4. End points and transversality conditions

AGEC 642 - 2022

Recall from Lecture 3 that a typical optimal control problem is to maximize $\int_0^T F(t,x,z)dt$ subject to $\dot{x} = f(t,x,z)$, the state equation. In that lecture we stated that if $H(t,x,z,\lambda) = F(t,x,z) + \lambda f(t,x,z)$, the conditions that characterize the optimum are

1. $\max_z H(t,x,z,\lambda)$ for all $t \in [0,T]$
2. $\dfrac{\partial H}{\partial x} = -\dot{\lambda} = -\dfrac{\partial \lambda}{\partial t}$
3. $\dfrac{\partial H}{\partial \lambda} = \dot{x}$
4. *Transversality condition* (such as, $\lambda(T) = 0$).

In this lecture we focus primarily on condition 4, the transversality condition. On our way to deriving the transversality condition, we also provide a somewhat more formal derivation of conditions 1-3.

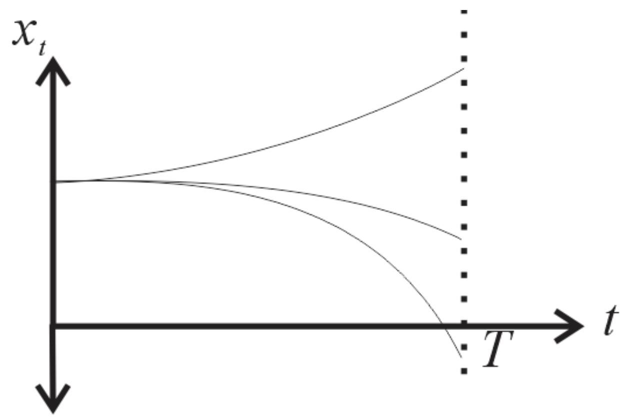
A transversality condition describes what must be satisfied at the end of the time horizon. i.e., as we *transverse* to the world beyond the planning horizon. The nature of the transversality condition depends greatly on the statement of the problem. For example, it might be that the state variable, *x*, must equal zero at the terminal time *T*, i.e., x_T=0, or it might be that it must be less than some function of *t*, $x_T \le \phi(T)$. We also consider problems where the ending time is flexible or $T\to\infty$.

I. Transversality conditions for a variety of ending points

(Based on Chiang pp. 181-184)

A. Vertical or free-endpoint problems

In a vertical end point type problem, *T* is fixed but x_T can take on any value. This would be appropriate, for example, if you are managing an asset or set of assets over a fixed horizon and it you have no restrictions on the condition of the assets when you reach *T*. We have considered this case previously. When consider the value that x_T from the perspective of the beginning of the planning horizon, x_T is free (unconstrained). Moreover, since *T* is the end of the planning horizon, we are unconcerned with and effect that might have on the future. Hence, we can treat x_T as an unconstrained choice variable and we want to maximize *V* w.r.t. x_T. Hence, it follows that $\dfrac{\partial V(\cdot)}{\partial x_T} = 0$, which means that the shadow price of x_T must equal zero, giving us our transversality condition, λ_T =0.



We will now confirm this intuition by deriving the transversality condition for this particular problem and at the same time giving a more formal presentation of Pontryagin's maximum principle introduced in Lecture 3.

The objective function is

$$V \equiv \int_0^T F(t,x,z) \quad dt \quad .$$

Now, setting up an equation as a Lagrangian with the state-equation constraint, we have

$$L = \int_0^T \left[F(t,x,z) + \lambda_t \left(f(t,x,z) - \dot{x}_t \right) \right] dt \, .$$

We put the constraint inside the integral because it must hold at every point in time. Note that the shadow price variable, λ_t, is not constant, but instead can vary over the planning horizon, possibly taking on a different value a every point in time in the interval 0 to T. Since the state equation must be satisfied at each point in time, at the optimum, it follows that $\lambda_t \left(f(t,x,z) - \dot{x}_t \right) = 0$ at each instant t, so that the value of L must equal the value of V. Hence, we might write instead

$$V = \int_0^T \left[F(t,x,z) + \lambda_t \left(f(t,x,z) - \dot{x}_t \right) \right] dt$$

or

$$V = \int_0^T \left[\underbrace{\left\{ F(t,x,z) + \lambda_t f(t,x,z) \right\}} - \lambda_t \dot{x}_t \right] dt$$

$$V = \int_0^T \left[\overbrace{H(t,x,z,\lambda)} - \lambda_t \dot{x}_t \right] dt$$

It will be useful to reformulate the last term in the integrand, $\lambda_t \dot{x}_t$, using integration by parts. For me, I have to remember the rule, $\int u dv = vu - \int v du$. If we let $\lambda = u$ and $x = v$ so that $dv = \dot{x} dt$ we know that

$$-\int_0^T \lambda_t \dot{x}_t dt = -[\lambda_t x_t]_0^T + \int_0^T \dot{\lambda}_t x_t dt$$

$$= \int_0^T \dot{\lambda}_t x_t dt + \lambda_0 x_0 - \lambda_T x_T .$$

Hence, we can rewrite V as

$$L = V = \int_0^T \left[H\left(t,x,z,\lambda\right) + \dot{\lambda}_t x_t \right] dt + \lambda_0 x_0 - \lambda_T x_T \qquad (1).$$

Pay attention to equation 1; we will use it repeatedly in these notes and refer to it often.

Derivation of the maximum conditions (Based on Chiang (1992) chapter 7)

Using 1, we can derive the first three conditions of the maximum principal which must hold over the interval $t \in (0, T)$.

Assuming an interior solution and twice-differentiability, a necessary condition for an optimum is that the first derivatives of choice variables are equal to zero.

First consider our choice variable, z_t. At each point in time, it must be that $\partial V / \partial z_t = 0$. This reduces to $\partial H / \partial z = 0$, which is the first of the conditions stated without proof in Lecture 3.

Second, for all $t \in (0,T)$, x_t is also a choice variable in 1 (though clearly a constrained one), so it must also hold that $\partial V / \partial x_t = 0$. This reduces to $H_x + \dot{\lambda} = 0$ or $H_x = -\dot{\lambda}$, the second of the conditions stated in Lecture 3.

Third, the FOC with respect to λ_t is more directly derived from the Lagrangian before carrying out integration by parts: $\partial L / \partial \lambda_t = f\left(t,x,z\right) - \dot{x}_t$, so this implies that $\partial L / \partial \lambda_t = 0 \Rightarrow \dot{x}_t = f\left(t,x,z\right)$, which simply means that the state equation must be satisfied.

In the second condition in the box above, we evaluated the values of x_t for $t \in (0,T)$, i.e., excluding the end points x_0 and x_T.[1] The initial value, x_0, is fixed, so clearly we cannot maximize over that. But x_T, the value of x at the terminal time, is flexible when considered at t=0. If the terminal condition is that x_T can take on any value, then it must be that the marginal value of a change in x_T must equal to zero, i.e., $\partial V / \partial x_T = 0$. Hence, the first-order condition of 1 with respect to x_T is $\frac{\partial V}{\partial x_T} = -\lambda_T = 0$.

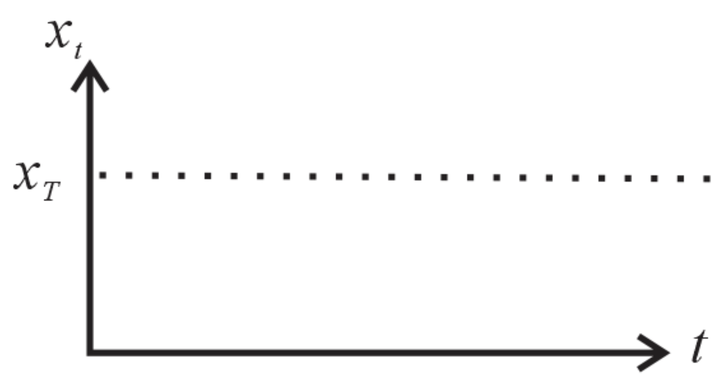
The intuitive interpretation of the minus sign on λ_T is that it reflects the marginal cost of leaving a marginal unit of the stock at time T. Hence if $\lambda_T > 0$, we could increase V by increasing x_T. Setting this FOC equal to zero, we obtain the transversality condition, $\lambda_T = 0$.

[1] To see why $t \in (0,T)$, consider a discrete time problem in which each time step is divided into $1/\Delta$ pieces: $\max \sum_{i=0}^{T/\Delta} F\left(x_{i\Delta}, z_{i\Delta}\right) + S\left(x_{T+\Delta}\right)$. In this case, the salvage value $S(\cdot)$ is after the choices in the sum. As we take the limit as $\Delta \rightarrow 0$, the salvage value is still outside the planning horizon; even as $\Delta \rightarrow 0$ there's a difference between x_T and $x_{T+\Delta}$.

Alternatively, we know that λ_T is the shadow price of the state variable at *T*, $\frac{\partial V}{\partial x_T}$. So, to some extent, the equation above requires that $\lambda_T = -\lambda_T$, which only happens when $\lambda_T = 0$.

This confirms our intuition that since we are attempting to maximize *V* over our planning horizon, from the perspective of the beginning of that horizon x_T is a variable to be chosen, it must hold that λ_T, the marginal value of x_T, must equal zero. Note that this is the marginal value to *V*, i.e., to the sum of all benefits over time for 0 to *T*, not the value to the benefit function, $F(\cdot)$. Although an additional unit may add value if it arrived at time *T*, i.e., $\partial F(\cdot)/\partial x_T > 0$, $\frac{\partial V}{\partial x_T} = 0$ means that the marginal benefit of x_T must be balanced by the costs necessary to increase x_T.

B. Horizontal Terminal Line



In this case, there is no fixed endpoint as in A; the ending state variables must have a given level but there is no specific ending time. For example, for example, consider an asset that you can use as long as you wish, but at the end of your use, it must be in a certain state. Again, we will use equation 1:

$$V = \int_0^T \left[H(t,x,z,\lambda) + \dot{\lambda}_t x_t\right]dt + \lambda_0 x_0 - \lambda_T x_T .$$

In this case, however, we are not only choosing the path of z_t, but also the terminal time, *T*. To understand how we might optimize over *T*, consider a problem in which the objective function is

$$V\left(\{z_t\},T\right) = \max_{z_t,T} \int_0^T F(t,x,z)\,dt \qquad (2)$$

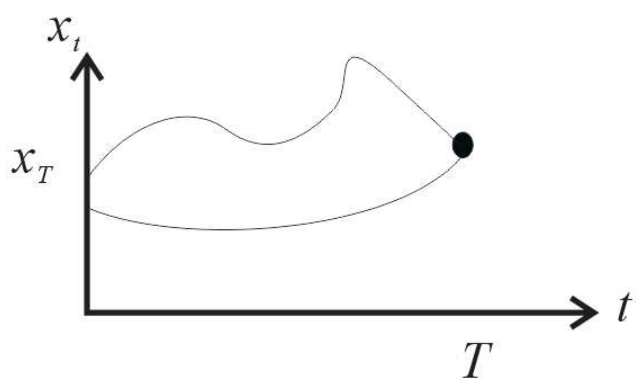
and *T* is a choice variable. Among the FOC's for the problem would be $\partial V/\partial T=0$. If $\partial V/\partial T>0$ we would want to increase the time horizon, and if $\partial V/\partial T<0$ it should be shortened. (Note that this is a necessary but not sufficient condition – we will address the sufficient condition when we introduce the infinite horizon framework below). Using Leibniz's rule, the FOC of (2) with respect to *T* would be $F(T,x_T,z_T)=0$.

Now, apply this to 1. Using Leibniz's rule for the integral and the product rule for the last term, we get

$$\frac{\partial V}{\partial T} = \left[H\left(T, x_T, z_T, \lambda_T\right) + \dot{\lambda}_T x_T \right] - \left(\dot{\lambda}_T x_T + \lambda_T \dot{x}_T \right) = 0 .$$

The second and third terms in this equation cancel and, since we are restricted to have x_T equal to a specific value, it follows that $\dot{x}_T = 0$. Hence, the transversality condition is what remains, i.e., the first term: $H\left(T, x_T, z_T, \lambda_T\right) = 0$. Expanding this, can provide a little more intuition: $F\left(T, x_T, z_T\right) + \lambda_T f\left(T, x_T, z_T\right) = 0$ -- you keep going until the value of your output, taking into account current output, $F(\cdot)$, plus the benefit (positive or negative) of changes in the state variable, $\lambda_T f\left(T, x_T, z_T\right)$, sum to zero. Note importantly that this holds only at t=T.

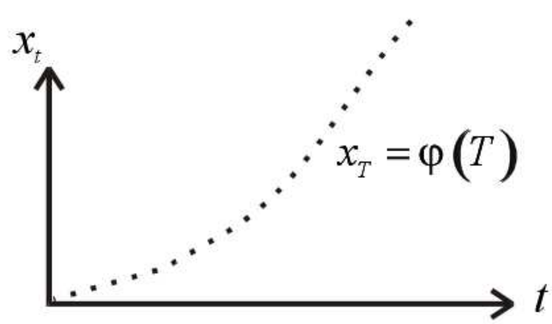
C. Fixed Terminal Point (fixed x_T <u>and</u> T)



In this case both x_T and T are fixed. Such would be the case if you are managing the asset and, at the end of a fixed amount of time you have to have the asset in a specified condition. A simple case: you rent a car for 3 days and at the end of that time the gas tank has to have 5 gallons in it. There is nothing complicated about the transversality condition here; it is satisfied by the constraints on T and x_T , i.e., x_3=5.

When added to the other optimum criteria, this transversality equation gives you enough equations to solve the system and identify the optimal path.

D. Terminal Curve



In this case the terminal condition is a function, $x_T = \varphi(T)$. Substituting this into 1 we get

1 $\quad V = \int_0^T \left[H(t, x, z, \lambda) + \dot{\lambda}_t x_t \right] dt \ + \lambda_0 x_0 - \lambda_T \varphi(T)$.

Since there is no fixed end time, we optimize over T by taking the derivative with respect to T and setting that equal to zero:

$$\frac{\partial V}{\partial T} = H(T, x_T, z_T, \lambda_T) \quad + \dot{\lambda}_T x_T - \dot{\lambda}_T x_T - \lambda_T \phi'(T) = 0\,,$$

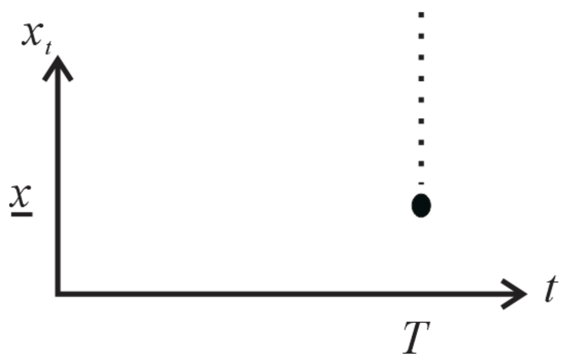
which can be simplified to the transversality condition,

$$\frac{\partial V}{\partial T} = H(T, x_T, z_T, \lambda_T) - \lambda_T \phi'(T) = 0\,.$$

Hence, the transversality condition becomes $H(T, x_T, z_T, \lambda_T) = \lambda_T \varphi'(T)$ or

$F(T, x_T, z_T) + \lambda_T f(T, x_T, z_T) = \lambda_T \varphi'(T)$.

E. Truncated Vertical Terminal Line



In this case the terminal time is fixed, but x_T can only take on a set of values, e.g., $x_T \geq \underline{x}$. This would hold, for example, in a situation where you are using a stock of inputs to generate a valuable output, but at the end of the period you cannot hold a negative balance, $x_T \geq 0$.

For such problems there are two possible transversality conditions. If $x_T > \underline{x}$, then the transversality condition $\lambda_T = 0$ applies. On the other hand, if the optimal path is to reach the constraint on x, then the terminal condition would be $x_T = \underline{x}$. In general, the Kuhn-Tucker specification is what we want. That is, our maximization objective is the same, but we now have an inequality constraint, i.e., we are seeking to maximize

$V = \int_0^T \left[H(t, x, z, \lambda) + \dot{\lambda}_t x_t \right] dt \ + \lambda_0 x_0 - \lambda_T x_T$ s.t. $x_T \geq \underline{x}$.

The Kuhn-Tucker conditions for the optimum then are:

$\lambda_T \leq 0$, $x_T \geq \underline{x}$, and $(x_T - \underline{x})\lambda_T = 0$,

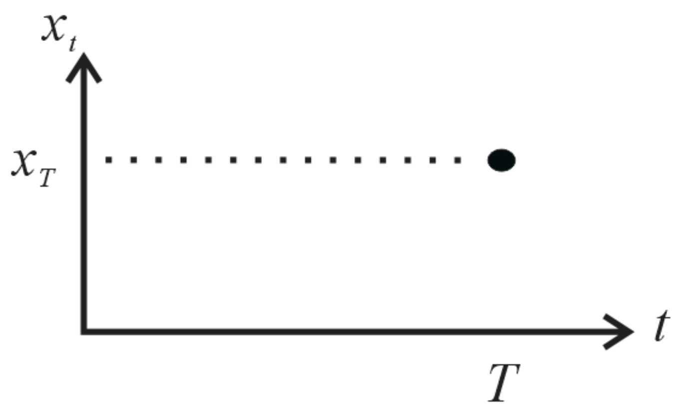
where the last of these is the complementary slackness condition of the Kuhn-Tucker conditions.

Note that if the constraint were instead, $x_T \leq \bar{x}$, i.e., there is an upper bound, then the Kuhn-Tucker conditions would be $\lambda_T \geq 0$, $x_T \leq \bar{x}$, and $(x_T - \underline{x})\lambda_T = 0$.

As a practical matter, rather than burying the problem in calculus and algebra, it usually works to take an educated guess. If at the end of your planning horizon you would like to diminish x if you could, then the constraint will bind, and you can solve the problem assuming that $x_T = \underline{x}$. You can then test, your intuition by ensuring that $\lambda_T < 0$.

If, on the other hand, you believe that x_T will optimally be greater than $\underline{x}$, then solve, the problem first using $\lambda_T = 0$. If your solution leads to $x_T \geq \underline{x}$, you are done. This algorithm of guessing and trying will usually work.

F. Truncated Horizontal Terminal Line



In this case the time is flexible up to a point, e.g., $T \leq T_{max}$, but the state is fixed at a given level, say x_T is fixed. Again, there are two possibilities, $T = T_{max}$ or $T < T_{max}$. Using the horizontal terminal line results from above, the transversality condition takes on a form similar to the Kuhn-Tucker conditions above,
$T \leq T_{max}$, $H(T, x_T, z_T, \lambda_T) \geq 0$, and $(T - T_{max})H_T = 0$.

II. A word on salvage value

The problems above have assumed that all benefits and costs accrue during the planning horizon. However, for finite horizon problems or horizontal end-point problems, it is often the case that there are benefits or costs that are functions of x_T at T. For example, owning and operating a car is certainly a dynamic problem and there is typically some value (perhaps negative) to your vehicle when you are finally finished with it. Similarly, farm production problems might be thought of as a dynamic optimization problem in which there are costs during the growing season, followed by a salvage value at harvest time.

Values that accrue to the planner outside of the planning horizon are referred to as *salvage values*. The general optimization problem with salvage value becomes

$$\max_z \int_0^T F(t,x,z)\,dt \;\; + S(x_T, T) \quad \text{s.t.}$$
$$\dot{x}_t = f(t,x,z)$$
$$x_0 \text{ fixed}$$

Rewriting equation 1 with the salvage value, we obtain:

1′ $V=\int_0^T\left[H(t,x,z,\lambda)+\dot{\lambda}_t x_t\right]dt+\lambda_0 x_0-\lambda_T x_T+S(T,x_T)$.

For the vertical end-point problem we again want to treat x_T as a choice variable and take the derivative with respect to that variable, in this case yielding

$$-\lambda_T+\frac{\partial S(T,x_T)}{\partial x_T}=0\Rightarrow\lambda_T=\frac{\partial S(T,x_T)}{\partial x_T}.$$

Intuitively, this makes sense: λ_T is the marginal value of the stock inside the planning horizon and $\frac{\partial S(T,x_T)}{\partial x_T}$ is the marginal value of the stock outside the planning horizon. When these are equal it means that these two values have, on the margin, been balanced.

Note that the addition of the salvage value does not affect the Hamiltonian, nor will it affect the first 3 of the criteria that must be satisfied.

What would be the transversality condition for a horizontal end-point problem with a salvage value? Include it in equation 1 and figure it out.

Thinking about salvage value can also help with the intuition for the vertical end-point problems. The objective function of a dynamic optimization problem can be written

$$V(x_t,t)=\int_t^T u(z_t,x_t,t)\,dt+S(x_T) \tag{3}$$

where the first term on the RHS indicates the benefits over the planning horizon from t to T, and the second term is the benefits that accrue at the end of the planning horizon. The $V(x_t,t)$ on the LHS is the value of the state x_t at time t and we know that, since λ_t is the shadow price of the state variable at time t, $\lambda_t=\partial V(x_t,t)/\partial x_t$. From (3) it is clear that $V(x_T, T)=S(x_T)$ so that $\frac{\partial V(x_T,T)}{\partial x_T}=\frac{\partial S(x_T)}{\partial x_T}$. Hence, we see that $\lambda_T=\frac{\partial S(x_T)}{\partial x_T}$. We can also use this to help understand the vertical end-point problems considered above; if $S(x_T)=0$, then

$$\lambda_T=\frac{\partial V(x_T,T)}{\partial x_T}=\frac{\partial S(x_T)}{\partial x_T}=0.$$

III. An important caveat

Most of the results above will not hold exactly if there are additional constraints on the problem or if there is a salvage value. However, you should be able to derive similar transversality conditions equation 1 or 1' and similar logic. We will consider questions with intratemporal constraints in Lecture 13.

IV. Infinite horizon problems

It is frequently the case (I would argue, usually the case) that the true problem of interest has an infinite horizon. The optimality conditions for an infinite horizon problem are identical to those of a finite horizon problem with the exception of the transversality condition. Hence, in solving the problem the most important change is how we deal with the need for the transversality conditions. [Obviously, in infinite horizon problems the mnemonic of *transversing* to the other side doesn't really work because there is no "other side" to which we might transverse.]

A. Fixed and finite target value for x

If we have a value of x to which we must arrive, i.e., $x_\infty = \lim_{t\to\infty} x_t = k$, then the problem is identical to the horizontal terminal line case considered above.

B. Flexible x_T

Recall from above that for the finite horizon problem we used equation 1:

$V = \int_0^T \left[H\left(t,x,z,\lambda\right) + \dot{\lambda}_t x_t \right] dt + \lambda_0 x_0 - \lambda_T x_T$.

In the infinite horizon case this equation is rewritten:

$V = \int_0^\infty \left[H\left(t,x,z,\lambda\right) + \dot{\lambda}_t x_t \right] dt + \lambda_0 x_0 - \lim_{T\to\infty} \lambda_T x_T$

and, for problem in which x_∞ is free, the condition analogous to the transversality condition in the finite horizon case is $\lim_{T\to\infty} \lambda_T = 0$.

Note that if our objective is to maximize the present-value of benefits, this means that the **present value** of the marginal value of an additional unit of x must go to zero as t goes to infinity. Hence, the current value (at time t) of an additional unit of x must either be finite or grow at a rate slower than r so that the discount factor, e^{-rt}, pushes the present value to zero.

In Lecture 5 we will combine the discount factor and the costate variable into one to obtain the "current value" costate variable, $\mu_t = e^{+rt} \lambda_t$.

One way that we frequently present the results of infinite horizon problems is to evaluate the equilibrium where $\dot{\lambda} = \dot{x} = 0$. Using these equations (and evaluating convergence and stability via a phase diagram) we can then solve the problem. See the fishery problem in Lecture 3.

V. Summary

The central idea behind all transversality conditions is that if there is any flexibility at the end of the time horizon, then the marginal benefit from taking advantage of that flexibility must be zero at the optimum and using equation 1 is the key here. You can apply this general principal to problems with more than one variable, to problems with constraints and, as we have seen, to problems with a salvage value.

VI. Reading for next class

Dorfman, Robert. 1969. An Economic Interpretation of Optimal Control Theory. *American Economic Review* 59(5):817-31.

VII. References

Chiang, Alpha C. 1992. *Elements of Dynamic Optimization*. McGraw Hill, New York