10 Oct.

Today: (1) Contraction Mapping Thm.
(2) Blackwell's Thm

Contraction:

Def: Let (S, d) be a metric space & $T: S \to S$.
An operator T is a contraction if
$$d(Tx, Ty) \leq \beta\, d(x,y) \quad \forall\ x, y \in S,\ \beta \in (0,1)$$
β is called modulus of contraction.

Lemma: (S, d) is a metric space & $T: S \to S$ is a contraction, then T is continuous.

Proof: SLP.

Contraction Mapping Theorem:-

Let (S, d) be a complete metric space, and T be an operator $T: S \to S$ which is a contraction with modulus $\beta \in (0, 1)$.

Then 1) There is unique fixed point of operator T

2) $d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$
where v^* is a fixed pt. of T.

Proof :-

1) $\{v_n\}_{n=1}^{\infty}$, where $v_n = T^n v_0$ converges to v^*

2) v^* is indeed a fixed point of T

3) It is unique.

$$d(v_n, v_{n-1}) = d(Tv_{n-1}, Tv_{n-2})$$

$$\leq \beta . d(v_{n-1}, v_{n-2}) \quad (\text{Bc. } T \text{ is a contraction.})$$

$$= \beta d(Tv_{n-2}, Tv_{n-3})$$

$$\leq \beta^2 d(v_{n-2}, v_{n-3})$$

$$\vdots$$

$$\leq \beta^n d(v_0, v_1) \quad (*)$$

Let's show that $\{v_n\}_{n=0}^{\infty}$ is Cauchy.

Take n, m (WLOG $m > n$);

$$d(v_n, v_m) \leq d(v_m, v_{m-1}) + d(v_{m-1}, v_n) \quad (\Delta \text{ inequality})$$

$$\leq d(v_m, v_{m-1}) + d(v_{m-1}, v_{m-2}) + \dots + d(v_{n+1}, v_n)$$

$$\leq \beta^m d(v_0, v_1) + \beta^{m-1} d(v_0, v_1) + \dots + \beta^{n+1} d(v_0, v_1)$$

$$\leq \beta^{n+1} d(v_0, v_1) (1 + \beta + \beta^2 + \dots + \beta^{m-n})$$

$$\leq \frac{\beta^{n+1}}{1-\beta} d(v_0, v_1)$$

$\Rightarrow \{v_n\}$ converges to v^*, where $v^* = \lim_{n\to\infty} T^n v_0$.

2) Need to show that $v^* = \lim_{n\to\infty} T^n v_0$ is a fixed pt.

$$Tv^* = v^*$$

$$Tv^* = T\left(\lim_{n\to\infty} T^n v_0\right) = \lim_{n\to\infty} \left(T(T^n v_0)\right) = \lim_{n\to\infty} v_{n+1} = v^*$$

3) Need to show that v^* is unique

suppose not $\Rightarrow \exists\ \tilde{v} \neq v^*$, s.t. $T\tilde{v} = \tilde{v}$

$$0 < \varepsilon = d(\tilde{v}, v^*) = d(Tv^*, T\tilde{v}) \leq$$

$$\leq \beta d(v^*, \tilde{v}) = \beta \varepsilon$$

Contradiction : v^* is a unique fixed point.

Finally, show $d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$

Proceed by math induction:

a) $n = 0 \Rightarrow d(v_0, v^*) \leq d(v_0, v^*)$ ✓

b) Assume it works for k.

$$d(T^k v_0, v^*) \leq \beta^k d(v_0, v^*)$$

c) Show it works for k+1 :

$$d(T^{k+1} v_0, v^*) = d(T(T^k v_0), Tv^*)$$

$$\leq \beta \cdot d(T^k v_0, v^*) \quad (T \text{ is contraction})$$

$$\leq \beta(\beta^k d(v_0, v^*))$$

$$= \beta^{k+1} d(v_0, v_1)$$

$\blacksquare$

Blackwell's Theorem :

Let $X \subseteq \mathbb{R}^l$, $B(x)$ is a space of bounded functions, $f: X \to \mathbb{R}$,

d is a supnorm.

Let $T: B \to B$, satisfying :

1) Monotonocity

$f, g \in B(x)$, s.t. $f \leq g$:

$\Rightarrow (Tf)(x) \leq (Tg)(x)$

2) Discounting : Let $(f+a)(x)$ denote $f(x)+a$, where a is a positive constant.
$\exists$ exists $\beta \in (0,1)$ s.t. $\forall f \in B(X)$
(bounded f^n) ;

$$T(f+a)(x) \leq Tf + \beta a$$

If T satisfies monotonocity & discounting
$\Rightarrow$ T is a contraction.

⊛ Blackwell's Thm gives an easy way to identify contraction but because of the assumptions, we lose some generality.

Proof :-

Fix $x \in X$
$f, g \in B(X)$

$$f(x) - g(x) \leq \sup_{y \in X} |f(y) - g(y)| = d(f,g)$$

$$f - g \leq d(f,g)$$

$$\Rightarrow f \leq g + d(f,g)$$

$$Tf \leq T(g + d(f,g)) \quad \text{(Monotonocity)}$$

$$\leq Tg + \beta d(f, g) \qquad \text{(discounting)}$$

$$Tf - Tg \leq \beta d(f, g)$$

Symmetrically, you can show:

$$Tg - Tf \leq \beta d(f, g)$$

$$\Rightarrow \underbrace{\sup_{x \in X} \left| Tf(x) - Tg(x) \right|}_{d(Tf,\, Tg)} \leq \beta d(f, g)$$

$\Rightarrow$ T is a contraction.

Neoclassical Growth Model.

output net of investment ↳ what is consumed

$$Tv(k) = \max_{0 \le k' \le f(k)} \left[u(\overbrace{f(k)-k'}) + \beta v(k') \right]$$

1) If $u(\cdot)$ is bounded $\Rightarrow T: B[0,\infty) \to B[0,\infty)$
(only interested in $\mathbb{R}^+$ because k cannot be negative).

2) Monotonicity

$v, w \in B[0,\infty)$, s.t. $v \le w$

Let $g_v(k)$ a policy function / is the argmax for $v(k)$ (the sol^n of $Tv(k)$)

$$\begin{aligned} T_v(k) &= u(f(k) - g_v(k)) + \beta v(g_v(k)) \\ &\le u(f(k) - g_v(k)) + \beta w(g_v(k)) \\ &\le \max_{0 \le k' \le f(k)} \left[u(f(k) - k') + \beta w(k') \right] \\ &= Tw(k) \end{aligned}$$

3) Discounting

$$T(v+a)(k) = \max_{0 \le k' \le f(k)} \left[u(f(k)-k') + \beta[v(k')+a] \right]$$

$$= \max_{0 \le k' \le f(k)} \left[u(f(k)-k') + \beta v(k') \right] + \beta a$$

$$= Tv(k) + \beta a$$

Discounting (✓)