Topic 3 – Security Markets Economy and Asset Pricing

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Security Market Economy

A Security Market Economy

1. Environment: build on stochastic two-period setting from earlier with two small modifications:
    - $i = \{1, \ldots, N\}$ agents indexed by i
    - S (finite) states of nature
2. Market Structure: our main focus today
3. Equilibrium: adjusted for changes in market structure

Securities

Security market structure:

- Securities $j \in \{1, 2, ..., J\}$
- Securities are defined by their return $x_j \in \mathbb{R}^S$, which lists the payoff of security j in each state s: $x_j = (x_{j1}, \ldots x_{jS})$
- Payoff matrix stacks all the payoff vectors vertically

$$\underset{J \times S}{X} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_J \end{pmatrix}$$

- A portfolio $h \in \mathbb{R}^J$ describes an agent's holdings of the J securities
- Payoff of portfolio is $z = \underset{1 \times J}{h} \underset{J \times S}{X} = \sum_{j=1}^{J} h_j x_j \in \mathbb{R}^S$

Prices and Returns

- Security prices: $p = (p_1, p_2, \dots, p_J)$
- Price of portfolio: $p \cdot h = \sum_{j=1}^{J} p_j h_j$ (inner product)
- Gross return of security j: $\underset{1\times S}{\mathrm{r_j}} = \frac{x_j}{p_j} = \left(\frac{x_{j1}}{p_j}, \frac{x_{j2}}{p_j}, \dots, \frac{x_{jS}}{p_j}\right)$

Types of Securities

Security = tradable financial asset

Examples:

- Bond - debt that can be traded
- Equity - ownership share of firm that pays dividends
- Asset-based security (ABS) - security whose value depends on the value of a bundle of assets
- Option - right to buy/sell in the future at a fixed price
- Futures - agreement to buy or sell in the future at a fixed price
- Swap - exchanges risky payment for safe payment (offloads risk)

Asset Span

- Asset span $\mathcal{M}$ of a security economy = set of payoffs that can be obtained using the available securities = row space of X:

$$\mathcal{M} = \{z \in \mathbb{R}^S : z = hX \text{ for some } h \in \mathbb{R}^J\}$$

- Markets complete $\iff \mathcal{M} = \mathbb{R}^S$
- Markets incomplete $\iff \mathcal{M} \subsetneqq \mathbb{R}^S$

Proposition

Markets are complete iff rank$(X) = S$.

- Market completeness means that all values in $\mathbb{R}^S$ are reachable
- This means we need at least S securities.
- $J > S$ is a possible case for this – if S of the J securities are linearly independent

Market Completeness

Proposition

Markets are complete iff rank$(X) = S$.

Proof.

$\Rightarrow$ If rank$(X) = S$, $\exists S$ linearly independent row vectors of length S in X. Then, they span $\mathbb{R}^S$ (form a basis for it). Hence, markets are complete.

$\Leftarrow$ If markets are complete, the row vectors in X span $\mathbb{R}^S$. Then, pick out the linearly independent ones. This gives us a basis for $\mathbb{R}^S$. Then, we get that rank$(X) = S$. $\square$

Proof.

$\mathcal{M} = \mathbb{R}^S$ iff $z = hX$ has a solution $\forall z \in \mathbb{R}^S$, i.e. $h = zX^{-1}$ (right inverse exists), which is equivalent to market completeness. $\square$

Redundant Securities

- Security j is redundant if its payoffs x_j can be replicated as a portfolio of the other securities, i.e. $\exists h$ s.t. $h_j = 0$ with $hX = x_j$
- A security that is not linearly independent of the other securities
- If $J > S$ then some securities must be redundant

Proposition

There are no redundant securities iff rank$(X) = J$.

Optimization

Strategy of each agent i:

$$\max_{C_1^i, C_2^{is}, h^i} u(C_1^i) + \beta E\left[u(C_2^i)\right]$$

$$\text{s.t.}\quad C_1^i + p \cdot h^i = Y_1^i$$

$$C_2^{is} = Y_2^{is} + \sum_{j=1}^{J} h_j^i x_j^s, \forall s, \text{ or equiv. } \underset{1\times S}{\mathrm{C}_2^{\mathrm{i}}} = \underset{1\times S}{\mathrm{Y}_2^{\mathrm{i}}} + \underset{1\times J}{\mathrm{h}^{\mathrm{i}}} \underset{J\times S}{\mathrm{X}}$$

$$\text{FOC}(h_j^i):\quad p_j u'(C_1^i) = \beta \sum_{s\in S} \pi^s u'(C_2^{is}) x_j^s \quad \forall j$$

$$\text{or:}\quad p_j = E\left[M^{is} x_j^s\right] \quad \text{where } M^{is} = \underbrace{\frac{\beta u'(C_2^{is})}{u'(C_1^i)}}_{\text{stochastic discount factor}}$$

Equilibrium

Definition (Security Market Equilibrium)

An equilibrium in the security markets economy is a set of allocations (C_1^i, C_2^i, h^i) and prices (p) such that

- all individuals optimize, and
- markets clear: $\sum_i h^i = 0$ (security markets clear), $\sum_i C_1^i = \sum_i Y_1^i$, and $\sum_i C_2^i = \sum_i Y_2^i$ (goods markets clear)

Definition (Law of One Price, LOOP)

If $hX = h'X, p \cdot h = p \cdot h'$.

- Note that this holds trivially if there are no redundant securities ($hX = h'X$ requires $h = h'$).

Arrow Securities

Arrow Securities (or State Claims):

- $e^s = (0, 0, \cdots, 1, \cdots, 0)$ is an Arrow security - a security that returns 1 unit of consumption in state s and nothing in any other state
- Denote cost of Arrow security e^s by *state price* q^s

If LOOP holds and markets are complete:

- Well-defined q_s for every state s
- The vector $q \equiv (q_1, q_2, \cdots . q_S)$ is a payoff pricing functional
- $q : \mathbb{R}^S \to \mathbb{R}$ map stochastic payoff vectors into the (scalar) price space
- For any $z \in \mathbb{R}^S, q \cdot z \equiv$ price of payoff z

Optimization Problem in Payoff Space

We can then express the optimization problem of agent i as directly choosing a payoff vector z^i (rather than a portfolio h^i of securities):

$$\max_{C_1^i, C_2^i, z^i \in \mathcal{M}} u(C_1^i) + \beta E\left[u(C_2^{is})\right]$$

$$\text{s.t.} \;\; C_1^i + q \cdot z^i = Y_1^i$$

$$C_2^i = Y_2^i + z^i$$

$$\text{FOC}(z^{is}) : q^s = \frac{\pi^s \beta u'(C_2^{is})}{u'(C_1^i)} =: \pi^s M^{is}$$

...State prices equal the probability-weighted MRS $\pi^s M^{is}$ for each agent i

$\rightarrow$ q is frequently called "pricing kernel" and is unique under complete markets.

Note: $p_j = q \cdot x_j = \sum_{s \in S} q_s X_{js} \forall j$ or $p = qX'$

Fundamental Theorem of Finance

Definition

An arbitrage is a portfolio with positive payoffs but zero price.

- Small arbitrage opportunities exist in practice, but they go away once engaged in.

Theorem (Fundamental Theorem of Finance)

$q \gg 0$ is strictly positive (i.e. $q^s > 0 \forall s$) iff there is no arbitrage.

Proof.

Under complete markets, $p = Xq$. Then, $q = Lp$, where L is the left inverse of X.

Note that $q \gg 0 \implies p \gg 0 \implies$ no arbitrage.

On the other hand, no arbitrage $\implies p \gg 0 \implies q \gg 0$. $\square$

Risk-neutral probabilities

- Let $\iota = (1, 1, \cdots, 1)$ be a risk-free bond
- Then, $q \cdot \iota = \sum_s q^s$ is price of a risk-free bond
- The risk-free return is then $\bar{r} = \frac{1}{q \cdot \iota} = \frac{1}{\sum_s q^s}$

Definition (Risk Neutral Probabilities)

$$\pi_s^* = \bar{r} q^s = \frac{q^s}{\sum_s q^s}$$

- Note that π_s^* are non-negative and sum up to 1
  $\rightarrow$ Use π_s^* to construct a probability measure $\Pi^*(\{s\}) = \pi_s^*$
- This measure differs from the objective probability measure Π because it reflects the market value of individual states of nature

Risk-neutral probability space

Definition (Risk-Neutral Probability Space)

The triple $(S, \mathcal{B}, \Pi^*)$ forms a probability space that we call the **risk-neutral probability space** (or Q-space)

- This differs from the objective probability space (P-space)
- The probability measure Π^* and the associated expectations operator $E^*[\cdot]$ allow us to express:

$$q \cdot z = \frac{E^*[z]}{\bar{r}}, \qquad p_j = \frac{E^*[x_j]}{\bar{r}}, \qquad \bar{r} = E^*[r_j], \forall j$$

Example: Consider $S = \{L, H\}$ with $\pi_L = \pi_H = 1/2$
Assume $C_2^L < C_2^H$: then, $q^L > q^H$, implying that $\pi_L^* > 1/2 > \pi_H^*$

Risk Aversion

Definition (Risk Aversion)

A utility function $u(\cdot)$ is risk-averse if $E[u(C^s)] < u(E[C^s])$ for any non-degenerate random variable C^s

- By Jensen's inequality, this definition is equivalent to

Definition (Risk Aversion)

A utility function $u(\cdot)$ is risk-averse if it is strictly concave.

Measures of Risk Aversion

- Risk Compensation
  - For risky payoff z at deterministic consumption level y, we define the *risk compensation* $\rho(y, z)$ s.t. $E[u(y+z)] = u(y - \rho + z)$
  - $\rho \equiv$ risk compensation (premium)
  - $y - \rho \equiv$ certainty equivalent
  - Note that ρ is a scalar, and not an RV like z.

- Arrow-Pratt Coefficient of Relative Risk Aversion

$$R(C) = -\frac{u''(C)}{u'(C)} \cdot C = \epsilon_{MU(C)}$$

  - Utility functions with constant relative risk aversion (CRRA): $u(C) = \frac{C^{1-\sigma}}{1-\sigma}$ for $\sigma \neq 1$ or $u(C) = \log C$ for $\sigma = 1$
  - **Exercise:** show that these functions satisfy $R(C) = \sigma =$ constant $\forall C$

The Sharpe Ratio

- From the agent's optimality condition

$$E[u'(C_2^s)(r_j - \bar{r})] = 0$$
$$\Rightarrow \rho_{u', r_j - \bar{r}} = \frac{Cov(u'(C_2^s), (r_j - \bar{r}))}{\sigma(u'(C_2^s))\sigma(r_j - \bar{r})}$$

- Since $|\rho| \leq 1$,

$$\sigma(u'(C_2^s)) \geq \frac{E[u'(C_2^s)]E[r_j - \bar{r}]}{\sigma(r_j)}$$

- Using $E[u'(C_2^s)] = \frac{u'(C_1)}{\beta \bar{r}}$,

$$\underbrace{\sigma\left(\frac{\beta u'(C_2^s)}{u'(C_1)}\right)}_{\text{std dev of MRS}} \geq \frac{1}{\bar{r}} \cdot \underbrace{\frac{|E[r_j] - \bar{r}|}{\sigma(r_j)}}_{\text{Sharpe ratio}}, \forall j$$

The Equity Premium Puzzle

- What is the puzzle?
  - $\sigma(MRS)$ is usually $\leq 5\%$
  - Sharpe Ratio for most securities is in the range $1 - 3$
    - For example, standard values of $E[r_j] = 1.07, \bar{r} = 1.02, \sigma(r_j) = 0.05$ imply a Sharpe Ratio ≈ 1.

  $\rightarrow$ The inequality clearly fails
- Consider the CRRA utility $u(c) = \frac{c^{1-\theta}}{1-\theta}$
  - 1st-order Taylor approximation: $\sigma(MRS) \approx \frac{\theta}{\bar{r}}\sigma\left(\frac{c_2^s}{E[c_2^s]}\right)$
  - Use standard values $\theta = 2$ (or $\theta = 5$ at most), $\bar{r} = 1.02 \approx 1$, and $\sigma\left(\frac{c_2^s}{E[c_2^s]}\right) \approx 0.01$.
  - Then, $\sigma(MRS) \approx 0.02 \geq 1 \approx$ Sharpe ratio according to inequality

  $\rightarrow$ The inequality clearly fails

Conclusion

- For the inequality to hold with CRRA utility, we would need
  - $\theta \geq \dfrac{\text{Sharpe Ratio}}{\text{\% Volatility of Consumption}} \approx 100$
  - This is far above empirical estimates of risk aversion
- How do we reconcile these differences?

Conclusion: Expected returns on risky assets in the data are far greater than they should be, based on observed consumption fluctuations and reasonable parameters of risk aversion

Introduction

- Most influential framework for asset pricing
- 1990 Nobel Prize - Sharpe, Markowitz, Miller

Consumption-based Asset Pricing

- Optimality (FOC) of portfolio implies

$$
\begin{aligned}
u'(C_1) &= \beta E[r_j^s u'(C_2^s)] \\
&= \beta E[r_j^s] E[u'(C_2^s)] + \beta cov(r_j^s, u'(C_2^s)) \\
\implies E[r_j^s] &= \frac{u'(C_1)}{\beta E[u'(C_2^s)]} - \frac{cov(r_j^s, u'(C_2^s))}{E[u'(C_2^s)]} \\
&= \bar{r} - \frac{\beta cov(r_j^s, u'(C_2^s))}{u'(C_1)} \bar{r} \\
\implies E[r_j^s] - \bar{r} &= -\bar{r} cov\left(\frac{\beta u'(C_2^s)}{u'(C_1)}, r_j^s\right)
\end{aligned}
$$

- This is the equation of consumption-based asset pricing.
- Expected excess return of asset is proportional to the negative covariance of risky return and the MRS of consumption.

Examples:

- Risk-free security r_j = constant
  - $\implies E[r_j^s] = \bar{r}$
- Typical risky securities:
  - Bad states imply High MRS and low $r_j^s \Rightarrow$ negative covariance
  - Therefore $E[r_j^s] \geq \bar{r}$ for a risky security
  - The excess return is the risk compensation.
- Lottery
  - r_j^s uncorrelated with C
  - No risk compensation despite level of risk
- Hedge
  - Insures against aggregate risk
  - Positive covariance between MRS and r_j^s
  - Hedges pay negative excess return (insurance against aggregate risk is costly)

CAPM Assumptions

Impose two assumptions (to simplify derivation):

1. Complete markets
   - Every imaginable asset (including market portfolio) is in asset span
   - Market portfolio is a portfolio including all risky assets in the economy
2. Quadratic utility
   - $u(C) = -\frac{1}{2}(\alpha - C)^2, C \leq \alpha$
   - So $u'(C) = \alpha - C$
   - Note that the MU of any utility function can be expressed in linear form from 1st-order Taylor approximation

   $\rightarrow$ Quadratic utility is not restrictive, only an approximation for any concave utility function

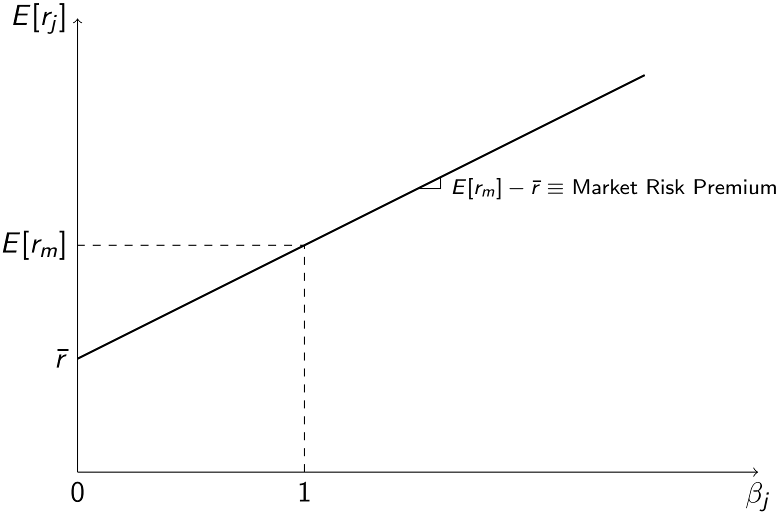
CAPM Fundamental Equation

- $cov(MRS, r_j) = -const \cdot cov(C_2^s, r_j)$ because of quadratic utility
- $cov(C_2^s, r_j) = const \cdot cov(r_m, r_j)$ where r_m is return on market portfolio because complete markets
- Re-express equation of consumption-based asset pricing:
  - $E[r_j] - \bar{r} = \bar{r} \cdot \gamma \cdot cov(r_m, r_j)$ holds $\forall j$, including market return m
  - $\rightarrow$ $E[r_m] - \bar{r} = \bar{r} \cdot \gamma \cdot var(r_m)$
- Dividing the two yields $\frac{E[r_j]-\bar{r}}{E[r_m]-\bar{r}} = \frac{cov(r_m,r_j)}{var(r_m)}$

$$\text{or} \quad E[r_j] = \bar{r} + \beta_j \cdot (E[r_m] - \bar{r}) \quad \text{where} \quad \beta_j = \frac{cov(r_m, r_j)}{var(r_m)}$$

- This the fundamental equation of CAPM:
  - $\beta_j = \frac{cov(r_m,r_j)}{var(r_m)}$ is the coefficient from the OLS regression of r_j on r_m
  - $E[r_m] - \bar{r}$ is the market risk premium

Graphical Representation of CAPM



CAPM - Cases

- Safe asset ($r_j =$ constant):
  - $\beta_j = 0 \Rightarrow E[r_j] = \bar{r}$
- Market portfolio ($r_j = r_m$):
  - $\beta_j = 1 \Rightarrow E[r_j] = E[r_m]$
- Relatively risky asset: $\beta_j > 1$
- Relatively "safe" asset: $\beta_j < 1$
- Hedge: $\beta_j < 0$

Decomposition of risk

The **CAPM** implies that risk can be decomposed into 2 parts:

1. market/systematic/aggregate/undiversifiable risk
2. asset-specific/"idiosyncratic"/diversifiable/zero-β risk

- Big insight of CAPM
  - Idiosyncratic risk can be diversified away hence no compensation
  - Investors compensated only for market risk since it cannot be diversified

β_j is obtained by regressing r_j on r_m:

- $r_j = \alpha_j + \beta_j r_m + \epsilon$
- β_j represents the security j's loading on market risk
- ϵ represents idiosyncratic risk
- Taking on more risk implies a greater exposure to market risk, and hence grants more risk compensation (i.e. a higher expected return).