Midterm, Problem 2

Problem 2, part a
First for clarity, let $n_t^i = 1 - \ell_t^i$
An ADE is a collection of allocations for consumers $\left\{c_t^i, \ell_t^i\right\}_{t\geq 0}$ (for $i = 1, 2$), allocations for firms $\left\{n_t^d\right\}_{t\geq 0}$ and prices $\{p_t, w_t\}_{t\geq 0}$ s.t

1. Given prices, $\left\{c_t^i, \ell_t^i\right\}$ solves agent i's problem

$$\max_{\left\{(c_t^i, \ell_t^i)\in\mathbb{R}_+\times[0,1]\right\}} \sum \beta^t \left[u(c_t^i) + \gamma_i v(\ell_t^i)\right] \text{ s.t } \sum p_t c_t^i \leq \sum w_t n_t^i$$

2. Given prices, $\left\{n_t^d\right\}$ solves the firm's problem

$$\max_{n_t} p_t A n_t - w_t n_t$$

3. Markets clear: for each t

$$\sum_{i=1}^{2} c_t^i = A n_t \text{ \underline{\textbf{and}} } n_t = n_t^d = \sum_{i=1}^{2} n_t^i$$

Problem 2, part b
The lagrangian for agent i is $\mathcal{L} = \sum \beta^t \left[u(c_t^i) + \gamma_i v(\ell_t^i)\right] + \lambda^i \sum p_t(w_t(1-\ell_t^i) - c_t^i)$. FOCs at t for i

$$\beta^t u'(c_t^i) = \lambda^i p_t \text{ \underline{\textbf{and}} } \beta^t \gamma_i v'(\ell_t^i) = \lambda^i w_t$$

Everyone struggled with this problem because these are generic utility functions and you can't directly substitute market clearing into marginal utilities. But the intuition is the same. Equate marginal utilities by

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\lambda^1}{\lambda^2}$$

This holds for all t. So the ratio of marginal utilities is not changing. You should be confident when you get here that consumption is not changing. A brief proof by contradiction: if consumption changed at a given $t > 0$, it changed in the perfect way such that the ratio remained constant, meaning c_t^1, c_t^2 either both increased or both decreased consumption (relative to $t-1$). c growing (or decaying) forever or c bouncing up and down can't be an equilibrium, given that the only asset is labor income (scroll all the way to bottom for more on this). Therefore, consumption is constant, which means the sum of marginal utilities are constant, so using $p_0 = 1$, we get the same $p_t = \beta^t$ as HW/notes. From firm's FOC, $w_t = Ap_t$.

I discuss the intuition and mechanics of this problem extensively below the solution for part c.

Problem 2, part c
All you needed for full credit: Agent 1 gets more utility from a given level of leisure because of their higher γ, and the two agents have the same utility function for consumption, so they value consumption the same, meaning absent additional frictions or differences (none in this question) agent 1 will have more leisure. With more leisure, there is less labor income, and therefore agent 1 has a smaller budget and less consumption.

If you want to use math. Constant consumption means leisure must be constant because constant consumption can only happen (be both optimal and feasible) with constant income ($\implies$ constant labor). Let i's constant allocation be (c_i, ℓ_i). Proof by contradiction: Suppose $c_1 \geq c_2$. So they have more labor income and $\ell_1 \leq \ell_2$. Using $\frac{u'(c_t^i)}{\gamma_i v'(\ell_t^i)} = \frac{p_t}{w_t} = \frac{1}{A}$, we have

$$\frac{u'(c_1)}{\gamma_1 v'(\ell_1)} = \frac{u'(c_2)}{\gamma_2 v'(\ell_2)}$$

By concavity, $c_1 \geq c_2 \implies u'(c_1) \leq u'(c_2)$ and $\ell_1 \leq \ell_2 \implies v'(\ell_1) \geq v'(\ell_2)$. But since $\gamma_1 > \gamma_2$, the LHS has a (weakly) smaller numerator and bigger denominator than the RHS, so LHS $<$ RHS, contradiction.

Problem 2, additional discussion

When you see an unfamiliar problem, it's easy to focus on the aspects you're unfamiliar with. But you should do the opposite: look for the ways it's the same as what you've seen in the past. Here, there is again no uncertainty, agents are infinity lived, and preferences only change over time through a common discount factor. Assume you have an allocation at $t = 0$. Now compare this to $t = 1$. Nothing about the environment has changed. If it's possible for either agent to have a different allocation at $t = 1$ they are happier with, why didn't they just do this at $t = 0$? No such change exists because of market clearing – neither agent can consume more unless the other agent consumes less, and neither has an incentive to do so in this problem. When you've done enough of these, your intuition will be strong enough that you can read the setup and know immediately that consumption is constant and prices are $p_t = \beta^t$. When setups are simple like this, you should always be looking for a reason consumption *isn't* constant. Ask yourself if there is any reason there would be a deviation from that baseline (an example of a departure was seen in HW2 – there, a preference gap exists that becomes larger as time goes on).

Now I will do this "the long way" to highlight different ways you can see that this problem isn't structurally different from the one's you had seen before. Let's take our FOCs and sum across i for consumption

$$\beta^t \left(u'(c_t^1) + u'(c_t^2)\right) = (\lambda^1 + \lambda^2)p_t$$

This holds at all t. So substituting out $\lambda^1 + \lambda^2$ using the $t + 1$ expression

$$u'(c_t^1) + u'(c_t^2) = \frac{\beta p_t}{p_{t+1}} \left(u'(c_{t+1}^1) + u'(c_{t+1}^2)\right)$$

Now we ask, is it possible that the sum of marginal utility *isn't* constant but consumption is? In other words, does there exist a continuously differentiable, strictly increasing, and strictly concave function such that $u'(a) + u'(b) = u'(c) + u'(d)$ for any $a, b, c, d \in \mathbb{R}_+$ such that $a + b = c + d$. I believe by strict concavity, this is impossible, but feel free to double check me on that. Of course, this doesn't prove consumption is in fact constant, but again you should be operating under the assumption that it likely is and be looking for ways to prove that. This should only confirm your intuition.

We could also go back to our FOC for consumption and notice that by substituting out λ^i

$$u'(c_t^i) = \beta \frac{p_t}{p_{t+1}} u'(c_{t+1}^i) \implies \frac{u'(c_t^i)}{u'(c_{t+1}^i)} = \beta \frac{p_t}{p_{t+1}}$$

This tells you that if $\beta \frac{p_t}{p_{t+1}}$ is just some constant, the ratio of marginal utilities for a consumer is constant. But if this constant is not 1, because this relationship must hold for both consumers, we would arrive at the same contradiction as we did in the body of our solutions.

If you get that an intertemporal ratio is constant for all t, until we get into some more complicated models, you should be looking to disprove the fact that consumption is constant. When there are multiple agents and the environment is simple, it's not easy for intertemporal relationships (both between and among the same agent) to be constant and the allocations themselves also be changing. There are plenty of ways to break this, but you should be building your core intuition from this starting point. Doing this will help you understand why the dynamics are different when you get to other models.

Finally, we formally prove that intertemporal consumption changes do not satisfy the Euler equations. For further intuition on this topic, take a look at the public good on discounting/consumption smoothing.

$$\textbf{(intratemporal) Euler}: \qquad Au'(c_t^i) = \gamma_i v'(\ell_t^i)$$

Let's say $c_t^i < c_{t+1}^i$ (remember: this must hold for both i). The LHS goes down $(t + 1)$, so the RHS must also go down for equality to hold. However, for more consumption to happen, agents have to work more. This means they will have less leisure, meaning the RHS goes up, contradiction. If $c_t^i > c_{t+1}^i$: LHS goes up, but agents have more leisure now (don't need as much labor income), so RHS goes down, contradiction. If Euler doesn't hold, not an equilibrium (agents are not maximizing utility). Intuitively, the only sensible possible departure from this is that consumption slowly decays to some steady state. But because prices bake in that the market (agents) value future consumption less, we are always at a "steady state" in some sense.