23 Oct, 2023

* Missing: Notes from last class.

$$Z_n = \frac{\bar{X}_n - \mu_x}{\frac{\delta_x}{\sqrt{n}}}$$ Standardized R.V.

$$P(|\bar{X}_n - \mu_x| > \varepsilon) \leq \frac{c}{\varepsilon^2}$$ (Chebeye's inequality gives us this right bend).

$$\sup_z |F_{z_1}(z) - \phi(z)| \to 0$$

$$\int_{|x|>\varepsilon} e^{-x^2/2} \cdot dx \sim \frac{e^{-\varepsilon^2/2}}{\varepsilon^2}$$

Area under the normal distribution

Proportional to this. (The order of the bound is going to look like this.)

$$Z_n \xrightarrow{d} N(0,1)$$

Theorem : (Continuous Mapping Theorem.)

$X_n \xrightarrow{d} X$,

(could be distn or random vectors)

$g(\cdot)$ is a measurable function continuous w.p. 1 in the support of X_n & X

Then $g(X_n) \xrightarrow{d} g(X)$

$Z_n \xrightarrow{d} N(0,1)$

Then, $Z_n^2 \xrightarrow{d} \chi^2_1 \rightarrow$ dof.

(Chi square is defined this way. Check Dennis' first set of notes.)

$$\begin{pmatrix} Z_n^1 \\ Z_n^2 \\ \vdots \\ Z_n^k \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, I\right)$$

Then, $(Z_n^1)^2 + \dots + (Z_n^k)^2 \xrightarrow{d} \chi^2_k \rightarrow$ dof

If $Z \sim N(\mu, \Sigma)$ $\rightarrow$ Covariance Matrix. (Positive semi-definite)

$\Sigma = \Sigma^{1/2}\Sigma^{1/2}$ (Product of 2 symmetric positive semi-definite matrix)

We can standardize this R.V. using the inverse of $\Sigma^{1/2}$.

$$\Sigma^{-1/2}(Z-\mu) \sim N(0, I)$$

Pre-multiplying by $\Sigma^{-1/2}$ makes the otherwise correlated R.V. independent.

Theorem :

- **Theorem:** Let X_n, X and Y_n be random vectors. Then

  (i) $X_n \xrightarrow{a.s.} X$ implies $X_n \xrightarrow{p} X$;

  (ii) $X_n \xrightarrow{p} X$ implies $X_n \xrightarrow{d} X$;

  (iii) $X_n \xrightarrow{p} c$ (c is a constant) iff $X_n \xrightarrow{d} c$;

  (iv) if $X_n \xrightarrow{d} X$ and $\|X_n - Y_n\| \xrightarrow{p} 0$, then $Y_n \xrightarrow{d} X$;

  (v) if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ (c is a constant), then $(X_n, Y_n) \xrightarrow{d} (X, c)$;

  (vi) if $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then $(X_n, Y_n) \xrightarrow{p} (X, Y)$

(ii) $X_n \xrightarrow{p} X$ is stronger than $\xrightarrow{d}$

Does not tell us about individual components

(Last HW covered this!).

(iii) When convergence in p $\Leftrightarrow$ convergence in d

(Put it in epsilon box of something)

(iv) Mixing 2 different types of converges.

* Marginal convergence does not imply joint convergence ?

?

Exception (Only exception)

(v)

$$\underbrace{X_n \xrightarrow{d} X}, \quad Y_n \xrightarrow{d} c$$

Marginal convergence.

$$\Rightarrow \underbrace{(X_n, Y_n) \xrightarrow{d} (X, c)}$$

Joint convergence

Also holds when X_n also $\xrightarrow{d} k$ (another constant)

(Any R.V. is going to be independent from a constant so you get independence for free → But how does independence feature in joint convergence?)

(vi) $X_n \xrightarrow{p} X$, $Y_n \xrightarrow{p} Y$

$$\Rightarrow (X_n, Y_n) \xrightarrow{p} (X, Y)$$

Examples:

1) Fischer Theorem

Theory of Degree of Freedom

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 \quad (1)$$

Sample Variance

Hypothesis. But comes out, $E[S_x^2] = \sigma_x^2$ when divided by $\frac{1}{n-1}$.

$$E[s_x^2] = \sigma_x^2.$$

$$t_n = \frac{\bar{x}_n - \mu_x}{\frac{s_x}{\sqrt{n}}}$$

What can we say about convergence here?

1. Apply LLN to (1)

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x + \mu_x - \bar{x}_n)^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2 + 2(\mu_x - \bar{x}_n)\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x) +$$

$\frac{n}{n-1}(\bar{x}_n - \mu_x)$

$$+ \frac{n}{n-1}(\mu_x - \bar{x}_n)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2 - \frac{n}{n-1}(\mu_x - \bar{x}_n)^2$$

$$= \underbrace{\frac{n}{n-1}\left(\frac{1}{n}\right)\sum_{i=1}^{n}(x_i - \mu_x)^2}_{} - \underbrace{\frac{n}{n-1}(\mu_x - \bar{x}_n)^2}_{}$$

Converges in probability to σ_x^2 (Law of large numbers)

By continuous mapping theorem converges to 0.

$\Rightarrow \xrightarrow{P} \sigma_x^2$

$$t_n = \frac{\bar{X}_n - \mu_x}{\frac{S_x}{\sqrt{n}}} = \underbrace{\frac{\bar{X}_n - \mu_x}{\frac{\sigma_x}{\sqrt{n}}}}_{= Z_n} \Bigg/ \underbrace{\frac{\sqrt{S_x^2}}{\sigma_x}}_{= Y_n}$$

Assuming variance exists & by CLT:—

$$\frac{\bar{X}_n - \mu_x}{\frac{\sigma_x}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

And,
$$\frac{\sqrt{S_x^2}}{\sigma_x} \xrightarrow{P} 1$$

Let $g(z,y) = \frac{z}{y}$

This is continuous except ε around origin.

$$(Z_n, Y_n) \xrightarrow{d} (N(0,1), 1)$$

$$t_n = g(Z_n, Y_n) \xrightarrow{d} N(0,1)$$

$$\left(\frac{Z}{Y}\right), \frac{0}{1}, \frac{1}{1} \rightarrow N(0,1)$$

If constant $= 1/2$

$$\frac{1}{2} N(0,1) = N\left(0, \frac{1}{2}\right)$$

My Question: When can you have sample variance but no popn variance?