Suggested Solutions: ECON 7710 HW IV

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Question 1

Solution 1

We know $X \sim N(0,1)$ and $Y \sim N(0,1)$ and X and Y are independent. We define $Z = \frac{X}{Y}$. CDF of Z is written as follows:

$$
\begin{aligned}
F_Z(z) = Pr(Z \leq z) = Pr(\frac{X}{Y} \leq z) &= Pr(X \leq zY | Y > 0)Pr(Y > 0) + Pr(X \geq zY | Y < 0)Pr(Y < 0) \\
&= Pr(X \leq zY, Y > 0) + Pr(X \geq zY, Y < 0) \\
&= \int_0^{\infty} \int_{-\infty}^{zy} f(x)f(y)dxdy + \int_{-\infty}^{0} \int_{zy}^{\infty} f(x)f(y)dxdy
\end{aligned}
$$

We know $\frac{\partial F_Z(z)}{\partial z} = f_Z(z)$. Leibniz rule tells us

$$
\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x,z)dx = f(b(z),z)b'(z) - f(a(z),z)a'(z) + \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} f(x,z)dx
$$

Therefore, pdf of Z is

$$
f_Z(z) = \frac{\partial F_Z(z)}{\partial z} = \int_0^{\infty} yf(zy)f(y)dy - \int_{-\infty}^{0} yf(zy)f(y)dy = \underbrace{2\int_0^{\infty} yf(zy)f(y)dy}_{\text{f() is symmetric}}
$$

We know pdf for $N(0,1)$ is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in \mathbb{R}$, then

$$
f_Z(z) = \frac{1}{\pi} \int_0^{\infty} ye^{-y^2(\frac{z^2+1}{2})}dy = \frac{1}{2\pi} \int_0^{\infty} 2ye^{-y^2(\frac{z^2+1}{2})}dy = \frac{1}{2\pi} \int_0^{\infty} e^{-y^2(\frac{z^2+1}{2})}dy^2
$$

Denote $u = y^2$, $t = \frac{z^2+1}{2}$, then

$$
f_Z(z) = \frac{1}{2\pi} \int_0^{\infty} e^{-tu}du = -\frac{1}{2\pi t}[e^{-tu}]_0^{\infty} = \frac{1}{2\pi t} = \frac{1}{\pi(z^2+1)}
$$

Clearly, this is a **standard Cauchy Distribution**. And we know none of n-moments exist for $n \geq 1, n \in \mathbb{N}$ in Cauchy distribution.

Solution 2

We know $X \sim N(0,1)$ and $Y \sim N(0,1)$ and X and Y are independent and we define $Z = \frac{X}{Y}$. Following **Example 4.3.6** on Casella & Berger(p162). We define $V = |Y|$. Then we can derive a partition A of $\mathbb{R}^2$ where:

- $A_1 = \{(x,y) : y > 0\}$
- $A_2 = \{(x,y) : y < 0\}$
- $A_0 = \{(x,y) : y = 0\}$

Then since $U = X/Y, P((X,Y) \in A_0) = P(Y = 0) = 0$.
For a fixed value of $v = |y|$, we know u can be any real number. So denote $\mathbb{B} = \{(u,v) : v > 0\}$ be the image of both A_1 and A_2 under the transformation. We also know that the inverse transformation from $\mathbb{B}$ to A_1 and from $\mathbb{B}$ to A_2 are given by :

- $x = h_{11}(u,v) = uv$
- $y = h_{21}(u,v) = v$
- $x = h_{12}(u,v) = -uv$
- $y = h_{22}(u,v) = -v$

Then we know Jacobians from two inverses $J_1 = J_2 = v$
Since we know the joint pdf of x,y

$$
\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi} e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2}} \\
\Rightarrow f_{U,V}(u,v) &= \frac{1}{2\pi} e^{\frac{-(uv)^2}{2}} e^{\frac{-v^2}{2}} |v| + \frac{1}{2\pi} e^{-\frac{(-uv)^2}{2}} e^{-\frac{(-v)^2}{2}} |v| \\
&= \frac{v}{\pi} e^{\frac{-(u^2+1)v^2}{2}}, \quad u \in (-\infty, \infty), \quad v \in (0, \infty)
\end{aligned}
$$

Then we know marginal pdf of u is

$$
f_U(u) = \int_0^\infty \frac{v}{\pi} e^{\frac{-(u^2+1)v^2}{2}} dv = \frac{1}{\pi(u^2+1)}, \quad u \in (-\infty, \infty)
$$

We land on a Cauchy distribution at the end and none of n-moments ($n \geq 1, n \in \mathbb{N}$) exist.

Question 2

We know $\{X_n\}_{n=1}^{\infty}$ and $X_n \xrightarrow{d} X$, where $X \sim N(0,1)$. We suppose $Y_n = X_n$ for all $n \geq 1$

2.(a)

Since $Y_n = X_n, \forall n \geq 1$, we know $F_{X_n}(x) = F_{Y_n}(x), \forall n \geq 1, \forall x \in \mathbb{R}$. So we know since $X_n \xrightarrow{d} X$, we can derive:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x$$

We can also get

$$\lim_{n\to\infty} F_{Y_n}(x) = \lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x$$

Therefore if we denote Y as the distribution limit of $\{Y_n\}_{n=1}^{\infty}$, we know $Y \sim N(0,1)$.

2.(b)

$Y_n \xrightarrow{d} Y$. We claim that $X_n + Y_n \xrightarrow{d} X + Y$ **doesn't always hold**. A counterexample is when we consider $Y = -X$. Then by symmetry of normal distribution, we can still have $F_Y = F_{-X} = F_X$ and now $X + Y = 0$.

- For Y_n, still we have
$$\lim_{n\to\infty} F_{Y_n}(x) = \lim_{n\to\infty} F_{X_n}(x) = F_X(x) = F_Y(x), \forall x$$
, where $Y = -X$. We proved $Y_n \xrightarrow{d} Y$ still holds.
- Since $X_n + Y_n = 2X_n$ For $\lim_{n\to\infty} F_{2X_n} = \lim_{n\to\infty} P(2X_n \leq x) = \lim_{n\to\infty} P(X_n \leq \frac{x}{2})$.
We know $X_n \xrightarrow{d} X$ So $\lim_{n\to\infty} P(X_n \leq \frac{x}{2}) = P(X \leq \frac{x}{2}) = P(2X \leq x)$.
We know if $X \sim N(0,1)$, then $2X \sim N(0,4)$. So $X_n + Y_n \xrightarrow{d} Z, Z \sim N(0,4) \neq X + Y = 0$

Therefore, we found a counterexample when $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, $Y_n = X_n \forall n \geq 1$, $X_n + Y_n \xrightarrow{d} X + Y$ may fail as well.

Question 3

We define the median of the distribution of random variable X is the number $q_{0.5}$ that solves

$$\inf_q \{P(X \leq q) \geq \frac{1}{2}\}$$

We know there exists a numeric sequence a_n such that $X_n - a_n \xrightarrow{p} 0$. We let $q_{0.5}^n$ be the median of the distribution of X_n

3.(a)

We want to prove that $\lim_{n\to\infty}(q_{0.5}^n - a_n) = 0$.
We already know that $X_n - a_n \xrightarrow{p} 0$. In other words, we have

$$\lim_{n\to\infty} P(|X_n - a_n| \geq \epsilon) = 0 \Rightarrow \lim_{n\to\infty} P(X_n \geq \epsilon + a_n) + \lim_{n\to\infty} P(X_n \leq a_n - \epsilon) = 0$$

Since for any probability p, we have $0 \leq p \leq 1$, then we know $\lim_{n\to\infty} P(X_n \geq \epsilon + a_n) = 0$ and $\lim_{n\to\infty} P(X_n \leq a_n - \epsilon) = 0$
In other words, the possible range of X_n is $(a_n - \epsilon, a_n + \epsilon)$. Then $q_{0.5}^n$, the median of distribution of this random variable x_n, we will have $q_{0.5}^n \in (a_n - \epsilon, a_n + \epsilon)$ In other words, $|q_{0.5}^n - a_n| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we know $\lim_{n\to\infty}(q_{0.5}^n - a_n) = 0$ is proved.

3.(b)

For $\lim_{n\to\infty}(E[X_n] - a_n) = 0$, we claim it is false and a counterexample is given below:

$$X_n = \begin{cases} n & p = 1/n \\ 0 & p = 1 - 1/n \end{cases}$$

Let $a_n = \{0, 0, 0, ...\}$, a numerical sequence only contains constant number 0. Then we know:

$$\lim_{n\to\infty} P(|X_n - a_n| > \epsilon) = \lim_{n\to\infty} P(|X_n| > \epsilon) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Then we know we successfully formulated $X_n - a_n \xrightarrow{p} 0$.
By defination, $\lim_{n\to\infty} q_{0.5}^n = 0$.
For $\lim_{n\to\infty}(E[X_n] - a_n)$. We know $\forall n, E[X_n] = n * \frac{1}{n} + 0 * (1 - \frac{1}{n}) = 1$

$$\lim_{n\to\infty}(E[X_n] - a_n) = \lim_{n\to\infty} E[X_n] = 1$$

Then we found for X_n and a_n we constructed, $\lim_{n\to\infty}(E[X_n] - a_n) \neq 0$.

Question 4

We know X_1, X_2...is a sequence of independent and identically distributed random variables. We also know $X_n \xrightarrow{p} X$. We want to prove that X has a degenerate distribution.
We consider two subsequences of X_n, which are

- $\{X_i\}, \quad i = 1, 3, 5...$
- $\{X_j\}, \quad j = 2, 4, 6...$

Since $X_n \xrightarrow{p} X$, we know $X_i \xrightarrow{p} X$ and $X_j \xrightarrow{p} X$. We also know all X_n are i.i.d. Then we know for the characteristic function of X_i and X_j, we will have $\phi_{X_i+X_j}(t) = \phi_{X_i}(t)\phi_{X_j}(t)$. We also know that convergence in probability implies convergence of characteristic function. Therefore, we know:

- $X_i \xrightarrow{p} X \Rightarrow \lim_{i\to\infty} \phi_{X_i}(t) = \phi_X(t)$
- $X_j \xrightarrow{p} X \Rightarrow \lim_{j\to\infty} \phi_{X_j}(t) = \phi_X(t)$

So we found that $\lim_{(i,j)\to(\infty,\infty)} \phi_{X_i+X_j}(t) = [\phi_X(t)]^2$
On the other hand, as $X_i \xrightarrow{p} X$ and $X_j \xrightarrow{p} X$, we know $X_i+X_j \xrightarrow{p} 2X$, therefore we have $\lim_{(i,j)\to(\infty,\infty)} \phi_{X_i+X_j}(t) = \phi_{2X}(t) = \phi_X(2t)$
Therefore we land on the conclusion that X's characteristic function must satisfy:

$$\phi_X(t)^2 = \phi_X(2t)$$

For a degenerate distribution with probability $P(X = c) = 1$. We know $\phi_X(t) = e^{itc}$. So we know: $\phi_X(t)^2 = e^{2itc}$ and $\phi_X(2t) = e^{2itc}$. Then we found our conclusion holds for a degenerate distribution and we proved that X has a degenerate distribution.