Oct 23, 2023

Slutsky Equation.

$$\underbrace{\frac{\partial x_i}{\partial p_k}}_{\text{Total}} = \underbrace{\frac{\partial h_i}{\partial p_k}}_{\text{substitution}} - \underbrace{\frac{\partial x_i}{\partial m} x_k(p,m)}_{\text{Income}}$$

For own-price effects $(k=i)$

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial m} x_i(p,m)$$

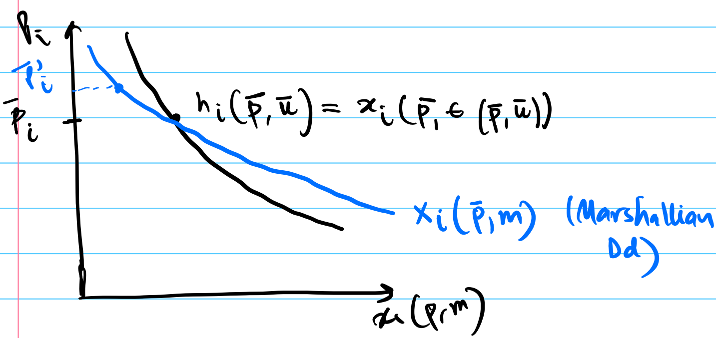
If good is normal $\frac{\partial x_i}{\partial m} \geq 0$

$\Rightarrow \frac{\partial x_i}{\partial p_i} \leq 0$

If good i is inferior $\frac{\partial x_i}{\partial m} \leq 0$

SE & IE are "opposed"

$\frac{\partial x_i}{\partial p_i} \geq 0$ only if good i is strongly inferior.



Normal Good

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial m} x_i(p, m)$$

$$x_i(\bar{p}, e(\bar{p}, \bar{u})) = h_i(\bar{p}, \bar{u})$$

Redraw for inferior good.

Hicksian Dd: Min. expenditure for given utility

Hicksian Matrix.

$$Dh(p) = \begin{pmatrix} \partial h_1/\partial p_1 & \cdots & \partial h_1/\partial p_n \\ \vdots & \ddots & \\ \partial h_n/\partial p_1 & & \partial h_n/\partial p_n \end{pmatrix}$$

- symmetric
- n, s, d
- $Dh(p) \cdot (p) = 0$

Consider:

$$S(p, m) = \begin{bmatrix} s_{11} & s_{12} & \cdots & \\ s_{21} & \ddots & & \\ & & \ddots & \\ & & & s_{nn} \end{bmatrix}$$

Slutsky Substitution Matrix

$$s_{j,k} = \frac{\partial x_j(p, m)}{\partial p_k} + \frac{\partial x_j(p, m)}{\partial m} \cdot x_k(p, m) \rightarrow \text{how?}$$

By slutsky eq:

$S(p, m) = Dh(p)$ should also satisfy property of hicksian matrix.

* Special case (No income effect)

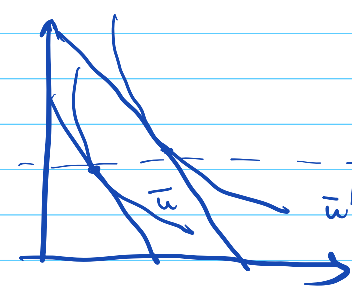
$$x_i(p, m) = x_i(p)$$

Duality identity:

$$h_i(p, \bar{u}) = x_i(p, e(p, \bar{u})) = x_i(p)$$

$$\Rightarrow h_i(p, \bar{u}) = h_i(p)$$

say $h_2(p, \bar{u}) = h_2(p)$



slutsky eq^n

$$\frac{\partial h_i}{\partial p_i} = \frac{\partial x_i}{\partial p_i} \quad \text{(Income effects go away)}$$

* $\max\ x_1 + v(x_2)$ s.t. $x_1 + p_2 x_2 = m$

$$\max_{x_2}\ m - p_2 x_2 + v(x_2)$$

FOC: $v'(x_2) = p_2$

$$x_2^*(p, m) = (v')^{-1}(p_2)$$

$$\frac{\partial x_2^*}{\partial m} = 0$$

From BC,

$$x_1^*(p, m) = m - p_2 \left[(v')^{-1} (p_2) \right]$$

$$\frac{\partial x_1^*}{\partial m} = 1$$

$\Rightarrow$ Quasilinear prefs. have no income effects.

Integrability

Q Given functions $x(p, m)$ that satisfy:

- → hd-0
- → Walras' law
- → symmetric & n.s.d slutsky matrix

Can we recover utility? YES.

$$x_i(p, m) = \frac{-\partial v / \partial p_i}{\partial v / \partial m}$$

Roy's identity in reverse:

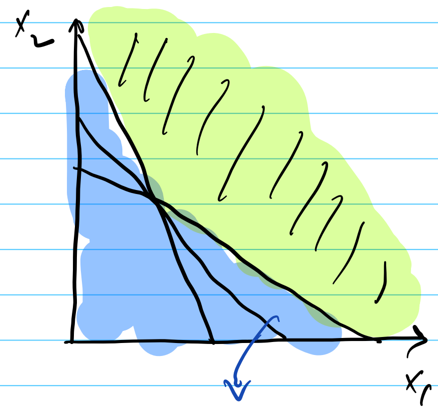
Proof strategy:

demand function $\xrightarrow{1}$ expenditure function $\xrightarrow{2}$ utility

Start w 2 :—

Say we know $e(p, \bar{u})$

We want $V(\bar{u}) = \{$ bundles that give utility atleast $\bar{u}\}$



$\{x : p \cdot x = e(p, \bar{u})\}$

Step 1: $x(p, m) \longrightarrow e(p, \bar{u})$

Pick a starting pt. (p^0, m^0)

let $u(x(p^0, m^0)) = u^0$

Goal: solve for $e(p, u^0)$

Write as $e(p)$

By shephard:

$$\frac{\partial e(p)}{\partial p_i} = h_i(p, u^0)$$

$$\frac{\partial e(p)}{\partial p_i} = x_i(p, e(p))$$

initial cond^n

$e(p^0) = m^0$

Does a sol^n to this partial differential eq^n even exist?

A necessary & sufficient cond^n for a system of equation $\frac{\partial f}{\partial x_i} = g_i(x)$ to have a sol^n is

$$\frac{\partial g_i(x)}{\partial x_j} = \frac{\partial g_j(x)}{\partial x_i}$$

$$\left(\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

In our problem:

$$\frac{\partial x_i(p, e(p))}{\partial p_j} = \frac{\partial x_j(p, e(p))}{\partial p_i}$$

$$S = [S_{i,j}]$$

S must be symmetric.

Symmetry of S is necessary & sufficient for system of PDEs to have a solution.

For the solution $e(p)$ to be a valid expenditure f^n, it also must be concave.

Concavity means $DE(p) = S$ is negative semi-definite.

Bottom Line: –

Given some $x(p, m)$, if S is symmetric & n.s.d, we can solve for $e(p, \bar{u})$. Then use earlier technique to go from

$$e(p, \bar{u}) \to v(p, m)$$

Ex

$$x_1(p, m) = \frac{\alpha m}{p_1} \quad ; \quad x_2(p, m) = \frac{\beta m}{p_2}$$

Find $e(p, \bar{u})$ by solving :

$$\frac{\partial e(p)}{\partial p_1} = x_1(p) \qquad \frac{\partial e(p)}{\partial p_2} = x_2(p)$$

Check symmetry :—

$$\frac{\partial^2 e(p)}{\partial p_2 p_1} = 0 = \frac{\partial^2 e(p)}{\partial p_1 \partial p_2}$$

$$\frac{\partial e(p)}{\partial p_1} = \frac{\alpha e(p)}{p_1}$$

$$\frac{\partial e(p)/\partial p_1}{e(p)} = \frac{\alpha}{p_1}$$

$$\int \frac{\partial \log e(p) \cdot dp_1}{\partial p_1} = \int \frac{\alpha}{p_1} dp_1$$

$$\log e(p) = \alpha \log p_1 + c_1(p_2, \bar{u})$$

Similarly,

$$\log e(p) = \beta \log p_2 + c_2(p_1, \bar{u})$$

$$\log e(p) = \alpha \log p_1 + \beta \log p_2 + f(\bar{u})$$

$$e(p) = e^{f(a)} p_1^{\alpha} p_2^{\beta}$$

$$= \tilde{u}\, p_1^{\alpha} p_2^{\beta}$$

To find $v(p, m)$:

$$e(p, v(p, m)) = m$$

$$v(p, m)\, p_1^{\alpha} p_2^{\beta} = m$$

$$v(p, m) = \frac{m}{p_1^{\alpha} p_2^{\beta}}$$