problem at target output $\hat{y}$

$$c(w,\hat{y}) = w \cdot (\hat{y} \cdot x^*) = \hat{y} \cdot (w \cdot x^*)$$
$$= \hat{y} \cdot c(w,1)$$



Cost function : $C(w,y) = \min \; w \cdot x \quad s.t \quad f(x) \geq y$ Sept 25, 2023

for now, just $C(y)$

Average cost : $AC(y) = \dfrac{C(y)}{y}$

Mg. cost : $MC(y) = C'(y)$

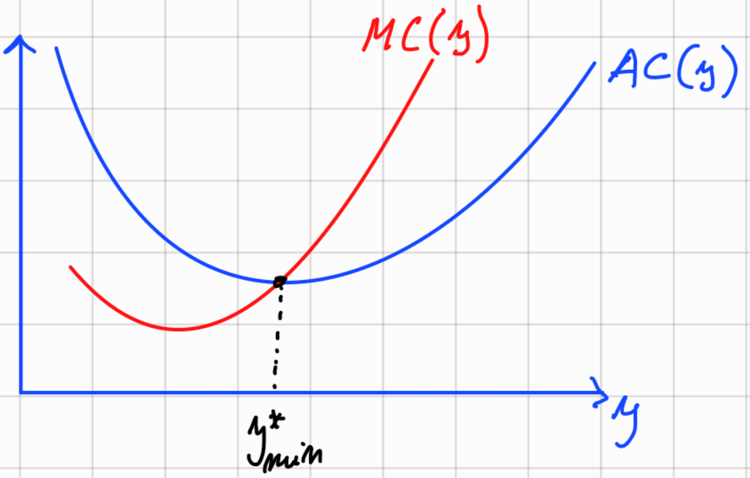
<u>Marginal - average relationship</u> :

$$AC'(y) = \frac{C'(y)}{y} - \frac{C(y)}{y^2} = \frac{C'(y) \cdot y - C(y)}{y^2} \quad , \quad y^2 > 0$$

AC incr $\longleftrightarrow$ $C'(y) > C(y)/y$, or $MC(y) > AC(y)$

AC decr $\longleftrightarrow$ $C'(y) < C(y)/y$, or $MC(y) < AC(y)$

$AC'(y) = 0 \longleftrightarrow MC(y) = AC(y)$



y^*_{min} : Minimum efficient Scale (MES)

$$\max_{y \geq 0} \; py - c(y)$$

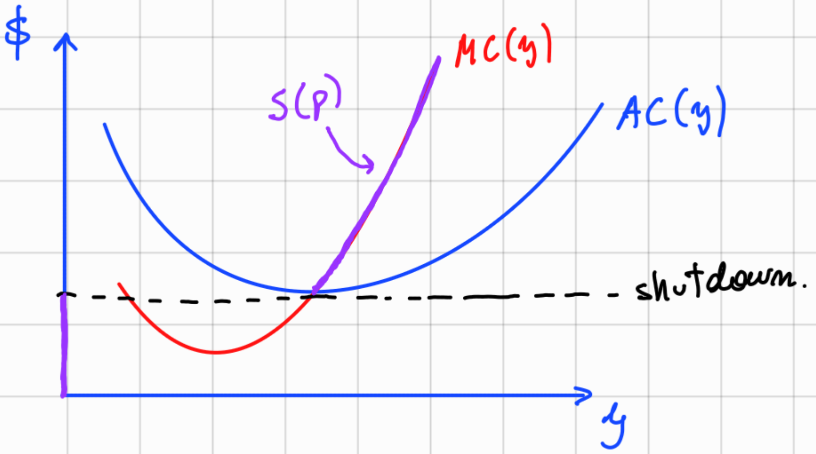
FONC $\quad p \leq c'(y^*)$, with equality if $y^* > 0$

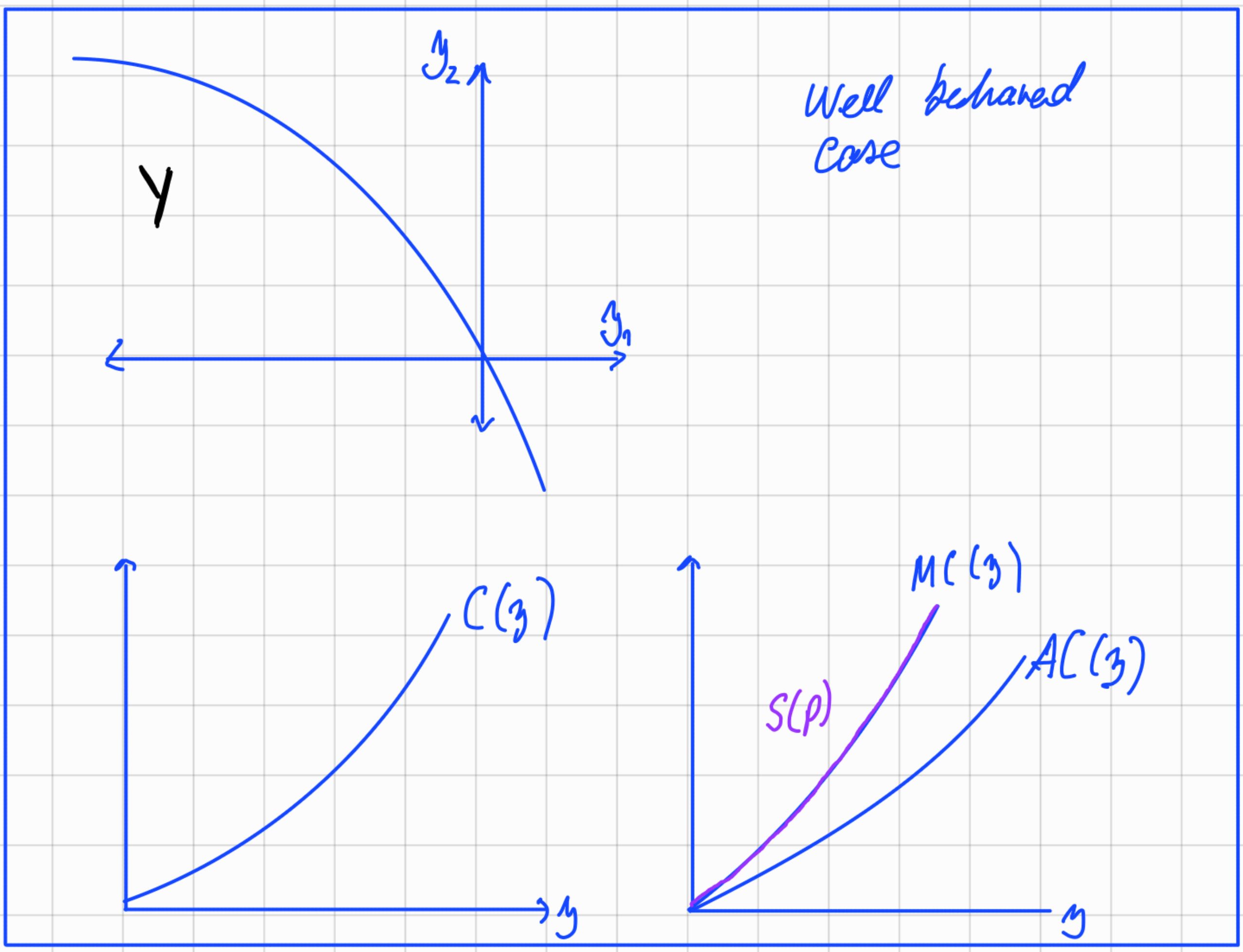
If $c(y)$ is convex, this is also sufficient

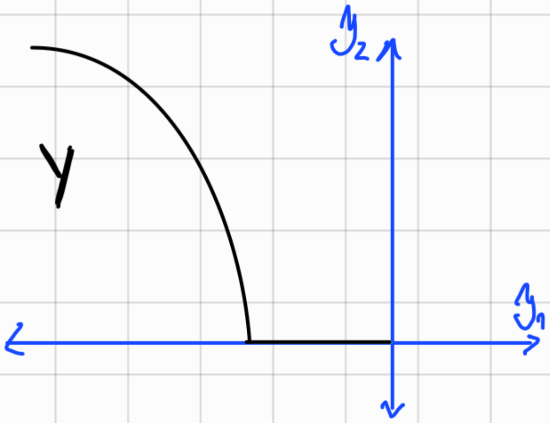
Price = mc if firm chooses to produce a strictly positive amount

Shutdown Rule: Check that at y^* where $p = c'(y^*)$,

$py^* - c(y^*) \geq 0$ or $p \geq AC(y^*)$

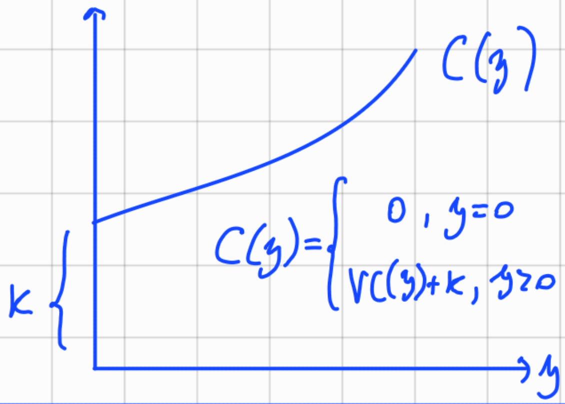


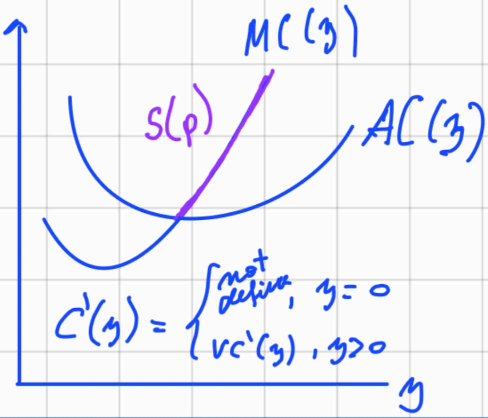




Fixed cost case

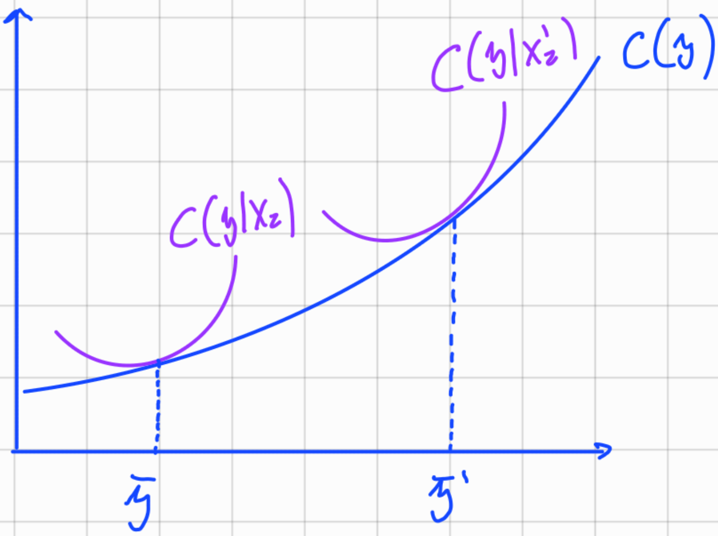
* VC: Variable cost





Long-Run vs Short-Run costs

- 2 inputs (x_1, x_2)
- $C(y)$ be the LR cost function
- Let $\bar{x}_2$ be optimal x_2 when long-run output $\bar{y}$
- $C(y|\bar{x}_2)$. short run cost, when x_2 fixed at $\bar{x}_2$.



Can we go from $c(w,y) \to f(x)$?
- Yes, they are "duals" of each other.

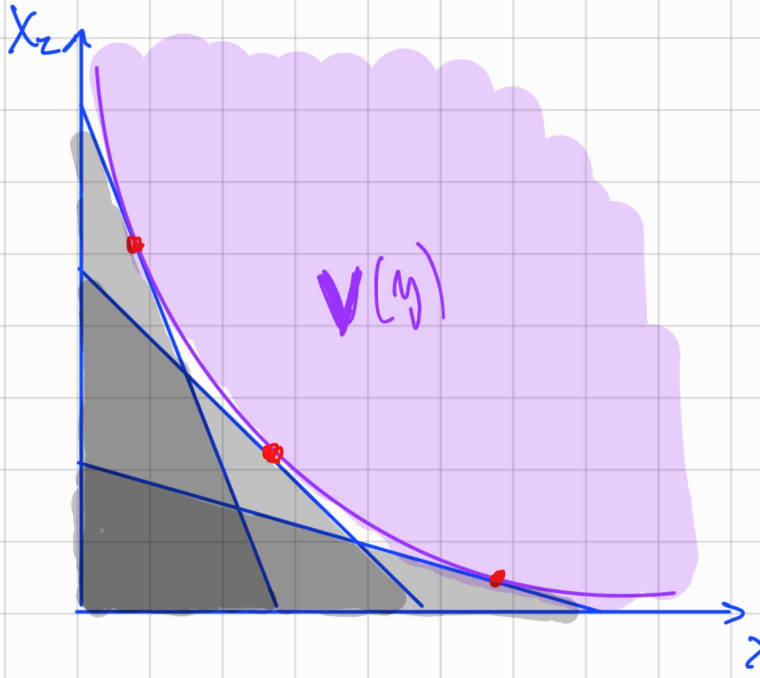
Observe $c(w,y)$ $\forall (w,y)$

$V(y) = \{x : f(x) \geq y\}$

If we know $V(y)$ $\forall y$, then we know $f(x)$.

Can we recover $V(y)$? Define an "outer bound"

$V^o(y) = \{x : w \cdot x \geq c(w,y),\ \forall w \geq 0\}$ at $\bar{w}$, cost is $\bar{c} = c(\bar{w},y)$



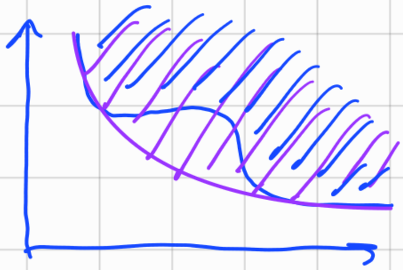
firm chose $\bar{x}$

$\bar{w}_1 x_1 + \bar{w}_2 x_2 = c$

Black Shaded Area: Ruled-out points because of Revealed preferences.

Claim: Suppose $V(y)$ is the input requirement set for a firm with a convex technology with free disposal. Then, $V(y) = V^o(y)$

Proof: Homework (use sep. hyperplane thm).



Presupposes the function we start with is a valid cost function

Question: Given an arbitrary function $\phi(w,y)$, how do we know if ϕ is a valid cost function? that is, does

$$\phi(w,y) = \min wX \quad \text{s.t } x \in V(y)$$

where $V(y) = \{X \geq 0 : w \cdot x \geq \phi(w,y) \; \forall w\}$.

Showed before that if ϕ is a cost function, then:

i) hd-1 in prices: $\phi(tw,y) = t\phi(w,y)$

ii) $\phi(w,y) \geq 0$ (positivity)

iii) $\phi(w',y) \geq \phi(w,y)$, $w' \geq w$ (monotonic)

iv) Concave in w

Theorem: Let $\phi(w,y)$ be a C^2 function satisfying i)-iv), then, $\phi(w,y)$ is the cost function for the technology

$$V(y) = \{x : w \cdot x \geq \phi(w,y), \forall w\}$$

proof: Define

$$x(w,y) = \left(\frac{\partial \phi(w,y)}{\partial w_1}, \ldots, \frac{\partial \phi(w,y)}{\partial w_n} \right)$$

By monotonicity, $x(w,y) \geq 0$

By hd-1, $\phi(w,y) = \sum_{i=1}^{n} w_i \frac{\partial \phi(w,y)}{\partial w_i} = w \cdot x(w,y)$

Euler's Law

we want to show for any $w \geq 0$

$$\phi(w,y) = w \cdot x(w,y) \leq wx \quad \forall x \in V(y)$$

this will imply $\phi(w,y) = \min \; w \cdot x \;$ s.t $x \in V(y)$

To show $x(w,y)$ is feasible: By concavity,

$$\phi(w',y) \leq \phi(w,y) + \underbrace{\nabla \phi(w,y)}_{x(w,y)}(w'-w) \quad \text{for any } w'$$

$$\leq \phi(w,y) + w' \cdot x(w,y) - \underbrace{w \cdot x(w,y)}_{\phi(w,y)}$$

$$\phi(w',y) \leq w' x(w,y), \quad \forall w'$$

$$\Rightarrow x(w,y) \in V(y) \quad (\text{feasible})$$

Now notice that for any $x \in V(y)$, $w \cdot x \geq \phi(w,y) = w \cdot x(w,y)$

or $w \cdot x(w,y) \leq w \cdot x \quad \forall x \in V(y)$

i.e, $x(w,y) = \underset{x \in V(y)}{\operatorname{argmin}} \; w \cdot x$

$$\phi(w,y) = w \cdot x(w,y)$$

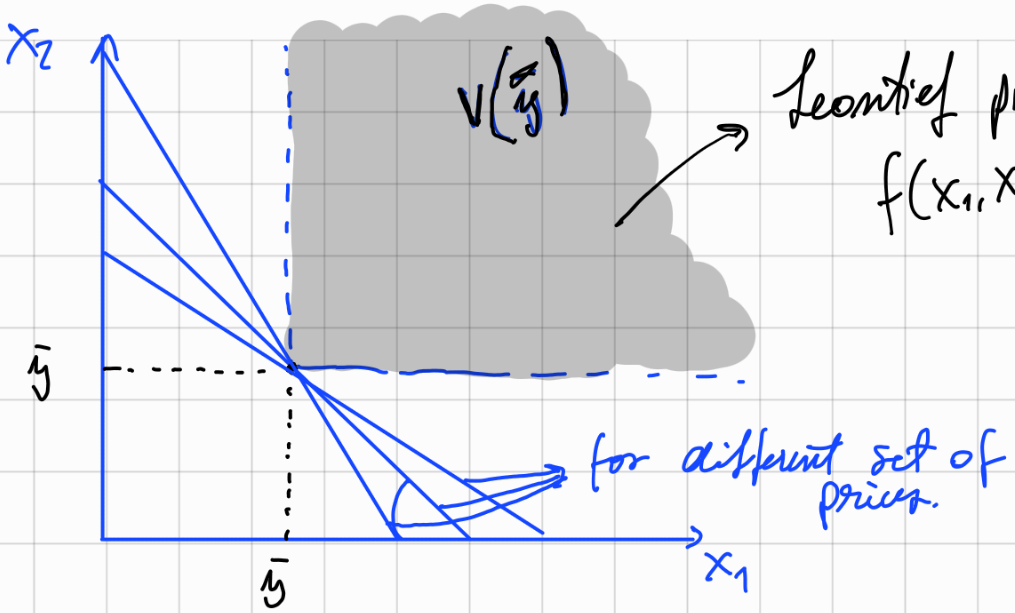
Example: $c(w,y) = y \cdot (w_1 + w_2)$

Fix an output level $\bar{y}$. Graphically, we want (x_1, x_2) s.t

$wx \geq c(w, \bar{y}) \quad \forall (w_1, w_2)$

$w_1 x_1 + w_2 x_2 \geq w_1 \bar{Y}_1 + w_2 \bar{Y}_2$

$\rightarrow x_2 \geq \frac{w_1 \bar{Y}_1}{w_2} + \bar{Y}_1 - \frac{w_1 x_1}{w_2}$



Leontief production function
$f(x_1, x_2) = \min\{x_1, x_2\}$

Algebraically

$\frac{\partial C}{\partial w_1} = x_1^*$; $\frac{\partial C}{\partial w_2} = x_2^*$ (Shepard's Lemma)

Here $\bar{y} = x_1^*(w, y)$; $\bar{y} = x_2^*(w, y)$

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$c(w,y) = (w_1 + w_2)y$ $\leftrightarrow$ $f(x_1,x_2) = \min\{x_1, x_2\}$

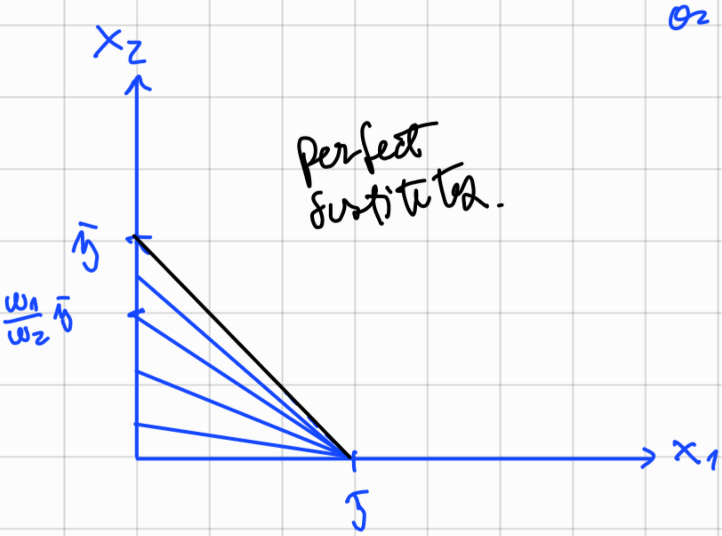
$c(w,y) = y \cdot \min\{w_1, w_2\}$ $\leftrightarrow$ $f(x_1,x_2) = x_1 + x_2$

3 cases:

① $w_1 < w_2$; $c(w,\bar{y})\ \bar{y}w_1$

we want the points (x_1,x_2) s.t. $w_1x_1 + w_2x_2 \geq \bar{y}w_1$

or $x_2 \geq \frac{w_1}{w_2}(\bar{y} - x_1)$ $\forall\ \frac{w_1}{w_2} \in (0,1)$



② $w_1 = w_2 = w$; $c(w,y) = \bar{y}w$

$w(x_1 + x_2) \geq \bar{y}w$

$x_2 \geq \bar{y} - x_1$

③ $w_1 > w_2$

Same as (1).

Algebraically:

$w_1 < w_2$

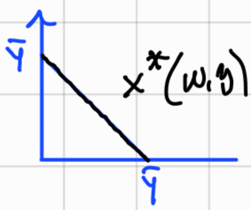
by Shepard's Lemma

$x_1(w,y) = \frac{\partial c}{\partial w_1} = \bar{y}$; $x_2(w,y) = \frac{\partial c}{\partial w_2} = 0$

$w_1 > w_2$

$x_1(w,y) = \partial c/\partial w_1 = 0$; $x_2(w,y) = \partial c/\partial w_2 = \bar{y}$

at $w_1 = w_2$, Shepard's lemma doesn't apply



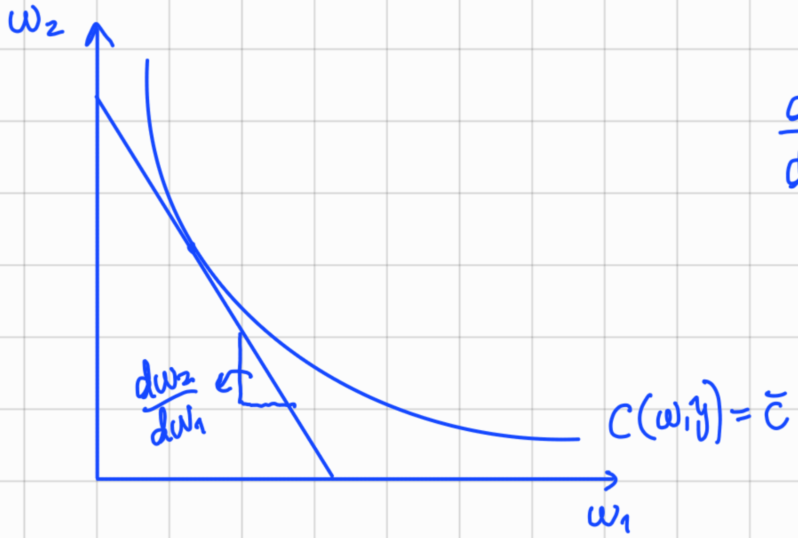
In general, take an isocost curve

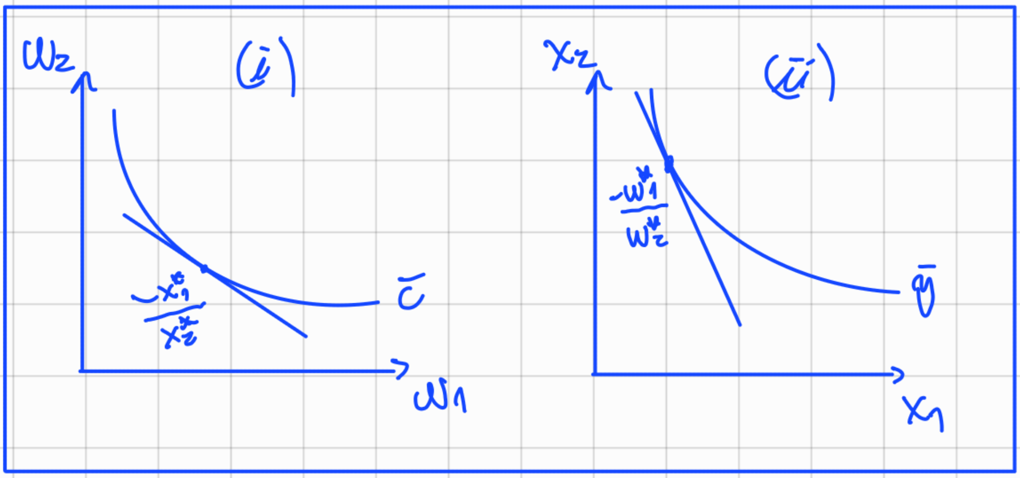
$c(w,y) = \bar{c}$

$$0 = d\bar{c} = \frac{\partial c}{\partial w_1} dw_1 + \frac{\partial c}{\partial w_2} \cdot dw_2$$

$$\frac{dw_2}{dw_1} = -\frac{\partial c / \partial w_1}{\partial c / \partial w_2} \overset{☆}{=} -\frac{x_1^*(w,y)}{x_2^*(w,y)}$$

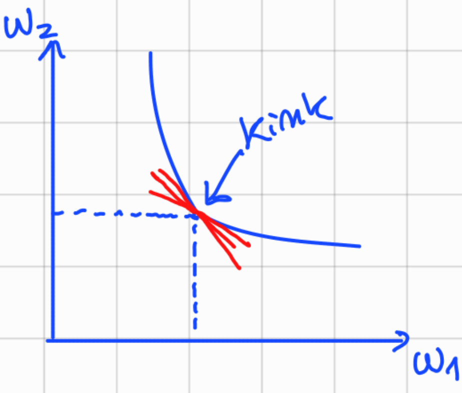
☆ Shepard Lemma

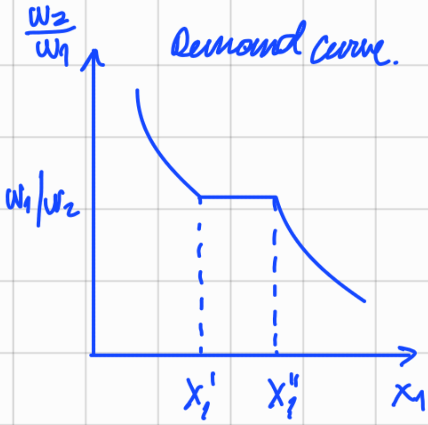




Duality between (i) & (ii)

Isoquant with a flat spot

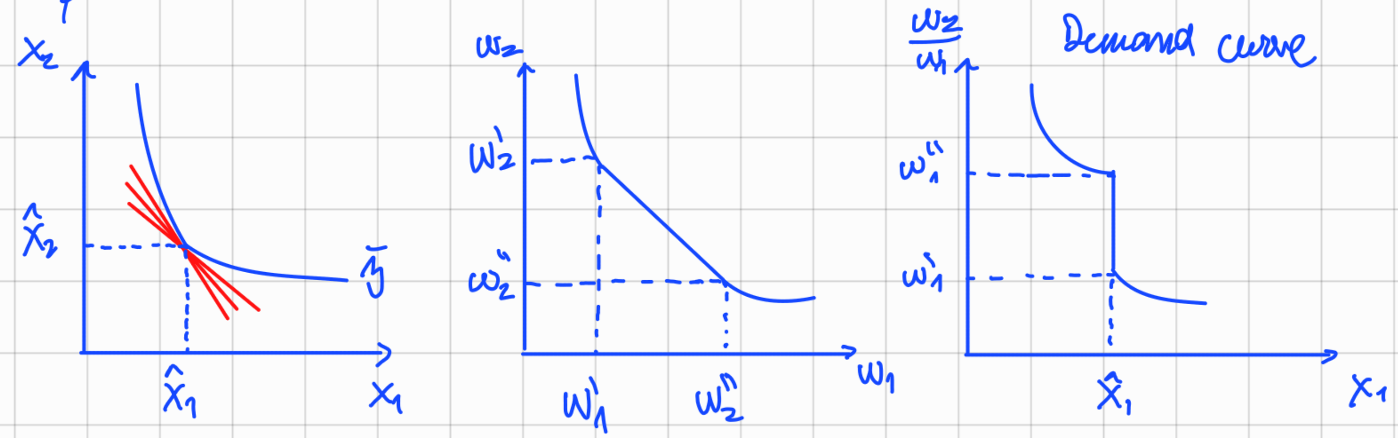




multiply optimal bundles

multiple tangents at the optimal.

Isoquant with a kink



Isoquant and isocost slopes are inversely related

Aggregation: J firms, production sets $Y_1, \dots, Y_J$ (non empty closed, FD).

$\pi_j(p)$, $y_j(p)$ solution functions for firm j

$$y(p) = \sum_{j=1}^{\infty} y_j(p) = \{ y \in \mathbb{R}^n : y = \sum_{j=1}^{J} y_j', \text{ for some } y_j' \in y_j(p) \ \forall j \}$$

Recall for a single j, we had the law of supply

$$(p-p')\cdot(y_j(p) - y_j(p')) \geq 0$$

Add over j:
$$\sum_{j=1}^{J} (p-p')(y_j(p) - y_j(p')) = (p-p')\cdot \sum_{j=1}^{J} (y_j(p) - y_j(p'))$$
$$= \underbrace{(p-p')(y(p) - y(p')) \geq 0}_{\text{Agg. Law of Supply}}$$

Agg substitution matrix, $Dy(p)$, also inherits properties such as symmetry, PSD.

Strengthen this to a "Representative firm" result.

Agg. production set:

$$Y^* = Y_1 + \dots + Y_J = \{ y \in \mathbb{R}^n : y = \sum_{j=1}^{J} y_j \text{ , for some } y_j \in Y_J \ \forall j \}$$

a single firm with production set Y^*. Let $\pi^*(p)$, $y^*(p)$ be solution functions for a firm, Y^*.

<u>Theorem</u>: i) $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$ (This result does not hold for consumers)

ii) $y^*(p) = \sum_{j=1}^{J} y_j(p)$

proof of ii):
want to show

① $\sum_j y_j(p) \subseteq y^*(p)$

② $y^*(p) \subseteq \sum_j y_j(p)$

For ①: Let $y_j' \in y_j(p) \ \forall j$

$$p \cdot \underbrace{\left(\sum_j y_j'\right)}_{\text{so } \bar{y} \in y^*(p)} = \sum_j \underbrace{p \cdot y_j'}_{\pi_j(p)} = \sum_j \pi_j(p) \underset{\text{By (i)}}{=} \pi^*(p)$$

for ②: $y \subseteq \sum_j y_j(p)$

Take $y \in y^*(p)$ By construction, $y = \sum_j y_j'$ for some selection $y_j' \in Y_j$

$$p \cdot y = p \cdot \left(\sum_j y_j\right) = \pi^*(p) \underset{\text{By (i)}}{=} \sum \pi_j(p)$$

$$\boxed{\sum_j p \cdot y_j = \sum_j \pi_j(p)} \quad (☆)$$

Note that $p_j \cdot y_j^* \leq \pi_j(p) \quad \forall j$

$$\sum p_j y_j^* \leq \sum \pi_j(p) \quad (\star\star)$$

By $(\star)$ we know that $(\star\star)$ holds with strict equality $\forall j$.

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...

Costs (y, θ)

F.O.C $p'(y^*)y^* + p(y^*) - \frac{\partial C(y^*, \theta)}{\partial y} = 0$

write y^* implicitly as a function of θ.

$$p'(y^*(\theta))\, y^*(\theta) + p(y^*(\theta)) - \frac{\partial\, C(y^*(\theta), \theta)}{\partial y} = 0$$

Now differentiate w.r.t θ

$$p''(y^*(\theta), \theta)\, y^*(\theta) \frac{d y^*(\theta)}{\partial \theta} + p'(y^*(\theta)) \frac{d y^*(\theta)}{d\theta} + p'(y^*(\theta)) \frac{d y^*(\theta)}{d\theta}$$

$$- \frac{\partial^2 C(y^*(\theta), \theta)}{\partial y^2} \frac{d y^*}{d\theta} - \frac{\partial^2 C(y^*(\theta), \theta)}{\partial y\, \partial \theta}$$

Solve for $dy^*/d\theta$

$$\frac{dy^*}{d\theta} = \frac{\partial^2 C(y^*, \theta) / \partial y \partial\theta}{p''(y^*)\, y^* + 2p'(y^*) - \partial^2 C(y^*, \theta)/\partial^2 y}$$

Example: CES demand, constant mg cost

$D(p) = A p^{-\beta}$, $\beta > 1$

$C(y) = \tilde{c} y$

why is it called CES?

Calculate price elasticity of demand : $\sigma_{D,p} = \frac{d \log D(p)}{d \log p}$

$\log D(p) = \log A - B \log p$

$\sigma_{D,p} = -\beta$

Inverse demand: $y = A p^{-\beta} \to p(y) = \left(\frac{y}{A}\right)^{-1/\beta}$

$p'(y) \cdot y + p(y) = c'(y)$

$p'(y) = -\frac{1}{\beta} \underbrace{\left(\frac{y}{A}\right)^{-\frac{1}{\beta}-1}}_{p(y)} \left(\frac{1}{A}\right)$

$= -\frac{1}{\beta} p(y) \frac{1}{y}$

F.O.C becomes $\left(-\frac{1}{\beta} p(y) \frac{1}{y}\right) y + p(y) = \tilde{c}$

$p(y)\left[1 - \frac{1}{\beta}\right] = \tilde{c}$

$p(y) = \frac{\tilde{c}}{1 - 1/\beta}$

What is dy^*/dc?

$y^*(c)$ defined implicitly by : $p(y^*(c))\left[1 - \frac{1}{\beta}\right] - \tilde{c} = 0$

Imp. func. T.:

$p'(y^*(\tilde{c}))\left[1 - \frac{1}{\beta}\right] \frac{dy^*}{d\tilde{c}} - 1 = 0$

$\frac{dy^*}{dc} = \frac{1}{\underbrace{p'(y^*(c))}_{<0} \underbrace{\left\{1 - \frac{1}{\beta}\right]}_{>0}} < 0$

Consumer theory:

More complicated for two reasons:
→ Rationality axioms on preferences
→ Cannot use optimization techniques on preferences. Need to "construct" a utility function.
→ Consumer's problem has prices in the constraints income/substitution effects.

Widely used in economics
i) Normatively useful
ii) Positive predictions
iii) Widely applicable
iv) Simple/sparse model.

Two approaches:
1) preferences ⟶ choices
2) choices → preferences (observed behaviour)

X: (abstract) set of alternatives
elements $x \in X$ are mutually exclusive

A choice structure $(\mathcal{B}, C(\cdot))$ is:
- $\mathcal{B}$ is a family of nonempty subsets of X
- $C(\cdot)$ is a choice rule s.t $C(B) \subseteq B \ \forall B \in \mathcal{B}$, $C(B) \neq 0$

$X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$

$C(\{x,y\}) = \{x\}$, $C(\{x,y\}) = \{x, z\}$

What are some "reasonable" restrictions on behaviour?

Weak axiom of Revealed preference (WARP)

If, for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any other $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C(B')$, we also have $x \in C(B')$

$X = \{a, b, c\}$ $\quad \mathcal{B} = \{ \underbrace{\{a, b\}}^{B}, \underbrace{\{a, b, c\}}^{B'} \}$

starting from choices to preferences...

we observe $C(\{a,b,c\}) = \{b\}$
WARP implies that $C(\{a,b\}) = \{b\}$

Why? Assume $a \in C(\{a,b\})$. Then, since a was chosen when b was available, we must have $a \in C(\{a,b,c\})$. Contradiction.

we observe $C(\{a,b\}) = \{b\}$ (we know nothing about c).
WARP implies $C(\{a,b,c\}) = \{b\}, \{c\}$ or $\{b, c\}$

Preference-based approach

X: set of alternatives.
Primitive is a binary relation on X, $\succsim$.
$x \succsim y$: "x is at least as good as y"
$x \succ y$: $x \succsim y \wedge y \not\succsim x$: "x is strictly preferred to y"
$x \sim y$: $x \succsim y \wedge y \succsim x$: "x is indifferent to y"

A preference relation is rational if
i) Complete: $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both).
ii) Transitivity: $\forall x, y \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$

Violation of transitivity: Framing (Kanheman, Tversky).
Best buy to purchase \$125 stereo and \$15 calculator.
A) Calculator on sale for \$5 off at store 20 min away — YES
B) Stereo " " " " — NO
C) Both are out of stock at current store, but we'll give you a \$5 copon for one item. Does it matter which one? — Indiferent.

X: Travel to store 2 for \$5 calculator.
Y: " " " stereo
Z: Buy both items at store 1.

Reponse imply:

$x \succ z$ (A)
$z \succ y$ (B)
$x \sim y$ (C)

What are the links between choice & preference approach?
Q2. ¿Does WARP imply rationality? NO
Q1. ¿Does Rationality imply Warp? Yes.

For Q1. Given $\succsim$, let $C(B, \succsim) = \{x \in B : x \succsim y, \forall y \in B\}$
Theorem: Suppose $\succsim$ are rational. Then, the choice structure $(\mathcal{B}, C(B, \succsim))$ satisfies WARP.
Proof: Suppose for some $B \in \mathcal{B}$, we have $x, y \in B$ and $x \in C(B, \succsim)$ by definition of C, $x \succsim y$.

Check Warp: Suppose $x, y \in B'$, and $y \in C(B', \succsim)$. So, $y \succsim z$ for all $z \in B'$. By transitivity $x \succsim y \succsim z$, so, $x \succsim z$, for all $z \in B'$. Thus $x \in C(B', \succsim)$ and WARP holds.

For Q2: Counterexample.

$X = \{a, b, c\}$, $\mathcal{B} = \{\{a,b\}, \{b,c\}, \{a,c\}\}$

assume we observe: $C(\{a,b\}) = a$; $C(\{a,c\}) = c$; $C(\{b,c\}) = b$

Can check that $(\mathcal{B}, C())$ satisfies WARP.

Observed choices implies: $\left.\begin{array}{l} a \succ b \\ a \succ c \\ b \succ c \end{array}\right\}$ this violates transitivity.

<u>Theorem (Arrow, 1959):</u>

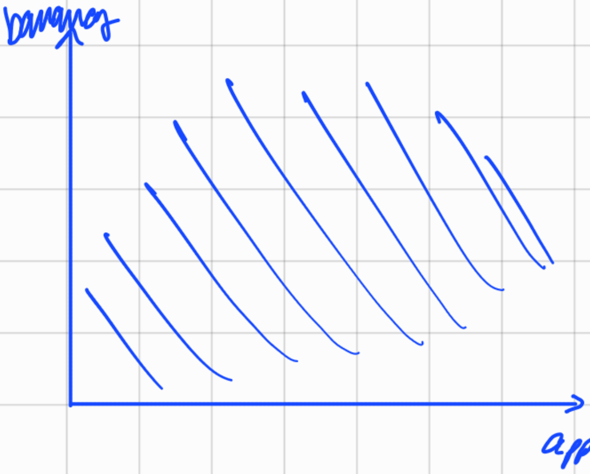
If $(\mathcal{B}, C(\cdot))$ is s.t:

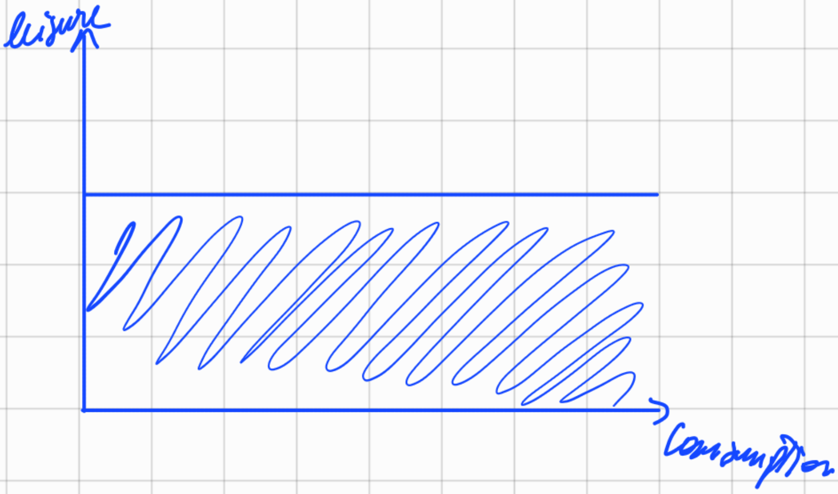
i) WARP is satisfied

ii) $\mathcal{B}$ includes all subsets of X with up to 3 elements.

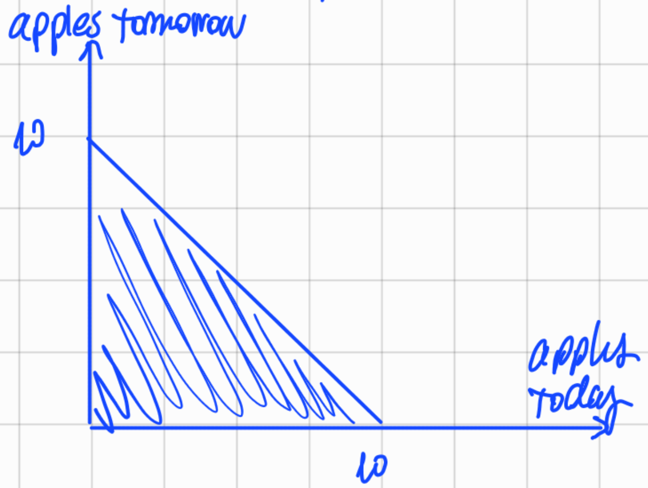
Then, there is a (unique) rational preference relation $\succsim$ that rationalizes $C(\cdot)$.

Restrict $X \subseteq \mathbb{R}^n_+$





Goods can be timed:



State dependent

Properties of preferences on $X \subseteq \mathbb{R}^n_+$

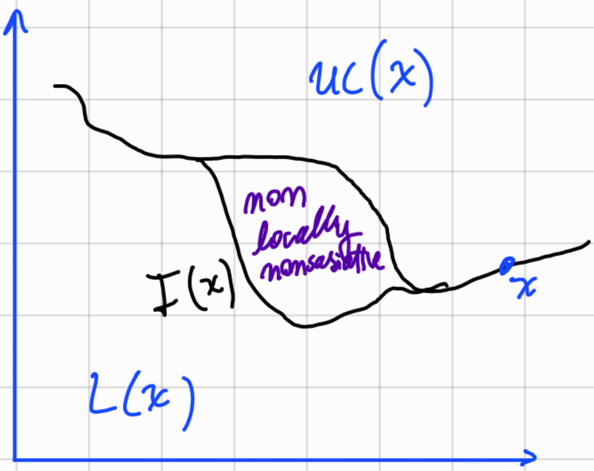
→ Preference relation $\succsim$ is monotone if $y \gg x$ implies $y \succ x$

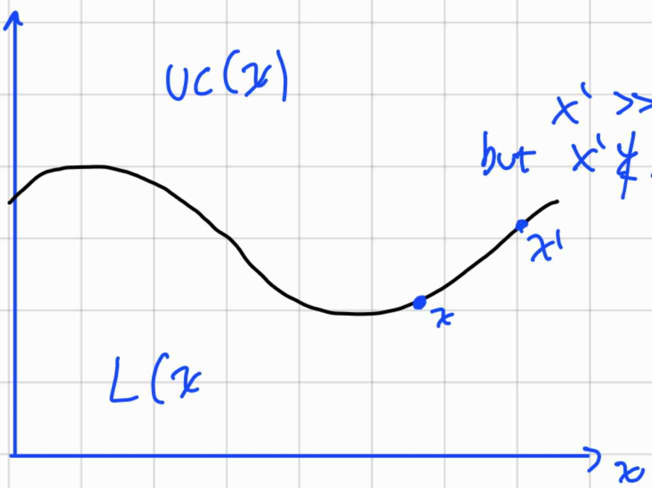
→ $\succsim$ is strongly monotone if $y \geq x$ and $y \neq x$, implies $y \succ x$

→ $\succsim$ is locally nonsatiative if for every $x \in X$ (bundle) and $\varepsilon > 0$ there is some $y \in X$ s.t $\|y - x\| < \varepsilon$ and $y \succ x$.

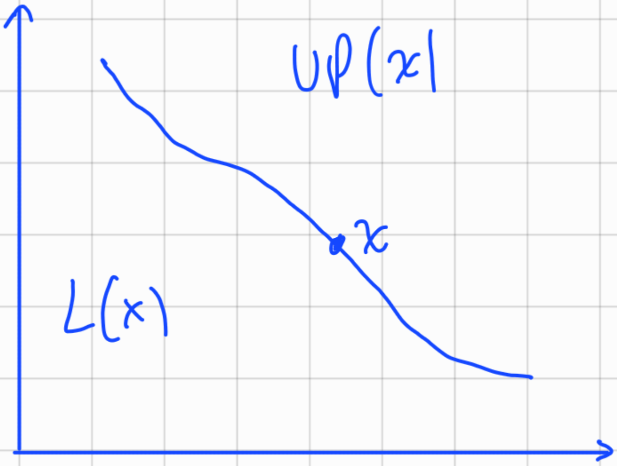
upper contour of x

$$uC(x) = \{y \in X : y \succsim x\}$$
$$\mathcal{L}C(x) = \{y \in X : x \succsim y\}.$$
$$I(x) = \{y \in X : x \sim y\}$$

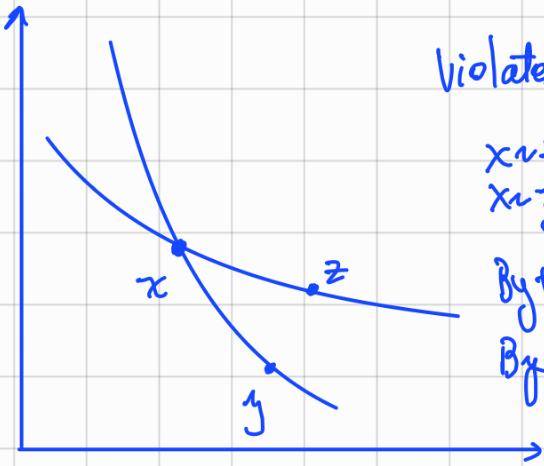




Monotonicity → Downward slope



IC cannot cross



Violates transitivity

$x \sim z$
$x \sim y$

By trans $x \sim z$

By monotone: $z > y$

contradiction.