Oct 9, 2023

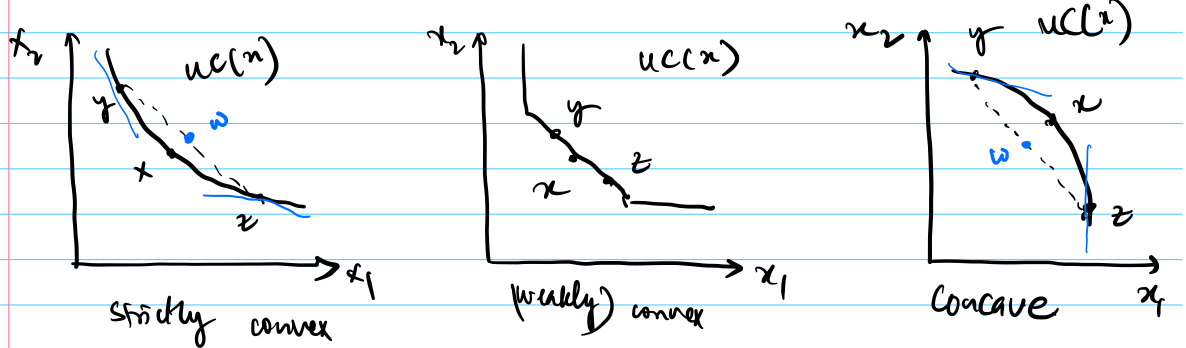
Convexity

$\succsim$ is convex if the upper contour set

$$UC(x) = \{ y \in X : y \succsim x \} \text{ is convex.}$$

Equivalently, $y \succsim x$ & $z \succsim x$, then

$$\alpha y + (1-\alpha) z \succsim x, \quad \forall \alpha \in [0, 1]$$



Interpretation of convexity :-

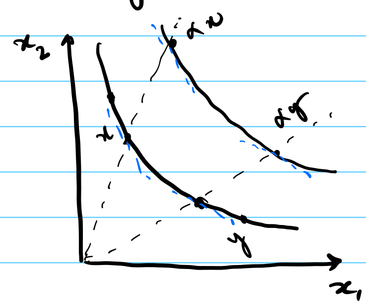
1) Diminishing marginal rate of substitution (slope becomes flatter)

2) Preference for diversification (Average bundle over extremes) (pt w)

(*) Useful "classes" of preferences

1. Homothetic

$x \sim y \Rightarrow \alpha x \sim \alpha y \quad \forall \ \alpha \geq 0$



All ICs are blown up version of the other.

(2) Quasilinear preferences

Let $X = (-\infty, \infty) \times \mathbb{R}_+^{n-1}$

Write $x = (t, y)$ where $t \in \mathbb{R}$ & $y \in \mathbb{R}_+^{n-1}$

$\succsim$ are quasilinear if :

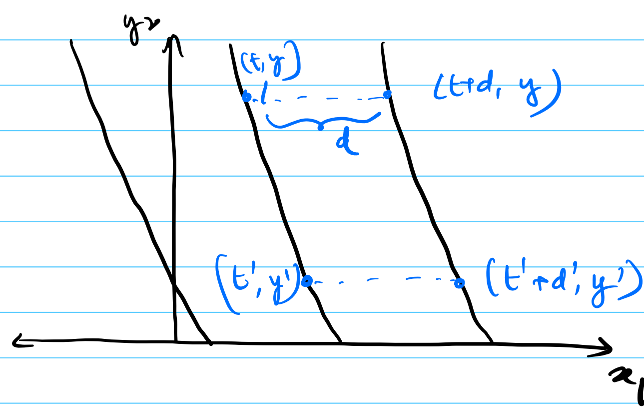
1) Good 1 is desirable : $t' > t$
then $(t', y) \succ (t, y)$

2) "No wealth effects"
If $(t, y) \sim (t', y')$, then $\forall \ d \in \mathbb{R}$,
$(t+d, y) \sim (t'+d, y')$

→ Good 1 often called a "numeraire" (think of as money, debt if negative)

→ Will allow us to write utility as
$u(t, y_2, \ldots, y_n) = t + v(y_2, \ldots, y_n)$

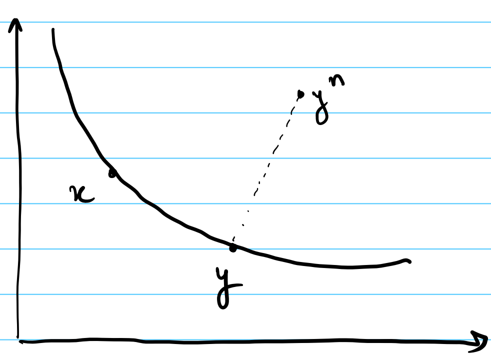
→ Also called "transferable utility"



<u>Continuity of preferences</u>:

$\succsim$ is continuous if for any sequence, $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$, and $x = \lim_{n\to\infty} x^n$, $y = \lim_{n\to\infty} y^n$, then we have $x \succsim y$.

Equivalently; $UC(x)$ & $LC(x)$ are closed sets.



$y^n \to y$,
Let $x^n = x \ \forall n$
$y^n \succsim x$

$y^n \in UC(x)$, continuity implies $y \in UC(x)$

Example: Lexicographic Preferences

$X = \mathbb{R}^2_+$

Define $\succsim$ as follows: $x \succsim y$ if either:

1. $x_1 > y_1$
2. $x_1 = y_1$ & $x_2 \geq y_2$

$\succsim$ is: complete
transitive
strongly monotone
strictly convex

$\succsim$ is not continuous

$x^n = \left(\frac{1}{n}, 0\right) \qquad y^n = (0, 1) \quad \forall n$

$x^n \succ y^n \quad \forall n$

$x = \lim\limits_{n\to\infty} x^n = (0,0)$

$y = \lim\limits_{n\to\infty} y^n = (0,1) \qquad$ (limit jumps)

and so $y \succ x$.

Utility Functions :—

A preference relation $\succsim$ is represented by a utility function $u: X \to \mathbb{R}$ if, $\forall\, x, y$

$$x \succsim y \iff u(x) \geq u(y)$$

Utility f^n are not unique.

Let $g: \mathbb{R} \to \mathbb{R}$ be any strictly increasing f^n

Define $\hat{u}(x) = g(u(x))$

$$\tilde{u}(x) \geq \tilde{u}(y) \iff u(x) \geq u(y) \iff x \succsim y$$

$\tilde{u}(x)$ also represents $\succsim$

$\boxed{\text{how?}}$ *?

Q When can we find a utility f^n to represent $\succsim$?

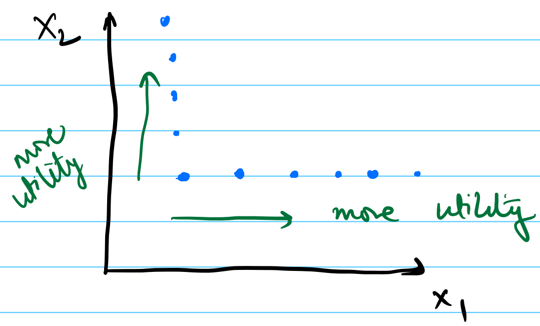
Theorem : $\succsim$ can be represented by a utility f^n only if it is rational.

Proof : Homework (Irrational $\to$ non transitivity $\to$ can't come up with utility f^n)

Harder Q: Is the following true? NB

"Theorem" : If $\succsim$ is rational, then $\exists$ a utility f^n representing it.

Example: For lexicographic prefs (rational), indifference sets are singletons



No indifference curves. They are just pt.

Sufficient conditions for a utility representation:-

① Finiteness: If X is finite, $\exists$ exists a utility representation.

or

② Continuity of $\succsim$

Theorem: If preferences are rational & continuous, then $\exists$ exists a utility f^n that represents them. Furthermore, $\exists$ exists – continuous $u(\cdot)$ that represents them.

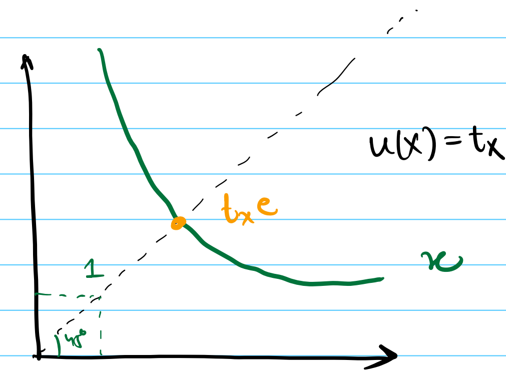
Proof: Show for case that $\succsim$ is strongly monotone.

Idea: ① Construct a "candidates" $u(c)$

② Check it works

Let $e = (1, \dots, 1)$

Take a bundle $x \in X$



$A^+(x) = \{t \in \mathbb{R}_+ : te \succsim x\}$

(t is a scalar)

$A^-(x) = \{t \in \mathbb{R}_+ : x \succsim te\}$ all that are less preferred than x (below the x curve)

By strong monotonicity;

$A^+(x) \neq \phi$ (take $\max\{x_1, \dots, x_n\} e \succsim x$)

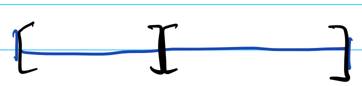
$A^-(x) \neq \phi$ ($x \succsim 0$)

Completeness of $\succsim \Rightarrow \mathbb{R}_+ = A^+(x) \cup A^-(x)$

By continuity, $A^+(x)$ & $A^-(x)$ are closed sets.

Since $\mathbb{R}^+$ is connected, $A^+(x) \cap A^-(x) \neq \phi$

(Since they are both closed sets)



(They have something in common).

$\exists$ some $t_x \in A^+(x) \cap A^-(x)$ s.t. $t_x e \sim x$

Define $u(x) = t_x$

Last step: Show proposed $u(\cdot)$ represents $\succsim$

1. $u(y) \geq u(x) \Rightarrow y \succsim x$
2. $y \succsim x \Rightarrow u(y) \geq u(x)$

For (1):

$$t_y \geq t_x \quad (\text{def}^n \text{ of } u)$$
$$t_y e \geq t_x e \quad (\text{monotonicity})$$

$$y \sim t_y e \succsim t_x e \sim x \quad (\text{def}^n \text{ of } t)$$
$$y \succsim x \quad (\text{transitivity})$$

For (2): Take $y \succsim x$

$$(t_y e) \sim y \succsim x \sim (t_x e) \quad (\text{def}^n \text{ of } t)$$

$$t_y e \succsim t_x e \quad (\text{transitivity})$$
$$t_y \geq t_x \quad (\text{monotonicity})$$

$$u(y) \geq u(x) \quad (\text{def}^n \text{ of } u)$$

- Also true that a continuous $u(\cdot)$ exists (will skip)

- Further conditions on $\succsim$ that guarantee a differentiable $u(\cdot)$ exists (also skip)

- We have constructed <u>one</u> utility representation, there are (many) others, not all are continuous.

$$g(x) = \begin{cases} x, & x \leq 1 \\ 2x, & x > 1 \end{cases}$$

$\tilde{u}(x) = g(u(x))$ also represents $\succsim$.