Econ 7010 DQ (10) November 5th

Slutsky Equation

$h_i(p, \bar{u}) = x_i(p, e(p, \bar{u}))$ by duality.

Differentiate wrt p_k

$$\frac{\partial h_i(p,\bar{u})}{\partial p_k} = \frac{\partial x_i(p, e(p,\bar{u}))}{\partial p_k} + \frac{\partial x_i(p, e(p,\bar{u}))}{\partial m} \cdot \frac{\partial e(p,\bar{u})}{\partial p_k}$$

At $\bar{u} = v(p,m)$, duality applies, and we know

$$\frac{\partial e(p,\bar{u})}{\partial p_k} = \frac{\partial e(p, v(p,m))}{\partial p_k} \underset{\text{Shephard's lemma}}{=} h_k(p, v(p,m)) \underset{\text{duality}}{=} x_k(p,m)$$

Thus, $\frac{\partial h_i(p, v(p,m))}{\partial p_k} = \frac{\partial x_i(p,m)}{\partial p_k} + \frac{\partial x_i(p,m)}{\partial m} \cdot x_k(p,m)$

Slutsky equation

$$\Rightarrow \boxed{\frac{\partial x_i(p,m)}{\partial p_k} = \underbrace{\frac{\partial h_i(p, v(p,m))}{\partial p_k}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_i(p,m)}{\partial m} \cdot x_k(p,m)}_{\text{income effect}}}$$

* Hicksian demand is usually unobservable since we don't observe $\bar{u}$, but it is well-behaving: $Dh(p)$ is symmetric, NSD, and $\frac{\partial h_i}{\partial p_i} \le 0$.

$$Dh(p) = \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 e(p,\bar{u})}{\partial p_1^2} & \cdots & \frac{\partial^2 e(p,\bar{u})}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 e(p,\bar{u})}{\partial p_n \partial p_1} & \cdots & \frac{\partial^2 e(p,\bar{u})}{\partial p_n^2} \end{bmatrix}$$

With the slutsky equation, we can construct $\underline{Dh(p)}$ as

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ \vdots & & \ddots & \vdots \\ s_{n1} & & \cdots & s_{nn} \end{bmatrix} \quad \text{where} \quad \underline{s_{jk} = \frac{\partial x_j}{\partial p_k} + \frac{\partial x_j}{\partial m} \cdot x_k}$$

Slutsky matrix.

Exercise # 1

A consumer lives for two periods and has total wealth w. She consumes c_1 in period 1 and c_2 in period 2. Her BC is $p_1 c_1 + p_2 c_2 \leq w$.

Assume an econometrician has given you the following observed demand functions for c_2 (and $\alpha > 0$ and $\beta > 0$ are given)

$$c_2(p,w) = \begin{cases} 0 & \text{if } w < \alpha(p_1+p_2) \\ \alpha + \beta/p_2 \, (w - \alpha p_1 - \alpha p_2) & \text{if } w \geq \alpha(p_1+p_2) \end{cases}$$

(a) Assume local nonsatiation. What is her implied demand for c_1?

(b) What's the most natural interpretation of $\bar{w} = \alpha(p_1+p_2)$?

(c) Assume the consumer always has the wealth level $w < \bar{w}$.

Are these restricted demand functions consistent with utility maximization for some utility function $u(c_1, c_2)$? If so, provide a utility function that rationalizes her behavior.

(d) Now assume the consumer always has $w > \bar{w}$.

Use the Slutsky substitution matrix to determine any additional restrictions on parameters α and β that are necessary for her behavior to be consistent with the deman

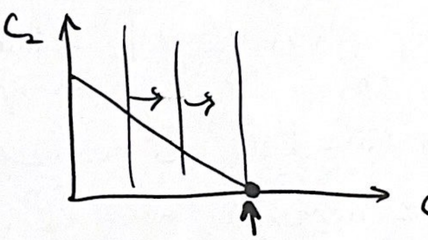
Sol)

(a) LNS $\Rightarrow$ Walras' Law! $p \cdot c = w. \rightarrow c_1 = \dfrac{w - p_2 c_2}{p_1}$

$$\therefore c_1(p,w) = \begin{cases} w/p_1 & \text{if } w < \alpha(p_1+p_2) \\[2ex] \dfrac{w - \alpha p_2 - \beta w + \alpha\beta(p_1+p_2)}{p_1} & \\[2ex] = \alpha + \dfrac{1-\beta}{p_1}(w - \alpha p_1 - \alpha p_2) & \text{if } w \geq \alpha(p_1+p_2) \end{cases}$$

(b) A minimal level of wealth required to survive both periods. If $w < \bar{w}$, the consumer cannot afford α in both periods so will "die".

(c) $u(c_1, c_2) = c_1$



(d) $(c_1(p,w), c_2(p,w)) = \left(\alpha + \dfrac{1-\beta}{p_1}(w - \alpha p_1 - \alpha p_2),\ \alpha + \dfrac{\beta}{p_2}(w - \alpha p_1 - \alpha p_2)\right)$

$$[S_{ij}] = \left[\frac{\partial c_i(p,w)}{\partial p_j} + \frac{\partial c_i(p,w)}{\partial w} c_j(p,w)\right] \Rightarrow$$ ① symmetric

② NSD

$$S = \begin{bmatrix} -\dfrac{(1-\beta)\beta(w - \alpha p_1 - \alpha p_2)}{p_1^2} & \dfrac{(1-\beta)\beta(w - \alpha p_1 - \alpha p_2)}{p_1 p_2} \\[3ex] \dfrac{(1-\beta)\beta(w - \alpha p_1 - \alpha p_2)}{p_1 p_2} & -\dfrac{(1-\beta)\beta(w - \alpha p_1 - \alpha p_2)}{p_2^2} \end{bmatrix}$$

→ symmetric

→ $S_{11} \leq 0 \Leftrightarrow \beta \in [0$

→ $D = 0$

$\Rightarrow \beta \in [0,1)$

Exercise #2

Consider an indirect utility function

$$v(p, m) = \frac{m - \gamma p_x}{p_x^{\alpha} p_y^{1-\alpha}}$$

where $\alpha \in (0,1)$ and $m > p_x \gamma$.

(a) Marshallian demand $x(p,m)$ and $y(p,m)$?

(b) What is the expenditure function $e(p, \bar{u})$?

(c) What are the hicksian demand $h_x(p, \bar{u})$ and $h_y(p, \bar{u})$?

(d) Consider a change in the price of good x.
Among two possible measures of change in consumer welfare
ΔCS and EV. which measure will result in a larger change of welfare?

(e) consider an economy populated by I individual consumers with different wealth levels $(m_1, \cdots, m_I)$, each with indirect utility $v_i(p, m_i) = v(p, m_i)$. Let $\bar{m} = \sum_i m_i$ denote the aggregate wealth.
What are the aggregate demand functions $\bar{x}(p, m)$ and $\bar{y}(p, m)$?
Does a representative consumer exist in this economy?
If so, what is the representative consumer's indirect utility function $\bar{v}(p, \bar{m})$?

Sol)

(a) Roy's Identity! $\left(v = m p_x^{-\alpha} p_y^{\alpha-1} - \gamma p_x^{1-\alpha} p_y^{\alpha-1}\right)$

$$\begin{cases} x(p,m) = -\dfrac{\partial v/\partial p_x}{\partial v/\partial m} = -\dfrac{-\alpha p_x^{-\alpha-1} p_y^{\alpha-1} m - \gamma(1-\alpha) p_x^{-\alpha} p_y^{\alpha-1}}{p_x^{-\alpha} p_y^{\alpha-1}} = \dfrac{\alpha m}{p_x} + \gamma(1-\alpha) \\ y(p,m) = -\dfrac{\partial v/\partial p_y}{\partial v/\partial m} = -\dfrac{(\alpha-1) m p_x^{-\alpha} p_y^{\alpha-2} - \gamma(\alpha-1) p_x^{1-\alpha} p_y^{\alpha-2}}{p_x^{-\alpha} p_y^{\alpha-1}} = \dfrac{(1-\alpha)m}{p_y} + (\alpha-1)\gamma \cdot \dfrac{p_x}{p_y} \end{cases}$$

(b) $e(p,u)$? Use duality!

$$\underline{v(p, e(p,u)) = u} \Rightarrow \frac{e(p,u) - \gamma p_x}{p_x^{\alpha} p_y^{1-\alpha}} = u \Rightarrow \underline{e(p,u) = p_x^{\alpha} p_y^{1-\alpha} \cdot u + \gamma \cdot p_x} \;//$$

(c) Use Shephard's lemma!

$$\begin{cases} h_x(p,u) = \dfrac{\partial e(p,u)}{\partial p_x} = \alpha p_x^{\alpha-1} p_y^{1-\alpha} \cdot u + \gamma \\ h_y(p,u) = \dfrac{\partial e(p,u)}{\partial p_y} = (1-\alpha) p_x^{\alpha} p_y^{-\alpha} \cdot u \end{cases}$$

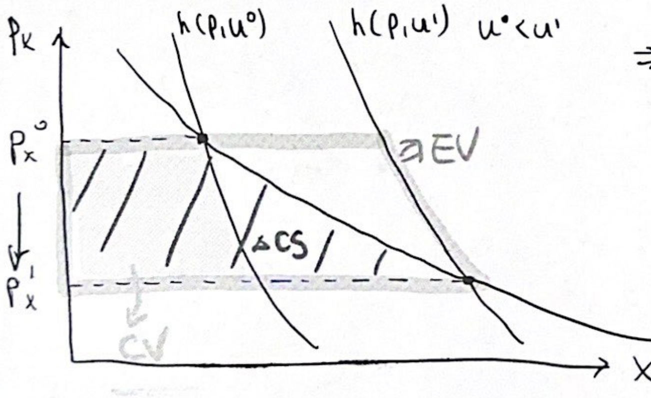
(d) By the slutsky equation. $\dfrac{\partial x(p,m)}{\partial p} = \underbrace{\dfrac{\partial h(p,u)}{\partial p}}_{\text{always } \ominus} \underbrace{- \dfrac{\partial x(p,m)}{\partial m} \cdot x(p,m)}_{\ominus \text{ for normal goods}}$

for x and similarly for y.

Let's first check if x & y are normal goods or inferior goods

$\dfrac{\partial x}{\partial m} = \dfrac{\alpha}{p_x} > 0$, $\dfrac{\partial y}{\partial m} = \dfrac{(1-\alpha)}{p_y} > 0$: normal goods.

So, we know $\left|\dfrac{\partial x}{\partial p}\right| > \left|\dfrac{\partial h_x}{\partial p}\right|$



$x(p,m)$: Marshallian

$$\Rightarrow \begin{cases} EV = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - m \quad (\text{at } p^0) \\ CV = e(p^1, u^1) - e(p^1, u^0) = m - e(p^1, u^0) \quad (\text{at } p^1) \end{cases}$$

$\Rightarrow \therefore$ **$CV \le \Delta CS \le EV$**

(e)

$$\Rightarrow \begin{cases} \bar{x}(p,m) = \sum_i \left(\frac{\alpha m_i + (1-\alpha)\gamma p_x}{p_x}\right) = \frac{\alpha \bar{m} + I(1-\alpha)\gamma p_x}{p_x} \\ \bar{y}(p,m) = \sum_i \left(\frac{(1-\alpha)(m_i - \gamma p_x)}{p_y}\right) = \frac{(1-\alpha)\bar{m} - I(1-\alpha)\gamma p_x}{p_y} \end{cases}$$

$\Rightarrow$ A representative consumer exists if and only if all individual indirect utility function have the Gorman form

: $v_i(p, m_i) = a_i(p) + b(p) m_i$

In this case, $a_i(p) = -\gamma p_x / p_x^{\alpha} p_y^{1-\alpha}$ & $b(p) = 1/p_x^{\alpha} p_y^{1-\alpha}$

Thus, the representative consumer's indirect utility function is

$$v(p, \bar{m}) = \sum_i a_i(p) + b(p)\bar{m} = -\frac{I\gamma p_x}{p_x^{\alpha} p_y^{1-\alpha}} + \frac{\bar{m}}{p_x^{\alpha} p_y^{1-\alpha}} \;//$$

Exercise #3

Let $u(x_1, x_2)$ be a consumer's utility, and consider a change in the price of good 1 (p_2 is fixed).

(a) Hicksian compensation: the net number of dollars we must give the consumer to be able to afford the same utility as she achieved before the price change.

Slutsky compensation: the net number of dollars we must give the consumer to achieve the same bundle she purchased before the price change.

$\Rightarrow$ Prove $HC \leq SC$.

(b) Let $u(x_1, x_2) = v(x_1) + x_2$ for some function v.

Prove $EV = CV$. when there was a change in price 1.

(ignore corner solutions)

(sol)

(a) p^{old} & p^{new} ; u^{old} & u^{new} ; x^{old} & x^{new}

$$\begin{cases} HC = e(p^{new}, u^{old}) - e(p^{new}, u^{new}) \\ SC = \quad p^{new} \cdot x^{old} \quad - \underset{\underset{m}{\shortparallel}}{e(p^{new}, u^{new})} \end{cases}$$

$\therefore$ $HC - SC = e(p^{new}, u^{old}) - p^{new} \cdot x^{old} \leq 0$ always

since the <u>expenditure function</u> is the optimal value.

$\therefore HC \leq SC$.

(b) $\Rightarrow$ EV is the area to the left of $h_1(p, u^{new})$ and

CV is the area to the left of $h_1(p, u^{old})$.

However, since the utility is quasilinear in good 2,

we know that $h_1(p, u^{new}) = h_1(p, u^{old}) = h_1(p) = x_1(p)$,

since good 1 is not affected by wealth.

Thus, those two hicksian demand curves are the same,

$CV = EV$ □