Econ 7010 - Microeconomics I
University of Virginia
Fall 2023

Problem Set 1
Solutions

1. *Review the section on Properties of Production Sets starting on pg.130 of MWG. Draw a production set for which all of the following hold:*

   (a) *No free lunch is satisfied*

   (b) *Irreversibility is not satisfied (i.e. production is reversible) over some regions of the production process.*

   (c) *The production set is not everywhere closed, and*

   (d) *The possibility of inaction is violated.*

   (e) *The production set exhibits nondecreasing returns to scale.*

   Answer Many pictures will work. A very simple two good example is $Y = \left\{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 + y_2 \leq 0\right\} \setminus \{(0,0)\}$ (where $\setminus$ represents set subtraction):

   a) NFL: If $y \in Y$ and $y' \leq y$, then $y'_1 \leq y_1$, $y'_2 \leq y_2$, hence $y'_1 + y'_2 \leq y_1 + y_2 \leq 0$, so $y' \in Y$.

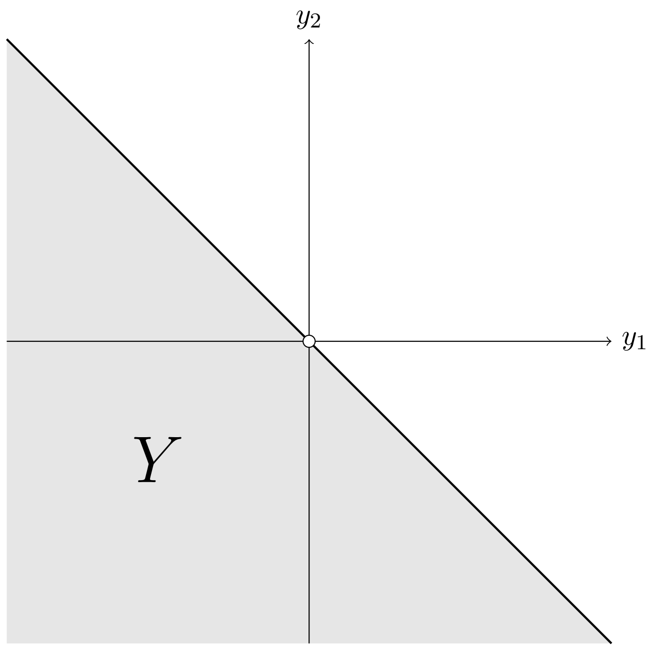
   b) Reversibility. $(-\gamma, \gamma)$ and $(\gamma, -\gamma)$ are both in Y for any $\gamma > 0$ (notice that $-\gamma + \gamma \leq 0$ for any γ, and so any such points are in Y)

   c) $(0,0)$ is in the closure of Y, but isn't in Y.

   d) $(0,0) \notin Y$

   e) If $y_1 + y_2 \leq 0$, then $\alpha y_1 + \alpha y_2 \leq 0$ for any $\alpha \geq 1$ (actually, for any $\alpha > 0$), so nondecreasing returns holds.

   This is shown in the following picture:



2. *A production set Y is* additive *if $y, y' \in Y$ implies that $y + y' \in Y$.*

   (a) *Give a brief description in words of what this condition means economically.*

   (b) *Give two examples of single-input, single output production functions, one of which satisfies additivity, and one of which does not.*

   (c) *Let Y be a general (multi-input, multi-output) production set that exhibits nonincreasing returns to scale (i.e., for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha \in [0, 1]$). Show that if in addition Y is additive, then Y is convex and exhibits constant returns to scale.*

Answer

(a) In words, this simply means that if y and y' are two possible production plans, then it is possible for the firm run each plan independently and sum the (net) outputs (e.g., by opening two separate plants which do not interfere with each other).

(b) An example of a production function that does satisfy additivity is $f(x) = x$. To see this, consider two production plans $(-x, y), (-x', y') \in Y$. We want to show that $(-(x + x'), y + y') \in Y$, or, equivalently,

$y + y' \le x + x'$. Since $(-x, y), (-x', y') \in Y$, we must have $y \le x$ and $y' \le x'$. Summing these two inequalities gives the desired result.

An example of a production function that does not satisfy additivity is $f(x) = \sqrt{x}$. To see this, note that the vectors $(-1, 1)$ and $(-4, 2)$ both belong to Y, but their sum, $(-5, 3)$ does not (since the maximum y that can be produced is $f(5) = \sqrt{5} = 2.23 < 3$).

(c) We first show that nonincreasing returns + additivity implies convexity. Take two points $y, y' \in Y$ and a scalar $\alpha \in [0, 1]$. By nonincreasing returns to scale, $\alpha y, (1 - \alpha)y' \in Y$. By additivity, $\alpha y + (1 - \alpha)y' \in Y$, which implies that Y is convex.

For constant returns to scale, we only need to show that $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \ge 1$ (the nonincreasing returns part, $\alpha \in [0, 1]$, holds by assumption). So, assume that $y \in Y$, and consider some $\alpha > 1$. Let $k = \lfloor \alpha \rfloor$ be the integer part of α, and $\delta = \alpha - k$ be the decimal part.[1] Then, by nonincreasing returns, $\delta y \in Y$. By additivity, $\alpha y = \overbrace{y + y + \cdots + y}^{k} + \delta y \in Y$.

3. *Consider a single-output production function $f(x)$, and let Y be the associated production set: $Y = \{(-x, y) : x \ge 0$ and $y \le f(x)\}$. In class, we gave the following two definitions of non-increasing returns to scale:*

   (a) $y \in Y$ *implies* $\alpha y \in Y$ *for all* $\alpha \in [0, 1]$

   (b) $f(tx) \le tf(x)$ *for all* $t \ge 1$

   Prove formally that definitions (a) and (b) are equivalent; that is, show that (a) implies (b), and (b) implies (a). [Hint: consider $\alpha = \frac{1}{t}$. For extra practice, you can also show that the two definitions of non-decreasing returns to scale and and constant returns to scale are also equivalent.]

   Answer

   We begin by showing that (a) implies (b). Thus, assume that (a) holds. Take $(-tx, f(tx)) \in Y$ for some $t \ge 1$. Let $\alpha = \frac{1}{t}$, and note that $\alpha \in [0, 1]$. Thus, by (a), $(-\alpha tx, \alpha f(tx)) \in Y$, or $\alpha f(tx) \le f(\alpha tx)$. Replacing $\alpha = \frac{1}{t}$, this becomes $\frac{1}{t} f(tx) \le f(x)$, or $f(tx) \le tf(x)$, which is (b).

[1] For example, if $\alpha = 10.5$, then $k = 10$ and $\delta = 0.5$.

Now for the other direction ((b) implies (a)). Assume that (b) holds, and choose some $(-x, y) \in Y$. Given any $\alpha \in [0, 1]$, we want to show that $(-\alpha x, \alpha y) \in Y$. Define $t = \frac{1}{\alpha}$, and note that $t \geq 1$ and $t\alpha = 1$. Since $(-x, y) \in Y$, we have $y \leq f(x)$. Since $t\alpha = 1$, we can write this as $t\alpha y \leq f(t\alpha x) \leq tf(\alpha x)$, where the last inequality follows by (b). Canceling the t's in the first and last terms, we have $\alpha y \leq f(\alpha x)$, or, in other words, $(-\alpha x, \alpha y) \in Y$, which is (a).

4. *For each of the following production functions (i) calculate the marginal rate of technical substitution and use this to sketch several isoquants (a very small, very rough picture is fine), (ii) determine whether the production function exhibits increasing, constant, or diminishing returns to scale, and (iii) derive the expression for the elasticity of substitution:*

   (a) $f(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)}$ *where* $\alpha \in (0, 1)$. *[Cobb-Douglas]*

   (b) $f(x_1, x_2) = [\alpha x_1^\rho + (1-\alpha) x_2^\rho]^{\frac{1}{\rho}}$ *for* $\alpha \in (0, 1)$. *[CES]. Draw isoquants for three separate values of* $\rho = 0, 1, -\infty$.

   (c) $f(x_1, x_2) = (x_1 + x_2)^\alpha$ *where* $\alpha > 0$.

   (d) $f(x_1, x_2) = \max\{x_1, 2x_2\}$.

   Answers

   (i) The MRTSs are: (a) $\frac{\alpha}{1-\alpha}\frac{x_2}{x_1}$ (b) $\frac{\alpha}{1-\alpha}(\frac{x_1}{x_2})^{\rho-1}$ (c) 1 (d) if $x_2 > \frac{x_1}{2}$, MRTS=0; if $x_2 < \frac{x_1}{2}$, MRTS=∞; if $x_2 = \frac{x_1}{2}$, the MRTS is undefined.

   The sketches of the isoquants follow from the MRTSs, i.e., the slopes. Part (a) is a standard Cobb-Douglas.

   For (b), when $\rho = 0$, the MRTS reduces to the same as part (a), and so the CES production function reduces to the Cobb-Douglas production function when $\rho = 0$. When $\rho = 1$, the MRTS becomes $\alpha/(\alpha - 1)$, which means the isoquants are just straight lines, i.e., the CES production function reduces to the perfect substitutes production function ($f(x_1, x_2) = \alpha x_1 + (1-\alpha)x_2$). When $\rho \to -\infty$, the CES production function approaches a Leontief production function $f(x_1, x_2) = \min\{x_1, x_2\}$, so the isoquants are L-shaped. See Varian, pages 19-20 for more discussion.

   For (c), the MRTS is constant, all isoquants are straight lines, i.e., this is a case of perfect substitutes.

For (d), the shape of the isoquants is an "inverted Leontief" with kink points along the line $x_2 = x_1/2$.

(ii a) C.R.S. $f(\lambda x_1, \lambda x_2) = (\lambda x_1)^\alpha (\lambda x_2)^{1-\alpha} = \lambda x_1^\alpha x_2^{1-\alpha} = \lambda f(x_1, x_2)$.

(ii b) C.R.S. $f(\lambda x_1, \lambda x_2) = ((\lambda x_1)^\rho + (\lambda x_2)^\rho)^{\frac{1}{\rho}} = (\lambda^\rho (x_1^\rho + x_2^\rho))^{\frac{1}{\rho}} = \lambda (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = \lambda f(x_1, x_2)$.

(ii c) Depends on value of α. $f(\lambda x_1, \lambda x_2) = (\lambda x_1 + \lambda x_2)^\alpha = \lambda^\alpha (x_1 + x_2)^\alpha = \lambda^\alpha f(x_1, x_2)$

D.R.S. for $\alpha < 1$. $\lambda^\alpha f(x_1, x_2) < \lambda f(x_1, x_2)$

C.R.S. for $\alpha = 1$. $\lambda^\alpha f(x_1, x_2) = \lambda f(x_1, x_2)$

I.R.S. for $\alpha > 1$. $\lambda^\alpha f(x_1, x_2) > \lambda f(x_1, x_2)$

(ii d) C.R.S. $f(\lambda x_1, \lambda x_2) = \max\{\lambda x_1, \lambda 2x_2\} = \lambda \max\{x_1, 2x_2\} = \lambda f(x_1, x_2)$.

(iii a) $MRTS = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1} \Leftrightarrow \ln\left(\frac{x_2}{x_1}\right) = -\ln\left(\frac{\alpha}{1-\alpha}\right) + \ln MRTS$. $\sigma = \frac{d\ln\left(\frac{x_2}{x_1}\right)}{d\ln MRTS} = 1$.

(iii b) $MRTS = \frac{\alpha}{1-\alpha}\left(\frac{x_2}{x_1}\right)^{1-\rho} \Leftrightarrow \ln\left(\frac{x_2}{x_1}\right) = -\ln\left(\frac{\alpha}{1-\alpha}\right) + \frac{1}{1-\rho}\ln MRTS$. $\sigma = \frac{1}{1-\rho}$.

(iii c) Since the isoquants of this technology are identical to that of the linear production function (except for a relabeling of output), they have the same elasticity of substitution: $\sigma = \infty$.

(iii d) This production function is C.E.S. as $\rho \to \infty$. Therefore, $\sigma = 0$.

5. *(MWG 5.B.3) Show that for a single-output technology with free disposal, Y is convex if and only if the production function $f(z)$ is concave. Determine which parts of this equivalence, if any, still hold if free disposal is violated.*

Answer

1. Y convex $\Rightarrow f(z)$ concave. We need to show that $\lambda f(z_1) + (1-\lambda) f(z_2) \leq f(\bar{z})$ (where $\bar{z} = \lambda z_1 + (1-\lambda) z_2$) for all z_1 and z_2 and all $\lambda \in [0,1]$. For brevity, write $\bar{y} = \lambda f(z_1) + (1-\lambda) f(z_2)$. Then, writing net output vectors in the form $(y, -z)$, we know that $(f(z_1), -z_1) \in Y$ and $(f(z_2), -z_2) \in Y$. Convexity of Y then implies

that $(\bar{y}, -\bar{z}) \in Y$. But then since $f(\bar{z}) = \max\{y : (y, -\bar{z}) \in Y\}$, $f(\bar{z})$ must be at least as large as $\bar{y}$, so we are done.

2. $f(z)$ concave $\Rightarrow Y$ convex. Fix arbitrary output vectors $(y_1, -z_1) \in Y$ and $(y_2, -z_2) \in Y$ and $\lambda \in [0,1]$. We need to show that $(\bar{y}, -\bar{z}) \in Y$, where $\bar{y} = \lambda y_1 + (1-\lambda) y_2$ and $\bar{z} = \lambda z_1 + (1-\lambda) z_2$. Observe that $y_1 \leq f(z_1)$ and $y_2 \leq f(z_2)$, so $\bar{y} \leq \lambda f(z_1) + (1-\lambda) f(z_2)$. Next, concavity of $f(z)$ implies $\lambda f(z_1) + (1-\lambda) f(z_2) \leq f(\bar{z})$, so we conclude that $\bar{y} \leq f(\bar{z})$. Since $(f(\bar{z}), -\bar{z}) \in Y$, free disposal implies that $(\bar{y}, \bar{z}) \in Y$ also.

Part 1 doesn't rely on free disposal, but Part 2 does.

6. *Consider the following production function:* $f(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ *where* $\alpha \in (0,1)$. *[Cobb-Douglas]*

   (a) *Formulate the profit maximizing problem when the price per unit is* p *and the factor prices are* w_1 *and* w_2 *and derive the profit maximizing first order conditions.*

   (b) *Now suppose that the pair* x_1, x_2 *solves the first order conditions. Multiply the first order condition for factor* i *by* x_i *for* $i \in \{1,2\}$. *Add the two conditions. What does this tell you about profit at the profit maximizing pair,* x_1, x_2*?*

   (c) *Finally, define the output* $y = x_1^{\alpha} x_2^{1-\alpha}$. *Show using the first order conditions that it is possible to express the factor demands for* x_1 *and* x_2 *as a function of* y, w_1, *and* w_2 *(the output price* p *should not be a part of these functions). Using these factor demands, write down the profit maximization problem where the choice variable is now the level of output* y. *Let* $\alpha = 1$ *and* $w_1 = w_2 = 1$. *For which prices* p *of the output good is there a solution the profit maximization problem? If you find such a price, what level(s) of output maximize profit?*

   (d) *Think about your answer to (c). Which of Varian's "Difficulties" of profit maximization does the Cobb-Douglas production function face?*

Answer

(a)

$$\begin{aligned}
\pi &= \max_{x_1,x_2} px_1^{\alpha}x_2^{(1-\alpha)} - w_1x_1 - w_2x_2 \\
\frac{\partial \pi}{\partial x_1} &= \alpha p x_1^{\alpha-1}x_2^{1-\alpha} - w_1 = 0 \\
\frac{\partial \pi}{\partial x_2} &= (1-\alpha)\, px_1^{\alpha}x_2^{-\alpha} - w_2 = 0.
\end{aligned}$$

(b) From the FOCs, we have:

$$\begin{aligned}
\alpha p x_1^{\alpha}x_2^{1-\alpha} &= w_1x_1 \\
(1-\alpha)px_1^{\alpha}x_2^{1-\alpha} &= w_2x_2.
\end{aligned}$$

So that:

$$px_1^{\alpha}x_2^{1-\alpha} = w_1x_1 + w_2x_2.$$

Thus we have that profit is 0 for any interior solution.

(c) Conditional factor demands are functions of y, w_1, and w_2 (and NOT a function of the final goods price p). Dividing the FOC for x_1 by that for x_2 and solving for x_1 gives

$$x_1 = \frac{\alpha}{1-\alpha}\frac{w_2}{w_1}x_2$$

Now, using $y = x_1^{\alpha}x_2^{1-\alpha}$ and solving and substituting for x_2 in the above equation gives

$$x_1(y) = y\left(\frac{\alpha}{\alpha-1}\frac{w_2}{w_1}\right)^{1-\alpha}.$$

Similarly, we find

$$x_2(y) = y\left(\frac{\alpha}{\alpha-1}\frac{w_2}{w_1}\right)^{-\alpha}.$$

The profit function is $\Pi(y) = \max_y py - w_1x_1(y) - w_2x_2(y)$. and will be zero for any interior solution (see part b. above). If $\alpha = 1$, then $f(x_1, x_2) = x_1$. Then if $p < 1$ and $w_1 = 1$, $y = 0$ is the unique corner solution. Conversely, if $p > 1$, then the profit maximizing output level, y, is infinite. Only if $p = 1$ can there be an interior solution (which yields zero profits, of course), but then *any* $y \geq 0$ maximizes profit. Notice that the h.d.1 Cobb-Douglas production function thus illustrates three of Varian's 4 difficulties: boundary solutions, existence, and uniqueness. The moral of the story– be careful with CRS!