**Econ 7010 - Microeconomics I**
**University of Virginia**
**Fall 2023**

**Problem Set 4**
Due November 3rd

1. The UVA Economics Department is looking to hire a new professor next year. For simplicity, assume there are only 3 new Ph.D. graduates on the market, named $\mathcal{X} = \{Ann, Bob, Charlie\}$. It may be that only some (and not all) of the candidates apply. For any potential set of applicants $Y \subset \mathcal{X}$, the department has a choice rule $C(Y)$ that determines who will be given an offer. The following condition, known as Sen's $\alpha$, is a (partial) alternative to WARP. Let $Y, Z$ be two sets of potential applicants.

   - *Sen's $\alpha$:* If $x \in Y$, $Y \subseteq Z$, and $x \in C(Z)$, then $x \in C(Y)$.

   (a) Translate this condition into words. (It may be helpful, though not necessary, to draw a Venn diagram.)

   Consider the following two alternative choice procedures:

   - *Second-best:* The department has the following preference relation: $Ann \succ Bob \succ Charlie$. However, rather than choose the best alternative, they choose the second-best applicant from any applicant pool $Y$ (perhaps because they think the best applicant is unlikely to accept).
   - *Satisficing:* The department assigns a value $v(A)$, $v(B)$, and $v(C)$ to each possibility, a threshold $v^*$, and a fixed ordering of the candidates $O$. For any $Y$, the candidates are considered in in this order $O$, and, the first candidate with a value exceeding $v^*$ is chosen; if no candidate exceeds the threshold, the choice is the last candidate in the list. For example, assume that $v(A) = 5$, $v(B) = 3$, and $v(C) = 10$, and the threshold value is $v^* = 4$. If $Y = \{Ann, Bob, Charlie\}$ and the candidates are ordered *Bob*, followed by *Ann*, followed by *Charlie*, $C(Y) = Ann$, because she is the first one to exceed the threshold. If the order were *Bob*, *Charlie*, *Ann*, then $C(Y) = Charlie$.

(b) Show that the second-best procedure does *not* satisfy Sen's $\alpha$. Does it satisfy WARP?

(c) Show that satisficing *does* satisfy Sen's $\alpha$. (A formal "proof" is not necessary, but make sure you provide a clear, logical argument.)

(d) Show that in general, WARP implies Sen's $\alpha$.[1]

2. Prove the following proposition:

   > A preference relation, $\succeq$, can be represented by a utility function only if it is rational.

3. Let $x(p, m)$ be an agent's Marshallian demand function. Prove that if the agent's preferences are convex, then $x(p, m)$ is convex, and further, if the agent's preferences are strictly convex, then $x(p, m)$ is a singleton. (Note: you may assume a continuous utility representation $u(x)$ exists, but do not assume it is differentiable. Let $x', x' \in x(p, m)$, and define $\bar{x} = \alpha x + (1 - \alpha)x'$. To prove the first part, it is sufficient to show the following: (i) $\bar{x}$ is feasible at $(p, m)$ and (ii) $\bar{x}$ gives at least as much utility as $x, x'$. Use the definition of quasiconcavity that says $u(\bar{x}) \geq \min\{u(x), u(x')\}$.)

4. MWG 3.D.4 (Do not assume differentiability for part (a) - it is actually easier to show directly, without FOCs; for part (c), you may assume that $\eta$ is differentiable and strictly increasing)

5. MWG 2.D.1, 2.D.2,

6. Consider the utility function

$$u(x_1, x_2) = (x_1 - a_1)^{b_1}(x_2 - a_2)^{b_2},$$

   where $b_1, b_2 > 0, b_1 + b_2 < 1$, and $a_1, a_2 > 0$. Find:

   (a) the Marshallian demands,

   (b) the expenditure function,

   (c) the indirect utility function, and

   (d) the money metric indirect utility function.

   (e) Confirm the identities $e(p, v(p, m)) = m$ and $v(p, e(p, \bar{u})) = \bar{u}$.

[1] The converse, that Sen's $\alpha$ implies WARP, is false, but you are not required to show this.

(f) Assume you are given the expenditure function and indirect utility function from parts (b) and (c), but have no other knowledge of the consumer (in particular, you do not know the direct utility function $u(x_1, x_2)$). Calculate calculate the consumer's Hicksian demand using only this information. Then, calculate the consumer's Marshallian demand *without using Roy's identity* (Roy's identity is of course a valid way to calculate the Marshallian demand given knowledge of $v(p, m)$, but this question is asking you to do this in another way).

7. Consider the indirect utility function,

$$v(p, m) = m p_1^{-1/2} p_2^{-1/2}.$$

Find expressions for the expenditure function, the Hicksian and Marshallian demands, and the (direct) utility function.

8. Show that Roy's identity is a direct consequence of the duality identities (those that relate the expenditure function with the indirect utility function and the Marshallian demand function with the Hicksian demand function) and Shepard's lemma. (Hint: Start with the identity $v(p, e(p, \bar{u})) = \bar{u}$ and differentiate this equation with respect to $p_i$.)

9. MWG 3.E.1

10. MWG 3.G.4, parts (a)-(c). The wording of this problem is a bit unclear. For part (a), you need to show that if $u$ is additively separable, and $g(x)$ is a *linear* monotonic transformation (i.e., $g(x) = ax + b$ for some constants $a > 0$ and $b$), then, the transformed utility $v(x) = g(u(x))$ has an additively separable representation (i.e., we can write $v(x) = \sum_{i=1}^{N} g_i(x_i)$ for some single-variable functions $g_i(\cdot)$). For part (b), partition the consumption space $X = Y \times Z$, where $Y = \mathbb{R}^m$ and $Z = \mathbb{R}^{n-m}$. Then, consider consumption bundles $(a, c), (a, d), (b, c)$ and $(b, d)$, where $a, b \in Y$ and $c, d \in Z$. The question is asking you to prove that $(a, c) \succeq (b, c)$ if and only if $(a, d) \succeq (b, d)$.[2] For part (c), you just need to show that there are no inferior goods (i.e., the wealth effects for all goods are nonnegative).

[2] That is, the ordering between bundles $a$ and $b$ in $Y$ is independent of the bundle you have from $Z$. This part should not be very hard.