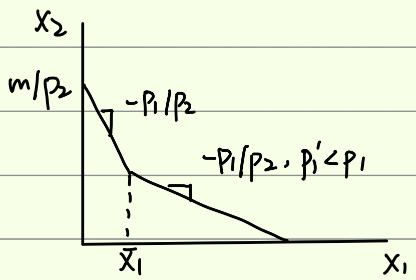
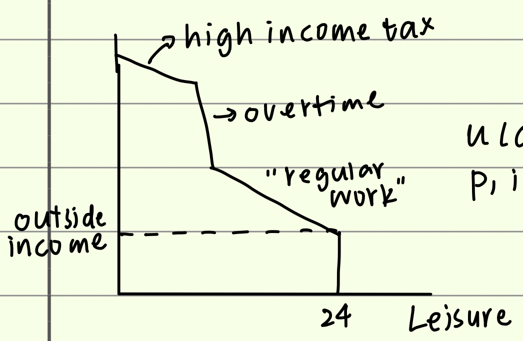
- Walrasian budget set is convex

  Nonconvex budget sets come up all the time.
  - Progressive taxation
  - Welfare payments
  - Nonlinear pricing



Discount on x_1 if you purchase $\geq \bar{x}_1$ units



<u>labor supply model</u>

$u(c,t)$

p_1 is the "price" of leisure (wage)

"Classic" Utility Maximization Problem (UMP)

$$\max u(x)$$
$$x_i \geq 0$$
$$p \cdot x \leq m$$

- Solution functions: $x(p,m) = \underset{\substack{x_i \geq 0 \\ p\cdot x \leq m}}{\arg\max}\ u(x)$ ← Marshallian Demand Correspondence

  $v(p,m) = \underset{\substack{x_i \geq 0 \\ p\cdot x \leq m}}{\max}\ u(x)$ Indirect utility function

- 1st question: existence?

<u>Theorem:</u> 1) If $p \gg 0$, + $u(\cdot)$ is continuous then UMP has a solution

2) If in addition $\succsim$ are locally nonsatiated then $p \cdot x^* = m$ for all $x^* \in x(p,m)$ (Walras' Law).

Proof: 1) $p >> 0$ ensures $B_{p,m}$ is compact (closed & bounded)

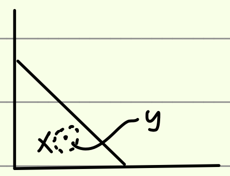
Continuous functions on compact sets have a max

2) Assume $p \cdot x^* < m$

Then x^* is in the interior of my budget set. ($B_{p,m}$). By LNS,

$\exists$ a $y \in B_{p,m}$ s.t. $u(y) > u(x^*)$

This contradicts that x^* is optimal.



Characterizing Solutions

$$\max_{\substack{x_i \geq 0 \\ p \cdot x \leq m}} u(x)$$

- $\mathcal{L} = u(x) - \lambda(p \cdot x - m) + \mu_i x_i$
- K-T: $\frac{\partial u}{\partial x_i} - \lambda p_i + \mu_i = 0 \rightsquigarrow \partial u / \partial x_i \leq \lambda p_i$ with equality if $x_i^* > 0$

  $\lambda(p \cdot x - m) = 0$

  $\mu_i x_i = 0$

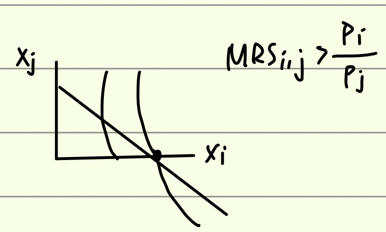
  $\lambda_i, u_i \geq 0$

- Take 2 x's : $x_i, x_j > 0$

  Divide i by j : $\left.\frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j} \right] \underbrace{\frac{\partial u / \partial x_i}{p_i}}_{\text{marginal utility per dollar}} = \frac{\partial u / \partial x_j}{p_j}$

  $MRS_{i,j} = \frac{p_i}{p_j}$

- If $x_j^* = 0$, then could have $MRS_{i,j} \neq p_i / p_j$



Properties of v(p,m)

$v(p,m) = u(x^*(p,m))$

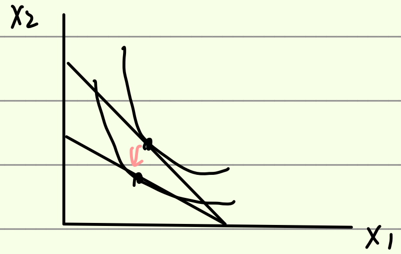
1) $v(p,m)$ is hd-0 in (p,m)

   Proof: $B_{p,m} = B_{\alpha p, \alpha m}$

   same objective, same constraint $\Rightarrow$ same solution

2) $v(p,m)$ is nonincreasing in p_i, nondecreasing in m

Proof: Let $p' \gneq p$, then $B_{p',m} \subseteq B_{p,m}$



$p_1' = p_1$, $p_2' > p_2$, slope becomes flatter

Max over a smaller set is less than max over a larger set.

3) $v(p,m)$ is quasiconvex in (p,m), i.e. the set $\{(p,m) : v(p,m) \le \bar{v}\}$ is convex.

Proof: Take $v(p,m) \le \bar{v}$, $v(p',m') \le \bar{v}$.

Define $(p'', m'') = (\alpha p + (1-\alpha) p', \alpha m + (1-\alpha) m')$

WTS: $v(p'', m'') \le \bar{v}$

Sufficient to show: $p'' \cdot x \le m'' \Rightarrow u(x) \le \bar{v}$

$\Rightarrow \alpha p \cdot x + (1-\alpha) p' \cdot x \le \alpha m + (1-\alpha) m'$

One of the following holds:

① $\cancel{\alpha} p \cdot x \le \cancel{\alpha} m$

② $\cancel{(1-\alpha)} p' \cdot x \le \cancel{(1-\alpha)} m'$

$\Rightarrow$ In words, x is affordable at (p,m), or (p',m'), or both

If ① holds: $u(x) \le v(p,m) \le \bar{v}$

If ② holds: $u(x) \le v(p',m') \le \bar{v}$

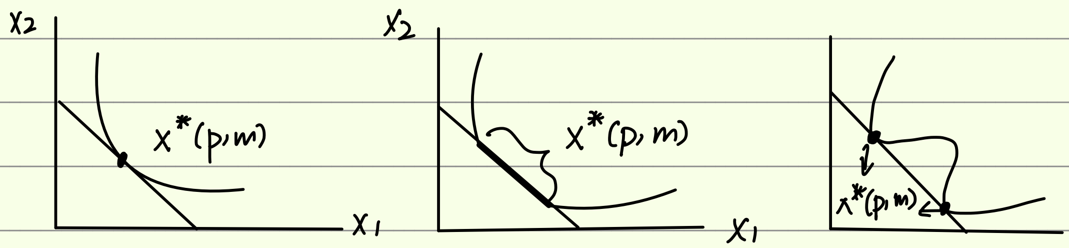
Properties of $x(p,m)$

1) $x(p,m)$ is hd-0

Proof: same as before

2) $p \cdot x = m$ for all $x \in x(p,m)$

3) If $\succsim$ is convex (quasiconcave $u(\cdot)$), then $x(p,m)$ is a convex set. If $\succsim$ is strictly convex, then $x(p,m)$ is a singleton.



strictly convex — weakly convex — nonconvex

Roy's Identity

- Recall Hotelling's Lemma: $\partial \pi / \partial p_i = y_i(p)$
- Use Envelope on UMP:

$$\mathcal{L} = u(x) - \lambda(p \cdot x - m)$$

$$\frac{\partial v(p,m)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i^*(p,m)$$

- Use again on m:

$$\frac{\partial v}{\partial m} = \lambda$$

$\Rightarrow$ Roy's identity: $x_i^*(p,m) = -\dfrac{\partial v / \partial p_i}{\partial v / \partial m}$

- Example:

$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, $\alpha + \beta = 1$ — same

Equivalently, $u(x_1, x_2) = \alpha \log x_1 + \beta \log x_2$

$$\max \ \alpha \log x_1 + \beta \log x_2$$

$$\text{s.t.} \ \cancel{x_1, x_2 \geq 0}, \quad p_1 x_1 + p_2 x_2 \leq m$$

$\log 0 \to -\infty$

$$\mathcal{L} = \alpha \log x_1 + \beta \log x_2 - \lambda (p_1 x_1 + p_2 x_2 - m)$$

FOC: ① $\alpha / x_1 = \lambda p_1$

② $\beta / x_2 = \lambda p_2$

③ $p_1 x_1 + p_2 x_2 = m$

solve and get

$x_1(p, m) = \boxed{\alpha m / p_1}$ proportion of your money you want to spend on x_1

$x_2(p, m) = \beta m / p_2$

$$\Rightarrow v(p, m) = \alpha \log (x_1(p,m)) + \beta \log (x_2(p,m))$$
$$= \alpha \log (\alpha m / p_1) + \beta \log (\beta m / p_2)$$

Value function should just be a function of p, m.

- Check Roy's identity:

$$\partial v / \partial p_1 = \alpha \cdot \frac{p_1}{\alpha m} \left(-\frac{\alpha m}{p_1^2}\right) = -\frac{\alpha}{p_1}$$

$$\partial v / \partial m = \alpha \cdot \frac{p_1}{\alpha m} \cdot \frac{\alpha}{p_1} + \frac{\beta p_2}{\beta m} \frac{\beta}{p_2} = \frac{\alpha + \beta}{m} = \frac{1}{m}$$

$$-\frac{\partial v / \partial p_1}{\partial v / \partial m} = \frac{-\alpha / p_1}{1/m} = \boxed{\frac{\alpha m}{p_1}}$$

$u(\cdot)$: direct utility

$v(\cdot)$: indirect utility