ECON 7710 TA Session

Week 2

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Outline

Definition Matters

- It is normal to see gaps on same concepts between denis' notes and textbook(C & B)
- For example:
  - **P4 from Denis' Notes**
    - Distribution function $F(x)$ of r.v. X has the following properties
      1. Monotonicity: If $x_1 \leq x_2$ then $F(x_1) \leq F(x_2)$
      2. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$
      3. Left-continuity: $\lim_{x \uparrow x_0} F(x) = F(x_0)$
  - **P31 from C& B**

    Theorem 1.5.3 *The function $F(x)$ is a cdf if and only if the following three conditions hold:*

    a. $\lim_{x \to -\infty} F(x) = 0$ *and* $\lim_{x \to \infty} F(x) = 1$.

    b. $F(x)$ *is a nondecreasing function of* x.

    c. $F(x)$ *is right-continuous; that is, for every number* x_0, $\lim_{x \downarrow x_0} F(x) = F(x_0)$.
- Why? Because in denis' notes, he defined $F_X(x) = P(X < x)$, while in C& B, it is defined as $F_X(x) = P(X \leq x)$.

A Clarification of Concepts based on Wiki's Definition

- **Probability Theory:**
  - We use a set of axioms to formalise probability in terms of a **probability space**, which assigns a **measure** taking values between 0 and 1, termed the **probability measure**, to a set of outcomes called the **sample space**.
  - Our Purpose is to develop a theory that measures the probabilities systematically.
- **Concepts are related here**

Set → Sigma Algebra → Measurable Space
→ Measure → Measure Space → Probability Space

A Summary on Related Concepts

- **Set** is an arbitrary collection of items. Subset of items A of set S is called **Subset**. $A \subset S$
  - S can be the subset of itself, but **proper subsets** are all the subsets of S that not equal to S itself, $A \subsetneqq S$
  - Set with 0 elements is called **empty set**, $\varnothing$, also a subset and proper subset of S.
  - Set with n elements, $n \in \mathbb{N}^+$ have
    - 2^n subsets.
    - $2^n - 1$ non-empty subsets.
    - $2^n - 1$ proper subsets.
    - $2^n - 2$ non-empty proper subsets.
- **Sample Space**, a.k.a. **Sample Description Space**, **Possibility Space**, **Outcome Space**, is the set of all possible outcomes, usually as $\mathbf{\Omega}$, S, U.
- **Event** is a subset of sample space to which a probability is assigned, usually as **E**. If the outcome of an experiment is included in **E**, then event **E** has occurred.
- $\mathbf{\Omega}$ can be **E** as well.

A Summary on Related Concepts

- **Sigma Algebra**
  - A **collection** of subsets of S, denoted by $\mathcal{F}$(notes), $\mathcal{B}$(CB) that:
    - $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$. Closed under complement
    - $A_1, A_2, ..., \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. Closed under countable unions.
    - $\emptyset \in \mathcal{F}$, Empty set as an element.
  - **Example:** $S = \{a, b, c, d\}$, then one possible sigma algebra on S is $\{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$
- **Borel set**
  - **A Borel Set:** Any set in a topological space that can be formed from open sets(, or equivalently, from closed sets) through the operations of countable union, countable intersection and relative complement.
  - **Example:** (),[],(],[) $+ \cap, \cup, {}^C$
- Sigma algebra induced by all segments of real line is called **Borel Sigma Algebra**.

A Summary on Related Concepts

- **Measurable Space** or **Borel Space** $(\boldsymbol{\Omega}, \mathcal{F})$.
  - A **Measurable Space** consists of a **set** and a **sigma algebra**, which defines the subsets that will be measured.
  - **Example:** Define a set $A = \{1, 2, 3\}$, and one possible sigma algebra is $\mathcal{A}_1 = \{A, \varnothing\}$, then $(A, \mathcal{A}_1)$ is a measurable space.
- **Measurable Function** $f : (A, \mathcal{A}) \to (B, \mathcal{B})$
  - Given two measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$, function $f : A \to B$ is **measurable** if for every $S_a \in \mathcal{B}$, the pre-image of S_a under f is in $\mathcal{A}$. Or

    $$\forall S_a \in \mathcal{B}, f^{-1}(S_a) := \{x \in A | f(x) \in S_a\} \in \mathcal{A}$$

    That is $\sigma(f) \subseteq \mathcal{A}$, where $\sigma(f)$ is the sigma algebra generated by f.
- **Measure**
  - A **Measure** on a set is a systematic way to assign a number to each suitable subset of that set, intuitively interpreted it as size.
  - A measure is defined on a measurable space.
  - **Examples:** **Probability Measure**(notes) or **Probability Function**(CB)

    Length, area, volume, magnitude, mass, probability...

A Summary on Related Concepts

- **Measure Space:** $(\mathcal{X}, \mathcal{A}, \mu)$
  - A **Measure Space** consists of a **measurable space** and a **measure**.
  - A measure space is a triple $(\mathcal{X}, \mathcal{A}, \mu)$ where
    - X is a set.
    - $\mathcal{A}$ is a sigma algebra on set X.
    - μ is a measure on $(X, \mathcal{A})$.
- **Example: Probability space:** $(\boldsymbol{\Omega}, \mathcal{F}, P)$, where
  - $\boldsymbol{\Omega}$ is a **sample space** that is the set of **all possible outcomes**.
  - $\mathcal{F}$ is an **event space**, which is a set of subsets of $\boldsymbol{\Omega}$, such that it satisfy the properties of sigma algebra.
  - **Probability measure**: P is a set function on a probability space that:
    - $\forall A \in \mathcal{F},\ P(A) \geq 0$
    - $P(\boldsymbol{\Omega}) = 1$
    - $\forall \{A_i\}_{i=1}^{\infty}$ s.t. $A_i \cap A_j = \varnothing$, $P(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1} P(A_i)$
- **Random variable** is a **measurable function** from a **probability space** to a **measure space**

Enough for these boring concepts?

Lets' play Poker Cards!

Introduction to Poker Cards

- Probability and Combinatorics of Poker Cards is one type of question that Denis loves a lot.
- Basic Facts about Poker Cards.
  - 4 Suits: ♥ ♦ ♣ ♠
    I know that the ♠ are the swords of a soldier. I know that the ♣ are weapons of war. I know that ♦ mean money for this art. But that's not the shape of my ♥.- Shape of My Heart, Sting, 1993
  - 13 Ranks. A 2 3 4 5 6 7 8 9 10 J Q K
  - In all, there are $4*13 = 52$ different cards.
  - Jokers? Sorry, you are banned unless Denis wants to take you out.

An Introduction on Poker Card Game from C & B P16-17

Consider choosing a **five-card** poker hand from a standard deck of 52 cards.

- To specify possible outcomes, obviously it is the case of without replacement. What we care about is **ordered or unordered** here.
    - If we think of the hand as being dealt **sequentially**, use **ordered** without replacement $\frac{n!}{(n-k)!}$. Say we want to know the probability of an ace in the **first two cards**.
    - If we think of the hand as being dealt **at once**, use **unordered** without replacement $\binom{n}{k}$ Say we want to know the result of **this 5 cards draw.**
- How many possible hands? $\binom{52}{5}$. If we believe deck is well shuffled and cards are randomly dealt, each possible hand's probability is $\frac{1}{\binom{52}{5}}$

An Introduction on Poker Card Game from C & B P16-17

Consider choosing a **five-card** poker hand from a standard deck of 52 cards.

- What's the probability of having four aces from this five-card draw?

$$P(4As) = \frac{48}{\binom{52}{5}}$$

- What's the probability of having four of a kind? 2 steps
  1. Choose which rank from A to K is the chosen one with 4 suits collected
  2. Choose the 5th freely specifying card
- **Fundamental Theorem of Counting**: If a job consists of k separate tasks, the ith of which can be done in n_i ways, $i = 1, 2, ..., k$, then the entire job can be done in $n_1 \times n_2 \times ... \times n_k$ ways. Proof on CB P14

$$P(4 \text{ of a kind}) = \frac{13 * 48}{\binom{52}{5}}$$

An Introduction on Poker Card Game from C & B P16-17

Consider choosing a **five-card** poker hand from a standard deck of 52 cards.

- What's the probability of having **exactly one pair**? 4 steps:
  - Which rank we use to pair? A to K, 13 ways.
  - For this pair, which are the two suits we choose to use? $\binom{4}{2}$
  - For, the rest three positions, what are the available ranks now? $\binom{12}{3}$.
  - What are the suits that the remaining three cards can use? 4^3

  Therefore, we have:

$$P(\text{Exactly one pair}) = \frac{13 * \binom{4}{2} * \binom{12}{3} * 4^3}{\binom{52}{5}}$$

Practice Question on Poker Card Game

2019 Midterm Q1 13 cards were randomly pulled from the deck of 52 cards. Find the probability that these 13 cards contains exactly k pairs "ace and king" from the same suit.

Hint:

1. What are the values k can be.
2. How should I specify each tasks in this job.
3. What is the probability I am calculating.

Practice Question on Poker Card Game

2019 Midterm Q1 13 cards were randomly pulled from the deck of 52 cards. Find the probability that these 13 cards contains exactly k pairs "ace and king" from the same suit.

♥ AK ♦AK ♣AK ♠AK

Ans:

- Obviously, k can be 1,2,3,4 as there are 4 units. We need to calculate 4 probabilities: $P(k = 1)$, $P(k = 2)$, $P(k = 3)$, $P(k = 4)$
- First, we can easily know the number of ways that 13 cards can be chosen from the deck: $\binom{52}{13}$
- Starting from the extreme case, $k = 4$.
  - All 4 suits are used which is $\binom{4}{4} = 1$. After that 8 cards are gone, we are free to choose 5 cards from the remaining 44 cards. So:

$$P(k = 4) = \frac{\binom{4}{4}\binom{44}{5}}{\binom{52}{13}}$$

Practice Question on Poker Card Game

2019 Midterm Q1 13 cards were randomly pulled from the deck of 52 cards. Find the probability that these 13 cards contains exactly k pairs "ace and king" from the same suit.

♥ AK ♦AK ♣AK ♠AK

Ans:

- We know:

$$P(k=4) = \frac{\binom{4}{4}\binom{44}{5}}{\binom{52}{13}}$$

- Now we move on to k=3. Now there are $\binom{4}{3}$ ways of selecting suits. After 6 cards are gone, we are free to choose 7 cards from the remaining 46 cards. So

$$P(k=3) = \frac{\binom{4}{3}\binom{46}{7}}{\binom{52}{13}}$$

- Is that correct?

Practice Question on Poker Card Game

2019 Midterm Q1 13 cards were randomly pulled from the deck of 52 cards. Find the probability that these 13 cards contains exactly k pairs "ace and king" from the same suit.

$$\heartsuit \text{ AK } \diamondsuit\text{AK } \clubsuit\text{AK } \spadesuit\text{AK}$$

Ans:

- No, this is not $P(k=3)$ but $P(k \geq 3)$. Why? Because, we double counted $k=4$ here. We can get additional one pair when we are choosing from the 7 remaining cards freely.

$$P(k \geq 3) = \frac{\binom{4}{3}\binom{46}{7}}{\binom{52}{13}}$$

So $P(k=3) = P(k \geq 3) - P(k > 3) = \frac{\binom{4}{3}\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{4}{4}\binom{44}{5}}{\binom{52}{13}}$

Practice Question on Poker Card Game

2019 Midterm Q1 13 cards were randomly pulled from the deck of 52 cards. Find the probability that these 13 cards contains exactly k pairs "ace and king" from the same suit.

♥ AK ♦AK ♣AK ♠AK

Ans:

- We know:

$$P(k=4) = \frac{\binom{4}{4}\binom{44}{5}}{\binom{52}{13}} \quad P(k=3) = \frac{\binom{4}{3}\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{4}{4}\binom{44}{5}}{\binom{52}{13}}$$

- Likewise:

$$P(k=2) = \frac{\binom{4}{2}\binom{48}{9}}{\binom{52}{13}} - \frac{\binom{4}{3}\binom{46}{7}}{\binom{52}{13}}$$

$$P(k=1) = \frac{\binom{4}{1}\binom{50}{11}}{\binom{52}{13}} - \frac{\binom{4}{2}\binom{48}{9}}{\binom{52}{13}}$$

C& B Coverage on Notes

ECON 7710

Econometrics I

Lecture notes 1.

Elements of set theory:

- Set is arbitrary collection of items; subset of items A of set S is called its subset, denote $A \subset S$
- The set with no elements is called an empty set (denoted $\emptyset$)

  Sample space is the set of all possible outcomes
- In probability usually deal with sets of "outcomes" (subsets of sample space Ω) called events

  Any subset of S including itself is called event.
- Set operations $A \cup B = \{x : x \in A \text{ or } x \in B\}$ $A \cap B = \{x : x \in A \text{ and } x \in B\}$
  1. $A \cup B = B \cup A$, $A \cap B = B \cap A$
  2. $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
  3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \setminus B = \{x : x \in A, x \notin B\}$
- For each $A \subset \Omega$, complement $A^c = \Omega \setminus A$
- Sigma-algebra defines "order" of sets
  1. $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$.
  2. $A_1, A_2, \ldots, \in \mathcal{F} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.
  3. $\emptyset \in \mathcal{F}$.

  CB P6

  Question: Show that the above conditions imply $\Omega \in \mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.
- Sigma algebra induced by all segments of real line is called Borel sigma-algebra
- $(\Omega, \mathcal{F})$ is called measurable space

1

C& B Coverage on Notes

- Function $f : A \mapsto B$ with $\mathcal{A}$- sigma algebra on A and $\mathcal{B}$- sigma algebra on B such that for any $S_a \in \mathcal{B}$, $f^{-1}(S_a) \in \mathcal{A}$ is called measurable function

Probability Space: $(\Omega, \mathcal{F}, P)$

- Random variable is a measurable function on algebra of events
- Probability measure P is a set function on $\mathcal{F}$ such that
  1. $\forall A \in \mathcal{F}$, $P(A) \geq 0$
  2. $P(\Omega) = 1$
  3. $\forall \{A_i\}_{i=1}^{\infty}$ such that $A_i \cap A_j = \emptyset$, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
- Properties of probability measure

  Proof on p10 CB

  1. $P(\emptyset) = 0$, $P(A) \leq 1$, $P(A^c) = 1 - P(A)$
  2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, $P(A) \leq P(B)$ if $A \subseteq B$
  3. $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$

  Proof on p12 CB

- A measurable space with probability measure is called the probability space

Basic combinatorics

- Factorial $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ (convention $0! = 1$)
- Binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
  - $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
- Basic problem: an urn has n different balls. How many possible combinations of k balls can be drawn from the urn.
  1. Ordered with replacement n^k

     Proof on p16 CB

  2. Ordered without replacement $n!/(n-k)!$
  3. Unordered with replacement $\binom{n+k-1}{k}$
  4. Unordered without replacement $\binom{n}{k}$

2