Homework 1

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1. A die is tossed n times. Find the probability of events where:

(i) Atleast one of the outcomes is equal to 6.

= 1 − none of the outcomes are equal to 6.

$= 1 - \left(\frac{5}{6}\right)^n$

(ii) An outcome equal to 6 is observed exactly once.

① None of the times

Atleast once
② Exactly once
③ Exactly twice
⋮
ⓝ All the time.

$n = 1 \quad = \left(\frac{1}{6}\right)$

$n = 2 \quad = \frac{1}{6}\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$

$n = 3 = \frac{1}{6}\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$

$${}^{n}C_{1} \left(\frac{1}{6}\right)^{1} \left(\frac{5}{6}\right)^{n-1}$$

4. n marbles are placed in n urns.

P(no urn is empty) - ?

$n = 2$

There is a marble in each urn

1/2

1/2.

1^{st} marble

No urn is empty ?

There is a marble in each urn

1 marble → 1 urn.

1.

2 marble → 2 urns

Possible cases →

	Marble		
A	1	2	0
B	1	0	2

$\frac{1}{3}$ (2)(2)

3 marbles

Marbles

A	3	0	0
B	0	3	0
C	0	0	3

Total cases:-

$\boxed{\text{All Combinations of placing the marbles in three urns}}$

__ __ __

✓ ✓ ✓

3(3)

2(2)

n marbles are placed in n urns

2 marbles are placed in 2 urns:-

5. Let $F(\cdot)$ and $G(\cdot)$ be two distribution functions. Find necessary and sufficient conditions that $H(x) = F(G(x))$ is a distribution function.

1) If $x_1 \leq x_2$, then $F_X(x_1) \leq F_X(x_2)$

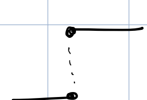
2) (cadlag) : $\lim_{x \downarrow x_0} F_X(x) = F(x_0)$

(Approaching x_0 from right)

$\lim_{x \uparrow x_0} F_X(x)$ exists

(Approaching x_0 from left)

⇒ the function can exhibit weird jumps.



jumps are only one way ⇒ post jump, the function has to behave properly.

3) $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to +\infty} F_X(x) = 1$.

- Distribution function $F(x)$ of r.v. X has the following properties

  1. Monotonicity: If $x_1 \leq x_2$ then $F(x_1) \leq F(x_2)$
  2. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$
  3. Left-continuity: $\lim_{x \uparrow x_0} F(x) = F(x_0)$

Q. 5

For $H(x)$ to be a distribution function, it has to fulfill the following properties :—

i) Monotonicity

$$\text{If } x_1 \leq x_2 \Rightarrow H(x_1) \leq H(x_2)$$

Assume $x_1 \leq x_2$

$$\Rightarrow G(x_1) \leq G(x_2) \quad \left(\text{As } G(\cdot) \text{ is a distn } f^n\right)$$

Let $y_1 = G(x_1)$ & $y_2 = G(x_2)$

$$\Rightarrow y_1 \leq y_2$$

$$\Rightarrow f(y_1) \leq f(y_2) \quad \left(\text{As } f(\cdot) \text{ is a distn function}\right)$$

$$\Rightarrow F(G(x_1)) \leq F(G(x_2))$$

$$\Rightarrow H(x_1) \leq H(x_2) \qquad \square$$

No conditions required.

2) $\lim_{x \to -\infty} H(x) = 0$ and $\lim_{x \to +\infty} H(x) = 1$

$H(x) = F(G(x))$

$$\lim_{x \to -\infty} H(x) = \lim_{x \to -\infty} F(G(x))$$

$$= F\left(\lim_{x \to -\infty} G(x)\right)$$

$$= F(0)$$ (As $G(\cdot)$ is a distn fn)

For $\lim_{x \to -\infty} H(x) = 0$;

(*) $F(0)$ should be continous &
$F(0) = 0$

$$\lim_{x \to +\infty} H(x) = \lim_{x \to +\infty} F(G(x))$$

$$= F\left(\lim_{x \to +\infty} G(x)\right)$$

$$= F(1)$$ (As $G(\cdot)$ is a distn function)

For $\lim_{x \to +\infty} H(x) = 1$:

(*) $F(1)$ should be continoeus & $F(1) = 1$.

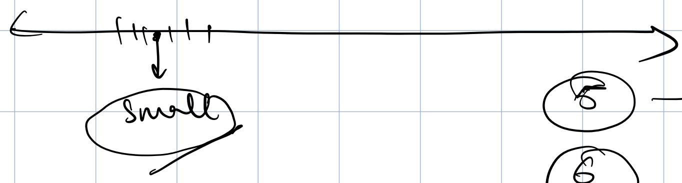
3) Left continuity: $\lim_{x \uparrow x_0} H(x) = H(x_0)$

$$\lim_{x \uparrow x_0} F(G(x))$$

$$\Rightarrow F(\lim_{x \uparrow x_0} (G(x)))$$

$$= F(G(x_0))$$

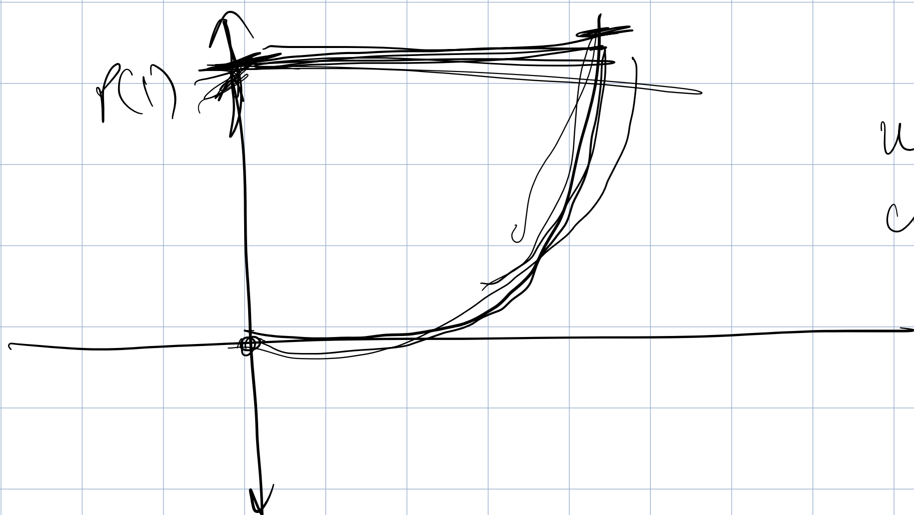
- F should be continous at every $G(x_0)$



$\delta \to$ mo

$\varepsilon \to n$

unbounded on a bounded domain

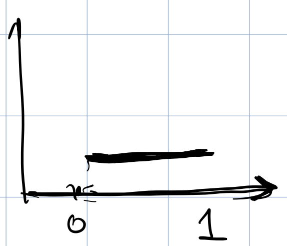


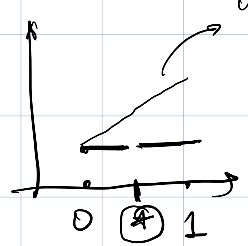
uniformly continuous

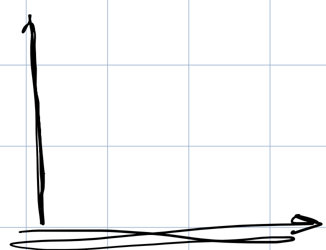
$\boxed{\text{infinite domain}}$

Q3. If random variable $x \in [a,b]$ has density almost everywhere on this segment

$\Rightarrow$ there is atleast a point ~~where~~ in $[a,b]$ where x does not have a density.







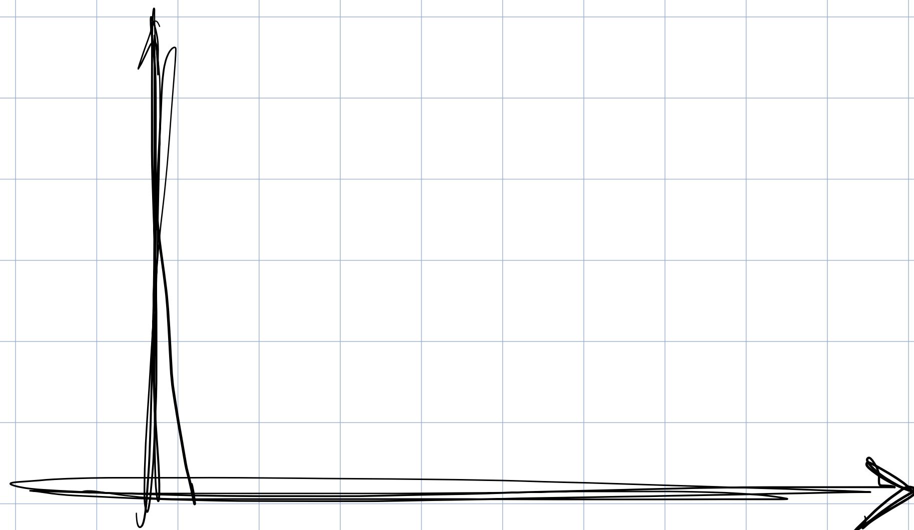
$\frac{1}{p} \in [0,2)$ $\quad$ $0, 2$

$[0,2]$

$0, 2$

Continous.

Uniformly continous



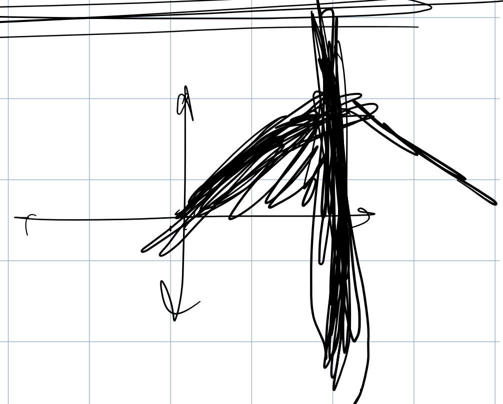
$$\frac{1}{p} = \int \text{[illegible]}$$

$$\frac{d}{dp}\left(\frac{1}{p}\right) = -\frac{1}{p^2}$$

p^{-1}

$$\boxed{= -\frac{1}{p^2}}$$

x^2



$2x$

PDF : $2p$. $\longrightarrow$ ( )

CDF : p^2

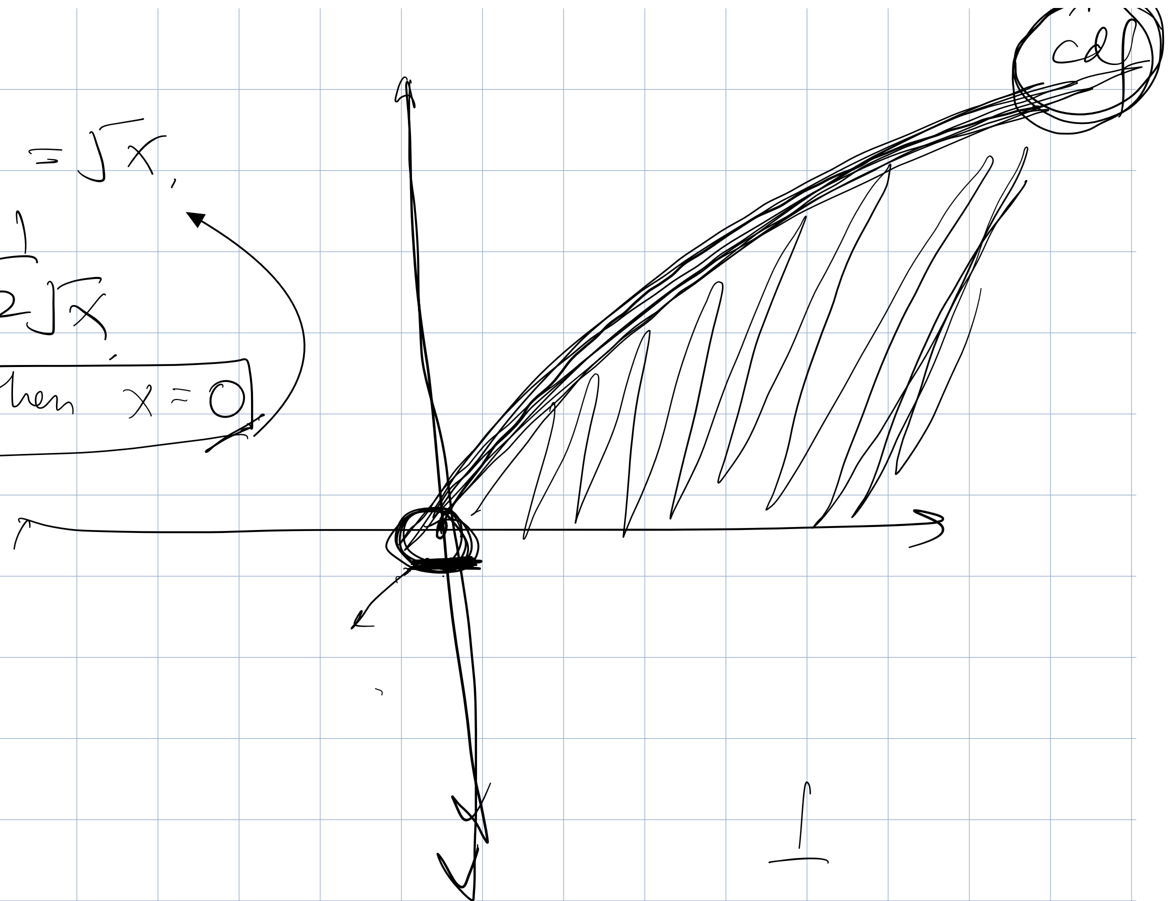
If at a pt. is $cdf' = \infty$ ← NOT uniformly continuous

$f(x) = \sqrt{x}$

$f'(x)$ $\frac{1}{2\sqrt{x}}$

$\downarrow$

$\boxed{= \infty \text{ when } x = 0}$



$\frac{1}{2\sqrt{x}}$

Uniformly continuous

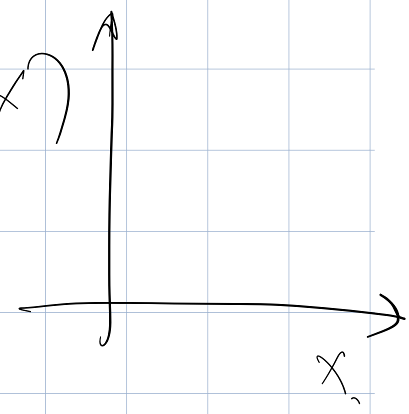
$x^2 \to [0,1]$

$x^2 \not\to \mathbb{R}$

$\sqrt{x}$

$$\begin{cases} p = 0 & \text{if } x = 0 \\ p = \dfrac{1}{2\sqrt{x}} & \text{if } x \neq 0. \end{cases}$$

CDF = $\sqrt{x}$



$\underline{\text{PDF}}$ = $\dfrac{1}{2\sqrt{x}}$

[0, 1] where

$$\text{PDF} \begin{cases} p = 0 & \text{if } x = 0 \\ p = \dfrac{1}{2\sqrt{x}} & \text{if } x \neq 0 \end{cases}$$

$$\frac{d}{dx}(\text{CDF}) \text{ at } x = 0 = \underline{\infty}$$

So its not absolutely continuous

To make it absolutely continuous, redefine domain from [0, 1]