Nov 1, 2023

Price Indices

$p^0 = (4, 2.5, 0.41)$

$p^1 = (2.66, 3.5, 0.49)$

Ideal: Fix a reference utility, u^R

$$I(u^R) = \frac{e(p^1, u^R)}{e(p^0, u^R)}$$

$I(u^r) > 1$: "cost of living" has increased

Natural choices for u^r : u^0 & u^1

Take $u^r = u^0$

$$I(u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)}$$

$$e(p^0, u^0) = p^0 \cdot x^0$$

In practice, fix a reference bundle, x^R

$$\frac{p^1 \cdot x^R}{p^0 \cdot x^R}$$

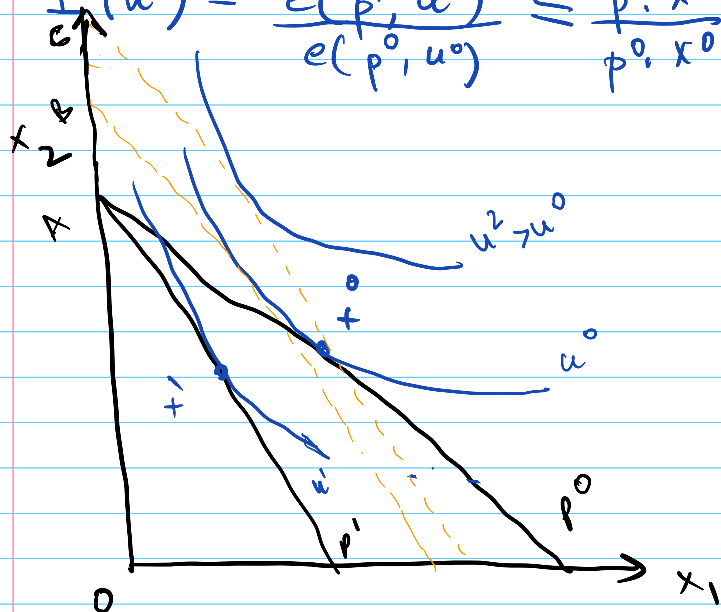
Natural choices for x^R: x^0 & x^1

We can atleast bound $e(p^1, u^0)$

By revealed preferences;

$$e(p^1, u^0) \leq p^1 \cdot x^0$$

$$I(u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} \leq \frac{p^1 \cdot x^0}{p^0 \cdot x^0} = \underbrace{L(p^0, p^1, x^0)}_{\text{Laspeyres Index}}$$



$p_2^1 = p_2^0 = 1$

$p_1^1 > p_1^0$

Ideal : $\frac{OB}{OA} = \frac{e(p^1, u^0)}{p^0 \cdot x^0}$

$L = \frac{OC}{OA} = \frac{p^1 \cdot x^0}{p^0 \cdot x^0}$

$L \geq I$

* Could also use $x^R = x^1$ as reference

$$I(u^1) = \frac{e(p^1, u^1) = p^1 \cdot x^1 \longrightarrow \text{observed}}{e(p^0, u^1) \longrightarrow \text{Not observed}}$$

Paasche Index : $P(p^0, p^1, x^1) = \frac{p^1 \cdot x^1}{p^0 \cdot x^1} \leq \frac{e(p^1, u^1)}{e(p^0, u^1)} = I(u^1)$

$e(p^0, u^1) \leq p^0 \cdot x^1$

Aggregation

For producers :—

K firms $y_1, \ldots, y_K$

"Representative firm"

$$y = y_1 + \ldots + y_K$$

$$y^*(p) = y_1^*(p) + \ldots + y_K^*(p)$$

* For consumer

consumer

price

$$x_k^i(p, m_i)$$

good

income of the consumer

* Aggregate Demand

$$\bar{x}_k(p, \underbrace{m_1, \ldots, m_I}_{\text{wealth/income dist}^n}) = \sum_{i=1}^{I} x_k^i(p, m_i)$$

We would like to write

$$\bar{x}_k(p, m_1, \ldots, m_I) \stackrel{?}{=} D_k(p, \bar{m})$$

$$\bar{m} = m_1 + \ldots + m_I$$

(Can we do this?

As it is lot easier to work with aggregate income/wealth.

Questions :-

1. When does aggregate demand depend only on aggregate income (& not the distribution)?

2. Can we interpret aggregate demand as arising from the UMP of a single "representative consumer" with income $\bar{m}$?

Example

2 consumers a, b

$$u_a(x_1, x_2) = u_b(x_1, x_2) = \log x_1 + x_2$$

impose $x_1, x_2 \geq 0$

$$x_1^i(p_1, m_i) = \begin{cases} p_2/p_1 & , p_2 \leq m_i \\ \frac{m_i}{p_1} & , p_2 > m_i \end{cases}$$

$$= \frac{\min(p_2, m_i)}{p_1}$$

Aggregate Dd

$$\bar{x}_1(p, m_a, m_b) = \frac{1}{p_1}\left(\min(p_2, m_a) + \min(p_2, m_b)\right)$$

$$\bar{m} = m_a + m_b$$

Can we write $\bar{x}(p, m_a, m_b) = D_1(p_1, \bar{m})$?

- If $p_2 < m_a, m_b$

$$\bar{x}_1(p, m_a, m_b) = \frac{2p_2}{p_1}$$

$$\bar{x}_2(p, m_a, m_b) = \underbrace{\frac{m_a - p_1\left(\frac{p_2}{p_1}\right)}{p_2}}_{\text{Agent a}} + \underbrace{\frac{m_b - p_1\left(\frac{p_2}{p_1}\right)}{p_2}}_{\text{agent B}}$$

$$= \frac{m_a + m_b}{2} - 2$$

$$= \frac{\bar{m}}{p_2} - 2$$

- If $m_a < p_2 < m_b$

$$\bar{x}(p, m_a, m_b) = \underbrace{\frac{m_a}{p_1}}_{x_1^a} + \underbrace{\frac{p_2}{p_1}}_{x_1^b}$$

We cannot write $D_1(p, \bar{m})$

Are there general conditions that allow aggregation?

$$\bar{x}_k(p, \bar{m}) = x^1_k(p, m_1) + \dots\dots + x_k^I(p, m_I)$$

$$\bar{m} = m_1 + \dots\dots + m_I$$

Diff. wrt m_1:

$$\frac{\partial \bar{x}_k(p, \bar{m})}{\partial \bar{m}} \cdot \frac{\partial \bar{m}}{\partial m_1} = \frac{\partial x^1_k(p, m_1)}{\partial m_1}$$

wrt m_2:

$$\frac{\partial \bar{x}_k(p, \bar{m})}{\partial \bar{m}} \frac{\partial \bar{m}}{\partial m_2} = \frac{\partial x_k^2(p, m_2)}{\partial m_2}$$

$$\frac{\partial x_k^1(p, m_1)}{\partial m_1} = \frac{\partial x_k^2(p, m_2)}{\partial m_2}$$

Diff again wrt m_1

$$\frac{\partial^2 x_k^1(p, m_1)}{\partial m_1^2} = 0$$

Necessary condition for aggregation is

$x_1^i(p, m_i)$ is :-

1. linear in income
2. All agents have same slope coefficient

$$(m_1, \dots, m_I) \rightarrow (m_1 + dm_1, \dots, m_I + dm_I)$$

$$\sum_{i=1}^{I} dm_i = 0$$

This sort of transformation doesn't change anything; still linear & same slope

Theorem:— Aggregate demand can be written as a function of aggregate income if & only if the indirect utility of the consumers all have the following Gorman form :

$$v^i(p, m_i) = a^i(p) + b(p)\, m_i$$

a^i can vary across i

$b(\cdot)$ is same for all i

Proof (sufficiency)

By Roy's identity :—

$$x^i_k(p, m_i) = \frac{-\partial v^i / \partial p_k}{\partial v^i / \partial m_i}$$

$$= \frac{\partial a^i(p)/\partial p_k + \frac{\partial b(p)}{\partial p_k} \cdot m_i}{b(p)}$$

Aggregate :—

$$\bar{x}_k(p, m_1, \dots, m_I) = \underbrace{\frac{1}{-b(p)} \sum_{i=1}^{I} \frac{\partial a^i(p)}{\partial p_k}}_{\alpha_k(p)} \underbrace{- \frac{\frac{\partial b(p)}{\partial p_k}}{b(p)}}_{\beta_k(p)} \underbrace{\sum_{i=1}^{I} m_i}_{\bar{m}}$$

$$= \alpha_k(p) + \beta_k(p)\, \bar{m}$$

$$= P_k(p, m)$$

Check (at home) this dd f^n is the dd f^n for a single consumer with income $\bar{m}$ & indirect utility.

$$v(p, \bar{m}) = A(p) + B(p)\bar{m}$$

where $A(p) = \sum_{i=1}^{I} a^i(p)$

$B(p) = b(p)$

What preferences fit Gorman form?

- Linear income effects $\Rightarrow$ straight line Engel curves
  - Identical homothetic pref. (Cobb-Douglas)
  - Quasi linear preferences

Representative consumer is basically defined here as an aggregation of the economy.

WARP & GARP (with consumer budget sets)

Q: Given some observed data (x^t, p^t), can we find a rationalising & locally non-satiated $u(\cdot)$?

Throughout, look for $u(\cdot)$ that are locally nonsatiated.

Definitions:

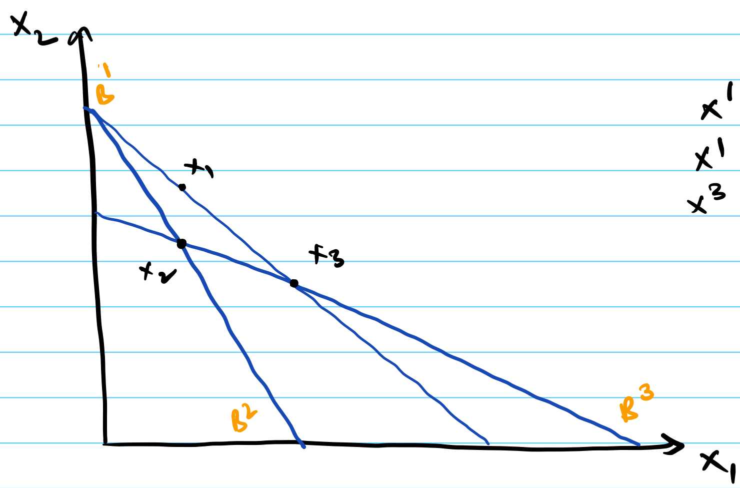
① $p^t \cdot x^t \geq p^t \cdot z$: x^t is directly revealed preferred to z; $x^t R^D z$

② $p^t \cdot x^t > p^t \cdot z$: x^t is strictly directly revealed preferred to z; $x^t P^D z$

Idea:

Data is generated by utility maximization!

$$x^t R^D z \Rightarrow u(x^t) \geq u(z)$$
$$x^t P^D z \Rightarrow u(x^t) > u(z)$$

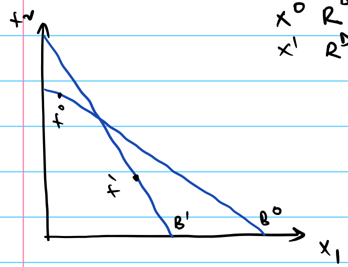


$x^1 R^D x^3$
$x^1 P^D x^2$
$x^3 R^D x^2$

WARP: If $x^t R^D x^s$ and $x^t \neq x^s$, then not $x^s R^D x^t$

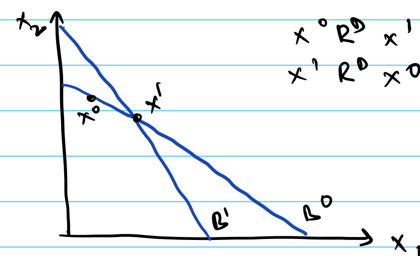
Another way: $p^t \cdot x^t \geq p^t \cdot x^s \Rightarrow p^s \cdot x^t > p^s \cdot x^s$

Violations of WARP



$x^0 R^D x^1$

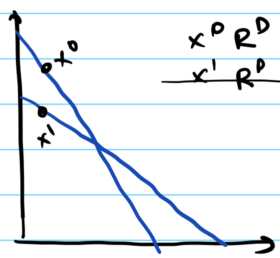
$x^1 R^D x^0$



$x^0 R^D x^1$

$x^1 R^D x^0$

WARP satisfied



$x^0 R^D x^1$

$\cancel{x^1 R^D x^0}$

Q: If data satisfy WARP, is there always a rationalizing $u(\cdot)$?

A: No.

$t =$	A	B	C
p^t	(1,2,2)	(2,2,1)	(2,1,2)
x^t	(1,2,0)	(2,0,1)	(0,1,2)
$p^t \cdot x^t$	5	5	5

Note:

$x^A R^D x^B$: $p^A \cdot x^A = 5 \geq p^A \cdot x^B = 4$ and

$x^B \cancel{R^D} x^A$: $p^B \cdot x^A = 6 > p^B \cdot x^B = 5$

Can also check:

- $x^B R^D x^C$ & $x^C \cancel{R^D} x^B$
- $x^C R^D x^A$ & $x^A R^D x^C$

No violations of WARP

Claim : No rationalizing $u(\cdot)$ exists

Why? Assume it did.

$p^A \cdot x^B = 4 < p^A \cdot x^A = 5$

By LNS, $\exists$ some $\hat{x}$ s.t.

$p^A \cdot \hat{x} < p^A \cdot x^A$ & $u(\hat{x}) > u(x^B)$

$\Rightarrow u(x^A) \geq u(\hat{x}) > u(x^B)$

$\Rightarrow u(x^A) > u(x^B)$

Do same to show:–

$u(x^B) > u(x^C)$; $u(x^C) > u(x^A)$

$\Rightarrow u(x^A) > u(x^A)$ CONTRADICTION!

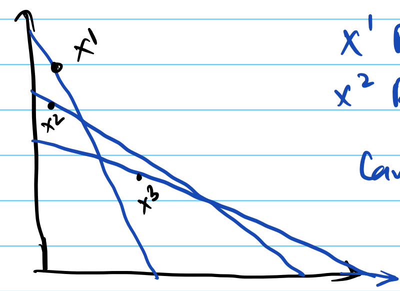
Problem with WARP: only has pairwise comparisons

Definition:

let $x^1 R^D x^2$, $x^2 R^D x^3$, ..., $x^{n-1} R^D x^n$

we say x^1 is indirectly revealed preferred to x^n

Write $x^1 R x^n$



$x^1 R^D x^2$
$x^2 R^D x^3$

Can indirectly compare x^1 to x^1:
$x^1 R x^3$

Generalized Axiom of Revealed Preference

$\{(p^t, x^t)\}_{t=1}^T$, satisfy GARP if

$x^t R x^s \Rightarrow$ not $x^s P^D x^t$

$x^t R x^s \Longrightarrow p^s \cdot x^t \geq p^s \cdot x^s$

$x^A R^D x^B$

$x^B R^D x^C \Longrightarrow x^A R x^C$
$x^C R^D x^A$

$p^c \cdot x^A = 4 < p^c \cdot x^c = 5$ Violates GARP.

& LNS

GARP is a necessary condition for a rational $u(\cdot)$ ^

Assume GARP did not hold

$$x^1 R^D x^2 R^D x^3 R^D \ldots R^D x^n$$

$$u(x^1) \geq u(x^2) \geq \ldots \geq u(x^n)$$

$$\Rightarrow u(x^1) \geq u(x^n)$$

& $x^n P^D x^1$

$$\Downarrow$$

$$p^n x^n > p^n x^1$$

Same LNS argument implies $u(x^n) > u(x^1)$

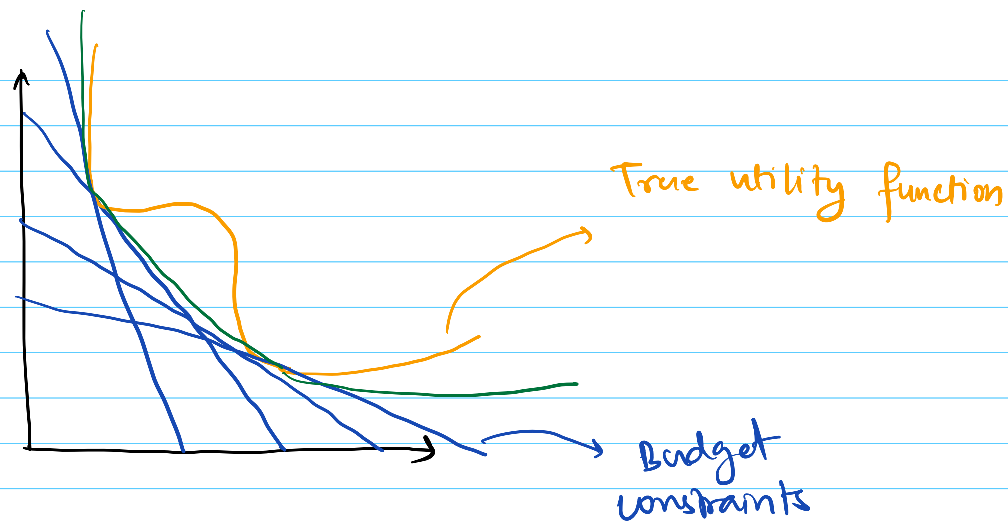
CONTRADICTION!

* GARP is necessary. Is it also sufficient?

<u>Afriat's Theorem</u>:

A finite set of demand data $\{(p^t, x^t)\}$ satisfies GARP $\Longleftrightarrow$

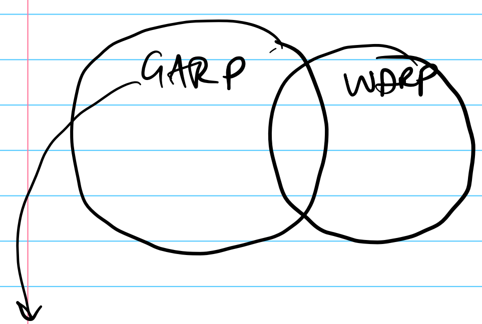
$\exists$ a continuous, monotonic & concave $u(\cdot)$ that rationalizes the data.



As many budget constraints you plot here, you'll never be able to recover the hump of the true utility function.

WARP $\not\Rightarrow$ GARP

Also true that GARP $\not\Rightarrow$ WARP



Rationalizable data.