Suggested Solutions: ECON 7710 HW III

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September 29, 2023

Question 1

We know density function

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}(1 + x_1x_2), & \text{if } (x_1, x_2) \in [-1, 1] \times [-1, 1] \\ 0, & \text{if } (x_1, x_2) \notin [-1, 1] \times [-1, 1] \end{cases}$$

We first derive the marginal density of $f(x_1, x_2)$, denoted by f_{x_1} and f_{x_2}.
For f_{x_1}:
If $x_1 \in [-1, 1]$, $f_{x_1} = \int_{-1}^{1} \frac{1}{4}(1 + x_1x_2)dx_2 = \frac{1}{2}$.
If $x_1 \notin [-1, 1]$, $f_{x_1} = 0$
Therefore we know:

$$f_{x_1} = \begin{cases} \frac{1}{2}, & \text{if } x_1 \in [-1, 1] \\ 0, & \text{Otherwise} \end{cases}$$

By symmetry, we know

$$f_{x_2} = \begin{cases} \frac{1}{2}, & \text{if } x_2 \in [-1, 1] \\ 0, & \text{Otherwise} \end{cases}$$

For an ideal $g(x_1, x_2)$, we want $g_{x_1} = f_{x_1}$ and $g_{x_2} = f_{x_2}$. Notice that these two marginal densities with constants look like those of a uniform distribution on $[-1, 1]$. We also know a bivariate distribution of two uniform random variables has uniform marginal densities.

Therefore, a joint distribution with joint density $g(x_1, x_2)$ that satisfies our need is

$$g(x_1, x_2) = \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2) \in [-1, 1] \times [-1, 1] \\ 0, & \text{if } (x_1, x_2) \notin [-1, 1] \times [-1, 1] \end{cases}$$

We can verify that for g_{x_1} :
If $x_1 \in [-1, 1]$, $g_{x_1} = \int_{-1}^{1} \frac{1}{4} dx_2 = \frac{1}{2}$.
If $x_1 \notin [-1, 1]$, $g_{x_1} = 0$
Therefore we know:

$$g_{x_1} = \begin{cases} \frac{1}{2}, & \text{if } x_1 \in [-1, 1] \\ 0, & \text{Otherwise} \end{cases}$$

By symmetry, we know

$$g_{x_2} = \begin{cases} \frac{1}{2}, & \text{if } x_2 \in [-1, 1] \\ 0, & \text{Otherwise} \end{cases}$$

We also know $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{x_1,x_2}(x_1, x_2) = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} dx_1 dx_2 = 1$ and $g_{x_1,x_2}(x_1, x_2) \geq 0$.
Then this joint distribution with joint density $g(x_1, x_2) = \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2) \in [-1, 1] \times [-1, 1] \\ 0, & \text{if } (x_1, x_2) \notin [-1, 1] \times [-1, 1] \end{cases}$ is what we want.

Question 2

We know the discrete random variable X with probability mass function

$$P(X = 2^n) = \frac{1}{en!}, n = 0, 1, 2...$$

2. (a)

The rth moment of random variable X can be written as

$$E(X^r) = \sum_{n=0}^{\infty} (2^n)^r \frac{1}{en!} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(2^r)^n}{n!} = e^{2^r - 1}$$

We know r is arbitrarily picked and finite. So we proved that this random variable has moments of all orders and derived their formula. (Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

2. (b)

We know the characteristic function of this random variable is:

$$\phi_X(t) = E[e^{itX}]$$

We also know $e^{itX} = \sum_{n=0}^{\infty} \frac{(itX)^n}{n!}$ So characteristic function of this random variable is:

$$\phi_X(t) = E(e^{itX}) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} e^{2^n - 1}$$

2. (c)

We know the moment generating function of this random variable is:

$$M_X(t) = E[e^{tX}]$$

We know $e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$ So

$$E(e^{tX}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{2^n - 1}$$

But we have to check if moment generating function exists. By definition,

$$M_X(t) = E(e^{tX}) \quad \forall t \in \{t \in \mathbb{R} | -h \leq t \leq h\}, \;\; h > 0$$

We know if there is an $h > 0$ such that for all t in $-h \leq t \leq h$, $E[e^{tX}]$ exists, then moment generating function exists. In other words, we need $E[e^{tX}]$ to be finite for all real values of t is a subset of real line that includes zero but not only zero.

Then we carry out **Ratio Test** (**D'Alambert's criterion**) to see if a series converges. For sequence $\{a_n\} = \frac{t^n}{n!}e^{2^n-1}$, we know $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{e^{2^n}|t|}{n+1}$. We know exponential growth grows much faster than linear growth when n is very large. So we know $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{e^{2^n}|t|}{n+1} = \infty$. Therefore moment generating function does not exist for any $t \neq 0$.

Question 3

We know X_1 and X_2 are independent $N(0,1)$ random variables and we know:

$$\mathbf{Y} = (Y_1, Y_2) = \begin{cases} (X_1, |X_2|), \text{if } X_1 \geq 0 \\ (X_1, -|X_2|), \text{if } X_1 < 0 \end{cases}$$

(a)

We know both X_1 and X_2 are $N(0,1)$, then the pdf of these two random variables are $\begin{cases} f_{X_1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2} \\ f_{X_2}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2} \end{cases}$

Note: Symmetry of standard normal distribution can be veryhelpful here. $F(-z) = 1 - F(z)$.
We want to derive the marginal distributions of (Y_1, Y_2)
For Y_1, $F_{Y_1} = \lim_{y_2 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2) = F_{X_1}(y_1)$, which is just $N(0,1)$.
For Y_2, $F_{Y_2} = \lim_{y_1 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2)$, we need to consider two cases now:

- If $y_2 \geq 0$,

$$\lim_{y_1 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2) = P(X_1 \geq 0)P(Y_2 \leq y_2 | X_1 \geq 0) + P(X_1 < 0)P(Y_2 \leq y_2 | X_1 < 0)$$
$$= \frac{1}{2}P(|X_2| \leq y_2) + \frac{1}{2}P(-|X_2| \leq y_2) = \frac{1}{2}(F_{X_2}(y_2) - F_{X_2}(-y_2)) + \frac{1}{2} = \frac{1}{2}(F_{X_2}(y_2) - (1 - F_{X_2}(y_2))) + \frac{1}{2} = F_{X_2}(y_2)$$

- If $y_2 < 0$,

$$\lim_{y_1 \to \infty} P(Y_1 \leq y_1, Y_2 \leq y_2) = P(X_1 \geq 0)P(Y_2 \leq y_2 | X_1 \geq 0) + P(X_1 < 0)P(Y_2 \leq y_2 | X_1 < 0)$$
$$= 0 + \frac{1}{2}P(-|X_2| \leq y_2) = \frac{1}{2}(P(X_2 \geq -y_2) + P(X_2 \leq y_2)) = \frac{1}{2}[1 - F_{X_2}(-y_2) + F_{X_2}(y_2))] = \frac{1}{2}[2F_{X_2}(y_2)] = F_{X_2}(y_2)$$

In both cases, we see $F_{Y_2} = F_{X_2}(y_2)$, which means it is also a $N(0,1)$.

(b)

(Y_1, Y_2) cannot be a jointly normally distributed although the marginal distribution of both Y_1, Y_2 are $N(0,1)$. Suppose (Y_1, Y_2) are jointly normally distributed but neither perfectly correlated nor independent. In other words, there is a correlation coefficient between X and Y: $\rho \in (0,1)$. It has to be that $P(Y_1 \leq 0, Y_2 > 0) \neq 0$. Basically, you can get a sense of that by looking at the pdf of standard bivariate normal distribution with correlation coefficient ρ. The joint PDF is

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\{-\frac{1}{2(1-\rho^2)}[x^2 - 2\rho xy + y^2]\}$$

A formal discussion can be found here. The probability is $P(Y_1 \leq 0, Y_2 > 0) = \frac{cos^{-1}\rho}{2\pi}$. But following our definition, we have $P(Y_1 \leq 0, Y_2 > 0) = 0$. Then it is a contradiction and (Y_1, Y_2) are not jointly normally distributed.

Question 4

For function:

$$\phi(t) = \begin{cases} 1-t^2, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

Since we know for a continuous function $\phi(x)$ with $\phi(0) = 1$ is a characteristic function iff it is positive semi-definite, i.e. for any $t_1, ..., t_n \in \mathbb{R}$ and any $\lambda_1, ..., \lambda_n \in \mathbb{C}$

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k\lambda_j^* \geq 0$$

Here, λ_j^* is the complex conjugate of λ_j, when you simply change the sign of the imaginary part of the complex number. In other words, to prove it is not a characteristic function, we need to find a group of λ and t that makes the inequality above fail.
We take three points of t:

$$\begin{cases} t_1 = -\frac{1}{2} \\ t_2 = 0 \\ t_3 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} \phi(t_1) = \frac{3}{4} \\ \phi(t_2) = 1 \\ \phi(t_3) = \frac{3}{4} \end{cases}$$

Then we know a bunch of $\phi(t_k - t_j)$

$$\begin{cases} \phi(t_1 - t_2) = \frac{3}{4} \\ \phi(t_1 - t_3) = 0 \\ \phi(t_2 - t_3) = \frac{3}{4} \\ \phi(t_2 - t_1) = \frac{3}{4} \\ \phi(t_3 - t_1) = 0 \\ \phi(t_3 - t_2) = \frac{3}{4} \\ \phi(t_1 - t_1) = \phi(t_2 - t_2) = \phi(t_3 - t_3) = 1 \end{cases}$$

$$\Rightarrow \sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k\lambda_j^* = \frac{3}{4}\lambda_1\lambda_2^* + \frac{3}{4}\lambda_2\lambda_3^* + \frac{3}{4}\lambda_2\lambda_1^* + \frac{3}{4}\lambda_3\lambda_2^* + \lambda_1\lambda_1^* + \lambda_2\lambda_2^* + \lambda_3\lambda_3^*$$

Let:

$$\begin{cases} \lambda_1 = a_1 + b_1 i \\ \lambda_1^* = a_1 - b_1 i \end{cases} \begin{cases} \lambda_2 = a_2 + b_2 i \\ \lambda_2^* = a_2 - b_2 i \end{cases} \begin{cases} \lambda_3 = a_3 + b_3 i \\ \lambda_3^* = a_3 - b_3 i \end{cases}$$

Then we plug in the equation above and we will get:

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k \lambda_j^* = (a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2) + \frac{3}{2}(a_1a_2 + b_1b_2 + a_2a_3 + b_2b_3)$$
$$= (a_1 + a_2)^2 + (b_1 + b_2)^2 + (a_2 + a_3)^2 + (b_2 + b_3)^2 - (a_2^2 + b_2^2) - \frac{1}{2}(a_1a_2 + b_1b_2 + a_2a_3 + b_2b_3)$$

Specifically, we can pick:

$$a_1 = b_1 = a_3 = b_3 = 0.9$$
$$a_2 = b_2 = -1$$

Then the equation above will be

$$\sum_{k,j=1}^{n} \phi(t_k - t_j)\lambda_k \lambda_j^* = 4 * 0.1^2 - 2 - \frac{1}{2} * 4 * (-0.9) = -0.16 < 0$$

Then we found a specific groups of t_1, t_2, t_3 and $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_1^*, \lambda_2^*, \lambda_3^*$ that breaks the inequality. So we proved $\phi(t)$ cannot be a characteristic function.