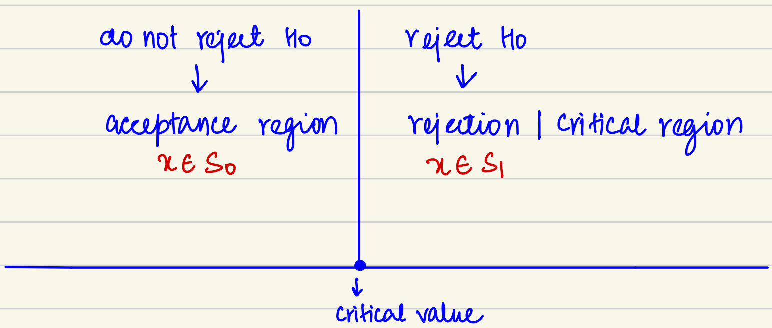
testing:

- statistical decision : function $d = S(x)$; where x is an observation
- define the problem as accepting or rejecting a certain hypothesis regarding P_θ
- parameter space generates a class of distributions $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$; then
  $\mathcal{P} = H \cup K$ (where hypothesis is true for H and not true for K
  with $\Theta = \Theta_H \cup \Theta_K$
- the hypothesis mathematically is a statement : $P_\theta \in H$ (hypothesis $P_\theta \in H_0$)
  ↳ K is the class of alternatives (H_1)
- d_0 is the decision to not reject H & d_1 is the decision to reject H
- if $X \in S$ is the r.v. corresponding to the sample ; then $S = S_0 \cup S_1$, s.t.
  H_0 isn't rejected if $x \in S_0$ (acceptance region)
  H_0 is rejected if $x \in S_1$ (rejection / critical region)



test-stat > critical value
↳ reject H_0

- two types of errors :

  ① Type I error : $\alpha = \text{prob}(\text{ reject } H_0 \mid H_0 \text{ is true})$
  $\alpha = \text{prob}(x \in S_1 \mid H_0)$

  ② Type II error : $\beta = \text{prob}(\text{ do not reject } H_0 \mid H_1 \text{ is true})$
  $= \text{prob}(x \in S_0 \mid H_1)$

Q: $f(x;\theta) = \begin{cases} 1/\theta & ; \ 0 \le x \le \theta \\ 0 & ; \ \text{ow} \end{cases}$ $\quad H_0 : \theta = 1$, $H_1 : \theta = 2$ $\quad$ single value of x

what would be the sizes of Type I & Type II errors if you choose $x \geq 0.5$ as the critical region ?

Ans. $\alpha = p(x \in S_1 \mid H_0) = p(x \in \text{critical region} \mid H_0) = p(x \geq 0.5 \mid H_0) = p(0.5 \le x < \theta \mid \theta = 1)$
$= \left(\int_{0.5}^{\theta} \frac{1}{\theta} \, dx \,\middle|\, \theta = 1 \right) = \int_{0.5}^{1} \frac{1}{1} \, dx = 0.5$

$\beta = p(x \in S_0 \mid H_1) = p(x < 0.5 \mid \theta = 2) = \int_{0}^{0.5} \frac{1}{\theta} \, dx = \int_{0}^{0.5} \frac{1}{2} \, dx = 0.25$

power $\propto \frac{1}{\beta}$

- Power of the test $= 1 -$ prob(type II error) $= 1-\beta = 1 -$ prob$(x \in S_0 \mid H_1)$
  $=$ prob$(x \in S_1 \mid H_1)$

power is the probability of correctly rejecting H_0 - we can increase the power of the test by inc^n sample size; decreasing standard error, increasing the difference between sample statistic & hypothesized parameter, or increase α level

in the above example; power $=$ prob$(x \geq 0.5 \mid \theta = 2)$

$$= \left(\int_{0.5}^{\theta} \frac{1}{\theta} \, dx \;\Big|\; \theta = 2 \right) = \int_{0.5}^{2} \frac{1}{2} \, dx = 0.75$$

or, simply power $= 1 - 0.25 = 0.75$

- ideally, we want both errors to be small; but that's impossible
  $\rightarrow$ dec^n α will increase β - to decrease the likelihood of both errors; we should increase the sample size

- Level of significance is a number $0 < \alpha < 1$ such that

$$P_\theta(\delta(X) = d_1) = P_\theta(X \in S_1) \leq \alpha \quad \forall\, \theta \in \Theta_H$$

$\downarrow$

$\delta(X) = d_1$; decision is to reject $H \rightarrow X \in S_1$ i.e. prob (reject H)

- Size of the test $-$ $\sup_{\theta \in \Theta_H} P_\theta(X \in S_1)$

size isn't necessarily equal to the level of significance (until $\alpha = 0$ maybe) because size is supremum $\sup_{\theta \in \Theta_H} P_\theta(X \in S_1)$ and level of significance is $P_\theta(X \in S_1)$

note: if the test you construct takes the form:

$$\phi(x) = \begin{cases} 1 & , \text{ if } x_i > 1 \\ 0 & \text{o/w} \end{cases}$$

then; size $= \sup_{\theta \in \Theta} (\phi(x) = 1)$ where $\phi(x)$ is the prob^y of rejection

and level of significance $= P_\theta(\phi(x) = 1) \leq \alpha$

- Probability $\beta(\theta) = P_\theta(\delta(X) = d_1)$ considered as a function of θ is called the power function of the test

simple v/s composite hypothesis :

if a random sample is taken from a disb^n with parameter θ, a hypothesis is said to be simple hypothesis if the hypothesis uniquely specifies the disb^n of the population from which the sample is taken ; o/w composite

ex $X_1, X_2, \ldots$ is a random sample from an exponential disb^n with parameter θ ; which of the following hypothesis is simple or composite ?

$H_0 : \theta = 3$

$\downarrow$

$f(x) = \theta e^{-\theta x}$

under $H_0 : \theta = 3$;

$f(x) = 3e^{-3x}$

pdf uniquely determined

simple

$H_0 : \theta > 4.5$

$\downarrow$

$f(x) = \theta e^{-\theta x}$

$\downarrow$

under $H_0 : \theta > 4.5$

$f(x)$ could be an infinite # of things

$\therefore$ pdf not uniquely determined

composite

ex: $N \sim (\mu, \sigma^2)$; variance is unknown

$H_0 : \mu = 550$

not simple $\because$ σ could be anything because pdf not uniquely determined

ex: X is a single observation with pdf $f(x) = \theta x^{\theta-1}$ for $0 < x < 1$
check whether $H_0 : \theta = 3$ and $H_1 : \theta = 2$ are simple or composite hypothesis

Ans: $f(x) = \theta x^{\theta - 1}$

under H_0 : $3x^2$

under H_1 : $2x$

$\therefore$ pdf of the disb^n is uniquely determined under both H_0 & H_1,

$\therefore$ simple hypothesis

ex: X is a single observation with pdf $f(x) = \theta x^{\theta-1}$ for $0 < x < 1$
check whether $H_0 : \theta = 3$ and $H_1 : \theta \neq 3$ are simple or composite hypothesis

Ans: $f(x) = \theta x^{\theta-1}$

$H_0 : \theta = 3$ $\quad f(x) = 3x^2$

$H_1 : \theta \neq 3$ $\quad \theta$ can be anything

$\hookrightarrow$ infinite #s

only uniquely determined for H_0 ; not H_1

H_0 : simple ; H_1 : composite

Constructing most powerful test :

test is constructed by maximising $P_\theta(S(X) = d_1) = P_\theta(X \in S_1)$ (the power of the test) over $\theta \in \Theta_K$ setting the significance level to fixed α

randomized test : $\phi(x)$ is the probability of rejection ; use this as an indicator for critical region

construction of test : select $\phi(\cdot)$ to maximize

$$\beta_\phi(\theta) = E_\theta[\phi(X)], \quad \forall \theta \in \Theta_K$$

s.t.

$$E_\theta[\phi(X)] \le \alpha, \quad \forall \theta \in \Theta_H$$

a test $\phi(\cdot)$ for testing H v/s K (or, H_0 v/s H_1) is said to be of size-α if $E_\theta[\phi(x)] \le \alpha$

↳ most powerful test - a test $\phi(\cdot)$ is said to be most powerful (best) test for testing H_0 and H_1 if it is size α and has maximum power in the class of all level α tests

Neyman-Pearson lemma : let P_1 and P_0 be probability distributions with densities p_1 and p_0 -

1. Existence : For testing $H : p_0$ against the alternative $K : p_1$, there exists a test $\phi(\cdot)$ and a constant k, such that :

$$E_\theta(\phi(X)) = \alpha \qquad \text{(a)}$$

and

$$\phi(x) = \begin{cases} 1, & \text{when } p_1(x) > k\, p_0(x) \\ 0, & \text{when } p_1(x) < k\, p_0(x) \end{cases} \qquad \text{(b)}$$

2. sufficient condition for most powerful test : if the test satisfies (a) and (b) for some k then it is most powerful for testing p_0 against p_1 at level α

3. necessary condition for most powerful test : if ϕ is most powerful at level α for testing p_0 against p_1, then for some k it satisfies (b) almost everywhere ; it also satisfies (a) unless there exists a test of size $<\alpha$ with power 1

ex: $X \sim N(\mu, 1)$ $\quad H_0 : \mu = 1 \;$ v/s $\; H_0 : \mu = \mu' > 0$

likelihood ratio $\quad \dfrac{p_1(x)}{p_0(x)} = \dfrac{e^{-\frac{(x-\mu')^2}{2}}}{e^{-\frac{x^2}{2}}} = e^{x\mu' - \frac{\mu'^2}{2}}$

$\dfrac{p_1(x)}{p_0(x)}$ is monotone increasing in x ; $\dfrac{p_1(x)}{p_0(x)}$ corresponds to $x > c$ for some c

set $P_0(X > c) = \alpha$, find c

Q. $Y_1, Y_2, \ldots, Y_n$ is an iid sample from an exponential (θ) population, $\theta > 0$ derive the most powerful α- level test for

$$H_0 : \theta = \theta_0 \qquad \text{where } \theta_1 < \theta_0$$
$$H_1 : \theta = \theta_1$$

Ans. $f_Y(y|\theta) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & ; y > 0 \\ 0 & \text{ow} \end{cases}$

likelihood f^n is given by : $\prod_{i=1}^{n} \frac{1}{\theta} e^{-y/\theta} = \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} y_i / \theta}$

likelihood ratio is given by $\frac{L(\theta_1)}{L(\theta_0)} = \frac{1/\theta_1^n \, e^{-\sum_{i=1}^{n} y_i / \theta_1}}{1/\theta_0^n \, e^{-\sum y_i / \theta_0}} = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-\sum_{i=1}^{y_i} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}$

where the sufficient statistic is $t = \sum_{i=1}^{n} y_i$

the neyman pearson lemma says the most powerful α test uses

$$T(x) = \frac{L(\theta_1)}{L(\theta_0)} > k \Rightarrow \left(\frac{\theta_0}{\theta_1}\right)^n e^{-t\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)} > k$$

$\frac{L_1(x)}{L_0(x)}$ is monotone increasing in x ; define test as -

$$\phi(x) = \begin{cases} 1 & ; T(x) > k \\ 0 & ; \text{ow} \end{cases}$$

where k satisfies $\alpha = E_\theta(\phi(x)) = \text{prob}(\text{reject } H_0 \mid H_0)$; find k

how to construct most powerful test $\phi(x)$?

STEP 1. write likelihood f^n ; start from $\frac{L_1}{L_0} > k$; find critical region S_1

$$L = \prod_{i=1}^{n} f(x_i, \theta) \quad \begin{matrix} \nearrow L(\theta_1) = L_1 \\ \searrow L(\theta_0) = L_0 \end{matrix}$$

on solving we get $x <> k^*$

STEP 2. define test function as

$$\phi(x) = \begin{cases} 1 & ; x >< k^* \\ 0 & ; \text{ow} \end{cases} \Rightarrow \frac{L(H_1)}{L(H_0)} > k$$

STEP 3. find k^* using $E_{H_0}(\phi(X)) = \alpha \Rightarrow P(x \in S_1 \mid H_0) = p(\text{reject } H_0 ; x <> k^* \mid H_0) = \alpha$

solve this to get k^*

neyman-pearson lemma : most powerful test

let $k > 0$ be constant and ω be a critical region of size α such that

$$S_1 = \left\{ x \in S : \frac{L_1}{L_0} > k \right\}$$

$$S_0 = \left\{ x \in S : \frac{L_1}{L_0} \leq k \right\}$$

where L_0 & L_1 are likelihood functions of the sample observations under H_0 & H_1 respectively ; then S_1 is the most powerful critical region of the simple hypothesis $H_0 : \theta = \theta_0$ v/s $H_1 : \theta = \theta_1$

note : if a level α test's rejection region (critical region) doesn't involve a sufficient statistic ; we know it can't be most powerful

sufficient statistic :

Q. suppose X is single observation sample from a population with pdf $f(x) = \theta x^{\theta - 1}$ for $0 < x < 1$; find a test with the best critical region i.e. find the most powerful test with $\alpha = 0.05$ for testing $H_0 : \theta = 3$ v/s $H_1 : \theta = 2$

Ans. single ; $L = \prod_{i=1}^{1} f(x_i, \theta)$

null & alternate hypothesis are simple, so we can apply the neyman-pearson lemma i.e. $\frac{L(\theta_1)}{L(\theta_0)} > k$

$$L(\theta) = \prod_{i=1}^{1} f(x_i)$$

$$= \theta x^{\theta - 1}$$

$$L(\theta_1) = L(2) = 2x$$

$$L(\theta_0) = L(3) = 3x^2$$

so, $\frac{L(\theta_1)}{L(\theta_0)} = \frac{L(2)}{L(3)} = \frac{2x}{3x^2} > k \implies \frac{2}{3x} > k \implies \frac{2}{3} > kx$

$$\therefore \quad x \leq \frac{2}{3k} = k^*$$

neyman-pearson lemma tells us that the form of the rejection region for the most powerful test is :

critical region $S_1 = \{ x \mid x < k^* \}$

we need to find x^* s.t. size of critical region is $\alpha = 0.05$

$$\alpha = P(x \in S_1 \mid H_0)$$
$$0.05 = P(x < k^* \mid \theta = 3)$$
$$0.05 = P(0 < x < k^* \mid \theta = 3)$$
$$0.05 = \int_0^{k^*} 3x^2\,dx \Rightarrow 0.05 = \frac{3}{3} k^3$$
$$k^3 = 0.05$$
$$k^* = (0.05)^{1/3} = 0.368$$

$\therefore$ neyman-pearson lemma tells us that the rejection region of the most powerful test is $x < 0.368$

Q. $X_1, X_2, \ldots, X_n$ is a random sample from a normal population with mean μ and variance 16. Find the test with the best critical region i.e. find the most powerful test with a sample size of $n = 16$ and $\alpha = 0.05$ to test $H_0: \mu = 10$ v/s $H_1: \mu = 15$. Also find the power of the test.

Ans. $X_i \sim N(\mu, 16)$ Likelihood f^n : $L = \prod_{i=1}^{16} f(x_i, \theta)$

both null & alternate hypothesis are simple hypothesis; we can use neyman-pearson lemma to find the most powerful test :

$$\frac{L_1}{L_0} > K \Rightarrow L = \prod_{i=1}^{16} = \frac{1}{\sqrt{2\pi \times 16}} \times e^{-\frac{(x-\mu)^2}{2\times 16}} =$$

$$\frac{1}{(32\pi)^8} e^{-\frac{1}{32}\sum_{i=1}^{16}(x_i-\mu)^2}$$

$$\prod_{i=1}^{16} \frac{1}{\sqrt{32\pi}} = \frac{1}{(32\pi)^{16/2}}$$

$$\frac{L_1}{L_0} = \frac{L(\mu=15)}{L(\mu=10)} = \frac{\cancel{\frac{1}{(32\pi)^8}} e^{-\frac{1}{32}\sum_{i=1}^{16}(x_i-15)^2}}{\cancel{\frac{1}{(32\pi)^8}} e^{-\frac{1}{32}\sum_{i=1}^{16}(x_i-10)^2}} > K$$

$$= e^{-\frac{1}{32}\sum_{i=1}^{16}(x_i-15)^2 + \frac{1}{32}\sum_{i=1}^{16}(x_i-10)^2} > K$$

$$= e^{-\frac{1}{32}(\cancel{\sum x_i^2} + \underbrace{16\times 225}_{=3600} - 30\sum x_i) + \frac{1}{32}(\cancel{\sum x_i^2} + 1600 - 20\sum x_i)} > K$$

$$= e^{-\frac{2000}{32} + \frac{10}{32}\sum x_i} > K$$

$$= -\frac{2000}{32} + \frac{10}{32}\sum x_i > \ln(K)$$

$$10\sum_{i=1}^{16} x_i > 32\ln(K) + 2000$$

divide by 16 ; $10 \frac{\sum_{i=1}^{16} x_i}{16} > 2\ln(K) + \frac{2000}{16}$

$$\bar{x} \geq \frac{200}{16} + 2\ln(K) = K^*$$

neyman pearson lemma tells us that the rejection region for the most powerful test under the normal probability model is of the form : $\bar{x} > k^*$

critical region : $S_1 = \{ x \mid \bar{x} > k^* \}$

find k^*.

$$\alpha = P(x \in S_1 \mid H_0)$$
$$0.05 = P(\bar{x} > k^* \mid \mu = 10)$$

$X \sim N(\mu, 16)$

$\bar{X} \sim N\left(\mu, \frac{16}{n}\right) \sim N(\mu, 1)$

$$0.05 = P\left(\frac{\bar{x} - \mu}{1} > \frac{k^* - \mu}{1} \,\Big|\, \mu = 10\right)$$

$Z = \frac{\bar{X} - \mu}{1}$

$$0.05 = P\left(Z \geq \frac{k^* - 10}{1}\right)$$
$$P\left(Z \leq -(k^* - 10)\right) = 0.05$$
$$-(k^* - 10) = -1.645$$
$$\therefore k^* = 11.645$$

MP test : $\bar{x} > 11.645$

now, $\beta = P(\bar{x} \in S_0 \mid H_1)$
$= 1 - P(\bar{x} \in S_0 \mid H_1)$
$\beta = 1 - P(\bar{x} > k^* \mid \mu = 15)$
Power $= 1 - \beta = P(\bar{x} > 11.645 \mid \mu = 15)$
$= P\left(Z \geq \frac{11.645 - 15}{1}\right)$

$= P(Z \geq 3.36$
$= 1 - P(Z \leq -3.36)$
$= 1 - 0.0004$
$= 0.996$

Uniformly most powerful test :

$H_0 : \theta \leq \theta_0$

$H_1 : \theta > \theta_0$

remember : $H_0 : \theta = \theta_0$; $H_1 : \theta = \theta_1$ → simple hypothesis → use most powerful test

$H_0 : \theta >< \theta_0$; $H_1 : \theta >< \neq \theta_0 ; \theta_1$ → compound hypothesis → use UMP test

we would like to determine the most powerful test when H_1 is composite ; however now we would like our test / rejection region to be "most powerful for all values of θ which satisfy H_1" – we call these uniformly most powerful tests

monotone likelihood ratio property :

monotone likelihood ratio - ratio of two pdfs (or pmfs)

$$\frac{\partial}{\partial x}\left(\frac{f(x)}{g(x)}\right) \geq 0 \qquad \text{non-dec}^n \text{ or inc}^n \quad (f'(\,) > 0)$$

how to use in hypothesis testing – if the family of RV has the monotone likelihood ratio in $T(x)$, then a UMP test can be determined

$T(x) = \Sigma x_i$ or $\prod x_i$ or $\Sigma \log x_i \ldots$

sufficient statistic

monotone likelihood ratio in $T(x)$ if the density $f(x|\theta)$ exists such that whenever $\theta_1 < \theta_2$, the likelihood ratio $\frac{f(x|\theta_2)}{f(x|\theta_1)}$ is a non-dec^n f^n of $T(x)$

not conditional ; $f(x;\theta)$

how to check if family of distributions follow monotone likelihood ratio –

$\theta_1 < \theta_2$ and $\lambda(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)}$

where $f(x|\theta)$ is the likelihood function i.e. $f(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$

we need to show $\lambda(x)$ is non-dec^n in $T(x)$

↳ $\lambda(x) > 0$ or $\lambda'(x) > 0$

Q. Consider a random sample of size n drawn from Normal $N(\mu, 1)$; show that family has monotone likelihood ratio.

Ans. pdf of $N(\mu, 1) = f(x|\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}$

likelihood f^n is $f(x|\mu) = \prod_{i=1}^{n} f(x_i|\mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\Sigma(x_i-\mu)^2}$

take $\mu_1 < \mu_2$; $\lambda(x) = \frac{f(x|\mu_2)}{f(x|\mu_1)} = e^{(\quad)} > 0 \quad \because \mu_1 < \mu_2$

non-dec^n in $T(x) = \Sigma x_i$

UMP test exists in $\Sigma x_i = t$ (sufficient statistic)

<u>definition of UMP test</u> – the real-parameter distribution family $\{p_\theta(x) ; \theta \in \Theta\}$ is said to have monotone likelihood ratio if there exists a real-valued function $T(x)$ such that for any $\theta < \theta'$ the distributions P_θ and $P_{\theta'}$ are distinct and the ratio $p_{\theta'}(x) \,/\, p_\theta(x)$ is a non-dec^n f^n of $T(x)$.

Whenever Φ satisfies monotone likelihood ratio property, UMP exists

<u>Karlin and Rubin Theorem</u> : Consider one-sided testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ for some $\theta_0 \in \Theta$. Let the family of distribution of X, $\{f(x, \theta), \theta \in \Theta\}$, have the monotone likelihood ratio in $T(x)$

Any test of the form
$$\phi(x) = \begin{cases} 1 & ; T(x) > k \\ \gamma & ; T(x) = k \\ 0 & ; T(x) < k \end{cases}$$

has non-decreasing power function and is UMP of its size α is positive; where k is determined such that

$$E_{H_0}(\phi(X)) = \alpha$$

different forms of UMP tests for one-sided problems –
$\theta_1 < \theta_2$ and $\lambda(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)}$

$\frac{f(x|\theta_2)}{f(x|\theta_1)}$ is <u>non-dec^n in T</u>, then UMP size-α test for

$H_0 : \theta \leq \theta_0$ v/s $H_1 : \theta > \theta_0$ is

$$\phi(x) = \begin{cases} 1 & ; T(x) > k \\ \gamma & ; T(x) = k \\ 0 & ; T(x) < k \end{cases}$$

so that $E_{H_0}(\phi(x)) = \alpha$

$H_0 : \theta \geq \theta_0$ v/s $H_1 : \theta < \theta_0$ is

$$\phi(x) = \begin{cases} 1 & ; T(x) < k \\ \gamma & ; T(x) = k \\ 0 & ; T(x) > k \end{cases}$$

so that $E_{H_0}(\phi(x)) = \alpha$

how to construct uniformly most powerful test-

compound hypothesis $H_0 : \theta \leq \theta_0$
$H_1 : \theta > \theta_0$

STEP 1. Check monotone likelihood property in $T(x)$
i.e. for $\theta_1 < \theta_2$

$$\lambda(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)}$$ and show $\lambda(x)$ is monotonic in $T(x)$

STEP 2. construct the UMP test function ϕ as

$$\phi(x) = \begin{cases} 1 & ; \; T(x) > K \\ 0 & ; \; \text{ow} \end{cases}$$

TEP 3. find K such that $E_{H_0}(\phi(x)) = \alpha$