Suggested Solutions: ECON 7710 HW I

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Question 1

i

Event i is defined as we toss a die n times and at least one of the outcomes is equal to 6.
We want to calculate the probability of no 6 through n toss, which is $(\dfrac{5}{6})^n$. Because each single toss, the probability of no 6 is $\frac{5}{6}$ and we independently repeated it n times.
So we know the probability of event i.

$$P(i) = 1 - (\frac{5}{6})^n$$

ii

Event ii is defined as we toss a die n times and an outcome equal to 6 is observed exactly once.

- There are 6^n ways of outcome through n times' toss. That's the denominator of probability.
- For the numerator
  - First, there are $\binom{n}{1}$ ways of choosing the one "6", which are n ways.
  - Once the "6" is chosen, it shows up exactly 1 time.
  - For the rest $n-1$ times, "1" to "5" can be freely chosen, so in all 5^{n-1} ways.

  Therefore, the numerator is : $n * 5^{n-1}$. So,

$$P(ii) = \frac{n * 5^{n-1}}{6^n}$$

Question 2

a)

Since the time for potential arrival is between 7:00 pm to 8:00 pm and time to wait cannot exceed 10 minutes, we can normalize time on interval $[0, 1]$ by simply taking "hour" as our unit. Then 10 minutes will turn $1/6$. 7:00 pm will be point 0 and 8:00 pm will be point 1.
Meanwhile, we know the arrival of each people is uniformly random distribution. So we denote husband's distribution of time of arrival as $H \sim U[0, 1]$ and wife's distribution of time of arrival as $W \sim U[0, 1]$.
We denote the probability density function(PDF) of H and W with $h(x)$ and $w(x)$ respectively. Then we know,

$$h(x) = w(x) = \begin{cases} 1, & \text{For } x \in [0, 1] \\ 0, & \text{Otherwise} \end{cases}$$

Then we know for the event that date will occur, denoted by D, there are two cases:

- D_H: Husband arrives earlier than wife and wife shows up within 10 minutes.
- D_W: Wife arrives earlier than husband and husband shows up within 10 minutes.

Since H , W follow the same distribution, by symmetry event D_H and event D_W has the same probability. WLOG, $P(D) = P(D_H) + P(D_W) = 2P(D_H)$
Event D_H can be further discussed into two cases:

- D_{H1}: H shows up in the first 5/6 of the interval, or between 7:00 pm to 7:50 pm. To make the date happen, W only needs to show up with in ten minutes after H's arrival. The probability is $\int_0^{5/6} dH \int_H^{H+1/6} dW$.
- D_{H2}: H shows up in the last 1/6 of the interval or between 7:50pm to 8:00 pm. Then W can still arrive later than H, but she cannot arrive later than 8:00. The probability is: $\int_{5/6}^1 dH \int_H^1 dW$.

Note: Here we calculated joint probability. Joint PDF is $h(x)w(x) = 1 * 1 = 1$. As you can see 1[8:00pm] is our boundary here so we need to discuss two sub-cases as above. So probability of D_H is:

$$P(D_H) = \int_0^{5/6} dH \int_H^{H+1/6} dW + \int_{5/6}^1 dH \int_H^1 dW = \frac{5}{36} + \frac{1}{72} = \frac{11}{72}$$

Therefore,

$$P(D) = P(D_H) + P(D_W) = 2 * \frac{11}{72} = \frac{11}{36} \approx 0.306$$

. **b).i**

Now we consider the arrival of ex boyfriend, the distribution is denoted by $E \sim U[0, 1]$. Likewise, $e(x) = h(x) = w(x) = 1$. And the joint pdf is also 1. We define M as the event all three meet together.
Meanwhile, if all three meet together, then E has to arrive before D happens. Otherwise, the couples will

disappear as it makes no sense to stay if H and W already met. In that case, E cannot get them caught. Since E is indifferent to whoever arrives the latest, by symmetry, there are two sub-events here with the same probability:

- M_H where husband arrives latest.
- M_W where wife arrives latest.

Again, the probability of all three meet is $P(M) = P(M_H) + P(M_W) = 2P(M_H)$. [1] , WLOG we just calculate $P(M_H)$.
Likewise, there are two sub-cases:

- M_{H1} : H arrives between $[\frac{1}{6}, 1]$, then the earliest time for either E or W to arrive is $H - 1/6$, because they at most wait $1/6$ to meet H. The latest time for either E or W to arrive is H, because here we assumed H is the latest one to arrive.
- M_{H2} : H arrives between $[0, \frac{1}{6}]$, then the earliest time for either E or W to arrive is 0, because they arrive after 7:00 pm. The latest time for either E or W to arrive is H, because here we assumed H is the latest one to arrive.

As you can see, 0 is our boundary here so we need to discuss two sub-cases above. Then we know $P(M_H)$ is

$$P(M_H) = \int_{1/6}^{1} dH \int_{H-1/6}^{H} dW \int_{H-1/6}^{H} dE + \int_{0}^{1/6} dH \int_{0}^{H} dW \int_{0}^{H} dE = \frac{15+1}{648} = \frac{2}{81}$$

Therefore, we know

$$P(M) = P(M_H) + P(M_W) = 2 * \frac{2}{81} = \frac{4}{81} \approx 0.049$$

b).ii

There are 3 different cases that H and E meet with each other, denote this event by F, short for fight.

- D happens and E meets both of W and H, which is M
  We already know $P(M) = \frac{4}{81}$.
- D happens and E meets H only. Denote this event by M'. In this case, E must arrives the earliest and E leaves before W comes. W has to arrive the latest.
  There are three sub-cases here:
  - M_1': If H arrives in $[\frac{1}{6}, \frac{5}{6}]$
    * For E, the earliest time to arrive is $H - \frac{1}{6}$, to be the earliest; The latest time to arrive is $W - \frac{1}{6}$, to avoid meeting W.

[1] Since these three meet anyway, we need to use whoever arrives the latest as an anchor to carry out backward induction to derive the probability of M_H

    * For W, the earliest time to arrive is H to meet H, the latest time to arrive is $H + \frac{1}{6}$ as H's waiting time is $\frac{1}{6}$.
  - M_2': If H arrives in $[0, \frac{1}{6}]$
    * For E, the earliest time to arrive is 0, to be the earliest for sure; the latest time to arrive is $W - \frac{1}{6}$, to avoid meeting W.
    * For W, the earliest time to arrive is $\frac{1}{6}$ to meet H and make sure $W - \frac{1}{6} \geq 0$. The latest time to arrive is $H + \frac{1}{6}$ as husband's waiting time is $\frac{1}{6}$.
      Here 0 is the boundary.
  - M_3': If H arrives in $[\frac{5}{6}, 1]$
    * For E, the earliest time to arrive is $H - \frac{1}{6}$, to be the earliest; the latest time to arrive is $W - \frac{1}{6}$, to avoid meeting W.
    * For W, the earliest time to arrive is H to meet H and make sure $H + \frac{1}{6} \leq 1$. The latest time to arrive is 1. Here 1 is the boundary.

Therefore the probability of M' is

$$
\begin{aligned}
P(M') = \int_0^{1/6} dH \int_{1/6}^{H+1/6} dW \int_0^{W-1/6} dE + \int_{1/6}^{5/6} dH \int_H^{H+1/6} dW \int_{H-1/6}^{W-1/6} dE + \int_{5/6}^{1} dH \int_H^{1} dW \int_{H-1/6}^{W-1/6} dE \\
= \frac{1}{6^4} + \frac{1}{108} + \frac{1}{6^4} \\
= \frac{7}{648}
\end{aligned}
$$

- D does not happen and E meets H. Denote this event by M''. There are two sub-events that wife came too late, M_W'' or husband came too late M_H''. We know $P(M'') = P(M_H'') + P(M_W'')$ and by symmetry $P(M_H'') = P(M_W'')$.
  Here we consider the case when wife came too late, M_W'':
  - If H arrives in $[\frac{1}{6}, \frac{5}{6}]$,
    * For E, the earliest time to arrive is $H - \frac{1}{6}$, to meet husband. The latest time to arrive is $H + \frac{1}{6}$. Note here whether E meets W doesn't matter.
    * For W, the earliest time to arrive is $H + \frac{1}{6}$, to miss the date. The latest time to arrive is just 1.
  - If H arrives in $[0, \frac{1}{6}]$,
    * For E, the earliest time to arrive is just 0, as 0 is a boundary here. The latest time to arrive is $H + \frac{1}{6}$, to meet H.
    * For W, the earlist time to arrive is $H + \frac{1}{6}$, to miss the date. The latest time to arrive is just 1.
  - If H arrives in $[\frac{5}{6}, 1]$, this doesn't work here as W will for sure meet H.

  Therefore the probability of M_W'' is

$$
P(M_W'') = \int_0^{1/6} dH \int_{H+1/6}^{1} dW \int_0^{H+1/6} dE + \int_{1/6}^{5/6} dH \int_{H+1/6}^{1} dW \int_{H-1/6}^{H+1/6} dE
$$

$$= \frac{5}{162} + \frac{2}{27} = \frac{17}{162}$$

$$P(M'') = P(M''_W) + P(M''_M) = \frac{17}{81}$$

So $P(F) = P(M) + P(M') + P(M'') = \frac{4}{81} + \frac{7}{648} + \frac{17}{81} = \frac{175}{648} \approx 0.270$

Question 3

This is false.

Think about a random variable X, which is a Bernoulli distribution on point $x = 0$ and $x = 1$ with mass 0.5 and with the rest mass 0.5 being a uniform distribution on $[0, 1]$

$$X = \begin{cases} P(X) = \begin{cases} 0.5 & \text{If X=0} \\ 0.5 & \text{If X=1} \end{cases} & \text{(With mass 0.5)} \\ X \sim U[0,1] \quad \text{(With mass 0.5)} \end{cases}$$

Then it has density on almost every point on $[0, 1]$, except for point $x = 0$ and $x = 1$. The corresponding CDF is:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.25 & x = 0 \\ 0.5x + 0.25 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Clearly, this CDF is not continuous, let alone uniformly continuous.

By Heine-Cantor theorem, a sufficient condition to make it true is that if random variable X, which distributed on a compact set $[a, b]$ has a continuous CDF, then we have a uniformly continuous CDF.

Question 4

We define the event that no urn is empty as N. Since each marble is equally likely to be placed in each of n urns, we know it is assumed that each urn always has space to hold additional marbles. Meanwhile, since no urn is empty, we need every urn contains exactly one marble since there are n marbles and n urns.

- Under the assumption of **throwing n marbles sequentially each time** [Ordered with Replacement]. To calculate the probability, we know:
  - There are n^n ways of throwing the balls.
  - To make one in each urn, we have $n!$ ways.
    So the answer to $P(N) = \frac{n!}{n^n}$.
- Under the assumption of **throwing n marbles simultaneously each time** [Unordered with Replacement]. To calculate the probability, we know:
  - There are $\binom{2n-1}{n-1}$ ways of outcome.
  - But there's only one way of putting exactly one marble in every urn.
    So the answer to $P(N) = \frac{1}{\binom{2n-1}{n-1}}$

Note: Conceptually, first one is like distinguishable balls and the second one is like indistinguishable balls. But whether the balls are not actually distinguishable or not doesn't matter as we are focusing on probability(frequency) here. In other words, **what makes the difference is not whether the marbles are labeled or not but how we throw them.**

Question 5

Since $F(\cdot)$and $G(\cdot)$ are two distribution functions. From lecture notes we also know if $H(x) = F(G(x))$ is a distribution function, it has to satisfy three properties.

- **Monotonicity:**
  If $x_1 \leq x_2$, then $H(x_1) \leq H(x_2) \Rightarrow F(G(x_1)) \leq F(G(x_2))$.
  Since if $x_1 \leq x_2$ then $G(x_1) \leq G(x_2)$ as $G(x)$ is a distribution function. If $G(x_1) \leq G(x_2)$ then $F(G(x_1) \leq F(G(x_2))$ as $F(\cdot)$ is a distribution function. So if $x_1 \leq x_2$, then $H(x_1) \leq H(x_2)$.
  Monotonicity is satisfied for $H(x)$.

- $\lim_{\mathbf{x} \to -\infty} \mathbf{H}(\mathbf{x}) = \mathbf{0}$ **and** $\lim_{\mathbf{x} \to +\infty} \mathbf{H}(\mathbf{x}) = \mathbf{1}$
  $\lim_{x \to -\infty} H(x) = \lim_{x \to -\infty} F(G(x))$
  As we know $\lim_{x \to -\infty} G(x) = 0$ for $G(x)$ is a distribution function. We also know $G(x)$ as a distribution function satisfies monotonicity. So when $x \to -\infty$, $G(x)$ is non-increasing. [Here $G(x)$ is the "x" in $F(\cdot)$].
  Hence, we will need to set $\lim_{x \downarrow 0} F(x) = 0$ to make H satisfies $\lim_{x \to -\infty} H(x) = 0$
  Likewise, as we know $\lim_{x \to +\infty} G(x) = 1$ as $G(x)$ is a distribution function. Then we will need $\lim_{x \uparrow 1} F(x) = 1$ to make H satisfies $\lim_{x \to +\infty} H(x) = 1$
  Additionally, we want the support of $F(x)$ is $[0, 1]$

- **Right-continuity** $\lim_{\mathbf{x} \downarrow \mathbf{x_0}} \mathbf{H}(\mathbf{x}) = \mathbf{H}(\mathbf{x_0})$
  As we know, $\lim_{x \downarrow x_0} H(x) = H(x_0) \Leftrightarrow \lim_{x \downarrow x_0} F(G(x)) = F(G(x_0))$
  Since we know $\lim_{x \downarrow x_0} G(x) = G(x_0)$ as $G(\cdot)$ is a distribution function.
  Then we need $\lim_{x \downarrow G(x_0)} F(G(x)) = F(G(x_0))$. Clearly this is satisfied as $F(\cdot)$ is a distribution function.

In conclusion, since both $F(x)$ and $G(x)$ are distribution function, to make $H(x) = F(G(x))$ a distribution function, the extra necessary and sufficient conditions we need are:

$$\lim_{x \downarrow 0} F(x) = 0$$
$$\lim_{x \uparrow 1} F(x) = 1$$
$$supp(F(x)) = [0, 1]$$