12 Oct 2023

Topic : Principle of Optimality

Sequential Problem

$$w(x_0) = \sup_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

s.t $x_{t+1} \in \Gamma(x_t)$ (i.e. is feasible)

x_0 is given

Recursive formulation

$$v(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$$

s.t. x_0 is given

Note : We are using sup instead of max because max may not always exist. sup make it more general.

Q: 1) Under which conditions $v(x) = w(x)$?

2) when $\{x_{t+1}\}$ from SP be the same as the one implied by FE?

functional equation

Preliminaries

X - set of values, that state x_t can take

$\Gamma : X \Rightarrow X$ is feasibility correspondence.

function is a single valued. A correspondence can result in a set.

$c + k' = f(k)$

correspondence spits out entire set of k that's feasible, not just a number.

gamma

Def^n: A graph of Γ, A, is

$$A = \{(x, y) : y \in \Gamma(x)\}$$

Period return $F(x, y)$

$F : A \to \mathbb{R}$

Period Return

Discount factor.

Fundamentals : $(\underbrace{\Gamma, X}_{\text{production technology}}, \underbrace{F, \beta}_{\text{Preferences}})$

$\Gamma \to$ what is feasible

$X \to$ values x_t can take

Def^n: A sequence of states $\{x_t\}_{t=0}^{\infty}$ is a plan.

Def^n: Given x_0, $\Pi(x_0) = \{\{x\}_{t=1}^{\infty} : x_{t+1} \in \Gamma(x_t)\}$ is a set of feasible plans.

$\bar{x}$ be a generic element of $\Pi(x_0)$.

Assumptions:

A 1) $\forall x_0 \in X$, $\Pi(x_0)$ is not empty

A 2) $\forall x_0$, for any $\bar{x} \in \Pi(x_0)$

$$\lim_{n \to \infty} \sum_{t=1}^{n} \beta^t F(x_t, x_{t+1}) \text{ exists.}$$

(Can be $+\infty$ or $-\infty$)

(s.P. is convergent)

Sufficient Conditions for A 2:

(any one of these conditions are sufficient)

1) $\beta \in (0,1)$, F is bounded.

(Just having F bounded w/o any restriction on β is not sufficient)

$$\beta = 1 \quad F(x_t, x_{t+1}) = \begin{cases} -1, & t \text{ is even} \\ 1, & t \text{ is odd} \end{cases}$$

It never converges

$\hookrightarrow$ $1, (-1+1), 1, (-1+1) + \dots$

2) Define $F^+ = \max\{0, F\}$ (bounded above)

$F^- = \max\{0, -F\}$ (bounded below)

If $\lim_{n \to \infty} \sum_{t=1}^{n} \beta^t F^+(x_t, x_{t+1}) < +\infty$

or $\lim_{n \to \infty} \sum_{t=1}^{n} \beta^t F^-(x_t, x_{t+1}) < +\infty$

$\Rightarrow$ A2 holds

(Can this not oscillate, how would it converge then?)

3) $\forall x_0 \in X$ & any $\bar{x} \in \Pi(x_0)$,
$\exists\ \theta \in (0, 1/\beta)$, $c \in (0, +\infty)$
s.t.
$F(x_t, x_{t+1}) \leq c\theta^t$

why?!

Def^n : $\{u_n\}$ is a sequence of functions (why make this def^n?)

$$u_n(\bar{x}_t) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

Vlad: No reason

Given A2, limit

$\lim_{n\to\infty} u_n$ exists

$$u(\bar{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) \text{ exists.}$$

$$w(x_0) = \sup_{\bar{x} \in \Pi(x_0)} u(\bar{x})$$

Principle of Optimality :-

Thm : Let (X, Γ, F, β) st. A1 and A2 hold.

$\Rightarrow$ 1) w (solution to SP) satisfies FE.

2) If $\forall x_0 \in X$, and $\forall \bar{x} \in \Pi(x_0)$

$$\lim_{n\to\infty} \beta^n v(x_n) = 0$$

$$\Rightarrow v = w$$

< notes about the above thm >

Thm: Suppose (X, Γ, F, β) satisfy A1, A2:

1) Let $\bar{x} \in \Pi(x_0)$ attain supremum in SP
$\Rightarrow \forall t \geq 0$
$$w(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta w(\bar{x}_{t+1})$$

2) Let $\hat{x} \in \Pi(x_0)$ be a feasible plan that satisfies $w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$
and additionally $\limsup_{t\to\infty} \beta^t w(\hat{x}_t) \leq 0$

$\Rightarrow \{\hat{x}_t\}$ attains supremum in S.P. for given x_0.