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$\mathcal{L} = \{e^{itx}, t \in \mathbb{R}\}$

Theorem $F_n \Rightarrow F$, $F \in \mathcal{G}$

$\phi_n(t) \longrightarrow \phi(t)$ (point wise convergence of characteristic f^n to a limit)

$\phi(t) = \int_{-\infty}^{+\infty} e^{itx} \, dF(x)$

Limit of the characteristic f^n

Theorem: Suppose $\phi_n(t) \longrightarrow \phi(t)$

The following are equivalent:

1. $F \in \mathcal{F}$
2. $\phi(t)$ is a characteristic function.
3. $\phi(0) = 1$ & $\phi(t)$ is continuous at 0 (origin)
4. $\{F_n\}$ is asymp. tight

* Denegerate R. V. $\rightarrow$ equal to some constant with probability 1.

Theorem (Weak LLN in Khinchine's form)

Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of i.i.d r.v.
and $E[X_i] = \mu_X$,

then $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} \mu_X$

Proof. $\phi_{X_i}(0) = 1$, there exists a fixed neighborhood of 0, where $|\phi_{X_i}(t) - 1| < \frac{1}{2}$, then in this neighborhood

$l(t) = \log \phi_{X_i}(t)$

Given $\phi'_{X_i}(0) = i\mu_X \Rightarrow \ell'(0) = i\mu_X$

Further
given $\phi_{aX}(t) = \phi_X(at) \Rightarrow \phi_{\frac{X_i}{n}}(t) = \phi_{X_i}\left(\frac{t}{n}\right)$

$$\phi_{\bar{X}}(t) = \phi_{\sum_{i=1}^{n}\frac{X_i}{n}}(t) = \phi_{\frac{X_i}{n}}(t)^n = \phi_{X_i}\left(\frac{t}{n}\right)^n = e^{n\ell\left(\frac{t}{n}\right)}$$

$$= e^{\frac{\ell\left(\frac{t}{n}\right) - \ell(0)}{t/n} \cdot t}$$

[$\ell(0) = 0$]
Multiplying by t.

As. $\lim_{\varepsilon \to 0} \frac{\ell(\varepsilon) - \ell(0)}{\varepsilon} = \ell'(0)$,

The above expression

$$= e^{\ell'(0)t} = e^{i\mu_X t}$$

So it converges to a denegerate characteristic function.

And as we know, $Z_n \xrightarrow{d} C \Rightarrow Z_n \xrightarrow{P} C$

$$P(|Z_n - C| > \varepsilon) \to P(|0| > \varepsilon) = 0$$

- <u>Sampling</u>: clones of R.V.