# Econ7020: Macroeconomic Theory

Fall 2023

Problem Set 5. Due date: 12PM on December 6.

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## Problem 1

Consider a version of the neoclassical growth model with human capital and exogenous labor supply:

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\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\
c_t + x_{kt} + x_{ht} &= A k_t^{\alpha} z_t^{1-\alpha} \\
k_{t+1} &= (1-\delta_k)k_t + x_{kt} \\
h_{t+1} &= (1-\delta_h)h_t + x_{ht} \\
z_t &\leq n_t h_t \\
0 &\leq l_t + n_t \leq 1 \\
h_0, k_0 &\text{ given.}
\end{aligned}
$$

1. Show that the value function for this problem is homogeneous of degree $1-\sigma$ in the initial capital stocks $(k_0, h_0)$.

2. Suppose now that the government has a fixed sequence of expenditures that it must finance, $\{g_t\}_{t=0}^{\infty}$, and that it can use taxes on capital and labor incomes, $\tau_{kt}$ and $\tau_{zt}$. Define a competitive equilibrium for this environment.

3. What is the Ramsey problem here for a benevolent government? In particular, carefully derive and explain the implementability constraint for this environment.

4. Assume $\delta_k = \delta_h$. What is $\frac{\tau_{kt}}{\tau_{zt}}$ in this case?

5. Assume $\delta_k = \delta_h$. What can you say about $\lim_{t\to\infty} \tau_{kt}$? About $\lim_{t\to\infty} \tau_{zt}$?

## Problem 2

Consider an infinite horizon growth model with a representative consumer, with log utility and inelastic labor supply:

$$U(\{c_t\}, \{n_t\}) = \sum_{t=0}^{\infty} \beta^t \log c_t.$$

Assume that there is a representative firm with Cobb-Douglas production function

$$y_t = Ak_t^{\alpha} n_t^{1-\alpha}$$

and full depreciation $d = 1$.

For parts 2-5 assume that there is also a government that taxes all income at a uniform rate $\tau$ in every period, and uses the revenue generated to purchase $g_t$ in each period. Assume that the government balances the budget in every period.

1. Find the policy functions for the associated planning problem and use these to give an expression for $y_t$, output in period $t$. What is $\lim_{t\to\infty} y_t$?
2. Set up and define a TDCE for this economy.
3. Show that the TDCE allocation solves the planner's problem and give that problem.
4. Find expression for $y_t$ in this economy.
5. Which is larger, output when $\tau = 0$, or output when $\tau > 0$? Show your work.

## Problem 3

Consider a version of the neoclassical growth model with $Ak$ production technology, whereby a representative consumer has preferences

$$U = \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \right].$$

Capital depreciates fully $\delta = 1$.

Assume that the government taxes all income in period $t$ at the rate $\tau_t \in (0,1)$ where $\tau_t$ is stochastic. Assume that all revenue generated in period $t$ is used in that period to purchase $g_t$:

$$p_t g_t = \tau_t r_t k_t.$$

1. Show that the TDCE allocation for this model solves the planner's problem. Be sure to carefully and clearly state what this planning problem is.

2. Assume that the tax rates $\tau_t$ are i.i.d. Derive an expression for the expected consumption growth rate $\gamma^c_{t,t+1} = \mathbb{E}_t\left[\frac{c_{t+1}}{c_t}\right]$.

   *Hint: first show that the value function is homogeneous of degree $1-\sigma$. Next, argue that it is optimal to allocate constant fractions of output to consumption $c$ and investment $k'$ (denoted $\varphi$ and $1-\varphi$, respectively). Finally, derive expression for $\gamma^c_{t,t+1}$ involving only parameters of the model, $\varphi$ and $\tau_{t+1}$.*

# Problem 4

Consider an RBC model in which output $y_t$ is produced with technology

$$y_t = z_t k_t^\alpha n_t^{1-\alpha},$$

where $k_t$ denotes beginning-of-period capital, and $n_t$ is labor input. $z_t$ is an i.i.d. technology shock. Assume full depreciation $\delta = 1$.

Representative household maximizes lifetime utility:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t, n_t)\right].$$

1. Consider an economy with $U(c_t, n_t) = \ln c_t - \frac{1}{2}n_t^2$. Express maximization problem of the social planner and write down the associated Bellman equation. Solve for the equilibrium policy functions for consumption, investment and labor.

2. Now suppose $U(c_t, n_t) = \ln\left(c_t - \frac{1}{2}n_t^2\right)$. Express maximization problem of the social planner and write down the associated Bellman equation. Solve for the equilibrium policy functions for consumption, investment and labor.

3. Compare equilibrium behavior in both economies and provide an explanation for the differences.