$\boxed{\text{September 18}}$

Mid Term : Around Oct 11

Properties of profit & supply functions

$\pi(p) = \max_{y \in Y} p \cdot y$

$y^*(p) = \arg\max_{y \in Y} p \cdot y$

$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}$ (function)

$y^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (correspondence)

Note: Results do not rely on regularity assumptions (convexity, differentiability etc.) unless stated differently.

① $\pi(p)$ is non increasing in input prices and non-decreasing in output prices.

Proof: Take 2 price vectors p, p' and choose $y \in y^*(p)$ and $y' \in y^*(p')$

Let $p_i' \geq p_i$ for all outputs, $p_i' \leq p_i$ for all inputs.

By WAPM; $p' \cdot y' \geq p' \cdot y$

consider $(p'-p) \cdot y = (p_1'-p_1) y_1 + \dots + (p_n'-p_n) y_n$

$y_i \geq 0$: $p_i' \geq p_i$ $\quad$ $y_i \leq 0$: $p_i' \leq p_i$

$$(p'-p) \cdot y \geq 0 \Rightarrow p' \cdot y \geq p \cdot y$$

$$\pi(p') = p' \cdot y' \geq p' \cdot y \geq p \cdot y = \pi(p)$$

$$\pi(p') \geq \pi(p)$$

② $\pi(p)$ is homogeneous of degree 1.

WTS: $\pi(tp) = t\pi(p)$

<u>Proof</u>: $$\pi(tp) = \max_{y \in Y} (tp) \cdot y$$

$$= \max_{y \in Y} t(p \cdot y)$$

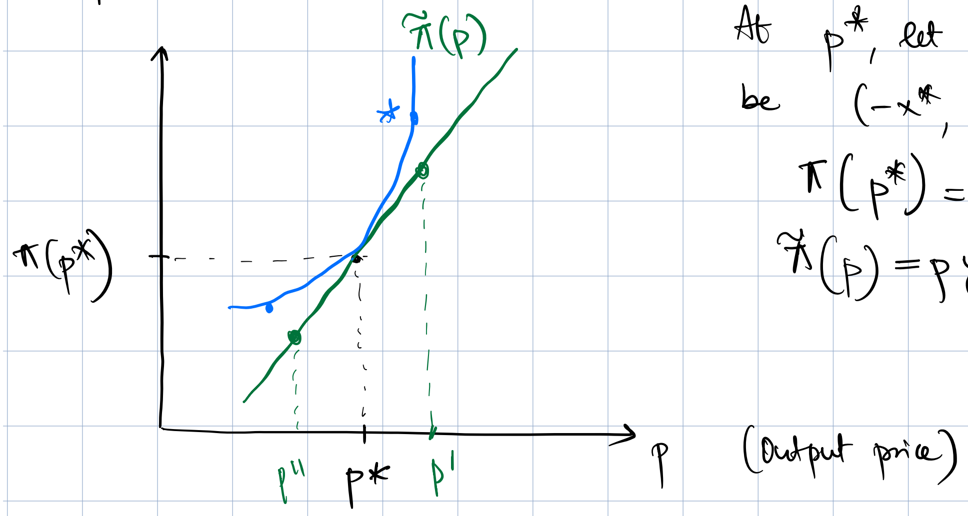
$$= t \max_{y \in Y} p \cdot y$$

$$= t\pi(p)$$

③ $\pi(p)$ is convex in p

i.e. $\pi(\alpha p + (1-\alpha)p') \leq \alpha \pi(p) + (1-\alpha)\pi(p')$

Graphical intuition



Single Output Case

At p^*, let optimal prodⁿ be $(-x^*, y^*)$

$$\pi(p^*) = p^* y^* - w^* \cdot x^*$$

$$\tilde{\pi}(p) = p y^* - w^* \cdot x^*$$

⊛ Re-optimized points. When the price changes, either the profit is on the tangent line or above it [

If we do assume π is differentiable:

$$H(p) = D^2\pi(p) = \begin{bmatrix} \frac{\partial^2 \pi}{\partial p_1^2} & \cdots & \frac{\partial^2 \pi}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial p_n \partial p_1} & \cdots & \frac{\partial^2 \pi}{\partial p_n \partial p_n} \end{bmatrix}$$

$D^2\pi(p)$ is positive semi-definite.

<u>Properties of $y(p)$</u> :—

① $y(p)$ is homogenous of degree 0.

Proof :—

$$\max_{y \in Y^*} (\alpha p) \cdot y$$

$$= \alpha \max_{y \in Y} p \cdot y$$

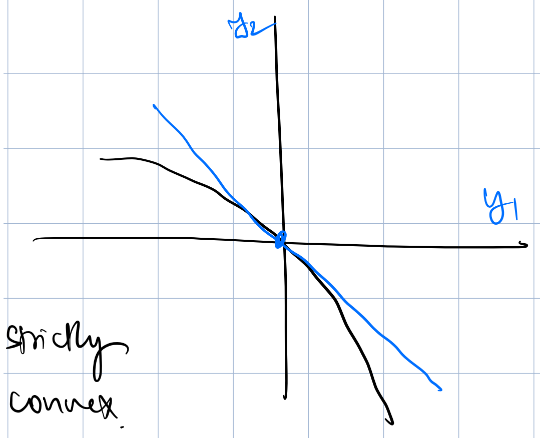
* includes both inputs & outputs whenever you see cap Y.

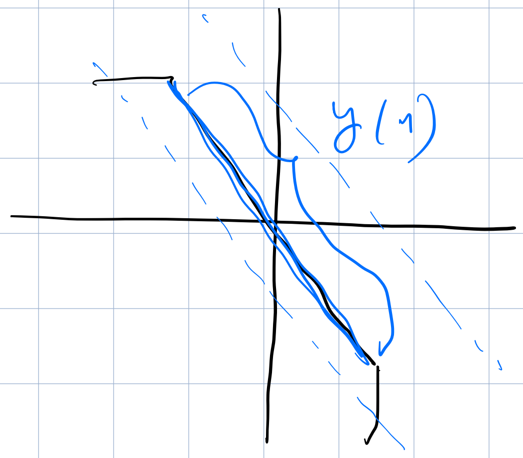
(Scaling all the prices shouldn't change your optimal choices)

~~②~~ (a) If Y is convex, then $y(p)$ is convex set.

(b) If Y is strictly convex, then $y(p)$ is a singleton — (?).

Proof by graph: -





⊛ Envelope Theorem → V. imp.

Q: How does the optimal value of a maximization problem changes as a parameter changes?

$$\max_x f(x, \theta)$$

x: choice variable

θ: parameter eg. prices

Define the value function → f^n of the parameters

$$V(\theta) = \max_x f(x, \theta)$$

[What is $\frac{dV}{d\theta}$?]

One way:

$$V(\theta) = f(x^*(\theta), \theta)$$

A change in θ has 2 effects :-

- Direct Effect : θ changes f directly through the second argument.
- Indirect Effect : θ changes x^*, which changes f.

$$\frac{dV}{d\theta} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=x^*(\theta)}}_{=0 \text{ as optimal pt. (FOC)}} \cdot \frac{dx^*}{d\theta} + \underbrace{\left.\frac{\partial f}{\partial \theta}\right|_{x=x^*(\theta)}}_{\text{evaluating at optimal pt.}}$$

$$\Rightarrow \frac{dV}{d\theta} = \left.\frac{\partial f}{\partial \theta}\right|_{x=x^*(\theta)}$$

- Found all over economics! -
  - Hotelling's lemma
  - Shepard's lemma.
  - Myerson's lemma.
  - Benveniste - Scheinkman formula (macro)

When there are constraints:

$$\max_x f(x, \theta)$$

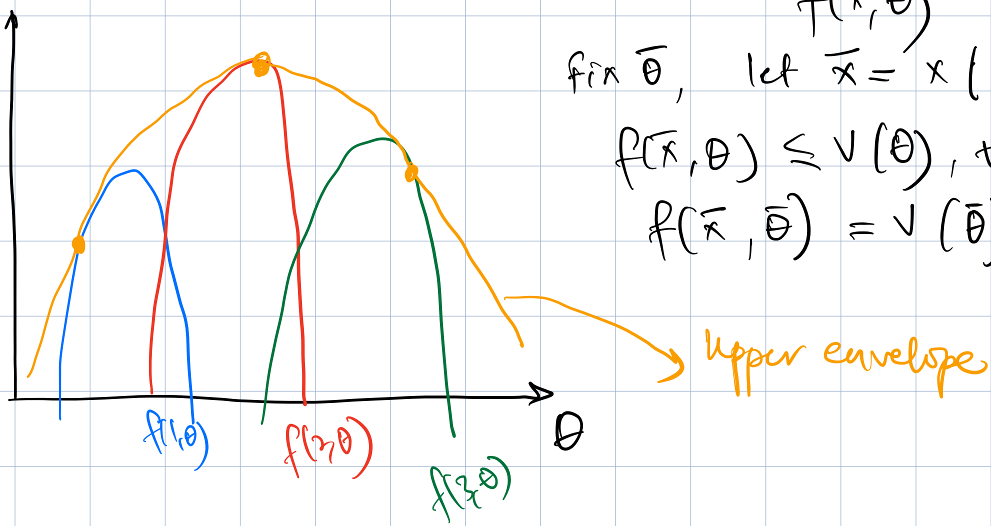
constraints

$$s.t \quad g_r(x, \theta) \leq c_r \quad \text{for } r = 1, \ldots, k$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial f}{\partial \theta} - \lambda_1 \frac{\partial g_1}{\partial \theta} - \ldots - \lambda_k \frac{\partial g_k}{\partial \theta}$$

To remember: $\mathcal{L} = f(x, \theta) + \lambda_1 (c_1 - g_1(x, \theta)) + \ldots$
$\ldots + \lambda_k (c_k - g_k(x, \theta))$

$$\frac{dV}{d\theta} = \frac{\partial \mathcal{L}}{\partial \theta}$$



$f(x,\theta)$

fix $\bar{\theta}$, let $\bar{x} = x(\bar{\theta})$

$f(\bar{x},\theta) \leq V(\theta), \forall \theta$

$f(\bar{x},\bar{\theta}) = V(\bar{\theta})$

Hotelling's Lemma :-

If $\pi(p)$ is differentiable at $\bar{p}$, $\bar{p}_i > 0$, then

$$\frac{\partial \pi(\bar{p})}{\partial p_i} = y_i(\bar{p})$$

Proof: Restrict to single output case.
(assume interior sol^n)

$$\pi(p,w) = \max_{x \geq 0} \underbrace{pf(x) - w \cdot x}_{h(x)}$$

By the envelope theorem

$$\frac{d\pi}{dp}(p,w) = \left.\frac{\partial h}{\partial p}\right|_{x=x^*(p,w)}$$

$$= f(x^*(p,w)) = y^*(p,w)$$

$$\frac{d\pi}{dw_i} = \frac{\partial h}{\partial w_i} = -x_i \Big|_{x=x^*(p,w)} = -x_i^*(p,w)$$

Example :

$$f(x_1, x_2) = 30 x_1^{2/5} x_2^{2/5}$$

$$\max \; 30 p x_1^{2/5} x_2^{2/5} - w_1 x_1 - w_2 x_2$$

FOCs x_1 : $12 p x_1^{-3/5} x_2^{2/5} = w_1$

$$12 p x_1^{2/5} x_2^{-3/5} = w_2$$

(✲✲)

$$x_1^*(p,w) = \frac{12^5 p^5}{w_1^3 w_2^2} \qquad x_2^*(p,w) = \frac{12^5 p^5}{w_1^2 w_2^3}$$

$$\pi(p,w) = 30\,p \cdot \frac{12^4 p^4}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2} - \frac{12^5 p^5}{(w_1 w_2)^2}$$

$$= \frac{1}{2}\left[\frac{12^5 p^5}{w_1^2 w_2^2}\right]$$

To find x_1^*

$$\frac{\partial \pi}{\partial w_1} = -\frac{2}{2}\left[\frac{12^5 p^5}{w_1^3 w_2^2}\right]$$

This is same as (✲✲)

So envelope theorem gives you a way to recover x_1^*.

(Recover supply from profit f^n)

④ Integral form of Hotelling's Lemma

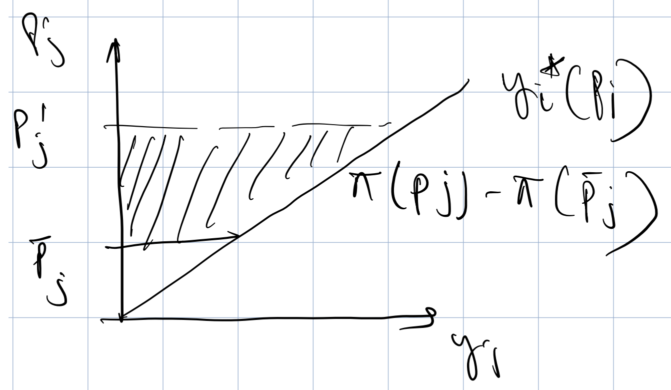
Hold all prices other than p_j fixed at $\bar{p}_{-j}$.

Then, $\pi(p_j, \bar{p}_{-j}) - \pi(\bar{p}_j, \bar{p}_{-j}) = \int_{\bar{p}_j}^{p_j} y_i(s, \bar{p}_{-j})\, ds$

By original Hotelling:

$$\int_{\bar{p}_j}^{p_j} y_j(s, \bar{p}_{-j})\, ds = \int_{\bar{p}_j}^{p_j} \frac{d\pi(s, \bar{p}_{-j})}{dp_j} \cdot ds$$

$$= \pi(p_j, \bar{p}_{-j}) - \pi(\bar{p}_j, \bar{p}_{-j})$$



September 20

Hotelling's Lemma:

$$\frac{\partial \pi}{\partial P_i} = y_i(P)$$

$$Dy(p) = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \cdots & \frac{\partial y_n}{\partial p_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial p_n} & \cdots & \frac{\partial y_n}{\partial p_n} \end{bmatrix}$$

[Jacobian / Substitution matrix]

↳ $\boxed{\text{In recon}}$

$$\frac{\partial y_i(p)}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_j \partial p_i}$$

$$Dy(p) = \begin{bmatrix} \frac{\partial^2 \pi}{\partial p_1^2} & \cdots & \frac{\partial^2 \pi}{\partial p_1 \partial p_n} \\ \vdots & & \vdots \\ \frac{\partial^2 \pi}{\partial p_n \partial p_1} & \cdots & \frac{\partial^2 \pi}{\partial p_n^2} \end{bmatrix}$$

This matrix is positive semi-definite.

This implies:—

① $\dfrac{\partial y_i}{\partial p_i} = \dfrac{\partial^2 \pi}{\partial p_i^2} \geq 0$ (Law of supply)

② $\dfrac{\partial y_i}{\partial p_j} = \dfrac{\partial^2 \pi}{\partial p_j \, \partial p_i} = \dfrac{\partial^2 \pi}{\partial p_i \, \partial p_j} = \dfrac{\partial y_j}{\partial p_i}$

Substitution matrix is symmetric. (Not intuitive ⇒ math shows this.)

LeChatelier Principle

"Firms should respond more to price changes in the long run than the short run."

long run ↳ All the inputs are variable.

short run ↳ Some inputs are fixed (factory, contracts etc.)

Suppose, (y^*, x^*) (Single input model)

$= (y(p^*, w^*), x(p^*, w^*))$

at price (p^*, w^*)

Lets fix one input z_i^* in the short run variable in the long run.

z_i^* is optimal at (p^*, w^*)

Q: How does x_j depend on w_j in the short vs. the long run?

$$\pi(p, w) = \max \ pf(x) - w \cdot x \quad \text{s.t. } x_j \geq 0$$

$$\left. \frac{\partial x_j(p,w)}{\partial w_j} \right|_{(p,w) = (p^*, w^*)}$$

$$\pi^s(p,w) = \max \; pf(x) - w.x$$

s.t. $x_j \geq 0$

$x_i = z_i^*$

$$\left.\frac{\partial x_j^s(p,w)}{\partial w_j}\right|_{(p,w)=(p^*,w^*)}$$

Let $h(p,w) = \pi(p,w) - \pi^s(p,w)$

Profits in the LR always have to be greater than in SR as more constraint can only reduce

By def., $h(p,w) \geq 0$ & $h(p^*,w^*) = 0$

(p^*,w^*) is a local minima.

$$\Rightarrow \quad \frac{\partial^2 h(p^*,w^*)}{\partial w_j^2} \geq 0$$

Diagonal element of Hessian matrix has to be nonnegative for locally convex.

$$\frac{\partial^2 \pi(p^*,w^*)}{\partial w_j^2} - \frac{\partial^2 \pi^s(p^*,w^*)}{\partial w_j^2} \geq 0$$

Hotelling :- $\dfrac{\partial \pi(p,w)}{\partial w_j} = -x_j(p,w)$

$$-\frac{\partial x_j(p^*,w^*)}{\partial w_j} + \frac{\partial x_j^s(p^*,w^*)}{\partial w_j} \geq 0$$

These are negative so when we take absolute values we flip the signs.

$$\left|\frac{\partial x_j^s(p^*,w^*)}{\partial w_j}\right| \leq \left|\frac{\partial w_j(p^*,w^*)}{\partial w_j}\right|$$

$\boxed{\text{Cost Minimization}}$

- Single output good firm.
- Pricing power in output market, but not input markets.
- Previous analysis does not work but we can do the following :-

cost minimization problem.

① Find the cheapest way to make <u>any</u> target output Y.

② Use the "cost function" $c(y)$ to choose optimal p/y combination.

CMP (Cost Minimization Problem) :—

- Fix a target level of output y.

$$\underbrace{c(w, y)}_{\boxed{\text{parameters}}} = \min_{x} \; w \cdot x$$

s.t.

$x_i \geq 0$

$f(x) \geq y$

Ignore non-negativity for now :—

$$\mathcal{L} = w \cdot x - \lambda\left(f(x) - y\right)$$

FOCs:

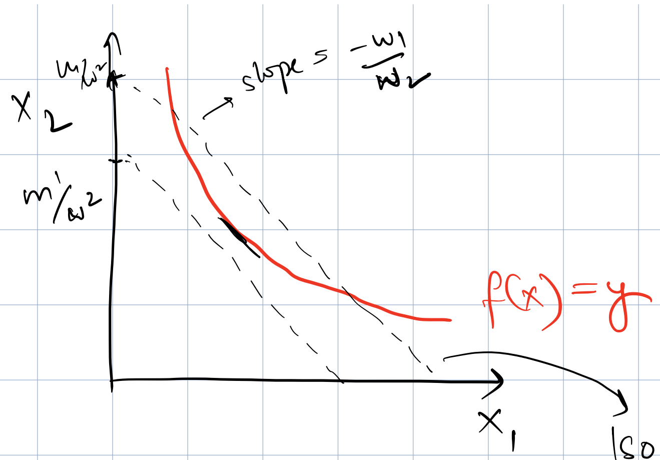
$$w_i - \lambda \frac{\partial f}{\partial x_i} = 0$$

or

$$w_i = \lambda \frac{\partial f}{\partial x_i} \quad \forall i$$

Divide i by j

$$\underbrace{\frac{w_i}{w_j}}_{\text{Economic rate of substitution}} = \underbrace{\frac{\partial f / \partial x_i}{\partial f / \partial x_j}}_{\boxed{\text{MRTS}}} \qquad (\text{for } x_i^*, x_j^* > 0)$$



Constraint: Isoquant

Objective: Isocost

$w_1 x_1 + w_2 x_2 = m$

$x_2 = \frac{m}{w_2} - \frac{w_1}{w_2} x_1$

Isocost line.

The slope remains the same

Looking for tangency as the optimal point.

$x^*(w,y)$ conditional factor demand correspondence

$$c(w,y) = w.\, x^*(w,y) = \min_{\substack{s.t.\\ x_i \geq 0\\ f(x) \geq y}} w.x$$

Envelope Theorem $\frac{dV}{d\theta} = \frac{\partial L}{\partial \theta}$

Here set $V = C$, $\theta = y$

$$\frac{\partial c}{\partial y} = \frac{\partial L}{\partial y}$$

$$\boxed{\frac{\partial c}{\partial y} = \lambda}$$

$\Rightarrow$ λ is the marginal cost of increasing prod^n by one unit.

"Shadow Prices" $\longrightarrow$ Cost of tightening the constraint.
$[f(x) \geq y]$

$\boxed{\text{Relationship with Profit Maximization Problem (PMP)}}$

$$\max_{y \geq 0} \; py - c(w, y)$$

FOC: $\quad p = \dfrac{\partial c}{\partial y} = (\lambda \text{ from CMP})$

Things to keep in mind :-

- We have assumed a differentiable f, interior solutions.
- General FOCs :

$$\lambda \frac{\partial f}{\partial x_i}(x^*) - w_i \leq 0 \quad \forall i,$$

with equality if $x_i^* > 0$.

- KT conditions are necessary but not sufficient in general. They will be sufficient if $f(x)$ is concave.
  $\rightarrow$ prodⁿ fⁿ
- Existence / uniqueness issues.

Weak Axiom of Cost Minimization (WACM)

$\{(w^1, x^1, y^1), \ldots\ldots, (w^T, x^T, y^T)\}$

WACM: what you did.

$\underbrace{w^t . x^t}_{\text{the cost you paid in a month}} \leq w^t . x^s$ could have done for all $\underbrace{y^s \geq y^t}$.

Produced what you needed to produce in that month.

- Any firm that violates WACM is "irrational" / Are not minimizing costs.

[It should be less than any other cost plan for that month.]

If WACM does not hold ⇒ you are not cost minimizing.

Implication of WACM: — downward sloping demand.

Take $y^s = y^t$, WACM gives 2 inequalities

① $w^t . x^t \leq w^t . x^s$

② $w^s . x^s \leq w^s . x^t$

Add ① & ② $\quad w^t x^t + w^s x^s \leq w^t x^s + w^s x^t$

$\Rightarrow w^t x^t + w^s x^s - w^t x^s - w^s x^t \leq 0$

$\Rightarrow (w^t - w^s) . (x^t - x^s) \leq 0$

$\Delta w . \Delta x \leq 0$ (Talks about vector of price changes & not just a price change)

$$c(w,y) = \min_{x} \; w \cdot x \quad \text{s.t. } f(x) \geq y$$

① $c(w,y)$ is nondecreasing in w.

Proof: Take $w' \geq w$ & $x \in x^*(w,y)$, $x' \in x^*(w',y)$

$$c(w,y) = \underbrace{w \cdot x \leq}_{\text{WACM}} w \cdot x' \underbrace{\leq}_{w' > w} w' \cdot x' = c(w'y)$$

$\Rightarrow \quad c(w,y) < c(w',y)$

② $c(w,y)$ is nondecreasing in y

Let $y'' \geq y'$, $x' \in x^*(w,y')$, $x'' \in x^*(w,y'')$

$$c(w,y') = \min_{\substack{\text{s.t.} \\ f(x) \geq y'}} w \cdot x = w \cdot x' \underbrace{\leq}_{\text{WACM}} w \cdot x''$$

$$= \min_{\substack{\text{s.t.} \\ f(x) \geq y''}} w \cdot x = c(w, y'')$$

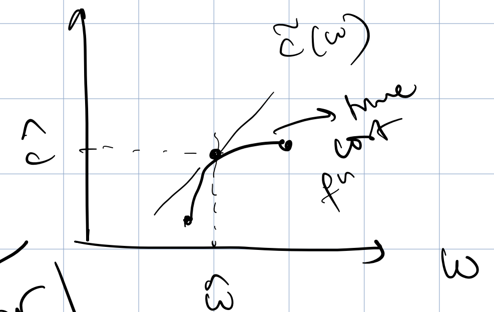
Inequality works because $f(x'') \geq y'' \geq y'$.

③ $c(w,y)$ is hd-1 in w.

$\iff c(tw, y) = tc(w,y)$

④ $c(w,y)$ is concave in w. NOT true for y.

(Proof is same as that of the π maximization being convex.)



When price increases, costs increases linearly & vice versa.

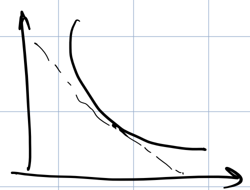
↓

don't let the firm respond.

When the firm responds, its cost would be equal or lower than the $\tilde{c}(w)$.

H.W. for formal proof

(5) If $v(y)$ is convex, then $x^*(w, y)$ is convex valued.
If $v(y)$ is strictly convex, then $x^*(w, y)$ is single-valued.



(6) $x^*(w, y)$ is hd-0 in w.

$$x^*(tw, y) = x^*(w, y)$$

→ Relative prices of the 2 goods are the same.

Implication :- (Looking for one good)!

$$x_i^*(t, w, y) = x_i^*(w, y)$$

Differentiate wrt t, eval @ $t = 1$.

$$\nabla_w x_i^*(tw, y) \cdot w = 0$$

$$\sum_{j=1}^{n} \frac{\partial x_i^*(tw, y)}{\partial w_j} \cdot w_j = 0$$

We know : $\frac{\partial x_i^*}{\partial w_i} \leq 0$

Say $n=2$

negative

$$\frac{\partial x_1^*}{\partial w_1} w_1 + \frac{\partial x_1^*}{\partial w_2} w_2 = 0$$

this is positive

$$\frac{\partial x_1^*}{\partial w_2} \geq 0$$ $\therefore$ goods 1 & 2 are substitutes.

If $n > 2$, then $\frac{\partial x_i^*}{\partial w_j} \geq 0$ for some good j.

Shepherd's Lemma

If $c(w,y)$ is differentiable, and $\bar{w} > 0$, then:

$$x_i(\bar{w},y) = \frac{\partial c(\bar{w},y)}{\partial w_i}$$

Proof 1: Envelope Theorem

$$\mathcal{L} = w \cdot x - \lambda(f(x) - y)$$

$$\frac{\partial c(w,y)}{\partial w_i} = \frac{\partial \mathcal{L}}{\partial w_i}\bigg|_{x = x^*(w,y)}$$

$$\boxed{\frac{\partial c}{\partial w_i} = x_i^*(w,y)}$$

Proof 2: $C(w, y) = w \cdot x^*(w, y)$

$$(*) \quad \frac{\partial c}{\partial w_i} = x_i^*(w, y) + \sum_{j=1}^{n} w_j \frac{\partial x_j^*(w, y)}{\partial w_i}$$

Recall the FOCs of CMP:—

$$(1) \quad \lambda \frac{\partial f}{\partial x_i} = w_i \qquad (2) \quad f(x^*(w, y)) = y$$

lets differentiate (2) wrt w_i:

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j^*}{\partial w_i} = 0$$

let's plug (1) into (*)

$$\frac{\partial c}{\partial w_i} = x_i^*(w, y) + \sum_{j=1}^{n} \lambda \underbrace{\frac{\partial f}{\partial x_j} \frac{\partial x_j^*}{\partial w_i}}_{= 0}$$

$$\boxed{\frac{\partial c}{\partial w_i} = x_i^*(w, y)}$$

Substitution Matrix:

$$\begin{bmatrix} \frac{\partial x_i^*}{\partial w_i} & \cdots & \frac{\partial x_i^*}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^*}{\partial w_1} & \cdots & \frac{\partial x_n^*}{\partial w_n} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial^2 c}{\partial w_1^2} & \cdots & \frac{\partial^2 c}{\partial w_n \partial w_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 c}{\partial w_1 \partial w_n} & \cdots & \frac{\partial^2 c}{\partial w_n^2} \end{bmatrix}}_{\text{Hessian of } c \text{ which is concave.}}$$

* Law of Demand : $\frac{\partial x_i^*}{\partial w_i} \leq 0 \quad \forall i$ (Diagonal elements of Hessian matrix of concave f^n is negative!)

* Symmetric cross-price effects

$$\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$$

Theorem: If f is CRS, then $c(w, y) = y \underbrace{c(w,1)}_{\text{unit cost of } f^n}$

Proof:-

Assume f is differentiable, interior solution

FOCs at $y = 1$.

$$\frac{w_i}{w_j} = \frac{F_i(x^*)}{F_j(x^*)} \quad \text{and} \quad F(x^*) = 1$$

For general $\hat{y}$:

$$\frac{w_i}{w_j} = \frac{f_i(\hat{x})}{f_i(\hat{x}_j)} \qquad f(\hat{x}) = \hat{y}$$

Consider inputs $\hat{x} = \hat{y}\, x^*$

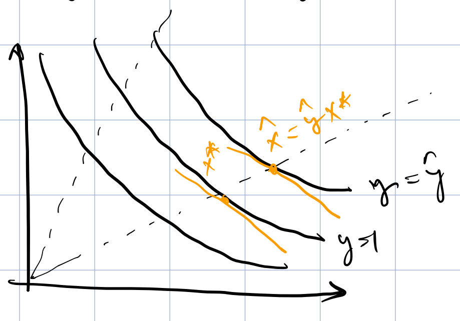
Claim: $\hat{x}$ solves the FOCs at $\hat{y}$.

$$\frac{f_i(\hat{x})}{f_j(\hat{x})} = \frac{f_i(\hat{y}\, x^*)}{f_j(\hat{y}\, x^*)} \underset{\text{CRS}}{=} \frac{f_i(x^*)}{f_j(x^*)} = \frac{w_i}{w_j}$$

$$f(\hat{y}\, x^*) \underset{\text{by CRS}}{=} \hat{y}\, f(x^*) = \hat{y}$$

So, $\hat{x} = \hat{y}\, x^*$ solves the problem at target output $\hat{y}$.

$$c(w, \hat{y}) = w \cdot (\hat{y}\, x^*) = \hat{y}\,(w \cdot x^*) = \hat{y}\, c(w, 1)$$



The slope of the tangent lines are the same.