Suggested Solutions: ECON 7710 HW V

Author:
Jiarui (Jerry) Qian

October 27, 2023

Question 1

We are asked to find characteristic function of a uniformly distributed random variable on $[0,1]$ We have $X \sim U[0,1]$ and we know for a uniform distribution function:

$$\phi_X(t) = E[e^{itX}] = \int_0^1 e^{itx}dx = \frac{[e^{itx}]_0^1}{it} = \frac{e^{it}-1}{it}$$

If $t=0$, then $\phi_X(t) = \lim\limits_{t\to 0} \frac{e^{it}-1}{it} = 1$ by L'Hospital rule.

Question 2

X_1, X_2... are Bernoulli random variables with parameter $p = \frac{1}{2}$. Using the method of characteristic functions find the distribution of random variables:

$$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}$$

For X_k, we have $X_k = \{0,1\}$ and $p_k = p = 1/2$. Define $Z_k = \frac{X_k}{2^k}$, then the characteristic function of $Z_k = \frac{X_k}{2^k}$ is:

$$\phi_{Z_k}(t) = E[e^{itZ_k}] = \frac{1}{2}e^{\frac{it}{2^k}} + \frac{1}{2} = \frac{1}{2}[e^{\frac{it}{2^k}} + 1]$$

We know $Y_n = \sum_{k=1}^{n} Z_k$, **assuming X_k's are independent**. we can further derive that the characteristic function of Y_n is:

$$\phi_{Y_n}(t) = \phi_{\sum_{k=1}^{n} Z_k}(t) = \phi_{Z_1}(t) \times \phi_{Z_2}(t) ... \times \phi_{Z_n}(t) = \prod_{k=1}^{n} \frac{1}{2}[e^{\frac{it}{2^k}} + 1] = \frac{1}{2^n} \frac{e^{it}-1}{e^{\frac{it}{2^n}}-1}$$

When $n \to \infty$, we use L'Hospital's rule to derive that:

$$\lim_{n\to\infty} \phi_{Y_n}(t) = \lim_{n\to\infty} \frac{1}{2^n} \frac{e^{it}-1}{e^{\frac{it}{2^n}}-1} = \frac{e^{it}-1}{it}$$

This is the characteristic function of distribution $U[0,1]$.
Then we know the distribution of Y is $U[0,1]$.

Question 3

This problem asks under which conditions imposed on random variables X, random variables X and $sin(X)$ are independent.
Claim: For any continuous measurable function f, X and $f(X)$ are independent if and only if at least one of them is a constant.

- Sufficiency
  If X and $f(X)$ are independent, then let $Y = f(X)$ for some Borel measurable function f such that X and Y are independent and define a new set $A(y) = \{\omega : f(X(\omega)) \leq y\}$. Then we can write $A(y) = \{\omega : X(\omega) \in f^{-1}((-\infty, y])$. Clearly, $A(y) \in \sigma(X)$.
  Then we know $P(X \in A(y), Y \leq y) = P(X \in A(y))P(Y \leq y) = P(f(X) \leq y)P(Y \leq y)$, which holds for all $y \in \mathbb{R}$.
  But by definition of $A(y)$, $P(X \in A(y), Y \leq y) = P(f(X) \leq y, Y \leq y) = P(f(X) \leq y)$.
  In other words, we must have $P(f(X) \leq y)P(Y \leq y) = P(f(X) \leq y)$ and since $Y = f(X)$, This is equal to say:
  $$P(f(X) \leq y)^2 = P(f(X) \leq y) \Rightarrow P(f(X) \leq y) = 0 \text{ or } 1$$
  So we must have some constant $c \in \mathbb{R}$ such that distribution function $f(X)$ jumps from zero to one at c, which means $f(X) = c$ almost surely.
  Then go back to our case when $f() = sin()$, it is straight forward to say only two cases would work:
  1 X is a constant. $X = c$
  2 $sin(X)$ is a constant. $sin(X) = c$. Then due to the property of periodic function, $X = sin^{-1}(c) + 2\pi K$ for any $|c| \leq 1$ and K is an integer will work.

  So ultimately, all the X that makes $X \perp f(X)$ is when X is a constant c. Or any random variable X with support on $X = sin^{-1}(c) + 2\pi K$ for any $|c| \leq 1$ and any subset of integer K.

- Necessary:
  This is much easier to check. When X is a degenerate distribution, $P(X \in B_1) = \{0,1\}$ and $P(sin(X) \in B_2) = \{0,1\}$. The joint probability $P(X \in B_1, sin(X) \in B_2) = \{0,1\}$.
  Clearly, the following equation holds:
  $$P(X \in B_1, sin(X) \in B_2) = P(X \in B_1) \times P(sin(X) \in B_2)$$

which implies random variables X and $sin(X)$ are independent.
Formally, if X is a degenerate distribution, with $P(X = c) = 1$, then the characteristic function is

$$\phi_X(t) = e^{itc}$$

Likewise, $sinX$ is also a degenerate distribution with $P(sinX = sin(c)) = 1$, the characteristic function is

$$\phi_{sinX}(t) = e^{itsin(c)}$$

Then we know for distribution, $X + sin(X)$ it is also a degenerate distribution with $P(X + sin(X) = c + sin(c)) = 1$. So

$$\phi_{X+sinX} = e^{it(c+sin(c))} = \phi_X(t)\phi_{sinX}(t) = e^{itc} * e^{itsin(c)} = e^{it(c+sin(c))}$$

The discussion on $sin(X) = c$ follows the same steps as above.