ECON 7710 TA Session

Week 3

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Outline

Office Hour Next Week & Seminar Sources

- Next week, I need to move my Office Hour to **Thursday(Sep 14) 2:00 pm to 3:00 pm @ Monroe B-16**
- Why? Because I want to attend the workshop by Melanie Wallskog **Wednesday(Sep 13) 3:30 pm to 5:00 pm @ Monroe 120**
- (Actually, also need to meet at Thursday for the next next week...
- Seminars/Workshops help you take a breath from problem sets. It also helps you know the research frontier and hopefully ignite your own research idea.
- Seminar Sources I collected that is on ground at UVA
  - Dept of Econ
  - Darden Finance
  - Law & Econ
  - Politics
  - Batten Public Policy
- There are virtual seminars held by NBER, CEPR, etc. as well.

Brief Introduction to Python

- Python is a free and open source programming language that is quite popular in (certain) economic research.
- It can help you do a lot include:
  - Numerical Programming
  - Symbolic Algebra
  - Statistics
  - Graphics...
- There are many self-tutorials on Youtube, Coursera, etc. One that I highly recommend for economists is QuantEcon.
  - Learn the introduction at your free time.
  - They will guide you from the very basic tasks: from how to install python to how to carry out basic analysis like what I have showed in the sample code.
  - **ChatGPT** is actually a very cool reference as well. In Anton's words: View it as your intern who is
    - Super smart and enthusiastic!
    - But know zero about the context....

Brief Review

- Random Variables are i.i.d.?
  - independent $X \perp Y$.
    I flip a coin twice. H/T, I got from the first flip and the second flip doesn't rely on each other.
  - identically distributed For every set $A \in \mathcal{B}^1, P(X \in A) = P(Y \in A)$ or $F_X(x) = F_Y(y)$.
    I got 50-50 chance of H/T in the first and second flip.

Brief Review - Probability Functions

- $f_X(x) = P(X = x)$.
  - Probability Mass Function(pmf) for **Discrete** r.v.
  - Probability Density Function(pdf) for **Continuous** r.v.
  - A function is pdf or pmf of a r.v. if and only if:
    a. $f_X(x) \geq 0$ for all x.
    b. $\sum_x f_X(x) = 1$(pmf) or $\int_{-\infty}^{\infty} f_X(x)dx = 1$(pdf)
- $F_X(x) = P(X \leq x)$, either **discrete** or **continuous** or **mixed**: We call it Cumulative Distribution Function(cdf)
  - **Discrete**: Sum over pmf to get cdf
  - **Continuous**: $P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt$ for all x.[1]
    $\frac{d}{dx}F_X(x) = f_X(x)$
  - Pdf of **continuous** r.v. is 0 at any point by continuity of $F_X(x)$.
    $f_X(x) = P(X = x) = 0$.
    In other words, for continuous case we have:

$$P(a < X < b) = P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b)$$

$$= F_X(b) - F_X(a) = \int_a^b f_X(x)dx$$

[1] Technically, r.v. has to be *absolutely continuous*

Practice Question - Resolved Calls

Past HW The probability that k customers in a given hour call United Airlines call center is $\frac{\lambda^k e^{-k}}{k!}(\lambda > 0)$. For each call, the probability that customer's issue is not resolved is p. Find the probability that s people in a given hour will have the issue not resolved.

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- Note $k \geq s$ as you cannot have more people unresolved than people actually called in.
- Define events: X is # unresolved calls. Y is # calls in one hour.
- There are 2 steps to complete this task under given k:
  - Exactly s from k calls being not resolved: $P(X = s) = \binom{k}{s} p^s (1-p)^{k-s}$. Binomial, notice p is not resolved.
  - Receiving exactly k calls. $P(Y = k) = \frac{\lambda^k e^{-\lambda}}{k!}$. Possion.
- Since X and Y are independent, combining these gives us the probability of s out of given k calls are not resolved:

$$P(\text{s out of k calls not resolved}) = \frac{\lambda^k e^{-\lambda}}{k!} \binom{k}{s} p^s (1-p)^{k-s}$$

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- Account for all possible number of calls k, then we know the probability goes:

$$P(\text{s calls not resolved}) = \sum_{k=s}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \binom{k}{s} p^s (1-p)^{k-s}$$

$$= \sum_{k=s}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{k!}{s!(k-s)!} p^s (1-p)^{k-s} = \frac{e^{-\lambda} p^s}{s!} \sum_{k=s}^{\infty} \frac{\lambda^k}{(k-s)!} (1-p)^{k-s}$$

$$= \frac{e^{-\lambda} p^s \lambda^s}{s!} \sum_{k=s}^{\infty} \frac{\lambda^{k-s}}{(k-s)!} (1-p)^{k-s}$$

Practice Question - Resolved Calls

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- Denote $t = k - s$. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (from Taylor expansion around $x = 0$). Then we know

$$P(\text{s calls not resolved}) = \frac{e^{-\lambda} p^s \lambda^s}{s!} \sum_{k=s}^{\infty} \frac{\lambda^{k-s}}{(k-s)!}(1-p)^{k-s}$$

$$= \frac{e^{-\lambda} p^s \lambda^s}{s!} \sum_{t=0}^{\infty} \frac{(\lambda(1-p))^t}{t!} = \frac{e^{-\lambda} p^s \lambda^s}{s!} e^{\lambda(1-p)} = \frac{(\lambda p)^s e^{-\lambda p}}{s!}$$

This is simply the Possion probability mass function with the rate parameter of λp.

Practice Question - White and Black Marbles

Midterm 2017 Q1 The first urn has N_1 white marbles and M_1 black marbles. The second urn has N_2 white marbles and M_2 black marbles. We move one randomly selected marble from the first urn to the second one. After careful mixing we randomly draw a marble from the second urn. What is the probability that this marble is white?

Hint:

1. Use conditional probability and law of total probability.
2. How the probability changes when you draw a white ball from the first urn versus you draw a black ball from the first urn?

Practice Question - White and Black Marbles

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Define a series of events as follows:

- w_1 = {Draw a white ball from the first urn}, $P(w_1) = \frac{N_1}{N_1+M_1}$
- b_1 = {Draw a black ball from the first urn, $P(b_1) = \frac{M_1}{N_1+M_1}$
- w_2 = {Draw a white ball from the second urn}, $P(w_2) =?$

Use Law of total probability, we know:

$$
\begin{aligned}
P(w_2) = P(w_2, w_1) + P(w_2, b_1) &= P(w_2|w_1)P(w_1) + P(w_2|b_1)P(b_1) \\
&= \frac{N_2+1}{M_2+N_2+1}P(w_1) + \frac{N_2}{M_2+N_2+1}P(b_1) \\
&= \frac{N_2+1}{M_2+N_2+1}\frac{N_1}{N_1+M_1} + \frac{N_2}{M_2+N_2+1}\frac{M_1}{N_1+M_1}
\end{aligned}
$$

Practice Question - Poker Card Swap

Midterm 2022 Q1 Four players randomly draw 4 cards each from the standard deck of 52 cards (4 units). Suppose that each player is allowed to swap exactly one card in his/her set of 4 cards with exactly one other player (they don't have to swap if they don't need to). What is the probability that after the swap all 4 players have all their 4 cards from different suits?

Hint:

1. Discuss how many swaps can happen. Clear notations will be helpful for effective discussion.
2. Write some simple examples for yourself to understand difficult cases.
3. View three tasks, picking players, picking suits, picking rankings separately. Where we land on?

Practice Question - Poker Card Swap

Four players randomly draw 4 cards each from the standard deck of 52 cards (4 units). Suppose that each player is allowed to swap exactly one card in his/her set of 4 cards with exactly one other player (they don't have to swap if they don't need to). What is the probability that after the swap all 4 players have all their 4 cards from different suits?

Setup:

- We define event S as after the swap all 4 players, denoted by P_1, P_2, P_3, P_4 have their 4 cards one each from: ♥ ♦ ♣ ♠.
- We use $\{A, B, C, D\}$ to represent ♥ ♦ ♣ ♠, and 1-13 to represent $A - K$. So ♥$A \Rightarrow A1$
- How many swaps can happen?
  - **0** if no need; **1 Swap**: $P_1 \Leftrightarrow P_2$; **2 Swaps**: $P_1 \Leftrightarrow P_2$ & $P_3 \Leftrightarrow P_4$
  - Denote these three sub-events as S_0, S_1, S_2
  - **Note:** Swap is only between **pairs**.
    In other words: $\cancel{P_1 \Leftrightarrow P_2 \Leftrightarrow P_3}$ or $\cancel{P_1 \Leftrightarrow P_2 \Leftrightarrow P_3 \Leftrightarrow P_4}$

Practice Question - Poker Card Swap

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- We start from the most extreme case which is S_0.
- Each player got a perfect draw on suits $\{A, B, C, D\}$ at the first draw.
- For the denominator, how many different ways we can allocate 4 players each with 4 cards? [Unordered with replacement].

$$\text{Total Combination} = \binom{52}{4} * \binom{48}{4} * \binom{44}{4} * \binom{40}{4} = \frac{52!}{(4!)^4 * 36!}$$

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- For numerator, it will be a ordered with replacement as P_1 can choose 13 numbers from each 4 colors, it will be 13^4
- For P_2, 12 numbers from each 4 colors, it will be 12^4.
- So on and so forth we have:

$$\# \text{ of ways } S_0 \text{ can happen} = (13)^4 * (12)^4 * (11)^4 * (10)^4$$

Hence, $P(S_0) = \dfrac{(13)^4 * (12)^4 * (11)^4 * (10)^4}{\frac{52!}{(4!)^4 * 36!}} \approx 0.0001$

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- Now we move on to think about when 1 swap is needed.
  - **Players**: One and only one **pair** is needed. And other two people has to be lucky dog

$$P_1 \Leftrightarrow P_2 \quad P_3 : \checkmark \, P_4 : \checkmark$$

  - **Suits**: To make a Swap work, they each have to have 3 different suits in hand. And the suits they have are <u>complementary</u>: Once one is settled, the other as well.

$$P_1 : \{\mathbf{AA}\underline{BC}\} \Leftrightarrow P_2 : \{\underline{BC}\mathbf{DD}\} \Downarrow$$
$$P_1 : \{ABCD\}\checkmark \quad P_2 : \{ABCD\}\checkmark$$

  - **Rankings**: Same as before. Ultimately, I want 4 numbers out of 13 got drawn from each suit.

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Four players randomly draw 4 cards each from the standard deck of 52 cards (4 units). Suppose that each player is allowed to swap exactly one card in his/her set of 4 cards with exactly one other player (they don't have to swap if they don't need to). What is the probability that after the swap all 4 players have all their 4 cards from different suits?

- **Players**: $\binom{4}{2}$

$$\underbrace{P_1 \quad P_2 \quad P_3 \quad P_4}_{\text{Pick 2 players from 4 to form a Swap Pair}}$$

- **Suits**: $\binom{4}{1} * \binom{3}{1}$

$$\underbrace{A \quad B \quad C \quad D}_{\text{Pick 1 suit from 4 to miss and another 1 from the chosen 3 to repeat}}$$

- **Rankings**: Same as # of ways S_0 can happen.
- $P(S_1) = \dfrac{\binom{4}{2} * \binom{4}{1} * \binom{3}{1} * (13)^4 * (12)^4 * (11)^4 * (10)^4}{\frac{52!}{(4!)^4 * 36!}} \approx 0.01$

Practice Question - Poker Card Swap

Four players randomly draw 4 cards each from the standard deck of 52 cards (4 units). Suppose that each player is allowed to swap exactly one card in his/her set of 4 cards with exactly one other player (they don't have to swap if they don't need to). What is the probability that after the swap all 4 players have all their 4 cards from different suits?

- Now we move on to think about when 2 swaps are needed.
  - **Players**: Two pairs are needed.

$$P_1 \Leftrightarrow P_2 \quad P_3 \Leftrightarrow P_4$$

  - **Suits**: Same logic as before.
    - For each pair, they can swap in the same way or differently.
    - Within pair they have to be complementary as well.

$$P_1 : \{\mathbf{AA}\underline{BC}\} \Leftrightarrow P_2 : \{\underline{BC}\mathbf{DD}\} \mid P_3 : \{\mathbf{AA}\underline{BC}\} \Leftrightarrow P_4 : \{\underline{BC}\mathbf{DD}\}$$

Or

$$P_1 : \{\mathbf{AA}\underline{BC}\} \Leftrightarrow P_2 : \{\underline{BC}\mathbf{DD}\} \mid P_3 : \{\mathbf{BB}\underline{AD}\} \Leftrightarrow P_4 : \{\underline{AD}\mathbf{CC}\}$$

  - **Rankings**: Same as before.

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- **Players**: 3 or $\frac{\binom{4}{2}}{2}$ or $\frac{4!}{2^2*2!}$.

$$\underbrace{P_1 \quad P_2 \quad P_3 \quad P_4}_{\text{Pick 2 pairs from 4 players}}$$

- **Suits**: $\binom{4}{1}^2 * \binom{3}{1}^2$

$$\underbrace{A \quad B \quad C \quad D}_{\text{Each Pair Pick 1 suit from 4 to miss and another 1 from the chosen 3 to repeat}}$$

- **Rankings**: Same as # of ways S_0 can happen.
- $P(S_2) = \dfrac{3 * \binom{4}{1}^2 * \binom{3}{1}^2 * (13)^4 * (12)^4 * (11)^4 * (10)^4}{\frac{52!}{(4!)^4 * 36!}} \approx 0.057$

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- For S

$$P(S) = P(S_0) + P(S_1) + P(S_2) \approx 0.0001 + 0.01 + 0.057 \approx 0.0671$$