Assignment 2

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1.

X & Y are independent random variables with dist^n f^n $F(\cdot)$ & $G(\cdot)$

(a) $\max\{X, Y\}$

Let $Z = \max\{X, Y\}$

$$P(Z < z) = P(X < z, Y < z)$$
$$= P(X < z)\, P(Y < z)$$
$$= F(z)\, G(z)$$

(b) $\min\{X, Y\}$

~~This is wrong~~

$P(\min\{X, Y\} > b)$

$\boxed{= 1 - \{P(\min\{X, Y\} < b)\}}$

Let $\min\{x, y\} \geq z$

$$\Rightarrow P(X > z)\, P(Y > z)$$
$$\Rightarrow (1 - P(X \leq z))(1 - P(Y \leq z))$$
$$\Rightarrow (1 - F(z))(1 - G(z))$$
$$\Rightarrow 1 - G(z) - F(z) + F(z)G(z)$$

(c) $\max\{2X, Y\}$

Let $\max\{2X, Y\} \leq z$.

$\Rightarrow P(2X \leq z)\, P(Y \leq z)$

$\Rightarrow F(z/2)\, G(z)$

(d) $\min\{X^3, Y\}$

Let $\min\{X^3, Y\} > z$

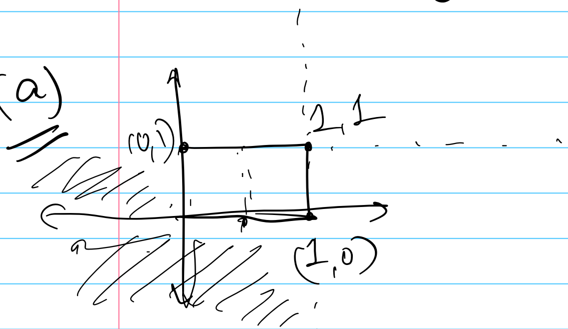
$\Rightarrow P(X^3 > z)\, P(Y^3 > z)$

$\Rightarrow \left(1 - F\left(z^{1/3}\right)\right)\left(1 - G(z)\right)$

(2) cdf of (X, Y) on the unit square is $\frac{1}{2}(xy + \min\{x,y\})$

$$\Rightarrow \quad F(x,y) = \begin{cases} \frac{1}{2}(xy + \min\{x,y\}), & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\ \end{cases}$$

(a)



Evaluating $F(x,y)$ from above on end points

$F(0,y) = 0$

$F(x,0) = 0$

$F(1,1) = \frac{1}{2}(1+1) = 1$

$F(0.5, 1) = \frac{1}{2}(0.5 + 0.5) = \frac{1}{2} = \frac{1}{2}(2x) = x$

$F(1, 0.5) = \frac{1}{2}(0.5 + 0.5) = \frac{1}{2} = \frac{1}{2}(2y) = y$

CDF behaviour outside the unit square:
(x or $y < 0$ / $x > 1$ / $y > 1$)

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ x & \text{if } x < 1 \text{ \& } y > 1 \\ y & \text{if } x > 1 \text{ \& } y < 1 \\ 1 & \text{if } x > 1 \text{ \& } y > 1 \end{cases}$$

did not bound this to be between 0 & 1

(b) Marginal cdf:

when y goes to infinity, the f^n is x.

$$F_X(x) = F_{X,Y}(x,\infty) = \lim_{y\to\infty} F_{X,Y} = \begin{cases} 0 & \text{if } x < 0 \\ x, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}$$

$$F_Y(y) = F_{X,Y}(\infty, y) = \lim_{x\to\infty} F_{X,Y} = \begin{cases} 0, & \text{if } y \leq 0 \\ y, & \text{if } 0 \leq y < 1 \\ 1, & \text{if } y \geq 1 \end{cases}$$

(c) If $U = \log X$ & $V = \log Y$, find cdf (U, V)

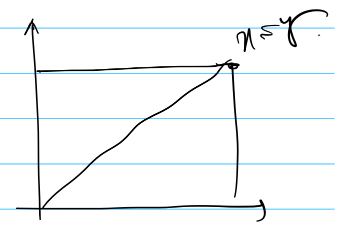
$F(u) = P(\log x \leq u) = P(x \leq e^u) = e^u$

$F(v) = P(\log y \leq v) = P(y \leq e^v) = e^v$

$F(u,v) = e^u e^v$ as u & v are independent events.

⊛ Need to write out the CDF while as well as in part (a)!

(d) Inside the unit square box



cdf = $\frac{1}{2}(xy + \min\{x, y\})$

$F(x,y) = \frac{1}{2}(xy + \min\{x,y\})$

- for $x < y$:-

$\frac{\partial F(x,y)}{\partial x} = \frac{1}{2}(y+1)$; $\frac{\partial f(x,y)}{\partial y} = \frac{1}{2}(x)$

$\frac{\partial F(x,y)}{\partial x \partial y} = \frac{1}{\partial y}\left(\frac{1}{2}(y+1)\right) = \frac{1}{2}$ — (1)

- for $x > y$

$\frac{\partial F(x,y)}{\partial y} = \frac{1}{2}(x+1)$; $\frac{\partial f(x,y)}{\partial x} = \frac{1}{2}(y)$

$\frac{\partial f(x,y)}{\partial y \partial x} = \frac{1}{\partial x}\left(\frac{1}{2}(x+1)\right) = \frac{1}{2}$ —(2)

(1) + (2) = $\frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow$ The density is concentrated on points $x > y$ or $x < y$ & none of the density reside at $x = y$ $\Rightarrow$ no density on lebesgue measure inside the graph.

From the cdf achieved in part 1, we can see that no density exists on line $x = y$ outside the unit box as well.

$\Rightarrow$ Joint distn of X & Y does not have a density wrt lebesgue measure.

(*) Marginal distributions of X and Y:

$\frac{\partial F_x}{\partial x} = 1$ | $\frac{\partial F_y}{\partial y} = 1.$

$\Rightarrow$ $\mu([a, b]) = 0$ $\Rightarrow$ measure zero. $\Rightarrow$ no density on lebesgue measure.

(*) ~~U~~ U & V distribution.

$f(u, v) = e^u e^v$

$\frac{\partial f}{\partial v \, du} = \frac{\partial f}{dv} e^u e^v = e^u e^v$ $\Rightarrow$ There is density on lebesgue measure.

(e) find $E[Y|X]$, $E[U|V]$, $E[V|U]$

$E[Y|X]$

$$= \int_0^1 y \cdot \frac{f_{XY}(x,y)}{f_X(x)} \cdot dy = \int_0^1 y \cdot \left(\frac{1}{2}\right)$$

$$= \cancel{\frac{y^2}{4}}$$

$E[U|V]$

$$= \int_{-\infty}^{0} u \, \frac{f_{U,V}(u,v)}{f_V(v)} = \int_{-\infty}^{0} \frac{u \, e^{u+\cancel{v}}}{\cancel{e^{v}}} \, du$$

$$= e^{u}[u-1]\Big|_{-\infty}^{0} = -1 - 0 = -1$$

$E[V|U]$

$$= \int_{-\infty}^{0} v \cdot \frac{f_{U,V}(u,v)}{f_U(u)} = \int_{-\infty}^{0} v \cdot \frac{\cancel{e^{u}} \cdot e^{v}}{\cancel{e^{u}}} \, dv$$

$$= e^{v}(v-1)\Big|_{-\infty}^{0} = -1$$