Econ 7010 - Microeconomics I
University of Virginia

Problem Set 6 Solutions

1. *MWG 2.F.3*

   Solution

   ———-

   Let y be the missing quantity.

   (a) The weak axiom is violated if

   $$\begin{aligned} 100 \cdot 120 + 100y &\leq 100 \cdot 100 + 100 \cdot 100 \text{ and} \\ 100 \cdot 100 + 80 \cdot 100 &\leq 100 \cdot 120 + 100 \cdot 80y \end{aligned}$$

   which is true if $y \in [75, 80]$.

   (b) The bundle in year 1 is revealed preferred if

   $$\begin{aligned} 100 \cdot 120 + 100y &\leq 100 \cdot 100 + 100 \cdot 100 \text{ and} \\ 100 \cdot 100 + 80 \cdot 100 &> 100 \cdot 120 + 100 \cdot 80y \end{aligned}$$

   which is true if $y < 75$.

   (c) The bundle in year 2 is revealed preferred if

   $$\begin{aligned} 100 \cdot 120 + 100y &> 100 \cdot 100 + 100 \cdot 100 \text{ and} \\ 100 \cdot 100 + 80 \cdot 100 &\leq 100 \cdot 120 + 100 \cdot 80y \end{aligned}$$

   or in other words, if $y > 80$.

   (d) For any value of y, we have sufficient information to justify exactly one of (a), (b), and (c).

   (e) By assumption, $y < 75$ or $y > 80$. Suppose $y < 75$ – we will prove in this case that good 1 is inferior. By (b) we have $u^1 = u(100, 100) \geq u^2 = u(120, y)$. We know $h_1\left(100, 100, u^1\right) = 100$ and

$h_1\left(100, 80, u^1\right) \leq 100$ because with only two goods, a decline in p_2 must reduce h_1, holding utility fixed. But $h_1\left(100, 80, u^2\right) = 120$. So $h_1\left(\vec{p}^2, u^2\right) - h_1\left(\vec{p}^2, u^1\right) \geq 20 > 0$ (where $\vec{p}^1 = (100, 100)$ and $\vec{p}^2 = (100, 80)$). So, $x_1\left(\vec{p}^2, m\right) - x_1\left(\vec{p}^2, m'\right) > 0$ where $m = e\left(\vec{p}^2, u^2\right)$ and $m' = e\left(\vec{p}^2, u^1\right)$. This is a pure income change. But $m - m' \leq 0$ (because $u^1 \geq u^2$ and $e()$ is increasing), so good 1 is inferior (for this change in income).

(f) Suppose $y \in (80, 100)$. By (c), we have $u^2 \geq u^1$. We have $h_2\left(\vec{p}^1, u^1\right) = 100$ and so $h_2\left(\vec{p}^2, u^1\right) \geq 100$ (because $\frac{\partial h_2}{\partial p_2} \leq 0$). Furthermore, we have $h_2\left(\vec{p}^2, u^2\right) = y < 100$, so $h_2\left(\vec{p}^2, u^2\right) - h_2\left(\vec{p}^2, u^1\right) = y - 100 < 0$. Define m and m' as in (e). In this case, we have $m - m' \geq 0$ (for the same reasons as in (e)), and $x_2\left(\vec{p}^2, m\right) - x_2\left(\vec{p}^2, m'\right) = h_2\left(\vec{p}^2, u^2\right) - h_2\left(\vec{p}^2, u^1\right) < 0$. Thus good 2 is inferior.

2. *Clarence is a well known legal scholar who worries about his professional image. He has hired you to make sure that he does not generate a trail of consumption evidence that would suggest that he is irrational. He chooses bundle (1,1) when prices are (3,6), but he could have afforded bundle (a,b). Prices move to (4,4). Find the set of choices (a,b) at prices (4,4) that will maintain his image (i.e. the set of bundles that satisfy GARP).*

Answer

We want (a, b) to satisfy the following conditions:

$$
\begin{array}{ll}
(I) & (a,b)\cdot(3,6) \leq (1,1)\cdot(3,6) \quad \text{(This is by assumption.)} \\
(II) & \text{If } (a,b)\cdot(3,6) \leq (1,1)\cdot(3,6)\text{, then } (1,1)\cdot(4,4) \geq (a,b)\cdot(4,4) \\
 & \quad \text{(This is GARP: if } (1,1) \succsim_{rp} (a,b)\text{, then ...)} \\
(III) & \text{If } (1,1)\cdot(4,4) \leq (a,b)\cdot(4,4)\text{, then } (a,b)\cdot(3,6) \geq (1,1)\cdot(3,6) \\
 & \quad \text{(This is also GARP: if } (a,b) \succsim_{rp} (1,1)\text{, then ...)}
\end{array}
$$

(I) gives us

$$a + 2b \leq 3$$

(II) gives us (since we know that the "if" clause holds),

$$
\begin{aligned}
4a + 4b &\leq 8\text{, or} \\
a + b &\leq 2
\end{aligned}
$$

And (III) gives us

$$\text{If } a + b \geq 2, \text{ then } a + 2b \geq 3$$

When (II) is satisfied strictly, we don't have to worry about (III). Thus, any (a, b) satisfying

$$a + 2b \leq 3 \text{ and } a + b < 2$$

are valid. If (II) binds, then for all three conditions to be satisfied, we must have

$$\begin{aligned} a + 2b &\leq 3 \text{ and } a + b = 2 \text{ and } a + 2b \geq 3, \text{ or therefore,} \\ a + 2b &= 3 \text{ and } a + b = 2 \end{aligned}$$

These equations have one solution, $(a, b) = (1, 1)$. So in summary, the possible values of (a, b) that do not violate GARP are

$$(1, 1) \cup \{(a, b) \mid a + 2b \leq 3 \cap a + b < 2\}$$

(If you've had trouble with GARP, WARP, and WACM, it might be a good idea to sketch this set for practice.)

3. *In class, we showed that WARP does not imply GARP by giving an example of price-consumption data $\{(p^t, x^t)\}_{t=1}^T$ that satisfy WARP, but do not satisfy GARP. Construct an example of data points such that GARP is satisfied and yet WARP is not, thereby proving that GARP does not imply WARP either.*

   Answer

   Consider the following: $(p^1, x^1) = ((1, 1), (2, 1))$ and $(p^2, x^2) = ((1, 1), (1, 2))$. Then, it is easy to check that $x^1 R^D x^2$ and $x^2 R^D x^1$, and so WARP is violated. However, neither $x^1 P^D x^2$ nor $x^2 P^D x^1$ hold, and so GARP is satisfied. (This example is a bit trivial because the price vectors are the same for each data point, but it highlights an important distinction between WARP and GARP: GARP allows there to be multiple "optimal" bundles for a given budget, while WARP implicitly restricts to a unique optimal bundle.)

4. MWG 6.B.4, 6.B.7

   Solution

   6.B.4 There are many possible utility functions that are of the expected utility form for the agent. One possibility is the following:

   $$U\left(L\right)=\left[\begin{array}{c}1, \text{if } L=A\\ p, \text{if } L=B\\ pq, \text{if } L=C\\ 0, \text{if } L=D\end{array}\right]$$

   Criterion 1 results in the following probabilities: P(A)=0.891, P(B)=0.099, P(C)=0.009, and P(D)=0.001. Criterion 2 results in the following probabilities: P(A)=0.9405, P(B)=0.0495, P(C)=0.0095, and P(D)=0.0005.

   Using the utility function above, we find that:

   $$U\left(L_1\right)=0.891+p0.099+pq0.009+0$$
   $$U\left(L_2\right)=0.9405+p0.0495+pq0.0095+0$$

   Comparing these two equations, we find that the agency prefers Criterion 2 to Criterion 1.

   6.B.7 We need to show that individuals must prefer L to L' if $x_L > x_{L'}$, given monotonic and transitive preferences. So, taking $x_L > x_{L'}$, we find:

   $$U\left(x_L\right)>U\left(x_{L'}\right)$$
   $$U\left(L\right)=U\left(x_L\right)>U\left(x_{L'}\right)=U\left(L'\right)$$
   $$U\left(L\right)>U\left(L'\right)$$

5. *A Heisman Trophy candidate quarterback has his final game of the season on Saturday. He believes that three states of the world are equally likely:*

   A. The game is rained out.

   B. The game is played and he passes for 500 yards.

   C. The game is played and he breaks his leg.

   If B happens, he'll be able to sign an NFL contract worth $D+x$ dollars. If C happens, teams will be spooked, and he'll only be able to sign for $D-x$ dollars. If A happens, the teams learn nothing new, and he'll sign *for D dollars. Suppose that the quarterback has decreasing absolute risk aversion and total wealth equal to w_0 plus his football income.*

(a) Suppose that he is given the chance to sign a contract today, before finding out what will happen on Saturday. Let d be the smallest contract offer that he would accept today. How does d vary with w_0? You can appeal to results in the textbook, but be rigorous.

(b) Let's change the situation slightly. Suppose that w_0 is income that he could earn by going to his part-time job on Saturday. He can earn this money if the game is rained out, but not if he has to play. d is defined just as in (a). Once again, determine how d varies with w_0. Compare to your result in (a) and explain.

Solution

(a) I'm going to let $\tilde{D}$ be the random variable equal to either $D+x$, $D-x$, or D with equal probability. Then $d(w_0)$ is defined by

$$u(w_0 + d(w_0)) = Eu\left(w_0 + \tilde{D}\right)$$

In other words, $w_0 + d(w_0)$ is the certainty equivalent of uncertain wealth $w_0 + \tilde{D}$. Applying Prop. 6.C.3(iii) in MWG, we have (because preferences are DARA) $w_0 - (w_0 + d(w_0))$ is decreasing in w_0. That implies that $d(w_0)$ is increasing in w_0. To put the same point differently, note that $E\left(w_0 + \tilde{D}\right) = w_0 + D$. So the risk premium $D - d(w_0)$ that he is willing to pay in order to eliminate the risk he faces is decreasing as he grows wealthier.

(b) If he doesn't take d, his wealth in states A, B, and C will be $(w_0 + D, D + x, D - x)$. If he does take d, his wealth in those three states will be $(w_0 + d, d, d)$. Now $d(w_0)$ is defined by the indifference condition

$$\frac{1}{3}\left(u(w_0 + d(w_0)) + u(d(w_0)) + u(d(w_0))\right) \equiv \frac{1}{3}\left(u(w_0 + D) + u(D + x) + u(D - x)\right)$$

Differentiate to get (compressing $d(w_0)$ to d for readability):

$$u'(w_0 + d) + \left(u'(w_0 + d) + 2u'(d)\right)d' = u'(w_0 + D)$$

or

$$d'(w_0) = \frac{u'(w_0 + D) - u'(w_0 + d)}{u'(w_0 + d) + 2u'(d)}$$

The denominator is positive. Furthermore, since the quarterback is risk averse, we know that $d < E\left(\tilde{D}\right) = D$. Then, because marginal utility is decreasing, we have $u'(w_0 + D) < u'(w_0 + d)$, so the numerator is negative. We conclude that $d'(w_0) < 0$. Equivalently, $\frac{d}{dw_0}(D - d(w_0)) > 0$. That is, as the quarterback's outside wage income rises, he is willing to accept less and less favorable deals today to avoid the uncertainty of the game on Saturday – despite his DARA preferences, and in constrast with our result from (a).

What is the intuitive difference here? In case (b), his wealth increase is stochastic – as w_0 rises, he gets that extra income in some states of the world but not others. In particular, he is not wealthier in the states (B and C) where he faces the contract risk, so his distaste for that risk is still strong. Furthermore, he buys off that contract risk using dollars in all three states. As w_0 rises, $u'(w_0 + D)$ falls, so the utility cost of using an state A dollars to buy off risk in states B and C goes down.

6. *A risk-averse bakery owner faces an uncertain price $\tilde{p}$ for her bread. Her production function for bread is $y = x^a$, where $a \in (0,1)$ and x is labor. The wage rate is $w = 1$ with certainty. The bakery owner has CRRA utility $u(w) = \frac{1}{1-\sigma} w^{1-\sigma}$ over her wealth, which comes entirely from bakery profits. (Assume $\sigma \in [0, \infty)$.)*

   (a) *Suppose that the price of bread is equally likely to be p_1 or p_2 (with $p_1 < p_2$). The owner must make her production decision before the price uncertainty is resolved. Let x_0 be her optimal choice of x if she is risk neutral ($\sigma = 0$) and x_σ her optimal choice if she is strictly risk averse ($\sigma > 0$). Determine whether $x_0 > x_\sigma$ or vice versa. Provide some brief intuition for your result.*

   (b) *Now suppose that $\tilde{p}$ is distributed according to cdf $F(p)$ and, unlike (a), suppose that the owner gets to observe the realized value of p before making her production decision. Suppose that prior to all of this, the chairman of the Fed proposes a policy that will stabilize prices, replacing the uncertain $\tilde{p}$ with its expectation $\bar{p} = E_F(\tilde{p})$. For which values of σ, if any, would the bakery owner oppose this policy? Explain.*

Solution

(a) Her expected utility if she uses labor x to produce $y = x^a$ will be

$$\frac{1}{2}u\left(p_1x^a - x\right) + \frac{1}{2}u\left(p_2x^a - x\right)$$

so her optimal production level satisfies the FOC

$$\left(p_1ax^{a-1} - 1\right)u'\left(\pi_1\right) + \left(p_2ax^{a-1} - 1\right)u'\left(\pi_2\right) = 0 \qquad (*)$$

where $\pi_i = p_ix^a - x$, or equivalently,

$$\frac{p_2ax^{a-1} - 1}{1 - p_1ax^{a-1}} = \frac{u'\left(\pi_1\right)}{u'\left(\pi_2\right)} = \left(\frac{\pi_2}{\pi_1}\right)^{\sigma}$$

Note that $\pi_2 > \pi_1$ (since $p_2 > p_1$), so for $\sigma > 0$, the LHS is strictly greater than 1. Also note that $\frac{p_2ax^{a-1}-1}{1-p_1ax^{a-1}}$ is decreasing in x (remember, $a - 1 < 0$) when the numerator and denominator are positive (as they are here). Finally, note that if $\sigma = 0$, we have $\left(\frac{\pi_2}{\pi_1}\right)^{\sigma} = 1$. In summary, we have

$$\left[\frac{p_2ax^{a-1} - 1}{1 - p_1ax^{a-1}}\right]_{x=x_\sigma} > 1 \text{ and } \left[\frac{p_2ax^{a-1} - 1}{1 - p_1ax^{a-1}}\right]_{x=x_0} = 1$$

Since the expression in brackets is decreasing in x, we must have $x_\sigma < x_0$, so the baker makes less bread when she is risk averse. Intuitively, if she knew the price would be $\bar{p}$, she would set MR-MC $= \bar{p}ax^{a-1} - 1 = 0$. Under uncertainty, she sets x so as to balance the loss from overproducing when the price is low $\left(p_1ax^{a-1} - 1 < 0\right)$ and the loss from underproducing when the price is high $\left(p_2ax^{a-1} - 1 > 0\right)$. If she is risk neutral, she weights these two losses equally. However, if she is risk averse, lost profit hurts more in the low price p_1 state of the world where she is poorer. This leads her to shift production toward the level that is optimal for p_1, which turns out to mean a reduction in x.

(b) For a realized price p, the baker maximizes her wealth by maximizing profits:

$$\max_x px^a - x$$

so her optimal production level solves the FOC $pax^{a-1} - 1 = 0$, and therefore, $x\left(p\right) = \left(ap\right)^{\frac{1}{1-a}}$. This implies profits (and wealth) of

$$w\left(p\right) = p\left(ap\right)^{\frac{a}{1-a}} - \left(ap\right)^{\frac{1}{1-a}} = Kp^{\frac{1}{1-a}} \quad \text{where } K = \left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)$$

(Note that $K = a^{\frac{a}{1-a}}(1-a) > 0$.) The baker's expected utility over all of the realizations of p is therefore

$$E_F\left(u\left(w\left(\tilde{p}\right)\right)\right) = \frac{K^{1-\sigma}}{1-\sigma}\int p^{\frac{1-\sigma}{1-a}}dF\left(p\right) = \frac{K^{1-\sigma}}{1-\sigma}E_F\left(\tilde{p}^{\frac{1-\sigma}{1-a}}\right)$$

If price variability were eliminated and replaced with $\bar{p}$, then the baker would earn wealth $w\left(\bar{p}\right)$ and enjoy utility $u\left(w\left(\bar{p}\right)\right) = \frac{K^{1-\sigma}}{1-\sigma}\bar{p}^{\frac{1-\sigma}{1-a}}$. So she strictly prefers that price variability be eliminated iff

$$\begin{aligned} \bar{p}^{\frac{1-\sigma}{1-a}} &> E_F\left(\tilde{p}^{\frac{1-\sigma}{1-a}}\right) \quad \text{or equivalently, if} \\ \left(E_F\left(\tilde{p}\right)\right)^{\frac{1-\sigma}{1-a}} &> E_F\left(\tilde{p}^{\frac{1-\sigma}{1-a}}\right) \end{aligned}$$

By Jensen's inequality, this is true (for any F) if $p^{\frac{1-\sigma}{1-a}}$ is a strictly concave function, or in other words, if $\sigma > a$. Conversely, if $\sigma < a$, then $p^{\frac{1-\sigma}{1-a}}$ is a strictly convex function, and Jensen's inequality implies that she strictly prefers the status quo with price uncertainty. (And if $\sigma = a$, she is indifferent.)

Recall that we can think of a as an elasticity of scale for this production function. Also, remember from earlier in the course that profits are convex in prices, reflecting the fact that price variability is beneficial if we have the flexibility to fine-tune our production plan to the different prices. In this setting, a gives us some sense of how valuable that flexibility is (i.e. how drastically we change our plans in response to price changes). So as long as we can act *after* observing the outcome of the price uncertainty, expected *profits* benefit from the uncertainty. However, because our baker dislikes variability in her profits ($u'' < 0$), her feelings about the price uncertainty ultimately depend on whether she dislikes variable profits more or less than she benefits from the flexibility to adjust her plans to different prices ($\sigma \gtrless a$).

7. *Suppose that an agent with CARA utility is offered Game 1: a gamble over the outcome of one toss of a loaded coin: Heads, he wins X, Tails, she loses X. (Heads has probability p.) Alternatively, suppose she is offered Game 2: the chance to play Game 1 100 times (that is, 100 independent coin flips), earning $\pm X$ each time. Let her initial wealth be denoted w_0. Are there conditions under which she is willing to play Game 1 but not Game 2? Or willing to play Game 2 but not Game 1? Prove your answer.*

Solution

She has the same preferences over the two games: she will play one iff she will also play the other. More generally, let Game n be the $n-$fold repetition of Game 1, and let $\tilde{X}_i$ be the outcome of the i^{th} coin flip, and let w_0 be the agent's initial wealth. Then,

$$\begin{aligned} Eu\,(\text{Game } n) &= -e^{-rw_0} E\left(e^{-r\left(\tilde{X}_1+\tilde{X}_2+\ldots+\tilde{X}_n\right)}\right) \\ &= -e^{-rw_0} E\left(\left(e^{-r\tilde{X}_1}\right)\left(e^{-r\tilde{X}_2}\right)\ldots\left(e^{-r\tilde{X}_n}\right)\right) \\ &= -e^{-rw_0} E\left(e^{-r\tilde{X}_1}\right)^n \end{aligned}$$

The last step follows from the fact that the coin flips are i.i.d. Not playing provides utility $-e^{-rw_0}$, so the agent strictly prefers playing Game n to not playing iff

$$E\left(e^{-r\tilde{X}_1}\right)^n < 1 \quad \text{(remember the negative sign on utility)}$$

But this is true iff $E\left(e^{-r\tilde{X}_1}\right) < 1$, so the agent also strictly prefers Game 1 to not playing.

8. *Suppose we have a prize space in dollars of* $X = \{1, 2, 3, 4, 5\}$.

   (a) *Suppose a risk-averse expected utility maximizer is comparing the following two gambles:* $p = (1/5, 1/5, 1/5, 1/5, 1/5)$ *and* $q = (2/5, 0, 1/5, 0, 2/5)$. *Can you say unambiguously which she would prefer?*

   (b) *Suppose a risk-averse expected utility maximizer is is comparing* $p' = (1/5, 1/5, 1/5, 1/5, 1/5)$ *with* $q' = (2/5, 0, 0, 1/5, /2/5)$. *Can you say unambiguously which she would prefer?*

   (c) *Consider a decision maker with utility function* $u(x) = x - ax^2$, *defined on* $x \in [1, 5]$, *where* $0 < a < 1/5$.

      i. *Calculate the decision maker's coefficient of absolute risk aversion and coefficient of relative risk aversion. Does she have decreasing/increasing/constant absolute and/or relative risk aversion?*

      ii. *Show that for this Bernoulli utility function* u, *the corresponding expected utility function* $U(F)$ *depends on only the mean and variance of* F; *that is, show that we can write*

$U(F) = V(\mu_F, \sigma_F^2)$, where μ_F and σ_F^2 are the mean and variance of F, respectively (such utility functions are sometimes called mean-variance utility).

iii. *How will this decision maker rank p vs. q and p' vs. q'?*

Solution

(a) Note that q is a mean-preserving spread of p, where we take the "weight" placed on the outcomes 2 and 4 and "spread" it to the outcomes 1 and 5, respectively (equivalently, p second-order stochastically dominates q; this can be seen by drawing graphs of the CDFs and showing that the area under the CDF for p is always less than the area for that under q). Since risk averse agents do not like mean-preserving spreads because it is more risk, our decision maker will certainly prefer p to q.

(b) Note that here, q' can be obtained by starting with q and applying a first-order upward shift of probability mass from \$3 to \$4 (i.e., q' first-order stochastically dominates q). Now, even risk averse agents prefer such first-order probability shifts, and so the agent will prefer q' to q. However, we cannot determine her preferences between q' and p: q' can be achieved by starting from p and applying a mean-preserved spread (which the risk averse agent dislikes), followed by a first-order dominance shift (which the risk averse agent likes). Without knowing more about her utility function, it is impossible to say which effect dominates.

(c) i. The coefficient of absolute risk aversion is $r(x) = \frac{2a}{1-2ax}$. We can then calculate

$$\frac{dr(x)}{dx} = \frac{4a^2}{(1-2ax)^2} > 0$$

so the agent has increasing absolute risk aversion. The coefficient of relative risk aversion is $\rho(x) = xr(x)$. Taking a derivative of this, we see that

$$\rho'(x) = r(x) + xr'(x) = \frac{2a}{(1-2ax)^2} > 0$$

so this agent also has increasing relative risk aversion.

ii. Given any distribution F, we can write

$$U(F) = \int (x - ax^2) dF(x) = E(x) - aE(x^2)$$

where $E(\cdot)$ denotes the expectation operator. Then, letting $\mu_F = E(x)$ and recalling that the variance can be written as $\sigma_F^2 = E(x^2) - E(x)^2$, we can substitute in the above to get

$$U(F) = E(x) - aE(x^2) = \mu_F - a\mu_F^2 - a\sigma_F^2,$$

which implies $V(\mu_F, \sigma_F^2) = \mu_F - a\mu_F^2 - a\sigma_F^2$.

iii. The expected utility of p is $U(p) = 3 - 11a$. The expected utility of q is $U(q) = 3 - 61a/5$. Then, $U(p) - U(q) = \frac{6a}{5} > 0$ (since $a > 0$), which means p is preferred to q. This confirms part (a), where we found that a risk averse decision maker must prefer p to q.
The expected utility of q' is $U(q') = -\frac{4}{5}(17a - 4)$, and $U(p') - U(q') = \frac{13a-1}{5}$ (note that p' is the same lottery as p). This expression equals 0 at $a = 1/13$. So, for $a < 1/13$, q' is preferred, while for $a > 1/13$, p' is preferred. Once again, this confirms part (b): some risk averse decision makers will prefer p', while some will prefer q'.

9. *Mom would burst with pride if her two kids (creatively named 1 and 2) became astronauts. The probability that kid i becomes an astronaut is $p_i = \theta_i x_i$, where θ_i is kid i's natural talent, and x_i is how much time Mom spends sticking glow-in-the-dark stars to i's bedroom ceiling. (These probabilities are independent across kids.) Mom has a time constraint on her effort: $x_1, x_2 \geq 0$ and $x_1 + x_2 \leq 1$.*[1] *Her Bernoulli utility over the number of astronaut children is $u()$. You can assume that the talent levels satisfy $0 < \theta_1 < \theta_2 < 1$.*

   (a) *Suppose that all that Mom cares about is raising at least one astronaut. That is, $u(2) = u(1) > u(0)$. Determine her optimal allocation of effort between her kids. (Be careful!) How does kid i's maternal attention x_i vary with his talent θ_i?*

   (b) *Alternatively, suppose that Mom knows that either kid would be miserable if he failed while his sibling succeeded, and she wants to avoid this. Her preferences are now $u(2) = u(0) > u(1)$. Once again, determine her optimal allocation of effort and comment on the relationship between θ_i and x_i.*

[1] She can also also choose to spend some of her time with neither kid and just drink Chardonnay. However, this does not help either kid become an astronaut, and hence does not enter her utility function.

Solution

(a) $\Pr(\text{at least 1 astronaut}) = 1 - \Pr(0 \text{ astronauts}) = 1 - (1-\theta_1 x_1)(1-\theta_2 x_2) = x_1\theta_1 + x_2\theta_2 - x_1 x_2 \theta_1 \theta_2$. Mom's expected utility is

$$\begin{aligned} & \Pr(0 \text{ astronauts})\, u(0) + \Pr(>0 \text{ astronauts})\, u(1) \\ = \;& (1 - \Pr(>0 \text{ astronauts}))u(0) + \Pr(>0 \text{ astronauts})u(1) \\ = \;& u(0) + \Pr(>0 \text{ astronauts})(u(1) - u(0)) \\ = \;& u(0) + (x_1\theta_1 + x_2\theta_2 - x_1 x_2 \theta_1\theta_2)(u(1) - u(0)) \end{aligned}$$

Mom's maximization problem is (note that we can drop the constants $u(0)$ and $u(1) - u(0)$)

$$\max_{x_1, x_2} \{x_1\theta_1 + x_2\theta_2 - x_1 x_2 \theta_1 \theta_2\} \quad \text{s.t. } x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1$$

Note that the objective is everywhere increasing in both x_1 and x_2 (double-check – this follows from $\theta_1 x_1 < 1$ and $\theta_2 x_2 < 1$), so $x_1 + x_2 \leq 1$ binds. We could proceed in different ways here, but let's substitute in $x_2 = 1 - x_1$ to get

$$\max_{x_1 \in [0,1]} \left\{\theta_2 + (\theta_1 - \theta_2 - \theta_1\theta_2)\, x_1 + x_1^2 \theta_1 \theta_2\right\}$$

This is clearly convex in x_1, so the solution will be on a boundary. We have

$$\begin{aligned} \left[\theta_2 + (\theta_1 - \theta_2 - \theta_1\theta_2)\, x_1 + x_1^2\theta_1\theta_2\right]_{x_1 = 0} &= \theta_2 \quad \text{and} \\ \left[\theta_2 + (\theta_1 - \theta_2 - \theta_1\theta_2)\, x_1 + x_1^2\theta_1\theta_2\right]_{x_1 = 1} &= \theta_1 \end{aligned}$$

so Mom chooses $x_1 = 1, x_2 = 0$ if $\theta_1 > \theta_2$ and $x_1 = 0, x_2 = 1$ if $\theta_1 < \theta_2$. (We assumed the latter in the statement of the problem.) Thus Mom puts all of her effort into the more promising kid.

(b) We have $\Pr(2 \text{ astronauts}) = \theta_1\theta_2 x_1 x_2$ and $\Pr(0 \text{ astronauts}) = (1 - \theta_1 x_1)(1 - \theta_2 x_2)$. This time, let $\tilde{\Delta} = u(2) - u(1) = u(0) - u(1)$. Using similar logic to (a), we can write Mom's expected utility as

$$\begin{aligned} & u(1) + \tilde{\Delta} \Pr(0 \text{ or } 2 \text{ astronauts}) \\ = \;& u(1) + \tilde{\Delta}\,(1 - x_1\theta_1 - x_2\theta_2 + 2x_1x_2\theta_1\theta_2) \end{aligned}$$

Again dropping constants, she solves

$$\max_{x_1, x_2} \{-x_1\theta_1 - x_2\theta_2 + 2x_1x_2\theta_1\theta_2\} \quad \text{s.t. } x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1$$

Unlike (a), the objective is no longer everywhere increasing/decreasing in x_1 and x_2 (the derivative w.r.t. x_1, for example, is $-\theta_1 + 2\theta_1\theta_2 x_2$, which may be positive or negative). So, this time, let's solve the problem in its full Lagrangian glory. Let

$$L = (2x_1x_2\theta_1\theta_2 - x_1\theta_1 - x_2\theta_2) - \lambda(x_1 + x_2) + \mu_1 x_1 + \mu_2 x_2$$

The FOCs are

$$\begin{aligned} 2\theta_1\theta_2 x_2 - \theta_1 - \lambda + \mu_1 &= 0 \text{ and} \\ 2\theta_1\theta_2 x_1 - \theta_2 - \lambda + \mu_2 &= 0 \end{aligned}$$

giving us

$$\begin{aligned} x_1 &= \frac{1}{2}\frac{1}{\theta_1} + \frac{\lambda - \mu_2}{2\theta_1\theta_2} \text{ and} \\ x_2 &= \frac{1}{2}\frac{1}{\theta_2} + \frac{\lambda - \mu_1}{2\theta_1\theta_2} \end{aligned}$$

I'll claim that if either the μ_1 or μ_2 constraint binds, then both do. Why? Suppose $\mu_1 > 0$. Then we must have $x_1 = \frac{1}{2}\frac{1}{\theta_1} + \frac{\lambda - \mu_2}{2\theta_1\theta_2} = 0$. Since the first term is positive and $\lambda \geq 0$, we must have $\mu_2 > 0$, and therefore $x_2 = 0$ also. The same argument applies starting with μ_2. So prospective solutions must have x_1 and x_2 both positive or both zero. Consider the positive cases first. We have

$$\begin{aligned} x_1 &= \frac{1}{2}\frac{1}{\theta_1} + \frac{\lambda}{2\theta_1\theta_2} \quad , \quad x_2 = \frac{1}{2}\frac{1}{\theta_2} + \frac{\lambda}{2\theta_1\theta_2} \quad \text{, and therefore,} \\ x_1 + x_2 &= \frac{1}{2}\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) + \frac{\lambda}{\theta_1\theta_2} \end{aligned}$$

But by assumption, $\theta_1 < 1$ and $\theta_2 < 1$, and we also have $\lambda \geq 0$. (We were careful to get λ set up with the proper sign.) But then the last line tells us that $x_1 + x_2 > 1$, which is inconsistent with our constraint. We conclude that the solution to Mom's optimization must be to set $x_1 = x_2 = 0$: she puts no effort into either kid to ensure that they have the same outcome. (Note that in general the Kuhn-Tucker conditions are only necessary, and not sufficient. In this case, however, $(x_1, x_2) = (0, 0)$ is the unique point satisfying the KT-conditions, and, since we know that a maximum exists, this must be it.)

Intuitively, Mom can ensure that $\Pr(0 \text{ or } 2 \text{ astronauts}) = 1$ by setting $x_1 = x_2 = 0$. By exerting positive effort, there is some chance of an unequal outcome, making her strictly worse off. Given $\theta_1 < \theta_2 < 1$, she can't guarantee that they both succeed, so she is best off doing nothing.