# Macroeconomic Theory*

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# 1 Introduction

These notes cover the first semester of the Ph.D.-level course in macroeconomic theory taught at the University of Virginia. On a high level, the objective of the course is to familiarize students with modern macroeconomic theory along three key dimensions:

1. **Technical** (e.g., Markov processes, (stochastic and deterministic) dynamic programming, log-linearization);

2. **Language** (Arrow-Debreu Equilibrium, Sequential Markets Equilibrium, Recursive Competitive Equilibrium, Pareto Optimality);

3. **Substantive** (neoclassical growth model, stochastic growth model, complete and incomplete markets, overlapping-generations model).

I start with a simple, dynamic endowment economy with two infinitely-level agents and introduce notions of a competitive equilibrium and a Pareto efficial allocation. In a simple exchange economy like that it is simple to show that the competitive allocation is optimal due to the First Welfare Theorem.

Subsequently, I introduce production in the economy and simultaneously assume there is one representative household for simplicity; this gives rise to the workhorse model of modern macroeconomics, the Neoclassical Growth Model. I consider a Social Planner's problem in this environment and motivate dynamic programming (DP). At first, we will brush many critical technical details under the rug to see a big picture. Subsequently, I revisit DP and fill in the details on the principle of optimality, contraction mapping theorem, Blackwell's sufficient conditions and other important results.

# 2 Kaldor Facts and the Solow-Swan Growth Model

We are going to start thinking through the process of model building with the simplest model—the Solow-Swan growth model. We will motivate our model by a salient set of facts, which will dictate what we put in and what we leave out.

The key facts we need are going to be related to growth, especially the so-called Kaldor facts:

– real output grows at roughly a constant rate over time, as does real output per capita;

– the capital/output ratio remains nearly constant;

– total labor hours grow at a rate smaller than that of capital;

– the return on capital does not grow over time.

We will think of a world where time is infinite (both forward and backward) and is discrete—there is a meaningful way to speak of "today" and "tomorrow"; note that the definition of a period is also flexible, so that today and tomorrow could be as close or as far apart as we wish (depending on the question and the data). In continuous time, there is no tomorrow; for some questions continuous time can be easier, but we do not have the time to do both in one semester. Our model economy will be populated by a large number of individual households (current size $N_t$) and a large number of firms (normalized to be 1), along with markets for capital, labor, and goods.

**Notation** We will denote the derivative of function $f(x)$ as $Df(x)$; similarly, $D^2f(x)$ denotes the second derivative of $f(x)$. In a multivariate case, $D_1F(x,y)$ and $D_xF(x,y)$ denote the derivative of $F(x,y)$ with respect to $x$, while $D_2F(x,y)$ and $D_yF(x,y)$ denote the derivative of $F(x,y)$ with respect to $y$.

## 2.1 Model Setup

The Solow-Swan model starts with a description of production. Let $K$ be the total amount of capital in the economy, $N$ be the total population, and $X$ be the "productivity" of workers. The Solow model says that total output is connected to these inputs by the aggregate production function:

$$Y_t = F(K_t, X_tN_t),$$

where $XN$ is the total effective units of labor. We will assume that $F$ is continuously differentiable (for strictly positive input levels), strictly increasing in both arguments, strictly concave, satisfies

$$F(0, XN) = F(K, 0) = 0,$$

and satisfies Inada conditions:

$$\lim_{K \to 0} D_1 F = \lim_{XN \to 0} D_1 F = \infty$$
$$\lim_{K \to \infty} D_1 F = \lim_{XN \to \infty} D_1 F = 0.$$

Furthermore, we assume that $F(\cdot)$ is homogeneous of degree 1 (i.e., $F$ exhibits constant returns to scale):

$$F(\lambda K, \lambda XN) = \lambda F(K, XN)$$

for any $\lambda > 0$.

**Remark** Note that if we set $\lambda = \frac{1}{XN}$, then it follows

$$F(K, XN) = \frac{1}{NX} F(K, XN).$$

We now specify the dynamics of our model economy. Specifically, we assume that $X$ and $N$ grow exogenously at constant rates $x$ and $n$, respectively:

$$X_{t+1} = X_t(1 + x)$$
$$N_{t+1} = N_t(1 + n).$$

The law of motion for aggregate capital is

$$K_{t+1} = (1 - \delta)K + I,$$

where $\delta \in [0, 1]$ is depreciation rate, and $I$ denotes investment. This accumulation equation embodies an assumption called perpetual inventory, whereby increments to the capital stock (investment) disappear asymptotically at rate $\delta$.

The consumer's side of the model is kept deliberately simply; it is assumed that there is a representative household who consumes a constant fraction of income:

$$C_t = (1-\beta)Y_t.$$

Savings is defined as the residual (not consumed) income:

$$S_t = \beta Y_t.$$

**Remark** In this simple model, the consumption/savings decision is reduced-form, and later in the course we will consider models in which we will derive $C_t$ and $S_t$ explicitly as an optimal choice of the household.

We also have a *resource constraint* (or *feasibility constraint*) that states:

$$C_t + I_t = F(K_t, X_t N_t).$$

It follows that savings equals investment:

$$\begin{aligned} S_t &= \beta Y_t \\ &= Y_t - (1-\beta)Y_t \\ &= F(K_t, X_t N_t) - C_t \\ &= I_t. \end{aligned}$$

Now, combining this result with the law of motion of aggregate capital, we obtain:

$$K_{t+1} = (1-\delta)K_t + \beta F(K_t, X_t N_t).$$

## 2.2 Analysis of the BGP

We will look for the "long-run" outcomes. The Kaldor facts suggest we should look for a *balanced growth path*, whereby $K$ and $Y$ (and, therefore, $C, I$ and $S$) grow at the same constant rate over time. Note that resource constraint requires that constant growth rate must be equal. We will now find this growth rate by exploiting the CRS property of the production function:

$$\frac{K_{t+1}}{X_t N_t} = (1-\delta)\frac{K_t}{N_t X_t} + \beta F\left(\frac{K_t}{N_t X_t}, 1\right).$$

Now, substitute in equations describing the dynamics of $X$ and $N$:

$$(1+x)(1+n)\frac{K_{t+1}}{X_{t+1}N_{t+1}} = (1-\delta)\frac{K_t}{N_t X_t} + \beta F\left(\frac{K_t}{N_t X_t}, 1\right).$$

If we define $k_t := \frac{K_t}{N_t X_t}$ (capital per effective worker), we can rewrite the equation above more compactly:

$$(1+x)(1+n)k_{t+1} = (1-\delta)k_t + \beta F\left(k_t, 1\right).$$

The equation above is the first-order non-linear different equation, since (1) it involves capital dated time $t$ and $t+1$, and (2) it involves a non-linear function of $k_t$. A balanced growth path is a generalized version of a steady-state in which variables are constants. In this model, some variables will be constant, while others will grow at a constant rate (not necessarily common).
Along the balanced growth path, $k_t$ will be constant, thus:

$$(1+n)(1+x)k = (1-\delta)k + \beta F(k, 1).$$

Rearranging, we obtain:

$$(n + x + nx + \delta)k = \beta F(k, 1).$$

The LHS is the *capital widening line*, which is increasing and linear in $k$. The RHS is also increasing, and strictly concave. Given Inada conditions, it follows:

$$\beta D_1 F(0, 1) > n + x + nx + \delta$$

$$\beta D_1 F(\infty, 1) < n + x + nx + \delta.$$

From this, it follows that there exists a single crossing point for the two curves, called $k^*$, which is the steady-state capital/effective worker ratio:

$$(1+n)(1+x)k^* = (1-\delta)k^* + \beta F(k^*, 1).$$

**Remark** $k^{**} = 0$ is also a steady state given our assumptions on $F$. However, $k^{**} = 0$ is not approached from any initial positive capital stock value. Thus, if we exclude 0 from the domain of our problem, $k^*$ becomes globally attractive. Furthermore, if the capital stock includes land, then no economy has zero land, thus we will ignore this steady state.

Suppose $k_t < k^*$, then

$$(n + x + nx + \delta)k_t < \beta F(k_t, 1),$$

so that the amount of investment exceeds the capital widening amount, implying that $k_{t+1} > k_t$. Over time, $k_t \to k^*$. Similar arguments apply for $k_t > k^*$.

**Balanced Growth Path** Along the BGP, the growth of $k$ is 0. Thus, the growth rate of $\frac{K_t}{N_t}$, the per capita capital stock, is $1 + x$, and the growth rate of $K_t$ is $(1 + n)(1 + x)$. Output grows at the same rate as capital, and therefore so do consumption, savings and investment. Thus, the capital-to-output ratio is constant, and capital grows faster than total hours. The first three Kaldor facts are replicated (i.e., consistent with the model).

The last Kaldor fact states that returns to capital are constant. Think of a firm that runs the aggregate production technology by renting capital and labor from the household. This firm will solve the following problem:

$$\Pi = F(K_t, X_t N_t) - r_t K_t - w_t X_t N_t \to \max.$$

By Euler's theorem, if $F$ is homogeneous of degree 1, then $D_1 F$ is homogeneous of degree 0. Therefore,

$$r_t = D_1 F(k_t, 1).$$

The return on capital is $r_t - \delta$ which is clearly constant at $k = k^*$. The fourth Kaldor fact is satisfied. Also, notice that the wage per unit of labor $W_t$ grows with $X_t$ along the BGP, while the wage per unit of efficiency labor $w$ is constant:

$$w_t = D_2 F(K_t, X_t N_t) = D_2 F(k_t, 1)$$

$$W_t = D_2 F(K_t, X_t N_t) X_t = X_t w_t.$$

An interesting point to make here is that the growth rate in the long run $(1+n)(1+x)$ is independent of the savings rate $\beta$. Thus, countries with higher savings rate will not converge to higher growth paths.

### 2.3 Application to Growth Accounting

We can use the model to ask which fraction of the growth in per capita GDP is due to various factors. Let $Y, N, L, K$ denote GDP, working age population, total working hours and capital, respectively. The growth accounting equation is:

$$\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\frac{Y_t}{N_t} &= A_t \left(\frac{K_t}{N_t}\right)^{\alpha} \left(\frac{L_t}{N_t}\right)^{1-\alpha} \\
\frac{Y_t}{N_t} &= A_t \left(\frac{K_t}{Y_t N_t}\right)^{\alpha} \left(\frac{L_t}{Y_t N_t}\right)^{1-\alpha} Y_t \\
\frac{Y_t}{N_t} &= A_t \left(\frac{K_t}{Y_t}\right)^{\alpha} \left(\frac{L_t}{N_t}\right)^{1-\alpha} \left(\frac{Y_t}{N_t}\right)^{\alpha} \\
\frac{Y_t}{N_t} &= A_t^{\frac{1}{1-\alpha}} \left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{L_t}{N_t}\right).
\end{aligned}$$

Taking logs, and noting that $\log(1+x) \approx x$ for $x$ small enough, we can obtain:

$$g_{Y/N} = \frac{1}{1-\alpha} g_A + \frac{\alpha}{1-\alpha} g_{K/Y} + g_{L/N}.$$

Thus, depending on the growth rates of $\frac{K}{Y}$ and $\frac{L}{N}$, we can determine how much growth is being driven by the residual factor $A$. It turns out that the vast majority of growth is driven by $A$, and not changes in factor inputs.

# 3 A Simple (Endowment) Dynamic Economy

## 3.1 Model Setup

Consider a simple economy with 2 agents. Time is discrete and runs forever, $t = 0, 1, \ldots$. There is no production, and agents are simply endowed each period with some amount of a (perishable) consumption good. There is no uncertainty in this model and both agents know their endowment pattern perfectly in advance. All information is public, i.e. all agents know everything.

**Definition 1.** *An allocation is a sequence $\{c_t^1, c_t^2\}_{t=0}^{\infty}$, where $c_t^i$ is the amount of consumption of agent $i \in \{1, 2\}$ in period $t$.*

**Preferences** Individuals have preferences over allocations which are captured by the following utility function:

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \log(c_t^i),$$

where $\beta$ is a subjective discount factor. For now, we assume that both agent have the same discount factor, i.e., they are equally impatient.

**Production technology** There is no production in this economy.

**Endowments** Agents face deterministic sequences of endowments $\{e_t^1, e_t^2\}_{t=0}^{\infty}$, where $e_t^i$ is the endowment of agent $i \in \{1, 2\}$ in period $t$. For simplicity, let us assume that agent 1 receives 2 units of consumption good in odd periods (2,0,...) and nothing in even periods. Conversely, agent 2 receives 2 units of the good in even periods, and nothing in odd periods.

## 3.2 Arrow-Debreu (AD) Equilibrium

At period 0, before endowments are received and consumption takes place, the two agents meet at a central market place and trade all commodities, i.e. trade consumption for all future dates. Let $p_t$ denote the price, in period 0, of one unit of consumption to be delivered in period $t$. We will see later that prices are only determined up to a constant, so we can always normalize the price of one commodity to 1 and make it the numeraire. Both agents are assumed to behave competitively in that they take the sequence of prices $\{p_t\}_{t=0}^{\infty}$ as given.

**Definition 2.** *A competitive Arrow-Debreu equilibrium consists of prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *and allocations* $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ *such that:*

1. *given the prices* $\{\hat{p}_t\}_{t=0}^{\infty}$, $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ *solve the problem of agents:*

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \log(c_t^i) \to \max$$
$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$
$$c_t^i \geq 0 \quad \forall t$$

2. *the market clears in each period* $t$*:*

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 \quad \forall t.$$

**Question:** Why is market clearing condition written as equality?

### 3.3 Solving for the Equilibrium

In order to solve for the equilibrium, let us form a Lagrangian:

$$\mathcal{L}_i = \sum_{t=0}^{\infty} \beta^t \log(c_t^i) + \lambda^i \left( \sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \right) \to \max_{\{c_t^i\}_{t=0}^{\infty}}.$$

The first order conditions with respect to $c_t^i$ and $c_{t+1}^i$ yield:

$$\frac{\beta^t}{c_t^i} = -\lambda^i p_t$$
$$\frac{\beta^{t+1}}{c_{t+1}^i} = -\lambda^i p_{t+1}.$$

Combine the two optimality conditions and obtain:

$$\beta p_t c_t^i = p_{t+1} c_{t+1}^i$$

for all $t$ and $i$. Summing the equation above across two agents, yields:

$$\beta p_t(c_t^1 + c_t^2) = p_{t+1}(c_{t+1}^1 + c_{t+1}^2).$$

Now, invoke market clearing conditions for both periods:

$$\beta p_t(e_t^1 + e_t^2) = p_{t+1}(e_{t+1}^1 + e_{t+1}^2).$$

Recall that the total endowment (across both agents) is constant and equals 2 in each period. Thus, $p_{t+1} = \beta p_t$. Interating in back to $t = 0$ yields $p_t = \beta^t p_0$. Provided that scaling prices by any positive constant does not change the optimal allocation (show this!), we can normalize $p_0 = 1$, thus $p_t = \beta^t$. It directly follows that

$$\beta\beta^t c_t^i = \beta^{t+1} c_{t+1}^i,$$

thus the consumption stream of both agents is constant over time, $c_t^i = c_{t+1}^i = c_0^i$.

**Equilibrium Allocations** Invoke the budget constraint of agent $i$:

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t e_t^i$$
$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t e_t^i.$$

We have established that both agents will consume the same amount each period. Thus,

$$\sum_{t=0}^{\infty} \beta^t c_0^i = \sum_{t=0}^{\infty} \beta^t e_t^i.$$

Therefore, for agent 1 we have $c_0^i \sum_{t=0}^{\infty} \beta^t = \frac{c_0^1}{1-\beta} = \frac{2}{1-\beta^2}$. That is, agent 1 will consume $c_0^1 = \frac{2}{1+\beta}$ every period. Agent 2 will consume $c_0^2 = \frac{2\beta}{1+\beta}$ every period (check that markets indeed clear at this allocation).

**Remark** Note that agent 1 consumes more than agent 2 in every single period simply because agent 1 has a bigger endowment in the first period. Besides, both agents are better off in this competitive equilibirum relative to autarky because of the assumed utility function ($\lim_{c \to 0} \log(c) = -\infty$). Thus, the trade is mutually beneficial.

### 3.4 Pareto Optimality

**Definition 3.** *An allocation is Pareto efficient if it is feasible and there is no other feasible allocation which makes at least someone strictly better off and others at least no worse off.*

In the context of the model laid out in this section, an allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ will be Pareto efficient if is feasible, and there is no other feasible $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$, such that:

$$u(\tilde{c}^i) \geq u(c^i) \quad \forall i$$
$$u(\tilde{c}^i) > u(c^i) \quad \exists i.$$

**Remark** Pareto efficiency has nothing to do with fairness. If one of the agents eats the entire endowment each period (and the second agent consumes 0), this also constitutes a Pareto efficient allocation, because if we redistribute any amount of the good toward the second agent, we will strictly reduce the utility of the first agent.

**Theorem 1.** *First Welfare Theorem: If $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ is a competitive equilibrium, then it is also Pareto efficient.*

*Proof.* The proof is by contradiction. Suppose $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ is an equilibrium, which is not PO. That is, suppose there is another feasible allocation $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$ which Pareto dominates $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$. Without loss of generality, assume that agent 1 is strictly better off at $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$, and agent 2 is no worse. It follows that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1.$$

If not, then

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1.$$

This implies that $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$ is affordable by agent 1, and delivers a strictly higher utility. This contradicts with $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ being a CE.

As per agent 2, it has to be the case that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2.$$

If not, then

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2.$$

This, in turn, implies that one can slightly increase consumption of agent 2 in some period (and keep consumption in other period the same), without violating his budget constraint. This will strictly increase his utility, and, thus, contradict with $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ being a CE.

Adding the two equations, we obtain:

$$\sum_{t=0}^{\infty} \hat{p}_t (\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t (\hat{c}_t^1 + \hat{c}_t^2).$$

Invoking market clearing, we finally arrive at contradiction:

$$\sum_{t=0}^{\infty} \hat{p}_t (e_t^1 + e_t^2) > \sum_{t=0}^{\infty} \hat{p}_t (e_t^1 + e_t^2).$$

□

### 3.5 Negishi Method to Compute Equilibria

In an endowment economy considered in this section, solving for the CE is relatively straightforward. However, in general, it is not the case; the CE will comprise dynamic problems for households, firms, market clearing conditions.

The problem (solving for CE) can be greatly simplified in economies where welfare theorems hold. In this case, we can compute PO allocations by solving an appropriate social planner's (SP) problem. The SP problem is simpler as it does not involve any prices (optimization does not take into account agents' budget constraints, but only the resource constraint). Negishi's algorithm is a procedure to solve for all PO allocations, and then pick the one which constitutes the CE.

It is important to emphasize that this method works for a broad class of models where welfare theorems hold. In general, we will solve a SP problem, invoke welfare theorems to argue that we have solved for all CE allocations. Subsequently, we need to find prices which will support these

PO allocations as CE. There are economies where welfare theorems do not hold (e.g., models with externalities, borrowing constraints, etc), which we will touch later in the course. Negishi's method will not work in such environments.

**Definition 4.** *Social planner maximizes*

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \alpha u(c^1) + (1-\alpha) u(c^2),$$

*s.t.*

$$\begin{aligned} c_t^1 + c_t^2 =& e_t^1 + e_t^2 \quad \textit{for all } t \\ c_t^i \geq 0 & \quad \textit{for all } i, t \end{aligned}$$

Note that the solution of the SP problem will be a function of the Pareto weight $\alpha \in [0,1]$: $\{c_t^1, c_t^2\} = \{c_t^1(\alpha), c_t^2(\alpha)\}$. In other word, the solution to the SP problem is the entire frontier of PO allocations.

Let us solve the SP problem. We first form the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} [\alpha \log(c_t^1) + (1-\alpha)\log(c_t^2)] + \sum_{t=0}^{\infty} \mu_t [e_t^1 + e_t^2 - c_t^1 - c_t^2] + \sum_{t=0}^{\infty} \lambda_t c_t^1 + \sum_{t=0}^{\infty} \gamma_t c_t^2 \to \max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}}.$$

The first-order conditions with respect to $c_t^1$, $c_t^2$, $\mu_t$, $\lambda_t$ and $\gamma_t$:

$$\begin{aligned} \beta^t \frac{\alpha}{c_t^1} &= \mu_t - \lambda_t \\ \beta^t \frac{1-\alpha}{c_t^2} &= \mu_t - \gamma_t \\ e_t^1 + e_t^2 - c_t^1 - c_t^2 &\geq 0 \\ c_t^1 &\geq 0 \\ c_t^2 &\geq 0 \\ \mu_t [e_t^1 + e_t^2 - c_t^1 - c_t^2] &= 0 \\ \lambda_t c_t^1 &= 0 \\ \gamma_t c_t^2 &= 0. \end{aligned}$$

Given that zero consumption cannot be a solution to the problem ($\lim_{c\to 0}\log(c) = -\infty$), then $\lambda_t = \gamma_t = 0$ (from the complementary slackness conditions). The FOCs with respect to $c_t^1$ and $c_t^2$ imply $c_t^1 = \frac{\alpha}{1-\alpha}c_t^2$. By substituting this into the resource constraint, we obtain the following solution to the SP problem:

$$c_t^1 = c_t^1(\alpha) = 2\alpha$$
$$c_t^2 = c_t^2(\alpha) = 2(1-\alpha).$$

Note that the optimal allocation does not depend on time, but only depends on $\alpha$. If the social planner only cares about the agent 1 ($\alpha = 1$), then agent 1 consumes 2 units every period (i.e., the entire endowment), and agent 2 consumes nothing. And vice verse, if $\alpha = 0$, agent 2 consumes the entire endowment every period.

Thus, the set of all PO allocations for this economy is

$$\{\{(c_t^1, c_t^2)\}_{t=0}^{\infty} : c_t^1 = 2\alpha, c_t^2 = 2(1-\alpha), \alpha \in [0,1]\}.$$

Summarizing, we obtained an infinite number of PO allocations, and the CE is one of them. Why is the set of PO allocations bigger than the set of CE allocations? The reason is that CE allocation has to (additionally) satisfy the budget constraint. Thus, the last step in the Negishi's algorithm is to find a PO allocation which will be affordable to the households if they were to face market prices which are equal to Lagrange multipliers from SP problem (establishing this fact is more involved, so we take this for granted for now). Define transfer functions $t^i(\alpha)$ as

$$t^i(\alpha) = \sum_{t=0}^{\infty} \mu_t[c_t^i(\alpha) - e_t^i].$$

Transfer functions define the amount of the numeraire good (i.e. period 0 good) that the household will additionally need as a transfer in order to be able to afford the PO allocation indexed by $\alpha$.

From the first order conditions to the Social Planner's problem, we know that $\mu_t = \frac{\beta^t}{2}$. Thus, for our economy, the value of the transfers are:

$$\begin{aligned}
t^1(\alpha) &= \sum_t \mu_t[c_t^1(\alpha) - e_t^1] \\
&= \frac{1}{2}\sum_t \beta^t[2\alpha - e_t^1] \\
&= \frac{\alpha}{1-\beta} - \frac{1}{1-\beta^2} \\
t^2(\alpha) &= \frac{(1-\alpha)}{1-\beta} - \frac{\beta}{1-\beta^2}
\end{aligned}$$

The CE requires the allocation to be affordable with zero transfers; thus, we need to find $\alpha^*$ such that $t^1(\alpha^*) = t^2(\alpha^*) = 0$. It is straightforward to see that

$$\alpha^* = \frac{1}{1+\beta},$$

and the allocations are:

$$\begin{aligned}
c_t^1\left(\frac{1}{1+\beta}\right) &= \frac{2}{1+\beta} \\
c_t^2\left(\frac{1}{1+\beta}\right) &= \frac{2\beta}{1+\beta}.
\end{aligned}$$

We have found the CE allocations. The prices will be given by $\mu_t = \frac{\beta^t}{2}$ (note that budget sets are homogeneous of degree 0, so these prices are consistent with the prices we found earlier in the direct solution for the CE).

### 3.6 Sequential Markets (SM) Equilibrium

The market structure of Arrow-Debreu equilibrium in which all agents meet only once, at the beginning of time, to trade claims to future consumption may seem empirically implausible. In this section we show that the same allocations as in an Arrow-Debreu equilibrium would arise if we let agents trade consumption and one-period bonds in each period. We will call a market structure in which markets for consumption and assets open in each period Sequential Markets and the corresponding equilibrium Sequential Markets (SM) equilibrium.

Let $r_{t+1}$ denote the interest rate on one period bonds from period $t$ to period $t+1$. A one period bond is a promise (contract) to pay 1 unit of the consumption good in period $t+1$ in exchange for

$\frac{1}{1+r_{t+1}}$ units of the consumption good in period $t$. We can interpret $q_t = \frac{1}{1+r_{t+1}}$ as the relative price of one unit of the consumption good in period $t+1$ in terms of the period $t$ consumption good. Let $a^i_{t+1}$ denote the amount of such bonds purchased by agent $i$ in period $t$ and carried over to period $t+1$. If $a^i_{t+1} < 0$ we can interpret this as the agent taking out a one-period loan at interest rate $r_{t+1}$. Household $i$'s budget constraint in period $t$ reads as:

$$c^i_t + q_t a^i_{t+1} \leq a^i_t + e^i_t.$$

Agents start out their life with initial bond holdings $a^i_0$ (remember that period 0 bonds are claims to period 0 consumption).

**Definition 5.** *A sequential markets equilibrium consists of allocations* $\{\hat{c}^1_t, \hat{c}^2_t, \hat{a}^1_{t+1}, \hat{a}^2_{t+1}\}^\infty_{t=0}$ *and interest rates* $\{\hat{r}_{t+1}\}^\infty_{t=0}$ *such that:*

1. *given the interest rates* $\{\hat{r}_{t+1}\}^\infty_{t=0}$, $\{\hat{c}^1_t, \hat{c}^2_t, \hat{a}^1_{t+1}, \hat{a}^2_{t+1}\}^\infty_{t=0}$ *solve the problem of agents:*

$$\begin{aligned} u(c^i) = \sum_{t=0}^{\infty} \beta^t \log(c^i_t) &\rightarrow \max \\ c^i_t + \frac{a^i_{t+1}}{1+r_{t+1}} &\leq a^i_t + e^i_t \\ c^i_t &\geq 0 \quad \forall t \\ a^i_{t+1} &\geq -\bar{A}^i \\ a^i_0 &= 0. \end{aligned}$$

2. *the market clears in each period* $t$*:*

$$\begin{aligned} \hat{c}^1_t + \hat{c}^2_t &= e^1_t + e^2_t \quad \forall t \\ \sum_{i=1}^{2} \hat{a}^i_{t+1} &= 0. \end{aligned}$$

Note that one of equilibrium conditions states that the amount of borrowing cannot exceed some pre-specified limit $\bar{A}^i$. We impose this restriction in order to prevent agents from running Ponzi schemes, whereby agents increase consumption by borrowing more and subsequently rolling over on their debt. We assume that this limit is sufficiently large so that it is never binding in equilibrium.

**Remark** There are models with the binding borrowing constraints; in this case, the AD and SM allocations may be different.

### 3.7 Equivalence of AD and SM Equilibria

In this section, we state and prove an important result which establishes an equivalence of AD and SM equilibria.

**Theorem 2.** *Let allocations* $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ *and prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *constitute an AD equilibrium. Then there exist* $(\bar{A}^i)_{i=1,2}$ *and an associated SM equilibrium with allocations* $\{(\tilde{c}_t^i, \tilde{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ *such that* $\tilde{c}_t^i = \hat{c}_t^i$ *for all* $i$ *and* $t$*.*

*And the other way around, if allocations* $\{(\tilde{c}_t^i, \tilde{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ *constitute a SM equilibrium, then there is an AD equilibrium* $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ *and prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *such that* $\tilde{c}_t^i = \hat{c}_t^i$ *for all* $i$ *and* $t$*.*

The proof will proceed in three steps:

1. Show that the budget constraint for SM equilibrium implies AD budget constraint (at certain prices);

2. Show that if an allocation is an AD equilibrium, then there is a SM equilibrium with the same consumption allocation;

3. Argue that if an allocation is a SM equilibrium, then it will be an AD equilibrium.

*Proof.* **Step 1.** Normalize $\hat{p}_0 = 1$ and relate equilibrium prices and interest rates as

$$1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}.$$

Now consider a sequence of budget constraints:

$$c_0^i + \frac{a_1^i}{1+\tilde{r}_1} = e_0^i$$
$$c_1^i + \frac{a_2^i}{1+\tilde{r}_2} = e_1^i + a_1^i$$
$$...$$
$$c_t^i + \frac{a_{t+1}^i}{1+\tilde{r}_{t+1}} = e_t^i + a_t^i.$$

By substituting $a_1^i$ from equation 2 in equation 1, we obtain:

$$c_0^i + \frac{c_1^i}{1+\tilde{r}_1} + \frac{a_2^i}{(1+\tilde{r}_1)(1+\tilde{r}_2)} = e_0^i + \frac{e_1^i}{1+\tilde{r}_1}.$$

Repeating the substitution, one can get:

$$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=0}^{t}(1+\tilde{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\prod_{j=0}^{t}(1+\tilde{r}_j)}.$$

Note that
$$\prod_{j=1}^{t}(1+\tilde{r}_j) = \frac{\hat{p}_0}{\hat{p}_1}\frac{\hat{p}_1}{\hat{p}_2}\frac{\hat{p}_2}{\hat{p}_3}\dots\frac{\hat{p}_{t-1}}{\hat{p}_t} = \frac{1}{\hat{p}_t}.$$

Take the limits $T \to \infty$ and obtain:

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i + \lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t e_t^i.$$

Provided that we started off with the SM equilibrium, the NPG condition implies:

$$\lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} \geq \lim_{T\to\infty} \frac{-\bar{A}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = 0.$$

Thus,
$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i.$$

Thus any allocation that satisfies the SM budget constraints and the NPG condition, satisfies the AD budget constraint when AD prices and SM interest rates are related by $1+\tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$.

**Step 2.** Suppose we have an AD equilibrium $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$, $\{\hat{p}_t\}_{t=0}^{\infty}$. We want to show that there exist a SM equilibrium with the same allocation:

$$\tilde{c}_t^i = \hat{c}_t^i.$$

Provided that this allocation is an AD equilibrium, it satisfies the market clearing. Furthermore, define asset holdings as:

$$\tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(\hat{c}_{t+\tau}^i - e_{t+\tau}^i)}{\hat{p}_{t+1}}.$$

Note that the consumption and asset allocation so constructed satisfies the SM budget constraint since, recalling $1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$:

$$\hat{c}_t^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(\hat{c}_{t+\tau}^i - e_{t+\tau}^i)}{\hat{p}_{t+1}(1+\tilde{r}_{t+1})} = e_t^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t-1+\tau}(\hat{c}_{t-1+\tau}^i - e_{t-1+\tau}^i)}{\hat{p}_{t-1+1}}.$$

Simplifying the expression above, we obtain:

$$\hat{c}_t^i = e_t^i + \frac{\hat{p}_t(e_t^i - \hat{c}_t^i)}{\hat{p}_t} = \hat{c}_t^i.$$

We need to show that we can find borrowing limits $\{\bar{A}^i\}$ large enough that NPG holds. Note that (assuming that the price growth and endowment stream is bounded)

$$\tilde{a}_{t+1}^i \geq \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(-e_{t+\tau}^i)}{\hat{p}_{t+1}} > -\infty.$$

It remains to show that allocation $\{\tilde{c}_t^i\}$ will actually be chosen subject to the SM budget constraint and the NPG condition. Suppose not, then there is another allocation that delivers to the agent a higher utility. Based on Step 1, this allocation will also be affordable in AD equilibrium, but it was not chosen. Thus, $\{\tilde{c}_t^i\}$ will indeed be chosen in SM equilibrium at interest rates $1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$.

**Step 3.** Now suppose that $\{(\tilde{c}_t^i, \tilde{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ and interest rates $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ constitute a SM equilibrium. We need to show that there is an AD equilibrium $\{(\hat{c}_t^i)\}$, $\{\hat{p}_t\}$ such that $\tilde{c}_t^i = \hat{c}_t^i$ for all $i$ and $t$.

Clearly, given that $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ is a (SM) equilibrium, it has to satisfy the market clearing. It remains to be shown that it maximizes utility within the set of allocations satisfying the AD budget constraint, for prices $\hat{p}_0 = 1$ and $\frac{\hat{p}_t}{\hat{p}_{t+1}} = 1 + \tilde{r}_{t+1}$. For any other allocation satisfying the

AD budget constraint we could construct asset holdings such that this allocation together with the asset holdings satisfies the SM-budget constraints. The only complication is that in the SM household maximization problem there is an additional constraint, the NPG condition. Thus, the set over which we maximize in the AD case is larger, since the borrowing constraints are absent in the AD formulation, and we need to rule out that allocations that would violate the NPG are optimal choices in the AD household problem, at the equilibrium prices. However, by assumption the NPG are not binding at the SM equilibrium allocation. But for maximization problems with concave objective and convex constraint set (such as the SM household maximization problem) if the additional constraints are not binding, then this argmax is also an argmax of the relaxed problem with the constraint removed. Hence $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ is optimal for household $i$ within the set of allocations satisfying only the AD budget constraint.

$\square$

# 4 Neoclassical Growth Model

We now consider a neoclassical growth model which is widely used in various applications in modern macro, including business cycles, growth and public finance. Our plan is as follows:

1. description of the environment;
2. formulation of the Social Planner's problem and Pareto optimal allocation;
3. solving for the optimal allocation: Introduction to Dynamic Programming;
4. Competitive (Decentralized) Equilibrium and its characterization;
5. Recursive Competitive Equilibrium.

## 4.1 Environment

The model is set in discrete time $t = 0, 1, \ldots,$ and there are 3 goods in this economy: labor, capital and final output.

**Technology** There is an aggregate production technology

$$y_t = F(k_t, n_t),$$

that is, capital $k_t$ and labor $n_t$ can be combined to produce the final good $y_t$. Final good $y_t$ can be either consumed or invested:

$$y_t = c_t + i_t.$$

Investment can change the aggregate capital stock, which, in turn, depreciates at a rate $\delta \in [0, 1]$:

$$k_{t+1} = i_t + (1 - \delta)k.$$

Note that capital stock cannot be negative $k_t \geq 0$, while investment can. This means that capital stock can be reduced (by dis-investing) and consumed.

**Preferences** There is a large mass of identical, infinitely-lived household with instantaneous utility function $U(c_t)$. Preferences of the representative household are assumed to be given the timeseparable utility function:

$$u(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t U(c_t),$$

where $\beta \in (0,1)$ is a discount factor. Thus, $u(\cdot)$ returns the utility from the consumption stream $\{c_t\}_{t=0}^{\infty}$.

**Endowments** Each household is endowed with starting capital $\bar{k}_0$ at time 0. Moreover, every period each household has an allocation of 1 unit of time which can be spent on either work or leisure.

**Information** There is no uncertainty in this economy, and agents have full information about the structure of the economy.

## 4.2 Social Planner's Problem

Before we delve into the competitive equilibrium that requires additional knowledge on who owns what in this environment, we start with formulating the SPP. The solution to the SPP is the Pareto optimal allocation.

The problem of the social planner is to maximize the utility of the representative household subject to the technological constrains of the environment.

**Definition 6.** *Social Planner solves the following problem:*

$$v(\bar{k}_0) = \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t$$

$$c_t \geq 0, k_{t+1} \geq 0$$

$$n_t \in [0,1], k_0 \leq \bar{k}_0.$$

Function $v(\cdot)$ gives the lifetime utility of the household if the social planner chooses the feasible allocation $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ optimally. Also note that under some standard assumptions:

- $U(\cdot)$ is continuously differentiable, stricly concave, strictly increasing, satisfies Inada conditions;

– $F(\cdot)$ is continuously differentiable, homogenous of degree 1, stricly concave, strictly increasing, satisfies Inada conditions. Additionally, assume that $F(0, n) = F(k, 0) = 0$ for all $k$ and $n$.

then $v(k)$ is a strictly increasing function of $k$. That is, $k_0$ will be equal to the capital endowment in the inital period ($k_0 = \bar{k}_0$) and households will devote all their time endowment to work because they do not get disutility from work.

### 4.3 Euler Equation and Transversality Condition

Let us form the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \lambda_t [c_t + k_{t+1} - (1-\delta)k_t) - F(k_t, 1))].$$

Note that in equilibrium, $n_t = 1$ following the argument presented above. $\{\lambda_t\}$ denotes a sequence of Lagrange multipliers.

There are two choice variables, $k_{t+1}$ and $c_t$. Take first-order conditions with respect to those variables:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t U'(c_t) + \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \lambda_t + \lambda_{t+1}(-(1-\delta) - F'_k(k_{t+1}, 1)) = 0.$$

By combining the first-order conditions, one obtains the following:

$$\beta^t U'(c_t) = \beta^{t+1} U'(c_{t+1})(F'_k(k_{t+1}, 1) + (1-\delta)),$$

or simply

$$U'(c_t) = \beta U'(c_{t+1})(F'_k(k_{t+1}, 1) + (1-\delta)).$$

The equation above is the Euler equation. It is also easy to find the steady-state for the economy by working on the Euler equation; the steady-state is an allocation which does not change over time, $c_t = c_{t+1} = c$ and $k_t = k_{t+1} = k$. By imposing the time-invariance on consumption, we get:

$$1 = \beta(F'(k, 1) + 1 - \delta).$$

This is one equation in one unknown; its solution is the steady-state value of the capital stock, $k_{ss}$.

**Transversality Condition** The Euler equation is a second-order difference equation (because it contain capital stock in periods $t, t+1$ and $t+2$). To properly characterize the optimal allocation, one needs two boundary conditions. The model specifies the initial capital stock, $\bar{k}_0$, but we need one more. Transversality condition serves the purpose of the second boundary condition. Mathematically, it is stated as:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0,$$

where $\lambda_t$ is the Lagrange multiplier on the constraint. The multiplier on the constrain has interpretation of the "shadow price" of capital, so the transverality condition means that the value of the capital stock has to converge to 0 over time. It does not mean that the capital stock itself has to be 0 in the limit, however.

We know from the first-order conditions that $\lambda_t = \beta^t U'(c_t) = \beta^t U'(f(k_t) - k_{t+1})$, thus we can re-write the transversality condition for our problem as follows:

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1})k_{t+1} = 0.$$

Now, the Euler equation along with the initial condition and the transversality condition fully characterize the optimal allocation in this economy.

## 4.4 Recursive Formulation

Next, denote $f(k) = F(k, 1) + (1-\delta)k$. Then the SPP can be rewritten as follows:

$$\begin{aligned} v(\bar{k}_0) = & \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ & 0 \le k_{t+1} \le f(k_t) \\ & k_0 = \bar{k}_0. \end{aligned}$$

Now the problem of the planner became simpler as it only solves for one sequence $\{k_{t+1}\}_{t=0}^{\infty}$; in words, it decides how much of the output needs to be consumed now vs. how much needs to be invested to increase consumption in the future.

We typically solve this kind of problems using *dynamic programming* (DP). Note that the original problem is hard since the solution is an infinite sequence $\{k_{t+1}^*\}_{t=0}^{\infty}$. The idea behind DP is to represent that problem differently exploiting the stationarity of the environment. Subsequently, one needs to show that the solution to the easier problem is the same as to the original one.

To illustrate this, notice that:

$$\begin{aligned}
v(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t. } 0 \le k_{t+1} \le f(k_t)} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t. } 0 \le k_{t+1} \le f(k_t)} \left[ U(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1}) \right] \\
&= \max_{k_1 \text{ s.t. } 0 \le k_1 \le f(k_0)} \left[ U(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty} \text{ s.t. } 0 \le k_{t+1} \le f(k_t)} \left( \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1}) \right) \right] \\
&= \max_{k_1 \text{ s.t. } 0 \le k_1 \le f(k_0)} \left[ U(f(k_0) - k_1) + \beta \max_{\{k_{t+2}\}_{t=0}^{\infty} \text{ s.t. } 0 \le k_{t+2} \le f(k_{t+1})} \left( \sum_{t=0}^{\infty} \beta^{t} U(f(k_{t+1}) - k_{t+2}) \right) \right].
\end{aligned}$$

Now notice the similarity of the object inside the brackets with the original formulation of function $v(\cdot)$. As a result, we have:

$$v(k_0) = \max_{k_1 \text{ s.t. } 0 \le k_1 \le f(k_0)} \left[ U(f(k_0) - k_1) + \beta v(k_1) \right].$$

**Is this progress?** Yes, because now we are not solving for an infinite sequence $\{k_{t+1}^*\}_{t=0}^{\infty}$, but only one number $k_1$. However, we still are not ready to solve for it because we have an unknown function $v(\cdot)$ both on the left- and right-hand sides.

**Language** A few words about jargon. We call $v(\cdot)$ a value function, and $k$ is a state variable. It is called so because we only need to know the stock of capital in the given time period to decide what is an optimal thing to do. Capital stock in the next period $k_{t+1}$ is our choice (or control) variable. The solution to the equation above is a function $k' = g(k)$ which we call a decition rule, or policy function. It tells us what is the optimal thing to do today (how much capital to invest and how much to consume) given the stock of capital $k$. Provided that the solution is a function, the general name for the type of equation above is functional equation.

**Some additional questions** How do we know the solution to function equation exists, and if so, how do we know it is unique? What are the properties of the value and policy functions $v(\cdot)$, $g(\cdot)$? How do we know that the solution induced by the policy function $g(\cdot)$ is the same as the solution to the sequential problem $\{k_{t+1}^*\}_{t=0}^{\infty}$? We leave these questions to for later.

## 4.5 Dynamic Programming: Example

In practice, functional equations of the form outlined above (also called Bellman equations) are solved numerically on the computer. Math behind this (i.e., why this works) will be discussed later in the course. The conceptual algorithm is as follows:

1. guess $v_0$, which is a function of $k$;

2. obtain $v_{j+1}(k)$ which is

$$v_{j+1}(k) = \max_{k' \text{ s.t. } 0 \leq k' \leq f(k)} \left[ U(f(k) - k') + \beta v_j(k') \right].$$

   Subscript $j$ denotes the iteration number;

3. check if $||v_{j+1}(k) - v_j(k)|| \leq \varepsilon$ where $\varepsilon$ is a tolerance level (small positive number). If this holds, then stop. If not, go back to step 2.

In rare cases (like the one we will consider now), it is possible to obtain an analytical solution. Assume the following:

- log utility function $u(c) = \log(c)$;

- full depreciation $\delta = 1$;

- power production function $y_t = f(k_t) = \theta k_t^{\alpha}$.

Under these assumptions, our Bellman equation takes the following form:

$$v(k) = \max_{c,k'} \log c + \beta v(k')$$

$$\theta k^{\alpha} = c + k'$$

$$k \text{ is given.}$$

We guess that the solution takes the form $v(k) = a_0 + a_1 \log k$, where $a_0, a_1$ are some constants. Form the Lagrangian:

$$\mathcal{L} = \log c + \beta(a_0 + a_1 \log k') + \lambda[\theta k^\alpha - c - k'].$$

The first-order conditions imply:

$$\lambda = \frac{1}{c}$$
$$\lambda = \beta a_1 \frac{1}{k'}.$$

Combining and re-arranging, obtain:

$$\beta a_1(\theta k^\alpha - k') = k'.$$

Thus, the decision rules (with yet-to-be-determined coefficients) are:

$$\begin{cases} k'(k) = \frac{\beta a_1}{1+\beta a_1}\theta k^\alpha \\ c(k) = \frac{1}{1+\beta a_1}\theta k^\alpha. \end{cases}$$

The next step is to plug in policy functions with unknown coefficients into the Bellman equation and apply the method of indeterminate coefficients:

$$a_0 + a_1 \log k = \log \frac{\theta k^\alpha}{1+\beta a_1} + \beta\left(a_0 + a_1 \log \frac{\beta a_1 \theta k^\alpha}{1+\beta a_1}\right).$$

By way of balancing coefficients on the terms with $\log k$ and constants, it is immediate that $a_0 = \alpha + \alpha\beta a_1$, and $a_1 = \frac{\alpha}{1-\alpha\beta}$. Thus, the resulting policy functions are:

$$\begin{cases} k'(k) = \alpha\beta\theta k^\alpha \\ c(k) = (1-\alpha\beta)\theta k^\alpha. \end{cases}$$

### 4.6 A Version with Population Growth

So far, the model abstracted from population technology growth. In this section, I demonstrate that the model is flexible and requires only a small modification to allow for these features.

Assume that population grows at the rate $n$, so that population at time $t$ is $N_t = (1+n)^t$. Furthermore, assume that the production function features a labor-augmenting technological progress such that output at time $t$ is given by $F(K_t, N_t(1+g)^t)$. Assume that the objective of the social planner is to maximize the per capita discounted flow of utilities:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \to \max_{\{c_t\}_{t=0}^{\infty}} .$$

The resource constraint for this economy is as follows:

$$(1+n)^t c_t + K_{t+1} = F(K_t, (1+n)^t (1+g)^t) + (1-\delta)K_t.$$

Define the growth-adjusted per capital consumption and capital:

$$\begin{aligned} \tilde{c}_t &:= \frac{c_t}{(1+g)^t} \\ \tilde{k}_t &:= \frac{k_t}{(1+g)^t} = \frac{K_t}{(1+n)^t(1+g)^t}. \end{aligned}$$

Divide all terms in the resource constraint by $(1+n)^t(1+g)^t$ to obtain:

$$\tilde{c}_t + (1+n)(1+g)\tilde{k}_{t+1} = F(\tilde{k}_t, 1) + (1-\delta)\tilde{k}_t.$$

For convenience, assume that the instantaneous utility function is CRRA, $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, then the objective of the social planner is to maximize:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t \frac{[\tilde{c}_t(1+g)^t]^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(1+g)^{1-\sigma})^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}.$$

Therefore, technically we are back to the version of the model without population growth with re-defined discount factor.

The social planner's problem thus becomes:

$$\max \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\left(f(\tilde{k}_t) - (1+g)(1+n)\tilde{k}_{t+1}\right)^{1-\sigma}}{1-\sigma}$$

$$0 \leq (1+g)(1+n)\tilde{k}_{t+1} \leq f(\tilde{k}_t)$$

$$\bar{k}_0 \text{ given.}$$

We are back to the same social planner's problem as before, and we can apply the same techniques to solve for the optimal allocation.

## 4.7 Competitive Equilibrium

So far in this chapter we have been discussing an optimal allocation; and from the perspective of the social planner the ownership structure (who owns what in the economy) does not matter. In the decentralized equalibrium we will consider now the ownership structure is crucial.

We assume that households own capital and rent it out to firms. Firms also belong to household; this means that household own claims to firms' profits. In what follows, we will consider sequential markets and Arrow-Debreu equlibriums of the economy and show that those allocations are optimal.

### 4.7.1 Arrow-Debreu Equilibrium

An Arrow-Debreu Equilibrium for the economy is an allocation for households, $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$, an allocation for firms, $\{y_t, k_t^f, l_t^f\}_{t=0}^{\infty}$ and a price system, $\{p_t, r_t, w_t\}_{t=0}^{\infty}$, such that:

1. given prices, $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ solve

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leq \sum_{t=0}^{\infty} (w_t l_t + r_t k_t + p_t(1-\delta)k_t)$$

$$l_t \leq \bar{l}, k_{t+1} \geq 0, k_0 \leq \bar{k}_0.$$

2. given prices and $\{y_t\}_{t=0}^{\infty}$, $\{k_t^f, l_t^f\}_{t=0}^{\infty}$ sovles

$$\min \sum_{t=0}^{\infty} w_t l_t^f + r_t k_t^f$$
$$F(k_t, l_t) \geq y_t \quad \forall t$$

3. markets clear

$$c_t + k_{t+1} - (1-\delta)k_t = y_t$$
$$k_t = k_t^f, l_t = l_t^f, \text{firms earn zero profits.}$$

**What if firms own capital?** In this case, the household's budget constraint is

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} w_t l_t + \underbrace{p_0(1-\delta+r_0)k_0}_{\text{selling capital in the first period}}.$$

In turn, the firm's problem becomes:

$$\max \sum_{t=0}^{\infty} [p_t(F(k_t, l_t) - i_t) - w_t l_t^f] - \underbrace{p_0(1-\delta+r_0)k_0}_{\text{can be shown to be equal to } \sum_{t=0}^{\infty} r_t k_t}$$
$$k_{t+1} = (1-\delta)k_t + i_t.$$

### 4.7.2 Sequential Markets Equilibrium

A Sequential Markets equilibrium for this economy is an allocation for households, $\{c_t, k_{t+1}, l_t, b_{t+1}\}_{t=0}^{\infty}$, an allocation for firms, $\{y_t, k_t^f, l_t^f\}_{t=0}^{\infty}$, and a price system, $\{r_t^b, r_t^k, w_t\}_{t=0}^{\infty}$ (we normalize all prices in terms of the consumption good since it is SME), such that:

1. given the prices, $\{c_t, k_{t+1}, l_t, b_{t+1}\}_{t=0}^{\infty}$ solves:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \to \max$$

$$\text{s.t. } c_t + i_t + b_{t+1} \leq w_t l_t + r_t^k k_t + \underbrace{(1 + r_t^b)}_{\text{convention}} b_t$$

$$0 \leq l_t \leq 1$$

$$k_{t+1} \leq i_t + (1 - \delta)k_t$$

$$k_0 = \bar{k}, b_0 = 0$$

$$\text{NPGC}$$

2. given prices and production level $\{y_t\}_{t=0}^{\infty}$, $\{k_t^f, l_t^f\}_{t=0}^{\infty}$ solves

$$\sum_{t=0}^{\infty} w_t l_t + r_t k_t \to \min$$

$$F(k_t, l_t) \geq y_t$$

$$k_t, l_t \geq 0.$$

3. Markets clear:

$$c_t + i_t = y_t$$

$$l_t = l_t^f$$

$$k_t = k_t^f$$

$$b_t = 0.$$

**Why do we include bonds if they are in zero supply?** We have a mass of identical households, thus everyone makes the same decision regarding bonds, and thus they have to be in zero next supply. It is common in macro to include assets which in equilibrium are not traded in order to find their equilibrium prices.

**Characterizing the SME** Let us form the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[ (1+r_t^b)b_t + w_t - c_t - k_{t+1} + (1-\delta+r_t^k)k_t - b_{t+1} \right].$$

We have to assets, $k$ and $b$, thus we need two transversality conditions:

$$\lim_{t\to\infty} \beta^t U'(c_t)k_{t+1} = \lim_{t\to\infty} \beta^t U'(c_t)b_{t+1} = 0.$$

Consider the first-order conditions (assume interior solution):

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t U'(c_t) - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \lambda_t - \lambda_{t+1}[(1-\delta) + r_{t+1}^k] = 0 \\
\frac{\partial \mathcal{L}}{\partial b_{t+1}} &= \lambda_t - \lambda_{t+1}(1 + r_{t+1}^b) = 0
\end{aligned}$$

By combining the first and second FOCs, obtain:

$$U'(c_t) = \beta U'(c_{t+1})[1 - \delta + r_{t+1}^k].$$

By combining the second and third FOCs, obtain the no-arbitrage condition (NAC):

$$1 + r_{t+1}^b = 1 - \delta + r_t^k.$$

Now, the firm's problem:

$$\mathcal{L}^f = \sum_{t=0}^{\infty} (r_t^k k_t + w_t l_t^f) - \sum_{t=0}^{\infty} \mu_t (F(k_t, l_t^f) - y_t).$$

The system of FOCs:

$$r_t^k = \mu_t F_k'(k_t, l_t)$$

$$w_t = \mu_t F_l'(k_t, l_t).$$

Our production function is constant returns to scale; Euler's theorem implies $F(k_t, l_t) = k_t F'_k + l_t F'_l$. Recall also that from resource and budget constraints:

$$c_t + i_t = F(k_t, l_t) = r_t^k k_t + w_t l_t.$$

Thus, it has to be the case that $\mu_t = 1$. Therefore, we can re-write the Euler's equation from above as follows:

$$U'(c_t) = \beta U'(c_{t+1})[1 - \delta + F'_k(k_{t+1}, 1)].$$

Note that this coincides with the Euler's equation for the planner's problem; since the initial conditions are the same, the competitive equilibrium allocation will be Pareto optimal.

# 5 Dynamic Programming

## 5.1 Mathematical preliminaries

We want to study functional equations of the following form:

$$v(x) = \max_{y \in \Gamma(x)} \left[ F(x, y) + \beta v(y) \right],$$

where $F(\cdot)$ is the per-period return function and $\Gamma(\cdot)$ is the feasibility correspondence.

In case of the neoclassical growth model, period return function is utility function, $x$ is the current capital stock $k$, and $y$ is the amount of capital brought into the next period, $k'$. The feasibility correspondence is $\Gamma(k) = \{k' \in \mathbb{R}_+ : 0 \leq k' \leq f(k)\}$; that is, capital investment cannot be negative, and it cannot exceed the amount of resources available in the economy $f(k)$.

In order to solve the functional equation, we define an operator $T$:

$$(Tv)(x) = \max_{y \in \Gamma(x)} \left[ F(x, y) + \beta v(y) \right].$$

That is, the input for this operator is a function $v$, and it returns another function $Tv$. The solution to the functional equation is a function $v^*$ which represents a fixed point of operator $T$:

$$v^* = Tv^*.$$

We would like to have answers to the following questions:

- Under what conditions there exists the solution to the functional equation? [existence]
- Under what conditions this solution is unique? [uniqueness]
- Under what conditions we can find the solution to the functional equation by repeatedly applying the operator $T$ to an arbitrarily chosen starting function $v_0$? That is, will $\lim_{i\to\infty} v_i = v^*$?

To answer these questions, we need to be precise with what the domain and range of the operator $T$ are; this requires the introduction of the notion of complete metric spaces. Our plan for this chapter is as follows:

1. State and prove the contraction mapping theorem (CMT). According to this theorem, an operator $T$ has a unique fixed point if this operator is a contraction (to be defined). This theorem also assures that the fixed point can be obtained by repeatedly applying the operator to any starting function $v_0$.

2. State and prove Blackwell's theorem (aslo known as Blackwell's sufficient conditions). This theorem states sufficient conditions for an operator to be a contraction.

3. We will show that the operator $T$ for the neoclassical growth model is a contraction; hence, it has the unique fixed point.

### 5.1.1 Complete Metric Spaces

**Definition 7.** *A metric space is a set $S$ and a function $d : S \times S \to \mathbb{R}$ (called metric), such that for any $x, y, z \in S$:*

1. $d(x, y) \geq 0$,

2. $d(x, y) = 0$ *iff* $x = y$,

3. $d(x, y) = d(y, x)$,

4. $d(x, z) \leq d(x, y) + d(y, z)$.

**Example** Consider $S = \mathbb{R}$ along with metric $d(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$
Clearly, the first 3 conditions are satisified. If $x = z$, the fourth condition follows immediately. If $x \neq z$, then $d(x, z) = 1$. Either $x \neq y$, or $y \neq z$, or both. In either case, $d(x, y) + d(y, z)$ will be at least 1.

**Example** Let $X \subseteq \mathbb{R}^l$, and $S = C(X)$ is a set of all continuous bounded functions $f : X \to \mathbb{R}$. Define a metric $d : C(X) \times C(X) \to \mathbb{R}$ as $d(f, g) := \sup_{x \in X} |f(x) - g(x)|$. This is a metric space.
Provided that functions are bounded, then $\sup_{x \in X} |f(x)| < \infty$ and $\sup_{x \in X} |g(x)| < \infty$; from this it follows that $\sup_{x \in X} |f(x) - g(x)| < \infty$. The first 3 properties are obvious, and to show the fourth property, one can apply the supnorm to the both sides of the following inequality $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$.

### 5.1.2 Convergence of Sequences

**Definition 8.** *For an arbitrary metric space $(S, d)$, a sequence $\{x_n\}_{n=0}^{\infty}$ with $x_n \in S$ converges to $x \in S$ if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$ $d(x_n, x) < \varepsilon$. Notation: $\lim_{n\to\infty} x_n = x$.*

**Exercise** Prove that if the sequence converges, then the limit is unique.

Note that the definition relies on the knowledge of what the limit of the sequence is. In some simple examples one can just guess what the limit is. In more involved cases, it may be not so easy. There is another convergence criterion, according to Cauchy.

**Definition 9.** *A sequence $\{x_n\}_{n=0}^{\infty}$ with $x_n \in S$ is a Cauchy sequence if for all $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$, such that for all $n > N_\varepsilon$ $d(x_n, x_m) < \varepsilon$ for any $n, m > N_\varepsilon$.*

There is a connection between convergence of a sequence and it being Cauchy. It is formalized by the following theorem.

**Theorem 3.** *Let $(S, d)$ be a metric space, and a sequence $\{x_n\}$ converge to $x \in S$. Then this sequence is a Cauchy sequence.*

*Proof.* Since the sequence converges to $x$, then for any $\varepsilon > 0$ there exists $M_{\frac{\varepsilon}{2}}$, such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq M_{\frac{\varepsilon}{2}}$. Thus, if $n, m \geq M_{\frac{\varepsilon}{2}}$ it follows (using the triangle inequality) $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Then it follows that for any $\varepsilon > 0$ pick $N_\varepsilon = M_{\frac{\varepsilon}{2}}$ and it will be true that $d(x_n, x_m) < \varepsilon$ for $n, m > N_\varepsilon$. Therefore, the sequence is Cauchy. ☐

**Example** Let's go back to a metric space with $S = \mathbb{R}$ and metric $d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$
Consider the following sequence: $\{x_n\} = \frac{1}{n}$. The distance between adjacent elements of the sequence will always by 1 (according to that metric). Thus, it is not a Cauchy sequence. It then also follows from the theorem above that this sequence does not converge.

Our objective is to prove the CMT, and that theorem deals with operators $T : S \to S$ where $(S, d)$ is required to be a complete metric space. Thus, we have the following important definition.

**Definition 10.** *A metric space $(S, d)$ is complete if every Cauchy sequence $\{x_n\}$ with $x_n \in S$ for all $n \in \mathbb{N}$ converges to some $x \in S$.*

**Remark** Note that limits of sequences need to lie in $S$.

**Example** Let $S$ be a set of all continuous, strictly decreasing functions on $X = [1, 2]$, and let $d$ be a supnorm. This will not be a complete metric space, because one can come up with a counterexample of a Cauchy sequence which, however, does not converge to an element of $S$. For example, consider a sequence of functions $\{f_n(x)\}$ where $f_n(x) = \frac{1}{nx}$. Clearly, all elements of the sequence are elements of $S$ since they are all continuous and strictly decreasing on $[1, 2]$. It is a Cauchy sequence, too. Pick any $\varepsilon > 0$ and take $N_\varepsilon = \frac{2}{\varepsilon}$. For any $n, m > N_\varepsilon$ (without loss of generality let $m > n$) the following holds:

$$
\begin{aligned}
d(f_n, f_m) &= \sup_{x \in [1,2]} |f_n(x) - f_m(x)| \\
&= \sup_{x \in [1,2]} \left| \frac{1}{nx} - \frac{1}{mx} \right| = \sup_{x \in [1,2]} \frac{1}{nx} - \frac{1}{mx} \\
&= \sup_{x \in [1,2]} \frac{m-n}{mnx} = \frac{m-n}{mn} \\
&= \frac{1 - \frac{n}{m}}{n} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} = \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}
$$

Hence, the sequence is indeed Cauchy. However, this sequence converges pointwise to a zero function, $\lim_{n \to \infty} f_n(x) = 0$, which is not a strictly decreasing function. Thus, the metric space is not complete.

**Remark** Note that, in the example above, the set of all continous increasing (or decreasing) functions along with the supnorm is a complete metric space.

**Example** Consider the following example which is very important from the perspective of applications we are interested in. Let $X \subseteq \mathbb{R}^L$, and $C(X)$ is a set of all continuous bounded functions $f : X \to \mathbb{R}$. Let this set be equipped with the supnorm $d$. The tuple $(C(X), d)$ is a complete metric space.

*Proof.* For proof, see SLP. $\square$

### 5.1.3 Contraction Mapping Theorem

Now we are ready to state the theorem that will give us the existence and uniqueness of a fixed point of th eoperator $T$; i.e. existence and uniquencess of a function $v^*$ satisfying $v^* = Tv^*$. Let $(S, d)$ be a metric space. An operator we are interested in maps functions into functions. We start with a definition of what a contraction mapping is.

**Definition 11.** *Let $(S, d)$ be a metric space and $T : S \to S$ be a function mapping $S$ into itself. The function $T$ is a contraction mapping if there exists a number $\beta \in (0, 1)$ satisfying*

$$d(Tx, Ty) \leq \beta d(x, y) \quad \forall x, y \in S.$$

*The number $\beta$ is called the modulus of the contraction mapping.*

The following lemma will be handy when proving the CMT.

**Theorem 4.** *Let $(S, d)$ be a metric space and $T : S \to S$ be a function mapping $S$ into itself. If $T$ is a contraction mapping, then $T$ is continuous.*

*Proof.* We need to show that for all $s_0 \in S$ and all $\varepsilon > 0$, there exists $\delta(\varepsilon, s_0)$ such that if $s \in S$ and $d(s, s_0) < \delta(\varepsilon, s_0)$ then $d(Ts, Ts_0) < \varepsilon$.

Fix an arbitrary $s_0 \in S$ and $\varepsilon > 0$, and let $\delta(\varepsilon, s_0) = \varepsilon$. Then

$$d(Ts, Ts_0) \leq \beta d(s, s_0) \leq \beta\delta(\varepsilon, s_0) = \beta\varepsilon < \varepsilon.$$

□

We now can state and prove the CMT. Let $v_n = T^n v_0 \in S$ denote the element in $S$ that is obtained by applying the operator $T$ $n$-times to $v_0$, i.e. the $n$-th element in the sequence starting with an arbitrary $v_0$ and defined recursively by $v_n = Tv_{n-1} = T(Tv_{n-2}) = \cdots = T^n v_0$.

**Theorem 5.** *Let $(S, d)$ be a complete metric space and suppose that $T : S \to S$ is a contraction mapping with modulus $\beta$. Then (1) the operator $T$ has exactly one fixed point $v^* \in S$, and (2) for any $v_0 \in S$ and any $n \in \mathbb{N}$ we have*

$$d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*).$$

**Remarks** The first part of the theorem states that there is a unique $v^*$ satisfying $v^* = Tv^*$. The second part of the theorem states that the sequence $\{v_n\}_{n=0}^{\infty}$ defined recursively converges to $v^*$ at a geometric rate $\beta$ from any starting point $v_0$. This is especially important from the computational perspective.

*Proof.* Let us start with an arbitrary $v_0$. Let $v^* = \lim_{n\to\infty} v_n$ be our candidate for the fixed point. We need to establish 3 things:

1. we need to show that the sequence $\{v_n\}_{n=0}^{\infty}$ indeed converges to function $v^*$;

2. we need to show that $v^*$ is a fixed point, i.e. $v^* = Tv^*$;

3. we need to show that there is no other fixed point.

By assumption $T$ is a contraction:

$$\begin{aligned} d(v_{n+1}, v_n) &= d(Tv_n, T_{n-1}) \le \beta d(v_n, v_{n-1}) \\ &= \beta d(Tv_{n-1}, Tv_{n-2}) \le \beta^2 d(v_{n-1}, v_{n-2}) \\ &= \cdots = \beta^n d(v_1, v_0). \end{aligned}$$

Without loss of generality, assume that $m > n$; invoke the triangle inequality:

$$\begin{aligned} d(v_m, v_n) &\le d(v_m, v_{m-1}) + d(v_{m-1}, v_n) \\ &\le d(v_m, v_{m-1}) + d(v_{m-1}, v_{m-2}) + \cdots + d(v_{n_1}, v_n) \\ &\le \beta^m d(v_1, v_0) + \beta^{m-1} d(v_1, v_0) + \cdots + \beta^n d(v_1, v_0) \\ &= \beta^n (\beta^{m-n-1} + \cdots + \beta + 1) d(v_1, v_0) \\ &\le \frac{\beta^n}{1-\beta} d(v_1, v_0). \end{aligned}$$

We can take $n$ sufficiently large such that $d(v_m, v_n)$ can become arbitrarily small. Thus, $\{v_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Provided that by assumption $(S, d)$ is a complete metric space, the limit of the sequence is well-defined.

Next, we need to show that $v^*$, the limit of the sequence so defined, is a fixed point of the operator $T$, i.e. $v^* = Tv^*$. Note that

$$Tv^* = T\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} T(v_n) = \lim_{n\to\infty} v_{n+1} = v^*.$$

Note that the second equality follows from the continuity of the operator $T$.

Finally, we need to show that the fixed point is unique. Suppose not, i.e. there is another $\tilde{v} \in S$, such that $\tilde{v} = T\tilde{v}$, and $\tilde{v} \neq v^*$. Then there is $\varepsilon$, such that

$$0 < \varepsilon = d(\tilde{v}, v^*) = d(T\tilde{v}, Tv^*) \leq \beta d(\tilde{v}, v^*) = \beta\varepsilon.$$

A contradiction.

What is left to show is that one can obtain $v^*$ by iteratively applying operator $T$ from an arbitrary starting point. We prove this by induction. For $n = 0$, the statement automatically holds. Now let us assume the claim holds for an arbitrary $k$:

$$d(T^k v_0, v^*) \leq \beta^k d(v_0, v^*).$$

We need to show now that the claim is true for $k + 1$:

$$d(T^{k+1} v_0, v^*) \leq \beta^{k+1} d(v_0, v^*).$$

Note now that

$$d(T^{k+1} v_0, v^*) = d(T(T^k v_0), Tv^*) \leq \beta d(T^k v_0, v^*) \leq \beta^{k+1} d(v_0, v^*),$$

where the last inequality follows from the induction hypothesis.

□

### 5.1.4 Blackwell's Theorem

The Contraction Mapping Theorem that we have just proved is very powerful; however, it is not very operarational from the practical perspective because we need to prove that a particular operator is a contraction in order to apply the CMT. The Blackwell's theorem (aslo known as Blackwell's sufficient conditions) fill in this gap.

**Theorem 6.** *Let $X \subseteq \mathbb{R}^L$ and $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with $d$ being the supnorm. Let $T : B(X) \to B(X)$ be an operator satisfying the following two conditions:*

1. *Monotonicity. If $f, g \in B(X)$ such that $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*

2. *Discounting. Let function $f + a$ for $f \in B(X)$ and $a \geq 0$ be defined as $(f + a)(x) = f(x) + a$. There is $\beta \in (0, 1)$ such that for all $f \in B(X)$ and $a \geq 0$, the following is true:*

$$[T(f + a)](x) \leq \beta[Tf](x) + \beta a.$$

*If these two conditions are met, then the operator $T$ is a contraction.*

*Proof.* Fix $x \in X$. By the definition of the supnorm, $f(x) - g(x) \leq \sup_{y \in X} |f(y) - g(y)| = d(f, g)$. This holds for any $x$, thus $f \leq g + d(f, g)$.

Invoking monotonicity, obtain:

$$Tf \leq T[g + d(f, g)] \leq Tg + \beta d(f, g).$$

The last inequality follows from the discounting assumption. Thus,

$$Tf - Tg \leq \beta d(f, g).$$

Simmetrically, one can also show that

$$Tg - Tf \leq \beta d(f, g).$$

Therefore,

$$\sup_{x \in X} |(Tf)(x) - (Tg)(x)| = d(Tf, Tg) \leq \beta d(f, g).$$

Hence, $T$ is a contraction with modulus $\beta$.

$\square$

### 5.1.5 Neoclassical Growth Model: Example

Now let us apply the developed mathematical tools to the neoclassical growth model which we discussed earlier. Recall the functional equation:

$$(Tv)(k) = \max_{0 \le k' \le f(k)} [U(f(k) - k') + \beta v(k')].$$

Define the metric space as $(B[0,\infty), d)$, where $B[0,\infty)$ is a space of bounded functions on $[0,\infty)$ and $d$ is the supnorm. We would like to invoke the CMT to argue that $T$ has a unique fixed point; to do so we need to argue that $T$ is a contraction. For that, we need to show that $T$ satisfies monotonicity and discounting properties.

1. Recall that we need to show that $T$ maps from $B[0,\infty)$ into itself. If we take that $v$ is bounded, then since $U$ is bounded, $Tv$ will be bounded, too.

2. Monotonicity. Let $v \le w$. Let $g_v(k)$ denote the optimal policy with respect to value function $v$. Then it follows:

$$\begin{aligned}
(Tv)(k) &= U(f(k) - g_v(k)) + \beta v(g_v(k)) \\
&\le U(f(k) - g_v(k)) + \beta w(g_v(k)) \\
&\le \max_{0 \le k' \le f(k)} [U(f(k) - k') + \beta w(k')] \\
&= (Tw)(k).
\end{aligned}$$

3. Discounting.

$$\begin{aligned}
T(v + a)(k) &= \max_{0 \le k' \le f(k)} [U(f(k) - k') + \beta(v(k') + a)] \\
&= \max_{0 \le k' \le f(k)} [U(f(k) - k') + \beta v(k')] + \beta a \\
&= Tv(k) + \beta a.
\end{aligned}$$

Hence, $T$ is a contraction, and CMT applies. In particular, we can obtain the unique fixed point by recursively applying the operator $T$ from any starting point.

## 5.2 Principle of Optimality

So far, we understand under what conditions the solution to the functional equation **(FE)** exists and is unique:

$$v(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)].$$

However, what we are really interested in is the sequential problem **(SP)**:

$$\begin{aligned} w(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ x_{t+1} &\in \Gamma(x_t) \\ x_0 &\in X \text{ given} \end{aligned}$$

We would like to understand under what conditions will $w$ and $v$ be equal, and when will the optimal plan $\{x_{t+1}\}_{t=0}^{\infty}$ from the SP be equivalent to the decision rule $y = g(x)$ from the FE. It turns out that those conditions are pretty weak.

In this section, my objective is to provide key results omitting proofs. Readers looking for a rigorous treatment of the principle of optimality are advised to consult Stokey and Lucas.

### 5.2.1 Preliminaries

Let $X$ denote the set of possible values that the state today can take. It can be a subset of Euclidean space, a set of functions, as a few examples. The correspondence $\Gamma : X \rightrightarrows X$ describes the set of feasible states next period given state $x$ today.

**Definition 12.** *The graph of $\Gamma$, $A$ is defined as:*

$$A = \{(x, y) \in X \times X : y \in \Gamma(x)\}.$$

The period return function $F : A \to \mathbb{R}$ maps the set of feasible combinations of today's and tomorrow's states into the set of real numbers. The set of fundamentals for us is $(X, \Gamma, \beta, F)$. For the neoclassical growth model, $F$ and $\beta$ describe preferences, and $X, \Gamma$ describe technology.

**Definition 13.** *A sequence of states $\{x_t\}_{t=0}^{\infty}$ is a plan.*

**Definition 14.** *Given the initial condition* $x_0$*, the set of feasible plants* $\Pi(x_0)$ *is*

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} : x_{t+1} \in \Gamma(x_t)\}.$$

For the Principle of Optimality, we need just 2 assumptions which state that (1) for any initial condition $x_0$, the set of feasible plans is non-empty, and (2) the total return from all feasible plans can be evaluated.

**Assumption 1** $\Gamma(x)$ is non-empty for all $x \in X$.

**Assumption 2** For any initial condition $x_0$ and for any feasible plan from $\Pi(x_0)$,

$$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

exists (it may, however, be $\infty$ or $-\infty$).

**Remark** Condition 1 is clear: we do not want to deal with situations when the social planner (or whoever is solving the dynamic program) cannot do anything. The second condition is more subtle, and there are several ways to verify that it indeed holds (i.e. sufficient conditions for Condition 2).

1. $F$ is bounded and $\beta \in (0,1)$. Note that boundedness of $F$ is not enough. To illustrate, consider the following example. Let $\beta = 1$ and $F(\cdot) = \begin{cases} 1, & \text{if } t \text{ is even} \\ -1, & \text{if } t \text{ is odd} \end{cases}$. Clearly, $F$ is bounded, but the limit of the sequence does not exist.

2. Define $F^+ = \max\{F, 0\}$ and $F^- = \max\{-F, 0\}$. Condition 2 is satisfied if for all $x_0 \in X$ and all feasible plans, either of the following two conditions is satisfied:

$$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F^+(x_t, x_{t+1}) < +\infty$$
$$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F^-(x_t, x_{t+1}) < +\infty$$

3. For all $x_0 \in X$ and all feasible plans, there are $\theta \in (0, 1/\beta)$ and $c \in (0, +\infty)$ such that

$$F(x_t, x_{t+1}) \leq c\theta^t$$

for all $t$. That is, $F$ can be unbounded, but returns cannot grow too fast.

I introduce one more object.

**Definition 15.** *Define a sequence of functions $u_n : \Pi(x_0) \to \mathbb{R}$:*

$$u_n(\bar{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}).$$

*That is, for each feasible plan $\bar{x}$, $u_n$ is equal to the total discounted utility up time $n$.*

If Condition 2 is satisfied then the function $u(\bar{x}) : \Pi(x_0) \to \bar{\mathbb{R}}$

$$u(\bar{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

is also well-defined (i.e. limit exists). The range of $u$ is the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, since we allowed the limit to be infinite, too. From the definition of $u$ it follows that

$$w(x_0) = \sup_{\bar{x}\in\Pi(x_0)} u(\bar{x}).$$

By construction, if $w$ exists, it is unique (because supremum is unique).

### 5.2.2 Principle of Optimality

**Theorem 7.** *Suppose $(X, \Gamma, F, \beta)$ satisfy assumptions 1 and 2. Then:*

1. *Function $w$ satisfies the FE;*

2. *If for all $x_0 \in X$ and all $\bar{x} \in \Pi(x_0)$ the solutions to FE $v$ satisfies*

$$\lim_{n\to\infty} \beta^n v(x_n) = 0$$

*then $v = w$.*

**Remark** The first result is nice but not exactly what we need. We know how to solve the FE, and we want conditions under which a solution to FE will also solve the SP. Thus, the second result is really important for us. The FE may have many solutions, and only one of those can satisfy the property in the theorem (because in this case, $w = v$, and $w$ is unique because it is supremum).

We have established the conditions under which the SP and FE have the same solution. Now, we would like to establish equivalence with respect to decision rules. Note that the solution to SP is a feasible plan, $\{x_n\}_{n=0}^{\infty}$. The solution to FE is $y = g(x)$, where $g$ can be a correspondence. One can reconstruct the feasible plan by recursively applying the decision rule starting from $x_0$. That is, $x_1 = g(x_0)$, $x_2 = g(x_1)$ and so on. How does a plan recovered from the FE decision rule relate to the solution of the SP?

**Theorem 8.** *Suppose $(X, \Gamma, F, \beta)$ satisfy assumptions 1 and 2.*

1. *Let $\bar{x} \in \Pi(x_0)$ be a feasible plan that attains the supremum in the SP. Then for all $t \geq 0$*

$$w(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta w(\bar{x}_{t+1}).$$

2. *Let $\hat{x} \in \Pi(x_0)$ be a feasible plan that satisfies (for all $t \geq 0$)*

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

*and additionally*

$$\lim_{t\to\infty} \sup \beta^t w(\hat{x}_t) \leq 0.$$

*Then $\hat{x}$ attains the supremum in the SP for the initial condition $x_0$.*

**Discussion** The first part of the theorem states, loosely speaking, that the optimal plan from the SP is the optimal policy for the FE if the value function is the right one. The second part states that for the particular fixed point of the FE $w$ (which satisfies the additional limit condition), the optimal plan generated by the FE decision rule $g$ attains the maximum in the SP.

These two theorems outline a procedure to follow. First, pick the right fixed point (if there are more than one, pick the one that satisfies the limit condition) and then construct the plan corresponding to that fixed point. Check the limit condition to make sure that the plan so constructed is indeed optimal for the SP.

### 5.3 Global Dynamics: Characterizing the Policy Function, $g(k)$

Now let's discuss dynamic properties of the Neoclassical growth model:

$$v(k) = \max_{k' \geq 0}[U(c_t) + \beta v(k')]$$

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t)$$

$$k_t \geq 0$$

$$c_t \geq 0$$

$$k_0 \text{ given.}$$

**Remark** Under certain conditions on preferences and technology (boundedness), it can be shown that there is a finite valued solution to the problem for all initial conditions and, hence, $v^*$ is a real-valued function. If, additionally, we assume that $\beta \in (0,1)$, $U(\cdot)$ is continuous, strictly increasing and strictly concave, whereas $F(\cdot)$ is continuous, strictly increasing and strictly quasi-concave, then $v^*$ is continuous, strictly increasing and strictly concave.

**Remark** If labor is supplied inelastically $U_l' = 0$ and if $F(0,\bar{n}) = 0$, then it can be shown that there are 2 steady states for the system, i.e. values of $k$ such that $g_k(k) = k$. One of them is $k = 0$. The other one is strictly positive and (under differentiability of $U(\cdot)$ and $F(\cdot)$) is the solution to

$$\frac{1}{\beta} = 1 - \delta + F_k'(k, \bar{n}).$$

Recall that $g(k)$ is the $y$ that solves:

$$\max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y),$$

where $f(k)$ is the total output available for consumption and investment.

The first-order condition and the envelope condition for this problem are:

$$U'(f(k) - g(k)) = \beta v'(g(k)) \qquad [FOC]$$

$$v'(k) = U'(f(k) - g(k))f'(k). \qquad [ENV]$$

### 5.3.1 Envelope Condition

Now we take a moment to derive the envelope condition.

**General Case** Let $f(x, \alpha)$ be a function, and we treat $\alpha$ as a parameter. Suppose we would like to find the maximum of $f(\cdot)$ as a function of parameter $\alpha$. The first-order condition is:

$$\frac{\partial f}{\partial x} = 0.$$

If the second-order conditions are met (i.e., if we actually found the extremum of $f$), then define the solution to be $x^*(\alpha)$. Furthermore, define the maximum value function as $V(\alpha) = f(x^*(\alpha), \alpha)$. If we differentiate $V(\alpha)$ with respect to $\alpha$, we obtain:

$$\frac{\partial V}{\partial \alpha} = f'_x \frac{\partial x^*}{\partial \alpha} + f'_\alpha.$$

Note, however, that $f'_x = 0$; thus:

$$\frac{\partial V}{\partial \alpha} = f'_\alpha.$$

Therefore, the envelope condition for the Bellman equation is:

$$v'(k) = U'(f(k) - g_k(k))f'(k).$$

**Alternative Derivation** Consider the problem from above:

$$\max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y).$$

The first-order condition is

$$U'(f(k) - g(k)) = \beta v'(g(k)).$$

Define

$$W(k) = U(f(k) - g(k)) + \beta v(g(k)).$$

Function $W(k)$ represents the indirect utility of having the state $k$ today. Take the derivative of $W(k)$ with respect to $k$:

$$W'(k) = U'(f(k) - g(k))(f'(k) - g'(k)) + \beta v'(g(k))g'(k).$$

Plug in the FOC to obtain:

$$W'(k) = U'(f(k) - g(k))(f'(k) - g'(k)) + U'(f(k) - g(k))g'(k).$$

Thus,

$$W'(k) = U'(f(k) - g(k))f'(k).$$

Finally, since we know from Bellman Equation that $W = v$, we obtain the final result:

$$v'(k) = U'(f(k) - g(k))f'(k).$$

**Theorem 9.** *Suppose $W(z)$ is strictly concave and differentiable. Then $(W'(z) - W'(\hat{z}))(z - \hat{z}) \leq 0$ with equality if and only if $z = \hat{z}$.*

*Proof.* There are 3 possible cases:

- If $z < \hat{z}$, then $W'(z) - W'(\hat{z}) > 0$;
- If $z > \hat{z}$, then $W'(z) - W'(\hat{z}) < 0$;
- if $z = \hat{z}$, then $W'(z) - W'(\hat{z}) = 0$;

□

Now, since $v(k)$ is strictly concave (if the utility function is strictly concave and strictly increasing, and the production function is strictly increasing and quasi-concave) and differentiable on $(0, \bar{k}]$, $z = k$, $\hat{z} = g(k)$, one gets the following:

$$[v'(k) - v'(g(k))][k - g(k)] \leq 0 \quad \forall k \in (0, \bar{k}]$$

with equality iff $k = g(k)$.

From the envelope condition it follows that:

$$v'(k) = U'(f(k) - g(k))f'(k).$$

From the FOC we get:

$$v'(g(k)) = \frac{1}{\beta}U'(f(k) - g(k)).$$

Thus, once we put it all together:

$$[U'(f(k) - g(k))f'(k) - \frac{1}{\beta}U'(f(k) - g(k))][k - g(k)] \leq 0$$

with equality iff $k = g(k)$.

Or, equivalently,

$$\left[f'(k) - \frac{1}{\beta}\right][k - g(k)] \leq 0$$

with equality iff $k = g(k)$.

At the steady-state, $f'(k_{ss}) = \frac{1}{\beta}$, and thus $k_{ss} = g(k_{ss})$. Since we know there is only one positive steady-state, it follows:

$$\left[f'(k) - \frac{1}{\beta}\right][k - g(k)] < 0 \quad k \neq k_{ss}.$$

1. If $k < k_{ss}$, then $f'(k) > \frac{1}{\beta}$ and thus $k < g(k)$. Since the decision rule is monotone, $k < g(k) < g(k_{ss}) = k_{ss}$, $g(k) \in (k, k_{ss})$.

2. If $k > k_{ss}$, then $f'(k) < \frac{1}{\beta}$ and thus $k > g(k)$. Since the decision rule is monotone, $k > g(k) > g(k_{ss}) = k_{ss}$, $g(k) \in (k_{ss}, k)$.

**Discussion** The model can generate growth only if the starting level of capital is lower than $k_{ss}$, and this growth will only be temporary. Moreover, it turns out that in reasonably calibrated models the steady-state is achieved pretty fast. This is in stark contrast with the U.S. data which displays no slow down of growth.

# 6 Overlapping Generations Model

So far, we have been discussing models with infinitely lived agents; clearly, this may not be appropriate for certain applications where the focus is on, for example, the life-cycle dynamics of households. That is, we may want to capture the income profile of people over their lifes whereby they become richer as they age. The infinite horizon model cannot capture this.

To address this, we develop an overlapping generations model (OG), another workhorse model of the modern macro. The plan is to start with the simple setup where agents live only 2 periods and there is no production (an exchange economy). We will define an Arrow-Debreu and Sequential Markets equilibria in this framework, and characterize them. We will then proceed with a version of the model with production.

## 6.1 A Pure Exchange OG Model

Consider an economy in discrete time $t = 1, 2, \ldots$ where in each period there are 2 types of agents: young and old. Each generation lives for 2 periods, while the economy lasts forever. We label each generation by the birth time period. Each generation $t$ has endowments in $t$ and $t+1$, $(e_t^t, e_{t+1}^t)$. Correspondingly, generation $t$ consumes $(c_t^t, c_{t+1}^t)$ in periods $t$ and $t+1$, correspondingly. Thus, in each time period $t$ there is an old generation with endowment $e_t^{t-1}$ and consumption $c_t^{t-1}$, and a young generation with endowment $e_t^t$ and consumption $c_t^t$ (see incidence matrix in table 1). In the first period, there is a generation of initially old agents with endowment $e_1^0$ and consumption $c_1^0$.

Table 1: Incidence Matrix

| **Generation** \ **Period** | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| 0 | $(c_1^0, e_1^0)$ | | | | |
| 1 | $(c_1^1, e_1^1)$ | $(c_2^1, e_2^1)$ | | | |
| 2 | | $(c_2^2, e_2^2)$ | $(c_3^2, e_3^2)$ | | |
| 3 | | | $(c_3^3, e_3^3)$ | $(c_4^3, e_4^3)$ | |
| $\vdots$ | | | | | |

In some cases, we will consider a version of the model where initially old generation is endowed with some amount of outside money $m$. Cases with $m \geq 0$ are interpreted as fiat money, and $m < 0$ may carry an interpretation of some debt that initially old borrowed in the past and now have to repay.

Agents derive utility from their consumption profiles according to the utility function

$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t),$$

while the initially old generation gets the following utility

$$u_0(c) = U(c_1^0).$$

We will routinely assume that $U(\cdot)$ is strictly increasing, strictly concave and twice continuously differentiable.

**Definition 16.** *An allocation is consumption for initially old $c_1^0$ and a sequence of consumption allocations for generations $t \geq 1$, $(c_t^t, c_{t+1}^t)_{t=1}^{\infty}$.*

**Definition 17.** *An allocation $c_1^0, (c_t^t, c_{t+1}^t)_{t=1}^{\infty}$ is feasible if for all $t \geq 1$,*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t.$$

**Definition 18.** *An allocation $c_1^0, (c_t^t, c_{t+1}^t)_{t=1}^{\infty}$ is Pareto Optimal if it is feasible and there is no other feasible allocation $\hat{c}_1^0, (\hat{c}_t^t, \hat{c}_{t+1}^t)_{t=1}^{\infty}$, such that*

$$u_t(\hat{c}_t^t, \hat{c}_{t+1}^t) \geq u_t(c_t^t, c_{t+1}^t)$$
$$u_0(\hat{c}_1^0) \geq u_0(c_1^0),$$

*with at least one of inequalities being strict.*

### 6.1.1 Arrow-Debreu Equilibrium

The AD equilibrium in this model is conceptually the same as in the case of the model with infinitely-lived consumers.

**Definition 19.** *An ADE for the OG model with money is an allocation $c_1^0, (c_t^t, c_{t+1}^t)_{t=1}^{\infty}$ and the price system $\{p_t\}_{t=1}^{\infty}$, such that:*

1. *given $m$ and the $p_1$, $c_1^0$ must solve the problem of generation 0:*

$$\max_{c_1^0} u_0(c_1^0)$$
$$p_1 c_1^0 \leq p_1 e_1^0 + m$$
$$c_1^0 \geq 0;$$

2. *given the price system, $(c_t^t, c_{t+1}^t)_{t=1}^{\infty}$ must solve generation's $t$ problem:*

$$\max_{c_t^t, c_{t+1}^t} U(c_t^t) + \beta U(c_{t+1}^t)$$
$$p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t$$
$$c_t^t \geq 0, c_{t+1}^t \geq 0;$$

3. *markets clear. For all $t \geq 1$*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t.$$

**Remark** Without outside money there will be no trade between generations. To see this, note that initially old will simply consume their endowment $e_1^0$ because they will be dead next period, and generation 1 will thus consume their endowment at time $t = 1$. At time $t = 2$, generation 1 will also simply consume their endowment $e_2^1$ since generation 2 will not give anything to them ($t = 2$ is the last time period for generation 1). And so on. In other words, outside money allows to violate the budget constraint for initially old.

### 6.1.2 Sequential Markets Equilibrium

Suppose trade takes place sequentially, and households can make savings. Specifically, let $s_{t+1}^t$ be the amount of resources generation $t$ saves (or borrows) at time $t$ into period $t+1$. Let $r_{t+1}$ be the interest rate from period $t$ to period $t+1$.

**Definition 20.** *A SME for the OG economy with money consists of allocations $c_1^0, (c_t^t, c_{t+1}^t, s_{t+1}^t)_{t=1}^{\infty}$ and interest rates $\{r_{t+1}\}_{t=0}^{\infty}$, such that:*

1. *given* $r_1$, $c_1^0$ *solves:*

$$\max_{c_1^0} u_0(c_1^0)$$
$$c_1^0 \leq e_1^0 + (1+r_1)m$$
$$c_1^0 \geq 0;$$

2. *given the interest rate* $r_{t+1}$, $(c_t^t, c_{t+1}^t, s_{t+1}^t)$ *must solve generation's t problem:*

$$\max_{c_t^t, c_{t+1}^t, s_{t+1}^t} U(c_t^t) + \beta U(c_{t+1}^t)$$
$$c_t^t + s_{t+1}^t \leq e_t^t$$
$$c_{t+1}^t \leq (1+r_{t+1})s_{t+1}^t + e_{t+1}^t$$
$$c_t^t \geq 0, c_{t+1}^t \geq 0;$$

3. *markets clear. For all* $t \geq 1$

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t.$$

**Remark** In case of an infinitely-lived consumer, we also added a No-Ponzi Game condition preventing the household from rolling over on debt. In case of a finitely-lived household this condition is redundant since whichever debt it is making at time $t$ needs to be repaid in $t+1$. This implicitly uses perfect enforceability of contracts.

**Equilibrium on the asset market** Since the utility function is strictly increasing, the budget constraints in equilibrium hold with equality. The budget constrain for initiall old and the budget constraint at $t = 1$ for generation 1 are:

$$c_1^0 = e_1^0 + (1+r_1)m$$
$$c_1^1 + s_2^1 = e_1^1.$$

Add the above budget constraints to obtain:

$$c_1^0 + c_1^1 + s_2^1 = e_1^0 + e_1^1 + (1+r_1)m.$$

Note, however, that the market clearing implies $c_1^0+c_1^1 = e_1^0+e_!^1$. Therefore, we obtain $s_2^1 = (1+r_1)m$. Repeat the same steps for period $t = 2$:

$$
\begin{aligned}
c_2^1 &= e_2^1 + (1+r_2)s_2^1 \\
c_2^2 + s_3^2 &= e_2^2.
\end{aligned}
$$

This implies (again invoking market clearing) $s_3^2 = (1+r_2)s_2^1$. Recalling that $s_2^1 = (1+r_1)m$ and interating forward, one can get that

$$
s_{t+1}^t = \prod_{k=1}^{t}(1+r_k)m.
$$

Clearly, if $1 + r_t > 1$, then value of money in terms of the consumption good explodes, and this cannot be part of equilibrium. Strictly speaking, this should be part of the equilibrium definition. We, however, can omit this due to the Walras law.

### 6.1.3 Equivalence between ADE and SME

For generation $t$ we have the following 2 budget constraints (in periods $t$ and $t+1$):

$$
\begin{aligned}
c_t^t + s_{t+1}^t &\leq e_t^t \\
c_{t+1}^t &\leq (1+r_{t+1})s_{t+1}^t + e_{t+1}^t.
\end{aligned}
$$

Using the second equation, solve for $s_{t+1}^t$:

$$
s_{t+1}^t = \frac{c_{t+1}^t}{1+r_{t+1}} - \frac{e_{t+1}^t}{1+r_{t+1}}.
$$

Plug in the first equation:

$$
c_t^t + \frac{c_{t+1}^t}{1+r_{t+1}} = e_t^t + \frac{e_{t+1}^t}{1+r_{t+1}}.
$$

Now, use the ADE budget constraint for generation $t$ and divide both sides by $p_t$:

$$
c_t^t + \frac{p_{t+1}c_{t+1}^t}{p_t} = e_t^t + \frac{p_{t+1}e_{t+1}^t}{p_t}.
$$

Similarly, for the initially old, we have

$$c_1^0 = e_1^0 + \frac{m}{p_1}.$$

We can now establish the following result:

**Theorem 10.** *Let the allocation* $\hat{c}_1^0, (\hat{c}_t^t, \hat{c}_{t+1}^t)_{t=1}^{\infty}$ *and the price system* $\{p_t\}_{t=1}^{\infty}$ *with* $p_t > 0$ *for all* $t$ *constitute an Arrow-Debreu equilibrium. Then there is a corresponding Sequential Market equilibrium with allocations* $\tilde{c}_1^0, (\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_{t+1}^t)_{t=1}^{\infty}$ *and interest rates* $\{r_t\}_{t=1}^{\infty}$*, such that*

$$\hat{c}_t^t = \tilde{c}_t^t \text{ for all } t \geq 1$$
$$\hat{c}_t^{t-1} = \tilde{c}_t^{t-1} \text{ for all } t \geq 1.$$

*And the other way around, suppose* $\tilde{c}_1^0, (\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_{t+1}^t)_{t=1}^{\infty}$ *and interest rates* $\{r_t\}_{t=1}^{\infty}$ *constitute a Sequential Markets equilibrium with* $r_t + 1 > 0$ *for all* $t$*. Then there is an Arrow-Debreu equilibrium with allocations* $\hat{c}_1^0, (\hat{c}_t^t, \hat{c}_{t+1}^t)_{t=1}^{\infty}$ *and the price system* $\{p_t\}_{t=1}^{\infty}$*, such that:*

$$\tilde{c}_t^t = \hat{c}_t^t \text{ for all } t \geq 1$$
$$\tilde{c}_t^{t-1} = \hat{c}_t^{t-1} \text{ for all } t \geq 1.$$

*Proof.* Given a sequence of AD prices $\{p_t\}_{t=1}^{\infty}$, construct a sequence of interest rates as follows:

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}}$$
$$1 + r_1 = \frac{1}{p_1}.$$

The savings for SME can be constructed as $\tilde{s}_{t+1}^t = e_t^t - \hat{c}_t^t$.

The other way around. Suppose $\{r_t\}_{t=1}^{\infty}$ are the interest rates in the Sequential Markets equilibrium. Construct Arrow-Debreu prices as:

$$p_1 = 1,$$
$$p_{t+1} = \frac{p_t}{1 + r_{t+1}}.$$

□

### 6.1.4 Analysis using Offer Curves

Now let us characterize the equilibrium; we will work with the ADE formulation. Assuming $U(c) = \log c$, which is a strictly increasing function, we can drop non-negativity constraints from the generation $t$'s problem. The solution will be interior.

$$\max_{c_t^t, c_{t+1}^t} \log c_t^t + \beta \log c_{t+1}^t$$

$$p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t.$$

Let $\lambda$ be a Lagrange multiplier on the budget constraint:

$$\mathcal{L} = \log c_t^t + \beta \log c_{t+1}^t + \lambda(-p_t c_t^t - p_{t+1} c_{t+1}^t + p_t e_t^t + p_{t+1} e_{t+1}^t).$$

The system of first-order conditions is:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_t^t} & = \frac{1}{c_t^t} - p_t \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{t+1}^t} & = \frac{\beta}{c_{t+1}^t} - p_{t+1} \lambda = 0. \end{cases}$$

These FOCs imply $c_{t+1}^t = c_t^t \beta \frac{p_t}{p_{t+1}}$. Next, divide both sides of the budget constraint by $p_t$ to get:

$$c_t^t + c_{t+1}^t \frac{p_{t+1}}{p_t} = e_t^t + e_{t+1}^t \frac{p_{t+1}}{p_t}.$$

Substituting the FOC:

$$c_t^t (1 + \beta) = e_t^t + e_{t+1}^t \frac{p_{t+1}}{p_t}.$$

For initially old, normalizing the budget constraint by $p_t$ yields:

$$c_1^0 = e_1^0 + \frac{m}{p_1}.$$

**Remark** Note that we cannot normalize the price system because it will change consumption of the initially old.

**Definition 21.** *Excess demand of consumer $i$ in an endowment economy is the difference between demand and endowment of consumer $i$.*

**Example** The excess demand of the initially old in $t = 1$ is $z_1^0 = \frac{m}{p_1}$. The excess demand of the initially old in $t = 2$ is $z_2^0 = 0$.

The excess demand of generation $t$ at time $t$ and $t+1$ is characterized by the following system of equations:

$$\begin{cases} z_t^t & = c_t^t - e_t^t = \frac{1}{1+\beta}\left(e_t^t + \frac{p_{t+1}}{p_t}e_{t+1}^t\right) - e_t^t \\ z_{t+1}^t & = c_{t+1}^t - e_{t+1}^t = \frac{\beta}{1+\beta}\left(\frac{p_t}{p_{t+1}}e_t^t + e_{t+1}^t\right) - e_{t+1}^t \end{cases}$$

**Remark** The system of excess demand functions fully characterize the behavior of generation $t$. (1 equation for the initially old, 2 equations for generations $t \geq 1$).

**Definition 22.** *An ADE with money is a price system $\{p_t\}_{t=1}^{\infty}$ and the corresponding allocations such that the excess demand is equal to excess supply:*

$$z_t^{t-1} + z_t^t = 0 \textit{ for all } t \geq 1.$$

**Algorithm to solve for equilibrium** In order to solve for the equilibrium using the derived excess demand functions, proceed as follows:

1. choose some arbitrary $p_1$ and find $z_1^0$ according to $z_1^0 = \frac{m}{p_1}$. Recall that the choice of $p_1$ will affect the allocation;
2. find $z_1^1$ using the equality of excess demand to excess supply;
3. find $p_2$ using the excess demand function $z_t^t$, and $z_2^1$ using the excess demand function $z_{t+1}^t$;
4. once the price sequence is recovered, find the consumption path.

**Offer curve** We can provide visual interpretation on the model's equilibrium using the offer curve. To derive the offer curve, first use expression for $z_t^t$ to solve for $\frac{p_{t+1}}{p_t}$:

$$\frac{p_{t+1}}{p_t} = \frac{(1+\beta)(z_t^t + e_t^t) - e_t^t}{e_{t+1}^t}.$$

Next, plug that into expression for $z_{t+1}^t$:

$$z_{t+1}^t = \frac{\beta}{1+\beta}\left[\frac{e_t^t e_{t+1}^t}{(1+\beta)(z_t^t + e_t^t) - e_t^t} + e_{t+1}^t\right] - e_{t+1}^t.$$

This expression is the so-called offer curve; it is a decreasing function in $z_t^t$. The set of all stationary equilibria (i.e., those where the excess demand is not changing) is represented by intersections of the offer curve with the diagonal line $z_t^{t-1} + z_t^t = 0$ in $(z_t^t, z_{t+!}^t)$ space. Also, note that the offer curve passes through the origin where excess demands are both 0; this is an autarky equilibrium (e.g., in the model without money).

**Remark** While equations above are derived in general form with endowments potentially being time-variant, the graphical analysis of the two period OG model requires stationary endowment streams, $e_t^t = e_1$ and $e_{t+1}^t = e_2$.

**Remark** It is also understood that the OG model with money has an infinite number of equilibria, depending on the choice of $p_1$.

### 6.1.5 Equilibrium May Not Be Efficient

When we discussed economies where households are infinitely-lived, we were able to show that the First Welfare theorem holds: the equilibrium allocation is Pareto efficient. In OG models, however, this may not necessarily be the case.

To show the potential failure of the FWT, it suffices to construct a feasible allocation which Pareto dominates the equilibrium one. For example, consider an OG model without money $m = 0$ where the endowment stream is time-invariant $(e_t^t, e_{t+1}^t) = (2, 1)$. As we have argued above, the equilibrium will be an autarky whereby agents consume their endowments, $(c_t^t, c_{t+1}^t) = (2, 1)$.

Consider another feasible allocation $(\tilde{c}_t^t, \tilde{c}_{t+1}^t) = (1.5, 1.5)$. Clearly, the initially old are strictly better off because

$$u_0(1.5) > u_0(1).$$

But all other generations are also better off in this alternative allocation, since:

$$u_t((2, 1)) = \log 2 + \log 1 < \log \frac{9}{4} = u_t((1.5, 1.5)).$$

## 6.2 OG Model with Production

Consider an economy with infinite horizon in which at every instance of time there are two generations, the young and the old. Each generation values consumption and leisure. There is a representative firm in this economy with access to technology $F(k, l)$ with constant returns to scale. Initially old generation chooses $(c_1^0, l_1^0)$; they are endowed with $\bar{l}_2$ units of time and $\bar{k}$ units of capital. Their utility function is $u(c_1^0, l_1^0)$. The initially old generation is also endowed with some outside fiat money $m$, which may be positive, zero or negative. The utility function of the generation $t$ is

$$u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t).$$

We also typically assume $u_c' > 0$ and $u_l' < 0$. Each generation $t$ is endowed with $(\bar{l}_1, \bar{l}_2)$ units of time. Capital in the economy depreciates at a rate $\delta$.

### 6.2.1 Sequential Markets Equilibrium

**Definition 23.** *A Sequential Markets equilibrium for this economy is an allocation for the initially old, $(\hat{c}_1^0, \hat{l}_1^0)$, an allocation for generations $t \geq 1$, $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{l}_t^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t, \hat{b}_{t+1}^t\}_{t=1}^{\infty}$, an allocation for firms, $\{\hat{y}_t, \hat{k}_t^f, \hat{l}_t^f\}_{t=1}^{\infty}$, and a price system $\{\hat{r}_t^k, \hat{w}_t, \hat{r}_t^b\}_{t=1}^{\infty}$, such that:*

1. *given $(\hat{r}_1^k, \hat{r}_1^b, \hat{w}_1)$, $(\hat{c}_1^0, \hat{l}_1^0)$ solves:*

$$\begin{aligned} &\max u(c_1^0, l_1^0) \\ &c_1^0 \leq \hat{w}_1 l_1^0 + (1 - \delta + \hat{r}_1^k)\bar{k}_1 + (1 + \hat{r}_1^b)m \\ &0 \leq l_1^0 \leq \bar{l}_2; \end{aligned}$$

2. *given the prices, $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{l}_t^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t, \hat{b}_{t+1}^t\}$ solves:*

$$\begin{aligned} &\max u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) \\ &c_t^t + k_{t+1}^t + b_{t+1}^t \leq \hat{w}_t l_t^t \\ &c_{t+1}^t \leq \hat{w}_{t+1} l_{t+1}^t + (1 - \delta + \hat{r}_{t+1}^k) k_{t+1}^t + (1 + \hat{r}_{t+1}^b) b_{t+1}^t \\ &0 \leq l_t^t \leq \bar{l}_1, 0 \leq l_{t+1}^t \leq \bar{l}_2; \end{aligned}$$

*3. given prices,* $\{\hat{y}_t\}_{t=1}^{\infty}$, $\{\hat{l}_t^f, \hat{k}_t^f\}$ *sovles*

$$\max_{\{l_t,k_t\}} y_t - \hat{w}_t l_t - \hat{r}_t^k k_t$$
$$F(k_t, l_t) \geq y_t;$$

*4. markets clear*

$$\begin{aligned} \hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} &= \hat{y}_t \\ \hat{k}_t^f &= \hat{k}_t^{t-1} \\ \hat{l}_t^f &= \hat{l}_t^f + \hat{l}_t^{t-1} \\ \hat{b}_{t+1}^t &= (1+\hat{r}_t^b)b_t^{t-1} \\ \hat{b}_2^1 &= (1+r_1^b)m. \end{aligned}$$

### 6.2.2 Arrow-Debreu Equilibrium

**Definition 24.** *An Arrow-Debreu equilibrium for this economy is an allocation for the initially old,* $(\tilde{c}_1^0, \tilde{l}_1^0)$, *an allocation for generations* $t \geq 1$, $\{\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{l}_t^t, \tilde{l}_{t+1}^t, \tilde{k}_{t+1}^t\}_{t=1}^{\infty}$, *an allocation for firms,* $\{\tilde{y}_t, \tilde{k}_t^f, \tilde{l}_t^f\}_{t=1}^{\infty}$, *and a price system* $\{\tilde{p}_t, \tilde{r}_t, \tilde{w}_t\}_{t=1}^{\infty}$, *such that:*

*1. given* $(\tilde{r}_1, \tilde{w}_1)$, $(\tilde{c}_1^0, \tilde{l}_1^0)$ *solves:*

$$\max u(c_1^0, l_1^0)$$
$$\tilde{p}_1 c_1^0 \leq \tilde{w}_1 l_1^0 + \tilde{r}_1 \bar{k}_1 + \tilde{p}_1(1-\delta)\bar{k}_1 + m$$
$$0 \leq l_1^0 \leq \bar{l}_2;$$

*2. given the prices,* $\{\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{l}_t^t, \tilde{l}_{t+1}^t, \tilde{k}_{t+1}^t\}$ *solves:*

$$\max u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t)$$
$$\tilde{p}_t c_t^t + \tilde{p}_{t+1} c_{t+1}^t + \tilde{p}_t k_{t+1}^t \leq \tilde{w}_t l_t^t + \tilde{w}_{t+1} l_{t+1}^t + \tilde{r}_{t+1} k_{t+1}^t + \tilde{p}_{t+1}(1-\delta)k_{t+1}^t$$
$$0 \leq l_t^t \leq \bar{l}_1, 0 \leq l_{t+1}^t \leq \bar{l}_2;$$

*3. given prices,* $\{\tilde{y}_t\}_{t=1}^{\infty}$, $\{\tilde{l}_t^f, \tilde{k}_t^f\}$ *sovles*

$$\max_{\{l_t, k_t\}} \tilde{p}_t y_t - \tilde{w}_t l_t - \tilde{r}_t k_t$$

$$F(k_t, l_t) \geq y_t;$$

*4. markets clear*

$$\begin{aligned} \tilde{c}_t^{t-1} + \tilde{c}_t^t + \tilde{k}_{t+1}^t - (1-\delta)\tilde{k}_t^{t-1} &= \tilde{y}_t \\ \tilde{k}_t^f &= \tilde{k}_t^{t-1} \\ \tilde{l}_t^f &= \tilde{l}_t^t + \tilde{l}_t^{t-1}. \end{aligned}$$

### 6.2.3 Characterizing Equilibrium

Let us focus on the SME formulation. The Lagrangian for the generation $t$ takes the form:

$$\begin{aligned} \mathcal{L} = u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) + \lambda_t^t [w_t l_t^t - c_t^t - k_{t+1}^t - b_{t+1}^t] + \\ + \lambda_{t+1}^t [w_{t+1} l_{t+1}^t + (1 - \delta + r_{t+1}^k) k_{t+1}^t + (1 + r_{t+1}^b) b_{t+1}^t - c_{t+1}^t]. \end{aligned}$$

The system of first-order conditions is:

$$\begin{aligned} u_c'(c_t^t, l_t^t) &= \lambda_t^t \\ \beta u_c'(c_{t+1}^t, l_{t+1}^t) &= \lambda_{t+1}^t \\ \lambda_t^t &= (1 - \delta + r_{t+1}^k) \lambda_{t+1}^t \\ \lambda_t^t &= (1 + r_{t+1}^b) \lambda_{t+1}^t. \end{aligned}$$

From these FOCs we can obtain conditions relating consumption and leisure choices:

$$\begin{aligned} u_l'(c_t^t, l_t^t) &= -\lambda_t^t w_t = -u_c'(c_t^t, l_t^t) w_t \\ u_l'(c_{t+1}^t, l_{t+1}^t) &= -\lambda_{t+1}^t w_{t+1} = -\beta u_c'(c_{t+1}^t, l_{t+1}^t) w_{t+1}. \end{aligned}$$

From the firm's problem, we know that

$$
\begin{aligned}
w_t &= F_l'(k_t^f, l_t^f) \\
r_t^k &= F_k'(k_t^f, l_t^f).
\end{aligned}
$$

We can then express the Euler's equation in terms of capital:

$$
u_c'(c_t^t, l_t^t) = \beta u_c'(c_{t+1}^t, l_{t+1}^t)(1 - \delta + r_{t+1}^k).
$$

Suppose, for simplicity, that consumers do not value leisure (i.e., they supply it inelastically) and $m = 0$. Thus, in equilibrium, $l_t^t = \bar{l}_1$, $l_{t+1}^t = \bar{l}_2$, $c_t^t = F_l'(k_t^f, l_t^f)l_t^t - k_{t+1}^f$ and $c_{t+1}^t = F_l'(k_{t+1}^f, l_{t+1}^f) + (1 - \delta + F_k'(k_{t+1}^f, l_{t+1}^f))k_{t+1}$. By plugging that into the Euler equation, one can obtain an expression in $k$, which along with the initial condition $k_1^0 = \bar{k}$, characterizes the dynamics of capital.

**No-Arbitrage Condition** The first-order condition with respect to bond holdings delivers a relationship between two interest rates in this model, $r_t^b$ and $r_t^k$:

$$
r_t^b = r_t^k - \delta.
$$

This condition ensures that the two assets in the model are priced consistently in a sense that neither of the assets delivers a return higher than the other one.

# 7 Aggregation

In this section, we will first lay out a simple model of aggregate time-series featuring many consumers and producers. Subsequently, we will show which assumptions are typically utilized to simplify the analysis of that model. Specifically, we will consider how to represent the aggregate dynamics in that model using the framework with one consumer and one producer.

## 7.1 A Simple First-Cut Model of Aggregate Time-Series

Consider an economy populated by three types of agents:

1. Households: make consumption, labor supply, saving/investment decisions. Households are indexed by $i = 1, \dots, I$;

2. Consumption firms: buy labor, capital services from the household and produce the consumption good. These firms are indexed by $j_c = 1, \dots, J_c$;

3. Investment firms: buy labor, capital services from the household and produce the investment good. These firms are indexed by $j_x = 1, \dots, J_x$.

We assume that investment firms have access to production technology $F_{xt}$ in each period $t$:

$$x_{jt} \leq F^j_{xt}(k^j_{xt}, n^j_{xt}).$$

Consumption firms, in turn, have access to production technology $F_{ct}$ in each period $t$:

$$c_{jt} \leq F^j_{ct}(k^j_{ct}, n^j_{ct}).$$

Households are characterized by:

– strictly increasing and strictly concave utility function $U_i(\tilde{c}, \tilde{l})$, where $\tilde{c} = \{c_t\}_{t=0}^{\infty}$ and $\tilde{l} = \{l_t\}_{t=0}^{\infty}$;

– endowment of capital good in the initial period $k^i_0$ and the stream of labor endowments $\{\bar{n}^i_t\}_{t=0}^{\infty}$;

– firm ownership $\{\theta^c_{ij}\}$ and $\{\theta^x_{ij}\}$ such that $\sum_i \theta^c_{ij} = 1$ and $\sum_i \theta^x_{ij} = 1$.

**Definition 25.** *A competitive equilibrium in this economy consists of prices* $\{p_{ct}, p_{xt}, w_t, r_t\}_{t=0}^{\infty}$ *and quantities for households* $\{c_t^i, x_t^i, l_t^i, n_t^i, k_t^i\}_{t=0}^{\infty}$*, consumption firms* $\{c_t^j, k_{ct}^j, n_{ct}^j\}_{t=0}^{\infty}$ *and investment firms* $\{x_t^j, k_{xt}^j, n_{xt}^j\}_{t=0}^{\infty}$*, such that:*

1. *given prices, households' allocations solve their maximization problems:*

$$\max_{\tilde{c},\tilde{l},\tilde{k},\tilde{x},\tilde{n}} U_i(\tilde{c},\tilde{l})$$
$$\sum_{t=0}^{\infty}[p_{ct}c_t + p_{xt}x_t] \leq \sum_{t=0}^{\infty}[r_t k_t + n_t w_t] + \Pi_i$$
$$k_{t+1} \leq (1-\delta)k_t + x_t$$
$$l_t + n_t \leq \bar{n}_t^i$$

2. *given prices, consumption firms solve:*

$$\max_{c,k,n} \sum_{t=0}^{\infty}[p_{ct}c_t - w_t n_t - r_t k_t]$$
$$c_t \leq F_{ct}^j(k_t, n_t)$$

3. *given prices, investment firms solve:*

$$\max_{x,k,n} \sum_{t=0}^{\infty}[p_{xt}x_t - w_t n_t - r_t k_t]$$
$$x_t \leq F_{xt}^j(k_t, n_t)$$

4. *profits household $i$ gets are:*

$$\Pi_i = \sum_{j_c} \theta_{ij}^c \sum_t [p_{ct}c_t^j - w_t n_t^j - r_t k_t^j] + \sum_{j_x} \theta_{ij}^x \sum_t [p_{xt}c_t^j - w_t n_t^j - r_t k_t^j].$$

5. *markets clear* $\forall t$*:*

$$\sum_i c_{it} = \sum_{j_c} c_{j_c t}$$
$$\sum_i x_{it} = \sum_{j_x} x_{j_x t}$$
$$\sum_i k_{it} = \sum_{j_c} k_{ct}^{j_c} + \sum_{j_x} k_{xt}^{j_x}$$
$$\sum_i n_{it} = \sum_{j_c} n_{ct}^{j_c} + \sum_{j_x} n_{xt}^{j_x}.$$

## 7.2 Simplifying the Model to Make it Tractable

We will simplify the model developed in the previous section by aggregating both the firm and household sides.

### 7.2.1 Simplifying the Firm Side

There are 3 steps to simplify the firm side dramatically.

1. A standard assumption to get rid of profits is to assume production technology with constant returns to scale (CRS): $F(\lambda k, \lambda n) = \lambda F(k, n)$;

2. Furthermore, we can make a representative firm assumption $F_{ct}^j = F_{ct}^{j'}$ and $F_{xt}^j = F_{xt}^{j'}$. As a result, there will only be 1 firm of each type;

3. Finally, we can collapse two sectors into one: $F_{xt} = F_{ct}$. As a result, the two first market clearing conditions become:
$$\sum_i (c_{it} + x_{it}) = F_t(k_t, n_t).$$

### 7.2.2 Simplifying the Household Side

There are two ways to simplify the household's side:

1. assume a representative household;

2. perform a homothetic aggregation.

**Definition 26.** *Preferences are homothetic, if for any elements $x$ and $y$ in the consumption set and $\lambda > 0$:*

$$u(x) = u(y) \Leftrightarrow u(\lambda x) = u(\lambda y).$$

**Representative Household Version** Assume that every household is identical (same preferences and endowments); in this case, all households will be making the same decisions. Note that we need strict concavity of preferences for this result.

Recall that resource constraint in our model is:

$$\sum_i (c_{it} + x_{it}) = F_t(k_t, n_t).$$

If $c_{it} = c_{i't}$ and $x_{it} = x_{i't}$, then

$$\sum_i (c_{it} + x_{it}) = I(c_{1t} + x_{1t}).$$

It follows then from the original resource constraint (assuming CRS):

$$c_{1t} + x_{1t} = \frac{1}{I} F_t(n_t^f, k_t^f) = F_t\left(\frac{n_t^f}{I}, \frac{k_t^f}{I}\right) = F(n_{1t}, k_{1t}).$$

Thus, the competitive equilibrium solves:

$$\begin{aligned} &\max_{\tilde{c}, \tilde{k}, \tilde{l}, \tilde{n}, \tilde{x}} U_i(\tilde{c}, \tilde{l}) \\ &c_t + x_t \leq F(k_t, n_t) \\ &\quad k_{t+1} = (1 - \delta) k_t + x_t \\ &l_t + n_t \leq \bar{n}_{it}. \end{aligned}$$

**Homothetic Aggregation** Homothetic aggregation allows us to weaken the equal wealth assumption; this, however, comes at a cost of stronger assumptions on utility.

**Theorem 11.** *If the utility function is homogenous of any degree, then the preferences it represents are homothetic.*

*Proof.* Suppose $U(x, y)$ is homogenous of degree $\eta$; then

$$U(\lambda x, \lambda y) = \lambda^\eta U(x, y).$$

Now, take $(x_1, y_1)$ and $(x_2, y_2)$ such that $U(x_1, y_1) = U(x_2, y_2)$. Scale up both consumption sets by $\lambda > 0$:

$$U(\lambda x_1, \lambda y_1) = \lambda^\eta U(x_1, y_1) = \lambda^\eta U(x_2, y_2) = U(\lambda x_2, \lambda y_2).$$

$\square$

**Remark** A partial converse is true. Specifically, as per MWG (Chapter 3), suppose preferences are rational and monotone. Then preferences are homothetic if and only if there is a homogeneous of degree one utility function representing them.

**Example** The CES utility function

$$U(x, y) = \frac{1}{1-\sigma} x^{1-\sigma} + \frac{1}{1-\sigma} y^{1-\sigma}$$

is homogenous of degree $1 - \sigma$; thus, the preferences it represents are homothetic.

- if $\sigma \to 1$, then the utility is log;
- if $\sigma = 0$, then $(x, y)$ are perfect substitutes;
- if $\sigma \to \infty$, then $(x, y)$ are perfect complements.

**Example** An example of non-homothetic preferences is the following utility function:

$$U(x, y) = \alpha x^{a_1} + (1 - \alpha) y^{a_2}.$$

**Theorem 12.** *If the preferences are homothetic, then the income expansion path is a straight line. In other words, cosider the following problem:*

$$\max_{x,y} U(x, y)$$

$$p_x x + p_y y \leq W$$

*Let $(x(W), y(W))$ be the solution to that problem. The claim of the theorem can be reformulated as*

$$(x(\lambda W), y(\lambda W)) = \lambda(x(W), y(W)).$$

*Proof.* First, note that $\lambda(x(W), y(W))$ is a feasible choice set under wealth $\lambda W$, because the budget constraint is homogenous of degree 1.

Next, we need to show that $\lambda(x(W), y(W))$ maximizes utility of the agent. We prove this claim by contradiction. Suppose not. It means that there is another allocation $(\hat{x}, \hat{y})$ which is feasible $p_x\hat{x} + p_y\hat{y} \leq \lambda W$; this allocation also attains a higher utility than $\lambda(x(W), y(W))$:

$$U(\hat{x}, \hat{y}) > U(\lambda x(W), \lambda y(W)).$$

Consider $(\tilde{x}, \tilde{y}) = \left(\frac{\hat{x}}{\lambda}, \frac{\hat{y}}{\lambda}\right)$. Note that it is feasible: $p_x\tilde{x} + p_y\tilde{y} \leq \lambda W$. Then the homogeneity of $U$ implies:

$$U(\tilde{x}, \tilde{y}) = U\left(\frac{\hat{x}}{\lambda}, \frac{\hat{y}}{\lambda}\right) = \frac{1}{\lambda^\eta}U(\hat{x}, \hat{y}) > \frac{1}{\lambda^\eta}U(\lambda x(W), \lambda y(W)) = \frac{\lambda^\eta}{\lambda^\eta}U(x(W), y(W)) = U(x(W), y(W)).$$

This is a contradiction with $x(W), y(W)$ being a solution to the maximization problem. □

Suppose the preferences are homothetic. If $\{\tilde{c}_i, \tilde{l}_i\}_{i=1}^I$ are the equilibrium allocations, then

$$\tilde{c}_1 + \tilde{c}_2 + \cdots + \tilde{c}_I$$

and

$$\tilde{l}_1 + \tilde{l}_2 + \cdots + \tilde{l}_I$$

are the same as what one would get with one consumer having initial endowment of $\sum_i \bar{k}_0$ and labor endowment stream $\{\sum_i \bar{n}_{it}\}_{t=0}^\infty$.

**Discussion** So how do we solve for the equilibrium in the first-cut model of aggregate time-series? The general recipe is as follows:

1. Assuming homothetic preferences, solve for the planner's problem (we have shown earlier that the competitive equilibrium is optimal in the Neoclassical growth model). This way we can find physical allocations without solving for prices.

2. Use first-order conditions from the first-cut model to recover prices, given the obtained allocations.

3. Calculate shares of all individuals in the total distribution of wealth.

4. Each agent gets the corresponding share of aggeregate series of consumption and leisure.

Add discussion about CRS in the investment production function $k' = (1-\delta)k + x$, so it does not matter who makes investment.

# 8 Optimal Fiscal Policy

In this section, we explore what happens to equilibrium when we add fiscal policy (taxes, transfers and government spending) to our baseline model. Importantly, we will also discuss Ramsey Problem, i.e. what is the best tax system to finance a given sequence of government expenditures $\{g_t\}_{t=0}^{\infty}$.

**Notation** Throughout the section, we will use the following notation:

1. consumption tax $\tau_{ct}$;
2. investment tax $\tau_{xt}$;
3. labor income tax $\tau_{nt}$;
4. capital income tax $\tau_{kt}$;
5. lump-sum taxes and transfers $T_t^i$ (for household $i$);
6. government purchases of goods and services $g_t$.

## 8.1 Tax-Distorted Competitive Equilibrium

We first extend the notion of competitive equilibrium to this setting, in which households and firms will take prices, taxes, transfers and government spending as given.

**Definition 27.** *A Tax-Distorted Competitive Equilibrium (TDCE) given a fiscal policy $\{(\tau_{ct}, \tau_{xt}, \tau_{nt}, \tau_{kt}, g_t, T_t)\}_{t=0}^{\infty}$ is a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocation chosen by household $\{c_t, x_t, n_t, l_t, k_t\}_{t=0}^{\infty}$, and allocation chosen by firms $\{c_t^f, x_t^f, g_t^f, n_t^f, k_t^f\}_{t=0}^{\infty}$, such that:*

1. *household's allocation solves*

$$\max_{\tilde{c},\tilde{l}} U(\tilde{c}, \tilde{l})$$

$$\sum_{t=0}^{\infty} [(1+\tau_{ct})p_t c_t + (1+\tau_{xt})p_t x_t] \leq \sum_{t=0}^{\infty} [(1-\tau_{nt})n_t w_t + (1-\tau_{kt})r_t k_t + T_t]$$

$$k_{t+1} = (1-\delta)k_t + x_t$$

$$n_t + l_t \leq \bar{n}$$

*2. firm's allocation solves:*

$$\max_{c,x,n,g,k} \sum_{t=0}^{\infty} p_t(c_t + x_t + g_t) - \sum_{t=0}^{\infty} (r_t k_t + w_t n_t)$$

$$c_t + x_t + g_t \le F_t(k_t, n_t)$$

$$k_0 \textit{ given}$$

*3. Markets clear:*

$$c_t = c_t^f, x_t = x_t^f, n_t = n_t^f, k_t = k_t^f, g_t = g_t^f.$$

*4. Government budget is balanced:*

$$\sum_{t=0}^{\infty} [p_t g_t + T_t] = \sum_{t=0}^{\infty} [p_t \tau_{ct} c_t + p_t x_t \tau_{xt} + r_t k_t \tau_{kt} + w_t n_t \tau_{nt}].$$

**Remark** We need to make explicit several things:

1. Tax rates are flat, i.e. there is no progressivity/regressivity;

2. We assume that households pay taxes. This can be relaxed (you will be asked to analyze it in one of the problem sets);

3. Prices of the consumption, investment and government goods are assumed to be the same and equal to $p_t$. This is based on the perfect substitutability of these goods in the production process. In any equilibrium with strictly positive amounts of $c_t$, $x_t$ and $g_t$ it has to be the case, because otherwise it will not be optimal for the firm to produce a good with a lower price;

4. This model is sort of "throw it in the ocean" model of government spending. Government expenditures enter neither the utility function $U$ (parks and schools), nor the production function $F$ (roads and bridges). Analysis of models with these features will be more involved, clearly;

5. In this model, it is helpful to think about $\{g_t\}$ as given from outside of the model, i.e. the government for some reason has to have $g_t$ every time period.

**Remark** Government budget constraint is redundant in this problem. Let us assume that the solution is interior and all constraints are satisfied with equality. To demonstrate this, start with the household's budget constraint:

$$\sum_{t=0}^{\infty}\left[(1+\tau_{ct})p_t c_t + (1+\tau_{xt})p_t x_t\right] = \sum_{t=0}^{\infty}\left[(1-\tau_{nt})n_t w_t + (1-\tau_{kt})r_t k_t + T_t\right].$$

Rewrite it as:

$$\sum_{t=0}^{\infty}\left[\tau_{ct}p_t c_t + \tau_{xt}p_t x_t + \tau_{nt}n_t w_t + \tau_{kt}r_t k_t\right] = \sum_{t=0}^{\infty}\left[n_t w_t + r_t k_t - p_t c_t - p_t x_t + T_t\right].$$

If the production function is CRS, then $p_t y_t = p_t F_t(k_t, n_t) = r_t k_t + w_t n_t$. Thus, the term under the summation operator on the right-hand side is

$$n_t w_t + r_t k_t - p_t c_t - p_t x_t + T_t = p_t y_t - p_t(c_t + x_t) + T_t = p_t g_t + T_t.$$

The government budget constraint immediately follows.

**Exercise** *Prove the same claim relaxing the assumption of constant returns to scale.*

*Proof.* In this case, the budget constraint of the household is:

$$\sum_{t=0}^{\infty}\left[(1+\tau_{ct})p_t c_t + (1+\tau_{xt})p_t x_t\right] = \sum_{t=0}^{\infty}\left[(1-\tau_{nt})n_t w_t + (1-\tau_{kt})r_t k_t + T_t + \Pi_t\right],$$

where firm's profits are:

$$\Pi_t = p_t y_t - w_t n_t - r_t k_t = p_t(c_t + g_t + x_t) - w_t n_t - r_t k_t.$$

The second equality follows from the market clearing condition. Repeating the same steps as in case with the CRS production function, it is now straightforward to derive the government budget constraint. □

## 8.2 Solving the Model with Representative Consumer

In general, there is no corresponding planning problem which would deliver the same allocations as in TDCE. The reason is that TDCE is not PO in general. It is PO if $\tau = 0$ and the government only uses lump-sum taxes to raise revenue. With distortionary taxes this typically will not work.

However, there are some exceptions when it works.

**Example 1** Suppose $\tau_{ct} = \tau_{xt} = \tau_{nt} = \tau_{kt} = 0$ for all $t$, and government can only use lump-sum taxes and transfers. In this case, the TDCE allocation solves:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} u(c_t, n_t) \\ c_t + x_t &= \widehat{F}(k_t, n_t) \\ k_{t+1} &= (1-\delta)k_t + x_t \\ 0 \leq l_t &+ n_t \leq 1 \\ k_0 & \text{ given,} \end{aligned}$$

where $\widehat{F}(k_t, n_t) = F(k_t, n_t) - g_t$.

*Proof.* The set of FOCs for the planning problem is: The system of FOCs is:

$$\begin{aligned} \frac{u_c'(t)}{\beta u_c'(t+1)} &= \hat{F}_k'(t+1) + 1 - \delta \\ \frac{u_n'(t)}{u_c'(t)} &= \hat{F}_n'(t). \end{aligned}$$

It is straightforward to show that TDCE will feature the same set of optimality conditions, and the feasibility constraint will also be the same. □

**Example 2** Assume that there is a representative consumer and $\tau_{ct} = \tau_{xt} = T_t = 0$ but $\tau_{nt} = \tau_{kt} = \tau_t > 0$ and $p_t g_t = w_t \tau_t n_t + r_t \tau_t k_t$, i.e. government balances budget every period.

Then the TDCE allocation solves:

$$\max \quad \sum_{t=0}^{\infty} u(c_t, n_t)$$

$$c_t + x_t = (1 - \tau_t)F(k_t, n_t)$$

$$k_{t+1} = (1 - \delta)k_t + x_t$$

$$0 \leq l_t + n_t \leq 1$$

$$k_0 \text{ given.}$$

*Proof.* First consider the planning problem outlined above:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) + \sum_{t=0}^{\infty} \lambda_t \left[(1 - \tau_t)F(k_t, n_t) - k_{t+1} + (1 - \delta)k_t - c_t\right].$$

The system of FOCs is:

$$\frac{u_c'(t)}{\beta u_c'(t+1)} = (1 - \tau_{t+1})F_k'(t+1) + 1 - \delta$$
$$\frac{u_n'(t)}{u_c'(t)} = (1 - \tau_t)F_n'(t).$$

Now consider the TDCE problem. The system of the FOCs is:

$$\frac{u_c'(t)}{\beta u_c'(t+1)} = (1 - \tau_{t+1})\frac{r_{t+1}}{p_{t+1}} + 1 - \delta$$
$$\frac{u_n'(t)}{u_c'(t)} = (1 - \tau_t)\frac{w_t}{p_t}.$$

Clearly, the FOCs coincide with those in the planning problem once the FOCs of the firms are substituted. Now, we need to show that the feasibility constraint in the TDCE will match the feasibility constraint in the planning problem.

$$
\begin{aligned}
c_t + k_{t+1} - (1-\delta)k_t + g_t &= F(k_t, n_t) \\
p_t c_t + p_t k_{t+1} - p_t(1-\delta)k_t + p_t g_t &= p_t F(k_t, n_t) \\
p_t c_t + p_t k_{t+1} - p_t(1-\delta)k_t + [w_t \tau_t n_t + r_t \tau_t k_t] &= p_t F(k_t, n_t) \\
p_t c_t + p_t k_{t+1} - p_t(1-\delta)k_t &= p_t F(k_t, n_t) - [w_t \tau_t n_t + r_t \tau_t k_t] \\
p_t c_t + p_t k_{t+1} - p_t(1-\delta)k_t &= (1-\tau_t) p_t F(k_t, n_t) + \tau_t [p_t F(k_t, n_t) - w_t n_t + r_t k_t] \\
p_t c_t + p_t k_{t+1} - p_t(1-\delta)k_t &= (1-\tau_t) p_t F(k_t, n_t) \\
c_t + k_{t+1} - (1-\delta)k_t &= (1-\tau_t) F(k_t, n_t).
\end{aligned}
$$

We have shown that TDCE gives rise to the same set of optimality conditions and feasibility constraint. Note that it is crucial for the government to balance budget every period for this result to hold. □

**Example 3** Consider an infinite horizon economy where consumer's preferences are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The household has an initial stock of capital $k_0$; capital is subject to the law of motion $k_{t+1} = x_t$. That is, assume full depreciation. Assume that labor endowment is one unit per period, and it is supplied inelastically. Feasibility constrain is $c_t + x_t = F(k_t, n_t)$, and $F(\cdot)$ satisfies all standard conditions. Assume that government taxes capital income at a rate $\tau$ every period (rate is constant), and all revenue is rebated lump-sum to the household. There are no other taxes, and no other government spending.

1. Define a TDCE for this economy.

   Standard. The government budget constraint is $\tau r_t k_t = T_t$.

   The feasibility and the Euler equation determine allocations:

$$
\begin{aligned}
u_c'(t) &= (1-\tau)\beta u_c'(t+1) F_k'(t+1) \\
c_t + k_{t+1} &= F(k_t, 1) \\
l_t &= 1.
\end{aligned}
$$

2. Show that the allocation that results from the TDCE you defined above can also be obtained by solving a Planner's problem. Lay out the SPP and show that its solution matches the TDCE allocation.

   Given a stream of policy variables $\{T_t, \tau\}_{t=0}^{\infty}$, the planner solves:

$$\sum_{t=0}^{\infty}((1-\tau)\beta)^t u(c_t) \to \max$$
$$c_t + k_{t+1} = F(k_t, 1)$$
$$c_t, k_{t+1} \geq 0.$$

   It is straightforward to show that the Euler equation (as well as the feasibility constraint) are the same as in the TDCE. The equilibrium prices can then be recovered using the TDCE conditions:

$$p_t = p_{t-1}\frac{\beta u_c'(t)}{u_c'(t-1)}$$
$$r_t = p_t F_k'(t)$$
$$w_t = p_t F_n'(t),$$

   with an arbitrary $p_0$.

### 8.3 A Detailed Example

Suppose you start in a situation where $\tau_{nt} = \tau_{kt} = 0.2$ and that this has been true for a long time so that you start in a steady state. Question: what happens if you suddenly raise the tax rate to 0.3? We will analyze a surprise change in taxes; if it were foreseen, then the analysis would be more involved.

We will use the planning problem equivalence result outlined above. Moreover, we will assume that the labor supply is inelastic to simplify the analysis.

In this case, the TDCE allocation can be obtained by solving the following planning problem:

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$c_t + k_{t+1} - (1-\delta)k_t = (1-\tau)f(k_t).$$

The Euler equation is

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + (1-\tau)f'(k_{t+1})].$$

This implies that the steady-state variables are characterized by the following set of equations:

$$1 = \beta[1 - \delta + (1-\tau)f'(k_{ss})]$$
$$x_{ss} = \delta k_{ss}$$
$$c_{ss} = (1-\tau)f(k_{ss}) - \delta k_{ss}.$$

Now suppose that production technology $f(k) = Ak^{\alpha}$; this implies that the steady-state level of capital is

$$k_{ss} = [(1-\tau)\alpha A]^{\frac{1}{1-\alpha}} \left[\frac{1}{\beta} - (1-\delta)\right]^{\frac{1}{\alpha-1}}.$$

Therefore, the ration of old steady-state capital to new one is

$$\frac{k_{ss}^1}{k_{ss}^2} = \frac{(1-\tau_1)^{\frac{1}{1-\alpha}}}{(1-\tau_2)^{\frac{1}{1-\alpha}}}.$$

Assuming that $\alpha = \frac{1}{3}$, a typical estimate of the share of capital income in the GDP, we can say that:

$$\frac{k_{ss}^2}{k_{ss}^1} = \frac{(1-0.3)^{1.5}}{(1-0.2)^{1.5}} \approx 0.82.$$

Thus, in the aftermath of the tax change, the capital stock will gradually fall.

## 8.4 Ramsey Problem

We have defined a competitive equilibrium for a given fiscal policy. An interesting question to ask is which tax system is the best to finance the given stream of government expenditures $\{g_t\}_{t=0}^{\infty}$? This is the Ramsey problem.

Suppose the government has a sequence of expenditures $\{g_t\}_{t=0}^{\infty}$ to finance, and let taxes on consumption and investment as well as lump-sum transfers be zero $\tau_{ct} = \tau_{xt} = T_t = 0$, i.e. the policy-maker does not have access to these instruments. What will the optimal sequence of labor and capital income taxes be to finance those expenditures?
Conceptually, the problem is to find a sequence of $\{\tau_{kt}, \tau_{nt}\}_{t=0}^{\infty}$ so as to maximize

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t),$$

subject to

$$\{c_t(\tau), l_t(\tau), n_t(\tau), x_t(\tau), k_t(\tau)\} \text{ are part of a TDCE allocation,}$$

where

$$\sum_{t=0}^{\infty} p_t(\tau) g_t = \sum_{t=0}^{\infty} \left[\tau_{nt} w_t(\tau) n_t(\tau) + \tau_{kt} r_t(\tau) k_t(\tau)\right].$$

**Remark** It is understood that allocations are functions of the tax sequence $\{\tau_{kt}, \tau_{nt}\}_{t=0}^{\infty}$.
Now, consider the household's problem. Attach the Lagrange multiplier $\lambda$ to the budget constraint, and $\beta^t \mu_t$ to the law of motion for capital.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) + \lambda \left[\sum_{t=0}^{\infty} \left[(1-\tau_{nt}) n_t w_t + (1-\tau_{kt}) r_t k_t\right] - \sum_{t=0}^{\infty} \left[p_t c_t + p_t x_t\right]\right]$$
$$+ \sum_{t=0}^{\infty} \beta^t \mu_t \left[(1-\delta) k_t + x_t - k_{t+1}\right].$$

The first-order conditions are:

$$\begin{aligned}
\beta^t U_c'(t) &= \lambda p_t \\
\beta^t U_l'(t) &= \lambda(1-\tau_{nt})w_t \\
\beta^t \mu_t &= \lambda p_t \\
\beta^t \mu_t &= \lambda(1-\tau_{kt+1})r_{t+1} + \beta^{t+1}\mu_{t+1}(1-\delta)
\end{aligned}$$

Plug in the third equation into the last one to obtain:

$$\lambda p_t = \lambda(1-\tau_{kt+1})r_{t+1} + \lambda p_{t+1}(1-\delta).$$

Divide both sides by $\lambda p_{t+1}$ and invoke $\frac{p_t}{p_{t+1}} = \frac{U_c'(t)}{\beta U_c'(t+1)}$:

$$U_c'(t) = \beta U_c'(t+1)[1-\delta+(1-\tau_{kt+1})F_k'(k_{t+1}, n_{t+1})].$$

This is the Euler equation for the TDCE.

Now, consder the firm's problem. The first-order conditions are standard:

$$\begin{aligned}
p_t F_k'(t) &= r_t, \\
p_t F_n'(t) &= w_t.
\end{aligned}$$

### 8.4.1 Implementability Constraint

Therefore, a TDCE is the set of prices $\{p_t, w_t, r_t\}$, allocations for household and firms which must satisfy the following set of equations:

1. $p_t = \frac{\beta^t U_c'(t)}{U_c'(0)}$ under normalization $p_0 = 1$;
2. $\frac{r_t}{p_t} = F_k'(t)$;
3. $\frac{w_t}{p_t} = F_n'(t)$;
4. $\frac{U_c'(t)}{U_l'(t)} = \frac{1}{(1-\tau_{nt})F_n'(t)}$;
5. $U_c'(t) = \beta U_c'(t+1)[1-\delta+(1-\tau_{kt+1})F_k'(k_{t+1}, n_{t+1})]$;

6. $k_t = k_t^f$;

7. $n_t = n_t^f$;

8. $c_t + x_t + g_t = F(k_t, n_t)$;

9. $\sum_{t=0}^{\infty} [p_t c_t + p_t x_t] = \sum_{t=0}^{\infty} [(1-\tau_{nt})n_t w_t + (1-\tau_{kt})r_t k_t]$;

10. $k_{t+1} = (1-\delta)k_t + x_t$.

Now our objective is to combine some of these equations above into what is known as implementability constraint. We start with the budget constraint:

$$\begin{aligned}
\sum_{t=0}^{\infty} [p_t c_t + p_t x_t] &= \sum_{t=0}^{\infty} [(1-\tau_{nt})n_t w_t + (1-\tau_{kt})r_t k_t] \\
\sum_{t=0}^{\infty} p_t [c_t + k_{t+1} - (1-\delta)k_t] &= \sum_{t=0}^{\infty} [(1-\tau_{nt})n_t w_t + (1-\tau_{kt})r_t k_t] \\
\sum_{t=0}^{\infty} p_t c_t &= \sum_{t=0}^{\infty} (1-\tau_{nt})n_t w_t + k_0(1-\delta)p_0 + (1-\tau_{k0})r_0 k_0 \\
&+ \sum_{t=0}^{\infty} k_{t+1} \underbrace{[-p_t + (1-\delta)p_{t+1} + (1-\tau_{kt+1})r_{t+1}]}_{=0 \text{ under FOC}}
\end{aligned}$$

We are almost there. Now we need to eliminate prices. Use Equations 1 and 4. Specifically,

$$(1-\tau_{nt})w_t = \frac{U_l'(t)}{U_c'(t)F_n'(t)}F_n'(t)p_t = \frac{U_l'(t)}{U_c'(t)F_n'(t)}F_n'(t)\frac{\beta^t U_c'(t)}{U_c'(0)} = \beta^t \frac{U_l'(t)}{U_c'(0)}.$$

Thus, our implementability constraint takes the following form:

$$\sum_{t=0}^{\infty} \beta^t \left(U_c'(t)c_t - U_l'(t)n_t\right) = U_c'(0)[k_0(1-\delta)p_0 + (1-\tau_{k0})F_k'(0)k_0].$$

<u>Thus, the following equations must by satisfied in TDCE:</u>

1. feasibility constraint;

2. implementability constraint;

3. law of motion for capital.

Also note that once physical allocations for the TDCE are found, one can recover prices supporting those allocations from the equations above (1-10).

### 8.4.2 Chamley-Judd Result

We have shown that if $\{\tilde{c}, \tilde{n}, \tilde{l}, \tilde{x}, \tilde{k}\}$ and $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ constitute a TDCE for a given fiscal policy $\{\tau_{nt}, \tau_{kt}, g_t\}_{t=0}^{\infty}$, then those 10 equations above must be satisfied.

**Conversely** If $\{\tilde{c}, \tilde{n}, \tilde{l}, \tilde{x}, \tilde{k}\}$ satisfy

1. feasibility constraint;
2. implementability constraint;
3. law of motion for capital,

then there is a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and a tax system $\{\tau_{nt}, \tau_{kt}\}_{t=0}^{\infty}$ such that together they constitute a TDCE. In other words, the set of allocations that can be implemented as a TDCE with $\{\tau_{nt}, \tau_{kt}\}_{t=0}^{\infty}$ and raise enough revenue in equilibrium to pay for the sequence $\{g_t\}_{t=0}^{\infty}$ is the set of allocations satisfying feasibility, implementability constraints and the law of motion for capital.

Lut us formulate and solve the Ramsey Problem:

$$\max_{c,l,n,x,k} \sum_{t=0}^{\infty} \beta^t U(c_t, \bar{n}_t - n_t)$$

$$k_{t+1} = (1-\delta)k_t + x_t$$

$$c_t + x_t + g_t = F(k_t, n_t)$$

$$\sum_{t=0}^{\infty} \beta^t \left( U_c'(t) c_t - U_l'(t) n_t \right) = W_0$$

$$\bar{n}_t = \bar{n}, F_t(k_t, n_t) = F(k_t, n_t).$$

The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t, \bar{n}_t - n_t) + \lambda \left[ W_0 - \sum_{t=0}^{\infty} \beta^t \left( U_c'(t) c_t - U_l'(t) n_t \right) \right] + \sum_{t=0}^{\infty} \beta^t \mu_t \left[ (1-\delta)k_t + x_t - k_{t+1} \right] + \sum_{t=0}^{\infty} \beta^t \gamma_t \left[ F(k_t, n_t) - c_t - x_t - g_t \right].$$

Now, pretend that $\lambda$ is known, and combine the first two terms on the right-hand side:

$$\mathcal{L} = \lambda W_0 + \sum_{t=0}^{\infty} \beta^t \left[ U(c_t, \bar{n}_t - n_t) - U_c'(t)\lambda c_t + \lambda U_l'(t) n_t \right] \\ + \sum_{t=0}^{\infty} \beta^t \mu_t \left[ (1-\delta)k_t + x_t - k_{t+1} \right] + \sum_{t=0}^{\infty} \beta^t \gamma_t \left[ F(k_t, n_t) - c_t - x_t - g_t \right].$$

Define $V(c, n; \lambda) = U(c_t, \bar{n}_t - n_t) - U_c'(t)\lambda c_t + \lambda U_l'(t) n_t$. With this, we can rewrite the Lagrangian as follows:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t V(c_t, n_t; \lambda) + \\ + \sum_{t=0}^{\infty} \beta^t \mu_t \left[ (1-\delta)k_t + x_t - k_{t+1} \right] + \sum_{t=0}^{\infty} \beta^t \gamma_t \left[ F(k_t, n_t) - c_t - x_t - g_t \right].$$

**Remark** This is the same Lagrangian as in case $U(\cdot) = V(c, n; \lambda)$ and $\hat{F} = F - g$.

Using first-order conditions, we can obtain the following set of equations:

$$\frac{V_l'(t)}{V_c'(t)} = F_n'(k_t, n_t)$$
$$V_c'(t) = \beta V_c'(t+1)[1 - \delta + F_k'(k_t, n_t)].$$

**Assumption** Assume that the solution to the Ramsey problem coverges to the steady-state.

$$c_t^{RP} \to c_\infty^{RP}$$
$$x_t^{RP} \to x_\infty^{RP}$$
$$k_t^{RP} \to k_\infty^{RP}$$
$$n_t^{RP} \to n_\infty^{RP}.$$

Note that in this case, $g_t$ has to converge to some constant $g_t \to g_\infty$.

Under these assumptions, we have:

$$\frac{V_l'(c_\infty^{RP}, n_\infty^{RP})}{V_c'(c_\infty^{RP}, n_\infty^{RP})} = F_n'(k_\infty^{RP}, n_\infty^{RP})$$
$$1 = \beta[1 - \delta + F_k'(k_\infty^{RP}, n_\infty^{RP})].$$

Now, invoke the Euler equation for the TDCE:

$$U_c'(c_t^{RP}, n_t^{RP}) = \beta U_c'(c_{t+1}^{RP}, n_{t+1}^{RP})[1 - \delta + (1 - \tau_{kt+1}^{RP})F_k'(k_t^{RP}, n_t^{RP})].$$

In the limit, it collapses to

$$1 = \beta[1 - \delta + (1 - \tau_{k\infty}^{RP})F_k'(k_\infty^{RP}, n_\infty^{RP})].$$

Thus, for the two optimality conditions to be consistent across the Ramsey problem and the TDCE, the tax on capital has to converge to 0 over time, $\tau_{k\infty}^{RP} = 0$. This is the Chamley-Judd result.

### 8.4.3 Detailed Example with $\tau_{nt}$ and $\tau_{ct}$

Consider an infinite horizon economy in which there is a representative consumer and a representative firm. The utility function of the consumer is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \to \max .$$

The feasibility constraint is

$$c_t + x_t + g_t = F(k_t, n_t).$$

The law of motion for capital is standard.

Suppose the government has at its disposal only labor income and consumption taxes for financing expenditures, but can freely borrow and lend (i.e., it faces a present value budget constraint). The consumer takes $\{g_t, \tau_{ct}, \tau_{nt}\}$ as given when making decisions.

1. Set up and define a TDCE.

   [Standard.]

2. Assuming interiority of the equilibrium, give the FOCs that characterize the equilibrium.

   The system of FOCs (omitting obvious steps) is:

$$\frac{u_l'(t)}{u_c'(t)} = \frac{1 - \tau_{nt}}{1 + \tau_c t} F_n'(t)$$
$$\frac{u_c'(t)}{\beta u_c'(t+1)} \frac{1 + \tau_{ct+1}}{1 + \tau_{ct}} = F_k'(t+1) + 1 - \delta.$$

3. What is the implementability constraint for this economy? Compare this to what you would get if the government could choose labor and capital income taxes, but not consumption taxes.

$$
\begin{aligned}
\sum_{t=0}^{\infty}((1+\tau_{ct})p_t c_t + p_t k_{t+1}) &= \sum_{t=0}^{\infty}(k_t(r_t + p_t(1-\delta)) + (1-\tau_{nt})w_t n_t) \\
\sum_{t=0}^{\infty}((1+\tau_{ct})p_t c_t + p_t k_{t+1}) &= \sum_{t=0}^{\infty}\left(k_t(r_t + p_t(1-\delta)) + (1+\tau_{ct})p_t \frac{u_l'(t)}{u_c'(t)} n_t\right) \\
\sum_{t=0}^{\infty}\left((1+\tau_{ct})p_t c_t - (1+\tau_{ct})p_t \frac{u_l'(t)}{u_c'(t)} n_t\right) &= \sum_{t=0}^{\infty}(k_t(r_t + p_t(1-\delta)) - p_t k_{t+1}) \\
\sum_{t=0}^{\infty}\left((1+\tau_{ct})p_t c_t - (1+\tau_{ct})p_t \frac{u_l'(t)}{u_c'(t)} n_t\right) &= \sum_{t=0}^{\infty}(k_{t+1}(r_{t+1} + p_{t+1}(1-\delta)) - p_t k_{t+1}) + k_0(r_0 + p_0(1-\delta)) \\
\sum_{t=0}^{\infty}\left((1+\tau_{ct})p_t c_t - (1+\tau_{ct})p_t \frac{u_l'(t)}{u_c'(t)} n_t\right) &= k_0(r_0 + p_0(1-\delta))
\end{aligned}
$$

Now, from the FOC we have:

$$
(1+\tau_{ct})p_t = \beta^t \frac{u_c'(t)}{u_c'(0)}(1+\tau_{c0})p_0.
$$

Invoke this, and rewrite the above as follows:

$$
\begin{aligned}
\sum_{t=0}^{\infty}\left(\beta^t \frac{u_c'(t)}{u_c'(0)}(1+\tau_{c0})p_0 c_t - \beta^t \frac{u_c'(t)}{u_c'(0)}(1+\tau_{c0})\frac{u_l'(t)}{u_c'(t)} p_0 n_t\right) &= k_0(r_0 + p_0(1-\delta)) \\
\sum_{t=0}^{\infty}\beta^t\left(u_c'(t)c_t - u_l'(t)n_t\right) &= \frac{u_c'(0)}{(1+\tau_{c0})p_0} k_0(r_0 + p_0(1-\delta)).
\end{aligned}
$$

If instead of the consumption tax we had the capital income tax, then the implementability constraint takes the following form:

$$
\sum_{t=0}^{\infty}\beta^t\left(u_c'(t)c_t - u_l'(t)n_t\right) = \frac{u_c'(0)}{p_0} k_0((1-\tau_{k0})r_0 + p_0(1-\delta)).
$$

4. Formulate the Ramsey problem. Will it be true that $\tau_{ct} \to 0$? How about $\tau_{nt} \to 0$? Assume that in Ramsey allocation, all quantities converge to constant levels.

The Ramsey problem takes the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \to \max$$

$$c_t + x_t + g_t = F(k_t, n_t)$$

$$\sum_{t=0}^{\infty} \beta^t \left(u_c'(t)c_t - u_l'(t)n_t\right) = \frac{u_c'(0)}{(1+\tau_{c0})p_0} k_0(r_0 + p_0(1-\delta)).$$

The system of FOCs is:

$$\beta^t u_c'(t) = \lambda_t + \beta^t [u_c'(t) + u_{cc}''(t)c_t - u_{lc}''(t)n_t]\mu$$

$$\beta^t u_l'(t) + \beta^t u_{lc}''(t)c_t\mu = \lambda_t F_n'(t) + \beta^t [u_l'(t) + u_{ll}''(t)n_t]\mu$$

$$\lambda_t = \lambda_{t+1}[F_k'(t+1) + 1 - \delta].$$

This implies:

$$\frac{u_c'(t) - [u_c'(t) + u_{cc}''(t)c_t - u_{lc}''(t)n_t]\mu}{\beta\left(u_c'(t+1) - [u_c'(t+1) + u_{cc}''(t+1)c_{t+1} - u_{lc}''(t+1)n_{t+1}]\mu\right)} = F_k'(t+1) + 1 - \delta.$$

In the limit, the RP Euler equation becomes:

$$\frac{1}{\beta} = F_k'(\lim k, \lim n) + 1 - \delta.$$

Compare this to the Euler equation in the TDCE and conclude that $\tau_{ct} = \tau_{ct+1}$.

From the optimality condition relating consumption and leisure, we obtain:

$$\frac{u_l'(t) - [u_l'(t) + u_{ll}''(t)n_t - u_{cl}''(t)c_t]\mu}{u_c'(t) - [u_c'(t) + u_{cc}''(t)c_t - u_{cl}''(t)n_t]\mu} = F_n'(t) = \frac{u_l'(t)(1+\tau_{ct})}{u_c'(t)(1-\tau_{nt})}.$$

We can conclude that since $\tau_{ct} \to \tau_c$, then at the limit $\tau_{nt} \to \tau_n$ where the exact value is determined by the limit allocations. Note that neither of the taxes has to go necessarily to zero, although in the case where $\mu = 0$ it has to be the case that $\tau_{c_t} = -\tau_{nt}$.

# 9 Endogenous Growth Model

## 9.1 Introduction

We have dealt with the neoclassical growth model which demonstrates that the economy grows only:

– during transition to the steady-state;

– is initial capital stock is lower than the steady-state level.

This is in sharp contrast with the data: GDP per capita grows with no sign of slowing down among developed economies.

There are standard "fixes" to the neoclassical growth model. Specifically, the exogenous technological change. Recall that we briefly considered a model with exogenous labor-augmenting productivity growth along the lines:

$$y_t = F(k_t, (1+g)^t n_t).$$

Other versions include the Harrod-neutral growth (when $1+g$ multiplies the entire production function). These class of models implicitly assumes that this growth is exogenous to the efforts, decisions of the agents in the economy, is free and does not require any resources.

We have seen that under standard assumptions on the utility function and production technology, one can de-trend variables and work with a time-invariant model. The model then implies that macroeconomic aggregates grow at a constant rate. Note that even in a counterfactual scenario with a constant capital, the output would grow; even more so, it is impossible for the economy not to grow. It is difficult to understand that in world in which some countries have still not started to grow. Does it mean that those countries had shrinking capital stocks? At the very least, this seems implausible.

It is also difficult to understand things like productivity slowdown which took place in many developed countries in the second half of the 20th century. Besides, how can one explain "crossings" in levels of GDP per capita across countries?

Paul Romer in 1986 essentially started a new literature on endogenous growth. In simple terms, we now would like agents to choose $(1+g)$, and we want this choice to be costly. In what follows, we will consider the simplest model of endogenous growth.

## 9.2 A Simple Model with Human Capital: $A(k_t, h_t)$

**Remark** This model is known as $A(k_t, h_t)$ because it is "linear" in $(k_t, h_t)$. Consider a Planner's Problem of the following form:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, \underbrace{(1-n_t)h_t}_{\text{quality-adj leisure}})$$

$$c_t + x_{kt} + x_{ht} \leq F(k_t, \underbrace{n_t h_t}_{\text{quality-adj labor}})$$

$$k_{t+1} = (1-\delta_k)k_t + x_{kt}$$

$$h_{t+1} = (1-\delta_h)h_t + x_{ht}.$$

**Remark** You can easily incorporate (and inevitably complicate the analysis) spillover (external) effects. For example, consider the law of motion for human capital:

$$h_{it+1} = (1-\delta_h)h_{it} + G\left(x_{ih}, \int_I x_{i'ht} \mathrm{d}i'\right).$$

However, note that if $G(\cdot)$ does not depend on $x_{ih}$ then nobody will invest anything into human capital. Thus, these kind of modifications have to be done carefully.

### 9.2.1 Simplifying Assumptions

Let's assume that labor is supplied inelastically. Then the planner's problem (given $k_0, h_0$) is:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + x_{kt} + x_{ht} \leq F(k_t, h_t)$$

$$k_{t+1} = (1-\delta_k)k_t + x_{kt}$$

$$h_{t+1} = (1-\delta_h)h_t + x_{ht}.$$

The associated Euler equations (for capital and human capital):

$$u_c'(t) = \beta u_c'(t+1)\left[1 - \delta_k + F_k'(t+1)\right]$$

$$u_c'(t) = \beta u_c'(t+1)\left[1 - \delta_k + F_h'(t+1)\right].$$

**CRS Assumption** Under the assumption that the production technology is CRS (i.e., homogenous of degree 1), then marginal products are homogenous of degree 0:

$$F'_k(k,h) = F'_k\left(\frac{k}{h},1\right)$$
$$F'_h(k,h) = F'_h\left(\frac{k}{h},1\right).$$

Graphically, one can show that the right-hand sides of [EEK] and [EEH] are decreasing and increasing functions in $\frac{k}{h}$, let us call the intersection $\phi = \left(\frac{k}{h}\right)^*$. With this:

$$F(k_t,h_t) = F(k_t,\phi k_t) = \underbrace{A}_{F(1,\phi)} k_t.$$

**Remark** Note that standard Inada conditions do not hold here ($\lim_{k\to 0} F'(k) \neq \infty$ and $\lim_{k\to\infty} F'(k) \neq 0$), thus what we saw in the standard case about dynamics and stability need not hold.

**Cobb-Douglas Assumption** Additionally, suppose that $F(\cdot)$ is Cobb-Douglas and $\delta_k = \delta_h$. We know that for C-D technology, the following is true:

$$k_t F'_k(t) = \alpha F(k_t,h_t)$$
$$h_t F'_h(t) = (1-\alpha) F(k_t,h_t).$$

In equilibrium, $F'_k(t) = F'_k(t)$, thus yielding:

$$\frac{h_t}{k_t} = \frac{1-\alpha}{\alpha} = \phi.$$

Now, consider the law of motion for human capital:

$$h_{t+1} = (1-\delta_h)h_t + x_{ht} \iff$$
$$\phi k_{t+1} = (1-\delta_h)\phi k_t + x_{ht} \iff$$
$$\phi[k_{t+1} - (1-\delta_h)k_t] = x_{ht} \iff$$
$$\phi[k_{t+1} - (1-\delta_k)k_t] = x_{ht} \iff$$
$$\phi x_{kt} = x_{ht}.$$

Equipped with this, we can rewrite the problem without $h_t$ and $x_{ht}$:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$c_t + (1+\phi)x_{kt} \leq Ak_t$$
$$k_{t+1} = (1-\delta_k)k_t + x_{kt}.$$

This looks like a standard dynamic programming problem. To make progress, let's specify the utility function:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}.$$

**Remark** Note that if $\{c_t, x_{kt}, k_t\}$ is feasible from $k_0$, then $\{\lambda c_t, \lambda x_{kt}, \lambda k_t\}$ is feasible from $\lambda k_0$. In other words, the feasibility correspondence is homogeneous of degree 1:

$$\Gamma(\lambda k_0) = \lambda \Gamma(k_0).$$

On top of that, the utility function is strictly increasing, homogeneous of degree $1-\sigma$ and, thus, represents homothetic preferences. This implies that policy functions are homogeneous of degree 1 in $k_0$. That is, if $k_0^*, c_0^*, x_{k0}^*, k_1^*, c_1^*, x_{k1}^*, \ldots$ solves the problem with $k_0$ as initial capital ($P(k_0)$), then $\lambda k_0^*, \lambda c_0^*, \lambda x_{k0}^*, \lambda k_1^*, \lambda c_1^*, \lambda x_{k1}^*, \ldots$ solves $P(\lambda k_0)$.

**Remark** Note that the value function is homogeneous of degree $1-\sigma$:

$$V(\lambda k) = \sum_{t=0}^{\infty} \beta^t \frac{(\lambda c_t)^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} V(k).$$

### 9.2.2 Recursive Formulation

The Bellman equation takes the following form:

$$P(k_0) : V(k) = \max_{c, x_k, k'} \left[ \frac{c^{1-\sigma}}{1-\sigma} + \beta V(k') \right]$$
$$c + (1+\phi)x_k = Ak$$
$$k' = (1-\delta)k + x_k$$
$$k_0 \text{ given}; h_0 = \phi k_0$$

**Remark** Note that both physical and human capitals are states, so, in principle, we should carry two states. However, starting from $t = 1$ the ratio will be right. To simplify the matters, let's assume that $h_0 = \phi k_0$, thereby, it suffices to carry only one state variable in the Bellman equation. Since the policy functions are homogeneous of degree one, then:

$$\begin{aligned}
g_c(k) &= k g_c(1) \\
g_x(k) &= k g_x(1) \\
g_k(k) &= k g_k(1).
\end{aligned}$$

We now need to find $g_c(1), g_x(1), g_k(1)$. Let's go back to sequential formulation:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$c_t + (1+\phi)(k_{t+1} - (1-\delta_k)k_t) = Ak_t.$$

The Euler equation implies:

$$\begin{aligned}
\beta^t c_t^{-\sigma}(1+\phi) &= \beta^{t+1} c_{t+1}^{-\sigma}[A + (1+\phi)(1-\delta_k)] \\
\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} &= \beta\left[\frac{A}{1+\phi} + 1 - \delta_k\right] \\
\left(\frac{g_c(1)k_{t+1}}{g_c(1)k_t}\right)^{\sigma} &= \beta\left[\frac{A}{1+\phi} + 1 - \delta_k\right] \\
\left(\frac{g_k(k_t)}{k_t}\right)^{\sigma} &= \beta\left[\frac{A}{1+\phi} + 1 - \delta_k\right] \\
\left(\frac{k_t g_k(1)}{k_t}\right)^{\sigma} &= \beta\left[\frac{A}{1+\phi} + 1 - \delta_k\right] \\
g_k(1) &= \left[\beta\left(\frac{A}{1+\phi} + 1 - \delta_k\right)\right]^{\frac{1}{\sigma}}.
\end{aligned}$$

We now can find the $g_x(1)$ and $g_c(1)$.

$$
\begin{aligned}
k_{t+1} &= (1-\delta_k)k_t + x_{kt} \\
g_k(k_t) &= (1-\delta_k)k_t + g_x(k_t) \\
g_k(1)k_t &= (1-\delta_k)k_t + k_t g_x(1) \\
g_x(1) &= g_k(1) - (1-\delta_k).
\end{aligned}
$$

As per $g_c(1)$:

$$
\begin{aligned}
c_t + (1+\phi)x_{kt} &= Ak_t \\
g_c(k_t) + (1+\phi)g_x(k_t) &= Ak_t \\
g_c(1)k_t + (1+\phi)g_x(1)k_t &= Ak_t \\
g_c(1) &= A - (1+\phi)g_x(1).
\end{aligned}
$$

### 9.2.3 Implications for Growth

The growth rate of capital is:

$$
\gamma_{t,t+1}^k = \frac{k_{t+1}}{k_t} = \frac{g_k(k_t)}{k_t} = g_k(1).
$$

In other words, $g_k(1)$ governs how fast the economy grows. If $g_k(1) > 1$, then growth in $\sigma$ causes $g_k(1)$ to fall (when risk-aversion rises).

The consumption growth is:

$$
\gamma_{t,t+1}^c = \frac{c_{t+1}}{c_t} = \frac{g_c(k_{t+1})}{g_c(k_t)} = \frac{k_{t+1}g_c(1)}{k_t g_c(1)} = g_k(1).
$$

We can also express capital in period $t$ as follows:

$$
k_t = [g_k(1)]^t k_0.
$$

Taking logs, we obtain a linear (in logs) expression:

$$
\log k_t = \log k_0 + t\log[g_k(1)].
$$

Invoking the formula for $g_k(1)$, we conclude that as agents become more patient ($\beta \uparrow$), the capital stock grows faster.

## 9.3 Adding Fiscal Policy

Consider a situation when the government has access to labor and capital income taxes. Then the household's problem takes the form:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\sum_{t=0}^{\infty} p_t(c_t + x_{kt} + x_{ht}) \leq \sum_{t=0}^{\infty} [(1-\tau_{kt})r_t k_t + (1-\tau_{nt})w_t h_t]$$
$$k_{t+1} = (1-\delta_k)k_t + x_{kt}$$
$$h_{t+1} = (1-\delta_h)h_t + x_{ht}.$$

If the utility function is CRRA, then the Euler equations are:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta[1-\delta_k + (1-\tau_{kt+1})F_k'(k_{t+1}, h_{t+1})]$$
$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta[1-\delta_h + (1-\tau_{nt+1})F_h'(k_{t+1}, h_{t+1})].$$

Consider several cases:

1. Suppose $\delta_k = \delta_h$ and $\tau_{kt} = \tau_{nt} = \tau$. From EEK and EEH, it follows that

$$F_k'(k_{t+1}, h_{t+1}) = F_h'(k_{t+1}, h_{t+1}).$$

   Thus, this gives the same $\frac{k}{h}$ ratio as without taxes: $h = \phi k$.

   Moreover, since the Euler equation is now:

$$\left(\gamma^c_{t,t+1}\right)^{\sigma} = \beta[1-\delta_k + (1-\tau)F_k'(k_{t+1}, h_{t+1})],$$

   hence $\gamma^c$ is decreasing in $\tau$.

2. Suppose $\delta_k = \delta_h$ and $\tau_{kt} = \tau_k \neq \tau_n = \tau_{nt}$. In this case:

$$(1 - \tau_k)F'_k(k_{t+1}, h_{t+1}) = (1 - \tau_n)F'_h(k_{t+1}, h_{t+1}).$$

One case use the graphical device to gauge what happens with the equilibrium ratio $\frac{k}{h}$.

# 10 Stochastic Neoclassical Growth Model

So far, we have considered deterministic models, whereby we could reconstruct the entire equilibrium allocation given the initial conditions and optimality conditions. We saw that the neoclassical growth model gives rise to monotone dynamics (depending on the initial capital endowment).

A big part of modern macroeconomics deals with business cycles, i.e., the aggregate fluctuations of the economy. Regular deterministic models cannot generate complex dynamics, in which the dynamics of output, investment or other quantities exhibit wiggles around the trend.

There are two broad modelling techniques which can add "wiggles" to the aggregate time-series.

1. Deterministic Complex Dynamics. This method has been shown to not reproduce well the time-series we observe in the data.

2. Aggregate Shocks. In these course, we will consider shocks to (aggregate) productivity with understanding that there are many other ways to introduce aggregate shocks in the model economy.

Before we dive into the substantive material, we need to introduce some additional concepts and notation.

## 10.1 Some Additional Concepts and Notation

Imagine an environment where each period $t$ it can be in one of the states $\mathcal{S}$, i.e. $s_t \in \mathcal{S}$ for all $t$. For example, the state can be weather, and $\mathcal{S} = \{\text{Sunny}, \text{Rainy}\}$, or $\mathcal{S}$ can contain different levels of aggregate productivity, say $\mathcal{S} = \{\text{Low}, \text{HIgh}\}$. We will call $s_t$ a *realization* of state in period $t$.

**Definition 28.** *$s^t = (s_t, s_{t-1}, s_{t-2}, \ldots, s_0) \in \mathcal{S}^t$ is a history of states up to date $t$.*

Let $\Pi(s^t)$ be an unconditional probability of history $s^t$. The conditional probability of observing history $s^t$ given history $s^\tau$ with $\tau \leq t$ is $\Pi(s^t|s^\tau)$. We will routinely assume that the initial state $s_0$ is revealed at the beginning of time, $\Pi(s^0) = 1$.

**Remark** Note that once we introduce uncertainty, all allocations become functions of histories. In particular, consumption allocation following some history $s^t$, $c(s^t)$, will in general be different from consumption allocation following another history $\widehat{s}^t$, $c(\widehat{s}^t)$.

## 10.2 Arrow-Debreu Equilibrium

For the purpose of illustration what changes in the definition of competitive equilibrium in the simplest exchange economy we considered at the start of the course, we now consider the Arrow-Debreu equilibrium in an environment with uncertainty. The source of uncertainty is the endowment stream of agent $i$, $\{y_t^i(s^t)\}_t^{s^t}$.

When we had no uncertainty, in the Arrow-Debreu market structure agents were assumed to trade time-contingent claims, i.e., perfectly enforceable contracts that promise delivery of consumption goods in period $t$. We used the following notation: $\{c_t^i\}_{t=0}^{\infty}$. With uncertainty, we assume that agents now trade state-contingent claims that promise delivery of consumption goods in case some history $s^t$ is realized. We denote those claims as $c_t^i : \mathcal{S}^t \to \mathbb{R}_+$ for each period $t$. The entire consumption allocation of agent $i$ is then captured by $\{c_t^i(s^t)\}_t^{s^t \in \mathcal{S}^t}$.

In order to define the Arrow-Debreu equilibrium, we need to introduce prices of consumption claims. Let $q_t^0(s^t)$ denote the price of a consumption claim that promises a delivery of one unit of the consumption good if history $s^t$ gets realized. The problem of the household $i$ is then formulated as follows:

$$\max_{\{c_t^i(s^t)\}_t^{s^t}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t U(c_t^i(s^t)) \right]$$
$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t), c_t^i(s^t) \geq 0 \text{ for all } t \text{ and } s^t.$$

**Remark** As you learned it from the microeconomic sequence, in case we used the Von Neumann–Morgenstern utility, we could rewrite the objective of the household as:

$$\max_{\{c_t^i(s^t)\}_t^{s^t}} \sum_{t=0}^{\infty} \sum_{s^t} \left[ \beta^t \Pi(s^t) U(c_t^i(s^t)) \right].$$

**Definition 29.** *An Arrow-Debreu equilibrium is an allocation $\{c_t^i(s^t)\}_t^{s^t}$ and a price system $\{q_t^0(s^t)\}_t^{s^t}$, such that:*

1. *Given prices, the allocation solves the household's problem;*

2. *Market clears: $\sum_i c_t^i(s^t) = \sum_i y_t^i(s^t)$ for all $t$ and $s^t$.*

**Discussion** It can be shown that the ADE is efficient, thus the equilibrium allocation coincides with the solution to the social planner's problem (with appropriate welfare weights). This is a good exercise to show this (you can assume that the state space $\mathcal{S}$ is finite for convenience). Besides, Kenneth Arrow showed in 1964 that *"one period securities are enough to implement ADE allocations if markets re-open in each period."* Those securities are called Arrow securities.

## 10.3 Real Business Cycle Model

We now consider the Real Business Cycle model (RBC) which is the bread and butter of the modern business cycle research. This is essentially the neoclassical growth model we studies earlier augmented with aggregate TFP shocks. The planner's problem is formulated as follows:

$$\max \ \mathbb{E}\sum_{t=0}^{\infty} \beta^t u(c(s^t), 1 - n(s^t))$$

$$c(s^t) + x(s^t) \leq As_t(k_t(s^{t-1}))^{\alpha}(n_t(s^t))^{1-\alpha}$$

$$k_{t+1}(s^t) \leq (1-\delta)k_t(s^{t-1}) + x_t(s^t).$$

**Remark** First, $s_t$ in front of the production function is the productivity shock; all allocations are functions of histories of shocks until period $t$, $s^t$. Moreover, capital at time $t$, $k_t$ is decided upon in period $t-1$; therefore, it is a function of history until period $t-1$, $k_t(s^{t-1})$.

Typically, RBC models are analyzed numerically by applying dynamic programming techniques; this is something you could learn in the computational macro course. Computational aspects, however, are beyond the scope of this course. Instead, we will consider and analyze a simplified version of the model. Specifically, let us assume that labor is supplied inelastically, $u_l'(\cdot) = 0$, production function is of $Ak$-type, $\alpha = 1$, and capital fully depreciates from period to period, $\delta = 1$. With this, we can rewrite the social planner's problem as follows:

$$P(k_0, s_0) = \max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c(s^t))\right]$$

$$c(s^t) + x(s^t) \leq As_t(k_t(s^{t-1}))$$

$$k_{t+1}(s^t) \leq x(s^t).$$

**Assumption** Let us assume that the utility function is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Here $\sigma$ measures both the intertemporal elasticity of substitution and the risk-aversion (i.e., how much variation in consumption the agent is willing to accept across times and states). Under assumptions made so far, the value function is homogeneous of degree $1-\sigma$, i.e.

$$V(\lambda k, s_0) = \lambda^{1-\sigma} V(k, s_0) \text{ for all } t, s_0$$

and

$$(\tilde{c}^*(\lambda k_0, s_0), \tilde{k}^*(\lambda k_0, s_0)) = \lambda(\tilde{c}^*(k_0, s_0), \tilde{k}^*(k_0, s_0)).$$

**Definition 30.** *A stochastic process $s_0, s_1, \ldots$ is called first-order Markov process if*

$$P(s_{t+1}|s_0, s_1, \ldots, s_t) = P(s_{t+1}|s_t).$$

Some examples of Markov processes include AR(1), MA(1). An i.i.d. process is a 0-th order Markov process.

Now, let us assume that the process for $\{s_t\}$ is Markov. Then the Bellman equation becomes:

$$\begin{aligned} V(s,k) &= \sup_{c,k'} \left[u(c) + \beta \mathbb{E}[V(k',s')|s]\right] \\ c + k' &= Ask. \end{aligned}$$

Invoke the homogeneity property and obtain:

$$V(k,s) = \sup_{c,k'} \left[\frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}[(k')^{1-\sigma} V(1,s')|s]\right].$$

Furthermore, we can pull capital next period $k'$ out of the expectation operator, because capital is pre-determined in period $t$:

$$V(k,s) = \sup_{c,k'} \left[\frac{c^{1-\sigma}}{1-\sigma} + \beta (k')^{1-\sigma} \underbrace{\mathbb{E}[V(1,s')|s]}_{D(s)}\right].$$

In other words, we are maximizing a concave objective function over a linear constraint $c + k' = Ask$.

Therefore, consumption and capital next period are some fractions over production:

$$
\begin{aligned}
c &= \varphi A s k \\
k' &= (1-\varphi) A s k.
\end{aligned}
$$

In order to fully characterize the solution to the social planner's problem we need to find $\varphi^*$. Let us first find the Euler equation. Consider the Lagrangian:

$$
\mathcal{L} = \sum_t \sum_{s^t} \beta^t \Pi(s^t) u(c(s^t)) + \sum_t \sum_{s^t} \lambda_t(s^t) \left[ A s_t k_t(s^{t-1}) - c(s^t) - k_{t+1}(s^t) \right].
$$

The first-order conditions with respect to $c(s^t)$ and $k_{t+1}(s^t)$ are as follows:

$$
\begin{aligned}
\beta^t \Pi(s^t) u'(c(s^t)) &= \lambda_t(s^t) \\
-\lambda_t(s^t) + \sum_{s_{t+1}} \lambda_{t+1}(s^t, s_{t+1}) A s_{t+1} &= 0.
\end{aligned}
$$

Combine the first-order conditions to obtain:

$$
\beta^t \Pi(s^t) u'(c(s^t)) = \sum_{s_{t+1}} \beta^{t+1} \Pi(s^t, s_{t+1}) A s_{t+1} U'(c(s^t, s_{t+1})).
$$

Simplifying the expression, we obtain the Euler equation:

$$
u'(c(s^t)) = \beta \underbrace{\sum_{s_{t+1}} \Pi(s_{t+1}|s^t) A s_{t+1} u'(c(s^t, s_{t+1}))}_{\mathbb{E}[A s_{t+1} u'(s^t, s_{t+1})|s^t]}.
$$

Recall that the instantaneous utility function is $c^{1-\sigma}/(1-\sigma)$; with this, the Euler equation further simplifies to:

$$
1 = \beta \mathbb{E} \left( A s_{t+1} \left[ \frac{c(s^t)}{c(s^{t+1})} \right]^\sigma \middle| s^t \right).
$$

**Optimal Decisions** Now recall that the structure of optimal decisions is $c = \varphi Ask$ and $k' = (1-\varphi)Ask$. Plug them into Euler equation.

$$
\begin{aligned}
1 &= \beta \mathbb{E}\left( As_{t+1}\left[\frac{\varphi As_t k(s^{t-1})}{\varphi As_{t+1}k(s^t)}\right]^{\sigma} \middle| s^t \right) \iff \\
1 &= \beta \mathbb{E}\left( As_{t+1}\left[\frac{s_t k(s^{t-1})}{s_{t+1}(1-\varphi)As_t k(s^{t-1})}\right]^{\sigma} \middle| s^t \right) \iff \\
1 &= \beta \mathbb{E}\left( (1-\varphi)^{-\sigma}(As_{t+1})^{1-\sigma} \middle| s^t \right) \iff \\
(1-\varphi)^{\sigma} &= \beta \mathbb{E}\left[ (As_{t+1})^{1-\sigma} \middle| s^t \right].
\end{aligned}
$$

If the stochastic process is i.i.d., then

$$
(1-\varphi)^{\sigma} = \beta \mathbb{E}\left[(As)^{1-\sigma}\right].
$$

Thus, $1-\varphi = \left(\mathbb{E}\left[(As)^{1-\sigma}\right]\right)^{\frac{1}{\sigma}}$.

**Analysis of the Solution** What is the growth rate of output?

$$
\frac{y_{t+1}}{y_t} = \frac{As_{t+1}k(s^t)}{As_t k(s^{t-1})} = \frac{As_{t+1}(1-\varphi)As_t k(s^{t-1})}{As_t k(s^{t-1})} = As_{t+1}(1-\varphi).
$$

Thus, the expected growth of output is:

$$
\mathbb{E}\left(\frac{y_{t+1}}{y_t}\right) = (1-\varphi)A\mathbb{E}(s_{t+1}).
$$

Now let us explore how $\varphi$ and $1-\varphi$ are affected by uncertainty. Consider two distributions with the same mean:

- Distribution 1: $s_t = 1$ with probability 1;
- Distribution 2: $\mathbb{E}[s_t] = 1$ and $\mathbb{V}[s_t] > 0$.

In case of the first distribution, $1-\varphi = A^{\frac{1-\sigma}{\sigma}} = 1$ if $A = 1$. On the other hand,the effect of going from the first distribution to the second one depends on the sign of $\sigma$.

- If $\sigma \in [0,1)$, then $1-\varphi$ will decrease.
- If $\sigma > 1$, then $1-\varphi$ will increase.

# 11 Log-Linearization

Log-linearization is a solution to the problem of reducing computational complexity for systems of numerically specified equations that need to be solved simultaneously. Such systems can be found in macroeconomics and, increasingly, also in microeconomics as numerical simulation methods are becoming more popular throughout economics. Log-linearization converts a non-linear equation into an equation that is linear in terms of the log-deviations of the associated variables from their steady state values. For small deviations from the steady state, log-deviations have a convenient economic interpretation: they are approximately equal to the percentage deviations from the steady state.

**Remark** Log-linearization is a powerful tool, but it is important to keep in mind its limitations.

1. The method involves a Taylor expansion around the steady-state, thus the approximation is accurate only in the vicinity of the steady-state, and could potentially provide a poor approximation far away from the steady-state;

2. This method is inappropriate for the analysis of risk, uncertainty and anything else that depends on higher-order moments, because log-linearization takes into account only the mean of the variable (i.e., the first moment).

## 11.1 Substitution Method

We start off with the recap of the first-order Taylor expansion of function $f(x)$ at some point $x = a$:

$$f(x) \approx f(a) + Df(x)|_{x=a}(x - a).$$

Let $\tilde{x}_t$ denote the log-deviation of some variable $x_t$ from its steady-state $x$:

$$\tilde{x}_t = \log x_t - \log x.$$

The right-hand side of this equation can be written as

$$\log \frac{x_t}{x} = \log\left(1 + \frac{x_t - x}{x}\right).$$

Consider the first-order Taylor expansion of the right-hand side at the steady-state value $x_t = x$:

$$\log\left(1 + \frac{x_t - x}{x}\right) \approx \log 1 + \frac{1}{x}(x_t - x).$$

It follows that $\tilde{x}_t \approx \frac{x_t}{x} - 1$, or, equivalently, $x_t = x(1 + \tilde{x}_t)$. It turns out that using this approach works well with linear equations.

**Example** Consider the following equation:

$$y_t = c_t + i_t.$$

Use the result above to substitute each variable with $x(1 + \tilde{x}_t)$ to obtain:

$$y(1 + \tilde{y}_t) = c(1 + \tilde{c}_t) + i(1 + \tilde{i}_t).$$

It follows:

$$y + y\tilde{y}_t = c + c\tilde{c}_t + i + i\tilde{i}_t.$$

We know that at the steady-state, $y = c + i$. Thus, we arrive at $y\tilde{y}_t = c\tilde{c}_t + i\tilde{i}_t$, or, dividing both sides by the steady-state value of $y_t$,

$$\tilde{y}_t = \frac{c}{y}\tilde{c}_t + \frac{i}{y}\tilde{y}_t.$$

### 11.2 More General Approach

The substitution method described in the previous subsection works well with linear equation, but does not work well with non-linear equations (e.g., involving products of variables, or their ratios). We now consider a more general approach.

The starting point is the identity $\tilde{x}_t = \log x_t - \log x$. Now exponentiate this equation to obtain:

$$e^{\tilde{x}_t} = \frac{x_t}{x}.$$

Thus,

$$x_t = xe^{\tilde{x}_t}.$$

Consider a first-order Taylor expansion of $e^{\tilde{x}_t}$ around the steady-state of $\tilde{x}_t$ which is 0:

$$e^{\tilde{x}_t} \approx e^0 + e^1(\tilde{x}_t - 0) = 1 + \tilde{x}_t.$$

Therefore,

$$x_t \approx x(1 + \tilde{x}_t).$$

Note that this is identical to the approximation we derived in the substitution method; however, the advantage will become apparent in case of more complicated equations.

**Example** Perform the log-linearization of the expression $k_t^\alpha$. Note that if we directly apply the substitution method, we get

$$[k(1 + \tilde{k}_t)]^\alpha,$$

which is clearly non-linear in deviation $\tilde{k}_t$. The trick now is to first express $k_t$ as $ke^{\tilde{k}_t}$, and plug that in the original expression:

$$k_t^\alpha = (ke^{\tilde{k}_t})^\alpha = k^\alpha e^{\alpha\tilde{k}_t}.$$

It is straightforward to show that $e^{\alpha\tilde{k}_t}$ is approximately $1 + \alpha\tilde{k}_t$ (convince yourselves). Thus, $k_t^\alpha \approx k^\alpha(1 + \alpha\tilde{k}_t)$.

The crucial point to remember from the above example is that the exponent form of the log-linearization procedure makes it possible to turn the exponent $\alpha$ into a multiplier before the Taylor approximation is applied. This simplification is missed if the substitution approximation is applied directly to the original function.

**Example** Perform log-linearization of $\frac{x_t}{y_t}$.

$$\frac{x_t}{y_t} = \frac{xe^{\tilde{x}_t}}{ye^{\tilde{x}_t}} = \frac{x}{y}e^{\tilde{x}_t}e^{-\tilde{y}_t} \approx \frac{x}{y}(1+\tilde{x}_t)(1-\tilde{y}_t) \approx \frac{x}{y}(1+\tilde{x}_t-\tilde{y}_t).$$

Note that we ignored the cross-term $\tilde{x}_t\tilde{y}_t$ as this is negligible for small deviations from the steady-state.

## 11.3 Equations in Logs

Frequently in macro you will need to deal with equations involving logarithms. For example, the stochastic process for productivity can take the following form:

$$\log z_t = \rho \log z_{t-1} + \varepsilon_t.$$

We proceed with exponential substitution:

$$\log ze^{\tilde{z}_t} = \rho \log ze^{\tilde{z}_{t-1}} + \varepsilon_t.$$

Apply properties of logarithms to obtain:

$$\log z + \tilde{z}_t = \rho \log z + \rho\tilde{z}_{t-1} + \varepsilon_t.$$

At the steady-state, $\log z = \rho \log z$, thus we arrive at:

$$\tilde{z}_t = \rho\tilde{z}_{t-1} + \varepsilon_t.$$

## 11.4 Equations with Expectation Terms

In economic models with uncertainty, some equilibrium conditions may involve expectation. For example, the Euler equation in a standard growth model will look as follows:

$$\frac{1}{c_t} = \beta\mathbb{E}\left[\frac{1+r_{t+1}}{c_{t+1}}\right].$$

Since the expectation of the logarithm is different from the logarithm of expectation due to Jensen's inequality, cases like that need to be handled carefully. Let's start with exponential substitution:

$$\frac{1}{ce^{\tilde{c}_t}} = \beta \mathbb{E}\left[\frac{(1+r)e^{\widetilde{1+r}_{t+1}}}{ce^{\tilde{c}_{t+1}}}\right].$$

Cancelling out $c$ and doing some re-arrangements, we obtain:

$$e^{-\tilde{c}_t} = \beta(1+r)\mathbb{E}[e^{\widetilde{1+r}_{t+1}}e^{-\tilde{c}_{t+1}}].$$

Recall that at the steady-state $1 = \beta(1+r)$. Therefore,

$$1 = \mathbb{E}[e^{\widetilde{1+r}_{t+1}}e^{-\tilde{c}_{t+1}}e^{\tilde{c}_t}].$$

Now, replace each term with an approximation:

$$1 = \mathbb{E}[(1+\widetilde{1+r}_{t+1})(1-\tilde{c}_{t+1})(1+\tilde{c}_t)].$$

Expanding the expression inside the brackets, we obtain:

$$0 = \mathbb{E}[\tilde{c}_t - \tilde{c}_{t+1} + \widetilde{1+r}_{t+1}].$$

The final step is to convert $\widetilde{1+r}_{t+1}$ into $\widetilde{r}_{t+1}$. The trick is to consider the following identities:

$$\begin{aligned}\widetilde{1+r}_{t+1} &= \frac{1+r_{t+1}-1-r}{1+r} = \frac{r_{t+1}-r}{1+r}\\ \widetilde{r}_{t+1} &= \frac{r_{t+1}-r}{r}.\end{aligned}$$

Noting that the numerators are the same, it follows that

$$(1+r)\widetilde{1+r}_{t+1} = r\widetilde{r}_{t+1},$$

or

$$\widetilde{1+r}_{t+1} = r\widetilde{r}_{t+1}/(1+r).$$

Thus, the final answer is:

$$0 = \mathbb{E}[\tilde{c}_t - \tilde{c}_{t+1} + r\widetilde{r}_{t+1}/(1+r)].$$

## 11.5 Multivariate Case

Suppose you need to log-linearize a function of two variables $x_{t+1} = f(x_t, y_t)$. Consider a first-order Taylor expansion around the steady-state $x_t = x$ and $y_t = y$:

$$f(x_t, y_t) \approx f(x, y) + D_x f(x, y)(x_t - x) + D_y f(x, y)(y_t - y).$$

At the steady-state $x = f(x, t)$, thus:

$$x_{t+1} \approx x + D_x f(x, y)(x_t - x) + D_y f(x, y)(y_t - y).$$

Divide both sides by $x$ to get

$$\frac{x_{t+1}}{x} \approx 1 + D_x f(x, y)(x_t - x)/x + D_y f(x, y)(y_t - y)/x,$$

$$\tilde{x}_{t+1} \approx D_x f(x, y)\tilde{x}_t + D_y f(x, y)\tilde{y}_t \frac{y}{x}.$$

Multiply both sides by $x$ to obtain the final answer:

$$x\tilde{x}_{t+1} \approx D_x f(x, y)x\tilde{x}_t + D_y f(x, y)\tilde{y}_t y.$$

**Example** Log-linearize $k_{t+1} = sz_t k_t^\alpha + (1 - \delta)k_t$.

Using the formula above, we directly obtain:

$$k\tilde{k}_{t+1} \approx [\alpha szk^{\alpha-1} + (1 - \delta)]k\tilde{k}_t + [sk^\alpha]z\tilde{z}_t.$$

## 12 Heterogeneity in Macro

TBA

# 13 Concluding Remarks