Micro theory Aug 23, 2023

Grading: 40% exam1
40% exam2
20% Problem sets

Content:
I) Firm theory (Rec: Varian)
II) Consumer theory (Rec:Mas-collel)
III) Choice under uncertainty (Rec: Mas-collel)

Firm Theory: Assumptions
- Firms are price takers (no market power)
- Technology is exogenous
- Firms maximize profits

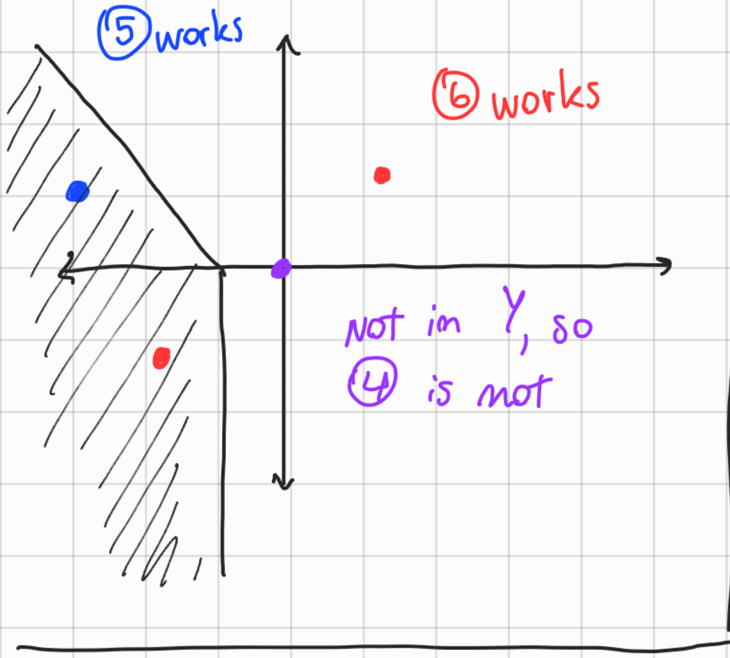
Technology:
n commodities y =(y_1 , y_2, ... , y_n)
y_i < 0 : Input
y_i > 0 : Output
Production set: Y subset R^n

Properties of production sets

1. $Y \neq 0$
2. Y is closed (includa bandury)
3. if $y \in Y$ and $y \geq 0$, then $y = 0$

always assumed

4. Shutdown is possible: $0 \in Y$
5. Free disposal: $y \in Y$ and $y' \in y_i$ then $y' \in Y$
6. Irreversibility: $y \in Y$, $y \neq 0$, then $y \notin Y$. $y(-1, 1)$ in out not the opposite $-y(1,-1)$



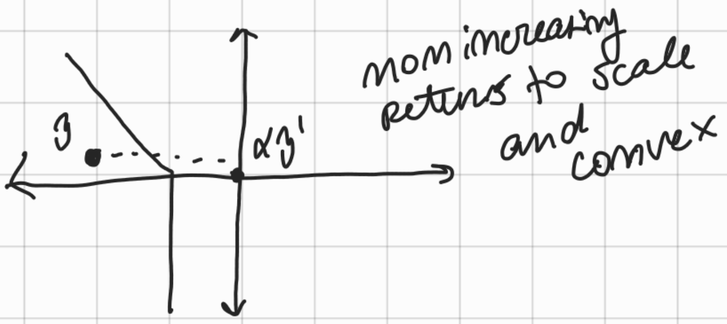
Returns to scale and convexity

i) Y has nonincreasing returns to scale if $y \in Y$
$\Rightarrow \alpha y \in Y \ \forall \alpha \in [0,1]$

ii) Nondecreasing: $y \in Y$
$\Rightarrow \alpha y \in Y \ \forall \alpha \geq 1$.

iii) Constant returns to scale: $y \in Y \to \alpha y \in Y \ \forall \alpha \geq 0$

Y is convex if $y, y' \in Y \Rightarrow \alpha y + (1-\alpha) y' \in Y \ \forall \alpha \in [0,1]$



<u>Theorem</u>: For any convex Y with $0 \in Y$, there is a constant returns production set $Y' \subseteq \mathbb{R}^{n+1}$ s.t
$Y = \{ y \in \mathbb{R}^n : (y, -1) \in Y' \}$

Idea: The farmer controls a subset of possible inputs

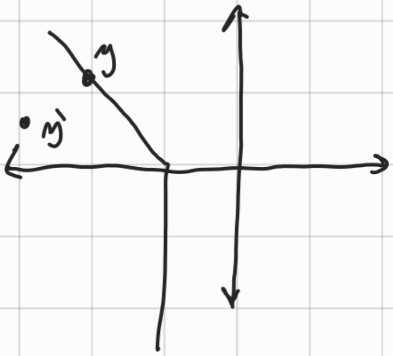
<u>proof</u>: Define $Y' = \{ y' \in \mathbb{R}^{n+1} : y' = \alpha(y, -1) \text{ for some } y \in Y \wedge \alpha \geq 0 \}$

Check: Y' is constant returns

Technological efficiency:

Production plan $y \in Y$ is tech efficient if there doesn't exist $y' \in Y$ s.t $y' > y$.

$\hookrightarrow y'_i \geq y_i \ \forall i$ and $y'_i > y$ for some i



lecture #2:

Aug 28, 2023

* Homework (on Canvas).

Transformation frontier (function):

→ Firm is a production set Y

→ Transformation function $T: \mathbb{R}^n \to \mathbb{R}$

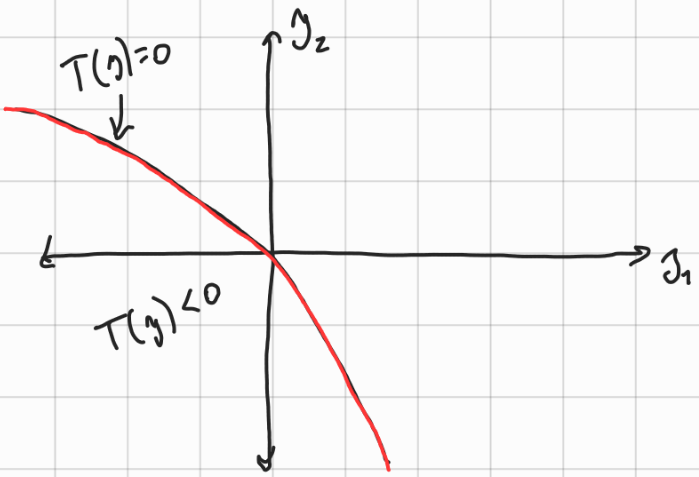
$T(y) < 0 \longleftrightarrow y$ is inefficient

$T(y) = 0 \longleftrightarrow y$ is efficient

$T(y) > 0 \longleftrightarrow y$ is infeasible.

$\Rightarrow Y = \{y \in \mathbb{R}^n : T(y) \leq 0\}$ production set.

$\Rightarrow \{y \in \mathbb{R}^n : T(y) = 0\}$ production frontier.



Lets fix goods $\bar{y}_3, \dots, \bar{y}_n$. Consider y_1, y_2

Define a function $y_2(y_1)$ by

$$T(y_1, \underbrace{y_2(y_1)}_{\text{implicit function of } y_1}, \bar{y}_3, \dots, \bar{y}_n) = 0$$

Differentiate both sides w.r.t. y_1

$$\underbrace{\frac{\partial T}{\partial y_1} + \frac{\partial T}{\partial y_2} \cdot \frac{dy_2}{dy_1}}_{\text{Chain Rule.}} = 0 \quad \Rightarrow \quad \frac{dy_2}{dy_1} = -\frac{\partial T / \partial y_1}{\partial T / \partial y_2}$$

* for any good : $$\frac{dy_k}{dy_j} = -\frac{\partial T / \partial y_j}{\partial T / \partial y_k} = \underbrace{-MRT_{j,k}(\bar{y})}_{\text{marginal rate of Transformation.}}$$

Special case : many inputs, one output

Inputs : $(x_1, \dots, x_n) \geq 0$

Output : $y \geq 0$

production function : $f(x)$

production set : $Y = \{ \underbrace{-x_1, \dots, -x_n, y}_{n+1.} \} : x_i \geq 0 \ \forall i, \ y \leq f(x_1, \dots, x_n)$

$T(z) = y - f(x)$, $z = (-x, y)$

eff. frontier is

$T(z) = 0$

$y - f(x) = 0$

$y = f(x)$

Example: Cobb-Douglas

→ 2 inputs, capital x_K and labor x_L

→ Output y.

→ $f(x_K, x_L) = x_K^{\alpha} x_L^{\beta}$, $\alpha + \beta = 1$.

→ $Y = \{(-x_K, -x_L, y): x_K, x_L \geq 0, y \leq x_K^{\alpha} x_L^{\beta}\}$

→ $T(-x_K, -x_L, y) = y - x_K^{\alpha} x_L^{\beta}$.

$$MRT_{(-x_K, y)} = -\frac{\partial T / \partial(-x_K)}{\partial T / \partial y} = -\frac{-\alpha x_K^{\alpha-1} x_L^{\beta}}{1} = \alpha x_K^{\alpha-1} x_L^{\beta} = \underbrace{\alpha \left(\frac{x_L}{x_K}\right)^{\beta}}_{\substack{\text{mg prod of } K \\ (\text{MRK})}}$$

MRT for capital and Labor

→ output remains constant.

$$MRT_{(-x_K),(-x_L)} = -\frac{\partial T / \partial(-x_K)}{\partial / \partial(-x_L)} = \frac{\alpha}{\beta}\left(\frac{x_L}{x_K}\right)$$

$$\frac{d(-x_L)}{d(-x_K)} = -MRT_{(-x_K)(-x_L)} = \frac{dx_L}{dx_K} = \frac{-\alpha}{\beta}\left(\frac{x_L}{x_K}\right)$$

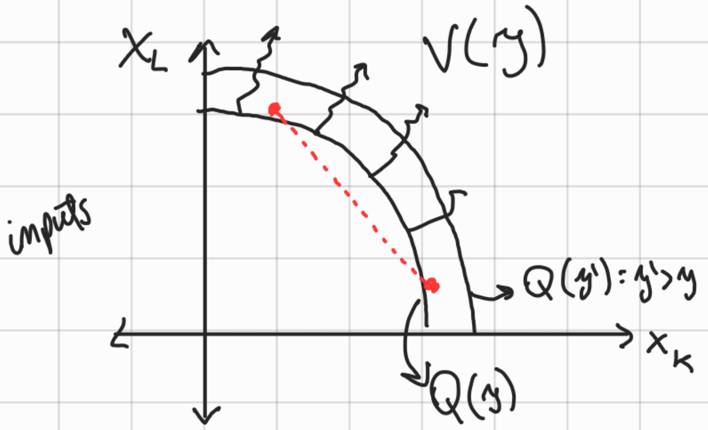
slightly Different interpretation (?).

Single output Fix y

$V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$ → Input Requirement set

$Q(y) = \{x \in \mathbb{R}^n : f(x) = y\}$ → Isoquant

Example: $f(x_1, x_2) = x_K^2 + x_L^2 = y$ (constant)



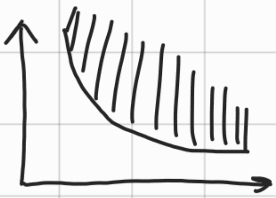
with a single output, we can define the mg rate of technical sustitution between 2 inputs

$$MRTS_{x_K, x_L} = \frac{\partial f / \partial x_K}{\partial f / \partial x_L}$$

$$\frac{\partial x_L}{\partial x_K} = -MRTS_{x_K, x_L} = -\frac{f_{x_K}}{f_{x_L}}$$

In our example, the production function is increasing in K (not usual) because V it is not convex.

$\Rightarrow$



Theorem: If Y is convex, then $V(y)$ is also convex

$(P \Rightarrow Q \iff \neg Q \Rightarrow \neg P)$.

proof (Direct): $(-x, y) \in Y$, $(-x', y) \in Y$ (two differents inputs that yield the same output)

Y is convex implies: $t(-x, y) + (1-t)(-x', y) \in Y$ (most of proofs will be by contradiction)

$$(-tx - (1-t)x', ty + (1-t)y) \in Y$$

$$(-tx - (1-t)x', y) \in Y$$

$$(-x'', y) \in Y. \quad (-x'' \text{ produces } y)$$

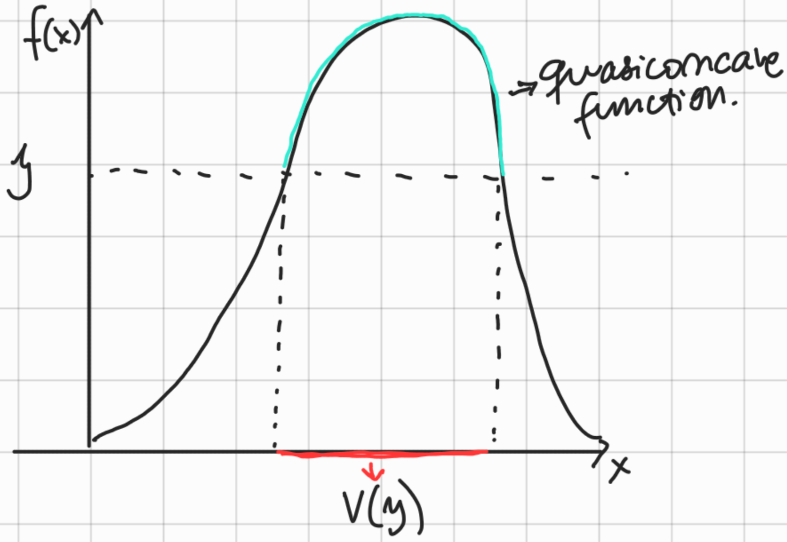
$\Rightarrow x'' \in V(y)$. meaning $V(y)$ is convex

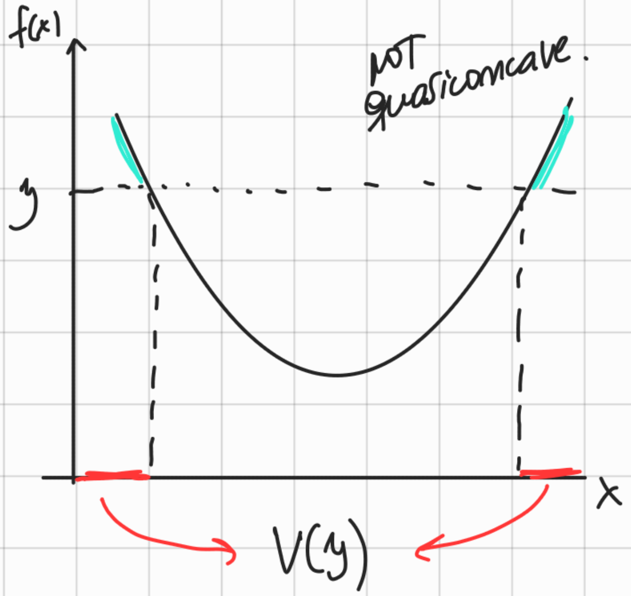
Theorem: $V(y)$ is convex IFF $f(x)$ is quasiconcave.

"**proof**": $V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$

↳ upper contour set of f

Def: Quasiconcave function is one that has convex upper contour sets.





Homogeneity and homotheticity.

A function $f: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree K if $f(tx) = t^k f(x)$ $\forall t > 0$

$k=0$: $f(tx) = f(x)$

$k=1$: $f(tx) = t\, f(x)$.

Example: Cobb Douglas $f(x_K, x_L) = x_K^{\alpha} x_L^{\beta}$, $\alpha + \beta = 1$

$$f(tK, tL) = (tK)^{\alpha} (t \cdot L)^{\beta}$$
$$= t^{\alpha} (K)^{\alpha} t^{\beta} \cdot (L)^{\beta}$$
$$= t\, K^{\alpha} L^{\beta}$$
$$\underbrace{f(tK, tL) = t \cdot f(K, L)}_{\text{constant Returns to scale}}$$

(if I double the input, I get double output).

$$MRTS = \frac{\alpha}{\beta} \frac{L}{K}$$

$$\frac{\alpha}{\beta} \frac{t \cdot L}{t \cdot K} = \frac{\alpha L}{\beta K}$$

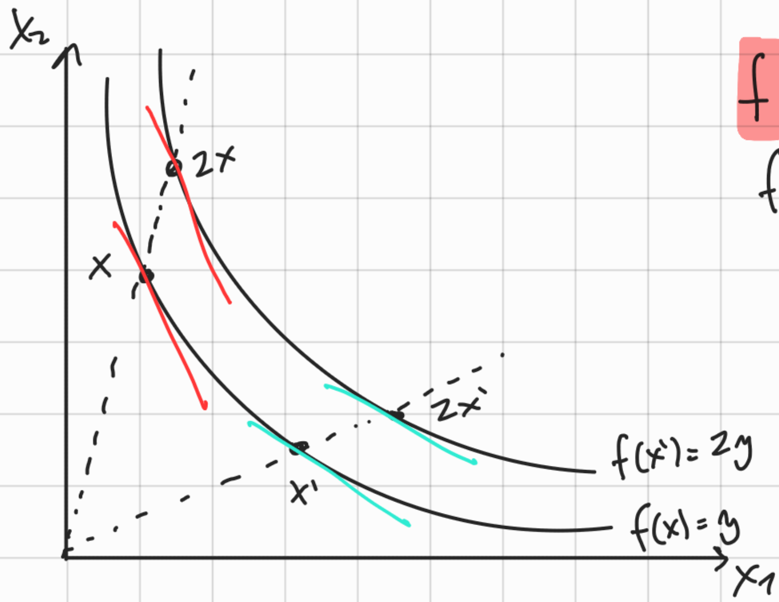
↓

MRTS doesn't change

⇒ MRTS is homogeneous of degree 0.

Theorem: If is hd-k, then the partial derivative f_{x_j} is hd-(k-1).

Proof:
$$f(t\cdot x) = t^k f(x)$$
$$t\cdot f_{x_i}(t\cdot x) = t^k f_{x_i}(x)$$
$$f_{x_i}(t\cdot x) = t^{k-1} f_{x_i}(x). //.$$



$\boxed{f \text{ is hd-1}}$

$f(2x) = 2f(x)$.

→ partial derivatives does not change.

Theorem: If a production function $f(x)$ is hd-1, then the associated MRTS is hd-0.

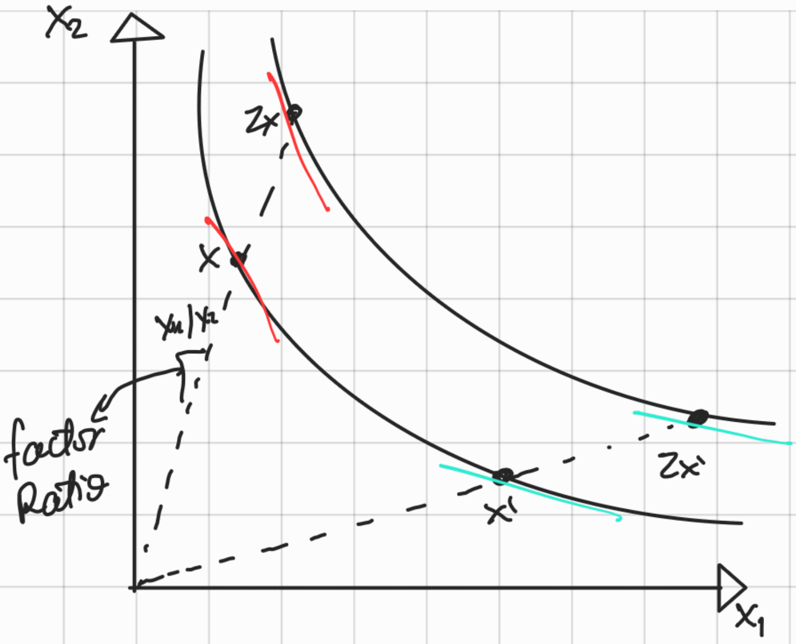
{Aug 30, 2023}

*Hd-1: $f(tx) = t\cdot f(x)$ $\forall t>0$

Proof:
$$MRTS_{ij}(\bar{x}) = \frac{f_{x_i}(\bar{x})}{f_{x_j}(\bar{x})}$$

$$MRTS_{ij}(t\cdot\bar{x}) = \frac{f_{x_i}(t\cdot\bar{x})}{f_{x_j}(t\cdot\bar{x})} = \frac{t\cdot f_{x_i}(\bar{x})}{t\cdot f_{x_j}(\bar{x})} = MRTS_{ij}(\bar{x})$$

Euler's Theorem.



$f(x) = f(x') = 10$
$f(2x) = f(2x') = 20$

A homothetic function: Is a positive monotonic transformation of an hd-1 function i.e., $f: \mathbb{R}^n \to \mathbb{R}$ is homothetic if $\exists$ a $g: \mathbb{R} \to \mathbb{R}$ strictly increasing and and $h: \mathbb{R}^n \to \mathbb{R}$ that is hd-1 s.t $f(x) = g(h(x))$.

$$\text{MRTS of } f = \frac{f_{x_j}}{f_{x_i}} = \frac{g'(h(x))\,h_{x_i}}{g'(h(x))\,h_{x_j}} = \frac{h_{x_i}}{h_{x_j}} \to hd\text{-}0$$

Returns to scale with a single output.

Production function f exhibits:

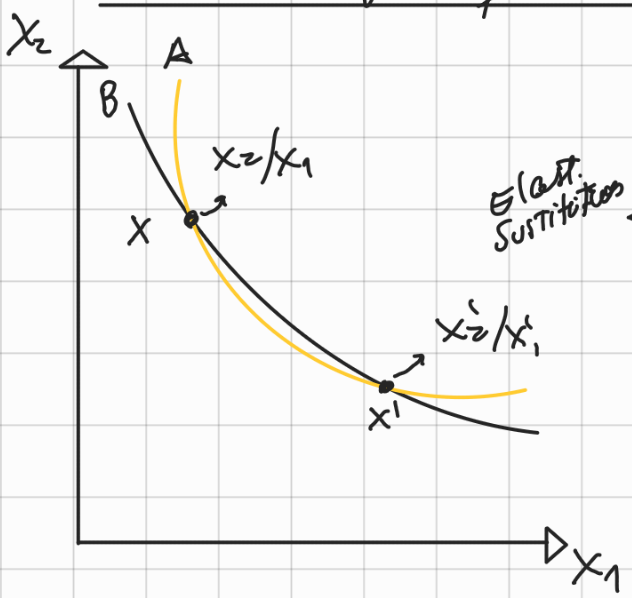
- Decreasing Returns to scale
  if $f(tx) \leq t\,f(x) \quad \forall t \geq 1$

- Increasing returns to scale
  if $f(tx) \geq t \cdot f(x) \quad \forall t \geq 1$

- Constant returns to scale
  if $f(tx) = t \cdot f(x) \quad \forall t \geq 0$

Curvature of isoquants:

Elasticity of complementarity. (for "large" jumps).



Elast. Substitution

$$\frac{1}{\sigma} = \frac{\dfrac{\Delta MRTS}{MRTS}}{\dfrac{\Delta(x_2/x_1)}{x_2/x_1}}$$

* Denominators are the same for A & B in example.

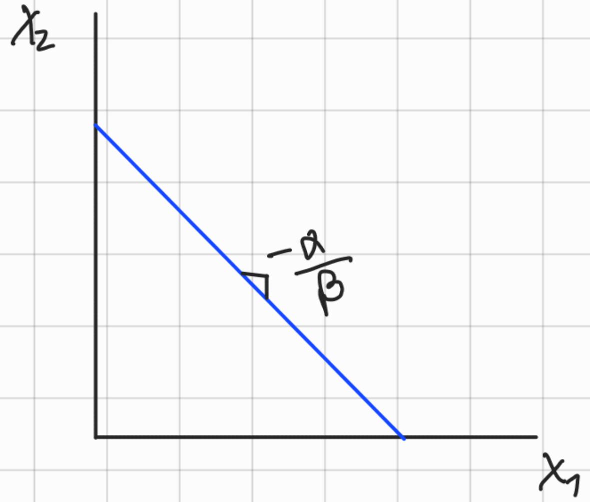
** Numerator is much higher for A

$$\frac{1}{\sigma_A} > \frac{1}{\sigma_B}$$

In the limit:

$$\frac{1}{\sigma} = \frac{x_2/x_1}{MRTS} \cdot \frac{d\,MRTS}{d(x_2/x_1)}$$

Perfect substitution: $f(x_1, x_2) = \alpha x_1 + \beta x_2$



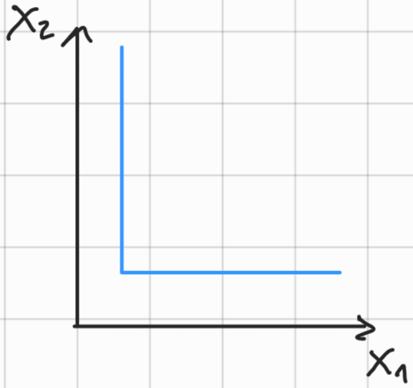
$\frac{1}{\sigma} = 0$, because MRTS is a fixed number.

Elasticity of substitution: $\sigma = \frac{MRTS}{x_2/x_1} \cdot \frac{d(x_2/x_1)}{d(MRTS)}$

↳ for perfect substitutes: $\sigma = \infty$

↳ Less curvature → bigger σ.

Perfect Complements: $f(x_1,x_2) = \min\{x_1, x_2\}$



$\sigma = 0$

For any two quantities, y, x, the elasticity of y with respect to x: $\boxed{\varepsilon_{yx} = \dfrac{d\log(y)}{d\log(x)}}$

* way to see this: $d\log(y) = \frac{1}{y}\, dy$ (Total derivative).

$d\log(x) = \frac{1}{x}\cdot dx$.

$\hookrightarrow$ $\varepsilon_{yx} = \dfrac{x}{y}\cdot\dfrac{dy}{dx}$ but $\dfrac{d\log(y)}{d\log(x)}$ is easier to remember.

<u>Example</u>: σ for Cobb-Douglas

MRTS: $\dfrac{\alpha}{\beta}\cdot\dfrac{x_L}{x_K}$ $\Rightarrow$ $\sigma = \dfrac{MRTS}{x_L/x_K}\;\dfrac{d\,(x_L/x_K)}{d\,MRTS}$.

a change variable to readability.

$\theta = \dfrac{\alpha}{\beta}\,\dfrac{x_L}{x_K} = MRTS$

$$\sigma = \frac{\theta}{\frac{\beta}{\alpha}\theta}\cdot\frac{d\left(\frac{\beta}{\alpha}\theta\right)}{d\theta} = \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1$$

Property of Cobb-Douglas.

(NOT the same as slope)

CES : Constant elasticity of subst. production.

$$f(x) = \left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{\frac{1}{\rho}} \quad , \quad \rho = \underbrace{0, 1, \infty}_{\text{subcases.}}$$

The Profit maximization function (PMP)

Recall assumptions
- Price takers
- max profits

$$\pi(p) = \max_{\text{s.t. } y \in Y} p \cdot y, \quad \text{where } p \cdot y = \sum_{i=1}^{n} p_j y_j$$

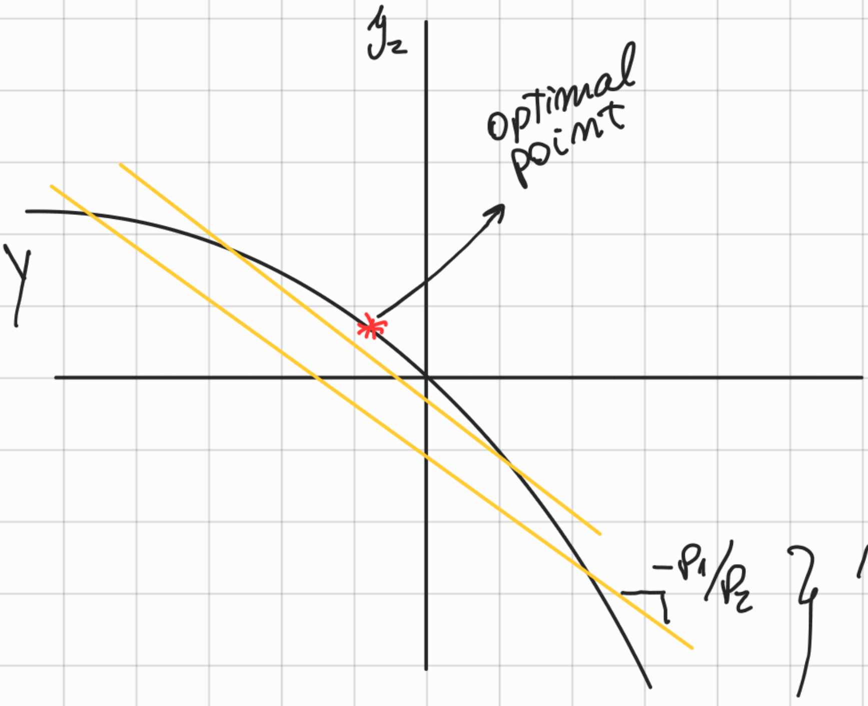
positive $y \Rightarrow$ output
negative $y \Rightarrow$ inputs

Equivalently, $\pi(p) = \max p \cdot y$
s.t. $T(y) \leq 0$

$\pi : \mathbb{R}^n \to \mathbb{R}$ is also called profit function.
Example of value function in constrained optimization
$p \cdot y$ is called the objective function.

$y^*(p) = \{ y \in Y : p \cdot y = \pi(p) \}$ is called the supply correspondence
↓
set valued mapping.

$y(p)$ could be multivalued (or $y(p) = \emptyset$).

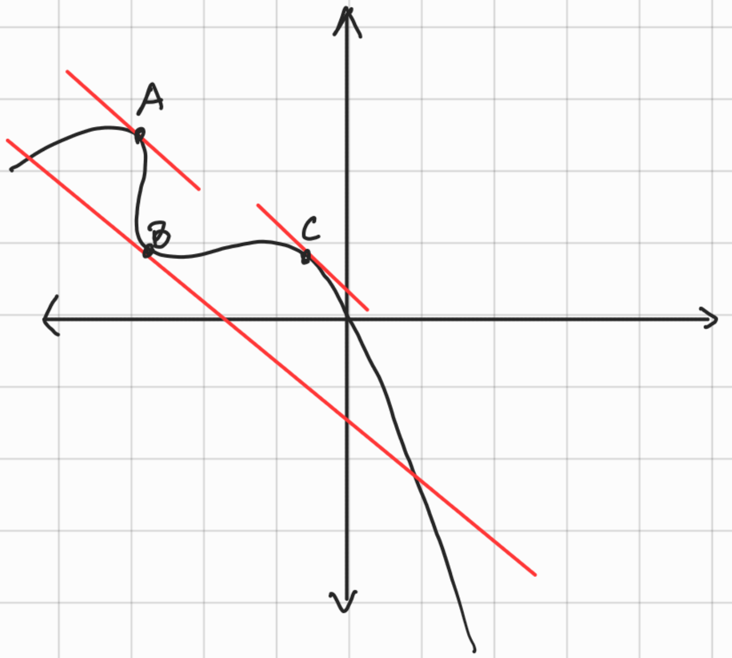


isoprofit lines

Objective: $p_1 y_1 + p_2 y_2$
Level sets: Fix an $m \in \mathbb{R}$
set of (y_1, y_2) s.t. $p_1 y_1 + p_2 y_2 = m$

$$y_2 = \frac{m}{p_2} - \frac{p_1}{p_2} y_1$$

In more dimensions: Solution is hyperplane.



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max $p \cdot y$

s.t $T(y) \leq 0$

Sept 4, 2023

$$\mathcal{L} = p \cdot y - \lambda T(y)$$

K.T. Conditions:

$$\frac{\partial \mathcal{L}}{\partial y_i} = 0 \Rightarrow p_i = \frac{\partial T}{\partial y_i} \cdot \lambda$$

c.s.c. $\lambda \cdot T(y) = 0$

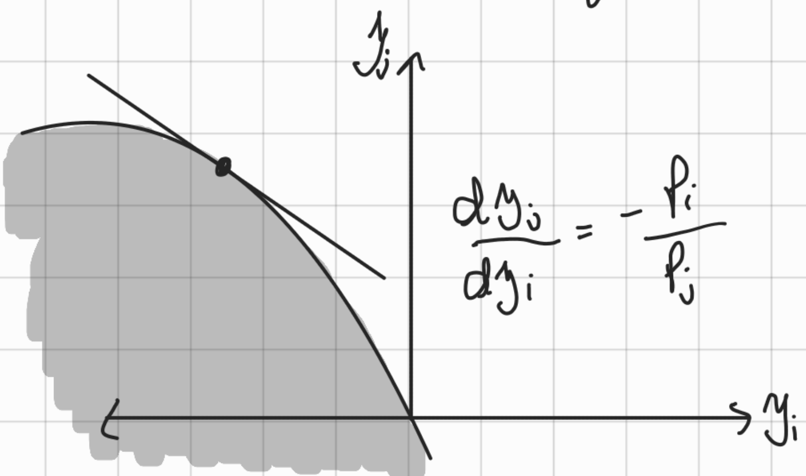
$\lambda \geq 0$

Implication of the f.o.c

$$\frac{p_i}{p_j} = \underbrace{\frac{\partial T / \partial y_i}{\partial T / \partial y_j}}_{MRT_{ij}} \quad \forall i,j$$

* Recall $\frac{dy_j}{dy_i} = -MRT$

$\Rightarrow$ Optimum at $\frac{p_i}{p_j} = -\frac{dy_j}{dy_i}$



* Economic interpretation in: $\frac{\partial T/\partial y_i}{P_i} \gtreqless \frac{\partial T/\partial y_j}{P_j}$

Single output case: Production function $f(y)$

$$\max \; p \cdot f(x) - \omega \cdot x$$
s.t. $x_j \geq 0$

ω: vector of input prices
p: output price
μ: multiplier.

$$\mathcal{L} = p \cdot f(x) - \omega x + \mu \cdot x$$

alternatively.
s.t $-x_j \leq 0$
$\dots \; -\mu(-x)$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \Rightarrow p \cdot \frac{\partial \mathcal{L}}{\partial x_i} - \omega_i + \mu_i = 0 \quad (\star)$$

csc. $\mu_i x_i = 0 \quad \forall i$
$\mu_i \geq 0 \quad \forall i$

Another way for $(\star)$: $\underbrace{p \cdot \frac{\partial f}{\partial x_i}}_{\text{mg. Revenue of add. } x_i} \leq \underbrace{\omega_i}_{\text{mg. cost of add. } x_i} \; \forall i$, with equality if $x_i^* > 0$

Divide i by j: $\underbrace{\frac{\partial f/\partial x_i}{\partial f/\partial x_j}}_{\text{MRTS}} = \frac{\omega_i}{\omega_j} \quad \forall \; x_i^*, x_j^* > 0$

$$\frac{\partial f/\partial x_i}{\omega_i} = \frac{\partial f/\partial x_j}{\omega_j}$$

* NOTE
$f(x) \leq 0$
$-\mu f(x)$
$f(x) \geq 0$
$\mu f(x)$

Notation for simple output problem:

$x^*(p,w)$: factor demand correspondence (or function).

$y^*(p,w)$: Supply correspondence

$\pi(p,w) = p \cdot f(x^*(p,w)) - w \cdot x^*(p,w)$

* K.T. cond. are necessary but not sufficient (in general).

** S.O.C are informative but locally.

Back: $\max p \cdot y$
s.t $T(y) \leq 0$

Theorem: Assume $T(y)$ is convex. Then, if a point y^* satisfies the KT conditions for some $\lambda \geq 0$, then y^* is a solution to maximization.

$$\max \underbrace{g(x)}_{\text{quasiconcave}}$$
$$\text{st. } \underbrace{h(x) \geq 0}_{\text{quasiconvex}}$$

Then KT are necessary and sufficient.

Single output: $\max\ p \cdot f(x) - wx$
s.t $x_i \geq 0$

Theorem: Assume f is concave. Then, if x^* satisfies

$$p \cdot \frac{\partial f}{\partial x_i} \leq w_i \quad \forall i \text{, with equality if } x_j^* > 0,$$

Then x^* is a solution to the problem.

Caveats: KT/FOC are useful but be careful with
- nondifferentiabilities
- nonconvexities
- Existence / uniqueness
- Corner solution (some variable can be optimal at 0).

Example: single-output, single-input $f(\cdot)$

$$y = f(x) = \frac{1}{2}x$$

$$\max_{x \geq 0} \; p f(x) - \omega x$$

$$\mathcal{L} = p\left(\frac{1}{2}x\right) - \omega x + \mu x$$

$$\mathcal{L}_x = \frac{1}{2}p - \omega + \mu = 0$$

CSC: $\mu x = 0$
$\mu \geq 0.$

Solve by cases:

I) $x = 0$, $\mu > 0$.

$\rightarrow \mu = \omega - \frac{1}{2}p > 0$?

what if $\omega - \frac{1}{2}p < 0$?

Economic interpretation:
$2\omega < p$
Mg cost < mg I
Firm should produce inf. So, no solution.

II) $x > 0$, $\mu = 0$.

$$\underbrace{\frac{1}{2}p = \mu}$$

Economic sense.

Solution:

$$x(p,\omega) = \begin{cases} 0, & p < 2\omega \\ [0,\infty), & p = 2\omega \\ \emptyset, & p > \omega \end{cases}$$

Example 2: $f(x_1, x_2) = \log(1 + x_1 + x_2)$

$x_1 \geq 0$
$x_2 \geq 0$

$$\mathcal{L} = p \cdot \log(1 + x_1 + x_2) - \omega_1 x_1 + \mu_1 x_1 + \mu_2 x_2$$

$$\partial\mathcal{L}/\partial x_1 = 0 \longrightarrow \frac{p}{1 + x_1 + x_2} - \omega_1 + \mu_1 = 0$$

$$\partial\mathcal{L}/\partial x_2 = 0 \longrightarrow \frac{p}{1 + x_1 + x_2} - \omega_2 + \mu_2 = 0$$

CSC
$\mu_1 x_1 = 0$
$\mu_2 x_2 = 0$
$\mu_1 \geq 0$, $\mu_2 \geq 0$

Case I: $\mu_1 = 0,\ \mu_2 = 0,\ x_1 > 0,\ x_2 > 0$

$$w_1 = \frac{p}{1+x_1+x_2} \ ; \ w_2 = \frac{p}{1+x_1+x_2} \rightarrow w_1 = w_2 = \bar{w}.$$

$$\rightarrow x_1 + x_2 = \frac{p}{\bar{w}} - 1.$$

implicitly. $\frac{p}{\bar{w}} - 1 > 0$

$p > \bar{w}$ condition

Case II: $\mu_1 = 0,\ \mu_2 \geq 0,\ x_1 \geq 0,\ x_2 = 0$

$$\frac{p}{1+x_1} = w_1 \Rightarrow x_1^* = \frac{p}{w_1} - 1$$

$w_2 \geq w_1$ condition

$$\mu_2 = w_2 - \frac{p}{1+x_1} = w_2 - w_1 \geq 0$$

Case III: $\mu_1 \geq,\ \mu_1 = 0,\ x_1 = 0,\ x_2 \geq 0$

$w_1 \geq w_2$ condition

$$\rightarrow \mu_1 = w_1 - \frac{p}{1+x_2} = w_1 - w_2 \geq 0$$

wrapping all together: Factor demands

assume $p \geq \min\{w_1, w_2\}$

$$x_1(p,w) = \begin{cases} \frac{p}{w_1} - 1, & w_1 < w_2 \\ \alpha\left(\frac{p}{\bar{w}} - 1\right), & w_1 = w_2 = \bar{w},\ \alpha \in [0,1] \\ 0, & w_1 > w_2 \end{cases}$$

$$x_2(p,w) = \begin{cases} \frac{p}{w_2} - 1, & w_2 > w_1 \\ (1-\alpha)\left(\frac{p}{\bar{w}} - 1\right), & w_1 = w_2 = \bar{w},\ \alpha \in [0,1] \\ 0, & w_2 > w_1 \end{cases}$$

$$y(p,w) = \begin{cases} 0, & p \le \hat{\omega} \\ \log\left(\frac{p}{\hat{\omega}}\right), & p > \hat{\omega} \end{cases}, \quad \hat{\omega} = \min\{w_1, w_2\}$$

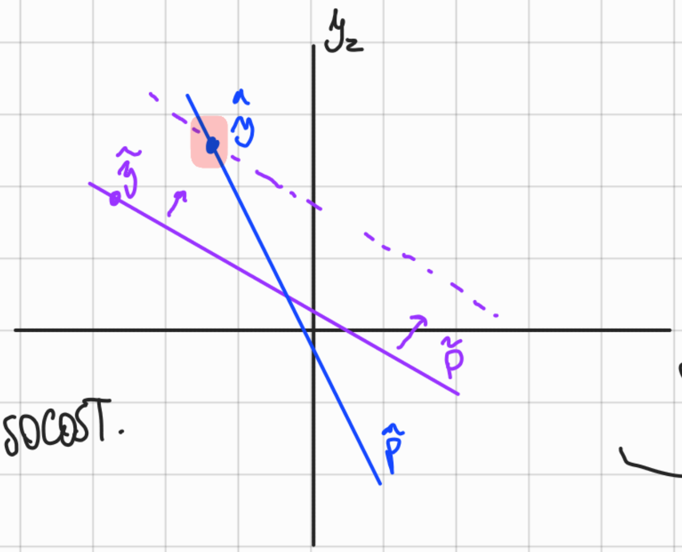
$$\pi(p,w) = \begin{cases} 0, & p \le \hat{\omega} \\ p \cdot \log\left(\frac{p}{\hat{\omega}}\right) - p + \hat{\omega}, & p > \hat{\omega} \end{cases}$$

Sept 06, 2023

Rationalizability:

Q: Given some data $\{(p^1, y^1), \dots, (p^T, y^T)\}$ vectors of data

Can we "learn" about the firm's technology?

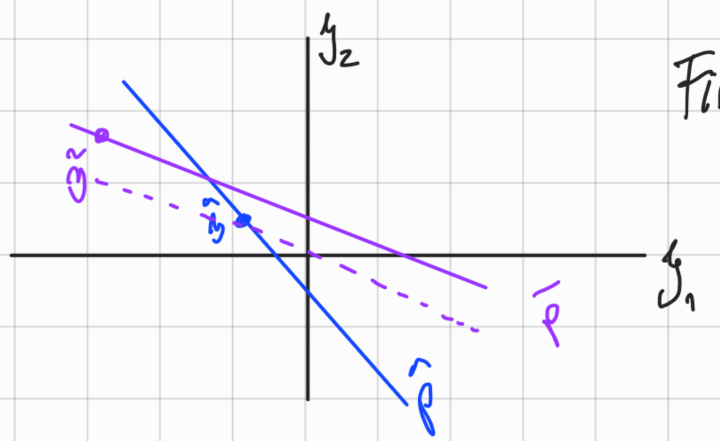


ISOCOST.

Firm is irrational: Violate the weak axiom of profit maximization: (WAPM)

$$p^t y^t \ge p^t y^s, \quad \forall s,t = 1, \dots, T$$

$\longrightarrow \tilde{p}\hat{y} \ge \tilde{p}\tilde{y}$ violates.



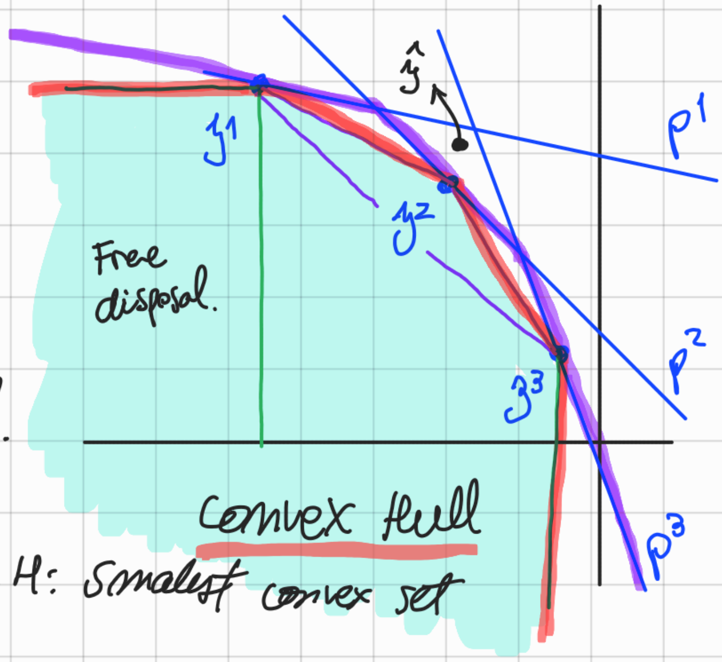
Firm could've chosen $\hat{y}$, but at a price.

What other production sets could "rationalize" the data?

A production set Y rationalizes the data if

$$y^t \in y^*(p^t) \quad \forall\, t = 1, \dots, T$$

$$y^*(p) = \arg\max_{y \in Y} p \cdot y$$



make some additional assumptions.

- Data satisfies WAMP
- Y is convex & free disposal.

$$Y^I = \text{conv}_{fd}(\{y^1, \dots, y^T\})$$

*C.H: smallest convex set

Does Y^I rationalize the data?

Upper bound.

claim: yes.

Proof: Contradiction: Then $\exists$ a point $y \in Y^F$ s.t. $p^t \cdot y > p^t y^t$ for some t.

By convexity: $y = \alpha^1 y^1 + \alpha^2 y^2 + \dots + \alpha^T y^T$, $\alpha^1 + \dots + \alpha^T = 1$.

① $p^t(\alpha^1 y^1 + \dots + \alpha^T y^T) > p^t \cdot y^t$

② $$\alpha^1 p^t y^1 + \dots + \alpha p^t y^T \leq \alpha^1 p^t \cdot y^t + \dots + \alpha^T p^t \cdot y^t$$
$$= (\alpha^1 + \dots + \alpha^T) p^t \cdot y^t$$
$$= p^t \cdot y^t$$

contradiction.

$\rightarrow$ Y^F is one production set that rationalizes the data. By def. for any other convex Y that rationalizes the data $Y^F \subseteq Y$

$Y^o = \{ y \in \mathbb{R}^n : p^t \cdot y \leq \pi(p^t), \ \forall t = 1, \dots, T \}$

$\hat{y} \notin Y^o$ because $p^2 \cdot \hat{y} > p^2 \cdot y^2 = \pi(p^2)$

$\mathbb{R}^n \setminus Y^o$ bundles that yield higher profits than some observed choice $= \{ y \in \mathbb{R}^n : p^t y > \pi(p^t) \text{ for some } t \}$.

Does Y^o rationalize the data?
claim. yes.

Proof: Assume not. Then, $\exists$ some $y \in Y^o$ s.t $p^t \cdot y > p^t \cdot y^t$
Directly contradicts definition of Y^o.

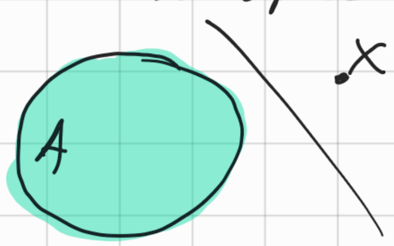
Theorem: For any production set Y that rationalizes the data, we have $Y^F \subseteq Y \subseteq Y^o$

What if we had infinite data? we would pin down the exact production function.

Suppose we observe $y(p)$ $\forall p$. Let $\pi(p) = p \cdot y(p)$ be the observed profit function.

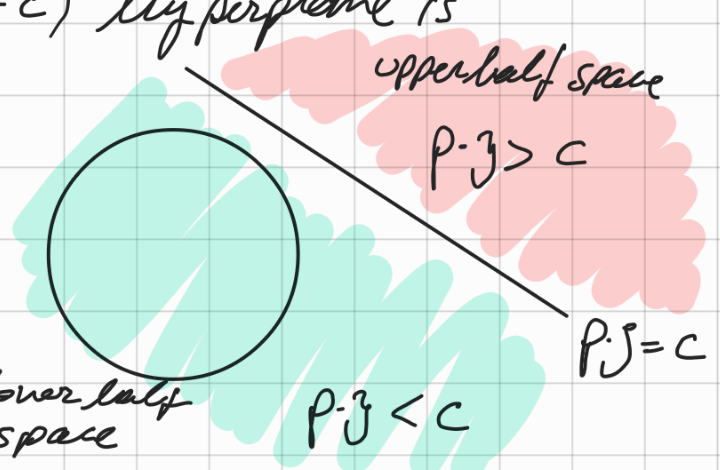
Theorem: Suppose Y is convex, closed & satisfies free disposal.
Then $Y = Y^o = \{ y \in \mathbb{R}^n : p y(p) \leq \pi(p) \ \forall p \in \mathbb{R}^n_+ \}$

Detour: Separating hyperplane theorem. In $\mathbb{R}^n$, a hyperplane is a subspace of dim $(n-1)$.



Given $p \in \mathbb{R}^n$, $c \in \mathbb{R}$, the $(p-c)$ hyperplane is

$$H_{p,c} = \{ y \in \mathbb{R}^n : p \cdot y = c \}$$



Theorem: Suppose $A \subseteq \mathbb{R}^n$ is convex and closed, and $x \notin A$. Then, there is a $p \in \mathbb{R}^n$, $p \neq 0$, and $c \in \mathbb{R}$ s.t. $p \cdot x < c$ & $p \cdot y < c$ $\forall y \in A$.

* Convexity is important:

Proof: $Y = Y^o$

NTS ① $Y \subseteq Y^o$ & ② $Y^o \subseteq Y$

For ②: Take some $\hat{y} \in \mathbb{R}^n \setminus Y$

By S.H.T., $\exists$ some $p \in \mathbb{R}^n$ & c s.t.

$$p \cdot \hat{y} > c > p \cdot y \quad \forall y \in Y$$

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y \quad (*)$$

By free disposal, $p \geq 0$ if $p_i < 0$ for some i, then $\max p \cdot y = \infty$ which contradicts $(*)$

Therefore, there exists $p \geq 0$ s.t. $p \cdot \hat{y} \;\; \max_{y \in Y} p \cdot y = \pi(p)$
i.e., $\hat{y} \notin Y^o$, or $Y^o \subseteq Y$.

Sept 18, 2023

Properties of profit & Supply functions

$$\pi(p) = \max_{y \in Y} p \cdot y \qquad y(p) = \arg\max_{y \in Y} p \cdot y$$

$$\underbrace{\pi : \mathbb{R}^n}_{\text{space of prices}} \rightarrow \underbrace{\mathbb{R}}_{\text{profit}} \qquad y : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$$

Note: Results do not rely on regularity assumptions (convexity, differentiability, etc..) unless stated otherwise

(1) $\pi(p)$ is nonincreasing in input prices, nondecreasing in output prices.

Proof: Take a price vectors p, p' and choose $y \in y^*(p)$ and $y' \in y^*(p')$. Let $p_i' \geq p_i$ $\forall$ outputs, $p_i' \leq p_i$ for all inputs

By WAPM, $p' \cdot y' \geq p' \cdot y$

consider $(p' - p) \cdot y = (p_1' - p_1) y_1 + \dots + (p_n' - p_n) y_n$

$y_i \geq 0 : p_i' \geq p_i \qquad y_i \leq 0 : p_i' \leq p_i \Rightarrow (p' - p) y \geq 0$

$\Rightarrow p' y \geq p \cdot y$

$\pi(p') = p' \cdot y' \geq p' \cdot y \geq p \cdot y = \pi(p)$

$\Rightarrow \quad \pi(p') \geq \pi(p)$

② $\Pi(p)$ is hd-1. WTS: $\pi(tp) = t\pi(p)$

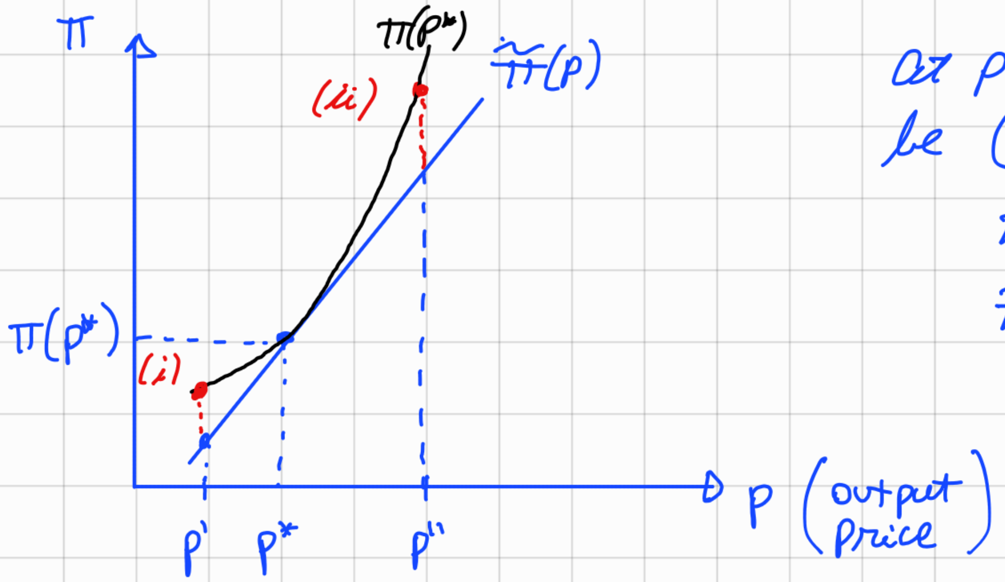
Proof: $\pi(tp) = \max_{y \in Y} (tp)\cdot y$

$$= t \cdot \left[\max_{y\in Y} p\cdot y \right] = t\cdot \pi(p)$$

③ $\pi(p)$ is convex in p, i.e,

$\pi(\alpha p + (1-\alpha)p') \le \alpha\pi(p) + (1-\alpha)\pi(p')$, $\forall \alpha \in [0,1]$.

Graphical intuition: [Do the proof...].



at p^*, let optimal production be $(-x^*, y^*)$.

$\pi(p^*) = p^* y^* - w^* \cdot x^*$

$\tilde{\pi}(p) = p\, y^* - w^* x^*$

(i) and (ii) are reoptimized decisions of the firm, that will be above the first optimized point.

If we do assume π is differentiable

$$H(p) = D^2\pi(p) = \begin{bmatrix} \partial^2\pi/\partial p_1^2 & \cdots & \partial^2\pi/\partial p_1\partial p_n \\ \vdots & \ddots & \vdots \\ \partial^2\pi/\partial p_n\partial p_1 & \cdots & \partial^2\pi/\partial p_n^2 \end{bmatrix}$$

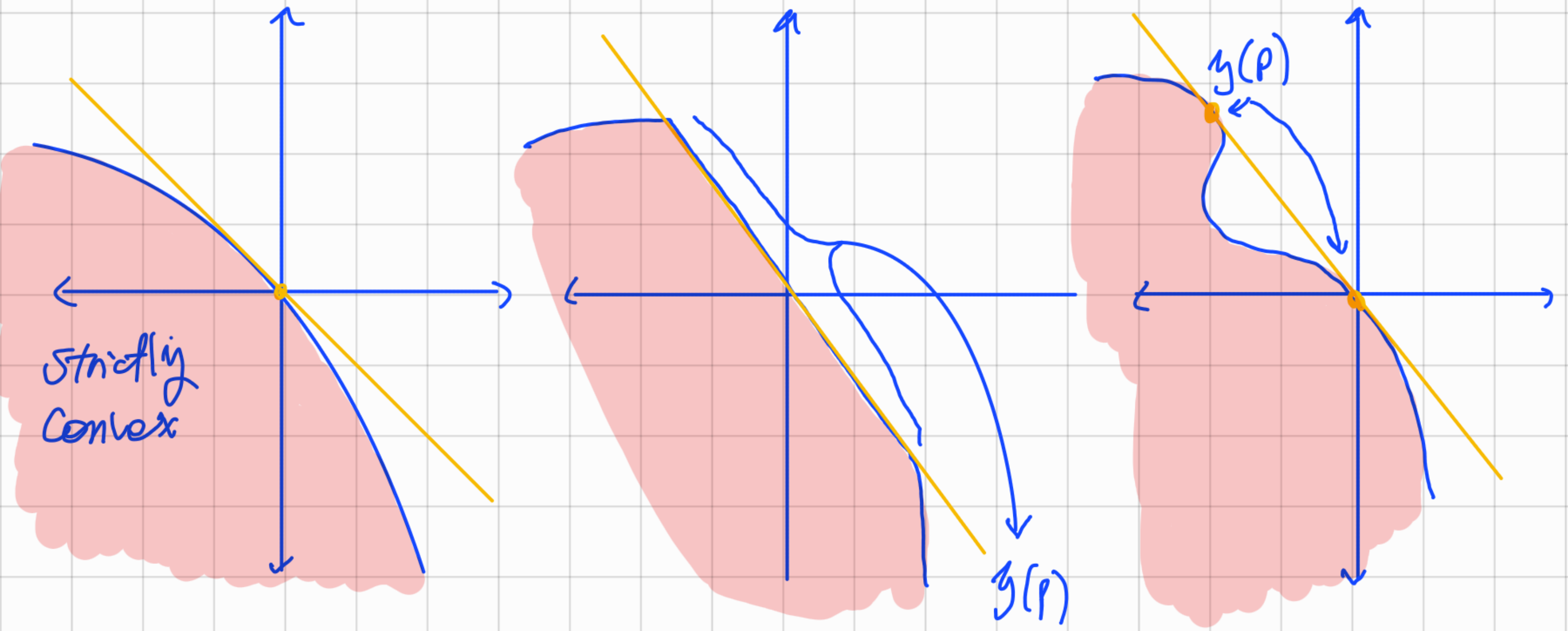
④ positive semidefinite.

Properties of $y(p)$

① $y(p)$ is homogeneous of degree 0. (change in prices don't affect decisions)

Proof: $\max_{y \in Y} (\alpha p) \cdot y = \max_{y \in Y} p \cdot y$ $\quad y(\alpha p) = y(p)$

② a) If Y is convex, then $y(p)$ is in convex set
b) If Y is strictly convex then $y(p)$ is a singleton



Detour: Envelope theorem.

Q: How does the optimal value of a maximization problem change as a parameter changes?

$\max_x f(x, \theta)$

Define the value function

x: choice variable
θ: parameter

$$V(\theta) = \max_x f(x, \theta)$$

$$V(\theta) = f(x^*(\theta), \theta)$$

A change in θ has 2 effects:

- Direct effect: θ changes f directly through and argument
- Indirect effect: θ changes x^*, which changes f.

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial x}\bigg|_{x=x^*(\theta)}}_{0} + \frac{\partial f}{\partial \theta}\bigg|_{x=x^*(\theta)}$$

$$\Rightarrow \boxed{\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta}\bigg|_{x=x^*(\theta)}}$$

Found all over economics:

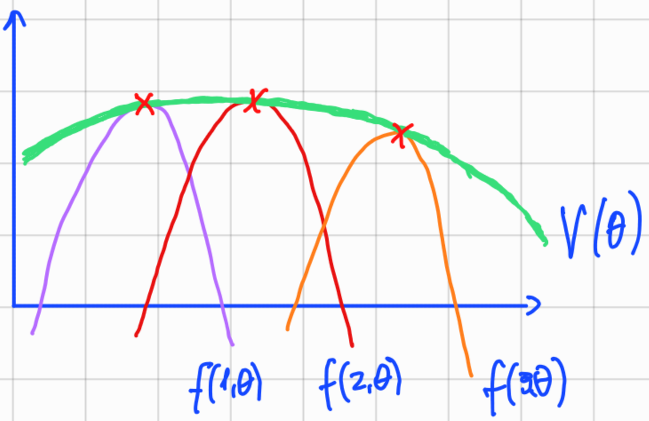
- Hotelling' Lemma
- Shepard's Lemma
- Myerson's Lemma
- Benveniste-Scheinkman formula
- ...

When there are constraints

$$\max_x f(x,\theta) \quad s.t \quad g_k(x,\theta) \leq c_k, \; k = 1, \dots, K.$$

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} - \lambda_1 \frac{\partial g}{\partial \theta} - \dots - \lambda_k \frac{\partial g}{\partial \theta}$$

$$\frac{dV}{d\theta} = \frac{\partial \mathcal{L}}{\partial \theta}\bigg|_{x=x^*(\theta)}$$



Hotelling's Lemma : If $\pi(p)$ is differentiable at $\bar{p}$, and $\bar{p}_i > 0$, then $\frac{\partial \pi}{\partial p_i} = y_i^*(\bar{p})$

Proof : For single output case

$$\pi(p,w) = \max_{x \geq 0} \underbrace{p \cdot f(x) - w \cdot x}_{h(x)}$$

By envelope theorem,

$$\frac{d\pi}{dp} = \frac{\partial h}{\partial p}\bigg|_{x = x^*(p,w)} = f(x^*(p,w)) = y^*(p,w)$$

$$\frac{d\pi}{dw_i} = \frac{\partial h}{\partial w_i}\bigg|_{x = x^*(p,w)} = -x_i^*(p,w)$$

Example: $f(x_1, x_2) = 30p\, x_1^{2/5} x_2^{2/5}$

$$\max\ 30p x_1^{2/5} x_2^{2/5} - w_1 x_1 - w_2 x_2$$

f.o.c. :
$$\left.\begin{aligned} 12p\, x_1^{-3/5} x_2^{2/5} &= w_1 \\ 12p\, x_1^{2/5} x_2^{-3/5} &= w_2 \end{aligned}\right\} \Rightarrow \boxed{x_1^* = \frac{12^5 p^5}{w_1^3 w_2^2} \quad x_2^* = \frac{12^5 p^5}{w_1^2 w_2^3}}$$

$$\pi(p,w) = 30p \frac{12^4 p^4}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} = \frac{p}{2}\left[\frac{12^5 p^5}{w_1^2 w_2^2}\right]$$

$$\boxed{\frac{\partial \pi}{\partial w_1} = -\frac{2}{2}\left[\frac{12^5 p^5}{w_1^3 w_2^2}\right]}$$

Integral form for Hotelling's Lemma

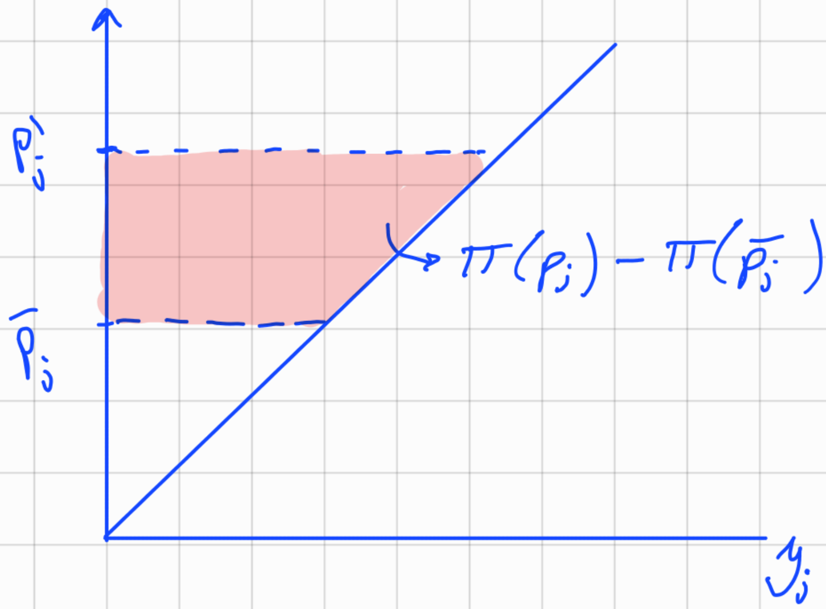
$-j$: everything else.

Hold all prices other than p_j fixed at $\bar{p}_{-j}$.
Then

$$\pi(p_j, \bar{p}_{-j}) - \pi(\bar{p}_j, \bar{p}_{-j}) = \int_{\bar{p}_j}^{p_j} y_j(s, \bar{p}_{-j})\, ds$$

Proof:

$$\int_{\bar{p}_j}^{p_j} y_j(s, \bar{p}_{-j})\, ds = \int_{\bar{p}_j}^{p_j} \frac{d\pi(s, \bar{p}_{-j})}{dp_j}\, ds = \underbrace{\pi(p_j, \bar{p}_{-j}) - \pi(\bar{p}_j, \bar{p}_{-j})}_{\text{Surplus.}}$$



Lecture

Sept 20, 2023

Hotelling's Lemma

$$\frac{\partial \pi}{\partial p_i} = y_i(p) \longrightarrow \frac{\partial y_i(p)}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_i \partial p_j}$$

$$Dy(p) = \begin{bmatrix} \partial y_1/\partial p_1 & \cdots & \partial y_n/\partial p_1 \\ \vdots & \ddots & \vdots \\ \partial y_1/\partial p_n & \cdots & \partial y_n/\partial p_n \end{bmatrix} = \begin{bmatrix} \partial^2\pi/\partial p_1^2 & \cdots & \frac{\partial^2 \pi}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \partial^2\pi/\partial p_n \partial p_1 & \cdots & \partial^2\pi/\partial p_n^2 \end{bmatrix}$$

Jacobian or substitution matrix.

π is convex, pos semidef

This implies:

① $\frac{\partial y_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2} \geq 0$ (Law of supply)

② $\underbrace{\frac{\partial y_i}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_j \partial p_i} = \frac{\partial^2 \pi}{\partial p_j \partial p_i}}_{\text{Young's Law}} = \frac{\partial y_j}{\partial p_i}$

Lechatelier Principle: "Firms should respond more to price changes in the long-Run than in the short-Run."

Suppose $(y^*, x^*) = (y(p^*, w^*), x(p^*, w^*))$ at price (p^*, w^*)

Fix one input z_i^* in the short run, variable in the long-Run. z_i^* is optimal at (p^*, w^*)

Q: How does x_j depend on w_j in the SR vs L-R?

Proof: $\longrightarrow$

$$\pi(p,w) = \max_{\substack{\text{s.t. } x_j \geq 0}} p \cdot f(x) - w \cdot x \; ; \; \pi^s(p,w) = \max_{\substack{\text{s.t. } x_j \geq 0 \\ x_i = z_i^*}} p \cdot f(x) - w \cdot x$$

Compare

$$\left. \frac{\partial x_j(p,w)}{\partial w_j} \right|_{(p,w)=(p^*,w^*)} \quad \text{vs} \quad \left. \frac{\partial x_j^s(p,w)}{\partial w_j} \right|_{(p,w)=(p^*,w^*)}$$

$$h(p,w) = \pi(p,w) - \underbrace{\pi^s(p,w)}_{\text{this is lower because it is a more restricted problem}}$$

By definition, $h(p,w) \geq 0$ and $h(p^*,w^*) = 0$, (p^*,w^*) is a local minimum.

$$\frac{\partial^2 h(p^*,w^*)}{\partial w_j^2} \geq 0$$

$$\parallel$$

$$\frac{\partial^2 \pi(p^*,w^*)}{\partial w_j^2} - \frac{\partial^2 \pi^s(p^*,w^*)}{\partial w_j^2} \geq 0$$

Hotelling : $\dfrac{\partial \pi(p,w)}{\partial w_j} = -x_j(p,w)$

$$-\frac{\partial x_j(p^*,w^*)}{\partial w_j} + \frac{\partial x_j^s(p^*,w^*)}{\partial w_j} \geq 0$$

$$\left| \frac{\partial x_j(p^*,w^*)}{\partial w_j} \right| \geq \left| \frac{\partial x_j^s(p^*,w^*)}{\partial w_j} \right|$$

because $\dfrac{\partial x_j}{\partial w_j}$ are negative values $\left(\dfrac{\Delta \text{Demand}}{\Delta \text{price input}} \right)$

<u>Cost minimization:</u>

- Single output good
- pricing power in output markets, but not input markets
- Previous analysis doesn't work, but we can:
  1. Find the cheapest way to make any target output y (cost minimization problem).
  2. Use this "cost f motion" $c(y)$ to choose optimal p/y combination.

<u>CMP:</u>

- Fix a target level of output, y.
- Cost: $\underbrace{C(w,y)}_{\text{parameters}} = \min\ w\cdot x$ s.t $x_i \geq 0$, $f(x) \geq y$ ← *Ignore this for now...

$$\mathcal{L} = wx - \lambda(f(x) - y)$$

<u>F.O.C:</u> $w_i - \lambda \dfrac{\partial f}{\partial x_i} = 0 \quad \forall i$

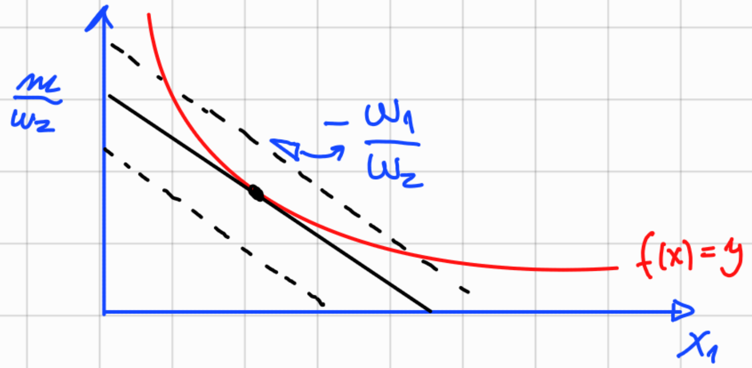
Divide i by j:

$$\underbrace{\frac{w_i}{w_j}}_{\text{economic rate of sustitution}} = \underbrace{\frac{\partial f/\partial x_i}{\partial f/\partial x_j}}_{\text{MRTS}}, \text{ for } (x_i^*, x_j^* > 0)$$

Constraint: Isoquant

objective: Isocost $w_1x_1 + w_2x_2 = m$

$$x_2 = \frac{m}{w_2} - \frac{w_1}{w_2}x_1$$



$x^*(w,y)$ conditional factor demand correspondence.

$$c(w,y) = w \cdot x^*(w,y) = \min w \cdot x \quad , \quad \begin{array}{l} x_i \geq 0 \\ f(x) \geq y. \end{array}$$

Apply Envelope theorem.

$$\boxed{\frac{\partial c}{\partial y} = \frac{\partial \mathcal{L}}{\partial y} = \lambda}$$

$\rightarrow$ λ is the mg cost of increasing production by 1 unit

Relationship with PMP

$$\max_{y \geq 0} \; p \cdot y - c(w,y)$$

F.O.C) $\quad p = \frac{\partial c}{\partial y} \; (= \lambda \text{ from CMP})$

Things to keep in mind:

- We have assumed, differentiable f, interior solutions
- General F.O.C : $\lambda \cdot \frac{\partial f(x^*)}{\partial x_i} - w_i \leq 0 \quad \forall i$

  with equality if $x_i^* > 0$
- KT conditions are necessary, but not sufficient in general
- will be sufficient if $f(\cdot)$ is concave
- Existence / uniqueness issues

Weak Axiom of Cost minimization (WACM)

$\{(w^1, x^1, y^1), \ldots, (w^T, x^T, y^T)\}$

WACM: $\underbrace{w^t x^t}_{\text{cost}} \leq \underbrace{w^t x^s}_{\substack{\text{other choices}\\ \text{are more expensive.}}} \quad \forall \; y^s \geq y^t$

Any firm that violates WACM is not Rational.

Implication of WACM: Downward sloping Demand

→ Take $y^s = y^t$ WACM gives 2 inequalities

1. $w^t x^t \leq w^t x^s$
2. $w^s \cdot x^s \leq w^s x^t$

1 + 2 : $w^t \cdot x^t + w^s x^s \leq w^t x^s + w^s x^t$

$$(w^t - w^s)(x^t - x^s) \leq 0$$

$$\Delta w \cdot \Delta x \leq 0$$

$$C(w, y) = \min_x x \cdot y \qquad \text{s.t } f(x) \geq y$$

1. $c(w, y)$ is nondecreasing in w. Proof:

Take $w' \geq w$ and $x \in x^*(w, y)$, $x' \in x^*(w', y)$

$$c(w, y) = \underbrace{w \cdot x \leq w \cdot x'}_{\text{WACM}} \underbrace{\leq}_{w' \geq w} w' \cdot x' = c(w', y)$$

$$c(w, y) \leq c(w', y)$$

② $C(w,y)$ is nondecreasing in y.

Let $y'' \geq y'$, $x' \in x^*(w,y')$, $x'' \in x^*(w,y'')$

$$c(w,y') = \min_{f(x)\geq y'} w\cdot x = w\cdot x' \underset{\text{WACM}}{\leq} w\cdot x'' = \min_{f(x)\geq y''} w\cdot x = c(w,y'')$$

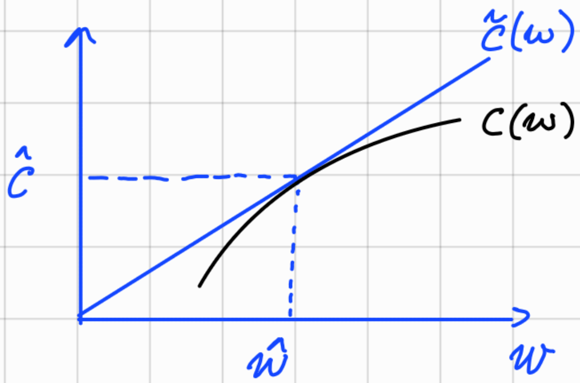
Inequality works because $f(x'') \geq y'' \geq y'$

③ $c(w,y)$ is hd-1 in w

$$c(tw,y) = t\,c(w,y)$$

④ $c(w,y)$ is <u>concave</u> in w (NOT TRUE for y)

(Same argument as in PMP problem)



⑤ If $V(y)$ is convex, then $x^*(w,y)$ is convex-valued.
If $V(y)$ is strictly convex, then $x^*(w,y)$ is single valued

⑥ $x^*(w,y)$ is hd-0 in w: $x^*(t\cdot w,y) = x^*(w,y)$

→ Implications:

$$x_i^*(tw,y) = x_i^*(w,y)$$

differentiate w.r.t. t, eval at $t=1$

$$\nabla_w \cdot x_i^*(tw,y)\cdot w = 0$$

$$\sum_{j=1}^{\infty} \frac{\partial x_i^*(tw,y)}{\partial w_j}\cdot w_j = 0$$

We know $\frac{\partial x_i^*}{\partial w_j} \leq 0$

Say $\boxed{n=2}$ $\frac{\partial x_i^*}{\partial w_i}\cdot w_i + \frac{\partial x_1^*}{\partial w_2}\cdot w_2 = 0$

$\frac{\partial x_1^*}{\partial w_2} \geq 0$: goods are substitutes

If $n>2$, then $\frac{\partial x_i^*}{\partial w_j} \geq 0$ for some good j

<u>Shepard's Lemma</u>: If $c(w,y)$ is C^1, and $\bar{w}>0$, then

$$x_i(\bar{w},y) = \frac{\partial c(\bar{w},y)}{\partial w_i}$$

<u>Proof 1</u>: Envelope Theorem.

$$\mathcal{L} = w\cdot x - \lambda(f(x)-y)$$

$$\frac{\partial c}{\partial w_i} = \frac{\partial \mathcal{L}}{\partial w_i}\Big|_{x=x^*(w,y)} \rightarrow \boxed{\frac{\partial c}{\partial w_i} = x_i^*(w,y)}$$

Proof 2: $c(w,y) = w \cdot x^*(w,y)$

$$\frac{\partial c}{\partial w_i} = x_i^*(w,y) + \sum^n w_j \cdot \frac{\partial x_i^*(w,y)}{\partial w_i} \qquad (*)$$

Recall the f.o.c of CMP

$$(1)\quad \lambda \frac{\partial f}{\partial x_i} = w_i \qquad\qquad (2)\quad f(x^*(w,y)) = y$$

Differentiate (2) wrt w_i:

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j^*}{\partial w_i} = 0$$

plug (1) into (*)

$$\frac{\partial c}{\partial w_i} = x_i^*(w,y) + \overbrace{\sum^n \lambda \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j^*}{\partial w_i}}^{0}$$

$$\boxed{\frac{\partial c}{\partial w_i} = x_i^*(w,y)}$$

Substitution matrix.

$$\begin{bmatrix} \partial x_1^*/\partial w_1 & \cdots & \partial x_1^*/\partial w_n \\ \vdots & \ddots & \vdots \\ \partial x_n^*/\partial w_1 & \cdots & \partial x_n^*/\partial w_n \end{bmatrix} = \begin{bmatrix} \partial^2 C/\partial w_1^2 & \cdots & \partial C/\partial w_n \partial w_1 \\ \vdots & \ddots & \vdots \\ \partial^2 C/\partial w_1 \partial w_n & \cdots & \partial^2 C/\partial w_n^2 \end{bmatrix}$$

Shepard's Lemma

$\partial^2 C$, which is concave.
(Hessian of C)

- Law of demand: $\frac{\partial x_i^*}{\partial w_i} \leq 0$, $\forall i$

- Symmetric cross-price effect: $\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$

<u>Theorem:</u> If f is constant returns to scale, then
$$c(w,y) = y \cdot \underbrace{c(w,y)}_{\text{unit cost function}}$$

<u>Proof:</u> Assume f C^1, int solution.

F.o.c at $y=1$:

$$\frac{w_i}{w_j} = \frac{f_i(x^*)}{f_j(x^*)} \quad \text{and} \quad f(x^*) = 1$$

For general $\hat{y}$: $\frac{w_i}{w_j} = \frac{f_i(\hat{x})}{f_j(\hat{x})}$, $f(\hat{x}) = y$

Consider input $\hat{x} = \hat{y}x^*$

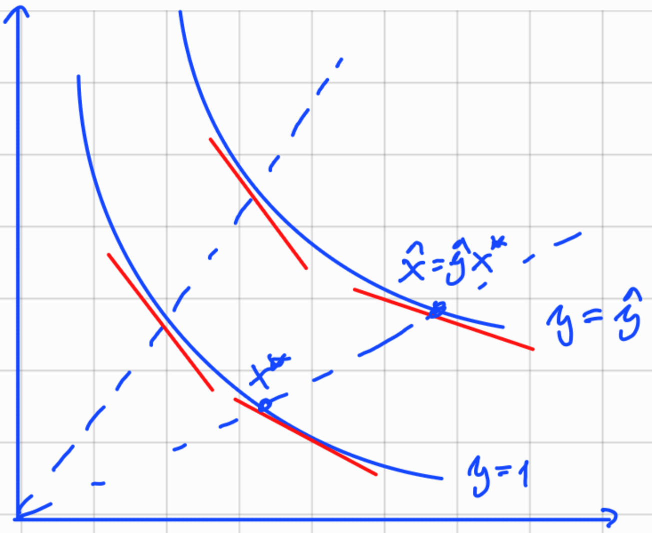
claim: $\hat{x}$ solves the f.o.c. at $\hat{y}$

$$\frac{f_i(\hat{x})}{f_j(\hat{x})} = \frac{f_i(\hat{y}, x^*)}{f_j(\hat{y}, x^*)} \underset{\text{C.R.S.}}{=} \frac{f_i(x^*)}{f_j(x^*)} = \frac{w_i}{w_j}$$

$$f(\hat{y}x^*) \underset{\text{by C.R.S.}}{=} \hat{y} f(x^*) = \hat{y}$$

$\Rightarrow$ so $\hat{x} = \hat{y}x^*$ solves the problem at target output $\hat{y}$

$$c(w,\hat{y}) = w \cdot (\hat{y} \cdot x^*) = \hat{y} \cdot (w \cdot x^*)$$
$$= \hat{y} \cdot c(w,1)$$



Cost function: $C(w, y) = \min\ w \cdot x \quad \text{s.t. } f(x) \geq y$ Sept 25, 2023

for now, just $C(y)$

Average cost: $AC(y) = \dfrac{C(y)}{y}$

Mg. cost: $MC(y) = C'(y)$

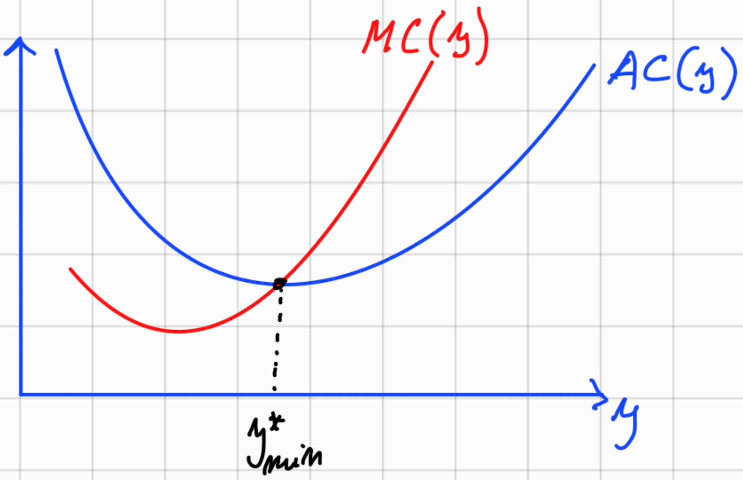
Marginal - average relationship:

$$AC'(y) = \frac{C'(y)}{y} - \frac{C(y)}{y^2} = \frac{C'(y) \cdot y - C(y)}{y^2}, \quad y^2 > 0$$

AC incr $\longleftrightarrow$ $C'(y) > C(y)/y$, or $MC(y) > AC(y)$

AC decr $\longleftrightarrow$ $C'(y) < C(y)/y$, or $MC(y) < AC(y)$

$AC'(y) = 0 \longleftrightarrow MC(y) = AC(y)$



y^*_{min}: Minimum efficient Scale (MES)

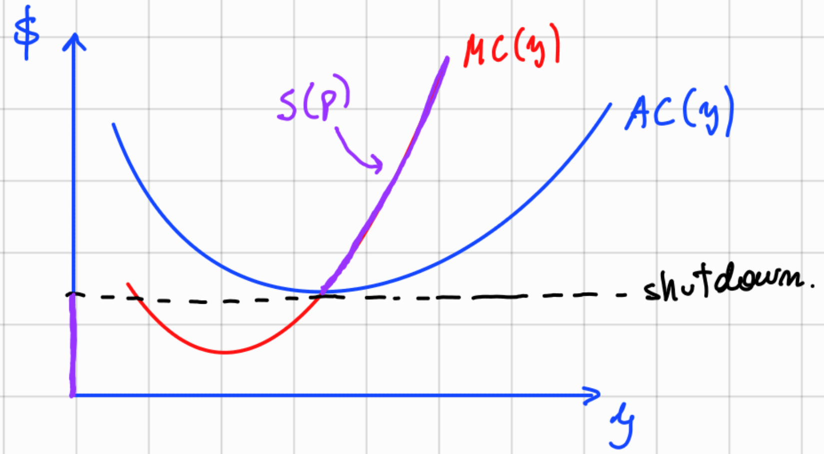
$$\max_{y \geq 0} \; py - c(y)$$

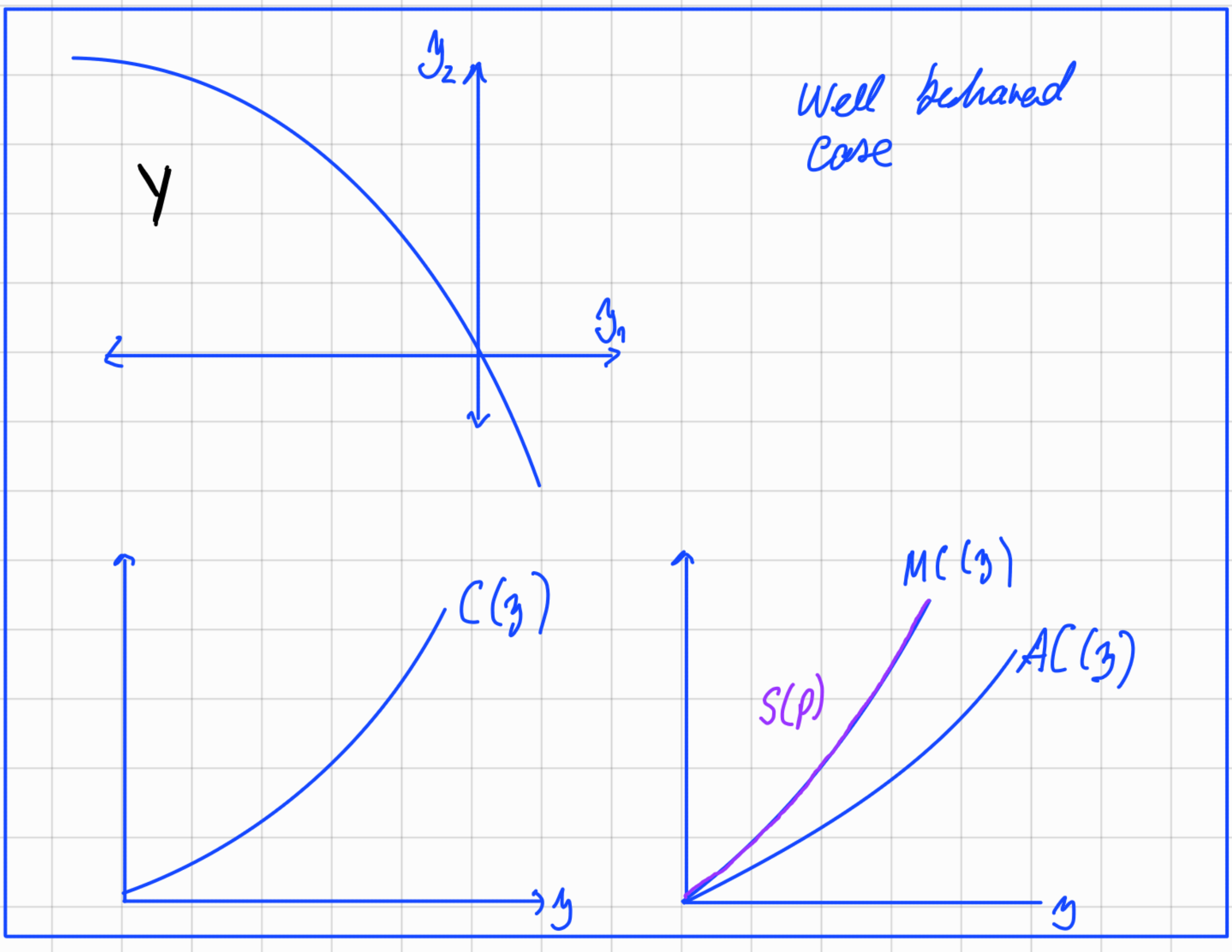
FONC $\quad p \leq c'(y^*)$, with equality if $y^* > 0$

If $c(y)$ is convex, this is also sufficient

Price = mc if firm chooses to produce a strictly positive amount

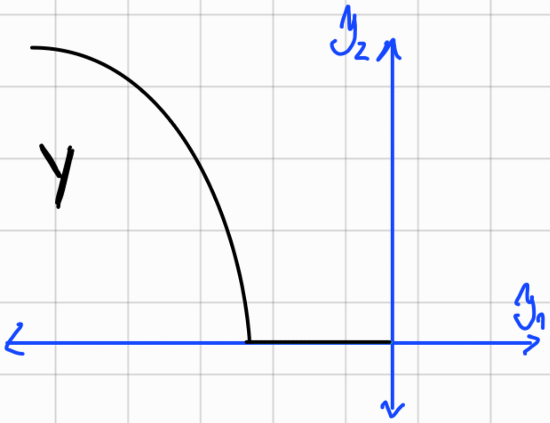
Shutdown Rule: Check that at y^* where $p = c'(y^*)$,
$py^* - c(y^*) \geq 0$ or $p \geq AC(y^*)$

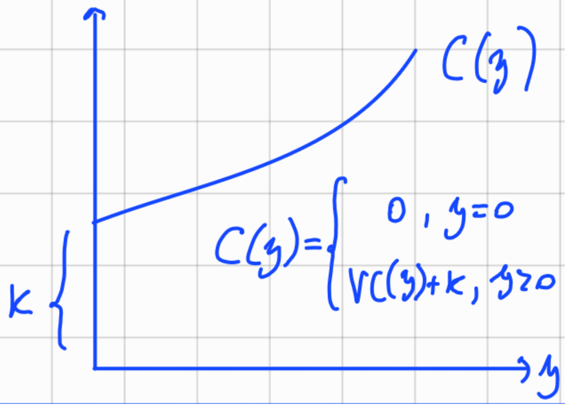




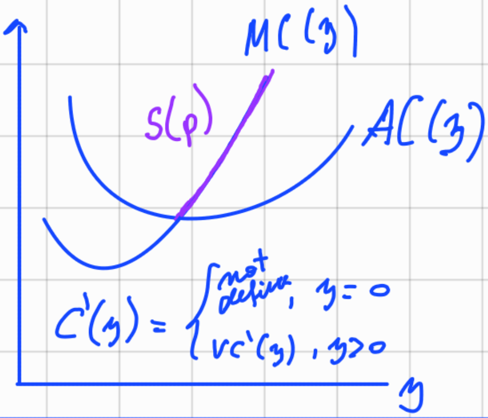
Fixed cost case

* VC: Variable cost





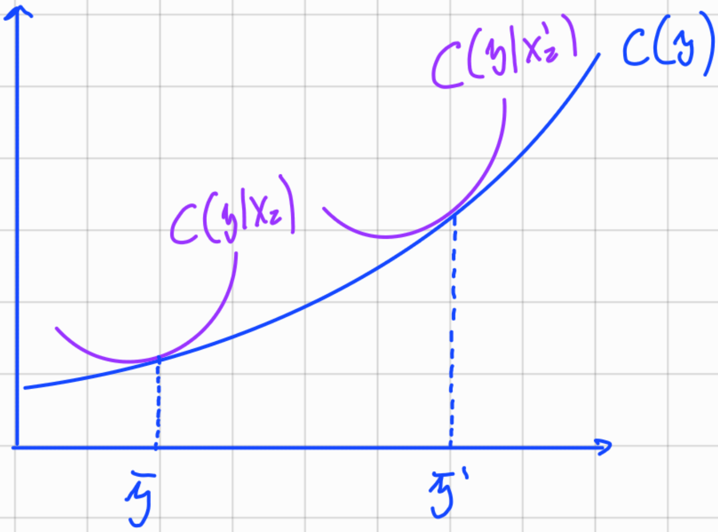
$$C(y) = \begin{cases} 0, & y = 0 \\ VC(y) + k, & y > 0 \end{cases}$$



$$C'(y) = \begin{cases} \text{not define}, & y = 0 \\ VC'(y), & y > 0 \end{cases}$$

Long-Run vs Short-Run costs

- 2 inputs (x_1, x_2)
- $C(y)$ be the LR cost function
- Let $\bar{x}_2$ be optimal x_2 when long-run output $\bar{y}$
- $C(y \mid \bar{x}_2)$. short run cost, when x_2 fixed at $\bar{x}_2$.



Can we go from $c(w,y) \to f(x)$?

- Yes, they are "duals" of each other.

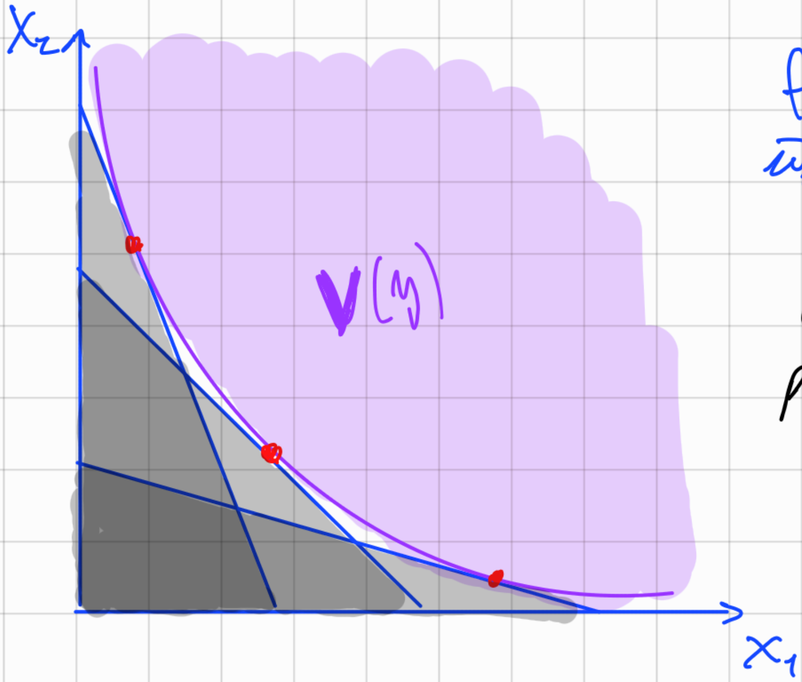
Observe $c(w,y)$ $\forall (w,y)$

$V(y) = \{x : f(x) \geq y\}$

If we know $V(y)$ $\forall y$, then we know $f(x)$.

Can we recover $V(y)$? Define an "outer bound"

$V^o(y) = \{x : w \cdot x \geq c(w,y),\ \forall w \geq 0\}$ at $\bar{w}$, cost is $\bar{c} = c(\bar{w},y)$



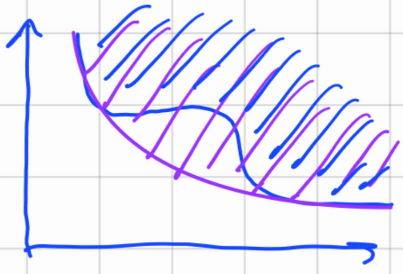
firm chose $\bar{x}$

$\bar{w}_1 x_1 + \bar{w}_2 x_2 = c$

Black Shaded Area: Ruled-out points because of Revealed preferences.

Claim: Suppose $V(y)$ is the input requirement set for a firm with a convex technology with free disposal. Then, $V(y) = V^o(y)$

Proof: Homework (use sep. hyperplane thm).



Presupposes the function we start with is a valid cost function

Question: Given an arbitrary function $\phi(w,y)$, how do we know if ϕ is a valid cost function? that is, does

$$\phi(w,y) = \min wx \quad \text{s.t } x \in V(y)$$

where $V(y) = \{x \geq 0 : w \cdot x \geq \phi(w,y) \; \forall w\}$.

Showed before that if ϕ is a cost function, then:

i) hd-1 in prices: $\phi(tw,y) = t\phi(w,y)$

ii) $\phi(w,y) \geq 0$ (positivity)

iii) $\phi(w',y) \geq \phi(w,y)$, $w' \geq w$ (monotonic)

iv) Concave in w

Theorem: Let $\phi(w,y)$ be a C^2 function satisfying i)-iv), then, $\phi(w,y)$ is the cost function for the technology

$$V(y) = \{x : w \cdot x \geq \phi(w,y), \forall w\}$$

proof: Define

$$x(w,y) = \left(\frac{\partial \phi(w,y)}{\partial w_1}, \dots, \frac{\partial \phi(w,y)}{\partial w_n} \right)$$

By monotonicity, $x(w,y) \geq 0$

By hd-1, $\phi(w,y) = \sum_{i=1}^{n} w_i \frac{\partial \phi(w,y)}{\partial w_i} = w \cdot x(w,y)$

Euler's Law

we want to show for any $w \geq 0$

$$\phi(w,y) = w \cdot x(w,y) \leq wx \quad \forall x \in V(y)$$

this will imply $\phi(w,y) = \min \ w \cdot x \quad s.t \ x \in V(y)$

To show $x(w,y)$ is feasible: By concavity,

$$\phi(w',y) \leq \phi(w,y) + \underbrace{\nabla \phi(w,y)}_{x(w,y)}(w'-w) \quad \text{for any } w'$$

$$\leq \phi(w,y) + w' \cdot x(w,y) - \underbrace{w \cdot x(w,y)}_{\phi(w,y)}$$

$$\phi(w',y) \leq w' x(w,y), \quad \forall w'$$

$$\Rightarrow x(w,y) \in V(y) \quad (\text{feasible})$$

Now notice that for any $x \in V(y)$, $w \cdot x \geq \phi(w,y) = w \cdot x(w,y)$

or $w \cdot x(w,y) \leq w \cdot x \quad \forall x \in V(y)$

i.e, $x(w,y) = \underset{x \in V(y)}{\arg\min} \ w \cdot x$

$$\phi(w,y) = w \cdot x(w,y)$$

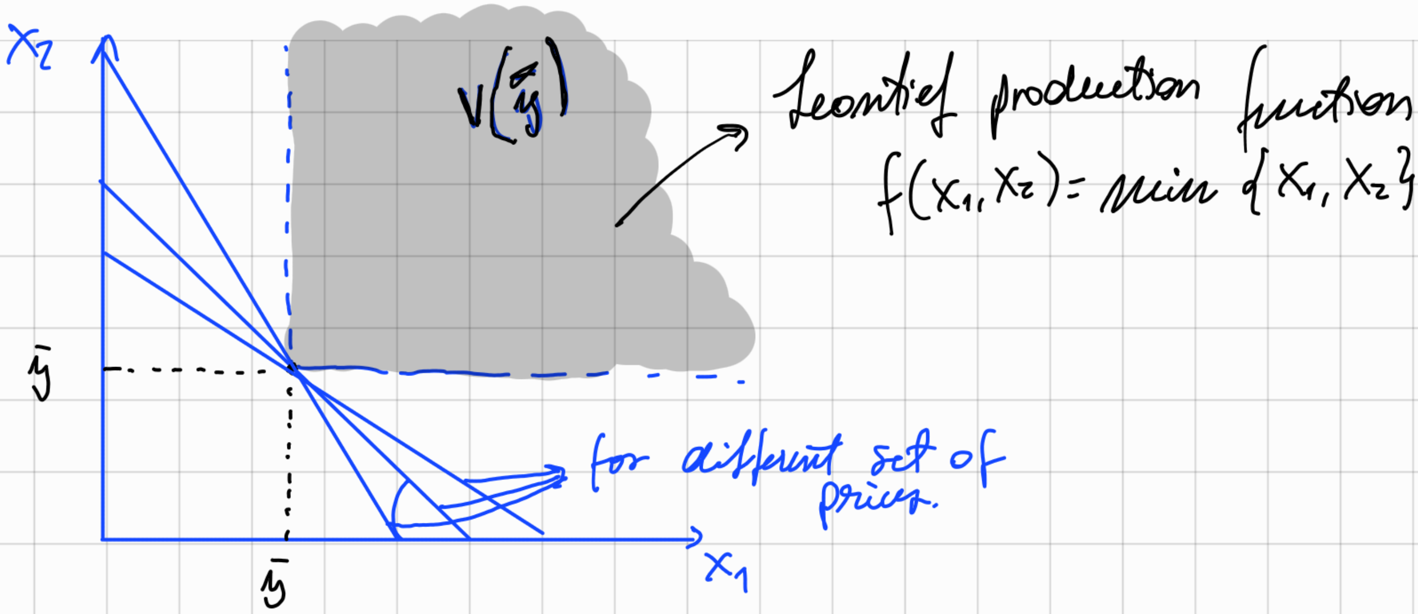
Example: $c(w,y) = y \cdot (w_1 + w_2)$

Fix an output level $\bar{y}$. Graphically, we want (x_1, x_2) s.t

$wx \geq c(w, \bar{y}) \quad \forall (w_1, w_2)$

$w_1 x_1 + w_2 x_2 \geq w_1 \bar{y}_1 + w_2 \bar{y}_2$

$\rightarrow x_2 \geq \frac{w_1 \bar{y}_1}{w_2} + \bar{y}_2 - \frac{w_1 x_1}{w_2}$



Algebraically

$$\frac{\partial C}{\partial w_1} = x_1^* \quad ; \quad \frac{\partial C}{\partial w_2} = x_2^* \qquad \text{(Shepard's Lemma)}$$

Here $\bar{y} = x_1^*(w, y)$; $\bar{y} = x_2^*(w, y)$

Lecture: Sept 27, 2023

$c(w,y) = (w_1 + w_2)y$ $\qquad$ $c(w,y) = y \cdot \min\{w_1, w_2\}$

$\updownarrow$ $\qquad$ $\updownarrow$

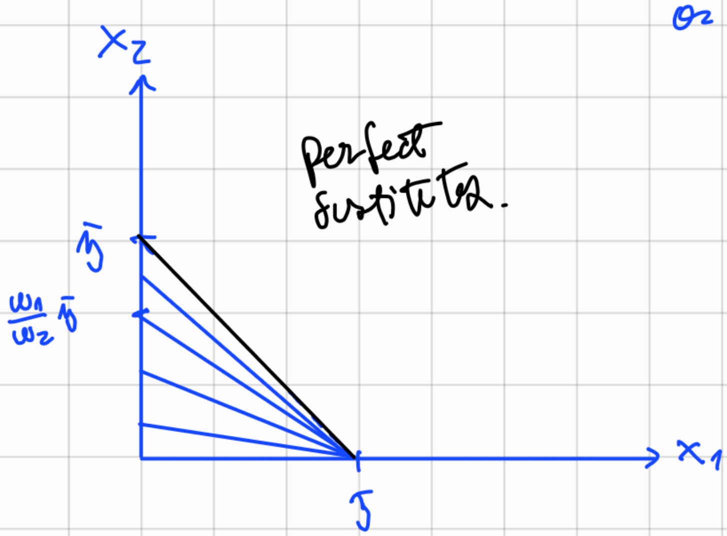
$f(x_1,x_2) = \min\{x_1, x_2\}$ $\qquad$ $f(x_1,x_2) = x_1 + x_2$

3 cases:

① $w_1 < w_2$; $c(w,\bar{y})\ \bar{y}w_1$

we want the points (x_1,x_2) s.t $w_1x_1 + w_2x_2 \geq \bar{y}w_1$

or $x_2 \geq \frac{w_1}{w_2}(\bar{y} - x_1)$ $\forall$ $\frac{w_1}{w_2} \in (0,1)$



② $w_1 = w_2 = w$; $c(w,y) = \bar{y}w$

$w(x_1 + x_2) \geq \bar{y}w$

$x_2 \geq \bar{y} - x_1$

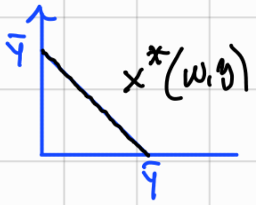
③ $w_1 > w_2$

Same as (1).

Algebraically:

$\underline{w_1 < w_2}$

at $w_1 = w_2$, Shepard's Lemma doesn't apply



by Shepard's Lemma

$x_1(w,y) = \frac{\partial c}{\partial w_1} = \bar{y}$; $x_2(w,y) = \frac{\partial c}{\partial w_2} = 0$

$\underline{w_1 > w_2}$

$x_1(w,y) = \partial c/\partial w_1 = 0$; $x_2(w,y) = \partial c/\partial w_2 = \bar{y}$

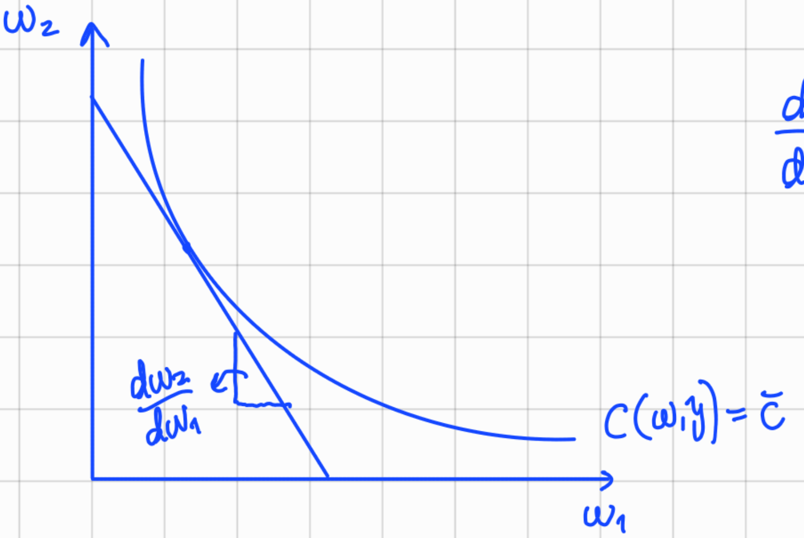
In general, take an isocost curve

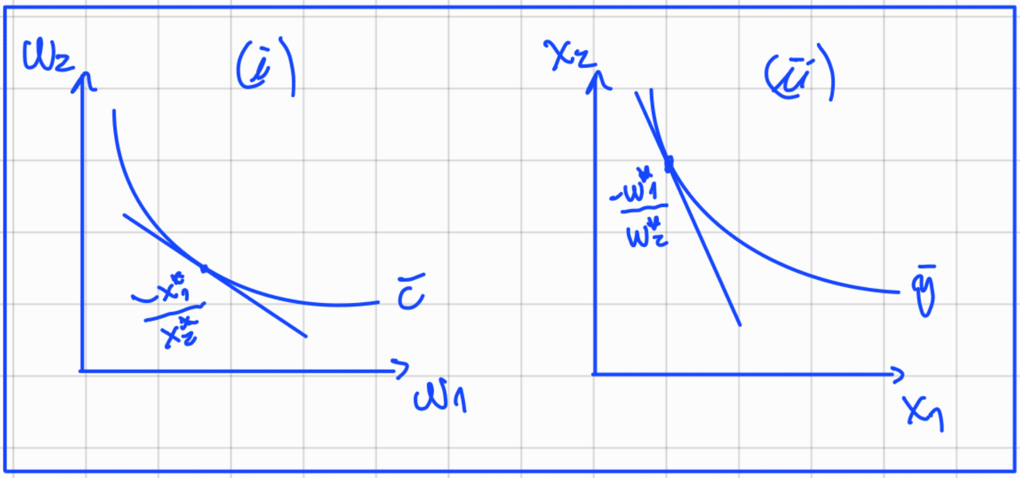
$$c(w,y) = \bar{c}$$

$$0 = d\bar{c} = \frac{\partial c}{\partial w_1} dw_1 + \frac{\partial c}{\partial w_2} \cdot dw_2$$

$$\frac{dw_2}{dw_1} = -\frac{\partial c/\partial w_1}{\partial c/\partial w_2} \overset{☆}{=} -\frac{x_1^*(w,y)}{x_2^*(w,y)}$$

☆ Shepard Lemma

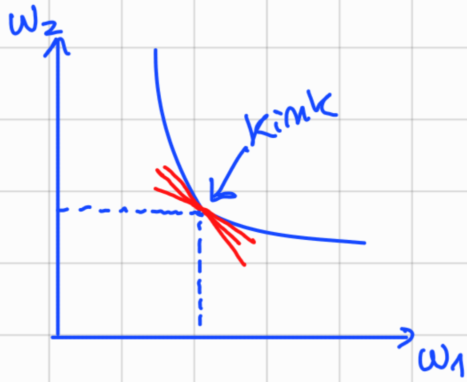




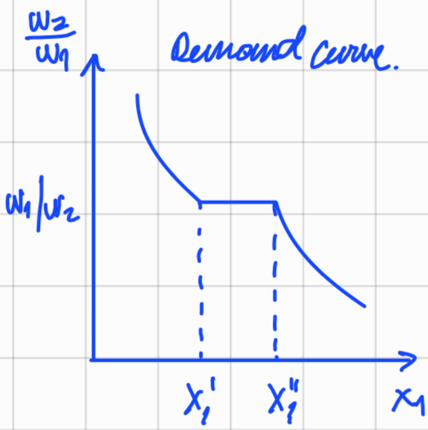
Duality between (i) & (ii)

Isoquant with a flat spot

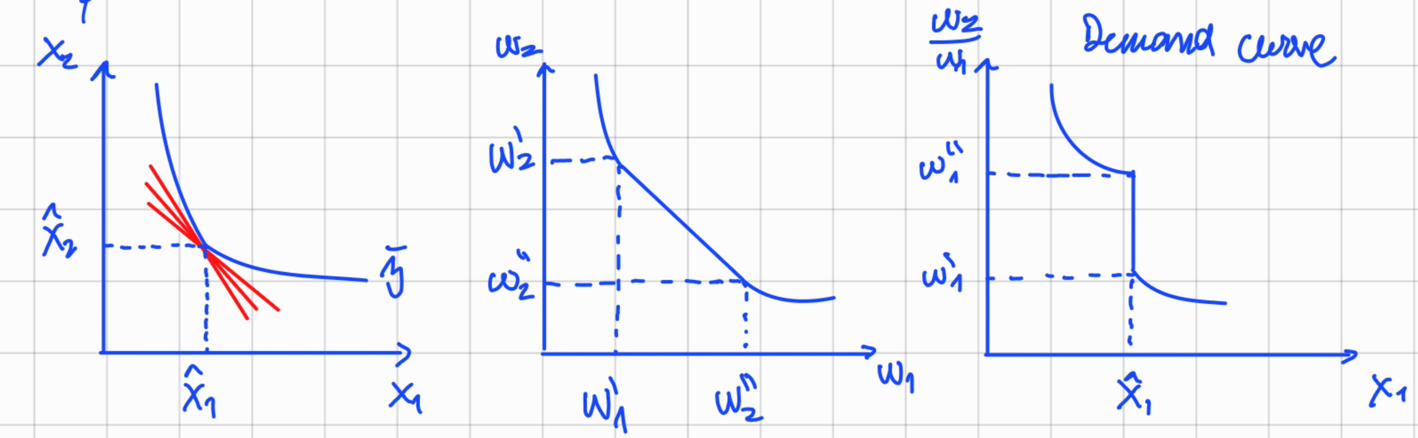
multiply optimal bundles



multiple tangents at the optimal.



Isoquant with a kink



Isoquant and isocost slopes are inversely related

Aggregation: J firms, production sets $Y_1, \dots, Y_J$ (non empty closed, $\ni 0$).

$\pi_j(p)$, $y_j(p)$ solution functions for firm j

$$y(p) = \sum_{j=1}^{\infty} y_j(p) = \{ y \in \mathbb{R}^n : y = \sum_{j=1}^{J} y_j', \text{ for some } y_j' \in y_j(p) \; \forall j \}$$

Recall for a single j, we had the law of supply

$$(p - p') \cdot (y_j(p) - y_j(p')) \geq 0$$

Add over j: $\displaystyle\sum_{j=1}^{J} (p-p')(y_j(p) - y_j(p')) = (p-p') \cdot \sum_{j=1}^{J} (y_j(p) - y_j(p'))$

$$= \underbrace{(p-p')(y(p) - y(p')) \geq 0}_{\text{Agg. Law of Supply}}$$

Agg substitution matrix, $Dy(p)$, also inherits properties such as symmetry, PSD.

Strengthen this to a "Representative firm" result.

Agg. production set:

$$Y^* = Y_1 + \dots + Y_J = \{ y \in \mathbb{R}^n : y = \sum_{j=1}^{J} y_j \text{, for some } y_j \in Y_J \ \forall j \}$$

a single firm with production set Y^*. Let $\pi^*(p)$, $y^*(p)$ be solution functions for a firm, Y^*.

<u>Theorem</u>: i) $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$ (This result does not hold for consumers)

ii) $y^*(p) = \sum_{j=1}^{J} y_j(p)$

proof of ii):
want to show

① $\sum_j y_j(p) \subseteq y^*(p)$

② $y^*(p) \subseteq \sum_j y_j(p)$

For ①: Let $y_j' \in y_j(p) \ \forall j$

$$p \cdot \underbrace{\left(\sum_j y_j'\right)}_{\text{so } \bar{y} \in y^*(p)} = \sum_j \underbrace{p \cdot y_j'}_{\pi_j(p)} = \sum_j \pi_j(p) \underset{\text{By (i)}}{=} \pi^*(p)$$

for ②: $y \subseteq \sum_j y_j(p)$

Take $y \in y^*(p)$ By construction, $y = \sum_j y_j'$ for some selection $y_j' \in Y_j$

$$p \cdot y = p \cdot \left(\sum_j y_j\right) = \pi^*(p) \underset{\text{By (i)}}{=} \sum \pi_j(p)$$

$$\boxed{\sum_j p \cdot y_j = \sum_j \pi_j(p)} \quad (☆)$$

Note that $p_j \cdot y_j' \leq \pi_j(p) \quad \forall j$

$$\sum p_j y_j' \leq \sum \pi_j(p) \quad (\star\star)$$

By $(\star)$ we know that $(\star\star)$ holds with strict equality $\forall j$.

Lecture : Monopoly — Oct 4, 2023

- Can "choose" output price (p)
- still face some constraint
  - technological constraint (cost function $C(y)$)
  - Consumer constraint (demand function $D(p)$)

Monopolist's problem : $\max\limits_{p,y} \; py - c(y)$ s.t $D(p) = y$

$\Rightarrow \max\limits_{p} \; p \cdot D(p) - C(D(p))$ (Single Variables)

Another way uses inverse demand function

$p(y) = D^{-1}(y)$

$$\max_{y \geq 0} \; p(y) \cdot y - C(y)$$

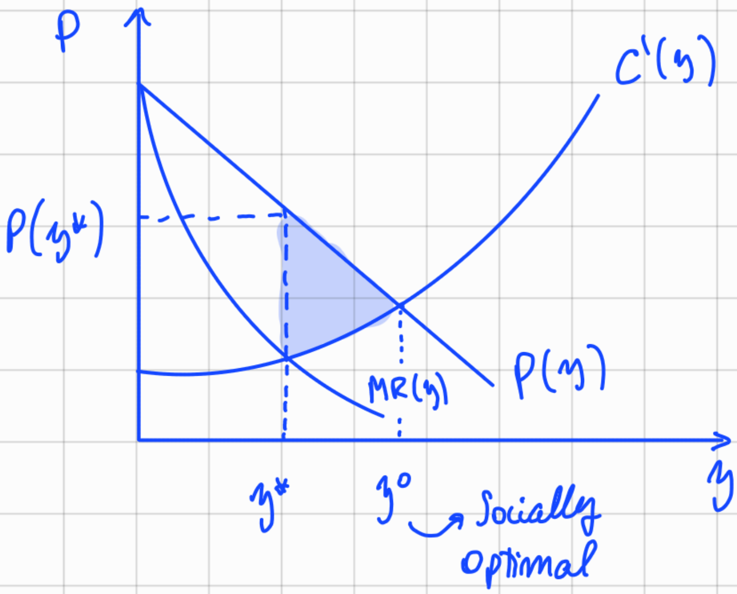
$\underline{\text{F.O.C}}$ $\quad \underbrace{p'(y^*)\, y^* + p(y^*)}_{\text{marginal revenue}} \leq \underbrace{C'(y^*)}_{\text{marginal cost}}$, with equality if $y^* > 0$

Assume $y^* > 0$, so $MR = MC$

$$p'(y^*)\, y^* + p(y^*) = C'(y^*)$$

Usual case : $p'(y) < 0$ (downward sloping demand)

$\Rightarrow p(y^*) > c'(y)$: monopolist prices above mg Cost



$$MR(y) = \underbrace{p'(y)}_{\text{This is negative}}y + p(y)$$

Q: How does monopolists price/output change as a parameter changes?

Tool here is the implicit function theorem: How does a solution function change as a parameter changes Contract with the envelope theorem which applies to the value function.

IFT "Recipe" :

- Start with f.o.c from max problem
- write solution function $x^*(\theta)$ implicitly as a function of θ
- Differentiate f.o.c wrt to θ (chain Rule)
- Solve resulting equation for $\dfrac{dx^*(\theta)}{d\theta}$

04 Oct, 2023

Back to monopolist...

Costs: $c(y,\theta)$

F.O.C $p'(y^*)y^* + p(y^*) - \frac{\partial c(y^*,\theta)}{\partial y} = 0$

write y^* implicitly as a function of θ.

$$p'(y^*(\theta))y^*(\theta) + p(y^*(\theta)) - \frac{\partial c(y^*(\theta),\theta)}{\partial y} = 0$$

Now differentiate w.r.t θ

$$p''(y^*(\theta),\theta)y^*(\theta)\frac{d y^*(\theta)}{\partial \theta} + p'(y^*(\theta))\frac{dy^*(\theta)}{d\theta} + p'(y^*(\theta))\frac{dy^*(\theta)}{d\theta}$$

$$-\frac{\partial^2 c(y^*(\theta),\theta)}{\partial y^2}\frac{dy^*}{d\theta} - \frac{\partial^2 c(y^*(\theta),\theta)}{\partial y\,\partial\theta}$$

Solve for $dy^*/d\theta$

$$\frac{dy^*}{d\theta} = \frac{\partial^2 c(y^*,\theta)/\partial y\partial\theta}{p''(y^*)y^* + 2p'(y^*) - \partial^2 c(y^*,\theta)/\partial^2 y}$$

Example: CES demand, constant mg cost

$D(p) = Ap^{-\beta}$, $\beta > 1$

$c(y) = \bar{c}y$

Why is it called CES?

Calculate price elasticity of demand: $\sigma_{D,p} = \dfrac{d \log D(p)}{d \log p}$

$$\log D(p) = \log A - B \log p$$

$$\sigma_{D,p} = -\beta$$

Inverse demand: $y = A p^{-\beta} \rightarrow p(y) = \left(\frac{y}{A}\right)^{-1/\beta}$

$$p'(y) \cdot y + p(y) = c'(y)$$

$$p'(y) = -\frac{1}{\beta} \underbrace{\left(\frac{y}{A}\right)^{-\frac{1}{\beta}-1}}_{p(y)} \left(\frac{1}{A}\right)$$

$$= -\frac{1}{\beta} p(y) \frac{1}{y}$$

F.O.C becomes $\left(-\frac{1}{\beta} p(y) \frac{1}{y}\right) y + p(y) = \tilde{c}$

$$p(y)\left[1 - \frac{1}{\beta}\right] = \tilde{c}$$

$$p(y) = \frac{\tilde{c}}{1 - 1/\beta}$$

What is dy^*/dc?

$y^*(c)$ defined implicitly by: $p(y^*(c))\left[1 - \frac{1}{\beta}\right] - \tilde{c} = 0$

Imp. func. T.:

$$p'(y^*(\tilde{c}))\left[1 - \frac{1}{\beta}\right] \frac{dy^*}{d\tilde{c}} - 1 = 0$$

$$\frac{dy^*}{dc} = \frac{1}{\underbrace{p'(y^*(c))}_{<0} \underbrace{\left\{1 - \frac{1}{\beta}\right]}_{>0}} < 0$$

Consumer theory:

More complicated for two reasons:
→ Rationality axioms on preferences
→ Cannot use optimization techniques on preferences. Need to "construct" a utility function.
→ Consumer's problem has prices in the constraints income/substitution effects.

widely used in economics
i) Normatively useful
ii) Positive predictions
iii) widely applicable
iv) Simple/sparse model.

Two approaches:
1) preferences ⟶ choices
2) choices → preferences (observed behaviour)

X: (abstract) set of alternatives
elements $x \in X$ are mutually exclusive

a choice structure $(\mathcal{B}, C(\cdot))$ is:
- $\mathcal{B}$ is a family of nonempty subsets of X
- $C(\cdot)$ is a choice rule s.t $C(B) \subseteq B \ \forall B \in \mathcal{B}$, $C(B) \neq 0$

$X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$

$C(\{x, y\}) = \{x\}$, $C(\{x, y\}) = \{x, z\}$

What are some "reasonable" restrictions on behaviour?

Weak axiom of Revealed preference (WARP)

If, for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any other $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C(B')$, we also have $x \in C(B')$

$X = \{a, b, c\}$ $\quad \mathcal{B} = \{ \underbrace{\{a, b\}}^{B}, \underbrace{\{a, b, c\}}^{B'} \}$

starting from choices to preferences...

we observe $C(\{a,b,c\}) = \{b\}$
WARP implies that $C(\{a,b\}) = \{b\}$

Why? Assume $a \in C(\{a,b\})$. Then, since a was chosen when b was available, we must have $a \in C(\{a,b,c\})$. Contradiction.

we observe $C(\{a,b\}) = \{b\}$ (we know nothing about c).
WARP implies $C(\{a,b,c\}) = \{b\}, \{c\}$ or $\{b, c\}$

Preference-based approach

X: set of alternatives.
Primitive is a binary relation on X, $\succsim$.

$x \succsim y$: "x is at least as good as y"

$x \succ y$: $x \succsim y \wedge y \not\succsim x$: "x is strictly preferred to y"

$x \sim y$: $x \succsim y \wedge y \succsim x$: "x is indifferent to y"

A preference relation is rational if
i) Complete: $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both).
ii) Transitivity: $\forall x, y \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$

Violation of transitivity: Framing (Kanheman, Tversky).
Best buy to purchase \$125 stereo and \$15 calculator.
A) Calculator on sale for \$5 off at store 20 min away — YES
B) Stereo " " " " — NO
c) Both are out of stock at current store, but we'll give you a \$5 copon for one item. Does it matter which one? — Indiferent.

X: Travel to store 2 for \$5 calculator.
Y: " " " stereo
Z: Buy both items at store 1.

Reponse imply:

$x \succ z$ (A)
$z \succ y$ (B)
$x \sim y$ (c)

What are the links between choice & preference approach?
Q2. ¿Does WARP imply rationality? NO
Q1. ¿Does Rationality imply WARP? Yes.

For Q1. Given $\succsim$, let $C(B, \succsim) = \{x \in B : x \succsim y, \forall y \in B\}$
Theorem: Suppose $\succsim$ are rational. Then, the choice structure $(\mathcal{B}, C(B \succsim))$ satisfies WARP.
Proof: Suppose for some $B \in \mathcal{B}$, we have $x, y \in B$ and $x \in C(B, \succsim)$ by definition of C, $x \succsim y$.

Check WARP: Suppose $x, y \in B'$, and $z \in C(B', \succsim)$. So, $y \succsim z$ for all $z \in B'$. By transitivity $x \succsim y \succsim z$, so, $x \succsim z$, for all $z \in B'$. Thus $x \in C(B', \succsim)$ and WARP holds.

For Q2: Counterexample.

$X = \{a, b, c\}$, $\mathcal{B} = \{\{a,b\}, \{b,c\}, \{a,c\}\}$

assume we observe : $C(\{a,b\}) = a$; $C(\{a,c\}) = c$; $C(\{b,c\}) = b$

Can check that $(\mathcal{B}, C())$ satisfies WARP.

Observed choices implies: $\left.\begin{array}{l} a \succ b \\ a \succ c \\ b \succ c \end{array}\right\}$ this violates transitivity.

<u>Theorem (Arrow, 1959):</u>

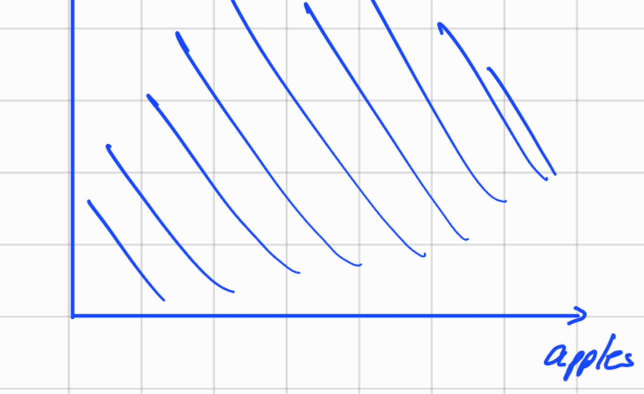
If $(\mathcal{B}, C(\cdot))$ is s.t:

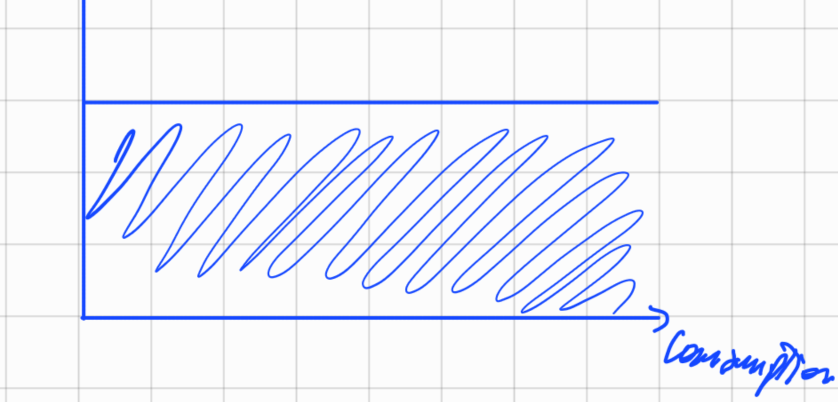
i) WARP is satisfied

ii) $\mathcal{B}$ includes all subsets of X with up to 3 elements.

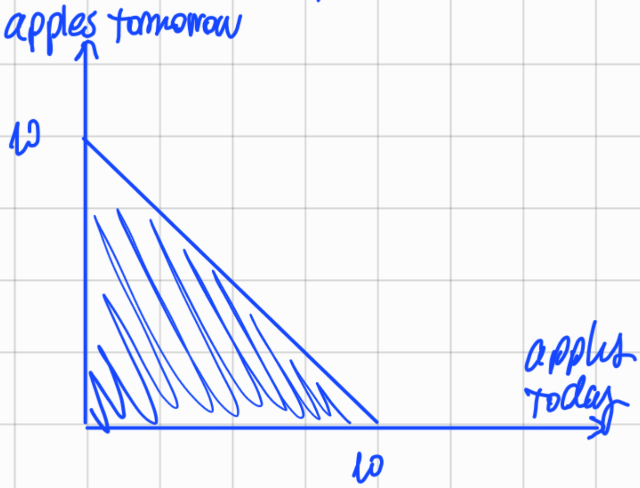
Then, there is a (unique) rational preference relation $\succsim$ that rationalizes $C(\cdot)$.

Restrict $X \subseteq \mathbb{R}^n_+$





Goods can be timed:



State dependent

Properties of preferences on $X \subseteq \mathbb{R}^n_+$

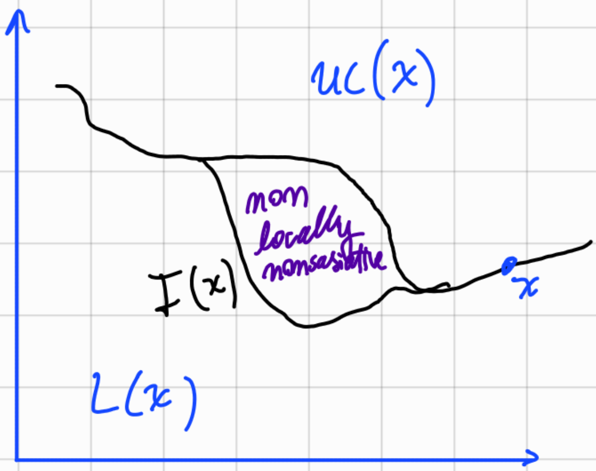
- → Preference relation $\succsim$ is monotone if $y \gg x$ implies $y \succ x$
- → $\succsim$ is strongly monotone if $y \geq x$ and $y \neq x$, implies $y \succ x$
- → $\succsim$ is locally nonsatiative if for every $x \in X$ (bundle) and $\varepsilon > 0$ there is some $y \in X$ s.t $\|y - x\| < \varepsilon$ and $y \succ x$.

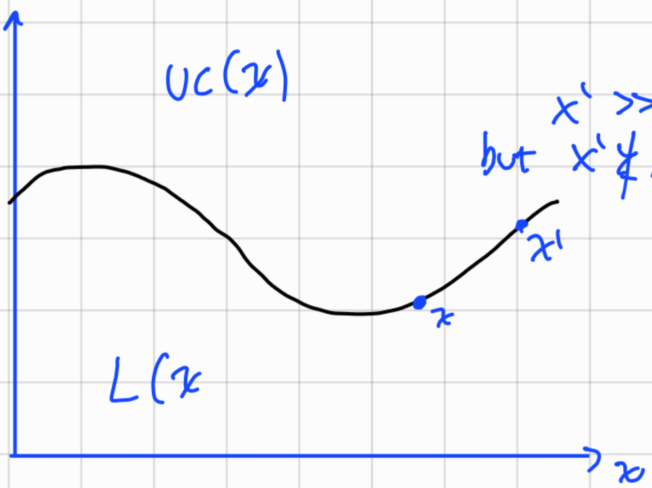
upper contour of x

$uC(x) = \{y \in X : y \succsim x\}$

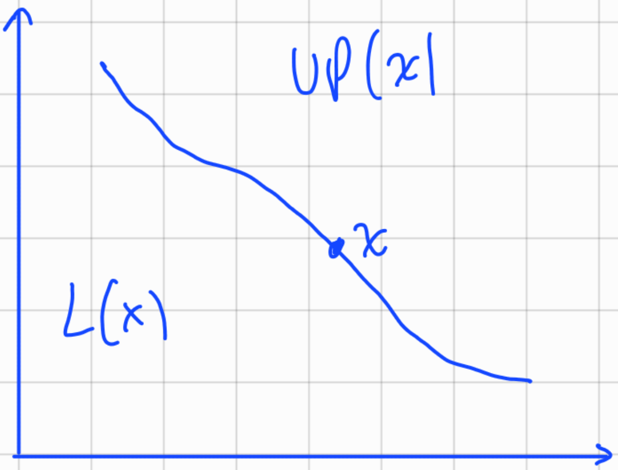
$\mathcal{L}C(x) = \{y \in X : x \succsim y\}$.

$I(x) = \{y \in X : x \sim y\}$

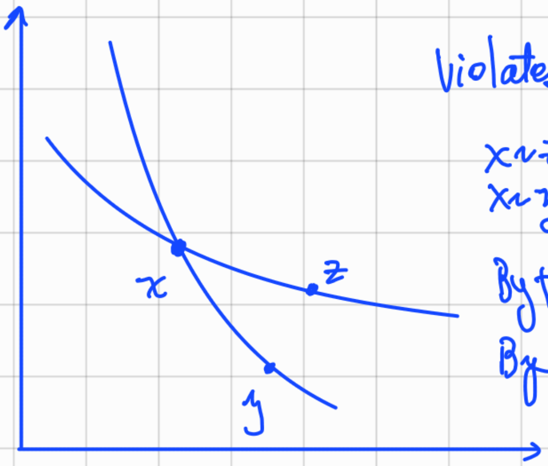




Monotonicity → Downward slope



IC cannot cross



Violates transitivity

$x \sim z$
$x \sim y$

By trans. $x \sim z$

By monotone: $z > y$

contradiction.

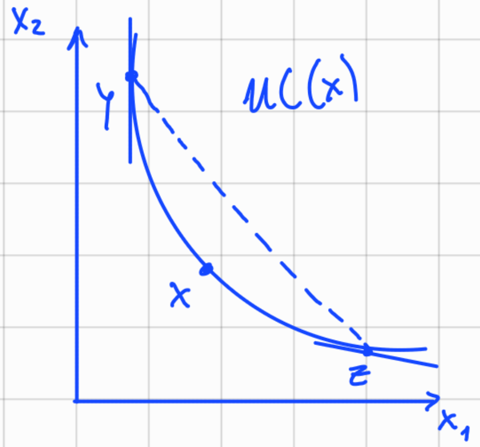
Lecture

Oct 09, 2023

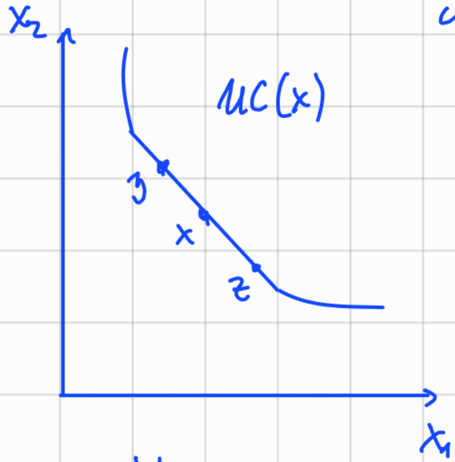
Convexity

$\succsim$ is convex if the upper contour set $UC(x) = \{ y \in X : y \succsim x \}$ is convex

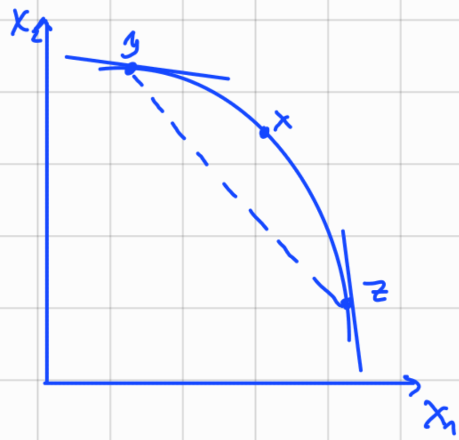
or equivalently, $y \succsim x$ and $z \succsim x$, then $\underbrace{\alpha y + (1-\alpha) z}_{w} \succsim x$, $\forall \alpha \in [0,1]$



strictly convex



weakly convex



not convex

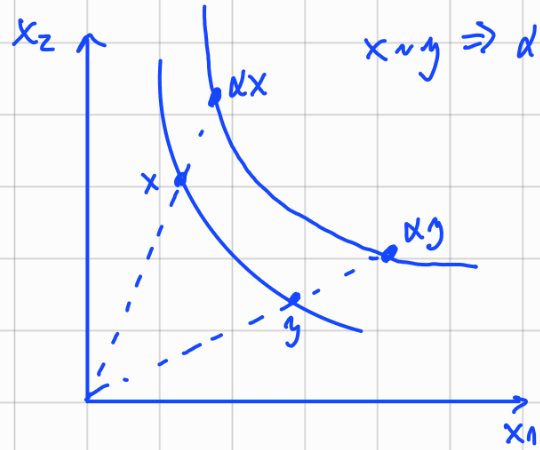
Interpretation of convexity:

① Diminishing marginal rate of substitution

② Preferences for "diversification"

Useful "classes" of preferences

1) Homothetic:



$x \sim y \Rightarrow \alpha x \sim \alpha y \quad \forall \alpha \geq 0$

2) Quasilinear preferences:

Let $X = (-\infty, \infty) \times \mathbb{R}_+^{n-1}$ (1st good can be negative; and there are other $n-1$ goods)

write $x = (t, y)$ where $t \in \mathbb{R}$ and $y \in \mathbb{R}_+^{n-1}$

$\succsim$ is quasilinear if:

i) Good 1 is desirable. $t' > t$, then $(t', y) \succ (t, y)$

ii) "No wealth effect"

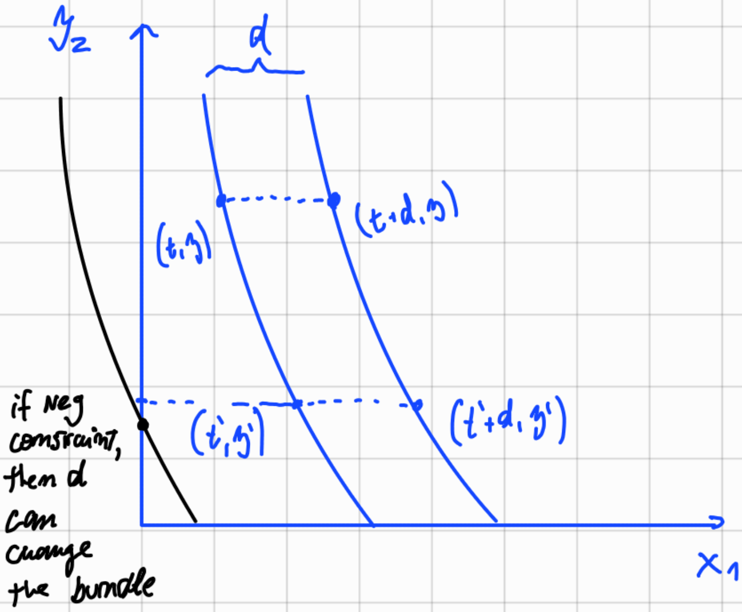
If $(t, y) \sim (t', y')$, then for all $d \in \mathbb{R}$, $(t+d, y) \sim (t'+d, y')$

→ good 1 often called "numeraire" (think of as "money").

→ will allow us to write utility as

$$u(t, y_1, \dots, y_n) = t + V(y_1, \dots, y_n)$$

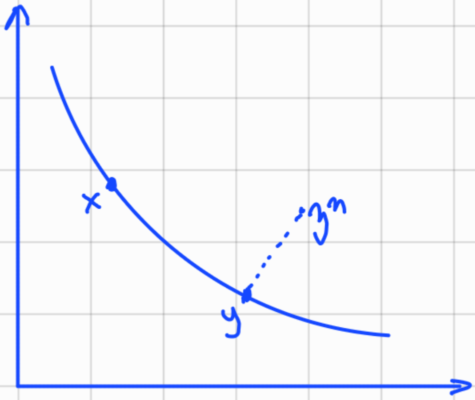
also called "transferable utility").



Continuity:

$\succsim$ is continuous if, for any sequence, $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n \; \forall n$, and $x = \lim_{n\to\infty} x^n$, $y = \lim_{n\to\infty} y^n$, then we have $x \succsim y$

An equivalent definition: $UC(x)$ and $LC(x)$ are closed sets



$z^n \to z$ Let $x^n = x$, $\forall n$

$z^n \succsim x$, $z^n \in UC(x)$

and so by continuity implies $z \in UC(x)$

Example: Lexigraphic

$X = \mathbb{R}^2_+$. Define $\succsim$ as follows: $x \succsim y$ if either:

i) $x_1 > y_1$

ii) $x_1 = y_1$ & $x_2 \geq y_2$

$\succsim$ is:

complete

transitive

strongly monotone

strongly convexity

$\succsim$ is not continuous: $x^n = \left(\frac{1}{n}, 0\right)$, $z^n = (0, 1)$ $\forall n$

$x^n \succ y^n \; \forall n$

$x = \lim_{n\to\infty} x^n = (0, 0)$

$z = \lim_{n\to\infty} y^n = (0, 1)$

and so $z \succ x$

Utility function: A preference relation is represented by a utility function $u: X \to \mathbb{R}$ if, $\forall x, y$

$$x \succsim y \longleftrightarrow u(x) \geq u(y)$$

utility functions are not unique.

Let $g: \mathbb{R} \to \mathbb{R}$ be any strictly increasing function.

Define $\hat{u}(x) = g(u(x))$

$$\hat{u}(x) \geq \hat{u}(y) \longleftrightarrow u(x) \geq u(y) \longleftrightarrow x \succsim y$$

$\Rightarrow \hat{u}(\cdot)$ also represents $\succsim$

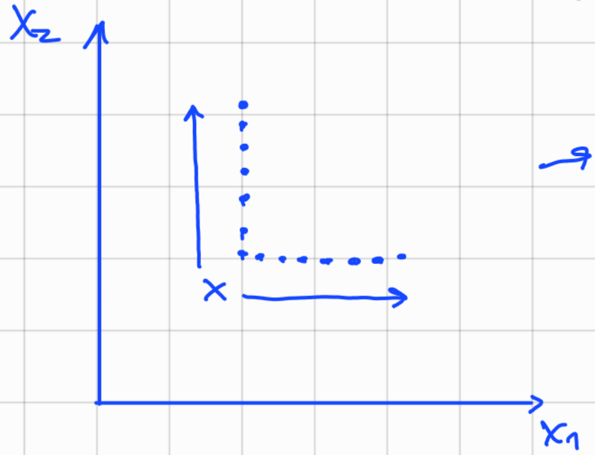
Q:

When can we find a utility function to represent $\succsim$?

<u>Theorem</u>: $\succsim$ can be represented by a utility function only if it is rational.

proof: Homework.

Q: Is the following true?: If $\succsim$ is rational, then there exists a utility function representing it. ANS: NO.



→ Is rational but not enough for a utility func. Representation.

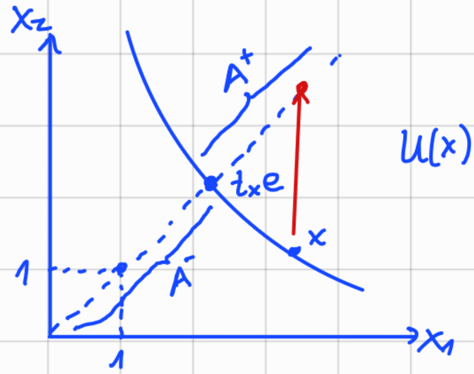
Sufficient conditions for a utility representation

① Finiteness: If X is finite, there exists a utility representation
② Continuity of $\succsim$

Theorem: If $\succsim$ is rational & continuous, then there exists a utility function that represents them. Furthermore, there exists a continuous $u(\cdot)$ that represents them

Proof: Show for case that $\succsim$ is strongly monotone.
sketch:
① Construct a candidate $u(x)$ & ② check it works.

Let $\underbrace{e=(1,\dots,1)}_{\text{unit vector}}$. Take a bundle $x \in X$



$U(x) = t_x$

$A^+(x) = \{t \in \mathbb{R}_+ : te \succsim x\}$
$A^-(x) = \{t \in \mathbb{R}_+ : x \succsim te\}$

By strong monotonicity: $A^+(x) \neq \phi$ (take $\max\{x_1,\dots,x_n\}e \succsim x$).
$A^-(x) \neq \phi$ $(x \succsim 0)$

Completeness of $\succsim \rightarrow \mathbb{R}_+ = A^+(x) \cup A^-(x)$
By continuity, $A^+(x)$ & $A^-(x)$ are closed sets
Since $\mathbb{R}_+$ is connected, $A^+(x) \cap A^-(x) \neq \phi \rightarrow$ this means that
$\exists t_x \in A^+ \cap A^-$ s.t. $t_x e \sim x$

Define $u(x) = t_x$

Last step, show proposed $u(x)$ represents $\succsim$ | Oct 11, 2023

① $u(y) \geq u(x) \Rightarrow y \succsim x$

② $y \succsim x \Rightarrow u(y) \geq u(x)$

For ①: $t_y \geq t_x$ (def of u)

$t_y e \geq t_x e$ (monotonicity)

$y \sim t_y e \succsim t_x e \sim x$ (def of t)

$y \succsim x$ (transitivity)

For ②: Take $y \succsim x$

$(t_y e) \sim y \succsim x \sim (t_x e)$

Def of t

$t_y e \succsim t_x e$ (transitivity)

$t_y \geq t_x$ (monotonicity)

$u(y) \geq u(x)$ (definition of u)

- Also true that a continuous $u(\cdot)$ exists (skip proof)
- Further conditions on $\succsim$ that guarantee a differentiable $u(\cdot)$ exists (also skip)
- We have constructed an utility representation; there are many others, not all are continuous.
- $g(x) = \begin{cases} x, & x \leq 1 \\ 2x, & x > 1 \end{cases}$, $\tilde{u}(x) = g(u(x))$ also represents $\succsim$

From conditions on preferences to conditions on $u(\cdot)$

① $\succsim$ convex $\Rightarrow \exists$ a quasiconcave $u(\cdot)$

proof: $\succsim$ convex means

$u(x) = \{y : y \succsim x\}$ is convex

But $y \succsim x$ is equivalent to $u(y) \geq u(x)$

$uc(x) = \{y : u(y) \geq u(x)\}$ is convex

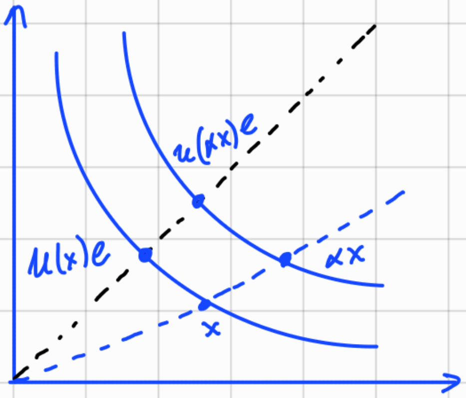
Which is a definition of quasiconcave $u(\cdot)$

Also true $\succsim$ strictly convex $\Rightarrow \exists$ a $u(\cdot)$ strictly quasiconcave

Not True: Convex $\succsim \Rightarrow \exists$ $u(\cdot)$ concave

② $\succsim$ homothetic (and monotone) $\Rightarrow \exists$ an hd-1 function $u(\cdot)$ that represents $\succsim$

Proof: Let $\succsim$ be homothetic & monotone



$u(x)\cdot e \sim x$ & $u(\alpha x)e \sim \alpha x$

$\alpha u(x)e \sim \alpha x$ (homotheticity)

$\underline{u(\alpha x)e} \sim \underbrace{\alpha u(x)}e$ (transitivity)

$u(\alpha x) = \alpha \cdot u(x)$ (monotonicity)

Classic homothetic utility: Cobb-Douglas

$u(x) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

It is wlog to assume $\alpha_1 + \cdots + \alpha_n = 1$

(why?) say $\alpha_1 + \cdots + \alpha_n = k$. Consider: $\tilde{u}(x) = (u(x))^{1/k}$. Then $\tilde{u}$ & u represent the same preferences, and $\tilde{u}(x) = x_1^{\alpha_1/k} \cdot x_2^{\alpha_2/k} \cdots x_n^{\alpha_n/k}$

$$\left(\frac{\alpha_1}{k} + \cdots + \frac{\alpha_n}{k}\right) = \frac{1}{k}\frac{(\alpha_1 + \cdots + \alpha_n)}{k} = 1.$$

Cobb-Douglas is a special case of CES function:

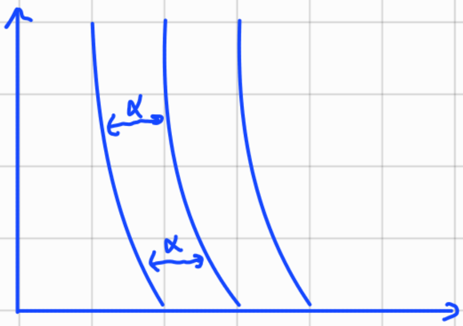
$u(x) = [\alpha_1 x_1^\rho + \cdots + \alpha_n x_n^\rho]^{1/\rho}$ (this is a hd-1)

$\rho = 1$: perfect substitutes

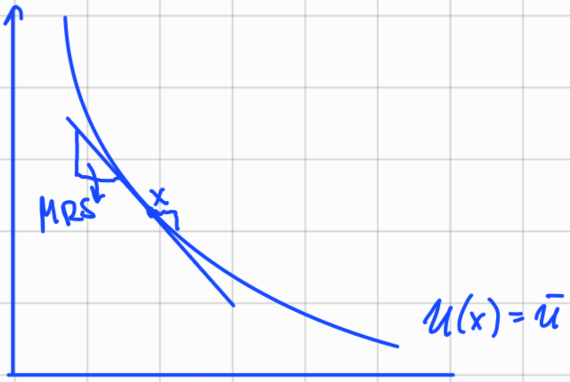
$\rho = 0$: Cobb-Douglas

$\rho = -\infty$: perfect complements / Leontief

③ $\succsim$ Quasilinear (in x_1) $\Rightarrow \exists$ a $u(\cdot)$ of the form $u(x_1,\dots,x_n) = x_1 + v(x_2,\dots,x_n)$



Marginal rate of substitution



$u(x) = \bar{u}$ | total derivative.

$$\frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} \cdot dx_j = d\bar{u} = 0$$ → BC we are the same curve.

$$\Rightarrow \frac{dx_j}{dx_i} = -\frac{\partial u/\partial x_i}{\partial u/\partial x_j}$$

where $MRS_{ij}(x) = \dfrac{\partial u/\partial x_i}{\partial u/\partial x_j}$

Budget Set

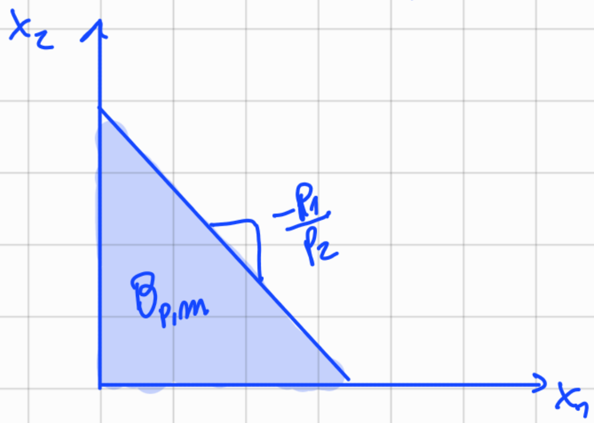
Consumption space: $X \in \mathbb{R}^n_+$

Standard (Walrasian) budget set: $B_{p,m} = \{ x \in X : p \cdot x \leq m \}$

prices (p), Income (m)

In 2-D: $\{ x \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 \leq m \}$

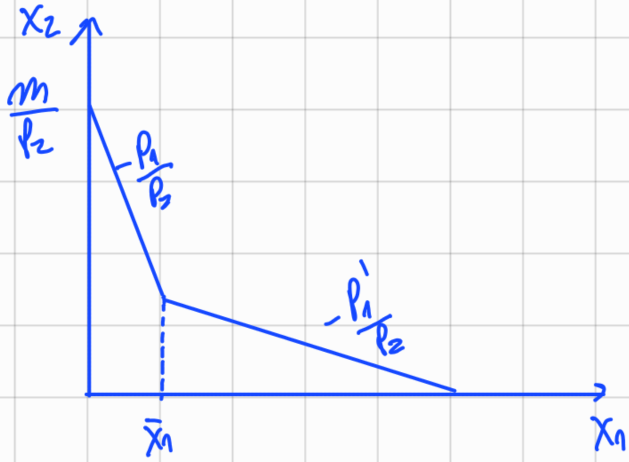
$$\Rightarrow x_2 \leq \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$



Walrasian budget set is convex

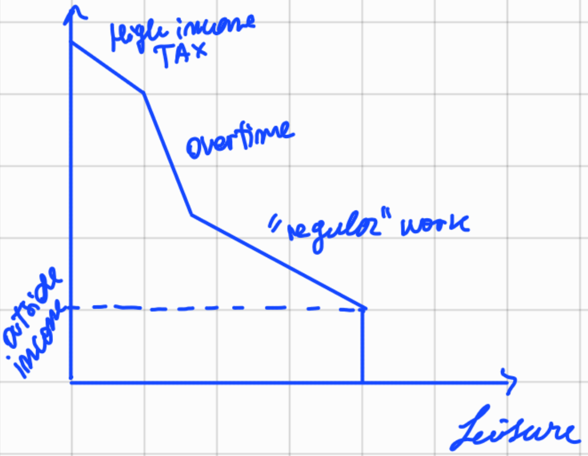
* Nonconvex budget sets come up all the time
  - progressive taxation
  - welfare payments
  - nonlinear pricing.

Non linear pricing : Example. Discount on X_1 if you purchase $\geq \bar{x}_1$ units.



* Kuhn tucker conditions are not sufficient in this case.

Labor supply model : $u(c, \ell)$, p is the price of leisure (wage).



"Classic" utility maximization problem (UMP)

$$\max_{\substack{x_i \geq 0 \\ p \cdot x \leq m}} u(x)$$

Solution functions: $X(p,m) = \underset{\substack{x_i \geq 0 \\ px \leq m}}{\operatorname{argmax}}\, u(x)$ (marshallian) demand correspondence

$V(p,m) = \max_{\substack{x_i \geq 0 \\ px \leq m}} u(x)$ Indirect utility function.

Question: Existence of solution to UMP?

Theorem:

i) If $p \gg 0$, & $u(\cdot)$ is continuous then UMP has a solution

ii) If, in addition, $\succsim$ are locally non satiative, then $p \cdot x^* = m$ forall $x^* \in x(p,m)$ (walras law)

Proof i): $p \gg 0$ ensures $B_{p,m}$ is compact (closed & bounded). Continuous function on compact sets have a maximum

proof ii): Assume $p \cdot x^* < m$, then x^* is in the interior of $B_{p,m}$ By LNS, $\exists\, y \in B_{p,m}$ s.t $u(y) > u(x^*)$.

This contradicts that x^* is optimal

Characterization Solutions

Oct 16, 2023

$$\max_x \; u(x) \qquad x_i \geq 0 \;, \; px \leq m$$

$$\mathcal{L} = u(x) - \lambda(px - m) + \mu_i x_i$$

KT conditions: $\frac{\partial u}{\partial x_i} - \lambda p_i + \mu = 0$ $\longrightarrow$ $\frac{\partial u}{\partial x_i} \leq \lambda p_i$, with equality if $x_i^* > 0$

$$\lambda(px - m) = 0 \;, \; \mu_i x_i = 0$$

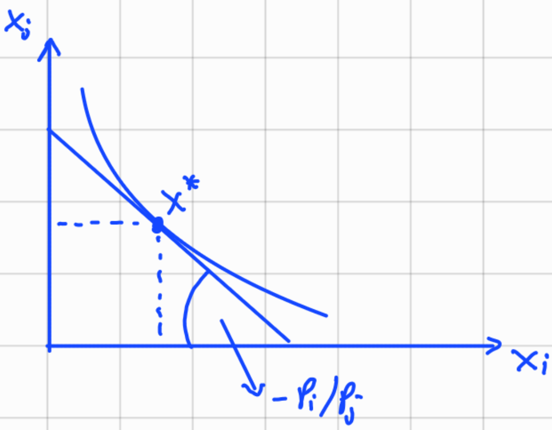
$$\lambda_i, \mu_i \geq 0$$

Take $\partial x_i^*, x_j^* > 0$. Divide i by j

$$\frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j}$$

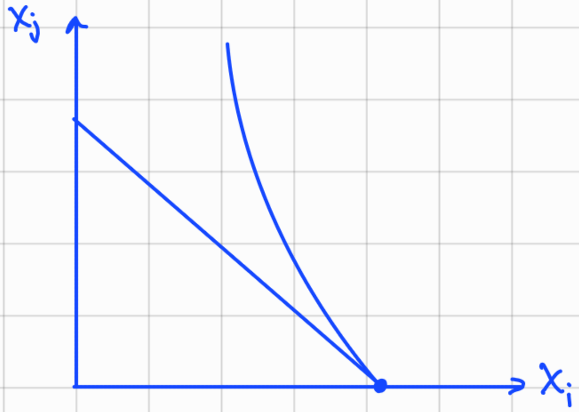
$$MRS_{ij} = \frac{p_i}{p_j}$$

$$\frac{\partial u / \partial x_i}{p_i} = \frac{\partial u / \partial x_j}{p_j}$$



Only works for interior solutions!

If $x_j^* = 0$, then could have $MRS_{ij} \neq p_i/p_j$



$$MRS_{ij} > \frac{p_i}{p_j}$$

Properties of $V(p,m)$ (value / indirect utility function)

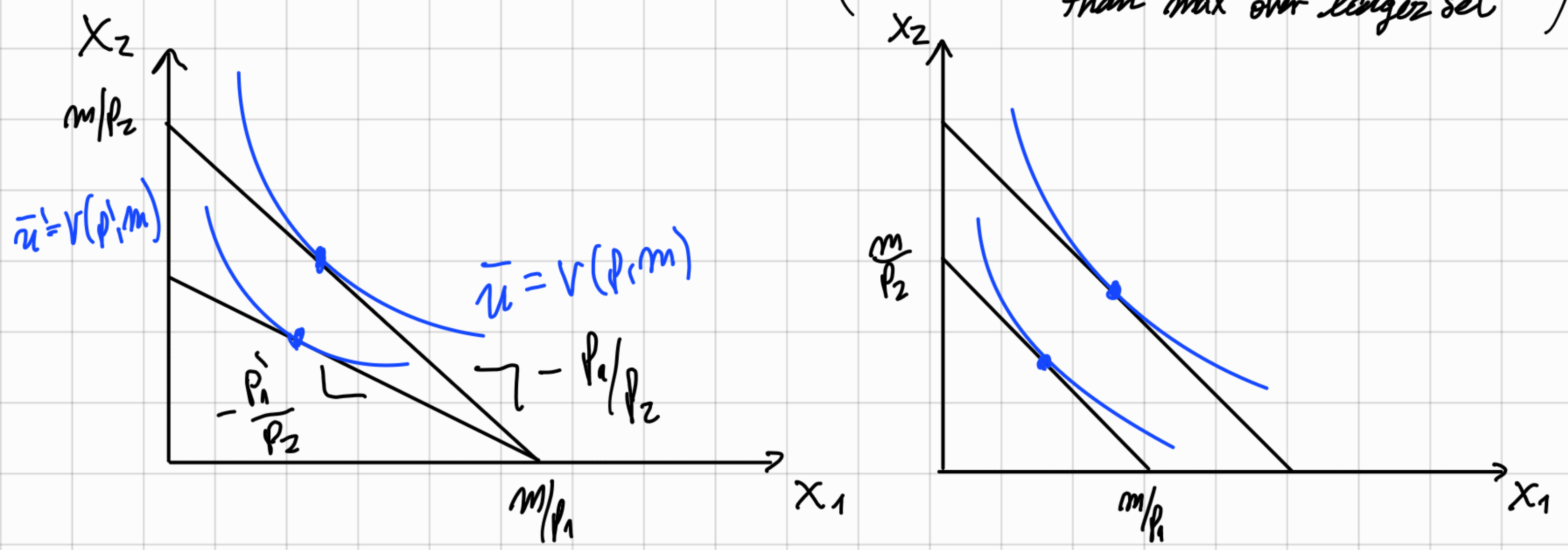
$V(p,m) = u(x^*(p,m))$ (analogous to profit function).

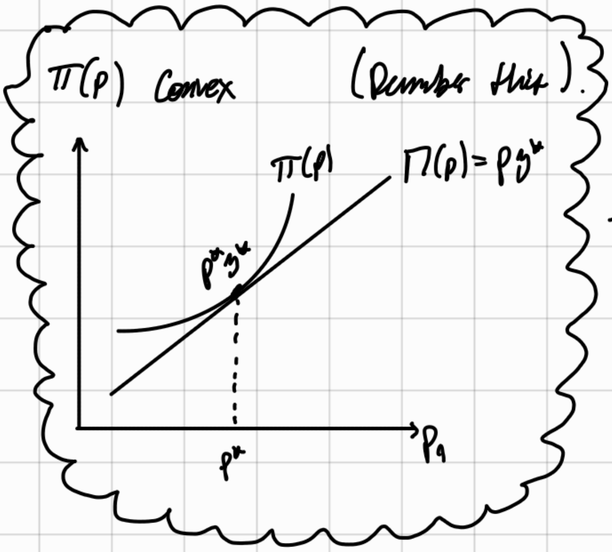
① $V(p,m)$ is hd-0 in (p,m)

proof: $B_{p,m} = B_{\alpha p, \alpha m}$ Same objective, same constraints $\Rightarrow$ same solution

② $V(p,m)$ is nonincreasing in p_i, nondecreasing in m

proof: let $p' \geq p$ then $B_{p',m} \subseteq B_{p,m}$ (max over a smaller set is (weakly) less than max over larger set)





$V(p,m)$ is quasiconvex in (p,m) that is, the set $\{(p,m) : V(p,m) \leq \bar{v}\}$ is convex.

* When prices change you may want to do what you were doing before (different from profit case).

Proof: Take $V(p,m) \leq \bar{V}$, $V(p',m') \leq \bar{V}$. We want to show $V(p'',m'') \leq \bar{V}$

Sufficient: $p'' \cdot x \leq m'' \Rightarrow u(x) \leq \bar{V}$

$\alpha p x + (1-\alpha) p' x \leq \alpha m + (1-\alpha) m'$ (*)

(...)

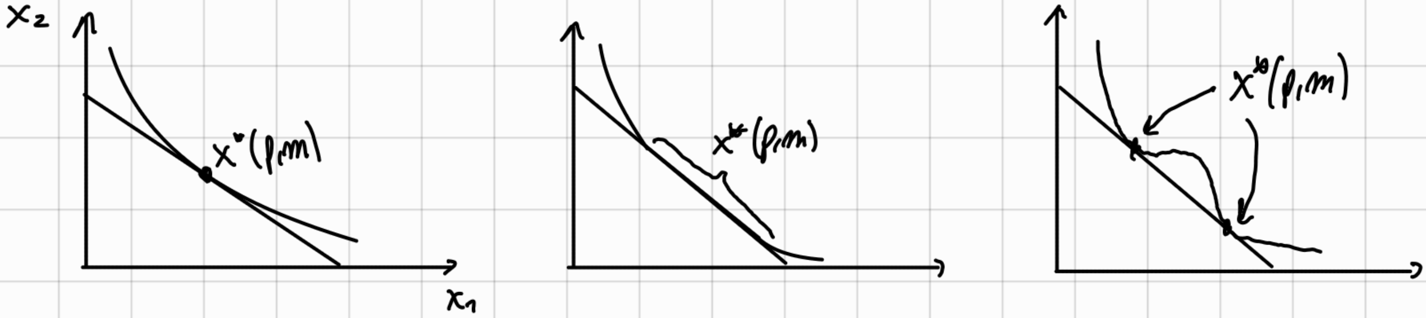
One of the following must hold:
i) $\alpha p x \leq \alpha m$ or ii) $(1-\alpha) p' x \leq (1-\alpha) m$ or (*) is false.

If x is affordable at (p, m) or (p', m') (or both)
If i) holds: $u(x) \leq V(p, m) \leq \bar{V}$
If ii) holds: $u(x) \leq V(p', m') \leq \bar{V}$

(Convex preferences $\leftrightarrow$ quasiconcave $u(x)$)

Properties of $x(p, m)$

① $x(p, m)$ is hd-0 (proof: Same as before)
② $p \cdot x = m$ for all $x \in X(p, m)$
③ If $\succsim$ is convex, then $x(p, m)$ is a convex set.
If $\succsim$ is strictly convex, then $X(p, m)$ is a singleton



strict convex $\succsim$ quasi convex $\succsim$ non convex $\succsim$

Roy's Identity: (Remember Hotelling's Lemma $\frac{\partial \pi}{\partial p_i} = y_i(p)$)

Result of the envelope theorem. Can do the same for UMP:

$$\mathcal{L} = u(x) - \lambda (p \cdot x - m)$$ (assume interior solution [NO KTC])

$$\frac{\partial V(p, m)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i^*(p, m)$$ (Envelope theorem)

Get rid of λ: Envelope theorem again on m $\frac{\partial V}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda$

Roy's Identity: $\boxed{x_i^*(p,m) = -\dfrac{\partial v/\partial p_i}{\partial v/\partial m}}$

Basic example: $u(x_1,x_2) = x_1^\alpha x_2^\beta$, $\alpha+\beta=1$.

Equivalent to $u(x_1,x_2) = \alpha \log x_1 + \beta \log x_2$

$$\max \ \alpha \log x_1 + \beta \log x_2$$

The domain is restricted so we can ignore $x_i>0$ constraint.

$$p_1x_1 + p_2x_2 \leq m$$

$$\mathcal{L} = \alpha \log x_1 + \beta \log x_2 - \lambda(p_1x_1 + p_2x_2 - m)$$

f.o.c

(1) $\dfrac{\alpha}{x_1} = \lambda p_1$ (2) $\dfrac{\beta}{x_2} = \lambda p_2$ (3) $p_1x_1 + p_2x_2 = m$

Solve for x_1, x_2, λ: $x_1(p,m) = \dfrac{\alpha m}{p_1}$; $x_2(p,m) = \dfrac{\beta m}{p_2}$

plug $x_i(p,m)$ in $u(\cdot)$

$$\Rightarrow v(p,m) = \alpha \log\left(\frac{\alpha m}{p_1}\right) + \beta \log\left(\frac{\beta m}{p_2}\right)$$ (indirect utility function)

Check Roy's identity:

$$\frac{\partial v}{\partial p_1} = \frac{\alpha p_1}{\alpha m}(-\alpha m) = -\alpha p_1 \qquad \frac{\partial v}{\partial p_2} = \frac{\beta p_2}{\beta m}(-\beta m) = -\beta p_2$$

$$\frac{\partial v}{\partial m} = \frac{\cancel{\alpha} \cancel{p_1}}{\cancel{\alpha} m} \cdot \frac{\alpha}{\cancel{p_1}} + \frac{\cancel{\beta} \cancel{p_2}}{\cancel{\beta} m} \cdot \frac{\beta}{\cancel{p_2}} = \frac{\alpha}{m} + \frac{\beta}{m} = \frac{1}{m}$$

$x_1(p,m) = \dfrac{\alpha m}{p_1}$; $x_2(p,m) = \dfrac{\beta m}{p_2}$ (same as before).

Expenditure minimization problem (EMP) (Analogous to CMP)

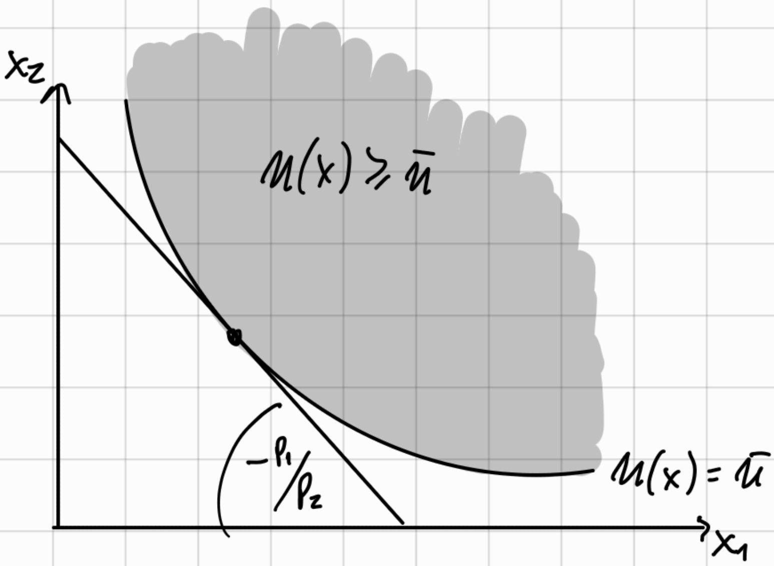
$$\min p \cdot x \quad \text{st} \quad u(x) \geq \bar{u}$$

This is exactly isomorphic to $\min w \cdot x$ st $f(x) \geq y$ (CMP)
→ properties and all the same as CMP

Solution functions: $\underbrace{h(p,\bar{u})}_{\text{Hicksian Demand}} = \arg\min p \cdot x$ s.t $u(x) \geq \bar{u}$

(optimal bundle) (conditional factor demand in CMP)

Value function: Expenditure function: $e(p,\bar{u}) = \min\limits_{\text{st } u(x) \geq \bar{u}} p \cdot x = p \cdot h(p,\bar{u})$ (cost function in CMP)



Properties of $e(p,\bar{u})$ (we did the proofs for CMP)
i) hd-1 in p
ii) nondecreasing in p & $\bar{u}$
iii) Concave in p

Properties of $h(p,\bar{u})$
i) hd-0 in p ; $h(\alpha p,\bar{u}) = h(p,\bar{u})$
ii) $\succsim$ convex → $h(p,\bar{u})$ is convex set
iii) $\succsim$ strictly convex → $h(p,\bar{u})$ is unique.

Shepard's Lemma : $\frac{\partial c}{\partial w_i} = x_i(w, z)$

for EMP : $\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u)$

Substitution matrix :

$$Dh(p,u) = \begin{bmatrix} \partial h_1/\partial p_1 & \cdots & \partial h_1/\partial p_n \\ \vdots & \ddots & \vdots \\ \partial h_n/\partial p_1 & \cdots & \partial h_n/\partial p_n \end{bmatrix} = \begin{bmatrix} \partial^2 e/\partial p_1^2 & \cdots & \partial^2 e/\partial p_1 \partial p_n \\ \vdots & \ddots & \vdots \\ \partial^2 e/\partial p_n^2 & \cdots & \partial^2 e/\partial p_n^2 \end{bmatrix}$$

Shepard's Lemma

Theorem : The substitution matrix $Dh(p, \bar{u})$ is
i) Negative semidefinite
ii) Symmetric
iii) Satisfies $Dh(p, \bar{u}) \cdot p = 0$

i) is the differential law of (compensated) demand

$\frac{\partial h_i}{\partial p_i} \leq 0$ $\left(\frac{\partial x_i}{\partial p_i} \nleq 0 \right.$ not true for marshallian, only for hicksian $h(p, \bar{u})$. $\left. \right)$.

Example : Cobb-Douglas

$\min \; p_1 x_1 + p_2 x_2 \quad \text{st} \quad \alpha \log x_1 + \beta \log x_2 \geq \bar{u}$

$\mathcal{L} : p_1 x_1 + p_2 x_2 - \lambda(\alpha \log x_1 + \beta \log x_2 - \bar{u})$

f.o.c $\frac{\partial \mathcal{L}}{\partial x_1} : p_1 = \frac{\alpha \lambda}{x_1}$ ① $\quad \frac{\partial \mathcal{L}}{\partial x_2} : p_2 = \frac{\beta \lambda}{x_2}$ ② $\quad$ csc. (slackness) $\lambda(\underbrace{\bar{u} - \alpha \log x_1 - \beta \log x_2}_{\text{will bind.}})$ ③

① / ② : $\frac{x_2}{x_1}\frac{\alpha}{\beta} = \frac{p_1}{p_2} \Rightarrow x_2 = \underbrace{\frac{p_1}{p_2}\frac{\beta}{\alpha}}_{C} x_1$

plug into ③ : $\alpha \log(x_1) + \beta \log(Cx_1) = \bar{u}$

$$\alpha \log(x_1) + \beta \log C + \beta \log x_1 = \bar{u}$$

$$\log(x_1) + \beta \log C = \bar{u}$$

$$\log(x_1) = \bar{u} - \log C^{\beta} \quad / \exp$$

$$x_1 = e^{\bar{u}} e^{-\log C^{\beta}} = e^{\bar{u}} C^{-\beta}$$

$$x_1 = e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha-1} \left[\frac{p_2}{\beta}\right]^{\beta}$$

$$\log(C^{\beta} \cdot x_1^{\alpha+\beta}) = \bar{u} \quad / \exp$$
$$C^{\beta} x_1 = e^{\bar{u}}$$
$$x_1 = e^{\bar{u}}/C^{\beta}$$
$$x_1 = e^{\bar{u}} \left(\frac{p_2}{p_1}\frac{\alpha}{\beta}\right)^{\beta}$$
$$x_1 = e^{u} \left[\frac{p_2}{\beta}\right]^{\beta} \left[\frac{p_1}{\alpha}\right]^{\alpha-1}$$

$$h_1(p,\bar{u}) = e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha-1} \left[\frac{p_2}{\beta}\right]^{\beta} \qquad h_2(p,\bar{u}) = e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha} \left[\frac{p_2}{\beta}\right]^{\beta-1}$$

Expenditure function: $e(p,\bar{u}) = p_1 h_1(p,\bar{u}) + p_2 h_2(p,\bar{u})$

$$e(p,\bar{u}) = \left[e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha} \left[\frac{p_2}{\beta}\right]^{\beta}\right]\alpha + \left[e^{\bar{u}} \left[\frac{p_1}{\alpha}\right]^{\alpha} \left[\frac{p_2}{\beta}\right]^{\beta}\right]\beta$$

$$e(p,\bar{u}) = e^{\bar{u}} \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta}$$

$x_1(p,m) = \frac{\alpha m}{p_1}$; $x_2(p,m) = \frac{\beta m}{p_2}$ $\Rightarrow$ what if we set $m = e(p,\bar{u})$ in marshallian demands?

$$x_1(p, e(p,\bar{u})) = \frac{\alpha e^{\bar{u}} \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta}}{p_1}$$

$x_1(p, e(p,\bar{u})) = e^{\bar{u}} \left(\frac{p_1}{\alpha}\right)^{\alpha-1} \left(\frac{p_2}{\beta}\right)^{\beta} = h_1(p,\bar{u}) \Rightarrow$ we get the hicksian demand again.

$\Rightarrow$ Duality between UMP & EMP.

(Local non satiation)

<u>Theorem</u>: Assume $u(\cdot)$ is continuous & LNS.

i) If x^* solves $\max u(x)$ s.t. $px \leq m$, $x_i \geq 0$ (UMP)

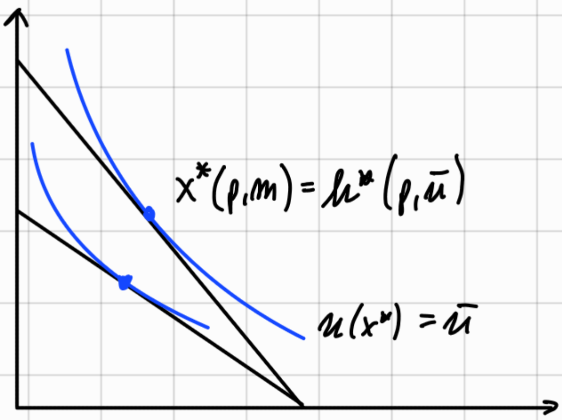
then, x^* also solves $\min p \cdot x$ st $u(x^*) \geq \bar{u}$, $x_i \geq 0$ (EMP)

Further, $e(p, u(x^*)) = m$

ii) If x^* solves $\min p \cdot x$ st $u(x) \geq \bar{u}$, $x_i \geq 0$ (EMP)

then x^* solves $\max u(x)$ s.t $p \cdot x \leq p \cdot x^*$, $x_i \geq 0$ (UMP)

Further, $v(p, p \cdot x^*) = \bar{u}$



<u>Proof of (i)</u>: Say x^* solves UMP, but not EMP at $\bar{u} = u(x^*)$

$\exists y$ st $u(y) \geq u(x^*)$
and $p \cdot y < p \cdot x^* = m$

By LNS, $\exists$ another y' s.t. $py' < m$ and $u(y') > u(y)$, so $u(y') > u(x^*)$
so x^* not optimal in UMP.

<u>Proof of (ii)</u>: Say x^* solves EMP, but not UMP at income $p \cdot x^*$

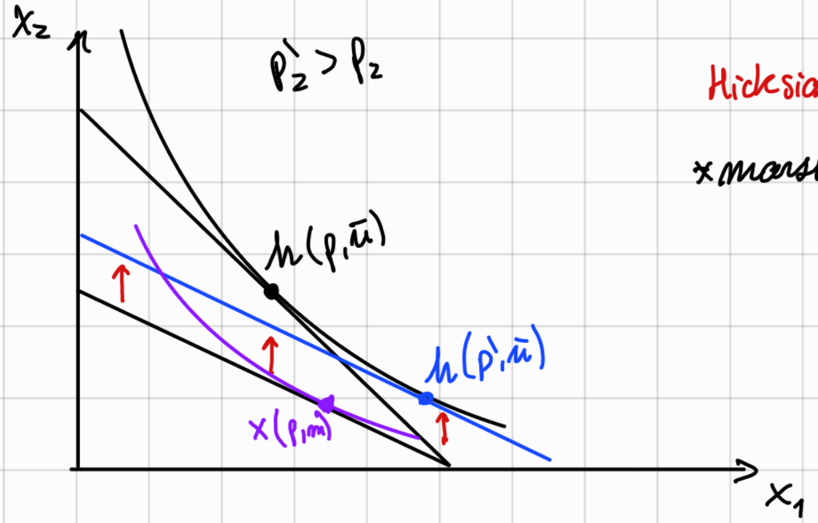
$\exists y$ st $u(y) > u(x^*)$
and $py \leq p \cdot x^*$

Let $y' = \alpha y$ for $\alpha \in (0,1)$. By continuity now, for $\alpha \approx 1$
$u(y') > u(x^*)$, and $p \cdot y' < p \cdot x^*$.
So x^* is not optimal in EMP.

Following 4 "duality identities":

① $e(p, v(p,m)) = m$
② $v(p, e(p,\bar{u})) = \bar{u}$
③ $x_i(p, e(p,\bar{u})) = h_i(p,\bar{u})$
④ $h_i(p, v(p,m)) = x_i(p,m)$

Why is Hicksian demand called "Compensated" demand?



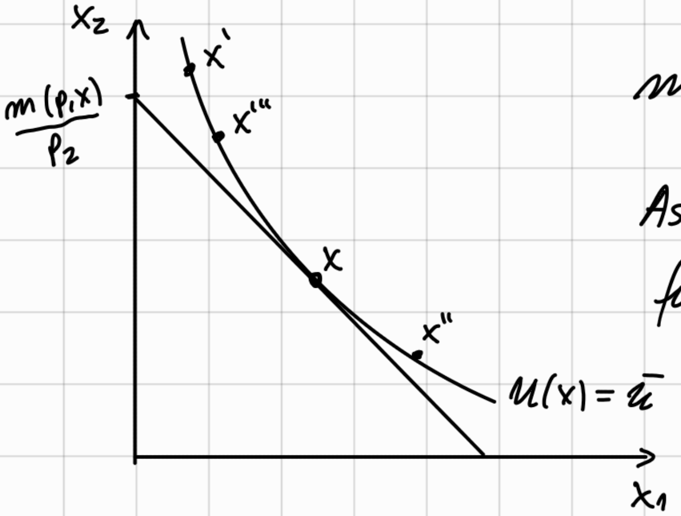
Hicksian Compensation: Δm needed to get $\bar{u}$

× marshallian absorb u value.

Money metric utility: Bundle x, utility $u(x)$, fix prices at p.

Q: How much money does the consumer need to achieve utility $u(x)$ at p?
NOT how much does it cost to buy x

$$m(p,x) = e(p, u(x))$$



fix p, view as a function of x
$m(p,\cdot) = e(p,\cdot)$

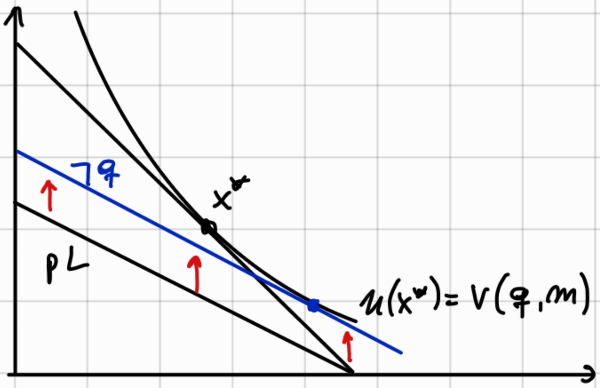
As a function of x, $m(p,x)$ is a utility function that represents the same preferences as $u(\cdot)$ (monotonic transformation)

$m(p,x) = m(p,x') = m(p,x'') = \dots$
$u(x) = \quad \dots \quad = u(x''') = \dots$

Also money metric indirect utility function:

$$u(p, e(q,m)) = e(p, v(q,m))$$

Answers question: How much money you need at prices p, to be just as well off as you were at prices q & income m?



For Cobb-Douglas: $u(x) = \alpha \log(x_1) + \beta \log(x_2)$

$$e(p, \bar{u}) = e^{\bar{u}} \cdot \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta}$$

$$m(p,x) = e(p, u(x))$$
$$= e^{\alpha \log x_1 + \beta \log x_2} \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta}$$

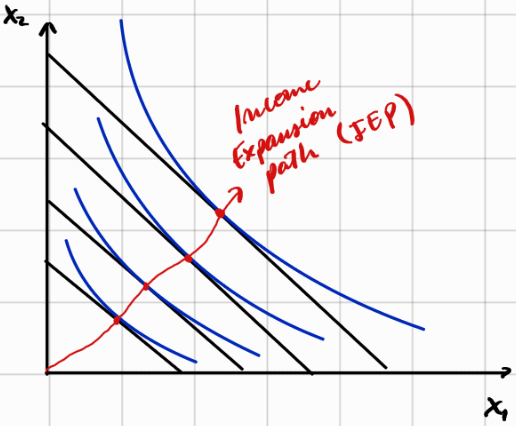
$$\boxed{m(p,x) = x_1^{\alpha} x_2^{\beta} \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta}} \rightarrow$$ Rescaled utility function and measured in dollars.

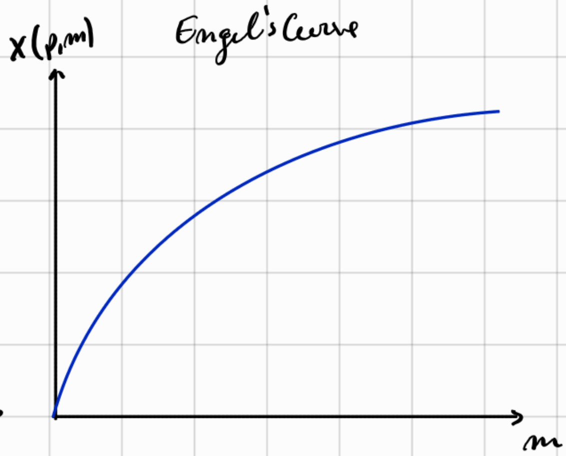
Comparative Statics of Demand $x(p,m)$

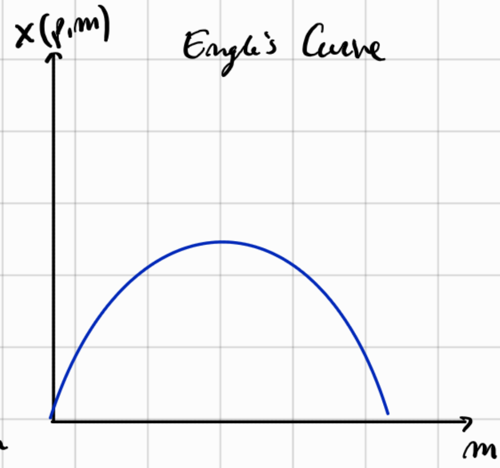
Oct 18, 2023

1) Income effects: varying m
2) Price effects: varying p

1) Income expansion path





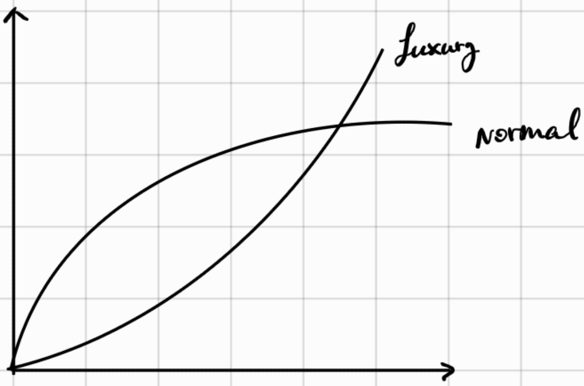


$\dfrac{\partial x_i(p,m)}{\partial m} \geq 0$: normal good $\qquad \dfrac{\partial x_i(p,m)}{\partial m} \leq 0$: inferior good

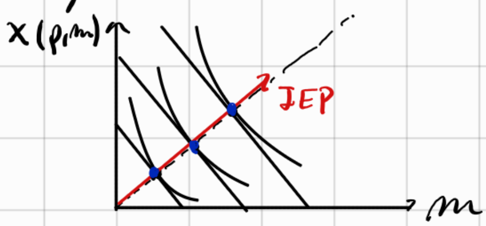
To measure the curvature, we use the income elasticity

$$\varepsilon_{im} = \frac{d\log(x_i(p,m))}{d\log(m)} = \frac{m}{x_i(p,m)} \cdot \frac{dx_i(p,m)}{dm}$$

$0 < \varepsilon_{im} < 1$: necessary good
$\varepsilon_{im} \geq 1$: luxury good



Special case: Homothetic preferences ($u(x)$ is hd-1)



More formally: If $u(\cdot)$ is hd-1, then $X(p, \alpha m) = \alpha X(p,m)$

proof: Let $x \in X(p,m)$

$$p \cdot x \leq m$$
$$\alpha p \cdot x \leq \alpha m$$
$$p(\alpha x) \leq \alpha m$$

Take any y s.t $p \cdot y \leq \alpha m$

$$p \cdot \frac{1}{\alpha} y \leq m$$

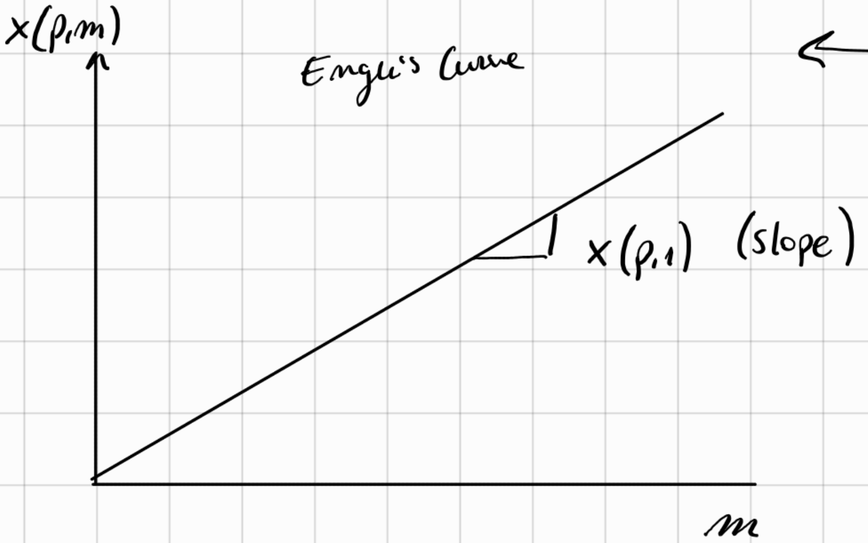
So $\frac{1}{\alpha} y$ is affordable at m, so $u\left(\frac{1}{\alpha} y\right) \leq u(x)$

$$\alpha u\left(\frac{1}{\alpha} y\right) \leq \alpha u(x)$$
$$u(y) \leq u(\alpha x) \quad \text{(by hd-1)}$$

$\therefore$ αx is optimal at income αm, so $X(p, \alpha m) = \alpha X(p,m)$

Set $\alpha = \frac{1}{m} \Rightarrow X(p,1) = \frac{1}{m} X(p,m) \Rightarrow$ $x(p,m) = m \cdot x(p,1)$

Equation



Elasticity:

$$\log x_i(p,m) = \log m + \log x_i(p,1)$$

$$\varepsilon_{im} = 1$$

For C.D., $u(x_1,x_2)$ $x_1^{\alpha} x_2^{\beta}$:

$$x_1(p,m) = \frac{\alpha m}{p_1}$$

Slope: $\frac{\alpha}{p_1}$

2) Price effects

* remember Hotelling's Lemma

(i) $y_i(p) = \frac{\partial \pi}{\partial p_i}$ $\qquad \frac{\partial y_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2}$

(ii) $\pi(p)$ convex $\longrightarrow$ $Dy(p) = D^2\pi(p)$ is pos. semi-def. $\left(\frac{\partial^2 \pi}{\partial p^2} \geq 0\right)$

(i) + (ii) $\Rightarrow \frac{\partial y_i}{\partial p_i} \geq 0$; Law of Supply

The analogous result for $x_i(p_i, m)$ would be $\frac{\partial x_i(p,m)}{\partial p_i} \leq 0$ "Law of Demand". Unfortunately, this is not TRUE

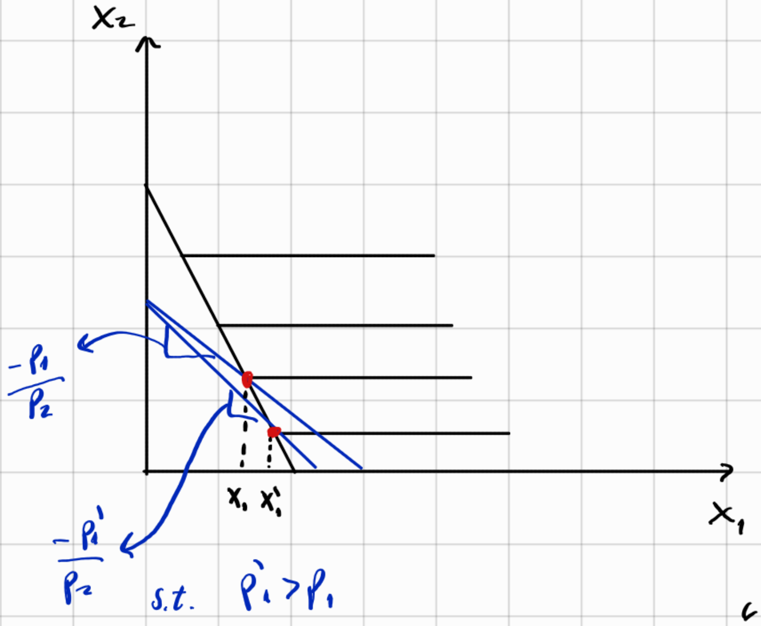
- $V(p,m)$ is only quasiconcave [(ii) doesn't work]

Example: Meat (x_2) + Potatoes (x_1)

- minimum calories requirement: C
- For any bundle (x_1, x_2): $C = 2x_1 + x_2$

$$u(x_1, x_2) = \begin{cases} x_2 & , 2x_1 + x_2 \geq 0 \\ -1 & , 2x_1 + x_2 < 0 \end{cases}$$

(After the minimum req, agent only wants meat)
(not enough calories, then the agent dies)



Indifference curves

Budget

Equilibrium: $\frac{m}{p_2} < C$, $\frac{m}{p_1} > C$

$\frac{\partial x_1}{\partial p_1} > 0$: Giffen Good

$\left(\frac{\partial x}{\partial p_1} \leq 0 : \text{Ordinary Good}\right)$

$\therefore$ "Law of Demand" is not TRUE

What about $h_i(p, \bar{u})$?

- In EMP, $e(p, \bar{u})$ is concave in p
- So, here: $De(p, \bar{u})$ is neg. semidef. matrix
- Shepard's Lemma applies

$$h_i(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_i}$$

$$\frac{\partial h_i(p, \bar{u})}{\partial p_i} = \frac{\partial^2 e(p, \bar{u})}{\partial p_i^2} \leq 0 \Rightarrow \text{"Law of 'compensated' demand"}$$

(But $h_i(p, \bar{u})$ are not observable)

But we can use the duality $h_i(p, \bar{u}) = x_i(p, e(p, \bar{u}))$

- differentiate wrt p_k

$$\frac{\partial h_i(p, \bar{u})}{\partial p_k} = \frac{\partial x_i(p, e(p, \bar{u}))}{\partial p_k} + \frac{\partial x_i(p, e(p, \bar{u}))}{\partial m} \cdot \overbrace{\frac{\partial e(p, \bar{u})}{\partial p_k}}^{= h_k(p, \bar{u}) \text{ by Shepard's Lemma}}$$

and $h_k(p, \bar{u}) = x_k(p, m)$

1. $m = e(p, \bar{u})$
2. $\bar{u} = v(p, m)$
3. $h_k(p, m) = x_k(p, e(p, \bar{u})) = x_k(p, m)$

Slutsky Equation

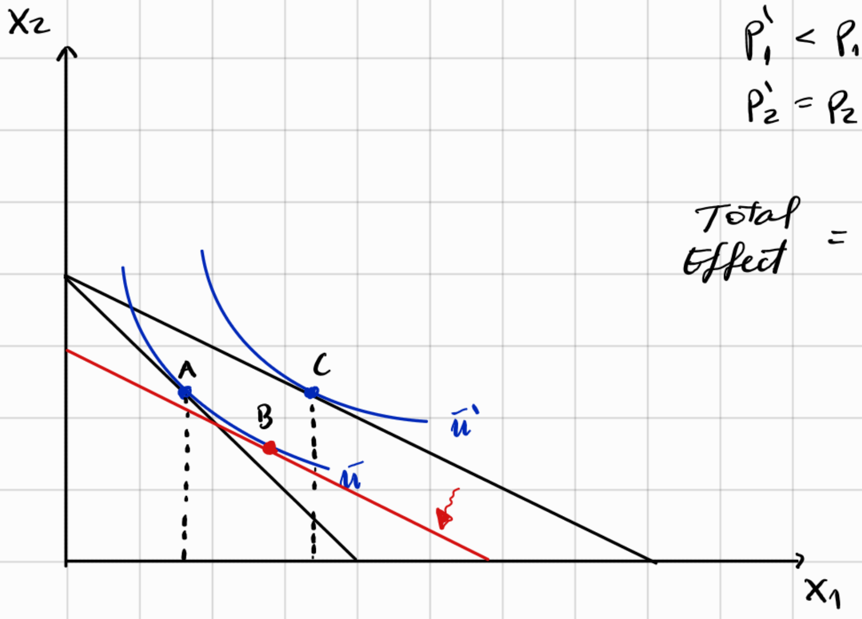
$$\frac{\partial x_i(p, e(p, \bar{u}))}{\partial p_k} = \underbrace{\frac{\partial h_i(p, \bar{u})}{\partial p_k}}_{\text{Substitution Effect}} - \underbrace{\frac{\partial x_i(p, e(p, \bar{u}))}{\partial m} \cdot x_k(p, \bar{u})}_{\text{Income Effect}}$$

$\partial h_i / \partial p_k \geq 0$: i, k are substitutes

$\partial h_i / \partial p_k \leq 0$: i, k are complements

$\partial x_i / \partial p_k \geq 0$: i, k are gross substitutes

$\partial x_i / \partial p_k \leq 0$: i, k are gross complements.



$p_1' < p_1$

$p_2' = p_2$

$$\text{Total Effect} = \underbrace{[B - A]}_{\text{sust effect}} + \underbrace{[C - B]}_{\text{income effect}}.$$

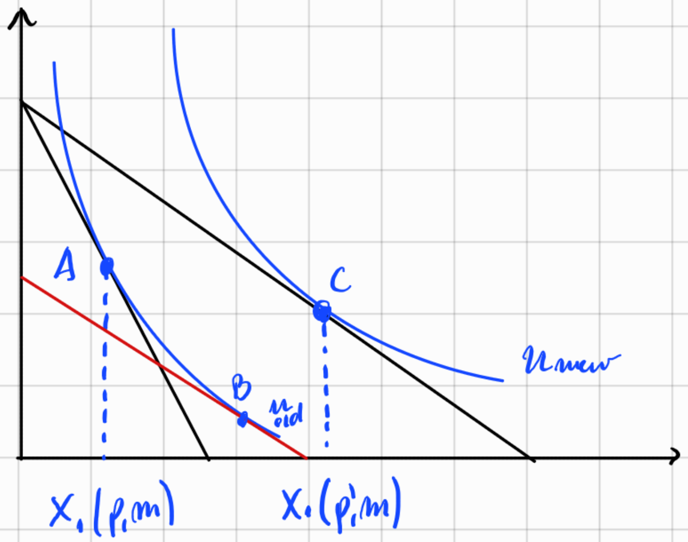
$$m^{comp} = e(p', \bar{u})$$

$$\underbrace{x_1(p', m) - x_1(p, m)}_{\text{Total Effect}} = \left[x_1(p', m^{comp}) - x_1(p, m)\right] + \left[x_1(p', m) - x_1(p', m^{comp})\right]$$

$$= \underbrace{\left[h(p', \bar{u}) - h(p, \bar{u})\right]}_{\text{Subst. Ef.}} + \underbrace{\left[x_1(p', m) - x_1(p', m^{comp})\right]}_{\text{Income Ef.}}$$

Lecture

Oct 23, 2023



$p_1' < p_1$

$p_2' < p_2$

$A \to B$: Sust. Effect

$B \to C$: Income Effect

$A \to C$: Total Effect.

Slutsky Equation :
$$\underbrace{\frac{\partial x_i}{\partial p_k}}_{(A \to C)} = \underbrace{\frac{\partial h_i}{\partial p_k}}_{(A \to B)} - \underbrace{\frac{\partial x_i}{\partial M} x_k(p, m)}_{(B \to C)}$$

For own-price effect $(k=i)$

$$\frac{\partial x_i}{\partial p_i} = \underbrace{\frac{\partial h_i}{\partial p_i}}_{\text{always } \leq 0} - \frac{\partial x_i}{\partial m} x_i(p,m)$$

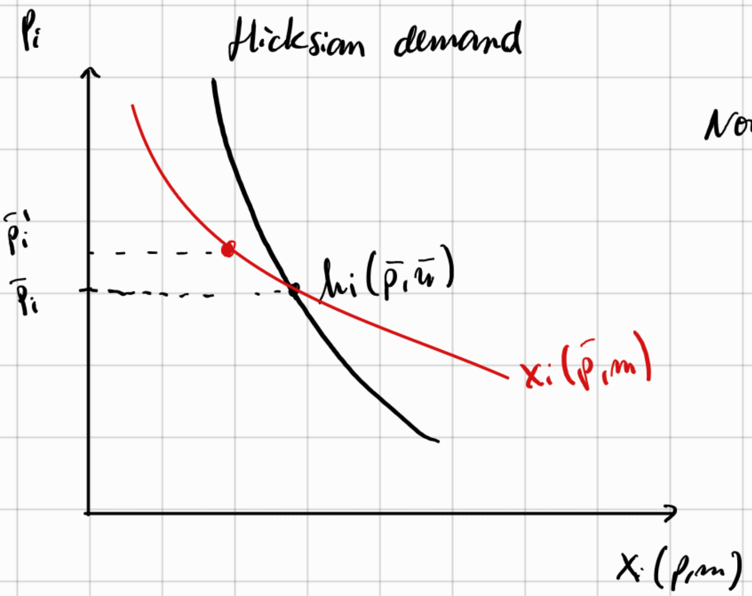
If good normal: $\frac{\partial x_i}{\partial m} \geq 0$

$\Rightarrow \frac{\partial x_i}{\partial p_i} \leq 0$

If good is inferior: $\frac{\partial x_i}{\partial m} \leq 0$

$\Rightarrow$ IE & SE are "opposed"

$\Rightarrow \frac{\partial x_i}{\partial p_i} \geq 0$ only if good is strongly inferior.



Normal good: $\frac{\partial x_i}{\partial p_i} = \underbrace{\frac{\partial h_i}{\partial p_i}}_{(-)} \underbrace{- \frac{\partial x_i}{\partial m} x_i(p,m)}_{(-)}$

$$x_i(\bar{p}, e(\bar{p},\bar{u})) = h_i(\bar{p},\bar{u})$$

$\big(x_i(\cdot)$ is flater than $h_i(\cdot)\big)$

marshallian Hickrian

$$Dh(p) = \begin{pmatrix} \partial h_1/\partial p_1 & \cdots & \partial h_1/\partial p_n \\ \vdots & \ddots & \vdots \\ \partial h_n/\partial p_1 & \cdots & \partial h_n/\partial p_n \end{pmatrix}$$

- Symmetric
- n.s.d
- $Dh(p)\cdot p = 0$

Problem: h_i is not observable

Consider: $S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$, $S_{jk} = \frac{\partial x_j}{\partial p_k} + \frac{\partial x_j}{\partial m} x_k$

By Slutsky equation: $S(p,m) = Dh(p)$ should also satisfy, and $S(\cdot)$ is observable.

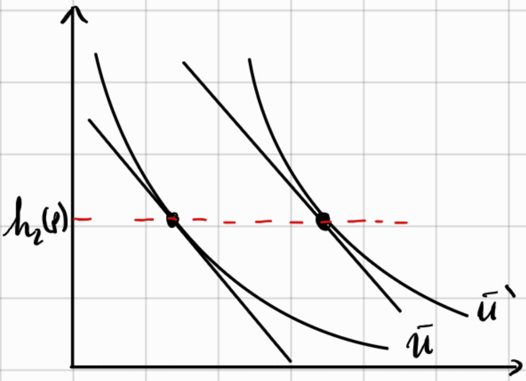
Special case: No income effects. $x_i(p,m) = x_i(p)$

Duality identity: $h_i(p,\bar{u}) = x_i(p, e(p,\bar{u})) = x_i(p)$

$$\Rightarrow h_i(p,\bar{u}) = h_i(p)$$

Say $h_2(p,\bar{u}) = h_2(p)$

$$\frac{\partial h_i}{\partial p_i} = \frac{\partial x_i}{\partial p_i}$$



$$\max_{x \in \mathbb{R}^2_+} x_1 + V(x_2) \quad \text{s.t} \quad x_1 + p_2 x_2 = m$$

[normalize $p_1 = 1$
quasilinear objective]

$$\hookrightarrow \max_{x_2} \; m - p_2 x_2 + V(x_2)$$

f.o.c: $V'(x_2) = p_2$, if invertible, then $x_2(p_2) = V'^{-1}(p_2)$ $(= x_2(p,m))$

$$\frac{\partial x_2}{\partial m} = 0 \quad \text{[No income effect]}$$

From budget constraint:

$$x_1^*(p,m) = m - p_2\left[(V')^{-1}(p_2)\right]$$

$$\frac{\partial x^*}{\partial m} = 1.$$

It is quasilinear only if m is high enough (problem set)
(solve with non-negativity constraints and then determine m is)

Integrability

Question: Given functions $x(p,m)$ that satisfy:

- hd-0
- Walras' law
- Symmetric n.s.d (Slutsky matrix)

Can we recover utility? Yes

$$x_i(p,m) = -\frac{\partial v/\partial p_i}{\partial v/\partial m}$$

Roy's Identity in reverse

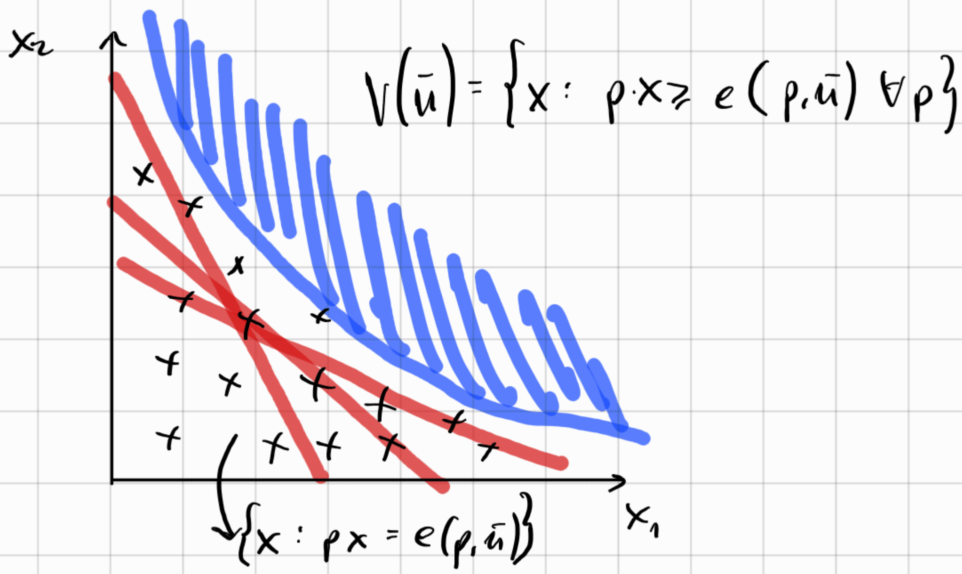
Proof Strategy:

Demand function $\xrightarrow{①}$ Expenditure function $\xrightarrow{②}$ utility

Start with ②

Say we know $e(p,\bar{u})$

we want: $V(\bar{u}) = \{$ bundles that give utility at least $\bar{u}\}$



Step 1: $x(p,m) \longrightarrow e(p,\bar{u})$

pick a starting point (p^0, m^0)

let $u(x(p^0,m^0)) = u^0$

Goal: solve for $e(p, m^0)$, write as $e(p)$

By Shephard's Lemma: $\frac{\partial e(p)}{\partial p_i} = h_i(e, u^0)$

$$\frac{\partial e(p)}{\partial p_i} = x_i(p, e(p)),\quad \text{initial condition } e(p^0) = m^0$$

Does the solution to this differential equations even exist?
A necessary and sufficient condition for a system

$$\frac{\partial f}{\partial x_i} = g_i(x)$$

to have a solution is

$$\underbrace{\frac{\partial g_i(x)}{\partial x_j} = \frac{\partial g_j(x)}{\partial x_i}}_{\text{Then Solution exists.}} \qquad \left(\text{Remember } \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \right)$$

In our problem: $$\frac{\partial x_i(p, e(p))}{\partial p_j} = \frac{\partial x_j(p, e(p))}{\partial p_i}$$

This condition says that is must be symmetric
(S must be symmetric).

Symmetry of S is N&S condition for System of PDE to have a solution.
For the solution $e(m)$ to be a valid expenditure function, it must be concave.

* Concavity: $De(p) = S$ is n.s.d.

Bottom line: Given some $x(p,m)$, if S is symmetric & n.s.d., we can solve for $e(p,\bar{u})$. Then use earlier technique to go from

$$e(p,\bar{u}) \longrightarrow v(p,m)$$

Say you have: $x_1(p,m) = \frac{\alpha m}{p_1}$, $x_2(p,m) = \frac{\beta m}{p_2}$

find $e(p,u)$ by solving: $$\frac{\partial e(p)}{\partial p_1} = x_1(p)$$

$$\frac{\partial e(p)}{\partial p_2} = x_2(p)$$

step 1: Check Symmetry $\frac{\partial^2 e(p)}{\partial p_i \partial p_j} = \frac{\partial^2 e(p)}{\partial p_j \partial p_i}$ satisfied ∴ Solution exists.

step 2: Solve $\frac{\partial e(p)}{\partial p_1} = \frac{\alpha e(p)}{p_1}$

$$\rightarrow \frac{\partial e(p)/\partial p_1}{e(p)} = \frac{\alpha}{p_1}$$

$$\rightarrow \frac{\partial \log e(p_1)}{\partial p_1} = \frac{\alpha}{p_1} \quad \Big/ \int dp_1$$

$$\rightarrow \int \frac{\partial \log e(p_1)}{\partial p_1} dp_1 = \int \frac{\alpha}{p_1} dp_1$$

$$\Rightarrow \log e(p) = \alpha \log p_1 + c_1(p_2, \bar{u})$$

Similarly:

$$\log e(p) = \beta \log p_2 + c_2(p_1, \bar{u})$$

$$\Rightarrow \log e(p) = \alpha \log p_1 + \beta \log p_2 + f(\bar{u}) \quad \Big/ \exp(\cdot)$$

$$e(p) = e^{f(\bar{u})} p_1^\alpha p_2^\beta$$

$$\boxed{e(p) = \tilde{u}\, p_1^\alpha p_2^\beta}$$

Finally, find $V(p,m)$ (use duality)

$$e(p, V(p,m)) = m$$

$$V(p,m)\, p_1^\alpha p_2^\beta = m$$

$$\boxed{V(p,m) = \frac{m}{p_1^\alpha p_2^\beta}}$$

Lecture: Welfare evaluations

Oct 30, 2023

Price change: $p^0 \to p^1$ (p: vector)

* we want to study $V(p^1, m) - V(p^0, m)$

Ordinal nature of $u(\cdot)$ makes this "problematic". We want a "standarized" welfare measure. $\Rightarrow$ Money metric indirect utility function

→ Fix some arbitrary base $\bar{p}$

→ $\mu(\bar{p}; p^0, m) = e(\bar{p}, V(p^0, m))$ (amount of money I need at $\bar{p}$ to get the utility I had at p^0).

$\mu(\bar{p}; p^1, m) = e(\bar{p}, V(p^1, m))$

Change in Welfare will be: $\Delta W = \mu(\bar{p}; p_1, m) - \mu(\bar{p}; p_0, m)$

* still needs reference prices, $\bar{p}$.

** Natural choices of $\bar{p}$: $\boxed{\bar{p} = p^0}$ or $\boxed{\bar{p} = p^1}$

$u^0 = V(p^0, m)$ $\quad$ $u^1 = V(p^1, m)$

Equivalent Variation (EV) $\quad$ Compensating Variation (CV)

1.) $EV = \mu(p^0; p^1, m) - \mu(p^0, p^0, m)$

$EV = e(p^0, u^1) - e(p^0, u^0)$

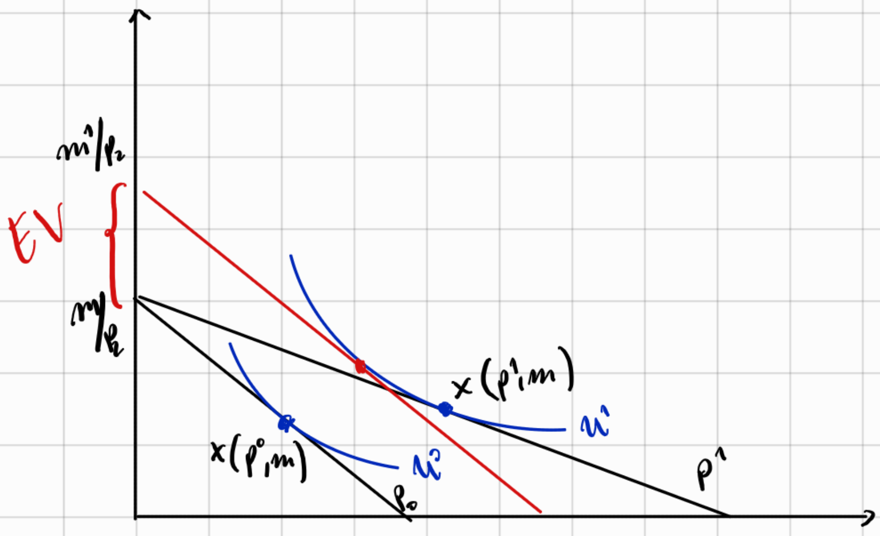
$EV = e(p^0, u^1) - m$

Two scenarios

i) price changes $p^0 \to p^1$

ii) prices fixed, consumer gets an extra \$EV (net).

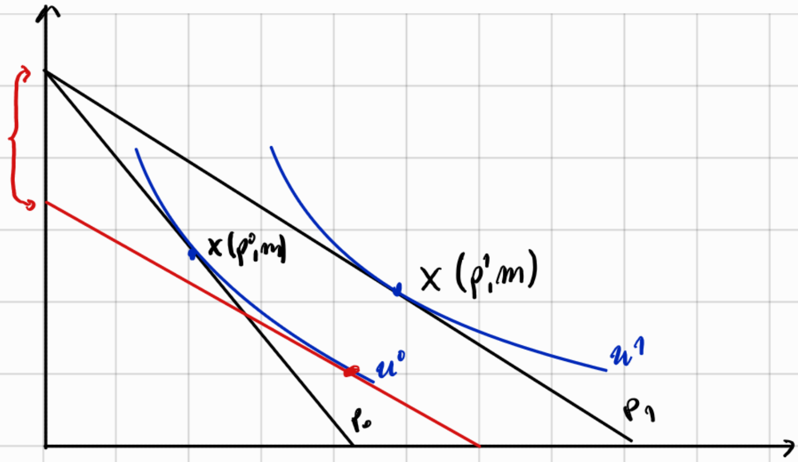
EV is the number that makes i) = ii)



2.) $CV = \mu(p^1; p^1, m) - \mu(p^1; p^0, m)$

$CV = e(p^1, u^1) - e(p^1, u^0)$

$CV = m - e(p^1, u^0)$



Comments:

→ EV and CV always have the same sign.

→ Many choices of utility function we could use for welfare

→ EV & CV are 2 most "natural"

→ In general $EV \neq CV$, but will always have same sign

→ In principle observable if we know marshallian demand (integrability).

Another way to view EV & CV (& CS)

Fix $p_i^0 = p_i^1 = \bar{p}_i \quad \forall i \neq 1$

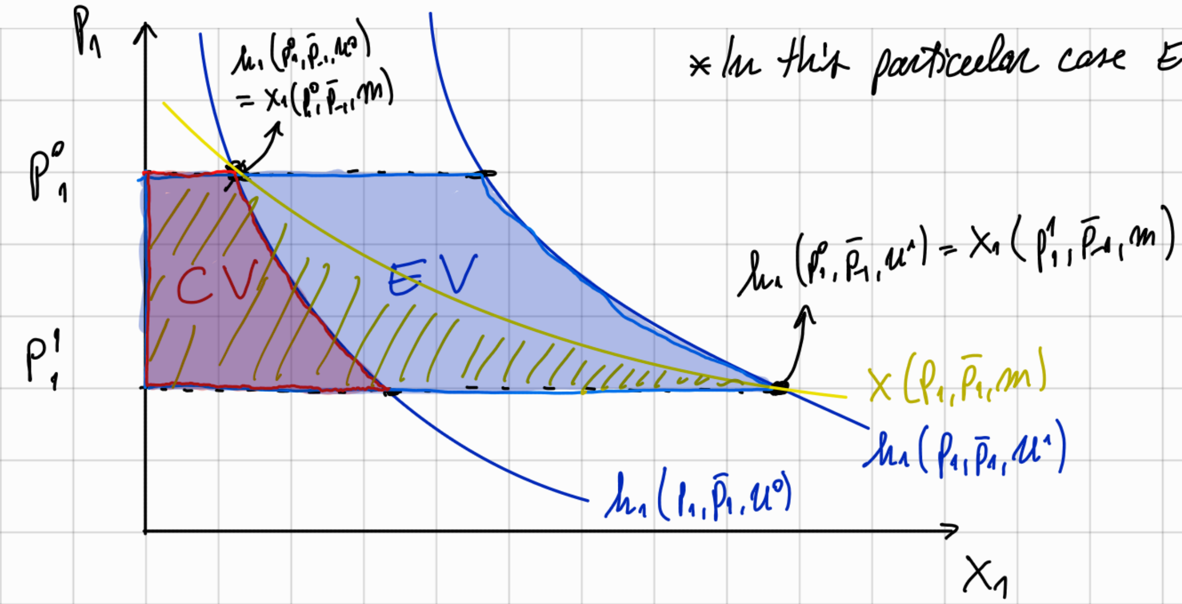
Recall $e(p^0, u^0) = e(p^1, u^1) = m$

$$EV = e(p^0, u^1) - e(p^1, u^1)$$
$$= e(p^0, u^1) - e(p^1, u^1)$$
$$= \int_{p_1^1}^{p_1^0} \frac{\partial e(\tilde{p}_1, \bar{p}_{-1}, u^1)}{\partial p_1} d\tilde{p}_1 = \int_{p_1^1}^{p_1^0} h_1(\tilde{p}_1, \bar{p}_{-1}, u^1)\, d\tilde{p}_1$$

by Shephard's Lemma.

For CV:

$$CV = e(p^1, u^1) - e(p^1, u^0)$$
$$= e(p^0, u^0) - e(p^1, u^0)$$
$$= \int_{p_1^1}^{p_1^0} \frac{\partial e(\tilde{p}_1, \bar{p}_{-1}, u^0)}{\partial p_1} d\tilde{p}_1 = \int_{p_1^1}^{p_1^0} h_1(\tilde{p}_1, \bar{p}_{-1}, u^0)\, d\tilde{p}_1$$



* In this particular case EV > CV

Consumer surplus:

$$\Delta CS = \int_{p_1^1}^{p_1^0} x_1(\tilde{p}_1,\bar{p}_{-1},m)\,d\tilde{p}_1$$

* for normal goods: $CV \leq \Delta CS \leq EV$

inferior goods: $EV \leq \Delta CS \leq CV$

quasilinear: $CV = \Delta CS = EV$

Example: Govt. wants to raise \$T tax revenue.

1. Linear tax on good 1 or 2. Lump-sum tax

$p^0 = (p_1^0, p_2^0) \rightarrow (p_1^0+\tau, p_2^0) = p^1$ $\qquad T = \tau \cdot x_1(p^1, m)$

Welfare comparison between 1 & 2? (answer by EV)

If $EV = e(p^0,u^1) - m < -T$, consumer is worse off under 1

$$DWL = -EV - T$$

$$= \underbrace{e(p^1,u^1)}_{m} - e(p^0,u^1) - T$$

$$= \int_{p_1^0}^{p_1^0+\tau} h_1(\tilde{p}_1, p_2^0, u^1)\,d\tilde{p}_1 - \tau \cdot \overbrace{h_1(p_1^0+\tau, p_2^0, u^1)}^{x_1(p^1,m)}$$

$$= \int_{p_1^0}^{p_1^0+\tau} \underset{(i)}{h_1(\tilde{p}_1,p_2^0,u^1)} - \underset{(ii)}{h_1(p_1^0+\tau,p_2^0,u^1)}\,d\tilde{p}_1 \geq 0$$

(because this is just a number)

because (i) is bigger than (ii) at any point.

$$DWL = -EV - T \geq 0 \Rightarrow EV \leq -T$$

//

DWL = Deadweight lost