Tanya Sethi

Q4. Prove that following is not a characteristic function

$$\phi(t) = \begin{cases} 1-t^2, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

To prove $\phi(t)$ is not a characteristic function, we check if the following theorem holds:—

Continuous f^n $\phi(t)$ with $\phi(0) = 1$ is a c.f. iff it is +ive semi definite for any $t_1, \ldots, t_n \in \mathbb{R}$ & any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k \lambda_j^* \geq 0 \qquad -(1)$$

Let $t_1 = 0$ & $t_2 = 1$, $n=2$

then simplifying (1):

$$\sum_{k,j=1}^{2} \phi(t_k - t_j)\lambda_k \lambda_j^* \geq 0$$

$\Rightarrow$ $\phi(t_1 - t_1)\lambda_k \lambda_j^* + \phi(t_2 - t_1)\lambda_k \lambda_j^* + \phi(t_1 - t_2)\lambda_k \lambda_j^* + \phi(t_2 - t_2)\lambda_k \lambda_j^* \geq 0$

$\Rightarrow$ $2\phi(0)\lambda_k \lambda_j^* + \phi(1)\lambda_k \lambda_j^* + \phi(-1)\lambda_k \lambda_j^* \geq 0$

$\Rightarrow$ $2\lambda_k \lambda_j^* \geq 0$

for some $\lambda_k = 0+2i$, & $\lambda_k^* = 0-2i$

$\lambda_j = 0-(-3)i$ $\quad \lambda_j^* = 0-3i$

$$2\lambda_k \lambda_j^* = 2(-6) = -12 < 0$$

$\Rightarrow$ The above is not a characteristic function.

Q3 X_1 and X_2 are independent $N(0,1)$ random variables.

$$\underline{Y} = (Y_1, Y_2) = \begin{cases} (X_1, |X_2|), & \text{if } X_1 \geq 0 \\ (X_1, -|X_2|), & \text{if } X_1 < 0 \end{cases}$$

- Find marginal distributions of (Y_1, Y_2)

$$F_{Y_1}(Y_1) = F_{Y_1,Y_2}(Y_1, \infty) = \lim_{Y_2 \to \infty} F_{Y_1,Y_2} = P(Y_1 \leq y_1, Y_2 \leq y_2)$$

$$\lim_{y_2 \to \infty} P(X_1 \leq y, Y_2 < y)$$

$$F_x(y_1)$$

$$F_{Y_2}(Y_2) = F_{Y_1,Y_2}(\infty, Y_2)$$

If $y_2 \geq 0$ $\Rightarrow \lim_{y_1 \to \infty} P(0 \leq Y_2 < y_2 \mid X_1 \geq 0) \cdot P(X_1 \geq 0) +$

$$\lim_{y_1 \to \infty} P(Y_2 < 0 < y_2 \mid X_1 < 0) \cdot P(X_1 < 0)$$

$$= \lim_{y_1 \to \infty} P(0 \leq |x_2| < y_2 \mid X_1 \geq 0) \cdot P(X_1 \geq 0) +$$

$$\lim_{y_1 \to \infty} P(-|X_2| < 0 < y_2 \mid X < 0) \cdot P(X_1 < 0)$$

$$= \frac{2}{2}\left(\frac{1}{\sqrt{2\pi}}\int_0^{y_2} e^{-x^2/2} \cdot dx\right) + \frac{1}{2} \cdot \left(\frac{1}{2\sqrt{\pi}}\right) \cdot \int_{-\infty}^{0} e^{-\frac{x^2}{2}} \cdot dx$$

(x & y are independent)

$$= \int_{-\infty}^{y_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= F_{Y_2}(y_2)$$

If $y_2 < 0$

$$\lim_{y_1 \to \infty} \left(P(|x_2| \leq y_2 \mid X_1 \geq 0) \, P(X_1 \geq 0) \right) +$$

$$\lim_{y_1 \to \infty} P(X_2 \geq -y_2 \text{ or } X_2 \leq y_2) \cdot P(X < 0)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{-\frac{x^2}{2}} \, dx + \int_{0}^{-y_2} e^{-x^2/2} \, dx$$

$$= F_{y_2}(-y_2)$$

- Are (Y_1, Y_2) jointly normally distributed?

$$F_{Y_1, Y_2}(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2) = 0 \; ; \; \begin{matrix} y_1 > 0 \\ y_2 < 0 \end{matrix}$$

As this is not a positive number, (Y_1, Y_2) are not jointly normally distributed.

Further, if we use the formula for 2-dimensional joint normal distribution:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\left(-\frac{1}{2[1-\rho^2]}\left[\left(\frac{x-\mu x}{\sigma x}\right)^2 - 2\rho\left(\frac{x-\mu x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu y}{\sigma_y}\right)^2\right]\right)}$$

Plugging in $\mu = 0$, $\sigma_x, \sigma_y = 1$, $\rho = 0$

$$= \frac{1}{2\pi} e^{\left(-\frac{1}{2}[x]^2 + [y]^2\right)}$$

This is always positive but the function Y does not always have density.

Q2 $\quad P(X=2^n) = \dfrac{1}{e\,n!}\quad , n = 0, 1, 2, \ldots$

(a) $\quad 1^{st}$ moment : $= E[X^1]$

$$= 2^0\left(\frac{1}{e\,0!}\right) + 2^1\left(\frac{1}{e\,1!}\right) + 2^2\left(\frac{1}{e\,2!}\right) + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{e\,n!}$$

Now, $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

$E[X^1] = \dfrac{e^2}{e} = e$ which exists.

k^{th} moment $= E[X^k]$

$$= \sum_{n=0}^{\infty} (2^n)^k \cdot \left(\frac{1}{e\,n!}\right)^k$$

$$= \sum_{n=0}^{\infty}\left(\frac{2^n}{e\,n!}\right)^k = \left(\frac{e^2}{e}\right)^k = e^k$$

k^{th} moment exists as well $\Rightarrow$ This r.v. has moments of all orders s.t. k^{th} moment $= e^k$

(b) $\phi_X(t) = E[e^{itX}]$

$\Rightarrow \int_{-\infty}^{\infty} e^{itX} f_X(x) \cdot dx$

$\Rightarrow \int_{-\infty}^{\infty} e^{itX} \cdot \frac{1}{en!} \cdot dx$

$\Rightarrow \sum_{n=0}^{\infty} \left[\frac{e^{it^{2^n}}}{it} \frac{1}{en!} \right]$

(c) Moment Generating Junction

$M_X(t) = E(e^{tX})$

$= \frac{1}{e0!}(e^{t2^0}) + \frac{1}{e1!}(e^{t2^1}) + \frac{1}{e2!}(e^{t2^2}) + \dots$

$= \sum_{k=0}^{\infty} e^{t2^k} \frac{1}{ek!}$

MGF of X exists if $\exists$ a +ive constant a s.t. $M_X(t)$ is finite $\forall\ t \in [-a, a]$

Using the ratio test to check if $M_x(t)$ is finite for any a.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\lim_{n\to\infty} \left| \frac{e^{t2^{n+1}} \; e(n)!}{e(n+1)! \; e^{t2^n}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{e^{t2^{n+1} - t2^n}}{n+1} \right) = \lim_{n\to\infty} \left(\frac{e^{2t}}{n+1} \right)$$

$$= e^{2t}$$

$\Rightarrow$ $M_x(t)$ diverges & thus MGF does not exist.

Q1

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}(1+x_1 x_2), \text{ if } (x_1, x_2) \in [-1,1] \times [-1,1] \\ 0 \text{ if } (x_1, x_2) \notin [-1,1] \times [-1,1] \end{cases}$$

$$f_{x_1}(x_1) = \int_{x_2} f(x_1, x_2) \cdot dx_2 \quad \forall x$$

$$= \int_{-\infty}^{\infty} f(x_1, x_2) \cdot dx_2$$

$$= \int_{-1}^{1} \frac{1}{4}(1 + x_1 x_2) \cdot dx_2$$

(it's 0 outside $-1, 1$)

$$= \frac{1}{4}\left[x_2 + x_1 \frac{x_2^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{2}$$

$$f_{x_2}(x_2) = \int_{x_1} f(x_1, x_2) \cdot dx_1 \quad \forall x$$

$$\int_{-\infty}^{\infty} f(x_1, x_2) \cdot dx_1$$

$$= \int_{-1}^{1} \left(\frac{1}{4} \left(1 + x_1 \, x_2 \right) \right) . dx_1$$

$$= \frac{1}{4} \left[x_1 + \frac{x_1^2}{2} . x_2 \right]_{-1}^{1}$$

$$= \frac{1}{2}$$

Let

$$g(x_1, x_2) = \begin{cases} \frac{1}{4} \left(1 + x_1^3 x_2^3 \right), & \text{if } (x_1, x_2) \in [-1,1] \times [-1,1] \\ 0, & \text{if } (x_1, x_2) \notin [-1,1] \times [-1,1] \end{cases}$$

$$g_{x_1}(x_1) = \int_{x_2} g(x_1, x_2) \, . dx_1 = \int_{-1}^{1} \frac{1}{4} (1 + x_1^3 . x_2^3 . dx_1$$

$$g_{x_2}(x_2) = \int_{x_1} g(x_1, x_2) . dx_2 = \int_{-1}^{1} \frac{1}{4} (1 + x^3 . x_2^3) . dx_2$$

$$g_{x_1}(x_1) = \frac{1}{2}$$

$$g_{x_2}(x_2) = \frac{1}{2}$$