Dec 1, 2023

Newmann Pearson Lemma

$H_0 : \{f_0\}$
$H_1 : \{f_1\}$

$$f_1(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}} e^{-1/2 \sum_{i=1}^{n} (x_i - \mu_1)^2}$$

$$\frac{f_1(x_1, \ldots, x_n)}{f_0(x_1, \ldots, x_n)} = e^{(\mu_1 - \mu_0) \underbrace{\sum_{i=1}^{n} x_i}_{\text{sufficient statistics}} + \frac{n}{2}(\mu_0^2 - \mu_1^2)} \geq k$$

$$\sum_{i=1}^{n} x_i > x^* n$$

$$\bar{x} > x^* \qquad x^* = \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$$

$$\bar{x} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

Definition: Family of distributions $\mathcal{G} = \{f_\theta, \theta \in \Theta\}$ has monotone likelihood ratio, if for $\theta' < \theta$ $f_{\theta'}(x) / f_\theta(x)$ is a monotone, increasing function of $\underbrace{T(\theta)}_{\text{sufficient statistic}}$

from the data to test the hypothesis

Theorem: Suppose that $\mathcal{P}$ is the distribution family with monotone likelihood ratio. For testing $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$

(i) There exist uniformly most powerful (UMP) test

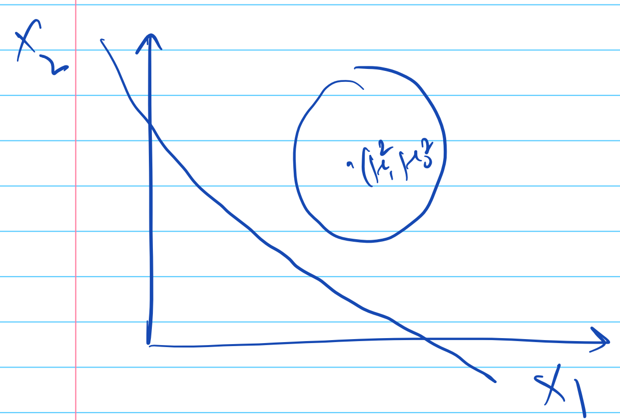
$$\phi(x) = \begin{cases} 1, \text{ if } T(x) > c & \therefore \text{(reject)} \\ \gamma, \text{ if } T(x) = c \\ 0, \text{ if } T(x) < c & \therefore \text{(accept)} \end{cases}$$

(you have to randomize them).

where $E_{\theta_0}[\phi(x)] = \alpha$

(ii) $\beta(\theta)$ is strictly increasing whenever $0 < \beta(\theta) < 1$

(iii) For any θ', test $\phi(\cdot)$ is UMP test for the new $H_0: \theta \leq \theta'$ vs $H_1: \theta > \theta'$, but with size $\alpha' = \beta(\theta')$



Multi dimensional optimization problem

$H_0: E[X_1] = \mu_0^1$

$E[X_2] = \mu_0^2$

$H_1: E[X_1] \neq \mu_1^1$

$E[X_2] \neq \mu_2^1$

Very difficult to calculate & this is only 2-dimensional. What is usually done are heuristic models.

$H_0 : \theta = \theta_0$

$H_1 : \theta \neq \theta_0$

Joint probability

$$\sqrt{n}\,(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Sigma)$$ still tough to evaluate

To make X_i independent:

$\Sigma^{1/2}\,\Sigma^{1/2} = \Sigma \longrightarrow$ covariance matrix

$$\sqrt{n}\,\Sigma^{-1/2}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, I)$$ Normal distⁿ I diagonal matrix.

$$\Rightarrow \quad n\,(\hat{\theta} - \theta_0)\,\Sigma^{-1}(\hat{\theta} - \theta_0) \xrightarrow{d} \chi^2_1$$

Allows us to use single dimensional distⁿ.

Q

Don't spend
too much
time

M

Find a test

optimal test

uniformly
most powerful

Neumann
Pearson

~~Test~~