# Suggested Solutions: ECON 7710 HW VII

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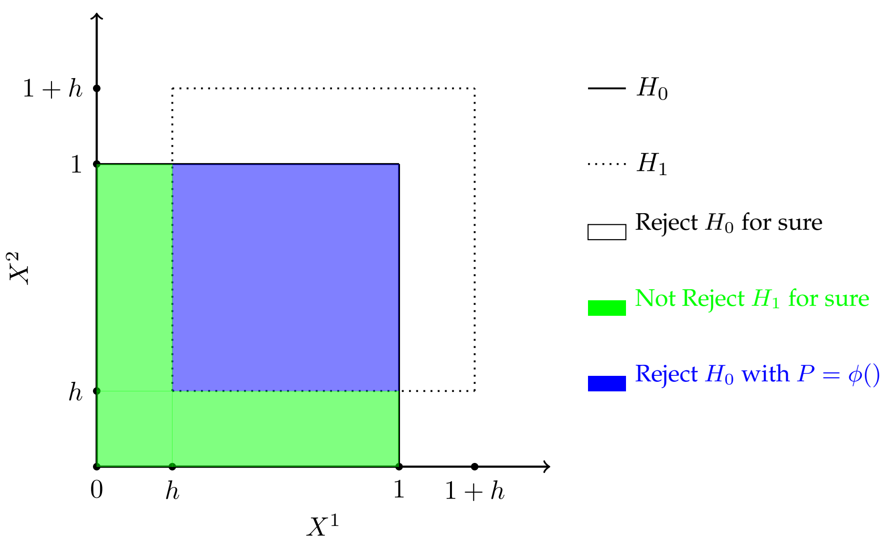
## Question 1

We have a sample of $n$ two-dimensional observations $\{X_i\}_{i=1}^n$ and we want to construct a test with significance level $\alpha$ for null hypothesis that this sample is generated by a uniformly distributed variable on $[0,1]\times[0,1]$ against the alternative hypothesis that sample is $[h,h+1]\times[h,h+1]$ for $h \geq 0$.
I think here we have a typo that $h > 0$, not $h \geq 0$.

### Part(a)

As long as your test explains itself well, you are good to go. Below is a UMP that Denis wants me to discuss.




Test rule we can formulate is:

- If any $X_i$ not in $[0,1]\times[0,1]$, we reject $H_0$ with probability 1;

- If any $X_i$ in "green" area, $\min\{X_i^1, X_i^2\} < h$, we don't reject $H_0$ with probability 1.
- If **all** $X_i$ in "blue" area,
  both $\max\{X_i^1, X_i^2\}$ and $\min\{X_i^1, X_i^2\} \in [h, 1] \forall i$,
  we reject $H_0$ with $P = \phi()$
- For level $\alpha$, we know $\phi(\cdot)(1-h)^{2n} = \alpha \Rightarrow \phi(\cdot) = \frac{\alpha}{(1-h)^{2n}}$

Therefore, when $0 < h < 1$, we have $\phi(x) =$

- $1$; $\exists i, \min\{X_i^1, X_i^2\} < 0$
  or $\max\{X_i^1, X_i^2\} > 1$
- 1 with $P = \frac{\alpha}{(1-h)^{2n}}$; $\forall i, \min\{X_i^1, X_i^2\} \geq h$
  and $\max\{X_i^1, X_i^2\} \leq 1$
- $0$; Otherwise

We know the size of our test is

- $0 < h < 1$,
  $\sup\limits_{\theta \in \Theta_0} = P_\theta(\phi(x) = 1) = (1-h)^{2n} * \frac{\alpha}{(1-h)^{2n}} = \alpha.$
- $h \geq 1$, $\sup\limits_{\theta \in \Theta_0} = 0 < \alpha.$

## Part(b)

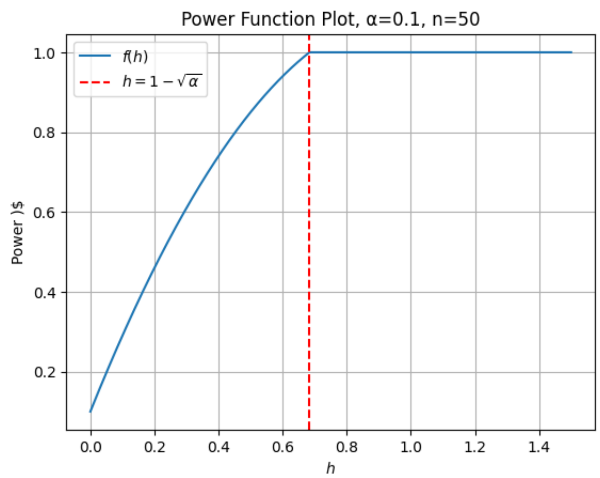
Move on to the discussion on power function:

- When $0 < h < 1$, power of our test is $E[\phi(x)|H_1] = (1-h)^{2n} * \frac{\alpha}{(1-h)^{2n}} + 1 * (1 - (1-h)^{2n}) = \alpha + 1 - (1-h)^{2n}$
- When $h \geq 1$, power of our test is just 1.

This is problematic as power goes above 1... We need to change our decision rule.

- We keep every other thing the same, just change the decision rule in blue area.
- When $\alpha > (1-h)^{2n}$, we reject with probability 1
- When $\alpha \leq (1-h)^{2n}$, we reject with probability $\frac{\alpha}{(1-h)^{2n}}$

Then the power function will look like this:




# Question 2

We have a classical simple regression model:

$$Y_i = \beta_0 + \beta_1 X_i + U_i$$

in which

$$E(U_i|X_i) = 0 \quad \text{and} \quad Var(U_i|X_i) = \sigma^2, Var(X_i) = \sigma_x^2$$

Let $(X_i, Y_i, i = 1, ..., n)$ be i.i.d. Instead of regressing $y_i$ on a constant and $x_i$, you regress $x_i$ on a constant and $y_i$:

$$\min_{\hat{\alpha}_0, \hat{\alpha}_1} \sum_{i=1}^{n} (x_i - \hat{\alpha}_0 - \hat{\alpha}_1 y_i)^2$$

## Part(a)

Is $\frac{1}{\hat{\alpha}_1}$ a consistent estimator of $\beta_0$? If not, what does it converge to?

I think there's a typo here. It should be $\beta_1$ instead of $\beta_0$?

- $\hat{\alpha}_1 = \dfrac{c\hat{o}v(x,y)}{v\hat{a}r(y)} = \dfrac{\bar{xy} - \bar{x}\bar{y}}{\bar{y^2} - (\bar{y})^2}$, where $\bar{xy} = \dfrac{\sum_{i=1}^{n} x_i y_i}{n}$; $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$; $\bar{y} = \dfrac{\sum_{i=1}^{n} y_i}{n}$; $\bar{y^2} = \dfrac{\sum_{i=1}^{n} y_i^2}{n}$
- By continuous mapping theorem, we know for the probability limit:

$$plim\hat{\alpha}_1 = \frac{plim(\bar{xy}) - plim(\bar{x})plim(\bar{y})}{plim(\bar{y^2}) - (plim(\bar{y}))^2}$$

- By LLN for i.i.d. random variables(given all the expectations exist), we know:

$$plim\hat{\alpha}_1 = \frac{E[XY] - E[X]E[Y]}{E[Y^2] - (E[Y])^2} = \frac{Cov(X,Y)}{Var(Y)}$$

- We know:

$$\hat{\alpha}_1 \xrightarrow{p} \frac{Cov(X,Y)}{Var(Y)} = \frac{Cov(X, \beta_0 + \beta_1 X + U)}{Var(\beta_0 + \beta_1 X + U)} = \frac{\beta_1 Var(X) + Cov(X,U)}{\beta_1^2 Var(X) + Var(U) + 2\beta_1 Cov(X,U)} = \frac{\beta_1 Var(X)}{\beta_1^2 Var(X) + Var(U)} = \frac{\beta_1 \sigma_x^2}{\beta_1^2 \sigma_x^2 + \sigma^2}$$

  Here

  - By Law of Iterated Expectation:

$$Cov(X,U) = E[XU] - E[X]E[U] = E[XE[U|X]] - E[X]E[E[U|X]] = 0$$

  - By Law of Total Variance:

$$Var(U) = E[Var(U|X)] + Var(E[U|X]) = \sigma^2$$

  Therefore, we know:

$$plim(\frac{1}{\hat{\alpha}_1}) = \frac{1}{plim(\hat{\alpha}_1)} = \frac{\beta_1^2 \sigma_x^2 + \sigma^2}{\beta_1 \sigma_x^2} = \beta_1 + \frac{\sigma^2}{\beta_1 \sigma_x^2} \neq \beta_0 \text{ or } \beta_1$$

  Hence $\frac{1}{\hat{\alpha}_1}$ is not a consistent estimator of $\beta_0$ or $\beta_1$, it converges to $\beta_1 + \frac{\sigma^2}{\beta_1 \sigma_x^2}$