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1) MWG 5.C.8

Total cost month 3 $3\cdot 40 + 1\cdot 50 = 170$, output 60

month 95 $2\cdot 55 + 2\cdot 40 = 190$, output 60

By month 3 we know that $f(40,50) = 60$

month 95 we know that $f(55,40) = 60$

The problem is that the firm is not minimizing costs in month 95 because it could have produced 60 units with a total cost of:

$2\cdot 40 + 2\cdot 50 = 180$

2) Input requirement set $V(y) = \{x : ax_1 + bx_2 + cx_3 \geq e^y - 1\}$. Find cost function

$y = \ln(ax_1 + bx_2 + cx_3 + 1)$

CMP: $\min \; w_1x_1 + w_2x_2 + w_3x_3$ s.t $y = \ln(ax_1 + bx_2 + cx_3 + 1)$

$\frac{\partial}{\partial x_1}$: $w_1 = \frac{a}{ax_1 + bx_2 + cx_3 + 1}$

$\frac{\partial}{\partial x_2}$: $w_2 = \frac{b}{ax_1 + bx_2 + cx_3 + 1}$

$\frac{\partial}{\partial x_3}$: $w_3 = \frac{c}{ax_1 + bx_2 + cx_3 + 1}$

$\frac{\partial}{\partial \lambda}$: $y = \ln(ax_1 + bx_2 + cx_3 + 1)$

$ax_1 + bx_2 + cx_3 + 1 = \frac{a}{w_1} = \frac{b}{w_2} = \frac{c}{w_3}$

$y = \ln\left(\frac{a}{w_1}\right) = \ln\left(\frac{b}{w_2}\right) = \ln\left(\frac{c}{w_3}\right)$

Firm will use the input with higher MgP/MgC. Therefore, the cost function is

$C(w,y) = (e^y - 1)\max\left\{\frac{a}{w_1}, \frac{b}{w_2}, \frac{c}{w_3}\right\}$

3) Consider $c(w,y) = \left(y\left(\sqrt{w_1} + \sqrt{w_2}\right)\right)^2$. Find firm production function

Conditional input demands:

$$\frac{\partial c}{\partial w_1} = 2y\left(\sqrt{w_1} + \sqrt{w_2}\right) \cdot \frac{y}{2} \cdot w_1^{-1/2} = y^2\left(\sqrt{w_1} + \sqrt{w_2}\right) / \sqrt{w_1} = x_1 \qquad (1)$$

$$\frac{\partial c}{\partial w_2} = y^2\left(\sqrt{w_1} + \sqrt{w_2}\right) / \sqrt{w_2} = x_2 \qquad (2)$$

$$x_1 = y^2\left(1 + \sqrt{\frac{w_2}{w_1}}\right) \Rightarrow \frac{w_2}{w_1} = \left(\frac{x_1}{y^2} - 1\right)^2 = \left(\frac{x_1 - y^2}{y^2}\right)^2 \qquad (3)$$

$$x_2 = y^2\left(1 + \sqrt{\frac{w_1}{w_2}}\right) \Rightarrow \frac{w_1}{w_2} = \left(\frac{x_2}{y^2} - 1\right)^2 \Rightarrow \frac{w_2}{w_1} = \left(\frac{y^2}{x_2 - y^2}\right)^2 \qquad (4)$$

Combining (3) & (4): $\dfrac{x_1 - y^2}{y^2} = \dfrac{y^2}{x_2 - y^2}$

$$(x_1 - y^2)(x_2 - y^2) = y^4$$

$$x_1 x_2 - x_1 y^2 - x_2 y^2 + y^4 = y^4$$

$$\boxed{\sqrt{\frac{x_1 x_2}{x_1 + x_2}} = y}$$ production function

4) Examine $C(w,y) = y\left(a_1 w_1 + a_2 w_2 + b w_3^{1/2} w_4^{1/2}\right)$. For what set of parameter values can this be a cost function?

Properties of cost function

1) non decreasing in w : therefore a_1, a_2 and b must be positive
2) Hd-1 in w: $C(\lambda w, y) = y\left(a_1 \lambda w_1 + a_2 \lambda w_2 + b \lambda w_3^{1/2} w_4^{1/2}\right) = \lambda C(w,y)$ holds
3) Concave in w:

$$y\left(a_1(t w_1 + (1-t) w_1') + a_2(t w_2 + (1-t) w_2') + b\left(t w_3 + (1-t) w_3'\right)^{1/2}\left(t w_4 + (1-t) w_4'\right)^{1/2}\right)$$

$$\geq$$

$$t y\left(a_1 w_1 + a_2 w_2 + b w_3^{1/2} w_4^{1/2}\right) + (1-t) y\left(a w_1' + a_2 w_2' + b w_3'^{1/2} w_4'^{1/2}\right)$$

$$=$$

$$y\left(a_1(t w_1 + (1-t) w_1') + a_2(t w_2 + (1-t) w_2') + b\left(t w_3^{1/2} w_4^{1/2} + (1-t) w_3'^{1/2} w_4'^{1/2}\right)\right)$$

$$\Rightarrow \left(t w_3 + (1-t) w_3'\right)^{1/2}\left(t w_4 + (1-t) w_4'\right)^{1/2} \geq \left(t w_3^{1/2} w_4^{1/2} + (1-t) w_3'^{1/2} w_4'^{1/2}\right) \quad /\wedge 2$$

$$\left(t w_3 + (1-t) w_3'\right)\left(t w_4 + (1-t) w_4'\right) \geq$$

$$\cancel{t^2 w_3 w_4} + (1-t) w_3' t w_4 + t w_3 (1-t) w_4' + \cancel{(1-t)^2 w_3' w_4'} \geq \cancel{t^2 w_3 w_4} + \cancel{(1-t)^2 w_3' \cdot w_4'} + 2t(1-t)\left(w_3 w_4 w_3' w_4'\right)^{1/2}$$

$$\cancel{(1-t)t}\left(w_3' w_4 + w_3 w_4'\right) \geq \cancel{2}t(1-t)\left(w_3 w_4 w_3' w_4'\right)^{1/2}$$

$$w_3' w_4 + w_3 w_4' \geq \left(w_3 w_4 w_3' w_4'\right)^{1/2} \quad /\wedge 2$$

$$(w_3' w_4)^2 + (w_3 w_4')^2 + \cancel{2}\left(w_3' w_4 w_3 w_4'\right) \geq \cancel{w_3 w_4 w_3' w_4'}$$

$$(w_3' w_4)^2 + (w_3 w_4')^2 + \left(w_3' w_4 w_3 w_4'\right) \geq 0$$

with strict inequality if $b>0$. with equality if $b=0$.

4) $C(w,y)$ is continuous. ✓

production function from $C(w,y) = y\left(a_1 w_1 + a_2 w_2 + b w_3^{1/2} w_4^{1/2}\right)$.

Conditional input demands. $\partial C / \partial w$ (Shepard's lemma)

$$\frac{\partial}{\partial w_1} = y a_1 = x_1 \quad ; \quad \frac{\partial}{\partial w_2} = y a_2 = x_2 \quad \Rightarrow \quad y = \frac{x_1}{a_1} = \frac{x_2}{a_2}$$

$$\frac{\partial}{\partial w_3} = \frac{y b w_4^{1/2}}{2\sqrt{w_3}} = x_3 \quad ; \quad \frac{\partial}{\partial w_4} = \frac{y b w_3^{1/2}}{2\sqrt{w_4}} = x_4$$

$$\frac{w_4}{w_3} = \left(\frac{2x_3}{yb}\right)^2 \qquad \frac{w_3}{w_4} = \left(\frac{2x_4}{yb}\right)^2 \Rightarrow \frac{w_4}{w_3} = \left(\frac{yb}{2x_4}\right)^2$$

$$\left(\frac{2x_3}{yb}\right)^2 = \left(\frac{yb}{2x_4}\right)^2$$ (only positive values in $(\cdot)^2$).

$$\frac{2x_3}{yb} = \frac{yb}{2x_4} \Rightarrow y = \frac{2\sqrt{x_3 x_4}}{b}$$

production function

$$\boxed{f(x) = \min\left\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \frac{2\sqrt{x_3 x_4}}{b}\right\}}$$

5) Cost function $C(q) = c(w, q)$

Ave. cost : $AC(q) = C(q)/q \quad , q > 0$

a) prove that if $(AC(q) \text{ is minimized at } \bar{q}) \wedge (C'(\bar{q}) \text{ exists}) \Rightarrow AC(\bar{q}) = C'(\bar{q})$.

$$AC(q) = \frac{C(q)}{q} \quad , \text{ let's minimize}$$

$$\frac{C'(\bar{q})\bar{q} - C(\bar{q})}{\bar{q}} = 0 \quad , \quad \bar{q} = \underset{q}{\operatorname{argmin}} \, AC(q)$$

$$\Rightarrow \boxed{C'(\bar{q}) = \frac{C(\bar{q})}{\bar{q}} = AC(\bar{q})} \quad (\star)$$

b) Prove that if $AC(q)$ is minimized, then

$$\underset{(i)}{\frac{C(\bar{q}) - C(\bar{q} - \varepsilon)}{\varepsilon}} \leq AC(\bar{q}) \leq \underset{(ii)}{\frac{C(\bar{q} + \varepsilon) - C(\bar{q})}{\varepsilon}} \quad , \quad \varepsilon \in [0, \bar{q}]$$

If $\varepsilon \to 0$, then (i) and (ii) are left and right derivatives, and because of continuity of $C(\cdot)$, they are equal. Then $\boxed{AC(\bar{q}) = C'(\bar{q})}$ which is the result in $(\star)$

if $\varepsilon \to \bar{q}$
$$\frac{C(\bar{q})}{\bar{q}} \leq AC(\bar{q}) \leq \frac{C(2\bar{q}) - C(\bar{q})}{\bar{q}} \quad , \text{ but } AC(\bar{q}) = \frac{C(\bar{q})}{\bar{q}}$$

then
$$C(\bar{q}) \leq C(2\bar{q}) - C(\bar{q})$$
$$\boxed{2\,C(\bar{q}) \leq C(2\bar{q})}$$

which holds because $C(\cdot)$ is increasing and convex in q.

if $\varepsilon \in (0, \bar{q})$: $\quad \dfrac{C(\bar{q}) - C(\bar{q}-\epsilon)}{\varepsilon} \leq AC(\bar{q}) \leq \dfrac{C(\bar{q}+\varepsilon) - C(\bar{q})}{\varepsilon}$

Since $0<\varepsilon<\bar{q}$, $\bar{q}-\varepsilon$ and $\bar{q}+\varepsilon$ are subotimal levels of output.

then $\quad C(\bar{q}) < C(\bar{q}-\varepsilon)$ and $C(\bar{q}) < C(\bar{q}+\varepsilon)$

Therefore $\quad \boxed{\dfrac{C(\bar{q}) - C(\bar{q}-\varepsilon)}{\varepsilon} < 0 < AC(\bar{q})}$

Now for the RHS we have. $\quad \dfrac{C(\bar{q})}{\bar{q}} \leq \dfrac{C(\bar{q}+\varepsilon) - C(\bar{q})}{\varepsilon}$

which holds with equality if $\epsilon \to 0$. $\left(\dfrac{C(\bar{q})}{\bar{q}} = C'(\bar{q}) \right)$. Then, for $\varepsilon > 0$

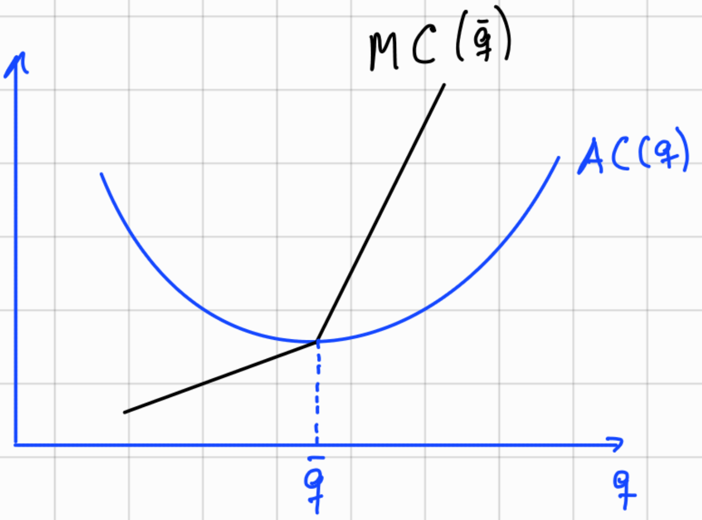
$$C(\bar{q}) \left(\frac{1}{\bar{q}} + \frac{1}{\varepsilon} \right) < \frac{C(\bar{q}+\varepsilon)}{\varepsilon}$$

$$C(\bar{q}) \left(\frac{\bar{q}+\varepsilon}{\bar{q}\varepsilon} \right) < \frac{C(\bar{q}+\varepsilon)}{\varepsilon}$$

$$C(\bar{q})(\bar{q}+\varepsilon) < C(\bar{q}+\varepsilon)\,\bar{q}$$

$$\boxed{\frac{\bar{q}+\varepsilon}{\bar{q}} < \frac{C(\bar{q}+\varepsilon)}{C(\bar{q})}}$$ which holds because of convexity of $C(\cdot)$

Case in that $C(q)$ is not differentiable at $\bar{q}$.



The inequality

$$\frac{C(\bar{q}) - C(\bar{q}-\varepsilon)}{\bar{q}} \leq AC(\bar{q}) \leq \frac{C(\bar{q}+\varepsilon) - C(\bar{q})}{\bar{q}}$$

still holds if the function is continuous at $\bar{q}$, because the LHS and RHS of the inequality approach to the same point as $\varepsilon \to 0$.

6) MWG 5.C.11.:

Show that $\partial z_\ell(w,q)/\partial q > 0$ IFF $MC(q)$ is increasing in w_ℓ

$$z_\ell = \frac{\partial C(w,q)}{\partial w_\ell} \quad \text{(Shepard's lemma)}.$$

$$\frac{\partial z_\ell}{\partial q} = \frac{\partial^2 C(w,q)}{\partial q\, \partial w_\ell} = \frac{\partial MC(q)}{\partial w_\ell}$$

$$\Rightarrow \frac{\partial z_\ell}{\partial q} = \frac{\partial MC(q)}{\partial w_\ell} \quad \therefore \left(\frac{\partial z_\ell}{\partial q} > 0\right) \longleftrightarrow \left(\frac{\partial MC(q)}{\partial w_\ell} > 0\right)$$

7) a) $\max_{L,k} \; p \cdot f(L,k) - wL - rk$

$\frac{\partial}{\partial L}: \quad p \cdot f_\ell' - w = 0 \qquad ; \quad \frac{\partial}{\partial k}: \quad p \cdot f_k' - r = 0$

$p \cdot f_\ell'(\ell^*) = w \quad (1) \qquad\qquad p \cdot f_k'(k^*) = r \quad (2)$

i) Since $f(\cdot)$ is strictly concave and differentiable, $f'(\cdot) > 0$, $f''(\cdot) < 0$.
ii) If p increases, the firm should increase the use of ℓ for (1) to hold true, because f_ℓ' is decreasing in ℓ $\left(\frac{\partial f_\ell'}{\partial \ell} < 0\right)$

b) Consider a labor-managed firm which rather than paying a fixed market wage w, takes the production surplus and distributes it equally among workers. Each worker receives $W = \dfrac{p f(L,k) - rK}{L}$

assume K is fixed to $\bar{K}$

How will a labor-managed firm respond to an increase in p?

$$\pi(L) = \max_L \frac{p \cdot f(L) - r\bar{k}}{L} = \frac{p \cdot f(L^*) - r\bar{k}}{L^*} \qquad \frac{dL}{dp} = ?$$

$$\frac{\partial}{\partial L}: \quad \frac{p \cdot f_e' \cdot L - (pf(L) - r\bar{k})}{L^2} = 0 \quad \Rightarrow \quad p \cdot f_e' \cdot L - pf(L) + r\bar{k} = 0$$

$$\Rightarrow \quad p \cdot f_e' \cdot L^* = pf(L^*) - r\bar{k}$$

$$\Rightarrow \boxed{p \cdot f_e'(L^*) = \frac{p \cdot f(L^*) - r\bar{k}}{L^*} = \pi(L^*) = W^*} \quad (*)$$

From f.o.c : $\quad p \cdot f_e' \cdot L^* - pf(L^*) + r\bar{k} = 0 \qquad \Big| \frac{d}{dp}$

$$f_e' L^* + p \cdot f_e'' \cdot \frac{dL}{dp} L^* + p \cdot f_e' \cdot \frac{dL}{dp} - \left(f(L^*) + p \cdot f_e' \cdot \frac{dL}{dp} \right) = 0$$

$$\frac{dL}{dp} = \frac{f(L^*) - f_e'(L^*) \cdot L^*}{p f_e''(L^*) \cdot L^*} \quad (**)$$

, we know $f > 0, f' > 0, f'' < 0$

If $|f(L^*)| > |f_e'(L^*) \cdot L^*|$, the firm responds increasing labor and output

If $|f(L^*)| < |f_e'(L^*) \cdot L^*|$, the firm responds decreasing labor and output

from $(*)$: $f_e'(L^*) \cdot L^* = (p \cdot f(L^*) - r\bar{k})/p$. Using in $(**)$

$$\frac{f(L^*)}{p f_e'' \cdot L^*} - \frac{(pf(L^*) - r\bar{k})}{p^2 f_e'' L^*} = \frac{f(L^*)}{p \cdot f_e'' \cdot L^*} - \frac{\left(f(L^*) - \frac{r\bar{k}}{p}\right)}{p f_e'' L^*} = \frac{(r\bar{k}/p)}{p f_e'' \cdot L^*}$$

which is negative due to $f'' < 0$. Therefore $\boxed{\frac{dL}{dp} < 0}$. The firm decrease production by unhiring labor

c) Consider an increase in r. How does the output respond? $\left(\frac{dL}{dr} = ?\right)$

from f.o.c.: $p \cdot f_e' \cdot L - pf(L) + r\bar{k} = 0 \qquad / \frac{d}{dr}$

$$L^* p f_e'' \cdot \frac{dL}{dr} + L \cdot f'(L^*) \underbrace{\cancel{\frac{dp}{dr}}}_{0} - f(L^*) \underbrace{\cancel{\frac{dp}{dr}}}_{0} + r \cdot \underbrace{\cancel{\frac{dk}{dr}}}_{0} + \bar{k} = 0$$

$$\frac{dL}{dr} = -\frac{\bar{k}}{L^* \cdot p \cdot f''(L^*)} > 0 \quad \left(\text{because of } f'' < 0\right)$$

Therefore, the firm respond increasing labor, and thus, increasing its output.

d) worker with $u(w)$. $u'>0$, $u''<0$, $u'(0)=\infty$. Firm maximizes $u(w)$ subject to being profitable ($pf(k,L)-wL-rk>0$)

$$\max\ u(W(k,L)) \quad s.t \quad p\cdot f(k,L) > wL + rk$$

$$\mathcal{L} = u(W(\bar{k},L)) + \lambda(pf(\bar{k},L) - wL - r\bar{k})$$

$$\frac{\partial}{\partial L} = u'\cdot W'_L + \lambda(p\cdot f'_L - w) = 0 \qquad (1)$$

$$\frac{\partial}{\partial \lambda} = pf(\bar{k},L) = wL + r\bar{k} \qquad (2)$$

$$W'_L = \frac{p\cdot f'_L\cdot L - (pf(L,\bar{k}) - r\bar{k})}{L^2}$$

$$u'\cdot\left[\frac{p\cdot f'_L\cdot L - (pf(L,\bar{k})-r\bar{k})}{L^2}\right] = -\lambda\underbrace{(p\cdot f'_L - w)}$$

This term is 0 in the optimum because of $M_gL = M_gC$

Therefore $$u'\cdot\left[\frac{p\cdot f'_L L - (pf(L,\bar{k})-r\bar{k})}{L^2}\right] = 0$$

becomes $$p\cdot f'_L\cdot L = pf(L,\bar{k}) - r\bar{k}$$

$$\boxed{p\cdot f'_L(L^*) = \frac{p\cdot f(L^*) - r\bar{k}}{L^*} = \pi(L^*) = W^*}$$

which is the same condition we found in (b). This result comes from the fact that $u(\cdot)$ is a monotonic transformation. Hence, the argmax is the same in both problems.

8) Inverse demand : $p(y, \lambda)$

y: liters of lemonade

λ: parameter (outside temperature) , $P_\lambda(y, \lambda) > 0$

Cost function: $c(y, \gamma)$

γ: parameter , $C_\gamma > 0$

a) monopolistic problem. (sufficient conditions for optimality and uniqueness).

$$\pi(y) = \max_{y} \; p(y, \lambda) \cdot y - c(y, \gamma)$$

$\boxed{\text{f.o.c}}$ $\frac{\partial}{\partial y}$: $P_y'(y, \lambda) y + p(y, \lambda) - C_y'(y, \gamma) = 0$ $\quad (mgR = mgC)$

Necessary conditions: $P(y, \lambda)$ and/or $C(y, \gamma)$ are differentiable at y^* (then $\pi(y)$ is differentiable) and $\nabla \pi = 0$ (i.e., f.o.c holds)

$\boxed{\text{s.o.c}}$ $\frac{\partial^2}{\partial y^2}$: $P_y''(y, \lambda) y + P_y'(y, \lambda) + P_y'(y, \lambda) - C_y''(y, \gamma) \leq 0$

$$P_y''(y, \lambda) y + 2 P_y'(y, \lambda) - C_y''(y, \gamma) \leq 0 \quad (mgR' \leq mgC')$$

necessary condition: $P(y, \lambda)$ and/or $C(y, \gamma)$ twice differentiable (then $\pi(y)$ is twice differentiable)

Sufficient condition: Hessian at y^* is negative definite. Then we have a maximum at y^*.

If $P(\cdot)$ is strictly concave and/or $C(\cdot)$ is strictly convex, then $\pi(\cdot)$ is strictly concave, and y^* is an unique global max.

Therefore, the conditions on P and C are: $P' > 0$, $P'' < 0$ and $C' > 0$, $C'' > 0$

b) Show how the optimal output respond to changes in λ and γ

from F.O.C, derive w.r.t λ.

$$\frac{\partial P_y'}{\partial \lambda} + \frac{\partial P_y}{\partial y} \cdot \frac{dy}{d\lambda} + \frac{\partial P}{\partial \lambda} + \frac{\partial P}{\partial y} \cdot \frac{dy}{d\lambda} = 0$$

$$\boxed{\frac{dy}{d\lambda} = - \frac{\left(P_{y\lambda}'' + P_\lambda'\right)}{\left(P_y'' + P_y'\right)}}$$

we know that $P_y' > 0$, $P_y'' < 0$ and $P_\lambda' > 0$

For $\frac{dy}{d\lambda} > 0$ we need

i) If $P_{y\lambda}''(y^*) \geq 0$, then $|P_y''(y^*)| > |P_y'(y^*)|$

ii) If $P_{y\lambda}''(y^*) < 0$, then $|P_y''(y^*)| > |P_y'(y^*)|$ and $|P_{y\lambda}''(y^*)| < |P_\lambda'(y^*)|$

or $|P_y''(y^*)| < |P_y'(y^*)|$ and $|P_{y\lambda}''(y^*)| > |P_\lambda'(y^*)|$

Same procedure for γ.

$$- C_{y\gamma}'' - \frac{\partial C_y'}{\partial y} \cdot \frac{dy}{d\gamma} = 0$$

$$\Rightarrow \boxed{\frac{dy}{d\gamma} = - \frac{C_{y\gamma}''}{C_y''}}$$

We know that $C_y'' > 0$

For $\frac{dy}{d\gamma}$ we need $C_{y\gamma}''(y^*) > 0$

c) Suppose $p(y,\lambda) = r(y) + s(\lambda)$ and $c(y,\gamma) = a(y) + b(\gamma)$

Answer (b) and explain intuition behind.

$$\frac{dy}{d\lambda} = -\frac{(P''_{y\lambda} + P'_{\lambda})}{(P''_{y} + P'_{y})} \quad \text{becomes} \quad -\frac{s'(\lambda)}{r''(y) + r'(y)}$$

, which is positive if $|r''(y^*)| > |r'(y^*)|$

$$\frac{dy}{d\gamma} = -\frac{C''_{y\gamma}}{C''_{y}} = 0 \quad \text{because} \quad C''_{y\gamma} = 0$$

Since $p(y,\lambda)$ and $c(y,\gamma)$ are separable functions, now mixed partial derivatives are zero. This means that now changes in λ and γ don't affect the marginal effect of y on $P(y,\lambda)$ and y on $C(y,\gamma)$.