# ECON 7710 TA Session

## Week 10


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# Outline

# Question 1

Find characteristic function of r.v. on $U[0,1]$

- We have $X \sim U[0,1]$ and we know for a uniform distribution function:

$$\phi_X(t) = E[e^{itX}] = \int_0^1 e^{itx} dx = \frac{[e^{itx}]_0^1}{it} = \frac{e^{it} - 1}{it}$$

- Specifically, if $t = 0$, then $\phi_X(t) = \lim_{t \to 0} \frac{e^{it}-1}{it} = 1$

# Question 2

$X_1$, $X_2$... are Bernoulli random variables with parameter $p = \frac{1}{2}$. Using the method of characteristic functions find the distribution of random variables:

$$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}$$

- Define $Z_k = \frac{X_k}{2^k}$, then the characteristic function of $Z_k = \frac{X_k}{2^k}$ is:

$$\phi_{Z_k}(t) = E[e^{itZ_k}] = \frac{1}{2}e^{\frac{it}{2^k}} + \frac{1}{2} = \frac{1}{2}[e^{\frac{it}{2^k}} + 1]$$

- We know $Y_n = \sum_{k=1}^{n} Z_k$, **assuming $X_k$'s are independent**.
- We can further derive that the characteristic function of $Y_n$ is:

$$\phi_{Y_n}(t) = \phi_{\sum_{k=1}^{n} Z_k}(t) = \phi_{Z_1}(t) \times \phi_{Z_2}(t) ... \times \phi_{Z_n}(t) = \prod_{k=1}^{n} \frac{1}{2}[e^{\frac{it}{2^k}} + 1] = \frac{1}{2^n} \frac{e^{it} - 1}{e^{\frac{it}{2^n}} - 1}$$

# Question 2

- $\phi_{Y_n}(t) = \frac{1}{2^n} \frac{e^{it}-1}{e^{\frac{it}{2^n}}-1}$
- When $n \to \infty$, we use L'Hospital's rule to derive that:

$$\lim_{n\to\infty} \phi_{Y_n}(t) = \lim_{n\to\infty} \frac{1}{2^n} \frac{e^{it}-1}{e^{\frac{it}{2^n}}-1} = \frac{e^{it}-1}{it}$$

This is the characteristic function of distribution $U[0,1]$.
Then we know the distribution of $Y$ is $U[0,1]$.

# Question 3

Under which conditions imposed on random variables $X$ such that random variables $X$ and $sin(X)$ are independent.

**Claim:** For any continuous measurable function f, $X$ and $f(X)$ are independent if and only if at least one of them is a constant.

- Sufficiency
  If $X$ and $f(X)$ are independent, let $Y = f(X)$ and define a new set

  $$A(y) = \{\omega : f(X(\omega)) \leq y\} \Leftrightarrow A(y) = \{\omega : X(\omega) \in f^{-1}((-\infty, y])$$

  Clearly, $A(y) \in \sigma(X)$ and under independence of $X$ and $Y$ we know:
  - $P(X \in A(y), Y \leq y) = P(X \in A(y))P(Y \leq y) = P(f(X) \leq y)P(Y \leq y), \forall y \in \mathbb{R}$.
  - But by definition of $A(y)$,
    $P(X \in A(y), Y \leq y) = P(f(X) \leq y, Y \leq y) = P(f(X) \leq y)$.
- In other words, we must have $P(f(X) \leq y)P(Y \leq y) = P(f(X) \leq y)$ and since $Y = f(X)$, This is equal to say:

  $$P(f(X) \leq y)^2 = P(f(X) \leq y) \Rightarrow P(f(X) \leq y) = 0 \text{ or } 1$$

- We must have some constant $c \in \mathbb{R}$ such that distribution function $f(X)$ jumps from zero to one at $c$.

# Question 3

- $f(X)$ jumps from zero to one at $c$ means $f(X) = c$ almost surely.
- Go back to our environment where $f() = sin()$, two cases would work.
  1. $X$ is a constant. $X = c$
  2. $sin(X)$ is a constant. $sin(X) = c$. Then due to the property of periodic function, $X = sin^{-1}(c) + 2\pi K$ for any $|c| \leq 1$ and $K$ is an integer will work.
- So ultimately, all the $X$ that makes $X \perp f(X)$ is when $X$ is a constant $c$.
  Or any random variable $X$ with support on $X = sin^{-1}(c) + 2\pi K$ for any $|c| \leq 1$ and any subset of integer $K$.

# Question 3

- Necessary conditions check are much easier. I just show the case when $X = c$ here.
  - When $X$ is a degenerate distribution, $P(X \in B_1) = \{0, 1\}$ and $P(sin(X) \in B_2) = \{0, 1\}$. The joint probability $P(X \in B_1, sin(X) \in B_2) = \{0, 1\}$.
    Clearly, the following equation holds:
    $$P(X \in B_1, sin(X) \in B_2) = P(X \in B_1) \times P(sin(X) \in B_2)$$
  - $X$ is a degenerate distribution, with $P(X = c) = 1$,
    $$\phi_X(t) = e^{itc}$$
    Likewise, $sinX$ is also a degenerate distribution with $P(sinX = sin(c)) = 1$,
    $$\phi_{sinX}(t) = e^{itsin(c)}$$
    Then we know for distribution, $X + sin(X)$ it is also a degenerate distribution with $P(X + sin(X) = c + sin(c)) = 1$. So
    $$\phi_{X+sinX} = e^{it(c+sin(c))} = \phi_X(t)\phi_{sinX}(t) = e^{itc} * e^{itsin(c)} = e^{it(c+sin(c))}$$

# Midterm 2022 Q3

## Modified from Core Aug 2021 Q3

Suppose that Econometrics Core exam can be re-taken infinitely many times. The probability that a student passes the exam at the first attempt is $\frac{1}{2}$ and the probability of passing at any subsequent attempt is $\frac{2}{3}$

a. What is the expected number times a given student needs to take the Core exam before passing it?

b. Suppose that N is the class size. Denote by $X_t$ a random variable equals to the number of students who *passed* the CORE exam by attempt $t$ (i.e., these students passed the exam at attempts 1,2,...,t including t). Does a sequence of random variables $\{X_t\}_{t=1}^{\infty}$ converge almost surely? If your answer is yes, prove it and find the limit of almost sure convergence. If your answer is no, formally prove the absense of almost sure convergence.

# Midterm 2022 Q3.a

a Denote $N$ as the number of Core exams a given student needs to take before passing it. Clearly $N = \{1, 2, 3, ...\}, N \in \mathbb{N}$

- $N = 1$, $P(N = 1) = \frac{1}{2}$, Passed in 1st try.
- $N = 2$, $P(N = 2) = \underbrace{\frac{1}{2}}_{\text{Failed in 1st try}} * \underbrace{\frac{2}{3}}_{\text{Passed in 2nd try}} = \frac{1}{3}$
- $N = 3$, $P(N = 3) = \underbrace{\frac{1}{2}}_{\text{Failed in 1st try}} * \underbrace{\frac{1}{3}}_{\text{Failed in 2nd try}} * \underbrace{\frac{2}{3}}_{\text{Passed in 3rd try}} = (\frac{1}{3})^2$

- Expectation should be:

$$E[N] = 1 * \frac{1}{2} + 2 * \frac{1}{2} * \frac{2}{3} + 3 * \frac{1}{2} * \frac{1}{3} * \frac{2}{3} ...$$

$$E[N] = \frac{1}{2} + 2 * \frac{1}{3} + 3 * (\frac{1}{3})^2 + 4 * (\frac{1}{3})^3 + ... = \frac{1}{2} + \sum_{n=2}^{\infty} n * (\frac{1}{3})^{n-1}$$

$$= \sum_{n=1}^{\infty} n * (\frac{1}{3})^{n-1} - \frac{1}{2} = 3 \sum_{n=1}^{\infty} n * (\frac{1}{3})^{n} - \frac{1}{2} = 3 * \frac{1/3}{(1 - 1/3)^2} - \frac{1}{2} = 1.75$$

# Midterm 2022 Q3.b

b Now we know $X_t$ is a random variable that equals to the number of students who passed core by attempt $t$. Clearly $X_t$ is monotone increasing.

- We want to know how $E[X_t]$ looks like.
  - $P(\text{Not Passing by t attempts}) = \frac{1}{2}(\frac{1}{3})^{t-1}$
  - $P(\text{Passing by t attempts}) = 1 - \frac{1}{2}(\frac{1}{3})^{t-1}$
  - $E[X_t] = [1 - \frac{1}{2}(\frac{1}{3})^{t-1}]N$
- Since $\lim_{t\to\infty} E[X_t] = \lim_{t\to\infty} [1 - \frac{1}{2}(\frac{1}{3})^{t-1}]N = N$.
- We know $\lim_{t\to\infty} E[|X_t - N|] = 0$, $X_t \xrightarrow{(1)} N$
- $X_t \xrightarrow{(1)} N \Rightarrow X_t \xrightarrow{p} N$ by Chebychev's inequality.
- Since $X_t$ is monotone increasing, $X_t \xrightarrow{p} N \Rightarrow X_t \xrightarrow{a.s.} N$

# Core June 2022 Q2

Suppose that $\{X_n\}$ is sequence of random variables with finite variances.

a Suppose that sequence $\{X_n\}$ converges in probability to a random variable $X$. Find the limit of the correlation $\rho_{X_n,X_{n+1}} = \frac{Cov(X_n,X_{n+1})}{\sqrt{Var(X_n)Var(X_{n+1})}}$ as $n \to \infty$

b Suppose that sequence $\{X_n\}$ converges in distribution to a random variable $X$. Does the correlation $\rho_{X_n,X_{n+1}}$ converge to the same limit as in the previous question? Prove or disprove.

# Core June 2022 Q2.a

a Since $X_n \xrightarrow{p} X$, $\forall \epsilon > 0$: $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$. We will have $X_n \xrightarrow{p} X$ and $X_{n+1} \xrightarrow{p} X$.

- For the correlation $\rho_{X_n,X_{n+1}} = \frac{Cov(X_n,X_{n+1})}{\sqrt{Var(X_n)Var(X_{n+1})}}$
  - Numerator: we know $Cov(X_n, X_{n+1}) \xrightarrow{p} Cov(X, X) = Var(X)$
  - Denominator: we also know $\sqrt{Var(X_n)} \xrightarrow{p} \sqrt{Var(X)}$, $\sqrt{Var(X_{n+1})} \xrightarrow{p} \sqrt{Var(X)}$ hence $\sqrt{Var(X_n)Var(X_{n+1})} \xrightarrow{p} Var(X)$
- Therefore, $\lim_{n\to\infty} \rho_{X_n,X_{n+1}} = \frac{Cov(X_n,X_{n+1})}{\sqrt{Var(X_n)Var(X_{n+1})}} = \lim_{n\to\infty} \frac{Var(X)}{Var(X)} = 1$.

# Core June 2022 Q2.b

b A counterexample is formulated below:

- Suppose we have a sequence of iid random variables $Z_n$ such that each $Z_n \sim N(0,1)$
- Denote $Y$ as another standard normal random variable which is independent to $Z_n$: $Y \sim N(0,1) Y \perp Z_n$
- A new sequence of random variables $X_n, n \in \mathbb{N}$ is formulated in this way:

$$X_n = \begin{cases} Z_n - Y & \text{n is odd} \\ Z_n + Y & \text{n is even} \end{cases}$$

Due to the linearity of normal distribution, we know $X_n \xrightarrow{d} X$, where $X \sim N(0,2)$.

But the correlation between $X_n$ and $X_{n+1}$ is 0.