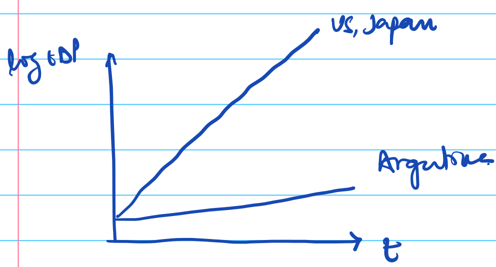
Nov 9, 2023

Today:–

1. ENDOG. GROWTH MODEL
2. ADDING FISCAL POLICY TO END. GR. MODEL

NGM :–

Growth happens but:
1. Temporary (transition to steady state)
2. $k_0 < k_{ST}$



Typical "fix"

→ exogenuous tech progress
→ $y = F(k_t, (1+g)^t n_t)$] → Growth is endogeneous, costless.

But here, everything is growing. No opportunity for stagnation. Growth only stops if K is shrinking which doesnt happen in real life.

A SIMPLE END. GROWTH MODEL.

* Social Planner's Problem:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, \underbrace{(1-n_t)h_t}_{\text{"quality adjusted leisure"}})$$

$h_t \to$ human capital
$n_t \to$ hours worked

$$F(k_t, n_t h_t)$$

$$c_t + x_{kt} + x_{nt} = F(k_t, n_t h_t)$$

$$k_{t+1} = (1-\delta_k)k_t + x_{kt}$$

$$h_{t+1} = (1-\delta_h)h_t + x_{nt}$$

no death here.

δ_h: depreciation on human capital (people forget things)

An alternative :-

$$h_{it+1} = (1-\delta_h)h_{it} + G\Big(\underbrace{x_{int}}_{\text{capital investment made by you}}, \underbrace{\int_I x_{i'nt}\, di'}_{\text{capital investment made by peers}}\Big)$$

* Inelastic labor supply

$n_t = 1$

Simplifying the model:—

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \rightarrow \max$$

$$c_t + x_{kt} + x_{ht} = f(k_t, h_t)$$

$$k_{t+1} = (1-\delta_k) k_t + x_{kt}$$

$$h_{t+1} = (1-\delta_h) h_t + x_{ht}$$

$$k_0, h_0 \rightarrow \text{given}$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[F(k_t, h_t) - c_t - k_{t+1} + (1-\delta_k) k_t - h_{t+1} + (1-\delta_h) h_t \right]$$

(derivative wrt k_{t+1})

$$k_{t+1}: \quad -\lambda_t + \lambda_{t+1} \left[F'_k(t+1) + 1 - \delta_k \right] = 0$$

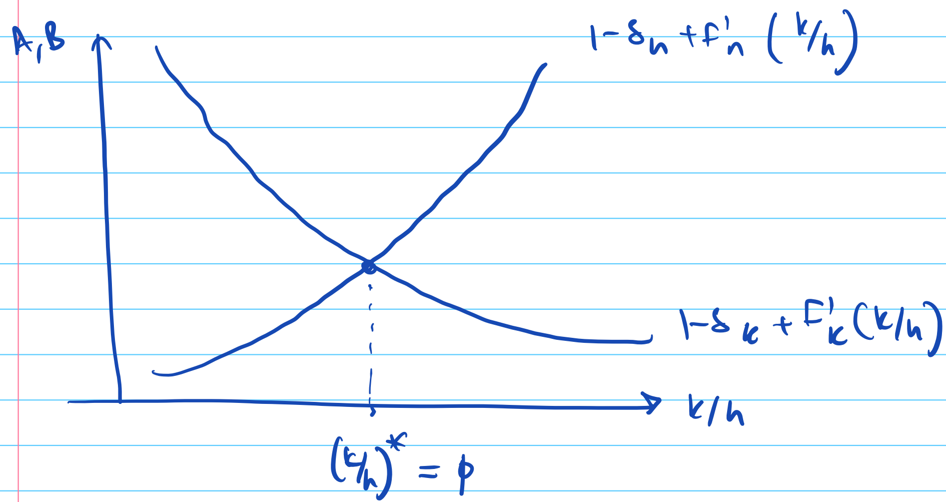
$$h_{t+1}: \quad -\lambda_t + \lambda_{t+1} \left[F'_h(t+1) + 1 - \delta_h \right] = 0$$

$$\Rightarrow \quad 1 - \delta_k + F'_k\left(\frac{k_{t+1}}{h_{t+1}}, 1\right) = 1 - \delta_h + f'_h\left(\frac{k_{t+1}}{h_{t+1}}, 1\right)$$

Assumption:— $f(\cdot)$ is CRS
Homogeneous of degree 1.
F'_k & f'_h are HD 0.

$$F'_k(k_{t+1}, h_{t+1}) = F'_k\left(\frac{k_{t+1}}{h_{t+1}}, 1\right)$$

$$F'_h(k_{t+1}, h_{t+1}) = f'_h\left(\frac{k_{t+1}}{h_{t+1}}, 1\right) = F'_h\left(1, \frac{h_{t+1}}{k_{t+1}}\right)$$



- If $F(k,h) = k^{\alpha} h^{1-\alpha}$

$$k f'_k = \alpha F(k,h)$$
$$h f'_h = (1-\alpha) F(k,h)$$

In eq^n (if $\delta_k = \delta_h$) $\Rightarrow f'_k = f'_h$

$$\Rightarrow \left(\frac{k}{h}\right)^* = \frac{\alpha}{1-\alpha} = \phi$$

This equality needs to hold in every time pd. It is not a steady state.

Law of motion for h:-

$$h_{t+1} = (1-\delta_h) h_t + x_{ht}$$

$$k = \phi h$$

$$\Rightarrow \frac{1}{\phi} k_{t+1} = (1-\delta_h) \frac{1}{\phi} k_t + x_{ht}$$

$$\Rightarrow \frac{1}{\phi}\left(k_{t+1} - (1-\delta_h)k_t\right) = x_{ht}$$

$$\Rightarrow \frac{1}{\phi}\left(k_{t+1} - (1-\delta_k)k_t\right) = x_{ht}$$

$$\left[\begin{array}{l}\text{Assumption} \\ \delta_h = \delta_k\end{array}\right]$$

if this is not made, then it won't be equal to x_{kt}, right?

$$\Rightarrow x_{kt} = \phi \, x_{ht}$$

SPP becomes.

$$\sum_{t=0}^{\infty} \beta_t$$

∘ Greek letters.

Recursive formulation

$$V(k_t, h_t) = \max_{\substack{c_t, x_{kt}, x_{ht}, \\ k_{t+1}, h_{t+1}}} \left[u(c_t) + \beta V(k_{t+1}, h_{t+1}) \right]$$

$$c_t + x_{kt}\left(1 + \frac{1}{\phi}\right) = F(k_t, h_t)$$

$$k_{t+1} = (1-\delta_k) k_t + x_{kt}$$

$k_0, h_0 \rightarrow$ given.

$$F(k_t, h_t) = k_t F\left(1, \frac{h_t}{k_t}\right)$$

$$= F(1, 1/\phi) k_t$$

$$= A k_t$$

Assumption: $\dfrac{h_0}{k_0} = \dfrac{1}{\phi}$

$\Rightarrow$ We can drop the second state variable.

2nd class. **Ak-model** ?

Consider CRRA utility $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$

SPP $\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$

$$c_t + (1+\phi) x_{kt} = A k_t \quad , \quad \phi = \frac{k_t}{h_t}$$

$$k_{t+1} = (1-\delta_k) k_t + x_t$$

k_0, h_0 given.

Remark: Inada cond^n do not hold.

Typically we had

$$\lim_{k \to 0} F'_k = \infty \quad \text{and}$$

$$\lim_{k \to \infty} F'_k = 0$$

Feasibility correspondence is HD-1

$$\Gamma(\lambda k_0) = \lambda \Gamma(k_0)$$

Remark:

$u(\cdot)$ is HD $(1-\sigma)$

$\Rightarrow$ $\succsim$ are homothetic

$\Rightarrow$ $c_0^*, x_{k0}^*, k_1^*, c_1^*, x_k^*, k_2^* \dots$ is optimal starting from k_0, then

$\Rightarrow$ $\lambda c_0^*, \lambda x_{k_0}^*, \lambda k_1^*, \lambda c_1^*, \lambda x_{k_1}^*, \lambda k_2^*$ is optimal starting from λk_0.

Proof :—

Suppose $\{c_t^*\}$ is optimal starting from k_0. and suppose that $\{\lambda c_t^*\}$ is not optimal if you start from (λk_0)

$$\Rightarrow \exists \; \{\tilde{c}_t\} \text{ s.t. } \sum_{t=0}^{\infty} \beta^t \cdot \frac{\tilde{c}_t^{\,1-\sigma}}{1-\sigma} > \sum_{t=0}^{\infty} \beta^t \frac{(\lambda c^*)^{1-\sigma}}{1-\sigma}$$

$$\Rightarrow \sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{\tilde{c}_t}{\lambda}\right)^{1-\sigma}}{1-\sigma} > \sum_{t=0}^{\infty} \beta^t \frac{(c^*)^{1-\sigma}}{1-\sigma}$$

$\Rightarrow$ CONTRADICTION (add why *)

Implication for value function:—

$$v(\lambda k) = \sum_{t=0}^{\infty} \beta^t \frac{(\lambda c_t)^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} v(k)$$

$\Rightarrow$ decision rules are HD-1

Consumption decision given capital

$$\left.\begin{aligned} g_c(k_t) &= k_t\, g_c(1) \\ g_k(k_t) &= k_t\, g_k(1) \\ g_{x_k}(k_t) &= k_t\, g_x(1) \end{aligned}\right\}$$

If we know the highlighted numbers, we'll know everything about tomorrow.

To get the Euler eq^n.

Sequential Formulation of SP

(This is different from recursive!)

$$\sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma}}{1-\sigma} \longrightarrow \max.$$

$$c_t + (1+\phi)(k_{t+1} - (1-\delta)k_t) = Ak_t \quad (*)$$

(*) where did we come with this?

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[Ak_t - c_t - (1+\phi)(k_{t+1} - (1-\delta)k_t) \right]$$

c_t : $\beta^t \cdot c_t^{-\sigma} = \lambda_t$

k_{t+1} : $-\lambda_t(1+\phi) + \lambda_{t+1} A + \lambda_{t+1}(1+\phi)(1-\delta) = 0$

$$\Rightarrow \lambda_t(1+\phi) = \lambda_{t+1}\left[A + (1+\phi)(1-\delta)\right]$$

$$\Leftrightarrow \beta^t \cdot c_t^{-\sigma}(1+\phi) = \beta^{t+1} c_{t+1}^{-\sigma} \left(A + (1+\phi)(1-\delta)\right)$$

$$\Rightarrow \left[\frac{c_{t+1}}{c_t}\right]^{\sigma} = \beta\left[\frac{A}{(1+\phi)} + (1-\delta)\right]$$

$$\Leftrightarrow \left(\frac{g_c(k_{t+1})}{g_c(k_t)}\right)^{\sigma} = \beta\left[\frac{A}{1+\phi} + 1 - \delta\right]$$

$$\Leftrightarrow \left(\frac{k_{t+1}\, g_c(1)}{k_t\, g_c(1)}\right)^{\sigma} = RHS$$

$$\Leftrightarrow \left(\frac{g_k(k_t)}{k_t}\right)^{\sigma} = RHS$$

$\mathfrak{X}$ $g_c(1) = 1$?
Oh, it just gets divided

$$\Rightarrow \left(\frac{k_t\, g_k(1)}{k_t}\right)^{\sigma} = RHS$$

$$\Rightarrow g_k(1) = \left[\beta\left(\frac{A}{1+\phi} + 1 - \delta\right)\right]^{1/\sigma}$$

Investment Decision:—

$$k_{t+1} = (1-\delta)k_t + x_{kt}$$

$$g_k(k_t) = (1-\delta)k_t + g_x(k_t)$$

$$k_t\, g_k(1) = (1-\delta)k_t + g_x(1) \cdot k_t$$

$$\Leftrightarrow g_x(1) = g_k(1) - (1-\delta)$$

$$g_c(1) = ?$$

$$c_t + (1+\phi)x_{kt} = Ak_t$$

$$g_c(k_t) + (1+\phi)g_x(k_t) = Ak_t$$

$$k_t\, g_c(1) + (1+\phi)k_t\, g_x(1) = Ak_t$$

$$g_c(1) = A - (1+\phi)g_x(1)$$

What was this?

Growth rate of capital

$$\gamma^{k}_{t,t+1} = \frac{k_{t+1}}{k_t} = g_k(1) = \left[\beta\left(\frac{A}{1+\phi} + 1 - \delta\right)\right]^{1/\sigma}$$

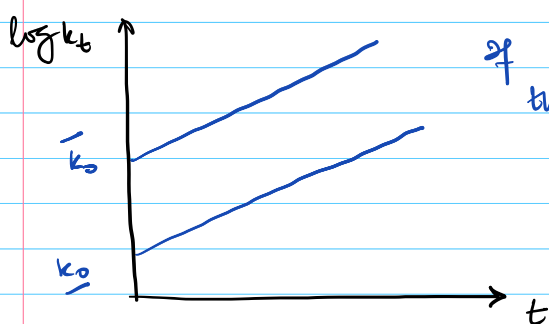
$$k_{t+1} = k_t\, g_t(1)$$

$$k_t = [g_k(1)]^t \cdot k_0$$

Economy grows iff $g_k(1) > 1$

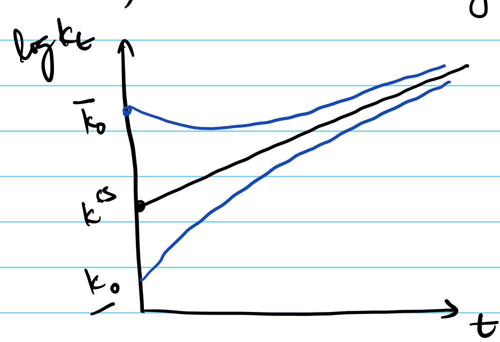
If $\beta \uparrow \Rightarrow$ economy grows faster

$$\log k_t = \log k_0 + t \log(g_k(1))$$



If parameters are the same.

Contrast; in the exog. growth model



$$r^c_{t,t+1} = \left(\frac{c_{t+1}}{c_t}\right) = \left(\frac{k_{t+1}\, g_c(1)}{k_t\, g_c(1)}\right) = g_k(1)$$

Adding fiscal Policy :-

Government, $\{g_t\}_{t=0}^{\infty}$, access to capital & labor income taxes

HH problem: $\sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma}}{1-\sigma} \to \max$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t c_t + p_t x_{kt} + p_t x_{ht} = \sum_{t=0}^{\infty} w_t h_t (1-\tau_{ht}) + (1-\tau_{kt}) r_t k_t$$

$$h_{t+1} = (1-\delta_h) h_t + x_{ht}$$

$$k_{t+1} = (1-\delta_k) k_t + x_{kt}$$

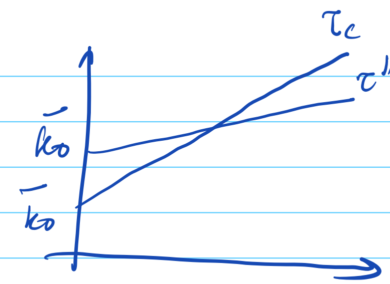
Euler Equations

[EEK] $$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[1-\delta_k + (1-\tau_{k\,t+1}) f_k'(k_{t+1}, h_{t+1})\right]$$

[EEh] $$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[1-\delta_H + (1-\tau_{ht+1}) F_h'(k_{t+1}, h_{t+1})\right]$$

Case 1:

Suppose $\delta_k = \delta_n$, $\tau_{kt} = \tau_{nt} = \tau$



$g_k(1) = ?$ If $\tau \uparrow$

$$g_k(1) = \left[\beta\left\{(1-\delta) + (1-\tau) F'_k(t+1)\right\}\right]^{1/\sigma}$$

$$\Delta \frac{k}{h} = 0 \quad \text{wrt } \tau$$

Case 2 $\quad \delta_n = \delta_k$

$$\tau_{kt+1} = \tau_k \neq \tau_n = \tau_{nt+1}$$