Econ 70010 - Microeconomics I
University of Virginia
Fall 2023

Problem Set 1
Due Friday, September 8th

1. Review the section on *Properties of Production Sets* starting on pg.130 of MWG. Draw a production set for which all of the following hold:

   (a) No free lunch is satisfied

   (b) Irreversibility is not satisfied (i.e. production is reversible) over some regions of the production process.

   (c) The production set is not everywhere closed, and

   (d) The possibility of inaction is violated.

   (e) The production set exhibits nondecreasing returns to scale.

2. A production set Y is *additive* if $y, y' \in Y$ implies that $y + y' \in Y$.

   (a) Give a brief description in words of what this condition means economically.

   (b) Give two examples of single-input, single output production functions, one of which satisfies additivity, and one of which does not.

   (c) Let Y be a general (multi-input, multi-output) production set that exhibits *nonincreasing returns to scale* (i.e., for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha \in [0, 1]$). Show that if in addition Y is additive, then Y is convex and exhibits constant returns to scale.

3. Consider a single-output production function $f(x)$, and let Y be the associated production set: $Y = \{(-x, y) : x \geq 0 \text{ and } y \leq f(x)\}$. In class, we gave the following two definitions of *non-increasing returns to scale*:

   (a) $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [0, 1]$

   (b) $f(tx) \leq tf(x)$ for all $t \geq 1$

Prove formally that definitions (a) and (b) are equivalent; that is, show that (a) implies (b), and (b) implies (a). [Hint: consider $\alpha = \frac{1}{t}$. For extra practice, you can also show that the two definitions of non-decreasing returns to scale and and constant returns to scale are also equivalent.]

4. For each of the following production functions (i) calculate the marginal rate of technical substation and sketch several isoquants (a very small, very rough picture is fine), (ii) determine whether the production function exhibits increasing, constant, or diminishing returns to scale, and (iii) derive the expression for the elasticity of substitution:

   (a) $f(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ where $\alpha \in (0, 1)$. [Cobb-Douglas]

   (b) $f(x_1, x_2) = [\alpha x_1^{\rho} + (1 - \alpha) x_2^{\rho}]^{\frac{1}{\rho}}$ for $\alpha \in (0, 1)$ [CES]. Draw isoquants for three separate values of $\rho = 0, 1, -\infty$. What more common production functions do these special cases correspond to?

   (c) $f(x_1, x_2) = (x_1 + x_2)^{\alpha}$ where $\alpha > 0$.

   (d) $f(x_1, x_2) = \max\{x_1, 2x_2\}$.

5. (MWG 5.B.3) Show that for a single-output technology with free disposal, Y is convex if and only if the production function $f(z)$ is concave. Determine which parts of this equivalence, if any, still hold if free disposal is violated.

6. Consider the following production function: $f(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ where $\alpha \in (0, 1)$. [Cobb-Douglas]

   (a) Formulate the profit maximizing problem when the price per unit is p and the factor prices are w_1 and w_2 and derive the profit maximizing first order conditions.

   (b) Now suppose that the pair x_1, x_2 solves the first order conditions. Multiply the first order condition for factor i by x_i for $i \in \{1, 2\}$. Add the two conditions. What does this tell you about profit at the profit maximizing pair, x_1, x_2?

   (c) Finally, define the output $y = x_1^{\alpha} x_2^{1-\alpha}$. Show using the first order conditions that it is possible to express the factor demands for x_1 and x_2 as a function of y, w_1, and w_2 (the output price p should *not* be a part of these functions). Using these factor demands, write down the profit maximization problem where the choice variable is now the level of output y. Let $\alpha = 1$ and $w_1 = w_2 = 1$. For which prices p of the output good is there a solution the profit maximization problem? If you find such a price, what level(s) of output maximize profit?

(d) Think about your answer to (c). Which of Varian's "Difficulties" of profit maximization does the Cobb-Douglas production function face?