ECON 7710 TA Session

Week 9

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Outline

Question 1

- We know $X \sim N(0,1)$ and $Y \sim N(0,1)$ and X and Y are independent. We define $Z = \frac{X}{Y}$. CDF of Z is written as follows: $F_Z(z) = Pr(Z \leq z) = Pr(\frac{X}{Y} \leq z)$

$$= Pr(X \leq zY | Y > 0)Pr(Y > 0) + Pr(X \geq zY | Y < 0)Pr(Y < 0)$$
$$= Pr(X \leq zY, Y > 0) + Pr(X \geq zY, Y < 0)$$
$$= \int_0^\infty \int_{-\infty}^{zy} f(x)f(y)dxdy + \int_{-\infty}^0 \int_{zy}^\infty f(x)f(y)dxdy$$

- By Leibniz rule,

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x,z)dx = f(b(z),z)b'(z) - f(a(z),z)a'(z) + \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} f(x,z)dx$$

we know the PDF of Z is

$$f_Z(z) = \frac{\partial F_Z(z)}{\partial z} = \int_0^\infty yf(zy)f(y)dy - \int_{-\infty}^0 yf(zy)f(y)dy = \underbrace{2\int_0^\infty yf(zy)f(y)dy}_{\text{f() is symmetric}}$$

Question 1

$f_Z(z) = 2\int_0^\infty yf(zy)f(y)dy$

- We know pdf for $N(0,1)$ is $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, x \in \mathbb{R}$, then
  $f_Z(z) = \frac{1}{\pi}\int_0^\infty ye^{-y^2(\frac{z^2+1}{2})}dy = \frac{1}{2\pi}\int_0^\infty 2ye^{-y^2(\frac{z^2+1}{2})}dy = \frac{1}{2\pi}\int_0^\infty e^{-y^2(\frac{z^2+1}{2})}dy^2$
- We know $f_Z(z) = \frac{1}{2\pi}\int_0^\infty e^{-y^2(\frac{z^2+1}{2})}dy^2$
- Denote $u = y^2$, $t = \frac{z^2+1}{2}$, then

$$f_Z(z) = \frac{1}{2\pi}\int_0^\infty e^{-tu}du = -\frac{1}{2\pi t}[e^{-tu}]_0^\infty = \frac{1}{2\pi t} = \frac{1}{\pi(z^2+1)}$$

- Clearly, this is a **standard Cauchy Distribution**. And we know none of n-moments exist for $n \geq 1, n \in \mathbb{N}$ in Cauchy distribution.
- Check $C\&B$ page 162 for another solution.

Question 2.a

We know $\{X_n\}_{n=1}^{\infty}$ and $X_n \xrightarrow{d} X$, where $X \sim N(0,1)$. We suppose $Y_n = X_n$ for all $n \geq 1$

a Since $Y_n = X_n, \forall n \geq 1$, we know $F_{X_n}(x) = F_{Y_n}(x), \forall n \geq 1, \forall x \in \mathbb{R}$. So we know since $X_n \xrightarrow{d} X$, we can derive:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x$$

We can also get

$$\lim_{n\to\infty} F_{Y_n}(x) = \lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x$$

Therefore if we denote Y as the distribution limit of $\{Y_n\}_{n=1}^{\infty}$, we know $Y \sim N(0,1)$.

Question 2

b $Y_n \xrightarrow{d} Y$. We claim that $X_n + Y_n \xrightarrow{d} X + Y$ **doesn't always hold**. Consider $Y = -X$. Then by symmetry of normal distribution, we can still have $F_Y = F_{-X} = F_X$ and now $X + Y = 0$.

- For Y_n, still we have

$$\lim_{n\to\infty} F_{Y_n}(x) = \lim_{n\to\infty} F_{X_n}(x) = F_X(x) = F_Y(x), \forall x$$

  , where $Y = -X$. We proved $Y_n \xrightarrow{d} Y$ still holds.
- Since $X_n + Y_n = 2X_n$ For $\lim_{n\to\infty} F_{2X_n}(x) = \lim_{n\to\infty} P(2X_n \leq x) = \lim_{n\to\infty} P(X_n \leq \frac{x}{2})$.
  We know $X_n \xrightarrow{d} X$ So $\lim_{n\to\infty} P(X_n \leq \frac{x}{2}) = F_X(\frac{x}{2}) = P(X \leq \frac{x}{2}) = P(2X \leq x)$.
  We know if $X \sim N(0,1)$, then $2X \sim N(0,4)$. So
  $X_n + Y_n \xrightarrow{d} Z, Z \sim N(0,4) \neq X + Y = 0$

Therefore, we found a counterexample when $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, $Y_n = X_n, \forall n \geq 1$, $X_n + Y_n \xrightarrow{d} X + Y$ may fail as well.

Question 3.a

We define the median of the distribution of random variable X is the number $q_{0.5}$ that solves

$$\inf_q \{P(X \leq q) \geq \frac{1}{2}\}$$

We know there exists a numeric sequence a_n such that $X_n - a_n \xrightarrow{p} 0$. We let $q_{0.5}^n$ be the median of the distribution of X_n

a We already know that $X_n - a_n \xrightarrow{p} 0$.

$$\lim_{n\to\infty} P(|X_n - a_n| \geq \epsilon) = 0 \Rightarrow \lim_{n\to\infty} P(X_n \geq \epsilon + a_n) + \lim_{n\to\infty} P(X_n \leq a_n - \epsilon) = 0$$

- Since for any probability p, we have $0 \leq p \leq 1$, then we know $\lim_{n\to\infty} P(X_n \geq \epsilon + a_n) = 0$ and $\lim_{n\to\infty} P(X_n \leq a_n - \epsilon) = 0$
  So the possible range of X_n is $(a_n - \epsilon, a_n + \epsilon)$. Then we will have the median $q_{0.5}^n \in (a_n - \epsilon, a_n + \epsilon)$
- In other words, $|q_{0.5}^n - a_n| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we know $\lim_{n\to\infty} (q_{0.5}^n - a_n) = 0$ is proved.

Question 3.b

b For $\lim_{n\to\infty}(E[X_n]-a_n)=0$, we claim it is false and a counterexample is given below:

$$X_n=\begin{cases} n & p=1/n \\ 0 & p=1-1/n \end{cases}$$

- Let $a_n=\{0,0,0,...\}$, a numerical sequence only contains 0. Then:

$$\lim_{n\to\infty}P(|X_n-a_n|>\epsilon)=\lim_{n\to\infty}P(|X_n|>\epsilon)=\lim_{n\to\infty}\frac{1}{n}=0$$

  So we successfully formulated $X_n-a_n\xrightarrow{p}0$.

* By defination, $\lim_{n\to\infty}q^n_{0.5}=0$.

- For $\lim_{n\to\infty}(E[X_n]-a_n)$. We know $\forall n, E[X_n]=n*\frac{1}{n}+0*(1-\frac{1}{n})=1$

$$\lim_{n\to\infty}(E[X_n]-a_n)=\lim_{n\to\infty}E[X_n]=1$$

  Then we found for X_n and a_n we constructed, $\lim_{n\to\infty}(E[X_n]-a_n)\neq 0$.

Question 4

We know X_1, X_2...is a sequence of independent and identically distributed random variables. We also know $X_n \xrightarrow{p} X$. We want to prove that X has a degenerate distribution.

- We consider two subsequences of X_n, which are
  - $\{X_i\}, \quad i = 1, 3, 5...$
  - $\{X_j\}, \quad j = 2, 4, 6...$

  Since $X_n \xrightarrow{p} X$, we know $X_i \xrightarrow{p} X$ and $X_j \xrightarrow{p} X$.
  We also know all X_n are i.i.d. Then we know for the characteristic function of X_i and X_j, we will have $\phi_{X_i+X_j}(t) = \phi_{X_i}(t)\phi_{X_j}(t)$.
- We also know that convergence in probability implies convergence in distribution implies convergence of characteristic function. Therefore, we know:
  - $X_i \xrightarrow{p} X \Rightarrow X_i \xrightarrow{d} X \Rightarrow \lim_{i\to\infty} \phi_{X_i}(t) = \phi_X(t)$
  - $X_j \xrightarrow{p} X \Rightarrow X_j \xrightarrow{d} X \Rightarrow \lim_{j\to\infty} \phi_{X_j}(t) = \phi_X(t)$

  So we found that $\lim_{(i,j)\to(\infty,\infty)} \phi_{X_i+X_j}(t) = [\phi_X(t)]^2$

Question 4

- On the other hand, as $X_i \xrightarrow{p} X$ and $X_j \xrightarrow{p} X$, we know $X_i + X_j \xrightarrow{p} 2X$
  Fix $\epsilon > 0$,

$$P(|X_i + X_j - 2X| \geq \epsilon) \leq P(|X_i - X| + |X_j - X| \geq \epsilon)$$
$$\leq \underbrace{P(|X_i - X| \geq \frac{\epsilon}{2})}_{\text{when } i \to \infty, \to 0} + \underbrace{P(|X_j - X| \geq \frac{\epsilon}{2})}_{\text{when } j \to \infty, \to 0}$$

  Then we have $\lim_{(i,j)\to(\infty,\infty)} \phi_{X_i+X_j}(t) = \phi_{2X}(t) = \phi_X(2t)$

- Therefore we land on the conclusion that X's characteristic function must satisfy:

$$\phi_X(t)^2 = \phi_X(2t)$$

- For a degenerate distribution with probability $P(X = c) = 1$. We know $\phi_X(t) = e^{itc}$. So we know: $\phi_X(t)^2 = e^{2itc}$ and $\phi_X(2t) = e^{2itc}$.
- Then we found our conclusion holds for a degenerate distribution and we proved that X has a degenerate distribution.

Core 2020 June Q2

Suppose that X_n is a sequence of real-valued random variables and suppose that sequence $(X_n)^2$ (containing the squares of the original random variable) converges in probability while sequence X_n does not converge in probability or in distribution.

a Characterize all possible limits X_*^2 of sequence of random variables $(X_n)^2$.

b Formally describe an algorithm to construct a subsequence of X_n that does converge in probability and characterize its possible limits.

2020 Jun Q2.a

a I claim that X_*^2 can be any positive real number as long as $X_*^2 \neq 0$
What we are given are:

- X_n does not converge
- $X_n^2 \xrightarrow{p} X_*^2$, $\lim_{n\to\infty} Pr(|X_n^2 - X_*^2| > \epsilon) = 0$

- Continuous Mapping Theorem tells you

$$X_n^2 \xrightarrow{p} X_*^2 \Rightarrow \begin{cases} \sqrt{X_n^2} \xrightarrow{p} \sqrt{X_*^2} \\ -\sqrt{X_n^2} \xrightarrow{p} -\sqrt{X_*^2} \end{cases} \Rightarrow |X_n| \xrightarrow{p} X_*$$

- We know $X_n \stackrel{p}{\nrightarrow} X_*$. So $X_* > 0$ but $X_* \neq 0$.
Why 0 not work? We can prove if $|X_n| \xrightarrow{p} 0$, then $X_n \xrightarrow{p} 0$, $\Rightarrow\Leftarrow$
  - $|X_n| \xrightarrow{p} 0 \Rightarrow \lim_{n\to\infty} Pr(|a_n| - 0 > \epsilon) = 0$ then we have
$\lim_{n\to\infty} Pr(|X_n - 0| > \epsilon) = 0 \Rightarrow X_n \xrightarrow{p} 0$
  - Or you can say if $X_* = 0$, then $\sqrt{X_*^2} = -\sqrt{X_*^2}$, which means we can remove the absolute value above...

2020 Jun Q2.b

b Obviously, you just need to split the X_n into two parts
Positive only or negative only.

- Formally, two algorithms to construct the subsequence:
  - Pick X_i from X_n such that all the $X_i > 0$
  - Pick X_j from X_n such that all the $X_j < 0$
- Following the result in part a, the all possible limits are:
  - For X_i, we have $X_i \xrightarrow{p} |X_*|$
  - For X_j, we have $X_j \xrightarrow{p} -|X_*|$
- Again, $|X_*|$ can be any positive real numbers but cannot be zero.