HW3, Problem 1

Problem 1, part a

Let $g(x_t) = c_t$, with $x_t = (l_t, k_t, k_{t+1})$. New problem: $\max\limits_{\{k_{t+1}, l_t\}} \sum \beta^t \left[\ln\left(g(x_t)\right) + \gamma \ln(1 - l_t)\right]$. We have

$$\textbf{Intra: } \frac{(1-\alpha)F(k_t, l_t)}{l_t c_t} = \frac{\gamma}{1 - l_t} \ \underline{\textbf{and}} \ \textbf{Inter: } \frac{1}{c_t} = \beta \frac{\alpha F(k_{t+1}, l_{t+1})}{k_{t+1} c_{t+1}} \ \underline{\textbf{and}} \ \textbf{TVC: } \lim_{t \to \infty} \beta^t \frac{k_{t+1}}{g(x_t)}$$

<u>Comments</u>: Several points to highlight here from previous discussion sections – I'll tackle the most content heavy ones first. The intratemporal Euler equation is within period relationships that must be satisfied at the optimum (relationships between t variables), while the intertemporal Euler is one that across period relationships must satisfy (relationships between variables of adjacent t). It seems like many of you are more familiar with the latter, but the former will always be there if you have leisure. The FOCs here lead you straight to the Eulers for reasons I'll go into below. Additionally, you may not like this formulation of the problem because there is now no lagrange multiplier, which I'm guessing was in your TVC definition. In a simple model, $\lambda_t = \beta^t u'(c_t)$ always. This can break: if you're really worried about it, briefly write out the constrained problem and take a derivative w.r.t c to make sure you get this relationship.

Additionally: simplicity is your friend. I always prefer to substitute out control variables when I can. Here, you can turn this into an unconstrained optimization pretty easily. Several benefits: less writing, less derivatives, Eulers pop out instantly. Further, use notation to your advantage. When I substituted consumption, I didn't rewrite the ARC, I just used the ARC to create a function (much less writing). I did the same for the production function. But once you've taken derivatives, feel free to swap them back in for clearer interpretation. One additional trick I exploited is the following property of Cobb-Douglass production: $F_K(\cdot) = \frac{\alpha F(\cdot)}{K}$ and $F_L(\cdot) = \frac{(1-\alpha)F(\cdot)}{L}$. I think you just learned this in Pete's course: C-D is also your friend!

Problem 1, part b

Controls: k', l. States: k. Substitute out ARC to get no constraints

$$V(k) = \max_{k', l \in (0,1)} \left\{\ln\left(g(x)\right) + \gamma \ln(1 - l) + \beta V(k')\right\}$$

Note: even if you guess labor choice is constant, you need labor FOC to find the specific *optimal* constant

Problem 1, part c

Solve the dynamic program = find V and policy functions. We will break this problem up into pieces. Our guess + capital FOC yield

$$\max_{k', l} \left\{\ln(g(x)) + \gamma \ln(1 - l) + \beta[a_0 + a_1 \ln(k')]\right\} \implies \frac{1}{c} = \beta \frac{a_1}{k'} \implies c = (a_1\beta)^{-1} k'$$

To find policy function for capital (and therefore consumption), substitute into ARC

$$(a_1\beta)^{-1}k' + k' = F(k, l) \implies k' = \frac{F(k, l)}{1 + (a_1\beta)^{-1}}$$

The next step is to plug back into value function and match terms. Because we are focusing on capital for now, we only currently care about the terms attached to $\ln(k)$. Decomposing:

$$g(x) = \theta k^\alpha l^{1-\alpha} \frac{a_1\beta}{1 + a_1\beta} \implies \ln(g(x)) = \alpha \ln(k) + \ldots$$

$$\beta a_1 \ln(k') = \beta a_1 \ln\left(\frac{F(k, l)}{1 + (a_1\beta)^{-1}}\right) = \beta a_1 \alpha \ln(k) + \ldots$$

$$\implies a_1 = \alpha(1 + \beta a_1) \implies \boxed{a_1 = \frac{\alpha}{1 - \beta\alpha}} \implies \boxed{k' = \alpha\beta F(k, l)} \ \underline{\textbf{and}} \ \boxed{c = (1 - \alpha\beta)F(k, l)}$$

The last step is sometimes hard to grasp. So I'll stop the problem to do a bit of discussion.

We need to satisfy

$$a_0 + a_1 \ln(k) = \ln(g(x)) + \gamma \ln(1-l) + \beta[a_0 + a_1 \ln(k')]$$

Because we are guessing that l will be constant, the only terms where a $\ln(k)$ will pop out are in red (the policy functions for c and k' are both functions of k). The sum of coefficients attached to $\ln(k)$ on the RHS must be equal to a_1. This is what we mean by "matching terms": when $\ln(k)$ is factored out on the RHS, the coefficient attached to $\ln(k)$ on the LHS must be equal to this sum of terms otherwise we get a contradiction. Fortunately, the log operator gets rid of pretty much everything attached to k, so it's not too messy.

$$a_0 + a_1 \ln(k) = \alpha \ln(k) + \beta a_1 \alpha \ln(k) + \cdots = \alpha(1 + \beta a_1) \ln(k) + \ldots$$

More discussion everything past the $\ldots$ later, but the matching principle is the same (just with a_0)

Now back to the problem. Take FOCs w.r.t l, impose guess of $l = l_0 \in \mathbb{R}$, and plug in policy function for c

$$\frac{\gamma}{1-l_0} = \frac{(1-\alpha)F(k,l_0)}{l_0 c} \implies \gamma(1-\alpha\beta)l_0 = (1-\alpha)(1-l_0) \implies \boxed{l_0 = \left(1 + \frac{1+\alpha\beta}{1-\alpha}\gamma\right)^{-1}}$$

If we plug everything back into the value function, we get a solution for a_0. These constant terms are relatively trivial because we don't really care about the numerical value of utility, we care about how it evolves and how it depends on choices. The constant terms are generally a mess: on an exam, don't be alarmed if it looks complicated and don't feel pressured to solve explicitly.

Problem 1, part d

This is just algebra. If you see a problem like this on an exam, feel happy because you are basically getting points for checking your work. If this doesn't happen, you've made a mistake somewhere (though be careful– the mistake could've occurred in the process of checking!)

Problem 1, part e

I'll write this out since you had ADE on your HW2/Midterm and leave in δ to be more general. I will exclude the $t \geq 0$ subscripts now that we're further in the semester – remember it's good to practice writing these quickly. There is a representative agent so bonds are in *zero net supply*
SME is a collection of allocations and prices $\{c_t, l_t^s, k_{t+1}^s, b_{t+1}\}$, $\{l_t^d, k_t^d\}$, and $\{r_t, w_t, i_{t+1}\}$ s.t

1. Given prices, $b_0 = 0$, and k_0, $\{c_t, l_t^s, k_{t+1}^s, b_{t+1}\}$ solves the HH problem

$$\max_{\{c_t, l_t^s \in (0,1), k_{t+1}, b_{t+1}\}} \sum \beta^t \left[\ln(c_t) + \gamma \ln(1-l_t)\right] \text{ s.t } c_t + b_{t+1} + k_{t+1}^s \leq w_t l_t^s + (1-\delta+r_t)k_t^s + (1+i_t)b_t$$

2. Given prices, $\{l_t^d, k_t^d\}$ maximizes profits at each t

$$\max_{k_t, l_t} F(k_t, l_t) - w_t l_t - r_t k_t$$

3. Markets clear at each t

$$F(K_t, L_t) = Y_t = C_t + I_t \ \underline{\textbf{and}}\ L_t = l_t^d = l_t^s \ \underline{\textbf{and}}\ K_t = k_t^d = k_t^s \ \underline{\textbf{and}}\ b_t = 0 \ \underline{\textbf{and}}\ I_t = K_{t+1} - (1-\delta)K_t$$

It's trivial that the allocations are the same as part c) (2nd Welfare Theorem). **Prices** are

$$r_t = F_K(K_t, L_t) = \alpha \frac{Y_t}{K_t} \ \underline{\textbf{and}}\ w_t = (1-\alpha)\frac{Y_t}{L_t} \ \underline{\textbf{and}}\ i_t = r_t - \delta$$

The third one comes from the no arbitrage condition (return on bonds and capital must be equal, otherwise I'd start investing more in whichever had a higher return and we wouldn't be at an equilibrium). It's fine to write this without explanation. To see explicitly, combine the HH FOCs for capital and bonds

$$\lambda_t = (r_{t+1} + 1 - \delta)\lambda_{t+1} \ \underline{\textbf{and}}\ \lambda_t = \lambda_{1+1}(1 + i_{t+1})$$

HW3, Problem 2

Problem 2, part a
To offer some variance in my solutions, I don't substitute out c here

$$v(k) = \max_{c,k'} \{\ln(c) + \beta v(k')\} \text{ s.t } c + k' - (1-\delta)k \le F(k_t)$$

Problem 2, part b
Since you haven't seen the envelope theorem in dynamic programming, here's a intuitive solution. Use your results from problem 1 part e), adjusted for inelastic labor (alternative interpretation: labor not an input)

$$\frac{c_{t+1}}{c_t} = \beta(\alpha A k_{t+1}^{\alpha-1} + 1 - \delta) \overset{\text{s.s}}{\Longrightarrow} \beta^{-1} = \alpha A k_{\text{s.s}}^{\alpha-1} + 1 - \delta \implies \boxed{k_{\text{s.s}} = \left(\frac{\beta^{-1}+\delta-1}{\alpha A}\right)^{\frac{1}{\alpha-1}}}$$

So from the ARC: $c_{\text{s.s}} = F(k_{s.s}) - \delta k_{\text{s.s}}$

Here's another solution. Since $k = k' = k_{\text{s.s}}$, FOCs of $v(k)$ w.r.t k and k' should be equal

$$\implies \frac{\beta^{-1}}{c_{\text{s.s}}} = \frac{A\alpha k_{\text{s.s}}^{\alpha-1} + 1 - \delta}{c_{\text{s.s}}}$$

This is a good preview for the envelope theorem! Overall, the point here is to look at the Euler equation

Problem 2, excel
See attachment. Your graphs should show value function and policy function convergence. The latter is a bit more difficult because the rate of convergence gets very slow the closer you get to the steady state. It's monotonic convergence, but the optimal step is smaller and smaller. This is why I had to play around with the gridpoints for a while to get a set of 10 that would show the convergence you want. I think the really important intuitive thing to get out of this exercise is that while the value function is of course a function, if we give it a given level of capital, it will return a number. So that's why we pull the $V(k')$ part from the previous iteration. Here, we can think of it like $V(k_i)$ where k_i is the ith point in the capital grid. When $k' = k_i$, we look back at our previous iteration. There, we looked at all the possibilities ($k = k_i$ and $k' = k_1, \ldots, k_n$). Let's say that $k' = k_j$ returned the highest value. Then $V(k_i) = \ln(F(k) - i) + \beta V(k_j)$. So in the next iteration, this gives us an actual number for $V(k')$ when $k' = k_i$.

HW3, Problem 3

$\mathcal{L}^i = \sum \beta_i^t \frac{c_{i,t}^{1-\sigma}}{1-\sigma} + \lambda^i \sum [r_t k_{i,t+1} - p_t(i_{i,t} + c_{i,t})]$, where $i_{i,t} = k_{i,t+1} - (1-\delta)k_{i,t}$. So FOCs:

$$\beta_i^t c_{i,t}^{-\sigma} = \lambda^i p_t \;\; \underline{\textbf{and}} \;\; p_t = r_{t+1} + p_{t+1}(1-\delta)$$

Continuing to manipulate consumption FOC (we'll use capital FOC later): notice that for any integer $k \in [0,t]$

$$\beta_i^{t-k} c_{i,t-k}^{-\sigma} = \lambda^i p_{t-k} \implies \frac{p_t}{p_{t-k}} = \beta_i^k \left(\frac{c_{i,t}}{c_{i,t-k}}\right)^{-\sigma} \implies \boxed{p_t = \beta_i^t \left(\frac{c_{i,t}}{c_{i,0}}\right)^{-\sigma}}$$

using $k = t$ and $p_0 = 1$. From firm FOC: $p_t = \frac{r_t}{A}$. Evaluating the household capital FOC at $t-1$

$$p_{t-1} = r_t + p_t(1-\delta) \implies 1 = \frac{r_t}{p_t}\frac{p_t}{p_{t-1}} + \frac{p_t}{p_{t-1}}(1-\delta) = (A+1-\delta)\frac{p_t}{p_{t-1}} = (A+1-\delta)\beta_i \left(\frac{c_{i,t}}{c_{i,t-1}}\right)^{-\sigma}$$

where we used a multiply/divide trick with p_t (get comfortable with these – very common in Macro).
Multiply both sides of the ending equality by $c_{i,t}^{\sigma}$, rasie everything to the power $\frac{1}{\sigma}$, then iteratively substitute

$$\implies c_{i,t} = [\beta_i(A+1-\delta)]^{\frac{1}{\sigma}} c_{i,t-1} = [\beta_i(A+1-\delta)]^{\frac{1}{\sigma}} \times [\beta_i(A+1-\delta)]^{\frac{1}{\sigma}} c_{i,t-2} = \cdots = [\beta_i(A+1-\delta)]^{\frac{t}{\sigma}} c_{i,0}$$

These last series of steps are to get the budget constraint in terms of time 0 variables
To recap: $p_t = \beta_i^t \left(\frac{c_{i,t}}{c_{i,0}}\right)^{-\sigma}$ and $c_{i,t} = [\beta_i(A+1-\delta)]^{\frac{t}{\sigma}} c_{i,0}$. This implies

$$\sum p_t c_{i,t} = \sum \beta_i^t [\beta_i(A+1-\delta)]^{-t} [\beta_i(A+1-\delta)]^{\frac{t}{\sigma}} c_{i,0} = c_{i,0} \sum \beta_i^t [\beta_i(A+1-\delta)]^{\frac{t(1-\sigma)}{\sigma}}$$

Careful with the last step: $x^{-a} \cdot x^{\frac{a}{b}} = x^{\frac{a(1-b)}{b}}$.
For our last simplifying step, let's go back to the capital FOC

$$p_t - r_{t+1} - p_{t+1}(1-\delta) = 0 \implies k_{i,t+1}[p_t - r_{t+1} - p_{t+1}(1-\delta)] = 0$$
$$\implies \sum \{r_t k_{i,t} - p_t i_{i,t}\} = r_0 k_{i,0} + p_0(1-\delta)k_{i,0} + \sum k_{i,t+1}[p_t - r_{t+1} - p_{t+1}(1-\delta)] = r_0 k_{i,0} + p_0(1-\delta)k_{i,0}$$

So now the ADE budget constraint is incredibly simple: $\sum p_t c_t = \sum \{r_t k_{i,t} - p_t i_{i,t}\}$ turns into

$$c_{i,0} \sum \left(\beta_i [\beta_i(A+1-\delta)]^{\frac{1-\sigma}{\sigma}}\right)^t = r_0 k_{i,0} + p_0(1-\delta)k_{i,0}$$
$$\implies \frac{c_{i,0}}{1 - \beta_i [\beta_i(A+1-\delta)]^{\frac{1-\sigma}{\sigma}}} = (A+1-\delta)k_{i,0}$$
$$\implies c_{i,0} = (A+1-\delta)g(\beta_i)k_{i,0}$$

Using $p_0 = 1 \implies r_0 = A$ and $g(\beta) = 1 - \beta[\beta(A+1-\delta)]^{\frac{1-\sigma}{\sigma}}$ (and **assuming** a valid geometric series).
We know both have the same initial capital stock. $g(\beta)$ is decreasing in β.
Therefore, $\beta_1 > \beta_2 \implies$ agent 1 consumes less at $t = 0$. This makes sense; they're more patient.

Comments: To have a valid geometric series, we must have $\beta_i [\beta_i(A+1-\delta)]^{\frac{1-\sigma}{\sigma}} \in (0,1)$. It's very possible this is not the case; β close to 1 is common and $\sigma < 1$ is reasonable in some applications. This was an oversight by Vladimir and I; ideally these don't happen, but sometimes you need to make assumptions to complete the problem. If you just leave this as an infinite sum, it isn't possible to answer the question: even if each term in the infinite sum is smaller for β_2, it's ultimately irrelevant if the sum doesn't converge.

Some of you found that prices take a different form using firm's FOC: $p_t = (A+1-\delta)^{-t}$, but this makes it more difficult to solve for a closed form for $c_{i,t}$. But an interesting question is does this alternative form for prices make other parts of the problem more doable, since the price form should be equivalent?

First so we're on the same page, I'll derive this alternate expression using $r_t = Ap_t$

$$r_t + p_t(1-\delta) = p_{t-1} \implies p_t = \frac{1}{(A+1-\delta)} p_{t-1} = ... = (A+1-\delta)^{-t} p_0$$

So if we used this expression in our BC, we'd get

$$\sum p_t c_{i,t} = (A+1-\delta)^{-t} [\beta_i(A+1-\delta)]^{\frac{t}{\sigma}} c_{i,0} = c_{i,0} \sum \left(\beta_i [(A+1-\delta)]^{1-\sigma}\right)^{\frac{t}{\sigma}}$$

which is equivalent to what we had earlier, which makes sense but unfortunately doesn't help us.

Finally, you'll notice that I jumped straight to the Lagrange. They all take the form utility + multiplier*constraint. You should have a goal with (eventually) feeling comfortable starting there once you've done enough of these problems. This will save you time on an exam. Remember that when the constraint is an infinite sum like ADE, just one LM for each agent. See the public good on lagrange stuff.