**Econ 7010 - Microeconomics I**
**University of Virginia**
**Fall 2022**

## Problem Set 3 Solutions

1. *MWG 5.C.8*

   **Answer**

   In month 95, the company could have produced the same output at lower cost using the input pair from month 3, so the data violate the WAPM. Formally,

$$\begin{aligned} \vec{p}^{95} \cdot \vec{y}^{95} - \vec{p}^{95} \cdot \vec{y}^{3} &= (4,2,2) \cdot (60,-55,-40) - (4,2,2) \cdot (60,-40,-50) \\ &= (4,2,2) \cdot (0,-15,10) = -10, \text{ so} \\ \vec{p}^{95} \cdot \vec{y}^{95} &< \vec{p}^{95} \cdot \vec{y}^{3} \end{aligned}$$

2. *A firm as an input requirement set of $V(y) = \{x : ax_1 + bx_2 + cx_3 \geq e^y - 1\}$. Find the firm's cost function.*

   **Answer**

   The production function is $f(x) = \ln(1 + ax_1 + bx_2 + cx_3)$. Note that this is simply a perfect substitutes production function with three goods (scaled by the natural logarithm). So, the relevant ratios are $\gamma_1 = w_1/a$, $\gamma_2 = w_2/b$, and $\gamma_3 = w_3/c$ (the "productivity" of each input relative to its price). The firm then uses the input $i$ with the lowest $\gamma_i$. For example, if $\gamma_1 < \gamma_2, \gamma_3$, then the firm purchases $\frac{e^y-1}{a}$ units of good 1 at price $w_1$. (If two or more of the $\gamma_i$'s are the same, any combination using those goods with the lowest $\gamma_i$'s that produces $y$ units of output will be optimal.)

   The firm's cost function can then be written

$$c(w,y) = (e^y - 1)\min\{w_1/a, w_2/b, w_3/c\}$$

3. *Consider a firm with cost function $c(w,y) = (y(\sqrt{w_1} + \sqrt{w_2}))^2$. Find the firm's production function $f(x)$.*

   **Answer**

By Shepard's Lemma, $x_1(w,y) = \partial c/\partial w_1 = \frac{\sqrt{w_1}+\sqrt{w_2}}{\sqrt{w_1}} y^2$ and $x_2(w,y) = \partial c/\partial w_2 = \frac{\sqrt{w_1}+\sqrt{w_2}}{\sqrt{w_2}} y^2$. Adding these two equations together, we see that

$$\left(\frac{1}{x_1} + \frac{1}{x_2}\right) y^2 = 1$$

Solving for $y$, we find that

$$y = f(x) = \left(\frac{1}{x_1} + \frac{1}{x_2}\right)^{-\frac{1}{2}} = \sqrt{\frac{x_1 x_2}{x_1 + x_2}}$$

4. *Examine the parametric form of a candidate cost function $C(y,w) = y(a_1 w_1 + a_2 w_2 + b w_3^{\frac{1}{2}} w_4^{\frac{1}{2}})$. For what set of parameter values can this be a cost function? (explain). Now assume that these parameter restrictions hold. Find the form of the production function that generates this form of cost function.*

   **Answer**

   Cost functions are non-decreasing in prices, so it must be true that $a_1, a_2, b \geq 0$. Note that $b \geq 0$ also ensures that the cost function is concave (a linear combination of concave functions is concave). This is clearly continuous and hd1 in w.

   The production function is $y = \min\left\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \frac{2x_3^{1/2}x_4^{1/2}}{b}\right\}$.

5. *Define $C(q) = c(w,q)$ for a firm's cost function (we will be holding $w$ fixed throughout the problem) and $AC(q) = C(q)/q$ for $q > 0$.*

   (a) *Prove that if $AC(q)$ is minimized at $\bar{q}$ and if $C'(\bar{q})$ exists, then $AC(\bar{q}) = C'(\bar{q})$.*

   (b) *Prove that if $AC(q)$ is minimized at $\bar{q}$, then*

   $$\frac{C(\bar{q}) - C(\bar{q} - \varepsilon)}{\varepsilon} \leq AC(\bar{q}) \leq \frac{C(\bar{q} + \varepsilon) - C(\bar{q})}{\varepsilon} \text{ for all } \varepsilon \in [0, \bar{q}]$$

   *Illustrating with a well-labeled picture, explain the implications of these inequalities as $\varepsilon \to 0$ for the case in which $C(q)$ is not differentiable at $\bar{q}$.*

**Answer**

a) $C$ differentiable at $\bar{q}$ implies $AC(q) = C(q)/q$ is differentiable at $\bar{q}$ also. But then, if $AC(q)$ is minimized at $\bar{q}$, we must have

$$AC'(\bar{q}) = \frac{C'(\bar{q})}{\bar{q}} - \frac{C(\bar{q})}{\bar{q}^2} = \frac{1}{\bar{q}}\left(C'(\bar{q}) - \frac{C(\bar{q})}{\bar{q}}\right) = \frac{1}{\bar{q}}\big(C'(\bar{q}) - AC(\bar{q})\big) = 0$$

b) Minimality of $AC(\bar{q})$ implies

$$\begin{aligned} AC(\bar{q}) &\leq AC(\bar{q}-\varepsilon) \\ \frac{C(\bar{q})}{\bar{q}} &\leq \frac{C(\bar{q}-\varepsilon)}{\bar{q}-\varepsilon} \end{aligned}$$

Clearing denominators and rearranging terms, we get

$$\begin{aligned} \bar{q}\left(C(\bar{q}) - C(\bar{q}-\varepsilon)\right) &\leq \varepsilon C(\bar{q}) \quad \text{, and hence} \\ \frac{C(\bar{q}) - C(\bar{q}-\varepsilon)}{\varepsilon} &\leq AC(\bar{q}) \quad \text{as claimed} \end{aligned}$$

The proof of the other case is virtually the same. Write $l = \lim_{\varepsilon\to 0} \frac{C(\bar{q})-C(\bar{q}-\varepsilon)}{\varepsilon}$ if the lefthand limit exists, and similarly $r = \lim_{\varepsilon\to 0} \frac{C(\bar{q}+\varepsilon)-C(\bar{q})}{\varepsilon}$. $C$ is differentiable at $\bar{q}$ iff both limits exist and $l = r$. If the limits exist but are not equal, then we must have $l < r$. The limit $l$ $(r)$ is the slope of $C(q)$ approaching $\bar{q}$ from the left (right), so $l < r$ implies a kink at $\bar{q}$. Furthermore, we have $l < AC(\bar{q}) < r$. (Try to draw this yourself.)

6. *MWG 5.C.11*

**Answer**

By Shepard's Lemma, $z_\ell(w,q) = \partial c(w,q)/\partial w_\ell$. So,

$$\frac{\partial z_\ell(w,q)}{\partial q} = \frac{\partial^2 c(w,q)}{\partial q \partial w_\ell} = \frac{\partial}{\partial w_\ell}\left(\frac{\partial c(w,q)}{\partial q}\right)$$

where the last equality follows from the symmetry of cross-partial derivatives. The term in parentheses is just the marginal cost $MC(w,q)$, and so $\frac{\partial z_\ell(w,q)}{\partial q} > 0$ if and only if $\frac{\partial MC(w,q)}{\partial w_\ell} > 0$, i.e., if and only if the marginal cost is increasing in $w_\ell$.

7. *Consider a firm that uses labor* $(L)$ *and capital* $(K)$ *to produce a single output according to a production function* $f(L,K)$. *You may assume that* $f$ *is differentiable, strictly increasing, strictly concave, and satisfies:* $f(L,0) = f(0,K) = 0$ *for all* $L, K$.

   *The standard profit-maximization problem for a neoclassical firm is*

$$\max_{L,K \geq 0} pf(L,K) - wL - rK,$$

   *where* $p$ *is the output price and* $w$ *and* $r$ *are input prices. In this problem, we focus on the short-run, where the level of capital is fixed at some* $\bar{K} > 0$, *and you may ignore corner solutions.*

   (a) *Show that if the output price* $p$ *rises, the neoclassical firm will increase its use of labor.*

   *Consider instead a* labor-managed firm, *which, rather than paying a fixed market wage* $w$, *takes the production surplus and distributes it equally among the workers; that is, the "wage" each worker receives is* $W = \frac{pf(L,K)-rK}{L}$, *and the goal of the firm is to maximize* $W$:

$$\max_{L,K \geq 0} \frac{pf(L,K) - rK}{L}.$$

   *Continue to assume that capital is fixed at* $\bar{K}$.

   (b) *How will a labor-managed firm respond to an increase in* $p$? *Prove your answer.*

   (c) *Consider an increase in the price of capital,* $r$. *How does the output of each firm—neoclassical and labor-managed—respond?*

   (d) *Consider the following microfoundation for the labor-managed firm's problem: there is a worker with utility function* $u(W)$, *where* $W$ *is the compensation remunerated to the worker per unit of labor (i.e., the "wage"). You may assume that* $u' > 0$, $u'' < 0$, *and* $u'(0) = \infty$. *The objective of the firm is to maximize the worker's utility, subject to the constraint that the firm is profitable (i.e., revenues cover both the total wages paid as well as the fixed cost* $r\bar{K}$). *Formulate this as a constrained optimization problem. Show that its solution leads to the same optimal choice of labor as you found in part (b).*

**Answer**

**Part (a)**

Suppressing the dependence on $\bar{K}$, the FOC is $pf_L(L^*(p)) - w = 0$. Using the implicit function theorem gives

$$\frac{dL^*(p)}{dp} = -\frac{f_L(L^*(p))}{pf_{LL}(L^*(p))}.$$

Since $f$ is concave, $dL^*/dp \geq 0$.

**Part (b)**

In this case, the FOC is (where $L^{**}$ denotes the optimal solution to the labor-managed firm's problem)

$$pf(L^{**}) - L^{**}pf_L(L^{**}) - r\bar{K} = 0.$$

Again applying the IFT, we get

$$\frac{dL^{**}(p)}{dp} = \frac{f(L^{**}) - L^{**}f_L(L^{**})}{pL^{**}f_{LL}(L^{**})}.$$

Rearranging the FOC, we get $f(L^{**}) - L^{**}f_L(L^{**}) = r\bar{K}/p$, and substituting in the numerator for the above equation:

$$\frac{dL^{**}(p)}{dp} = \frac{r\bar{K}/p}{pL^{**}f_{LL}(L^{**})}.$$

Note that in this case, the firm's supply of labor (and thus output) decreases as the price increases.

**Part (c)**

Since capital is fixed, and the firm is operational, the neoclassical firm's optimal choice of $L$—and hence, output— does not change. For the labor-managed firm, return to the FOC

$$pf(L^{**}) - L^{**}pf_L(L^{**}) - r\bar{K} = 0.$$

and use the IFT to get

$$\frac{\partial L^{**}}{\partial r} = -\frac{\bar{K}}{pf_{LL}(L^{**})L^{**}} > 0.$$

Since $\frac{df(L^{**}(r),\bar{K})}{dr} = f_L(L^{**}(r),\bar{K})\frac{\partial L^{**}}{\partial r}$ and $f_L > 0$, we have $df/dr > 0$.

**Part (d)**

We can write the problem as

$$\max_{(W,L)} u(W) \text{ subject to } pf(L,\bar{K}) - WL \geq r\bar{K}$$

Rewriting the constraint, it becomes $W \leq [pf(L,\bar{K}) - r\bar{K}]/L$, which clearly we want to hold with equality. Thus, the problem becomes

$$\max_{L\geq 0} u\left(\frac{pf(L,\bar{K}) - r\bar{K}}{L}\right)$$

This is just a positive monotonic transformation of the objective from (b), and so the solution will be the same.

8. *Sally runs a lemonade stand, and, as she is the only lemonade stand in the neighborhood, she is a monopoly. She faces an inverse demand curve of $p(y,\lambda)$, where $y$ denotes the liters of lemonade and $\lambda$ is a parameter that denotes the outside temperature, which can shift demand. Sally has cost function $c(y,\gamma)$, where $\gamma$ is a parameter that can shift the cost function. Assume that all functions are twice continuously differentiable, and that $p_\lambda(y,\lambda) > 0$ and $c_\gamma(y,\gamma) > 0$.*

   (a) *Write down the monopolist's problem, and take the first order condition. Provide conditions on the derivatives of $p$ and $c$ that guarantee that the first-order condition is sufficient for an optimum. Provide a second set of conditions that guarantee that the first order condition is sufficient for the optimum to be unique. Your conditions should be on $p$ and $c$ independently, and the second set of conditions should be stronger than the first.*

   *Assume that the conditions from part (a) hold.*

   (b) *Derive expressions showing how the optimal output of the monopolist responds to changes in $\lambda$ and $\gamma$ (consider each separately). Provide sufficient conditions under which the monopolist's output is increasing in $\lambda$. Answer the same for $\gamma$.*

   (c) *Consider the special case where $y, \lambda$, and $\gamma$ are separable, i.e., the inverse demand function can be written as $p(y,\lambda) = r(y) + s(\lambda)$ and the cost function can be written $c(y,\gamma) = a(y) + b(\gamma)$ for some (differentiable) functions $r(\cdot), s(\cdot), a(\cdot)$, and $b(\cdot)$. Answer part (b) in this case. Explain the intuition behind the conditions* *you derived. (If you have done part (b) correctly, this part should not take you very long.)*

**Answer**

(a) The monopolist's problem is

$$\max_{y \geq 0} p(y, \lambda)y - c(y, \gamma).$$

The first order condition reads $p_y(y, \lambda)y + p(y, \lambda) - c_y(y, \gamma) = 0$. Sufficient conditions to guarantee that the solution to the first order conditions are optimal are $p_{yy} \leq 0$ and $c_{yy} \geq 0$ for all $(y, \lambda, \gamma)$ (i.e., $p$ is concave in $y$ and $c$ is convex in $y$). Sufficient conditions to guarantee that the solution is unique are $p_{yy} < 0$ and $c_{yy} > 0$ for all $(y, \lambda, \gamma)$ (i.e., $p$ is *strictly* concave and $c$ is *strictly* convex).

(b) First, for $\lambda$, let $y(\lambda)$ be the implicit function of $y$ defined by the FOC. Then, taking a derivative of the FOC with respect to $\lambda$ to get:

$$p_{yy}y\frac{dy}{d\lambda} + p_{y\lambda}y + p_y\frac{dy}{d\lambda} + p_y\frac{dy}{d\lambda} + p_\lambda - c_{yy}\frac{dy}{d\lambda} = 0.$$

Solving for $dy/d\lambda$, we have

$$\frac{dy}{d\lambda} = -\frac{p_\lambda + p_{y\lambda}y}{p_{yy}y + 2p_y - c_{yy}}.$$

Under the sufficient conditions from part (a), the denominator is negative. Thus, the entire expression is positive when $p_\lambda + p_{y\lambda}y \geq 0$.

For $\gamma$, let $y(\gamma)$ be the implicit function defined by the FOC. Then, taking a derivative of the FOC with respect to $\gamma$, we have:

$$p_{yy}y\frac{dy}{d\gamma} + p_y\frac{dy}{d\gamma} - c_{yy}\frac{dy}{d\gamma} - c_{y\gamma} = 0.$$

Solving for $\frac{dy}{d\gamma}$, we get

$$\frac{dy}{d\gamma} = \frac{c_{y\gamma}}{p_{yy}y + p_y - c_{yy}}.$$

Under the sufficient conditions from part (a), the denominator is negative. Thus, the entire expression will be positive if $c_{y\gamma}$ is negative, i.e., if the marginal cost is decreasing in $\gamma$.

(c) Under these functional forms, the cross derivative terms are zero. Thus,

$$\frac{dy}{d\lambda} = -\frac{p_\lambda}{p_{yy}y + 2p_y - c_{yy}},$$

and this is increasing whenever $p_\lambda$ is increasing. Intuitively, this means that demand is increasing in $\lambda$ - i.e., as the temperature rises, the demand for lemonade also rises. When $c$ is separable, $c_{y\gamma} = 0$ and hence $dy/d\gamma = 0$, that is, the optimal output does not change in response to a change in $\gamma$. Intuitively, $c_{y\gamma}$ means that parameter $\gamma$ does not affect the *marginal* costs, which is what is used in determining the optimal quantity $y$.