HW4, Problem 1

(1.1) #1, part 1
Let $g(z) = c$, where $z = (k, k', n)$. Define $\Gamma(k) = \{(k', n) \in \mathbb{R}_{\geq 0} \times [0, 1] : k' \leq F(k, n) + (1 - \delta)k\}$

$$V(k) = \max_{k', n \in \Gamma(k)} \{u(g(z), 1 - n) + \beta V(k')\}$$

We make the following additional assumptions

- $\beta \in (0, 1)$, $\delta \in (0, 1]$
- $u(\cdot)$ is continuous, $F(\cdot)$ is continuous and non-negative

These assumptions imply $\Gamma(k)$ is well defined: non-empty, compact (closed and bounded subset of $\mathbb{R}^2$), and continuous. Now, $u(\cdot)$, $F(\cdot)$ are bounded because they are continuous on a compact set. Thus, our Bellman operator T will map bounded functions to bounded functions (Tv bounded if v bounded). Now, we need to show Blackwell's sufficient conditions hold (monotonicity and discounting), meaning T is a contraction map

- Monotonicity – let $\tilde{f}, f$ be bounded functions with $\tilde{f} \leq f$
  To make this a bit cleaner, we will substitute back in consumption. By $\tilde{f} \leq f$

  $$u(c) + \beta\tilde{f}(F(k, n) + (1 - \delta)k - c) \leq u(c) + \beta f(F(k, n) + (1 - \delta)k - c)$$

  maximize the LHS & RHS over the feasible set for c, n (it's the same $\Gamma(k)$). Since $T\tilde{f}, Tf$ bounded,

  $$\implies T\tilde{f} \leq Tf$$

  (when you take this maximization you get the definition of the Bellman operator)

- Discounting – again let f be bounded and fix arbitrary $a \geq 0$. Define $\tilde{f} = f + a$ and recall $\beta \in (0, 1)$

  $$T\tilde{f}(k) = \max_{k', n \in \Gamma(k)} \{u(g(z), 1 - n) + \beta f(k') + \beta a\} = \max_{k', n \in \Gamma(k)} \{u(g(z), 1 - n) + \beta f(k')\} + \beta a = Tf(k) + \beta a$$

We can now invoke the contraction map theorem and conclude the solution/fixed point is our $V(k)$

Comments: $\delta > 0$ is concerned with existence of a stable, interior equillibria as you saw in class after this HW was due, not as relevant for the "meat of this question". The Bellman operator will trivially always satisfy discounting, but monotonicity requires a bit more care. The way Vladimir proved it in class in special case was with the policy function, but you need restrictions on the objective function (e.g., concavity) for a unique policy function, which in principle aren't needed to answer this question. Also in general, checking monotonicity is "harder" since you need it to hold for a more general class of functions $\tilde{f} \leq f$ (continuity and compactness are working in the background). Finally, if you've written down the Bellman equation, the form of the Bellman operator is trivial; only write it if it helps you. Make sure you read problems fully. There's a difference between "what do you need" and "check that you have what you need".

(1.2) # 1, part 2
Definition: HW2 #2 with utility function $u(c, 1 - n_t)$ (I use ℓ there for labor – just replace with n_t)
This is what we talked about in discussion – given that we have a solution to the social planner's problem, we have policy functions that allow us to generate a sequence of allocations given k_0. By the second welfare theorem, these are the ADE allocations. We can get $\{p_t\}$ from the HH FOC w.r.t c. To be explicit

$$\beta^{t-k} u_c(t - k) = \lambda p_{t-k} \implies \frac{p_t}{p_{t-k}} = \beta^{t-k} \frac{u_c(t)}{u_c(t - k)}$$

Plug in $t = 1, k = 1$, normalize by $p_0 = 1$, then plug in $(c_0, n_0), (c_1, n_1)$ allowing you to solve for p_1, then iteratively solve for the rest. We can get the factor prices using marginal products times the price.

HW4, Problem 2

I again choose to keep only one labor-related variable floating around, n_t. Let 1-n_t be leisure.
Because there is no restriction on the government, I personally would set this up with sequential markets. However because Vladimir typically defaults to time-0 setup with taxes, I follow him. Feel free to not do this (if no government BC restrictions are listed), but also remember you need to add bonds.
Let $U(c_t, 1-n_t) = u(c_t) + v(1-n_t)$. Then $\mathcal{L} = \sum \beta^t U(t) + \lambda \sum \{r_t k_t + w_t n_t - p_t(c_t(1+\tau_t^c) - x_t)\}$. FOCs:

$$\beta^t U_c(t) = \lambda p_t(1+\tau_c^t) \ \underline{\textbf{and}} \ \beta^t U_n(t) = \lambda w_t \ \underline{\textbf{and}} \ r_{t+1} + (1-\delta)p_{t+1} = p_t$$

As with #1.1, this implies $p_t(1+\tau_t^c) = \beta^t \frac{(1+\tau_0^c)U_c(t)}{U_c(0)}$ when we normalize $p_0 = 1$. We can use $\lambda = \frac{U_c(0)}{1+\tau_0^c}$ from the FOC w.r.t c_0. So $w_t = \beta^t \frac{(1-\tau_0^c)U_n(t)}{U_c(0)}$. Our FOC for capital implies $\sum k_{t+1}\left[r_{t+1} + (1-\delta)p_{t+1} - p_t\right] = 0$. We use all this to simplify HH BC to get the implementability constraint (IC)

$$\sum \beta^t \left[U_c(t)c_t - U_n(t)n_t\right] = (1+\tau_0^c)^{-1} U_c(0)\left[r_0 + (1-\delta)k_0\right] = A_0$$

Note $U_c(t) = c_t^{-\sigma}$ and $U_n(t) = v'(1-n_t)$. Define $W(t,\Phi) = U(t) + \Phi\left[U_c(t)c_t - U_n(t)n_t\right]$.
The Ramsey planner picks allocations $\{c_t, n_t, k_{t+1}\}$ s.t

$$\{c_t, k_{t+1}, n_t\} = \underset{\{c_t, k_{t+1}, n_t \in [0,1]\}}{\text{argmax}} \sum \beta^t U(t) \ \text{ s.t } \ F(k_t, n_t) = c_t + x_t + g_t \ \underline{\textbf{and}} \ \text{IC}$$

Ramsey Lagrangian: $\mathcal{L}^R = \sum \{\beta^t W(t,\Phi) + \mu_t\left[F(k_t,n_t) - (1-\delta)k_t - c_t - g_t - k_{t+1}\right]\} - \Phi A_0$. FOCs ($t \geq 1$):

$$\beta^t W_c(t) = \mu_t \ \underline{\textbf{and}} \ \beta^t W_n(t) = -\mu_t F_n(t) \ \underline{\textbf{and}} \ \mu_t = \mu_{t+1}\left[F_k(t+1) + (1-\delta)\right]$$

Combining c_t and k_{t+1} FOC $\frac{W_c(t)}{\beta W_c(t+1)} = F_k(t+1) + (1-\delta)$. From decentralized setup

$$p_{t+1}\left[F_k(t+1) + (1-\delta)\right] = p_t \implies 1 = \beta(F_k(t+1)+1-\delta)\frac{U_c(t+1)}{U_c(t)}\frac{1+\tau_t^c}{1+\tau_{t+1}^c} = \frac{W_c(t)}{W_c(t+1)}\frac{U_c(t+1)}{U_c(t)}\frac{1+\tau_t^c}{1+\tau_{t+1}^c}$$

Based on functional form of U, F and the deterministic environment, we can infer that the economy converges to an interior steady state. From the equation above, we conclude that the optimal policy is $\tau_t^c = \tau_{t+1}^c$.

Comments: I have verified with Vladimir that this is how he wants you to conclude the problem – just be careful about your language (don't say "impose steady state" unless otherwise directed). Notice how many derivatives you can leave implicit and how much writing you can save with notation. For full credit in a definition, you must have explicit expressions somewhere, which is why I took the time to say $U_c(t) = c_t^{-\sigma}$ and $U_n(t) = v'(1-n_t)$ since they show up in the IC. Otherwise, we ended up never needing to actually evaluate any of these terms to get our answer. This is because their structure depends on allocations we know are converging. It won't always be this way, but don't take derivatives until you have to.

HW4, Problem 3

(3.1) # 3, part 1
By default, assume labor is not inelastic. You may be counted off on an exam if you assume otherwise
A TDCE, given a fiscal policy $\{g_t, \tau_t^f\}$, is a collection of HH allocations $\{c_t, n_t^s, k_{t+1}^s\}$, firm allocations $\{n_t^d, k_t^d\}$, and prices $\{p_t, r_t, w_t\}$ s.t

- Given prices, HH allocations $\{c_t, n_t^s, k_{t+1}^s\}$ solve

$$\max_{\{c_t, n_t, k_{t+1}\}} \beta^t U(t) \ \text{ s.t } \ \sum p_t(c_t + x_t) = \sum \{r_t k_t + w_t n_t\} \ \underline{\textbf{and}} \ x_t = k_{t+1} - (1-\delta)k_t$$

- Given prices, firm allocations $\{n_t^d, k_t^d\}$ maximize profits at each t

$$\max_{k_t, n_t} \ p_t(1-\tau_f^t)F(k_t, n_t) - r_t k_t - w_t n_t$$

- Markets clear at each t

$$F(K_t, N_t) = y_t = c_t + x_t + g_t \ \underline{\textbf{and}} \ K_t = k_t^s = k_t^d \ \underline{\textbf{and}} \ N_t = n_t^s = n_t^d$$

- Government balances its lifetime BC

$$\sum p_t g_t = \sum \tau_t^f p_t y_t$$

(3.2) #3, part 2
With firm tax: firm FOC yield $w_t = p_t(1-\tau_t^f)F_n(t)$ and $r_t = p_t(1-\tau_t^f)F_k(t)$. Now, consider the HH optimality conditions from #2, but instead we impose a 0 consumption tax. Plugging in these factor prices

$$\frac{U_n(t)}{U_c(t)} = \frac{w_t}{p_t} = (1-\tau_t^f)F_n(t) \ \underline{\textbf{and}} \ \frac{U_c(t)}{\beta U_c(t+1)} = \frac{r_{t+1}}{p_{t+1}} + 1 - \delta = (1-\tau_t^f)F_k(t+1) + 1 - \delta$$

Thus, all we need to do is construct a tax scheme that will yield that same equations and we know that implies the same allocations (provided U is strictly increasing and concave). Suppose we impose a capital and labor income tax on the HH equal to $\tau = \tau_t^f$. Then HH FOCs would yield these same optimality conditions by $\mathcal{L} = \sum \beta^t U(t) + \lambda \sum \{(1-\tau)(n_t w_t + r_t k_t) - p_t(c_t + x_t)\}$ and factor prices $\tilde{w} = p_t F_L(t)$ and $\tilde{r} = p_t F_k(t)$

HW4, Problem 4

Important Pedagogy: When a question asks for optimal taxation, it is not concerned with pareto optimality at all. Optimal taxation is derived under the implementability constraint, which is not a requirement of a PO allocation. Therefore, solving a simple social planning problem (different from Ramsey planner) will not tell you what the optimal taxes are. Moreoever, note the difference in language: in this problem we are concerned with fiscal policy that *delivers* PO allocations. In #2, we are only concerned with optimal policy itself. So these problems are fundamentally different. In general, when asked about PO, think about SPP, when asked about optimal taxation, think about Ramsey planner. Now back to your regularly scheduled solutions.

Definition is again the same as HW2 #2, with a utility function $u(c_t, 1-n_t)$ and the government budget constraint given in the problem:$T_{1,t} = -T_{2,t} \ \forall t$ – there is no g_t in this problem (or you could consider $g_t = 0$) and the government BC *cannot* be represented as an infinite sum. The definition of PO is the same as the one earlier in the semester, just now the utility function is $u(c_t, 1-n_t)$. We can use the same approach we did in Discussion #3 (and your midterm) to prove it's PO. However, since that was endowment economy OLG, I will provide the explicit solution. Proof by contradiction: If not PO, then $\exists \ \{\widetilde{c}_{i,t}, \widetilde{n}_{i,t}\}$ s.t one agent has strictly higher lifetime utility. WLOG say agent 1 is strictly better off, which means agent 2 is weakly better off. Because the original is a CE, agents are maximizing utility, which implies this new bundle must have not been affordable to agent 1 under CE prices. Similarly, the new bundle must be weakly affordable to agent 2. Adding these two facts together (sequence of consumption and labor implies a sequence of capital)

$$\sum p_t(\tilde{C}_t + \tilde{X}_t) > \sum \left\{ r_t \tilde{K}_t + w_t \tilde{N}_t \right\}$$

where $\{\tilde{C}, \tilde{X}, \tilde{K}, \tilde{N}\}$ are aggregates under this alternate allocation. Thus, this allocation is not feasible (aggregate consumer spending exceeds aggregate consumer income), contradiction. I provide an alternate proof working directly with optimality conditions below. Finally, the $T_{1,t} = T_{2,t}$ component isn't necessary because the relevant feasibility constraint is over an entire lifetime. These transfer terms canceled out when we summed the BCs together, but because we were taking a sum over all t, all we need is that they cancel out in an infinite sum. Therefore, we can relax this to an ADE w/ transfers environment and everything still holds. To be explicit, the assumption is that $\sum T_{1,t} = -\sum T_{2,t}$.

<u>Extra</u> <u>Proof for 4.3</u>: Sketch: if we hold capital fixed, we get the same thing as Midterm #2. If we allow capital to change, from the intertemporal optimality condition, this means the ratio of marginal utility of consumption must have changed for all $t \leq T$, which isn't feasible.

Detailed proof: Recall an argument we saw earlier in Midterm #2: agents cannot be changing their labor-leisure decision without violating their optimality conditions. Because transfers show up linearly in HH BC, they won't show up in our FOCs. So the optimality conditions are the same as what we've seen earlier

$$-\frac{u_n(t)}{u_c(t)} = F_n(t) \ \underline{\textbf{and}} \ \frac{u_c(t)}{\beta u_c(t+1)} = F_k(t+1) + 1 - \delta$$

(assume this is all for the agent whose utility has gone up). First, consider the case we hold capital constant. If we work more in T, then we consume more (since we have more income), marginal utility (MU) of leisure is going up, MU of c_T is going down, and MPL(T) is going down. So the LHS and RHS are moving in opposite directions, contradiction. A similar contradiction happens if we work less and consume less. Now consider $\tilde{c}_T > c_T$ but we allow capital to change. To match the fact that the LHS $\uparrow$, MPL(T) $\uparrow$. So we increase our capital and MPK(T) goes down. From the second optimality condition, this means that $\tilde{c}_{T-1} > c_{T-1}$ (MU c_{T+1} must have gone down). We now need MPL($T-1$) to go up; we're back where we started. Iterating, this means for all $t \leq T$ capital and consumption have all strictly gone up. Once we get to period 0, this will violate feasibility because we cannot adjust k_0. The other scenarios have a similar unraveling.

This is a bit more cumbersome to think through when there's capital. Here's the punchline: if we hold capital fixed, both hours and consumption must have changed because utility depends on both (and we know utility increased). If we have capital, it's possible only c or n change (or both change), but it necessitates an adjustment in all allocations, which isn't feasible. Here's all possibilities for utility increasing at T

- only consumption goes up. It's financed through higher capital income. This is the scenario I covered in detail; all the optimality conditions have to adjust and its not possible at $t = 0$.
- consumption goes up and hours go up. This is the exact same as the first scenario.
- consumption goes up and hours go down. So we have less labor income but enough capital income to compensate. This means MPK is still going up, landing us back in the unraveling in the first scenario.
- the other possibilities are hours go down or both hours and consumption go down. All require these iterative adjustments that won't be possible once we run into period 0.