**Econ 7010 - Microeconomics I**
**University of Virginia**

## Problem Set 5 Solutions

1. *MWG 2.E.2, 2.E.7, 2.F.17*

   **Answers:**

   **2.E.2** We need to show that

$$\sum_{l=1}^{L} b_l(p,w)\,\varepsilon_{lk}(p,w) + b_k(p,w) = 0$$

is equivalent to (2.E.4). Substituting for $b_l(p,w) = p_l x_l(p,w)/w$ and $\varepsilon_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)}$, we have

$$\sum_{l=1}^{L} \frac{p_l x_l(p,w)}{w}\frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)} + \frac{p_k x_k(p,w)}{w} = 0.$$

After canceling terms and multiplying by $w/p_k$, we have (2.E.4):

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0.$$

We need to show that

$$\sum_{l=1}^{L} b_l(p,w)\,\varepsilon_{lw}(p,w) = 1$$

is equivalent to (2.E.6). Substituting for $b_l(p,w) = p_l x_l(p,w)/w$ and $\varepsilon_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)}$, we have

$$\sum_{l=1}^{L} \frac{p_l x_l(p,w)}{w}\frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)} = 1.$$

After canceling terms, we have (2.E.6):

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1.$$

**2.E.7** From the budget constraint, $p_1 x_1(p,w) + p_2 x_2(p,w) = w$, we can find the demand for good 2:

$$x_2(p,w) = (1-\alpha)\frac{w}{p_2}.$$

The demand function is homogenous of degree 0:

$$x_2(\lambda p, \lambda w) = (1-\alpha)\frac{\lambda w}{\lambda p_2} = (1-\alpha)\frac{w}{p_2} = x_2(p,w).$$

**2.F.17** (a) The demand function is homogeneous of degree 0:

$$x_k(\lambda p, \lambda w) = \frac{\lambda w}{\sum \lambda p_l} = \frac{w}{\sum p_l} = x_k(p,w).$$

(b) The demand function satisfies Walras' law:

$$\sum p_k x_k(p,w) = \sum p_k \frac{w}{\sum p_l} = w.$$

(c) The weak axiom is satisfied. Suppose that

$$p'\mathbf{x}(p,w) \le w' \quad \text{and} \quad \mathbf{x}(p,w) \ne \mathbf{x}(p',w') \qquad (1)\ .$$

The weak axiom is satisfied if

$$\mathbf{px}(p',w') > w. \quad (2)$$

Substituting the demand function into (1), we have

$$\textstyle\sum p'_l \frac{w}{\sum p_l} \le w' \quad \text{and} \quad \frac{w}{\sum p_l} \ne \frac{w'}{\sum p'_l}\ ,$$

which imply

$$\textstyle\frac{w'}{\sum p'_l} > \frac{w}{\sum p_l} \quad (3)\ .$$

Substituting the demand function into (2), we have

$$\sum p_l \frac{w'}{\sum p'_l} > w \Longleftrightarrow \frac{w'}{\sum p'_l} > \frac{w}{\sum p_l},$$

which is equivalent to (3).

(d) Element $(l,k)\,th$ of the Slutsky subsitution matrix is

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) = -\frac{w}{\left(\sum p_l\right)^2} + \frac{1}{\sum p_l}\frac{w}{\sum p_l} = 0.$$

That is, the Slutsky matrix is a matrix of zeros. Since all of the principal minors are equal to zero as well, the Slutsky matrix is symmetric, negative semidefinite *and* positive semidefinite

2. *A consumer has utility function*

$$u(c,x) = \min\{c,x\},$$

*where $c$ is consumption and $x$ is leisure. The consumer has $T$ units of time and revives wage $w$ for each unit of time she works. Wages are the only source of income. Use L to denote the number of units of time spent working, so that $x = T - L$. The price of consumption is $p$.*

(a) *Find the Marshallian demands for consumption and leisure.*

(b) *Find the indirect utility function and the expenditure function.*

(c) *Draw a graph of the Marshallian labor supply curve. Labor supply is said to be backward bending if there is a point at which an increase in $w$ leads to a decline in $L$. Does this occur here?*

(d) *A politician says that "a proposed tax on income will lead to ruin! If we tax income, no one will work and we'll lose our cherished way of life." Is the politician right in the context of the model?*

**Answer**

The trick here is to recognize that the endowment is $T$ hours, rather than some fixed level of wealth. Or, another way to say it is, our consumer is endowed with $wT$ dollars, some of which he spends on leisure $x$ by not working (at price $w$), and the rest of which is spent on consumption $c$ (at price $p$). In other words The budget constraint can be written as:

$$pc = wL$$

and, substituting $L = T - x$, we get:

$$pc + wx = wT$$

This looks just like $p_1x_1 + p_2x_2 = m$, where $p_1 = p$, $x_1 = c$ $p_2 = w$, $x_2 = x$, and $m = wT$.

(a) Given the Leonteif preferences, we know that the consumer will set $c = x$. Substituting this into the budget constraint, we have $pc + wc = wT$, or $c^* = wT/(p + w)$. Thus,

$$c^*(p, w, T) = x^*(p, w, T) = \frac{wT}{p + w}$$

(b) Indirect utility we find by substituting the demands into the direct utility function:

$$v(p, w, T) = \frac{wT}{p + w}$$

For the expenditure function, we can use the duality identity $v(p, e(p, u)) = u$. However, there is a little bit of ambiguity in the units of the endowment: it could be either in hours ($T$) or dollars ($wT$). Both give valid expenditure functions.

If we think of the endowment in hours, we have

$$v(p, w, e(p, u)) = u \implies \frac{we(p, w, u)}{p + w} = u \implies \boxed{e(p, w, u) = \frac{(p + w)u}{w}}$$

If we think of the endowment in dollars, then it would be more appropriate to write the indirect utility as a function of $wT$, rather than $T$:

$$v(p, w, wT) = \frac{wT}{p + w}$$

Then, using the duality identity:

$$v(p, w, e(p, w, u)) = u \implies \frac{e(p, w, u)}{p + w} = u \implies \boxed{e(p, w, u) = u(p + w)}$$

The first expression says it "costs" the consumer $(p+w)u/w$ hours to achieve utility $u$, while the second says it costs her $u(p + w)$ dollars to achieve utility $u$.

(c)

$$L = \frac{p}{p + w} T$$

Since $\frac{\partial L}{\partial w} < 0$, the labor supply is negatively sloped.

(d) A tax will decrease the "take-home" wage, $(1 - t)\,w$. Since $L$ is decreasing in the wage, the amount of labor supplied will increase if an income tax is implemented. Our politician is mistaken.

3. *Darrell consumes two goods, beer and Stata packages, at prices $p_b$ and $p_s$ respectively. Let $h_b(p, \bar{u})$ be Darrell's Hicksian demand for beer for utility level $\bar{u}$ and $x_b(p, m)$ be his Marshallian demand for beer at prices $p$ and wealth $m$. Show that beer and Stata packages are substitutes, show that $\partial h_b(p, \bar{u})/\partial p_s \geq 0$. Are beer and state packages also gross substitutes?*

   **Answer** Hicksian demand functions are homogeneous of degree 0 in prices: $h_b(\lambda p, \bar{u}) = h_b(p, \bar{u})$ for $\lambda > 0$. Taking the derivative with respect to $\lambda$ and evaluating the expression at $\lambda = 1$, we have

$$\frac{\partial h_b\left(p, \bar{u}\right)}{\partial p_b} p_b + \frac{\partial h_b\left(p, \bar{u}\right)}{\partial p_s} p_s = 0.$$

   Own-price effects are non-positive because the expenditure function is concave in $p$ (so that diagonal elements are non-positive). Therefore, $\frac{\partial h_b(p,u)}{\partial p_s} \geq 0$.

   We cannot make the equivalent statement about Marshallian demands, due to wealth effects. By the Slutsky equation, we can say that

$$\frac{\partial h_b(p, \bar{u})}{\partial p_s} = \frac{\partial x_b(p, m)}{\partial p_s} + \frac{\partial x_b(p, m)}{\partial m} x_b(p, m)$$

   We know the LHS of this equation is positive, but without more information about the wealth effects (the last term in the above equation) we are unable to sign $\frac{\partial x_b}{\partial p_s}$.

4. *(Core June 2000) Assume that an individual has an expenditure function*

$$e\left(p_1, p_2, u\right) = 2u\sqrt{p_1 p_2} - 100 p_1$$

   *over the set of values $(p_1, p_2, u)$ such that this function is non-decreasing in $p_1$.*

   *Suppose that in the absence of government action this person has a Walrasian budget set with $p_1 = \$1$, $p_2 = \$1$, and $w = \$800$. Now suppose that the government offers this person a cash payment of* \$1700 *on the condition that the person abstains from consumption of the first good.*

   (a) *Will the person accept this grant? Explain.*

   **Answer**

   There are two states of the world:

(1) The consumer rejects the grant. This gives the consumer a utility $u^{rej}$.

(2) The consumer accepts the grant. This gives utility $u^{acc}$.

The consumer will choose whichever gives higher utility.

First, consider (1). We can use the expenditure function to solve for $u^{rej}$ as follows:

$$800 = 2u^{rej}\sqrt{1 \times 1} - 100 \times 1$$

Solving, we see that $u^{rej} = 450$.

Thus, she will accept the grant if $u^{acc} > 450$. In state (2), she has \$2500 (800 of her own income plus 1700 from the grant), but can only purchase good 2, which means she can consume 2500 units of good 2. But, what we don't (yet) know is if the utility of the bundle $(0, 2500)$ is greater than 450. How can we figure this out? We can use the Hicksian demands (which can be obtained via the expenditure function using Shepard's Lemma):

$$h_1(p, u) = u\left(\frac{p_2}{p_1}\right)^{1/2} - 100$$
$$h_2(p, u) = u\left(\frac{p_1}{p_2}\right)^{1/2}$$

We can use these to solve for the direct utility function $u(x_1, x_2)$. We have $h_1 = u \times (p_2/p_1)^{1/2} - 100$ and $h_2 = u \times (p_1/p_2)^{1/2}$. Solving the second equation for $p_1/p_2$ and plugging into the first, we find that $u = \sqrt{(h_1 + 100)h_2}$, which means that our utility function is

$$u(x_1, x_2) = \sqrt{(x_1 + 100)x_2}$$

If she accepts the grant, she will consumer 0 units of $x_1$ and 2500 units of $x_2$. It is then easy to check that $u(0, 2500) > 450$, and so she will take the grant.

(b) *If the participation in this program were mandatory, what would be the equivalent variation measure of the net benefit of the program to this person? Show your work.*

**Answer**

The equivalent variation is the amount of additional money she would need to get the same utility if she rejects the grant as she would have gotten if she had accept it. Using the utility function found in part (a), the utility from accepting the grant would be $u^1 = u(0, 2500) = 500$. Then, the equivalent variation is

$$EV = e(p, u^1) - m = 900 - 800 = 100$$

5. *Maggie has a utility function $u(x_1, x_2)$, but rather than a monetary income $m$, she is given an endowment $E = (E_1, E_2)$ of goods 1 and 2, respectively. She maximizes utility subject to her budget constraint.*

   (a) *If the prices of the goods are $p_1$ and $p_2$, what is her budget constraint, written in terms of the endowment $(E_1, E_2)$? Write down Maggie's utility maximization problem.*

   **Answer**

   Maggie's utility maximization problem is:

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) \quad \text{subject to} \quad x_1 p_1 + x_2 p_2 = E_1 p_1 + E_2 p_2$$

   (b) *Draw Maggie's budget set, highlighting the endowment point $E$. Show what happens to the budget set if the price of $p_1$ rises.*

   The original budget set looks just like the standard Walrasian budget set: a line with slope $-p_1/p_2$, passing through the endowment point $E$. When the price of $p_1$ rises, the slope of the budget line becomes steeper, but *still passes through the endowment point*. So, the new budget line will just be a counterclockwise rotation of the old budget line, rotating through the point $E$ (note that $E$ is always an "affordable" bundle for any prices, since our consumer can just not trade at all, and consume her endowment). See the graph in section for further details.

   (c) *Now suppose that you know that Maggie's demand for good 1 exceeds her endowment of good 1. What can you say about her choice of $x_2$?*

   **Answer**

   Re-arranging the budget constraint:

$$p_1 (x_1 - E_1) + p_2 (x_2 - E_2) = 0.$$

   Clearly, if Maggie's demand for good 1 exceeds her endowment of good 1, then her choice of $x_2$ must be less than her endowment of good 2.

(d) *Write out Maggie's expenditure minimization problem (be careful: the objective will now contain endowments). Show that the Hicksian demands do not depend on $E$, but the expenditure function does.*

**Answer**

The EMP is

$$\min_{x_1,x_2} p_1(x_1 - E_1) + p_2(x_2 - E_2) \text{ such that } u(x) \geq \bar{u}$$

or

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 - p_1E_1 - p_2E_2 \text{ such that } u(x) \geq \bar{u}$$

The objective function of this problem is the same as for the standard EMP we saw in class, just shifted by the constant $-p_1E_1 - p_2E_2$. Shifting all values of the objective by a constant does not change the set of maximizers, and so the Hicksian demands $h_i(p, \bar{u})$ are independent of the endowments $E$. However, note that the constants *do* shift the value function itself (i.e, the expenditure function must be written $e(p, \bar{u}, E)$). Intuitively, even though the cheapest bundle that achieves the target utility $\bar{u}$ does not depend on the endowment, the greater the endowment of the consumer, the less she has to spend out of her own pocket to actually purchase this bundle (and in fact, the expenditure function may take on a negative value if the target level of utility is very low).

(e) *Denote the expenditure function as $e(p, \bar{u}, E)$. What is the analogue of Shepard's Lemma in this setting? That is, find an expression for $\frac{\partial e(p,\bar{u},E)}{\partial p_i}$ in terms of the Hicksian demand and the endowments $E$.*

**Answer**

Again using the envelope theorem, the analogue to Shepard's Lemma is

$$\frac{\partial e(p, \bar{u}, E)}{\partial p_i} = h_i(p, \bar{u}) - E_i$$

(f) *Derive the Slutsky equation in this setting. (The derivation is similar to what we showed in class for the case of no endowments, but the final result will include the endowments.)*

**Answer**

The Hicksian and Marshallian are related as follows:

$$h_i(p, \bar{u}) = x_i(p, e(p, \bar{u}, E), E),$$

where $\bar{u} = v(p, m)$.

Differentiating the identity with respect to $p_j$, we have

$$\frac{\partial h_i(p,\bar{u})}{\partial p_j} = \frac{\partial x_i(p,e(p,\bar{u},E),E)}{\partial p_j} + \frac{\partial x_i(p,e(p,\bar{u},E),E)}{\partial m}\frac{\partial e(p,\bar{u},E)}{\partial p_j}.$$

Plugging in for the last term and using Shepard's Lemma and the identities $h_i(p,\bar{u}) = x_i(p, e(p,\bar{u},E),E)$ and $e(p,\bar{u},E) = m$ and rearranging we have:

$$\frac{\partial x_i(p,m,E)}{\partial p_j} = \frac{\partial h_i(p,\bar{u})}{\partial p_j} + \frac{\partial x_i(p,m,E)}{\partial m}(E_j - x_j(p,m,E)).$$

(g) *If $\frac{p_1}{p_2}$ increases is Maggie better off or worse off? You can argue this graphically if you would like.*

**Answer**

If Maggie is net supplier of good 2, then she could be better or worse off. To see this, draw a graph in which maggie is just barely a net supplier of 2 (she consumes very close to, but south east of, her endowment point). Then draw a new, much steeper price ray through her endowment point– this shows that the new budget constraint will allow her to increase utility by switching from a net seller to a net buyer of good 2 (the now cheaper good). FYI, this is what we'd call a trade pattern reversal in international trade.

If Maggie is a net demander of good 2, then she is unambiguously better off. (FYI, this is what we'd call an unambiguously positive terms-of-trade improvement in international trade.)

See the graph in section.

6. *Consider a consumer with wealth $m$ who consumes two goods, $x_1$ and $x_2$. Normalize the price $p_2 = 1$. Let $x_1(p,m)$ be the Marshallian demand, and $h_1(p, v(\bar{p}, m))$ be the corresponding Hicksian demand when the required utility level is $v(\bar{p}, m)$ for some fixed $\bar{p}$. Assume that good 1 is an inferior good.*

(a) *On a graph with demand on the horizontal axis and price $p_1$ on the vertical axis, draw a sketch showing the relationship between the Marshallian and Hicksian demands. The key features of the sketch are the points where the two curves cross and the relative slopes of the two curves.*

**Answer**

This should look just like the graph we drew in class for normal goods, except the labels of the two curves should be reversed because the good is now an inferior good. The two curves intersect at the point $\bar{p}_1$. See Figure 3.G.1 in MWG for further details.

(b) *Consider a price change from $p^0$ to $p^1$ where all prices are the same except for good 1, whose price decreases. Show algebraically that if good 1 is an inferior good, then the equivalent variation is less than or equal to the compensating variation. (Hint: what is the relationship between the Hicksian demand curves at the old utility $u^0$ and the new utility $u^1$? How do these demand curves relate to EV and CV?)*

**Answer**

As in class, let $u^0 = v(p^0, m)$ and $u^1 = v(p^1, m)$ be the initial and final utilities. Since the prices decrease, we know that $u^0 \leq u^1$, which implies that $e(p, u^0) \leq e(p, u^1)$ for all price vectors $p$. Thus, since good 1 is inferior, we have

$$x_1(p, e(p, u^1)) \leq x_1(p, e(p, u^0))$$

for all $p$. By the identities relating Marshallian and Hicksian demands, this is equivalent to

$$h_1(p, u^1) \leq h_1(p, u^0)$$

for all $p$. Recall from class that the equivalent variation and compensating variation are just the areas under the relevant Hicksian demands. Thus, fix all of the other prices at $\bar{p}_{-1}$, and lets integrate both sides of this equation from $p_1^1$ to $p_1^0$:

$$\int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1) dp_1 \leq \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1) dp_1$$

But, as we showed in class, the LHS is the equivalent variation, and the RHS is the compensating variation, and so we are done.

7. *Consider the candidate indirect utility function*

$$v(p_1, p_2, m) = p_1^a p_2^b m^c.$$

*where* $a = b = -\frac{1}{2}, c = 1$. *Find expressions for*

(a) *the expenditure function.*

**Answer**

$$e(\mathbf{p}, u) = p_1^{1/2} p_2^{1/2} u$$

(b) *the Hicksian and Marshallian demands,*

**Answers**

$$\begin{aligned} h_1(\mathbf{p}, \mathbf{u}) &= \tfrac{1}{2} p_1^{-1/2} p_2^{1/2} u \\ h_2(\mathbf{p}, \mathbf{u}) &= \tfrac{1}{2} p_1^{1/2} p_2^{-1/2} u \\ x_1(\mathbf{p}, \mathbf{m}) &= \tfrac{1}{2} \tfrac{m}{p_1} \\ x_2(\mathbf{p}, \mathbf{m}) &= \tfrac{1}{2} \tfrac{m}{p_2} \end{aligned}$$

(c) *the direct utility function, and*

**Answer**

$$u(x_1, x_2) = 2 x_1^{1/2} x_2^{1/2}$$

(d) *CV and EV for a price change from* $(p_1, p_2)$ *to* $(q_1, q_2)$.

**Answers**

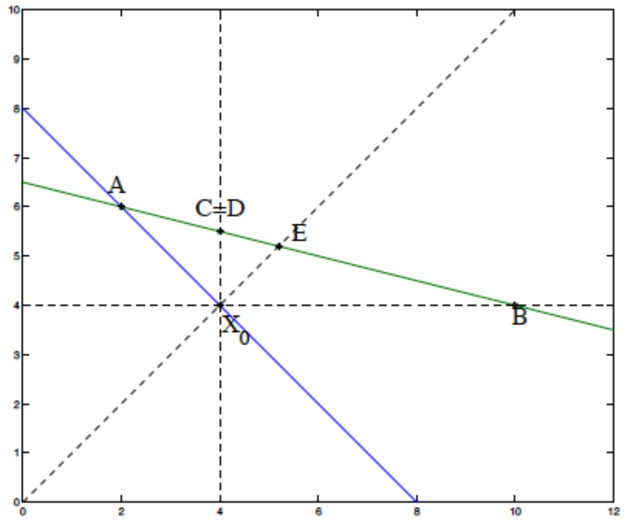
$$\begin{aligned} CV(\mathbf{p}, \mathbf{q}, m) &= \left(1 - q_1^{1/2} q_2^{1/2} p_1^{-1/2} p_2^{-1/2}\right) m \\ EV(\mathbf{p}, \mathbf{q}, m) &= \left(p_1^{1/2} p_2^{1/2} q_1^{-1/2} q_2^{-1/2} - 1\right) m \end{aligned}$$

8. *MWG 3.D.7, 3.G.15*

**Answer**

3.D.7 The figure referenced in the answers to this question is given below. The blue line through $x_0$ represents the old budget line corresponding to $p^0$ and $w^0$, while the green line represents the new budget line. We assume local non satiation throughout.

(a) Note that $x_0$ belongs to the new budget set. Any point on the new budget line to the right of point $A$ is consistent with preference maximization. Points to the left of $A$ are included in the original budget set, and hence have been revealed dispreferred to $x_0$. The point $A$ itself is NOT a permissible




choice. $A$ can be at most indifferent with $x_0$, which, by local non satiation, is strictly dispreferred to the new optimal bundle $x_1$ ($x_1$ is not pictured).

(b) Recall that for quaslinear preferences, indifference curves are all simply parallel horizontal shifts. First, imagine increasing wealth $m$ without changing prices. All extra income goes into good 1, so the income expansion path is simply a horizontal line from $x_0$ to $B$. Drawing an indifference curve should convince you that the point $B$ is then weakly preferred to all points on the budget line to the left of $B$. Thus, none of these points can be chosen under the new budget line. Everything to the right of point $B$ on the budget line may be chosen, depending on the shape of the indifference curve. (For a more mathematical explanation, under quasilinear utility, the first order condition reads $\partial u(x^t)/\partial x_2 = p_2^t/p_1^t$ for $t = 0, 1$. We have $p_2^0/p_1^0 < p_2^1/p_1^1$. Since $u$ is concave, this implies that $x_2^0 > x_2^1$, which implies that $x^1$ must lie to the right of point $B$).

(c) The reasoning is similar to part (b), only now the indifference curves are parallel vertical shifts. If we increase $m$ without changing prices, all extra income goes into good 2, and we find that point $C$ is an optimal point the old prices and some alternative income $m$. Again drawing an indifference curve, we see that all points on the green budget line to the left of $C$ are dispreferred to $C$, and so, under the new prices, the optimal bundle $x^1$ must lie to the right of point $C$. (Mathematically, an argument analogous to part (b) gives that $x_1^0 < x_1^1$.)

(d) Since both goods are normal, both have positive income effects. Since the relative price of good 1 has decreased, the substitution effect for good 1 is also positive. This means that both the income effect and substitution effect direct the consumer to purchase more $x_1$, and so $x_1^1 > x_1^0$. For good 2, however, we cannot be sure: the substitution effect tells her to consume less of $x_2$, but the income effect tells her to consume more. Thus, any point $x^1$ such that $x_1^1 > x_1^0$ works. In the picture, this is all points on the budget line to the right of $C$ (note that $x_1^1 > x_1^0$ is the same condition we found in part (c).)

(e) Homothetic preferences mean all marginal rates of substitution (slopes of indifference curves) are the same along the ray containing point $E$, and fall as the ray rotates to the right. Since the slope of the budget line has also fallen, we see from the picture that $x^1$ must lie along the budget line to the right of $E$.

**Note:** For parts (b)-(e), the relevant boundary points are included in the permissible set (this can be seen by drawing preferences with kink points in the appropriate places).

3.G.15 **(a)**

$$x_1(\vec{p}, w) = \frac{p_2}{p_1 p_2 + 4p_1^2} w \quad , \quad x_2(\vec{p}, w) = \frac{4p_1}{4p_1 p_2 + p_2^2} w$$

**(b)**

$$h_1(\vec{p}, u) = \left( \frac{p_2 u}{2(4p_1 + p_2)} \right)^2 \quad , \quad h_2(\vec{p}, u) = \left( \frac{p_1 u}{4p_1 + p_2} \right)^2$$

**(c)**

$$e(\vec{p}, u) = \frac{p_1 p_2 u^2}{4(4p_1 + p_2)}$$

It is straightforward to differentiate this to show that Shephard's Lemma holds.

**(d)**

$$v(\vec{p}, w) = 2\left(\frac{w}{p_1} + 4\frac{w}{p_2}\right)^{\frac{1}{2}}$$

For Roy's Identity, take

$$\begin{aligned}
\frac{\partial v(\vec{p}, w)}{\partial p_1} &= -\frac{w}{p_1^2}\left(\frac{w}{p_1} + 4\frac{w}{p_2}\right)^{-\frac{1}{2}} \\
\frac{\partial v(\vec{p}, w)}{\partial p_2} &= -4\frac{w}{p_2^2}\left(\frac{w}{p_1} + 4\frac{w}{p_2}\right)^{-\frac{1}{2}} \quad \text{, and} \\
\frac{\partial v(\vec{p}, w)}{\partial w} &= \left(\frac{1}{p_1} + \frac{4}{p_2}\right)\left(\frac{w}{p_1} + 4\frac{w}{p_2}\right)^{-\frac{1}{2}}
\end{aligned}$$

and proceed from there.

9. *Consider two consumers with utility functions $u_A(x_1, x_2) = x_1 + \alpha \ln(x_2)$ and $u_B(x_1, x_2) = x_1 + \beta\sqrt{x_2}$. Assume that the numeraire good, $x_1$, can take on any value in the set $(-\infty, \infty)$ (i.e., ignore the nonnegativity constraint on $x_1$). You may normalize $p_1 = 1$, and let the price of good 2 be $p_2$. The consumers have incomes $m_A$ and $m_B$ respectively.*

   (a) *Derive the individual demand functions for each consumer for each good.*

   (b) *What is the aggregate demand function?*

   (c) *Can you write the aggregate demand functions only as a function of aggregate wealth $\bar{m} = m_A + m_B$? Do you need any additional restrictions on the parameters of the utility functions for this to hold?*

   (d) *Do the consumers have preferences of the Gorman form? Prove your answer (i.e., if yes, find the functions $a_i(p)$ and $b(p)$).*

   **Answer**

   (a) The demands for agent $A$ are

   $$x_1^A(p, m_A) = m_A - \alpha \text{ and } x_2^A(p, m_A) = \alpha/p_2.$$

   For agent $B$, they are

   $$x_1^B(p, m_B) = m_B - \beta/2 \text{ and } x_2^B(p, m_B) = \beta^2/(2p_2)^2.$$

(b) The aggregate demands are

$$\bar{x}_1(p, m_A, m_B) = m_A + m_B - \alpha - \beta/2 = \bar{m} - \alpha - \beta/2$$

and

$$\bar{x}_2(p, m_A, m_B) = \alpha/p_2 + \beta^2/(2p_2)^2$$

(c) It is obvious from part (b) that this can be done. The aggregate demand for good 2 does not depend on either $m_A$ or $m_B$, and so can trivially be written as a function of $\bar{m}$. No further restrictions are required: if utility is quasilinear in the same good, agent preferences do not need to be identical in order for aggregation to be valid.

(d) Since we know that Gorman form preferences are both necessary and sufficient for aggregation, the answer here must be yes. We can easily calculate the indirect utility functions from the demand functions found in part (a). We have for agent $A$:

$$v_A(p, m) = -\alpha + \alpha \ln(\alpha/p_2) + m_A$$

And for agent $B$:

$$v_B(p, m) = -\frac{\beta}{2} + \frac{\beta^2}{2p_2} + m_B$$

By inspection, we see that $a_A(p) = -\alpha + \alpha \ln(\alpha/p_2)$ and $a_B(p) - \frac{\beta}{2} + \frac{\beta^2}{2p_2}$. The wealth coefficient is $b(p) = 1$, and is the same for both agent $A$ and $B$.

10. *[From the January 2009 core exam] Consider an economy with a continuum of consumers, indexed by $a \in [0, 1]$, and two goods. Consumer $a$'s utility function over the two goods is $u_a(x_1, x_2) = ax_1 + (1-a)x_2$. All consumers are endowed with the same income $m$.*

   (a) *Derive aggregate demand $X_1(\vec{p}, m)$ and $X_2(\vec{p}, m)$ (where $\vec{p} = (p_1, p_2)$).*

   (b) *Suppose that these aggregate demands can be interpreted as the bundle chosen by a single representative agent maximizing some function $U(X_1, X_2)$ subject to income $m$. Determine whether the representative agent's preferences are homothetic. Derive this agent's elasticity of substitution between the two goods and compare to an individual consumer.*

(c) *Now suppose that there are only two consumer types. Fraction $\lambda \geq 0$ of consumers have preferences $a = 1$ and fraction $1 - \lambda$ have preferences $a = 0$. Are there any values of $\lambda$ for which there is a representative agent whose utility function has a constant elasticity of substitution equal to 1?*

**Solution**

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**(a)**

$$\begin{aligned} \text{Consumer } a: \quad x_1(p,m) &= m/p_1 \;\text{ if } \frac{a}{1-a} \geq \frac{p_1}{p_2}, \text{ 0 otw.} \\ x_2(p,m) &= 0 \;\text{ if } \frac{a}{1-a} \geq \frac{p_1}{p_2}, \; m/p_2 \text{ otw.} \end{aligned}$$

The cutoff consumer is $\frac{a}{1-a} = \frac{p_1}{p_2} \Leftrightarrow a = \frac{p_1}{p_1+p_2}$ so

$$\begin{aligned} X_1(p,m) &= \int_{\frac{p_1}{p_1+p_2}}^{1} \frac{m}{p_1} da = \left(1 - \frac{p_1}{p_1+p_2}\right)\frac{m}{p_1} = \frac{p_2}{p_1}\frac{1}{p_1+p_2}m \quad \text{by symmetry,} \\ X_2(p,m) &= \frac{p_1}{p_2}\frac{1}{p_1+p_2}m \end{aligned}$$

**(b)**

$$\frac{X_1(p,m)}{X_2(p,m)} = \left(\frac{p_2}{p_1}\right)^2$$

so the income expansion path doesn't depend on $m$: $U$ is homothetic.

$$\begin{aligned} \ln(X_1/X_2) &= -2\ln(p_1/p_2) \quad \text{so} \\ \frac{d\ln(X_1/X_2)}{d\ln(p_1/p_2)} &= -2 \end{aligned}$$

so the representative agents elasticity of substitution is constant and equal to 2. Thus aggregate preferences are more convex than any individual's preferences are.

**(c)** In this case, we have

$$X_1(p,m) = \lambda\frac{m}{p_1} \text{ and } X_2(p,m) = (1-\lambda)\frac{m}{p_2}$$

These represent Cobb-Douglas preferences for any $\lambda \in (0, 1)$, so the answer is yes.