

Micro

(Wed Aug 23)

Individual agent

↓
single decision
problem

1. Firm \rightarrow Varian.
2. Consumer Theory \rightarrow MWG
3. Choice under uncertainty.

① Firm Theory :-

Assumptions:-

1. firms are price takers
2. Tech is exogenous
3. firms maximize profits.

Technology :-

- n commodities, $y = (y_1, \dots, y_n)$ ($y \in \mathbb{R}^n$)
 - * $y_i \leq 0$, i is an input
 - * $y_i \geq 0$, i is an output.

- Production Set : $Y \subseteq \mathbb{R}^n$

what the firm
can do.

$$n=3$$

$$y = (-2, 1, -3)$$

Firm's
function : 2 units
of something
as input

3 units of something
as input
1 Output

Properties of Production Sets! —

1. $Y \neq \emptyset$

2. Y is closed

3. No free lunch. If $y \in Y$ and $y \geq 0$, then $y = 0$

The following are sometimes assumed! —

4. Shutdown is possible: $0 \in Y$

5. Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$.

(You can take more input, and not lead to increased output, say you get rid of it.)

6. Irreversibility: $y \in Y$, $y \neq 0$, then $-y \notin Y$

You can't take the output to make the input.

Returns to scale & Convexity

Y has:

- non increasing returns to scale if

$$y \in Y \Rightarrow \alpha y \in Y \text{ & } \alpha \in [0, 1]$$

- non decreasing returns to scale if.

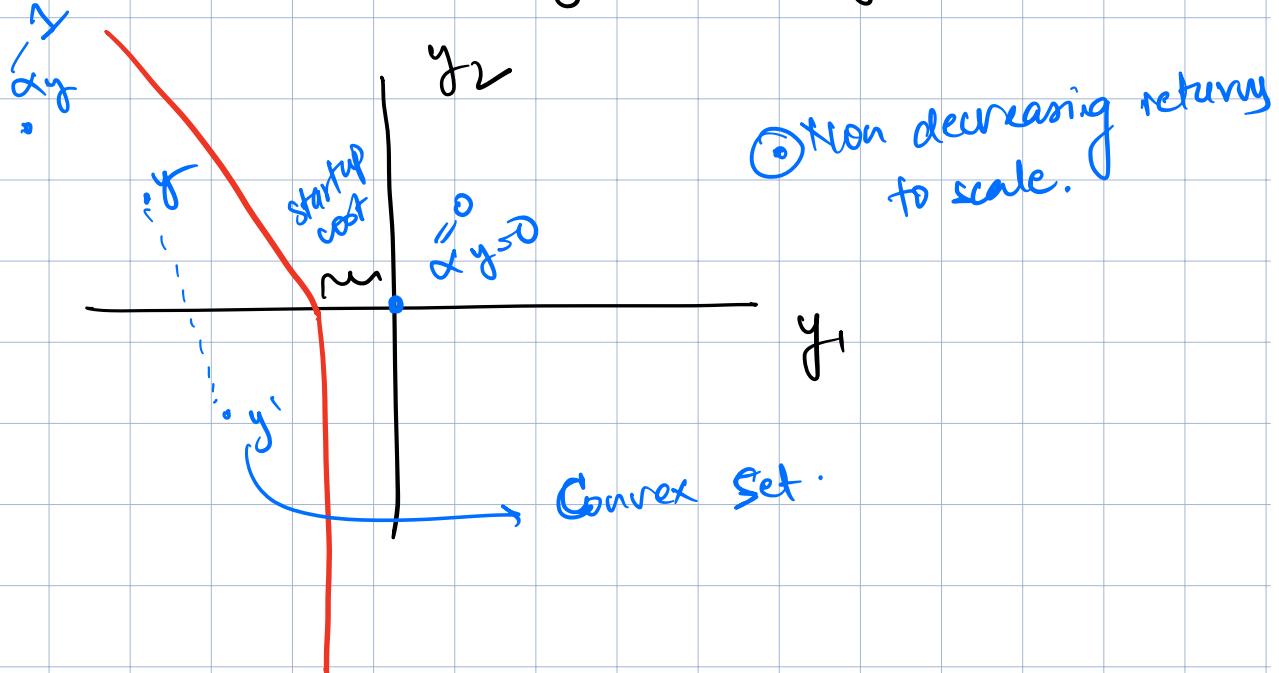
$$y \in Y \Rightarrow \alpha y \in Y \text{ & } \alpha \geq 1$$

- Constant returns to scale

$$y \in Y \Rightarrow \alpha y \in Y \text{ & } \alpha \geq 0$$

Y is convex if:

$$y, y' \in Y \Rightarrow \alpha y + c(1-\alpha)y' \in Y \text{ if } \alpha \in [0, 1]$$



Why do we "usually" assume non increasing returns?

- Modelling only inputs under our control.

Theorem

There is a constant returns production set

$$Y' \subseteq \mathbb{R}^{n+1} \text{ s.t. } Y = \{y \in \mathbb{R}^n : (y-1) \in Y'\}$$

Proof Define $Y' = \{y' \in \mathbb{R}^{n+1} : y' = \alpha(y - 1)$
for some $y \in Y$ & $\alpha > 0\}$

Check that Y' is constant returns.

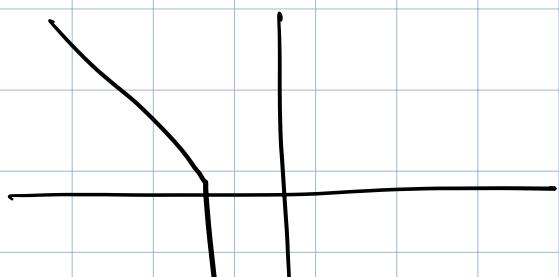
Efficiency :-

Technological efficiency:

A production plan y is technologically efficient if there does not exist

$$y' \in Y \text{ s.t. } \underbrace{y' > y}_{\downarrow}$$

$y'_i \geq y_i$ for all i & $y'_i > y_i$ for some i .



} efficient pfs are
going to be on the }

boundary.

Aug 28 (second class)

Transformation frontier.

firm, production set Y

The transformation function $T: \mathbb{R}^n \rightarrow \mathbb{R}$

$T(y) < 0 \Leftrightarrow y$ is inefficient

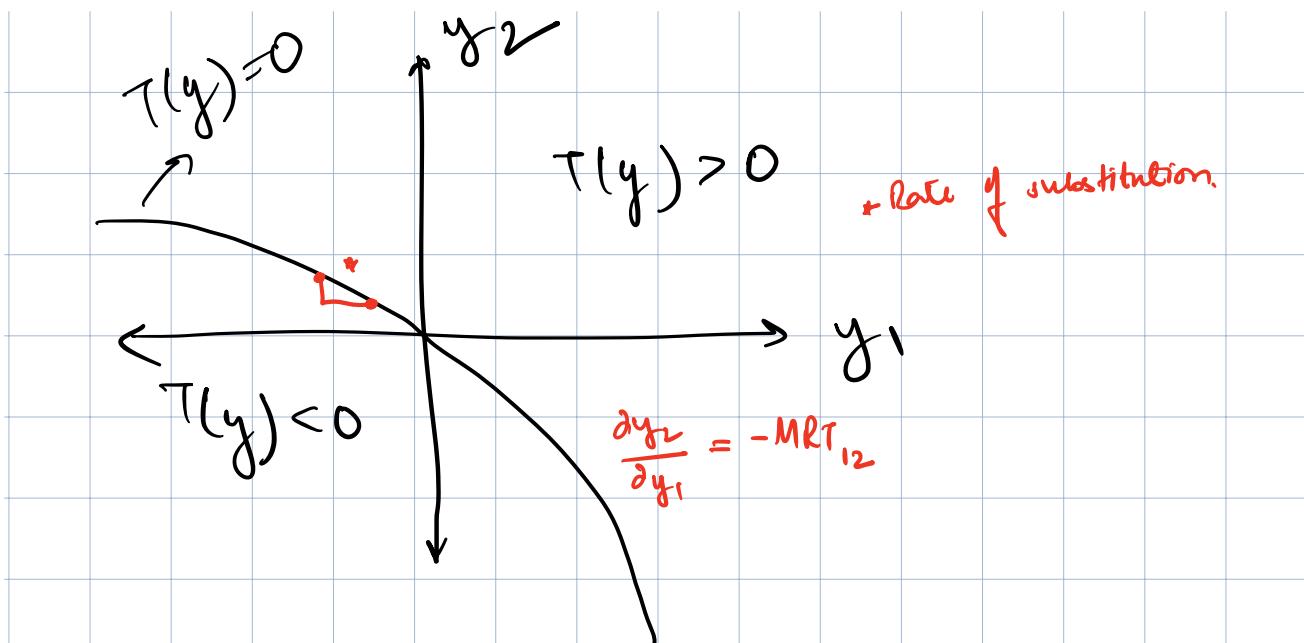
$T(y) = 0 \Leftrightarrow y$ is efficient

$T(y) > 0 \Leftrightarrow y$ is infeasible.

$$\Rightarrow Y = \{y \in \mathbb{R}^n : T(y) \leq 0\} \quad | \text{To check}$$

Production frontier

$$\{y \in \mathbb{R}^n : T(y) = 0\}$$



Let's fix $\bar{y}_3, \dots, \bar{y}_n$. Consider y_1, y_2, \dots

Define a function $y_2(y_1)$ by

$$T(y_1, y_2(y_1), \bar{y}_3, \dots, \bar{y}_n) = 0 \quad \text{constants}$$

implicit fn of y_1

Pick some y_1 , what y_2 gives me $T(y) = 0$

Let's differentiate both sides wrt y_1

$$\frac{\partial T}{\partial y_1} + \frac{\partial T}{\partial y_2} \cdot \frac{\partial y_2}{\partial y_1} = 0$$

slope in the picture

$$\frac{\partial y_2}{\partial y_1} = -\frac{\partial T / \partial y_1}{\partial T / \partial y_2}$$

Works for any 2 goods

$$\frac{\partial y_K}{\partial y_j} = -\frac{\partial T / \partial y_j}{\partial T / \partial y_K} \Big|_{y=\bar{y}} = -MRT_{jk}(\bar{y})$$

(Marginal Rate of Transformation)

My
Ques:

Special: Many inputs, one output.

Care

Inputs $(x_1, \dots, x_m) \geq 0$

Output: $y \geq 0$

Production function: $f(x)$

Production set: $Y = \{ \underbrace{(-x_1, -x_2, \dots, -x_m, y)}_{m} : y \leq f(x_1, \dots, x_m) \}$

$x_i \geq 0 \forall i \in \{1, \dots, m\}$

$y \leq f(x_1, \dots, x_m)$

$\tau(z) = y - f(x), z = (-x, y)$

Efficient frontier is
 $\tau(z) = 0$

$\Rightarrow y - f(x) = 0$

$$\rightarrow y = f(x)$$

Example

Cobb-Douglas

- 2 inputs, capital (x_k) + labor (x_l)

- Output y

$$- f(x_k, x_l) = x_k^\alpha x_l^\beta, \alpha + \beta = 1$$

$$- Y = \{(-x_k, -x_l, y) : x_k, x_l \geq 0, y \leq x_k^\alpha x_l^\beta\}$$

$$- T = (-x_k, -x_l, y) = y - x_k^\alpha x_l^\beta$$

MRT between capital + output.

$$MRT_{(-x_k), y} = \frac{\partial T / \partial (-x_k)}{\partial T / \partial y}$$

$$= - \frac{\partial T / \partial x_k}{\partial T / \partial y}$$

$$= - \frac{-\alpha x_k^{\alpha-1} x_l^\beta}{1}$$

$$= \alpha \left(\frac{x_l}{x_k} \right)^\beta$$

Marginal product of capital, MPK .

$$MPK = f_{x_K}(x_K, x_L) \rightarrow \text{from undergad.}$$

$$= \frac{\partial f}{\partial x_K}$$

$$= \alpha x_K^{\alpha-1} x_L^\beta$$

$$= \alpha \left(\frac{x_L}{x_K} \right)^\beta$$

The method now is more a general way of finding sol'n.

MRT for capital & labor

$$MRT(-x_K), (-x_L) = \frac{\frac{\partial T}{\partial (-x_K)}}{\frac{\partial T}{\partial (-x_L)}}$$

$$= \frac{\frac{\partial T}{\partial x_K}}{\frac{\partial T}{\partial x_L}}$$

$$= \frac{-\alpha x_K^{\alpha-1} x_L^\beta}{-\beta x_K^\alpha x_L^{\beta-1}}$$

$$= \frac{\alpha}{\beta} \left(\frac{x_L}{x_K} \right)^\beta$$

$$\frac{d(-x_L)}{d(-x_K)} = -MRT_{(-x_K, -x_L)}$$

$$\frac{dx_L}{dx_K} = -\frac{\alpha}{\beta} \frac{x_L}{x_K}$$

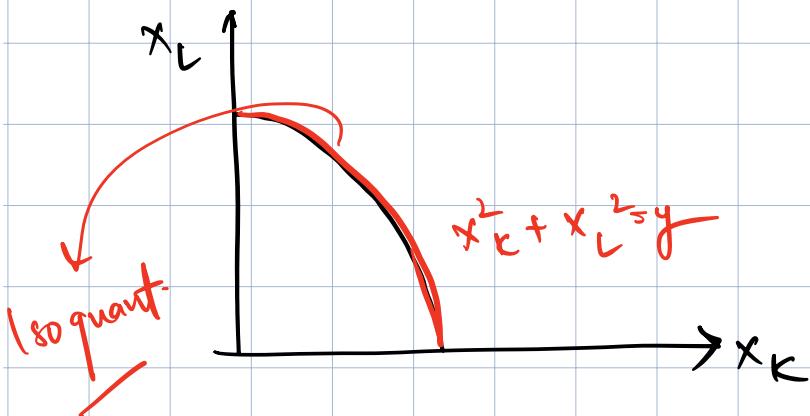
single output, fix y

$$V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\} \rightarrow \text{Input requirement set.}$$

$$Q(y) = \{x \in \mathbb{R}^n : f(x) = y\} \rightarrow \text{Isquant.}$$

Example:

$$f(x_K, x_L) = x_K^2 + x_L^2 = y$$



This is all fine as the inputs here are considered positive as we are only working with inputs.

With a single output, we can define the marginal rate of technical substitution between two inputs

$$MRTS = \frac{\frac{\partial f}{\partial x_K}}{\frac{\partial f}{\partial x_L}}$$

$$\frac{\partial x_L}{\partial x_K} = -MRTS_{x_K, x_L} = -\frac{f'_{x_K}}{f'_{x_L}}$$

In the example

$$\frac{dx_L}{dx_K} = -\frac{\frac{\partial x_K}{\partial x_L}}{\frac{\partial x_L}{\partial x_K}} = -\frac{x_K}{x_L}$$

This is a funny example because the slope is increasing (MRTS is usually decreasing). This is because $V(y)$ is not convex.

Theorem: If γ is convex, then $v(y)$ is also convex.

$$(P \Rightarrow Q \Leftrightarrow \text{not } Q \Rightarrow \text{not } P)$$

Proof

$$(-x, y) \in \gamma$$

$$(-x', y) \in \gamma$$

$$\gamma \text{ convex} \Rightarrow t(-x, y) + (1-t)(-x', y) \in \gamma$$

$\underbrace{_{\text{linear combination.}}}$

$$\Rightarrow (-tx - (1-t)x', ty + (1-t)y) \in \gamma$$

$$\Rightarrow (-tx - (1-t)x', y) \in \gamma$$

$$\Rightarrow (-x'', y) \in \gamma \text{ where } x'' = tx + (1-t)x'$$

$$\Rightarrow x'' \in v(y)$$

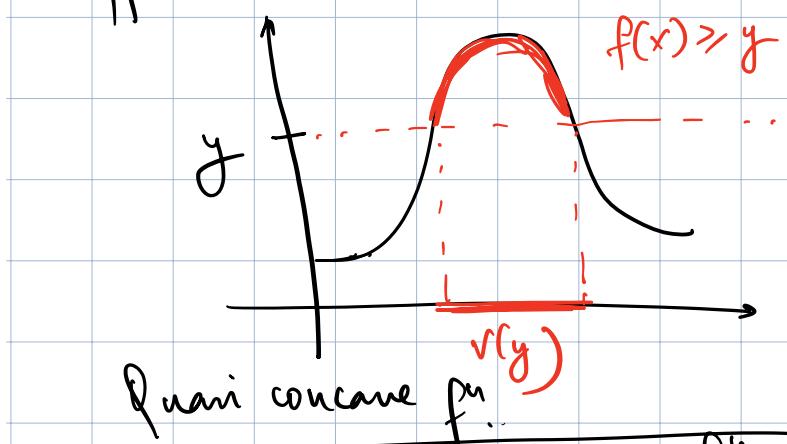
$$\Rightarrow v(y) \text{ is } \underline{\text{convex.}}$$

Theorem : $V(y)$ is convex iff $f(x)$ is quasi-concave.

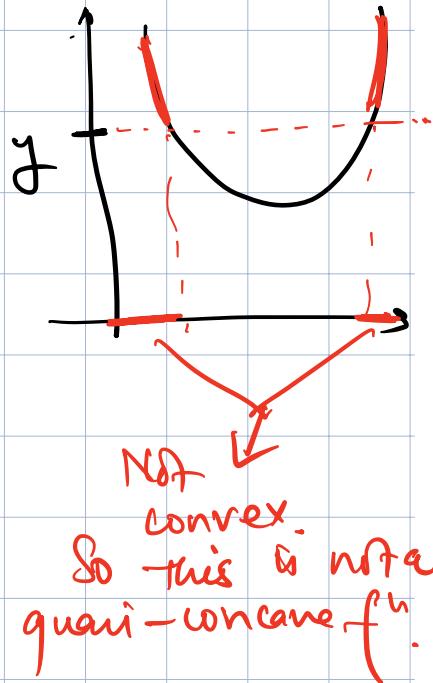
Proof : $V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$

(It's more of a defⁿ)

Definition of a quasi-concave f^n that has convex upper contour sets.



Concave f^n are quasi-concave f^n .



Homogeneity + Homotheticity

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree k

if $f(tx) = t^k f(x)$ & $t = 0$.

$$k=0 : f(tx) = f(x)$$

$$k=1 : f(tx) = tf(x)$$

Cobb-Douglas :

$$f(x_k, x_L) = x_k^\alpha x_L^\beta \quad \alpha + \beta = 1.$$

$$\begin{aligned} f(tx_k, tx_L) &= (tx_k)^\alpha (tx_L)^\beta \\ &= t^{\alpha+\beta} x_k^\alpha x_L^\beta \xrightarrow{\text{when } \alpha+\beta=1.} \\ &= t x_k^\alpha x_L^\beta = tf(x) \end{aligned}$$

What about MRTS?

$$\begin{aligned} \text{MRTS} &= \frac{\alpha}{\beta} \frac{x_L}{x_K} \\ &= \frac{\alpha}{\beta} \frac{(tx_L)}{(tx_K)} = \frac{\alpha}{\beta} \frac{x_L}{x_K} \end{aligned}$$

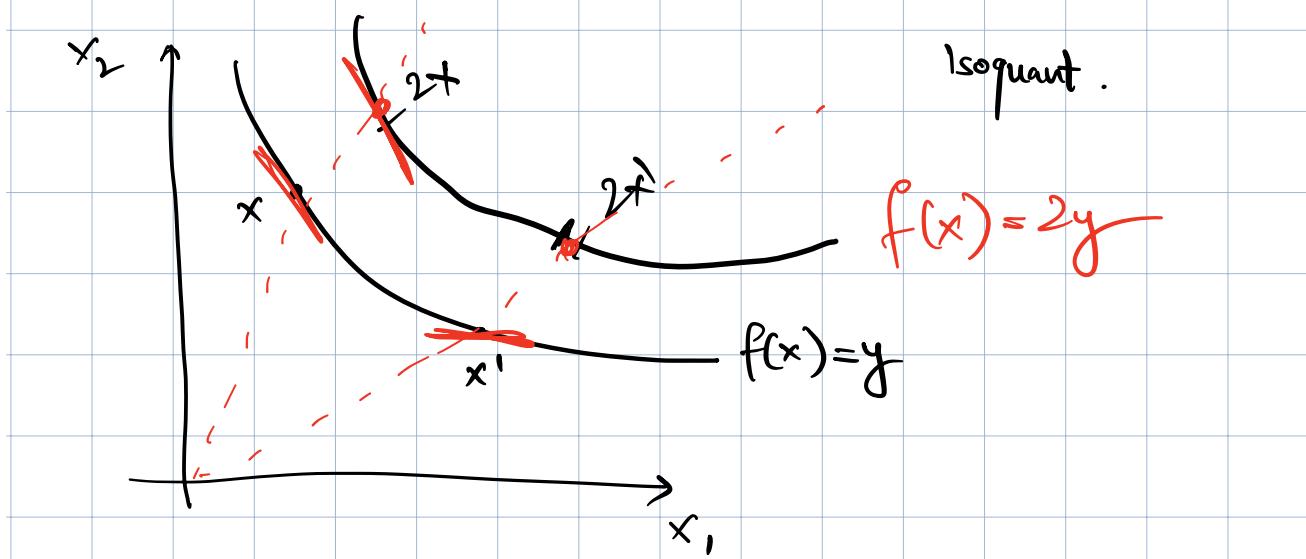
MRTS is homogeneous of degree 0.

Theorem: If f is $hd-k$, then the partial derivative f_{x_i} is $hd-(k-1)$.

Proof: $f(tx) = t^k f(x)$

$$t f_{x_i}(tx) = t^k f_{x_i}(x)$$

$$f_{x_i}(tx) = t^{k-1} f_{x_i}(x)$$



Say f is $hd-1$

$$f(2x) = 2f(x) = 2y$$

The slopes would be the same too -

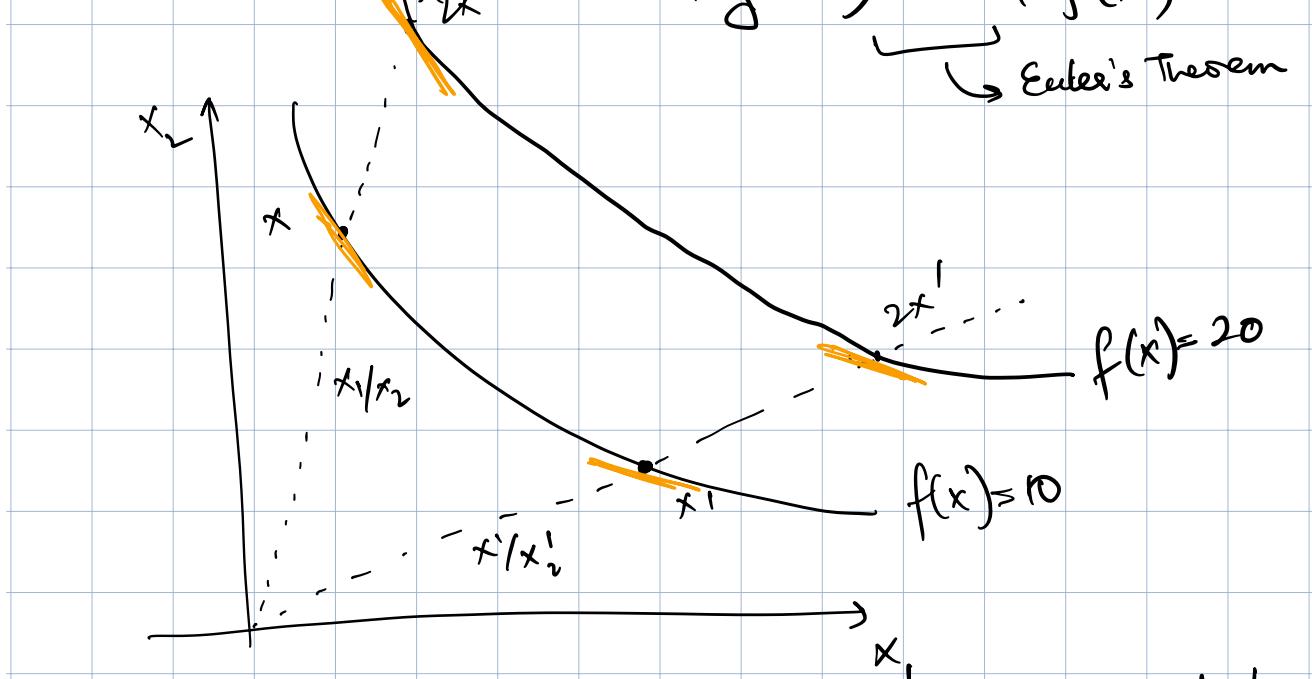
⇒ Every isoquant is just a blown up version of the first one.

30 Aug 2023

If a production f^u $f(x)$ is homogeneous of degree 1 ($hd = 1$), then the associated MRTS is $hd = 0$.

Proof : $MRTS_{ij}(\bar{x}) = \frac{f'x_i(\bar{x})}{f'x_j(\bar{x})}$

$$MRTS_{ij}(t\bar{x}) = \frac{f'x_i(t\bar{x})}{f'x_j(t\bar{x})} = \frac{f'x_i(\bar{x})}{f'x_j(\bar{x})} = MRTS_{ij}(\bar{x})$$



The slopes of the isoquants would be the same.

Correct later
factor ratio: x_2/x_1

Homothetic functions.

is a positive monotonic transformation
of an $h^d - 1$ function.

i.e. $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is homothetic if \exists a
 $g: \mathbb{R} \rightarrow \mathbb{R}$ strictly increasing and an
 $h: \mathbb{R}^n \rightarrow \mathbb{R}$ that \exists $h^d - 1$ s.t. $f(x) = g(h(x))$

MRTS of $f = \frac{f_{x_i}}{f_{x_j}} = \frac{g'(h(x)) h_{x_i}}{g'(h(x)) h_{x_j}} = \frac{h_{x_i}}{h_{x_j}}$

(\hookrightarrow homothetic)

The shape of the isoquant would be the same but the labelling could be different. Output would increase to 80 instead of 20 in the last example.

Returns to scale wrt a single output

Production function f exhibits:

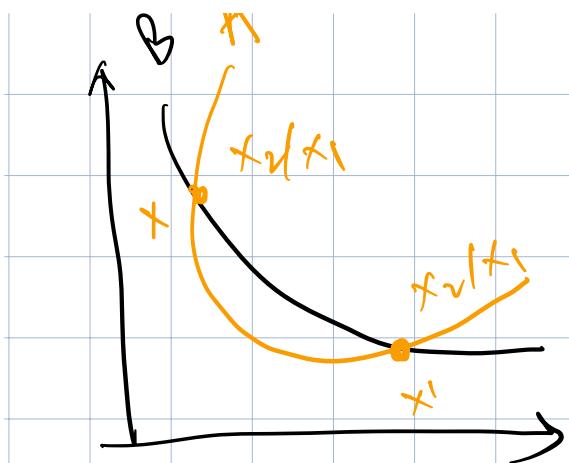
- decreasing returns to scale
if $f(tx) \leq t f(x)$ & $t \geq 1$.
- increasing returns to scale
if $f(tx) \geq t f(x)$ & $t \geq 1$
- constant returns to scale if *.
 $f(tx) = t f(x)$ & $t \geq 0$

Curvature of Isoquants.

④ Elasticity of complementarity
(for "large" jumps)

$$\frac{1}{\sigma} = \frac{\frac{\Delta MRTS}{MRTS}}{\frac{\Delta (x_2/x_1)}{x_2/x_1}}$$





Denominator is the same for A, B. In example.

Numerator much bigger for A.

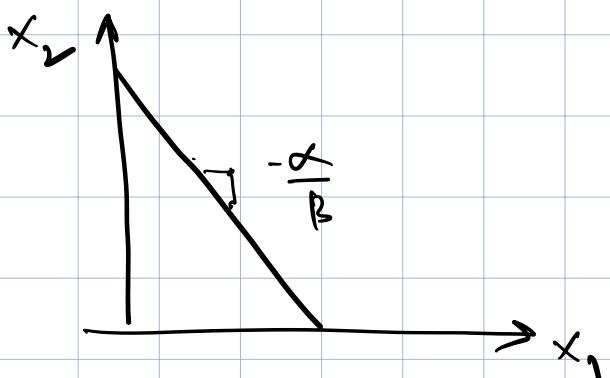
$$\frac{1}{\sigma_A} > \frac{1}{\sigma_B}$$

Def'n through limits : —

$$\frac{1}{\sigma} = \frac{x_2/x_1}{MRTS} \cdot \frac{dMRTS}{d(x_2/x_1)}$$

~~Example~~

Perfect Substitution $f(x_1, x_2) = \alpha x_1 + \beta x_2$



$$\frac{1}{\sigma} = 0$$

The slope does not change.

Elasticity of substitution

$$\sigma = \frac{\text{MRTS}}{x_2/x_1} \frac{d(x_2/x_1)}{d \text{ MRTS}}$$

For perfect substitutes, $\sigma = \infty$ (as $\gamma_a = 0$)

less curvature \rightarrow bigger elasticity of substitution

Perfect Complements

$$f(x_1, x_2) = \min \{x_1, x_2\} \quad \sigma = 0$$



For any 2 quantities, y, x the elasticity of y w.r.t x

$$E_{yx} = \frac{d \log(y)}{d \log(x)}$$

Way to see this:

$$\frac{d \log y}{y} = \frac{1}{y} dy \quad (\text{Total derivative})$$

$$\frac{d \log x}{x} = \frac{1}{x} dx$$

$$= \frac{\frac{d \log(y)}{y}}{\frac{d \log(x)}{x}} = \frac{x}{y} \frac{dy}{dx}$$

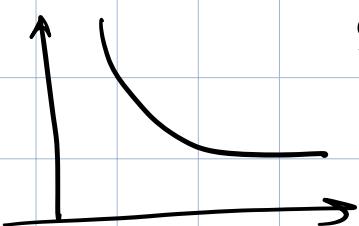
for σ :

$$y = x_2/x_1$$

$$x = MRTS$$

④ σ for Cobb-Douglas

Cobb Douglas :



Smoothly diminishing graphs

$$MRTS = \frac{\alpha}{\beta} \cdot \frac{x_L}{x_K}$$

$$\sigma = \frac{MRTS}{x_L/x_K} \cdot \frac{d(x_L/x_K)}{d(MRTS)}$$

$$\text{let } \theta = \frac{\alpha}{\beta} \frac{x_L}{x_K}$$

$$\text{or } \frac{x_L}{x_K} = \frac{\beta}{\alpha} \theta$$

$$= MRTS = \theta$$

$$\sigma = \frac{\theta}{\beta \theta} \cdot \frac{d(\frac{x_L}{x_K})}{d\theta} \xrightarrow{\text{constant}} \frac{1}{\alpha}$$

$$= \frac{\beta/\alpha}{\beta/\alpha} = 1.$$

C-1, perf complements / substitutes are all
special cases of constant elasticity of
substitution (CES) production.

$$f(x) = [\alpha_1 x_1^s + \alpha_2 x_2^s]^{1/s} \xrightarrow{\text{row}}$$

$s = 0, 1, \infty \rightarrow$ leads to special cases.

Recall our assumption that firms are price takers
f₂ maximize profits.

The profit maximization problem (PMP)

let p be a vector of n prices
 $\max p \cdot y$ s.t. $y \in Y$, where $p \cdot y = \sum_{i=1}^n p_i y_i$ is ^{is def profit}

Equivalently,

$$\begin{aligned} & \max p \cdot y \\ & \text{s.t. } T(y) \leq 0 \end{aligned}$$

$\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ is also called the profit function.

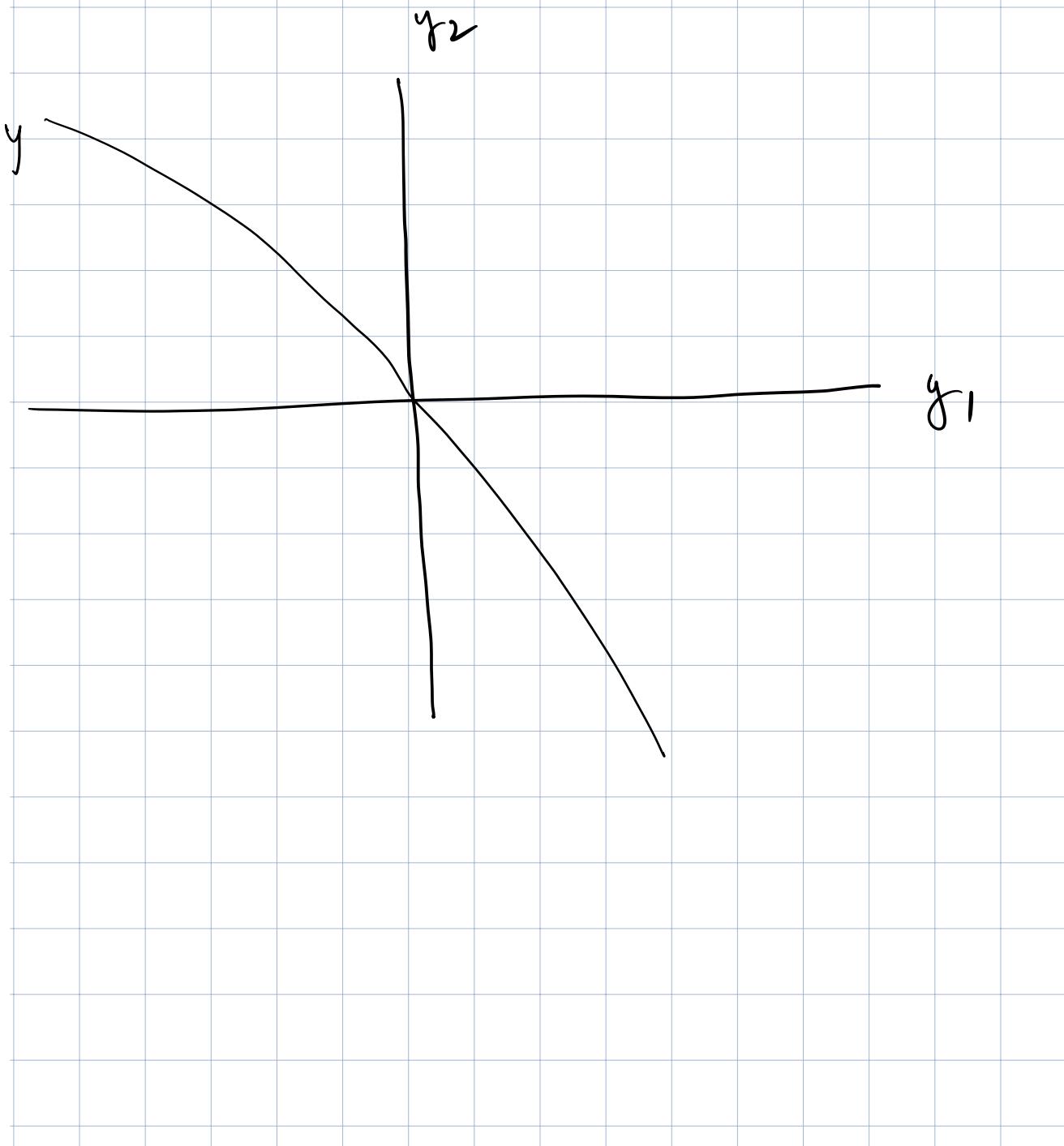
Example of a value function in constrained optimization.

$p \cdot y$ is called the objective function.

$y^*(p) = \{y \in Y : p.y = \pi(p)\}$ is called the supply correspondence.

(set valued mapping (function))

$y(p)$ could be multivalued (or $\{y(p) = \emptyset\}$)



Sep 4

$$\max p \cdot y$$

$$\text{s.t. } T(y) \leq c$$

$$L = p \cdot y - \lambda T(y)$$

KT conditions:

$$\frac{\partial L}{\partial y_i} = 0 \Rightarrow p_i - \lambda \frac{\partial T}{\partial y_i} = 0$$

$$\text{CSC : } \lambda T(y) = 0$$

$$\lambda \geq 0$$

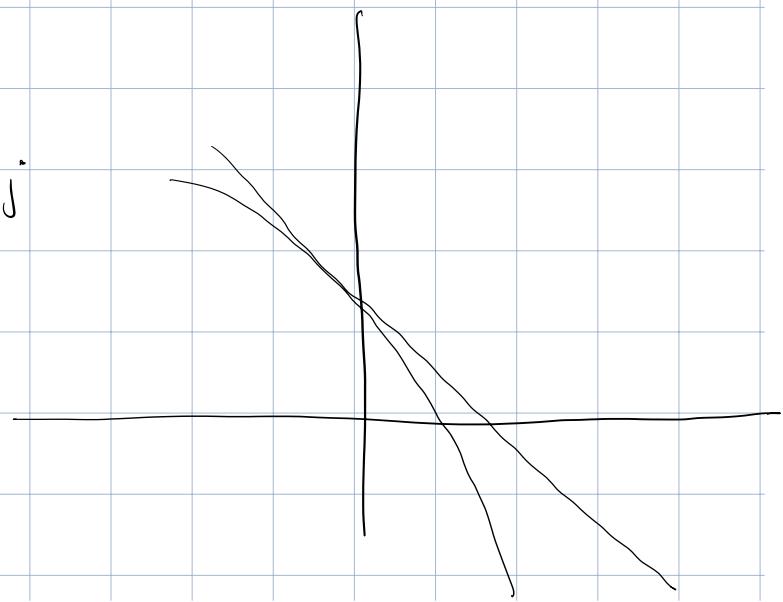
Implications of the FOC:

$$p_i = \frac{\partial T / \partial y_i}{\lambda} \quad \text{for any } i, j$$

$$p_j = \frac{\partial T / \partial y_j}{\lambda}$$

$$MRT_{ij}$$

$$\text{Recall } \frac{dy_j}{dy_i} = -MRT_{ij}$$



$$\frac{\partial T / \partial y_i}{P_i} = \frac{\partial T / \partial y_j}{P_j}$$

\Leftrightarrow Marginal product per dollar is the same

* Single Output Case

Production function $f(x)$

$$\text{Max } p f(x) - w \cdot x$$

$$\text{s.t. } x_i \geq 0$$

p : output price.

w : (vector A) input prices

$$L = p f(x) - w \cdot x + \mu_i x_i$$

μ_i is multiplier for
 $x_i \geq 0$ constraint.

KT conditions:

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow p \frac{\partial f}{\partial x_i} - w_i + \mu_i = 0 \quad (*)$$

$$\text{CSC: } \mu_i x_i = 0 \quad \forall i$$

$$\mu_i \geq 0 \quad \forall i$$

Another way to write (*)

$$p \frac{\partial f}{\partial x_i} \leq w_i \quad \text{with equality if } x_i > 0.$$

Marginal revenue
of addition x_i

Marginal cost of addition x_i

Divide i by j

$$\frac{\partial f / \partial x_i}{\partial f / \partial x_j} = \frac{w_i}{w_j} \quad \text{for any } 2 x_i^*, x_j^* > 0$$

$$\frac{\partial f / \partial x_i}{w_i} = \frac{\partial f / \partial x_j}{w_j}$$

Notation for single output problems:

$x^*(p, w)$ factor demand correspondence (or function)

$y^*(p, w) = f(x^*(p, w))$ supply correspondence

$$\begin{aligned}\Pi(p, w) &= p f(x^*(p, w)) - w \cdot x^*(p, w) && \text{profit}\\ &= \max_x p f(x) - w \cdot x^*\end{aligned}$$

$$x_i \geq 0$$

KT conditions are necessary but not (in general)
sufficient. $\Rightarrow \underline{\text{SOC}}$

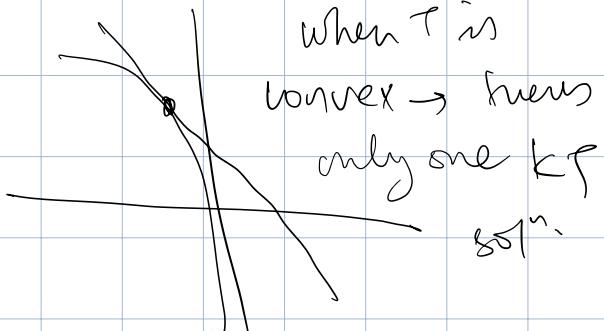
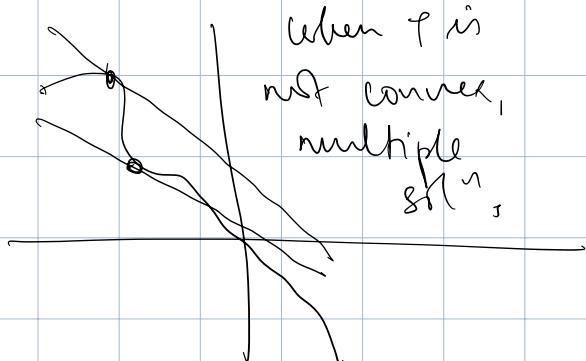
SOCs are also local. They don't tell you
globals.

Going back to original formulation to make FOCs sufficient:

$$\max p(y) \quad \text{s.t. } T(y) \leq 0$$

Theorem: Assume that $T(y)$ is convex. If a point y^* satisfies the KT conditions for some $\lambda \geq 0$, then y^* is a solⁿ to the maximization problem.

If your objective f is quasiconcave, if all the constraints are quasiconvex \Rightarrow the KT conditions are sufficient.



Single Output

$$\begin{aligned} \max \quad & p f(x) - w \cdot x \\ \text{st.} \quad & x_i \geq 0 \end{aligned}$$

Theorem: Assume f is concave, then if x^* satisfies

$$p \cdot \underline{f}(x_i^*) \leq w_i + \varepsilon \quad \text{with equality if } x_i^* > 0$$

then x^* is a solution to the problem.

Caveats :-

• KT/FOCs are very useful, but be careful about following :-

- 1) Non differentiability
- 2) Non convexities
- 3) Corner solutions
- 4) Existence/Uniqueness issues

One input - one output

$$y = f(x) = \frac{1}{2}x$$

$$\max_{x \geq 0} p f(x) - w x$$

$$\mathcal{L} = p\left(\frac{1}{2}x\right) - wx + \mu x$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow \frac{1}{2}p - w + \mu = 0$$

$$CSC = x \mu = 0$$

$$\mu > 0$$

$x = 0$, $\mu = w - \frac{1}{2}p$ is a sol^o if

$$\mu = w - \frac{1}{2}p \geq 0$$

What if $w - \frac{1}{2}p < 0 \Rightarrow w < \frac{1}{2}p$

The output price is too high. Then you keep producing \Rightarrow no answer.

$$\pi(p, w) = \begin{cases} 0 & , p < 2w \\ (0, \infty) & , p = 2w \\ \varnothing & , p > 2w \end{cases}$$

infinity, whatever you want.

$$\pi(p, w) = \begin{cases} 0 & , p \leq 2w \\ +\infty & , p > 2w \end{cases}$$

$\hookrightarrow f(x_1, x_2) = \log(1 + x_1 + x_2)$

Output price = p

Input price $w = (w_1, w_2)$

$$\max p \log(1 + x_1 + x_2) - w_1 x_1 - w_2 x_2$$

$$L = p \log(1 + x_1 + x_2) - w_1 x_1 - w_2 x_2 + \mu_1 x_1 + \mu_2 x_2$$

$$\frac{\partial L}{\partial x_1} \geq 0 \Rightarrow \frac{p}{1 + x_1 + x_2} - w_1 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow \frac{p}{1+x_1+x_2} - \omega_2 + \mu_2 = 0$$

case: $\mu_1 x_1 = 0$, $\mu_2 x_2 = 0$
 $\mu_1, \mu_2 \geq 0$

Case 1° Both inputs are used

$$x_1, x_2 > 0$$

$$\Rightarrow \mu_1 = \mu_2 = 0$$

$$\omega_1 = \frac{p}{1+x_1+x_2}$$

$$\omega_2 = \frac{p}{1+x_1+x_2}$$

$$\Rightarrow \omega_1 = \omega_2 = \bar{\omega}$$

$$\bar{\omega} = \frac{p}{1+x_1+x_2}$$

$$\Rightarrow 1+x_1+x_2 = \frac{p}{\bar{\omega}}$$

$$x_1+x_2 = \frac{p}{\bar{\omega}} - 1$$

Implicitly, we need $\frac{p}{w} > 1$

$$\Rightarrow p > \bar{w}$$

Case 2 : One input (say x_i) is used

$$x_2 = 0$$

$$\mu_1 = 0, \mu_2 \geq 0$$

$$\frac{p}{1+x_1} = w_1 \Rightarrow x_1^* = \frac{p}{w_1} - 1$$

$$\mu_2 = w_2 - \frac{p}{1+x_1} = w_2 - w_1 \geq 0$$

Case 3 Only x_2 used.

Same, switch x_1, x_2

$$x_1 = 0 \quad x_2 = \frac{p}{w_2} - 1$$

$$w_2 \leq w_1$$

factor demands are (Assume $p \geq \min\{w_1, w_2\}$)

$$x_1(p, w) = \begin{cases} \frac{p}{w_1} - 1, & w_1 < w_2 \\ \alpha \left(\frac{p}{\bar{w}} - 1 \right), & w_1 = w_2 = \bar{w} \\ 0, & w_1 > w_2 \end{cases}$$

for some $\alpha \in [0, 1]$

$$x_2(p, w) = \begin{cases} 0, & w_1 < w_2 \\ (1-\alpha) \left(\frac{p}{\bar{w}} - 1 \right), & w_1 = w_2 = \bar{w} \\ \frac{p}{w_2} - 1, & w_1 > w_2 \end{cases}$$

$$y(p, w) = f(x_1(p, w), x_2(p, w))$$

$$= \begin{cases} 0, & p \leq \hat{w} \\ \log \left(\frac{p}{\hat{w}} \right), & p > \hat{w} \end{cases}$$

$$\hat{w} = \min^{\circ} \{ w_1, w_2 \}$$

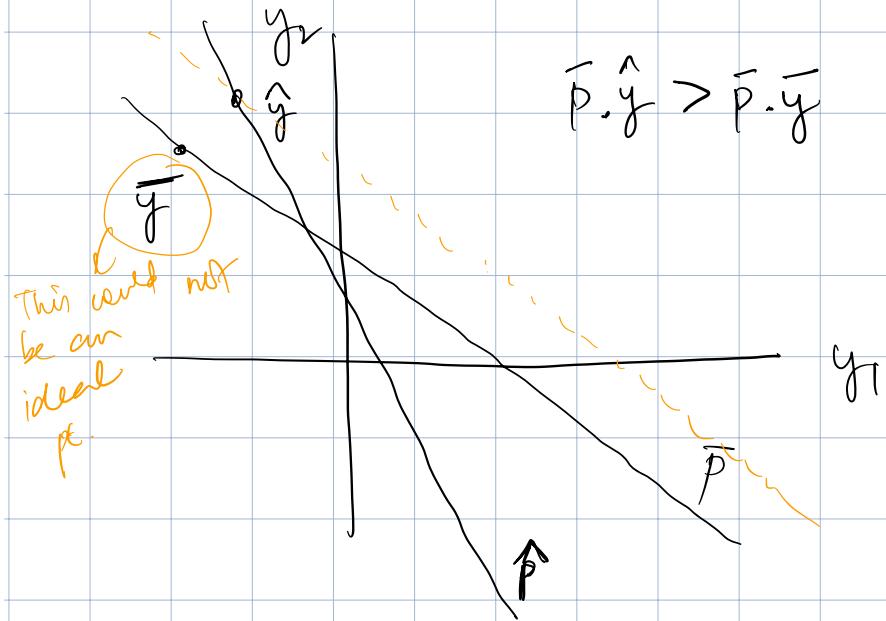
$$\pi(p, w) = \begin{cases} 0, & p \leq \hat{w} \\ \log\left(\frac{p}{\hat{w}}\right), & p > \hat{w} \end{cases}$$

6 September

Rationalizability :

Q: Given some data $\{(p^1, y^1), \dots, (p^T, y^T)\}$

Can we learn about the firm's technology?

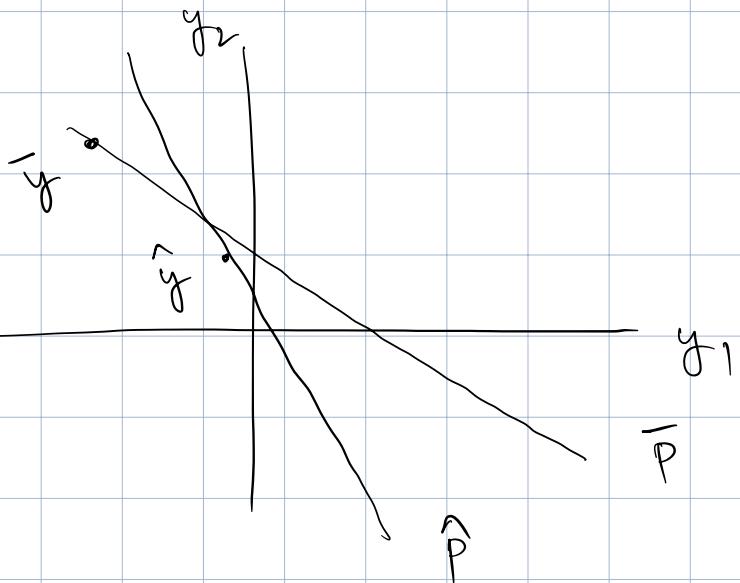


Weak Axiom of Profit Maximization

$$p^t \cdot y^t \geq p^t \cdot y^s$$

$$p^i \cdot y^i \geq p^i \cdot y^s \quad \forall s = 1, \dots, T$$

WAPM is a necessary condition for profit maximization.



What other production sets could "rationalize" the data?

A production set γ rationalize the data if
 $y_t \in \gamma^*(p^t)$ & $t = 1, \dots, T$

$$\gamma^*(p) = \arg \max_{y \in \gamma} p \cdot y$$

(optimal choices of the problem.)

Make some additional assumptions:

- data satisfies WAPM
- γ is convex & has free disposal

$$Y^I = \text{conv}_{\text{fd}}(\{y^1, \dots, y^T\})$$

Does Y^I rationalize the data?

Claim: Yes.

Proof: Assume w.l.o.g. Then \exists a point $y \in Y^I$ s.t. $p^t \cdot y > p^t \cdot y^t$ for some t .

By convexity b.F.O.,

(Need to add notes from photo)

$$Y^0 = \{ y \in \mathbb{R}^n : p^t y \leq \pi(p^t), \forall t=1, \dots, T \}$$

$\hat{y} \notin Y^0$ because $p^2 \cdot \hat{y} > p^2 \cdot y^2 = \pi(p^2)$

$\mathbb{R}^n \setminus Y^0$: bundles that yield higher profits than some observed choice.

$$= \{ y \in \mathbb{R}^n : p^t \cdot y > \pi(p^t) \text{ for some } t \}$$

Does Y^0 rationalize the data?

Claim: Yes.

Proof: Let's assume not. Then, \exists some $y \in Y^0$ s.t.
 $p^t \cdot y > p^t \cdot y^t$

Directly contradicts the definition of Y^0 .

Theorem: For any production set Y that rationalizes the data, we have

$$Y^I \subseteq Y \subseteq Y^0$$

(Proof in the HW → draw picture & look for a contradiction.)

What if we had infinite data?

Suppose we observe $y(p)$ for all p .

Let $\Pi(p) = p \cdot y(p)$ be the (observed) profit function.

Theorem: Suppose \mathcal{Y} is convex, closed & satisfies F.D. (free disposal)

Then, $\mathcal{Y} = \mathcal{Y}^0 = \{y \in \mathbb{R}^n : p \cdot y(p) \leq \Pi(p)$

$$\forall p \in \mathbb{R}_{+}^n \}$$

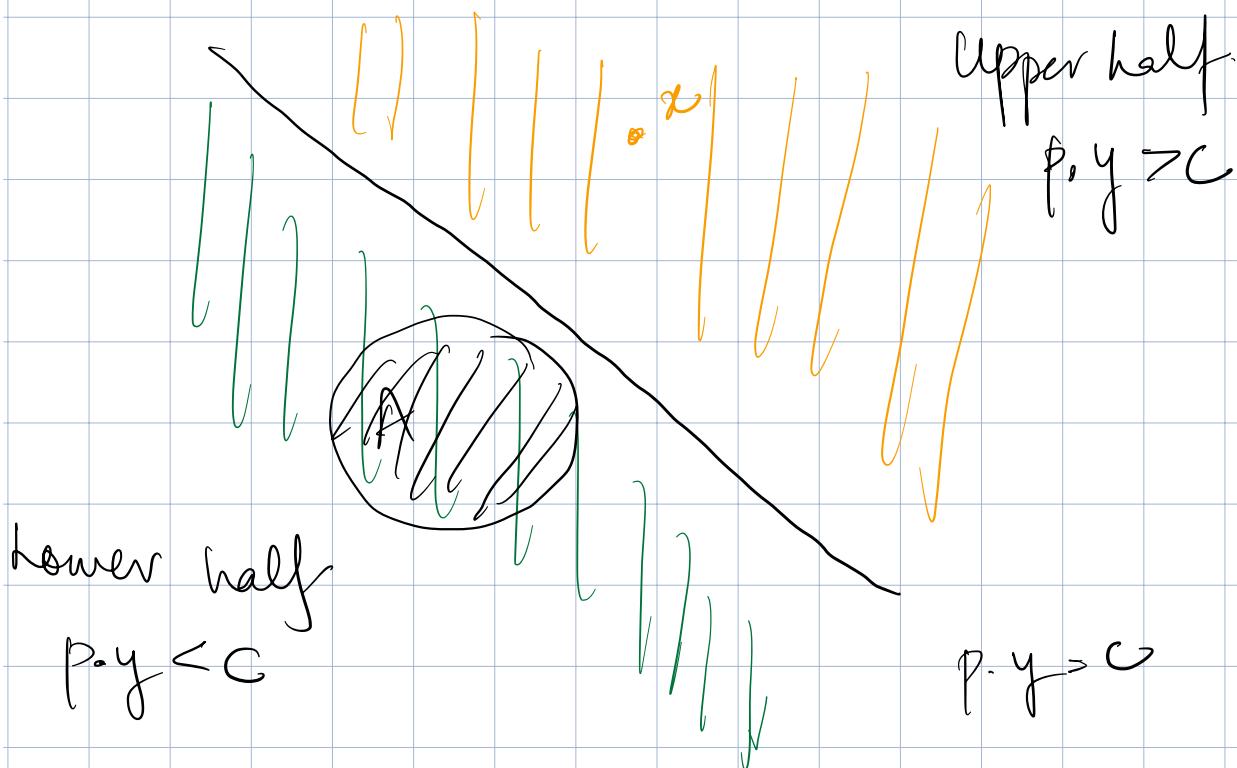
Detour: Separating Hyperplane Theorem

In \mathbb{R}^n , a hyperplane is a subspace of $\dim(n-1)$.

Given $p \in \mathbb{R}^n$, $c \in \mathbb{R}$, the

$(p-c)$ -hyperplane is

$$H_{p,c} = \{y \in \mathbb{R}^n : p \cdot y = c\}$$



Separating Hyperplane Theorem :-

Suppose $A \subseteq \mathbb{R}^n$ is convex & closed,
b $\in \mathbb{R}^n \setminus A$. Then, there is a $p \in \mathbb{R}^n$,

$p \neq 0$ & $c \in \mathbb{R}$ s.t.
 $\exists x > 0$ to $p \cdot y < c + y \in A$

Back to the theorem:-

Theorem: Suppose γ is convex, closed &
satisfies F.D. (free disposal)

Then, $\gamma = \gamma^0 = \{y \in \mathbb{R}^n : p \cdot y(p) \leq \alpha(p)\}$
 $+ p \in \mathbb{R}_+^n\}$

Proof $\gamma = \gamma^0$

NTS (need to show)

(1) $y \leq y^0$ & (2) $y^0 \subseteq \gamma$

For (1) : Take some $\hat{y} \in \mathbb{R}^n \setminus \gamma$

By SHT, if some $p \in \mathbb{R}^n$ & c s.t.

$$p \cdot \hat{y} > c > p \cdot y \quad \forall y \in Y$$

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y \quad \textcircled{*}$$

By free disposal, $p \geq 0$ &
 $p_i < 0$ for some i ,

then $\max_{y \in Y} p \cdot y = \infty$

which contradicts $\textcircled{*}$

Therefore, there exists $p \geq 0$ s.t.

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y = \pi(p)$$

i.e. $\hat{Y} \not\subseteq Y^0$ or $Y^0 \subseteq Y$

25 Sep

$$C(w, y) = \min_{\substack{\text{c.t.} \\ f(x) \geq y}} w \cdot x$$

For now write as $c(y)$

Average Cost : $AC(y) = \frac{c(y)}{y}$

Marginal Cost : $MC(y) = c'(y)$

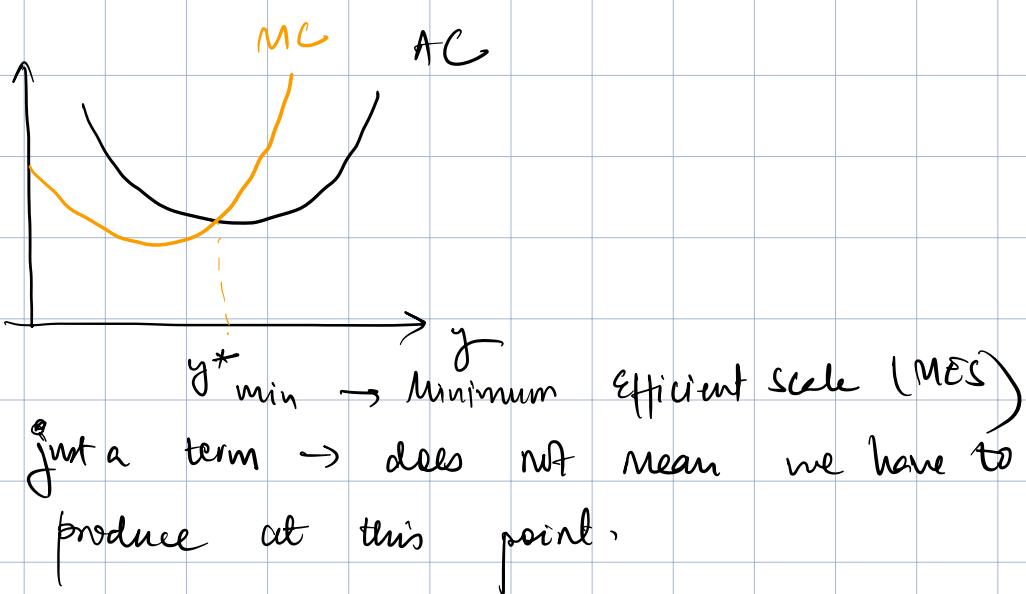
Marginal-Average Relationship

$$\begin{aligned} AC'(y) &= \frac{c'(y)}{y} - \frac{c(y)}{y^2} \\ &= \frac{c'(y)y - c(y)}{y^2} \quad \text{Denom} \geq 0 \end{aligned}$$

$$AC \text{ incr} \Leftrightarrow c'(y) > \frac{c(y)}{y} \text{ or } MC(y) > AC(y)$$

$$AC \text{ decr} \Leftrightarrow c'(y) < \frac{c(y)}{y} \text{ or } MC(y) < AC(y)$$

$$AC'(y) = 0 \Leftrightarrow MC(y) = AC(y)$$



①

$$\max_{y \geq 0} py - c(y)$$

FONC: (First Order Necessary Condition)

$p \leq c'(y^*)$, with equality if $y^* > 0$

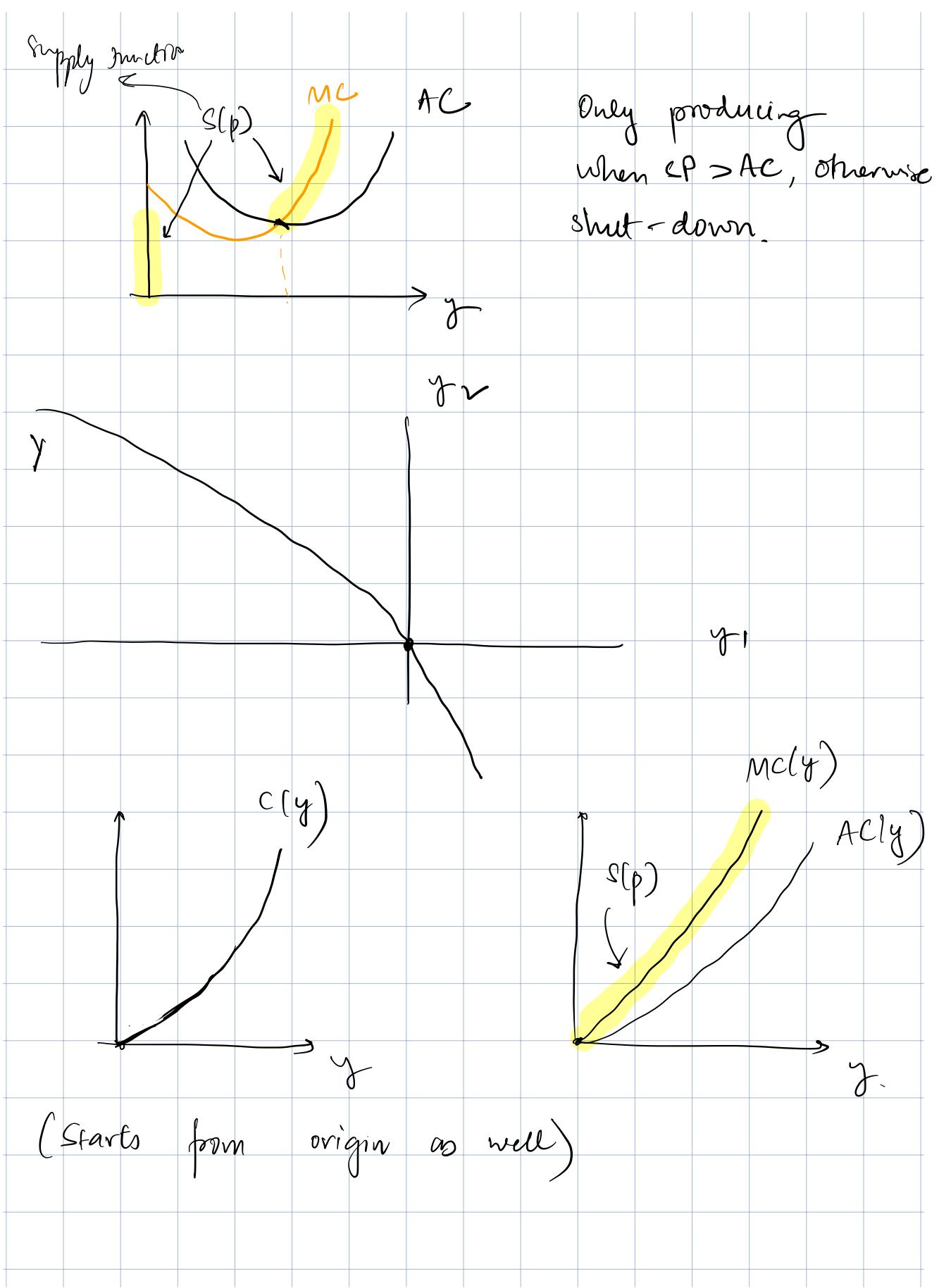
If $c(y)$ is convex, this is also sufficient.

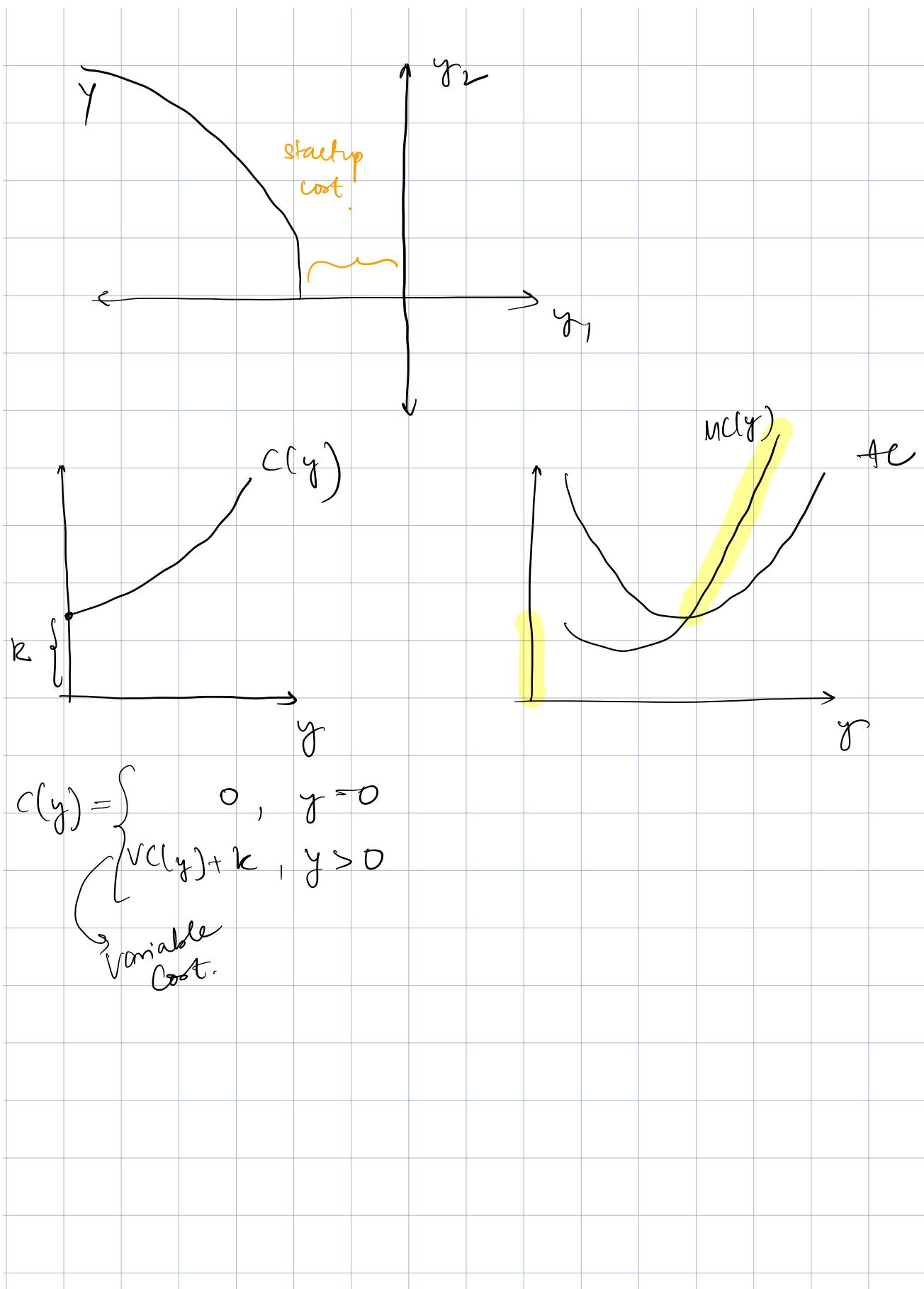
price = MC if firm chooses to produce a strictly positive amount.

Shut Down Rule: Check that at y^* where $p = c'(y^*)$,

$$py^* - c(y^*) \geq 0$$

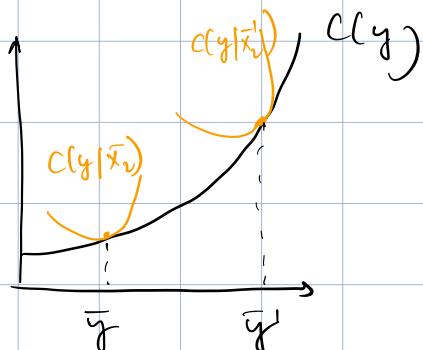
$$\text{or } p \geq AC(y^*)$$





Long-run vs short-run costs

- 2 inputs (x_1, x_2)
- $c(y)$ be the long run cost function
- Let \bar{x}_2 be optimal x_2 when long run output is \bar{y} .
- $c(y|\bar{x}_2)$ short run cost, when x_2 fixed at \bar{x}_2 .



The long run cost curve will be lower than the short run cost curves except at the optimal pts. \rightarrow It's like a lower envelope.

- ① $f(x) \rightarrow c(w,y)$. Can we go from $c(w,y) \rightarrow f(x)$?

Yes - they are duals of each other.

- Observe $c(w,y) \nabla (w,y)$

- $V(y) = \{x : f(x) \geq y\}$

If we know $V(y)$ for all y , then we know $f(x)$

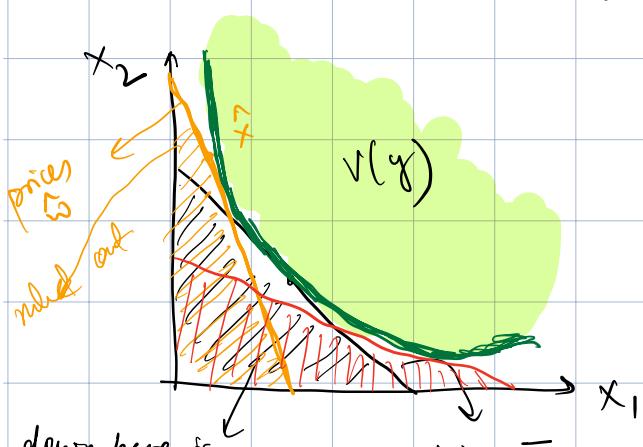
Can we recover $V(y)$?

Define an "outer bound".

$$V^0(y) = \{x : w \cdot x \geq c(w,y), \forall w \geq 0\}$$

at \bar{w}_1 , cost is $c(\bar{w}, y) = \bar{c}$. Firm chose \bar{x}

$$\bar{w}_1 x_1 + \bar{w}_2 x_2 = \bar{c}$$



down here is
cheaper but firm chose
to not produce it because
it does not produce enough.

As we draw more & more lines, we recover $V(y)$.

Claim: Suppose $V(y)$ is the input requirement set for a firm with a convex technology with free disposal. Then $V(y) = V^o(y)$

Proof: Show (use separating hyperplane theorem)

- Presupposes the function we start with is a "valid" cost function.
- Question - Given an arbitrary function $\Phi(w, y)$ how do we know if Φ is a valid cost function? That is, does:

$$\Phi(w, y) = \min_{\text{st. } x \in V(y)} w \cdot x$$

$$\text{where } V(y) = \{x \geq 0, w \cdot x \geq \Phi(w, y) + w^y\}$$

• Showed before that if Φ is a cost function, then:

$$(1) \text{ hd-1 in prices: } \Phi(tw, y) = t\Phi(w, y)$$

$$(2) \Phi(w, y) \geq 0 \text{ (positivity)}$$

$$(3) \Phi(w', y) \geq \Phi(w, y) \text{ for } w' \geq w$$

(4) Concave in w

Theorem: Let $\Phi(w, y)$ be a differentiable function satisfying 1-4, then $\Phi(w, y)$ is the cost function for the technology

$$V(y) = \{x : w \cdot x \geq \Phi(w, y), +w\}$$

Proof :- Define

$$x(w, y) = \left(\frac{\partial \Phi(w, y)}{\partial w_1}, \dots, \frac{\partial \Phi(w, y)}{\partial w_n} \right)$$

By monotonicity, $x(w, y) \geq 0$

$$\text{By hd-1, } \Phi(w, y) = \sum_{i=1}^n w_i \underbrace{\frac{\partial \Phi(w, y)}{\partial w_i}}_{\text{Euler's law}} = w \cdot x(w, y)$$

WTS for any $w \geq 0$,

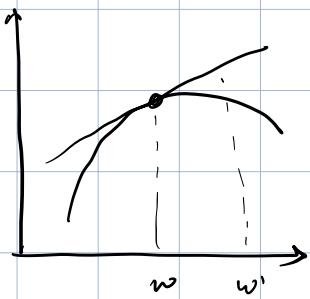
$$\phi(w, y) = w \cdot x(w, y) \leq w \cdot x \quad \forall x \in V(y)$$

This will imply:

$$\phi(w, y) = \min_{\text{st. } x \in V(y)} w \cdot x \quad (\text{so long as } x(w, y) \text{ is feasible.})$$

To show: $x(w, y)$ is feasible

$$\phi(w', y) \leq \phi(w, y) + \underbrace{\nabla_w \phi(w, y)(w' - w)}_{x(w, y)} \quad \text{for any } w'$$



$$\phi(w', y) \leq \phi(w, y) + w' \cdot x(w, y) - \underbrace{w \cdot x(w, y)}_{\phi(w, y)}$$

$$\phi(w', y) \leq w' \cdot x(w, y) \quad \text{for all } w'.$$

$$\text{So, } x(w, y) \in V(y)$$

$x(w, y)$ is feasible

Now notice that for any $x \in V(y)$

$$w \cdot x \geq \phi(w, y) = w \cdot x(w, y)$$

$$\text{or } w \cdot x(w, y) \leq w \cdot x \quad \forall x \in V(y)$$

$$\text{i.e. } x(w, y) = \arg \min_{x \in V(y)} w \cdot x$$

$$\phi(w, y) = w \cdot x(w, y)$$

Example :-

$$c(w, y) = y(w_1 + w_2)$$

\Rightarrow satisfies the 4 criteria.

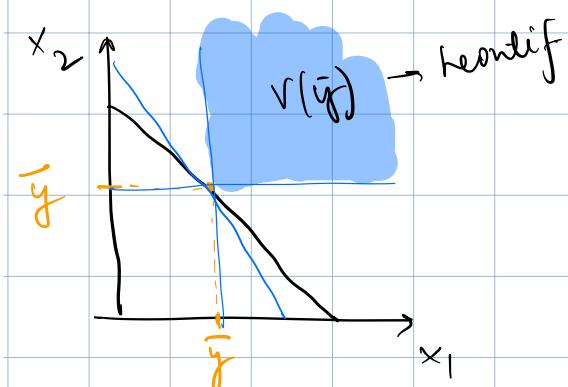
Fix an output \bar{y} . Graphically

Want points (x_1, x_2) s.t.

$$w \cdot x \geq c(w, \bar{y}) + (w_1, w_2)$$

$$w_1 x_1 + w_2 x_2 \geq w_1 \bar{y} + w_2 \bar{y}$$

$$x_2 \geq \frac{w_1}{w_2} (\bar{y} - x_1) + \bar{y}$$



Algebraically :-

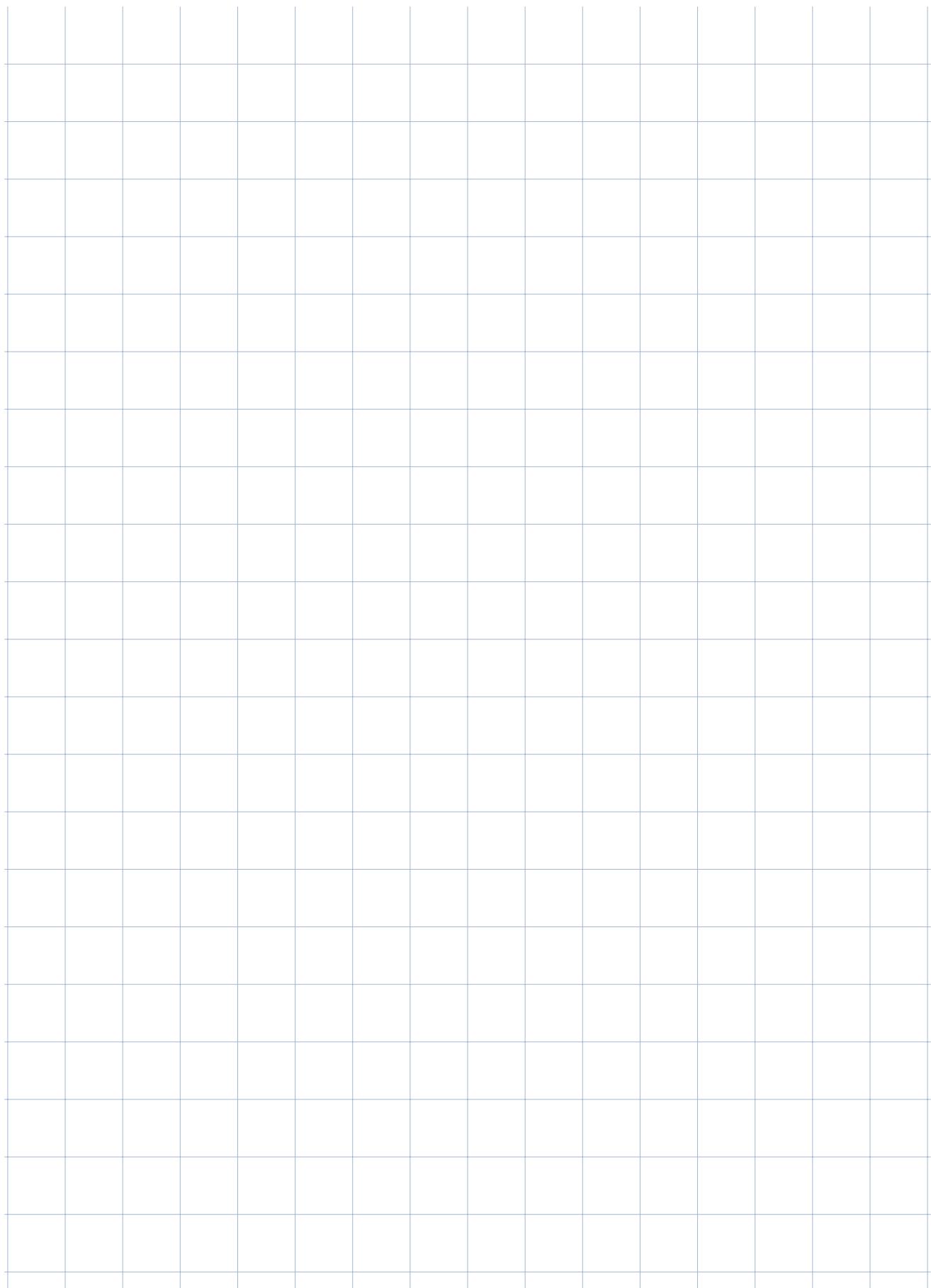
By Shephard's lemma:

$$\frac{\partial c}{\partial w_1} = x_1^*, \quad \frac{\partial c}{\partial w_2} = x_2^*$$

$$\text{Here: } \bar{y} = x_1^*(w, y)$$

$$\bar{y} = x_2^*(w, y)$$

\Rightarrow Leontif production function



4th Oct

Monopoly

- Can "choose" output price (p)
- still face some constraints
 - Technological constraints (cost function $c(y)$)
 - consumer constraints (demand $f^n D(p)$)

Monopolist's problem:

$$\begin{aligned} & \max_{p,y} p y - c(y) \\ & \text{s.t. } D(p) = y \end{aligned}$$

$$\max_p p D(p) - c(D(p)) \quad (\text{single variable})$$

Another way uses inverse demand function

$$p(y) = D^{-1}(y)$$
$$\max_{y \geq 0} p(y) y - c(y)$$

FOC: $p'(y^*) y^* + p(y^*) \leq c'(y^*)$, with equality if $y^* > 0$

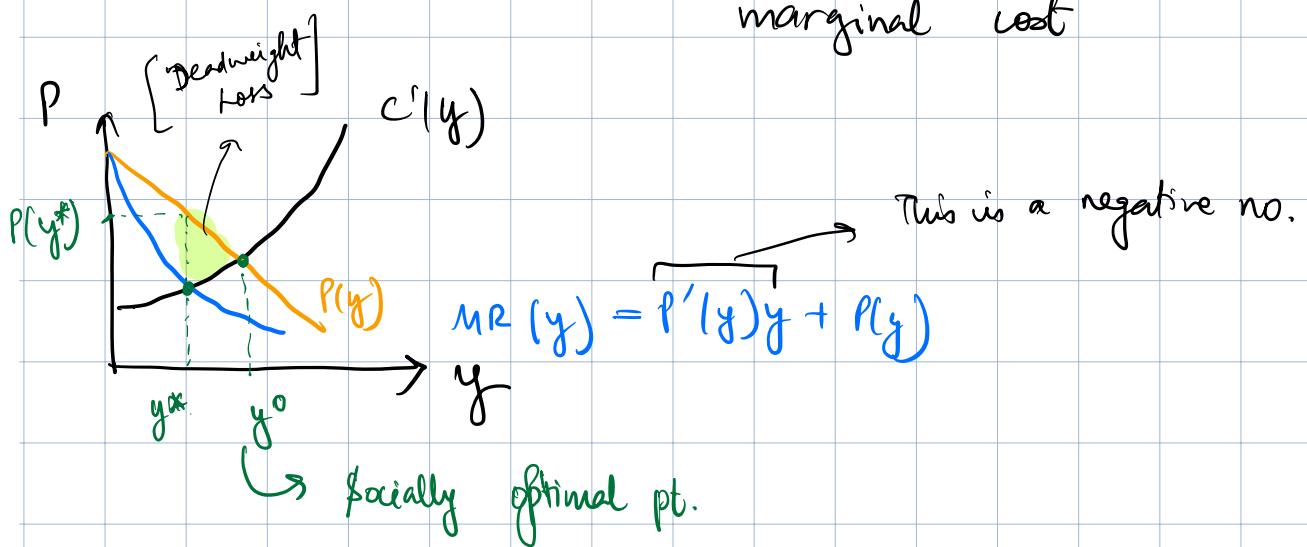
marginal revenue marginal cost

Assume $y^* > 0$, so $MR = MC$

$$p'(y^*) \cdot y^* + p(y^*) = c'(y^*)$$

Usual case: $p'(y) < 0$ (downward sloping demand)

$\Rightarrow p(y^*) > c'(y^*)$ Monopolist prices above marginal cost



Q how does monopolists price/output change as a parameter changes?

Tool here is the implicit function theorem. How does a solⁿ fn change as a parameter changes

Contrast with the envelope theorem which applies to the value function.

LFT "recipe" :

- Start w/ FOC from max. problem
- Write solution function $x^*(\theta)$ implicitly as a fn of parameter θ
- Differentiate FOC wrt to θ (chain rule)
- Solve resulting eqn for $\frac{dx^*(\theta)}{d\theta}$

Back to monopolist

Costs : $c(y, \theta)$

$$\text{FOC} : p'(y^*)y^* + p(y^*) - \frac{\partial c(y^*, \theta)}{\partial y} = 0$$

Write y^* implicitly as a fn of θ :

$$p'(y^*(\theta))y^*(\theta) + p(y^*(\theta)) - \frac{\partial c(y^*(\theta), \theta)}{\partial \theta} = 0$$

$$p''(y^*(\theta)) \frac{dy^*(\theta)}{d\theta} + p'(y^*(\theta)) \cdot \frac{dy^*(\theta)}{d\theta} + p'(y^*(\theta)) \cdot \frac{dy^*(\theta)}{d\theta}$$

$$-\frac{\partial^2 c(y^*(\theta), \theta)}{\partial y^2} \cdot \frac{dy^*}{d\theta} - \frac{\partial^2 c(y^*(\theta), \theta)}{\partial y \partial \theta} = 0$$

Solve for $\frac{dy^*}{d\theta}$

$$\frac{dy^*}{d\theta} = \frac{\partial^2 c(y^*, \theta) / \partial y \partial \theta}{p''(y^*) y^* + 2p'(y^*) - \frac{\partial^2 c(y^*, \theta)}{\partial y^2}}$$

Example: CES demand, constant marginal cost

$$D(p) = A p^{-\beta}, \quad \beta > 1$$

$$C(y) = \tilde{c} y$$

Why is it called CES?

Calculate price elasticity of demand:

$$\sigma_{D,p} = \frac{d \log D(p)}{d \log p}$$

$$\log D(p) = \log A - \beta \cdot \log p$$

$$\sigma_{D,p} = -\beta$$

Inverse Demand

$$y = A p^{-\beta} \Rightarrow p(y) = \left(\frac{y}{A}\right)^{-1/\beta}$$

FOC: for monopolist:

$$p'(y)y + p(y) = c'(y)$$
$$p'(y) = -\frac{1}{\beta} \left(\frac{y}{A}\right)^{-1/\beta-1} \cdot \left(\frac{1}{A}\right)$$

$$= -\frac{1}{\beta} p(y) \frac{1}{y}$$

FOC becomes:

$$\left(-\frac{1}{\beta} p(y) \frac{1}{y}\right) y + p(y) = \tilde{c}$$

$$p(y) \left[1 - \frac{1}{\beta}\right] = \tilde{c}$$

$$p(y) = \frac{\tilde{c}}{1 - 1/\beta}$$

Monopolist's price markup
depends on β which is
elasticity of dd.

$c \rightarrow$ here is \tilde{c} (a constant)

what is $\frac{dy^*}{d\tilde{c}}$?

$y^*(\tilde{c})$ defined implicitly by

$$p(y^*(\tilde{c})) \left[1 - \frac{1}{\beta} \right] - \tilde{c} = 0$$

(FT:

$$p'(y^*(\tilde{c})) \left[1 - \frac{1}{\beta} \right] \frac{dy^*}{d\tilde{c}} - 1 = 0$$

$$\frac{dy^*}{d\tilde{c}} = \frac{1}{p'(y^*(\tilde{c})) \left[1 - \frac{1}{\beta} \right]} < 0$$

$\underbrace{p'(y^*(\tilde{c}))}_{<0}$ $\underbrace{\left[1 - \frac{1}{\beta} \right]}_{>0}$

Consumer Theory

More complicated for few reasons:-

- Need to impose "rationality" axioms on preferences
- Can't use optimization techniques on preferences
Need to construct a utility fn.
- Consumer problem has forces in the constraint
income / substitution effects .

Widely used in economics:

- ① Normatively useful
- ② Positive predictions
- ③ Widely applicable
- ④ Simple / sparse model

Two approaches:-

- ① Preferences → Choices
- ② Choices → Preferences

Starting with ② :-

Let X : (abstract) set of alternatives

elements $x \in X$ are mutually exclusive

A choice structure $(\mathcal{B}, C(\cdot))$ is :-

- \mathcal{B} is a family of nonempty subsets of X
- $C(\cdot)$ is a choice rule s.t. $C(B) \subseteq B \neq \emptyset$

$$X = \{x, y, z\}$$

$$\mathcal{B} = \{\{x, y\}, \{x, z\}\}$$

$$C(\{x, y\}) = \{x\}, \quad C(\{x, z\}) = \{x\}$$

acceptable alternatives

What are some "reasonable" restrictions on behaviour?

° Weak Axiom of Revealed Preference (WARP)

If for some $B \in \mathcal{B}$ with $x, y \in B$, we have

$x \in C(B)$, then for any other $B' \in \mathcal{B}$ with

$x, y \in B'$, and $y \in C(B')$ we also have
 $x \in C(B')$.

$$X = \{a, b, c\} \quad B = \{\{a, b\}, \{a, b, c\}\}$$

We observe $C(\{a, b, c\}) = \{b\}$

WARP implies that $C(\{a, b\}) = \{b\}$

Why? Assume $a \in C(\{a, b\})$, then since a was chosen when b was available, we must have $a \in C(\{a, b, c\})$. Contradiction.

$$C(\{a, b\}) = \{b\}$$

WARP implies $C(\{a, b, c\}) = \{b\}, \{c\}; \{b, c\}$
(doesn't say anything about C)~~or~~.

Preference-based Approach :-

starting pt. of the model.

X set of alternatives

Primitive is a binary relation on X, \geq

$x \geq y$: "x is at least as good as y"

$x \succ y$: $x \geq y$ & $y \neq x$ "x is strictly preferred to y".

$x \sim y$: $x \geq y$ & $y \geq x$ "x is indifferent to y"

A preference relation is rational if it is:-

1. Complete: $\forall x, y \in X$, either $x \geq y$ or $y \geq x$ (or both)
2. Transitive: $\forall x, y, z \in X$, if $x \geq y$ & $y \geq z$, then $x \geq z$.

Violation of transitivity : framing (Kahneman, Tversky)

Best buy to purchase a \$125 stereo & \$15 calculator

- (A) Calculator on sale for \$5 off at a store 20 min. away Yes
- (B) Stereo in store A is \$125
Calculator in store B is \$15 No
- (C) Both are out of stock, but the store gives you \$5 coupon for either items. Does it matter which?
(still travel to store B for purchase). No

x = travel to store 2 for \$5 off calculator

y = \$5 off stereo

z = buy both items at store 1.

Responses imply: $x \geq z$ (A)

$z \geq y$ (B)

$x \geq y$ (C)

What are the links b/w choice & preference approach?

Q1 Does rationality imply WARP? Yes

Q2 Does WARP imply rationality? No

For Q1: Given \succsim , let $C(B, \succsim) = \{x \in B : x \succsim y \text{ & } y \in B\}$

Theorem: Suppose \succsim are rational. Then, the choice structure $(B, C(B, \succsim))$ satisfies WARP.

Proof: Suppose for some $B \subseteq \mathbb{B}$, we have $x, y \in B$ & $x \in C(B, \succsim)$. By def'n of C , $x \succsim y$.

Check WARP:

Suppose $x, y \in B'$, & $y \in C(B', \succsim)$. So, $y \succsim z \text{ & } z \in B'$.

By transitivity, $x \succsim y \succsim z$, so $x \succsim z \text{ & } z \in B'$.

Thus $x \in c(\mathcal{B}', \succ)$ and WARP holds.

Counterexample for Q2:

$$X = \{a, b, c\} \quad \mathcal{B} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$$

Assume we observe re'

$$c(\{a, b\}) = a \quad c(\{a, c\}) = c \quad c(\{b, c\}) = b$$

Can check that $(\mathcal{B}, c(\cdot))$ satisfies WARP:

Observed choices imply:-

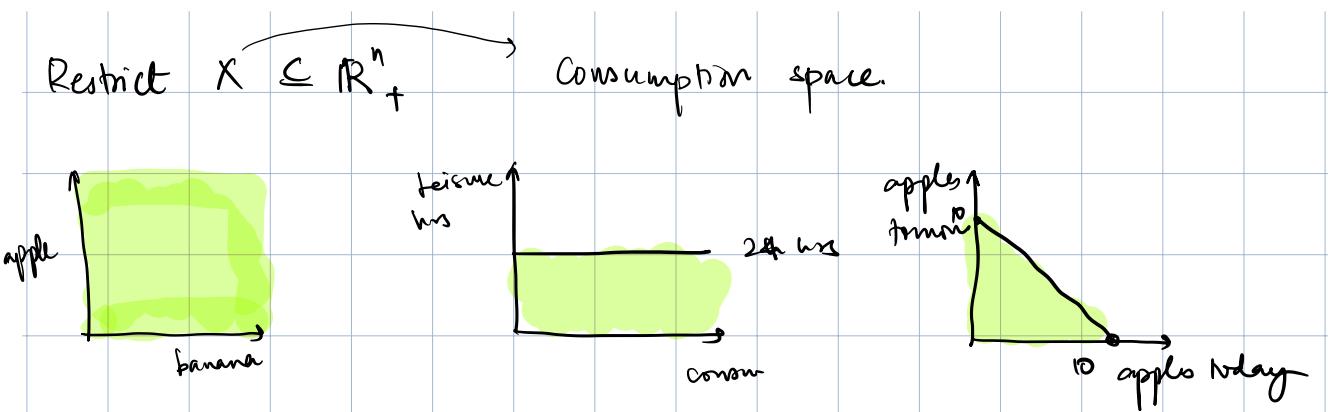
$$\begin{array}{l} a > b \\ b > c \\ a > c \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{violates transitivity}$$

Theorem (Arrow, 1959):

If $(\mathcal{B}, c(\cdot))$ is s.t.:

- i) WARP is satisfied.
- ii) \mathcal{B} includes all subsets of X with upto 3 elements

then, there is a (unique) rational preference relation \succ that rationalizes $c(\cdot)$



State dependent cons^m



Properties of preferences on $X \subseteq \mathbb{R}_+^n$

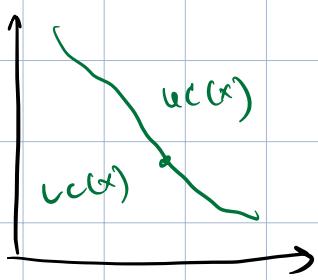
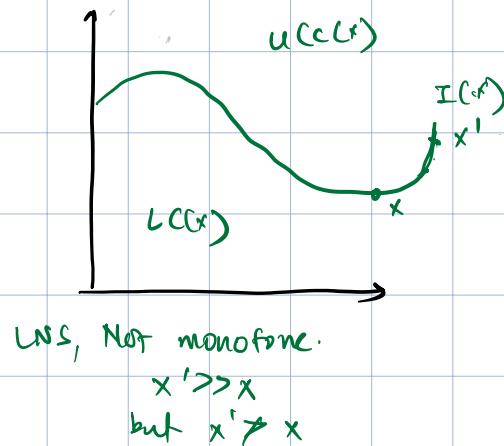
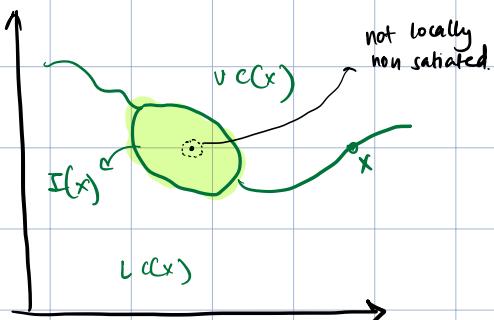
- Preference relation \succsim is monotone if
 $y \succsim x$ implies $y > x$
- \succsim is strongly monotone if $y \succsim x$ & $y \neq x$
implies $y > x$.
- \succsim is locally non-satiated if for every $x \in X$ &

$\epsilon > 0$, \exists some $y \in X$ s.t. $|y - x| \leq \epsilon$ b/c $y > x$.

$UC(x) \rightarrow$ upper contour of x
 $= \{y \in X : y \geq x\}$

$LC(x) = \{y \in X : x \geq y\}$ (lower contour set)

$I(x) = \{y \in X : x \sim y\}$ (indifference set)



Monotonicity \Rightarrow downward sloping ICs.

ICs cannot cross

$x \sim z$
 $x \sim y$
 $\Rightarrow z \sim y$ (transitivity)
But $z > y$ (monotonicity)
 \Rightarrow contradiction!

