Econ 7710
Assignment 5

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The due date for this assignment is Friday, October 27th.

1. Find characteristic function of a uniformly distributed random variable on [0, 1]
2. **(past midterm problem)** $X_1, X_2, \ldots$ are Bernoulli random variables with parameter $p = \frac{1}{2}$. Using the method of characteristic functions find the distribution of random variable
$$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}.$$
3. **(past midterm problem)** Under which conditions imposed on random variable X, random variables X and $\sin(X)$ are independent?

1. Find characteristic function of a uniformly distributed random variable on [0, 1]

1) Characteristic f^n of a uniformly distributed r.v. :-

$$\phi(t) = \frac{1}{it(b-a)}(e^{itb} - e^{ita})$$

Evaluating on [0, 1]

$$\phi(t) = \frac{(e^{it} - e^{0})}{it}$$

$$\phi(t) = \frac{e^{it} - 1}{it}$$

Rationalizing the denominator ;

$$\phi(t) = \frac{-i(e^{it} - 1)}{t} \quad \text{if } t \neq 0$$

$$\phi(t) = 1 \quad \text{if } t = 0$$

2. **(past midterm problem)** $X_1, X_2, \ldots$ are Bernoulli random variables with parameter $p = \frac{1}{2}$. Using the method of characteristic functions find the distribution of random variable

$$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}.$$

$X_1, X_2 \ldots$ are Bernoulli r.v. with $p = 1/2$

X_k are independent

$$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}$$

Let $Z_k = \frac{X_k}{2^k}$

& $Y_n = \sum_{n=1}^{k} Z_n$

$$\phi_{Z_k}(t) = E\left[e^{itZ_k}\right]$$

$$= E\left[e^{it\frac{X_k}{2^k}}\right]$$

$$= \frac{1}{2}\left(e^{it/2^k} - 1\right) + 1 \quad \text{(CF of a Bernoulli r.v)}$$

$$= \left(\frac{e^{it/2^k} + 1}{2}\right)$$

$$\phi_{Y_n}(t) = \phi_{\sum_{k=1}^{n} Z_k}(t) = \phi_{\sum_{k=1}^{n} \frac{X_k}{2^k}}(t)$$

$$= \prod_{k=1}^{n} \phi_{\frac{X_k}{2^k}}(t)$$ (As X_i are independent)

$$= \frac{1}{2^n} \prod_{k=1}^{n} \left(e^{it/2^k} + 1\right)$$

Multiplying both sides by $\left(e^{it/2^n} - 1\right)$

$$\phi_{Y_n}(t)\left(e^{it/2^n} - 1\right) = \frac{1}{2^n} \prod_{k=1}^{n} \left(e^{it/2^k} + 1\right) \cdot \left(e^{it/2^n} - 1\right)$$

$$= \frac{1}{2^n} \prod_{k=1}^{n-1} \left(e^{it/2^k} + 1\right) \left(e^{it/2^n} + 1\right) \left(e^{it/2^n} - 1\right)$$

$$= \frac{1}{2^n} \prod_{k=1}^{n-1} \left(e^{it/2^k} + 1\right) \left(e^{it/2^{n-1}} - 1\right)$$

$$\vdots$$

$$= \frac{e^{it} - 1}{2^n}$$

$$\phi_{Y_n}(t) = \frac{e^{it} - 1}{2^n \left(e^{it/2^n} - 1\right)}$$

Now, find $\lim_{n \to \infty} \phi_{Y_n}(t)$

$$\lim_{n \to \infty} \phi_{Y_n}(t) = \lim_{n \to \infty} \frac{\left(e^{it} - 1\right) \cdot 2^{-n}}{\left(e^{it/2^n} - 1\right)}$$

Using L'hopital rule:

$$\lim_{n\to\infty} -\ln(2)\frac{(e^{it}-1)}{2^n}\left(-\frac{2^n}{\ln(2)\, it\, e^{it/2^n}}\right)$$

$$= \lim_{n\to\infty} \frac{e^{it}-1}{it\, e^{it/2^n}}$$

$$= \frac{e^{it}-1}{it}$$

From Q1, we know that the above cf is of a rv $U[0,1]$. (cf's are unique)

Thus $X \sim U[0,1]$

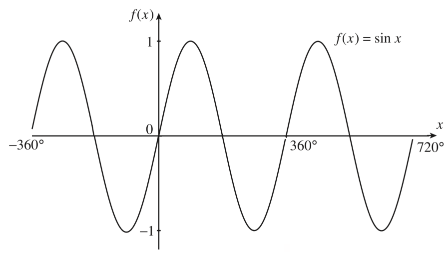
3. **(past midterm problem)** Under which conditions imposed on random variable X, random variables X and $\sin(X)$ are independent?

x and f(x) are independent if f(x) is a constant.

In the case of X and f(X), this is true if :

1) X is a degenerate distribution, or

2) As sin(X) is periodic, then any distribution of x that gives the same value of sin x.
Example, x = 0, 180°, 360°, 540°, 720°...



$\sin x = c$ where c is a constant

$x = \arccos(c)$

- Further, since sin x has a periodicity at 360°, $x = c + n\,360$ for $n = 0, 1, 2, \ldots$ would have the same constant value for sin x.