# Estimation of the mean

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Mean estimation is a statistical inference problem in which a sample is used to produce a point estimate of the mean of an unknown distribution.

The problem is typically solved by using the sample mean as an estimator of the population mean.

In this lecture, we present two examples, concerning:

1. normal IID samples;
2. IID samples that are not necessarily normal.

For each of these two cases, we derive the expected value, the variance and the asymptotic properties of the mean estimator.

## Normal IID samples

In this example of mean estimation, which is probably the most important in the history of statistics, the sample is drawn from a normal distribution.

Specifically, we observe the realizations of $n$ independent random variables $X_1$, ..., $X_n$, all having a normal distribution with unknown mean $\mu$ and variance $\sigma^2$.

### The estimator

As an estimator of the mean $\mu$, we use the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

### Expected value of the estimator

The expected value of the estimator $\bar{X}_n$ is equal to the true mean $\mu$.

This can be proved by using the linearity of the expected value:

$$\begin{aligned} \mathrm{E}[\bar{X}_n] &= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}[X_i] \\ &= \frac{1}{n}\sum_{i=1}^{n}\mu \\ &= \frac{1}{n}n\mu = \mu \end{aligned}$$

How?

Isn't this the sample & not the pop? how is it then equal to μ?

Therefore, the estimator $\bar{X}_n$ is unbiased.

### Variance of the estimator

The variance of the estimator $\bar{X}_n$ is equal to $\sigma^2/n$.

This can be proved by using the formula for the variance of an independent sum:

$$\begin{aligned} \mathrm{Var}[\bar{X}_n] &= \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}[X_i] \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 \\ &= \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n} \end{aligned}$$

leave question here

Therefore, the variance of the estimator tends to zero as the sample size $n$ tends to infinity.

### Distribution of the estimator

The estimator $\bar{X}_n$ has a normal distribution:

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Proof

### Risk of the estimator

The mean squared error of the estimator is

$$\begin{aligned} \mathrm{MSE}(\bar{X}_n) &= \mathrm{E}\left[\|\bar{X}_n - \mu\|^2\right] \\ &= \mathrm{E}\left[|\bar{X}_n - \mu|^2\right] && \text{(Euclidean norm in one dimension is equal to absolute value)} \\ &= \mathrm{E}\left[(\bar{X}_n - \mu)^2\right] \\ &= \mathrm{Var}[\bar{X}_n] && \text{(By the definition of variance, because } \mathrm{E}[\bar{X}_n] = \mu\text{)} \\ &= \frac{\sigma^2}{n} \end{aligned}$$

$V[X] = E([X - E[X]]^2)$

$E([(\bar{X}_n - E[\bar{X}_n]]^2)$

### Consistency of the estimator

The sequence $\{X_n\}$ is an IID sequence with finite mean.

Therefore, it satisfies the conditions of Kolmogorov's Strong Law of Large Numbers.

Hence, the sample mean $\bar{X}_n$ converges almost surely to the true mean $\mu$:

$$\bar{X}_n \xrightarrow{a.s.} \mu$$

that is, the estimator $\bar{X}_n$ is strongly consistent.

The estimator is also weakly consistent because almost sure convergence implies convergence in probability:

$$\operatorname*{plim}_{n\to\infty} \bar{X}_n = \mu$$

## IID samples

In this example of mean estimation, we relax the previously made assumption of normality.

The sample is made of the realizations of $n$ independent random variables $X_1$, ..., $X_n$, all having the same distribution with mean $\mu$ and variance $\sigma^2$.

### The estimator

How do we come to this conclusion?

Again, the estimator of the mean $\mu$ is the sample mean:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

### Expected value of the estimator

The expected value of the estimator $\bar{X}_n$ is equal to the true mean:

$$\mathrm{E}[\bar{X}_n] = \mu$$

Therefore, the estimator is unbiased.

The proof is the same found in the previous example.

### Variance of the estimator

The variance of the estimator $\bar{X}_n$ is

$$\mathrm{Var}[\bar{X}_n] = \frac{\sigma^2}{n}$$

Also in this case the proof is the same found in the previous example.

### Distribution of the estimator

Unlike in the previous example, the estimator $\bar{X}_n$ does not necessarily have a normal distribution: its distribution depends on those of the terms of the sequence $\{X_n\}$.

However, we will see below that $\bar{X}_n$ has a normal distribution asymptotically, that is, it converges to a normal random variable when $n$ becomes large.

What is n here?

### Risk of the estimator

The mean squared error of the estimator is

$$\mathrm{MSE}(\bar{X}_n) = \mathrm{Var}[\bar{X}_n] = \frac{\sigma^2}{n}$$

The proof is the same found in the previous example.

### Consistency of the estimator

Since the sequence $\{X_n\}$ is an IID sequence whose terms have finite mean, it satisfies the conditions of Kolmogorov's Strong Law of Large Numbers.

Therefore, the estimator $\bar{X}_n$ is both strongly consistent and weakly consistent (see example above).

### Asymptotic normality

The sequence $\{X_n\}$ is an IID sequence with finite mean and variance.

Therefore, it satisfies the conditions of Lindeberg-Lévy Central Limit Theorem.

?

Hence, the sample mean $\bar{X}_n$ is asymptotically normal:

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \xrightarrow{d} Z$$

where $Z$ is a standard normal random variable and $\xrightarrow{d}$ denotes convergence in distribution.

In other words, the sample mean $\bar{X}_n$ converges in distribution to a normal random variable with mean $\mu$ and variance $\frac{\sigma^2}{n}$.

## Solved exercises

Below you can find some exercises with explained solutions.

### Exercise 1

Consider an experiment that can have only two outcomes: either success, with probability $p$, or failure, with probability $1-p$.

The probability of success is unknown, but we know that

$$p \in \left[\frac{1}{10}, \frac{1}{5}\right]$$

Suppose that we can independently repeat the experiment as many times as we wish and use the ratio

$$\frac{\text{Successes obtained}}{\text{Total experiments performed}}$$

as an estimator of $p$.

What is the minimum number of experiments needed in order to be sure that the standard deviation of the estimator is less than 1/100?

Solution

### Exercise 2

Suppose that you observe a sample of 100 independent draws from a distribution having unknown mean $\mu$ and known variance $\sigma^2 = 1$.

How can you approximate the distribution of their sample mean?

Solution

## How to cite

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