

Grading: 40% exam1
 40% exam2
 20% Problem sets

Content:

- I) Firm theory (Rec: Varian)
- II) Consumer theory (Rec: Mas-collel)
- III) Choice under uncertainty (Rec: Mas-collel)

Firm Theory: Assumptions

- Firms are price takers (no market power)
- Technology is exogenous
- Firms maximize profits

Technology:

n commodities $y = (y_1, y_2, \dots, y_n)$

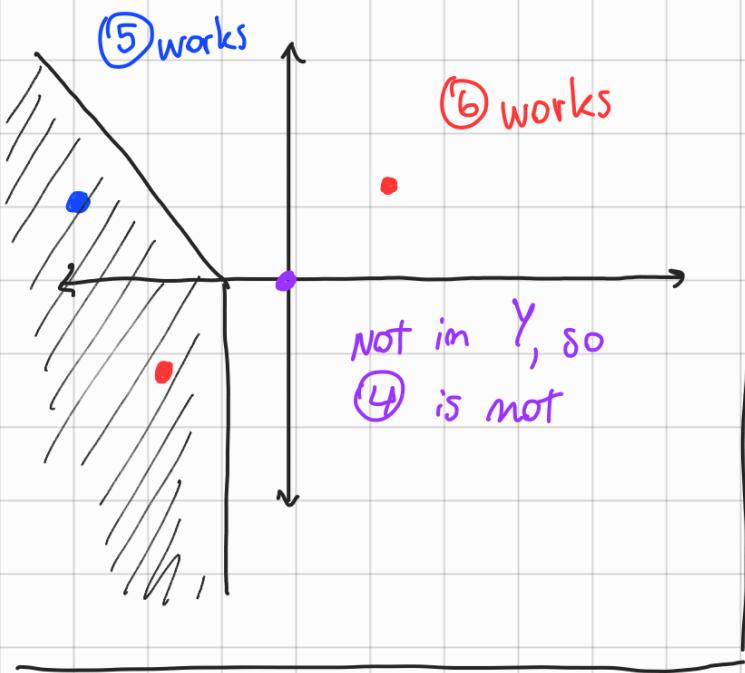
$y_i < 0$: Input

$y_i > 0$: Output

Production set: $Y \subset \mathbb{R}^n$

Properties of production sets

1. $y \neq 0$
2. Y is closed (includes boundary)
3. if $y \in Y$ and $y \geq 0$, then $g = 0$
4. Shutdown is possible: $0 \in Y$
5. Free disposal: $y \in Y$ and $y' \in Y$, then $y' \in Y$
6. Irreversibility: $y \in Y$, $y \neq 0$, then $y \notin Y$. $y^{(1,-1)}$ - $y^{(1,-1)}$
not the opposite



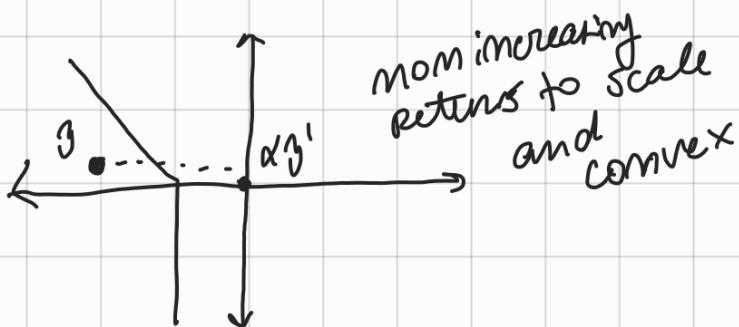
Returns to scale and convexity

i) Y has nonincreasing returns to scale if $y \in Y$
 $\Rightarrow \alpha y \in Y \wedge \alpha \in [0, 1]$

ii) Nondecreasing: $y \in Y$
 $\Rightarrow \alpha y \in Y \wedge \alpha \geq 1$.

iii) Constant Returns to scale: $y \in Y \rightarrow \alpha y \in Y \wedge \alpha \geq 0$

Y is convex if $y, y' \in Y \Rightarrow \alpha y + (1-\alpha)y' \in Y \wedge \alpha \in [0, 1]$



Theorem: For any convex Y with $0 \in Y$, there is a constant returns production set $Y^* \subseteq \mathbb{R}^{m+1}$ s.t

$$Y^* = \{y \in \mathbb{R}^m : (y, -1) \in Y\}$$

→ Idea: The farmer controls a subset of possible inputs

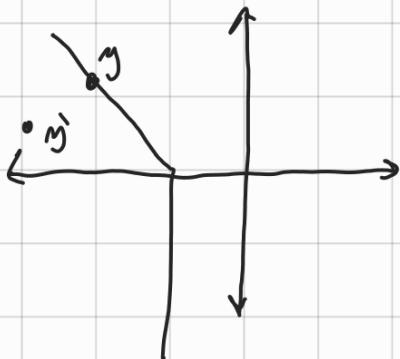
proof: Define $Y^* = \{y^* \in \mathbb{R}^{m+1} : y^* = \alpha(y, -1) \text{ for some } y \in Y \wedge \alpha \geq 0\}$

check: Y^* is constant returns

Technological efficiency:

Production plan $y \in Y$ is tech efficient if there doesn't exist $y' \in Y$ s.t. $y' > y$.

$\hookrightarrow y'_i \geq y_i \forall i$ and $y'_i > y_i$ for some i :



Lecture #2:

Aug 28, 2023

Transformation frontier (function):

* Homework (on Canvas).

\rightarrow Firm is a production set Y

\rightarrow Transformation function $T: \mathbb{R}^n \rightarrow \mathbb{R}$

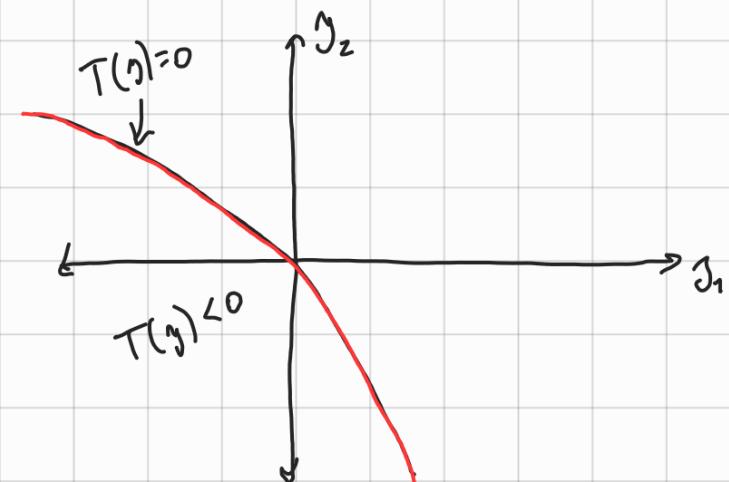
$T(y) < 0 \iff y$ is inefficient

$T(y) = 0 \iff y$ is efficient

$T(y) > 0 \iff y$ is infeasible.

$\Rightarrow Y = \{y \in \mathbb{R}^n : T(y) \leq 0\}$ production set.

$\Rightarrow \{y \in \mathbb{R}^n : T(y) = 0\}$ production frontier.



Let's fix goods $\bar{y}_3, \dots, \bar{y}_m$. Consider y_1, y_2

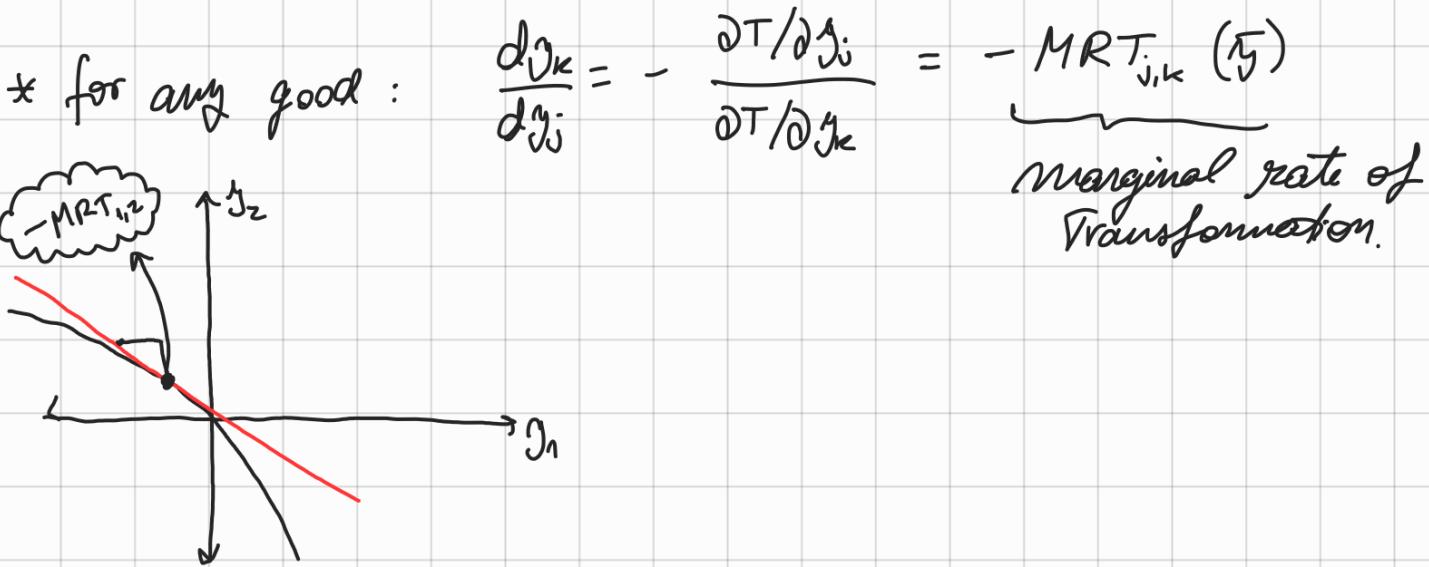
Define a function $\bar{y}_2(y_1)$ by

$$T(\bar{y}_1, \bar{y}_2(y_1), \bar{y}_3, \dots, \bar{y}_m) = 0$$

implicit
function
of y_1 .

Differentiate both sides wrt. g_i .

$$\underbrace{\frac{\partial T}{\partial g_i} + \frac{\partial T}{\partial g_z} \cdot \frac{dy_z}{dy_i}}_{\text{chain rule.}} = 0 \Rightarrow \frac{dy_z}{dy_i} = - \frac{\partial T / \partial g_i}{\partial T / \partial g_z}$$



Special case: many inputs, one output

Inputs : $(x_1, \dots, x_m) \geq 0$

Output : $y \geq 0$

production function : $f(x)$

production set : $Y = \underbrace{\{(-x_1, \dots, -x_m, y)\}}_{m+1.} : x_i \geq 0 \quad \forall i, \quad y \leq f(x_1, \dots, x_m)$

$$T(z) = y - f(x), \quad z = (-x, y)$$

eff. frontier is

$$T(z) = 0$$

$$y - f(x) = 0$$

$$y = f(x)$$

Example: Cobb-Douglas

→ 2 inputs, capital X_K and labor X_L

→ Output y .

$$\rightarrow f(X_K, X_L) = X_K^\alpha X_L^\beta, \quad \alpha + \beta = 1.$$

$$\rightarrow Y = \{(-X_K, -X_L, y) : X_K, X_L \geq 0, y \leq X_K^\alpha X_L^\beta\}$$

$$\rightarrow T(-X_K, -X_L, y) = y - X_K^\alpha X_L^\beta.$$

$$MRT_{(-X_K, y)} = - \frac{\partial T / \partial (-X_K)}{\partial T / \partial y} = - \frac{-\alpha X_K^{\alpha-1} X_L^\beta}{1} = \alpha X_K^{\alpha-1} X_L^\beta = \alpha \left(\frac{X_L}{X_K} \right)^\beta$$

mg prod of K
(MRK)

MRT for capital and labor

→ output remains constant.

$$MRT_{(-X_K), (-X_L)} = - \frac{\partial T / \partial (-X_K)}{\partial T / \partial (-X_L)} = \frac{\alpha}{\beta} \left(\frac{X_L}{X_K} \right)$$

slightly
different
interpretation (?)

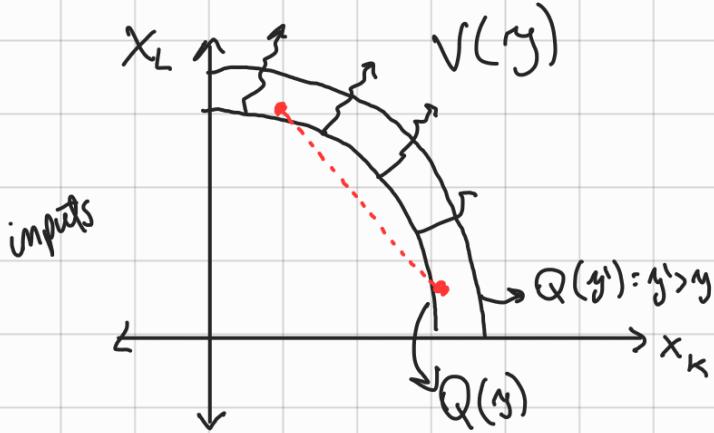
$$\frac{d(-X_L)}{d(-X_K)} = -MRT_{(-X_K), (-X_L)} = \frac{dX_L}{dX_K} = -\frac{\alpha}{\beta} \left(\frac{X_L}{X_K} \right)$$

Single output Fix y

$$V(y) = \{x \in \mathbb{R}^m : f(x) \geq y\} \rightarrow \text{Input requirement set}$$

$$Q(y) = \{x \in \mathbb{R}^m : f(x) = y\} \rightarrow \text{Isoquant}$$

Example: $f(x_k, x_L) = x_k^2 + x_L^2 = y$ (constant)



with a single output, we can define the **marginal rate of technical substitution** between 2 inputs

$$MRTS_{x_k, x_L} = \frac{\partial f / \partial x_k}{\partial f / \partial x_L}$$

$$\frac{\partial x_L}{\partial x_k} = -MRTS_{x_k, x_L} = -\frac{f_{x_k}}{f_{x_L}}$$

In our example, the production function is increasing in k (^{not} reversal) because V it is not convex.



Theorem: If Y is convex, then $V(y)$ is also convex
 $(P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P)$.

Proof: $(-x, y) \in Y, (-x', y) \in Y$ (two different inputs that yield the same output)
(Direct).

Y is convex implies: $t(-x, y) + (1-t)(-x', y) \in Y$

(most of proofs will be)
by contradiction

$$(-tx - (1-t)x', ty + (1-t)y) \in Y$$

$$(-tx - (1-t)x', y) \in Y$$

$$(-x'', y) \in Y. (-x'' \text{ produces } y)$$

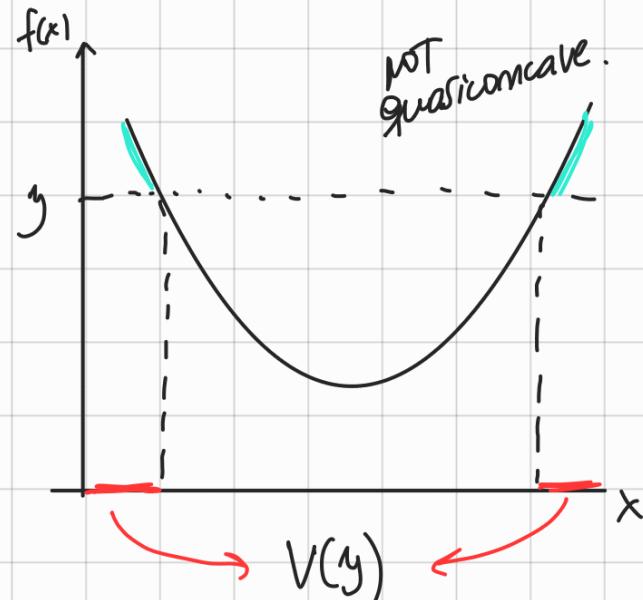
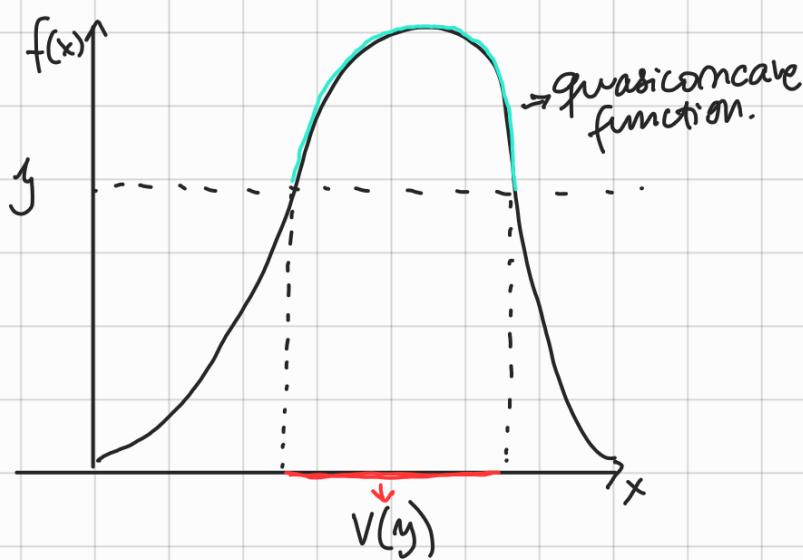
$\Rightarrow x'' \in V(y)$. meaning $V(y)$ is convex

Theorem: $V(y)$ is convex IFF $f(x)$ is quasiconcave.

"proof": $V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$

↳ upper contour set of f

Def: Quasiconcave function is one that has convex upper contour sets.



Homogeneity and Homothecy:

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree K if $f(tx) = t^K f(x)$

$$K=0 : f(tx) = f(x)$$

$$K=1 : f(tx) = t f(x).$$

Example: Cobb Douglas $f(x_k, x_L) = x_k^\alpha x_L^\beta$, $\alpha + \beta = 1$

$$\begin{aligned} f(tk, tL) &= (tk)^\alpha (tL)^\beta \\ &= t^\alpha (k)^\alpha t^\beta (L)^\beta \\ &= t^{1-\alpha} k^\alpha L^\beta \end{aligned}$$

$$f(tk, tL) = t \cdot f(k, L)$$

constant returns

(if I double the input, I get double output).

$$MRTS = \frac{\alpha}{\beta} \frac{L}{K}$$

$$\frac{\alpha}{\beta} \frac{t \cdot L}{t \cdot K} = \frac{\alpha L}{\beta K}$$

MRTS doesn't change

⇒ MRTS is homogeneous of degree 0.

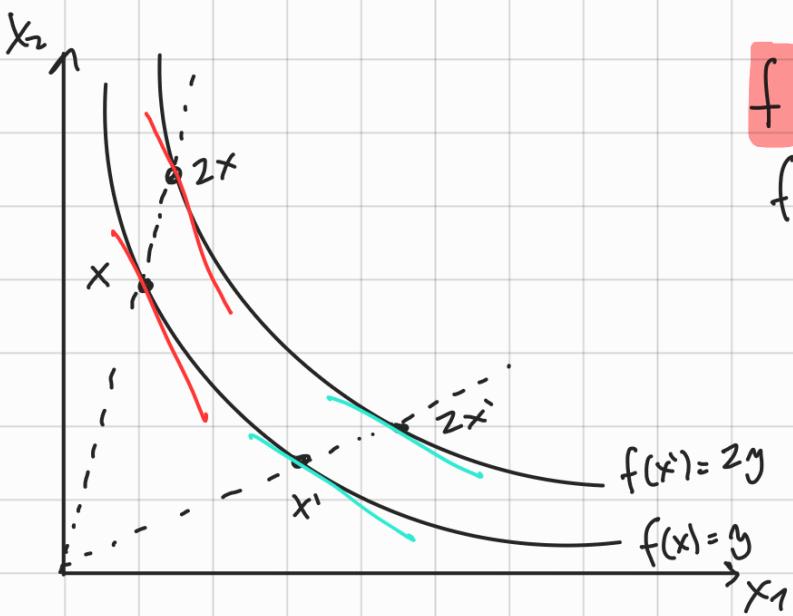
Theorem: If f is $hd-k$, then the partial derivative f_{x_i} is $hd-(k-1)$.

Proof:

$$f(tx) = t^k f(x)$$

$$t \cdot f_{x_i}(tx) = t^{k-1} f_{x_i}(x)$$

$$f_{x_i}(tx) = t^{k-1} f_{x_i}(x) //$$



f is $hd-1$

$$f(2x) = 2f(x).$$

→ partial derivatives does not change.

Theorem: If a production function $f(x)$ is $hd-1$, then the associated MRTS is $hd-0$.

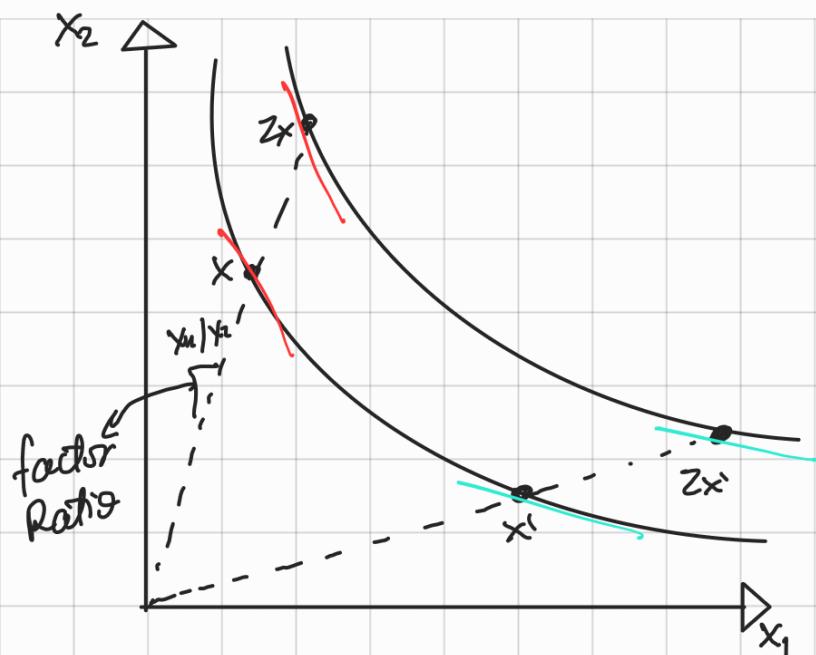
Aug 30, 2023 }
Euler's Theorem

* $hd-1$: $f(tx) = t \cdot f(x) \quad \forall t > 0$

Proof: $MRTS_{ij}(\bar{x}) = \frac{f_{x_i}(\bar{x})}{f_{x_j}(\bar{x})}$

$$MRTS_{ij}(t \cdot \bar{x}) = \frac{f_{x_i}(t \cdot \bar{x})}{f_{x_j}(t \cdot \bar{x})} = \frac{t \cdot f_{x_i}(\bar{x})}{t \cdot f_{x_j}(\bar{x})} = MRTS_{ij}(\bar{x})$$

↓
Euler's Theorem.



$$f(x) = f(x^*) = 10$$

$$f(zx) = f(zx^*) = 20$$

A homothetic function: is a positive monotonic transformation of an hd-1 function i.e., $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is homothetic if $\exists \alpha g: \mathbb{R} \rightarrow \mathbb{R}$ strictly increasing and $h: \mathbb{R}^n \rightarrow \mathbb{R}$ that is hd-1 s.t $f(x) = g(h(x))$.

MKTS of f =
$$\frac{f_{x_j}}{f_{x_i}} = \frac{g'(h(x)) h_{x_j}}{g'(h(x)) h_{x_i}} = \frac{h_{x_j}}{h_{x_i}} \rightarrow \text{hd-0}$$

Returns to scale with a single output.

Production function f exhibits:

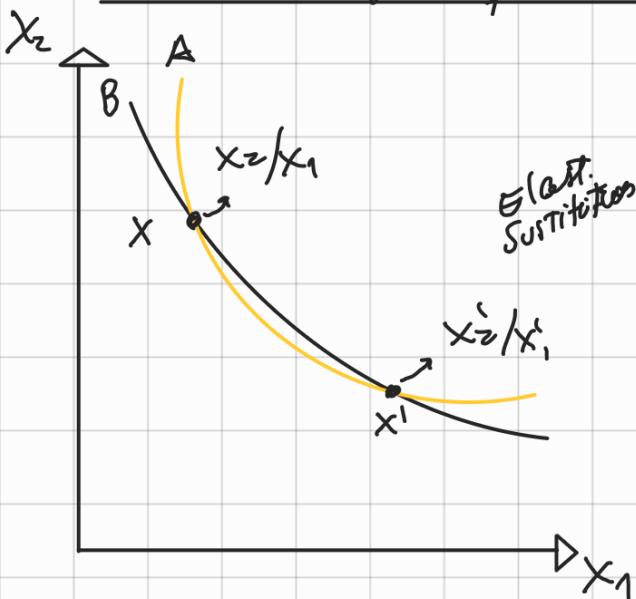
- Decreasing returns to scale
if $f(tx) < t \cdot f(x)$ $\forall t > 1$

- Increasing returns to scale
if $f(tx) > t \cdot f(x)$ $\forall t > 1$

- Constant returns to scale
if $f(tx) = t \cdot f(x)$ $\forall t > 0$

Curvature of isoquants:

Elasticity of complementarity.
(for "large" jumps).



$$\frac{1}{\sigma} = \frac{\Delta MRTS}{MRTS} \cdot \frac{\Delta(x_2/x_1)}{x_2/x_1}$$

* Denominators are the same for A & B in example.

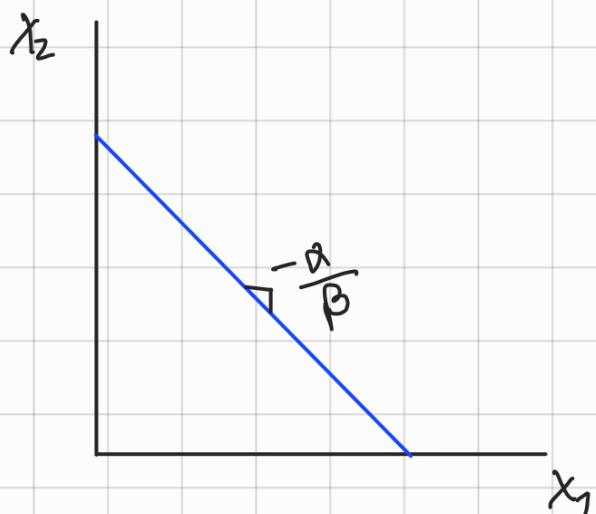
** Numerator is much higher for A

In the limit:

$$\frac{1}{\sigma} = \frac{x_2/x_1}{MRTS} \cdot \frac{\partial MRTS}{\partial(x_2/x_1)}$$

$$\frac{1}{\sigma_A} > \frac{1}{\sigma_B}$$

Perfect substitution: $f(x_1, x_2) = \alpha x_1 + \beta x_2$



$\frac{1}{\sigma} = 0$, because MRTS is a fixed number.

Elasticity of substitution: $\sigma = \frac{MRTS}{x_2/x_1} \cdot \frac{d(x_2/x_1)}{d(MRTS)}$

↳ for perfect substitutes: $\sigma = \infty$

↳ Less curvature \rightarrow bigger σ .

Perfect Complements: $f(x_1, x_2) = \min\{x_1, x_2\}$



For any two quantities, y, x , the elasticity of y with respect to x : $E_{y/x} = \frac{d \log(y)}{d \log(x)}$

* way to see this: $d \log(y) = \frac{1}{y} dy$ (Total derivative).

$$d \log(x) = \frac{1}{x} dx.$$

↳ $E_{y/x} = \frac{x}{y} \cdot \frac{dy}{dx}$ but $\frac{d \log(y)}{d \log(x)}$ is easier to remember.

Example: σ for Cobb-Douglas

$$MRTS: \frac{\alpha}{\beta} \cdot \frac{x_L}{x_K} \Rightarrow \sigma = \frac{MRTS}{x_L/x_K} \frac{d(x_L/x_K)}{d MRTS}$$

a change variable to readability.

$$\theta = \frac{\alpha}{\beta} \frac{x_L}{x_K} = MRTS$$

Property of
Cobb-Douglas.

$$\sigma = \frac{\theta}{\frac{\beta}{\alpha} \theta} \cdot \frac{d(\frac{\beta}{\alpha} \theta)}{d \theta} = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1 \nearrow \begin{matrix} \text{(NOT the same)} \\ \text{(as slope)} \end{matrix}$$

CES : Constant elasticity of subst. production.

$$f(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{\frac{1}{\rho}} \quad , \quad \rho = \underbrace{0, 1, \infty}_{\text{subcases.}}$$

The Profit maximization function (PMP)

$$\Pi(p) = \max_{s.t. y \in Y} p \cdot y, \quad \text{where } p \cdot y = \sum_{i=1}^m p_i y_i$$

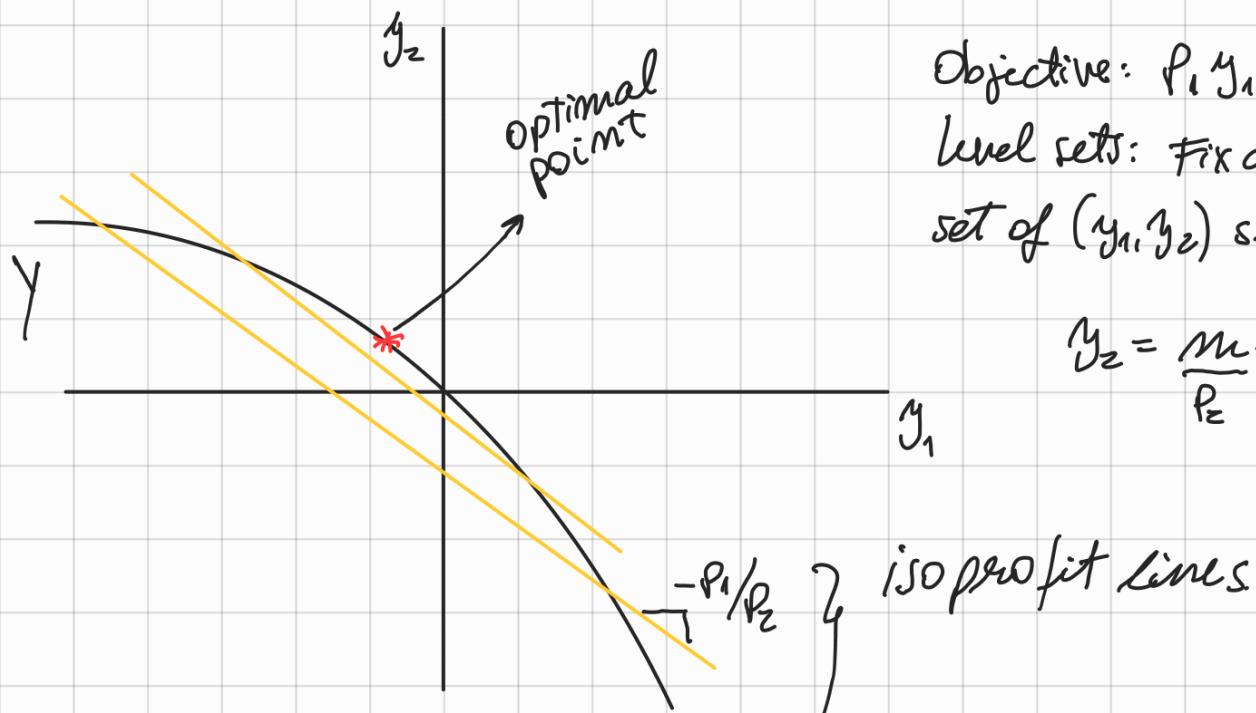
positive $y \Rightarrow$ output
negative $y \Rightarrow$ inputs

$$\text{Equivalently, } \Pi(p) = \max_{s.t. T(y) \leq 0} p \cdot y$$

$\Pi: \mathbb{R}^n \rightarrow \mathbb{R}$ is also called profit function.

Example of value function in constrained optimization
 $p \cdot y$ is called the objective function.

$y^*(p) = \{y \in Y : p \cdot y = \Pi(p)\}$ is called the supply correspondence
 $y(p)$ could be multivalued (or $y(p) = \emptyset$). set \downarrow valued mapping.



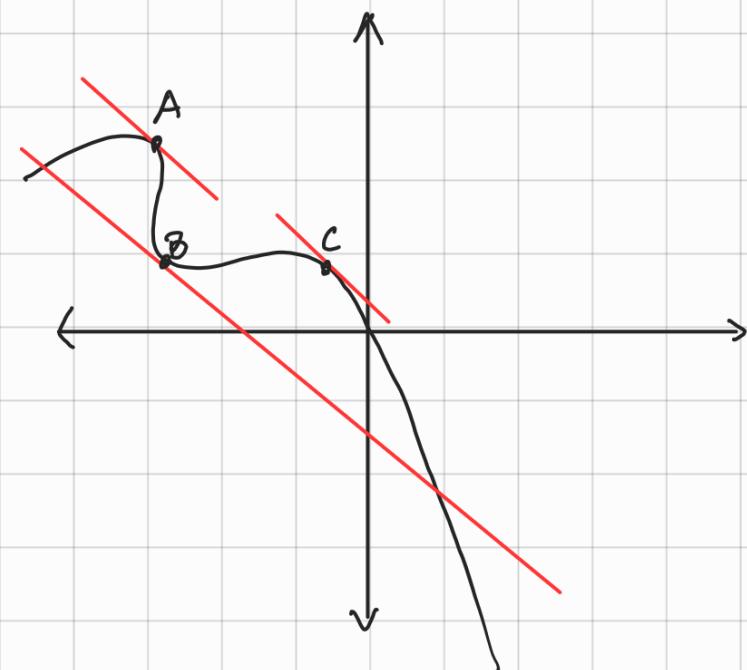
Objective: $p_1 y_1 + p_2 y_2$

level sets: Fix an $m \in \mathbb{R}$

set of (y_1, y_2) s.t. $p_1 y_1 + p_2 y_2 = m$

$$y_2 = \frac{m}{p_2} - \frac{p_1}{p_2} y_1$$

In more dimensions: solution is hyperplane.



$$\begin{aligned} & \max p \cdot y \\ \text{s.t. } & T(y) \leq 0 \end{aligned}$$

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$$L = p \cdot y - \lambda T(y)$$

K.T. Conditions :

$$\frac{\partial L}{\partial y_i} = 0 \Rightarrow p_i = \frac{\partial T}{\partial y_i} \cdot \lambda$$

$$\text{c.s.c. } \lambda \cdot T(y) = 0$$

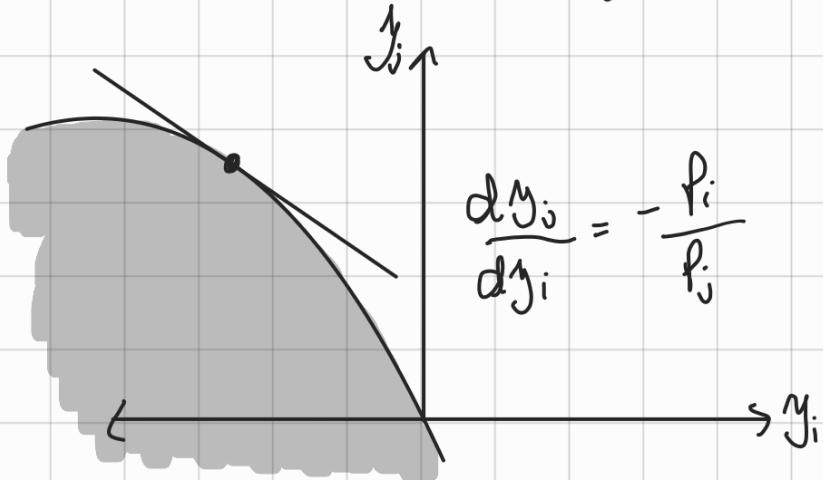
$$\lambda \geq 0$$

Implication of the f.o.c

$$\frac{p_i}{p_j} = \frac{\partial T / \partial y_i}{\underbrace{\partial T / \partial y_j}_{MRT_{ij}}} \quad \forall i, j$$

* Recall $\frac{dy_j}{dy_i} = -MRT$

$$\Rightarrow \text{Optimum at } \frac{p_i}{p_j} = -\frac{dy_j}{dy_i}$$



* Economic interpretation im: $\frac{\partial T / \partial y_i}{P_i} \geq \frac{\partial T / \partial y_j}{P_j}$

Single output case: Production function $f(x)$

$$\max P \cdot f(x) - w \cdot x \\ \text{s.t. } x_j \geq 0$$

w : Vector of input prices

P : Output price

μ : Multiplier.

$$L = P \cdot f(x) - w \cdot x + \mu \cdot x$$

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow P \cdot \frac{\partial f}{\partial x_i} - w_i + \mu_i = 0 \quad (\star)$$

$$\text{csc. } \mu_i x_i = 0 \quad \forall i \\ \mu_i \geq 0 \quad \forall i$$

Alternatively.
 s.t. $-x_j \leq 0$
 $\dots -\mu(-x)$

Another way for (\star) : $P \cdot \underbrace{\frac{\partial f}{\partial x_i}}_{\text{M.R.P.}} \leq w_i \quad \forall i$, with equality if $x_i^* > 0$

M.R.P. Revenue M.R.C.
 of add. x_i of add. x_i

Divide i by j:

$$\frac{\frac{\partial f / \partial x_i}{\partial f / \partial x_j}}{w_i / w_j} = \frac{w_i}{w_j} \quad \forall x_i^*, x_j^* > 0$$

$\underbrace{\frac{\partial f / \partial x_i}{\partial f / \partial x_j}}_{\text{M.R.T.S.}}$

$$\frac{\partial f / \partial x_i}{w_i} = \frac{\partial f / \partial x_j}{w_j}$$

* NOTE
 $f(x) \leq 0$
 $-\mu f(x)$
 $f(x) \geq 0$
 $\mu f(x)$

Notation for simple output problem:

$x^*(p, w)$: factor demand correspondence (or function).

$y^*(p, w)$: Supply correspondence

$$\pi(p, w) = p \cdot f(x^*(p, w)) - w \cdot x^*(p, w)$$

* K.T. cond. are necessary but not sufficient (in general).

** S.O.C are informative but locally.

$$\text{Max } p \cdot y \\ \text{s.t. } T(y) \leq 0$$

Theorem: Assume $T(y)$ is convex. Then, if a point y^* satisfies the K.T condition for some $\lambda \geq 0$, then y^* is a solution to maximization.

$$\max g(x) \\ \text{s.t. } \begin{cases} h(x) \geq 0 \\ \text{quasiconcave} \end{cases}$$

} Then K.T are necessary and sufficient.
quasiconvex

$$\text{Single output: Max } p \cdot f(x) - w \cdot x \\ \text{s.t. } x_i \geq 0$$

Theorem: Assume f is concave. Then, if x^* satisfies

$$p \cdot \frac{\partial f}{\partial x_i} \leq w_i \quad \forall i, \text{ with equality if } x_j^* > 0,$$

Then x^* is a solution to the problem.

Caveats: K.T / FOC are useful but be careful with

- nondifferentiability

- nonconvexities

- Existence / uniqueness

- corner solution (some variable can be optimal at 0).

Example: Single-output, single-input $f(\cdot)$

$$y = f(x) = \frac{1}{z}x$$

$$\max_{x \geq 0} p f(x) - \omega x$$

$$L = p \left(\frac{1}{z}x \right) - \omega x + \mu x$$

$$L_x = \frac{1}{z}p - \omega + \mu = 0 \quad \text{csc: } \begin{cases} \mu x = 0 \\ \mu \geq 0. \end{cases}$$

Solve by cases:

I) $x=0, \mu > 0.$

$$\rightarrow \mu = \omega - \frac{1}{z}p > 0?$$

what if $\omega - \frac{1}{z}p < 0?$

Economic interpretation:

$$\begin{aligned} z\omega &< p \\ \text{My cost} &< \text{My I} \end{aligned} \quad \begin{cases} \text{firm should} \\ \text{produce int.} \\ \text{so, no solution,} \end{cases}$$

II) $x > 0, \mu = 0.$

$$\underbrace{\frac{1}{z}p}_{\text{Economic sent.}} = \mu$$

Economic sent.

Solution:

$$x(p, \omega) = \begin{cases} 0, & p < z\omega \\ [0, \infty), & p = z\omega \\ \emptyset, & p > z\omega \end{cases}$$

Example 2: $f(x_1, x_2) = \log(1 + x_1 + x_2)$

$$\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \end{array}$$

$$L = p \log(1 + x_1 + x_2) - \omega_1 x_1 + \mu_1 x_1 + \mu_2 x_2$$

$$\frac{\partial L}{\partial x_1} = 0 \rightarrow \frac{p}{1 + x_1 + x_2} - \omega_1 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \rightarrow \frac{p}{1 + x_1 + x_2} - \omega_2 + \mu_2 = 0$$

csc

$$\mu_1 x_1 = 0$$

$$\mu_2 x_2 = 0$$

$$\mu_1 \geq 0, \mu_2 \geq 0$$

Case I: $\mu_1 = 0, \mu_2 = 0, x_1 > 0, x_2 > 0$

$$w_1 = \frac{P}{1+x_1+x_2} ; w_2 = \frac{P}{1+x_1+x_2} \rightarrow w_1 = w_2 = \bar{\omega}.$$

$$\rightarrow x_1 + x_2 = \frac{P}{\bar{\omega}} - 1. \quad \text{implicitly. } \frac{P}{\bar{\omega}} - 1 > 0 \\ P > \bar{\omega} \quad \text{condition}$$

Case II: $\mu_1 = 0, \mu_2 \geq 0, x_1 \geq 0, x_2 = 0$

$$\frac{P}{1+x_1} = w_1 \Rightarrow x_1^* = \frac{P}{w_1} - 1$$

$$\mu_2 = w_2 - \frac{P}{1+x_1} = w_2 - w_1 \geq 0$$

$w_2 \geq w_1$ condition

Case III: $\mu_1 \geq 0, \mu_2 = 0, x_1 = 0, x_2 \geq 0$

$w_1 \geq w_2$ condition

$$\rightarrow \mu_1 = w_1 - \frac{P}{1+x_2} = w_1 - w_2 \geq 0$$

Wrapping all together: Factor demands

Assume $P \geq \min\{w_1, w_2\}$

$$x_1(p, w) = \begin{cases} \frac{P}{w_1} - 1, & w_1 < w_2 \\ \alpha \left(\frac{P}{\bar{\omega}} - 1 \right), & w_1 = w_2 = \bar{\omega}, \alpha \in [0, 1] \\ 0, & w_1 > w_2 \end{cases}$$

$$x_2(p, w) = \begin{cases} \frac{P}{w_2} - 1, & w_2 > w_1 \\ (1-\alpha) \left(\frac{P}{\bar{\omega}} - 1 \right), & w_1 = w_2 = \bar{\omega}, \alpha \in [0, 1] \\ 0, & w_2 > w_1 \end{cases}$$

$$y(p, w) = \begin{cases} 0, & p \leq \hat{\omega}, \hat{\omega} = \min\{w_1, w_2\} \\ \log\left(\frac{p}{\hat{\omega}}\right), & p > \hat{\omega} \end{cases}$$

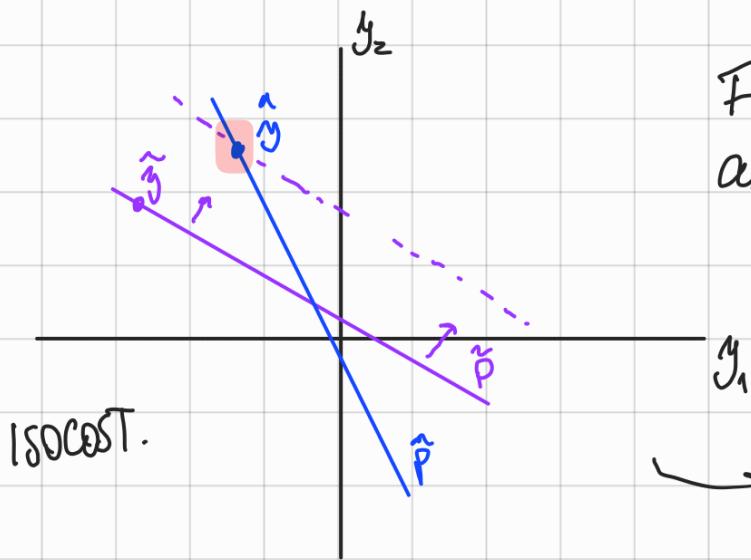
$$\pi(p, w) = \begin{cases} 0, & p \leq \hat{\omega} \\ p \cdot \log\left(\frac{p}{\hat{\omega}}\right) - p + \hat{\omega}, & p > \hat{\omega} \end{cases}$$

Rationalizability:

[Sept 06, 2023]

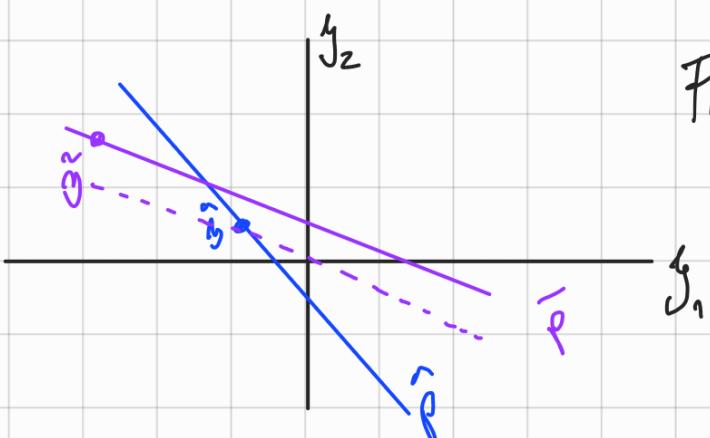
Q: Given some data $\{(p^i, y^i), \dots, (p^T, y^T)\}$
 ↓ vectors of data

Can we "learn" about the firm's technology?



Firm is irrational: Violate the weak axiom of profit maximization: (WAPM)
 $p^t y^t \geq p^s y^s, \forall s, t = 1, \dots, T$

$\hat{p} \hat{y} \geq \tilde{p} \tilde{y}$ violates.



Firm could've chosen \tilde{y} , but at price.

What other production sets could "rationalize" the data?

A production set \mathcal{Y} rationalizes the data if

$$y^t \in \mathcal{Y}^*(p^t) \quad \forall t = 1, \dots, T$$

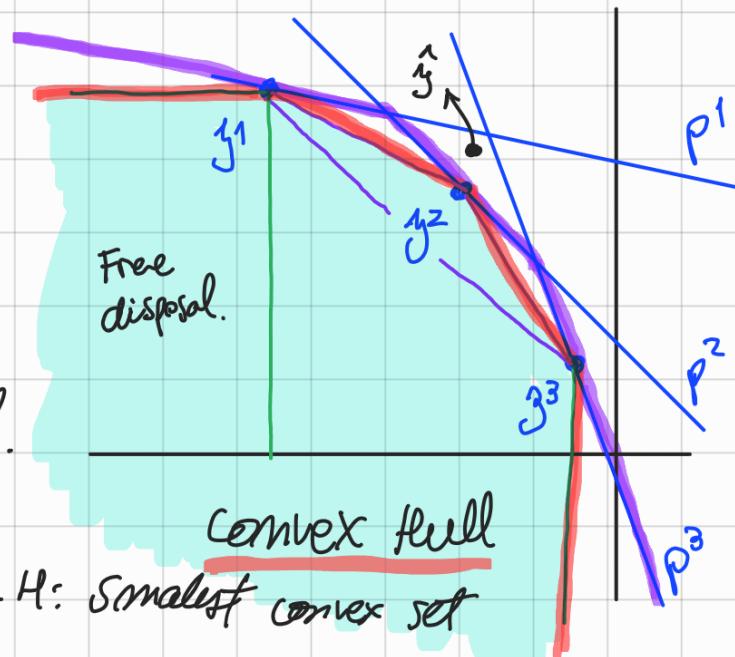
$$\mathcal{Y}^*(p) = \arg \max_{y \in \mathcal{Y}} p \cdot y$$

Make some additional assumptions.

- Data satisfies WAMP
- \mathcal{Y} is convex & free disposal.

$$\mathcal{Y}^F = \text{Conv}_{\text{fd}}(\{y^1, \dots, y^T\})$$

* C.H: Smallest convex set



Does \mathcal{Y}^F rationalize the data?

Upper bound.

claim: Yes.

Proof: Contradiction: Then \exists a point $y \in \mathcal{Y}^F$ s.t. $p^t \cdot y > p^t \cdot y^t$ for some t .

By convexity: $y = \alpha^1 y^1 + \alpha^2 y^2 + \dots + \alpha^T y^T$, $\alpha^1 + \dots + \alpha^T = 1$.

$$\textcircled{1} \quad p^t (\alpha^1 y^1 + \dots + \alpha^T y^T) > p^t \cdot y^t$$

$$\begin{aligned} \textcircled{2} \quad \alpha^1 p^t y^1 + \dots + \alpha^T p^t y^T &\leq \alpha^1 p^t \cdot y^t + \dots + \alpha^T p^t \cdot y^t \\ &= (\alpha^1 + \dots + \alpha^T) p^t \cdot y^t \\ &= p^t \cdot y^t \end{aligned}$$

contradiction.

$\rightarrow \mathcal{Y}^F$ is one production set that rationalizes the data. By def. for any other convex \mathcal{Y} that rationalizes the data $\mathcal{Y}^F \subseteq \mathcal{Y}$

$$Y^o = \{y \in \mathbb{R}^n : p^t \cdot y \leq \pi(p^t), \forall t=1, \dots, T\}$$

$\hat{y} \notin Y^o$ because $p^z \cdot \hat{y} > p^z \cdot y^o = \pi(p^z)$

$\mathbb{R}^n \setminus Y^o$ bundles that yield higher profits than some observed choice = $\{y \in \mathbb{R}^n : p^t \cdot y > \pi(p^t) \text{ for some } t\}$.

Does Y^o rationalize the data?
claim. Yes.

Proof: Assume not. Then, \exists some $y \in Y^o$ s.t. $p^t \cdot y > p^t \cdot y^t$
Directly contradicts definition of Y^o .

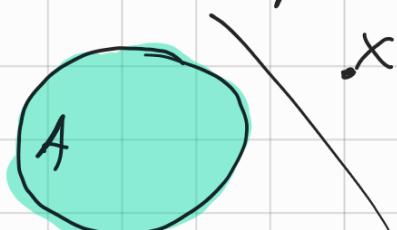
Theorem: For any production set Y that rationalizes the data, we have $Y^F \subseteq Y \subseteq Y^o$

What if we had infinite data? we would pin down the exact production function.

Suppose we observe $y(p)$ $\forall p$. Let $\pi(p) = p \cdot y(p)$ be the observed profit function.

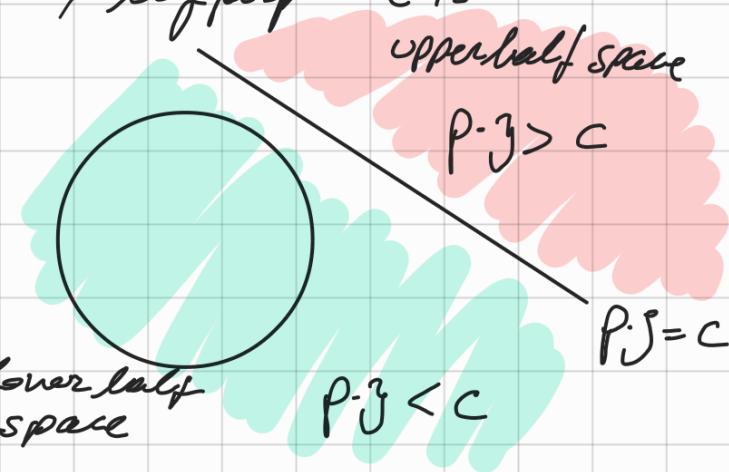
Theorem: Suppose Y is convex, closed & satisfies free disposal.
Then $Y = Y^o = \{y \in \mathbb{R}^n : p \cdot y(p) \leq \pi(p) \forall p \in \mathbb{R}_+^n\}$

Detour: Separating hyperplane theorem. In \mathbb{R}^n , a hyperplane is a subspace of dim $(n-1)$.



Given $p \in \mathbb{R}^n$, $c \in \mathbb{R}$, the $(p-c)$ hyperplane is

$$H_{p,c} = \{y \in \mathbb{R}^n : p \cdot y = c\}$$



Theorem: Suppose $A \subseteq \mathbb{R}^n$ is convex and closed, and

$x \notin A$. Then, there is a $p \in \mathbb{R}^n$, $p \neq 0$, and $c \in \mathbb{R}$ s.t. $p \cdot x < c$ & $p \cdot y < c \quad \forall y \in A$.

* Convexity is important:



Proof: $Y = Y^\circ$

NTS ① $Y \subseteq Y^\circ$ & ② $Y^\circ \subseteq Y$

For ② : Take some $\hat{y} \in \mathbb{R}^n \setminus Y$

By S.H.T., \exists some $p \in \mathbb{R}^n$ & c s.t.

$$p \cdot \hat{y} > c > p \cdot y \quad \forall y \in Y$$

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y \quad (*)$$

By free disposal, $p_i \geq 0$ if $p_i < 0$ for some i , then $\max p \cdot y = \infty$ which contradicts $(*)$

Therefore, there exists $p \geq 0$ s.t. $p \cdot \hat{y} \geq \max_{y \in Y} p \cdot y = \pi(p)$
i.e., $\hat{y} \notin Y^\circ$, or $Y^\circ \subseteq Y$.

Properties of profit & supply functions

Sept 18, 2023

$$\Pi(p) = \max_{y \in Y} p \cdot y \quad y(p) = \arg \max_{y \in Y} p \cdot y$$

$$\begin{array}{c} \Pi: \underbrace{\mathbb{R}^n}_{\substack{\text{Space} \\ \text{of prices}}} \rightarrow \underbrace{\mathbb{R}}_{\text{Profit}} \\ \text{Profit} \end{array} \quad y: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$$

Note: Results do not rely on regularity assumptions (convexity, differentiability, etc..) unless stated otherwise

(1) $\Pi(p)$ is nonincreasing in input prices, nondecreasing in output prices.

Proof: Take a price vectors p, p' and choose $y \in y^*(p)$ and $y' \in y^*(p')$. Let $p_i' \geq p_i \forall$ outputs, $p_i' \leq p_i$ for all inputs

By WAPM, $p \cdot y' \geq p \cdot y$

$$\text{consider } (p' - p) \cdot y = (p_1' - p_1)y_1 + \dots + (p_m' - p_m)y_m$$

$$y_i > 0 : p_i' \geq p_i \quad y_i \leq 0 : p_i' \leq p_i \Rightarrow (p' - p) \cdot y \geq 0$$

$$\Rightarrow p' \cdot y \geq p \cdot y$$

$$\Pi(p') = p' \cdot y' \geq p \cdot y \geq p \cdot y = \Pi(p)$$

$$\Rightarrow \Pi(p') \geq \Pi(p)$$

② $\pi(p)$ is $hd-1$. WTS: $\pi(tp) = t\pi(p)$

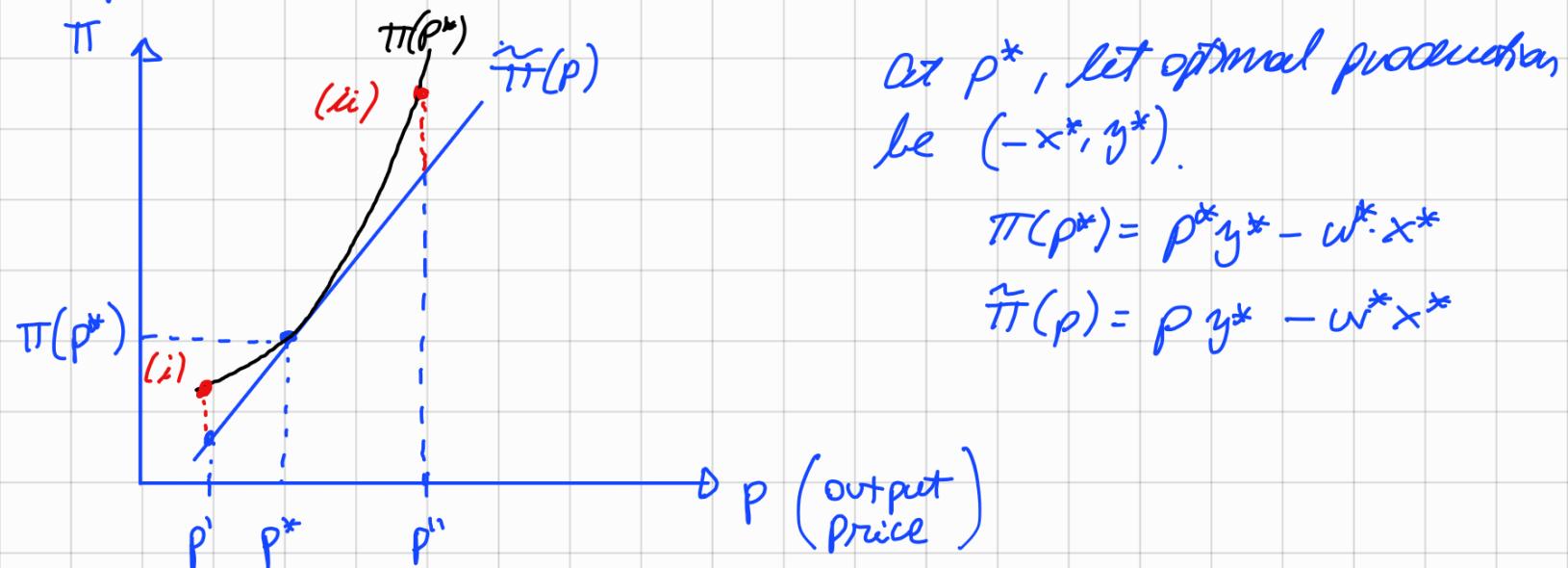
Proof: $\pi(tp) = \max_{y \in Y} (tp) \cdot y$

$$= t \cdot \left[\max_{y \in Y} p \cdot y \right] = t \cdot \pi(p)$$

③ $\pi(p)$ is convex in p , i.e.,

$$\pi(\alpha p + (1-\alpha)p') \leq \alpha \pi(p) + (1-\alpha)\pi(p') , \forall \alpha \in [0,1].$$

Graphical intuition: [Do the proof...].



(i) and (ii) are reoptimized decisions of the firm, that will be above the first optimized point.

If we do assume π is differentiable

$$H(p) = D^2\pi(p) = \begin{bmatrix} \frac{\partial^2 \pi}{\partial p_1^2} & \dots & \frac{\partial^2 \pi}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial p_n \partial p_1} & \dots & \frac{\partial^2 \pi}{\partial p_n^2} \end{bmatrix}$$

(4)
positive
semidefinite.

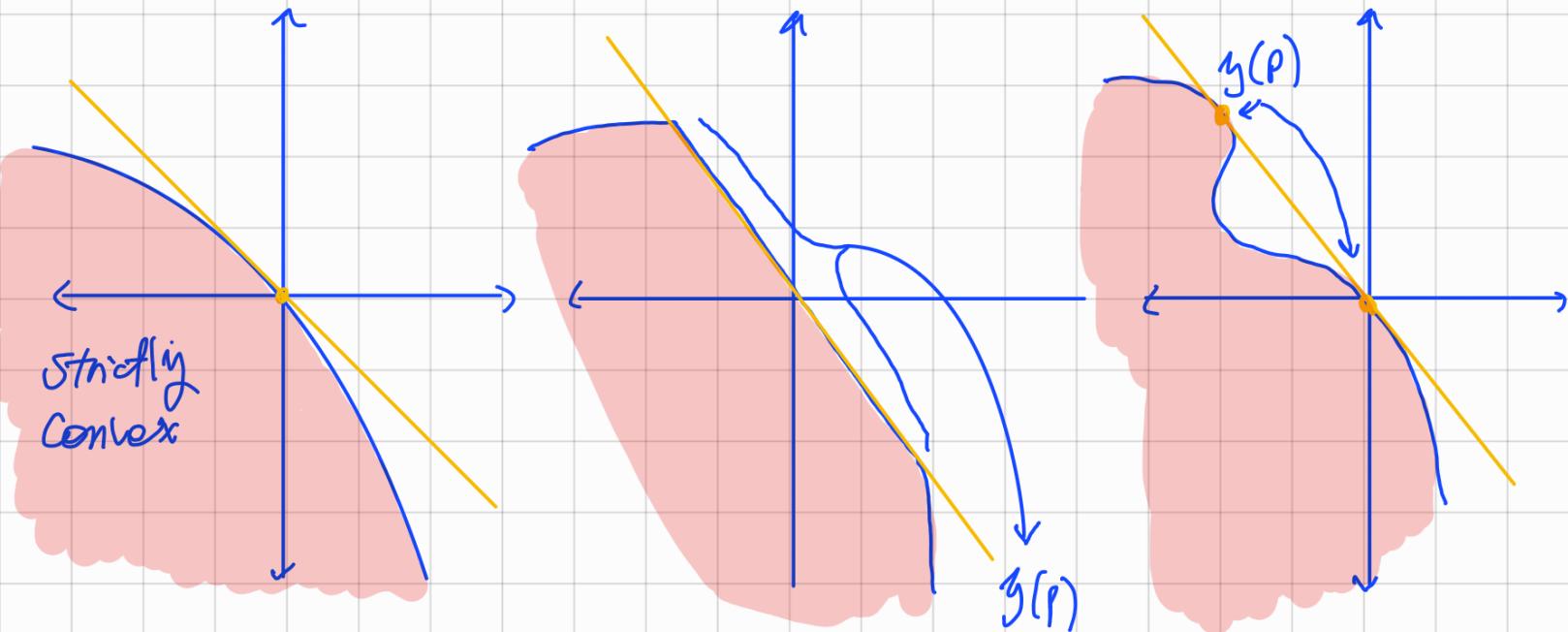
Properties of $y(p)$

① $y(p)$ is homogeneous of degree 0. (change in prices don't affect decisions)

$$\text{Proof: } \max_{y \in Y} (\alpha p) \cdot y = \max_{y \in Y} p \cdot y \quad \left\{ \begin{array}{l} y(\alpha p) = y(p) \\ \dots \end{array} \right.$$

② a) If Y is convex, then $y(p)$ is in convex set

b) If Y is strictly convex then $y(p)$ is a singleton



Detour: Envelope Theorem.

Q: How does the optimal value of a maximization problem change as a parameter changes?

$$\max_x f(x, \theta)$$

x : choice variable

θ : parameter

Define the value function

$$V(\theta) = \max_x f(x, \theta)$$

$$V(\theta) = f(x^*(\theta), \theta)$$

A change in θ has 2 effects:

- Direct effect: θ changes f directly through and argument
- Indirect effect: θ changes x^* , which changes f .

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial x} \Big|_{x=x^*(\theta)}}_{0} + \frac{\partial f}{\partial \theta} \Big|_{x=x^*(\theta)}$$

$$\Rightarrow \boxed{\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} \Big|_{x=x^*(\theta)}}$$

Fund all over economics:

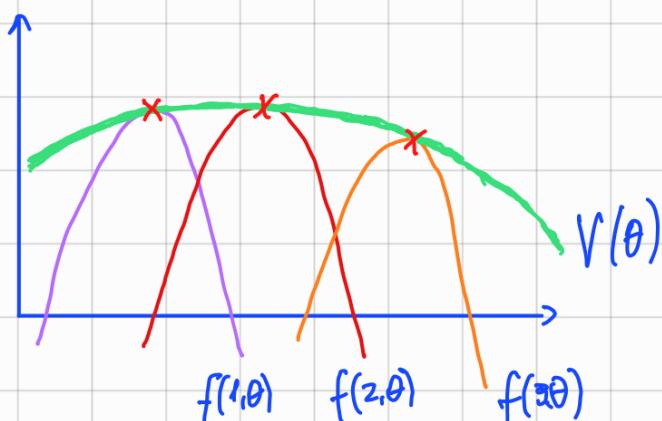
- Hotelling's Lemma
- Slutsky's Lemma
- Myerson's Lemma
- Deveniste-Schenkman formula
-

When there are constraints

$$\max_x f(x, \theta) \quad \text{s.t.} \quad g_k(x, \theta) \leq c_k, \quad k=1, \dots, K.$$

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} - \lambda_1 \frac{\partial g}{\partial \theta} - \dots - \lambda_K \frac{\partial g}{\partial \theta}$$

$$\frac{dV}{d\theta} = \frac{\partial L}{\partial \theta} \Big|_{x=x^*(\theta)}$$



Höelling's Lemma: If $\pi(p)$ is differentiable at \bar{p} , and $\bar{p}_i > 0$, then $\frac{\partial \pi}{\partial p_i} = y_i^*(\bar{p})$

Proof: For single output case

$$\pi(p, w) = \max_{x \geq 0} \underbrace{p \cdot f(x) - w \cdot x}_{h(x)}$$

By envelope theorem,

$$\frac{d\pi}{dp} = \left. \frac{\partial h}{\partial p} \right|_{x=x^*(p, w)} = f(x^*(p, w)) = y^*(p, w)$$

$$\frac{d\pi}{dw_i} = \left. \frac{\partial h}{\partial w_i} \right|_{x=x^*(p, w)} = -x_i^*(p, w)$$

Example: $f(x_1, x_2) = 30p x_1^{2/5} x_2^{2/5}$

$$\max 30p x_1^{2/5} x_2^{2/5} - w_1 x_1 - w_2 x_2$$

$$\text{f.o.c.: } \left. \begin{array}{l} 12p x_1^{-3/5} x_2^{2/5} = w_1 \\ 12p x_1^{2/5} x_2^{-3/5} = w_2 \end{array} \right\} \Rightarrow \boxed{x_1^* = \frac{12^5 p^5}{w_1^3 w_2^2}; x_2^* = \frac{12^5 p^5}{w_1^2 w_2^3}}$$

$$\pi(p, w) = 30p \frac{12^4 p^4}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} = \frac{p}{2} \left[\frac{12^5 p^5}{w_1^2 w_2^2} \right]$$

$$\boxed{\frac{\partial \pi}{\partial w_1} = -\frac{2}{2} \left[\frac{12^5 p^5}{w_1^3 w_2^2} \right]}$$

Integral form for Hotelling's Lemma

$-j$: everything else.

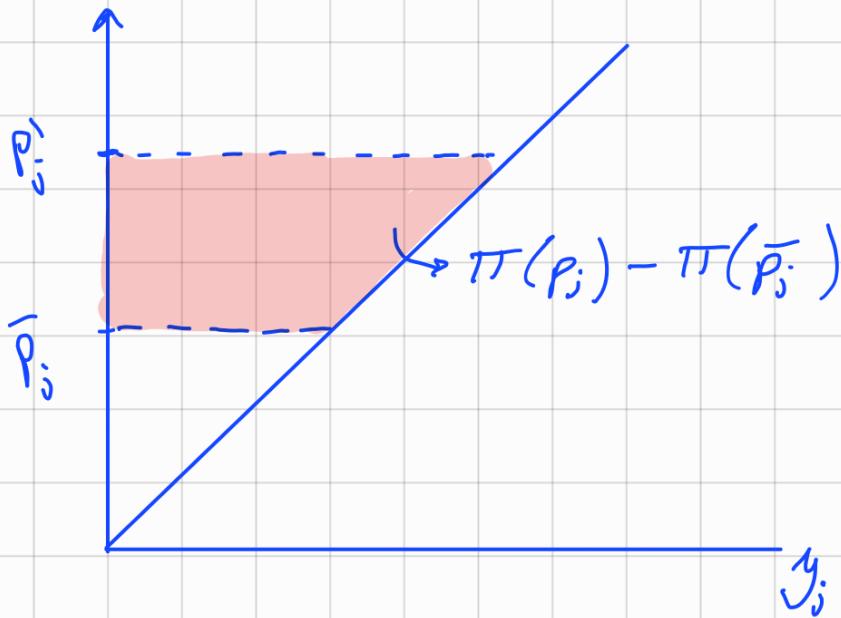
Hold all prices other than P_j fixed at \bar{P}_j .

Then

$$\pi(P_j, \bar{P}_j) - \pi(\bar{P}_j, \bar{P}_j) = \int_{\bar{P}_j}^{P_j} y_j(s, \bar{P}_j) ds$$

Proof:

$$\int_{\bar{P}_j}^{P_j} y_j(s, \bar{P}_j) ds = \int_{\bar{P}_j}^{P_j} \frac{d\pi(s, \bar{P}_j)}{dP_j} ds = \underbrace{\pi(P_j, \bar{P}_j) - \pi(\bar{P}_j, \bar{P}_j)}_{\text{Surplus.}}$$



Hotelling's Lemma

$$\frac{\partial \pi}{\partial p_i} = y_i(p) \rightarrow \frac{\partial y_i(p)}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_i \partial p_j}$$

$\mathcal{D}y(p) =$ Jacobian or substitution matrix.

$$\begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \dots & \frac{\partial y_m}{\partial p_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial p_m} & \dots & \frac{\partial y_m}{\partial p_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial p_1^2} & \dots & \frac{\partial^2 \pi}{\partial p_1 \partial p_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial p_m \partial p_1} & \dots & \frac{\partial^2 \pi}{\partial p_m^2} \end{bmatrix}$$

π is convex, pos semidef

This implies:

$$\textcircled{1} \quad \frac{\partial y_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2} \geq 0 \quad (\text{Law of supply})$$

$$\textcircled{2} \quad \frac{\partial y_i}{\partial p_j} = \underbrace{\frac{\partial^2 \pi}{\partial p_j \partial p_i}}_{\text{Young's Law}} = \frac{\partial^2 \pi}{\partial p_j \partial p_i} = \frac{\partial y_j}{\partial p_i}$$

Le Chatelier Principle: "Firms should respond more to price changes in the long-run than in the short-run."

Suppose $(y^*, x^*) = (y(p^*, w^*), x(p^*, w^*))$ at price (p^*, w^*)

Fix one input z_i^* in the short-run, variable in the long-run.

z_i^* is optimal at (p^*, w^*)

Q: How does x_j depend on w_j in the SR w L-R?

Proof: \rightarrow

$$\pi(p, w) = \max_{\text{s.t. } x_j \geq 0} p \cdot f(x) - w \cdot x ; \quad \pi^s(p, w) = \max_{x_i = z_i^*} p \cdot f(x) - w \cdot x$$

Compare

$$\left| \frac{\partial x_j(p, w)}{\partial w_j} \right| \quad \text{vs} \quad \left| \frac{\partial x_j^s(p, w)}{\partial w_j} \right|$$

$(p, w) = (p^*, w^*) \quad (p, w) = (p^*, w^*)$

$$h(p, w) = \pi(p, w) - \pi^s(p, w)$$

this is lower because it is a more restricted problem

By definition, $h(p, w) \geq 0$ and $h(p^*, w^*) = 0$, (p^*, w^*) is a local minimum.

$$\frac{\partial^2 h(p^*, w^*)}{\partial w_j^2} \geq 0$$

||

$$\frac{\partial^2 \pi(p^*, w^*)}{\partial w_j^2} - \frac{\partial^2 \pi^s(p^*, w^*)}{\partial w_j^2} \geq 0$$

$$\text{Hotelling : } \frac{\partial \pi(p, w)}{\partial w_j} = -x_j(p, w)$$

$$-\frac{\partial x_j(p^*, w^*)}{\partial w_j} + \frac{\partial x_j^s(p^*, w^*)}{\partial w_j} \geq 0$$

$$\left| \frac{\partial x_j(p^*, w^*)}{\partial w_j} \right| \geq \left| \frac{\partial x_j^s(p^*, w^*)}{\partial w_j} \right|$$

because $\frac{\partial x_j}{\partial w_j}$ are negative values ($\frac{\Delta \text{Demand}}{\Delta \text{Price input}}$)

Cost minimization:

- single output good
- pricing power in output markets, but not input markets
- Previous analysis doesn't work, but we can:
 - ① Find the cheapest way to make any target output y (cost minimization problem).
 - ② Use this "cost function" $C(y)$ to choose optimal p/y combination.

CMP:

- Fix a target level of output, y .
- Cost: $C(w, y) = \min_{\mathbf{x}} w \cdot \mathbf{x}$ s.t. $x_i \geq 0$
 $f(\mathbf{x}) \geq y$

* ignore this
for now...

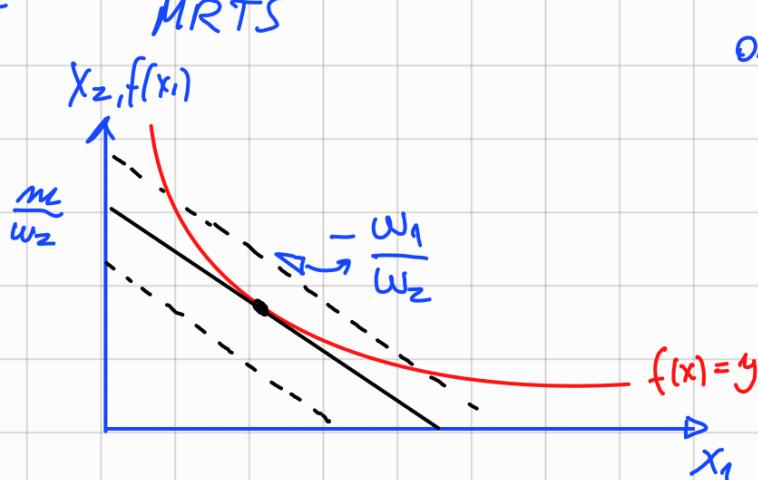
$$L = w \cdot x - \lambda (f(x) - y)$$

$$\text{F.O.C: } w_i - \lambda \frac{\partial f}{\partial x_i} = 0 \quad \forall i$$

Divide i by j :

$$\frac{w_i}{w_j} = \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}, \text{ for } (x_i^*, x_j^* > 0)$$

Economic
Rate of
Substitution



Constraint: Isoquant
Objective: Isocost $w_1 x_1 + w_2 x_2 = m$

$$x_2 = \frac{m}{w_2} - \frac{w_1}{w_2} x_1$$

$x^*(w, y)$ conditional factor demand correspondence.

$$C(w, y) = w \cdot x^*(w, y) = \min w \cdot x \quad , \quad x_i > 0 \\ f(x) > y.$$

Apply Envelope theorem.

$$\frac{\partial C}{\partial y} = \frac{\partial L}{\partial y} = \lambda$$

→ λ is the marginal cost of increasing production by 1 unit

Relationship with PMP

$$\max_{y \geq 0} p \cdot y - C(w, y)$$

F.O.C] $p = \frac{\partial C}{\partial y} (= \lambda \text{ from CMP})$

Things to keep in mind:

- We have assumed, differentiable f , interior solutions
- General FOC: $\lambda \cdot \frac{\partial f(x^*)}{\partial x_i} - w_i \leq 0 \quad \forall i$
with equality if $x_i^* > 0$
- KT conditions are necessary, but not sufficient in general
- will be sufficient if $f(\cdot)$ is concave
- Existence / uniqueness issues

Weak Axiom of Cost minimization (WACM)

$$\{(w^t, x^t, y^t), \dots, (w^s, x^s, y^s)\}$$

WACM: $\frac{w^t x^t}{\text{cost}} \leq \frac{w^s x^s}{\text{cost}}$ & $y^s \geq y^t$
other choices are more expensive.

Any firm that violates WACM is not Rational.

Implication of WACM: Downward sloping Demand

→ Take $y^s = y^t$ WACM gives 2 inequalities

$$\begin{aligned} \textcircled{1} \quad w^t x^t &\leq w^s x^s \\ \textcircled{2} \quad w^s x^s &\leq w^t x^t \end{aligned}$$

$$\textcircled{1} + \textcircled{2}: w^t x^t + w^s x^s \leq w^t x^s + w^s x^t$$

$$(w^t - w^s)(x^s - x^t) \leq 0$$

$$\Delta w \cdot \Delta x \leq 0$$

$$C(w, y) = \min_x x \cdot y \quad \text{s.t. } f(x) \geq y$$

(1) $C(w, y)$ is nondecreasing in w . Proof:

Take $w' \geq w$ and $x \in x^*(w, y)$, $x' \in x^*(w', y)$

$$C(w, y) = w \cdot x \leq w' \cdot x \stackrel{\text{WACM}}{\leq} w' \cdot x' = C(w', y)$$

$$C(w, y) \leq C(w', y)$$

② $C(w, y)$ is nondecreasing in y .

Let $y'' \geq y'$, $x' \in X^*(w, y')$, $x'' \in X^*(w, y'')$

$$C(w, y') = \min_{\substack{f(x) \geq y' \\ f(x) \in X^*(w, y')}} w \cdot x = w \cdot x' \leq w \cdot x'' = \min_{\substack{f(x) \geq y'' \\ f(x) \in X^*(w, y'')}} w \cdot x = C(w, y'')$$

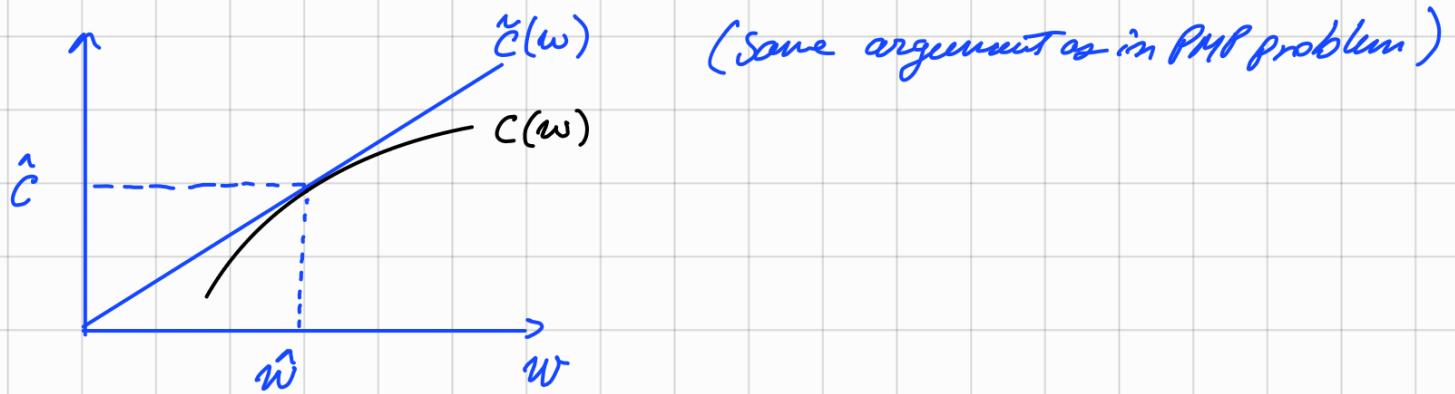
WACM

Inequality works because $f(x'') \geq y'' \geq y'$

③ $C(w, y)$ is lid-1 in w

$$C(tw, y) = tC(w, y)$$

④ $C(w, y)$ is concave in w (NOT TRUE for y)



⑤ If $V(y)$ is convex, then $X^*(w, y)$ is convex-valued.
If $V(y)$ is strictly convex, then $X^*(w, y)$ is single valued



b) $x^*(w, y)$ is homogenous in w : $x^*(t \cdot w, y) = x^*(w, y)$

→ implications:

$$x_i^*(t \cdot w, y) = x_i^*(w, y)$$

differentiate w.r.t. t , eval at $t=1$

$$\nabla_w \cdot x_i^*(t \cdot w, y) \cdot w = 0$$

$$\sum_{j=1}^{\infty} \frac{\partial x_i^*(t \cdot w, y)}{\partial w_j} \cdot w_j = 0$$

We know $\frac{\partial x_i^*}{\partial w_j} \leq 0$

Say $n=2$ | $\frac{\partial x_i^*}{\partial w_1} \cdot w_1 + \frac{\partial x_i^*}{\partial w_2} \cdot w_2 = 0$

$\frac{\partial x_i^*}{\partial w_2} \geq 0$: goods are substitutes

If $n > 2$, then $\frac{\partial x_i^*}{\partial w_j} \geq 0$ for some good j

Shepard's Lemma: If $C(w, y)$ is C^1 , and $w > 0$, then
 $x_i(\bar{w}, y) = \frac{\partial C(\bar{w}, y)}{\partial w_i}$

Proof 1: Envelope Theorem.

$$\mathcal{L} = w \cdot x - \gamma (f(x) - y)$$

$$\frac{\partial C}{\partial w_i} = \frac{\partial \mathcal{L}}{\partial w_i} \Big|_{x=x^*(w, y)}$$

$$\rightarrow \boxed{\frac{\partial C}{\partial w_i} = x_i^*(w, y)}$$

Proof 2: $C(w, y) = w \cdot x^*(w, y)$

$$\frac{\partial C}{\partial w_i} = x_i^*(w, y) + \sum_{j=1}^m w_j \cdot \frac{\partial x_j^*}{\partial w_i} \quad (\star)$$

Recall the f.o.c of CMP

$$\textcircled{1} \quad \lambda \frac{\partial f}{\partial x_i} = w_i \quad \textcircled{2} \quad f(x^*(w, y)) = y$$

Differentiate \textcircled{2} wrt w_i :

$$\sum_{j=1}^m \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j^*}{\partial w_i} = 0 \quad \xrightarrow{\text{plug (1) into } (\star)}$$

plug (1) into (\star)

$$\frac{\partial C}{\partial w_i} = x_i^*(w, y) + \sum_{j=1}^m \lambda \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j^*}{\partial w_i}$$

$$\boxed{\frac{\partial C}{\partial w_i} = x_i^*(w, y)}$$

Substitution matrix.

$$\begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} & \dots & \frac{\partial x_1^*}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m^*}{\partial w_1} & \dots & \frac{\partial x_m^*}{\partial w_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 C}{\partial w_1^2} & \dots & \frac{\partial^2 C}{\partial w_1 \partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 C}{\partial w_m \partial w_1} & \dots & \frac{\partial^2 C}{\partial w_m^2} \end{bmatrix}$$

Shepard's Lemma

$\frac{\partial^2 C}{\partial w_i^2}$, which is
concave.
(Hessian of C)

- Law of demand: $\frac{\partial x_i^*}{\partial w_i} \leq 0$, $\forall i$

- Symmetric cross-price effect: $\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$

Theorem: If f is constant returns to scale, then
 $c(w, y) = y \cdot \underbrace{c(w, 1)}_{\text{unit cost function}}$

Proof: Assume $f \in C^1$, int solution.

F.O.C at $y=1$:

$$\frac{w_i}{w_j} = \frac{f_i(x^*)}{f_j(x^*)} \quad \text{and} \quad f(x^*) = 1$$

For general \hat{y} : $\frac{w_i}{w_j} = \frac{f_i(\hat{x})}{f_j(\hat{x})}$, $f(\hat{x}) = \hat{y}$

Consider input $\hat{x} = \hat{y}x^*$

claim: \hat{x} solves the F.O.C. at \hat{y}

$$\frac{f_i(\hat{x})}{f_j(\hat{x})} = \frac{f_i(\hat{y}, x^*)}{f_j(\hat{y}, x^*)} = \frac{f_i(x^*)}{f_j(x^*)} = \frac{w_i}{w_j}$$

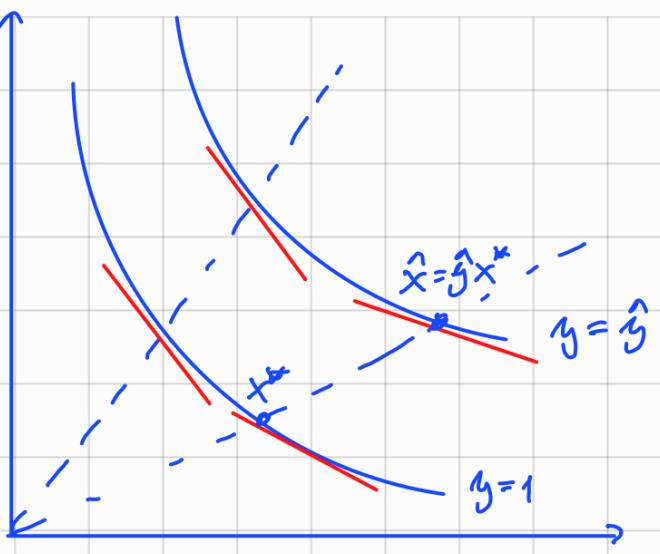
C.R.S.

\Rightarrow so $\hat{x} = \hat{y}x^*$ solves the problem at target output \hat{y}

$$\begin{aligned} c(w, \hat{y}) &= w \cdot (\hat{y} \cdot x^*) = \hat{y} \cdot (w \cdot x^*) \\ &= \hat{y} \cdot c(w, 1) \end{aligned}$$

$$f(\hat{y}x^*) = \hat{y} f(x^*) = \hat{y}$$

by C.R.S.



Cost function: $C(w, g) = \min w \cdot x \text{ s.t. } f(x) \geq g$

Sept 25, 2023

for now, just $C(g)$

Average cost: $AC(g) = \frac{C(g)}{g}$

Marginal cost: $MC(g) = C'(g)$

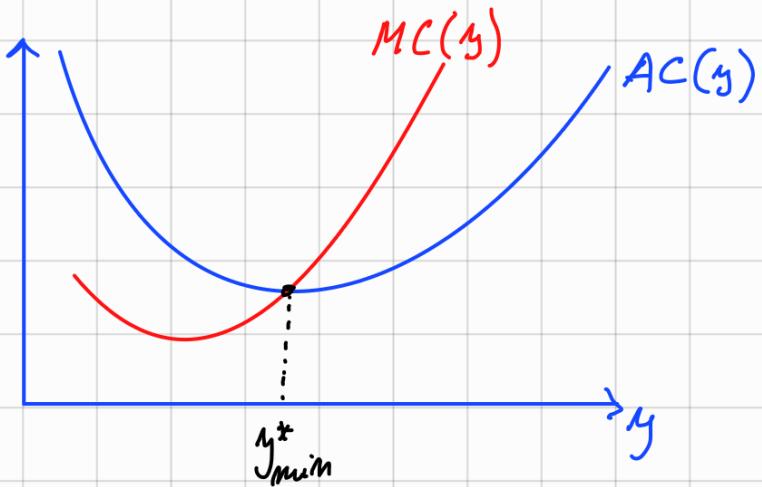
Marginal - average relationship:

$$AC'(g) = \frac{C'(g)}{g} - \frac{C(g)}{g^2} = \frac{C'(g) \cdot g - C(g)}{g^2}, \quad g^2 > 0$$

AC. incr $\leftrightarrow C'(g) > C(g)/g$, or $MC(g) > AC(g)$

AC. decr $\leftrightarrow C'(g) < C(g)/g$, or $MC(g) < AC(g)$

$AC'(g) = 0 \leftrightarrow MC(g) = AC(g)$



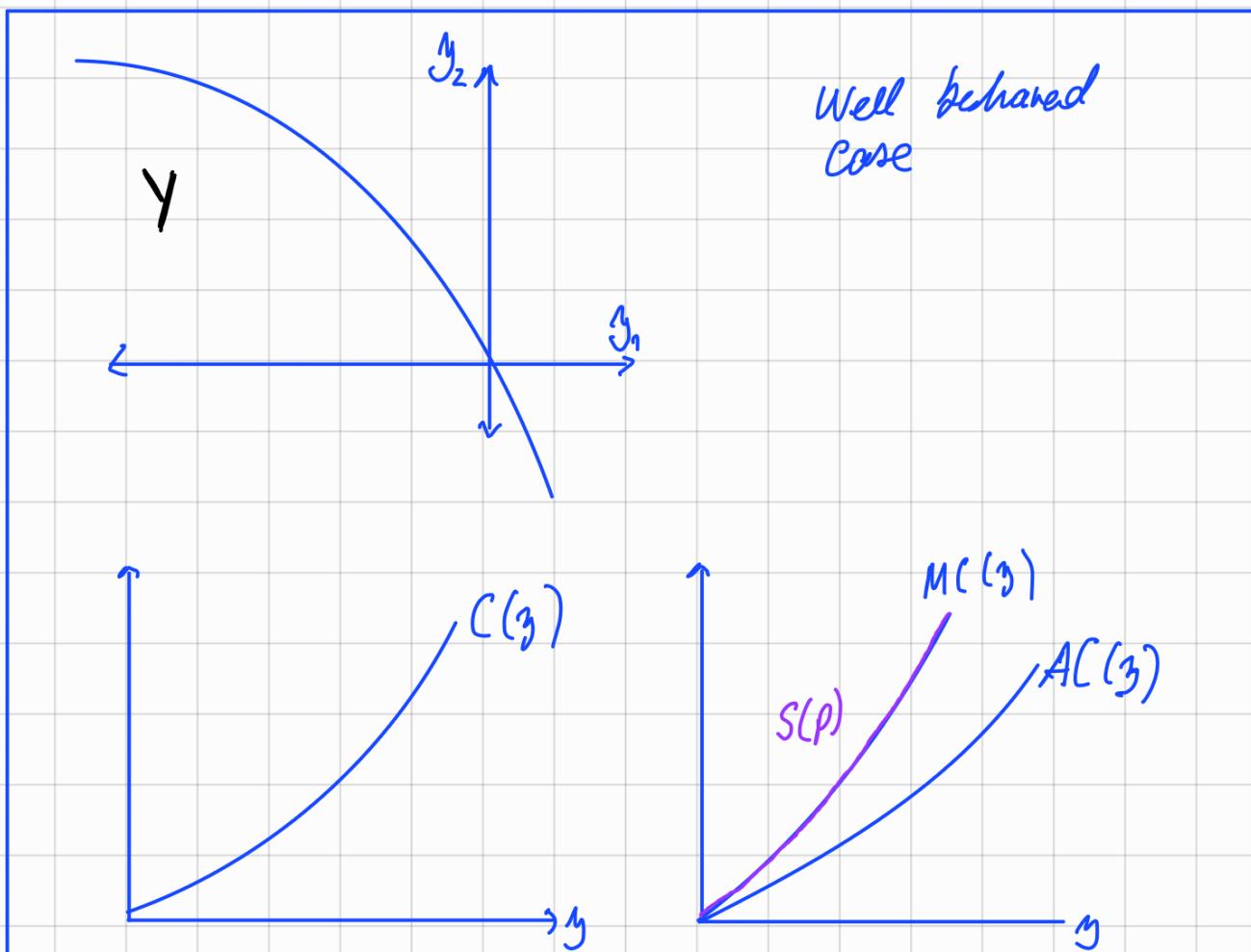
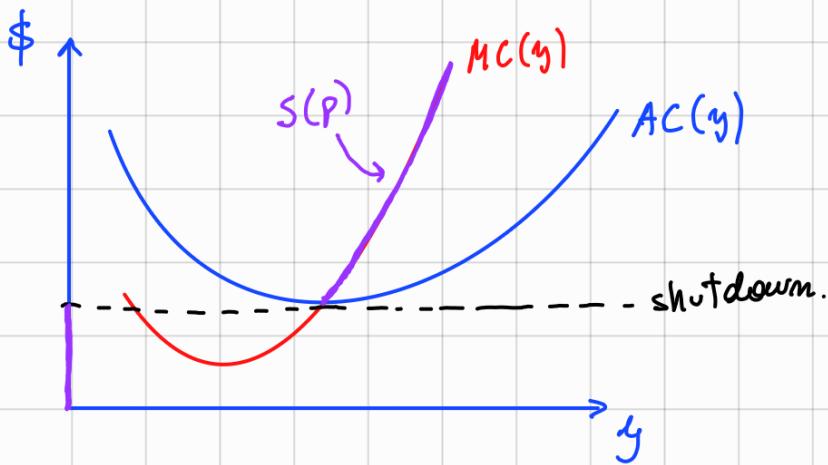
y^*_{\min} : Minimum efficient scale (MES)

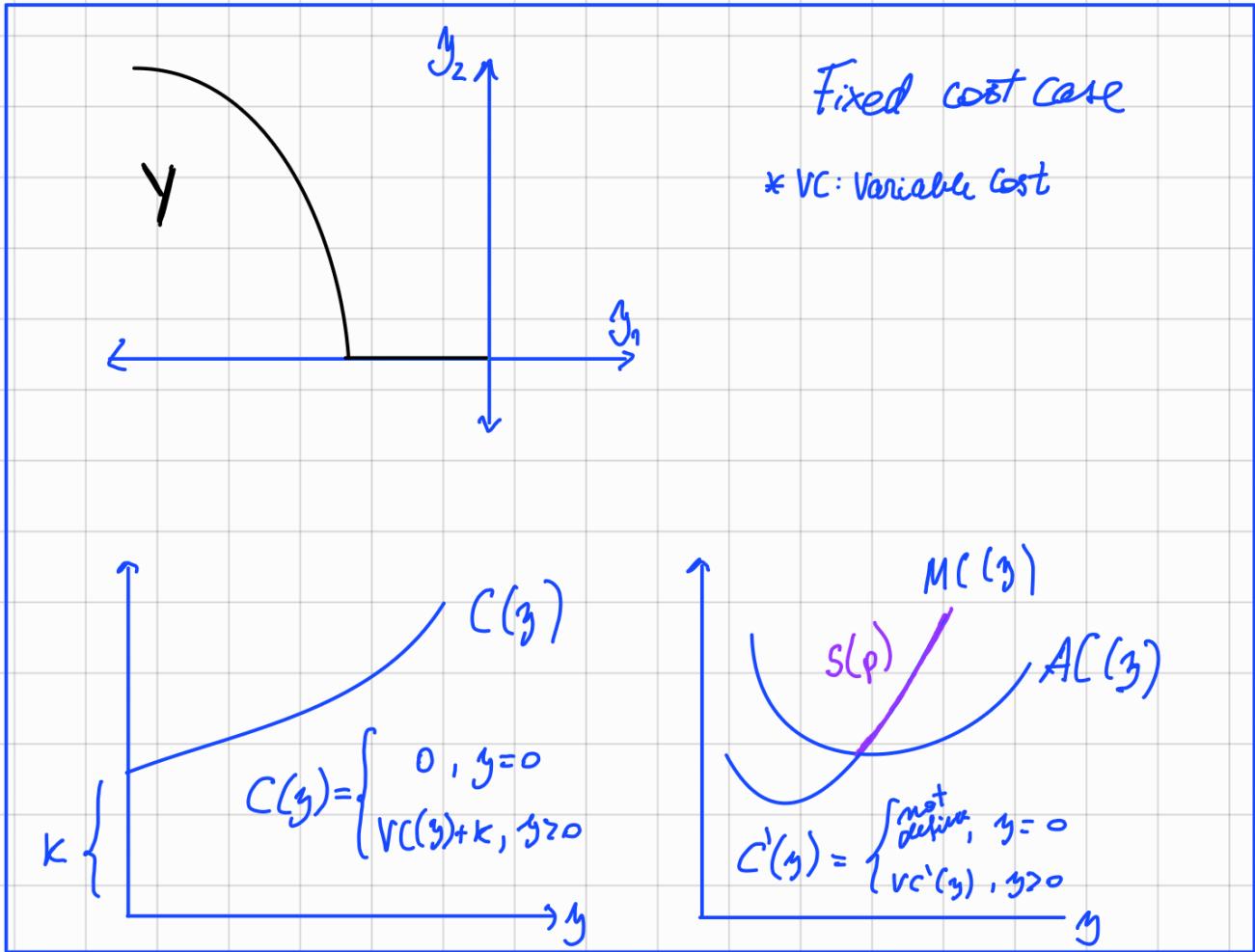
$$\max_{y \geq 0} py - C(y)$$

FONC $P \leq C'(y^*)$, with equality if $y^* > 0$
 If $C(y)$ is convex, this is also sufficient

Price = mc if firm chooses to produce a strictly positive amount

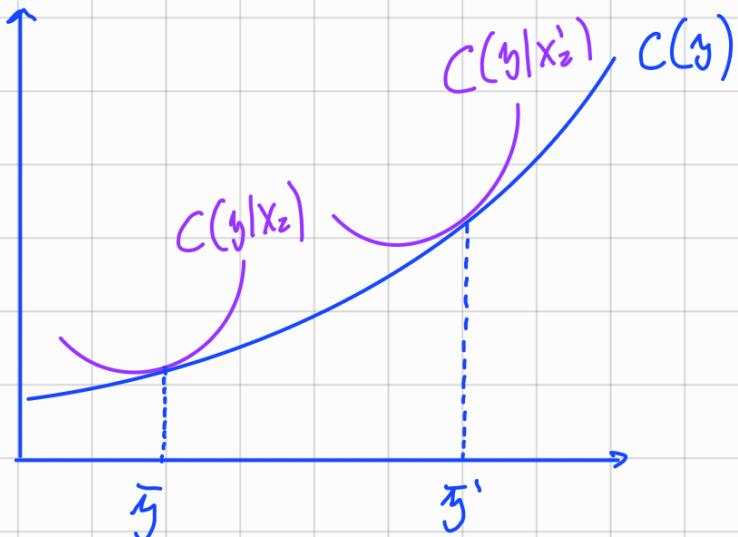
Shutdown Rule: Check that at y^* where $p = c'(y^*)$,
 $py^* - C(y^*) \geq 0$ or $p \geq AC(y^*)$





Long-Run vs Short-Run costs

- 2 inputs (x_1, x_2)
- $C(y)$ be the LR cost function
- Let \bar{x}_2 be optimal x_2 when long-run output \bar{y}
- $C(y|\bar{x}_2)$. short run cost, when x_2 fixed at \bar{x}_2 .



Can we go from $c(w, y) \rightarrow f(x)$?
 - Yes, they are "duals" of each other.

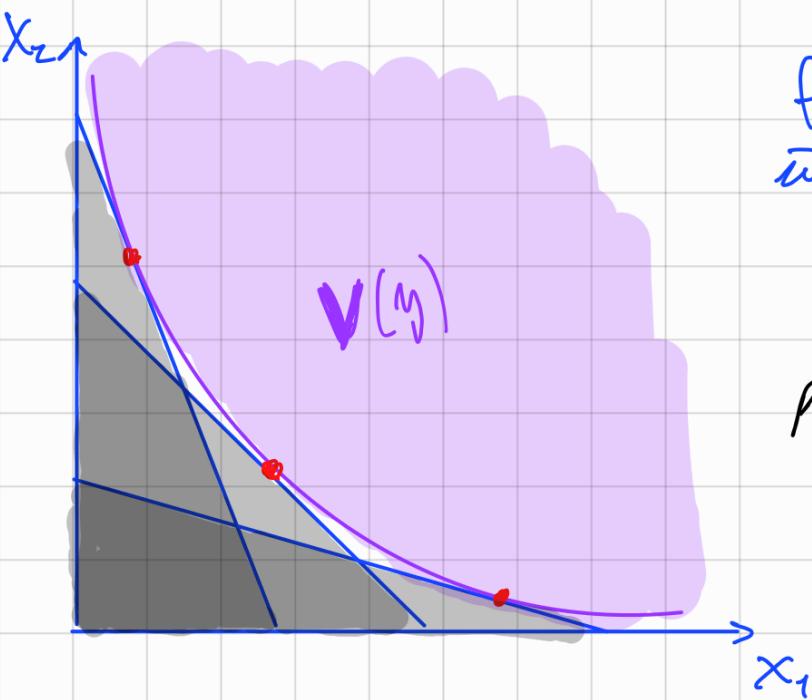
Observe $c(w, y) \nvdash (w, y)$

$$V(y) = \{x : f(x) \geq y\}$$

If we know $V(y) \nvdash y$, then we know $f(x)$.

Can we recover $V(y)$? Define an "outer bound"

$$V^*(y) = \{x : w \cdot x \geq c(w, y), \forall w \geq 0\} \text{ at } \bar{w}, \text{ cost is } \bar{c} = c(\bar{w}, y)$$

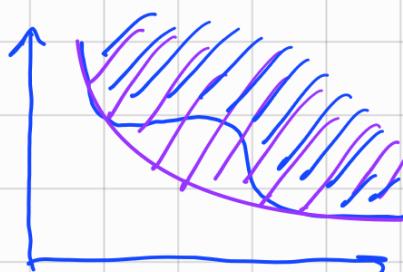


firm chose \bar{x}
 $\bar{w}_1 x_1 + \bar{w}_2 x_2 = c$

Black Shaded Area: Revealed output points because of Revealed preferences

Claim: Suppose $V(y)$ is the input requirement set for a firm with a convex technology with free disposal. Then, $V(y) = V^*(y)$

Proof: Homework (use sep. hyperplane thm).



Presupposed the function we start with is a valid cost function

Question: Given an arbitrary function $\phi(w, y)$, how do we know if ϕ is a valid cost function? That is, does

$$\phi(w, y) = \min_{x \in V(y)} w \cdot x$$

$$\text{where } V(y) = \{x \geq 0 : w \cdot x \geq \phi(w, y) \text{ for all } w\}.$$

Showed before that if ϕ is a cost function, then:

- i) add-1 in prices: $\phi(tw, y) = t\phi(w, y)$
- ii) $\phi(w, y) \geq 0$ (positivity)
- iii) $\phi(w, y) \geq \phi(w', y)$, $w' \geq w$ (monotonic)
- iv) Concave in w

Theorem: Let $\phi(w, y)$ be a C^2 function satisfying i)-iv), then, $\phi(w, y)$ is the cost function for the technology

$$V(y) = \{x : w \cdot x \geq \phi(w, y), \forall w\}$$

Proof: Define

$$x(w, y) = \left(\frac{\partial \phi(w, y)}{\partial w_1}, \dots, \frac{\partial \phi(w, y)}{\partial w_m} \right)$$

By monotonicity, $x(w, y) \geq 0$

$$\text{By add-1, } \phi(w, y) = \sum_{i=1}^m w_i \frac{\partial \phi(w, y)}{\partial w_i} = w \cdot x(w, y)$$

Euler's Law

We want to show for any $w \geq 0$

$$\phi(w, y) = w \cdot x(w, y) \leq wx \quad \forall x \in V(y)$$

This will imply $\phi(w, y) = \min w \cdot x \text{ s.t } x \in V(y)$

To show $x(w, y)$ is feasible: By concavity,

$$\phi(w, y) \leq \phi(w, y) + \underbrace{\nabla \phi(w, y)(w - w)}_{x(w, y)} \quad \text{for any } w'$$

$$\leq \phi(w, y) + w \cdot x(w, y) - \underbrace{w \cdot x(w, y)}_{\phi(w, y)}$$

$$\phi(w, y) \leq w \cdot x(w, y), \quad \forall w'$$

$$\Rightarrow x(w, y) \in V(y) \quad (\text{feasible})$$

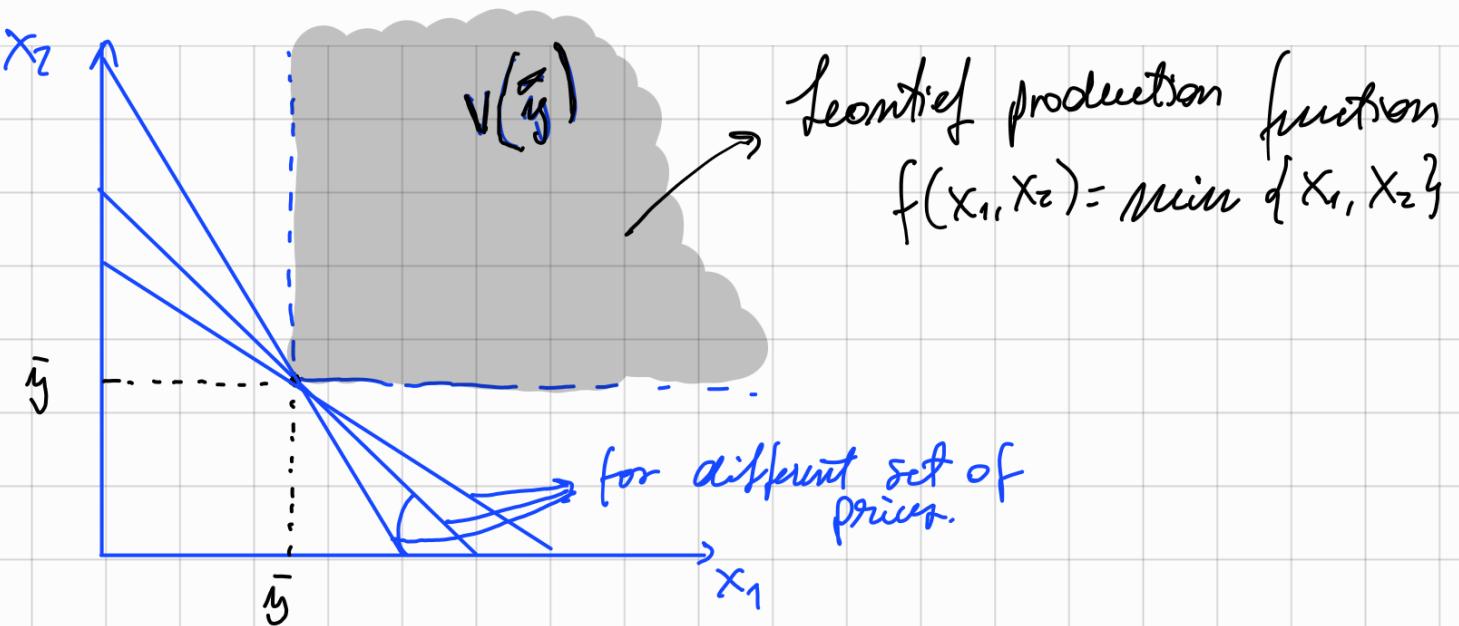
Now notice that for any $x \in V(y)$, $w \cdot x \geq \phi(w, y) = w \cdot x(w, y)$
or $w \cdot x(w, y) \leq w \cdot x \quad \forall x \in V(y)$

$$\text{i.e., } x(w, y) = \underset{x \in V(y)}{\operatorname{argmin}} w \cdot x$$

$$\phi(w, y) = w \cdot x(w, y)$$

Example: $C(w, y) = y(w_1 + w_2)$

Fix an output level \bar{y} . Graphically, we want (x_1, x_2) s.t
 $wx \geq C(w, \bar{y}) \quad \forall (w_1, w_2)$
 $w_1 x_1 + w_2 x_2 \geq w_1 \bar{y}_1 + w_2 \bar{y}_2$
 $\rightarrow x_2 \geq \frac{w_1 \bar{y}_1}{w_2} + \frac{\bar{y}_2 - w_1 \bar{y}_1}{w_2}$



Algebraically

$$\frac{\partial C}{\partial w_1} = x_1^* \quad ; \quad \frac{\partial C}{\partial w_2} = x_2^* \quad (\text{Shepard's Lemma})$$

$$\text{Here } \bar{y} = x_1^*(w, y) \quad ; \quad \bar{y} = x_2^*(w, y)$$

$$C(w, y) = (w_1 + w_2)y$$

$$\downarrow$$

$$f(x_1, x_2) = \min\{x_1, x_2\}$$

$$C(w, y) = y \cdot \min\{w_1, w_2\}$$

$$\downarrow$$

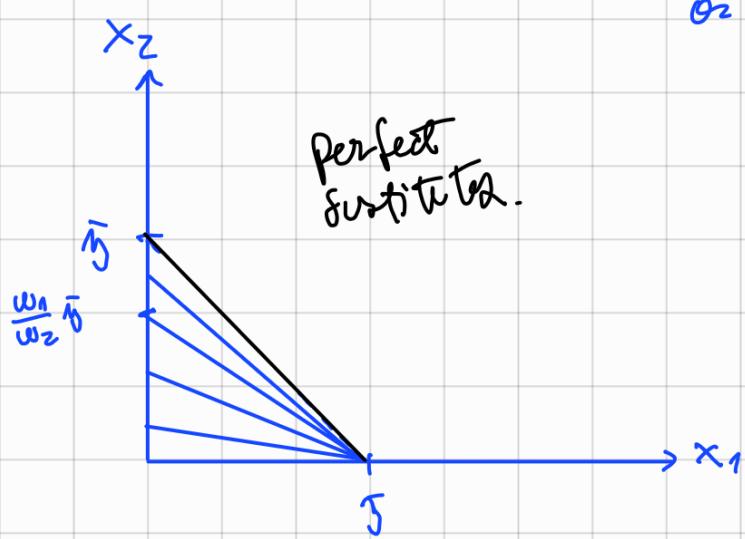
$$f(x_1, x_2) = x_1 + x_2$$

3 cases :

① $w_1 < w_2$; $C(w, \bar{y}) \leq w_1$

we want the points (x_1, x_2) s.t. $w_1 x_1 + w_2 x_2 \geq \bar{y} w_1$

$$\text{or } x_2 \geq \frac{w_1}{w_2}(\bar{y} - x_1) \quad \text{and} \quad \frac{w_1}{w_2} \in (0, 1)$$



② $w_1 = w_2 = w$; $C(w, y) = \bar{y}w$

$$w(x_1 + x_2) \geq \bar{y}w$$

$$x_2 \geq \bar{y} - x_1$$

③ $w_1 > w_2$

Same as (1).

Algebraically :

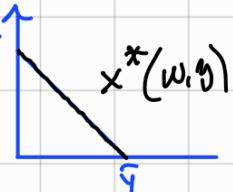
$$w_1 < w_2$$

by Shephard's Lemma

$$x_1(w, y) = \frac{\partial C}{\partial w_1} = \bar{y} \quad ; \quad x_2(w, y) = \frac{\partial C}{\partial w_2} = 0$$

$$\frac{w_1 > w_2}{x_1(w, y) = \frac{\partial C}{\partial w_1} = 0} \quad ; \quad x_2(w, y) = \frac{\partial C}{\partial w_2} = \bar{y}$$

at $w_1 = w_2$, Shephard's Lemma
doesn't apply



In general, take an iso-cost curve

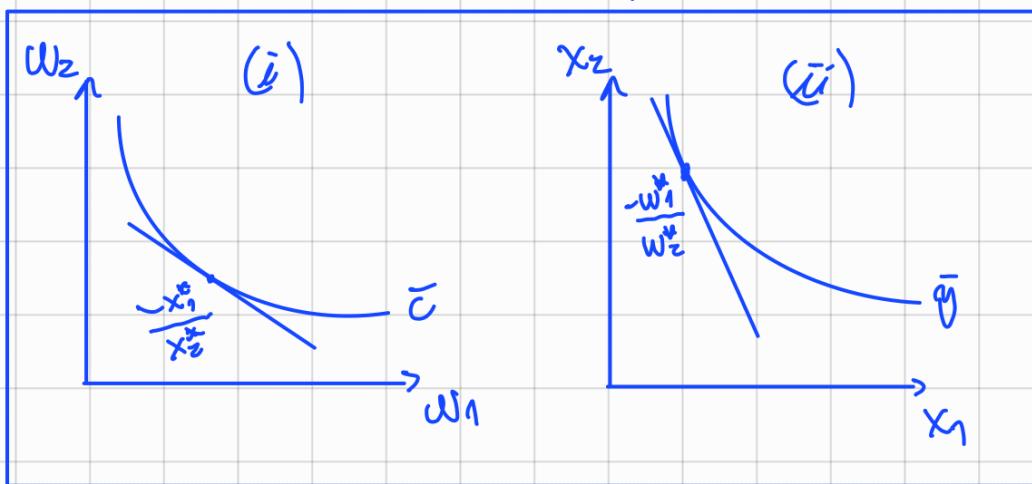
$$C(w, y) = \bar{c}$$

$$\partial = d\bar{c} = \frac{\partial C}{\partial w_1} dw_1 + \frac{\partial C}{\partial w_2} dw_2$$



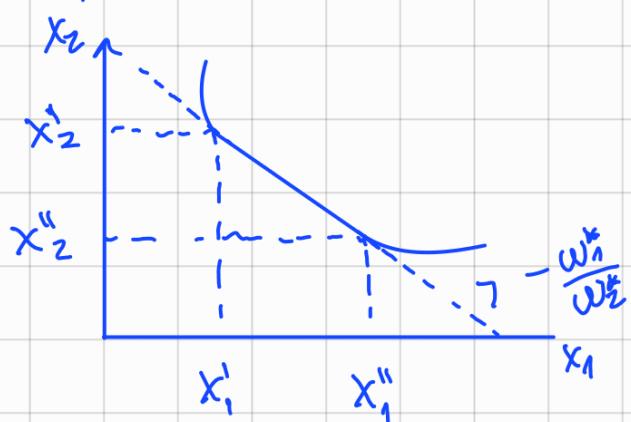
$$\frac{dw_2}{dw_1} = - \frac{\partial C / \partial w_1}{\partial C / \partial w_2} \stackrel{*}{=} - \frac{x_1^*(w, y)}{x_2^*(w, y)}$$

\star Shephard Lemma

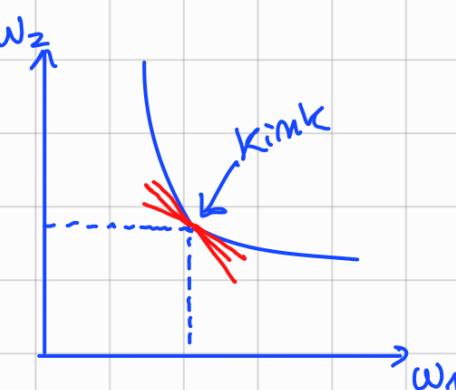


Duality between
(i) & (ii)

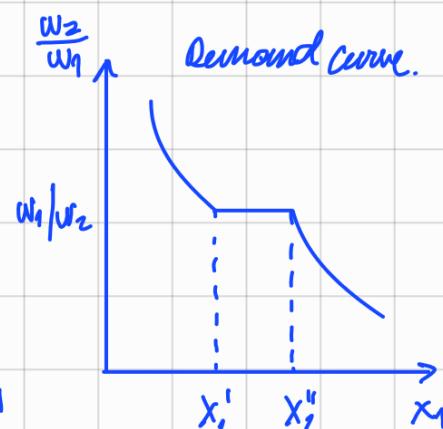
Isoquant with a flat spot



multiple
optimal
bundles

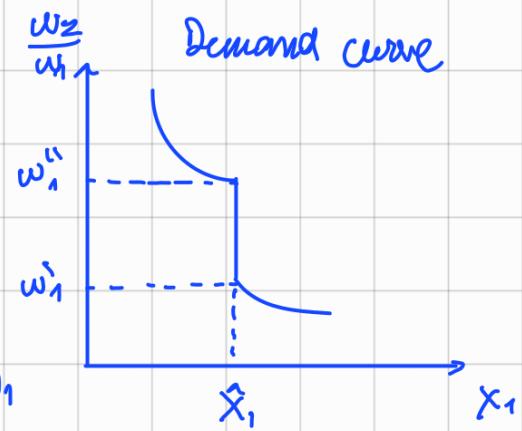
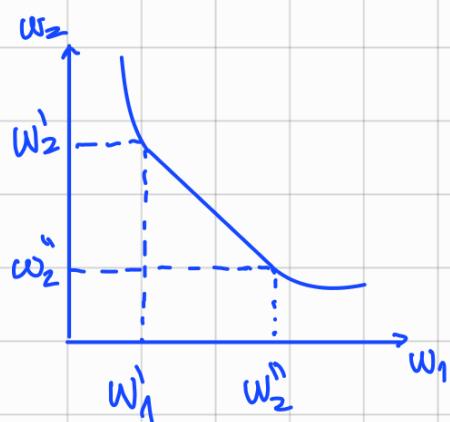
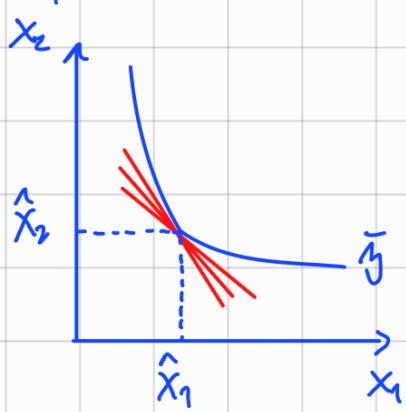


multiple tangents
at kink optimal.



Demand curve.

Isoquant with a kink



Isoquant and iso-cost slopes are inversely related

Aggregation: J firms, production sets Y_1, \dots, Y_J (non empty closed, FO).

$T\bar{y}_j(p)$, $\bar{y}_j(p)$ solution functions for firm j

$$y(p) = \sum_{j=1}^{\infty} y_j(p) = \left\{ y \in \mathbb{R}^n : y = \sum_{j=1}^J y_j^j, \text{ for some } y_j^j \in y_j(p) \forall j \right\}$$

Recall for a single j , we had the law of supply

$$(p - p') \cdot (y_j(p) - y_j(p')) \geq 0$$

$$\begin{aligned} \text{Add over } j : \sum_{j=1}^J (p - p') \cdot (y_j(p) - y_j(p')) &= (p - p') \cdot \sum_{j=1}^J (y_j(p) - y_j(p')) \\ &= \underbrace{(p - p') (y(p) - y(p'))}_{\text{Agg. Law of Supply}} \geq 0 \end{aligned}$$

Agg substitution matrix, $D_y(p)$, also inherits properties such as symmetry, PSD.

Strengthen this to a "representative firm" result.

Agg. production set:

$$Y^* = Y_1 + \dots + Y_J = \left\{ y \in \mathbb{R}^m : y = \sum_{j=1}^J y_j' , \text{ for some } y_j' \in Y_j \forall j \right\}$$

A single firm with production set Y^* . Let $\pi^*(p)$, $y^*(p)$ be solution functions for a firm, Y^* .

Theorem: i) $\pi^*(p) = \sum_{j=1}^J \pi_j(p)$ (This result does not hold for consumers)

$$\text{ii)} \quad y^*(p) = \sum_{j=1}^J y_j(p)$$

Proof of ii):

want to show

$$\textcircled{1} \quad \sum_j y_j(p) \subseteq y^*(p)$$

$$\textcircled{2} \quad y^*(p) \subseteq \sum_j y_j(p)$$

For $\textcircled{1}$: Let $y_j' \in y_j(p) \forall j$

$$p \cdot \left(\underbrace{\sum_j y_j'}_0 \right) = \sum_j \underbrace{p \cdot y_j'}_{\pi_j(p)} = \sum_j \pi_j(p) = \pi^*(p)$$

↓
By (i)

so $y \in y^*(p)$

For $\textcircled{2}$: $y \subseteq \sum_j y_j(p)$

Take $y \in y^*(p)$ by construction, $y = \sum_j y_j'$ for some selection $y_j' \in Y_j$

$$p \cdot y = p \cdot \left(\sum_j y_j' \right) = \pi^*(p) = \sum_j \pi_j(p)$$

↓
By (i)

$$\boxed{\sum_j p \cdot y_j' = \sum_j \pi_j(p)} \quad (\star)$$

Note that $p_j \cdot y_j \leq \pi_j(p) \quad \forall j$

$$\sum p_j y_j \leq \sum \pi_j(p) \quad (\star\star)$$

By (\star) we know that $(\star\star)$ holds with strict equality $\forall j$.

Lecture : Monopoly

Oct 4, 2023

- Can "choose" output price (p)
- still face some constraint
 - technological constraint (cost function $C(y)$)
 - consumer constraint (demand function $D(p)$)

Monopolist's problem : $\max_{p,y} p \cdot y - C(y) \quad \text{s.t. } D(p) = y$

$$\Rightarrow \max_p p \cdot D(p) - C(D(p)) \quad (\text{Single Variables})$$

Another way uses inverse demand function

$$p(y) = D^{-1}(y)$$

$$\max_{y \geq 0} p(y) \cdot y - C(y)$$

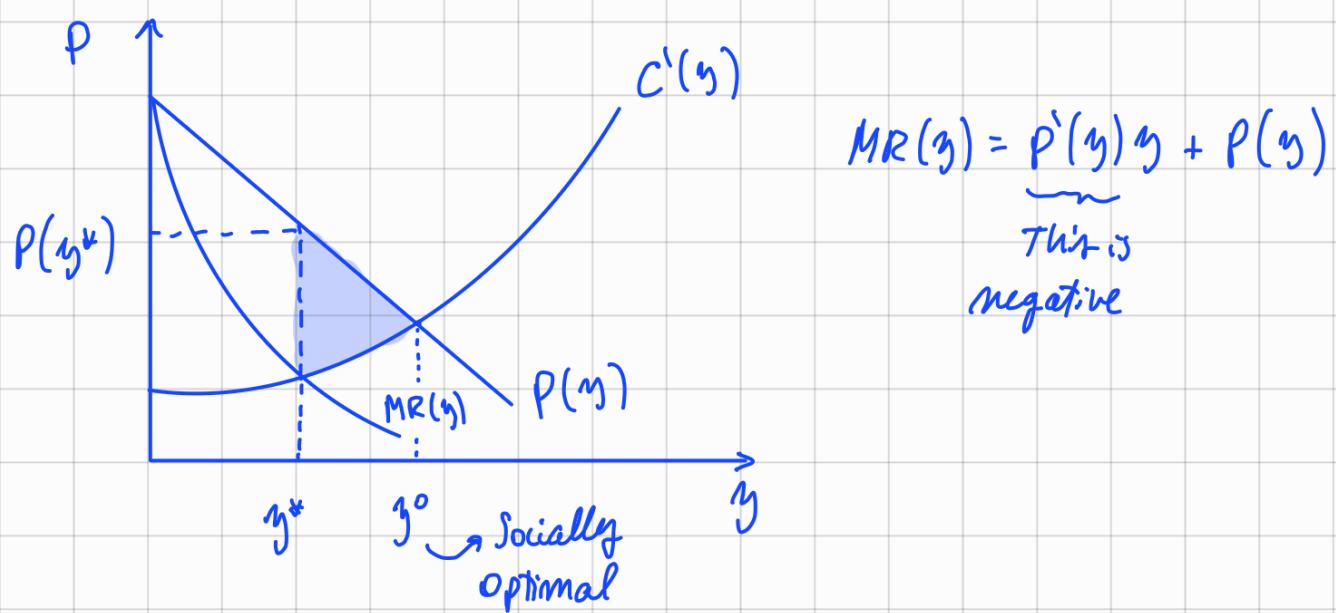
F.O.C

$$\underbrace{p'(y^*) y^* + p(y^*)}_{\text{marginal revenue}} \leq \underbrace{C'(y^*)}_{\text{marginal cost}}, \text{ with equality if } y^* > 0$$

Assume $y^* > 0$, so $MR = MC$

$$p'(y^*) y^* + p(y^*) = C'(y^*)$$

Usual case: $p'(y) < 0$ (downward sloping demand)
 $\Rightarrow p(y^*) > c'(y)$: monopolist prices above mg Cost



Q: How does monopolists price/output change as a parameter changes?

Tool here is the implicit function theorem: How does a solution function change as a parameter changes? Contract with the envelope theorem which applies to the value function.

IFT "Recipe":

- Start with f.o.c from max problem
- Write solution function $x^*(\theta)$ implicitly as a function of θ
- Differentiate f.o.c wrt to θ (chain rule)
- Solve resulting equation for $\frac{dx^*(\theta)}{d\theta}$

Back to monopolist...

Costs: $C(y, \theta)$

$$\text{F.O. } p'(y^*)y^* + p(y^*) - \frac{\partial C(y^*, \theta)}{\partial y} = 0$$

Write y^* implicitly as a function of θ .

$$p'(y^*(\theta))y^*(\theta) + p(y^*(\theta)) - \frac{\partial C(y^*(\theta), \theta)}{\partial y} = 0$$

Now differentiate w.r.t θ

$$p''(y^*(\theta), \theta)y^*(\theta) \frac{dy^*(\theta)}{\partial \theta} + p'(y^*(\theta)) \frac{dy^*(\theta)}{\partial \theta} + p'(y^*(\theta)) \frac{dy^*(\theta)}{\partial \theta}$$

$$- \frac{\partial^2 C(y^*(\theta), \theta)}{\partial y^2} \frac{dy^*}{\partial \theta} - \frac{\partial^2 C(y^*(\theta), \theta)}{\partial y \partial \theta}$$

Solve for $dy^*/d\theta$

$$\frac{dy^*}{d\theta} = \frac{\partial^2 C(y^*, \theta) / \partial y \partial \theta}{p''(y^*)y^* + \partial p'(y^*) - \partial^2 C(y^*, \theta) / \partial^2 y}$$

Example: CES demand, constant marginal cost

$$D(p) = A p^{-\beta}, \beta > 1$$

$$C(y) = \bar{c}y$$

Why is it called CES?

Calculate price elasticity of demand: $\sigma_{D,P} = \frac{d \log D(p)}{d \log P}$

$$\log D(p) = \log A - B \log p$$

$$\sigma_{D,P} = -B$$

$$\text{Inverse demand: } y = A p^{-\beta} \rightarrow p(y) = \left(\frac{y}{A}\right)^{1/\beta}$$

$$p'(y) \cdot y + p(y) = C'(y)$$

$$p'(y) = -\frac{1}{\beta} \underbrace{\left(\frac{y}{A}\right)^{\frac{1}{\beta}-1}}_{p(y)} \left(\frac{1}{A}\right)$$

$$= -\frac{1}{\beta} p(y) \frac{1}{y}$$

$$\text{F.O.C becomes } \left(-\frac{1}{\beta} p(y) \frac{1}{y}\right) y + p(y) = \tilde{C}$$

$$p(y) \left[1 - \frac{1}{\beta}\right] = \tilde{C}$$

$$p(y) = \frac{\tilde{C}}{1 - 1/\beta}$$

What is $\frac{dy^*(c)}{dc}$?

$y^*(c)$ defined implicitly by: $p(y^*(c)) \left[1 - \frac{1}{\beta}\right] - \tilde{C} = 0$

Fmp. func. T.:

$$p'(y^*(c)) \left[1 - \frac{1}{\beta}\right] \frac{dy^*}{dc} - 1 = 0$$

$$\frac{dy^*}{dc} = \frac{1}{p'(y^*(c)) \left[1 - \frac{1}{\beta}\right]} < 0$$

Consumer theory:

More complicated for two reasons:

- Rationality axioms on preferences
- Cannot use optimization techniques on preferences. Need to "construct" a utility function.
- Consumer's problem has prices in the constraints income / substitution effects.

Widely used in economics

- i) Normatively useful
- ii) Positive prediction
- iii) Widely applicable
- iv) Simple / sparse model.

Two approaches:

- 1) preferences \rightarrow choice
- 2) choices \rightarrow preferences (observed behaviour)

X : (abstract) set of alternatives
elements $x \in X$ are mutually exclusive

A choice structure $(\mathcal{P}, C(\cdot))$ is:

- \mathcal{P} is a family of nonempty subsets of X
- $C(\cdot)$ is a choice rule s.t $C(B) \subseteq B \quad \forall B \in \mathcal{P}, C(B) \neq \emptyset$

$$X = \{x, y, z\}, \quad \mathcal{P} = \{\{x\}, \{x, y, z\}\}$$

$$C(\{x, y\}) = \{x\}, \quad C(\{x, z\}) = \{x, z\}$$

What are some "reasonable" restrictions on behaviour?

Weak axiom of revealed preference (WARP)

If, for some $B \in \beta$ with $x, y \in B$ we have $x \in C(B)$, then for any other $B' \in \beta$ with $x, y \in B'$ and $y \in C(B')$, we also have $x \in C(B')$

$$X = \{a, b, c\} \quad \beta = \{ \underbrace{\{a, b\}}, \underbrace{\{a, b, c\}} \}$$

Starting from choices to preferences...

We observe $C(\{a, b, c\}) = \{b\}$

WARP implies that $C(\{a, b\}) = \{b\}$

Why? Assume $a \in C(\{a, b\})$. Then, since a was chosen when b was available, we must have $a \in C(\{a, b, c\})$. Contradiction.

We observe $C(\{a, b\}) = \{b\}$ (we know nothing about c).

WARP implies $C(\{a, b, c\}) = \{b\}, \{c\}$ or $\{b, c\}$

Preference-based approach

X : set of alternatives.

Primitive is a binary relation on X , \mathcal{Z} .

$x \succeq y$: " x is at least as good as y "

$x \succ y$: $x \succeq y \wedge y \not\succeq x$: " x is strictly preferred to y "

$x \sim y$: $x \succeq y \wedge y \succeq x$: " x is indifferent to y "

A preference relation is rational if

- i) Complete: $\forall x, y \in X$, either $x \succeq y$ or $y \succeq x$ (or both).
- ii) Transitivity: $\forall x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$

Violation of transitivity: Framing (Kahneman, Tversky).

Best buy to purchase \$125 stereo and \$15 calculator.

- A) Calculator on sale for \$5 off at store 20 min away YES
- B) Stereo " " " NO
- C) Both are out of stock at current store, but we'll give you a \$5 coupon for one item. Does it matter which one? Indifferent.

X: Travel to store 2 for \$5 calculator.

Y: " " " + Stereo

Z: Buy both items at store 1.

Response imply:

$X \succ Z$	(A)
$Z \succ Y$	(B)
$X \sim Y$	(C)

What are the links between choice & preference approach?

Q2. Does WARP imply rationality? NO

Q1. Does Rationality imply WARP? Yes.

For Q1. Given \succeq , let $C(B, \succeq) = \{x \in B : x \succeq y, \forall y \in B\}$

Theorem: Suppose \succeq are rational. Then, the choice structure $(B, C(B, \succeq))$ satisfies WARP.

Proof: Suppose for some $B \in \mathcal{B}$, we have $x, y \in B$ and $x \in C(B, \succeq)$ by definition of C , $x \succeq y$.

Check WARP: Suppose $x, y \in B'$, and $z \in C(B' \succeq)$. So, $y \succeq z$ for all $z \in B'$. By transitivity $x \succeq y \succeq z$, so, $x \succeq z$, for all $z \in B'$. Thus $x \in C(B', \succeq)$ and WARP holds.

For Q2: Counterexample.

$$X = \{a, b, c\}, \beta = \{\{a, b\}, \{b, c\}, \{a, c\}\}$$

Assume we observe: $C(\{a, b\}) = a$; $C(\{a, c\}) = c$; $C(\{b, c\}) = b$

Can check that $(\beta, C(\cdot))$ satisfies WARP.

Observed choice implies: $\begin{array}{l} a \succ b \\ a \succ c \\ b \succ c \end{array} \quad \left. \begin{array}{l} a \succ b \\ a \succ c \\ b \succ c \end{array} \right\} \text{this violates transitivity.}$

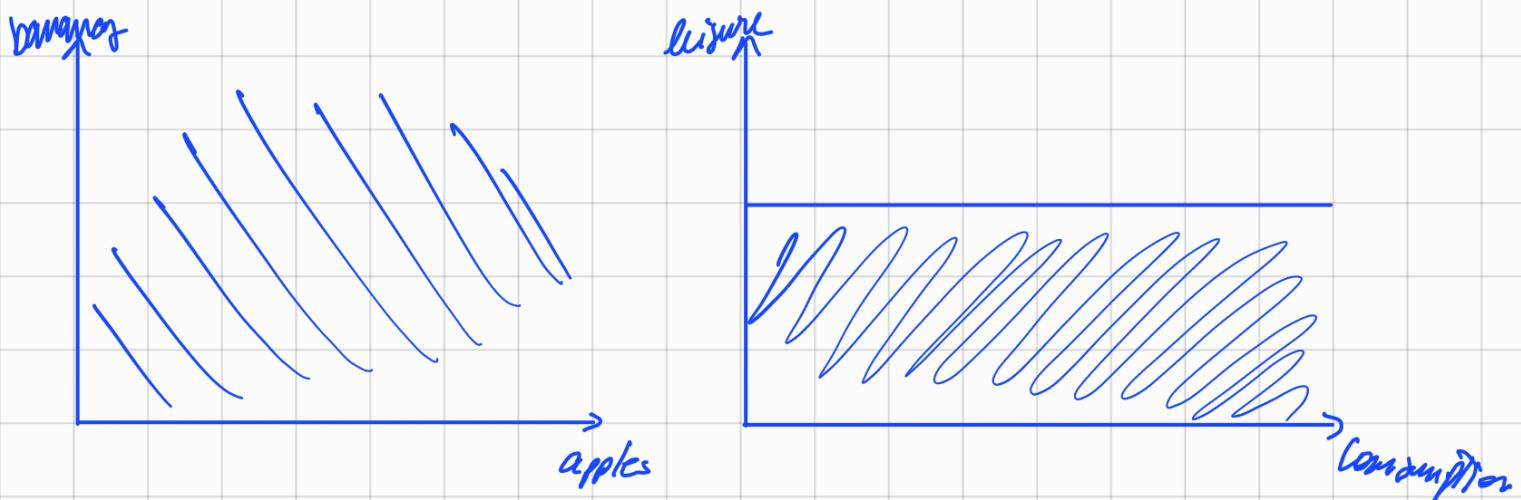
Theorem (Arrow, 1959):

If $(\beta, C(\cdot))$ is s.t:

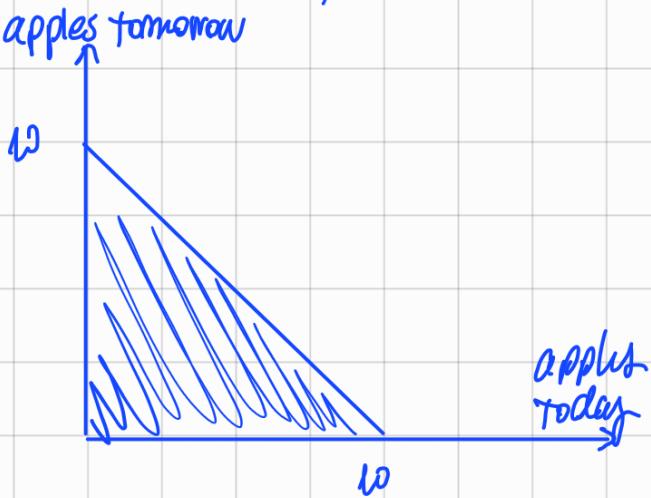
- i) WARP is satisfied
- ii) β includes all subsets of X with up to 3 elements.

Then, there is a (unique) rational preference relation \succeq that rationalizes $C(\cdot)$.

Restrict $X \subseteq \mathbb{R}_+^n$



Goods can be timed:



State dependent
icecream if cold

$\leftarrow x$

$x \rightarrow$

icecream
if hot.

Properties of preferences on $X \subseteq \mathbb{R}_+^n$

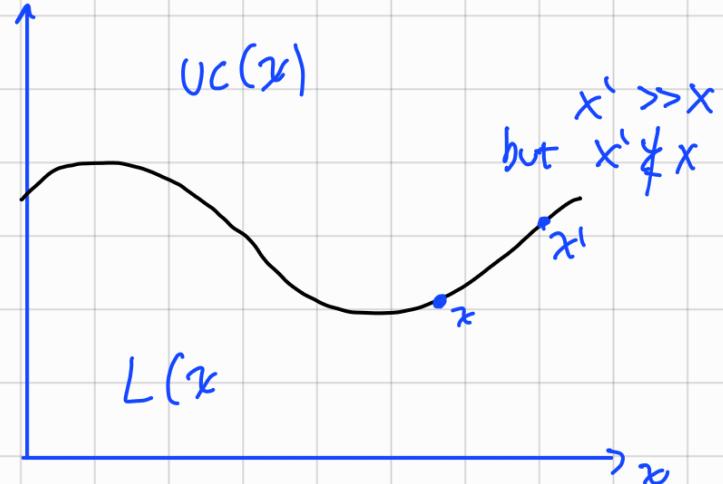
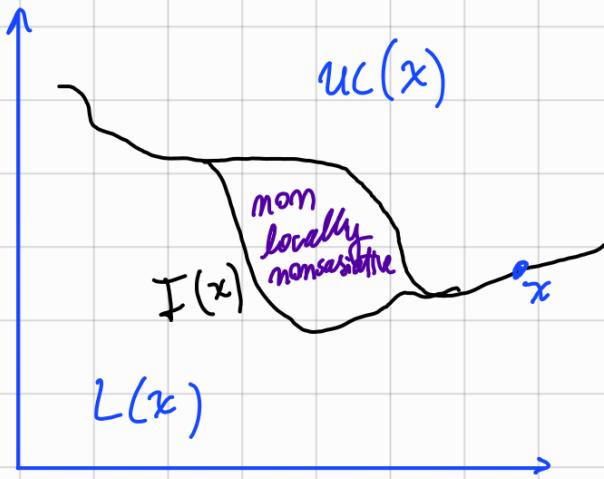
- Preference relation \succeq is monotone if $y \gg x$ implies $y \succ x$
- \succeq is strongly monotone if $y \geq x$ and $y \neq x$, implies $y > x$
- \succeq is locally non-sattractive if for every $x \in X$ (bundle) and $\epsilon > 0$ there is some $y \in X$ s.t $\|y - x\| < \epsilon$ and $y \succ x$.

upper contour of x

$$UC(x) = \{y \in X : y \succeq x\}$$

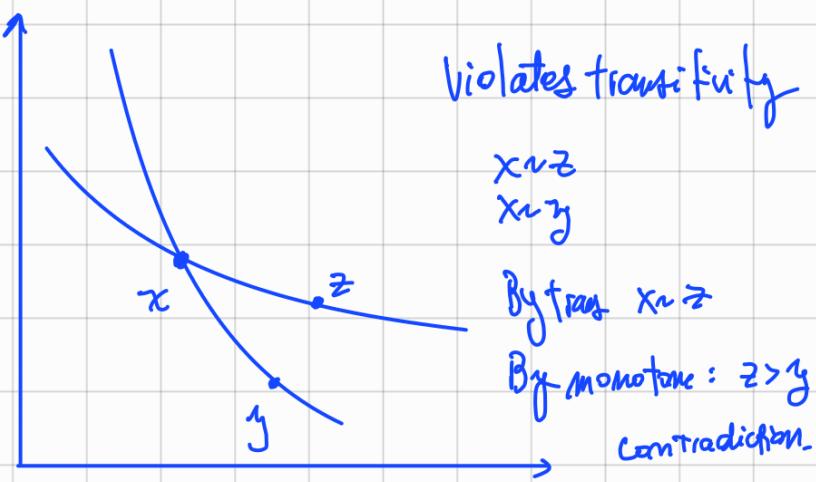
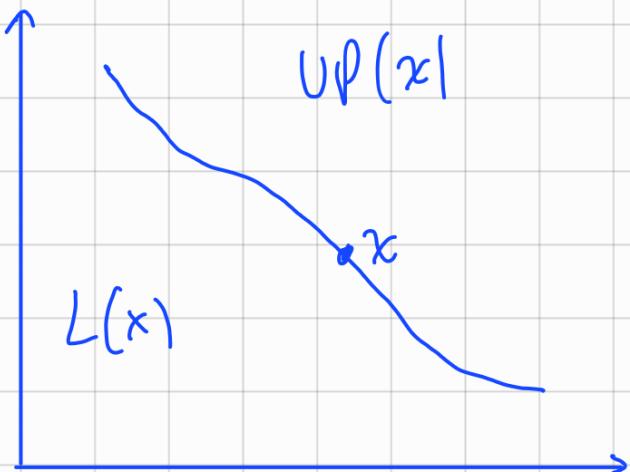
$$LC(x) = \{y \in X : x \succeq y\}.$$

$$I(x) = \{y \in X : x \sim y\}$$



Monotonicity \rightarrow Downward slope

IC cannot cross

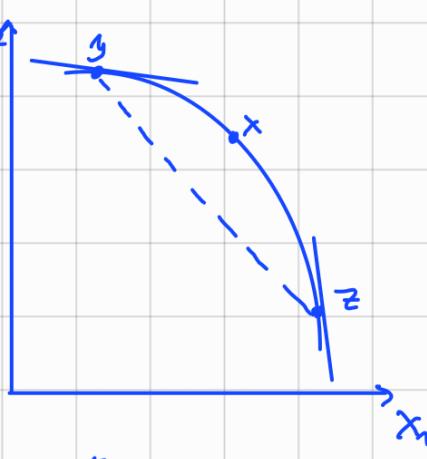
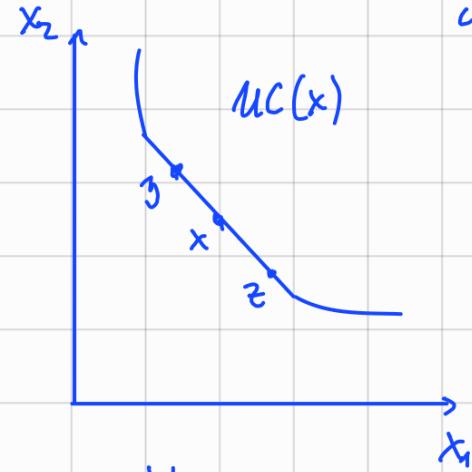
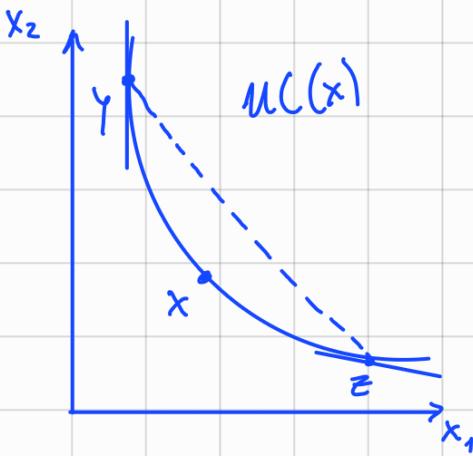


Lecture

Oct 09, 2023

Convexity

\succsim is convex if the upper contour set $UC(x) = \{y \in X : y \succeq x\}$ is convex or equivalently, $y \succsim x$ and $z \succsim x$, then $\underbrace{\alpha y + (1-\alpha)z}_{w} \succsim x$, $\forall \alpha \in [0,1]$

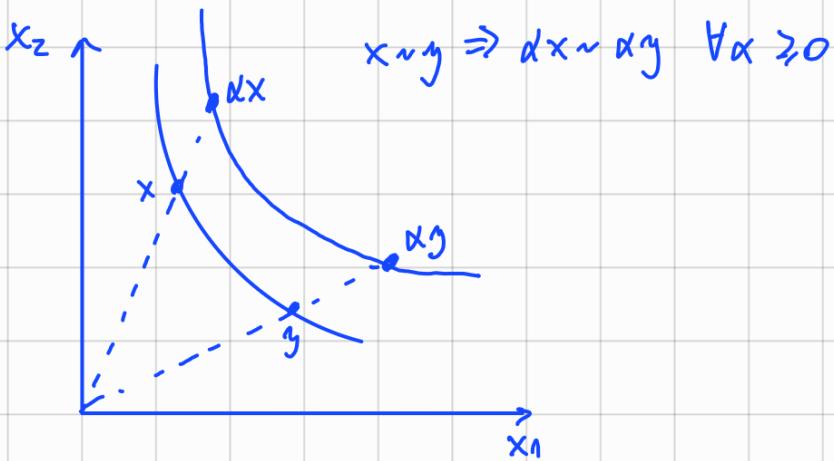


Interpretation of convexity:

- ① Diminishing marginal rate of substitution
- ② Preferences for "diversification"

Useful "classes" of preferences

1) Homothetic:



2) Quasilinear preferences:

Let $X = (-\infty, \infty) \times \mathbb{R}_+^{m-1}$ (1^{st} good can be negative; and there are other $m-1$ goods)
 write $x = (t, y)$ where $t \in \mathbb{R}$ and $y \in \mathbb{R}_+^{m-1}$

\succ is quasilinear if:

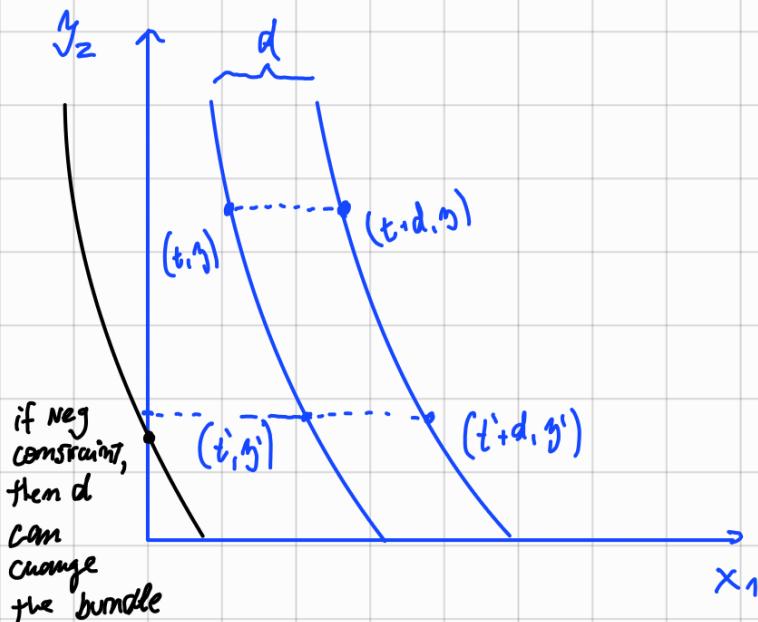
- i) Good 1 is desirable. $t' > t$, then $(t', y) \succ (t, y)$
- ii) "No wealth effect"

If $(t, y) \sim (t', y')$, then for all $d \in \mathbb{R}$, $(t+d, y) \sim (t'+d, y')$
 \Rightarrow good 1 often called "numeraire" (think of as "money").

\rightarrow will allow us to write utility as

$$u(t, y_1, \dots, y_m) = t + v(y_1, \dots, y_m)$$

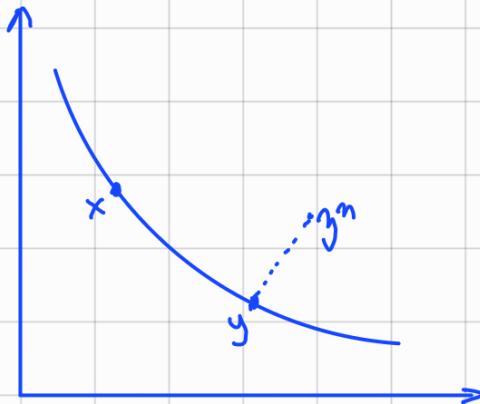
Also called "translatable utility".



Continuity:

\succeq is continuous if, for any sequence, $\{(x^n, y^n)\}_{n=1}^{+\infty}$ with $x^n \succsim y^n \forall n$, and $x = \lim_{n \rightarrow \infty} x^n$, $y = \lim_{n \rightarrow \infty} y^n$, then we have $x \succeq y$

An equivalent definition: $UC(x)$ and $LC(x)$ are closed sets



$y^n \rightarrow y$ Let $x^n = x, \forall n$
 $y^n \succsim x, y^n \in UC(x)$
 and so by continuity implies $y \in UC(x)$

Example: Lexicographic

$X = \mathbb{R}_+^2$. Define \succeq as follows: $x \succeq y$ if either:

- i) $x_1 > y_1$
- ii) $x_1 = y_1 \text{ & } x_2 \geq y_2$

\succeq is:

complete
transitive

strongly monotone

strongly convexity

\succeq is not continuous: $x^n = \left(\frac{1}{n}, 0\right), y^n = (0, 1) \forall n$

$x^n \succsim y^n \forall n$

$$x = \lim_{n \rightarrow \infty} x^n = (0, 0)$$

$$y = \lim_{n \rightarrow \infty} y^n = (0, 1)$$

and so $y \succ x$

Utility function: A preference relation is represented by a utility function $u: X \rightarrow \mathbb{R}$ if, $\forall x, y$

$$x \geq y \iff u(x) \geq u(y)$$

utility functions are not unique.

Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be any strictly increasing function.

Define $\tilde{u}(x) = g(u(x))$

$$\tilde{u}(x) \geq \tilde{u}(y) \iff u(x) \geq u(y) \iff x \geq y$$

$\Rightarrow \tilde{u}(\cdot)$ also represents \geq

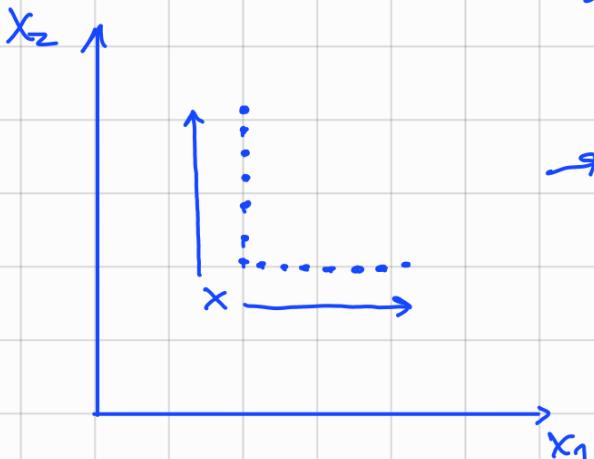
Q:

When can we find a utility function to represent \succeq ?

Theorem: \succeq can be represented by a utility function only if it is rational.

Proof: Homework.

Q: Is the following true?: If \succeq is rational, then there exists a utility function representing it. Ans: NO.



\rightarrow Is rational
but not enough for a utility func. Representation.

Sufficient conditions for a utility representation

- ① Finiteness: If X is finite, then exists a utility representation
- ② Continuity of \succsim

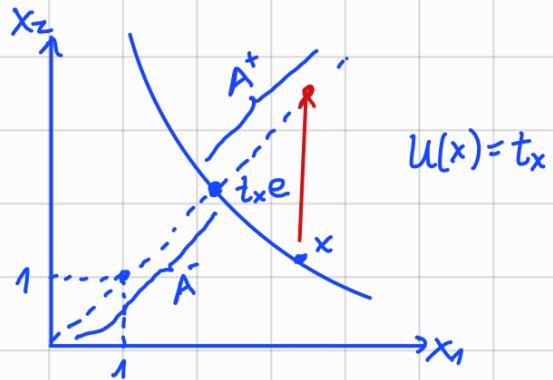
Theorem: If \succsim is rational & continuous, then there exists a utility function that represents them. Furthermore, there exists a continuous $u(\cdot)$ that represents them

proof: Show for case that \succsim is strongly monotone.

sketch:

- ① Construct a candidate $u(x)$ & ② check it works.

Let $e = (1, \dots, 1)$. Take a bundle $x \in X$



$$A^+(x) = \{t \in \mathbb{R}_+: t e \succsim x\}$$

$$A^-(x) = \{t \in \mathbb{R}_+: x \succsim t e\}$$

By strong monotonicity: $A^+(x) \neq \emptyset$ (take $\max\{x_1, \dots, x_n\}e \succsim x$).
 $A^-(x) \neq \emptyset$ ($x \geq 0$)

Completeness of $\succsim \rightarrow \mathbb{R}_+ = A^+(x) \cup A^-(x)$

By continuity, $A^+(x)$ & $A^-(x)$ are closed sets

Since \mathbb{R}_+ is connected, $A^+(x) \cap A^-(x) \neq \emptyset \rightarrow$ this means that
 $\exists t_x \in A^+ \cap A^- \text{ s.t. } t_x e \sim x$

Define $u(x) = t_x$

Last step, show proposed $u(x)$ represents \succsim

Oct 11, 2023

① $u(y) \geq u(x) \Rightarrow y \succsim x$

② $y \succsim x \Rightarrow u(y) \geq u(x)$

For ①: $t_y \geq t_x$ (def of u) For ②: Take $y \succsim x$
 $t_y e \succsim t_x e$ (monotonicity) $(t_y e) \sim y \succsim x \sim (t_x e)$
 $y \sim t_y e \succsim t_x e \sim x$ (def of t)
 $y \succsim x$ (transitivity)

\downarrow
def of t

$t_y e \succsim t_x e$ (transitivity)

$t_y \geq t_x$ (monotonicity)

$u(y) \geq u(x)$ (definition of u)

- Also true that a continuous $u(\cdot)$ exists (skip proof)

- Further conditions on \succsim that guarantee a differentiable $u(\cdot)$ exists (also skip)

- We have constructed an utility representation; there are many others, not all are continuous.

- $g(x) = \begin{cases} x, & x \leq 1 \\ 2x, & x > 1 \end{cases}$, $\tilde{u}(x) = g(u(x))$ also represents \succsim

From conditions on preferences to conditions on $u(\cdot)$

① \succsim convex $\Rightarrow \exists$ a quasiconcave $u(\cdot)$

proof: \succsim convex means.

$U(x) = \{y : y \succsim x\}$ is convex

But $y \succsim x$ is equivalent to $u(y) \geq u(x)$

$UC(x) = \{y : u(y) \geq u(x)\}$ is convex

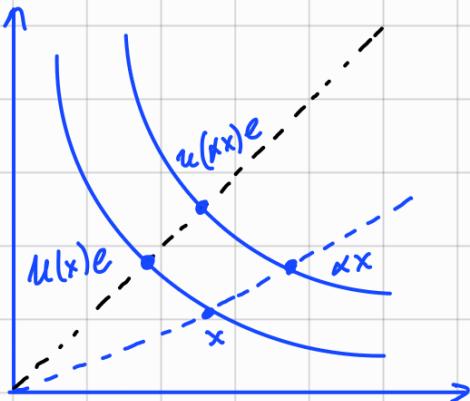
Which is a definition of quasiconcave $u(\cdot)$

Also true \succsim strictly convex $\Rightarrow \exists$ a $u(\cdot)$ strictly quasiconcave

Not True: Convex \succsim $\Rightarrow \exists$ $u(\cdot)$ concave

② \succsim homothetic (and monotone) $\Rightarrow \exists$ an hd-1 function $u(\cdot)$ that represents \succsim

Proof: Let \succsim be homothetic & monotone



$$\begin{aligned} u(x) \cdot e &\sim x \text{ & } u(ax) \sim \alpha x \\ \alpha u(x) e &\sim \alpha x \quad (\text{homothety}) \\ \underline{u(dx) e \sim \alpha u(x) e} &\quad (\text{transitivity}) \\ u(dx) &= \alpha \cdot u(x) \quad (\text{monotonicity}) \end{aligned}$$

Classic homothetic utility: Cobb-Douglas

$$u(x) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_m^{\alpha_m}$$

It is wlog to assume $\alpha_1 + \cdots + \alpha_m = 1$

(why?) say $\alpha_1 + \cdots + \alpha_m = k$. Consider: $\tilde{u}(x) = (u(x))^{1/k}$. Then \tilde{u} & u represent the same preference, and $\tilde{u}(x) = x_1^{\alpha_1/k} \cdot x_2^{\alpha_2/k} \cdots x_m^{\alpha_m/k}$

$$\left(\frac{\alpha_1}{k} + \cdots + \frac{\alpha_m}{k} \right) = \frac{1}{k} \underbrace{(\alpha_1 + \cdots + \alpha_m)}_k = 1.$$

Cobb-Douglas is a special case of CES function:

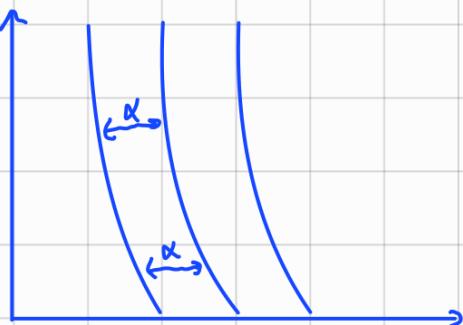
$$u(x) = [v_1 x_1^\rho + \cdots + v_m x_m^\rho]^{1/\rho} \quad (\text{this is a hd-1})$$

$\rho = 1$: perfect substitutes

$\rho = 0$: Cobb-Douglas

$\rho = -\infty$: perfect complements / Leontief

③ \Leftrightarrow Quasilinear (in x_1) $\Rightarrow \exists$ a $u(\cdot)$ of the form $u(x_1, \dots, x_n) = x_1 + v(x_2, \dots, x_n)$



Marginal rate of substitution



$$u(x) = \bar{u}$$

$$\frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} dx_j = d\bar{u} = 0 \rightarrow \text{BC we are on the same curve.}$$

$$\Rightarrow \frac{dx_j}{dx_i} = -\frac{\partial u / \partial x_i}{\partial u / \partial x_j}$$

$$\text{where } MRS_{ij}(x) = \frac{\partial u / \partial x_i}{\partial u / \partial x_j}$$

Budget Set

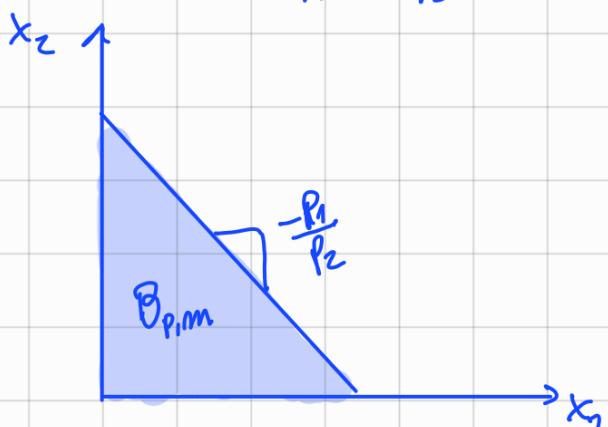
Consumption space: $X \in \mathbb{R}_+^n$

Standard (Walrasian) budget set: $B_{p,m} = \{x \in X : p \cdot x \leq m\}$

Price \leftrightarrow Income

In 2-D: $\{x \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 \leq m\}$

$$\Rightarrow x_2 \leq \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

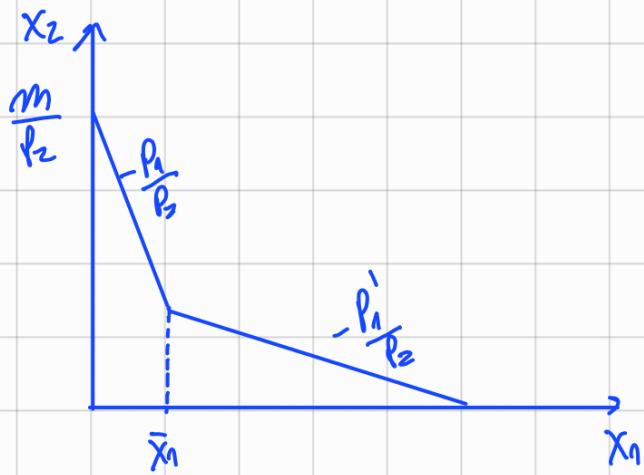


Walrasian budget set is convex

* Nonconvex budget sets come up all the time

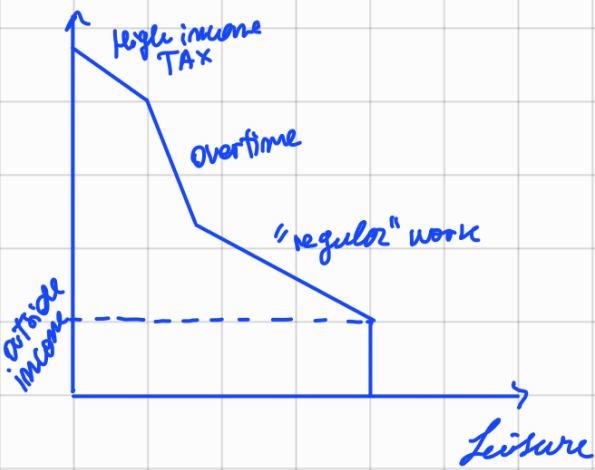
- progressive taxation
- welfare payments
- nonlinear pricing.

Non-Linear pricing : Example. Discount on x_1 if you purchase \bar{x}_1 units.



* Kuhn-Tucker conditions are not sufficient in this case.

Labor supply model : $u(c, l)$, p is the price of leisure (wage).



"Classic" utility maximization problem (UMP)

$$\max u(x) \\ x_i \geq 0 \\ p \cdot x \leq m$$

Solution functions : $X(p, m) = \underset{\substack{x_i \geq 0 \\ px \leq m}}{\operatorname{argmax}} u(x)$ (marshallian) demand correspondence

$$V(p, m) = \underset{\substack{x_i \geq 0 \\ px \leq m}}{\max} u(x) \quad \text{Indirect utility function.}$$

Question: Existence of solution to UMP?

Theorem :

- i) If $p > 0$, & $u(\cdot)$ is continuous then UMP has a solution
- ii) If, in addition, \tilde{u} are locally non satiatic, then $p \cdot x^* = m$ for all $x^* \in x(p, m)$ (walras law)

Proof i): $p > 0$ ensures $B_{p,m}$ is compact (closed & bounded).
continuous functions on compact sets have a maximum

Proof ii): Assume $p \cdot x^* < m$, then x^* is in the interior of $B_{p,m}$. By LNS, $\exists y \in B_{p,m}$ s.t $u(y) > u(x^*)$.
This contradicts that x^* is optimal

Characterization Solutions

Oct 16, 2023

$$\max_x u(x) \quad x \geq 0, \quad p \cdot x \leq m$$

$$L = u(x) - \lambda(p \cdot x - m) + \mu_i x_i$$

KT conditions: $\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i + \mu_i = 0 \rightarrow \frac{\partial u}{\partial x_i} \leq \lambda p_i, \text{ with equality if } x_i^* > 0$

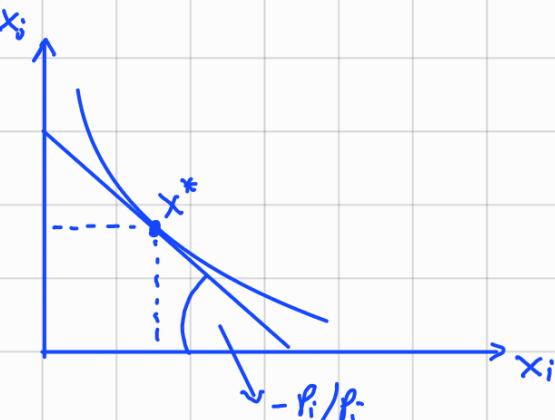
$$\lambda(p \cdot x - m) = 0, \quad \mu_i x_i = 0 \\ \lambda_i, \mu_i \geq 0$$

Take $\frac{\partial x_i^*}{\partial x_j}, x_j^* > 0$. Divide i by j

$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} = \frac{p_i}{p_j}$$

$$MRS_{ij} = \frac{p_i}{p_j}$$

$$\frac{\frac{\partial u}{\partial x_i}}{p_i} = \frac{\frac{\partial u}{\partial x_j}}{p_j}$$



Only works for interior solutions.

If $x_j^* = 0$, then could have $MRS_{ij} \neq p_i/p_j$

$$MRS_{ij} > \frac{p_i}{p_j}$$



Properties of $V(p, m)$ (value / indirect utility function)

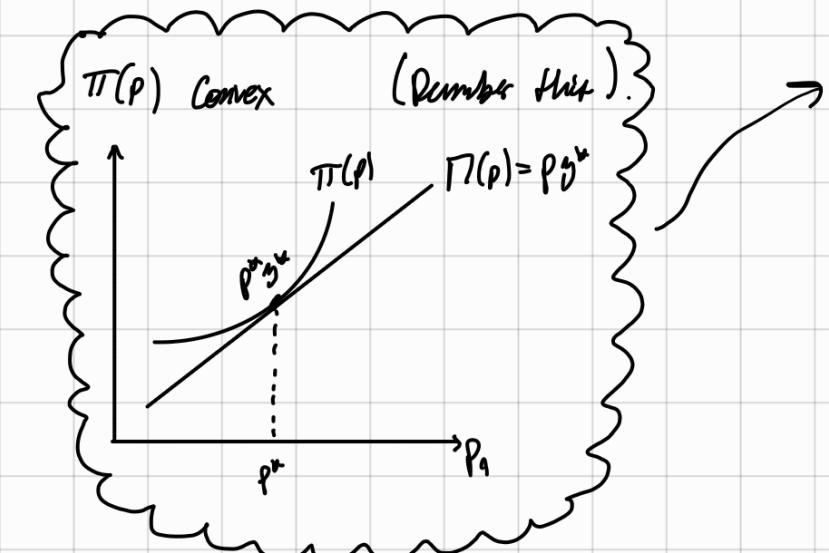
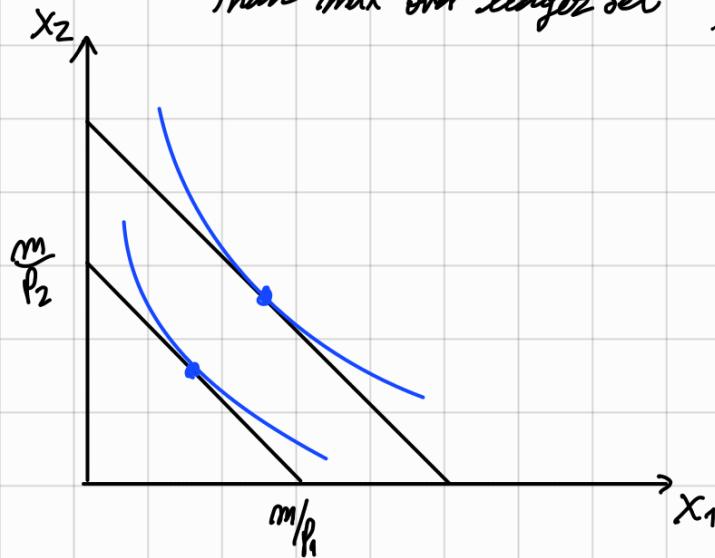
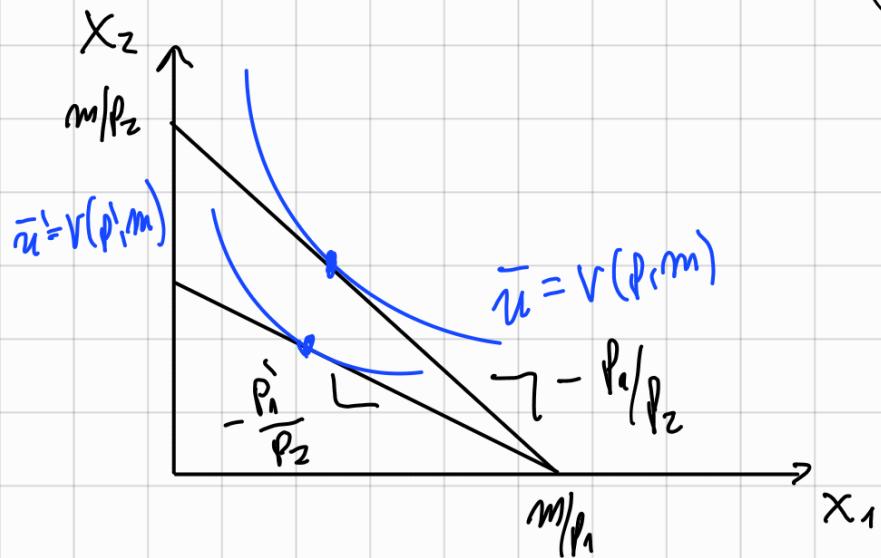
$$V(p, m) = u(x^*(p, m)) \quad (\text{analogous to profit function}).$$

① $V(p, m)$ is hom-0 in (p, m)

Proof: $B_{p, m} = B_{\alpha p, \alpha m}$ Same objective, same constraints \Rightarrow same solution

② $V(p, m)$ is monincreasing in p_i , mondecreasing in m

Proof: Let $p' \geq p$ Then $B_{p', m} \subseteq B_{p, m}$ (max over a smaller set is (weakly) less than max over larger set)



$V(p, m)$ is quasiconvex in (p, m) that is, the set $\{(p, m) : V(p, m) \leq \bar{V}\}$ is convex.

* When price change you may want to do what you were doing before (different from profit case).

Proof: Take $V(p, m) \leq \bar{V}$, $V(p', m') \leq \bar{V}$. We want to show $V(p'', m'') \leq \bar{V}$

Sufficient: $p'' \cdot x \leq m'' \Rightarrow u(x) \leq \bar{V}$

$$\rightarrow \alpha p x + (1-\alpha)p'x \leq \alpha m + (1-\alpha)m' \quad (\#)$$

(...)

One of the following must hold:

i) $\alpha p \cdot x \leq \alpha m$ or ii) $(1-\alpha)p \cdot x \leq (1-\alpha)m$ or (iii) is false.

If x is affordable at (p, m) or (p^*, m^*) (or both)

If i) holds: $u(x) \leq V(p, m) \leq \bar{V}$

If ii) holds: $u(x) \leq V(p^*, m^*) \leq \bar{V}$

(Convex preferences \hookrightarrow quasiconcave $u(x)$)

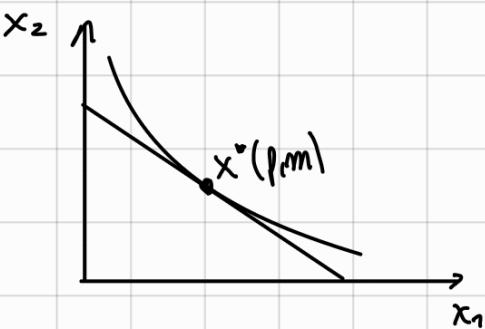
Properties of $x(p, m)$

① $x(p, m)$ is hd-0 (proof: same as before)

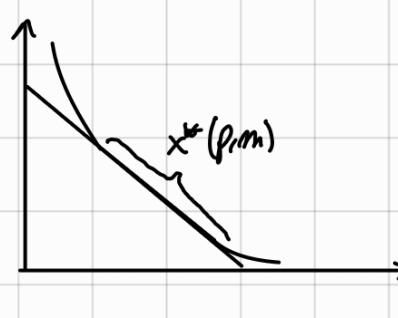
② $p \cdot x = m$ for all $x \in X(p, m)$

③ If \bar{u} is convex, then $X(p, m)$ is a convex set.

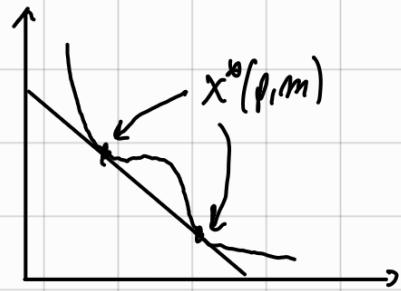
If \bar{u} is strictly convex, then $X(p, m)$ is a singleton



strict convex \bar{u}



quasi-convex \bar{u}



non-convex \bar{u}

Roy's Identity: (Reinhard Hotelling's Lemma $\frac{\partial \Pi}{\partial p_i} = g_i(p)$)

Result of the envelope theorem. Can do the same for UMP:

$$L = u(x) - \lambda(p \cdot x - m) \quad (\text{assume interior solution [No KTC]})$$

$$\frac{\partial V(p, m)}{\partial p_i} = \frac{\partial L}{\partial p_i} = -\lambda x_i^*(p, m) \quad (\text{Envelope theorem})$$

$$\text{Get rid of } \lambda: \text{Envelope theorem again on } m \quad \frac{\partial V}{\partial m} = \frac{\partial L}{\partial m} = \lambda$$

Roy's Identity:

$$x_i^*(p, m) = - \frac{\partial v / \partial p_i}{\partial v / \partial m}$$

Basic example: $u(x_1, x_2) = x_1^\alpha x_2^\beta$, $\alpha + \beta = 1$.

Equivalent to $u(x_1, x_2) = \alpha \log x_1 + \beta \log x_2$

$$\begin{aligned} \max \quad & \alpha \log x_1 + \beta \log x_2 \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \leq m \end{aligned}$$

The domain is restricted so we can ignore $x_i > 0$ constraint.

$$L = \alpha \log x_1 + \beta \log x_2 - \lambda (p_1 x_1 + p_2 x_2 - m)$$

f.o.c

$$\begin{aligned} \textcircled{1} \quad \frac{\alpha}{x_1} &= \lambda p_1 & \textcircled{2} \quad \frac{\beta}{x_2} &= \lambda p_2 & \textcircled{3} \quad p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

$$\text{Solve for } x_1, x_2, \lambda: \quad x_1(p, m) = \frac{\alpha m}{p_1} \quad ; \quad x_2(p, m) = \frac{\beta m}{p_2}$$

plug $x_i(p, m)$ in $u(\cdot)$

$$\Rightarrow v(p, m) = \alpha \log \left(\frac{\alpha m}{p_1} \right) + \beta \log \left(\frac{\beta m}{p_2} \right) \quad (\text{indirect utility function})$$

Check Roy's identity:

$$\frac{\partial v}{\partial p_1} = \frac{\alpha p_1 (-\alpha m)}{\alpha m} = -\alpha p_1 \quad \frac{\partial v}{\partial p_2} = \frac{\beta p_2 (-\beta m)}{\beta m} = -\beta p_2$$

$$\frac{\partial v}{\partial m} = \frac{\alpha p_1}{\alpha m} \cdot \frac{\alpha}{p_1} + \frac{\beta p_2}{\beta m} \cdot \frac{\beta}{p_2} = \frac{\alpha}{m} + \frac{\beta}{m} = \frac{1}{m}$$

$$x_1(p, m) = \frac{\alpha m}{p_1} \quad ; \quad x_2(p, m) = \frac{\beta m}{p_2} \quad (\text{same as before}).$$

Expenditure minimization problem (EMP)

(Analogous to CMP)

$$\min p \cdot x \quad \text{st} \quad u(x) \geq \bar{u}$$

This is exactly isomorphic to $\min w \cdot x \quad \text{st} \quad f(x) \geq y$ (CMP)
 → properties and all the same as CMP

Solution function: $\underline{h}(p, \bar{u}) = \arg \min p \cdot x \quad \text{s.t. } u(x) \geq \bar{u}$

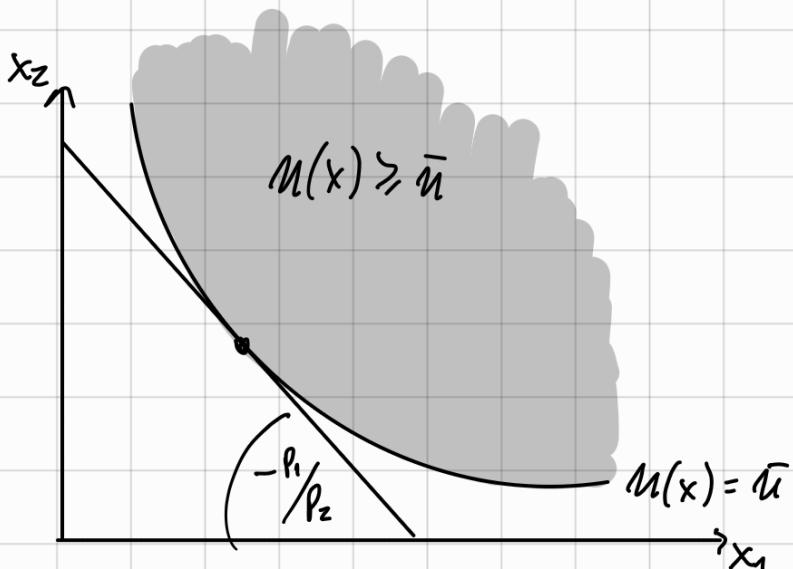
Hicksian demand (optimal bundle)

(conditional factor demand)
 in CMP

Value function: Expenditure function: $e(p, \bar{u}) = \min p \cdot x = p \cdot \underline{h}(p, \bar{u})$

st $u(x) \geq \bar{u}$

(cost function)
 in CMP



Properties of $e(p, \bar{u})$

(we did the proofs for CMP)

- i) hom-1 in p
- ii) nondecreasing in p & \bar{u}
- iii) concave in p

Properties of $\underline{h}(p, \bar{u})$

- i) hom-0 in p ; $\underline{h}(\alpha p, \bar{u}) = \underline{h}(p, \bar{u})$
- ii) Σ convex $\rightarrow \underline{h}(p, \bar{u})$ is convex set
- iii) Σ strictly convex $\rightarrow \underline{h}(p, \bar{u})$ is unique.

Shepard's Lemma : $\frac{\partial c}{\partial w_i} = x_i(w, u)$

for EMP : $\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u)$

Substitution matrix :

$$Dh(p, u) = \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \dots & \frac{\partial h_1}{\partial p_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial p_1} & \dots & \frac{\partial h_m}{\partial p_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 e}{\partial p_1^2} & \dots & \frac{\partial^2 e}{\partial p_1 \partial p_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 e}{\partial p_m^2} & \dots & \frac{\partial^2 e}{\partial p_m \partial p_1} \end{bmatrix}$$

Shepard's Lemma

Theorem : The substitution matrix $Dh(p, \bar{u})$ is

- i) Negative semi-definite
- ii) Symmetric
- iii) Satisfies $Dh(p, \bar{u}) \cdot p = 0$

i) is the differential law of (compensated) demand

$$\frac{\partial h_i}{\partial p_i} \leq 0 \quad \left(\frac{\partial x_i}{\partial p_i} \not\leq 0 \text{ not true for marshallian, } \right. \\ \left. \text{only for Hicksian } h(p, \bar{u}). \right)$$

Example : Cobb-Douglas

$$\min P_1 x_1 + P_2 x_2 \quad \text{st} \quad \alpha \log x_1 + \beta \log x_2 \geq \bar{u}$$

$$\mathcal{L} : P_1 x_1 + P_2 x_2 - \lambda (\alpha \log x_1 + \beta \log x_2 - \bar{u})$$

$$\text{f.o.c} \quad \frac{\partial \mathcal{L}}{\partial x_1} : P_1 = \frac{\alpha \lambda}{x_1} \quad \textcircled{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} : P_2 = \frac{\beta \lambda}{x_2} \quad \textcircled{2}$$

$$\text{csc. } \lambda \underbrace{(\bar{u} - \alpha \log x_1 - \beta \log x_2)}_{(\text{slackness})} \quad \textcircled{3}$$

will bind.

$$\textcircled{1}/\textcircled{2} : \frac{x_2}{x_1} \frac{\alpha}{\beta} = \frac{p_1}{p_2} \Rightarrow x_2 = \underbrace{\frac{p_1}{p_2} \frac{\beta}{\alpha}}_c x_1$$

plug into \textcircled{3} : $\alpha \log(x_1) + \beta \log(cx_1) = \bar{u}$

$$\alpha \log(x_1) + \beta \log c + \beta \log x_1 = \bar{u}$$

$$\log(x_1) + \beta \log c = \bar{u}$$

$$\log(x_1) = \bar{u} - \log c^\beta \quad / \exp$$

$$x_1 = e^{\bar{u}} e^{-\log c^\beta} = e^{\bar{u}} c^{-\beta}$$

$$x_1 = e^{\bar{u}} \left[\frac{p_1}{\alpha} \right]^{\alpha-1} \left[\frac{p_2}{\beta} \right]^\beta$$

$$h_1(p, \bar{u}) = e^{\bar{u}} \left[\frac{p_1}{\alpha} \right]^{\alpha-1} \left[\frac{p_2}{\beta} \right]^\beta \quad h_2(p, \bar{u}) = e^{\bar{u}} \left[\frac{p_1}{\alpha} \right]^\alpha \left[\frac{p_2}{\beta} \right]^{\beta-1}$$

Expenditure function: $e(p, \bar{u}) = p_1 h_1(p, \bar{u}) + p_2 h_2(p, \bar{u})$

$$e(p, \bar{u}) = \left[e^{\bar{u}} \left[\frac{p_1}{\alpha} \right]^\alpha \left[\frac{p_2}{\beta} \right]^\beta \right]^\alpha + \left[e^{\bar{u}} \left[\frac{p_1}{\alpha} \right]^\alpha \left[\frac{p_2}{\beta} \right]^\beta \right]^\beta$$

$$e(p, \bar{u}) = e^{\bar{u}} \left(\frac{p_1}{\alpha} \right)^\alpha \left(\frac{p_2}{\beta} \right)^\beta$$

$$x_1(p, m) = \frac{\alpha m}{p_1} ; x_2(p, m) = \frac{\beta m}{p_2} \Rightarrow \text{what if we set } m = e(p, \bar{u}) \text{ in marshallian demands?}$$

$$x_1(p, e(p, \bar{u})) = \frac{\alpha e^{\bar{u}} \left(\frac{p_1}{\alpha} \right)^\alpha \left(\frac{p_2}{\beta} \right)^\beta}{p_1}$$

$$x_1(p, e(p, \bar{u})) = e^{\bar{u}} \left(\frac{p_1}{\alpha} \right)^{\alpha-1} \left(\frac{p_2}{\beta} \right)^\beta = h_1(p, \bar{u}) \Rightarrow \text{we get the Hicksian demand again.}$$

\Rightarrow Duality between UMP & EMP.

$$\log(c^\alpha \cdot x_1^{\alpha \beta}) = \bar{u} \quad / \exp$$

$$c^\alpha x_1 = e^{\bar{u}}$$

$$x_1 = e^{\bar{u}} / c^\alpha$$

$$x_1 = e^{\bar{u}} \left(\frac{p_2}{p_1} \frac{\alpha}{\beta} \right)^\beta$$

$$x_1 = e^{\bar{u}} \left[\frac{p_2}{\beta} \right]^\beta \left[\frac{p_1}{\alpha} \right]^{\alpha-1}$$

(Focal max satisfaction)

Theorem: Assume $u(\cdot)$ is continuous & LNS.

i) If x^* solves $\max u(x)$ s.t. $p \cdot x \leq m$, $x_i \geq 0$ (UMP)
then, x^* also solves $\min p \cdot x$ s.t. $u(x^*) \geq \bar{u}$, $x_i \geq 0$ (EMP)

Further, $e(p, u(x^*)) = m$

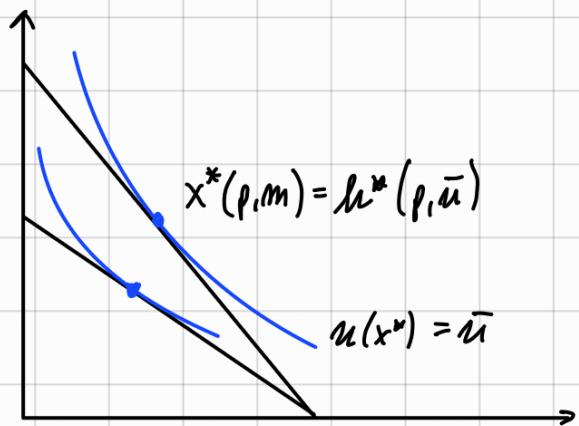
ii) If x^* solves $\min p \cdot x$ s.t. $u(x) \geq \bar{u}$, $x_i \geq 0$ (EMP)
then x^* solves $\max u(x)$ s.t. $p \cdot x \leq p \cdot x^*$, $x_i \geq 0$ (UMP)

Further, $V(p, p \cdot x^*) = \bar{u}$

Proof of (i): Say x^* solves UMP, but not EMP at $\bar{u} = u(x^*)$

$\exists y$ st $u(y) \geq u(x^*)$

and $p \cdot y < p \cdot x^* = m$



By LNS, \exists another y' s.t. $p \cdot y' < m$ and $u(y') > u(y)$, so $u(y') > u(x^*)$
so x^* not optimal in UMP.

Proof of (ii): Say x^* solves EMP, but not UMP at income $p \cdot x^*$

$\exists y$ st $u(y) \geq u(x^*)$

and $p \cdot y \leq p \cdot x^*$

Let $y' = \alpha y$ for $\alpha \in (0, 1)$. By continuity now, for $\alpha \approx 1$

$u(y') > u(x^*)$, and $p \cdot y' < p \cdot x^*$.

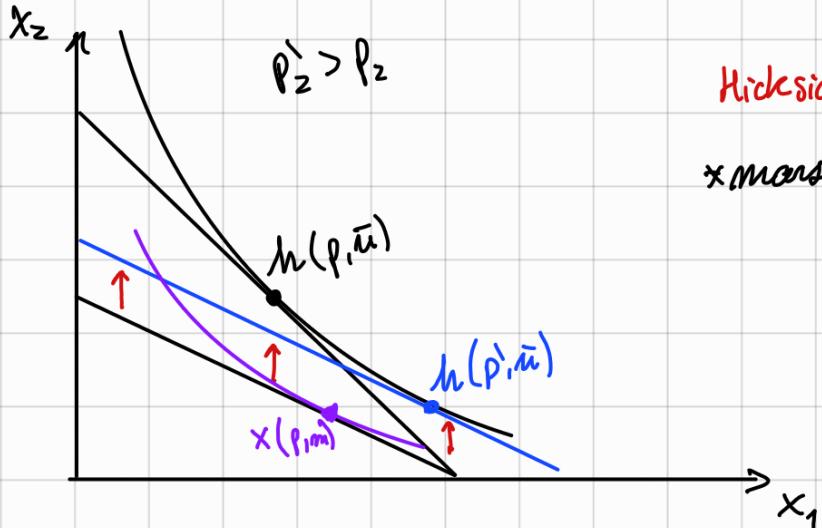
so x^* is not optimal in UMP.

Following 4 "duality identities"

$$\begin{aligned} \textcircled{1} \quad e(p, v(p, m)) &= m \\ \textcircled{2} \quad v(p, e(p, \bar{u})) &= \bar{u} \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad x_i(p, e(p, \bar{u})) &= h_i(p, \bar{u}) \\ \textcircled{4} \quad h_i(p, v(p, m)) &= x_i(p, m) \end{aligned}$$

Why is Hicksian demand called "Compensated" demand?



Hicksian Compensation: Δm needed to get \bar{u}
* marshallian absorb u value.

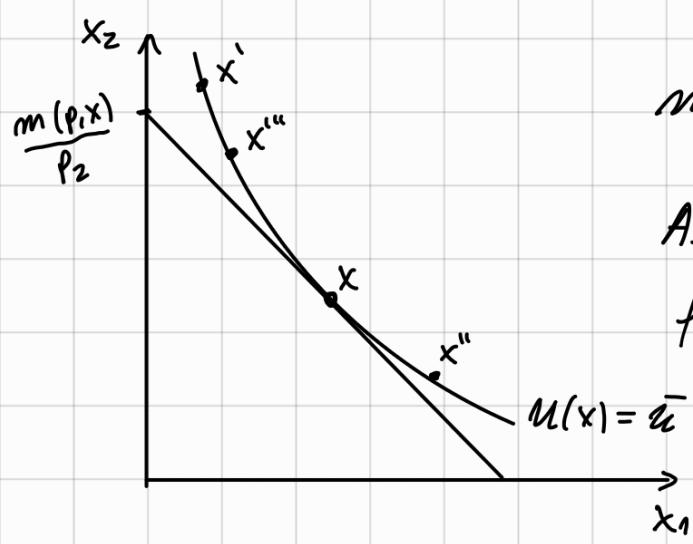
Money metric utility: Bundle x , utility $u(x)$, fix prices at p .

Q: How much money does the consumer need to achieve utility $u(x)$ at p ?

NOT how much does it cost to buy x

$$m(p, x) = e(p, u(x))$$

fix p , view as a function of x
 $m(p, \cdot) = e(p, \cdot)$



As a function of x , $m(p, x)$ is a utility function that represents the same preferences as $u(\cdot)$ (^{monotonic} transformation)

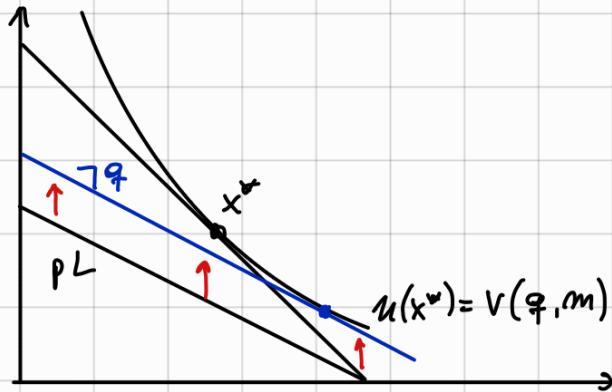
$$m(p, x) = m(p, x') = m(p, x'') = \dots$$

$$u(x) = \dots = u(x'') = \dots$$

Also money metric indirect utility function:

$$u(p, e(q, m)) = e(p, v(q, m))$$

Answers question: How much money you need at prices p , to be just as well off as you were at prices q & income m ?



For Cobb-Douglas: $u(x) = \alpha \log(x_1) + \beta \log(x_2)$

$$e(p, \bar{u}) = e^{\bar{u}} \cdot \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta$$

$$\begin{aligned} m(p, x) &= e(p, u(x)) \\ &= e^{\alpha \log x_1 + \beta \log x_2} \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \end{aligned}$$

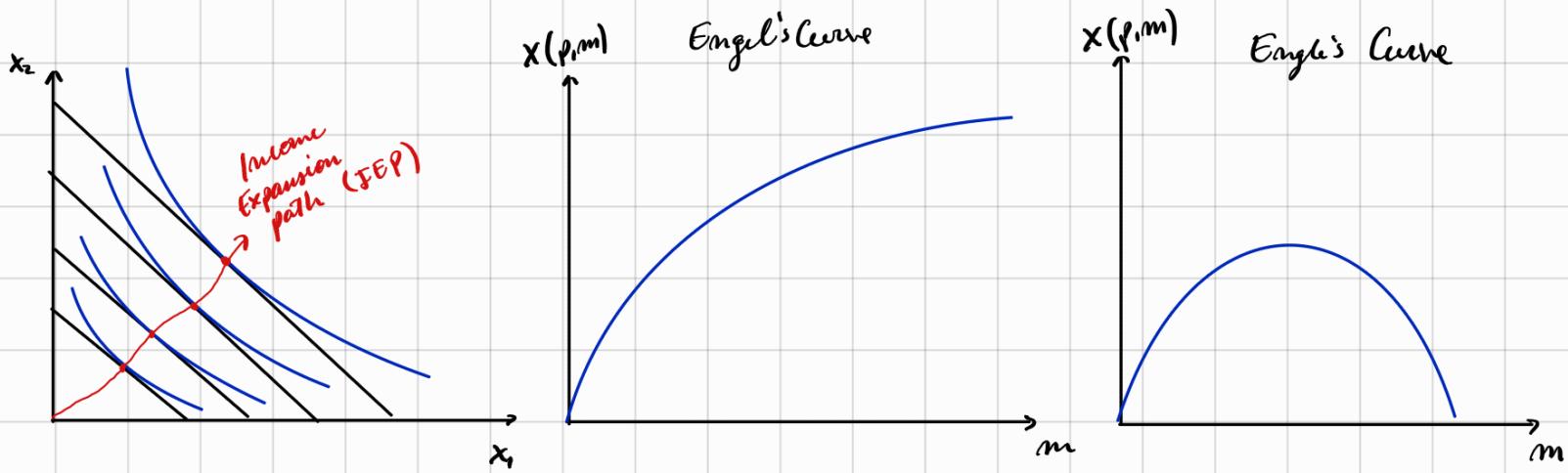
$$m(p, x) = X_1^\alpha X_2^\beta \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \rightarrow \text{Rescaled utility function and measured in dollars.}$$

Comparative Statics of Demand $x(p_i, m)$

Oct 18, 2023

- 1) Income effects: varying m
- 2) Price effects: varying p

- 1) Income expansion path

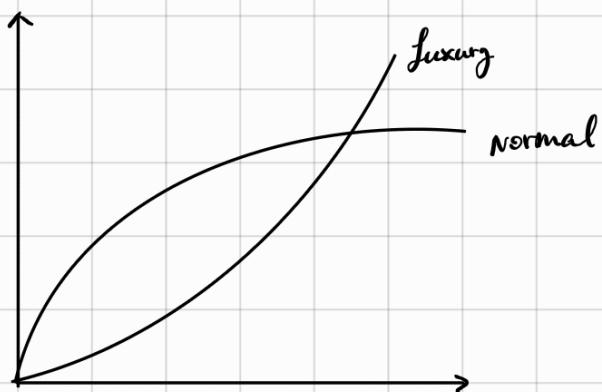


$$\frac{\partial x_i(p_i, m)}{\partial m} \geq 0 : \text{normal good} \quad \frac{\partial x_i(p_i, m)}{\partial m} \leq 0 : \text{inferior good}$$

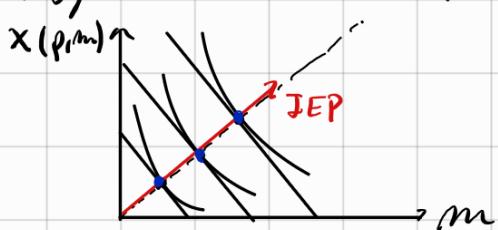
To measure the curvature, we use the income elasticity

$$E_{im} = \frac{d \log(x_i(p_i, m))}{d \log(m)} = \frac{m}{x_i(p_i, m)} \cdot \frac{dx_i(p_i, m)}{dm}$$

$0 < E_{im} < 1$: necessary good
 $E_{im} \geq 1$: luxury good



Special case: Homothetic preferences ($u(x)$ is $kd-1$)



More formally: If $u(\cdot)$ is hd-1, then $x(p, \alpha m) = \alpha x(p, m)$

proof: Let $x \in x(p, m)$

$$p \cdot x \leq m$$

$$\alpha p \cdot x \leq \alpha m$$

$$p(\alpha x) \leq \alpha m$$

Take any y s.t $p \cdot y \leq \alpha m$

$$p \cdot \frac{1}{\alpha} y \leq m$$

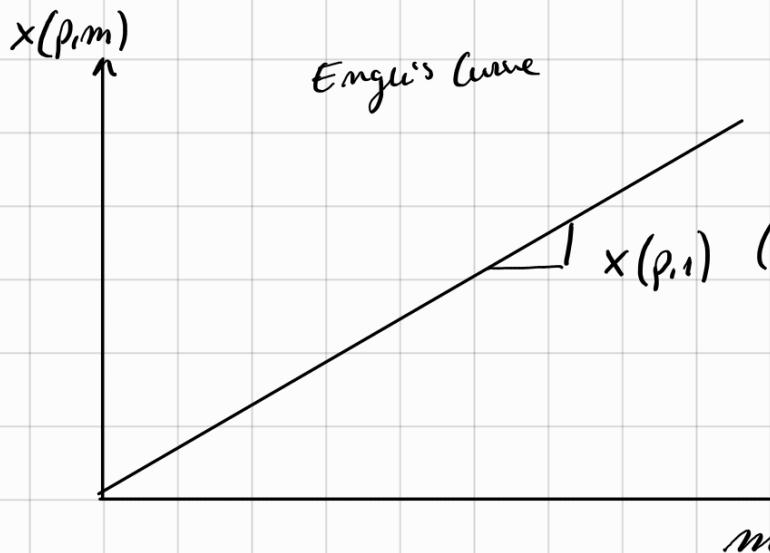
So $\frac{1}{\alpha} y$ is affordable at m , so $u\left(\frac{1}{\alpha} y\right) \leq u(x)$

$$\alpha u\left(\frac{1}{\alpha} y\right) \leq \alpha u(x)$$

$$u(y) \leq u(\alpha x) \quad (\text{by hd-1})$$

$\therefore \alpha x$ is optimal at income αm , so $x(p, \alpha m) = \alpha x(p, m)$

Let $\alpha = \frac{1}{m} \Rightarrow x(p, 1) = \frac{1}{m} x(p, m) \Rightarrow$



$$\underbrace{x(p, m)}_{\text{Equation}} = m \cdot x(p, 1)$$

Elasticity:

$$\log x_i(p, m) = \log m + \log x_i(p, 1)$$

$$\epsilon_{im} = 1$$

For C.D., $u(x_1, x_2) = x_1^\alpha x_2^\beta$:

$$x_1(p, m) = \frac{\alpha m}{p_1}$$

$$\text{slope} : \frac{\alpha}{p_1}$$

2) Price effects

* remember Hotelling's Lemma

$$(i) \quad y_i(p) = \frac{\partial \pi}{\partial p_i} \quad \frac{\partial y_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2}$$

(ii) $\pi(p)$ convex $\rightarrow D_y(p) = \frac{\partial^2 \pi}{\partial p^2}$ is pos. semi-def. $\left(\frac{\partial^2 \pi}{\partial p^2} \geq 0 \right)$

(i) + (ii) $\Rightarrow \frac{\partial y_i}{\partial p_i} \geq 0$; Law of Supply

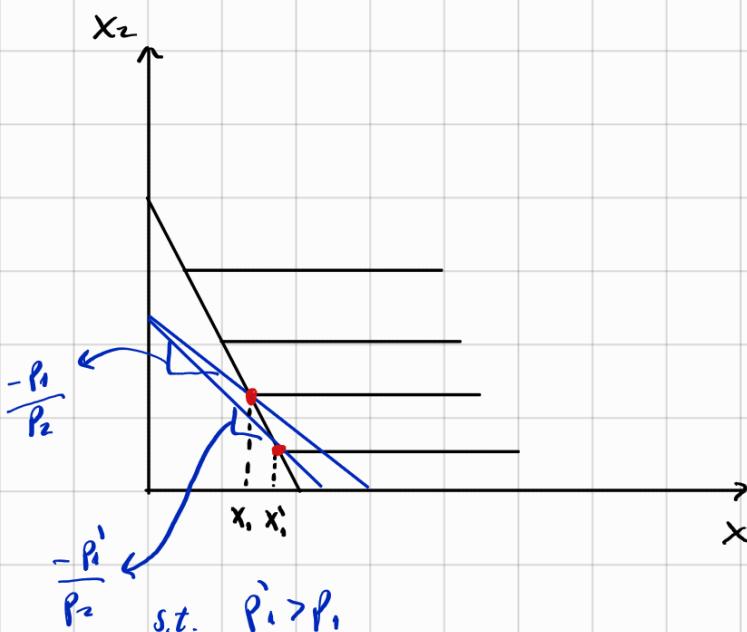
The analogous result for $x_i(p, m)$ would be $\frac{\partial x_i(p, m)}{\partial p_i} \leq 0$
 "Law of Demand". Unfortunately, this is not TRUE

- $V(p, m)$ is only quasiconcave [(ii) doesn't work]

Example: Meat (x_2) + Potatoes (x_1)

- minimum calories requirement: C
- For any bundle (x_1, x_2) : $C = 2x_1 + x_2$

$$u(x_1, x_2) = \begin{cases} x_2, & 2x_1 + x_2 \geq 0 \\ -1, & 2x_1 + x_2 < 0 \end{cases} \quad \begin{array}{l} \text{(After the minimum req, agent only)} \\ \text{wants meat} \\ \text{(not enough calories, then the agent dies)} \end{array}$$



Indifference curve

Budget

Equilibrium: $\frac{m}{P_2} < C, \frac{m}{P_1} > C$

$\frac{\partial x_1}{\partial p_1} > 0$: Giffen Good

$\left(\frac{\partial x_1}{\partial p_1} \leq 0 \right)$: Ordinary Good

\therefore "Law of Demand" is not TRUE

What about $h_i(p, \bar{u})$?

- In EMP, $e(p, \bar{u})$ is concave in p
- So, here: $D(e(p, \bar{u}))$ is neg. semidef. matrix
- Shepard's Lemma applies

$$h_i(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_i}$$

$$\frac{\partial h_i(p, \bar{u})}{\partial p_i} = \frac{\partial^2 e(p, \bar{u})}{\partial p_i^2} \leq 0 \Rightarrow \text{"Law of 'compensated' demand"} \\ (\text{But } h_i(p, \bar{u}) \text{ are not observable})$$

But we can use the duality $h_i(p, \bar{u}) = x_i(p_i, e(p, \bar{u}))$

- Differentiate wrt p_k

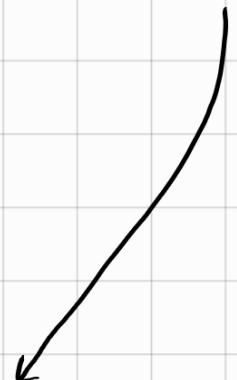
$$\frac{\partial h_i(p, \bar{u})}{\partial p_k} = \underbrace{\frac{\partial x_i(p_i, e(p, \bar{u}))}{\partial p_k}}_{\text{and } h_k(p, \bar{u}) = x_k(p, m)} + \underbrace{\frac{\partial x_i(p_i, e(p, \bar{u}))}{\partial m} \cdot \frac{\partial e(p, \bar{u})}{\partial p_k}}_{\text{by Shephard's Lemma}}$$

$$\text{and } h_k(p, \bar{u}) = x_k(p, m)$$

$$\textcircled{1} \quad m = e(p, \bar{u})$$

$$\textcircled{2} \quad \bar{u} = v(p, m)$$

$$\textcircled{3} \quad h_k(p, m) = x_k(p, e(p, \bar{u})) = x_k(p, m)$$



Slutsky Equation

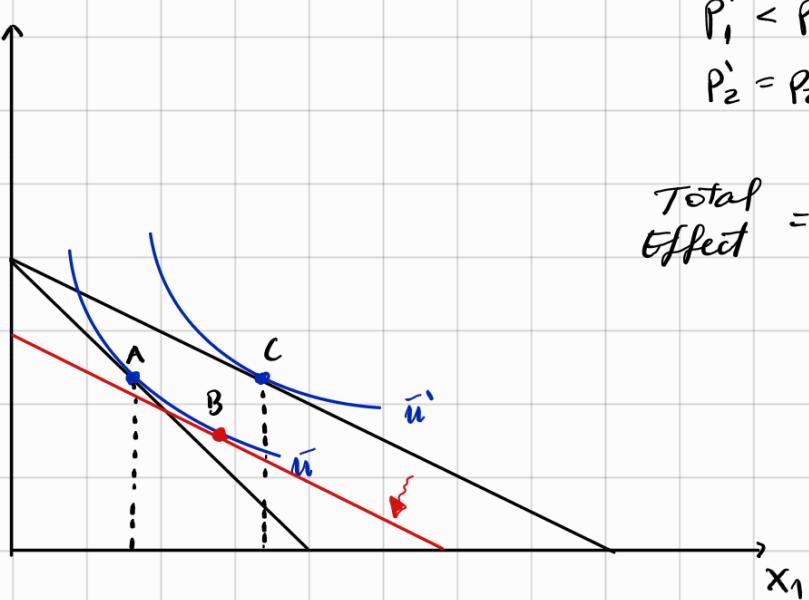
$$\frac{\partial x_i(p_i, e(p, \bar{u}))}{\partial p_k} = \underbrace{\frac{\partial h_i(p, \bar{u})}{\partial p_k}}_{\text{Substitution Effect}} - \underbrace{\frac{\partial x_i(p_i, e(p, \bar{u}))}{\partial m} \cdot x_k(p, \bar{u})}_{\text{Income Effect}}$$

$\frac{\partial h_i}{\partial p_k} \geq 0$: i, k are substitutes

$\frac{\partial h_i}{\partial p_k} \leq 0$: i, k are complements

$\frac{\partial x_i}{\partial p_k} \geq 0$: i, k are gross substitutes

$\frac{\partial x_i}{\partial p_k} \leq 0$: i, k are gross complements.



$$\text{Total Effect} = [B-A] + [C-B]$$

sust
effect income
effect.

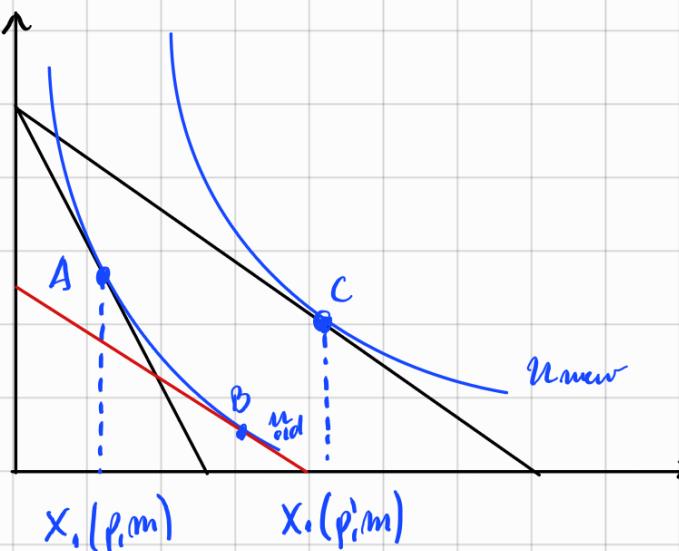
$$m^{\text{comp}} = e(p', \bar{u})$$

$$\underbrace{x_i(p'_i, m) - x_i(p_i, m)}_{\text{Total Effect.}} = \underbrace{[x_i(p'_i, m^{\text{comp}}) - x_i(p_i, m)]}_{\text{Subst. Ef.}} + \underbrace{[x_i(p'_i, m) - x_i(p'_i, m^{\text{comp}})]}_{\text{Income Ef.}}$$

$$= \underbrace{[h(p'_i, \bar{u}) - h(p_i, \bar{u})]}_{\text{Subst. Ef.}} + \underbrace{[x_i(p'_i, m) - x_i(p'_i, m^{\text{comp}})]}_{\text{Income Ef.}}$$

Lecture

Oct 23, 2023



$p'_i < p_i$
 $p'_2 < p_2$
 $A \rightarrow B : \text{Sust. Effect}$
 $B \rightarrow C : \text{Income Effect}$
 $A \rightarrow C : \text{Total Effect.}$

$$\text{Slutzky Equation} : \frac{\partial x_i}{\partial p_k} = \underbrace{\frac{\partial h_i}{\partial p_k}}_{(A \rightarrow C)} - \underbrace{\frac{\partial x_i}{\partial M} X_k(p_i, m)}_{(A \rightarrow B)} \underbrace{\left(B \rightarrow C \right)}_{(B \rightarrow C)}$$

For own-price effect ($k=i$)

If good normal: $\frac{\partial x_i}{\partial m} \geq 0$

$$\frac{\partial x_i}{\partial p_i} = \underbrace{\frac{\partial h_i}{\partial p_i}}_{\text{always} \leq 0} - \frac{\partial x_i}{\partial m} x_i(p_m)$$

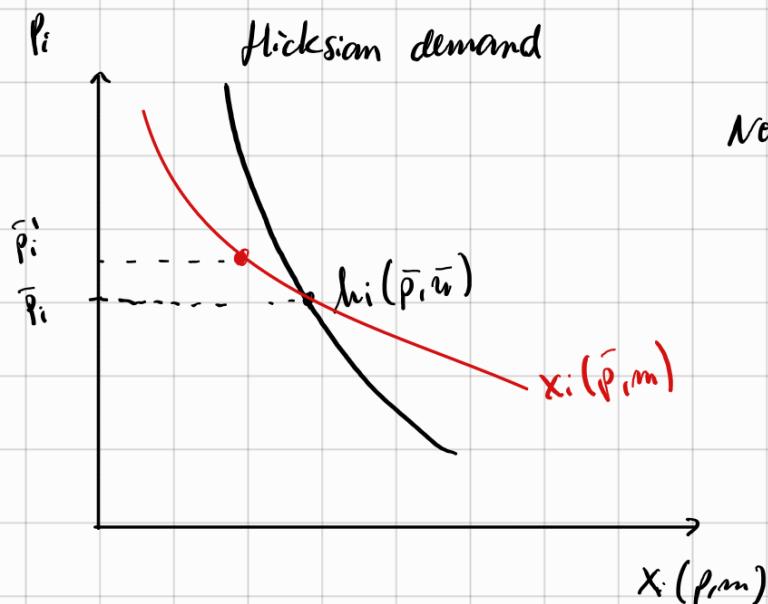
$$\Rightarrow \frac{\partial x_i}{\partial p_i} \leq 0$$

≤ 0

If good is inferior: $\frac{\partial x_i}{\partial m} \leq 0$

\Rightarrow IE & SE are "opposed"

$\Rightarrow \frac{\partial x_i}{\partial p_i} \geq 0$ only if good is strongly inferior.



$$\text{Normal good: } \frac{\partial x_i}{\partial p_i} = \underbrace{\frac{\partial h_i}{\partial p_i}}_{\leftarrow} - \underbrace{\frac{\partial x_i}{\partial m} x_i(p_m)}_{(-)}$$

$$x_i(\bar{p}_i, e(\bar{p}_m)) = h_i(\bar{p}, \bar{u})$$

($x_i(\cdot)$ is flatter than $h_i(\cdot)$)

Marshallian

Hicksian

$$Dh(p) = \begin{pmatrix} \frac{\partial h_i}{\partial p_1} & \dots & \frac{\partial h_i}{\partial p_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_i}{\partial p_1} & \dots & \frac{\partial h_i}{\partial p_m} \end{pmatrix}$$

- Symmetric

- n.s.d

- $Dh(p) \cdot p = 0$

problem: h_i is not observable

$$\text{Consider: } S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{nn} & s_{n2} & \dots & s_{nn} \end{bmatrix}$$

$$, s_{jik} = \frac{\partial x_j}{\partial p_k} + \frac{\partial x_j}{\partial m} x_{ik}$$

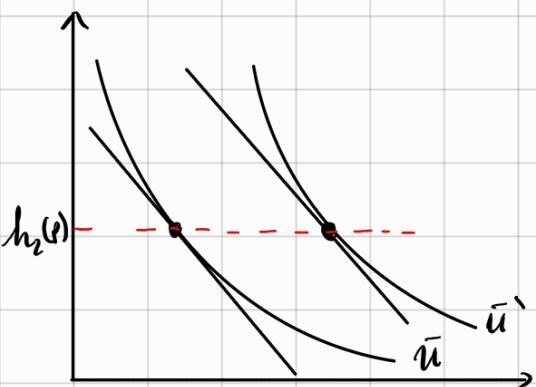
By Slutsky equation: $S(p, m) = Dh(p)$ should also satisfy, and $S(\cdot)$ is observable.

Special case: No income effects. $x_i(p, m) = x_i(p)$

Duality identity: $h_i(p, \bar{u}) = x_i(p, e(p, \bar{u})) = x_i(p)$
 $\Rightarrow h_i(p, \bar{u}) = h_i(p)$

Say $h_2(p, \bar{u}) = h_2(p)$

$$\frac{\partial h_i}{\partial p_i} = \frac{\partial x_i}{\partial p_i}$$



$$\max_{x \in \mathbb{R}_+^2} x_1 + V(x_2) \quad \text{s.t. } x_1 + p_2 x_2 = m$$

normalize $p_1 = 1$
quasilinear objective

$$\max_{x_2} m - p_2 x_2 + V(x_2)$$

f.o.c] $V'(x_2) = p_2$, if invertible, then $x_2(p_2) = V'^{-1}(p_2)$ ($= x_2(p, m)$)

$$\frac{\partial x_2}{\partial m} = 0 \quad \begin{bmatrix} \text{no income} \\ \text{effect} \end{bmatrix}$$

From budget constraint:

$$x_2^*(p, m) = m - p_2 [V'^{-1}(p_2)]$$

$$\frac{\partial x^*}{\partial m} = 1.$$

It is quasilinear only if m is high enough (problem set)
(solve with non-negativity constraint and the determine m in it)

Integrability

Question: Given function $x(p, m)$ that satisfy:

- Inv-0

- Walras' law

- Symmetric n.s.d (Slutsky matrix)

Can we recover utility? Yes

$$x_i(p, m) = - \frac{\partial v / \partial p_i}{\partial v / \partial m}$$

Rog's Identity in reverse

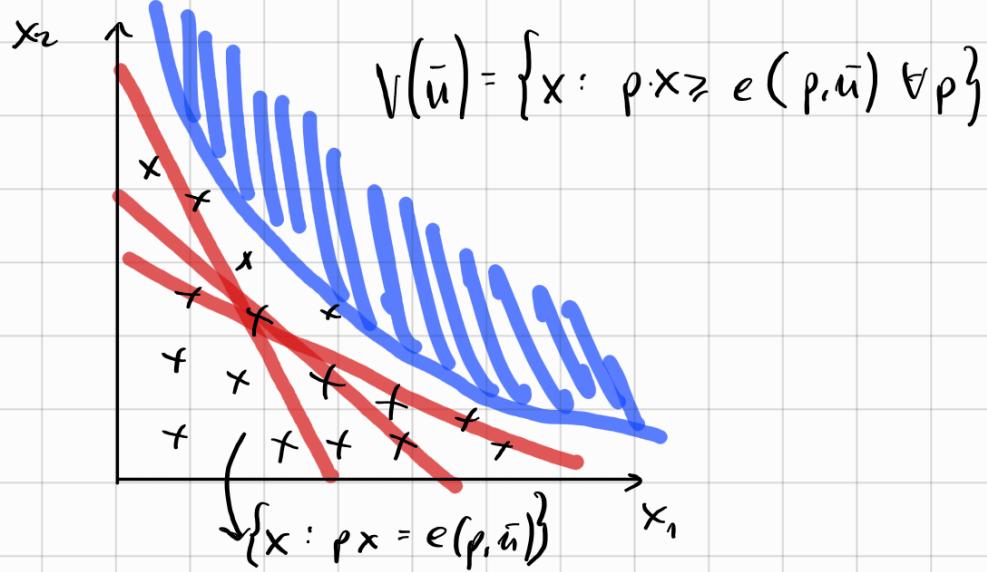
Proof Strategy:

Demand $\xrightarrow{(1)}$ Expenditure $\xrightarrow{(2)}$ Utility function

Start with (2)

Say we know $e(p, \bar{u})$

We want: $V(\bar{u}) = \{ \text{bundles that give utility at least } \bar{u} \}$



Step 1: $x(p, m) \rightarrow e(p, \bar{u})$

Pick a starting point (p^0, m^0)

Set $u(x(p^0, m^0)) = u^0$

Goal: solve for $e(p, m^0)$, write as $e(p)$

By Shephard's Lemma: $\frac{\partial e(p)}{\partial p_i} = h_i(e, u^0)$

$\frac{\partial e(p)}{\partial p_i} = x_i(p, e(p))$, initial condition $e(p^0) = m^0$

Does the solution to this differential equations even exist?

A necessary and sufficient condition for a System

$$\frac{\partial f}{\partial x_i} = g_i(x)$$

To have a solution is

$$\underbrace{\frac{\partial g_i(x)}{\partial x_j}}_{\text{Then Solution exists.}} = \frac{\partial g_j(x)}{\partial x_i} \quad \left(\text{Remember } \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \right)$$

In our problem : $\frac{\partial x_i(p, e(p))}{\partial p_j} = \frac{\partial x_j(p, e(p))}{\partial p_i}$


This condition says that S must be symmetric
(S must be symmetric).

Symmetry of S is N.S condition for system of PDE to have a solution.

For the solution $e(n)$ to be a valid expenditure function, it must be concave.

* Concavity : $\partial e(p) = S$ is n.s.d.

Bottom line: Given some $x(p, m)$, if S is symmetric & n.s.d., we can solve for $e(p, \bar{m})$. Then use earlier technique to go from $e(p, \bar{m}) \rightarrow v(p, m)$

Say you have : $x_1(p, m) = \frac{\alpha m}{p_1}$, $x_2(p, m) = \frac{\beta m}{p_2}$

find $e(p, \bar{m})$ by solving : $\frac{\partial e(p)}{\partial p_1} = x_1(p)$

$$\frac{\partial e(p)}{\partial p_2} = x_2(p)$$

Step 1: Check Symmetry $\frac{\partial^2 e(p)}{\partial p_i \partial p_j} = \frac{\partial^2 e(p)}{\partial p_j \partial p_i}$ satisfied \therefore solution exists.

Step 2: Solve $\frac{\partial e(p)}{\partial p_i} = \frac{\alpha e(p)}{p_i}$

$$\rightarrow \frac{\partial e(p)/\partial p_i}{e(p)} = \frac{\alpha}{p_i}$$

$$\rightarrow \frac{\partial \log e(p_i)}{\partial p_i} = \frac{\alpha}{p_i} \quad / \int dp_i$$

$$\rightarrow \int \frac{\partial \log e(p_i)}{\partial p_i} dp_i = \int \frac{\alpha}{p_i} dp_i$$

$$\Rightarrow \log e(p) = \alpha \log p_i + c_1(p_2, \bar{u})$$

Similarly:

$$\log e(p) = \beta \log p_2 + c_2(p_1, \bar{u})$$

$$\Rightarrow \log e(p) = \alpha \log p_1 + \beta \log p_2 + f(\bar{u}) \quad / \exp(\cdot)$$

$$e(p) = e^{f(\bar{u})} p_1^\alpha p_2^\beta$$

$$e(p) = \tilde{\mu} p_1^\alpha p_2^\beta$$

Finally, find $V(p, m)$ (use duality)

$$e(p, V(p, m)) = m$$

$$V(p, m) p_1^\alpha p_2^\beta = m$$

$$V(p, m) = \frac{m}{p_1^\alpha p_2^\beta}$$

Lecture: Welfare evaluations

Oct 30, 2023

Price change: $p^o \rightarrow p'$ (p : vector)

* we want to study $V(p', m) - V(p^o, m)$

Ordinal nature of $u(\cdot)$ makes this "problematic". We want a "standardized" welfare measure. \Rightarrow Money metric indirect utility function

\rightarrow Fix some arbitrary base \bar{p}

$$\mu(\bar{p}; p^o, m) = e(\bar{p}, V(p^o, m))$$

$$\mu(\bar{p}; p', m) = e(\bar{p}, V(p', m))$$

(Amount of money I need at \bar{p} to get the utility I had at p^o)

Change in Welfare will be: $\Delta W = \mu(\bar{p}; p', m) - \mu(\bar{p}; p^o, m)$

* still needs reference prices, \bar{p} .

** Natural choices of \bar{p} : $\bar{p} = p^o$ or $\bar{p} = p'$

$$u^o = V(p^o, m) \quad u' = V(p', m)$$

Equivalent Variation (EV)

Compensating Variation (CV)

$$1.) EV = \mu(p^o; p', m) - \mu(p^o, p^o, m)$$

Two scenarios

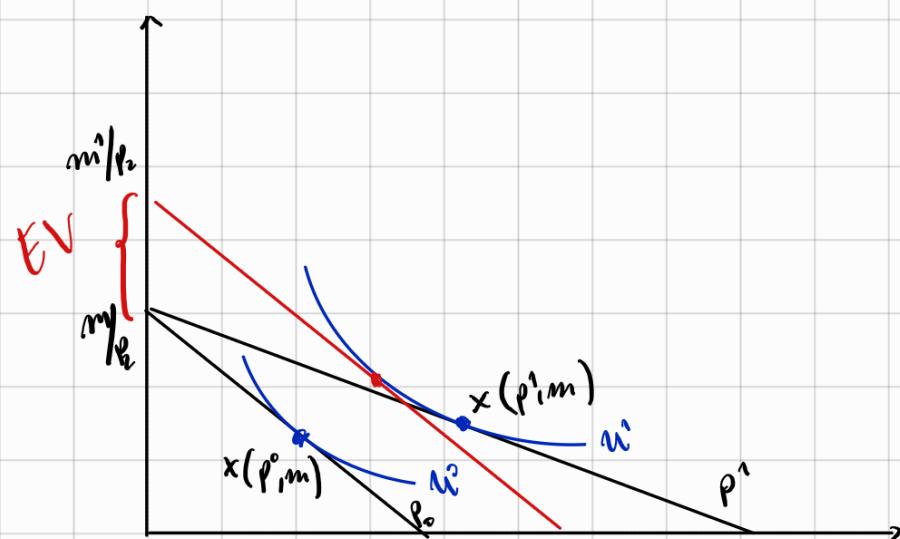
$$EV = e(p^o, u') - e(p^o, u^o)$$

i) price changes $p^o \rightarrow p'$

$$EV = e(p^o, u^o) - m$$

ii) prices fixed, consumer gets an extra \$EV (net).

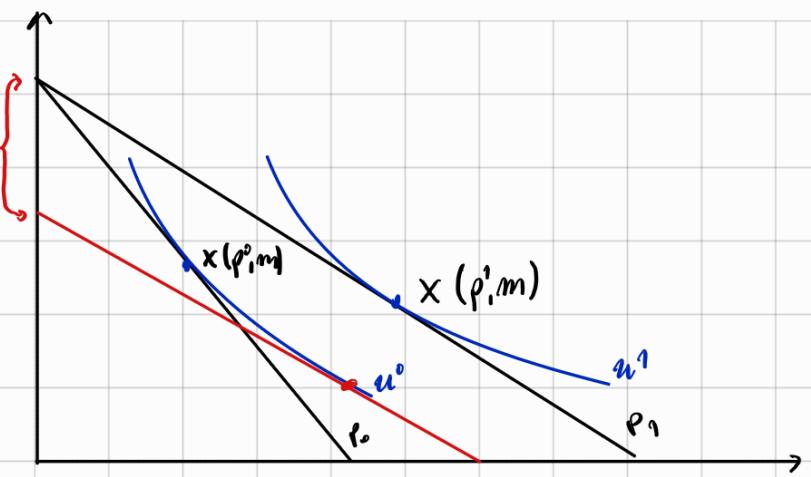
EV is the number that makes i) = ii)



$$2.) CV = \mu(p^*; p, m) - \mu(p^*; p^*, m)$$

$$CV = e(p^*, u^*) - e(p^*, u^*)$$

$$CV = m - e(p^*, u^*)$$



Comment:

→ EV and CV always have

the same sign.

→ Many choices of utility function we could use for welfare

→ EV & CV are 2 most "natural"

→ In general $EV \neq CV$, but will always have same sign

→ In principle observable if we know Marshallian demand (integrability).

Another way to view EV & CV (& CS)

Fix $p_i^* = p_i' = \bar{p}_i \quad \forall i \neq 1$

Recall $e(p^*, u^*) = e(p^*, u^1) = m$

$$EV = e(p^*, u^1) - e(p^*, u^*)$$

$$= e(p^*, u^1) - e(p^*, u^1)$$

$$= \int_{p_1'}^{p_1^*} \frac{\partial e(\tilde{p}_1, \bar{p}_{-1}, u^1)}{\partial p_1} d\tilde{p}_1 = \int_{p_1'}^{p_1^*} h_1(\tilde{p}_1, \bar{p}_{-1}, u^1) d\tilde{p}_1$$

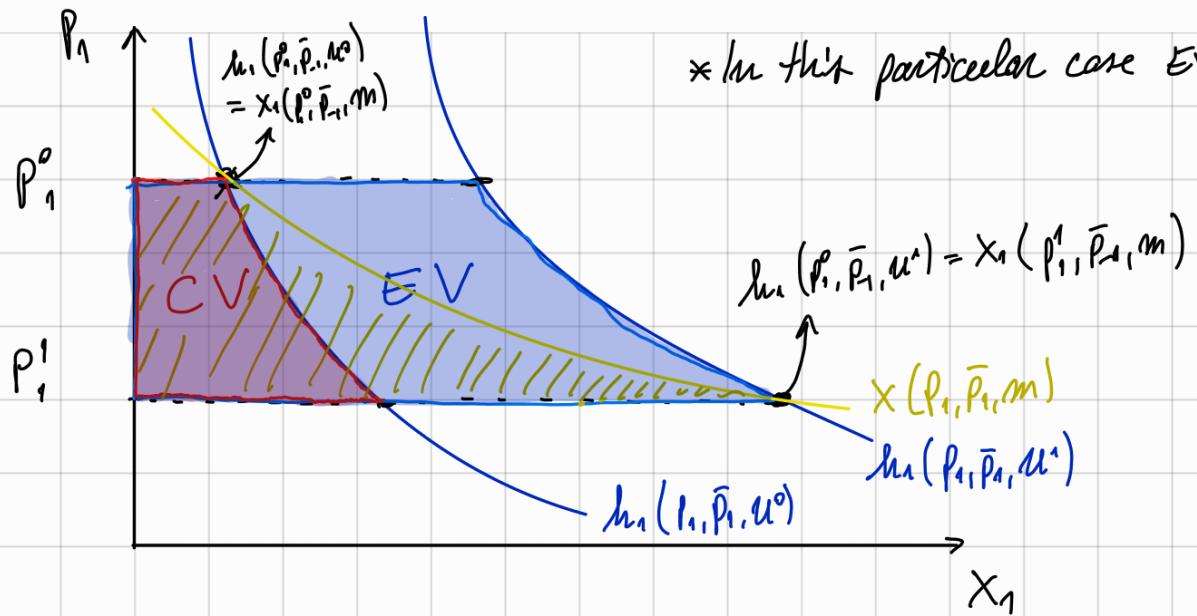
by Shephard's Lemma.

For CV:

$$CV = e(p^*, u^1) - e(p^*, u^*)$$

$$= e(p^*, u^*) - e(p^*, u^0)$$

$$= \int_{p_1'}^{p_1^*} \frac{\partial e(\tilde{p}_1, \bar{p}_{-1}, u^0)}{\partial p_1} d\tilde{p}_1 = \int_{p_1'}^{p_1^*} h_1(\tilde{p}_1, \bar{p}_{-1}, u^0) d\tilde{p}_1$$



Consumer Surplus:

$$\Delta CS = \int_{p_1^1}^{p_1^0} x_i(\tilde{p}_i, \bar{p}_i, m) d\tilde{p}_i$$

- * for normal goods: $CV \leq DCS \leq EV$
- inferior goods: $EV \leq DCS \leq CV$
- quasilinear: $CV = DCS = EV$

Example : Govt. wants to raise \$T tax revenue.

- $$1. \text{ Linear tax on good } 1 \quad \text{or} \quad 2. \text{ Lump-sum tax}$$

$$\rho^o = (P_1^o, P_2^o) \rightarrow (P_1^o + \tau, P_2^o) = p^1 \quad T = \tau \cdot X_1(p^1, m)$$

Welfare comparison between ① & ②? (answer by EV)

If $EV = e(p^*, u^*) - m < -T$, consumer is worse off under (1)

$$\begin{aligned}
 DWL &= -EV - T \\
 &= \underbrace{e(p^*, u^*) - e(p^1, u^*)}_{m} - T \\
 &= \int_{p_1^*}^{p_1^* + \tau} h_1(\tilde{p}_1, p_2^*, u^*) d\tilde{p}_1 - \tau \cdot \overbrace{h_1(p_1^* + \tau, p_2^*, u^*)}^{x_1(p^*, m)} \\
 &= \int_{p_1^*}^{p_1^* + \tau} h_1(\tilde{p}_1, p_2^*, u^*) - h_1(p_1^* + \tau, p_2^*, u^*) d\tilde{p}_1 \geq 0 \quad (\text{because this is just a number}) \\
 &\quad \text{(i) } \quad \text{(ii)} \\
 &\quad \text{because (i) is bigger than (ii) at any point.}
 \end{aligned}$$

$$DWL = -EV - T \geq 0 \Rightarrow EV \leq -T$$

DWL = Deadweight loss

