Macro - 7020: HW1

Paul Bousquet

University of Virginia

September 2023

HW1, #1c (prelude)

I will show a different way of doing the problem, but let's first briefly start with Vladimir's method. Will 2 agents and log-utility, you'll always get the following

$$\beta p_t(c_t^1 + c_t^2) = p_{t+1}(c_{t+1}^1 + c_{t+1}^2)$$

HW1, #1c (prelude)

I will show a different way of doing the problem, but let's first briefly start with Vladimir's method. Will 2 agents and log-utility, you'll always get the following

$$\beta p_t(c_t^1 + c_t^2) = p_{t+1}(c_{t+1}^1 + c_{t+1}^2)$$

Let k be even. By market clearing, $c_k^1 + c_k^2 = 3$ and $c_{k+1}^1 + c_{k+1}^2 = 6$. Therefore

$$\underbrace{\frac{1}{2}\beta p_k = p_{k+1}}_{\text{(even}\to\text{odd)}} \quad \underline{\textbf{and}} \quad \underbrace{2\beta p_{k+1} = p_{k+2}}_{\text{(odd}\to\text{even)}}$$

HW1, #1c (prelude)

I will show a different way of doing the problem, but let's first briefly start with Vladimir's method. Will 2 agents and log-utility, you'll always get the following

$$\beta p_t(c_t^1 + c_t^2) = p_{t+1}(c_{t+1}^1 + c_{t+1}^2)$$

Let k be even. By market clearing, $c_k^1 + c_k^2 = 3$ and $c_{k+1}^1 + c_{k+1}^2 = 6$. Therefore

$$\underset{\text{(even}\to\text{odd)}}{\frac{1}{2}\beta p_k = p_{k+1}} \quad \underline{\textbf{and}} \quad \underset{\text{(odd}\to\text{even)}}{2\beta p_{k+1} = p_{k+2}}$$

Let $p_0 = 1$. Plugging this into the above, you get the following

$$p_0 = 1 \implies p_1 = \frac{\beta}{2} \implies p_2 = \beta^2 \implies p_3 = \frac{\beta^3}{2} \implies p_4 = \beta^4 \implies \dots$$

$$\implies p_t = \begin{cases} \beta^t & t = 0, 2, 4, \dots \\ \frac{1}{2}\beta^t & t = 1, 3, 5, \dots \end{cases}$$

HW1, #1c (part 1)

Each consumer has Lagrangian $\mathscr{L}_i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t (e_t^i - c_t^i)\right]$

$$\textbf{FOC for } i = 1,2: \quad \frac{\beta^t}{c_t^i} = \lambda^i p_t \overset{\text{combine}}{\Longrightarrow} \frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1} \implies \boxed{c_t^2 = \frac{\lambda^1}{\lambda^2} c_t^1}$$

HW1, #1c (part 1)

Each consumer has Lagrangian $\mathscr{L}_i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t (e_t^i - c_t^i)\right]$

$$\textbf{FOC for } i = 1,2: \quad \frac{\beta^t}{c_t^i} = \lambda^i p_t \overset{\text{combine}}{\Longrightarrow} \frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1} \Longrightarrow \boxed{c_t^2 = \frac{\lambda^1}{\lambda^2} c_t^1}$$

Market clearing in even periods is $c_t^1 + c_t^2 = 3$. Substituting in the boxed equation

$$c_t^1 = 3\frac{\lambda^2}{\lambda^1 + \lambda^2} \quad t = 0, 2, 4, \ldots$$

HW1, #1c (part 1)

Each consumer has Lagrangian $\mathscr{L}_i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t (e_t^i - c_t^i)\right]$

$$\textbf{FOC for } i = 1,2: \quad \frac{\beta^t}{c_t^i} = \lambda^i p_t \overset{\text{combine}}{\Longrightarrow} \frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1} \implies \boxed{c_t^2 = \frac{\lambda^1}{\lambda^2} c_t^1}$$

Market clearing in even periods is $c_t^1 + c_t^2 = 3$. Substituting in the boxed equation

$$c_t^1 = 3\frac{\lambda^2}{\lambda^1 + \lambda^2} \quad t = 0, 2, 4, \ldots$$

Substituting in the FOC for $i = 1$

$$\lambda^1 p_t = \beta^t \frac{\lambda^1 + \lambda^2}{3\lambda^2} \implies p_t = \beta^t \frac{\lambda^1 + \lambda^2}{3\lambda^1 \lambda^2}$$

HW1, #1c (part 1)

Each consumer has Lagrangian $\mathscr{L}_i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t(e_t^i - c_t^i)\right]$

$$\textbf{FOC for } i = 1,2: \quad \frac{\beta^t}{c_t^i} = \lambda^i p_t \overset{\text{combine}}{\Longrightarrow} \frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1} \Longrightarrow \boxed{c_t^2 = \frac{\lambda^1}{\lambda^2} c_t^1}$$

Market clearing in even periods is $c_t^1 + c_t^2 = 3$. Substituting in the boxed equation

$$c_t^1 = 3\frac{\lambda^2}{\lambda^1 + \lambda^2} \quad t = 0, 2, 4, \ldots$$

Substituting in the FOC for $i = 1$

$$\lambda^1 p_t = \beta^t \frac{\lambda^1 + \lambda^2}{3\lambda^2} \Longrightarrow p_t = \beta^t \frac{\lambda^1 + \lambda^2}{3\lambda^1\lambda^2}$$

Normalizing $p_0 = 1 \Longrightarrow \frac{\lambda^1+\lambda^2}{3\lambda^1\lambda^2} = 1 \Longrightarrow \boxed{p_t = \beta^t \text{ for } t = 0, 2, 4, \ldots}$

HW1, #1c (part 1)

Each consumer has Lagrangian $\mathscr{L}_i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t(e_t^i - c_t^i)\right]$

$$\textbf{FOC for } i = 1,2: \quad \frac{\beta^t}{c_t^i} = \lambda^i p_t \overset{\text{combine}}{\Longrightarrow} \frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1} \Longrightarrow \boxed{c_t^2 = \frac{\lambda^1}{\lambda^2} c_t^1}$$

Market clearing in even periods is $c_t^1 + c_t^2 = 3$. Substituting in the boxed equation

$$c_t^1 = 3\frac{\lambda^2}{\lambda^1 + \lambda^2} \quad t = 0, 2, 4, \ldots$$

Substituting in the FOC for $i = 1$

$$\lambda^1 p_t = \beta^t \frac{\lambda^1 + \lambda^2}{3\lambda^2} \Longrightarrow p_t = \beta^t \frac{\lambda^1 + \lambda^2}{3\lambda^1 \lambda^2}$$

Normalizing $p_0 = 1 \Longrightarrow \frac{\lambda^1 + \lambda^2}{3\lambda^1\lambda^2} = 1 \Longrightarrow \boxed{p_t = \beta^t \text{ for } t = 0, 2, 4, \ldots}$ In odd periods

$$c_t^1 = 6\frac{\lambda^2}{\lambda^1 + \lambda^2} \quad t = 1, 3, 5, \cdots \Longrightarrow p_t = \beta^t \frac{\lambda^1 + \lambda^2}{6\lambda^1\lambda^2} \Longrightarrow \boxed{p_t = \frac{\beta^t}{2} \quad t = 1, 3, 5, \ldots}$$

HW1 #1c, part 2

This implies $p_t c_t^1 = 3\beta^t \frac{\lambda^2}{\lambda^1+\lambda^2}$.

Recall: if k is even, $p_{k+1} = \beta \frac{p_k}{2}$. Also, even numbers have form $2n$, odd $2n+1$. Thus

$$\sum p_t c_t^1 = \sum p_t e_t^1 \implies 3\frac{\lambda^2}{\lambda^1+\lambda^2}\sum \beta^t = 2\sum p_t$$

HW1 #1c, part 2

This implies $p_t c_t^1 = 3\beta^t \frac{\lambda^2}{\lambda^1+\lambda^2}$.

Recall: if k is even, $p_{k+1} = \beta \frac{p_k}{2}$. Also, even numbers have form $2n$, odd $2n+1$. Thus

$$\sum p_t c_t^1 = \sum p_t e_t^1 \implies 3\frac{\lambda^2}{\lambda^1+\lambda^2}\sum \beta^t = 2\sum p_t = 2\left\{\sum_{t=0}^{\infty} \underbrace{p_{2t} + p_{2t+1}}_{(1+\frac{1}{2}\beta)p_{2t}}\right\}$$

HW1 #1c, part 2

This implies $p_t c_t^1 = 3\beta^t \frac{\lambda^2}{\lambda^1+\lambda^2}$.

Recall: if k is even, $p_{k+1} = \beta \frac{p_k}{2}$. Also, even numbers have form $2n$, odd $2n+1$. Thus

$$\sum p_t c_t^1 = \sum p_t e_t^1 \implies 3\frac{\lambda^2}{\lambda^1+\lambda^2}\sum \beta^t = 2\sum p_t = 2\left\{\sum_{t=0}^{\infty} \underbrace{p_{2t}+p_{2t+1}}_{(1+\frac{1}{2}\beta)p_{2t}}\right\}$$

$$\implies \frac{3}{1-\beta}\frac{\lambda^2}{\lambda^1+\lambda^2} = \frac{2\left(1+\frac{\beta}{2}\right)}{1-\beta^2} = \frac{2+\beta}{(1+\beta)(1-\beta)}$$

HW1 #1c, part 2

This implies $p_t c_t^1 = 3\beta^t \frac{\lambda^2}{\lambda^1+\lambda^2}$.

Recall: if k is even, $p_{k+1} = \beta \frac{p_k}{2}$. Also, even numbers have form $2n$, odd $2n+1$. Thus

$$\sum p_t c_t^1 = \sum p_t e_t^1 \implies 3\frac{\lambda^2}{\lambda^1+\lambda^2}\sum \beta^t = 2\sum p_t = 2\left\{\sum_{t=0}^{\infty} \underbrace{p_{2t}+p_{2t+1}}_{(1+\frac{1}{2}\beta)p_{2t}}\right\}$$

$$\implies \frac{3}{1-\beta}\frac{\lambda^2}{\lambda^1+\lambda^2} = \frac{2\left(1+\frac{\beta}{2}\right)}{1-\beta^2} = \frac{2+\beta}{(1+\beta)(1-\beta)}$$

$$\implies \frac{\lambda^2}{\lambda^1+\lambda^2} = \frac{2+\beta}{3(1+\beta)}$$

HW1 #1c, part 2

This implies $p_t c_t^1 = 3\beta^t \frac{\lambda^2}{\lambda^1+\lambda^2}$.

Recall: if k is even, $p_{k+1} = \beta \frac{p_k}{2}$. Also, even numbers have form $2n$, odd $2n+1$. Thus

$$\sum p_t c_t^1 = \sum p_t e_t^1 \implies 3\frac{\lambda^2}{\lambda^1+\lambda^2}\sum \beta^t = 2\sum p_t = 2\left\{\sum_{t=0}^{\infty} \underbrace{p_{2t}+p_{2t+1}}_{(1+\frac{1}{2}\beta)p_{2t}}\right\}$$

$$\implies \frac{3}{1-\beta}\frac{\lambda^2}{\lambda^1+\lambda^2} = \frac{2\left(1+\frac{\beta}{2}\right)}{1-\beta^2} = \frac{2+\beta}{(1+\beta)(1-\beta)}$$

$$\implies \frac{\lambda^2}{\lambda^1+\lambda^2} = \frac{2+\beta}{3(1+\beta)}$$

To get c_t^2, we don't redo all the messy math and instead use

$$\frac{\lambda^1}{\lambda^1+\lambda^2} + \frac{\lambda^2}{\lambda^1+\lambda^2} = 1 \implies \frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{2\beta+1}{3(1+\beta)}$$

HW1 #2 (What happened?)

- First, I want to clear up what went wrong on 2c. My understanding was that Vladimir wanted you to use part b) + the formula he gave for transfers to solve this problem to show a shortcut for solving ADE+Transfers.
- His expectation was that you not use this shortcut, so that's my fault for misinterpreting. Further, this shortcut only works with proper normalization. This is because while the Lagrange multiplier in a social planning setup and prices in a competitive setup are *theoretically* equivalent if the welfare theorems hold, they are only *mathematically* proportional. This means that the time varying components are the same, but their may be a scaling constant (like $\mu_t = .5p_t$) of a difference. If you don't impose the $\alpha_1 + \alpha_2 = 1$ normalization, you don't get "proper scaling".
- I tried to make sure you all wouldn't have to think about this stuff because it's not what's important about this problem, but because Vladimir and I weren't on the same page, it didn't turn out like it should. In principle, normalization shouldn't matter
- Next slide shows what Vladimir was intending. Slide after shows the shortcut

HW1 #2c (Full Solution)

ADE+Transfers: i maximizes utility s.t $\sum p_t c_t^i = T^i + \sum p_t e_t^i$. FOC:

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t \implies \frac{1}{c_t^i p_t} = \frac{\beta}{c_{t+1}^i p_{t+1}} \implies p_t = \beta^t$$

This is the same as your notes: we just impose $p_0 = 1$. This implies that $\frac{1}{c_t^i} = \lambda^i \; \forall t$, meaning consumption is constant over time. Now we have

$$c^i \sum \beta^t = T^i + \sum \beta^t e_t^i$$

So for example this means person 1 has consumption

$$c^1 = (1-\beta)\left[T^1 + \frac{4+2\beta}{1-\beta^2}\right] \overset{\text{part b}}{\implies} T^1 = \frac{1}{1-\beta}\left(\frac{6\alpha_1}{\alpha_1+\alpha_2}\right) - \frac{4+2\beta}{1-\beta^2}$$

For part d), you can replace $\frac{6\alpha_1}{\alpha_1+\alpha_2}$ with 3 to get the answer.

HW1 #2 (Shortcut)

Let $\theta_1 = \alpha$ and $\theta_2 = 1-\alpha$ (normalize weights to sum to 1)

$$\textbf{FOC}: \quad \beta^t \frac{\theta_i}{c_t^i} = \mu_t \quad (i=1,2) \implies c_t^2 = \frac{1-\alpha}{\alpha} c_t^2$$

Substituting into market clearing

$$\left(1 + \frac{1-\alpha}{\alpha}\right) c_t^1 = 6 \implies \boxed{c_t^1 = 6\alpha} \implies \boxed{c_t^2 = 6(1-\alpha)}$$

HW1 #2 (Shortcut)

Let $\theta_1 = \alpha$ and $\theta_2 = 1-\alpha$ (normalize weights to sum to 1)

$$\textbf{FOC}: \quad \beta^t \frac{\theta_i}{c_t^i} = \mu_t \quad (i = 1,2) \implies c_t^2 = \frac{1-\alpha}{\alpha} c_t^2$$

Substituting into market clearing

$$\left(1 + \frac{1-\alpha}{\alpha}\right) c_t^1 = 6 \implies \boxed{c_t^1 = 6\alpha} \implies \boxed{c_t^2 = 6(1-\alpha)}$$

Let $c^i = c_t^i$ (it's constant). From FOC, $\mu_t = \frac{\beta^t}{6}$. So $\mu_t c_t^i = \mu_t c^i = \theta^i$

$$T^i(\theta_i) = \sum \mu_t [c_t^i - e_t^i] = \theta_i \sum \beta^t - \frac{1}{6}\left(e_0^i \sum_{t=0}^{\infty} \beta^{2t} + e_1^i \sum_{t=0}^{\infty} \beta^{2t+1}\right)$$

$$= \frac{\theta_i}{1-\beta} - \frac{e_0^i + \beta e_1^i}{6(1-\beta^2)}$$

Note the importance of who gets the big endowment first

Finally, note $(c_t^1, c_t^2) = (3,3)$ when $\alpha = .5$

HW1 #3

This is pretty much identical to the problem from discussion 3 (Vladimir is a really nice guy!). Look at your discussion section slides and notes for more detail. The minor differences in the problem are the following. We know we have autarky, so FOCs yield

$$\frac{p_{t+1}}{p_t} = \frac{c_t^t}{c_{t+1}^t} = \frac{w_1}{w_2}$$

The difference from the practice problem is we now have log-utility in both periods. So if $w_2 > w_1$, the steps we did in discussion are valid because we have a convergent infinite series: the ratio of $\frac{p_{t+1}}{p_t} < 1$, so p_t isn't blowing up (and we can sum across time). But if this isn't true, $p_t \to \infty$, so we have a divergent series (so we could do better w/ transfers).

Remember: an infinite series converges only if the terms of the series are approaching 0

If you have any other questions, email me