ECON 7710 TA Session

Week 7: 2023 Midterm Review

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Outline

Question 1.a

Two fair dice are rolled n times.

a. Find the probability that the maximum of the sum of outcomes of two dice over n rolls is **at least** 6.

- Setup: Define events:
  - A as maximum of the sum of outcomes of two dice over n rolls is at least 6.
  - A' as maximum of the sum of outcomes of two dice over n rolls is strictly less than 6.
    Then $P(A) = 1 - P(A')$.
  - A'_i as in ith time of rolling two dice, the sum of outcomes is strictly less than 6.
  - Maximum strictly less than 6 means every draw is strictly less than 6. And each draw are i.i.d.
  - Then we can focus on the probability in one draw.

Question 1.a

- In each single time of rolling two dice, there are 36 possible outcomes.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- The sum takes on the possible values $[2, 12]$. Summary statistics is listed below:

Sum	2	3	4	5	6	7	8	9	10	11	12
# of Cases	1	2	3	4	5	6	5	4	3	2	1

- So in each single time of rolling two dice, the probability that the sum of outcomes is strictly less than 6 is $P(A_i') = \frac{1+2+3+4}{36} = \frac{5}{18}$
- i.i.d. tells us $P(A') = \prod_{i=1}^{n} P(A_i') = (\frac{5}{18})^n$ and $P(A) = 1 - P(A') = 1 - (\frac{5}{18})^n$

Question 1.b

[b] Find the probability that one of the dice (suppose, that the two dice have different colors) has a higher outcome than the other one over each n rolls.

- WLOG, one red(R) dice one blue(B) dice.
- Then in each single draw of (R, B), we can listed the case when R has higher outcome than B in red and the case when B has higher outcome than R in blue as below:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- In each single draw we have:
  - $P(R_i > B_i) = P(B_i > R_i) = \frac{15}{36} = \frac{5}{12}$
- Then over each n time:
  - $P(R > B) = (\frac{5}{12})^n$ and $P(B > R) = (\frac{5}{12})^n$
- Either works. So $P = P(R > B) + P(B > R) = 2 * (\frac{5}{12})^n$
- $(\frac{30}{36})^n$ is wrong because over n times who is larger is fixed in our case.

Question 2.a

Random variables X and Y are independent Bernoulli random variables with parameter $\frac{1}{2}$(i.e.,
$P(X=1)=P(Y=1)=P(X=0)=P(Y=0)=\frac{1}{2}$). Random variable Z is constructed such that:

$$Z = \begin{cases} 1, & \text{if } X = Y, \\ 0, & \text{otherwise.} \end{cases}$$

a. Prove or disprove if pairs $\underbrace{(Y, Z)}_{①}$ and $\underbrace{(X, Z)}_{②}$ are independent.

It asks you to prove if pair (Y, Z) is independent **then** pair (X, Z) is independent. Not if these two pairs are independent. Sorry, but I took points off for a true but not expected answer.
Ask during the test if anything unclear. Adding your own assumptions is the last option.

Question 2.a

We know $X \perp Y$ and $X \sim B(\frac{1}{2})$, $Y \sim B(\frac{1}{2})$. Then we know:

X	Y	Prob	Z
0	0	$\frac{1}{4}$	1
1	0	$\frac{1}{4}$	0
0	1	$\frac{1}{4}$	0
1	1	$\frac{1}{4}$	1

- For pair (Y, Z), we list all the possible combinations
  - $P((Y=0, Z=0)) = \frac{1}{4}$ and $P(Y=0) = \frac{1}{2}$, $P(Z=0) = \frac{1}{2}$ $\Rightarrow P((Y=0, Z=0)) = P(Y=0) * P(Z=0)$.
  - $P((Y=1, Z=0)) = \frac{1}{4}$ and $P(Y=1) = \frac{1}{2}$, $P(Z=0) = \frac{1}{2}$ $\Rightarrow P((Y=1, Z=0)) = P(Y=1) * P(Z=0)$.
  - $P((Y=0, Z=1)) = \frac{1}{4}$ and $P(Y=0) = \frac{1}{2}$, $P(Z=1) = \frac{1}{2}$ $\Rightarrow P((Y=0, Z=1)) = P(Y=0) * P(Z=1)$.
  - $P((Y=1, Z=1)) = \frac{1}{4}$ and $P(Y=1) = \frac{1}{2}$, $P(Z=1) = \frac{1}{2}$ $\Rightarrow P((Y=1, Z=1)) = P(Y=1) * P(Z=1)$.

  So $P((Y, Z)) = P(Y)P(Z)$ and this pair is independent. Same for (X, Z) by symmetry. [Save your time]

Question 2.b

The joint triple (X, Y, Z) has the following probability distribution:

$$(X, Y, Z) = \begin{cases} (0, 0, 1) & \text{with probability } \frac{1}{4} \\ (1, 0, 0) & \text{with probability } \frac{1}{4} \\ (0, 1, 0) & \text{with probability } \frac{1}{4} \\ (1, 1, 1) & \text{with probability } \frac{1}{4} \end{cases}$$

We can derive the marginal probabilities of X, Y, Z:

$$f_X = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases} \qquad f_Y = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases} \qquad f_Z = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$$

Then if we plug in $(x = 0, y = 0, z = 1)$
$f_{X,Y,Z}(0, 0, 1) = \frac{1}{4} \neq f_X(0) * f_Y(0) * f_Z(1) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$. So (X, Y, Z) are not jointly independent.
Show me the numbers. (X, Y, Z) only have 4 cases not $2^3 = 8$.

Question 3

We know random variable X is symmetrically distributed about zero. ($i.e.$, $F_X(x) = F_{-X}(x)$) and A is Borel set symmetric about zero. Define random variable Y as:

$$Y = \begin{cases} X, & \text{if } X \in A, \\ -X, & \text{otherwise} \end{cases}$$

We want to derive the distribution of random variable Y from the distribution of random variable X. Your final result should be $F_X(\cdot)$
We know :

- If $X \in A$, $Y = X$, so $F_Y(y) = F_X(y)$.
- If $X \notin A$, then $Y = -X$, then $F_Y(y) = F_{-X}(y) = F_X(y)$

Therefore, we know the $F_Y = F_X$, the distribution of random variable Y and X are the same.
Again little y or x is just a number while X, Y, -X are random variables.

Overview

- I have finished the grading and will send your exams to Denis for review. You may get it back next week and happy to schedule an individual meeting.
- You guys are amazing. Everyone is higher than my midterm grade... We have:
  - Mean: 88.5
  - Median: 89
- Unfortunately, according to the current arrangement, final has nothing to do with Jerry.

  (I guess I will still be the proctor but not grader for sure)
- You will vote between taking final in
  a. 75 mins at last day of class, early Dec. Normally 3 Questions. Short exam where you can finish quickly and enjoy your winter break early with time constraint still be a problem.
  b. 3 hours at scheduled date, mid Dec. Normally 10 Questions. Long exam where you can see more beautiful econometrics questions and have more time to think deep through.

It's your choice ;)