ECON 7710 TA Session

Week 5

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Outline

1. Suggested Solutions to Homework II
2. Practice Questions

Question 1

Setup:

- We know X and Y are independent. Define the 4 random variables as follows:
  - $A = \max\{X, Y\}$
  - $B = \min\{X, Y\}$
  - $C = \max\{2X, Y\}$
  - $D = \min\{X^3, Y\}$
- For **Max**

$$P(\max\{X, Y\} \leq a) = P(X \leq a \text{ and } Y \leq a) = F_X(a) * F_Y(a)$$

- For **Min**

$$P(\min\{X, Y\} > b) = P(X > b \text{ and } Y > b) = (1 - F_X(b))(1 - F_Y(b))$$

- Since you flip the inequality of CDF:
  - $P(X > b) = 1 - F_X(b)$ and $P(Y > b) = 1 - F_Y(b)$
- $F_B(\cdot) = P(\min\{X, Y\} \leq b) = 1 - P(\min\{X, Y\} > b)$

Question 1

- $A = \max\{X, Y\}$
  - $F_A(a) = P(A \leq a) = P(\max\{X, Y\} \leq a) = F_X(a) * F_Y(a)$
- $B = \min\{X, Y\}$
  - $F_B(b) = P(B \leq b) = P(\min\{X, Y\} \leq b) = 1 - (1 - F_X(b))(1 - F_Y(b)) = F_X(b) + F_Y(b) - F_X(b) * F_Y(b).$
- $C = \max\{2X, Y\}$, notice $2x$ is monotonically increasing in x.
  - $F_C(c) = P(C \leq c) = P(\max\{2X, Y\} \leq c) = P(2X \leq c \text{ and } Y \leq c) = P(X \leq \frac{c}{2}) * P(Y \leq c) = F_X(\frac{c}{2}) * F_Y(c)$
- $D = \min\{X^3, Y\}$, notice x^3 is monotonically increasing in x.
  - $F_D(d) = P(D \leq d) = P(\min\{X^3, Y\} \leq d) = 1 - ((1 - F_X(d^{\frac{1}{3}}))(1 - F_Y(d))) = F_X(d^{\frac{1}{3}}) + F_Y(d) - F_X(d^{\frac{1}{3}}) * F_Y(d)$

Question 2.a

- **Setup:**
  - Unit Square: $S = \{0 \le x \le 1 \ , 0 \le y \le 1\}$
  - CDF: $F_{X,Y}(x,y) = P(X \le x, \ Y \le y) = \frac{1}{2}(xy + min\{x,y\})$

a How the CDF behave outside the unit square? 4 Cases:

  - When $x > 1, y > 1$:
    We already have $F(1,1) = 1$. Meanwhile, $F(x,y) \in [0,1]$ and it is nondecreasing. So we know $F(x,y) = 1$ when $x > 1$ and $y > 1$.
  - When $x \in [0,1]$ and $y > 1$:
    Then we know $F(x,y) = P(X \le x, \ Y \le y) = P(X \le x)$, when $0 \le x \le 1$ and $y = 1$. Then we know $F(x,y) = \frac{1}{2}(2x) = x$. (It depends on x only)
  - When $x > 1$ and $y \in [0,1]$:
    Then we know $F(x,y) = P(X \le x, \ Y \le y) = P(Y \le y)$, when $0 \le y \le 1$ and $x = 1$. Then we know $F(x,y) = \frac{1}{2}(2y) = y$. (It depends on y only)
  - When $x < 0$ **or** $y < 0$:
    We know $F(0,y) = 0, \forall y \ge 0$ and $F(x,0) = 0, \forall x \ge 0$. We also know CDF is nondecreasing and the value of CDF lies between 0 to 1. Therefore $F(x,y) = 0$ if $x < 0$ or $y < 0$.

Question 2.a

To sum up, we know when (x, y) lies outside the unit square, we know:

$$F(x,y) = \begin{cases} 1, & \text{For } x \in (1,\infty),\ y \in (1,\infty) \\ x, & \text{For } x \in [0,1],\ y \in (1,\infty) \\ y, & \text{For } x \in (1,\infty),\ y \in [0,1] \\ 0, & \text{For } x \in (-\infty,0) \text{ or } y \in (-\infty,0) \end{cases}$$

Question 2.b

b We know the CDF is

$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } x \in (-\infty, 0) \text{ or } y \in (-\infty, 0) \\ \frac{1}{2}(xy + min\{x,y\}), & \text{For } x \in [0,1],\ y \in [0,1] \\ x, & \text{For } x \in [0,1],\ y \in (1,\infty) \\ y, & \text{For } x \in (1,\infty),\ y \in [0,1] \\ 1, & \text{For } x \in (1,\infty),\ y \in (1,\infty) \end{cases}$$

- Denote marginal CDF of X as $F_X(x)$ and marginal CDF of Y as $F_Y(y)$. We know:

$$F_X(x) = \lim_{y \to +\infty} F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } x \in (-\infty, 0) \\ x, & \text{For } x \in [0,1] \\ 1, & \text{For } x \in (1,\infty) \end{cases}$$

Likewise, we know:

$$F_Y(y) = \lim_{x \to +\infty} F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } y \in (-\infty, 0) \\ y, & \text{For } y \in [0,1] \\ 1, & \text{For } y \in (1,\infty) \end{cases}$$

Question 2.c

- c We know $U = logX$, and $V = logY$, both are monotonically increasing functions.
- We denote CDF of (U, V) as $F_{U,V}(u, v)$. So we know $F_{U,V}(u, v) = P(U \leq u, V \leq v) = P(logX \leq u, logY \leq v) = P(X \leq e^u, Y \leq e^v)$ As we know from a), the CDF of (X, Y) is

$$F_{X,Y}(x, y) = \begin{cases} 0, & \text{For } x \in (-\infty, 0) \text{ or } y \in (-\infty, 0) \\ \frac{1}{2}(xy + min\{x, y\}), & \text{For } x \in [0, 1], y \in [0, 1] \\ x, & \text{For } x \in [0, 1], y \in (1, \infty) \\ y, & \text{For } x \in (1, \infty), y \in [0, 1] \\ 1, & \text{For } x \in (1, \infty), y \in (1, \infty) \end{cases}$$

Therefore, we know CDF of (U, V) is

$$F_{U,V}(u, v) = \begin{cases} \frac{1}{2}(e^{u+v} + min\{e^u, e^v\}), & \text{For } u \in (-\infty, 0], v \in (-\infty, 0] \\ e^u, & \text{For } u \in (-\infty, 0], v \in (0, \infty) \\ e^v, & \text{For } u \in (0, \infty), v \in (-\infty, 0] \\ 1, & \text{For } u \in (0, \infty), v \in (0, \infty) \end{cases}$$

Question 2.d

d We are asked if the following distributions
   1. joint distribution of X and Y
   2. marginal distribution of X and Y
   3. joint distribution of U and V

  has a density w.r.t. Lebesgue measure respectively.

- We also know that a distribution does not have a density w.r.t. the Lebesgue Measure if it places **positive probability on a set has Lebesgue Measure zero**.
- A formal discussion can be found here. In our case, you need to know any line in $\mathbb{R}^2$ has a Lebesgue measure zero.
- In other words, we want to see if we can find any line in R^2 with a positive probability.

Question 2.d.1

d.1 Joint distribution of X and Y **has no density** w.r.t Lebesgue measure
The CDF of (X, Y) is

$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } x \in (-\infty, 0) \text{ or } y \in (-\infty, 0) \\ \frac{1}{2}(xy + min\{x,y\}), & \text{For } x \in [0,1],\ y \in [0,1] \\ x, & \text{For } x \in [0,1],\ y \in (1,\infty) \\ y, & \text{For } x \in (1,\infty),\ y \in [0,1] \\ 1, & \text{For } x \in (1,\infty),\ y \in (1,\infty) \end{cases}$$

This CDF can tell us a a lot:

1 Within the unit square, it does not have a derivative at any point along the line when $x = y$ (kink of min function).

2

$$\frac{\partial F(x,y)}{\partial x \partial y} = \begin{cases} \frac{1}{2} & x \in [0,y),\ \ y \in (0,1] \\ \frac{1}{2} & x \in (y,1],\ \ y \in [0,1) \\ 0 & \text{Outside of the unit square} \end{cases}$$

3 Probability of drawing a pair of (x, y) outside of unit square is 0.

4

$$\left.\begin{array}{l} Pr(0 \leq X < Y, 0 < Y \leq 1) = \int_0^1 \int_0^y \frac{1}{2}\, dxdy = \frac{1}{4} \\ Pr(Y < X \leq 1, 0 \leq Y < 1) = \int_0^1 \int_y^1 \frac{1}{2}\, dxdy = \frac{1}{4} \end{array}\right\} \Rightarrow \frac{1}{2} \text{ of the missing mass is on line } X = y$$

d.2

- We already know marginal distribution of X is:

$$F_X(x) = \begin{cases} 0, & \text{For } x < 0 \\ x, & \text{For } x \in [0,1] \\ 1, & \text{For } x > 1 \end{cases}$$

And marginal distribution of Y is:

$$F_Y(y) = \begin{cases} 0, & \text{For } y < 0 \\ y, & \text{For } y \in [0,1] \\ 1, & \text{For } y > 1 \end{cases}$$

- For $F_X(x)$, if we pick an arbitrary line in unit square, the marginal probability will always be 0 because y is contained in the line but there is no y in $F_X(x)$. Same logic for $F_Y(y)$.
- So we know marginal distribution of X and Y have a density w.r.t. the Lebesgue Measure.

Question 2.d.3

d.3

- We already know the CDF of (U, V) is

$$F_{U,V}(u,v) = \begin{cases} \frac{1}{2}(e^{u+v} + min\{e^u, e^v\}), & \text{For } u \in (-\infty, 0],\ v \in (-\infty, 0] \\ e^u, & \text{For } u \in (-\infty, 0],\ v \in (0, \infty) \\ e^v, & \text{For } u \in (0, \infty),\ v \in (-\infty, 0] \\ 1, & \text{For } u \in (0, \infty),\ v \in (0, \infty) \end{cases}$$

- Similarly, we follow the steps in d.1 and set our line with Lebesgue Measure of zero as

$$\{(u,v) : u = v,\ \text{For, } u \in (-\infty, 0],\ v \in (-\infty, 0]\}$$

- Then we will get a positive probability on a set has Lebesgue measure zero, which means the joint distribution of U and V does not have a density w.r.t. the Lebesgue Measure.

- In summary, joint distributions(d.1, d.3) don't have a density. Marginal distributions(d.2) have a density.

Question 2.e

- 2.d tells us when x, y are in unit square:
  - With mass $= \frac{1}{2}$, $\frac{\partial F(x,y)}{\partial x \partial y} = \frac{1}{2}$, when $x \neq y$
  - With mass $= \frac{1}{2}$, when $x = y$
- This tells us the distribution of X, Y should be a **mixture** of
  - a bivariate uniform distribution (X, Y are independent)
  - and uniform on the line $x = y$ (X, Y are perfectly correlated).
- In other words, half of the time, pair (x, y) is drawn uniformly from the unit square, half of the time it is drawn from the line $x = y$.
- For the conditional distribution $X|Y$, it should also be a **mixture**.
  - Half of the time
    X will be drawn independently of the value of Y from $[0, 1]$ [Continuous Part].
  - Half of the time
    X will be equal to the value of Y [Discrete Part].

Question 2.e.1, 2.e.2

Then we know:

- Partial density function for the continuous part:

$$f_{X|Y}(x|y) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

- Partial probability mass function for the discrete part:

$$g_{X|Y}(x|y) = \begin{cases} 1 & x = y, \ \ (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{Otherwise} \end{cases}$$

Then the we know:

$$E[X|Y] = \frac{1}{2} \int_0^1 x dx + \frac{1}{2} y = \frac{1}{4} + \frac{1}{2} y$$

And by symmetry, we know:

$$E[Y|X] = \frac{1}{4} + \frac{1}{2} x$$

Question 2.e.3, 2.e.4

- Likewise, the distribution of U, V is also a mixture of bivariate uniform distribution [Continuous Part] and a uniform on the line $u = v$ [Discrete Part].
  Then we know:
  - Partial density function for the continuous part:

$$f_{U|V}(u|v) = \begin{cases} e^u & u \in (-\infty, 0] \\ 0 & \text{Otherwise} \end{cases}$$

  - Partial probability mass function for the discrete part:

$$g_{U|V}(u|v) = \begin{cases} 1 & u = v, \ \ (u,v) \in (-\infty, 0] \times (-\infty, 0] \\ 0 & \text{Otherwise} \end{cases}$$

  Then we know:

$$E[U|V] = \frac{1}{2}\int_{-\infty}^{0} ue^u du + \frac{1}{2}v = \frac{1}{2}([ue^u]_{-\infty}^{0} - \int_{-\infty}^{0} e^u du) + \frac{1}{2}v$$

Question 2.e.3, 2.e.4

$$E[U|V] = \frac{1}{2}\int_{-\infty}^{0} ue^u du + \frac{1}{2}v = \frac{1}{2}([ue^u]_{-\infty}^{0} - \int_{-\infty}^{0} e^u du) + \frac{1}{2}v$$

We know: $\lim_{u \to 0} ue^u = 0$. We also know $\lim_{u \to -\infty} ue^u = \lim_{u \to -\infty} \frac{u}{e^{-u}} = \lim_{u \to -\infty} \frac{1}{-e^{-u}} = 0$ by L'Hôpital's rule. So

$$E[U|V] = \frac{1}{2}(v - \int_{-\infty}^{0} e^u du) = \frac{1}{2}v - \frac{1}{2}$$

By symmetry, we know:

$$E[V|U] = \frac{1}{2}u - \frac{1}{2}$$

Practice Question

Deciding whether certain random variables are independent or not is one type of question that Denis likes a lot. Check $C\&B 4.2$ for details.

2016 Midterm Q3 & 2018 Midterm Q2

Random variables X and Y have a joint density

$$f(x,y) = \begin{cases} \frac{1}{2}(1+xy), & \text{if } |x| < 1 \text{ and } |y| < 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Are X and Y independent? Prove or disprove.
2. Are X^2 and Y^2 independent? Prove or disprove.

Hint:

1. Random variables are independent iff joint density $f(x,y) = f_X(x) * f_Y(y)$, where are $f_X(x)$ and $f_Y(y)$ are marginal densities. Or joint CDF $F(x,y) = F_X(x) * F_Y(y)$, where $F_X(x)$ and $F_Y(y)$ are CDF of X and Y
2. Think about how to find the joint CDF of (X^2, Y^2)

Practice Question

Random variables X and Y have a joint density

$$f(x,y) = \begin{cases} \frac{1}{2}(1+xy), & \text{if } |x| < 1 \text{ and } |y| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Marginal densities:

$$f_X(x) = \int_{-1}^{1} \frac{1}{2}(1+xy)dy = 1, \quad f_Y(y) = 1 \ \text{(By Symmetry)}$$

Clearly, $f_X(x) * f_Y(y) = 1 \neq f(x,y)$. So X and Y are not independent.

Practice Question

Random variables X and Y have a joint density

$$f(x,y) = \begin{cases} \frac{1}{2}(1+xy), & \text{if } |x| < 1 \text{ and } |y| < 1 \\ 0, & \text{otherwise} \end{cases}$$

The joint CDF for (X^2, Y^2) is $P(X^2 \leq x, Y^2 \leq y) =$

$$\begin{cases} P(-\sqrt{x} \leq X \leq \sqrt{x}, -\sqrt{y} \leq Y \leq \sqrt{y}) & x \in [0,\infty), y \in [0,\infty) \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \int_{-\sqrt{x}}^{\sqrt{x}} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}(1+uv)dudv^{1} & x \in [0,1), y \in [0,1) \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} 2\sqrt{x}\sqrt{y} & x \in [0,1), y \in [0,1) \\ 0 & \text{Otherwise} \end{cases}$$

[1] Distinguish variables and numbers

Practice Question

From part 1, Random variables X and Y have a marginal densities

$$\begin{cases} f_X(x) = 1 \\ f_Y(y) = 1 \end{cases}$$

Marginal distribution function of X^2:

$$P(X^2 \le x) = \begin{cases} P(-\sqrt{x} \le X \le \sqrt{x}) & x \in [0,1) \\ 0 & \text{Otherwise} \end{cases} = \begin{cases} \int_{-\sqrt{x}}^{\sqrt{x}} 1du & x \in [0,1) \\ 0 & \text{Otherwise} \end{cases}$$

$$\Rightarrow F_{X^2}(x) = \begin{cases} 2\sqrt{x} & x \in [0,1) \\ 0 & \text{Otherwise} \end{cases} \quad F_{Y^2}(y) = \begin{cases} 2\sqrt{y} & y \in [0,1) \\ 0 & \text{Otherwise} \end{cases}$$

$$\mathrm{F}_{X^2,Y^2}(x,y) = \begin{cases} 2\sqrt{x}\sqrt{y} & x \in [0,1), y \in [0,1) \\ 0 & \text{Otherwise} \end{cases}$$

X^2 and Y^2 are not independent since $F_{X^2,Y^2}(x,y) \neq F_{X^2}(x) * F_Y^2(y)$

Practice Question

2021 Midterm Q2

Let $(\Omega, \mathcal{F}, P)$ be the probability space. Provide a necessary and sufficient condition on P such that for *any* pair of events $A \neq B \in \mathcal{F}$, $P(A \cap B) = 0$. i.e. any pair of non-identical events on that probability space are independent.

Hints: What means $A \neq B$ in set theory? How to say independence of random variables.

Practice Question

2021 Midterm Q2

Let $(\Omega, \mathcal{F}, P)$ be the probability space. Provide a necessary and sufficient condition on P such that for *any* pair of events $A \neq B \in \mathcal{F}$, $P(A \cap B) = 0$. i.e. any pair of non-identical events on that probability space are independent.

- $A \neq B \Leftrightarrow A = B^C$. So $P(A \cap B) = P(B^C \cap B) = P(B^C) * P(B)$ by independence.
- Since $P(A \cap B) = 0$, we know $P(B^C \cap B) = P(B^C) * P(B) = 0$.
- So either
  - $P(B) = 0$
  - Or $P(B^C) = 0 \Leftrightarrow P(B) = 1$
- Then we know the necessary and sufficient condition on P is

$$P(\cdot) = 0 \quad \text{or} \quad P(\cdot) = 1$$