Recursive Formulation

Define $f(k) = F(k,1) + (1-\delta)k$

$$\Rightarrow \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \longrightarrow \max_{\{k_{t+1}\}_{t=0}^{\infty}}$$

$$\text{s.t.} \begin{cases} f(k_t) \geq k_{t+1} \geq 0 \quad \forall t \\ k_0 \text{ given} \end{cases}$$

$$\max_{\substack{\{k_{t+1}\}_{t=0}^{\infty} \\ 0 \leq k_{t+1} \leq f(k_t)}} \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) =$$

$$= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left[u\left(f(k_0) - k_1\right) + \beta \sum_{t=1}^{\infty} \beta^{t-1} \left(u\left(f(k_t) - k_{t+1}\right)\right) \right]$$

$$= \max_{k_1,\ 0 \leq k_1 \leq f(k_0)} \left[u\left(f(k_0) - k_1\right) + \beta \cdot \max_{\substack{\{k_{t+1}\}_{t=1}^{\infty} \\ 0 \leq k_{t+1} \leq f(k_t)}} \left(\sum_{t=1}^{\infty} \beta^{t-1} \left(u\left(f(k_t) - k_{t+1}\right)\right)\right) \right]$$

$$= \max_{0 \leq k_1 \leq f(k_0)} \left[u\left(f(k_0) - k_1\right) + \beta \cdot \max_{\{k_{t+2}\}_{t=0}^{\infty}} \left(\sum_{t=0}^{\infty} \beta^t \left(f(k_{t+1}) - k_{t+2}\right)\right) \right]$$

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$= \max_{\{k_1\}} [u(f$$

21 September

Sequential Problem of SP

$$\begin{cases} \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \to \max_{\{k_{t+1}\}_{t=0}^{\infty}} \\ \text{s.t. } f(k_t) \geq k_{t+1} \geq 0 \\ k_0 \text{ is given.} \end{cases}$$

Recursive Problem of the SP

$$v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \left[u\left(f(k_0) - k_1\right) + \underbrace{\beta v(k_1)}_{k_1 \text{ that maximizes utility for pd 1 / all the future}} \right]$$

(✱) k_0 is given.

The problem is we don't know $v(\cdot)$ yet.

Jargon:

$v(\cdot)$: the value function

k : state variable (if you know this, you know everything).

k' : choice variable (or control) → In the example above

$$\begin{pmatrix} k \to k_0 \\ k' \to k_1 \end{pmatrix}$$

$k' = g(k) \rightarrow$ Policy function (decision rule)

Algorithm to solve the Bellman Equation

(Value Function Iteration):

1) "Guess" $v_0(k)$
2) Use BE to update your guess

$$v_{j+1}(k) = \max_{0 \leq k' \leq f(k)} \left[U(f(k) - k') + \beta v_j(k') \right]$$

3) If $\| v_{j+1}(k) - v_j(k) \| < \varepsilon$,
then $v_{j+1}(k)$ is the solution.

Assume:

$u = \log(c)$ (by definition)
$\delta = 1$

$y_t = \theta k_t^{\alpha}$

$$v(k) = \max_{0 \leq k' \leq \theta k^\alpha} \left[\log(c) + \beta . v(k') \right]$$

s.t. $\quad c + k' = \theta k^\alpha$

$\quad\quad k_0$ is given

Guess (and later we will verify), that

$$v(k) = a_0 + a_1 \log k$$

$$\mathcal{L} = \log(c) + \beta\left(a_0 + a_1 \log k'\right) + \lambda\left[\theta k^\alpha - c - k'\right]$$

$\dfrac{\partial \mathcal{L}}{\partial c}$: $\dfrac{1}{c} = \lambda \qquad\qquad \dfrac{\partial \mathcal{L}}{\partial k}$: $\dfrac{\beta a_1}{k'} =$

$$\frac{1}{c} = \frac{\beta a_1}{k'}$$

$$\Rightarrow \frac{1}{\theta k^\alpha - k'} = \frac{\beta a_1}{k'}$$

$$\Rightarrow \beta a_1 \theta k^\alpha - \beta a_1 k' = k'$$

$$\Rightarrow k'(1 + \beta a_1) = \beta a_1 \theta k^\alpha$$

$$\Rightarrow \boxed{k' = \frac{\beta a_1}{1 + \beta a_1} \theta k^\alpha}$$

$$c = \theta k^{\alpha} - k' = \theta k^{\alpha} - \frac{\beta a_1}{1+\beta a_1} \theta k^{\alpha}$$

$$= \theta k^{\alpha} \left[1 - \frac{\beta a_1}{1+\beta a_1} \right]$$

$$\boxed{c = \theta k^{\alpha} \cdot \left(\frac{1}{1+\beta a_1} \right)}$$

Method of indeterminate coefs:

$$v(k) = \max \left[\log(c) + \beta v(k') \right]$$

$$a_0 + a_1 \log(k) = \log \left(\frac{\theta k^{\alpha}}{1+\beta a_1} \right) + \beta \cdot \left(a_0 + a_1 \log \left[\frac{\theta k^{\alpha} \beta a_1}{1+\beta a_1} \right] \right)$$

$$a_0 + a_1 \log k = -\log(1+\beta a_1) + \log \theta + \alpha \log(k) + \beta a_0 - \beta a_1 \log(1+\beta a_1) + \beta a_1 \log(\theta \beta a_1) + \beta a_1 \alpha \log(k)$$

$$a_1 = \alpha + \beta a_1 \alpha \iff a_1 (1 - \beta \alpha) = \alpha$$

$$\Rightarrow \boxed{a_1 = \frac{\alpha}{1-\beta\alpha}}$$

Decentralizing Equilibrium

Representative agent, $U(c)$

Firm with technology $F(k, l)$

Agent owns the firm

Agent is endowed with k_0.

Def :

An ADE is allocations for the consumer $\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}$, allocation for firm $\{y_t, k_t^f, l_t^f\}_{t=0}^{\infty}$ and a price system $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}$ s.t.

1) given the prices, $\{\hat{c}_t, \hat{k}_{t+1}, \hat{p}_t\}$ solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$\sum_{t=0}^{\infty} \left(\hat{p}_t c_t + \hat{p}_t \cdot i_t\right) \leq \sum_{t=0}^{\infty} \hat{w}_t l_t + \hat{r}_t k_t$$

$$i_t = k_{t+1} - (1-\delta) k_t$$

$$c_t \geq 0, \quad k_t \geq 0 \quad \forall t$$

2) Given prices, the firm chooses optimally $\{y_t, k_t^f, l_t^f\}$

s.t. $p_t F(k_t^f, l_t^f) - \hat{w}_t l_t^f - \hat{r}_t k_t \to \max$

3) Markets clear.

$$\hat{c}_t + \hat{i}_t = \hat{y}_t$$
$$\hat{k}_t^f = \hat{k}_t$$
$$\hat{l}_t^f = \hat{l}_t$$

Suppose hh sell "all the" capital to the firm in the first time period. $t=1$.

HH:

$$\sum_{t=0}^{\infty} \hat{p}_t c_t \leq \sum_{t=0}^{\infty} \hat{w}_t l_t + r_0 k_0 + p_0 (1-\delta) k_0$$

The firm:

$$\begin{cases} \max \sum_{t=0}^{\infty} \hat{p}_t \left(F(k_t, l_t) - i_t \right) - r_0 k_0 - p_0 (1-\delta) k - \sum \hat{w}_t l_t \\ i_t = k_{t+1} - (1-\delta) k_t . \end{cases}$$

Def: An SME is allocations $\{\tilde{c}_t, \tilde{b}_{t+1}, \tilde{k}_{t+1}, \tilde{l}_t\}$, allocation for firms $\{\tilde{y}_t^f, \tilde{k}_t^f, \tilde{l}_t^f\}$, prices $\{\tilde{w}_t, \tilde{r}_t^k, \tilde{r}_t^b\}$ s.t.

1) Given prices, HH solves:

$$\max_{\{c_t, l_t, b_{t+1}, k_{t+1}\}} U(\{c_t\})$$

s.t.

$$c_t + b_{t+1} + i_t = \tilde{w}_t l_t + \tilde{r}_t^k k_t + (1+\tilde{r}_t^b) b_t$$

$$i_t = k_{t+1} - (1-\delta) k_t$$

convention to write $(1+r)$ for interest rates from bond.

Tildas represent numbers we already have. Variables w/o tildas are the ones we are choosing.

2) Firms maximize profits.

$$y_t - \tilde{w}_t l_t^f - \tilde{r}_t^k k_t \to \max$$

$$\text{s.t.} \quad y_t \leq F(k_t^f, l_t^f)$$

3) Markets clear

$$\tilde{c}_t + \tilde{i}_t = y_t \qquad \tilde{b}_t = 0$$

$$\tilde{k}_t^f = \tilde{k}_t$$

$$\tilde{l}_t^f = \tilde{l}_t$$

we still include bonds even though it is 0, to find the equilibrium price of assets to convince HHs to not buy bonds.

Sep 26

Social Planner's Problem

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \rightarrow \max$$

s.t. $c_t + k_{t+1} - (1-\delta) k_t = y_t$

$c_t \geq 0, \quad k_{t+1} \geq 0 \quad , k_0$ is given

Euler Equation:

$$u_c'(t) = \beta u_c'(t+1)\left[1 - \delta + F_k'(k_{t+1})\right]$$

* Consumer (in SMO) seeks $\{c_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ to solve the following problem

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \rightarrow \max$$

s.t. $c_t + k_{t+1} + b_{t+1} \leq w_t l_t + (1 + r_t^b) b_t + (1-\delta) k_t + r_t^k k_t$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) + \sum_{t=0}^{\infty} \lambda_t \left[w_t l_t + (1+r_t^b) b_t + (1-\delta)k_t + r_t^k k_t - c_t - k_{t+1} - b_{t+1}\right]$$

FOC:

c_t: $\beta^t u'c = \lambda_t$

k_{t+1}: $-\lambda_t + \lambda_{t+1}\left[(1-\delta) + r_{t+1}^k\right] = 0$

b_{t+1}: $-\lambda_t + \lambda_{t+1}\left[1 + r_{t+1}^b\right]$

Combine FOCs wrt k and b:

$$1-\delta+r^k_{t+1} = 1+r^b_{t+1}$$

$$\Longleftrightarrow \boxed{r^b_{t+1} = r^k_{t+1} - \delta} \qquad \text{[No arbitrage cond}^n\text{]}$$

Transversality Condition:

$$\lim_{t\to\infty} \beta^t u'_c(t) \cdot k_{t+1} = 0$$

$$\lim_{t\to\infty} \beta^t u'_c(t) \cdot b_{t+1} = 0$$

Combine FOCs wrt c and k_{t+1}:

$$u'_c(t) = \beta u'_c(t+1)[1-\delta+r^k_{t+1}]$$

Firms: (Maximizing profits)

$$\Pi = y_t - w_t l_t - r^k_t k_t$$

$$y_t \leq F(k_t, l_t)$$

$$\Pi = F(k_t, l_t) - w_t l_t - r^k_t k_t$$

$$\frac{\partial \Pi}{\partial k} \Rightarrow F'_k = r^k_t$$

$$\frac{\partial \Pi}{\partial l} \Rightarrow F'_l = w_t$$

$F(k,l)$ is CRS $\iff \lambda F(k,l) = F(\lambda k, \lambda l)$

Take derivative wrt λ.

$$F(k,l) = \left.\frac{\partial F}{\partial k}\right|_{\lambda k, \lambda l} \cdot k + \left.\frac{\partial F}{\partial l}\right|_{\lambda k, \lambda l} \cdot l$$

Consider $\lambda = 1$

$$F(k,l) = F'_k \cdot k + F'_l \cdot l$$

From FOC:

$$F(k,l) = r^t_t k_t + w_t l_t$$

$\pi = 0 \rightarrow$ if $F(k,l)$ is CRS.

EE :-

$$u'_c(t) = \beta u'_c(t+1)\left[1 - \delta + r^k_{t+1}\right] \quad F'_k(t+1) \quad \text{(from FOC)}$$

EE, k_0 & tranversality are the same as in SPP.

$\Rightarrow$ CE is Pareto Optimal // First Welfare Theorem holds in this environment.

$$\boxed{\text{Neoclassical Growth Model with Pop. Growth.}}$$

$N_0 = 1 \rightarrow$ starting mass of consumers.

$n \rightarrow$ growth rate of population

$N_t = (1+n)^t N_0 = (1+n)^t$

Assume $F(K_t, \underbrace{N_t(1+g)^t}_{\text{labor augmented tech progress}})$

- small letters $\rightarrow$ per capita
- Capital $\rightarrow$ Aggregate

Notation:—

$C_t, K_t \rightarrow$ aggregate cons^m & Capital

$c_t, k_t \rightarrow$ per capita " " " "

Suppose SP maximizes utility per capita:—

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \rightarrow \max$$

Resource Constraint:

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, N_t(1+g)^t)$$

Define $\tilde{c}_t = \dfrac{c_t}{(1+g)^t}$ $\tilde{k}_t = \dfrac{k_t}{(1+g)^t}$

Divide RC by $(1+g)^t(1+n)^t$

$$\frac{C_t}{(1+g)^t(1+n)^t} + \frac{K_{t+1}}{(1+g)^t(1+n)^t} - (1-\delta)\frac{K_t}{(1+g)^t(1+n)^t}$$

$$= \frac{F(K_t, N_t(1+g)^t)}{(1+g)^t(1+n)^t}$$

Can take this inside the fn as CRS.

$$\tilde{c}_t + (1+g)(1+n)\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = F(\tilde{k}_t, 1)$$

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t \frac{(\tilde{c}_t(1+g)^t)^{1-\sigma}}{1-\sigma}$$

$$\boxed{u(c_t) = \frac{c^{1-\sigma}}{1-\sigma} \quad \text{(CRRA utility)}}$$

$$= \sum_{t=0}^{\infty} \left(\beta(1+g)^{1-\sigma}\right)^t \cdot \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}$$

$$= \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}$$

5th Oct.

Bellman Equation

$$v(k) = \max_{0 \le k' \le f(k)} \left[u(f(k) - k') + \beta v(k') \right]$$

More generally :

$$v(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y) \right]$$

$$(Tv)(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y) \right]$$

Solution to Bellman is a fixed point of operator T

$$v^* = T v^*$$

Question :—

1) When v^* exists?
2) Is v^* unique?
3) When is $v^* = \lim_{j \to \infty} T^j (v_0)$?

Contraction Mapping Theorem

Def: A metric space is a set S along with a metric $d(x,y)$, satisfying certain properties:-

for any $x,y \in S$:-

1) $d(x,y) \geq 0$
2) $d(x,y) = 0$ iff $x=y$
3) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in S$
4) $d(x,y) = d(y,x)$

Δ inequality

Example: $S = \mathbb{R}$

$$d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0, \text{ otherwise} \end{cases}$$

This is a metric space. (Add how Δ inequality holds)

Ex 2 Let $X \subseteq \mathbb{R}^\ell$, $S = C(X)$ set of all continuous & bounded functions $f: X \to \mathbb{R}$ equipped w/ supnorm d

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

This is a metric space.

Δ Inequality :-

Fix some x,

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

Take sup norm on both sides. ▨

Def : Let (S, d) be a metric space. A sequence $\{x_n\}$ with $x_n \in S$ $\forall n$ converges to x if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$. $d(x, x_n) < \varepsilon$ $\forall n > N$

Exercise : Prove that if limit exists $\Rightarrow$ it's unique (Proof by contradiction)

Cauchy Sequences: Let (S,d) be a metric space.

$\{x_n\}$ with $x_n \in S \ \forall n$ is Cauchy if $\forall \varepsilon > 0$

$\exists N$: s.t. $\forall n, m > N$, $d(x_n, x_m) < \varepsilon$.

Thm: If $\{x_n\}$ converges $\Rightarrow$ It's a cauchy sequence

Proof: Pick $\varepsilon > 0$, $\Rightarrow \exists M_{\frac{\varepsilon}{2}}$ s.t.

$d(x_n, x) < \frac{\varepsilon}{2} \ \forall n > M_{\frac{\varepsilon}{2}}$

(we take x_n s.t. $d(x_n, x) < \frac{\varepsilon}{2}$)

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

[Δ inequality applies as it metric space.]

Ex. $S = \mathbb{R}$, $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

$\{x_n\}$, where $x_n = \frac{1}{n}$

The distance b/w the elements would always be 1 $\Rightarrow$ not convergent $\Rightarrow$ not cauchy.

Def: A metric space (S, d) is complete if $\forall$ Cauchy sequence $\{x_n\}$ converges to any element $x \in S$. $(x_n \in S \; \forall n)$

(Limits might be different but all converges to a limit in set S).

ex: $X = [1, 2]$

$S = C(X)$ set of all continous, strictly decreasing on $[1,2]$

$d(\cdot)$ is supnorm. This is not a complete metric space. (why?)

Counter Example:

$\{f_n\} = \frac{1}{nx}$ Is it cauchy?

Pick $\varepsilon > 0$, & let $N_\varepsilon = \frac{2}{\varepsilon}$, $m > n > N_\varepsilon$

$$d(f_n, f_m) = \sup_{x \in [1,2]} \left| \frac{1}{n.x} - \frac{1}{mx} \right|$$

$$= \sup_{x \in [1,2]} \frac{1}{nx} - \frac{1}{mx}$$

$\frac{\cancel{0.4}}{\cancel{20}}$ 4 $\frac{1}{2}$

$$= \sup_{x \in [1,2]} \frac{m-n}{mnx}$$ (Max achieved at $x = 1$)

$$= \frac{m-n}{mn}$$

$$= \frac{1 - \frac{n}{m}}{mn} \leq \frac{1}{n} \leq \frac{1}{N_\varepsilon} = \frac{\varepsilon}{2} < \varepsilon$$

(Stofkey & Lucas)

Contraction :

Def : Let (S, d) be a metric space & $T: S \to S$.
An operator T is a contraction if
$$d(Tx, Ty) \leq \beta \, d(x, y) \quad \forall \; x, y \in S, \; \beta \in [0, 1)$$
β is called modulus of contraction.