Lecture #1: AD economy ~~m~~ and Exchange economy

Aug 24, 2023

Simple Dynamic exchange economy

- → 2 agents
- → live indefinitely
- → production
- → Endowments : agents 1 : $\{2, 0, 2, 0, \dots\}$
  agent 2 : $\{0, 2, 0, 2, \dots\}$
- → Preferences $u(c_i) = \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$
- → Perfect information.

Arrow Debreu Comp. Eq.

- → Competitive: price as given
- → Market structure: Agents meet at $t=0$ and trade consumption claims for every future t.

Def: AD Comp Eq is the price system $\{\hat{p}_t\}_{t=0}^{\infty}$ and an allocation $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$, s.t:

1) Given $\{\hat{p}_t\}_{t=0}^{\infty}$, the allocation solves agents' max problem

$$\max_{\{c_i\}} \sum_{t=0}^{\infty} \beta^t \log(c_t^i) \quad \text{s.t} \quad \underbrace{\sum_{t=1}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i}_{\text{Budget Constraint}}$$

2) Market clear in each period t

$$c_t^1 + c_t^2 = \underbrace{e_t^1 + e_t^2}_{\text{Resource constraint}} \quad \forall t$$

Characterize Equilibrium.

i) Lagrangian: $\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log(c_t^i) + \lambda^i \left[\sum_{t=0}^{\infty} p_t c_t^i - \sum_{t=0}^{\infty} p_t e_t^i \right]$

F.O.C) wrt c_t^i, c_{t+1}^i.

(c_t^i) $\frac{\beta^t}{c_t^i} + \lambda^i p_t = 0$ (1) | (c_{t+1}^i) $\frac{\beta^{t+1}}{c_{t+1}^i} + \lambda^i p_{t+1} = 0$ (2)

From (1) & (2): $\frac{\beta^t}{c_t^i p_t} = \frac{\beta^{t+1}}{c_{t+1}^i p_{t+1}}$. $\Rightarrow$ $\boxed{\beta^t p_{t+1} c_{t+1} = \beta^{t+1} p_t c_t}$ (3)

Adding (3) across agents: $\beta^t p_{t+1}(c_{t+1}^1 + c_{t+1}^2) = \beta^{t+1} p_t (c_t^1 + c_t^2)$

Invoke market clearing: $c_t^1 + c_t^2 = e_t^1 + e_t^2$

$$\beta^t p_{t+1} \underbrace{(e_{t+1}^1 + e_{t+1}^2)}_{2,\ \forall t} = \beta^{t+1} p_t \underbrace{(e_t^1 + e_t^2)}_{2,\ \forall t}$$

$\Leftrightarrow$ $2\beta^t p_{t+1} = 2\beta^{t+1} p_t$ $\quad / \frac{1}{2\beta^t}$

$\boxed{p_{t+1} = \beta p_t}$

Note $p_{t+1} = \beta\beta p_{t-1} = \beta\beta\beta p_{t-2} = \ldots = \beta^t p_0$

$\rightarrow p_{t+1} = \beta^t p_0$ (we can normalize p_0)

* Suppose we normalize the entire sequence of prices by a constant: z
$\sum^{\infty} z\hat{p}c \leq \sum^{\infty} z\hat{p}e \rightarrow \sum^{\infty} \hat{p}c \leq \sum^{\infty} \hat{p}e$

$\Rightarrow$ Doesn't affect the F.O.C

Invoke Budget Constraint: $\sum p_t c_t^i = \sum p_t e_t^i \Leftrightarrow \sum \beta^t c_t^i = \sum \beta^t e_t^i$

plug $p_t = \beta^t$ in (3): $\beta c_t^i \beta^t = c_{t+1}^i \beta^{t+1} \Leftrightarrow \boxed{c_t^i = c_{t+1}^i = c^i}$

Lecture #2:

Aug 29, 2023

→ Pareto optimality
→ Negishi method
→ Sequential markets eq.

Arrow Debreu Eq (ADE)

→ Agents 1 & 2
→ $e^1 = \{2, 0, 2, 0, \dots\}$
→ $e^2 = \{0, 2, 0, 2, \dots\}$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \right]$$

F.O.C

$$\frac{\partial \mathcal{L}}{\partial c_t^i} = \frac{\beta^t}{c_t^i} + \lambda^i p_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}^i} = \frac{\beta^{t+1}}{c_{t+1}^i} + \lambda^i p_{t+1} = 0$$

$\Rightarrow \beta c_t^i p_t = c_{t+1}^i p_{t+1}$

aggregating.

$$\beta (\underbrace{c_t^1 + c_t^2}_{2 \ \forall t}) p_t = p_{t+1} (\underbrace{c_{t+1}^1 + c_{t+1}^2}_{2 \ \forall t})$$

$$\beta p_t = p_{t+1}.$$

$$\Rightarrow p_t = \beta^t p_0$$

we can normalize $p_0 = 1$.

$$p_t = \beta^t$$

Budget Constraint Agent 1.

$$\sum p_t c_t^1 = \sum p_t e_t^1$$

from $\beta c_t^1 p_t = c_{t+1}^1 p_{t+1}$.

$$\beta c_t^1 \beta^t = c_{t+1}^1 \beta^{t+1}$$

$\rightarrow \boxed{c_t^1 = c_{t+1}^1}$ constant consumption.

Budget Agent 1

$$\underbrace{\sum \beta^t c_0^1}_{\text{geometric series.}} = \underbrace{\sum \beta^t e_t^1}_{2+0+2\beta^2+0+2\beta^4+\dots} = \frac{2}{1-\beta^2}$$

constant.

$$\frac{c_0^1}{1-\beta} = \frac{2}{1-\beta^2}$$

$$\boxed{c_0^1 = \frac{2}{1+\beta}}$$

Budget Agent 2

$$\sum \beta^t c_0^2 = \underbrace{\sum \beta^t e_t^2}_{0+2\beta+0+2\beta^3+0+2\beta^5+\dots}$$

$$\frac{c_0^2}{1-\beta} = \frac{2\beta}{1-\beta^2}$$

$$\boxed{c_0^2 = \frac{2\beta}{1+\beta}}$$

What agent consumes more?
→ agent 1 because he is rich earlier.

Pareto optimality

Def: An allocation is said to be said to be Pareto Optimal (or efficient) if there is no other feasible allocation which makes everyone no worse off, and at least one agent strictly better off.

In the context of the exchange economy $\{\hat{c}^1, \hat{c}^2\}$ is P.O. if $\nexists$ feasible $\{\tilde{c}^1, \tilde{c}^2\}$, s.t

$$u_i(\tilde{c}^i) \geq u_i(\hat{c}^i) \quad \forall i$$
$$u_j(\tilde{c}^j) > u_j(\hat{c}^j) \quad \exists j \qquad i \neq j$$

The First Welfare Theorem: The competitive eq is P.O.

Proof: By Contradiction.

Suppose not. Then $\exists$ feasible allocation $\{\tilde{c}^1, \tilde{c}^2\}$, such that

$$u_1(\tilde{c}^1) > u_1(\hat{c}^1)$$
$$u_2(\tilde{c}^2) \geq u_2(\hat{c}^2)$$

For agent 1 it follows that.

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

If not, then $\sum \hat{p}_t \tilde{c}_t^1 \leq \sum \hat{p} \hat{c}_t^1$

$\Rightarrow \tilde{c}$ is affordable, and not chosen by agent 1, and delivers strictly higher utility $\Rightarrow$ ⚡ with $\hat{c}$ being a CE.

For agent 2: $\sum \hat{p}_t \tilde{c}_t^2 \geqslant \sum \hat{p}_t \hat{c}_t^2$

if not, then

$$\sum \hat{p}_t \tilde{c}_t^2 < \sum \hat{p}_t \hat{c}_t^2$$

So agent can increase consumption slightly in (at least) one of the dates, and this will strictly increase utility $\Rightarrow$ W with $\hat{c}$ being a C.E. (contradiction).

$$\begin{cases} \sum \hat{p}_t \tilde{c}_t^1 > \sum \hat{p}_t \hat{c}_t^1 \\ \sum \hat{p}_t \tilde{c}_t^2 \geqslant \sum \hat{p}_t \hat{c}_t^2 \end{cases}$$

$$\rightarrow \sum \hat{p}_t \big(\underbrace{\tilde{c}_t^1 + \tilde{c}_t^2}_{=e_t^1 + e_t^2}\big) > \sum \hat{p}_t \big(\underbrace{\hat{c}_t^1 + \hat{c}_t^2}_{=e_t^1 + e_t^2}\big)$$

$$\sum \hat{p}_t (e_t^1 + e_t^2) > \sum \hat{p}_t (e_t^1 + e_t^2) \quad \text{W}$$

Negishi's Method to compute C.E.

Algorithm:

→ Solve for the Social Planner's Problem

→ Pick the PO allocation which is.

Social Planner's Problem.

→ Solve $\max\limits_{\{c^1_t, c^2_t\}} \alpha U_1 + (1-\alpha)U_2$, $\alpha \in [0,1]$

s.t $c^1_t + c^2_t = e^1_t + e^2_t$

Solution is $(\tilde{c}^1, \tilde{c}^2) = (\tilde{c}^1(\alpha), \tilde{c}^2(\alpha))$

$$\mathcal{L} = \alpha \sum_{t=0}^{\infty} \beta^t \log(c^1_t) + (1-\alpha)\sum_{t=0}^{\infty} \beta^t \log(c^2_t) + \sum_{t=0}^{\infty} \lambda_t \left[e^1_t + e^2_t - c^1_t - c^2_t\right]$$

F.O.C

$$\frac{\partial \mathcal{L}}{\partial c^1_t} = \frac{\alpha \beta^t}{c^1_t} - \lambda_t = 0 \quad (1) \; ; \quad \frac{\partial \mathcal{L}}{\partial c^1_{t+1}} = \frac{\alpha \beta^{t+1}}{c^1_{t+1}} - \lambda_{t+1} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial c^2_t} = \frac{(1-\alpha)\beta^t}{c^2_t} - \lambda_t = 0 \quad (3) \; ; \quad \frac{\partial \mathcal{L}}{\partial c^2_{t+1}} = \frac{(1-\alpha)\beta^{t+1}}{c^2_{t+1}} - \lambda_{t+1} = 0 \quad (4)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = e^1_t + e^2_t = c^1_t + c^2_t \quad (5)$$

(1) & (3) $\quad \dfrac{\alpha\beta^t}{c^1_t} = \dfrac{(1-\alpha)\beta^t}{c^2_t} \Rightarrow \alpha c^2_t = (1-\alpha)c^1_t$

using R.C. $\quad e_1 + e_2 = c^1_t + c^2_t$

$$e_1 + e_2 = c^1_t + \frac{(1-\alpha)c^1_t}{\alpha} = 2$$

$$\Leftrightarrow c^1_t\left(1 + \frac{1-\alpha}{\alpha}\right) = 2$$

$$\Leftrightarrow \frac{c^1_t}{\alpha} = 2 \Leftrightarrow \boxed{c^1_t = 2\alpha}$$

Lecture: Aug 31, 2023

→ Secuencial market equilibrium
→ Equivalence between ADE and SME

Social Planner's Problem

$* \lim_{c \to 0} \log c = -\infty$

$$\sum_{t=0}^{\infty} \beta^t \left(\alpha \log c_t^1 + (1+\alpha) \log c_t^2\right) \rightarrow \max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}}$$

$$\text{s.t} \quad c_t^1 + c_t^2 \leq e_t^1 + e_t^2$$
$$c_t^1 \geq 0$$
$$c_t^2 \geq 0$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\alpha \log c_t^1 + (1+\alpha) \log c_t^2\right) + \sum_{t=0}^{\infty} \mu_t \left[e_t^1 + e_t^2 - c_t^1 - c_t^2\right] + \sum_{t=0}^{\infty} \lambda_t c_t^1 + \sum_{t=0}^{\infty} \gamma_t c_t^2$$

F.O.C

$$\partial \mathcal{L} / \partial c_t^1 = \frac{\alpha \beta^t}{c_t^1} - \mu_t + \lambda_t = 0$$

$$\partial \mathcal{L} / \partial c_t^2 = \frac{(1+\alpha)\beta^t}{c_t^2} - \mu_t + \gamma_t = 0$$

$$(\mu_t): \; e_t^1 + e_t^2 = c_t^1 + c_t^2$$

$$\mu_t \left(e_t^1 + e_t^2 - c_t^1 - c_t^2\right) = 0$$
$$\lambda_t c_t^1 = 0$$
$$\gamma_t c_t^2 = 0 \qquad \mu_t, \lambda_t, \gamma_t \geq 0$$

We already know that $c_t^1, c_t^2 > 0$, so $\lambda_t = \gamma_t = 0$.

⇒ and we get the last class problem.

$$\boxed{\text{sol}: \left\{(c_t^1, c_t^2)\right\}_{t=0}^{\infty} = \left\{2\alpha, 2(1-\alpha), \alpha \in [0,1]\right\}}$$

Social planner's problem

Pick P.O. allocation which is affordable to agents, if the prices they faced were equal to Lagrange multiplier μ_t from S.P.P.

Define transfer: $t^i(\alpha) = \sum_{t=0}^{\infty} \mu_t \left(c_t^i(\alpha) - e_t^i \right)$ (lump sum transfer).

CE equilibrium:
(no transfers ; $t(\alpha)=0$)

$$\frac{\beta^t \alpha}{c_t^1} = \mu \xrightarrow{\text{using SPP}} \frac{\beta^t \alpha}{2\alpha} = \boxed{\mu_t = \frac{\beta^t}{2}}$$

the transfer for agent 1:
$$t^1(\alpha) = \sum_{t=0}^{\infty} \frac{\beta^t}{2} \left(c_t^1(\alpha) - e_t^1 \right)$$

$$= \sum_{t}^{\infty} \frac{\beta^t}{2}(2\alpha) - \sum_{t}^{\infty} \frac{\beta^t}{2} e_t^1 = \frac{\alpha}{1-\beta} - \frac{1}{2} \cdot \frac{2}{(1-\beta^2)}$$

$$= \frac{\alpha}{1-\beta} - \frac{1}{1-\beta^2} = 0 \iff \boxed{\alpha = \frac{1}{1+\beta}}$$

α that makes transfer $=0$.
so it is feasible in C.E.

* CE opt

$$c_t^1 = 2\alpha = \frac{2}{1+\beta}$$

$$c_t^2 = 2(1-\alpha) = \frac{2\beta}{1+\beta}$$

Sequential market equilibrium.

* Agents meet and trade 1-period bonds a_t^i
→ Agents 1 & 2, $t = 0, 1, 2, \dots$,
→ Endowments : $e_t^1 = (2, 0, 2, \dots)$; $e_t^2 = (0, 2, 0, \dots)$
→ r_{t+1} – interest rate from t to $t+1$.
* Bond is a promise/contract to pay 1 unit of consumption in $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ consumption today

a_{t+1}^i : # of bonds that agent i buys in t & carries to $t+1$
$a_{t+1}^i < 0$ agent borrows. in t

Budget constraint in SM: $\quad c_t^i + \frac{a_{t+1}^i}{1+r_{t+1}} \leq e_t^i + a_t^i$

Def: A SM eq. is $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$, assets holdings $\{(a_{t+1}^1, a_{t+1}^2)\}_{t=0}^{\infty}$ and interest rates $\{r\}_{t=0}^{\infty}$, such that

1) Given $\{r_{t+1}\}_{t=0}^{\infty}$, $\{(c_t^1, c_t^2, a_t^1, a_t^2)\}_{t=0}^{\infty}$ solves consumers' problems.

$$\sum_{t=0}^{\infty} \beta^t \log(c_i) \to \max_{\{(c_t^i, a_{t+1}^i)\}_{t=0}^{\infty}}$$

$$\text{s.t.} \quad c_t^i + \frac{a_{t+1}^i}{1+r_{t+1}} \leq e_t^i + a_t^i \quad \forall t$$

$$a_{t+1}^i \geq -\bar{A}^i \quad \forall t \quad \text{(No Ponzi Game Condition)}$$

$\bar{A}^i$ is some large positive number.

2) Market clearing $\quad c_t^1 + c_t^2 = e_t^1 + e_t^2 \quad \forall t$

$$\sum_{i=1}^{2} a_{t+1}^i = 0 \quad \forall t$$

Equivalence of ADE & SME

Let $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ and $\{\hat{p}_t\}_{t=0}^{\infty}$ constitute an ADE

Then there is an associated SME $\{\tilde{c}_t^1, \tilde{c}_t^2, \tilde{a}_{t+1}^1, \tilde{a}_{t+1}^2\}_{t=0}^{\infty}$ and $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ such that $\hat{c}_t^i = \tilde{c}_t^i \quad \forall i \, \forall t$

and vice versa — " —

Proof: 3 steps

I) SM budget constraint implies AD budget constraint (if the prices are right)

II) If there is a ADE $\Rightarrow \exists$ SME with the same allocation

III) If there is SME $\Rightarrow \exists$ ADE which supports the same allocation.

Step I: Let $1+\tilde{r}_{t+1} = \dfrac{\hat{p}_t}{\hat{p}_{t+1}}$, $\hat{p}_0 = 1$

$t=0)$ $$c_0^i + \frac{a_1^i}{1+\tilde{r}_1} = e_0^i$$

$t=1)$ $$c_1^i + \frac{a_2^i}{1+\tilde{r}_2} = e_1^i + a_1^i$$

$(\dots)$

Plug: $$c_0^i + \frac{c_1^i}{1+\tilde{r}_1} + \frac{a_2^i}{(1+\tilde{r}_1)(1+\tilde{r}_2)} = e_0^i + \frac{e_1^i}{(1+\tilde{r}_1)}$$

continuing until T:

$$\sum_{t=1}^{T} \frac{c_t^i}{\prod_{j=1}^{t}(1+\tilde{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{t+1}(1+\tilde{r}_j)} = \sum^{T} \frac{e_t^i}{\prod_{j=1}^{t}(1+\tilde{r}_j)}$$

Notice that: $$\prod_{j=1}^{t}(1+\tilde{r}_j) = \frac{\hat{p}_0}{\hat{p}_1} \cdot \frac{\hat{p}_1}{\hat{p}_2} \cdot \ldots \cdot \frac{\hat{p}_{t-1}}{p_t} = \boxed{\frac{1}{\hat{p}_t}}$$

$$\sum^{T} \hat{p}_t c_t^i + \hat{p}_t a_{t+1}^i = \sum^{T} \hat{p}_t e_t^i$$

AD Budget constraint.

Lecture Sept 05, 2023

→ Equivalence of SME and ADE

→ OLG model

Equivalence of SME and ADE

1) SM budget constraint ⇒ AD budget constraint (at certain prices).

$t=0 \qquad c_0^i + \frac{a_1^i}{1+\tilde{r}_t} = e_0^i$

$t=1 \qquad c_1^i + \frac{a_2^i}{1+\tilde{r}_{t+1}} = e_1^i + a_1^i$

$\vdots$

$$c_0^i + \frac{c_1^i}{1+\tilde{r}_t} + \frac{a_2^i}{(1+\tilde{r}_t)^2} = e_0^i + \frac{e_1^i}{(1+\tilde{r}_t)}$$

$$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=0}^{t}(1+\tilde{r}_j)} + \underbrace{\frac{a_{t+1}^i}{\prod_{j=1}^{t+1}(1+\tilde{r}_j)}}_{=\frac{1}{\hat{p}_t}}$$

Let $\hat{p}_0 = 1$, $1+\tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$

$$\prod_{j=1}^{t}(1+\tilde{r}_j) = \frac{\hat{p}_0}{\hat{p}_1}\cdot\frac{\hat{p}_1}{\hat{p}_2}\cdots\frac{\hat{p}_t}{\hat{p}_{t+1}} = \frac{\hat{p}_0}{\hat{p}_{t+1}}.$$

$$= \frac{1}{\hat{p}_{t+1}}$$

$$\Rightarrow \sum^{T} \hat{p}_t c_t^i + \frac{a_{T+1}^i}{\prod_{j=0}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{T} \hat{p}_t e_t^i$$

$$T\to\infty \Rightarrow \sum_{t=0}^{\infty} \hat{p} + c_t^i + \lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t + e_t^i$$

Since we started w/ SM $\rightarrow a_t^i \geq -\tilde{A}^i$

$$\rightarrow \lim_{T\to\infty} \frac{a^i_{T+1}}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} \geq \lim_{T\to\infty} \frac{-\tilde{A}^i}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} = 0$$

<u>step 2</u> If $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ and $\{\hat{p}_t\}_{t=0}^{\infty}$ is ADE

$\Rightarrow \exists \{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}, \{(\tilde{a}_{t+1}^1, \tilde{a}_{t+1}^2)\}_{t=0}^{\infty}$ and $\{\tilde{r}_t\}_{t=0}^{\infty}$, s.t $\hat{c}_t^i = \tilde{c}_t^i \ \forall i \ \forall t.$

a) Resource constraint holds in ADE $\Rightarrow$ it will hold in SME

b) Define

$$\tilde{a}^i_{t+1} = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(\hat{c}^i_{t+\tau} - e^i_{t+\tau})}{\hat{p}_{t+1}}$$

Show that consumption allocations $\hat{c}$ along with asset holdings so define & satisfy SM budget Constraint.

$$\hat{c}^i + \frac{\tilde{a}^i_{t+1}}{1+\tilde{r}_{t+1}} = e^i_t + \tilde{a}^i_t$$

$$\Rightarrow \hat{c}^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(\hat{c}^i_{t+\tau} - e^i_{t+\tau})}{\hat{p}_{t+1}(1+\tilde{r}_{t+1})} \overset{?}{=} e^i_t + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau+1}(\hat{c}^i_{t+\tau+1} - e^i_{t+\tau+1})}{\hat{p}_t}$$

$$(1+\tilde{r}_{t+1}) = \frac{\hat{p}_t}{\hat{p}_{t+1}}$$

$$\hat{c}^i + \underbrace{\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}(\hat{c}^i_{t+\tau} - e^i_{t+\tau})\cancel{\hat{p}_{t+1}}}{\cancel{\hat{p}_{t+1}}\ \hat{p}_t}}_{\frac{\hat{p}_{t+1}(\hat{c}^i_{t+1} - e^i_{t+1})}{\hat{p}_t} + \frac{\hat{p}_{t+2}(\hat{c}^i_{t+2} - e^i_{t+2})}{\hat{p}_t} + \cdots} = e^i_t + \underbrace{\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau+1}(\hat{c}^i_{t+\tau+1} - e^i_{t+\tau+1})}{\hat{p}_t}}_{\frac{\hat{p}_{t+2}(\hat{c}^i_{t+2} - e^i_{t+2})}{\hat{p}_t} + \frac{\hat{p}_{t+3}(\hat{c}^i_{t+3} - e^i_{t+3})}{\hat{p}_t}}$$

$$\Rightarrow \quad \hat{c}_t^i = e_t^i + \frac{\hat{p}_t\left(\hat{c}_t^i - \hat{e}_t^i\right)}{\hat{p}_t}$$

$$\Leftrightarrow \quad \hat{c}_t^i = e_t^i + \left(\hat{c}_t - \hat{e}_t^i\right) \checkmark$$

Show No Ponzi Scheme.

$$\hat{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_t} \geq \sum_{\tau=1}^{\infty} \underbrace{\frac{\hat{p}_{t+\tau}}{\hat{p}_t}}_{\text{if prices do not grow w/o limit.}} \overbrace{\left(-e_{t+\tau}^i\right)}^{\text{if they are bounded}} > -\infty$$

Wrapping all

Resource constraint ✓

Budget constraint ✓

NPG ✓

Is $\hat{c}$ maximizing utility of agent in SME?

suppose not. $\rightarrow \exists$ another affordable allocation which delivers strictly higher utility.

From step 1, we know that SM BC $\rightarrow$ AD BC

So that alternative allocation would be affordable in AD world, but it was not chosen (X) $\nexists$ such allocation.

Overlapping generations model

- Time is infinite $t = 1, 2, \ldots$
- Every period there are 2 generations living simultaneously: Young & Old.
- Endowment is (e^t_t, e^t_{t+1}).
- Consumption of generation t is (c^t_t, c^t_{t+1}).

gen \ Period	1	2	3
0	(c^0_1, e^0_1)		
1	(c^1_1, e^1_1)	(c^1_2, e^1_2)	
2		(c^2_2, e^2_2)	(c^2_3, e^2_3)
3			$\ldots$

Def: An allocation is consumption for the initially old and compsumption (c^t_t, c^t_{t+1}) for all gen $t \geq 1$.

Agents derive utility from $u_t(c^t_t, c^t_{t+1}) = U(c^t_t) + \beta U(c^t_{t+1})$

Def: An allocation $c^0_1, \{c^t_t, c^t_{t+1}\}^{\infty}_{t=1}$ is feasible if

$$c^{t-1}_t + c^t_t = e^{t-1}_t + e^t_t \quad \forall t.$$

Def: An allocation $\hat{c}^0_1, \{\hat{c}^t_t, \hat{c}^t_{t+1}\}^{\infty}_{t=1}$ is Pareto Efficient if it is feasible an there is no other feasible allocation that increases the general welfare.

$u_t(\tilde{c}^t_t, \tilde{c}^t_{t+1}) \geq u_t(\hat{c}^t_t, \hat{c}^t_{t+1}) \quad \forall t \quad u(\tilde{c}^0_1) \geq u(\hat{c}^0_1)$

with at least one inequality being strict.

Lecture:

Sept 07, 2023

→ OLG model

→ ADE and SME in

ADE of a OLG model

Def: An ADE is the price system $\{P_t\}_{t=0}^{\infty}$ and an allocation c^0_1, $\{c^t_t, c^t_{t+1}\}_{t=1}^{\infty}$, s.t.

1) For the initially old, given p_1, c^0_1 solves.

$$u_0(c^0_1) \quad \max$$

$$\text{s.t.} \quad p_1 c^0_1 \leq p_1 e^0_1 + m$$

$$c^0_1 \geq 0$$

← outside money "fiat money"

* Remark: w/o m equilibrium is autarky.

2) Given the prices, allocation c^t_t, c^t_{t+1} solves the generation's t problem:

$$u(c^t_t) + \beta u(c^t_{t+1}) \rightarrow \max_{c^t_t, c^t_{t+1}}$$

s.t

$$p_t c^t_t + p_{t+1} c^t_{t+1} \leq p_t e^t_t + p_{t+1} e^t_{t+1}$$

$$c^t_t, c^t_{t+1} \geq 0$$

3) Markets clear $\forall t$: $c^{t-1}_t + c^t_t = e^{t-1}_t + e^t_t$

SME of an OLG model

<u>Def</u>: A SME is the system of interest rates $\{r_{t+1}\}_{t=0}^{\infty}$ and allocations $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$, st.

1). Given r_1, the allocation for the initially old solves.

$$u(c_t^0) \to \max_{c_1^0}$$

$$\text{st} \quad c_1^0 \le e_1^0 + (1+r_1)m$$

$$c_1^0 \geq 0$$

2) Given $\{r_{t+1}\}_{t=0}^{\infty}$, c_t^t, c_{t+1}^t solves generation t's problem

$$u(c_t^t) + \beta u(c_{t+1}^t) \longrightarrow \max_{c_t^t, c_{t+1}^t}$$

$$\text{s.t} \quad c_t^t + s_{t+1}^t \le e_t^t$$

$$c_{t+1}^t \le e_{t+1}^t + (1+r_{t+1})$$

$$c_t^t, c_{t+1}^t \geq 0$$

3) Markets clear $\quad c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \quad \forall t$

* No need of "No Ponzi Scheme" condition because perfectable enforceable (Because the agent lives two periods)

<u>Equilibrium on asset market</u>

Budget constraint in $t=1$: $\quad c_1^0 = e_1^0 + (1+r_1)m. \quad$ (old)

$$c_1^1 + s_2^1 = e_1^1 \quad \text{(young)}.$$

add them.

$$\underbrace{c_1^0 + c_1^1}_{\text{market clear}} + s_2^1 = \underbrace{e_1^0 + e_1^1}_{\text{mkt clear.}} + (1+r_1)m$$

$$s_2^1 = (1+r_1)m$$

for $t=2$: $\quad c_2^1 = e_2^1 + (1+r_2)s_2^1$

$$c_2^2 + s_3^2 = e_2^2$$

add them: $\quad s_3^2 = (1+r_2)s_2^1 = (1+r_2)(1+r_1)m$

$$\dots \quad s_{t+1}^t = \underbrace{\prod_{j=1}^{t}(1+r_j)}\cdot m.$$

$\hookrightarrow$ if $1+r_j > 1$, savings explodes. $s_{t+1}^t \to \infty$

* Remark: Asset mkt clears (Walras' Law).

Equivalence of ADE and SME

ADE

SME

I.O: $p_1 c_1^0 = p_1 e_1^0 + m$

t: $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t$

$$c_1^0 = e_1^0 + \frac{m}{p_1}$$

$$c_t^t + \frac{p_{t+1}}{p_t} c_t^t = e_t^t + \frac{p_{t+1}}{p_t} e_{t+1}^t \implies$$

$$c_1^0 = e_1^0 + (1+r_1)m$$

$$s_{t+1}^t = \frac{c_{t+1}^t}{1+r_{t+1}} - \frac{e_{t+1}^t}{1+r_{t+1}}$$

$$c_t^t + \frac{c_{t+1}^t}{1+r_{t+1}} = e_t^t + \frac{e_{t+1}^t}{1+r_{t+1}}$$

I.O.: $c_1^0 = e_1^0 + (1+r_1)m.$

Theorem: $\exists$ ADE $\iff$ $\exists$ SME with the same consumption allocation

Proof: Suppose $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$, and $\{p_t\}_{t=0}^{\infty}$ is ADE then $\exists$ SME w/ the same allocation and

$$\boxed{\begin{aligned} 1+r_{t+1} &= p_t / p_{t+1} \\ 1+r_t &= 1/p_t \end{aligned}}$$

If $\{r_{t+1}\}_{t=0}^{\infty}$ and allocations constitute SME, then

$$\boxed{\begin{aligned} P_1 &= 1 \\ P_{t+1} &= \frac{P_t}{1+r_{t+1}} \end{aligned}}$$

Excess demand and offer curves

assume $u(c) = \log(c)$ (ADE)

$$\log(c_t^t) + \beta \cdot \log(c_t^t) \longrightarrow \max_{\{c_1^t, c_2^t\}}$$

$$\begin{cases} P_t c_t^t + P_{t+1} c_{t+1}^t = P_t e_t^t + P_{t+1} e_{t+1}^t \\ c_t^t \geq 0 \\ c_{t+1}^t \geq 0 \end{cases}$$

$$\mathcal{L} = \log(c_t^t) + \beta \log(c_{t+1}^t) + \lambda \left[P_t e_t^t + P_{t+1} e_{t+1}^t - P_t c_t^t - P_{t+1} c_{t+1}^t \right]$$

FOC

$$\frac{1}{c_t^t} = P_t \lambda$$

$$\frac{\beta}{c_{t+1}^t} = P_{t+1} \lambda$$

$\Rightarrow$

$$\beta \frac{c_t^t}{c_{t+1}^t} = \frac{P_{t+1}}{P_t}$$

$$c_{t+1}^t = \beta c_t^t \cdot \frac{P_t}{P_{t+1}}$$

Lecture

Sept 12, 2023

→ Excess demand

→ Offer curves

→ OLG with production.

* Describe : Who trades with who, when. concise.

* Define: prices, allocation that satisfies certain criteria

Excess of demand and offer curves

$$\left(e_t^t, e_{t+1}^t\right) = \left(e_1, e_2\right)$$

Utility function is $\log\left(c_t^t\right) + \beta \log\left(c_{t+1}^t\right)$

Consider ADE. from

$$\max \log\left(c_t^t\right) + \beta \log\left(c_{t+1}^t\right) \quad \text{st} \quad p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_1 + p_{t+1} e_2$$

$$c_{t+1}^t, c_t^t \geq 0$$

$$\mathcal{L} = \log\left(c_t^t\right) + \beta \log\left(c_{t+1}^t\right) + \lambda\left[p_t e_1 + p_{t+1} e_2 - p_t c_t^t - p_{t+1} c_{t+1}^t\right]$$

f.o.c

$$\left(c_t^t\right): \quad \frac{1}{c_t^t} = \lambda p_t \quad ; \quad \left(c_{t+1}^t\right): \quad \frac{\beta}{c_{t+1}^t} = \lambda p_{t+1}$$

$$\Rightarrow \boxed{c_{t+1}^t = \beta \cdot c_t^t \cdot \frac{p_t}{p_{t+1}}} \quad (\star)$$

Budget Constraint : $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_1 + p_{t+1} e_2 \quad / \ \frac{1}{p_t}$

$$c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t = e_1 + \frac{p_{t+1}}{p_t} e_2 \qquad \text{use } (\star)$$

$$c_t^t + \beta c_t^t = e_1 + \frac{p_{t+1}}{p_t} e_2$$

$$c_t^t (1+\beta) = e_1 + \frac{p_{t+1}}{p_t} e_2$$

For the initially old: $P_1 c_1^0 = P_1 e_2 + m$

$$c_1^0 = e_2 + \frac{m}{P_1}$$

Remark: we cannot normalize by $P_0 = 1$ because it'd affect the iniatially old

Def: An excess demand of consumer i in an endowment economy is the difference between the demand of that agent and his endowment.

EX: Excess demand for the initially old is $z_1^0 = c_1^0 - e_2 = \frac{m}{P_1}$

Derive excess demand functions z_t^t, z_{t+1}^t.

$$z_t^t = c_t^t - e_1 = \frac{1}{1+\beta}\left[e_1 + \frac{P_{t+1}}{P_t} e_2\right] - e_1 \qquad (\star)$$

$$z_{t+1}^t = \underbrace{c_{t+1}^t}_{\beta c_t^t \frac{P_t}{P_{t+1}}} - e_2 = \beta \frac{P_t}{P_{t+1}}\left[\frac{1}{(1+\beta)}\left(e_1 + \frac{P_{t+1}}{P_t} e_2\right)\right] - e_2 \qquad (\star\star)$$

Def: An ADE in OLG model is $\{P_t\}_{t=1}^{\infty}$ and allocations, s.t. $z_t^{t-1} + z_t^t = 0. \quad \forall t \geq 1.$

The algorithm to find equilibrium using excess demand functions.

1) Pick P_1. Find $z_1^0 = m/P_1$
2) Given z_1^0, find z_1^1 from def of equilibrium.
3) Find P_2 from $(\star)$
4) Find z_2^1 from $(\star\star)$

$\vdots$

(P_1 is arbitrary, so infinite solutions)

Offer curve : from (☆) $\frac{P_{t+1}}{P_t} = \frac{(1-\beta)(z_t^t+e_1)-e_1}{e_2}$

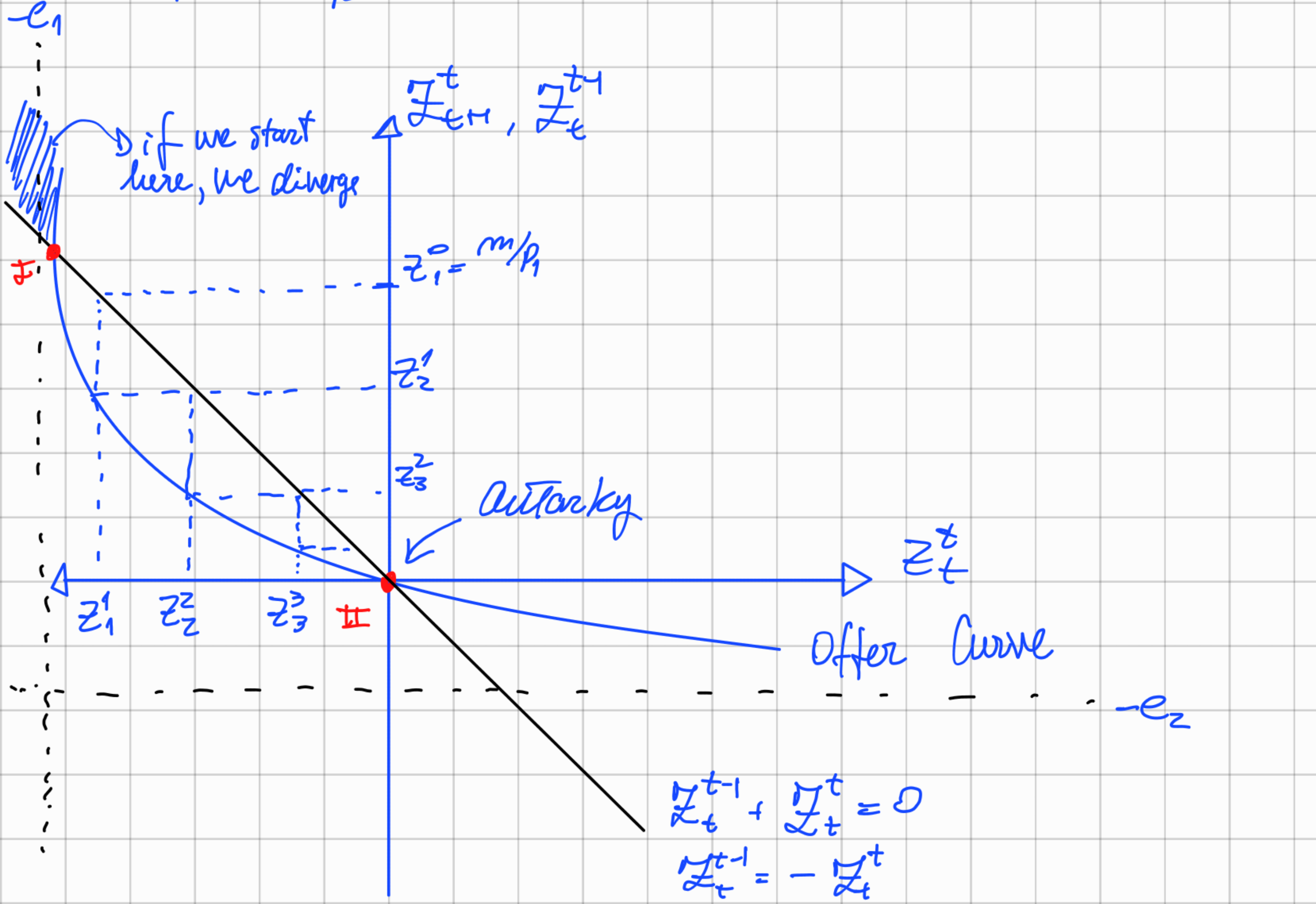
Plug into (☆☆) $\boxed{z_{t+1}^t = \frac{\beta}{1-\beta}\left(\frac{e_1e_2}{(1-\beta)(z_t^t+e_1)-e_1}+e_2\right)-e_2}$

Remark: O.C. passes through the origin

Suppose $z_t^t=0 \Rightarrow z_{t+1}^t = \frac{\beta}{1+\beta}\left(\frac{e_1e_2}{(1+\beta)e_1-e_2}+e_2\right)-e_2$

$$= \frac{\beta}{1+\beta}\left(\frac{\cancel{e_1}e_2}{\beta \cancel{e_1}}+e_2\right)-e_2 = \frac{\beta}{1+\beta}\left(\frac{e_2+\beta e_2}{\beta}\right)-e_2$$

$$= \frac{\beta}{1+\beta}\,\frac{(1+\beta)e_2}{\beta} - e_2 = 0$$



First Welfare thm doesn't apply: Proof by counterexample

$$u = \log(c_t^t) + \beta \log(c_{t+1}^t) \quad , \beta = 1.$$

$$(e_t^t, e_{t+1}^t) = (2, 1)$$

$m = 0$

$$C.E = \begin{cases} c_t^t = 2 \\ c_{t+1}^t = 1 \end{cases} \quad \text{autarky.}$$

Consider $(\tilde{c}_t^t, \tilde{c}_{t+1}^t) = \left(\frac{3}{2}, \frac{3}{2}\right)$, which is feasiable.

Initially old are better off at $\tilde{c}$

Any other generation:

$$u(2,1) = \log(2) + \log(1) = \log(2) < \log(9/4) = u\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$

$\Rightarrow$ We have found a feasible allocation that improves no one worse off and at least one is better off

OLG with production

Environment: $t = 1, 2, \ldots$

Generations live for 2 periods

$$U = u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) \qquad l: \text{labor.}$$

$$u_c' > 0 \quad , u_l' > 0$$

Time endowment $(\bar{l}_1, \bar{l}_2)$ time to use for work or as leisure.

Production: There is a representative firm with access to technology $Y_t \le f(k_t, l_t)$
capital leisure.

Remark: Technology is CRS
→ Capital is the only asset that agents use to transfer resources over time.

Initially old are endowed with $\bar{k} \geq 0$.

Lecture: Sept 14, 2023

→ OLG with production
→ Neoclassical Growth model (Intro).

Environment:
→ Gen lives 2 periods
→ utility: $u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t)$
Initially old: $u(c_1^0, l_1^0)$

Technology: $y \le F(k, l)$, $F(\cdot)$ is constant returns to scale
$F(\lambda k, \lambda l) = \lambda F(k, l)$

Assets: • physical capital, $k \geq 0$
$k_1^0 = \bar{k}$ (endowment of I.O.)
• bonds b_{t+1}^t

I.O. may have fiat money $m \gtreqless 0$

→ Capital depreciates at rate $\delta \in [0,1]$.

Def: SME for this economy is $\tilde{c}_1^0, \{\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{l}_t^t, \tilde{l}_{t+1}^t, \tilde{k}_{t+1}^t, \tilde{b}_{t+1}^t\}$, allocation for the firm $\{\tilde{y}_t, \tilde{k}_t^f, \tilde{l}_t^f\}_{t=1}^{\infty}$, and prices $\{\tilde{w}_t, \tilde{r}_t^k, \tilde{r}_t^b\}$, such that.

1) Given the prices, $\tilde{c}_1^0, \tilde{l}_1^0$, solves

$$\max_{c_1^0, l_1^0} u(c_1^0, l_1^0)$$

$$\text{s.t.} \begin{cases} c_1^0 \le \tilde{w}_1 l_1^0 + (1-\delta+r_1^k)\bar{k} + (1+r^b)m \\ c_1^0 \ge 0 \\ 0 \le l_1^0 \le \bar{l}_2 \end{cases}$$

2) Given the prices, $\{\tilde{c}_t^t, \tilde{c}_{t+1}, \tilde{l}_t^t, \tilde{l}_{t+1}^t, \tilde{k}_{t+1}^t, \tilde{b}_{t+1}^t\}_{t=1}^{\infty}$ solves the generation t's problem.

$$u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) \to \max$$

$$\text{s.t} \begin{cases} c_t^t + k_{t+1}^t + b_{t+1}^t \le \tilde{w}_t l_t^t \\ c_{t+1}^t \le \tilde{w}_{t+1} l_{t+1}^t + \tilde{r}_{t+1}^k k_{t+1}^t + (1-\delta)k_{t+1}^t + (1+\tilde{r}_{t+1}^b)b_{t+1}^t \\ c_t^t \ge 0, \quad c_{t+1}^t \ge 0 \\ 0 \le l_t^t \le \bar{l}_1, \quad 0 \le l_{t+1}^t \le \bar{l}_2 \end{cases}$$

3) Given the prices, $\{\tilde{y}_t, \tilde{k}_t^f, l_t^t\}$ solves the firm's problem

$$\begin{cases} y_t - \tilde{w}_t l_t^t - \tilde{r}_t^k k_t^f \to \max\limits_{k_t^f, l_t^t, y_t} \\ \text{st} \quad y_t \le F(k_t^f, l_t^t) \end{cases}$$

$k_t^f = k_t^{t-1}$

4) Markets clear $\tilde{c}_t^t + \tilde{c}_t^{t-1} + \tilde{k}_{t+1}^t - (1-\delta)\tilde{k}_t^{t-1} = \tilde{y}_t$

and $\tilde{l}_t^f = \tilde{l}_t^{t-1} + \tilde{l}_t^t$

$\hat{k}_t^f = \hat{k}_t^{t-1}$ $\tilde{b}_{t+1}^t = (1+\tilde{r}_{t+1}^b)\,\tilde{b}_t^{t-1}$, $\tilde{b}_2^1 = (1+\tilde{r}_1^b)\,m$

<u>Def</u>: An ADE is $\hat{c}_1^0, \hat{l}_1^0, \{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{l}_t^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t\}_{t=1}^{\infty}$, allocation for firms $\{\hat{y}_t, \hat{k}_t^f, \hat{l}_t^f\}_{t=1}^{\infty}$ and prices $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=1}^{\infty}$, s.t

1) Given prices, $\hat{c}_1^0, \hat{l}_1^0$ solves the I.O. problem

$$\max_{c_1^0, l_1^0} u(c_1^0, l_1^0)$$

$$\text{s.t} \begin{cases} \hat{p}_1 \hat{c}_1^0 \le \hat{w}_1 l_1^0 + \hat{r}_1 \cdot \bar{k} + \hat{p}_1 (1-\delta)\bar{k} \\ c_1^0 \ge 0 \\ 0 \le l_1^0 \le \bar{l}_2 \end{cases}$$

2) Given the prices, $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{l}_t^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t\}_{t=1}^{\infty}$ solves

$$u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) \longrightarrow \max$$

$$\text{s.t} \begin{cases} \hat{p}_t c_t^t + \hat{p}_{t+1} \hat{c}_{t+1}^t + \hat{p}_t k_t^t \le \hat{w}_t l_t^t + \hat{w}_{t+1} l_{t+1}^t + \hat{p}_{t+1}(1-\delta) k_{t+1}^t + \hat{r}_{t+1} k_{t+1}^t \\ c_t^t \ge 0,\ c_{t+1}^t \ge 0 \\ 0 \le l_t^t \le \bar{l}_1, \quad 0 \le l_{t+1}^t \le \bar{l}_2 \end{cases}$$

3) Given the prices, $\{\hat{y}_t, \hat{k}_t^f, \hat{l}_t^f\}$ solves the firm's problem

$$\max \begin{cases} \hat{p}_t y_t - \hat{w}_t l_t - \hat{r}_t^k k_t \longrightarrow \max\limits_{k,l,y} \\ \text{s.t} \quad y_t \le F(k_t, l_t) \end{cases}$$

4) Market Clear $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = \hat{y}_t$ and
$\hat{l}_t^f = \hat{l}_t^{t-1} + \hat{l}_t^t$ $\hat{k}_t^f = \hat{k}_t^{t-1}$

Eq. Characterization:

$$\mathcal{L} = u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t) + \lambda_t^t \left[w_t l_t^t - c_t^t - k_{t+1}^t - l_{t+1}^t \right]$$
$$+ \lambda_{t+1}^t \left[w_{t+1} l_{t+1}^t + (1-\delta) k_{t+1}^t + r_{t+1}^k k_{t+1}^t + (1+r_{t+1}^b) b_{t+1}^t - c_{t+1}^t \right]$$

$\boxed{\text{F.O.C}}$

$$(c_t^t) \quad u_c'(c_t^t, l_t^t) = \lambda_t^t \qquad (1)$$
$$(c_{t+1}^t) \quad \beta u_c'(c_{t+1}^t, l_{t+1}^t) = \lambda_{t+1}^t \qquad (2)$$
$$(k_t^t) \quad -\lambda_t^t + \lambda_{t+1}^t (1-\delta + r_{t+1}^k) = 0 \qquad (3)$$
$$(b_{t+1}^t) \quad -\lambda_t^t + \lambda_{t+1}^t (1 - r_{t+1}^b) = 0 \qquad (4)$$

$(1) \& (2) \longrightarrow (3)$: $u_c'(c_t^t, l_t^t) = \beta u_c'(c_{t+1}^t, l_{t+1}^t)(1-\delta + r_{t+1}^k)$ (Euler equation)

$(3)+(4)$: $r_{t+1}^b = \delta + r_{t+1}^k$ (no arbitrage condition)

Lecture: Last class for midterm. $\boxed{\text{Sept 19, 2023}}$

→ Neoclassical Growth model

→ Social planner's prob

→ Recursive formulation

Environment (NGM).

→ Agents one infinitely-lived agent

→ $u(\{c_t\}_{t=1}^{\infty}) = \sum^{\infty} \beta^t U(c_t)$ where, $U(\cdot)$ is the instant utility function.

→ Firms: $y_t = F(k_t, l_t)$

→ Resource constraint: $y_t = c_t + i_t$

→ Law of motion for capital stock: $k_{t+1} = (1-\delta) k_t + i_t$, $k_t \geq 0$

→ Endowments

1) $\bar{k}_0$ - endowment of capital at $t=0$

2) every period there is 1 unit of time endowment

→ Information: Full info.

→ Agents will supply labor inelastically. $l_t = 1$.

Social Planner's problem

$$\sum_{t=1}^{\infty} \beta^t U(c_t) \to \max_{\{c_t, l_t, k_t\}}$$

s.t. $c_t + i_t = F(k_t, l_t)$

$k_{t+1} = (1-\delta) k_t + i_t$

$0 \le l_t \le 1, \; k_t \geq 0, \; c_t \geq 0$

$k_0 = \bar{k}_0$

Assumptions on $U(\cdot)$ and $F(\cdot)$

→ $U(\cdot)$ is strictly concave and increasing. $u' > 0$, $u'' < 0$

→ Inada conditions:

$$\lim_{c \to 0} u' = \infty, \quad \lim_{c \to \infty} u' = 0$$

$F(k, l)$:

→ $F(0, l) = F(k, 0) = 1$

→ $F_k' > 0$, $F_l' > 0$, $F'' < 0$

→ $\lim_{i \to 0} F_i' = \infty$ $\lim_{i \to \infty} F_i' = 0$, $i = \{k, l\}$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[F(k_t, 1) + (1-\delta)k_t - k_{t+1} - c_t \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1}\left(F_k'(k_{t+1}, 1) + (1-\delta)\right) = 0 \qquad (2)$$

Combine (1) & (2): $\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1})\left[F_k'(k_{t+1}, 1) + (1-\delta)\right]$

* note : $\lambda_{t+1} = \beta^{t+1} u'(c_{t+1})$

$$u'(c_t) = \beta u'(c_{t+1})\left[F_k'(k_{t+1}, 1) + (1-\delta)\right] \qquad \text{euler eq.}$$

Steady State: $c_t = c_{t+1} = c^{ss}$

from e.e. : $u'(c^{ss}) = \beta u'(c^{ss})\left[F_k'(k^{ss}, 1) + (1-\delta)\right]$.

$$\Rightarrow \boxed{\frac{1}{\beta} = F_k'(k^{ss}, 1) + (1-\delta)}$$

from R.C: $c_t = F(k_t, 1) + (1-\delta)k_t - k_{t+1}$

$c_{t+1} = F(k_{t+1}, 1) + (1-\delta)k_{t+1} - k_{t+2}$

Plug in E.E: we get the 2^{nd} order diff. equation.

$$\boxed{u'\left(F(k_t,1) + (1-\delta)k_t - k_{t+1}\right) = \beta u'\left(c_{t+1} = F(k_{t+1},1) + (1-\delta)k_{t+1} - k_{t+2}\right)\left[F_k'(k_{t+1},1) + 1-\delta\right]}$$

we need 2 condition

i) initial condition $\bar{k}_0$

ii) Transversality Condition (TVC) : $\lim_{t\to\infty} \underbrace{\lambda_t k_{t+1}}_{\substack{\text{value of capital stock}\\ \text{must converge to 0.}}} = 0$

λ_t is the shadow value of k_{t+1}.

From FOC to SPP we know $\lambda_t = \beta^t u'(c_t)$

$\Rightarrow$ TVC is $\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0$

Recursive Formulation

Define $f(k) = F(k,1) + (1-\delta)k$

$$\Rightarrow \sum_{t=1}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \rightarrow \max_{\{k_{t+1}\}_{t=0}^{\infty}}$$

$$\text{s.t} \begin{cases} f(k_t) \geq k_{t+1} \geq 0 \quad \forall t \\ k_0 \text{ given.} \end{cases}$$

$$\underbrace{\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})}_{V(k_0)} = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left[u(f(k_0) + k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} (u(f(k_t) - k_{t+1})) \right]$$

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left[u(f(k_0) + k_1) + \underbrace{\beta \max_{\substack{\{k_{t+2}\}_{t=0}^{\infty} \\ 0 \leq k_{t+2} \leq f(k_{t+1})}} \sum_{t=1}^{\infty} \beta^{t-1} (u(f(k_t) - k_{t+1}))}_{\beta \cdot V(k_1)} \right]$$

$$\Rightarrow V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} u(f(k_0) + k_1) + \beta V(k_1)$$

$$\boxed{V(k_t) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} u(f(k_t) + k_{t+1}) + \beta V(k_{t+1})}$$

Lecture:

→ Intro to DP

→ Decentralized eqm

Sept 21, 2023

Sequential Problem of S.P.

$$\begin{cases} \sum_{t=0}^{\infty} \beta^t u(f(k_t)-k_{t+1}) \rightarrow \max_{\{k_{t+1}\}_{t=0}^{\infty}} \\ \text{st} \quad f(k_t) \geq k_{t+1} \geq 0 \\ \quad k_0 \text{ is given.} \end{cases}$$

Recursive Problem of S.P.

$$V(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \left\{ u(f(k_0)-k_1) + \beta V(k_1) \right\}$$

Problem: $V(\cdot)$ is not known (yet)

Jargon:

$V(\cdot)$ - Value function

k - state variable

k' - control variable

$k' = g(k)$ policy function (decision rule).

Algorithm to solve Bellman equation.

1) Guess $V_0(k)$
2) Use BE to update your guess. $V_{j+1} = \max \left[u(f(k)-k') + \beta V_j(k) \right]$
3) $\text{norm}(V_{j+1}, V_j) < \varepsilon$, then convergence
4) if not, Repeat.

Example: $u = \log(c)$, $\delta = 1$, $y_t = \theta k_t^\alpha$

$$V(k) = \max_{0 \le k' \le \theta k^\alpha} \left[\log(c) + \beta V(k')\right]$$

s.t. $c + k' = \theta k^\alpha$

k_0 is given

Guess (and later verify) that:

$$V(k) = a_0 + a_1 \log k$$

$$\mathcal{L} = \log c + \beta(a_0 + a_1 \log k') + \lambda(\theta k^\alpha - c - k')$$

$$\frac{\partial \mathcal{L}}{\partial c}: \frac{1}{c} = \lambda \quad (1) \qquad \frac{\partial \mathcal{L}}{\partial k'}: \beta \frac{a_1}{k'} = \lambda \quad (2)$$

Combining: $$\frac{1}{c} = \beta \frac{a_1}{k'} \quad (3)$$

Invoke resource constraint: $c = \theta k^\alpha - k'$

$$(4) \quad \frac{1}{\theta k^\alpha - k'} = \beta \frac{a_1}{k'} \Rightarrow \boxed{k' = \beta \frac{a_1 \theta k^\alpha}{1 + \beta a_1}}$$

Pluging back into R.C.

$$c = \theta k^\alpha - \frac{\beta a_1}{1 + \beta a_1} \theta k^\alpha$$

$$\boxed{c = \theta k^\alpha \frac{1}{1 + \beta a_1}}$$

* Method of indeterminated coefs.

$$V(k) = \max \left[\log C + \beta V(k') \right].$$

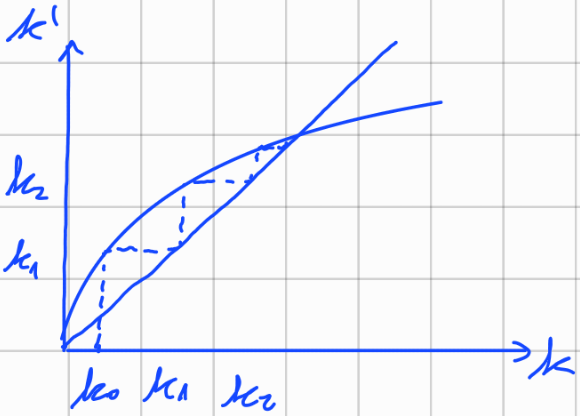
Plug our guess:

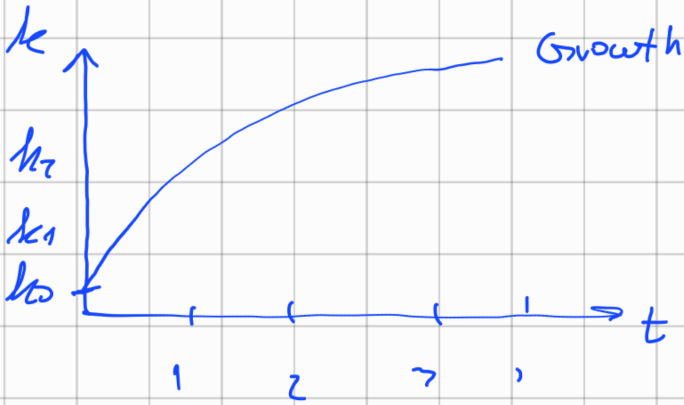
$$a_0 + a_1 \log k = \log\left(\frac{\theta k^\alpha}{1+\beta a_1}\right) + \beta\left(a_0 + a_1 \log\left[\frac{\theta k^\alpha \beta a_1}{1+\beta a_1}\right]\right)$$

$$a_0 + a_1 \log k = -\log(1+\beta a_1) + \log\theta + \alpha \log k + \beta a_0 - \beta a_1 \log(1+\beta a_1)$$
$$+ \beta a_1 \log(\theta\beta a_1) + \beta a_1 \alpha \log(k)$$

$$a_1 = \alpha + \beta a_1 \alpha \iff a_1(1-\beta\alpha) = \alpha \qquad a_1 = \frac{\alpha}{1-\beta\alpha}$$

* Growth:





<u>Decentralizing Eq.</u>

→ Representative agent, $u(c)$
→ Firm with technology $F(k,e)$
→ Agents owns the firm
→ Agent is endowed with k_0

Def: An ADE is allocations for the consumer $\{\hat{c}_t, \hat{k}_{t+1}, \hat{l}_t\}_{t=0}^{\infty}$, allocation for firm $\{\hat{y}_t, \hat{k}_t^f, \hat{l}_t^f\}_{t=0}^{\infty}$ and a price system $\{\hat{p}_t, \hat{r}_t, \hat{w}_t\}$, s.t.

1) Gives the prices $\{\hat{c}_t, \hat{k}_{t+1}, \hat{l}_t\}$ solves

$$\max_{\{c_t, k_{t+1}, l_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \sum_{t=0}^{\infty} (\hat{p}_t c_t + \hat{p}_t \cdot i_t) \le \sum_{t=0}^{\infty} (\hat{w}_t l_t + \hat{r}_t k_t)$$

$$i_t = k_{t+1} - (1-\delta) k_t$$

$$c_t \ge 0, \; k_t \ge 0 \quad \forall t$$

2) Given prices, the firm chooses optimally $\{y_t, k_t^f, l_t^f\}$ s.t. $p_t F(k_t^f, l_t^f) - \hat{w}_t l_t^f \to \max$

3) Markets clear:

$$\hat{c}_t + \hat{i}_t = \hat{y}_t$$
$$\hat{k}_t^f = \hat{k}_t$$
$$\hat{l}_t^f = \hat{l}_t^f$$

Suppose hh sell capital to the firm in the $t=1$.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t \le \sum_{t=0}^{\infty} \hat{w}_t l_t + r_0 k_0 + p_0 (1-\delta) k_0$$

<u>The firm</u>
$$\begin{cases} \max \sum^{\infty} \hat{p}_t (F(k_t, l_t) - i_t) - \sum^{\infty} \hat{w}_t l_t - \hat{r}_0 k_0 - p_0 (1-\delta) k \\ i_t = k_{t+1} - (1-\delta) k_t \end{cases}$$

def: An SME is allocations $\{\tilde{c}_t, \tilde{b}_{t+1}, \tilde{k}_{t+1}, \tilde{l}_t\}$, allocations for firms $\{\tilde{y}_t^f, \tilde{k}_t^f, \tilde{l}_t^f\}$ prices $\{\tilde{w}_t, \tilde{r}_t^k, \tilde{r}_t^l\}$, s.t

1) Given prices, HH solves.

$$\max_{\{c_t, l_t, b_{t+1}, k_{t+1}\}} u(c_t) \quad \text{s.t} \quad c_t + b_{t+1} + i_t = \tilde{w} l_t + \tilde{r}^k k_t + (1+\tilde{r}_t) b_t$$

$$i_t = k_{t+1} - (1-\delta) k_t$$

$$c_t \geq 0, \; k_t \geq 0, \; b_0 \geq 0$$

N.P.G.C.

2) Firms

$$y_t - \tilde{w}_t l_t^f - \tilde{r}_t^k k_t \longrightarrow \max \quad (\text{profit})$$

$$\text{s.t} \quad y_t \leq F(k_t^f, l_t^f)$$

3) Markets clear

$$\tilde{c}_t + \tilde{i}_t = y_t$$

$$\tilde{k}_t^f = \tilde{k}_t$$

$$\tilde{l}_t^f = \tilde{l}_t$$

$$\tilde{b}_t = 0$$

Lecture: Sept 26, 2023

→ Pareto Optimality in NGM

→ NGM with pop. growth

Social Planner's Problem

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \longrightarrow \max \quad \text{s.t} \quad c_t + k_{t+1} - (1-\delta) k_t = y_t$$

$$c_t \geq 0, \; k_{t+1} \geq 0$$

k_0 is given

Euler Eq: $u'(t) = \beta u'(t+1) \left[1 - \delta + F_k'(k_{t+1}) \right]$

Is CE optimal? Consumer (in SME) solves $\{c_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ to solve the following problem:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \rightarrow \max \quad \text{s.t} \quad c_t + k_{t+1} + b_{t+1} \leq w_t l_t + (1 + r_t^b) b_t + (1-\delta) k_t + r_t^k k_t$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) + \sum_{t=0}^{\infty} \lambda_t \left[w_t l_t + (1 + r_t^b) b_t + (1-\delta) k_t + r_t^k k_t - c_t - k_{t+1} - b_{t+1} \right\}$$

f.o.c:

* Characterization:
- f.o.c
- NAC
- TVC
- E.E.

$(c_t) \quad \beta^t u_c' = \lambda_t$

$$\left.\begin{aligned} &(k_{t+1}) \quad -\lambda_t + \lambda_{t+1} (1 - \delta + r_{t+1}^k) = 0 \\ &(b_{t+1}) \quad -\lambda_t + \lambda_{t+1} (1 + r_{t+1}^b) = 0 \end{aligned}\right\} \quad 1 - \delta + r_{t+1}^k = 1 + r_{t+1}^b$$

non arbitrage condition

Transversality condition: $\lim_{t \to \infty} \beta^t u_c' \cdot k_{t+1} = 0 \; ; \; \lim_{t \to \infty} \beta^t u_c' \cdot b_{t+1} = 0$

Combine foc (c_t) and (k_{t+1})

$$u_c'(t) = \beta u_c'(t+1) \cdot \left[1 - \delta + r_{t+1}^k \right]$$

Firms : $\Pi = y_t - w_t l_t - r_t^k k_t$, $y_t \le F(k_t, l_t)$

$\frac{\partial \Pi}{\partial k} = F_k' = r_t^k$; $\frac{\partial \Pi}{\partial l} = F_l' = w_t$

* F is CRS, which implies

$\lambda F(k,l) = F(\lambda k, \lambda l) \quad \Big| \frac{\partial}{\partial \lambda}$

$F(k,l) = \frac{\partial F}{\partial k}\Big|_{\lambda k, \lambda l} + \frac{\partial F}{\partial l}\Big|_{\lambda k, \lambda l}$

Take $\lambda = 1$:

from f.o.c : $F(k,l) = r_t^k k_t + w_t l_t$

E.E : $u_c'(t) = \beta u'(t+1)\left[1 - \delta + \underbrace{r_{t+1}^k}_{F_k'(t+1)}\right]$

EE, k_0 and TVC are the same as in SPP.
$\Rightarrow$ CE is P.O.

Neoclassical Growth model with population growth.

$N_0 = 1$: starting mass of consumers

n : growth of population.

$N_t = (1+n)^t N_0 = (1+n)^t$

g : Technological progress

$N_t(1+g)$: Labor augmented by T.P.

Assume $F(K_t, N_t(1+g)^t)$

Suppose SP maximizes utility per capita $\begin{pmatrix} C_t - \text{Agg. } C \\ c_t - \text{per capita} \end{pmatrix}$

$$\sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \longrightarrow \max$$

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, N_t(1+g)^t)$$

Define $\tilde{c}_t = \dfrac{c_t}{(1+g)^t}$, $\tilde{k}_t = \dfrac{k_t}{(1+g)^t}$

Divide R.C. by $(1+g)^t(1+n)^t$

$$\frac{C_t}{(1+g)^t(1+n)^t} + \frac{K_{t+1}}{(1+g)^t(1+n)^t} - (1-\delta)\frac{K_t}{(1+g)^t(1+n)^t} = \frac{F(K_t, N_t(1+g)^t)}{(1+g)^t(1+n)^t}$$

$$\Rightarrow \tilde{c}_t + (1+g)(1+n)\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = F(\tilde{k}_t, 1)$$

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t \frac{\left(\tilde{c}_t(1+g)^t\right)^{1-\sigma}}{1-\sigma}$$ CRRA

$$= \sum_{t=0}^{\infty} \left(\beta(1+g)^{1-\sigma}\right)^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}$$

Dynamic Programming:

math preliminaries:

$$v(x) = \max_{y \in \Gamma(x)} \left[\underbrace{F(x,y)}_{\text{period returns}} + \beta v(y) \right]$$

x: state (k_t)

y: control (k_{t+1})

F: Period Returns $(u(\cdot))$

Think as of an operator T s.t.

$$(Tv)(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y) \right].$$

Find fixed point of T: $v^* = Tv^*$

Lecture: $\boxed{\text{Oct } 5, 2023}$

→ metric spaces

→ Convergence of seq.

→ Cauchy sequences

→ Contraction mapping theorem.

Bellman Equation:

$$V(k) = \max_{0 \le k' \le f(k)} \left[u(f(k) - k') + \beta V(k') \right]$$

More generally: $v(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y) \right]$ ← solution of this is the fixed point of operator T

$$(Tv)(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y) \right]$$

Questions:

1) When v^* exists?
2) Is v^* unique?
3) When is $v^* = \lim_{j \to \infty} T^j(v_0)$?

} contraction mapping theory.

<u>Def</u>: A metric space is a set S along with a metric $d(x,y)$, satisfying:

1) $d(x,y) \ge 0$
2) $d(x,y) = 0$ IFF $x = y$
3) $d(x,z) \le d(x,y) + d(y,z)$, $\forall x,y,z \in S$ (triangle inequality)
4) $d(x,y) = d(y,x)$ (Symmetry).

Ex: $S = \mathbb{R}$, $d(x,y) = \begin{cases} 1 & , \text{if } x \neq y \\ 0 & , \text{o/w} \end{cases}$ all properties hold, then (S,d) is a metric space

Ex: Let $X \subseteq \mathbb{R}^{\ell}$, $S = C(x)$ set of all continuous and bounded functions $f: X \to \mathbb{R}$ equipped with supnorm d.

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

Fix some x

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

Take sup norm on both sides and we are done.

Def: Let (S,d) be a metric space. A sequence $\{x_n\}$ with $x_n \in S$ $\forall n$ converges to x if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$, $d(x, x_n) < \varepsilon$

Exercise: Prove that if limit exists $\Rightarrow$ it's unique

Def: Let (S,d) be a metric space $\{x_n\}$ with $x_n \in S$ $\forall n$ is Cauchy if $\forall \varepsilon > 0$ $\exists N : \forall n,m > N$ $d(x_n, x_m) < \varepsilon$

Theorem: If $\{x_n\}$ converges $\Rightarrow$ It's a Cauchy sequence.

Proof: Pick $\varepsilon > 0$, $\Rightarrow \exists M_{\frac{\varepsilon}{2}}$ s.t.

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \forall n > M_{\frac{\varepsilon}{2}}$$

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \{x_n\}$ is Cauchy

Ex: $S = \mathbb{R}$, $d(x,y) = \begin{cases} 1 , & x \neq y \\ 0 , & x = y \end{cases}$

$\{x_n\}$, where $x_n = \frac{1}{n}$. Is this Cauchy under this metric?

$1, \frac{1}{2}, \frac{1}{3}, \ldots$ } distance under $d(x,y)$ is always 1,
$\therefore$ Is not Cauchy, not converging.

Def: A metric space (S,d) is complete if $\forall$ Cauchy sequence $\{x_n\}$ converges to an element $x \in S$
$(x_n \in S, \forall n)$

EX: $X = [1,2]$
$S = C(X)$ set of cont, strictly decreasing on
$d(\cdot)$ is supnorm
This is not a complete metric space.
proof: Counterexample

$\{f_n\} = \frac{1}{nx}$, is it Cauchy? Pick $\varepsilon > 0$, and let $N_\varepsilon = \frac{2}{\varepsilon}$, $m > n > N_\varepsilon$

$$d(f_n, f_m) = \sup_{x \in [1,2]} \left| \frac{1}{nx} - \frac{1}{mx} \right| = \sup_{x \in [1,2]} \frac{1}{nx} - \frac{1}{mx}$$

$$= \sup_{x \in [1,2]} \frac{m-n}{mnx} = \frac{m-n}{mn} \quad \text{(maximum when } x=1\text{)}.$$

$$= \frac{1 - \frac{n}{m}}{n} \leq \frac{1}{n} \leq \frac{1}{N_\varepsilon} = \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow f_n, f_m$ is Cauchy.

Book: Stochey Lucas.

Def: Let (S, d) be a metric space and $T: S \to S$.
An operator T is a contraction if $d(Tx, Ty) \leq \beta d(x, y)$, $\beta \in (0,1)$ $\forall x, y \in S$. β is called modulus of contraction

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Lecture: | Oct 10, 2023

→ Contracting mapping theorem
→ Blackwell's theorem (sufficient conditions)

Lemma: (S, d) is a metric space and $T: S \to S$ is a contraction, then T is continuous (proof: SLP)

Contraction mapping theorem: Let (S, d) a complete metric space, and T be an operator $T: S \to S$ which is also a contraction with modulus $\beta \in (0,1)$. Then
i) There is unique fixed point of operator T
ii) $d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$, where v^* is a fixed point of T

Proof: steps
1) $\{v_n\}_{n=1}^{\infty}$, where $v_n = T^n v_0$, converges to v^*
2) v^* is indeed a fixed point of T
3) It is unique

* Intermediate step

$$d(v_n, v_{n-1}) = d(Tv_{n-1}, Tv_{n-2}) \underbrace{\leq}_{\text{because } T \text{ is a contraction}} \beta d(v_{n-1}, v_{n-2}) = \beta d(Tv_{n-2}, Tv_{n-3})$$

$$\leq \beta^2 d(v_{n-2}, v_{n-3})$$

$$\vdots$$

$$\leq \beta^n d(v_0, v_1)$$

Let's show that $\{v_n\}_{n=1}^{\infty}$ is Cauchy. Take n, m (WLOG $m>n$) and

$$d(v_m, v_n) \leq d(v_m, v_{m-1}) + d(v_{m-1}, v_n) \quad \text{(triangle inequality)}$$
$$\leq d(v_m, v_{m-1}) + d(v_{m-1}, v_{m-2}) + \cdots + d(v_{n+1}, v_n)$$
$$\leq \beta^m d(v_0, v_1) + \beta^{m-1} d(v_0, v_1) + \cdots + \beta^{n+1} d(v_0, v_1)$$
$$= \beta^{n+1} d(v_0, v_1)(1 + \beta + \beta^2 + \cdots + \beta^{m-n})$$
$$\leq \frac{\beta^{n+1}}{1-\beta} d(v_0, v_1) \implies \{v_n\} \text{ converges to } v^*, \text{ where}$$
$$v^* = \lim_{n\to\infty} T^n v_0$$

2) Need to show that $v^* = \lim_{n\to\infty} T^n v_0$ is a fixed point $Tv^* = v^*$

$$Tv^* = T\left(\lim_{n\to\infty} T^n v_0\right) \underset{\text{Continuity of } T}{=} \lim_{n\to\infty}\left(T(T^n v_0)\right) = \lim_{n\to\infty} v_{n+1} = v^*$$

$\therefore$ is indeed a fixed point

3) Need to show that v^* is unique.

Suppose not $\implies \exists\, \tilde{v} \neq v^*$ st $T\tilde{v} = \tilde{v}$

$$0 < \varepsilon = d(\tilde{v}, v^*) = d(T\tilde{v}, Tv^*) \leq \beta\, d(v^*, \tilde{v}) = \beta\varepsilon$$

contradiction $\therefore$ v^* is unique f.p.

Finally, show $d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$. Proceed by induction

i) $n=0$. $d(v_0, v^*) \leq d(v_0, v^*)$ ✓

ii) $n=k$. assume it works for k, $d(T^k v_0, v^*) \leq \beta^k d(v_0, v^*)$

iii) Show it works for $k+1$.

$$d(T^{k+1} v_0, v^*) = d(T(T^k v_0), Tv^*)$$
$$\leq \beta\, d(T^k v_0, v^*) \quad \text{(by contraction)}$$
$$\leq \beta^k d(v_0, v^*) \quad \text{(by (ii))}$$
$$= \beta^{k+1} d(v_0, v_1)$$

Blackwell's Theorem: Let $X \subseteq \mathbb{R}^l$, $B(x)$ is a space of bounded functions, $f: X \to \mathbb{R}$. d is a supnorm.

Let $T: B \to B$ satisfying

i) monotonicity: $f, g \in B(x)$ s.t $f \leq g \Rightarrow (Tf)(x) \leq (Tg)(x)$

ii) discounting: Let $(f+a)(x)$ denote $f(x)+a$, where $a>0$ is a constant

There exists $\beta \in (0,1)$, s.t $\forall f \in B(x)$, $T(f+a)(x) \leq Tf + \beta a$

$\Rightarrow$ If T satisfies (i) and (ii), then it is a contraction.

Proof: Fix $x \in X$, $f, g \in B(x)$

$$f(x) - g(x) \leq \sup_{y \in X} |f(y) - g(y)| = d(f,g)$$

$$f - g \leq d(f,g) \quad \text{(simplified notation)}$$

$$f \leq g + d(f,g)$$

$$Tf \underset{\text{monotonicity}}{\leq} T(g + d(f,g)) \underset{\text{discounting.}}{\leq} Tg + \beta d(f,g)$$

$$Tf - Tg \leq \beta d(f,g)$$

Symmetrically, you can show, $Tg - Tf \leq \beta d(f,g)$

$$\underbrace{\underbrace{\sup_{x \in X} |Tf(x) - Tg(x)|}_{d(Tf,Tg)} \leq \beta d(f,g)}_{\text{This is our def of contraction. } \therefore T \text{ is contraction}}$$

Neoclassical Growth model

$$Tv(k) = \max_{0 \le k' \le f(k)} \left[u\left(f(k) - k'\right) + \beta v(k') \right]$$

1) If $u(\cdot)$ is bounded $\Rightarrow$ $T: B[0,\infty) \to B[0,\infty)$ (is bounded because the Domain is bounded).

2) Monotonicity: Let's take $v, w \in B[0,\infty)$, st $v \le w$

Let $g_v(k)$ a policy function (is argmax for $v(k)$)

$$Tv(k) = u\left(f(k) - g_v(k)\right) + \beta v\left(g_v(k)\right)$$

$$\le u\left(f(k) - g_v(k)\right) + \beta w\left(g_v(k)\right)$$

$$\le \max_{0 \le k' \le f(k)} \left[u\left(f(k) - k'\right) + \beta w(k') \right] = Tw(k)$$

$\therefore$ monotonicity holds.

3) Check discounting: $T(v+a)(k) = \max_{0 \le k' \le f(k)} \left[u\left(f(k) - k'\right) + \beta \left[v(k') + a \right] \right]$

$= \max_{0 \le k' \le f(k)} \left[u\left(f(k) - k'\right) + \beta v(k') \right] + \beta a$ (since doesn't affect the maximization problem.

$= Tv(k) + \beta a$ $\therefore$ Discounting holds.

Lecture:

→ Principle of optimality

Oct 12, 2023

Sequential Problem:

$$w(x_0) = \sup_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{st} \quad x_{t+1} \in \Gamma(x_t), \; x_0 \text{ given}$$

Recursive formulation

$$v(x) = \sup_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y) \right]$$

Q. 1) Under which conditions $v(x) = w(x)$?

2) When $\{x_{t+1}\}$ from SP be the same as the one implied by FE?

Preliminaries:

X: set of values that state x_t can take

Γ: Correspondence $X \rightrightarrows X$ is feasibility correspondence.

Def: A graph of Γ, A, is $A = \{(x,y) : y \in \Gamma(x)\}$

Period return $F(x,y)$, $F: A \to \mathbb{R}$

Fundamentals: $(\underbrace{\Gamma, X}_{\text{Describe production technology}}, \underbrace{F, \beta}_{\text{describe preferences.}})$

Def: A sequence of states $\{x_t\}_{t=0}^{\infty}$ is a plan

Def: Given x_0, $\Pi(x_0) = \{ \{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t) \}$ is a set of feasible plans

$\bar{x}$ be a generic element of $\Pi(x_0)$

Assumption 1:

$\forall x_0 \in X$, $\Pi(x_0)$ is not empty

Assumption 2:

$\forall x_0$, for any $\bar{x} \in \Pi(x_0)$

$$\lim_{n\to\infty} \sum_{t=1}^{n} \beta^t F(x_t, x_{t+1}) \text{ exists } (\text{can be } +\infty \text{ or } -\infty$$

Sufficient conditions for A2:

1) $\beta \in (0,1)$, F is bounded

counterexample.

$$\beta = 1 \qquad F(x_t, x_{t+1}) = \begin{cases} -1, & t \text{ is even} \\ 1, & t \text{ is odd} \end{cases}$$

2) Define $F^+ = \max\{0, F\}$, $F^- = \min\{0, -F\}$

If $\lim_{n\to\infty} \sum_{t=1}^{n} \beta^t F^+(x_t, x_{t+1}) < +\infty$

or $\lim_{n\to\infty} \sum_{t=1}^{n} \beta^t F^-(x_t, x_{t+1}) < +\infty$

} is bounded above, A2 holds

3) $\forall x_0 \in X$ and any $\bar{x} \in \Pi(x_0)$, $\exists\, \theta \in (0, 1/\beta)$, $c \in (0, +\infty)$ s.t.

$F(x_t, x_{t+1}) \leq c \cdot \theta^t$

(any of these is sufficient).

Def: $\{u_n\}$ is a sequence of functions $u_n(\bar{x}_t) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$

Given A2, we know the limit

$\lim_{n\to\infty} u_n$ exists, $\quad u_n(\bar{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$

$$w(x_0) = \sup_{\bar{x} \in \Pi(x_0)} u(\bar{x})$$

Principle of optimality

Theorem: Let (X, Γ, F, β) s.t A1 and A2 hold

Recursive formulation.

$\Rightarrow$ 1) w (solution of seq problem) satisfy F.E. (function equation).

2) If $\forall x_0 \in X$, and $\forall \bar{x} \in \Pi(x_0)$, $\boxed{\lim_{n\to\infty} \beta^n v(x_n) = 0}$, $\Rightarrow v = w$

* If we get multiple values for w, we pick the one that meet the principle of optimality. (There can be multiple solution in the recursive formulation).

Theorem: Suppose (X, Γ, F, β) satisfy A1 & A2.

1) Let $\bar{x} \in \Pi(x_0)$ attain supremum in SP (seq. problem)

$\Rightarrow \forall t \geq 0 \quad w(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta w(\bar{x}_{t+1})$

} solution of SP solves the FE.

2) Let $\hat{x} \in \Pi(x_0)$ be a feasible plan that satisfies

$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$ and additionally $\limsup_{t\to\infty} \beta^t w(\hat{x}_t) \leq 0$

$\Rightarrow \{\hat{x}_t\}$ attains supremum in SP, for given x_0

Solution of FE solves SP if $\lim_{t\to\infty} \beta^t w(\hat{x}_t) \leq 0$

Lecture:

Oct 17, 2023

→ a simple first cut model of agg. time-series
→ Simplifying the model

a simple model of aggregate time series.

Households: make consumption, labor supply, investment decisions indexed by $i = 1, \dots, I$

Firms that produce consumption good use capital and labor indexed by $j_c = 1, \dots, J_c$

Firms that produce investment good use capital and labor indexed by $j_x = 1, \dots, J_x$

Consumption good firms : $c_{jt} \leq F_t^j(k_{ct}^j, n_{ct}^j) \quad \forall j \in \{1, \dots, J_c\}$

Investment good firms : $x_{jt} \leq F_t^j(k_{xt}^j, n_{xt}^j) \quad \forall j \in \{1, \dots, J_x\}$

Households : $U_i(\tilde{c}, \tilde{\ell})$, where $\tilde{c} = \{c_t\}_{t=0}^{\infty}$, $\tilde{\ell} = \{\ell_t\}_{t=0}^{\infty}$ ($\tilde{\ell}$ = leisure)

- $u > 0$, $u' > 0$, $u'' < 0$
- Endowments : k_0^i, $\{\bar{n}_t^i\}_{t=0}^{\infty}$ ($\bar{n}$: time endowment, n: labor)
- Ownership structure : $\{\theta_{ij}^c\}_{j=1}^{J_c}$, $\{\theta_{ij}^x\}_{j=1}^{J_x}$

<u>Def</u>: A CE in this environment is prices $\{p_{ct}, p_{xt}, w_t, r_t\}_{t=0}^{\infty}$, allocations for HH $\{c_t^i, n_t^i, l_t^i, x_t^i, k_t^i\}_{t=0}^{\infty}$, allocations for consumption-good firms $\{c_t^j, k_{ct}^j, n_{ct}^j\}_{t=0}^{\infty}$ $\forall j \in \{1, \dots, J_c\}$ & allocations for investment firms $\{x_t^j, k_{xt}^j, n_{xt}^j\}_{t=0}^{\infty}$ such that

i) Given prices, allocation for HH solves

$$u_i(\tilde{c}, \tilde{l}) \longrightarrow \max_{\{c, l, n, x, k\}}$$

s.t

$$\sum_{t=0}^{\infty} \left(p_{ct} c_t^i + p_{xt} x_t^i\right) \leq \sum_{t=0}^{\infty} \left(r_t k_t^i + w_t n_t^i\right) + \Pi_i$$

$$k_{t+1}^i = (1-\delta) k_t^i + x_t^i \quad , \quad 0 \leq n_t^i + l_t^i \leq \bar{n}_t^i$$

(labor + leisure) (Time endowment).

2) Given prices, cons good firms allocation solves:

$$\sum_{t=0}^{\infty} \left(p_{ct} c_t^j - w_t n_{ct}^j - r_t k_{ct}^j \right) \longrightarrow \max_{\{c^j, n^j, k^j\}_{t=0}^{\infty}}$$

s.t. $\quad c_t^j \leq F_t^j \left(k_{ct}^j, n_{ct}^j\right)$

3) Investment firms

$$\sum_{t=0}^{\infty} \left(p_{xt} x_t^j - w_t n_{xt}^j - r_t k_{xt}^j \right) \longrightarrow \max_{\{x_t^j, n_{xt}^j, k_{xt}^j\}}$$

s.t $\quad x_t^j \leq F_{xt}^j \left(k_{xt}^j, n_{xt}^j\right)$

<u>Profits</u>: $$\Pi_i = \sum_{j_c=1}^{J_c} \theta_{ij_c}^c \sum_{t=0}^{\infty} \left(p_{ct} \cdot c_t^{j_c} - w_t n_{ct}^{j_c} - r_t k_{ct}^{j_c} \right)$$

$$+ \sum_{j_x=1}^{J_x} \theta_{ij_x}^x \sum_{t=0}^{\infty} \left(p_{xt} \cdot x_t^{j_x} - w_t n_{xt}^{j_x} - r_t k_{xt}^{j_x} \right)$$

3) Market clear:

$$\sum_{i=1}^{I} c_t^i = \sum_{j_c=1}^{J_c} c_{ct}^{i} \qquad \sum_{i=1}^{I} n_t^i = \sum_{j_c=1}^{J_c} n_{ct}^{j_c} + \sum_{j_x=1}^{J_x} n_{xt}^{j_x}$$

$$\sum_{i=1}^{I} x_t^i = \sum_{j_x=1}^{J_x} x_t^{j_x} \qquad \sum_{i=1}^{I} k_t^i = \sum_{j_c=1}^{J_c} k_t^{j_c} \quad \sum_{j_x=1}^{J_x} k_t^{j_x}$$

Simplifying firm side:

1) CRS: To make profit zero.

$$\forall \lambda > 0 \quad F(\lambda k, \lambda n) = \lambda F(k, n)$$

$$\underbrace{F_x'}_{r} k + \underbrace{F_n'}_{} \cdot n = F(k, n)$$

2) Representative technology in each sector

$$\forall j_c, j_c' \quad F_{ct}^{j_c} = F_{ct}^{j_c'} = F_{ct}$$

$$\forall j_x, j_x' \quad F_{xt}^{j_x} = F_{xt}^{j_x'} = F_{xt}$$

3) Collapse two sectors into one

$$F_{ct} = F_{xt}$$

* New market clearing: $\sum_{i=1}^{T} (c_t^i + x_t^i) = F_t(k_t, n_t)$

Simplifying the consumer side

1) Representative agent ($u_i = u_j$, $k_0^i = k_0^j$)
2) Homothetic aggregation (need stronger assumptions on $\succsim$, no need to have the same endowment).

Remark: Strict concavity needed

$$\sum_{i=1}^{I} \left(c_t^i + x_t^i \right) = F_t \left(k_t, n_t \right)$$

$$\underbrace{I\left(c_t^1 + x_t^1 \right)} = F_t\left(k_t, n_t \right) \Rightarrow c_t^1 + x_t^1 = F_t\left(\frac{k_t}{I}, \frac{n_t}{I} \right)$$

because agents make same decisions i.e. $(c_t^i + x_t^i)$ is constant across i.

$$c_t^1 + x_t^1 = F_t\left(k_t^1, n_t^1 \right) \rightarrow \text{agent 1 problem}$$

*SPP: $u(\tilde{c}_1, \tilde{l}_1) \rightarrow \max$

s.t. $c_t^i + x_t^i = F(k_t^i, n_t^i)$

$k_{t+1}^i = (1-\delta) k_t^i + x_t^i$, $0 \leq n_t^i + l_t^i \leq \bar{n}_t^i$

k_0^i given $\forall i$

Homothetic Aggregation

$x \sim y \longleftrightarrow (\lambda x) \sim (\lambda y)$

Def: Preferences are homothetic if $\forall x, y$ $u(x) = u(y) \longleftrightarrow u(\lambda x) = u(\lambda y)$

Theorem: If u is homogeneous of any degree, then $\succsim$ it represents are homothetic

Proof : i) u is hd-η , $u(\lambda x, \lambda y) = \lambda^{\eta} u(x, y)$

ii) take two points (x_1, y_1) & (x_2, y_2) st $u(x_1, y_1) = u(x_2, y_2)$

iii) Because hd-η : $u(\lambda x_1, \lambda y_1) = \lambda^{n} u(x_1, y_1)$

iv) Given (ii) : $\lambda^{n} u(x_2, y_2)$

v) Given hd-η : $\lambda^{n} u(x_2, y_2) = u(\lambda x_2, \lambda y_2)$

$\therefore$ $u(\lambda x_1, \lambda y_1) = u(\lambda x_2, \lambda y_2)$ (u is homothetic or $\succsim$ are homothetic)

Lecture:

→ Homothetic aggregation

→ Government policy + Ramsey problem.

Oct 19, 2023

Last time: Representative agent.

Assume $\succsim$ (+ strict concavity)

$k_0^i = k_0^j$, $\{\bar{n}_t^i\} = \{\bar{n}_t^j\}$ $\forall t$ $\forall i,j$

$\Rightarrow$ Collapses to NGM

Homothetic Aggregation:

Assume homothetic $\succsim$

Ex: Need $u(\lambda x, \lambda y) = \lambda^{\eta} \cdot u(x,y)$ (homogeneous of degree η)

Ex: Constant elasticity of substitution utility function

$$u(x,y) = \frac{1}{1-\sigma} x^{1-\sigma} + \frac{1}{1-\sigma} y^{1-\sigma}$$

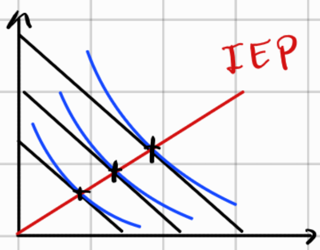
$$u(\lambda x, \lambda y) = \lambda^{1-\sigma} \cdot u(x,y) \rightarrow \text{homogenous of degree } 1-\sigma$$

If $\sigma = 1 \Rightarrow$ log preferences

$\sigma = 0 \Rightarrow$ perfect substitutes

$\sigma = \infty \Rightarrow$ perfect complements

Theorem: If $\succsim$ are homothetic the income expansion path is linear



$u(x,y) \longrightarrow \max$

$P_x X + P_y Y \leq W$ (income: W)

claim: $(X(\lambda W), Y(\lambda W)) = \lambda (X(W), Y(W))$

Proof: i) Is $\lambda(x(w), y(w))$ affordable?

$$P_x(\lambda x) + P_y(\lambda y) \leq \lambda w \Rightarrow \text{yes, affordable}$$

ii) Is $\lambda(x(w), y(w))$ optimum?

Suppose not, $\exists\ (\hat{x}, \hat{y})$ affordable, $P_x \hat{x} + P_y \hat{y} \leq \lambda w$

$u(\hat{x}, \hat{y}) > u(\lambda x(w), \lambda y(w))$

Consider $(\tilde{x}, \tilde{y}) = \left(\frac{\hat{x}}{\lambda}, \frac{\hat{y}}{\lambda}\right)$

$$\Rightarrow u(\tilde{x}, \tilde{y}) = u\left(\frac{\hat{x}}{\lambda}, \frac{\hat{y}}{\lambda}\right) = \frac{1}{\lambda^n} \cdot u(\hat{x}, \hat{y}) > \frac{1}{\lambda^n} u(\lambda x(w), \lambda y(w))$$

$$= \frac{\lambda^n}{\lambda^n} \cdot u(x(w), y(w)) = u(x(w), y(w)) \quad \text{(contradiction)}$$

Take time period t: $\quad C_t^1(w_1) + C_t^2(w_2) + \dots + C_t^I(w_I)$

Because of homothetic: $\quad w_1 C_t^1(1) + w_2 C_t^2(1) + \dots + w_I C_t^I(1)$

By aggregation: $\quad (w_1 + \dots + w_I)\, C_t \qquad (C_t = C_t^1 + \dots + C_t^I;\ C_t^1 = \dots = C_t^I)$

$\Rightarrow$ we solve for one guy and distribute the solution

Algorithm

1) $\sum_{t=1}^{I} k_0^i, \quad \sum_{t=1}^{I} \{\bar{n}_t^i\}$

2) Solve SPP $\Rightarrow$ get eqm allocations $\quad$ (* SPP only have allocations, not prices)

3) Recover prices from optimality conditions for CE

Example: $u(\tilde{c}, \tilde{\ell}) \longrightarrow \max \quad$ s.t $\sum_{t=0}^{\infty} P_t (C_t + X_t) \leq \sum_{t=0}^{\infty} [r_t k_t + w_t n_t]$

$$\tilde{k}_0 = \sum_{t=0}^{I} k_0^i$$

(X_t: Investment).

$$k_{t+1} = (1-\delta) k_t + X_t$$

$P_0 = 1.$ w_0, r_0 given

$$\mathcal{L} = \sum \beta^t u(c_t, l_t) + \lambda_t [BC]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u_c'(t) - \lambda P_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = -\beta^t u_l'(t) +$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -P_t + P_{t+1}(1-\delta) + r_{t+1} = 0$$

$$\boxed{\frac{u_c'(t)}{u_l'(c_t, l_t)} = \frac{P_t}{W_t}} \quad (\star)$$

$$\boxed{\frac{P_t}{P_{t+1}} = \frac{r_{t+1}}{P_{t+1}} + (1-\delta)} \quad (\star\star)$$

Firm: $\sum P_t F(k_t, n_t) - w_t n_t - r_t k_t$

$$P_t F_k' = r_t \longrightarrow P_{t+1} F_k'(t+1) = r_{t+1}$$

$$P_t F_n' = w_t \qquad P_{t+1} F_n'(t+1) = w_{t+1}$$

from $(\star\star)$

$$\frac{P_t}{P_{t+1}} = \underbrace{\frac{r_{t+1}}{P_{t+1}}}_{F_k'(t+1)} + 1 - \delta \Rightarrow$$

from $(\star)$

$$\frac{P_t}{P_{t+1}} = \frac{u_c'(t)}{\beta u_c'(t+1)} = F_k'(t+1) + (1-\delta)$$

Lecture:

→ Tax-distorted CE (TDCE)

→ Ramsey problem.

Oct 24, 2023

Notation:

$\{g_t\}_{t=0}^{\infty}$ - Sequence of govt. expenditures

τ_{ct} - Consumption tax

τ_{xt} - Investment tax

τ_{nt} - labor income tax

τ_{kt} - Capital income tax

T_t - Lump-sum tax ($T_t < 0$) or transfer ($T_t > 0$)

Def: A tax-distorted CE (TDCE) given the fiscal policy $\{g_t, \tau_{ct}, \tau_{xt}, \tau_{nt}, \tau_{kt}, T_t\}_{t=0}^{\infty}$ is the price system $\{p_{ct}, p_{xt}, w_t, r_t\}_{t=0}^{\infty}$, allocation for consumer $\{c_t, x_t, n_t, l_t, k_t\}_{t=0}^{\infty}$ and allocations for firms $\{y_t, k_t^f, n_t^f\}_{t=0}^{\infty}$, such that,

i) Given the prices, HH solves:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \longrightarrow \max_{\{c_t, l_t, n_t, k_t, x_t\}}$$

$$\text{s.t} \quad \sum_{t=0}^{\infty} \left[p_{ct}(1+\tau_{ct}) c_t + p_{xt}(1+\tau_{xt}) x_t \right] \leq \sum_{t=0}^{\infty} \left[(1-\tau_{kt}) r_t k_t + (1-\tau_{nt}) w_t n_t + T_t \right]$$

$$k_{t+1} = (1-\delta) k_t + x_t$$

$$0 \leq n_t + l_t \leq \bar{n}_t, \ k_0 \text{ is given} \qquad c_t, k_t, n_t, l_t \geq 0$$

ii) Given prices, $\{y_t, k_t^f, n_t^f\}_{t=0}^{\infty}$ solve the firm's problem

$$\max_{\{y_t, k_t, n_t\}} \sum_{t=0}^{\infty} p_t y_t - \sum_{t=0}^{\infty} (w n_t + r_t k_t) \qquad \text{s.t} \ y_t \leq F(k_t, n_t)$$

$$n_t^f, y_t, k_t^f \geq 0$$

iii) Market clear

$$n_t = n_t^f$$

$$k_t = k_t^f$$

$$c_t + x_t + g_t = F(k_t, n_t)$$

iv) Govt. Budget Constraint

$$\sum_{t=0}^{\infty} \left(p_t g_t + T_t \right) = \sum_{t=0}^{\infty} \left(\tau_{ct}\, p_{ct} c_t + \tau_{xt} x_t p_{xt} + w_t n_t + \tau_{kt} k_t r_t \right)$$

Showing that Govt. BC is redundant:

(HHBC)

$$\sum_{t=0}^{\infty} \left[p_t (1-\tau_{ct}) c_t + p_t (1+\tau_{xt}) x_t \right] = \sum_{t=0}^{\infty} \left[r_t k_t (1-\tau_{kt}) + w_t n_t (1-\tau_{nt}) + T_t \right]$$

Rearranging...

$$\sum_{t=0}^{\infty} \left[p_t c_t \tau_{ct} + p_t \tau_{xt} x_t + \tau_{kt} r_t k_t + w_t n_t \tau_{nt} \right] = \sum_{t=0}^{\infty} \left[r_t k_t + w_t n_t - p_t c_t - p_t x_t + T_t \right]$$

assume $F(\cdot)$ is CRS. Then $p_t \underbrace{y_t}_{c_t + x_t + g_t} = w_t n_t + r_t k_t$ (zero-profit).

Then

$$\sum_{t=0}^{\infty} \left[p_t c_t \tau_{ct} + p_t \tau_{xt} x_t + \tau_{kt} r_t k_t + w_t n_t \tau_{nt} \right] = \sum_{t=0}^{\infty} \left[p_t g_t + T_t \right]. \quad \text{(Govt. BC)}.$$

<u>Exercise</u>: Relax CRS assumption & show the same holds.

Ramsey Problem:

$\{g_t\}$ given

$\tau_{ct} = \tau_{xt} = 0$

$T_t = 0$

Optimal sequence of $\{\tau_{kt}, \tau_{nt}\}$?

$$\sum_{t=0}^{\infty} \beta^t U(c_t, n_t) \longrightarrow \max \quad \text{s.t.} \quad \{c_t(\tau), k_t(\tau), n_t(\tau), \ell_t(\tau), x_t(\tau)\} \text{ is TDCE}$$

where $\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} \left[r_t k_t \tau_{kt} + w_t n_t \tau_{nt} \right]$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t) + \lambda \underbrace{\left[\sum_{t=0}^{\infty} \left[r_t k_t (1-\tau_{kt}) + w_t n_t (1-\tau_{nt}) \right] - \sum_{t=0}^{\infty} \left[p_t c_t + p_t x_t \right] \right]}_{BC\ (ADE)}$$

$$+ \sum_{t=0}^{\infty} \beta^t \mu_t \underbrace{\left[(1-\delta) k_t + x_t - k_{t+1} \right]}_{\text{Law of motion for } k}$$

F.O.C:

(c_t): $\beta^t U_c'(t) = \lambda p_t$ (1)

(n_t): $-\beta^t U_n'(t) + \lambda w_t (1-\tau_{nt}) = 0$ (2)

(x_t): $-\lambda p_t + \beta^t \mu_t = 0$ (3)

(k_{t+1}): $-\beta^t \mu_t + \lambda \left[r_{t+1} (1-\tau_{kt+1}) \right] + \beta^{t+1} \mu_{t+1} (1-\delta) = 0$ (4)

Use (3) in (4)

$$\lambda p_t = \lambda \left[r_{t+1} (1-\tau_{kt+1}) \right] + \lambda p_{t+1} (1-\delta) \quad / \frac{1}{\lambda p_{t+1}}$$

$$\frac{p_t}{p_{t+1}} = \frac{r_t (1-\tau_{kt+1})}{p_{t+1}} + (1-\delta) \quad (*)$$

(1) + (2): $$\frac{U_c'(t)}{\beta U_c'(t+1)} = \frac{p_t}{p_{t+1}} \quad (**)$$

Combining (*) and (**)

$$u_c'(t) = \beta u_c'(t+1)\left[\frac{r_{t+1}}{p_{t+1}}\left(1-\tau_{k,t+1}\right)+\left(1-\delta\right)\right] \quad \text{(Euler Equation).}$$

Firm's Problem:

$$\sum_{t=0}^{\infty}\left[p_t F(k_t, n_t) - w_t n_t - r_t k_t\right] \longrightarrow \max$$

$$p_t F_n'(t) = w_t$$
$$p_t F_k'(t) = r_t$$

TDCE allocation (Given policy $\{\tau_{kt}, \tau_{nt}, g_t\}_{t=0}^{\infty}$)

1) normalize $p_0 = 1$: $p_t = \dfrac{\beta^t u_c'(t)}{u_c'(0)}$

2) $\dfrac{u_n'(t)}{u_c'(t)} = F_n'(t)\left(1-\tau_{nt}\right)$

3) $u_c'(t) = \beta u_c'(t+1)\left[\left(1-\tau_{kt}\right)F_k'(t+1) + \left(1-\delta\right)\right]$

4) $F_k' = r_t / p_t$

5) $F_n' = w_t / p_t$

6) $n_t = n_t^f$

7) $k_t = k_t^f$

8) $c_t + g_t + x_t = F(k_t, n_t)$

9) $\sum_{t=0}^{\infty}\left[p_t c_t + p_t x_t\right] = \sum_{t=0}^{\infty}\left[r_t k_t(1-\tau_{kt}) + w_t n_t\left(1-\tau_{nt}\right)\right]$

10) $k_{t+1} = (1-\delta)k_t + x_t$

These equations can be collapsed into 3 equations.
→ Implementability conditions.

Lecture:

→ Ramsey problem

→ Chamley - Judd Result

→ TDCE is not PO

(... continuation) Implementability Constraint

$$\sum_{t=0}^{\infty} P_t C_t + P_t (k_{t+1} - (1-\delta)k_t) = \sum_{t=0}^{\infty} (1-\tau_{nt}) w_t n_t + (1-\tau_{kt}) r_t k_t$$

$$\sum_{t=0}^{\infty} P_t C_t = \sum_{t=0}^{\infty} (1-\tau_{nt}) w_t n_t + (1-\tau_{k0}) r_0 k_0 + (1-\delta) k_0 P_0 + \sum_{t=0}^{\infty} k_{t+1} \underbrace{\left[-P_t + P_{t+1}(1-\delta) + (1-\tau_{kt+1}) r_{t+1}\right]}_{=0 \text{ (from FOC)}}$$

$$P_t \downarrow \frac{\beta^t u_c'(t)}{u_c'(0)}$$

$$(1-\tau_{nt}) w_t = \frac{u_n'(t)}{u_c'(t)\cancel{F_n'(t)}} \cdot \cancel{F_n'(t)} P_t = \frac{u_n'(t)}{u_c'(t)} \cdot \beta^t \frac{u_c'(t)}{u_c'(t)}$$

(IC): $$\sum_{t=0}^{\infty} \beta^t \frac{u_c'(t)}{u_c'(0)} C_t - \beta^t \frac{u_n'(t)}{u_c'(0)} n_t = (1-\tau_{k0}) F_k'(0) P_0 k_0 + (1-\delta) k_0 P_0$$

$$\Longleftrightarrow \sum_{t=0}^{\infty} \beta^t \left(u_c'(t) c_t - u_n'(t) n_t\right) = \underbrace{u_c'(0)\left[(1-\tau_{k0}) F_k'(0) P_0 k_0 + (1-\delta) k_0 P_0\right]}_{\text{constant. (call it } \Gamma_0\text{)}}$$

TDCE Allocation (Given the FP) must satisfy:

i) feasibility constraint: $C_t + X_t + g_t = F(k_t, n_t)$

ii) Law of motion: $k_{t+1} = (1-\delta) k_t + X_t$

iii) Implementability: $$\sum_{t=0}^{\infty} \beta^t \left(u_c'(t) c_t - u_n'(t) n_t\right) = \Gamma_0$$

↳ Government should be able to raise enough resources to implement its policy (?).

(it wold be different with lump-sum tax).

<u>Conversely</u>: If an allocation $\{c_t, k_{t+1}, x_t, l_t, n_t\}_{t=0}^{\infty}$ satisfies i)-iii), then $\exists$ $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and $\{\tau_{kt}, \tau_{nt}\}_{t=0}^{\infty}$ which TDCE where govt. raises enough to finance $\{g_t\}_{t=0}^{\infty}$

<u>Remark</u>: g_t is not a decision, it's given from outside of the model

Ramsey Problem:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \longrightarrow \max_{\{c_t, l_t, n_t, x_t, k_{t+1}\}_{t=0}^{\infty}}$$

s.t.
$$c_t + x_t + g_t = F(k_t, n_t)$$
$$k_{t+1} = (1-\delta)k_t + x_t$$
$$\sum_{t=0}^{\infty} \beta^t \left(u_c'(t) c_t - u_n'(t) n_t\right) = \Gamma_0$$

$$\mathcal{L} = \sum^{\infty} \beta^t u(c_t, 1-n_t) + \lambda \left[\Gamma_0 - \sum_{t=0}^{\infty} \beta^t \left(u_c'(t)c_t - u_n'(t) n_t\right)\right]$$
$$+ \sum_{t=0}^{\infty} \gamma_t \left[F(k_t, n_t) - c_t - x_t - g_t\right]$$
$$+ \sum_{t=0}^{\infty} \eta_t \left[(1-\delta)k_t + x_t - k_{t+1}\right]$$

$$\mathcal{L} = \lambda \Gamma_0 + B(c_t, n_t, \lambda) + \sum_{t=0}^{\infty} \gamma_t \left[F(k_t, n_t) - c_t - x_t - g_t\right]$$
$$+ \sum_{t=0}^{\infty} \eta_t \left[(1-\delta)k_t + x_t - k_{t+1}\right]$$

*

where
$$= \sum^{\infty} \beta^t u(c_t, 1-n_t) + \lambda \left[\Gamma_0 - \sum_{t=0}^{\infty} \beta^t \left(u_c'(t)c_t - u_n'(t) n_t\right)\right]$$
$$= \sum_{t=0}^{\infty} \beta^t \Big[\underbrace{u(c_t, 1-n_t) - \lambda\left(u_c'(t) c_t - u_n'(t) n_t\right)}_{B(c_t, n_t, \lambda_t)}\Big]$$

F.O.C. c: $B_c'(t) = \gamma_t$

m: $B_m'(t) = \gamma_t F_m'(t)$

x: $-\gamma_t + \eta_t = 0$

k_{t+1}: $-\eta_t + \eta_{t+1}(1-\delta) + \gamma_{t+1}(F_k'(t+1)) = 0$

$$\Rightarrow \frac{B_m'(t)}{B_c'(t)} = F_m'(t) \; ; \; \frac{\eta_t}{\eta_{t+1}} = F_k'(t+1) + 1 - \delta \; ; \; \frac{B_c'(t)}{B_c'(t+1)} = F_k'(t+1) + 1 - \delta$$

* Assumptions: (convergence to a constant)

$C_t^{RP} \to C_\infty^{RP}$

$X_t^{RP} \to X_\infty^{RP}$

$m_t^{RP} \to m_\infty^{RP}$

$l_t^{RP} \to l_\infty^{RP}$

* As $t \to \infty$

$$\frac{B_c'(t)}{B_m'(\infty)} = F_m'(\infty)$$

$$\boxed{\frac{1}{\beta} = 1 - \delta + F_k'(\infty)}$$

EE. Ramsey Problem

[* NGM: with $\delta = 0$ there is no SS]

TDCE Euler Eq (Given FP $\{\tau_{nt}, \tau_{kt}, g_t\}$):

$$u_c'(t) = \beta u_c'(t+1)\left[(1-\tau_{kt+1})F_k'(t+1) + 1 - \delta\right]$$

as $t \to \infty$: $$\boxed{\frac{1}{\beta} = (1-\tau_{k,\infty})F_k'(\infty) + 1 - \delta}$$

EE. TDCE

$\Rightarrow \boxed{\tau_{k\infty} = 0}$ Chamley - Judd Result (EE R.P. = EE.TDCE)

Lecture:

Oct 31, 2023

→ TDCE is not always PO

→ Some examples when it is

Example 1: Govt has lump-sum taxes and transfers, not distortionary taxes.

$\{g_t\}_{t=1}^{\infty}$, $\tau_{ct} = \tau_{kt} = \tau_{xc} = \tau_{nt} = 0$

Govt BC: $\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} T_t$

[In this setting, the TDCE is P.O.]

* Claim: TDCE is PO

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \longrightarrow \max$$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t c_t + p_t (k_{t+1} - (1-\delta) k_t) = \sum_{t=0}^{\infty} r_t k_t + w_t n_t + T_t$$

f.o.c.:

(c_t): $\beta^t u_c'(t) = \lambda p_t$

(n_t): $\beta^t u_n'(t) = \lambda w_t$

(k_{t+1}): $-p_t + r_{t+1} + (1-\delta) p_{t+1} = 0$

$$\Rightarrow \frac{p_t}{p_{t+1}} = \frac{r_{t+1}}{p_{t+1}} + (1-\delta)$$

$$\frac{u_c'(t)}{\beta u_c'(t+1)} = \boxed{\frac{r_{t+1}}{p_{t+1}}} + (1-\delta) \quad (1)$$

$F_k'(t+1)$

$$\frac{u_n'(t)}{u_c'(t)} = \boxed{\frac{w_t}{p_t}} \quad (2)$$

$F_n'(t)$

$$c_t + x_t + g_t = F(k_t, n_t) \quad (3)$$

Consider SPP:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \longrightarrow \max_{\{c_t, n_t\}_{t=0}^{\infty}}$$

s.t $\quad c_t + x_t = \hat{F}(k_t, n_t)$, $\hat{F}(k_t, n_t) = F(k_t, n_t) - g_t$

and $k_{t+1} = (1-\delta)k_t + x_t$, k_0 given.

f.o.c for SPP:

$(c_t) \quad \beta^t u_c'(t) = \lambda_t$

$(n_t) \quad \beta^t u_n'(t) = \lambda_t \hat{F}_n'(t)$

$(k_{t+1}) \quad -\lambda_t + \lambda_{t+1}\left[\hat{F}_k'(t+1) + (1-\delta)\right] = 0$

$$\Rightarrow \boxed{\frac{u_c'(t)}{\beta u_c'(t+1)} = \hat{F}_k'(t+1) + (1-\delta) \quad ; \quad \frac{u_n'(t)}{u_c'(t)} = \hat{F}_n'(t)}$$

This is the same as for TDCE $\therefore$ TDCE is P.O. (w/ Lump Sum Transfer/taxes)

Example 2: Stricter case (Because of the Govt BC written as ADE style)

$\tau_{kt} = \tau_{nt} = \tau_t \quad \forall t$

$\tau_{ct} = \tau_{kt} = 0 \quad \forall t$

$T_t = 0 \quad \forall t$

Govt. BC: $\quad p_t g_t = \tau_t r_t k_t + \tau_t w_t n_t \quad \forall t$

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \longrightarrow \max$$

$$\sum_{t=0}^{\infty} p_t c_t + p_t\left(k_{t+1} - (1-\delta)k_t\right) = \sum_{t=0}^{\infty}\left(r_t k_t (1-\tau_t) + w_t n_t (1-\tau_t)\right)$$

F.O.C.:

(c_t): $\beta^t u_c'(t) = \lambda p_t$

(n_t): $\beta^t u_n'(t) = \lambda w_t (1-\tau_t)$

(k_{t+1}): $-p_t + p_{t+1}(1-\delta) + r_{t+1}(1-\tau_{t+1}) = 0$

$$\Rightarrow \frac{p_t}{p_{t+1}} = \frac{r_{t+1}}{p_{t+1}}(1-\tau_t) + (1-\delta)$$

and

$$\boxed{\begin{aligned} \frac{u_c'(t)}{\beta u_c'(t+1)} &= \frac{r_{t+1}}{p_{t+1}}(1-\tau_t) + (1-\delta) \\ \frac{u_n'(t)}{u_c'(t)} &= \frac{w_t}{p_t}(1-\tau_t) \\ c_t + x_t + g_t &= F(k_t, n_t) \end{aligned}}$$

NOTE the trick

$c_t + x_t + \boxed{g_t} = F(k_t, n_t)$

$c_t + x_t + \boxed{\tau_t F(\cdot)} = F(k_t, n_t)$

↳ They are equal given Govt BC (feasibility constraint)

Now consider the SPP

$$\sum^{\infty} \beta^t u(c_t, n_t) \longrightarrow \max$$

s.t.

$$c_t + x_t = (1-\tau_t) F(k_t, n_t)$$

$$0 \leq n_t + l_t \leq 1$$

k_0 given

$\beta^t u_c'(t) = \lambda_t$

$\beta^t u_n'(t) = \lambda_n (1-\tau_t) F_n'(t)$

$-\lambda_t + \lambda_{t+1}\left[(1-\tau_{t+1}) F_k'(t+1) + 1-\delta\right] = 0$

$$\Rightarrow \boxed{\begin{aligned} \frac{u_c'(t)}{\beta u_c'(t+1)} &= (1-\tau_{t+1}) F_k'(t+1) + 1-\delta \\ \frac{u_n'(t)}{u_c'(t)} &= (1-\tau_t) F_n'(t) \end{aligned}}$$

We showed that the problems have the same f.o.c. Now we show that the result is feasible

$$c_t + x_t + g_t = F(k_t, n_t) \longrightarrow \text{FC in CE}$$

$$p_t c_t + p_t x_t + \underbrace{p_t g_t}_{r_t \tau_t k_t + \tau_t w_t n_t} = p_t F(k_t, n_t)$$

$$p_t c_t + p_t x_t + \tau_t (r_t k_t + w_t n_t) = p_t F(k_t, n_t)$$

under CRS:

$$p_t c_t + p_t x_t = (1-\tau_t) p_t F(k_t, n_t) + \tau_t (\underbrace{p_t F(k_t, n_t) - w_t n_t - r_t k_t}_{0})$$

$$p_t c_t + p_t x_t = (1-\tau_t) p_t F_t$$

$$c_t + x_t = (1-\tau_t) F \longrightarrow \text{F.C. in SPP}$$

<u>Example 3</u>: Same world as EX2

$$\tau_{nt} = \tau_{kt} = \tau_t = 0.2 \quad , \text{ assume } F(k_t) = A k_t^{\alpha} \ , \ \alpha = \frac{1}{3}$$

Suppose $\tau \uparrow$ to 30% and labor supply is inelastic $u_n' = 0$

$$EE: \ u_c'(t) = \beta u_c'(t+1) \left[F_k'(t+1)(1-\tau_{t+1}) + (1-\delta) \right]$$

Steady-State: $k_{0.2}^*$

$$k_{t+1} = (1-\delta) k_t + x_t \longrightarrow x_{0.2}^* = \delta k_{0.2}^*$$

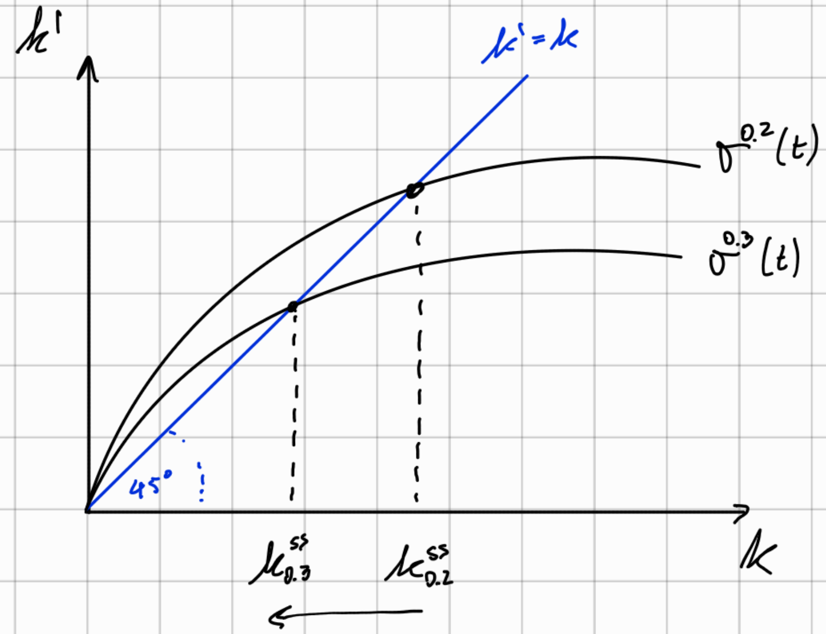
$$c_t = F(k_t) - k_{t+1} + (1-\delta) k_t \longrightarrow c_{0.2}^* = F(k_{0.2}^*) - \delta k_{0.2}^*$$

$$F_k'(k_t) = \alpha A k_t^{\alpha-1} \quad (*)$$

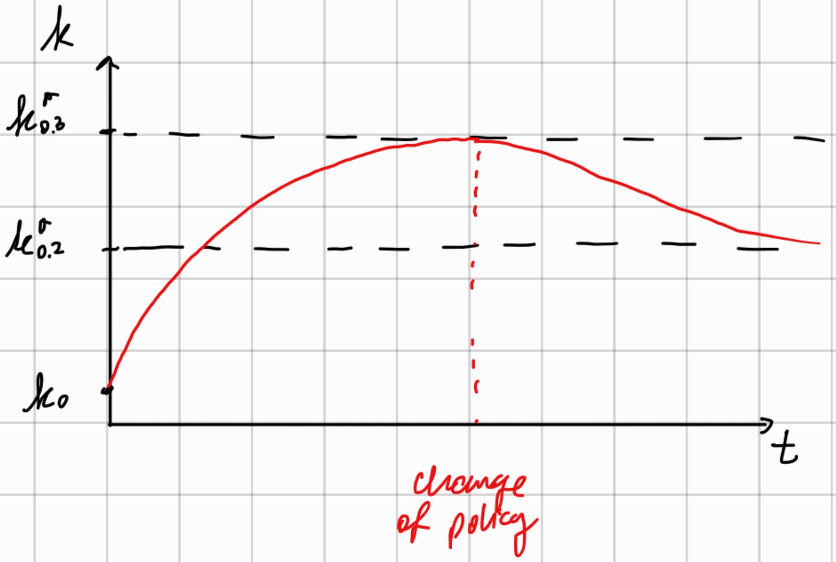
from EE: $u_{c_{0.2}^*}' = \beta u_{c_{0.2}^*}' \left[F_{k_{0.2}^*}'(1-0.2) + (1-\delta) \right]$

$$\frac{\left[\frac{1}{\beta} - (1-\delta)\right]}{(1-0.2)} = F_{k_{0.2}^*}' \overset{(*)}{=} \alpha A k_{0.2}^{*\,\alpha-1} \Rightarrow k_{0.2}^* = \left[\left[\frac{1/\beta - (1-\delta)}{1-0.2} \right] \frac{3}{A} \right]^{\frac{1}{\alpha-1}}$$

The excercise ask for $k^{ss}_{0.3}/k^{ss}_{0.2} = \left(\dfrac{1-0.2}{1-0.3}\right)^{\frac{1}{\alpha-1}} \approx 0.82$



$\sigma(t)$: policy function



Lecture: $\boxed{\text{Nov 2, 2023}}$

$\to$ Ramsey problem with consumption and labor income taxes.

TDCE:

Represented HH $\sum^{\infty} \beta^t u(c_t, n_t) \longrightarrow \max$

Rep. Firm $y_t \le F(k_t, n_t)$

Govt. has to finance $\{g_t\}_{t=0}^{\infty}$ Can use $\{\tau_{ct}, \tau_{nt}\}_{t=0}^{\infty}$ only

Govt. BC: $\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t c_t \tau_{ct} + w_t n_t \tau_{nt}$

HH BC: $\sum^{\infty} p_t(1+\tau_{ct})c_t + p_t k_{t+1} = \sum^{\infty} w_t n_t(1-\tau_{nt}) + r_t k_t + p_t(1-\delta)k_t \qquad (\lambda)$

assume interior solution

(c_t): $\beta^t u_c'(t) = \lambda p_t(1+\tau_{ct}) \qquad \Rightarrow \dfrac{u_c'(t)}{\beta u_c'(t+1)} = \dfrac{p_t}{p_{t+1}} \cdot \dfrac{(1+\tau_{ct})}{(1+\tau_{ct+1})}$

(n_t): $\beta^t u_n'(t) = \lambda w_t(1-\tau_{nt}) \qquad \Rightarrow \dfrac{u'(t)}{u'(t+1)} = \dfrac{w_t}{p_t} \cdot \dfrac{(1-\tau_{ct})}{1+\tau_{ct}}$

(k_{t+1}): $-p_t + p_{t+1}(1-\delta) + r_{t+1} = 0$

$$\Rightarrow \frac{p_t}{p_{t+1}} = \frac{r_{t+1}}{p_{t+1}} + (1-\delta)$$

Euler Eq:

$$\frac{u_c'(t)}{\beta u_c'(t+1)} \cdot \frac{1+\tau_{ct+1}}{1+\tau_{ct}} = \frac{r_{t+1}}{p_{t+1}} + 1 - \delta$$

Derive Implementability constraint for this economy (get rid of the prices).

Note that $w_t(1-\tau_{nt}) = \dfrac{u_n'(t)}{u_c'(t)} p_t (1+\tau_{ct})$

$$\frac{u_c'(t)}{\beta u_c'(t+1)} = \frac{p_t}{p_{t+1}} \frac{1+\tau_{ct}}{1+\tau_{ct+1}} \Rightarrow \beta^t \frac{u_c'(t)}{u'(0)} \cdot \frac{(1+\tau_{c0})}{(1+\tau_{ct})} = \frac{p_t}{p_0}$$

$$\Rightarrow (1+\tau_{ct})p_t = p_0 \cdot \frac{\beta^t u_c'(t)}{u_c'(0)} (1+\tau_{ct+1})$$

$$\sum_{t=0}^{\infty} p_0 \beta^t \frac{u_c'(t)}{u_c'(0)} (1+\tau_{c0}) c_t = \sum_{t=0}^{\infty} \frac{u_n'(t)}{u_c'(t)} \underbrace{p_t (1+\tau_{ct})}_{\frac{p_0 \beta^t u_c'(t)}{u_0'(t)}(1+\tau_{t0})} n_t + r_t k_t + p_t(1-\delta)k_t - p_t k_{t+1}$$

$$\Rightarrow \sum_{t=0}^{\infty} p_0 \beta^t \frac{1}{u_c'(0)} (1+\tau_{c0}) \left[u_c'(t) c_t - u_n'(t) n_t \right] = \sum_{t=0}^{\infty} r_t k_t + p_t (1-\delta) k_t - p_t k_{t+1}$$

Simplify RHS: [TRICK]

$$\sum_{t=0}^{\infty} \underbrace{r_{t+1} k_{t+1} + p_{t+1}(1-\delta) k_{t+1} - p_{t+1} k_{t+1}}_{k_{t+1}\left(\underbrace{r_{t+1} + (1-\delta) p_{t+1} - p_t}_{0 \text{ by F.O.C}}\right)} + r_0 k_0 + p_0 (1-\delta) k_0 - p_0 k_0$$

$$RHS = r_0 k_0 + p_0 (1-\delta) k_0 - p_0 k_0$$

$$\Rightarrow \text{I.C.}: \sum_{t=0}^{\infty} p_0 \beta^t \frac{1}{u_c'(0)} (1+\tau_{c0}) \left[u_c'(t) c_t - u_n'(t) n_t \right] = r_0 k_0 + p_0 (1-\delta) k_0 - p_0 k_0$$

$$\boxed{\sum_{t=0}^{\infty} \beta^t \left[u_c'(t) c_t - u_n'(t) n_t \right] = \frac{u_c'(0)}{p_0 (1+\tau_{c0})} \left[r_0 k_0 + p_0 (1-\delta) k_0 - p_0 k_0 \right]}$$

Is it optimal to have $\tau_{ct} \xrightarrow[t\to\infty]{} 0$? What about τ_{nt}?

$$\sum_{t=0}^{\infty} u(c_t, n_t) \longrightarrow \max \qquad \text{[Ramsey Problem]}$$

s.t. $\quad c_t + g_t + k_{t+1} - (1-\delta) k_t = F(k_t, n_t) \qquad (\lambda_t)$

$$\sum_{t=0}^{\infty} \beta^t \left[u_c'(t) c_t - u_n'(t) n_t \right] = \Gamma_0 \text{ (constant)}. \qquad (\mu)$$

f.o.c:

$$(c_t):\ \beta^t u_c'(t) - \lambda_t - \mu\left[\beta^t\left(u_{cc}''(t)c_t + u_c'(t) - u_{cn}''(t)n_t\right)\right] = 0$$

$$(n_t):\ \beta^t u_n'(t) - \lambda_t F_n'(t) - \mu\left[\beta^t\left(u_{cc}''(t)c_t - u_{nn}''(t)n_t - u_n'(t)\right)\right] = 0$$

$$(k_{t+1}):\ -\lambda_t + \lambda_{t+1}\left(F_k'(t+1) + 1-\delta\right) = 0$$

equate (c_t) & (n_t) using λ_t and then use it on (k_{t+1})

$$\left.\begin{aligned}(c_t)&:\ \beta^t u_c'(t) - \mu\left[\beta^t\left(u_{cc}''(t)c_t + u_c'(t) - u_{cn}''(t)n_t\right)\right] = \lambda_t \\ (n_t)&:\ \left\{\beta^t u_n'(t) - \mu\left[\beta^t\left(u_{cc}''(t)c_t - u_{nn}''(t)n_t - u_n'(t)\right)\right]\right\}\frac{1}{F_n'(t)} = \lambda_t\end{aligned}\right\}(*)$$

$$(k_{t+1}):\ \frac{\lambda_t}{\lambda_{t+1}} = F_k'(t+1) + (1-\delta)$$

Euler equation in RP:

$$\Rightarrow \boxed{\frac{\beta^t u_c'(t) - \mu\left[\beta^t\left(u_{cc}''(t)c_t + u_c'(t) - u_{cn}''(t)n_t\right)\right]}{\beta^{t+1} u_c'(t+1) - \mu\left[\beta^{t+1}\left(u_{cc}''(t+1)c_{t+1} + u_c'(t+1) - u_{cn}''(t+1)n_t\right)\right]} = F_k'(t+1) + (1-\delta)}$$

Suppose:

$$c_t^{RP} \to c_\infty^{RP}$$
$$n_t^{RP} \to n_\infty^{RP}$$
$$k_t^{RP} \to k_\infty^{RP}$$

$$\Rightarrow \text{EE in RP: } \frac{1}{\beta} = F_k'\left(k_\infty^{RP}, n_\infty^{RP}\right) + 1 - \delta$$

Have to be compatible.

$$\text{EE in TDCE: } \frac{u_c'(t)}{\beta u_c'(t+1)} = \underbrace{\frac{1+\tau_{ct}}{1+\tau_{ct+1}}}_{\text{Ratio} \to 1 \text{ so they are compatible}}\left[F_k'(t+1) + (1-\delta)\right]$$

($\tau_{ct} = \tau_{ct+1}$ in the limit)

$$\text{from } (*)\quad F_n'(t) = \frac{u_n'(t) - \mu\left[u_{nc}''(t)c_t - u_{nn}''(t)n_t - u_n'(t)\right]}{u_c'(t) - \mu\left[u_{cc}''(t)c_t + u_c'(t) - u_{nc}''(t)n_t\right]}$$

$$\text{and (TDCE)}\quad F_n'(t) = \frac{u_n'(t)}{u_c'(t)}\frac{(1+\tau_{ct})}{(1-\tau_{nt})}$$

$$\boxed{\begin{aligned}&\text{if } \mu = 0 \Rightarrow \text{it is optimal to } 1+\tau_{ct} = 1-\tau_{nt} \\ &\Rightarrow \tau_{ct} = -\tau_{nt}\end{aligned}}$$

Lecture:

NOV 9, 2023

→ Endogueous Growth Model

→ Adding Fiscal policy to EGM

Remark

i) Growth on NGM occurs when k_0 is below k^{ss}, and it is temporal until $k_t = k^{ss}$.

↳ Under regular condition it alway has a ss.

ii) Exogenous tech progress $F(k_t, (1+g)n_t)$. The problem is g is exogenous and there is no way for the economy to not grow. (not compatible with data).

Social Planner's Problem

$$\sum_{t=0}^{\infty} \beta^t u(c_t, \underbrace{(1-n_t)h_t}_{\text{"Quality adjusted leisure"}})$$

n_t: hours worked

h_t: human capital

$c_t + x_{kt} + x_{ht} = F(k_t, n_t \cdot h_t)$ — human and Physical capital: x_{ht} & x_{kt}

$h_{t+1} = (1-\delta_h)h_t + x_{ht}$ — Law of motion

$k_{t+1} = (1-\delta_k)k_t + x_{kt}$

* Alternative formulation: $h_{it+1} = (1-\delta_h)h_{it} + G\left(x_{iht}, \int_1 x_{i'ht}\, di'\right)$

Inelastic Labor Supply: $n_t = 1$

$$\Rightarrow \sum_{t=0}^{\infty} \beta^t u(c_t) \longrightarrow \max$$

$c_t + x_{kt} + x_{ht} = F(k_t, h_t)$

$k_{t+1} = (1-\delta_k)k_t + x_{kt}$

$h_{t+1} = (1-\delta_h)h_t + x_{ht}$

Optimality Conditions

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[F(k_t, h_t) - c_t - k_{t+1} + (1-\delta_k)k_t - h_{t+1} + (1-\delta_h)h_t\right]$$

(k_{t+1}) $\quad -\lambda_t + \lambda_{t+1}\left[F_k'(t+1) + 1 - \delta_k\right] = 0$

(h_{t+1}) $\quad -\lambda_t + \lambda_{t+1}\left[F_h'(t+1) + 1 - \delta_h\right] = 0$

$$\Rightarrow 1 - \delta_k + F_k'(t+1) = 1 - \delta_h + F_h'(t+1)$$

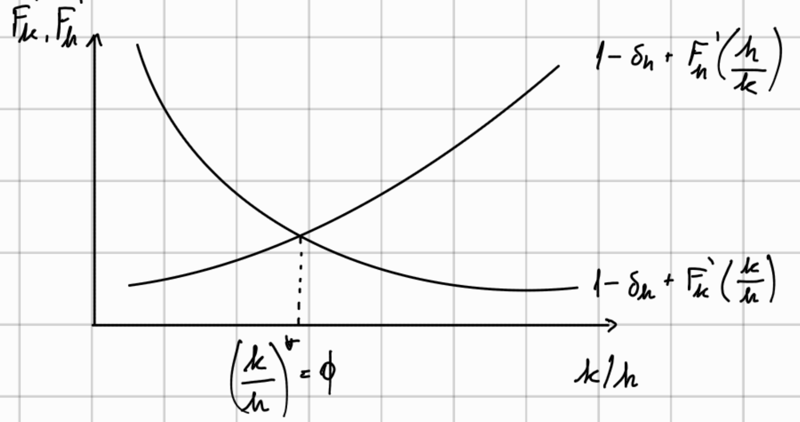
Assumption:

→ $F(\cdot)$ is CRS and ∴ hd-1

→ $F'(\cdot)$ is hd-0

$\Rightarrow F_k'(k_{t+1}, h_{t+1}) = F_k'\left(\frac{k_{t+1}}{h_{t+1}}, 1\right)$ & $F_h'(k_{t+1}, h_{t+1}) = F_h'\left(1, \frac{h_{t+1}}{k_{t+1}}\right)$

Our optimality condition: $1-\delta_k + F_k'\left(\frac{k_{t+1}}{h_{t+1}}\right) = 1-\delta_h + F_h'\left(\frac{k_{t+1}}{h_{t+1}}\right)$



(* Note $F_h'\left(1, \frac{h}{k}\right)$ is increasing in $\frac{h}{k}$ and decreasing in $\frac{k}{h}$)

If $F(k,h) = k^\alpha h^{1-\alpha}$

$$k F_k' = \alpha F(k,h)$$
$$h F_h' = (1-\alpha) F(k,h)$$

In eqm (and assuming $\delta_k = \delta_h$) $\Rightarrow F_k' = F_h' \quad \Rightarrow \left(\frac{k_t}{h_t}\right)^* = \frac{\alpha}{1-\alpha} = \phi$

Law of motion for h: $h_{t+1} = (1-\delta_h) h_t + x_{ht}$, we know $\Rightarrow k_t = \phi h_t$

$$\Rightarrow \frac{1}{\phi} k_{t+1} = (1-\delta_h)\frac{1}{\phi} k_t + x_{ht}$$

$$\Rightarrow x_{ht} = \frac{1}{\phi}\Big[\underbrace{k_{t+1} - (1-\delta_h) k_t}_{x_{kt}}\Big] \quad \Rightarrow \boxed{\phi\, x_{ht} = x_{kt}}$$

SPP becomes: $\sum^\infty \beta^t u(c_t) \longrightarrow \max_{\{c_t, x_{kt}, x_{ht}, k_{t+1}, h_{t+1}\}}$

$$\text{s.t.} \quad c_t + x_{kt} + \frac{1}{\phi} x_{kt} = F(k_t, h_t)$$
$$x_{kt} = k_{t+1} - (1-\delta_k) k_t$$
$$k_0, h_0 \text{ given}$$

Recursive formulation:

$$V(k_t, h_t) = \max \{ u(c_t) + \beta V(k_{t+1}, h_{t+1}) \}$$

$$c_t + x_{kt}(1 + 1/\phi) = F(k_t, h_t)$$

$$k_{t+1} = (1-\delta_k) k_t + x_{kt}$$

$$* \; F(k_t, h_t) = k_t F\left(1, \frac{h_t}{k_t}\right) \quad (\text{because of CRS})$$

$$= k_t F(1, 1/\phi)$$

$$= A \cdot k_t$$

This way we can drop a state variable

Lecture #2: AK - model | Nov 9, 2023

Consider CRRA utility $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$

SPP: $\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$ s.t. $c_t + (1+\phi) x_{kt} = A k_t, \quad \phi = k_t / h_t$

$$k_{t+1} = (1-\delta_k) k_t + x_{kt}$$

k_0, h_0 given

Remark: Inada conditions do not hold.

Typically we had

$\lim_{k \to 0} F_k' = \infty, \quad \lim_{k \to \infty} F_k' = 0$ (because F_k' is linear function [CRS])

Feasibility correspondence is hd-1: $\Gamma(\lambda k_0) = \lambda \Gamma(k_0)$

↳ * Double k_t allows double allocations of c_t and x_{kt}

$\boxed{\text{hd-1} \rightarrow \text{homothetic preferences}}$?

Remark #2: $u(\cdot)$ is hd-$(1-\sigma)$ ⇒ $\succsim$ are homothetic

⇒ $c_0^*, x_{k0}^*, k_1^*, c_1^*, x_{k1}^*, k_2^*, \ldots,$ is optimal starting from k_0, then

→ $\lambda c_0^*, \lambda x_{k0}^*, \lambda k_1^*, \lambda c_1^*, \lambda x_{k1}^*, \lambda k_2^*, \ldots,$ is optimal starting from λk_0

Sketch of a proof: Suppose $\{C_t^*\}$ is optimal starting from k_0 and $\{\lambda C_t^*\}$ is not, when starting from λk_0. Then $\exists\ \{\tilde{C}_t\}$ s.t $\sum_{t=0}^{\infty} \beta^t \frac{\tilde{C}_t^{1-\sigma}}{1-\sigma} > \sum_{t=0}^{\infty} \beta^t \frac{C_t^{*1-\sigma}}{1-\sigma}$

this implies that: $\underbrace{\sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{\tilde{C}}{\lambda}\right)^{1-\sigma}}{1-\sigma}}_{\text{This is feasibility from } k_0 \text{, because of FC is hd-1}} > \underbrace{\sum^{\infty} \beta^t \frac{(C^*)^{1-\sigma}}{1-\sigma}}_{\text{This is optimal starting from } k_0}$ $\therefore$ Contradiction

This implies: $V(\lambda k) = \sum_{t=0}^{\infty} \beta^t \frac{(\lambda c_t)^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} V(k)$

$\Rightarrow$ Decision rules are hd-1.

(c_t^*): $g_c(k_t) = k_t\, g_c(1)$

$g_k(k_t) = k_t\, g_k(1)$

$g_x(k_t) = k_t\, g_x(1)$

(boxed: $g_c(1)$, $g_k(1)$, $g_x(1)$) we want this objects

Sequential problem: $\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \to \max$ s.t. $c_t + (1+\phi)(k_{t+1} - (1-\delta)k_t) = Ak_t$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[Ak_t - c_t - (1+\phi)(k_{t+1} - (1-\delta)k_t) \right]$$

(c_t): $\beta^t c_t^{-\sigma} = \lambda_t$

(k_{t+1}): $-\lambda_t(1+\phi) + \lambda_{t+1} A + \lambda_{t+1}(1+\phi)(1-\delta) = 0$

$\Rightarrow \lambda_t(1+\phi) = \lambda_{t+1}\left[A + (1-\phi)(1-\delta)\right]$

$\Leftrightarrow \beta^t c_t^{-\sigma}(1+\phi) = \beta^{t+1} c_{t+1}^{-\sigma}\left(A + (1+\phi)(1-\delta)\right)$

$\Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[\frac{A}{1+\phi} + 1 - \delta\right] \longleftrightarrow \left(\frac{g_c(k_{t+1})}{g_c(k_t)}\right)^{\sigma} = \underbrace{\beta\left[\frac{A}{1+\phi} + 1 - \delta\right]}_{RHS}$

$\longleftrightarrow \left(\frac{k_{t+1}\, g_c(1)}{k_t\, g_c(1)}\right)^{\sigma} = RHS$

$\longleftrightarrow \left(\frac{g_k(k_t)}{k_t}\right)^{\sigma} = RHS \Rightarrow g_k(1) = \left[\beta\left(\frac{A}{1+\phi} + 1 - \delta\right)\right]^{1/\sigma}$

$*\ k_t\, g_k(1)/k_t \to g(1)$

Investment decision:

$$k_{t+1} = (1-\delta)k_t + X_{kt}$$

$$g_k(k_t) = (1-\delta)k_t + g_X(k_t)$$

$$k_t\, g_k(1) = (1-\delta)k_t + k_t\, g_X(1)$$

$$\Rightarrow \boxed{g_X(1) = g_k(1) - (1-\delta)}$$

and Consumption

$$C_t + (1+\phi) X_{kt} = A K_t$$

$$g_c(k_t) + (1+\phi)\, g_X(k_t) = A k_t$$

$$k_t\, g_c(1) + (1+\phi) k_t\, g_X(1) = A k_t$$

$$\boxed{g_c(1) = A - (1+\phi)\, g_X(1)}$$

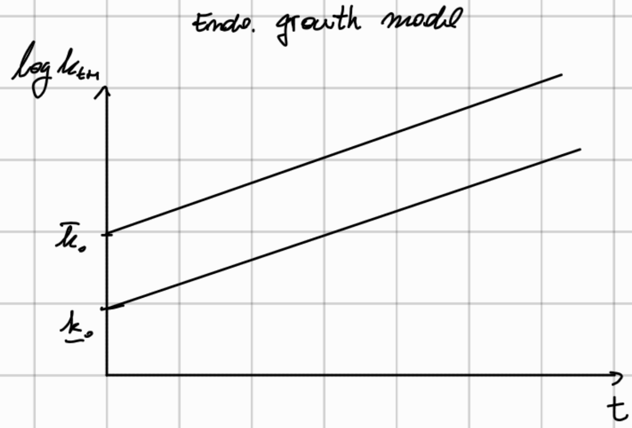
Growth Rate of Capital

$$\gamma^K_{t,t+1} = \frac{k_{t+1}}{k_t} = g_k(1) = \left[\beta\left(\frac{A}{1+\phi} + 1-\delta\right)\right]^{1/\sigma}$$

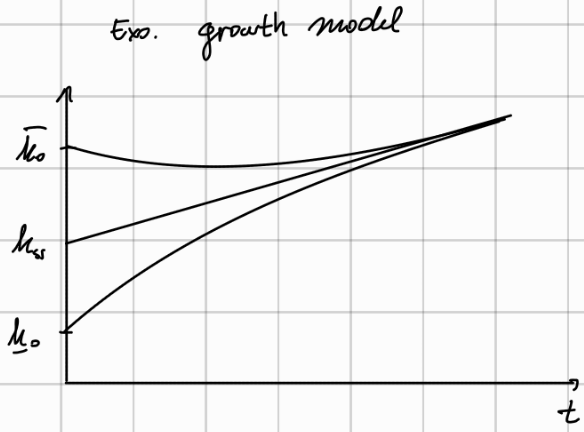
$\Rightarrow$ Economy grows if $g_k(1) > 1$.

- $\uparrow\beta$, $\uparrow A$: Economy grows faster

Taking log $\quad \log k_t = \log k_0 + t \log(g_k(1))$



$\hookrightarrow$ This fits the data better



Adding Fiscal policy

Goverment, $\{g_t\}_{t=0}^{\infty}$, accest to capital and labor income taxes

HH problem. $\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \longrightarrow \max$

s.t
$$\sum_{t=0}^{\infty} P_t C_t + P_t X_{kt} + P_t X_{ht} = \sum_{t=0}^{\infty} w_t h_t (1-\tau_{ht}) + (1-\tau_{kt}) r_t k_t$$
$$k_{t+1} = (1-\delta_k) k_t + X_{kt}$$
$$h_{t+1} = (1-\delta_h) h_t + X_{ht}$$

Euler Equations

$$[EEk]: \left[\frac{C_{t+1}}{C_t}\right]^{\sigma} = \beta\left[1-\delta_k + (1-\tau_{kt+1}) F_k'(k_{t+1}, h_{t+1})\right]$$

$$[EEh]: \left[\frac{C_{t+1}}{C_t}\right]^{\sigma} = \beta\left[1-\delta_h + (1-\tau_{ht+1}) F_h'(k_{t+1}, h_{t+1})\right]$$

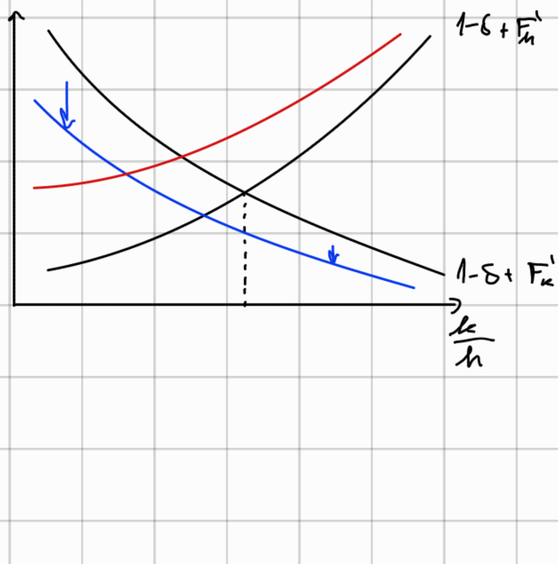
Case 1: $\delta_h = \delta_k$, $\tau_{kt} = \tau_{ht} = \tau$

$g_k(1)$? if $\tau \uparrow$

$$g_k(1) = \left[\beta\left[(1-\delta) + (1-\tau) F_k'(t+1)\right]\right]^{1/\sigma}$$

$\Delta \frac{k}{h}$? is the same : (equate EEk and EEh and we get same ratio)

Case 2: $\delta_h = \delta_k$, $\tau_{kt} \neq \tau_{ht}$



Lecture: NOV 14, 2023

→ Global Dynamics in NGM

→ Intro to NGM with uncertainty

Fiscal policy in model w/ human capital

$$[EEK] \quad \left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[1-\delta_K + (1+\tau_{kt})F_K'(t+1)\right]$$

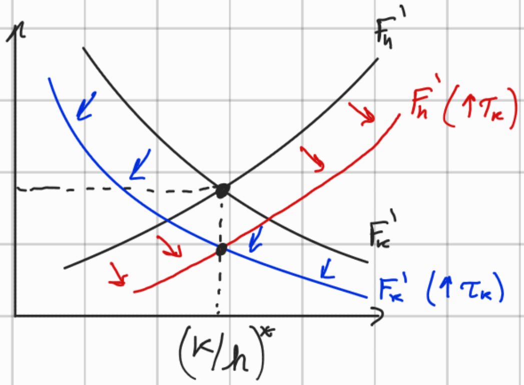
$$[EEH] \quad \left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[1-\delta_H + (1+\tau_{ht})F_h'(t+1)\right]$$

Case 1: $\delta_K = \delta_h$, $\tau_{ht} = \tau_{kt} = \tau$

- $k = \phi h$
- rate $\gamma^{c}_{t,t+1} \downarrow$ as $\tau \uparrow$
- $F_K' = F_h'$

Case 2: $\delta_K = \delta_h$, $\tau_{k,t} = \tau_{k,t+1} = \tau_k$, $\tau_{ht} = \tau_{ht+1} = \tau_h$

- $(1-\tau_k)F_K' = (1-\tau_h)F_h'$



Effect on $\left(\frac{k}{h}\right)^*$ is ambiguous

Global Dynamics in NGM

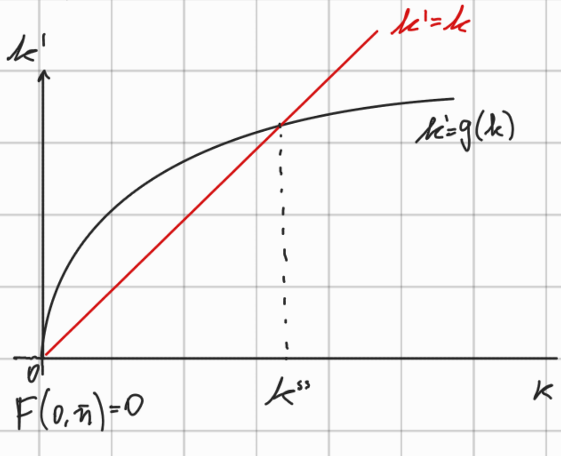
$$V(k) = \max_{0 \le k' \le f(k)} \left\{ u(f(k)-k') + \beta V(k') \right\}$$

(SLP for details)

Remark: if

- $u(\cdot)$ is strictly increasing and strictly concave
- $f(\cdot)$ is strictly increasing and strictly quasiconcave

$\Rightarrow V(k)$ is strictly increasing and strictly concave



F.O.C

$$-u'(f(k)-k') + \beta V_k'(k') = 0 \qquad (\text{use } k' = g(k))$$

$$-u'(f(k)-g(k)) + \beta V_k'(g(k)) = 0$$

* Envelope theorem: $f(x,\alpha) \longrightarrow \max_x$

f.o.c: $f_x'(x^*,\alpha) = 0$

Define VF: $V(\alpha) = f(x^*(\alpha),\alpha)$

$$\frac{\partial V(\alpha)}{\partial \alpha} = \frac{\partial f}{\partial \alpha} + \underbrace{\frac{\partial f}{\partial x}}_{0} \cdot \frac{\partial x}{\partial \alpha}$$

$$\Rightarrow \frac{\partial V(\alpha)}{\partial \alpha} = \frac{\partial f(x^*(\alpha),\alpha)}{\partial \alpha}$$

Envelope Condition:

$$\boxed{V'(k) = u'(f(k)-k') \cdot f'(k)}$$

* Derive envelope condition

$$V(k) = u(f(k)-g(k)) + \beta V(g(k))$$

$$\frac{\partial}{\partial k}: \quad V' = u' \cdot (f'-g') + \beta V' \cdot g'$$

Recall f.o.c: $u' = \beta V'$

$$\Rightarrow V' = u'(f'-g') + u'g'$$

$$\Rightarrow V' = u' \cdot f'$$

$$V_k'(k) = u_k'(f(k)-g(k)) \cdot f_k'(k)$$

Theorem: Let $T(x)$ be a strictly concave function $(T'' < 0)$

$\Rightarrow [T'(z) - T'(z')](z-z') \leq 0$, with $=$ IFF $z = z'$

Case I: $z < z' \Rightarrow T'(z) > T'(z')$

Case II: $z > z' \Rightarrow T'(z) < T'(z')$

Case III: $z = z' \Rightarrow T'(z) = T'(z')$

$$[V'(k) - V'(g(k))](k - g(k)) \leq 0 \quad \text{with } = \text{ IFF } k = g(k)$$

From

$$V'(k) = u'(f(k)-g(k)) \cdot f'(k) \qquad [\text{ENV}]$$

$$V'(g(k)) = \frac{1}{\beta} u'(f(k)-g(k)) \qquad [\text{FOC}]$$

$$\left[u'(f(k)-g(k)) \cdot f'(k) - \frac{1}{\beta} u'(f(k)-g(k))\right](k - g(k)) \leq 0$$

Since $u'(\cdot) > 0$ we can simplify to $\underbrace{\left[f'(k) - \frac{1}{\beta}\right](k - g(k)) \leq 0}$

This shows that the model is globally stable. (the model converges from any k).

If $f'(k) = 1/\beta \rightarrow k^{ss}$

at the SS : $k^{ss} = g(k^{ss})$

If $\underline{f'(k) > 1/\beta} \rightarrow k < g(k)$

this is below ss because k is not enough

(remember $f'(\cdot)$ is decreasing, so we need more k to reach $f'(k) = 1/\beta$)

$\Rightarrow k < g(k) < g(k_{ss}) = k_{ss}$ $\quad (g(k) \in (k, k^{ss}))$

If $f'(k) < 1/\beta \rightarrow k > g(k)$

$\Rightarrow k > g(k) > g(k_{ss}) = k_{ss}$ $\quad (g(k) \in (k^{ss}, k))$

Lecture:

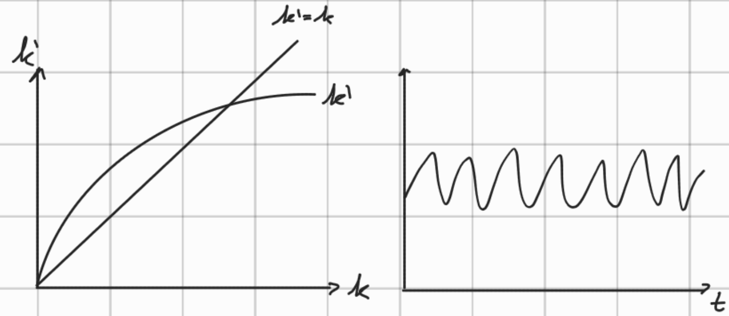
NOV 16, 2023

→ Adding uncertainty

→ ADE in a model w/ uncertainty

→ RBC model

Why uncertainty? Real world series have wiggles.



NOT succesful kind of model.

Two approaches:

1) Deterministic complex dynamics

2) Aggregate shocks

RBC = NGM + Agg productivity shock

Notation:

s_t : state of the economy

$s_t \in \{High, Low\}$

Def: $s^t = \{s_t, s_{t-1}, \ldots, s_0\}$ is a history of shocks at time t.

S^t, set of all histories up to time t.

Ex: $t=2$ (assuming you started at $t=1$)

$S^t = \{HH, HL, LH, LL\}$

Let $\Pi(s^t)$ be the unconditional probability of history s^t

Routinely assume $\Pi(s^0) = 1$

Let $\tau < t$, $\Pi(s^t \mid s^\tau)$ be the conditional probability of s^t given s^τ happened.

Deterministic Case: Allocations are function of time.

Uncertainty Case: Allocations are function of history.

Exchange economy with uncertainty

I : number of agents

$\{y^i(s^t)\}_t^{s^t}$: sequences of endowments.

An allocation is $\{c_t^i(s^t)\}_t^{s^t}$ $\quad (c_t^i : S^t \to \mathbb{R}_+)$

Arrow - Debreu Equilibrium.

$q_t^0(s^t)$: price of one unit of consumption good at time t after sequence s^t

Problem of agent i

y : endowment

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i(s^t)) \to \max_{\{c_t^i(s^t)\}_t^{s^t}} \quad \text{s.t.} \quad \sum_t^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)\, c_t^i(s^t) \leq \sum_t^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)\, y_t^i(s^t)$$

$$c_t^i(s^t) \geq 0 \quad \forall t \quad \forall s^t$$

If u is von Neumann - Morgenstein, then $\sum_t^{\infty} \sum_{s^t \in S^t} \Pi(s^t) \beta^t u(c_t^i(s^t))$

An ADE for this environment is the price system $\{q_t^0(s^t)\}_t^{s^t}$ and allocations $\{c_t^i(s^t)\}_t^{s^t}$, such that

i) given prices, the allocation solves the HH problem;

ii) and market clear

$$\sum_i^I c_t^i(s^t) = \sum_i^I y_t^i(s^t) \qquad \forall t, \forall s^t$$

Remark:

1) the AD allocation is PO (show this)

2) Kenneth Arrow: Same allocation will arise if market open every time period and agents trade Arrow security.

Lecture: a (simple) RBC model

$\boxed{\text{Nov 16, 2023}}$

A SPP: $\max \ \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{S_t} \beta^t u(c_t(s^t), n_t(s^t))$

Resource constraint: $c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1}) \leq A s_t F(k_t(s^{t-1}), n_t(s^t))$

k_0, S_0 given

Simplifying assumptions

1) Labor supply inelastic $u_\ell'(\cdot) = 0$
2) Production technology is $As_t k(s^{t-1})$
3) $\delta = 1$

$P(k_0, S_0)$: $\quad \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t(s^t)) \longrightarrow \max_{\{c_t(s^t), k_{t+1}(s^t)\}_t^{s^t}}$

s.t. $\quad c_t(s^t) + k_{t+1}(s^t) = As_t k_t(s^{t-1})$

$c_t(s^t) \geq 0, \ k_{t+1}(s^t) \geq 0 \quad \forall t, \forall s^t$

Let's assume $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$, σ - risk aversion $(\uparrow\sigma, \uparrow \text{risk averse})$

with this, $u(\cdot)$ & $V(\cdot)$ are hd $-(1-\sigma)$

$V(\lambda k_0, S_0) = \lambda^{1-\sigma} V(k_0, S_0)$

If $\left\{\tilde{c}^*(\lambda k_0, S_0), \tilde{k}^*(\lambda k_0, S_0)\right\} = \lambda \left\{\tilde{c}^*(k_0, S_0), \tilde{k}^*(k_0, S_0)\right\}$

$$\left[\ V(k_0, S_0) = \mathbb{E}_0 \sum \frac{(\lambda c_t)^{1-\sigma}}{1-\sigma} \right.$$
$$\lambda^{1-\sigma} \mathbb{E}_0 \sum \frac{c_t^{1-\sigma}}{1-\sigma}$$

Def: A stochastic process $S_0, S_1, \dots$ is first-order of Markov if
$P(S_{t+1} \mid S_t, \dots, S_0) = P(S_{t+1} \mid S_t)$

Ex: AR(1): $\quad S_{t+1} = \rho \boxed{S_t} + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \overset{iid}{\sim} N(0, \sigma^2)$

Markov degree 1.

Ex2: IID processes are Md-0

If $\{S_t\}$ is md-1, $V(k,s) = \max_{c,k'} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta\, \mathbb{E}\left[V(k',s')\,|\,s\right] \right\}$

s.t $c + k' = Ask$

k_0, s_0 given.

Invoke homogeneity of V. $V(k,s) = \max \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta\, \mathbb{E}\left[(k')^{1-\sigma} V(1,s')\,|\,s\right] \right\}$

$c + k' = Ask$

$$V(k,s) = \max \underbrace{\left\{ \frac{c^{1-\sigma}}{1-\sigma} + (k')^{(1-\sigma)} \beta\, \mathbb{E}\left[V(1,s')\,|\,s\right] \right\}}_{\text{Homothetic preference over } (c,k') \text{ with a linear restriction}}, \quad bc: c + k' = Ask$$

$$\left.\begin{aligned} c &= \varphi Ask \\ k' &= (1-\varphi) Ask \end{aligned}\right\} \text{From bc}$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \frac{c_t(s^t)^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \sum_{s^t} \lambda_t(s^t) \left[A s_t k_t(s^{t-1}) - c_t(s^t) - k_{t+1}(s^t) \right]$$

f.o.c:

$$(c) \quad \beta^t \Pi(s^t)\, c_t(s^t)^{-\sigma} = \lambda_t(s^t)$$

$$(k') \quad -\lambda_t(s^t) + \sum_{s_{t+1}} \lambda_{t+1}(s^t, s_{t+1}) \left[A s_{t+1}\right] = 0$$

$$\Rightarrow \quad \beta^t \Pi(s^t)\, c_t(s^t)^{-\sigma} = \sum_{s_{t+1}} \beta^{t+1} \Pi(s^t, s_{t+1})\, c_{t+1}^{-\sigma}(s^t, s_{t+1})\, A s_{t+1}$$

$$c_t(s^t)^{-\sigma} = \beta \sum \Pi(s_{t+1}\,|\,s^t)\, c_{t+1}^{-\sigma}(s^t, s_{t+1})\, A s_{t+1}.$$

$$c_t(s^t)^{-\sigma} = \beta\, \mathbb{E}\left[c_{t+1}^{-\sigma}(s_{t+1}) \cdot A s_{t+1} \,|\, s^t \right]$$

$$1 = \beta\, \mathbb{E}\left[A s_{t+1} \left(\frac{C_t(s^t)}{C_{t+1}(s^{t+1})} \right)^{\sigma} \Big| \, s^t \right] \qquad \text{Euler Equation}$$

* Recall $c = \varphi \cdot A s \cdot k$

$$1 = \beta\, \mathbb{E}\left[A s_{t+1} \left(\cancel{\varphi A} s_t k_t(s^{t-1}) / \cancel{\varphi A} s_{t+1} k_{t+1}(s^t) \right)^{\sigma} \Big| \, s^t \right]$$

$$1 = \beta\, \mathbb{E}\left[A s_{t+1} \left(\frac{s_t k_t(s^{t-1})}{s_{t+1} k_{t+1}(s^t)} \right)^{\sigma} \Big| \, s^t \right]$$

$$1 = \beta\, \mathbb{E}\left[A s_{t+1} \left(\frac{\cancel{s_t k_t(s^{t-1})}}{s_{t+1}(1-\varphi) A \cancel{s_t k_t(s^{t-1})}} \right)^{\sigma} \Big| \, s^t \right]$$

$$1 = \beta\, \mathbb{E}\left[A s_{t+1} \left(\frac{1}{A s_{t+1}(1-\varphi)} \right)^{\sigma} \Big| \, s^t \right] \Rightarrow (1-\varphi)^{\sigma} = \beta\, \mathbb{E}\left[(A s_{t+1})^{1-\sigma} \Big| \, s^t \right]$$

Assume $\{s_t\}_t^{\infty}$ is iid

$$\Rightarrow \boxed{(1-\varphi) = \left(\beta\, \mathbb{E}\left[A s^{1-\sigma} \right] \right)^{1/\sigma}}$$

Growth rate in consumption? $\gamma^c_{t,t+1} = \dfrac{C_{t+1}}{C_t} = \dfrac{\varphi A s_{t+1} k_{t+1}}{\varphi A s_t k_t}$

$$\gamma^c_{t,t+1} = \frac{s_{t+1}(1-\varphi) A s_t k_t}{s_t k_t} = (1-\varphi) A s_{t+1}$$

$$\mathbb{E}\left[\gamma^c_{t,t+1} \right] = (1-\varphi) A\, \mathbb{E}\, s_{t+1}$$

Effect of uncertainty:

<u>Case 1:</u> $\delta_t = 1$, $\forall t$

No shock, then NGM.

<u>Case 2:</u> $\mathbb{E}[\delta_t] = 1$, $VAR(\delta_t) > 0$

If $\sigma > 1 \Rightarrow V(s)\uparrow \Rightarrow (1-\varphi)\uparrow$ (Risky, then more saving)

If $\sigma \in [0,1] \Rightarrow V(s)\uparrow \Rightarrow (1-\varphi)\downarrow$

Lemma 4
E E
envelope
Characterize decision

Lecture: NOV 21, 2023

→ Kaldor facts
→ Solow-Swan model
→ Growth accounting

Kaldor stylized facts
1) Y grows at a roughly constant rate (so does $\frac{Y}{N}$)
2) K/Y is roughly constant
3) Total labor hours grow slower than K
4) Rate of return on K is roughly constant (it doesn't seem to grow).

<u>Solow-Swan model</u>

Technology: $Y_t = F(K_t, X_t N_t)$, X_t: "Productivity" of labor
N_t: number of people

$F(\cdot)$ is CRS

$F(0, XN) = F(K, 0) = 0$

INADA conditions: $F_K'(K, XN) \xrightarrow[K\to 0]{} \infty$ $F_K'(K, XN) \xrightarrow[K\to\infty]{} 0$

$F_{XN}' \xrightarrow[XN\to 0]{} \infty$ $F_{XN}' \xrightarrow[XN\to\infty]{} 0$

Feasibility constraint: $C_t + I_t = F(K_t, X_t N_t)$

Law of motion for capital: $K_{t+1} = (1-\delta) K_t + I_t$

Consumer: Saves s of Y, consumes $(1-s)$

$$C_t = (1-s) Y_t$$
$$S_t = s Y_t$$

$$S_t = s \cdot Y_t = Y_t - (1-s) Y_t = F(K_t, X_t N_t) - C_t = I_t$$

then, L.o.m: $K_{t+1} = (1-\delta) K_t + s F(K_t, X_t N_t)$ / divide both sides by $X_t N_t$.

$$\frac{K_{t+1}}{X_t N_t} = (1-\delta) \frac{K_t}{X_t N_t} + \frac{s \cdot F(K_t, X_t N_t)}{X_t N_t} \qquad * \underbrace{\frac{F(K_t, X_t N_t)}{X_t N_t} = F\left(\frac{K_t}{X_t N_t}, 1\right)}_{\text{because it is CRS}}$$

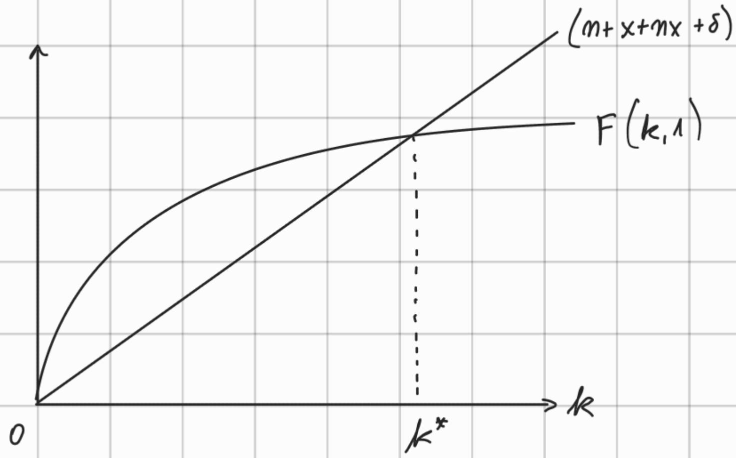
$$\hookrightarrow (1+n)(1+x)\, k_{t+1} = (1-\delta) k_t + s F(k_t, 1)$$

$$* \; k_{t+1} = \frac{K_{t+1}}{X_t N_t} = \frac{K_{t+1}}{X_t N_t (1+n)(1+x)} \implies \frac{K_{t+1}}{X_t N_t} = k_{t+1} (1+n)(1+x)$$

[Growth is exogenous]

Look at BGP: k doesn't grow, aggregates grow at a constant rate

$$(n + x + nx + \delta) k = s F(k, 1)$$



Growth Rates at BGP

- k doesn't grow
- ✓ K_t grows at $(1+n)(1+x)$ (faster than n_t)
- ✓ Y_t grows at $(1+n)(1+x)$ ($K_{t+1} = (1-\delta)K_t + s Y_t$)
- Y_t / N_t grows at a rate $(1+x)$
- ✓ K_t / Y_t is constant

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we need one more stylized fact...

$$\Pi_t = \max_{K_t, N} \left[F(K_t, N_t X_t) - w_t N_t X_t - r_t X_t \right]$$

$r_t = F_K'(K_t, X_t N_t) \rightarrow$ hd-0 because $F(\cdot)$ is CRS

$\rightarrow r_t = F_K'(k_t, 1) \rightarrow \{r_t\}$ is flat over BGP (Growth path)

$w_t = F_{XN}'(K_t, X_t N_t) = F_{XN}'(k_t, 1) \Rightarrow \{w_t\}$ is constant on BGP

$W_t = w_t X_t \Rightarrow \{W_t\}$ grows at a rate $(1+x)$

hd-1 = CRS
hd > 1 = IRS
hd < 1 = DRS

Homothetic: If preferences can be represented by a utility function which is hd-1. Ex: $X_1^{0.8} X_2^{0.5}$ is hd-1.3, but we can normalize the exponents to $X_1^{\frac{0.8}{1.3}} X_2^{\frac{0.5}{1.3}}$, and have a hd-1. $\therefore X_1^{0.8} X_2^{0.5}$ is homothetic

* quasilinear are not homothetic

Application to Growth Accounting

$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$

A_t: productivity

L_t: # of working hours.

$$\frac{Y_t}{N_t} = A_t \left(\frac{K_t}{N_t}\right)^{\alpha} \left(\frac{L_t}{N_t}\right)^{1-\alpha}$$

$$\frac{Y_t}{N_t} = A_t \left(\frac{K_t}{N_t Y_t}\right)^{\alpha} \left(\frac{L_t}{N_t Y_t}\right)^{1-\alpha} Y_t$$

$$\frac{Y_t}{N_t} = A_t \left(\frac{K_t}{Y_t}\right)^{\alpha} \left(\frac{L_t}{Y_t}\right)^{1-\alpha} \left(\frac{Y_t}{N_t}\right)^{\alpha} \longleftrightarrow \frac{Y_t}{N_t} = A_t^{\frac{1}{\alpha-1}} \left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{L_t}{N_t}\right)$$

Take log: $\log\left(\frac{Y_t}{N_t}\right) = \frac{1}{\alpha-1} \log(A_t) + \frac{\alpha}{1-\alpha} \log\left(\frac{K_t}{Y_t}\right) + \log\left(\frac{L_t}{N_t}\right) \quad (*)$

consider $\log\left(\frac{Y_{t+1}}{N_{t+1}}\right) = \frac{1}{\alpha-1} \log(A_{t+1}) + \frac{\alpha}{1-\alpha} \log\left(\frac{K_{t+1}}{Y_{t+1}}\right) + \log\left(\frac{L_{t+1}}{N_{t+1}}\right) \quad (**)$

Take difference $(**) - (*)$: $\boxed{g_{\frac{Y}{N}} \approx \frac{1}{1-\alpha} g_A + \frac{\alpha}{1-\alpha} g_{\frac{K}{N}} + g_{\frac{L}{N}}}$

Most of the growth comes from this factor

α usually 1/3 in data. (factor of K in Cobb-Douglas)

$\hookrightarrow \log(1+x) \approx x$, if x is small

$$\log\left(\frac{\frac{Y_{t+1}}{N_{t+1}}}{\frac{Y_t}{N_t}}\right) = \log\left(1 + g_{\frac{Y}{N}}\right) \approx g_{\frac{Y}{N}}$$