Econ 7020

OH: Th (2-4p)
Room 222

Today → 1) Admin
2) Arrow-Debreu equilibrium in our exchange economy.

Admin : Overall :-

→ Technical (Markov Processes,
Dynamic Programming
log-lin, Hodrick-Prescott filter)

→ Language (ADE, Sequential Markets Eq,
Pareto Optimization

→ Substantive (Neoclassical Growth Model,
Ramsey Problem,
Real Business cycle problem,
Overlapping G . . . model)

Grades :-

1. PS (30%) → 4/5 per semester.
2. MidTerm (30%) → (Sep 28)
3. Final Exam (40%) →
(All the material)

Simple Dynamic Exchange Economy :-

Agents : 2 agents, live indefinitely

$t = 0, 1, 2, \ldots$ (discrete, run till infinity)

Production: No production

Endowments: agent 1 gets 2 units of a perishable consumption good in odd periods & nothing in even periods.

Agent 2 gets 2 units of a perishable consumption good in even periods & nothing in odd periods.

Preferences : Agents value allocations according to the following utility function

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

discount factor

$\beta \in (0,1)$ – – – – –

$\beta \to 1$, more patient, values consⁿ in the future more.

Information : perfect information

Defⁿ : an allocation in this environment is a sequence $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$, where $c_t^i \to$ amount of consᵐ of agent i in period t.

Arrow - Debreu Competitive Equilibrium :-

Competitive $\Rightarrow$ agents take prices as given

Market Structure: agents meet at $t=0$ and trade consumption claims for every future t.

p_t – Price of consumption at time t.

<u>Def</u> an AD competitive equilibrium is a price system $\{\hat{p}_t\}_{t=0}^{\infty}$ and an allocation $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ s.t.:

1) given the price system $\{\hat{p}_t\}$, the allocation $\{(\hat{c}_t^1, \hat{c}_t^2)\}$ solves agent's maximization problem.

$$\sum \beta^t \log(c_t^i) \longrightarrow \max_{\{c^i\}}$$

{utility}

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} p_t e_t^i$$

(cons^m cannot exceed endowment)

$$c_t^i \geq 0$$

2) Markets clear in each period t

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 \quad \forall t$$

(resource constraint, feasibility)

$c_t^1 + c_t^2 \not< e_t^1 + e_t^2 \; \forall t$ because utility is increasing with increase of cons^m so this would mean that there the initial allocation was not optimal.

Characterizing the equilibrium (Lagrangean)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log(c_t^i) + \lambda^i \left[\sum_{t=0}^{\infty} p_t c_t^i - \sum_{t=0}^{\infty} p_t e_t^i \right]$$

Take FOC wrt e_t^i, e_{t+1}^i:

$$(e_t^i) : \frac{\beta^t}{c_t^i} + \lambda^i p_t = 0$$

$$(c_t^i) : \frac{\beta^t}{c_t^i} = -\lambda^i p_t \qquad [1]$$

$$(c_{t+1}^i) : \frac{\beta^{t+1}}{c_{t+1}^i} = -\lambda^i p_{t+1} \qquad [2]$$

(Answer & foc with the t+1 post script.)

From [1] & [2]:

$$\lambda^i = \frac{-\beta^t}{c_t^i p_t}$$

$$\lambda^i = \frac{-\beta^{t+1}}{c_{t+1}^i p_{t+1}}$$

$$\Rightarrow \frac{\beta^t}{c_t^i p_t} = \frac{\beta^{t+1}}{c_{t+1}^i p_{t+1}} \iff \beta c_t^i p_t = c_{t+1}^i p_{t+1} \qquad (*)$$

Adding (*) across agents yield:

$$\beta (c_t^1 + c_t^2) p_t = (c_{t+1}^1 + c_{t+1}^2) p_{t+1}$$

Invoke market clearing condition:

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 = c_{t+1}^1 + c_{t+1}^2$$

$\Rightarrow \beta p_t = p_{t+1}$

$\Rightarrow p_t = \beta^t p_0$ where p_0 is price in period 0.

Suppose multiply the entire sequence of prices by constant $\{2\hat{p}_t\}$ → it would not affect the allocation because of the budget constraint:

$$\sum \hat{p}_t c_t \le \sum \hat{p} e_t^i$$

Normalize $p_0 = 1$

$$p_t = \beta^t$$

Invoke the BC of agents

$$\sum p_t c_t^i = \sum p_t e_t^i$$

$$\Leftrightarrow \sum \beta^t c_t^i = \sum \beta^t e_t^i$$

Plug $p_t = \beta^t$ into $(*)$

β cancels out.

$$\beta c_t^i \beta^t = c_{t+1}^i \beta^{t+1}$$

$\Leftrightarrow c_t^i = c_{t+1}^i = c_0^i$ for both agents 1 & 2

$$\sum \beta^t c_0^1 = \sum \beta^t e_t^i.$$

$$\Rightarrow c_0^1 \underbrace{\sum_{t=0}^{\infty} \beta^t}_{\frac{1}{1-\beta} \to \text{G.P.}} = \underbrace{\sum \beta^t e_t^1.}_{2+0+2\beta^2+0+2\beta^4 \ldots}$$

G.P.

$$\Rightarrow \frac{c_0^1}{1-\beta} = \frac{2}{1-\beta^2}$$

$$\Leftrightarrow \boxed{c_0^1 = \frac{2}{1+\beta}}$$

for agent 2 : $0+2\beta+0+2\beta^3 \ldots$

$$c_0^2 \frac{1}{1-\beta} = \frac{2\beta}{1-\beta^2}$$

$$\boxed{c_0^2 = \frac{2\beta}{1+\beta}}$$

The first guy consumes more than the second person. This is because they have endowment in the first session.

Aug 29 / class 2.

Today: Pareto Optimality
Negishi method
Sequential Mkts equilibrium

Pareto Optimality :-

Def: An allocation is said to be Pareto Optimal (or efficient) if there is no other feasible allocation which makes everyone no worse off, and atleast one agent strictly better off.

In the context of the exchange economy, $\{\hat{c}^1, \hat{c}^2\}$ is PO if $\nexists$ feasible $\{\tilde{c}^1, \tilde{c}^2\}$, s.t.

$$u_i(\tilde{c}^i) \geq u_i(\hat{c}^i) \ \forall i$$

$$u_j(\tilde{c}_j) > u_j(\hat{c}_j) \ \exists j$$

theorem (The first welfare Theorem)

The competetive equilibrium allocation is pareto optimal

Proof by contradiction

Proof: Suppose not $\Rightarrow \exists$ feasible allocation $\{\tilde{c}^1, \tilde{c}^2\}$ s.t.

WLOG $u_1(\tilde{c}^1) > u_1(\hat{c}^1)$

$u_2(\tilde{c}^2) \geq u_2(\hat{c}^2)$

For the agent 1, it follows that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

If not, $\sum \hat{p} \tilde{c}^1 \leq \sum \hat{p}_t \hat{c}_t^1$

$\Rightarrow \tilde{c}$ is affordable and not chosen by agent 1, and delivers strictly higher utility $\Rightarrow$ ⨳ with $\hat{c}$ being a CE

contradiction sign

For agent 2

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

If not, then

$$\sum \hat{p} \tilde{c}_t^2 < \sum \hat{p}_t \hat{c}_t^2$$

So agent can increase consumption slightly in (at least) one of the dates, and this will strictly increase utility $\Rightarrow$ W w/ $\hat{c}$ being a CE.

$$\left[\begin{array}{l} \sum \hat{p}_t \tilde{c}_t^1 > \sum \hat{p}_t \hat{c}_t^1 \\ \sum \hat{p}_t \tilde{c}_t^2 \geq \sum \hat{p}_t \hat{c}_t \end{array}\right.$$

By summing these two,

$$\sum \hat{p}_t (\tilde{c}_t^1 + \tilde{c}_t^2) > \sum p^t (\hat{c}_t^1 + c_t^2)$$

$$\sum \hat{p}_t (e_t^1 + e_t^2) > \sum p^t (e_t^1 + e_t^2) \quad W$$

(contradiction)

Negishi's method to compute CE.

Algorithm

→ Solve for the social planner's problem

→ Pick the PO allocation which is CE.

Social Planner's Problem

$$\max \alpha U_1 + (1-\alpha) U_2 \to \max_{(\{c_t^1\},\{c^2\})}$$

s.t. $c_t^1 + c_t^2 = e_t^1 + e_t^2$

$\alpha \in [0, 1]$

$c_t^1, c_t^2 \geq 0$

$$\mathcal{L} = \alpha \sum_{t=0}^{\infty} \beta^t \log(c_t^1) + (1-\alpha) \sum_{t=0}^{\infty} \beta^t \log(c_t^2) + \sum_{t=0}^{\infty} \lambda_t [e_t^1 + e_t^2 - c_t^1 - c_t^2]$$

Solution is $(\tilde{c}^1, \tilde{c}^2) = (\tilde{c}^1(\alpha), \tilde{c}^2(\alpha))$

FOC

$$\frac{\partial \mathcal{L}}{\partial c_t^1} = \frac{\alpha \beta^t}{c_t^1} - \lambda_t = 0 \quad [1]$$

$$\frac{\partial \mathcal{L}}{\partial c_t^2} = \frac{(1-\alpha)(\beta^t)}{c_t^2} - \lambda_t = 0 \quad [3]$$

$$\frac{\partial \mathcal{L}}{\partial c^1_{t+1}} = \frac{\alpha \beta^{t+1}}{c^1_{t+1}} - \lambda_{t+1} = 0$$

Combine
[1] & [3]

$$\frac{\alpha \beta^t}{c_t^1} = \frac{(1-\alpha)\beta^t}{c_t^2}$$

$$\Rightarrow \quad (1-\alpha) c_t^1 = \alpha . c_t^2$$

Invoke resource constraint (RC)

$$c_t^1 + \underbrace{c_t^2}_{\frac{(1-\alpha)c^1 t}{\alpha}} = e_t^1 + e_t^2$$

$$c_t^1 + \frac{(1-\alpha)c^1 t}{\alpha} = e_t^1 + e_t^2 = 2$$

$$= c_t^1 \left(1 + \frac{1-\alpha}{\alpha}\right) = 2 \Leftrightarrow \frac{c_t^1}{\alpha} = 2$$

$$\Longleftrightarrow \quad c_t^1 = 2\alpha$$

$$c_t^1 = 2\alpha$$
$$c_t^2 = 2(1-\alpha)$$

Social Planner's Problem

$$\sum_{t=0}^{\infty} \beta^t \left(\alpha \log c_t^1 + (1-\alpha) \log c_t^2\right) \longrightarrow \max_{(c_t^1, c_t^2)_{t=0}^{\infty}}$$

s.t. $c_t^1 + c_t^2 \leq e_t^1 + e_t^2$

$c_t^1 \geq 0$

$c_t^2 \geq 0$

$\lim_{c \to 0} \log(c) = -\infty$ (so never choose this)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\alpha \log c_t^1 + (1-\alpha) \log c_t^2\right) + \sum_{t=0}^{\infty} \mu_t \left[e_t^1 + e_t^2 - c_t^1 - c_t^2\right] + \sum_{t=0}^{\infty} \lambda_t c_t^1 + \sum_{t=0}^{\infty} \gamma_t c_t^2$$

FOCs

$$(c_t^1) = \frac{\beta^t \alpha}{c_t^1} - \mu_t + \lambda_t = 0$$

$$(c_t^2) = \frac{\beta^t (1-\alpha)}{c_t^2} - \mu_t + \gamma_t = 0$$

$$\mu_t = e_t^1 + e_t^2 - c_t^1 - c_t^2 \geq 0$$

$$\left.\begin{aligned} \lambda_t \cdot c_t^1 &\geq 0 \\ \gamma_t c_t^2 &\geq 0 \end{aligned}\right\} \text{Complementary slackness}$$

$$\mu_t(e_t^1 + e_t^2 - c_t^1 - c_t^2) = 0$$

We know that

$c_t^1, c_t^2 > 0$

[This is because $\log 0 = \infty$ if its plugged in original f^n, social planner does not want that.

$\Rightarrow \gamma_t, \lambda_t = 0$ (from Complementary Slackness condition)

Solution is $\{(c_t^1, c_t^2)\}_{t=0}^{\infty} = \{2\alpha, 2(1-\alpha), \alpha \in [0,1]\}$

Pick PO allocation which is affordable to agents if the prices they faced were equal to Lagrange multiplier from SPD.

Define transfer $t^i(\alpha) = \sum_{t=0}^{\infty} \mu_t (c_t^i(\alpha) - e_t^i)$

CE allocation has to be affordable without transfers.

$$\frac{\beta^t \alpha}{c_t^1} = \mu_t \Rightarrow \mu_t = \frac{\beta^t}{2}$$

$$t^1(\alpha) = \sum_{t=0}^{\infty} \frac{\beta^t}{2} (\underbrace{c_t^1(\alpha)}_{2\alpha} - e_t^1)$$

$$= \sum \frac{\beta^t}{2} \cdot 2\alpha - \sum \frac{\beta^t}{2} e_t^1$$

$$= \frac{\alpha}{1-\beta} - \frac{1}{2}\cdot\frac{2}{1-\beta^2} = 0$$

$$\Rightarrow \frac{\alpha}{1-\beta} = \frac{1}{1-\beta^2} \Rightarrow \alpha^* = \frac{1}{1+\beta}$$

$$C_t^1 = 2\alpha = \frac{2}{1+\beta}$$

$$C_t^2 = 2(1-\alpha) = \frac{2\beta}{1+\beta}$$

Sequential Markets Equilibrium.

Agent 1 & 2 , $t = 0, 1, 2, \ldots$

Agents have endowments $e^1 = 2, 0, 2, 0 \ldots$

$e^2 = 0, 2, 0, 2 \ldots$

They trade one period bonds a^i_t.

r_{t+1} – interest rate from t to $t+1$.

A bond is a promise (contract) to pay 1 unit of consumption in $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ cons. today

$a^i_{t+1} \rightarrow$ # of bonds which agent buys in t & carries over to $t+1$

$a^i_{t+1} < 0 \Rightarrow$ borrows in t. ?

$$c^i_t + \frac{a^i_{t+1}}{1+r_{t+1}} \leq e^i_t + a^i_t$$

We will assume that $a^i_0 = 0 \ \forall i$

Def^n: A SM equilibrium is $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ asset holdings $\{(a^1_{t+1}, a^2_{t+1})\}_{t=0}^{\infty}$ and interest rates $\{r_{t+1}\}_{t=0}^{\infty}$, s.t.

1) given $\{r_{t+1}\}_{t=0}^{\infty}$, $\{(c_t^1, c_t^2, a_{t+1}^1, a_{t+1}^2)\}_{t=0}^{\infty}$

solves consumer's problems:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t^i) \longrightarrow \max_{\{(c_t^i, a_{t+1}^i)\}_{t=0}^{\infty}}.$$

s.t. $$c_t^i + \frac{a_{t+1}^i}{1+r_{t+1}} \leq e_t^i + a_t^i \ \forall t$$

$$c_t^i \geq 0$$

$$a_{t+1}^i \geq -\bar{A}^i \ \forall t \qquad \text{(No Ponzi Game Condition.)}$$

$\bar{A}^i$ is some large positive no.

This constraint is set to tell the agents that they can't keep borrowing. They'll have to pay back eventually.

2) Market Clearing

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 \ \forall t$$

$$\sum_{i=1}^{2} a_{t+1}^i = 0 \ \forall t$$

$i=1$ ...

Equivalence of ADE and SME :

Let $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ and $\{\hat{p}_t\}_{t=0}^{\infty}$ constitute an ADE.

Then, there is an associated SME $\{\tilde{c}_t^1, \tilde{c}_t^2, \tilde{a}_{t+1}^1, \tilde{a}_{t+2}^2\}_{t=0}^{\infty}$ and $\{r_{t+1}\}_{t=0}^{\infty}$ s.t. $\hat{c}_t^i = \tilde{c}_t^i$ $\forall i \; \forall t$.

And vice versa

Proof:- (Three Steps)

1) SM budget constraint implies AD budget constraint (if the prices are right).

2) If there is a ADE $\Rightarrow$ $\exists$ SME with the same allocation.

3) If there is a SME $\Rightarrow$ $\exists$ ADE which supports the same allocation.

Step 1 :

let $1+\tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$, $\hat{p}_0 = 1$

$$c_0^i + \frac{a_1^i}{1+\tilde{r}_1} = e_0^i$$ Period $t=0$.

$t=1$ $$c_1^i + \frac{a_2^i}{1+\tilde{r}_2} = e_1^i + a_1^i$$

[Putting the value of a_i in the previous equation.]

$$c_0^i + \frac{c_1^i}{1+\tilde{r}_1} + \frac{a_2^i}{(1+\tilde{r}_1)(1+\tilde{r}_2)} = e_0^i + \frac{e_1^i}{1+\tilde{r}_1}$$

Continuing until T :

$$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=1}^{t}(1+\tilde{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\prod_{j=1}^{t}(1+\tilde{r}_j)}$$

Note that $$\prod_{j=1}^{t}(1+r_j) = \frac{\hat{p}_0}{\hat{p}_1} \cdot \frac{\hat{p}_1}{\hat{p}_2} \cdots \frac{\hat{p}_{t-1}}{\hat{p}_t}$$

$$= \frac{\hat{p}_0}{\hat{p}_t} = \frac{1}{\hat{p}_t}$$

$$\sum_{t=0}^{T} p^t c_t^i + \frac{a^i_{T+1}}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{T} \hat{p}_t e_t^i$$

5 sep

$$T \to \infty \Rightarrow \sum_{t=0}^{\infty} \hat{p}_t e^i_t + \lim_{T\to\infty} \frac{a^i_{T+1}}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t e^i_t$$

Since we started with SM $\Rightarrow a_t^i \geq -\bar{A}_i$

$$\Rightarrow \lim_{T\to\infty} \frac{a^i_{T+1}}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} \geq \lim_{T\to\infty} \frac{-\bar{A}_i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = 0$$

$$\Rightarrow \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i \quad \text{if } \hat{p}_0 = 1$$

$$1+\tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$$

Step2 : If $\{\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ and $\{\hat{p}_t\}_{t=0}^{\infty}$ is ADE $\Rightarrow$

$\exists$ $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$, $\{(\tilde{a}_{t+1}^1, \tilde{a}_{t+1}^2)\}_{t=0}^{\infty}$ and

$\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$, s.t. $\hat{c}_t^i = \tilde{c}_t^i \ \forall t \ \forall i$.

(a) Resource constraint holds in ADE $\Rightarrow$ it will hold in SME

(b) Define

$$\tilde{a}_{t+1}^i = \frac{\sum_{\tau=1}^{\infty} \hat{p}_{t+\tau}(\hat{c}_{t+\tau}^i - e_{\tau+1}^i)}{\hat{p}_{t+1}}$$

How is this obtained in AD?

$\tau \to$ tao as time is fixed here so another indicator is used

Show that consumption allocations $\hat{c}$ along with asset holdings so defined satisfy SM budget constraint.

$$\hat{c}_t^i + \frac{\tilde{a}_{t+1}^i}{1+\tilde{r}_{t+1}} = e_t^i + \tilde{a}_t^i$$

$$\tilde{c}_t^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(\hat{c}_{t+\tau}^i - e_{t+\tau}^i)}{\hat{p}_{t+1}(1+\tilde{r}_{t+1})} = e_t + \frac{\sum_{\tau=1}^{\infty} \hat{p}_{t+\tau-1}(\hat{c}_{t+\tau-1}^i - \hat{e}_{t+\tau-1}^i)}{\hat{p}_t}$$

$$\Rightarrow \hat{c}^i_t + \frac{\sum_{\tau=1}^{\infty} \hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau} - e^i_{t+\tau}\right)\hat{p}_{t+1}}{\hat{p}_{t+1}}$$

$$= e^i_t + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau-1}\left(\hat{c}^i_{t+\tau-1} - e^i_{t+\tau-1}\right)}{\hat{p}_t}$$

$$\Rightarrow \hat{c}^i_t = e^i_t + \frac{\hat{p}_t\left(\hat{c}^i_t - e^i_t\right)}{\hat{p}_t}$$

$$\Longleftrightarrow \hat{c}^i_t = e^i_t + \hat{c}^i_t - e^i_t$$

We need to show NPG holds:

Recall $\tilde{a}^i_{t+1} = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau} - e^i_{t+\tau}\right)}{\hat{p}_t} \geq$

$$\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}}{\hat{p}_t}\left(-e^i_{t+\tau}\right) > -\infty$$

Step3 : Is $\hat{C}$ maximizing utility of agent in SME.

Suppose not $\Rightarrow \exists$ another affordable allocation which delivers strictly higher utility.

From step1: we know that

$$SM\ BC \Rightarrow AD\ BC$$

So that alternative allocation would be affordable in AD world but it was not chosen. $\otimes$

$\nexists$ such allocation.

Overlapping-generations model.

time is infinite $t = 1, 2, \ldots$

Every period there are 2 generations living simultaneously: young & old

Endowment of generation t is (e^t_t, e^t_{t+1})

Consumption of generation t is (c^t_t, c^t_{t+1})

generation \ period	1	2	3	4
0	(c^0_1, e^0_1)			
1	(c^1_1, e^1_1)	(c^1_2, e^1_2)		
2		(c^2_2, e^2_2)	(c^2_3, e^2_3)	
3			(c^3_3, e^3_3)	…
4				

Defintion: An allocation is cons^m for the initially old & cons^m (c_t^t, c_{t+1}^t) for all generations $t \geq 1$.

Agents derive utility from

$$u_t(c_t^t, c_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t), \quad \beta \in (0,1)$$

Initially old prefer more consumption to less.

<u>Def</u>: An allocation $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$ is feasible if

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \quad \forall t$$

Def: An allocation $\hat{c}_1^0, \{\hat{c}_t, \hat{c}_{t+1}\}_{t=1}^{\infty}$ is <u>pareto efficient</u> if it is <u>feasible</u> and there is no other feasible allocation $\tilde{c}_1^0, \{\tilde{c}_t^t, \tilde{c}_{t+1}^t\}_{t=1}^{\infty}$, such that

$$u_t(\tilde{c}_t^t, \tilde{c}_{t+1}^t) \geq u(\hat{c}_t^t, \hat{c}_{t+1}^t) \quad \forall t$$

$$u(\tilde{c}_1^0) \geq u(\hat{c}_1^0)$$

with atleast one inequality being strict.

Sep 7 :-

Arrow Debreu Equilibrium:-

Def: An ADE is the price system $\{p_t\}_{t=0}^{\infty}$ and an allocation $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$ s.t.

1) for the initially old, given p_1, e_1^0 solves

$$u_0(c_1^0) \to \max_{c_1^0}$$

s.t. $p_1 c_1^0 \le p_1 e_1^0 + m$

$c_1^0 \ge 0$

Outside money
"fiat money"
Can be negative or positive
↓
debt
Comes from outside the model.

2) Given the prices, allocation c_t^t, c_{t+1}^t solves the generation's t problem:

$$u(c_t^t) + \beta v(c_{t+1}^t) \to \max_{c_t^t\, c_{t+1}^t}$$

s.t. $p_t c_t^t + p_{t+1} c_{t+1}^t \le p_t e_t^t + p_{t+1} e_{t+1}^t$

$c_t^t,\ c_{t+1}^t \geq 0$

3) Markets clear $\forall t$

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t$$

Remark: Without fiat money, equilibrium is autarky: $\begin{cases} c_t^t = e_t^t \\ c_{t+1}^t = e_{t+1}^t \end{cases} \quad \forall t$

Sequential Markets Equilibrium

System of interest rates $\{r_{t+1}\}_{t=0}^{\infty}$, and allocations $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$ s.t.

1) Given r_1, the allocation for the initially old solves

$$u(c_1^0) \longrightarrow \max_{c_1^0}$$

$$\text{s.t.} \quad c_1^0 \leq e_1^0 + (1+r_1)\, m$$

$$c_1 \geq 0$$

2) Given $\{r_{t+1}\}_{t=1}^{\infty}$, c_t^t, c_{t+1}^t solves generation t's problem

$$u(c_t^t) + \beta u(c_{t+1}^t) \to \max_{c_t^t,\, c_{t+1}^t}$$

s.t. $c_t^t + s_{t+1}^t \leq e_t^t$ $\boxed{\text{savings}}$

$$c_{t+1}^t \leq e_t^t + (1+r_{t+1}) s_{t+1}^t$$

$$e_t^t,\ e_{t+1}^t \geq 0$$

3) Markets clear

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \quad \forall t$$

It is assumed that contracts are enforcable. i.e. if someone borrows in t, they will return the money in pd. t+1.

Equilibrium on asset market

Budget constraints in $t=1$

$$c_1^0 = e_1^0 + (1+r_1)m$$
$$c_1^1 + s_2^1 = e_1^1$$

Add: $\underbrace{c_1^0 + c_1^1}_{} + s_2^1 = e_1^0 + e_1^1 + (1+r_1)m$

$= e^0 + e^1$ (mkt clearing)

$\Leftrightarrow \quad s_2^1 = (1+r_1)m$

Repeat for $t=2$

$$c_2^1 = e_2^1 + (1+r_2)s_2^1$$

$$c_2^2 + s_3^2 = e_2^2$$

$\Rightarrow \quad s_3^2 = (1+r_2)s_2^1 = (1+r_2)(1+r_1)m$

$\Rightarrow \quad s_{t+1}^t = \prod_{j=1}^{t}(1+r_j)m$

Remark: if $1+r_p > 1$, $s^t_{t+1} \to +\infty$

(this cannot be part of eq^n)

Remark = asset market clears (Walras Law)

$\therefore$ Initially old use the fiat money to borrow endowments from the ~~young~~ next generation. And then they do it for the future generation etc.

Equivalence

Equivalence of ADE & SME.

ADE	SME
IO: $p_1 c_1^0 = p_1 e_1^0 + m$	IO: $c_1^0 = e_1^0 + (1+r_1) m$
t: $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t$	t: $c_t^t + s_{t+1}^t = e_t^t$ $c_{t+1}^t = e_{t+1}^t + (1+r_{t+1}) s_{t+1}^t$
$c_1^0 = e_1^0 + \frac{m}{p_1}$	$s_{t+1}^t = \frac{c_{t+1}^t}{1+r_{t+1}} - \frac{e_{t+1}^t}{1+r_{t+1}}$
$c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t = e_t^t + \frac{p_{t+1}}{p_t} \cdot e_{t+1}^t$	$\Rightarrow c_t^t + \frac{c_{t+1}^t}{1+r_{t+1}} = e_t^t + \frac{e_{t+1}^t}{1+r_{t+1}}$
	IO: $c_1^0 = e_1^0 + (1+r_1) m$

Theorem: If $\exists$ ADE $\Rightarrow$ $\exists$ SME, and if $\exists$ SME $\Rightarrow$ $\exists$ ADE with the same consumption allocation.

Proof: Suppose $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$ and $\{p_t\}$ is ADE. then $\exists$ SME with the same allocation, and

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} \qquad \Bigg| \qquad 1 + r_1 = \frac{1}{p_1}$$

If $\{r_{t+1}\}_{t=0}^{\infty}$ and allocations constitute SME, then

$$\boxed{\begin{aligned} &p_1 = 1 \\ &p_{t+1} = \frac{p_t}{1+r_{t+1}} \end{aligned}}$$

Excess Demand and offer curves:

Assume $u(c) = \log(c)$

$$\log(c_t^t) + \beta \log(c_{t+1}^t) \to \max_{c_t^t,\, c_{t+1}^t}$$

$$\begin{cases} p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t \\ c_t^t \geq 0 \\ c_{t+1}^t \geq 0 \end{cases}$$

$$\mathcal{L} = \log(c_t^t) + \beta \cdot \log(c_{t+1}^t) + \lambda\left[p_t e_t^t + p_{t+1} e_{t+1}^t - p_t c_t^t - p_{t+1} c_{t+1}^t\right]$$

FOC: $$\frac{1}{c_t^t} = p_t \lambda$$

$$\frac{\beta}{c_{t+1}^t} = p_{t+1} \lambda$$

Take the ratio: $$\frac{\beta c_t^t}{c_{t+1}^t} = \frac{p_{t+1}}{p_t}$$

$$\Rightarrow c_{t+1}^t = \beta c_t^t \cdot \frac{p_t}{p_{t+1}}$$

- who is ~~trading~~
- when
- what are they trading

Description of equilibrium.

$\boxed{\text{12 September}}$

Excess Demand & offer curves :-

$$(e^t_1, e^t_{t+1}) = (e_1, e_2)$$

Utility function is $\log(c^t_t) + \beta . \log(c^t_{t+1})$

Consider ~~A~~DE:

Problem for generation t :

$$\log(c^t_t) + \beta . \log(c^t_{t+1}) \to \max_{c^t_t,\, c^t_{t+1}}$$

s.t. $p_t c^t_t + p_{t+1} c^t_{t+1} \leq p_t e_1 + p_{t+1} e_2$

$c^t_t \geq 0, \quad c^t_{t+1} \geq 0$

$$\mathcal{L} = \log(c_t^t) + \beta . \log(c^t_{t+1}) + \lambda\left[p_t e_1 + p_{t+1} e_2 - p_t c_t^t - p_{t+1} c^t_{t+1}\right]$$

FOC: (c_t^t) : $\dfrac{1}{c_t^t} = \lambda p_t$

(c^t_{t+1}) : $\dfrac{\beta}{c^t_{t+1}} = \lambda p_{t+1}$

$$\Rightarrow \quad \frac{1}{c_t^t p_t} = \frac{\beta}{c^t_{t+1} p_{t+1}}$$

$$\Leftrightarrow \quad c^t_{t+1} = \beta . c^t . \frac{p_t}{p_{t+1}}$$

Invoke budget constraint :-

$$p_t c_t^t + p_{t+1} c^t_{t+1} = p_t e_1 + p_{t+1} e_2$$

Divide by p_t:

$$\Rightarrow \quad c_t^t + \frac{p_{t+1}}{p_t} \underbrace{c^t_{t+1}}_{\beta . c_t^t \frac{p_t}{p_{t+1}}} = e_1 + \frac{p_{t+1}}{p_t} . e_2$$

$$\Rightarrow \quad c_t^t + \beta c_t^t = e_1 + \frac{p_{t+1}}{p_t} e_2$$

$$\Rightarrow \quad c_t^t(1+\beta) = e_1 + \frac{p_{t+1}}{p_t} e_2$$

For initially old, $\quad p_1 c_1^0 = p_1 e_2 + m$

$$c_1^0 = e_2 + \frac{m}{p_1}$$

(*) We cannot normalize price system ~~p~~ here as we did in A-D. This is because of the fiat money. If there is ~~a~~ fiat money, then p_1 directly affects c_1^0, & thus we cannot just normalize it.

Excess Demand : An excess demand of consumer i in an endowment economy is the difference between the demand of that agent & his endowment.

Example : excess demand for initially old is ~~$\emptyset$~~

$$z_1^0 = c_1^0 - e_2 = \frac{m}{p_1}$$

generation born in pd 0.

initially old

Derive excess demand functions z_t^t, z_{t+1}^t

$$z_t^t = c_t^t - e_1 = \frac{1}{1+\beta}\left[e_1 + \frac{p_{t+1}}{p_t} e_2\right] - e_1$$

$$z_{t+1}^t = c_{t+1}^t - e_2 = \beta \frac{p_t}{p_{t+1}}\left[\frac{1}{1+\beta}\left(e_1 + \frac{p_{t+1}}{p_t} e_2\right)\right] - e_2$$

On simplifying: –

$$z_t^t = \frac{1}{1+\beta}\left[e_1 + \frac{p_{t+1}}{p_t} e_2\right] - e_1 \qquad (*)$$

$$z_{t+1}^t = \frac{\beta}{1+\beta}\left(\frac{p_t}{p_{t+1}} e_1 + e_2\right) - e_2 \qquad (**)$$

Def^n: An ADE in OG model is $\{p_t\}_{t=1}^{\infty}$ & allocations, s.t.:

$$z_t^{t-1} + z_t^t = 0 \qquad \forall t \geq 1.$$

The algorithm to find equilibrium using the excess demand functions:

Step 1: Pick p_1. Find $z_1^0 = \frac{m}{p_1}$

Step 2: Given z_1^0, find z_1^1 from $z_1^0 + z_1^1 = 0$

Step 3: find p_2 from $(*)$

Step 4: Find z_2^1 from $(**)$

⋮

Reconstruct the entire sequence of prices.

You can pick any p_1, so there are infinitely many solutions.

Offer Curve

from $(*)$

$$\frac{p_{t+1}}{p_t} = \frac{(1+\beta)(z_t^t + e_1) - e_1}{e_2}$$

Plug on to $(**)$

$$z_{t+1}^t = \frac{\beta}{1+\beta}\left(\frac{e_1 e_2}{(1+\beta)(z_t^t + e_1) - e_1} + e_2\right) - e_2$$

Remark: OC passes through the origin.

Suppose $z_t^t = 0 \Rightarrow z_{t+1}^t = \frac{\beta}{1+\beta}\left(\frac{e_1 e_2}{(1+\beta)e_1 - e_1} + e_2\right) - e_2$

$$= \frac{\beta}{1+\beta}\left(\frac{\cancel{e_1} e_2}{\beta \cancel{e_1}} + e_2\right) - e_2$$

$$= \frac{\beta}{1+\beta}\left(\frac{e_2 + \beta e_2}{\beta}\right) - e_2$$

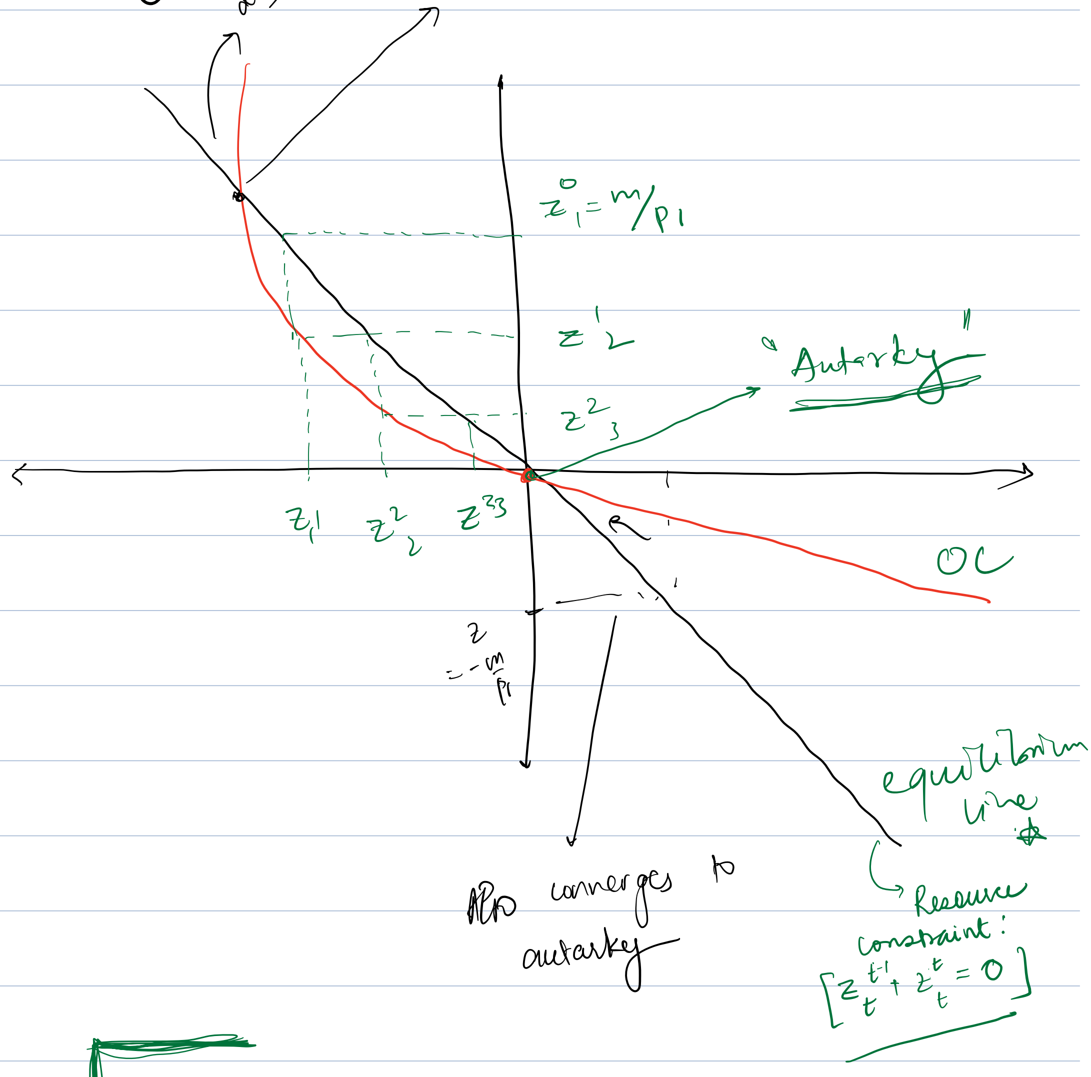
$$= \frac{\beta}{1+\beta} \frac{(1+\beta)e_2}{\beta} - e_2$$

$$= 0$$

diverges

This is equilibrium only if you start here

$z_t^{t-1} + z_t^t = 0$



Ne

Personal Note:

(1) Offer curve plots  equation i.e. r/p b/w z_t^t & z_{t+1}^t

(2) The budget line plots the r/p b/w z_t^t & z_t^{t-1}.

This is how we trace out the whole sequence.

Def^n of offer curve :→ A pt. on the offer curve is an optimal excess dd f^n for some $\frac{p_{t+1}}{p_t} \in (0,\infty)$.

First welfare theorem does not apply:

$$U = \log(c_t^t) + \log(c_{t+1}^t), \quad \beta = 1$$

$$(e_t^t, e_{t+1}^t) = (2, 1)$$

$$m = 0$$

$$\frac{3}{2} + \frac{3}{2} = \cancel{\frac{6}{2}}\ 3$$

$$CE = \begin{cases} c_t^t = 2 \\ c_{t+1}^t = 1 \end{cases}$$

Consider alternative $(\tilde{c}_{t+1}^t, \tilde{c}_{t+1}^t) = \left(\frac{3}{2}, \frac{3}{2}\right)$

Initially old are better off at $\tilde{c}$.

Any other generation:

$$u((2,1)) = \log(2) + \log(1) = \log(2) < \log\left(\frac{9}{4}\right)$$

$$= u\left(\left(\frac{3}{2}, \frac{3}{2}\right)\right)$$

Overlapping Generation with production :—

Environment $t = 1, 2, \ldots\ldots$

Generations live for 2 time periods

Utility function

$$U = u(c_t^t, \underbrace{l_t^t}_{\text{labour}}) + \beta u(c_{t+1}^t, l_{t+1}^t)$$

$u'_c > 0$

$u'_l < 0$ (people don't want to do labor)

Time endowment $(\underbrace{\bar{l}_1}_{\text{young}}, \underbrace{\bar{l}_2}_{\text{old}})$

Production : There is a representative firm with access to technology

$$y_t \leq F(\underbrace{k_t}_{\text{Capital}}, \underbrace{l_t}_{\text{labor}})$$

Technology has constant returns to scale. &

Capital is the only asset that agents use to transfer resources over time.

Initially old are endowed with $\bar{k} \geq 0$.

14 Sep

$$f(\lambda k, \lambda l) = \lambda F(k, l)$$

Assets : 2 assets.

- Physical Capital, $k \geq 0$

  $k_1^0 = \bar{k}$ (endowment of initially old)

- bonds b_{t+1}^t

Initially old may have fiat money m. $m \gtrless 0$

Capital depreciates at a rate $\delta \in [0, 1]$

Def: A SME for this economy is

$\tilde{c}_1^0, \tilde{l}_1^0, \{ \tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{l}_t^t, \tilde{l}_{t+1}^t, \tilde{k}_{t+1}^t, \tilde{b}_{t+1}^t \}$

allocation for the firm $\{ \tilde{y}_t, \tilde{k}_t^f, \tilde{l}_t^f \}_{t=1}^{\infty}$ and

a prices $\{ \tilde{w}_t, \tilde{r}_t^k, \tilde{r}_t^b \}$ s.t.

1) Given the prices, $\tilde{c}_1^0, \tilde{l}_1^0$ solves

$$u(c_1^0, l_1^0) \longrightarrow \max_{c_1^0, e_1^0}$$

depreciation of capital.

$$\text{s.t.} \begin{cases} c_1^0 \leq \tilde{w}_1 l_1^0 + \underbrace{(1-\delta)}_{}\bar{k} + \tilde{r}_1^k \bar{k} + (1+r_1^b) m \\ c_1^0 \geq 0 \\ 0 \leq l_1^0 \leq \bar{l}_2 \end{cases}$$

2) Given the prices, $\{\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{l}_t^t, \tilde{l}_{t+1}^t, \tilde{k}_{t+1}^t, \tilde{b}_{t+1}^t\}_{t=1}^{\infty}$ solves generation t's problem:

$$u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) \to \max$$

$$\text{s.t.} \quad c_t^t + k_{t+1}^t + b_{t+1}^t \leq \tilde{w}_t l_t^t$$

$$c_{t+1}^t \leq \tilde{w}_{t+1} l_{t+1}^t + \tilde{r}_{t+1}^k k_{t+1}^t + (1-\delta) k_{t+1}^t + (1+\tilde{r}_{t+1}^b) \cdot b_{t+1}^t$$

$$c_t^t \geq 0, \quad c_{t+1}^t \geq 0$$

$$0 \leq l_t^t \leq \bar{l}_1, \quad 0 \leq l_{t+1}^t \leq \bar{l}_2$$

3) Given the prices, $\{\tilde{y}_t, \tilde{k}^f_t, \tilde{\ell}^f_t\}$ solves the firm's problem :

$$\begin{cases} y_t - \tilde{w}_t \ell_t - \tilde{r}^k_t k_t \longrightarrow \max\limits_{k,\ell,y} \\ \text{s.t. } \; y_t \leq F(k_t, \ell_t) \end{cases}$$

4) Markets have to clear.

$$\tilde{c}^{t-1}_t + \tilde{c}^t_t + \tilde{k}^t_{t+1} - (1-\delta)\tilde{k}^{t-1}_t = \tilde{y}_t$$

$$\tilde{\ell}^f_t = \tilde{\ell}^{t-1}_t + \tilde{\ell}^t_t \quad \text{(labour demanded by firm} = \text{labour supplied by the 2 gens.)}$$

$$\tilde{b}^t_{t+1} = (1+\tilde{r}^b_{t+1})\,\tilde{b}^{t-1}_t, \quad \tilde{b}^1_2 = (1+\tilde{r}^b_2)\,m$$

$$\hat{k}^f_t = \hat{k}^{t-1}_t$$

Def: An ADE is $\hat{c}^0_1, \hat{\ell}^0_1, \{\hat{c}^t_t, \hat{c}^t_{t+1}, \hat{\ell}^t_t, \hat{\ell}^t_{t+1}, \hat{k}^t_{t+1}\}^{\infty}_{t=1}$, allocation for firms $\{\hat{y}_t, \hat{k}^f_t, \hat{\ell}^f_t\}_{t=1}^{\infty}$ and prices $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}^{\infty}_{t=1}$ s.t.

1) Given prices, $\hat{c}^0_1, \hat{\ell}^0_1$ solves the I-O (initially old) problem :

$$\max (c^0_1, \ell^0_1) \longrightarrow \max_{c^0_1\, \ell^0_1}$$

such that :-

$$\hat{p}_1 c_1^0 \leq \hat{w}_1 l_1^0 + \hat{r}_1 \bar{k} + \hat{p}_1 (1-\delta)\bar{k}$$

$$c_1^0 \geq 0$$

$$0 \leq l_1^0 \leq \bar{l}_2$$

2) Given the prices, $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{l}_t^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t\}_{t=1}^{\infty}$ solves $u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) \to \max$

s.t.

$$\hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t + \hat{p}_t k_{t+1}^t \leq \hat{w}_t l_t^t + \hat{w}_{t+1} l_{t+1}^t + \hat{p}_{t+1}(1-\delta) k_{t+1}^t + \hat{r}_{t+1} k_{t+1}^t$$

$$c_t^t \geq 0, \quad c_{t+1}^t \geq 0$$

$$0 \leq l_t^t \leq \bar{l}_1 \qquad 0 \leq l_{t+1}^t \leq \bar{l}_2$$

3) Given the prices $\{\hat{y}_t, \hat{k}_t^f, \hat{l}_t^f\}$ solves the firm's problem

$$\hat{p}_t y_t - \hat{w}_t l_t - \hat{r}_t^k k_t \to \max_{k, l, y}$$

s.t. $y_t \leq F(k_t, l_t)$

4. Market Clear:

- $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = \hat{y}_t$ (final good)
- $\hat{l}_t^f = \hat{l}_t^{t-1} + \hat{l}_t^t$ (labor)
- $\hat{k}_t^f = \hat{k}_t^{t-1}$ (capital market)

SME

$$\mathcal{L} = u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) + \lambda_t^t [w_t l_t^t - c_t^t - k_{t+1}^t - b_{t+1}^t] + \lambda_{t+1}^t [w_{t+1} l_{t+1}^t + (1-\delta) k_{t+1}^t + r_{t+1}^t k_{t+1}^t + (1+r_{t+1}^b) b_{t+1}^t - c_{t+1}^t]$$

FOC:

(c_t^t) : $u'_c(c_t^t, l_t^t) = \lambda_t^t$

(c_{t+1}^t) : $\beta u'_c(c_{t+1}^t, l_{t+1}^t) = \lambda_{t+1}^t$

(k_{t+1}^t) : $-\lambda_t^t + \lambda_{t+1}^t (1-\delta+r_{t+1}^k) = 0$

(b_{t+1}^t) : $-\lambda_t^t + \lambda_{t+1}^t (1+r_{t+1}^b) = 0$

[1] & [2] $\rightarrow$ [3] : Gives us Euler Equation

$$u'_c(c_t^t, l_t^t) = \beta u'_c(c_{t+1}^t, l_{t+1}^t)(1-\delta+r_{t+1}^k)$$

[3] + [4] $\Rightarrow$ $r_{t+1}^b = r_{t+1}^k - \delta$ (No Arbitrage cond^n)

19 Sep

{ ~~For~~ mid-term : until everything so far. In the class }

Neo-classical growth model

(stripped down version)

Environment :-

Agents : One infinitely lived agent

$$u(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$\beta \in (0,1)$

Firm : $y_t = F(k_t, l_t)$

Resource Constraint : $y_t = c_t + i_t$ (Output can be consumed or invested.)

Law of motion of capital : $k_{t+1} = (1-\delta)k_t + i_t$,

$k_t \geq 0$

i_t can be negative.

Endowments

1) $\bar{k}_0 \to$ endowment of capital at $t=0$.

2) Every pd there is 1 unit of time endowment.
↳ agent can spend on labor or leisure.

Information

1) Full information.

Environment:

Agents will supply labor inelastically.

$l_t = 1$. (As the utility f^n does not have leisure in it.)

Social Planner's Problem.

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \longrightarrow \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}}$$

[More general way of writing assuming we have a general utility f^n.]

s.t. $c_t + i_t = F(k_t, l_t)$

$k_{t+1} = (1-\delta) k_t + i_t$

$0 \leq l_t \leq 1, \quad k_t \geq 0, \quad c_t \geq 0$

$k_0 = \bar{k}_0$

Assumptions on $u(\cdot)$ and $F(\cdot)$

- $u(\cdot)$ is strictly increasing and strictly concave

$$u'_c(\cdot) > 0 \quad ; \quad u''_{cc}(\cdot) < 0$$

- Inada conditions :

$$\lim_{c \to 0} u'c = \infty \quad ; \quad \lim_{c \to \infty} u'c = 0$$

- Production Technology : $F(k, \ell)$

1) $F(0, \ell) = F(k, 0) = 0$

2) $F'_k > 0, \quad F'_\ell > 0$

3) $\lim_{k \to 0} F'_k = \infty \quad , \quad \lim_{\ell \to 0} F'_\ell = \infty$

$$\lim_{k \to \infty} F'_k = \lim_{k \to \infty} F'_\ell = 0$$

4) $F(\cdot)$ is strictly concave.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[F(k_t, 1) + (1-\delta) k_t - k_{t+1} - c_t \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) = \lambda_t \qquad [1]$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1} \left[F'_k(k_{t+1}, 1) + (1-\delta) \right] \qquad [2]$$

In the Lagrangian capital is infinite sum so we look at 2 eqns.

Combine [1] and [2];

$$\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1}) \left[F'_k(k_{t+1}, 1) + (1-\delta) \right]$$

$$\Longleftrightarrow \quad u'(c_t) = \beta u'(c_{t+1}) \left[F'_k(k_{t+1}, 1) + 1 - \delta \right]$$

<u>Steady state</u>:

Is an allocation where $c_t = c_{t+1} = c^{ss}$

$k_t = k_{t+1} = k^{ss}$

From Euler's eq^n:

$$u'(c^{ss}) = \beta u'(c^{ss}) \left[F'_k(k^{ss}, 1) + 1 - \delta \right]$$

$$\boxed{\frac{1}{\beta} = F'_k(k^{ss}, 1) + 1 - \delta}$$

From RC: $c_t = f(k_t, 1) + (1-\delta) k_t - k_{t+1}$

$$c_{t+1} = F(k_{t+1}, 1) + (1-\delta) k_{t+1} - k_{t+2}$$

Plug in EE:

$$u(F(k_t, 1) + (1-\delta) k_t - k_{t+1}) =$$
$$= \beta u'(F(k_{t+1}, 1) + (1-\delta) k_{t+1} - k_{t+2}) \times (F'_k(k_{t+1}, 1) + 1 - \delta)$$

This 2nd order difference ? equation ⇒ need 2 boundary conditions:

1) Initial condition $\bar{k}_0$
2) Transversality condition (TVC)

$$\lim_{t \to \infty} \lambda_t k_{t+1} = 0 \quad \left(\text{Value of capital converges to 0.} \right)$$

From FOC to SPP we know,

$$\lambda_t = \beta^t u'(c_t)$$

$\Rightarrow$ TVC is $\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0$