Final Exam - Econ 7010

December 7th, 2021

The exam contains 3 questions. **Use a separate bluebook for each question. Be sure to write your name on each, as well as the question number.**

Your answers to all questions must be fully justified to receive credit. Ensure that your final answers are presented in a clear and coherent manner. I should not have to (and will not) "search" for your answers to any question. If you cannot answer a question completely, a well-labeled diagram and/or clear explanation of the intuition behind the solution process can obtain partial credit.

You have 2 hours to complete the exam. The total number of points is ??. Budget your time wisely; you may find later questions easier than earlier ones.

Good luck!

Question 1

Sebastian has preferences over ice cream (x_1), donuts (x_2), and kale (x_3) described by the utility function

$$u(x) = \frac{x_1 x_2}{x_3 + 1}.$$

The price of good i is $p_i > 0$, and he has wealth m.

(a) For each of the following properties, (i) state its definition for a general utility function (ii) determine whether Sebastian's preferences satisfy it. If they do, prove it; if not, provide a counterexample.

- Local non-satiation
- Monotonicity
- Strong monotonicity
- Convexity

(b) Find Sebastian's Marshallian demand functions and indirect utility function.

(c) *Use the duality identities* to find Sebastian's Hicksian demand functions and expenditure function.

For the rest of the problem, let $p_1 = 1$, $p_2 = 1$, $p_3 = 2$, and $m = 10$. Concerned about its citizens eating too much junk food, the government decides to mandate that all households must buy at least one unit of kale (assume that the kale must also be consumed, i.e., it cannot be bought and just thrown away). Because of this, the price of kale has increased to $p_3' = 5$.

(d) Sebastian is incensed by this policy, and threatens to not vote for the current government in the next election unless he is compensated for his loss in welfare due to the kale mandate. Being knowledgable in microeconomic theory, he proposes to the government two possibilities: the equivalent variation, and the compensating variation. Calculate these measures. Presuming the government wants to spend as little money as possible, which will they choose?

(e) Concerned about mandates, the government decides to enact an alternative policy: for every unit of kale that he consumes, they will give Sebastian 1 dollar. Write down Sebastian's optimization problem, including the budget constraint.

(f) Find Sebastian's optimal bundle. Which policy is more effective in increasing kale consumption?

Answer

Part (a)

Local nonsatiation: Preferences are locally non-satiated if, for every $x \in X$, and every $\epsilon > 0$, there exists some y such that $||y - x|| \leq \epsilon$ and $u(y) > u(x)$.

Sebastian's preferences are locally non-satiated. Starting from any bundle $x = (x_1, x_2, x_3)$, consider the bundle $y = (x_1 + \varepsilon, x_2, x_3)$. Notice that $||y - x|| = \varepsilon$, and

$$u(y) = \frac{(x_1 + \varepsilon)x_2}{x_3 + 1} > \frac{x_1 x_2}{x_3 + 1} = u(x).$$

Thus, his preferences are locally non-satiated.

Monotonicity: Preferences are monotonic if for any $x \in X$, if $y >> x$, then $u(y) > u(x)$.

Sebastian's preferences are not monotonic. Consider $x = (x_1, x_2, x_3)$ and $y = (x_1 + \epsilon, x_2 + \epsilon, x_3 + \delta)$. It is clear that $y >> x$. Now, $u(x) = \frac{(x_1)(x_2)}{x_3+\delta+1}$ and $u(y) = \frac{(x_1+\epsilon)(x_2+\epsilon)}{x_3+\delta+1}$. By choosing δ sufficiently large, we will get $u(y) < u(x)$.

Strong monotonicity: Preferences are strongly monotonic if $y \geq x$ and $y \neq x$ implies $u(y) > u(x)$. Sebastian's preferences are not strongly monotonic. The same example as for monotonicity works.

Convexity: Preferences are convex if the utility function is strictly quasiconcave, i.e., if, for all $\alpha \in [0,1]$ and all x, y, we have:

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}.$$

Sebastian's preferences are not convex. Here is a counterexample: let $x = (1,0,1)$ and $y = (1,1,0)$. Then, $\alpha x + (1-\alpha)y = (1, 1-\alpha, \alpha)$. Thus, $u(x) = 1/2$ and $u(y) = 1$, and so for α sufficiently large, we have

$$u(\alpha x + (1-\alpha)y) = \frac{(1-\alpha)}{1+\alpha} < \min\{1, 1/2\} = 1/2,$$

and so quasiconcavity is violated.

Part (b)

It is clear that he will choose $x_3^* = 0$. After this, it is a standard Cobb-Douglas problem, and so the answer is

$$x_1^*(p,m) = \frac{m}{2p_1} \quad x_2^*(p,m) = \frac{m}{2p_2} \quad x_3^*(p,m) = 0.$$

Indirect utility is

$$v(p,m) = \frac{m^2}{4p_1p_2}.$$

Part (c)

First, we find the expenditure function using

$$v(p, e(p,u)) = u.$$

In this case, this becomes

$$\frac{e(p,u)^2}{4p_1p_2} = u.$$

Solving for e, we have

$$e(p,u) = 2\sqrt{up_1p_2}.$$

Then, we can find the Hicksian demands from

$$h(p,u) = x(p, e(p,u)).$$

Substituting, we get

$$h_1(p,u) = \frac{2\sqrt{up_1p_2}}{2p_1} = \sqrt{u\frac{p_2}{p_1}} \qquad h_2(p,u) = \frac{2\sqrt{up_1p_2}}{2p_2} = \sqrt{u\frac{p_1}{p_2}}$$

Part (d)

The compensating variation is

$$CV = m - e(p^1, u^0)$$

where p^1 is the prices after the policy, and u^0 is the old utility. Note here that $e(p^1, u^0)$ has a bit of a different interpretation, because not only does the price of kale rise, but there is also a mandate that 1 unit be purchased, which now enters into the denominator of the utility function. So, it is really the expenditure needed to achieve the same utility as before, with all of the "environmental changes" in place.

Using the indirect utility function we found above, he have that $u^0 = \frac{10^2}{4(1)(1)} = 25$. So, we need to know how much money m' he needs after the policy to reach a utility of 25. He must spend \$5 on kale, which leaves him \$$(m' - 5)$ dollars for donuts and ice cream. We need this to satisfy:

$$\frac{(m'-5)^2}{4(1)(1)} \times \frac{1}{1+1} = 25,$$

where the second term is the "disutility" corresponding to being forced to purchase one unit of kale. Solving, we get that $m' - 5 = 10\sqrt{2}$, or

$$m' - 5 = 10\sqrt{2} + 5 \approx 19.14.$$

Thus, for the CV, we get:

$$CV = 10 - 19.14 \approx -9.14.$$

For the equivalent variation, we want

$$EV = e(p^0, u^1) - m$$

We need to find u^1, which is the utility with just the kale mandate in place (and no subsidy). He will have \$5 to spend on ice cream and donuts, so

$$u^1 = \frac{5^2}{4} \times \frac{1}{1+1} = \frac{25}{8} = 3.125.$$

Then, we need to calculate how much money he would need to reach this utility under the old regime, i.e., we want to solve

$$\frac{(m')^2}{4(1)(1)} = \frac{25}{8}.$$

This gives $m' = 5/\sqrt{2} \approx 3.53$. So,

$$EV = 3.53 - 10 \approx -6.47.$$

In this case, $|EV| < |CV|$, and so the government would choose to implement the EV policy.

Part (e) His problem is

$$\max_{x_1,x_2,x_3} \frac{x_1 x_2}{x_3 + 1}$$

subject to

$$x_1 + x_2 + 2x_3 = 10 + x_3.$$

where the RHS of the budget constraint includes the extra income for purchasing x_3 units of kale.

Part (f)

Notice that the budget constraint on the new problem can be re-written as

$$x_1 + x_2 + x_3 = 10.$$

In other words, this looks just like the original problem, just with a slightly lower "price" for kale. But, it is clear that the solution to this problem will still have $x_3^* = 0$, and all money is spent on ice cream and donuts. The optimal bundle will be

$$x_1^*(p,m) = 5 \quad x_2^*(p,m) = 5 \quad x_3^*(p,m) = 0.$$

Thus, this does nothing for kale consumption. The problem is that the "kale subsidy" is not high enough - i.e., the effective price of kale is still positive, and so Sebastian won't buy any of it. If the subsidy was greater than 2, then the effective price of kale would be negative, and we might be able to get Sebastian to buy some.

Question 2

Paul Samuelson once defined a coward as someone who will not take a bet with 2-to-1 odds, even when you let him choose his side. You are wondering if your friend is a coward, so you decide to offer him such a bet. You will flip a loaded coin, that has a probability p_H of coming up heads and $p_T = 1 - p_H$ of coming up tails (you may assume both of you know the value of p_H, but $p_H \neq 1/2$). You allow your friend to choose heads or tails (this is what we mean by saying he can "choose his side"). After he makes his choice, you will flip the coin. If he wins, you will give him \$$2z$; if he loses, he must give you \$$z$, where $z > 0$.

Assume your friend is an expected utility maximizer with an increasing Bernoulli utility function over money $u(\cdot)$ with $u(0) = 0$. He has an initial wealth of w_0. For parts (a)-(c), you may assume $w_0 = 0$, but z can be any arbitrary positive number.

(a) Is it possible for your friend to be a Samuelson coward? In other words, is rejecting both sides of the bet consistent with expected utility maximization for some increasing utility function over money $u(\cdot)$? Either provide a utility function that is consistent with this behavior, or prove that no such function exists. (Your function can be defined only for certain points.)

(b) Assume now that you know your friend is in fact risk-neutral. Is it possible for him to be a Samuelson coward?

(c) Go back to an arbitrary $u(\cdot)$. "Prove" the following statement: If u is differentiable and concave, then for all z sufficiently small, an expected utility maximizer *cannot* reject both sides of the bet. (A complete formal proof is fine, but may be difficult; a clear graphical argument with an accompanying explanation of the intuition is sufficient for full credit. What you learned from the previous parts may be helpful.)

For the remainder of the question, consider the following utility function for your friend:

$$u(x) = \frac{x^\gamma}{\gamma}.$$

(d) Calculate the coefficient of relative risk aversion for this utility function. For which values of γ will the decision-maker be risk-averse? For which is he risk-loving?

(e) Let $w_0 = 11$, $z = 7$, $\gamma = 1/2$, and $p_H = 2/3$. Find your friend's certainty equivalent of this lottery when he chooses heads. How does it compare to the expected value of the lottery?

(f) Let $c^*(z)$ be the certainty equivalent as a function of z. *Use the implicit function theorem* to find an equation for dc^*/dz. Is it positive or negative? Provide intuition for the result.
(You may restrict to the range $z \leq 11$, so that the utility function is well-defined for all possible lottery outcomes; all other parameters are as in the previous part.)

Answer

(a) Yes. Basically, if $p_H > 1/2$, then obviously betting on heads is better for him; but, if he is sufficiently risk averse, he may not even want to take this bet, and so surely he will not take a bet on tails. The same logic applies to $p_H < 1/2$.

(More formally, for him to reject both bets, we need (normalizing $u(0) = 0$, and so $u(-x) < 0$ and $u(2x) > 0$) $p_H u(2x) + p_T u(-x) < 0$ and $p_H u(-x) + p_T u(2x) < 0$. Rewriting (and remembering $u(-x)$ is negative), this becomes $\frac{u(2x)}{u(-x)} > -\frac{p_T}{p_H}$ and $\frac{u(2x)}{u(-x)} > -\frac{p_H}{p_T}$. Now, just choose a $u(\cdot)$ such that both of these inequalities are satisfied.)

(b) No. Use the risk-neutral utility function $u(x) = x$. Then, the two inequalities that need to be satisfied for him to reject both bets are $p_H \times (2x) + p_T \times (-x) < 0$ and $p_H \times (-x) + p_T \times (2x) < 0$. The first simplifies to $p_H/p_T < 1/2$, while the second is $p_H/p_T > 2$. These obviously cannot both hold.

(c) The intuition is that a decision maker can be a "coward" if he is sufficiently risk averse, but a risk-neutral decision maker must accept one of the two bets (since one of them must have a positive expected value). However, even a risk-averse decision maker is "approximately" risk neutral over small gambles (assuming u is differentiable). Thus, as z becomes sufficiently small, he becomes approximately risk neutral, and must take one of the bets. Graphically, if z is small enough, then over the range $[-z, 2z]$, $u(x)$ is approximately linear.

(d) The coefficient of relative risk aversion is $\rho(x) = -xu''(x)/u'(x)$. Direct calculations show that $\rho(x) = 1 - \gamma$. Thus, the decision-maker is risk averse for $\gamma \leq 1$, risk-loving for $\gamma \geq 1$, and risk neutral for $\gamma = 1$.

(e) The certainty equivalent c is defined by

$$p_H u(w_0 + 2z) + (1 - p_H)u(w_0 - z) = u(w_0 + c).$$

Plugging in the numbers and doing the calculations gives $c = 5$. The expected value of the lottery is $2/3 \times 25 + 1/3 \times 4 = 54/3 = 18$. The certainty equivalent is much lower, as expected for a risk averse agent.

(f) We can implicitly define $c^*(\gamma)$ using the following equation:

$$\frac{2}{3}\frac{\sqrt{11+2z}}{1/2} + \frac{1}{3}\frac{\sqrt{11-z}}{1/2} = \frac{\sqrt{11+c(z)}}{1/2},$$

or, to simplify slightly:

$$2\sqrt{11+2z} + \sqrt{11-z} = 3\sqrt{11+c(z)}$$

It may be possible to solve explicitly for $c(z)$ here, but we are told to use the implicit function theorem. So, implicitly differentiating, we get

$$2(11+2z)^{-1/2} - \frac{1}{2}(11-z)^{-1/2} = \frac{3}{2}(11+c(z))^{-1/2}\frac{dc}{dz}.$$

Solving for dc/dz:

$$\frac{dc}{dz} = \frac{\sqrt{11+c(z)}}{3}\left(\frac{4}{\sqrt{11+2z}} - \frac{1}{\sqrt{11-z}}\right).$$

The first term is positive, and so the sign is determined by the term in parentheses. Thus,

$$\frac{dc}{dz} \geq 0 \iff z \leq \frac{55}{6} \approx 9.16.$$

Notice that for low z, the certainty equivalent is increasing, which means that as z increases, the DM is more willing to take the lottery (i.e., we have to pay him more for sure to get him to not take the lottery). But, for high z, $dc/dz < 0$, so the certainty equivalent *decreases,* so that for very high z, the DM is *less willing* to take the lottery.

What is going on here? Well, notice that as z goes up, the upside to the lottery, $+2z$, gets better, but the downside, $-z$, gets worse. Further, $u(x) = \sqrt{x}$ has decreasing marginal utility, which means that for high z, a little bit more doesn't do that much when he wins, while at the same time, when he loses, it is much worse. When z is large, this latter effect dominates, and the agent is less willing to take lottery (i.e., will accept a lower certainty equivalent).

Question 3

Consider a choice set with three alternatives, $\mathcal{X} = \{x, y, z\}$, and a choice function $C(\cdot)$ defined as follows:

$$C(\{x\}) = x \qquad C(\{y\}) = y \qquad C(\{z\}) = z$$

$$C(\{x,y\}) = y \quad C(\{y,z\}) = y \quad C(\{x,z\}) = x$$

$$C(\{x,y,z\}) = x$$

(a) Does this choice structure satisfy WARP?

Consider the following condition on choice structures, known as *Sen's* α:

Let $Y, Z \subseteq \mathcal{X}$ *be two subsets of* $\mathcal{X}$ *such that* $Y \subseteq Z$. *If* $x \in C(Z)$ *and* $x \in Y$, *then* $x \in C(Y)$.

(b) Translate this condition into words. (It may be helpful, though not necessary, to draw a Venn diagram.)

(c) Does the choice structure above satisfy Sen's α?

(d) Sen's α sometimes goes by the name of *independence of irrelevant alternatives*. Say that $x \in C(Z)$, and let $Y = Z \setminus \{w\}$ for some $w \neq x$. Using this notation, explain briefly why this alternative nomenclature makes sense.

(e) Prove formally that WARP implies Sen's α in general; that is, given an arbitrary choice function $C(\cdot)$ show that:

$C(\cdot)$ *satisfies WARP* $\implies$ $C(\cdot)$ *satisfies Sen's* α.

(Hint: To show that $P \implies Q$, it is equivalent to show that "not Q" $\implies$ "not P"; this is called proof by contrapositive.)

(f) Finally, consider the following condition, called Sen's β:

If $x, y \in Y$, $Y \subseteq Z$, $x, y \in C(Y)$, *and* $y \in C(Z)$, *then* $x \in C(Z)$.

Does the choice structure given above satisfy Sen's β? (Hint: Note that for any offer set $Y \subseteq \mathcal{X}$, $|C(Y)| = 1$.)

Answer

Part (a)

The choice structure does not satisfy WARP. Here is a violation: $C(\{x,y,z\}) = x$, and $C(\{x,y\}) = y$. In the first one, x was chosen when y was available, but the second is an example where x and y are available, x is chosen, and y is not. (There are other ways to show WARP is violated as well.)

Part (b)

Sen's α says if x is chosen from a larger set of alternatives, then it must be chosen from any smaller set that contains it.

Part (c)

Again, no. Let $Z = \{x,y,z\}$ and $Y = \{x,y\}$. Then, $Y \subseteq Z$ and $x \in C(Z)$, but we do not have $x \in C(Y)$.

Part (d)

Say x is chosen from some set Z, and now we remove some w to get $Y = Z \setminus \{w\}$. Sen's α says x should still be chosen, i.e., w is an alternative that was irrelevant, and removing it from the offer set should not affect whether or not the decision maker chooses x if it is still available.

Part (e)

Assume there exists $x \in Y$, $Y \subseteq Z$, $x \in C(Z)$, but $x \notin C(Y)$. Since $x \notin C(Y)$, there must be some other $y \in C(Y)$. Since $Y \subseteq Z$, it is clear that $y \in Z$. So, we have $x, y \in Y$ and $x, y \in Z$. So, x was chosen when y was available (from Z), but there is another choice set—Y— where x and y are available, y is chosen, and x is not, which is a violation of WARP.

Part (f)

This holds essentially vacuously. Since each $C(Y)$ is a single element, then $x, y \in C(Y)$ implies that $x = y$. Then, it is trivial to see that $y \in C(Z)$ implies $x \in C(Z)$ (because $x = y$).