ECON 7040 - Macroeconomic Theory

TA Session 2

January 27, 2024

Solving Models

- Model set up: environment and market structure
- Describe equilibrium: 1) optimality ; 2) market clearing
- Optimality: households, firms, any other agent in your model that solves an optimization problem
- Market clearing: supply = demand
- Optimality $\implies$ gives rules that agents follow to optimize **given prices** e.g. $c_t = F(x_t, p_t)$
- Market clearing $\implies$ pins down equilibrium prices given optimal behavior (rules) e.g. $c_t = y_t \implies p_t = G(x_t, y_t)$

Solving Linear Models

- In most cases we can't solve by hand, so we linearize
- You now have a linear model of the form:

$$\mathbb{E}_t y_{t+1} = \boldsymbol{A} y_t + \boldsymbol{B} \varepsilon_{t+1}$$

  y_{t+1} is the vector of all variables in your model, ε_{t+1} is a vector of exogenous shocks

- 2 'types' of variables in macro models:
  1. Jump (forward-looking) variables e.g.:

$$\mathbb{E}_t C_{t+1} = C_t$$

  2. State (backward-looking) variables e.g.:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

Blanchard-Kahn Conditions: Root Counting

$$\mathbb{E}_t y_{t+1} = \boldsymbol{A} y_t + \boldsymbol{B} \varepsilon_{t+1} \tag{1}$$

- Define N as the number of variables in vector y_{t+1} and J as the number of 'jump' (forward-looking) variables
- *Linear* system (1) will have N non-zero eigenvalues λ_i, $i = 1, \ldots, N$
- Let K be the number of eigenvalues that lie outside the interval $[-1, 1]$
- Blanchard-Kahn conditions say:
  - If $K > J$ then (1) **has no bounded solution**
  - If $K < J$ then (1) **has infinite number of solutions**
  - If $K = J$ then (1) **has a unique bounded solution**

Blanchard-Kahn: Existence and Uniqueness

- BK conditions tell us when a linear system of equations has a unique bounded solution
- This relates to two important features of linear systems: existence and uniqueness of a solution
- Existence requires $K \leq J$
- Uniqueness requires $K \geq J$

  $\therefore$ Unique bounded solution exists if and only if $K = J$

Sims' Method: Gensys

- General linear system in Gensys form:

$$\mathbf{\Gamma}_0 y_{t+1} = \mathbf{\Gamma}_1 y_t + \mathbf{\Psi}\varepsilon_{t+1} + \mathbf{\Pi}\eta_{t+1} \tag{2}$$

  where $\eta_{t+1} = y_{t+1} - \mathbb{E}_t[y_{t+1}]$ is the one-step-ahead forecast error

- Generalizes solution method going beyond "root counting" and checking "spanning conditions"
- Existence condition is:

$$\text{span}(\boldsymbol{P^U}\mathbf{\Psi}) \subset \text{span}(\boldsymbol{P^U}\mathbf{\Pi})$$

  all info in $\varepsilon_t, \varepsilon_{t-1}, \ldots$ can be obtained from $\eta_t, \eta_{t-1}, \ldots$

- Uniqueness condition is:

$$\text{span}(\mathbf{\Pi}'\boldsymbol{P^{S'}}) \subset \text{span}(\mathbf{\Pi}'\boldsymbol{P^{U'}})$$

  knowledge of $\boldsymbol{P^U}\mathbf{\Pi}\eta_{t+1}$ must also give us $\boldsymbol{P^S}\mathbf{\Pi}\eta_{t+1}$

Gensys Algorithm: Interpreting Results

- Matlab function:

$$\underbrace{[\boldsymbol{\Gamma}, \boldsymbol{C}, \boldsymbol{M}, fmat, fwt, ywt, gev, eu]}_{\text{output}} = gensys(\underbrace{\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1, c, \boldsymbol{\Psi}, \boldsymbol{\Pi}, div}_{\text{inputs}})$$

- Gensys gives you a solution of the form:

$$y_{t+1} = \boldsymbol{\Gamma} y_t + \boldsymbol{M}\varepsilon_{t+1} + \sum_{s=2}^{\infty} (ywt)(fmat)^s (fwt) \mathbb{E}_t[\varepsilon_{t+s}] \quad (3)$$

- If you assume $\varepsilon_t \sim iid$, then last term is 0 and you get a VAR-form solution:

$$y_t = \boldsymbol{\Gamma} y_{t-1} + \boldsymbol{M}\varepsilon_t$$

Gensys Algorithm: Interpreting results

$$\underbrace{[\boldsymbol{\Gamma}, \boldsymbol{C}, \boldsymbol{M}, fmat, fwt, ywt, gev, eu]}_{\text{output}} = gensys(\underbrace{\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1, c, \boldsymbol{\Psi}, \boldsymbol{\Pi}, div}_{\text{inputs}})$$

- eu is the existence/uniqueness vector
- It's a 2x1 vector: first element is existence (1 if true 0 if false) and the second uniqueness (1 if true 0 if false)
- We want to see [1;1]!
- Other common cases are [1,0] (indeterminacy) and [0,1] (non-existence of bounded eqm)
- gev is a Nx2 vector that contains the generalized eigenvalues of $\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Gamma}_1$
- Eigenvalues of the system can be obtained as $gev(:, 2)/gev(:, 1)$

Solving a Model: Steps

1. Write down your model's eqm conditions (optimality & M clearing)
2. Solve for the steady state
3. Linearize your model around the steady state
4. Reduce you linear system (optional, I prefer it)
5. Write your linear model in gensys form
6. Construct gensys matrices
7. Solve using gensys code in Matlab
8. Check existence & uniqueness
9. Create IRFs

RBC model

- Standard RBC model:

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} && \text{(Production function)}\\
A_{t+1} &= A_t^{\rho} e^{z_{t+1}}; \quad z_t \overset{iid}{\sim} N(0,1) && \text{(LoM for Productivity)}\\
K_{t+1} &= (1-\delta)K_t + I_t && \text{(LoM for K)}\\
\frac{1}{C_t} &= \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (r_{t+1} + (1-\delta)) \right] && \text{(HH FOCs)}\\
r_{t+1} &= A_{t+1} \alpha K_{t+1}^{\alpha-1} && \text{(Firm FOC)}\\
Y_t &= C_t + I_t && \text{(ARC)}
\end{aligned}
$$

RBC model: steady state

- In steady state the above become:

$$
\begin{aligned}
Y &= AK^{\alpha} \\
1 &= \beta(r + 1 - \delta) \\
K &= (1 - \delta)K + I \\
Y &= C + I \\
r &= A\alpha K^{\alpha - 1} \\
A &= A^{\rho}
\end{aligned}
$$

- Solution can be obtained by hand:

$$
K = \left[\frac{\alpha}{\beta^{-1} - (1 - \delta)}\right]^{\frac{1}{1-\alpha}}; \quad C = K^{\alpha} - \delta K; \quad I = \delta K; \quad Y = C + I
$$

- Do it yourselves!

Linearized RBC model

- Linearize using multiple techniques we went over last session:

$$
\begin{aligned}
y_t &= a_t + \alpha k_t \\
\mathbb{E}_t c_{t+1} &= c_t + (1 - \beta(1-\delta))\mathbb{E}_t r_{t+1} \\
k_{t+1} &= (1-\delta)k_t + \frac{I}{K} i_t \\
y_t &= \frac{C}{Y} c_t + \frac{I}{Y} i_t \\
r_{t+1} &= a_{t+1} - (1-\alpha)k_{t+1} \\
a_{t+1} &= \rho a_t + z_{t+1}
\end{aligned}
$$

- Do it yourselves!

RBC model (linearized & reduced)

- Can reduce model to only 3 equations:

$$
\begin{aligned}
\mathbb{E}_t c_{t+1} &= c_t + (1-\beta(1-\delta))(a_{t+1} - (1-\alpha)k_{t+1}) \\
k_{t+1} &= (1-\delta)k_t + \frac{Y}{K}(a_t + \alpha k_t) - \frac{C}{K}c_t \\
a_{t+1} &= \rho a_t + z_{t+1}
\end{aligned}
$$

- Again, do it yourselves!
- How many jump variables? how many state variables?
- Using BK condition: how many $|\lambda_i| > 1$ and $|\lambda_j| < 1$ do we need to ensure a unique solution exists?

RBC model in Gensys Form

- Write the system above in gensys form:

$$\mathbf{\Gamma}_0 y_{t+1} = \mathbf{\Gamma}_1 y_t + \mathbf{\Psi}\varepsilon_{t+1} + \mathbf{\Pi}\eta_{t+1}$$

- Define $y_{t+1} = [c_{t+1} \quad k_{t+1} \quad a_{t+1}]'$, $\varepsilon_{t+1} = z_{t+1}$, and $\eta_{t+1} = c_{t+1} - \mathbb{E}_t[c_{t+1}]$ then write:

$$\mathbf{\Gamma}_0 = \begin{bmatrix} 1 & \phi(1-\alpha) & -\phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{\Gamma}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{C}{K} & (1-\delta) + \frac{Y}{K}\alpha & \frac{Y}{K} \\ 0 & 0 & \rho \end{bmatrix}$$

where $\phi \equiv 1 - \beta(1-\delta)$

$$\mathbf{\Psi} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{\Pi} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$