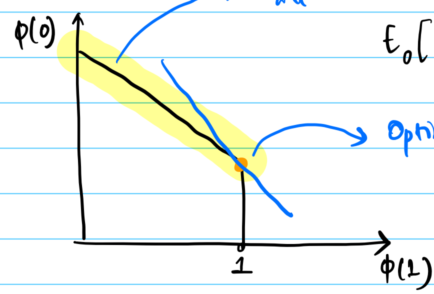
Nov 27, 2023

$H_0 : \{p_0\}$, $H_1 : \{p_1\}$

$p_1 > p_0$

Acceptable for all across the boundary

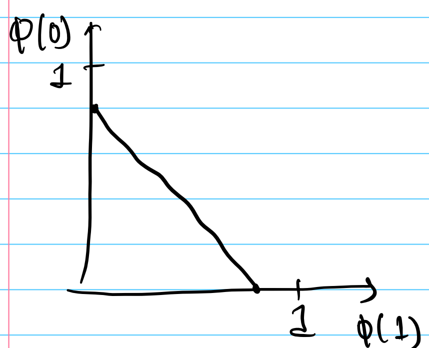


$E_0[\phi(x)] \leq \alpha$

Optimal test (highest IC)

$\beta(\theta) = E_\theta[\phi(x)]$ (rejection rate under the alternative)

$\inf_{\theta \in \Theta_1} \beta(\theta) \rightarrow$ power.



$\underset{\sim}{x} \in \mathcal{X}$

$\mathcal{X} = S_0 \cup S_1$

acceptance region

$$\phi(x) = \begin{cases} 1, & \text{if } x \in S_1 \\ 0, & \text{if } x \in S_0 \end{cases}$$

rejection region

$S_0 = \{0\}$
$S_1 = \{1\}$

is a distn

$H_0 = \{P_0(\cdot)\}$ $\quad H_1 : \{P_1\}$

Distribution's support co-incide.

when distribution is not common, it is much easier. say $4 \in H_0$ & not H_1. If $x = 4$, you know it is H_0 : imp for exam.

size of the test

Optimization Problem: significance level

size constraint

$$\sum_{x \in S_1} P_0(x) \leq \alpha \Big\} \rightarrow \text{Budget Constraint}$$

objective fn: $\left\{ \max_{S_1} \sum_{x \in S_1} P_1(x) \right\} \rightarrow$ utility fn

(p of rejection under the alt.)

This is the utility.

$$\left(\frac{P_1(x_i)}{P_0(x_i)} \right\}$$ Bank to their buck

This is the price.

we want ~~to~~ choose the elements that give us the max utility for given price.

$$S_1 = \left\{ x_i : \frac{P_1(x_i)}{P_0(x_i)} \geq \gamma \right\}$$

γ is chosen so that the following is satisfied:

$$\sum_{\frac{P_1(x_i)}{P_0(x_i)} \geq \gamma} P_0(x_i) \leq \alpha$$

Continuous Distribution

$H_0 : \{f_0(\cdot)\}$ $\qquad$ $H_1 : \{f_1(\cdot)\}$

Theorem (Neymann – Pearson's fundamental lemma)

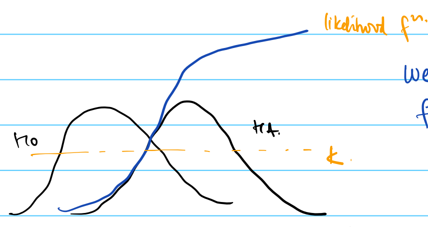
(i) (Existence): For H_0 against H_1 there exists test

$E_0[\phi(X)] = \alpha$ (*) and $\phi(x) = \begin{cases} 1, & f_1(x) \geq k f_0(x) \quad \text{(rejecting)} \\ 0, & \text{otherwise} \quad \text{(not rejecting)} \end{cases}$ (**)

(ii) (Sufficiency): if test $\phi(x)$ satisfies (*) and (**), then it is optimal.

(iii) (Necessity): If $\phi(x)$ is optimal, then it satisfies (**)

(it also satisfies (*) unless there is a test w. size $< \alpha$ and power $= 1$).



We reject ~~H_0~~ when likelihood f^n is above k.

Stylized Example

$H_0 : \{N(\mu_0, 1)\}$; $H_1 : \{N(\mu_1, 1)\}$, $\mu_1 > \mu_0$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_0)^2}{2}} \qquad f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2}}$$

$$\phi(x) = \mathbb{1}\left\{ \frac{f_1(x)}{f_0(x)} \geq k \right\}$$

Monotone f^n of x.

Likelihood ratio: $e^{-\frac{\mu_0^2 - \mu_1^2}{2} + (\mu_1 - \mu_0)x} > k$.

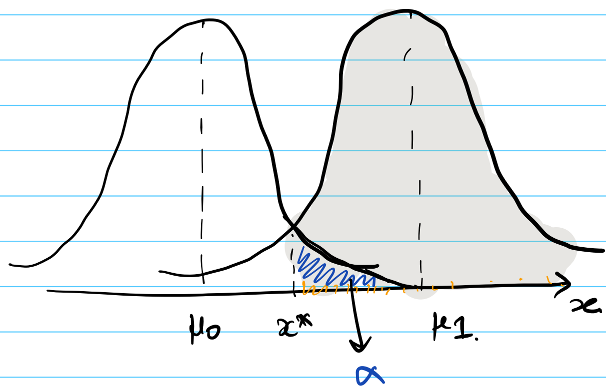
$\exists\, x^*$, we would be rejecting if $x \geq x^*$

$$\phi(x) = \begin{cases} 1, & \text{if } x > x^* \\ 0, & \text{otherwise} \end{cases}$$

CDF of standard normal distn

$$P_0(x > x^*) = 1 - \Phi(x^* - \mu)$$
$$= \Phi(\mu_0 - x^*) = \alpha$$

$\Phi(z_\alpha) = \alpha$: $\mu_0 - x^* = z_\alpha \Rightarrow x^* = \mu_0 - z_\alpha$ [we will be rejecting whenever we are at this pt.]



→ Probability of rejection

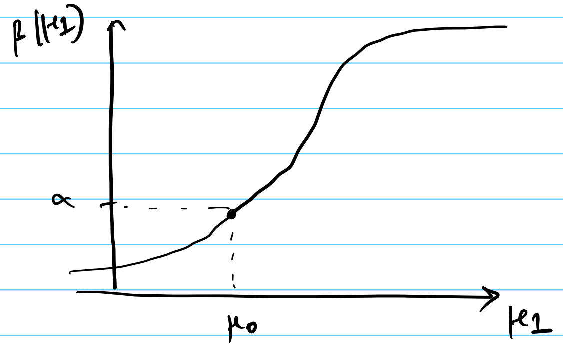
→ rejection region

→ Power → $P_1(x > x^*)$

$= 1 - \Phi(x^* - \mu_1)$

$= \Phi(z_\alpha + \mu_1 - \mu_0)$

another exam question: construct power function under the alternative :



* The more separation there is b/w null & alternative, greater power of the test.

$x^* = \mu_0 - z\alpha$.

(No role of μ_1 here)