Econ 7010 - Microeconomics I
University of Virginia

Problem Set 2 Solutions

1. *Mr. Jefferson, the owner of Hoos Incorporated, is worried about the quality of the manager he has hired to oversee production, and has hired you as a consultant. He is afraid the manager may not be acting in the firm's best interest when procuring inputs for production. View the firm as producing one aggregate output, y, using one aggregate input, x, with associated prices p and w, respectively. Last quarter, when $(p, w) = (1, 1)$, you observed the manager using two units of input to produce one unit of output. This quarter, the prices have risen to $(p', w') = (3, 4)$. What observable data (input-output pairs) would cause you to recommend to Mr. Jefferson that the manager be fired for not maximizing the profits of the firm? Sketch all such data points on a graph.*

   Answer

   In order for the manager to not be maximizing profits, the weak axiom of profit maximization must be violated at either the old prices (p, w) or the new prices (p', w'). That is, either:

   $$py - wx < py' - wx'$$

   or

   $$p'y' - w'x' < p'y - w'x$$

   Plugging in the numbers, the inequalities become $x' - 1 < y'$ and $y' < 4/3x' - 5/3$. So, any input-output combination (x', y') that satisfies either of these inequalities violates WAPM, and the manager should be fired. You should be able to graph these equations on your own.

2. *(Varian 2.6) Let $\{(p^1, y^1), \ldots, (p^T, y^T)\}$ be a (finite) set of observed choices of a firm that satisfy WAPM, and define $Y^I = conv_{fd}(y^1, \ldots, y^T)$ (the convex hull of the data points augmented by free disposal) and*

$Y^O = \{y \in \mathbb{R}^n : p^t \leq \pi(p^t)$ *for all* $t = 1, \ldots, T\}$ *(these are just the inner bounds and outer bounds defined in class). Let* $\pi^I(p)$ *and* $\pi^O(p)$ *be the profit functions associated with* Y^I *and* Y^O*, respectively, and* $\pi(p)$ *be the profit function associated with the true technology* Y*. Prove that* $\pi^O(p) \geq \pi(p) \geq \pi^I(p)$ *for all* p*.*

Answer

For the first inequality, let the optimal choices be $y(p)$ and $y^O(p)$, when the price is p and the production sets are Y and Y^O, respectively. Since $Y \subseteq Y^O$, it follows that $y(p) \in Y^O$, and hence $y(p)$ is a feasible choice under Y^O. But then, by definition of a maximum, we have $\pi^O(p) = \max_{y \in Y^O} p \cdot y \geq p \cdot y(p) = \max_{y \in Y} p \cdot y = \pi(p)$, and the inequality holds. The second inequality follows similar logic. (Intuitively, we are simply saying that expanding the firm's feasible set of choices must always weakly increase its profits.)

3. *Prove that* $\pi(p)$ *is convex (do* ***not*** *assume any type of differentiability for this problem).*

   Fix arbitrary p and p' and let $p'' = \alpha p + (1 - \alpha) p'$ for some arbitrary $\alpha \in [0, 1]$. Suppose that y'' is a profit-maximizing production plan for p'' (so $\pi(p'') = p'' \cdot y''$). We have $\pi(p) \geq p \cdot y''$ and $\pi(p') \geq p' \cdot y''$ (since y'' is feasible at p' and p, but not necessarily optimal). But that means that

   $$\alpha\pi(p) + (1 - \alpha)\pi\left(p'\right) \geq \alpha p \cdot y'' + (1 - \alpha) p' \cdot y'' = \left(\alpha p + (1 - \alpha) p'\right) \cdot y'' = p'' \cdot y'' = \pi\left(p''\right)$$

   which is what we set out to show.

4. *MWG 5.C.13*

   Answer

   $R()$ solves

   $$R(p, \vec{w}, C) = \max_{\vec{z}} pf(\vec{z}) \quad \text{s.t. } \vec{w} \cdot \vec{z} = C$$

   with maximizer $\vec{z}(p, \vec{w}, C)$. (Since f is increasing, it suffices to take the constraint to be binding.) Following Varian's terminology, if we write the Lagrangian

   $$L = pf(\vec{z}) - \lambda(\vec{w} \cdot \vec{z} - C)$$

then we have

$$\begin{aligned}
\frac{dR(p,\vec{w},C)}{dw_1} &= \left.\frac{\partial L}{\partial w_1}\right|_{\vec{z}=\vec{z}(p,\vec{w},C)} \\
&= \left.\frac{\partial(pf(\vec{z}))}{\partial w_1} - \lambda\vec{z}_1\right|_{\vec{z}=\vec{z}(p,\vec{w},C)} \\
-\frac{p\alpha}{w_1} &= -\lambda\vec{z}_1(p,\vec{w},C)
\end{aligned}$$

A second application of the constrained Envelope Theorem, differentiating wrt C, gives us

$$\begin{aligned}
\frac{dR(p,\vec{w},C)}{dC} &= \left.\frac{\partial L}{\partial C}\right|_{\vec{z}=\vec{z}(p,\vec{w},C)} \\
\frac{p}{C} &= \lambda
\end{aligned}$$

Plugging this into the first expression, we get

$$-\frac{p\alpha}{w_1} = -\frac{p}{C}\vec{z}_1(p,\vec{w},C)$$

and so

$$\vec{z}_1(p,\vec{w},C) = \alpha\frac{C}{w_1}$$

5. *Suppose a firm's production function is given by* $f(x_1,x_2) = min(2x_1+x_2, x_1+2x_2)$.

   (a) *Sketch the associated isoquant map.*

   (b) *Find the firm's conditional factor demand functions* $x_1(w,y)$ *and* $x_2(w,y)$.

Answer

(a) You should be able to draw this on your own. It looks like a perfect complements isoquant, except that the slopes on either side of the kink line are no longer horizontal/vertical.

(b) Throughout, assume that prices are strictly positive. Because the production function in this question is not differentiable, simple first-order conditions cannot be used to solve for the conditional factor demand functions. However, looking at the isoquants, it is clear that the firm will either choose to produce at the kink of the isoquant (where $x_1{=}x_2$) or at a corner (where either $x_1{=}0$ or $x_2{=}0$). Thus, in order to produce y units of output, the firm will use either an input combination of $(\frac{1}{3}y, \frac{1}{3}y)$, (0, y), or (y, 0). Because a little rigor now and then is good for the soul, I'm going to show this in a completely watertight way, without any handwaving. Based on the isoquants, we can anticipate that our solutions may be different for $w_1/w_2 < \frac{1}{2}$, $w_1/w_2 \in \left[\frac{1}{2}, 2\right]$, and $w_1/w_2 > 2$ respectively. Take each case in turn.

$\underline{w_1/w_2 < \frac{1}{2}}$

For a given y, inputs (x_1, x_2) are feasible if $2x_1 + x_2 \geq y$ and $x_1 + 2x_2 \geq y$ (and $x_1 \geq 0$ and $x_2 \geq 0$). At least one of the first two constraints must bind at the optimum (otherwise we could reduce whichever of x_1 or x_2 is strictly positive and strictly reduce cost). If (x_1, x_2) is feasible, then its cost must satisfy

$$\begin{aligned} w_1x_1 + w_2x_2 &\geq w_1x_1 + 2w_1x_2 \\ &= w_1(x_1 + 2x_2) \\ &\geq w_1y \end{aligned}$$

The inputs $(y, 0)$ are feasible and have cost equal to w_1y, so they must be optimal. To see that they are uniquely optimal, notice that if $x_2 > 0$, the first inequality above becomes strict, so any feasible (x_1, x_2) with $x_2 > 0$ has cost strictly greater than w_1y. Any $(x_1, 0)$ with $x_1 > y$ has cost greater than w_1y, and with $x_1 < y$ is infeasible. Thus, $x(\vec{w}, y) = (y, 0)$ is the conditional factor demand if $w_1 < w_2/2$.

$\underline{w_1/w_2 > 2}$

We can repeat the argument above, *mutatis mutandis.* Here, a feasible (x_1, x_2) must satisfy

$$\begin{aligned} w_1x_1 + w_2x_2 &\geq 2w_2x_1 + w_2x_2 = w_2(2x_1 + x_2) \\ &\geq w_2y \end{aligned}$$

Clearly $(0, y)$ is feasible and gives us the lower bound cost of $w_2 y$. Arguments just like those above show that it is the only feasible input bundle with cost $w_2 y$, so here we have $x(\vec{w}, y) = (0, y)$.

$\underline{w_1/w_2 \in \left(\frac{1}{2}, 2\right)}$

In this case, the isoquants make it pretty clear that optimal factor demands are going to be at the kink: $\left(\frac{1}{3}y, \frac{1}{3}y\right)$. To show this formally, note that $\left(\frac{1}{3}y, \frac{1}{3}y\right)$ is feasible and has cost $\frac{1}{3}y(w_1 + w_2)$. Now we show that any other feasible input set has strictly greater cost. This is obvious for $x_1 = x_2 > \frac{1}{3}y$. Consider input sets off the 45° line. Take the case of (x_1, x_2) for which $x_1 < x_2$. (The reverse case is similar.) In this case, $2x_1 + x_2 > x_1 + 2x_2$, so feasibility is satisfied if $x_1 + 2x_2 \geq y$. Suppose that we have $x_1 + 2x_2 = y$ (otherwise these inputs are certainly not optimal). Then their cost is

$$\begin{aligned} w_1 x_1 + w_2 x_2 &= w_1 (y - 2x_2) + w_2 x_2 \\ &= w_1 y - (2w_1 - w_2) x_2 \end{aligned}$$

Note that $2w_1 - w_2 > 0$ by assumption, and $x_2 < \frac{1}{3}y$ (implied by $x_1 + 2x_2 = y$ and $x_2 < x_1$). Therefore,

$$w_1 x_1 + w_2 x_2 > w_1 y - (2w_1 - w_2) \frac{y}{3} = \frac{1}{3} y (w_1 + w_2)$$

so (x_1, x_2) can't be optimal. After proceeding similarly to rule out input pairs with $x_2 > x_1$, we have $x(\vec{w}, y) = \left(\frac{1}{3}y, \frac{1}{3}y\right)$.

This leaves us with conditional factor demands:

$$x(\vec{w}, y) = \begin{cases} (y, 0) & \text{if } w_1 < \frac{1}{2}w_2 \\ \left(\frac{1}{3}y, \frac{1}{3}y\right) & \text{if } w_1 \in \left(\frac{1}{2}w_2, 2w_2\right) \\ (0, y) & \text{if } w_1 > 2w_2 \end{cases}$$

You'll notice that we've left out the knife-edge cases, $w_1 = w_2/2$ and $w_1 = 2w_2$. If $w_1 = w_2/2$, then all points on the lower branch of an isoquant (the points defined by $x_1 + 2x_2 = y$) have the same minimum cost. (That cost is $w_1 x_1 + w_2 x_2 = w_1 (x_1 + 2x_2) = w_1 y$.) Thus, all points satisfying $x_1 + 2x_2 = y$ and $x_1 \geq x_2$ are optimal. The $w_1 = 2w_2$ case is similar, so for those cases we have:

$$x(\vec{w}, y) = \begin{cases} \left\{(y - 2x, x) \mid x \in \left[\frac{1}{3}y, y\right]\right\} & \text{if } w_1 = \frac{1}{2}w_2 \\ \left\{(x, y - 2x) \mid x \in \left[0, \frac{1}{3}y\right]\right\} & \text{if } w_1 = 2w_2 \end{cases}$$

This may seem like overkill, given how easy it is to eyeball the correct answer from a graph of the isoquants. Most of the time when we deal with pathological problems (ones with boundary solutions, points of nondifferentiability, etc.), you won't need to be this pedantic in your solutions – substituting a coherent explanation and graphs for some of these details is typically OK. (And during an exam, you won't have time for anything more than this.) In deciding when to explain without proof and when to be completely rigorous, the general rules of thumb are:

1. You have to be sure beyond a shadow of a doubt that your claim is correct.

2. You have to convince your audience (me, someone reading your paper, etc.) just as thoroughly that

   a) you're correct, and

   b) you understand why you're correct.

6. *Consider a firm with a single-output production function* $f(x_1, x_2) = \ln(1+\min\{x_1, x_2\})$. *Let the output price be* p *and the input price vector be* $w = (w_1, w_2)$.

   (a) *Write the corresponding cost-minimization problem and solve for the conditional factor demands* $x_1^*(w, y)$, $x_2^*(w, y)$ *and cost function* $c(w, y)$.

   (b) *Using the cost function derived in part (a), write the firm's profit maximization problem (where the only choice variable should be the choice of output* y*). Solve this problem for the optimal input demands* $x_1(p, w)$ *and* $x_2(p, w)$ *and the profit function* $\pi(p, w)$.

   Let $\bar{x}_1 = x_1(\bar{p}, \bar{w})$ *and* $\bar{x}_2 = x_2(\bar{p}, \bar{w})$ *be the optimal input demands when the prices are* $(\bar{p}, \bar{w})$. *Suppose that in the short run,* x_1 *is fixed at* $\bar{x}_1$*, but* x_2 *is flexible.*

   (c) *Write the short-run profit maximization problem and find* $x_2^{SR}(p, w)$, *the short-run input demand function for* x_2. *(N.B.: watch out for kink points.)*

   (d) *Consider a small change in prices from* $(\bar{p}, \bar{w}_1, \bar{w}_2)$ *to* $(\bar{p}, \bar{w}_1, \bar{w}_2 + \epsilon)$, *for some small* $\epsilon > 0$. *What is the change in demand for* x_2 *in the short run? What is the change in demand for* x_2 *in the long run? Does the Le Chatelier principle hold?*

Answer

(a) The factor demands are $x_1^*(w,y) = x_2^*(w,y) = e^y - 1$. The cost function is $c(w,y) = (e^y - 1)(w_1 + w_2)$.

(b) The profit maximization problem is written as

$$\max_y py - c(w,y)$$

where $c(w,y)$ is obtained from part (a). Taking a first order condition and solving, we find $y = \ln\left(\frac{p}{w_1+w_2}\right)$. Then, the optimal factor demands are $x_1(p,w) = x_2(p,w) = \frac{p}{w_1+w_2} - 1$.

(c) The short run problem is

$$\max_{x_2} p\ln(1 + \min\{\bar{x}_1, x_2\}) - w_1\bar{x}_1 - w_2x_2$$

The derivative of the objective is $\frac{p}{1+x_2} - w_2$ if $x_2 < \bar{x}_1$ and is $-w_2$ if $x_2 > \bar{x}_1$; at $x_2 = \bar{x}_1$, the derivative is undefined. Profits are clearly decreasing for all $x_2 > \bar{x}_1$, and so choosing such an x_2 can never be optimal (this is intuitive due to the min function, because using more units of x_2 than the fixed units of $\bar{x}_1$ the firm has available would be wasteful). However, depending on the prices, it may be optimal to use less than $\bar{x}_1$ units of x_2. In particular, when $\frac{p}{1+\bar{x}_1} < w_2$, the first order condition has a solution to the left of the kink point. This solution occurs at $x_2 = \frac{p}{w_2} - 1$. So, the short-run demand function can be written:

$$x_2^{SR}(p,w) = \begin{cases} \bar{x}_1 = \frac{\bar{p}}{\bar{w}_1+\bar{w}_2} - 1, & \frac{p}{w_2} \geq \frac{\bar{p}}{\bar{w}_1+\bar{w}_2} \\ \frac{p}{w_2} - 1, & \frac{p}{w_2} < \frac{\bar{p}}{\bar{w}_1+\bar{w}_2} \end{cases}$$

(d) Evaluating $x_2^{SR}(p,w)$ at $(\bar{p}, \bar{w}_1, \bar{w}_2)$, we see that $x_2^{SR} = \bar{x}_1$. This implies that the short run demand for x_2 in this neighborhood has no response to price movements. In the long run, the response to a change is obtained by differentiating the long run input demand function obtained in part (b): $\frac{\partial x_2(\bar{p},\bar{w})}{\partial w_2} = -\frac{\bar{p}}{(\bar{w}_1+\bar{w}_2)^2}$. Thus, the magnitude of the change in demand is larger in the long run than the short run, and the Le Chatelier principle holds.

7. *Derive the cost function for the following technologies:*

   (a) $f(x_1, x_2) = \min\{a_1x_1, a_2x_2, ...a_nx_n\}$.

(b) $f(x_1, x_2) = \sum_{i=1}^{n} a_i x_i$.

(c) $f(x_1, x_2) = \min\{a_1 x_1, a_2 x_2 - k\}$, *where* $k > 0$ *is a constant. (There is a little problem with this formulation. Think about what that could be and how to fix it.)*

(d) $f(x_1, x_2) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$, *where* $\rho \leq -1$.

Answers

(a) $c(w, y) = y \sum_{i=1}^{n} \frac{w_i}{a_i}$.

(b) $c(w, y) = \min\left\{\frac{w_1}{a_1}, \cdots, \frac{w_n}{a_n}\right\} y$.

(c) Restriction: $x_2 \geq \frac{k}{a_2}$. $c(w, y) = y\left(\frac{w_1}{a_1} + \frac{w_2}{a_2}\right) + \frac{w_2 k}{a_2}$.

(d) See Varian p. 55.

8. *Let* $c(w, y)$ *and* $x(w, y)$ *be the cost function and conditional factor demand for a firm with a many input, single output production technology described by production function* $f(x)$ *that exhibits constant returns to scale.*

(a) *Prove that if* $c(\cdot)$ *and* $x(\cdot)$ *both exist (i.e., the cost minimization problem has a solution), they are homogeneous of degree 1 in* y.

(b) *Prove that if the production function* $f(x)$ *is concave, then the cost function* $c(w, y)$ *is convex in* y. *What is the economic interpretation of this result?*

Answer

(a) Consider two production levels y and $y' = ty$. We want to show that $c(w, y') = tc(w, y)$. We first show $c(w, y') \leq tc(w, y)$. To see this, note that since f is CRS, one way to produce y' is to use inputs $tx(w, y)$, which costs $tw \cdot x(w, y) = tc(w, y)$. Since $tc(w, y)$ is a feasible cost, the optimal level must be weakly smaller, i.e., $c(w, y') \leq tc(w, y)$.

To show $c(w, y') \geq tc(w, y)$, we use the same argument only in reverse. Define $y'' = \frac{1}{t} y'$. Then, by CRS, one way to produce output y'' is to use input vector $\frac{1}{t} x(w, y')$, which costs $\frac{1}{t} w \cdot x(w, y') = \frac{1}{t} c(w, y')$. Since this is one feasible cost to produce y'', the optimal cost must again be weakly less: $c(w, y'') \leq \frac{1}{t}c(w, y')$. Substituting $y'' = \frac{1}{t}y' = y$ and multiplying the inequality by t on both sides, we have $tc(w, y) \leq c(w, y')$. Combining this with the inequality from the previous paragraph, we have $c(w, y') = tc(w, y)$, which is what we wanted to show.

To show that $x(w, y)$ is also homogeneous of degree 1, note that input vector $tx(w, y)$ is feasible when the required output is ty (by CRS) and $w \cdot (tx(w, y)) = t(w \cdot x(w, y)) = tc(w, y) = c(w, ty)$, where the last equality follows from the fact that $c(w, y)$ is homogeneous of degree 1. Since $tx(w, y)$ is feasible, and achieves the minimum cost $c(w, ty)$, it must be an optimal production plan when the required output is ty, i.e., $x(w, ty) = tx(w, y)$.

(b) To simplify notation, suppress the dependence of costs on w and write $C(y) = c(w, y)$. Consider two points y', y'', and define $\bar{y} = \alpha y' + (1 - \alpha)y''$ for any $\alpha \in [0, 1]$. We want to show that $C(\bar{y}) \leq \alpha C(y') + (1 - \alpha)C(y'')$.

Let $x' \in x(w, y')$ and $x'' \in x(w, y'')$, be optimal input bundles conditional on output levels y' and y'', respectively, and define $\hat{x} = \alpha x' + (1 - \alpha)x''$. Thus, we know that $f(x') \geq y'$ and $f(x'') \geq y''$. By concavity of f, we know that $f(\hat{x}) \geq \alpha f(x') + (1 - \alpha)f(x'') = \bar{y}$. Note that the cost of producing $\hat{x}$ is $w \cdot \hat{x} = \alpha w \cdot x' + (1 - \alpha)w \cdot x'' = \alpha C(y') + (1 - \alpha)C(y'')$. But now, $\hat{x}$ is *one* feasible choice of inputs to produce $\bar{y}$, and so the *optimal* choice of inputs must cost weakly less: $C(\bar{y}) \leq w \cdot \hat{x} = \alpha C(y') + (1 - \alpha)C(y'')$, which is what we wanted to show.

The economic interpretation of this result is that the firm has increasing marginal costs. This can be most easily be seen by assuming that $C(y)$ is differentiable. Then, convexity implies that $C''(y) \geq 0$, i.e., $C'(y)$ is an increasing function.