# Suggested Solutions: ECON 7710 HW II

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## Question 1

We know $X$ and $Y$ are independent random variables and distribution functions are $F(x)$ and $G(y)$. We want to find the distribution functions of following 4 random variables:

- $\max\{X, Y\}$
  Define $A = \max\{X, Y\}$, then $F_A(a) = P(A \leq a) = P(\max\{X, Y\} \leq a) = P(X \leq a \text{ and } Y \leq a)$.
  Since $X$ and $Y$ are independent, we know

$$P(X \leq a \text{ and } Y \leq a) = P(X \leq a) * P(Y \leq a) = F_X(a) * F_Y(a)$$

- $\min\{X, Y\}$
  Define $B = \min\{X, Y\}$, then $F_B(b) = P(B \leq b) = P(\min\{X, Y\} \leq b) = 1 - P(\min\{X, Y\} > b) = 1 - P(X > b \text{ and } Y > b)$.
  Since $X$ and $Y$ are independent, we know

$$P(X > b \text{ and } Y > b) = P(X > b)*P(Y > b) = (1-P(X \leq b))*(1-P(X \leq b)) = (1-F_X(b))(1-F_Y(b))$$

  Therefore, we know:

$$F_B(b) = 1 - (1 - F_X(b))(1 - F_Y(b)) = F_X(b) + F_Y(b) - F_X(b) * F_Y(b)$$

- $\max\{2X, Y\}$
  Define $C = \max\{2X, Y\}$, then $F_C(c) = P(C \leq c) = P(\max\{2X, Y\} \leq c) = P(2X \leq c \text{ and } Y \leq c)$.
  Since $2X$ and $Y$ are independent, we know

$$P(2X \leq c \text{ and } Y \leq c) = P(X \leq \frac{c}{2}) * P(Y \leq c) = F_X(\frac{c}{2}) * F_Y(c)$$

- $\min\{X^3, Y\}$

  Define $D = \min\{X^3, Y\}$, then $F_D(d) = P(D \leq d) = P(\min\{X^3, Y\} \leq d) = 1 - P(\min\{X^3, Y\} > d) = 1 - P(X^3 > d \text{ and } Y > d)$.

  Notice that $x^3$ is a monotonically increasing function and since $X^3$ and $Y$ are independent, we know

$$
\begin{aligned}
P(X^3 > d \text{ and } Y > d) &= P(X^3 > d) * P(Y > d) = (1 - P(X^3 \leq d))(1 - P(Y \leq d)) \\
&= (1 - P(X \leq d^{\frac{1}{3}}))(1 - P(Y \leq d)) = (1 - F_X(d^{\frac{1}{3}}))(1 - F_Y(d))
\end{aligned}
$$

  Therefore, we know:

$$F_D(d) = F_X(d^{\frac{1}{3}}) + F_Y(d) - F_X(d^{\frac{1}{3}}) * F_Y(d)$$

# Question 2

In this question, we know the joint cumulative distribution function (CDF) of $(X, Y)$ on the unit square, denoted by $S = \{0 \leq x \leq 1\ , 0 \leq y \leq 1\}$, is $F(x, y) = \frac{1}{2}(xy + min\{x, y\})$.

### a)

Since joint CDF means $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$ and now our CDF is defined on $x \in [0, 1]$, $y \in [0, 1]$. To answer how the CDF behave outside the unit square, we discuss as follows:

- When $x > 1, y > 1$:
  We already have $F(1, 1) = 1$. Meanwhile, $F(x, y) \in [0, 1]$ and it is nondecreasing. So we know $F(x, y) = 1$ when $x > 1$ and $y > 1$.
- When $x \in [0, 1]$ and $y > 1$:
  Then we know $F(x, y) = P(X \leq x, Y \leq y) = P(X \leq x)$, when $0 \leq x \leq 1$ and $y = 1$. Then we know $F(x, y) = \frac{1}{2}(2x) = x$. (It depends on $x$ only)
- When $x > 1$ and $y \in [0, 1]$:
  Then we know $F(x, y) = P(X \leq x, Y \leq y) = P(Y \leq y)$, when $0 \leq y \leq 1$ and $x = 1$. Then we know $F(x, y) = \frac{1}{2}(2y) = y$. (It depends on $y$ only)
- When $x < 0$ or $y < 0$:
  As we already have $F(0, y) = 0, \forall y \geq 0$ and $F(x, 0) = 0, \forall x \geq 0$. We also know CDF is nondecreasing and the value of CDF lies between 0 to 1. Therefore $F(x, y) = 0$ if $x < 0$ or $y < 0$.

As we know when $0 \leq x \leq 1$ and $0 \leq y \leq 1$, $F(x, y) = \frac{1}{2}(xy + \min\{x, y\})$. When $(x, y)$ lies outside the unit square, we know:

$$F_{X,Y}(x, y) = \begin{cases} 1, & \text{For } x \in (1, \infty), y \in (1, \infty) \\ x, & \text{For } x \in [0, 1], y \in (1, \infty) \\ y, & \text{For } x \in (1, \infty), y \in [0, 1] \\ 0, & \text{For } x \in (-\infty, 0) \text{ or } y \in (-\infty, 0) \end{cases}$$

### b)

Denote marginal CDF of $X$ as $F_X(x)$ and marginal CDF of $Y$ as $F_Y(y)$. We know:

$$F_X(x) = \lim_{y\to+\infty} F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } x \in (-\infty,0) \\ x, & \text{For } x \in [0,1] \\ 1, & \text{For } x \in (1,\infty) \end{cases}$$

Likewise, we know:

$$F_Y(y) = \lim_{x\to+\infty} F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } y \in (-\infty,0) \\ y, & \text{For } y \in [0,1] \\ 1, & \text{For } y \in (1,\infty) \end{cases}$$

### c)

Since we know $U = logX$, and $V = logY$, both are monotonically increasing functions. We denote CDF of $(U,V)$ as $F_{U,V}(u,v)$. So we know $F_{U,V}(u,v) = P(U \le u, V \le v) = P(logX \le u, logY \le v) = P(X \le e^u, Y \le e^v)$

As we know from a), the CDF of $(X,Y)$ is

$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } x \in (-\infty,0) \text{ or } y \in (-\infty,0) \\ \frac{1}{2}(xy + min\{x,y\}), & \text{For } x \in [0,1], y \in [0,1] \\ x, & \text{For } x \in [0,1], y \in (1,\infty) \\ y, & \text{For } x \in (1,\infty), y \in [0,1] \\ 1, & \text{For } x \in (1,\infty), y \in (1,\infty) \end{cases}$$

Therefore, we know CDF of $(U,V)$ is

$$F_{U,V}(u,v) = \begin{cases} \frac{1}{2}(e^{u+v} + min\{e^u,e^v\}), & \text{For } u \in (-\infty,0], v \in (-\infty,0] \\ e^u, & \text{For } u \in (-\infty,0], v \in (0,\infty) \\ e^v, & \text{For } u \in (0,\infty), v \in (-\infty,0] \\ 1, & \text{For } u \in (0,\infty), v \in (0,\infty) \end{cases}$$

### d)

We are asked if the following distributions

1. joint distribution of $X$ and $Y$

2. marginal distribution of $X$ and $Y$

3. joint distribution of $U$ and $V$

has a density w.r.t. Lebesgue measure respectively.
We also know that a distribution does not have a density w.r.t. the Lebesgue Measure if it places positive probability on a set has Lebesgue Measure zero.

**d.1**

As we know from a), the CDF of $(X, Y)$ is

$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{For } x \in (-\infty, 0) \text{ or } y \in (-\infty, 0) \\ \frac{1}{2}(xy + min\{x,y\}), & \text{For } x \in [0,1], y \in [0,1] \\ x, & \text{For } x \in [0,1], y \in (1,\infty) \\ y, & \text{For } x \in (1,\infty), y \in [0,1] \\ 1, & \text{For } x \in (1,\infty), y \in (1,\infty) \end{cases}$$

First, we know within the unit square the CDF does not have a derivative at any point along the line when $x = y$ (kink of min function).
Second, we know

$$\frac{\partial F(x,y)}{\partial x \partial y} = \begin{cases} \frac{1}{2} & x \in [0,y), \quad y \in (0,1] \\ \frac{1}{2} & x \in (y,1], \quad y \in [0,1) \\ 0 & \text{Outside of the unit square} \end{cases}$$

Third, we know:

$$Pr(0 \leq X < Y, 0 < Y \leq 1) = \int_0^1 \int_0^y \frac{1}{2} dx dy = \frac{1}{4}$$

Likewise,

$$Pr(Y < X \leq 1, 0 \leq Y < 1) = \int_0^1 \int_y^1 \frac{1}{2} dx dy = \frac{1}{4}$$

Since the probability of drawing a pair of (x,y) from a region outside of the unit square is 0. And the probability above only gives us $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. It suggests that there is probability mass $\frac{1}{2}$ on the line $x = y$. We can also prove that there are no other mass lines in the unit square by contradiction. Basically, if other line has a mass, the corresponding distribution function we construct will not coincide with the CDF we are given.

Since lines in 2D square have Lebesgue measure zero and we know the distribution of $(X, Y)$ places positive probability on line $x = y$ Thus Joint distribution of $(X, Y)$ does not have density w.r.t. Lebesgue measure.

**d.2**

We already know marginal distribution of $X$ is:

$$F_X(x) = \begin{cases} 0, & \text{For } x < 0 \\ x, & \text{For } x \in [0, 1] \\ 1, & \text{For } x > 1 \end{cases}$$

And marginal distribution of $Y$ is:

$$F_Y(y) = \begin{cases} 0, & \text{For } y < 0 \\ y, & \text{For } y \in [0, 1] \\ 1, & \text{For } y > 1 \end{cases}$$

Then for $F_X(x)$, if we pick an arbitrary line in unit square, the marginal probability will always be 0 because $y$ is contained in the line but there is no y in $F_X(x)$. Same logic for $F_Y(y)$. So we know marginal distribution of $X$ and $Y$ have a density w.r.t. the Lebesgue Measure.

**d.3**

We already know the CDF of $(U, V)$ is

$$F_{U,V}(u, v) = \begin{cases} \frac{1}{2}(e^{u+v} + min\{e^u, e^v\}), & \text{For } u \in (-\infty, 0], v \in (-\infty, 0] \\ e^u, & \text{For } u \in (-\infty, 0], v \in (0, \infty) \\ e^v, & \text{For } u \in (0, \infty), v \in (-\infty, 0] \\ 1, & \text{For } u \in (0, \infty), v \in (0, \infty) \end{cases}$$

Similarly, we follow the steps in d.1 and set our line with Lebesgue Measure of zero as $\{(u, v) : u = v, \text{ For, } u \in (-\infty, 0], v \in (-\infty, 0]\}$.
Then we will get a positive probability on a set has Lebesgue Measure zero, which means the joint distribution of $U$ and $V$ does not have a density w.r.t. the Lebesgue Measure.
In summary, joint distributions(d.1, d.3) don't have a density w.r.t. Lebesgue Measure. Marginal distributions(d.2) have a density w.r.t. Lebesgue Measure.

## e)

### e.1

From previous analysis, we know with mass = $\frac{1}{2}$, $x = y$; with mass = $\frac{1}{2}$, $\frac{\partial F(x,y)}{\partial x \partial y} = \frac{1}{2}$ when $x, y$ are in unit square and $x \neq y$. This tells us the distribution of $X, Y$ should be a **mixture** of the bivariate uniform distribution ($X, Y$ are independent) and uniform on the line $x = y$ ($X, Y$ are perfectly correlated). In other words, half of the time, pair $(x, y)$ is drawn uniformly from the unit square, half of the time it is drawn from the line $x = y$.
Now for the conditional distribution $X|Y$, it should also be a mixture. Half of the time, $X$ will be drawn independently of the value of $Y$ from $[0, 1]$ [Continuous Part]. Half of the time, it will be exactly equal to the value of $Y$ [Discrete Part].
Then we know:

- Partial density function for the continuous part:

$$f_{X|Y}(x|y) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

- Partial probability mass function for the discrete part:

$$g_{X|Y}(x|y) = \begin{cases} 1 & x = y, \ \ (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{Otherwise} \end{cases}$$

Then the we know:

$$E[X|Y] = \frac{1}{2} \int_0^1 x dx + \frac{1}{2} y = \frac{1}{4} + \frac{1}{2} y$$

### e.2

By symmetry, we know:

$$E[Y|X] = \frac{1}{4} + \frac{1}{2} x$$

### e.3

Likewise, the distribution of $U, V$ is also a mixture of bivariate uniform distribution [Continuous Part] and a uniform on the line $u = v$ [Discrete Part].
Then we know:

- Partial density function for the continuous part:

$$f_{U|V}(u|v) = \begin{cases} e^u & u \in (-\infty, 0] \\ 0 & \text{Otherwise} \end{cases}$$

- Partial probability mass function for the discrete part:

$$g_{U|V}(u|v) = \begin{cases} 1 & u = v, \quad (u,v) \in (-\infty, 0] \times (-\infty, 0] \\ 0 & \text{Otherwise} \end{cases}$$

Then we know:

$$E[U|V] = \frac{1}{2}\int_{-\infty}^{0} ue^u du + \frac{1}{2}v = \frac{1}{2}([ue^u]_{-\infty}^{0} - \int_{-\infty}^{0} e^u du) + \frac{1}{2}v$$

We know: $\lim_{u \to 0} ue^u = 0$. We also know $\lim_{u \to -\infty} ue^u = \lim_{u \to -\infty} \frac{u}{e^{-u}} = \lim_{u \to -\infty} \frac{1}{-e^{-u}} = 0$ by L'Hôpital's rule. So

$$E[U|V] = \frac{1}{2}(v - \int_{-\infty}^{0} e^u du) = \frac{1}{2}v - \frac{1}{2}$$

**e.4**

By symmetry, we know:

$$E[V|U] = \frac{1}{2}u - \frac{1}{2}$$