ECON 7710 TA Session

Week 6

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Outline

Question 1

- We know density function

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}(1 + x_1 x_2), & \text{if } (x_1, x_2) \in [-1, 1] \times [-1, 1] \\ 0, & \text{if } (x_1, x_2) \notin [-1, 1] \times [-1, 1] \end{cases}$$

- We first derive the marginal density of $f(x_1, x_2)$, f_{x_1} and f_{x_2}.
  For f_{x_1}:
  If $x_1 \in [-1, 1]$, $f_{x_1} = \int_{-1}^{1} \frac{1}{4}(1 + x_1 x_2) dx_2 = \frac{1}{2}$.
  If $x_1 \notin [-1, 1]$, $f_{x_1} = \int_{-\infty}^{\infty} 0 dx_2 = 0$
  Therefore we know:

$$f_{x_1} = \begin{cases} \frac{1}{2}, & \text{if } x_1 \in [-1, 1] \\ 0, & \text{Otherwise} \end{cases} \quad f_{x_2} = \begin{cases} \frac{1}{2}, & \text{if } x_2 \in [-1, 1] \\ 0, & \text{Otherwise} \end{cases}$$

- For an ideal $g(x_1, x_2)$, we want $g_{x_1} = f_{x_1}$ and $g_{x_2} = f_{x_2}$.

Question 1

$$f_{x_1} = \begin{cases} \frac{1}{2}, & \text{if } x_1 \in [-1,1] \\ 0, & \text{Otherwise} \end{cases} \quad f_{x_2} = \begin{cases} \frac{1}{2}, & \text{if } x_2 \in [-1,1] \\ 0, & \text{Otherwise} \end{cases}$$

- Constant numbers... $\Rightarrow$ uniform!
  Try
  $$g_{x_1,x_2}(x_1,x_2) = \begin{cases} \frac{1}{4}, & \text{if } (x_1,x_2) \in [-1,1] \times [-1,1] \\ 0, & \text{if } (x_1,x_2) \notin [-1,1] \times [-1,1] \end{cases}$$
- Check if that gives you $g_{x_1} = f_{x_1}$ and $g_{x_2} = f_{x_2}$. Check either.
  $g_{x_1} : x_1 \in [-1,1]$, $g_{x_1} = \int_{-1}^{1} \frac{1}{4} dx_2 = \frac{1}{2}$. If $x_1 \notin [-1,1]$, $g_{x_1} = 0$ ✓
- Check if that is a PDF:
  We also know $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{x_1,x_2}(x_1,x_2) = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} dx_1 dx_2 = 1$ and $g_{x_1,x_2}(x_1,x_2) \geq 0$ ✓.

Therefore $g_{x_1,x_2}(x_1,x_2)$ ✓

Question 2.a

We know the discrete random variable X with probability mass function

$$P(X = 2^n) = \frac{1}{en!}, n = 0, 1, 2...$$

a Prove this random variable has moments of all orders and find them
The rth moment of random variable X can be written as

$$E(X^r) = \sum_{n=0}^{\infty} (2^n)^r \frac{1}{en!} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(2^r)^n}{n!} = e^{2^r - 1}$$

We know r is arbitrarily picked and finite. So we proved that this random variable has moments of all orders and derived their formula.
(Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

Question 2.b

We know the discrete random variable X with probability mass function

$$P(X = 2^n) = \frac{1}{en!}, n = 0, 1, 2...$$

b Find the characteristic function of this random variable
We know the characteristic function of this random variable is:

$$\phi_X(t) = E[e^{itX}]$$

We also know $e^{itX} = \sum_{n=0}^{\infty} \frac{(itX)^n}{n!}$ So characteristic function of this random variable is:

$$\phi_X(t) = E(e^{itX}) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} e^{2^n - 1}$$

Question 2.c

We know the discrete random variable X with probability mass function

$$P(X = 2^n) = \frac{1}{en!}, n = 0, 1, 2...$$

c Find the moment generating function of this random variable.

- Be careful of MGFs as they may not exist. By definition,

$$M_X(t) = E(e^{tX}) \quad \forall t \in \{t \in \mathbb{R} | -h \leq t \leq h\}, \;\; h > 0$$

  We know if there is an $h > 0$ such that for all t in $-h \leq t \leq h$, $E[e^{tX}]$ exists, then moment generating function exists.
  In other words, we need $E[e^{tX}]$ to be finite for all real values of t in a subset of real line that includes zero but not only zero.

- How we proceed?

Step 1 Get the candidate of MGF

Step 2 Check if it works

Question 2.c

We know the discrete random variable X with probability mass function

$$P(X = 2^n) = \frac{1}{en!}, n = 0, 1, 2...$$

c Find the moment generating function of this random variable.

Step 1 By $e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$, we know $E(e^{tX}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{2^n-1}$

Step 2 Ratio test for convergence of series $\sum_{n=0}^{\infty} a_n$

Formulate a sequence $\{a_n\} = \frac{t^n}{n!} e^{2^n-1}$,

Then we know $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{e^{2^n}|t|}{n+1}$.

Since exponential growth grows much faster than linear growth when n is very large. So $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{e^{2^n}|t|}{n+1} = \infty$

Our mgf **failed to pass Ratio Test.**

This is actually a good news! Therefore our moment generating function does not exist for any $t \neq 0$.

Question 3.a

We know X_1 and X_2 are independent $N(0,1)$ random variables and we know:

$$\mathbf{Y} = (Y_1, Y_2) = \begin{cases} (X_1, |X_2|), \text{if } X_1 \geq 0 \\ (X_1, -|X_2|), \text{if } X_1 < 0 \end{cases}$$

- We want to derive the marginal distributions of (Y_1, Y_2)
  For Y_1, $F_{Y_1} = \lim_{y_2 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2) = F_{X_1}(y_1)$ just $N(0,1)$.
  For Y_2, $F_{Y_2} = \lim_{y_1 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2)$, consider two cases now:
  - If $y_2 \geq 0$, $\lim_{y_1 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2) = P(X_1 \geq 0)P(Y_2 \leq y_2 | X_1 \geq 0) + P(X_1 < 0)P(Y_2 \leq y_2 | X_1 < 0) = \frac{1}{2}P(|X_2| \leq y_2) + \frac{1}{2}P(-|X_2| \leq y_2) = \frac{1}{2}(F_{X_2}(y_2) - F_{X_2}(-y_2)) + \frac{1}{2} = \frac{1}{2}(F_{X_2}(y_2) - (1 - F_{X_2}(y_2))) + \frac{1}{2} = F_{X_2}(y_2)$
  - If $y_2 < 0$, $\lim_{y_1 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2) = P(X_1 \geq 0)P(Y_2 \leq y_2 | X_1 \geq 0) + P(X_1 < 0)P(Y_2 \leq y_2 | X_1 < 0) = 0 + \frac{1}{2}P(-|X_2| \leq y_2) = \frac{1}{2}(P(X_2 \geq -y_2) + P(X_2 \leq y_2)) = \frac{1}{2}[1 - F_{X_2}(-y_2) + F_{X_2}(y_2))] = \frac{1}{2}[2F_{X_2}(y_2)] = F_{X_2}(y_2)$
- Then F_{Y_2} is also just $N(0,1)$

Question 3.b

We know X_1 and X_2 are independent $N(0,1)$ random variables and we know:

$$\mathbf{Y} = (Y_1, Y_2) = \begin{cases} (X_1, |X_2|), \text{if } X_1 \geq 0 \\ (X_1, -|X_2|), \text{if } X_1 < 0 \end{cases}$$

- (Y_1, Y_2) cannot be a jointly normally distributed although the marginal distribution of both Y_1, Y_2 are $N(0,1)$.
  Suppose (Y_1, Y_2) are jointly normally distributed but positively correlated with $\rho \in (0,1)$.
  It has to be that $P(Y_1 \leq 0, Y_2 > 0) \neq 0$. Basically, you can get a sense of that by looking at the pdf of standard bivariate normal distribution with correlation coefficient ρ. A formal discussion can be found here. The probability is $P(Y_1 \leq 0, Y_2 > 0) = \frac{cos^{-1}\rho}{2\pi}$.
- But following our definition, we have $P(Y_1 \leq 0, Y_2 > 0) = 0$. Then it is a contradiction and (Y_1, Y_2) are not jointly normally distributed.

Question 4

For function:

$$\phi(t) = \begin{cases} 1 - t^2, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

Since we know for a continuous function $\phi(x)$ with $\phi(0) = 1$ is a characteristic function iff it is positive semi-definite, i.e. for any $t_1, ..., t_n \in \mathbb{R}$ and any $\lambda_1, ..., \lambda_n \in \mathbb{C}$

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k \lambda_j^* \geq 0$$

Here, λ_j^* is the complex conjugate of λ_j, when you simply change the sign of the imaginary part of the complex number.
In other words, to prove it is not a characteristic function, we need to find a group of λ and t that makes the inequality above fail.

Question 4

We take three points of t:

$$\begin{cases} t_1 = -\frac{1}{2} \\ t_2 = 0 \\ t_3 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} \phi(t_1) = \frac{3}{4} \\ \phi(t_2) = 1 \\ \phi(t_3) = \frac{3}{4} \end{cases}$$

Question 4

Then we know a bunch of $\phi(t_k - t_j)$

$$\begin{cases} \phi(t_1) = \frac{3}{4} \\ \phi(t_2) = 1 \\ \phi(t_3) = \frac{3}{4} \end{cases} \Rightarrow \begin{cases} \phi(t_1 - t_2) = \frac{3}{4} \\ \phi(t_1 - t_3) = 0 \\ \phi(t_2 - t_3) = \frac{3}{4} \\ \phi(t_2 - t_1) = \frac{3}{4} \\ \phi(t_3 - t_1) = 0 \\ \phi(t_3 - t_2) = \frac{3}{4} \\ \phi(t_1 - t_1) = \phi(t_2 - t_2) = \phi(t_3 - t_3) = 1 \end{cases}$$

$$\Rightarrow \sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k \lambda_j^* =$$

$$\frac{3}{4}\lambda_1\lambda_2^* + \frac{3}{4}\lambda_2\lambda_3^* + \frac{3}{4}\lambda_2\lambda_1^* + \frac{3}{4}\lambda_3\lambda_2^* + \lambda_1\lambda_1^* + \lambda_2\lambda_2^* + \lambda_3\lambda_3^*$$

Question 4

Let:

$$\begin{cases} \lambda_1 = a_1 + b_1 i \\ \lambda_1^* = a_1 - b_1 i \end{cases} \quad \begin{cases} \lambda_2 = a_2 + b_2 i \\ \lambda_2^* = a_2 - b_2 i \end{cases} \quad \begin{cases} \lambda_3 = a_3 + b_3 i \\ \lambda_3^* = a_3 - b_3 i \end{cases}$$

Then we plug in the equation above and we will get:

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k \lambda_j^* =$$
$$(a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2) + \frac{3}{2}(a_1 a_2 + b_1 b_2 + a_2 a_3 + b_2 b_3)$$
$$= (\mathrm{a}_1 + a_2)^2 + (b_1 + b_2)^2 + (a_2 + a_3)^2 + (b_2 + b_3)^2 - (a_2^2 + b_2^2) -$$
$$\frac{1}{2}(a_1 a_2 + b_1 b_2 + a_2 a_3 + b_2 b_3)$$

Question 4

$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k\lambda_j^* =$
$(a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2) + \frac{3}{2}(a_1 a_2 + b_1 b_2 + a_2 a_3 + b_2 b_3)$
$= (a_1 + a_2)^2 + (b_1 + b_2)^2 + (a_2 + a_3)^2 + (b_2 + b_3)^2 - (a_2^2 + b_2^2) -$
$\frac{1}{2}(a_1 a_2 + b_1 b_2 + a_2 a_3 + b_2 b_3)$
Specifically, we can pick:

$$a_1 = b_1 = a_3 = b_3 = 0.9$$
$$a_2 = b_2 = -1$$

Then the equation above will be

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k\lambda_j^* = 4 * 0.1^2 - 2 - \frac{1}{2} * 4 * (-0.9) = -0.16 < 0$$

So we proved $\phi(t)$ cannot be a characteristic function.