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- **Convergence**

Lemma: Suppose that X_n is a monotone sequence of r.v. then,

$$X_n \xrightarrow{P} \iff X_n \xrightarrow{a.s.} X$$

Proof (By contradiction)

Suppose, $X_n \xrightarrow{P} X$, but $X_n \overset{a.s.}{\not\to} X$

This means that $\exists\ \varepsilon > 0$, δ set A s.t. $P(A) \geq \delta > 0$ and for all $\omega \in A$: $\sup_{k \geq n} |X_k(\omega) - X(\omega)| > \varepsilon$

wlog $X_n \geq X_{n+1}$ (monotonoicity), therefore

$$\sup_{k \geq n} |X_k(\omega) - X(\omega)| = |X_n(\omega) - X(\omega)| > \varepsilon$$

$$P(|X_n - X| > \varepsilon) \geq \delta > 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n - X| > \varepsilon) \geq \delta > 0 \Rightarrow X_n \overset{P}{\not\to} X$$

This is a contradiction.

This is monotone decreasing by construction

Theorem : $X_n \xrightarrow{a.s.} X \iff Y_n = \sup_{k \geq n} |X_k - X| \xrightarrow{P} 0$

$$P\left(\sup_{k \geq n} |X_k - X| \geq \varepsilon\right) \longrightarrow 0$$

Lemma (Borel-Cantelli) : Let $\{A_n\}_{n=1}^{\infty}$ are events on $(\Omega, \mathcal{F}, P)$ **

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k$$

intersection of the union (what is happening at the end of the universe).

Then if $\underbrace{\sum_{k=1}^{\infty} P(A_k) < \infty}$, then $P(A)=0$

if this is finite.

Example: Bernoulli

Theorem: If $\forall \varepsilon > 0$, $\sum_{k=1}^{\infty} P(|X_k - X| > \varepsilon) < \infty$, then

$$X_n \xrightarrow{a.s} X$$

$$P\left(\left\{\sup_{k \geq n} |X_k - X| > \varepsilon\right\}\right) \leq P\left(\bigcup_{k \geq n} \{|X_k - X| > \varepsilon\}\right)$$

$$\leq \sum_{k=n}^{\infty} P(|X_k - X| > \varepsilon) \to 0$$

Explanation:-

$$\sum_{k=1}^{\infty} a_k < \infty$$

$$S_n = \sum_{k=n}^{\infty} a_k \xrightarrow{n \to \infty} 0$$

<u>Lemma</u> : Suppose that $X_n \xrightarrow{P} X$, then $\exists$ n_k s.t. $X_{n_k} \xrightarrow{a.s.} X$

Set $\varepsilon > 0$, $a_k = \frac{c}{k^2}$ constant, set $n_1 = 1$,

$c = P(|X_1 - X| > \varepsilon)$

$P(|X_{n_1} - X| > \varepsilon) = a_1 = c$

$n = 2$, $P(|X_2 - X| > \varepsilon) < a_2 = \frac{c}{4} \Rightarrow X_{n_2} = X_2$

Otherwise, move to $n = 3$, $P(|X_3 - X| > \varepsilon) < a_2$

$\Rightarrow X_{n_2} = X_3$

Otherwise, keep going along the sequence

$P(|X_{n_k} - X| > \varepsilon) \leq a_k$

[Lebesgue space index r.]

Standard Functional Convergence

Definition: X_n converges to r.v. X in L_r if $E[|X_n - X|^r] \to 0$.

$X_n \xrightarrow{L_r} X$

(Same as real analysis def but in probabilistic terms!)

($r=2$: convergence in mean square)

r^{th} moment is not guaranteed to exist, so it's less powerful.

(*) Cauchy sequences help us to find the limit when we don't know what the limit is (as in the real theoretical problems).