HW5, Problem 1

(1.1) #1, part 1
See Discussion 11. I highly recommend doing this using approach 1 because if you get asked a question on the core exam, that's what Vladimir expects to see. The other way is fine but on those slides is just a sketch.

(1.2) #1, part 2
A TDCE, given a fiscal policy $\{g_t, \tau_t^k, \tau_t^h\}$, is a collection of HH allocations $\{c_t, k_{t+1}^s, h_{t+1}^s\}$, firm allocations $\{k_t^d, z_t^d\}$, and prices $\{p_t, r_t, w_t\}$ s.t

- Given prices, HH allocations $\{c_t, k_{t+1}^s, h_{t+1}^s\}$ solve

$$\max_{\{c_t, k_{t+1}, h_{t+1}\}} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$\text{s.t } \sum p_t(c_t + x_t) = \sum \left\{ r_t k_t (1-\tau_t^k) + w_t h_t (1-\tau_t^h) \right\} \ \underline{\textbf{and}} \ x_t = x_t^k + x_t^h = k_{t+1} - (1-\delta_k)k_t + h_{t+1} - (1-\delta_h)h_t$$

- Given prices, firm allocations $\{k_t^d, z_t^d\}$ maximize profits at each t

$$\max_{k_t, z_t} p_t F(k_t, n_t) - r_t k_t - w_t z_t$$

- Markets clear at each t

$$F(K_t, Z_t) = y_t = c_t + x_t + g_t \ \underline{\textbf{and}} \ K_t = k_t^s = k_t^d \ \underline{\textbf{and}} \ Z_t = z_t^d = h_t^s$$

- Government balances its lifetime BC

$$\sum p_t g_t = \sum \tau_t^h w_t h_t + \tau_t^k r_t k_t$$

(1.3) # 1, part 3

$$\textbf{FOCs: } \beta^t u'(c_t) = \lambda p_t \ \underline{\textbf{and}} \ r_{t+1}(1-\tau_{t+1}^k) - p_{t+1}(1-\delta_k) = p_t \ \underline{\textbf{and}} \ w_t(1-\tau_{t+1}^h) - p_{t+1}(1-\delta_h) = p_t$$

Normalize $p_0 = 1$ so $p_t = \beta^t \frac{u'(c_t)}{u'(c_0)}$. As usual, notice that $\sum k_{t+1}[r_{t+1}(1-\tau_{t+1}^k) - p_{t+1}(1-\delta_k) - p_t] = 0$ (similar for h_t). From firm FOC, $r_t k_t = \alpha y_t$ and $w_t h_t = (1-\alpha) y_t$ (Cobb-Douglas). So the IC is

$$\sum \beta^t u'(c_t) = u'(c_0)^{-1} \left\{ y_0 \left[\alpha(1-\tau_0^k) + (1-\alpha)(1-\tau_0^h) \right] + (1-\delta_k)k_0 + (1-\delta_h)h_0 \right\} = A_0$$

Note $u'(c_t)c_t = u(c_t)$. Define $W(t, \Phi) = (1+\Phi)u(c_t)$. The Ramsey planner picks $\{c_t, k_{t+1}, h_{t+1}\}$ to solve

$$\max_{\{c_t, k_{t+1}, h_{t+1}\}} \sum \beta^t u(c_t) \ \text{ s.t } \ F(k_t, h_t) = c_t + x_t + g_t \ \underline{\textbf{and}} \ \text{IC}$$

(1.4) #1, part 4
The capital and human capital Eulers in TDCE must be equal (other side of equation $= (c_{t+1}/c_t)^\sigma$, see **(3.1)**)

$$\implies 1 - \delta_k + (1-\tau_{t+1}^k)F_k(t+1) = 1 - \delta_h + (1-\tau_{t+1}^h)F_h(t+1) \implies \frac{1-\tau_{t+1}^k}{1-\tau_{t+1}^h} = \frac{\alpha k}{(1-\alpha)h}$$

Taking FOCs w.r.t k_{t+1}, h_{t+1} in the Ramsey problem

$$\mu_{t+1}\left[F_k(t+1) + (1-\delta_k)\right] = \mu_t \ \underline{\textbf{and}} \ \mu_{t+1}\left[F_h(t+1) + (1-\delta_h)\right] = \mu_t \implies F_k(t+1) = F_h(t+1) \implies \frac{\alpha k}{(1-\alpha)h} = 1$$

Combining results, $\frac{\tau_t^k}{\tau_t^h} = 1$

(1.5) #1, part 5
We can infer from the last part both taxes converge to the same thing. We haven't fundamentally changed the problem s.t the Chamley-Judd result would not hold. We know both of these equations must be satisfied

$$\text{(TDCE): } 1 = \beta[1 - \delta F_k(t+1)] \ \underline{\textbf{and}} \ \ \text{(RP): } 1 = \beta[1 - (1 - \tau^k_{t+1})F_k(t+1)]$$

Based on the functional form of u and the deterministic environment, we can infer that the economy converges to a steady state. To match those two expressions, we must have $\lim_{t\to\infty} \tau^h_t = \lim_{t\to\infty} \tau^k_t = 0$

HW5,Problem 2

(2.1) #2, part 1
Since labor is inelastic, the policy functions can be found in your notes using Guess and Verify. See **(2.4)** and **(4.1)** for a way to do this quicker w/ Envelope Theorem. We can infer the economy converges to a steady state (given the form of u, F). $k' = \alpha\beta A k^\alpha \implies k^* = (\alpha\beta A)^{\frac{1}{1-\alpha}} \implies y^* = \lim_{t\to\infty} y_t = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} A^{\frac{1+\alpha}{1-\alpha}}$

(2.2) #2, part 2
A TDCE, given a fiscal policy $\{g_t, \tau_t\}$, is a collection of HH allocations $\{c_t, k^s_{t+1}\}$, firm allocations $\{k^d_t, n^d_t\}$, and prices $\{p_t, r_t, w_t\}$ s.t

- Given prices, HH allocations $\{c_t, k^s_{t+1}\}$ solve

$$\max_{\{c_t, k_{t+1}\}} \beta^t \ln(c_t) \ \text{ s.t } \sum p_t(c_t + k_{t+1}) = \sum (1-\tau_t)\{r_t k_t + w_t\}$$

- Given prices, firm allocations $\{k^d_t, n^d_t\}$ maximize profits at each t

$$\max_{k_t, n_t} \ p_t F(k_t, n_t) - r_t k_t - w_t n_t$$

- Markets clear at each t

$$F(K_t, 1) = y_t = c_t + x_t + g_t \ \underline{\textbf{and}} \ K_t = k^s_t = k^d_t \ \underline{\textbf{and}} \ n^d_t = 1$$

- Government balances its per period BC at each t

$$p_t g_t = \tau_t(w_t n_t + r_t k_t)$$

(2.3) #2, part 3
Let $f(k_t) = F(k_{t+1}, 1)$. SPP: use new ARC (same as your notes) by $\frac{w_t}{p_t} = F_k = \alpha\frac{f}{k}$ and $\frac{r_t}{p_t} = F_n = (1-\alpha)\frac{f}{n}$.

$$\text{(ARC): } c_t = f(k_t) - k_{t+1} - g_t \implies p_t c_t = p_t(1-\tau_t)f(k_t) + \tau_t[p_t f(k_t) - w_t n_t - r_t k_t] - p_t k_{t+1} \implies c_t = (1-\tau_t)f(k_t) - k_{t+1}$$

$$\text{(SPP): } \frac{c_{t+1}}{c_t} = \beta\alpha(1-\tau_{t+1})A k^{\alpha-1}_{t+1} \ \underline{\textbf{and}} \ \text{(TDCE): } \frac{\beta^t}{\lambda c_t} = p_t = \beta(1-\tau_{t+1})r_{t+1} \implies \frac{c_{t+1}}{c_t} = \beta\alpha(1-\tau_{t+1})A k^{\alpha-1}_{t+1}$$

To combine TDCE FOCs, remember the multiply divide trick: $\frac{r_{t+1}}{p_t} = \frac{r_{t+1}}{p_{t+1}}\frac{p_{t+1}}{p_t}$; it comes up a lot (like **(3.1)**)

(2.4-5) #2, part 4 and 5
We need the policy function for capital. Let $g(x) = c$, where $x = (\tau, k, k')$, solve the new ARC from part 3

$$V(k) = \max_k \{\ln(g(x)) + \beta V(k')\}$$

$$\textbf{FOC: } \frac{1}{c} = \beta V_k(k')$$

$$\textbf{Envelope: } \ V_k(k) = \frac{(1-\tau)}{c}\alpha A k^{\alpha-1} \overset{\textbf{scroll}}{\implies} V_k(k') = \frac{(1-\tau')}{c'}\alpha A(k')^{\alpha-1}$$

$$\textbf{Euler: } \frac{c'}{c} = \beta(1-\tau')\alpha A(k')^{\alpha-1}$$

Economy still converges to a steady state. From the Euler, $k^*_{\text{FP}} = (\beta(1-\tau^*)\alpha A)^{\frac{1}{\alpha-1}}$. If $\tau^* = 0$, $k^*_{\text{FP}} = k^*$, so $y^*_{\text{FP}} = y^*$ (FP=fiscal policy). If not, $k^*_{\text{FP}} < k^*$ and $y^*_{\text{FP}} < y^*$ (f is increasing in k)

HW5, Problem 3

(3.1) #3, part 1 (ARC derived in previous part)

$$\text{(SPP):}\ \max_{\{k_{t+1}\}}\left\{\mathbb{E}_0\left[\sum\beta^t\frac{G(x_t)^{1-\sigma}}{1-\sigma}\right]\right\}\quad\text{where } G(x_t)=c_t(s^t)=(1-\tau_t(s^t))Ak_t(s^{t-1})-k_{t+1}(s^t)$$

$$\textbf{FOC/Euler:}\ c_t(s^t)^{-\sigma}=\beta A\mathbb{E}_t[(1-\tau_{t+1})c_{t+1}^{-\sigma}]$$

$$\text{(TDCE–HH):}\ \max_{\{c_t,k_{t+1}\}}\left\{\mathbb{E}_0\left[\sum\beta^t\frac{c_t(s^t)^{1-\sigma}}{1-\sigma}\right]\right\}\ \text{s.t}\ \sum_{t=0}^{\infty}\sum_{s^t}p_t(s^t)(c_t(s^t)+x_t(s^t))\le\sum_{t=0}^{\infty}\sum_{s^t}r_t(s^t)(1-\tau_t(s^t))k_t(s^{t-1})$$

$$\textbf{FOCs:}\ \beta^t c_t(s^t)^{-\sigma}=\lambda p_t(s^t)\ \underline{\textbf{and}}\ p_t(s^t)=\mathbb{E}_t[r_{t+1}(1-\tau_{t+1})]\implies c_t(s^t)^{-\sigma}=\beta A\mathbb{E}_t[(1-\tau_{t+1})c_{t+1}^{-\sigma}]$$

(3.2) #3, part 2
Here's a longer intuitive explanation – the first part of this question is straight from your notes.
We showed the this utility has a h.d 1-σ representation earlier. If the value function is h.d k (for any k), then preferences are homothetic. In the context of a problem with a stochastic process, this matters because it means that we know that the optimal solution at a given t will just be a properly scaled version of a general form (yields concave objective with linear constraint). Intuitively, think about the case of stochastic technology $z_t k_t^{\alpha} n_t^{1-\alpha}$. If there is a high realization of z_t you are richer because there's more stuff. Homothetic preferences tell you that your optimal decision is just to scale everything based on the actual realization of output; in other words your decision rule is not time varying. Let's say I tell you at output=1, the optimal decision was $c=.4$ and $k'=.6$ (because of full depreciation $c+k'=y$). This means you know the policy functions exactly – you will always split everything this way. For example, if $y=2$, then $c=.8$ and $k'=1.2$. Thus, your decision rule in general form is $c=(1-\varphi)(1-\tau)y$ and $k'=\varphi(1-\tau)y$.
Plugging this into the expression that was asked for, by iid $\mathbb{E}_t[1-\tau_{t+1}]=1-\mathbb{E}[\tau]$, so

$$\mathbb{E}_t\left[\frac{c_{t+1}}{c_t}\right]=\mathbb{E}_t\left[\frac{(1-\varphi)A(1-\tau_{t+1})\varphi(1-\tau_t)y}{(1-\varphi)(1-\tau_t)y}\right]=\varphi(1-\varphi)A(1-\mathbb{E}[\tau])$$

HW5, Problem 4

(scroll to the end for discussion on all three parts)
(4.1) #4, part 1
Let $g(x)=c=y-k'=F(k,n)-k'$, where $x=(k,k',n,z)$. Then

$$V(k,z)=\max_{k',n}\left\{\ln(g(x))-.5n^2+\beta\mathbb{E}[V(k',z')|z]\right\}$$

$$\textbf{FOCs:}\quad\frac{1}{c}=\beta\mathbb{E}[V_k(k',z')]\ \underline{\textbf{and}}\ \frac{(1-\alpha)y}{nc}=n\implies\boxed{n^2=\frac{(1-\alpha)y}{c}}\ (\textbf{note}\text{: not closed form})$$

$$\textbf{Envelope:}\quad V_k(k,z)=\frac{\alpha y}{kc}\overset{\textbf{scroll}}{\implies}V_k(k',z')=\frac{\alpha y'}{k'c'}=\frac{F_k(k',n')}{c'}$$

$$\textbf{Euler:}\quad 1=\beta\mathbb{E}\left[\frac{c}{c'}F_k(k',n')\right]$$

Because we have linear ARC, guess that $c=\phi y$ and $k'=(1-\phi)y$. Plug this into the Euler

$$1=\beta\mathbb{E}\left[\frac{y}{y'}\frac{\alpha y'}{(1-\phi)y}\right]\implies\boxed{\phi=1-\alpha\beta}$$

So our policy function for hours is $\boxed{n=\sqrt{\frac{1-\alpha}{\phi}}}$ and policy functions for c,k' are now well-defined (y is a function of state variables and parameters). In principle, our guess could include that the fraction is a time-varying ϕ_t, but by iid this is not possible because you would get that ϕ_{t+1} depends on ϕ_t from the Euler (can't hold under all possible realizations of state variables unless ϕ_t is constant).

(4.2) #4, part 2
Using the same definitions

$$V(k,z) = \max_{k',n} \left\{ \ln\left(g(x) - .5n^2\right) + \beta\mathbb{E}[V(k',z')|z] \right\}$$

$$\textbf{FOCs:} \quad \frac{1}{c - .5n^2} = \beta\mathbb{E}[V_k(k',z')] \ \underline{\textbf{and}} \ \frac{(1-\alpha)\frac{y}{n} - n}{c - .5n^2} = 0 \implies \boxed{n^2 = (1-\alpha)y} \ (\textbf{note}\text{: not closed form})$$

$$\textbf{Envelope:} \quad V_k(k,z) = \frac{\alpha y}{k(c - .5n^2)} \overset{\textbf{scroll}}{\implies} V_k(k',z') = \frac{\alpha y'}{k'(c' - .5n'^2)} = \frac{F_k(k',n')}{c' - .5n'^2}$$

$$\textbf{Euler:} \quad 1 = \beta\mathbb{E}\left[\frac{c - .5n^2}{c' - .5n'^2} F_k(k',n')\right]$$

Use the same guess before (turns out to be the same but we'll use ψ)

$$1 = \beta\mathbb{E}\left[\frac{(\psi - .5(1-\alpha))y}{(\psi - .5(1-\alpha))y'} \frac{y'}{(1-\psi)y}\right] \implies \boxed{\psi = 1 - \alpha\beta}$$

So our policy function for hours is $\boxed{n = [(1-\alpha)zk^\alpha]^{\frac{1}{1+\alpha}}}$ and the others (for k', c) are well-defined

(4.3) #4, part 3
In part 2, the labor supply is not constant. Specifically, it fluctuates based on the amount of resources in the economy. Agents work more when there is more output, or equivalently when labor is more useful (both occurr when the stochastic tech component has a higher realization). Intuitively, this is because of the lack of separability in the utility function: the effect on leisure can no longer be separated from the effect on wealth (when we have a higher z, we feel richer and don't want to work – this is called an income effect). Because we work more in the part 2 economy, our output is higher $\implies$ we also consume more.

Comments: If you do lots of practice on these kinds of problems, they will become very quick. To highlight a couple of things, because technology is i.i.d, you technically don't need to condition on it in the Bellman equation. I do that there just to give you an idea of what it might look like if it were Markov. Also, remember the Cobb-Douglas trick $F_k = y/k$ and $F_n = y/n$. It will come in handy often. Finally, notice that the requirement for a full answer is everything is written in terms of state variables and parameters. Don't waste time writing more than you need to (similarly, if you substitute out consumption, I recommend doing like I did - define a function to sub out c, and then after you take derivatives, replace the function with c.