Suggested Solutions: ECON 7710 HW VI

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Question 1

We have n i.i.d. draws from uniform distribution on $[0, \theta]$

1.(a)

We know for a uniform distribution on $[0, \theta]$, the pdf is:

$$f(x) = \begin{cases} \frac{1}{\theta} & x \in [0, \theta] \\ 0 & \text{Otherwise} \end{cases}$$

The likelihood function is written as:

$$\hat{L}(\theta; \mathbf{X}) = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbf{1}(0 \leq X_i \leq \theta)$$

Denote $X_m = max(X_1, X_2, ..., X_n)$, then we discuss the relationship between X_m and θ.

- If $\theta < X_m$, then $\mathbf{1}(0 \leq x_m \leq \theta) = 0$, which means $\hat{L}(\theta; \mathbf{X}) = 0$.
- If $\theta \geq X_m$, then $\hat{L}(\theta; \mathbf{X}) = \frac{1}{\theta^n}$. Clearly, when θ increases, $\frac{1}{\theta^n}$ decreases.

Therefore, we know the maximum of $\hat{L}(\theta; \mathbf{X})$ is achieved when $\theta = X_m$. In other words, we constructed:

$$\hat{\theta} = X_m$$

Notice that likelihood function here is not differentiable!

1.(b)

From 1.(a), the estimator we constructed is $\hat{\theta} = x_m$.
We have already wrote the pdf of each x_i in part 1.(a) and the cdf is:

$$F(x) = \begin{cases} 0 & \text{If } x \in (-\infty, 0) \\ \frac{x}{\theta} & \text{If } x \in [0, \theta] \\ 1 & \text{If } x \in (\theta, \infty) \end{cases}$$

For any $x \in [0, \theta]$, under the estimator we constructed, we know $F_{\hat{\theta}}(x) = P(\max\{x_1, ..., x_n\} \leq x) \Rightarrow P(x_1 \leq x)P(x_2 \leq x)...P(x_n \leq x) = (\frac{x}{\theta})^n$ Then the corresponding pdf is $f_{\hat{\theta}}(x) = \frac{n}{\theta}(\frac{x}{\theta})^{n-1}$. So we have already derived the exact distribution of our estimator.
To wrap it up, the pdf and cdf of $\hat{\theta}$ are:

$$cdf: F_{\hat{\theta}}(x) = \begin{cases} 0 & \text{If } x \in (-\infty, 0) \\ (\frac{x}{\theta})^n & \text{If } x \in [0, \theta] \\ 1 & \text{If } x \in (\theta, \infty) \end{cases} \qquad pdf: f_{\hat{\theta}}(x) = \begin{cases} \frac{n}{\theta}(\frac{x}{\theta})^{n-1} & \text{If } x \in [0, \theta] \\ 0 & \text{Otherwise} \end{cases}$$

Then we can construct $E(\hat{\theta}_{MLE}) = \int_0^\theta x f_{\hat{\theta}}(x)dx = \int_0^\theta n(\frac{x}{\theta})^n dx = \frac{n}{\theta^n}\frac{x^{n+1}}{n+1}|_0^\theta = \frac{n\theta}{n+1}$. Then we know $E(\hat{\theta}_{MLE}) - \theta = -\frac{\theta}{n+1}$

1.(c)

We know:

$$E[\hat{\theta}^2_{MLE}] = \int_0^\theta x^2 f_{\hat{\theta}}(x)dx = \int_0^\theta \frac{n}{\theta^n}x^{n+1}dx = \frac{n}{\theta^n}\frac{x^{n+2}}{n+2}|_0^\theta = \frac{n\theta^2}{n+2}$$

Therefore, we know the variance of our estimator is:

$$Var[\hat{\theta}_{MLE}] = E[\hat{\theta}^2_{MLE}] - E[\hat{\theta}_{MLE}]^2 = \frac{n\theta^2}{n+2} - (\frac{n\theta}{n+1})^2 = \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{(n+1)^2(1+\frac{2}{n})}$$

The rate of convergence quantifies how fast the estimation error decreases when increasing the sample size n. In other words, what we are interested in is when divided by n^{-r} or multiplied by n^r, what's the largest r such that $|\hat{\theta}_n - \theta|$ remains stochastic bounded.
Clearly, for $Var[\hat{\theta}] = \frac{\theta^2}{(n+1)^2(1+\frac{2}{n})}$, the convergence rate should be n^2. Therefore, for standard deviation of $\hat{\theta}$, convergence rate is $r_n = n$.
We already know $E(\hat{\theta}_{MLE}) = \frac{n\theta}{n+1} = \frac{\theta}{1+\frac{1}{n}}$. $\lim_{n\to\infty} E(\hat{\theta}_{MLE}) = \theta$. So it is an unbiased estimator and we also know:

$$\lim_{n\to\infty} Var(\hat{\theta}_{MLE}) = \lim_{n\to\infty} \frac{\theta^2}{n^2 + 4n + 5 + \frac{2}{n}} = 0$$

Since our estimator is unbiased, using Chebyshev's inequality we know $P(|\hat{\theta} - \theta| > \epsilon) \leq Var(\hat{\theta})/\epsilon^2$. Then for any $\epsilon > 0$, if $\lim_{n\to\infty} Var(\hat{\theta}) = 0$ we have $\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \epsilon) = 0$ as well, which means consistency is proved. ($\hat{\theta} \xrightarrow{p} \theta$).

1.(d)

Now we want to find the asymptotic distribution of our estimator. $\hat{\theta}_{MLE} = \max(X_1, ..., X_n) = X_m$ and $r_n = n$ Notice that:

$$Pr(n(X_m - \theta) < x) = Pr(X_m < \frac{x}{n} + \theta)$$

Then we can derive that:

$$Pr(X_m < \frac{x}{n} + \theta) = (1 + \frac{x}{n\theta})^n$$

Since $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$, we know the limit of probability we derived above is:

$$\lim_{n\to\infty}(1 + \frac{x}{n\theta})^n = e^{\frac{x}{\theta}} \quad (x \leq 0)$$

The cdf of asymptotic distribution of our estimator is

$$F(x) = \begin{cases} e^{\frac{x}{\theta}} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

, which tells us the asymptotic distribution is exponential, but on the negative side of the real line.