StatLect

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Estimation of the mean

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Mean estimation is a statistical inference problem in which a sample is used to produce a point estimate of the mean of an unknown distribution.

The problem is typically solved by using the sample mean as an estimator of the population mean. In this lecture, we present two examples, concerning:

1. normal IID samples;

- 2. IID samples that are not necessarily normal.
- For each of these two cases, we derive the expected value, the variance and the asymptotic properties of the mean estimator.

In this example of mean estimation, which is probably the most important in the history of statistics, the sample is drawn from a normal distribution.

Normal IID samples

Specifically, we observe the realizations of n independent random variables $X_1, ..., X_n$, all having a normal distribution with unknown mean μ and variance σ^2 .

The estimator

As an estimator of the mean μ , we use the sample mean

Expected value of the estimator

The expected value of the estimator
$$X_n$$
 is equal to the true mean μ .

 $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$

This can be proved by using the linearity of the expected value:

thow? I south this the sample &t not the folial to post then regular to post $=\frac{1}{n}n\mu=\mu$ $= \frac{1}{n^2} \operatorname{Var} \left[\sum_{i=1}^n X_i \right]$ $= \frac{1}{n^2} \sum_{i=1}^n X_i$

VIX) = E([X-E(X]])

The variance of the estimator X_n is equal to σ^2/n . This can be proved by using the formula for the variance of an independent sum:

Therefore, the estimator X_n is unbiased.

Variance of the estimator

 $=\frac{1}{n^2}\sum_{i=1}^n\sigma^2$

 $\operatorname{Var}\left[\overline{X}_{n}\right] = \operatorname{Var}\left[\frac{1}{n}\sum_{i=1}^{n}X_{i}\right]$

 $= \frac{1}{n^2} \sum_{i=1}^n \mathrm{Var}[X_i]$

 $=\frac{1}{n^2}n\sigma^2=\frac{\sigma^2}{n}$ Therefore, the variance of the estimator tends to zero as the sample size n tends to infinity. Distribution of the estimator The estimator X_n has a normal distribution: $X_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Proof

The mean squared error of the estimator is $MSE(X_n) = E[||X_n - \mu||^2]$

Risk of the estimator

E([CXn-E[xn]]) Consistency of the estimator The sequence $\{X_n\}$ is an IID sequence with finite mean. Therefore, it satisfies the conditions of Kolmogorov's Strong Law of Large Numbers. Hence, the sample mean X_n converges almost surely to the true mean μ :

IID samples

In this example of mean estimation, we relax the previously made assumption of normality.

The sample is made of the realizations of n independent random variables X_1 , ..., X_n , all having the

same distribution with mean μ and variance σ^2 .

Expected value of the estimator

Also in this case the proof is the same found in the previous example.

Therefore, the estimator is unbiased.

The proof is the same found in the previous example.

Distribution of the estimator

distribution depends on those of the terms of the sequence $\{X_n\}$.

The mean squared error of the estimator is

Consistency of the estimator Since the sequence $\{X_n\}$ is an IID sequence whose terms have finite mean, it satisfies the conditions

Asymptotic normality The sequence $\{X_n\}$ is an IID sequence with finite mean and variance.

mean μ and variance $\frac{\sigma^2}{n}$.

Solved exercises

Exercise 1

Hence, the sample mean X_n is asymptotically normal: $\sqrt{n}\left(\frac{\overline{X}_n-\mu}{\sigma}\right)\stackrel{d}{\to} Z$

Consider an experiment that can have only two outcomes: either success, with probability p, or failure, with probability 1 - p. The probability of success is unknown, but we know that

What is the minimum number of experiments needed in order to be sure that the standard deviation of the estimator is less than 1/100? Solution

mean μ and known variance $\sigma^2 = 1$.

Exercise 2

Solution

Please cite as:

statistics/mean-estimation.

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WHAT DOES IT SAY?

How can you approximate the distribution of their sample mean?

Suppose that you observe a sample of 100 independent draws from a distribution having unknown

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Expected value of the estimator

$$= \mathbb{E} \Big[|\overline{X}_n - \mu|^2 \Big] \qquad \text{(Euclidean norm in one dimension is equal to absolute value)}$$

$$= \mathbb{E} \Big[(\overline{X}_n - \mu)^2 \Big]$$

$$= \mathbb{V} \text{ar} \Big[\overline{X}_n \Big] \qquad \text{(By the definition of variance, because } \mathbb{E} \Big[\overline{X}_n \Big] = \mu \text{)}$$

$$= \frac{\sigma^2}{2}$$

 $X_n \stackrel{a.s.}{\rightarrow} \mu$ that is, the estimator X_n is strongly consistent. The estimator is also weakly consistent because almost sure convergence implies convergence in probability: $p\lim \bar{X}_n = \mu$

Again, the estimator of the mean μ is the sample mean: $\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$

The expected value of the estimator X_n is equal to the true mean: $\mathbb{E}[\overline{X}_n] = \mu$

Variance of the estimator The variance of the estimator X_n is $\operatorname{Var}[\overline{X}_n] = \frac{\sigma^2}{n}$

Unlike in the previous example, the estimator X_n does not necessarily have a normal distribution: its

However, we will see below that X_n has a normal distribution asymptotically, that is, it converges to a

 $MSE(\overline{X}_n) = Var[\overline{X}_n] = \frac{\sigma^2}{n}$

Whole is no here?

Risk of the estimator

The proof is the same found in the previous example.

normal random variable when n becomes large.

of Kolmogorov's Strong Law of Large Numbers. Therefore, the estimator X_n is both strongly consistent and weakly consistent (see example above).

where z is a standard normal random variable and $\stackrel{d}{\Rightarrow}$ denotes convergence in distribution. In other words, the sample mean X_n converges in distribution to a normal random variable with

Below you can find some exercises with explained solutions.

Therefore, it satisfies the conditions of Lindeberg-Lévy Central Limit Theorem.

Suppose that we can independently repeat the experiment as many times as we wish and use the ratio Successes obtained Total experiments performed as an estimator of p.

 $p \in \left[\frac{1}{10}, \frac{1}{5}\right]$

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