ECON 7710 TA Session

Week 12

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Outline

1. PS 6
2. Practice Questions

Question 1.a

We have n i.i.d. draws from uniform distribution on $[0, \theta]$, the pdf is:

$$f(x) = \begin{cases} \frac{1}{\theta} & x \in [0, \theta] \\ 0 & \text{Otherwise} \end{cases}$$

The likelihood function is written as:

$$\hat{L}(\theta; \mathbf{X}) = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbf{1}(0 \leq X_i \leq \theta)$$

Denote $X_m = max(X_1, X_2, ..., X_n)$, then we discuss the relationship between X_m and θ.

- If $\theta < X_m$, then $\mathbf{1}(0 \leq x_m \leq \theta) = 0$, which means $\hat{L}(\theta; \mathbf{X}) = 0$.
- If $\theta \geq X_m$, then $\hat{L}(\theta; \mathbf{X}) = \frac{1}{\theta^n}$. Clearly, when θ increases, $\frac{1}{\theta^n}$ decreases.

Therefore, we know the maximum of $\hat{L}(\theta; \mathbf{X})$ is achieved when $\theta = X_m$. In other words, we constructed:

$$\hat{\theta} = X_m$$

Question 1.b

- $\hat{\theta} = X_m = \max\{X_1, X_2, ..., \}$, the pdf of each X_i in part 1.(a) and the cdf is:

$$F(x) = \begin{cases} 0 & \text{If } x \in (-\infty, 0) \\ \frac{x}{\theta} & \text{If } x \in [0, \theta] \\ 1 & \text{If } x \in (\theta, \infty) \end{cases}$$

- We can easily construct the CDF and pdf of $\hat{\theta} = X_m$:

$$cdf : F_{\hat{\theta}}(x) = \begin{cases} 0 & \text{If } x \in (-\infty, 0) \\ (\frac{x}{\theta})^n & \text{If } x \in [0, \theta] \\ 1 & \text{If } x \in (\theta, \infty) \end{cases} \qquad pdf : f_{\hat{\theta}}(x) = \begin{cases} \frac{n}{\theta}(\frac{x}{\theta})^{n-1} & \text{If } x \in [0, \theta] \\ 0 & \text{Otherwise} \end{cases}$$

- $E(\hat{\theta}_{MLE}) = \int_0^\theta x f_{\hat{\theta}}(x) dx = \int_0^\theta n(\frac{x}{\theta})^n dx = \frac{n}{\theta^n} \frac{x^{n+1}}{n+1}\Big|_0^\theta = \frac{n\theta}{n+1}$.
- Then we know $E(\hat{\theta}_{MLE}) - \theta = -\frac{\theta}{n+1}$

Question 1.c

1. Variance

   We know:

$$E[\hat{\theta}^2_{MLE}] = \int_0^{\theta} x^2 f_{\hat{\theta}}(x)dx = \int_0^{\theta} \frac{n}{\theta^n} x^{n+1} dx = \frac{n}{\theta^n} \frac{x^{n+2}}{n+2}\Big|_0^{\theta} = \frac{n\theta^2}{n+2}$$

   Therefore, we know the variance of our estimator is:

$$Var[\hat{\theta}_{MLE}] = E[\hat{\theta}^2_{MLE}] - E[\hat{\theta}_{MLE}]^2 = \frac{n\theta^2}{n+2} - (\frac{n\theta}{n+1})^2 = \frac{\theta^2}{(n+1)^2(1+\frac{2}{n})}$$

2. Rate of convergence

   r_n quantifies how fast the estimation error decreases when increasing the sample size n. In other words, what we are interested in is when divided by n^{-r} or multiplied by n^r, what's the largest r such that $|\hat{\theta}_n - \theta|$ remains stochastic bounded. $r_n = (n^r)$ is rate of convergence. Clearly, for $Sd[\hat{\theta}] = \frac{\theta}{(n+1)\sqrt{(1+\frac{2}{n})}}$, the convergence rate should be n.

Question 1.c

3 Consistency of $\hat{\theta}$.
We already know $E(\hat{\theta}_{MLE}) = \frac{n\theta}{n+1} = \frac{\theta}{1+\frac{1}{n}}$. $\lim_{n\to\infty} E(\hat{\theta}_{MLE}) = \theta$. So it is an unbiased estimator.

- We also know:

$$\lim_{n\to\infty} Var(\hat{\theta}_{MLE}) = \lim_{n\to\infty} \frac{\theta^2}{n^2 + 4n + 5 + \frac{2}{n}} = 0$$

- Since our estimator is unbiased, using Chebyshev's inequality we know $P(|\hat{\theta} - \theta| > \epsilon) \leq Var(\hat{\theta})/\epsilon^2$. Then for any $\epsilon > 0$, if $\lim_{n\to\infty} Var(\hat{\theta}) = 0$ we have $\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \epsilon) = 0$ as well, which means consistency is proved. ($\hat{\theta} \xrightarrow{p} \theta$).

Question 1.d

- For asymptotic distribution of our estimator. $\hat{\theta}_{MLE} = \max(X_1, ..., X_n) = X_m$ and $r_n = n$
- Notice that:

$$Pr(n(X_m - \theta) < x) = Pr(X_m < \frac{x}{n} + \theta)$$

- Then we can derive that:

$$Pr(X_m < \frac{x}{n} + \theta) = (1 + \frac{x}{n\theta})^n$$

- Since $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$, we know the limit of probability:

$$\lim_{n\to\infty}(1 + \frac{x}{n\theta})^n = e^{\frac{x}{\theta}} \quad (x \leq 0)$$

- The cdf of asymptotic distribution of our estimator is

$$F(x) = \begin{cases} e^{\frac{x}{\theta}} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

- which tells us the asymptotic distribution is exponential, but on the negative side of the real line.

Core July 2022 Q2

Suppose X_1, X_2, ... ,X_n are iid realizations from Uniform distribution with support $[0, \theta]$, where θ is an unknown parameter.

a Show that $\hat{\theta}_n = 2 \times \frac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased and consistent estimator of θ.

b Is $\hat{\theta}_n = 2 \times X_1$ an unbiased and consistent estimator of θ? Why? Show your work.

c Show that $\hat{\theta}_n = \max\{X_1, ..., X_n\}$ is a biased estimator of θ.

Core July 2022 Q2.a

Suppose X_1, X_2, ... ,X_n are iid realizations from Uniform distribution with support $(0, \theta)$, where θ is an unknown parameter.

a Show that $\hat{\theta}_n = 2 \times \frac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased and consistent estimator of θ.

- Unbiased: $E[\hat{\theta}_n] = \frac{2}{n} \sum_{i=1}^{n} E[X_i] = \frac{2}{n} * n * \frac{\theta}{2} = \theta$. $E[\hat{\theta}_n] = \theta$, so unbiased.
- Consistent: By Law of Large Numbers, $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{p} E[X] = \frac{\theta}{2}$, hence $2\bar{X}_n \xrightarrow{p} \theta$, so consistent.

Core July 2022 Q2.b

Suppose X_1, X_2, ... ,X_n are iid realizations from Uniform distribution with support $(0, \theta)$, where θ is an unknown parameter.

b Is $\hat{\theta}_n = 2 \times X_1$ an unbiased and consistent estimator of θ? Why? Show your work.

- Unbiased: $E[\hat{\theta}_n] = 2E[X_1] = 2 * \frac{\theta}{2} = \theta$, so unbiased.
- Not Consistent:

$$P(|2X_1 - \theta| > \epsilon) = P(|X_1 - \frac{\theta}{2}| > \frac{\epsilon}{2}) = P(X_1 > \frac{\theta + \epsilon}{2}) + P(X_1 < \frac{\theta}{2} - \frac{\epsilon}{2})$$

Suppose ϵ is small and $\theta > \epsilon$, then we have

$$\begin{cases} P(X_1 > \frac{\theta+\epsilon}{2}) = 1 - \frac{\theta+\epsilon}{2\theta} \\ P(X_1 < \frac{\theta}{2} - \frac{\epsilon}{2}) = \frac{\theta-\epsilon}{2\theta} \end{cases}$$

$P(|2X_1 - \theta| > \epsilon) = 1 - \frac{\epsilon}{\theta} \neq 0$, so not consistent.

Core July 2022 Q2.c

Suppose X_1, X_2, ... ,X_n are iid realizations from Uniform distribution with support $(0, \theta)$, where θ is an unknown parameter.

c Show that $\hat{\theta}_n = \max\{X_1, ..., X_n\}$ is a biased estimator of θ.
We want to know $E[\hat{\theta}_n]$. To get this, we need to first derive the CDF and PDF of $\hat{\theta}_n = X_m = \max\{X_1, ..., X_n\}$.

- Clearly as you just derived in your problem set 6,

$$cdf : F_{\hat{\theta}_n}(x) = \begin{cases} 0 & \text{If } x \in (-\infty, 0) \\ (\frac{x}{\theta})^n & \text{If } x \in [0, \theta] \\ 1 & \text{If } x \in (\theta, \infty) \end{cases} \qquad pdf : f_{\hat{\theta}_n}(x) = \begin{cases} \frac{n}{\theta}(\frac{x}{\theta})^{n-1} & \text{If } x \in [0, \theta] \\ 0 & \text{Otherwise} \end{cases}$$

Then $E(\hat{\theta}_n) = \int_0^\theta x f_{\hat{\theta}_n}(x)dx = \int_0^\theta n(\frac{x}{\theta})^n dx = \frac{n}{\theta^n}\frac{x^{n+1}}{n+1}\Big|_0^\theta = \frac{n\theta}{n+1}$.
Then we know $E[\hat{\theta}_n] \neq \theta$. We proved this is a biased estimator.

* Although he didn't ask, but you just proved in PS6 1.c that $\hat{\theta}_n = X_m$ is a consistent estimator

Core Jun 2017 Q1

Suppose $X \sim N[e^{\alpha\beta}, 1]$ and $Y \sim N[e^{\alpha}, 1]$, independent of each other. Let $\{X_i, Y_i\}$, $i = 1, 2..., n$ be i.i.d. observations on (X, Y), and define $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ and $\bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i$. We are to estimate β by $\hat{\beta} = \frac{log \bar{X}}{log \bar{Y}}$.

Prove the consistency of $\hat{\beta}$(namely, $\hat{\beta} \xrightarrow{p} \beta$) and derive its asymptotic distribution.

Core Jun 2017 Q1

We know $X \sim N[e^{\alpha\beta}, 1], \ Y \sim N[e^{\alpha}, 1]$

- By LLN:

$$\begin{cases} \bar{X} \xrightarrow{p} E[X] = e^{\alpha\beta} \\ \bar{Y} \xrightarrow{p} E[Y] = e^{\alpha} \end{cases}$$

- By CMP:

$$\begin{cases} log\bar{X} \xrightarrow{p} loge^{\alpha\beta} = \alpha\beta \\ log\bar{Y} \xrightarrow{p} loge^{\alpha} = \alpha \end{cases}$$

- By property of convergence

$$\begin{cases} log\bar{X} \xrightarrow{d} \alpha\beta \\ log\bar{Y} \xrightarrow{d} \beta \end{cases}$$

- By Slutsky

$$\hat{\beta} = \frac{log\bar{X}}{log\bar{Y}} \xrightarrow{d} \frac{\alpha\beta}{\alpha} = \beta, \Rightarrow \hat{\beta} \xrightarrow{p} \beta$$

So we proved consistency of $\hat{\beta} : \hat{\beta} \xrightarrow{p} \beta$

Core Jun 2017 Q1

We know $\hat{\beta} = \frac{log(\bar{X})}{log(\bar{Y})} = f(\bar{X}, \bar{Y})$, a function of $(\bar{X}, \bar{Y})$.

- By CLT:

$$\begin{cases} \sqrt{n}(\bar{X} - e^{\alpha\beta}) \xrightarrow{d} N(0,1) \\ \sqrt{n}(\bar{Y} - e^{\alpha}) \xrightarrow{d} N(0,1) \end{cases}$$

- By independence of X and Y:

$$\sqrt{n}(\begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} - \begin{bmatrix} e^{\alpha\beta} \\ e^{\beta} \end{bmatrix}) \xrightarrow{d} N(0, \Sigma), \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- By multivariate delta method, considering $\hat{\beta} = \frac{log(\bar{X})}{log(\bar{Y})} = h(\bar{X}, \bar{Y})$

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \nabla h(\beta)' \Sigma \nabla h(\beta))$$

$$\nabla h(\beta)'|_{\mu_x, \mu_y} = \begin{bmatrix} \frac{\partial h(\beta)}{\partial \bar{X}} \\ \frac{\partial h(\beta)}{\partial \bar{Y}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\bar{X} log(\bar{Y})} \\ -\frac{log(\bar{X})}{(log(\bar{Y}))^2 \bar{Y}} \end{bmatrix}_{\mu_x, \mu_y} = \begin{bmatrix} \frac{1}{e^{\alpha\beta}\alpha} \\ -\frac{\beta}{\alpha e^{\alpha}} \end{bmatrix}$$

- Therefore, the asymptotic distribution is:
$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \frac{1}{\alpha^2 e^{2\alpha\beta}} + \frac{\beta^2}{\alpha^2 e^{2\alpha}})$