ECON 7710 TA Session

Final Review

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Question 1 (Core Aug 2019 Q2)

Stupid test: (Brute Force) We know each $\{X_i\}_{i=1}^n$ is a two-dimensional observation and X is a two-dimensional random variable. Wlog, we denote the two dimensions as P and Q.

Using $t_n = \frac{1}{n}\sum_{i=1}^{n} P_i Q_i$ to formulate our test statistic. $H_0 : h = 0$, $H_1 : h > 0$

- By CLT, we know when n goes large,

$$\frac{1}{n}\sum_{i=1}^{n} P_i Q_i \xrightarrow{d} N(E[PQ], \frac{Var(PQ)}{n}) \quad \Rightarrow \quad \frac{\frac{1}{n}\sum_{i=1}^{n} P_i Q_i - E[PQ]}{\sqrt{\frac{Var(PQ)}{n}}} \xrightarrow{d} N(0,1)$$

  For size and significance:

  - The significance level α of our test is $P_\theta(\text{Reject } H_0 | h = 0) \leq \alpha, \theta \in \Theta_0$, in our case Θ_0 is $\{0\}$, singleton.
  - The size of our test is $\sup_{\theta \in \{0\}} P(\text{Reject } H_0 | h = 0)$.

  Same in this test.

Question 1 (Core Aug 2019 Q2)

The power of our test is the probability of rejecting null hypothesis conditional on alternative hypothesis is true, $P(\text{Reject } H_0 | h > 0)$. When $h > 0$ is true, we know: $E[PQ] = \int_h^{h+1} \int_h^{h+1} PQdPdQ = \frac{1}{4}(2h+1)^2$ $Var[PQ] = E[(PQ - E[PQ])^2] = \int_h^{h+1} \int_h^{h+1} (PQ - \frac{1}{4}(2h+1)^2)dPdQ = \frac{1}{144}(24h^2 + 24h + 7)$ Then we know the power function could be written as:

$$Power = 1 - P(\text{Reject } H_0 | h > 0)$$

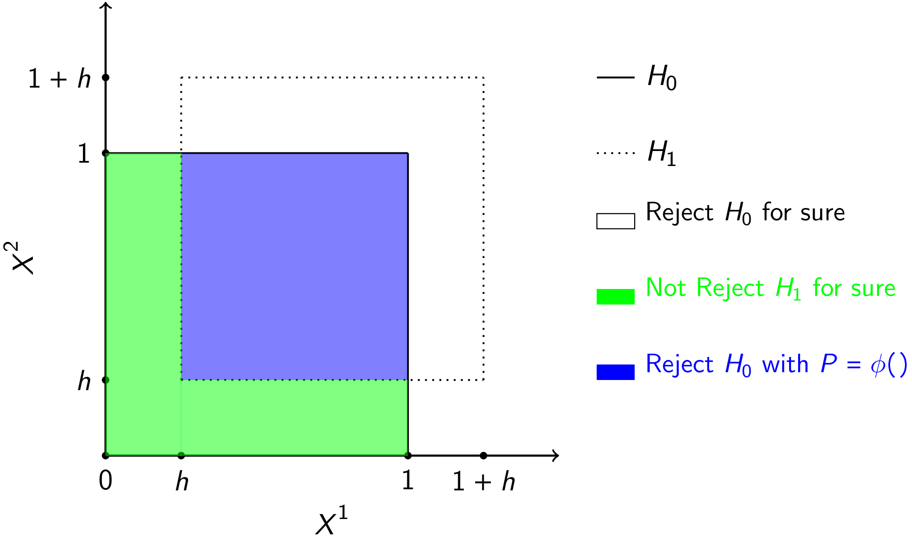
$$P(t_n > z_\alpha | h > 0) =$$

$$P(\frac{\frac{1}{n}\sum_{i=1}^{n} P_i Q_i - \frac{1}{4}}{\sqrt{\frac{7/144}{n}}} > z_\alpha | h > 0) = P(\frac{1}{n}\sum_{i=1}^{n} P_i Q_i > \frac{1}{4} + z_\alpha * \sqrt{\frac{7/144}{n}} | h > 0)$$

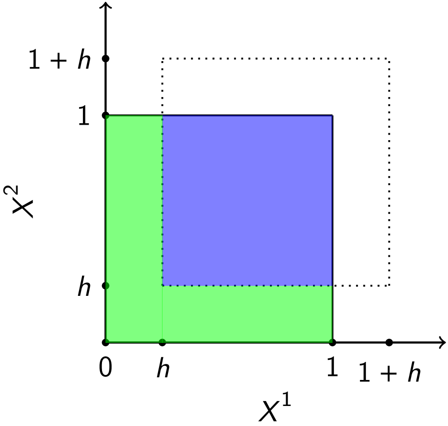
Nightmare to plot the function.

Question 1 (Core Aug 2019 Q2)

Smart test: (Kate's Test)



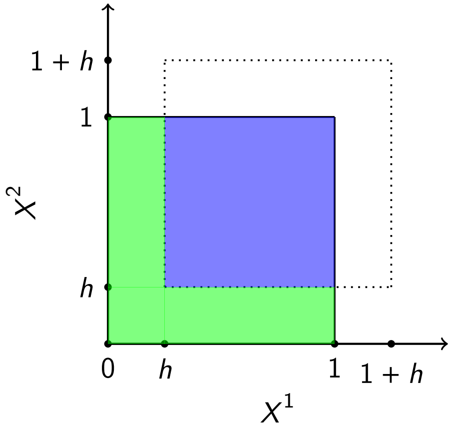
Question 1 (Core Aug 2019 Q2)



Test rule we can formulate is:

- If any X_i not in $[0,1]\times[0,1]$, we reject H_0 with probability 1;
- If any X_i in "green" area, $\min\{X_i^1, X_i^2\} < h$, we don't reject H_0 with probability 1.
- If **all** X_i in "blue" area, both $\max\{X_i^1, X_i^2\}$ and $\min\{X_i^1, X_i^2\} \in [h,1] \forall i$, we reject H_0 with $P = \phi()$
- For level α, we know $\phi(\cdot)(1-h)^{2n} = \alpha \Rightarrow \phi(\cdot) = \frac{\alpha}{(1-h)^{2n}}$

Question 1 (Core Aug 2019 Q2)



Therefore, when $0 < h < 1$, we have $\phi(x) =$

- 1; $\exists i, \min\{X_i^1, X_i^2\} < 0$ or $\max\{X_i^1, X_i^2\} > 1$
- 1 with $P = \frac{\alpha}{(1-h)^{2n}}$; $\forall i, \min\{X_i^1, X_i^2\} \geq h$ and $\max\{X_i^1, X_i^2\} \leq 1$
- 0; Otherwise

We know the size of our test is

- $0 < h < 1$, $\sup_{\theta \in \Theta_0} = P_\theta(\phi(x) = 1) = (1-h)^{2n} * \frac{\alpha}{(1-h)^{2n}} = \alpha$.
- $h \geq 1$, $\sup_{\theta \in \Theta_0} = 0 < \alpha$.

Question 1 (Core Aug 2019 Q2)

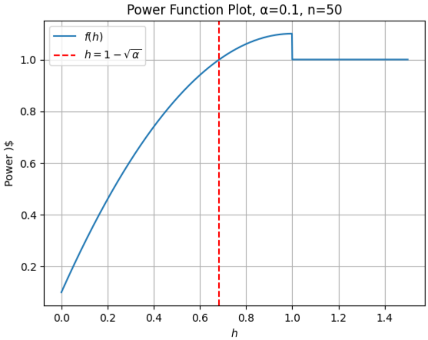


Figure: Power Function Plot

- When $0 < h < 1$, power of our test is $E[\phi(x)|H_1] = (1-h)^{2n} * \frac{\alpha}{(1-h)^{2n}} + 1 * (1 - (1-h)^{2n}) = \alpha + 1 - (1-h)^{2n}$
- When $h \geq 1$, power of our test is just 1.

This is problematic as power goes above 1... We need to change our decision rule.

Question 1 (Core Aug 2019 Q2)

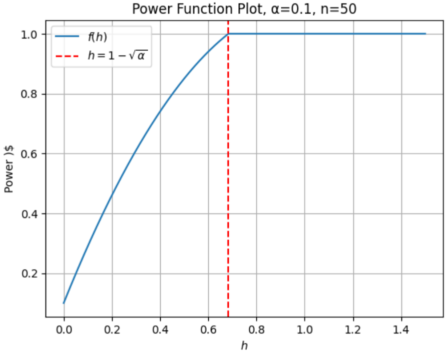


Figure: Corrected Power Function Plot

- We keep every other thing the same, just change the decision rule in blue area.
- When $\alpha > (1-h)^{2n}$, we reject with probability 1
- When $\alpha \leq (1-h)^{2n}$, we reject with probability $\frac{\alpha}{(1-h)^{2n}}$

Question 2

$$Y_i = \beta_0 + \beta_1 X_i + U_i$$

in which

$$E(U_i|X_i) = 0 \quad \text{and} \quad Var(U_i|X_i) = \sigma^2, Var(X_i) = \sigma_x^2$$

Let $(X_i, Y_i, i = 1, ..., n)$ be i.i.d. Instead of regressing y_i on a constant and x_i, you regress x_i on a constant and y_i:

$$\min_{\hat{\alpha}_0, \hat{\alpha}_1} \sum_{i=1}^{n} (x_i - \hat{\alpha}_0 - \hat{\alpha}_1 y_i)^2$$

- $\hat{\alpha}_1 = \frac{\hat{cov}(x,y)}{\hat{var}(y)} = \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{y^2} - (\bar{y})^2}$

  , where $\bar{xy} = \frac{\sum_{i=1}^{n} x_i y_i}{n}$; $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$; $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$; $\bar{y^2} = \frac{\sum_{i=1}^{n} y_i^2}{n}$

Question 2

$$\hat{\alpha}_1 = \frac{\hat{cov}(x,y)}{\hat{var}(y)} = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{y^2} - (\bar{y})^2}$$

- By continuous mapping theorem, we know for the probability limit:

$$plim\hat{\alpha}_1 = \frac{plim(\overline{xy}) - plim(\bar{x})plim(\bar{y})}{plim(\overline{y^2}) - (plim(\bar{y}))^2}$$

- By LLN for i.i.d. random variables(given all the expectations exist), we know:

$$plim\hat{\alpha}_1 = \frac{E[XY] - E[X]E[Y]}{E[Y^2] - (E[Y])^2} = \frac{Cov(X,Y)}{Var(Y)}$$

- We derived that: $\hat{\alpha}_1 \xrightarrow{p} \frac{Cov(X,Y)}{Var(Y)}$

Question 2

$\hat{\alpha}_1 \xrightarrow{p} \frac{Cov(X,Y)}{Var(Y)}$, $E(U|X) = 0$ and $Var(U|X) = \sigma^2$, $Var(X) = \sigma_x^2$

- $\frac{Cov(X,Y)}{Var(Y)} = \frac{Cov(X,\beta_0+\beta_1 X+U)}{Var(\beta_0+\beta_1 X+U)} = \frac{\beta_1 Var(X)+Cov(X,U)}{\beta_1^2 Var(X)+Var(U)+2\beta_1 Cov(X,U)} = \frac{\beta_1 Var(X)}{\beta_1^2 Var(X)+Var(U)} = \frac{\beta_1\sigma_x^2}{\beta_1^2\sigma_x^2+\sigma^2}$,

Here we use Law of Iterated Expectations and Law of Total Variances

- By Law of Iterated Expectation:
  $Cov(X, U) = E[XU] - E[X]E[U] = E[XE[U|X]] - E[X]E[E[U|X]] = 0$
- By Law of Total Variance:
  $Var(U) = E[Var(U|X)] + Var(E[U|X]) = \sigma^2$

Therefore, we know:

$$\frac{1}{\hat{\alpha}_1} \xrightarrow{p} \frac{\beta_1^2\sigma_x^2 + \sigma^2}{\beta_1\sigma_x^2} = \beta_1 + \frac{\sigma^2}{\beta_1\sigma_x^2} \neq \beta_0 \text{ or } \beta_1$$

Final 2022 Q3

Let $f(\cdot)$ and $g(\cdot)$ be two probability density functions.
Let the family of distributions be $P = \{\theta f(x) + (1-\theta)g(x), 0 \leq \theta \leq 1\}$.
Prove that for testing $H_0 : \theta \leq \theta_0$ or $\theta \geq \theta_1$(with $0 < \theta_0 < \theta_1 < 1$) against the alternative H_1: $\theta_0 < \theta < \theta_1$ the test with constant rejection probability $\phi(x) = \alpha$ is the uniformly most powerful test at level α.

Final 2022 Q3

We have a family of distributions: $\mathcal{P} = \{\theta f(x) + (1-\theta)g(x), 0 \le \theta \le 1\}$. $H_0 : \theta \le \theta_0$ or $\theta \ge \theta_1$(with $0 < \theta_0 < \theta_1 < 1$) against H_1: $\theta_0 < \theta < \theta_1$. We want to prove with constant rejection probability $\phi(x) = \alpha$ is the uniformly most powerful test at level α.

- If you have some test with the selection function $\phi()$, then you can define the rejection rate for each distribution in $\mathcal{P}$ as $\theta*$ [Rejection probability for $f(\cdot)$] $+(1-\theta)*$ [Rejection probability for $g(\cdot)$].
- Denote two probabilities with A and B, $A, B \in [0,1]$:
  - A as the probability if we reject distribution $f(\cdot)$
  - B as the probability if we reject distribution $g(\cdot)$.
- Then the problem becomes: can you decide if $A = B$ gives you the largest power given the size constraint?

$$\max_{A,B} \theta_a * A + (1-\theta_a) * B$$

$$s.t. \quad \theta_h * A + (1-\theta_h) * B \le \alpha$$

$\theta_h \in \Theta_H : (-\infty, \theta_0] \cup [\theta_1, \infty)$. $\theta_a \in \Theta_A : (\theta_0, \theta_1)$.

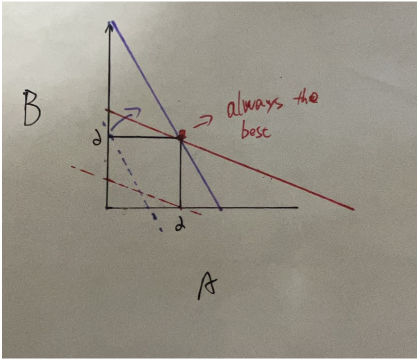
Final 2022 Q3

$$\max_{A,B} \theta_a * A + (1-\theta_a) * B$$
$$s.t. \quad \theta_h * A + (1-\theta_h) * B \leq \alpha$$

$\theta_h \in \Theta_H : (-\infty, \theta_0] \cup [\theta_1, \infty)$. $\theta_a \in \Theta_A : (\theta_0, \theta_1)$.

- Constraint part:
  $\theta_h * A + (1-\theta_h) * B \leq \alpha$, this is equivalent to say $A \leq \alpha$ and $B \leq \alpha$.
- Think about when fixing $\theta_h > 0$ and $1-\theta_h > 0$, and fix $A \leq \alpha$.
  Let B grow from 0, then when $B = \alpha$, we have $A = \alpha$.
  B cannot grow further as $B = \alpha + \epsilon$ will be make the $LHS > \alpha \Rightarrow\Leftarrow$.
- Same thing for A and other case when θ_h changes.

Final 2022 Q3



$$\max_{A,B} \theta_a * A + (1-\theta_a) * B$$

$$s.t. A \leq \alpha \text{ and } B \leq \alpha$$

$\theta_a \in \Theta_A : (\theta_0, \theta_1)$.
With the knowledge in linear programming, we know $\theta_a, 1-\theta_a > 0$. Let $z = \theta_a * A + (1-\theta_a) * B$, we have a bunch of lines with negative slope $B = -\frac{\theta_a}{1-\theta_a} A + \frac{z}{1-\theta_a}$.
The largest intercept with B axis is always got at the corner when $A = B = \alpha$. We proved!

At the End of Day...

- As you have seen, past finals are a little bit hard but don't worry about it. (Cores will be easier!)
- You guys have practiced a lot. Make sure you understand your notes, problem sets and questions we went over in TA sessions before you try to attempt sth new.
  Practice past finals which I have not discussed with you may not be efficient as I almost surely don't have a reliable answer in hand.
- Time management matters. Stop when you feel hard to move on.
- I appreciate your time and consideration in the past semester.
  I will TA undergrad's metrics(well metrics again...) in spring semester but keep me posted and wish you enjoy this unique journey.

* Please remember to submit your course evaluation on time:)