Nov 20, 2023

$\mathcal{P} = \{P_\theta, \theta \in \Theta\}$

$X \sim P_\theta, \quad P_\theta \in \mathcal{P}$

$\mathcal{P} = H_0 \cup H_1$

$\Theta = \Theta_0 \cup \Theta_1$

$x \longrightarrow \{0, 1\}$ {we need to make a decision if x gets mapped to 0 or 1}

If $x \in \mathcal{X}$

$\delta : \mathcal{X} \mapsto \{0, 1\}$

We need to commit to δ ex-ante: If we observe some value of x, it gets mapped to what. (Pre-commit to decision)

1 – rejection decision

0 – acceptance / non-rejection

* Probability of making the rejection decision: (as a function of θ)

POWER

$\beta(\theta) = P_\theta(\delta(X) = 1)$ is the power function of the test $\delta(\cdot)$

(test → decision rule.)

(Rule that is optimal for some θs would not be for others.)

The size of test $\delta(\cdot)$ is $\sup_{\theta \in \Theta_0} P_\theta(\delta(X) = 1)$

The worst case value of

(Probability of incorrectly rejecting the null hypothesis if it is true.)

The significance level of test $\delta(\cdot)$ is, $\alpha \in [0,1]$ s.t.
$P_\theta(\delta(x)=1) \leq \alpha \quad \forall\, \theta \in \Theta_0$.

$*$ $\sup_{\delta} \inf_{\theta \in \Theta_1} P_\theta(\delta(x)=1)$ (label them as bad when they are bad)

Optimization problem $P_\theta(\delta(x)=1) \leq \alpha, \ \forall\, \theta \in \Theta_0$

[we add this constraint so that bad are classified as bad, but not all good are classified as bad too)

$\to$ Optimization Problem

Bernoulli Ex.

$\Theta_0 = \{p_0\}, \quad \Theta_1 = \{p_1\}$; WLOG $p_1 > p_0$

$\delta_1(x) = x$
$\delta_2(x) = 1-x$
$\delta_3(x) = 0$
$\delta_4(x) = 1$

Decision Rule	$\beta(p^0)$	$\beta(p^1)$
1	p_0	p_1
2	$1-p_0$	$1-p_1$
3	0	0
4	1	1

* If $\alpha < \min\{p_0, 1-p_0\}$, the optimum is a unique decision rule → decision rule #3.
  then $\delta^*(x) = 0 \Rightarrow$ Always rejecting

Stochastic Test:

Define the problem differently (moving away from ML)

Binary classifier → distribution over a classifier

* **Stochastic Test:**
  $\phi(x) : \mathcal{X} \longmapsto [0, 1]$

- Incorporates the deterministic cases we saw & much more.
- More useful in discrete settings
- In continuous settings (when rv are continuous) → deterministic tests work better.

* Power function of stochastic test:

$$\beta(\theta) = E_\theta[\phi(X)]$$

$$\sup_{\phi} \inf_{\theta \in \Theta_1} \beta_\phi(\theta)$$

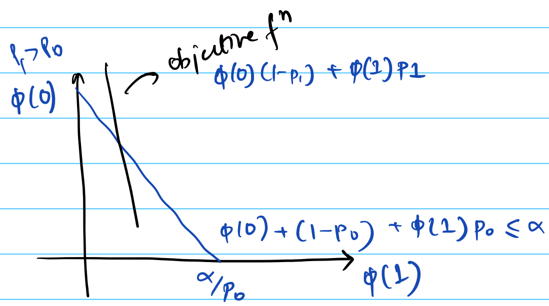
$$\text{s.t. } \beta_\phi(\theta) \leq \alpha, \quad \theta \in \Theta_0$$

$\Theta_0 = \{p_0\}$, $\Theta_1 = \{p_1\}$ → rv

$\phi(0)$, $\phi(1)$

$$\max_{\phi(0),\,\phi(1)} \ (1-p_1)\,\phi(0) + p_1\,\phi(1)$$

$$(1-p_0)\,\phi(0) + p_0\,\phi(1) \le \alpha$$



If $\frac{\alpha}{p_0} > 1 \Rightarrow$ not feasible (cannot reject with p greater than 1).

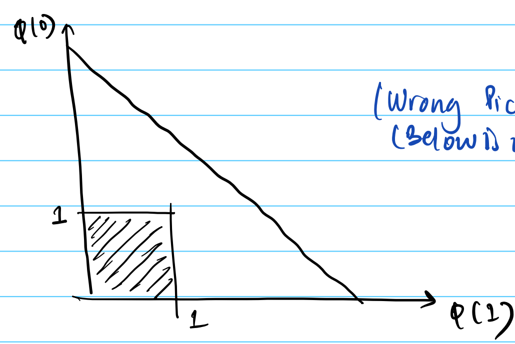
If $\frac{\alpha}{p_0} < 1$, then $\phi(1) = 1$
$\phi(0) = 0$

If $\frac{\alpha}{p_0} > 1$,

In general, additional constraints:—

$0 \le \phi(0) \le 1$, $0 \le \phi(1) \le 1$

making the initial constraint garbage/trivially satisfied.



(Wrong Picture)
(Below is correct)

Checking the boundary pts:

$\phi(1) = 1$, $(1-p_0)\,\phi(0) \le \alpha - p_0 \Rightarrow \phi(0) = \frac{\alpha - p_0}{1-p_0} \le 1$

When $\frac{\alpha}{p_0} > 1$: