HW2, Problem 1

Problem 1, part a)
Because working provides labor income and there is no added utility from lack of work (leisure), we know agents will use their full endowment of labor in a given period. This means labor supply is a redundant allocation object for consumers and is thus excluded.
Here, you'll see lots of small rotational adjustments I would've made if this was my HW assignment. One in particular is that I use (ℓ_o, ℓ_y) for young and old labor endowments, respectively.

A SME is a collection of allocations for consumers $\left\{c_1^0, \{c_t^t, c_{t+1}^t, b_{t+1}^t, k_{t+1}^t\}_{t\geq 1}\right\}$, allocations for firms $\{k_t^d, \ell_t^d\}_{t\geq 1}$, and prices $\{i_t, r_t, w_t\}_{t\geq 1}$ s.t

1. Given prices, initially old allocation c_1^0 solves

$$\max_{c_1^0\in\mathbb{R}_+} \ln(c_1^0) \text{ s.t } c_1^0 \leq w_1\ell_o + r_1\overline{k} + (1-\delta)\overline{k} + (1+i_1)m$$

2. Given prices, generation t allocation $\left\{c_t^t, c_{t+1}^t, k_{t+1}^t, b_{t+1}^t\right\}$ solves

$$\max_{(c_t^t, c_{t+1}^t)\in\mathbb{R}_+^2} \ln\left(c_t^t(c_{t+1}^t)^\beta\right) \text{ s.t } c_t^t + k_{t+1}^t + b_{t+1}^t \leq w_t\ell_y \ \underline{\textbf{and}}\ c_{t+1}^t \leq w_{t+1}\ell_o + (1+r_{t+1}-\delta)k_{t+1}^t + (1+i_{t+1})b_{t+1}^t$$

3. Given prices, $\{k_t^d, \ell_t^d\}$ maximizes firm profits at t by

$$\max_{\ell_t, k_t} f(k_t,\ell_t) - w_t\ell_t - r_tk_t$$

4. Markets clear:

$$\text{(ARC)}\ \ C_t + I_t = f(K_t, L_t)\ \underline{\textbf{and}}\ K_t = k_t^d = k_t^{t-1}\ \underline{\textbf{and}}\ L_t = \ell_t^d = \ell_o + \ell_y\ \underline{\textbf{and}}\ b_{t+1}^t = \prod_{k=1}^{t}(1+i_k)m$$

Important Note: I technically used "bad answer" practices in the market clearing to make a point. I didn't explicitly define C_t and I_t and seemingly inserted K_t, L_t out of nowhere. This is so you can conceptualize these equations beyond just alphabet soup. These letters are all capitalized because they correspond to *aggregate* values. The ARC (aggregate resource constraint) is just the GDP equation from intro to macro: $C + I + G + NX = Y$. Here, we're in a closed economy and don't have government spending. To be formal, $C_t = c_t^{t-1} + c_t^t$ and $I_t = K_{t+1} - (1-\delta)K_t$ (investment).

Other note: you may wonder why in the PMP (profit maximizing problem) we drop the superscripts. It's important to remember that each of these first 3 points is saying that the given allocation is the *argmax*. We can't really do this for the consumers problem because we have two separate periods of life (there's no way to consistently simplify the notation – they have to correspond to specific constraints with similar notation).

Problem 1, part b)
An ADE is a collection of allocations for consumers $\left\{c_1^0, \{c_t^t, c_{t+1}^t, k_{t+1}^t\}_{t\geq 1}\right\}$, allocations for firms $\{k_t^d, \ell_t^d\}_{t\geq 1}$, and prices $\{p_t, r_t, w_t\}_{t\geq 1}$ s.t

1. Given prices, initially old allocation c_1^0 solves

$$\max_{c_1^0\in\mathbb{R}_+} \ln(c_1^0) \text{ s.t } p_1c_1^0 \leq w_1\ell_o + r_1\overline{k} + (1-\delta)\overline{k} + m$$

2. Given prices, generation t allocation $\left\{c_t^t, c_{t+1}^t\right\}$ solves

$$\max_{(c_t^t, c_{t+1}^t)\in\mathbb{R}_+^2} \ln(c_t^tc_{t+1}^t) \text{ s.t } p_t\left(c_t^t + k_{t+1}^t\right) + p_{t+1}c_{t+1}^t \leq w_t\ell_y + w_{t+1}\ell_o + r_{t+1}k_{t+1}^t + p_{t+1}(1-\delta)k_{t+1}^t$$

3. Given prices, $\{k_t^d, \ell_t^d\}$ maximizes firm profits at t by

$$\max_{\ell_t, k_t} p_t f(k_t, \ell_t) - w_t \ell_t - r_t k_t$$

4. Markets clear:

$$\text{(ARC)} \;\; C_t + I_t = f(K_t, L_t) \;\; \underline{\textbf{and}} \;\; K_t = k_t^d = k_t^{t-1} \;\; \underline{\textbf{and}} \;\; L_t = \ell_t^d = \ell_o + \ell_y$$

Problem 1, part c)
Environment: we will define four stages of life: young, adult, middle-aged, old. Labor endowments $\mathbb{L}_t = \{\ell_t^{t-j}\}_{j=0:3}$ are generation-invariant (but not stage-invariant) and follow $(\ell_y, \ell_a, \ell_m, \ell_o)$ when you are young, adult, middle-aged, and old, respectively. Since we now have four total generations, we have an initially adult (generation 0) and initially middle-aged (generation -1), in addition to the intentionally old (generation -2). Like the standard two stage model, agents can only consume, work, and have utility in periods in which they are alive. We assume no discounting and initial old is the only generation with money endowment.

A SME is a collection of allocations for generations "initial" $\left\{c_1^{-2}, k_2^{-1}, b_2^{-1}, \{c_i^{-1}, c_i^0, k_{i+1}^0, b_{i+1}^0\}_{i=1:2}, c_3^0\right\}$ and "full" $\left\{c_t^t, \{c_{t+i}^t, k_{t+i}^t, b_{t+i}^t\}_{i=1:3}\right\}_{t\geq 1}$, allocations for firms $\{k_t^d, \ell_t^d\}_{t\geq 1}$, and prices $\{i_t, r_t, w_t\}_{t\geq 1}$ s.t

1. Given prices, initially old allocation c_1^{-2} solves

$$\max_{c_1^{-2}\in\mathbb{R}_+} \ln(c_1^{-2}) \;\text{ s.t }\; c_1^{-2} \leq w_1\ell_o + r_1\overline{k} + (1-\delta)\overline{k} + (1+i_1)m$$

2. Given prices, initially middle-aged allocation $\left\{c_1^{-1}, c_2^{-1}, k_2^{-1}, b_2^{-1}\right\}$ solves

$$\max_{(c_1^{-1}, c_2^{-1})\in\mathbb{R}_+^2} \ln(c_1^{-1}c_2^{-1}) \;\text{ s.t }\; c_1^{-1} + k_2^{-1} + b_2^{-1} \leq w_1\ell_m \;\; \underline{\textbf{and}} \;\; c_2^{-1} \leq w_2\ell_o + (1+r_2-\delta)k_2^{-1} + (1+i_2)b_2^{-1}$$

3. Given price, initially adult allocation $\left\{c_1^0, \{c_i^0, k_i^0, b_i^0\}_{i=2:3}\right\}$ solves

$$\max_{(c_1^0, c_2^0, c_3^0)\in\mathbb{R}_+^3} \ln\left(c_1^0c_2^0c_3^0\right) \;\text{ s.t}$$

$$c_1^0 + k_2^0 + b_2^0 \leq w_1\ell_a \;\; \underline{\textbf{and}} \;\; c_2^0 + k_2^0 + b_2^0 \leq w_2\ell_m + (1+r_2-\delta)k_2^0 + (1+i_2)b_2^0 \;\; \underline{\textbf{and}} \;\; c_3^0 \leq w_3\ell_o + (1+r_3-\delta)k_3^0 + (1+i_3)b_3^0$$

4. Given prices, generation t allocation $\left\{c_t^t, \{c_{t+i}^t, k_{t+i}^t, b_{t+i}^t\}_{i=1:3}\right\}$ solves

$$\max_{\mathbb{R}_+^4 \ni \{c_{t+j}^t\}_{j=0:3}} \ln\left(\prod_{j=0}^{3} c_{t+j}^t\right) \;\text{ s.t}$$

$$(j=0,1,2,3) \;\; c_{t+j}^t + \mathbb{1}_{j\neq 3}\left[k_{t+1+j}^t + b_{t+1+j}^t\right] \leq w_{t+j}\ell_{t+j}^t + \mathbb{1}_{j\neq 0}\left[(1+r_{t+j}-\delta)k_{t+j}^t + (1+i_{t+j})b_{t+j}^t\right]$$

5. Given prices, $\{k_t^d, \ell_t^d\}$ maximizes firm profits at t by

$$\max_{\ell_t, k_t} f(k_t, \ell_t) - w_t\ell_t - r_tk_t$$

6. Markets clear:

$$\text{(ARC)} \;\; C_t + I_t = f(K_t, L_t) \;\; \underline{\textbf{and}} \;\; K_t = k_t^d = \sum_{j=1}^{3} k_t^{t-j} \;\; \underline{\textbf{and}} \;\; L_t = \ell_t^d = \sum_{j=0}^{3} \ell_t^{t-j} \;\; \underline{\textbf{and}} \;\; \sum_{j=0}^{2} b_{t+1}^{t-j} = \prod_{k=1}^{t}(1+i_k)m$$

Important Note: Look how much writing you can save with good notation! Using indexes (like $j = 1:3$), indicator functions ($\mathbb{1}_{j\neq 0}$), and sums (both additive and product) can save lots of time while still being rigorous. I didn't do anything extra creative since this is a HW solution, but as we talked about in discussion, it's possible to write like 70% as much as me and get full credit! You just have to be "lazy" in the right way :)

HW2, Problem 2

Problem 2, part a)
(For reference, I've left labor in here, but it's inelastically supplied at 1 for all t and isn't needed)
An ADE is a collection of allocations for consumers $\{\{c_{i,t}, \ell_{i,t}, k_{i,t+1}\}_{i=1,2}\}_{t\geq 0}$, firm allocations $\{k_t^d, \ell_t^d\}_{t\geq 0}$, and prices $\{p_t, r_t, w_t\}_{t\geq 0}$ s.t

1. Given prices and $k_{i,0}$, consumer i allocations $\{c_{i,t}, \ell_{i,t}, k_{i,t+1}\}_{t\geq 1}$ solve

$$\max_{\{c_{i,t},\ell_{i,t},k_{i,t+1}\}_{t\geq 0}} \sum \beta_i^t \ln(c_{i,t}) \text{ s.t } \sum p_t\left(c_{i,t}+k_{i,t+1}\right) \leq \sum \left(r_t k_{i,t} + p_t(1-\delta)k_{i,t} + w_t \ell_{i,t}\right)$$

2. Given prices, $\{k_t^d, \ell_t^d\}$ maximizes firm profits at t by

$$\max_{\ell_t,k_t} p_t f(k_t, \ell_t) - w_t\ell_t - r_t k_t$$

3. Markets Clear

$$\text{(ARC)} \quad C_t + I_t = f(K_t, L_t) \ \underline{\textbf{and}} \ K_t = k_t^d = k_{1,t} + k_{2,t} \ \underline{\textbf{and}} \ L_t = \ell_t^d = \ell_{1,t} + \ell_{2,t}$$

Problem 2, part b)
Taking $c_{i,t}$ FOC

$$(i=1,2) \ \frac{\beta_i^t}{c_{i,t}} = p_t\lambda_i \implies (i=1,2) \ \frac{p_{t+1}}{p_t} = \beta_i \frac{c_{i.t}}{c_{i,t+1}} \implies \frac{c_{2,t}}{c_{2,t+1}} = \frac{\beta_1}{\beta_2}\frac{c_{1,t}}{c_{1,t+1}} \implies \frac{c_{1,t+1}}{c_{2,t+1}} = \frac{\beta_1}{\beta_2}\frac{c_{1,t}}{c_{2,t}}$$

We can iteratively substitute on the RHS of the last equation

$$\frac{c_{1,t+1}}{c_{2,t+1}} = \left(\frac{\beta_1}{\beta_2}\right)^2 \frac{c_{1,t-1}}{c_{2,t-1}} = \cdots = \left(\frac{\beta_1}{\beta_2}\right)^{t+1} \frac{c_{1,0}}{c_{2,0}} \implies \lim_{t\to\infty} \frac{c_{1,t}}{c_{2,t}} = \lim_{t\to\infty} \left(\frac{\beta_1}{\beta_2}\right)^t \frac{c_{1,0}}{c_{2,0}} = 0 \qquad \square$$

Comments: This works since $c_{2,0} \in \mathbb{R}_+ \implies \frac{c_{1,0}}{c_{2,0}}$ is just some constant. Also, note that constant terms have no effect on the indexing of an infinite limit. Explicitly – for $k \in \mathbb{Z}$, $\lim\limits_{t\to\infty} \alpha^t x_t = \lim\limits_{t\to\infty} \alpha^{t+k} x_{t+k}$

Problem 2, part c)
First, we extrapolate two key facts from b) $\lim_t \frac{c_{1,t}}{c_{2,t}} = 0 \implies$ **(i)** $\lim\limits_{t\to\infty} c_{1,t} = 0 \implies$ **(ii)** $\lim\limits_{t\to\infty} c_{2,t} = c_2 \in \mathbb{R}$. **(i)** is because the only other way that ratio goes to 0 is if $c_{2,t} \to \infty$, which is impossible (also couldn't be an optimal path). **(ii)** is because $c_{1,t} \in \mathbb{R}_+$ for each t, so the limit of the ratio could not exist if the limit of $c_{2,t}$ also did not exist. Now we use these facts to solve for what we want.

Normalize $L_t = 1$. Taking $k_{i,t+1}$ FOC and combining with Firm's FOC for capital ($r_t = \alpha p_t k_t^{\alpha-1}$)

$$\left[r_{t+1} + p_{t+1}(1-\delta)\right] k_{i,t+1} = p_t k_{i,t+1} \implies \frac{p_t}{p_{t+1}} = \frac{r_{t+1}}{p_{t+1}} + 1 - \delta = \alpha k_{t+1}^{\alpha-1} + 1 - \delta$$

We can plug in our equation for relative price we derived in b), take limits, and use **(ii)** from above

$$\lim_{t\to\infty} \alpha k_{t+1}^{\alpha-1} + 1 - \delta = \lim_{t\to\infty} \left(\beta_2 \frac{c_{2.t}}{c_{2,t+1}}\right)^{-1} = \frac{1}{\beta_2} \implies \boxed{\lim_{t\to\infty} k_t = \left[\frac{1}{\alpha}\left(\beta_2^{-1} + 1 - \delta\right)\right]^{\frac{1}{\alpha-1}}}$$

Important Note: Imposing a steady state is *not* the same thing as convergence. If a question asks about convergence (this includes in Econometrics) and you don't discuss limits, you're doing something wrong.

Other note: We normalize aggregate labor for simplicity since it's constant anyway. This is okay because all of the dynamics in this problem are about capital. However, if you were asked about a problem where you needed to make a calculation related to wages, it would likely not be appropriate to do this.