Micro (Wed Aug 23)

1. Firm → Varian.
2. Consumer Theory → MWG
3. Choice under uncertainty.

Individual agent
↓
Single decision problem

① Firm Theory :-

Assumptions :-

1. Firms are price takers
2. Tech is exogenous
3. Firms maximize profits.

Technology :-

- n commodities, $y = (y_1, \dots, y_n)$ $(y \in \mathbb{R}^n)$
    - $y_i < 0$, i is an input
    - $y_i > 0$, i is an output.
- Production Set : $Y \subseteq \mathbb{R}^n$ → what the firm can do.

$n = 3$

$y = (-2, 1, -3)$

Firm's function : 2 units of som as input; 1 Output; 3 units of somethig as input

Properties of Production Sets:—

1. $Y \neq \phi$
2. Y is closed
3. No free lunch. If $y \in Y$ and $\underbrace{y \geq 0}_{\text{there are no inputs.}}$, then $\underbrace{y = 0}_{\text{then you can't make anything.}}$

The following are sometimes assumed:—

4. Shutdown is possible: $0 \in Y$
5. Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$.
   (You can take more input, and not lead to increased output, say you get rid of it.)
6. Irreversibility: $y \in Y$, $y \neq 0$, then $-y \notin Y$
   You can't take the output to make the input.

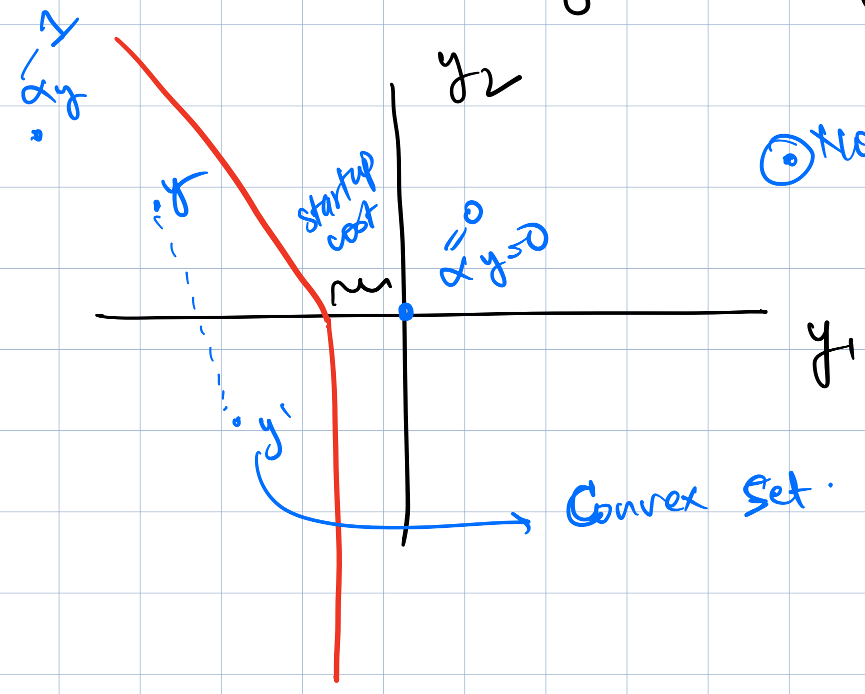
Returns to scale & Convexity

Y has:

- nonincreasing returns to scale if
  $y \in Y \Rightarrow \alpha y \in Y \quad \forall \quad \alpha \in [0, 1]$
- nondecreasing returns to scale if.
  $y \in Y \Rightarrow \alpha y \in Y \quad \forall \quad \alpha \geq 1$
- Constant returns to scale
  $y \in Y \Rightarrow \alpha y \in Y \quad \forall \quad \alpha \geq 0$

$\underline{Y \text{ is convex}}$ if :

$$y, y' \in Y \Rightarrow \alpha y + c(1-\alpha)y' \in Y \quad \forall\, \alpha \in [0,1]$$



⊙ Non decreasing returns to scale.

Why do we "usually" assume non increasing returns?

– Modelling only inputs under our control.

Theorem

There is a constant returns production set $Y' \subseteq R^{n+1}$ s.t. $Y = \{y \in \mathbb{R}^n : (y, -1) \in Y'\}$

Proof Define $Y' = \{y' = \mathbb{R}^{n+1} : y' = \alpha(y, -1)$ for some $y \in Y$ & $\alpha \geq 0\}$

Check that Y' is constant returns.

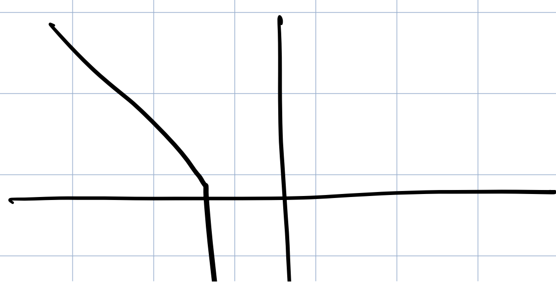
Efficiency :-

Technological efficiency:

A production plan y is technologically efficient if there does not exist $y' \in Y$ s.t. $\underbrace{y' > y}$

$\downarrow$

$y_i' \geq y_i$ for all i & $y_i' > y_i$ for some i.



{efficient pts. are going to be on the}

boundary.

Aug 28 (second class)

Transformation frontier.

firm, production set Y

The transformation function $T: \mathbb{R}^n \to \mathbb{R}$

$T(y) < 0 \iff y$ is inefficient

$T(y) = 0 \iff y$ is efficient

$T(y) > 0 \iff y$ is infeasible.

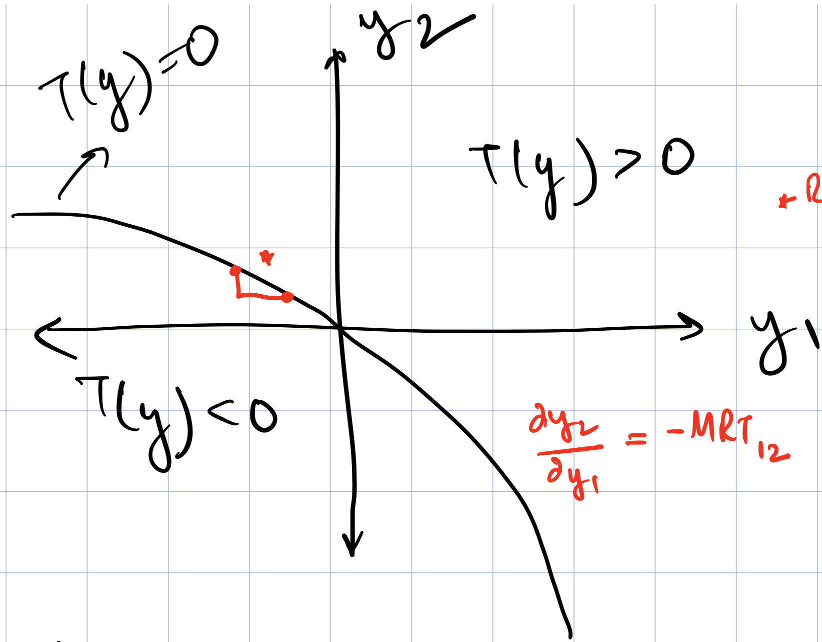
$\Rightarrow Y = \{y \in \mathbb{R}^n : T(y) \leq 0\}$

~~To check~~

Production frontier

$\{y \in \mathbb{R}^n : T(y) = 0\}$

* Rate of substitution



Lets fix $\bar{y}_3, \dots \bar{y}_n$. Consider $y_1, y_2, \dots$

Define a function $y_2(y_1)$ by

$$T(y_1, y_2(y_1), \underbrace{\bar{y}_3, \dots \bar{y}_n}_{\text{constants}}) = 0$$

implicit f^n of y_1

Pick some y_1, what y_2 gives me $T(y)=0$

Let's differentiate both sides wrt y_1

$$\frac{\partial T}{\partial y_1} + \frac{\partial T}{\partial y_2} \cdot \frac{\partial y_2}{\partial y_1} = 0$$

slope in the picture

$$\frac{\partial y_2}{\partial y_1} = \frac{-\partial T / \partial y_1}{\partial T / \partial y_2}$$

Works for any 2 goods

$$\frac{\partial y_k}{\partial y_j} = \left. \frac{-\partial T / \partial y_j}{\partial T / \partial y_k} \right|_{y=\bar{y}} = -MRT_{jk}(\bar{y})$$

(Marginal Rate of Transformation)

My Ques:

Special Case: Many inputs, one output.

Inputs $(x_1, \ldots, x_m) \geq 0$

Output: $y \geq 0$

Production function: $f(x)$

Production set: $Y = \{(\underbrace{-x_1, -x_2, \ldots, -x_m}_{m}, y) :$ Inputs.

$x_i \geq 0 \ \forall i$ & $\quad n = m+1$.

$y \leq f(x_1, \ldots x_m)\}$

$T(z) = y - f(x)$, $z = (-x, y)$

Efficient frontier is

$T(z) = 0$

$\Rightarrow y - f(x) = 0$

$\rightarrow y = f(x)$

Example

Cobb-Douglas

- 2 inputs, capital (x_K) + labor (x_L)
- Output y
- $f(x_K, x_L) = x_K^{\alpha} x_L^{\beta}$, $\alpha + \beta = 1$
- $Y = \{(-x_K, -x_L, y) : x_K, x_L \geq 0,\ y \leq x_K^{\alpha} x_L^{\beta}\}$
- $T = (-x_K, -x_L, y) = y - x_K^{\alpha} x_L^{\beta}$

MRT between capital + output.

$$MRT_{(-x_K), y} = \frac{\partial T / \partial(-x_K)}{\partial T / \partial y}$$

$$= - \frac{\partial T / \partial x_K}{\partial T / \partial y}$$

$$= - \frac{-\alpha x_K^{\alpha - 1} x_L^{\beta}}{1}$$

$$= \alpha \left(\frac{x_L}{x_K} \right)^{\beta}$$

Marginal product of capital, MPK.

$$MPK = f_{x_K}(x_K, x_L) \rightarrow$$ from undergrad

$$= \partial f / \partial x_K$$

$$= \alpha x_K^{\alpha-1} x_L^{\beta}$$

$$= \alpha \left(\frac{x_L}{x_K}\right)^{\beta}$$

The method now is more a general way of finding sol^ns.

MRT for capital & labor

$$MRT_{(-x_K),(-x_L)} = \frac{\partial T / \partial(-x_K)}{\partial T / \partial(-x_L)}$$

$$= \frac{\partial T / \partial x_K}{\partial T / \partial x_L}$$

$$= \frac{-\alpha x_K^{\alpha-1} x_L^{\beta}}{-\beta x_K^{\alpha} x_L^{\beta-1}}$$

$$= \frac{\alpha}{\beta}\left(\frac{x_L}{x_K}\right)$$

$$\frac{d(-x_L)}{d(-x_K)} = -MRT_{(-x_K, -x_L)}$$

$$\frac{dx_L}{dx_K} = -\frac{\alpha}{\beta}\frac{x_L}{x_K}$$

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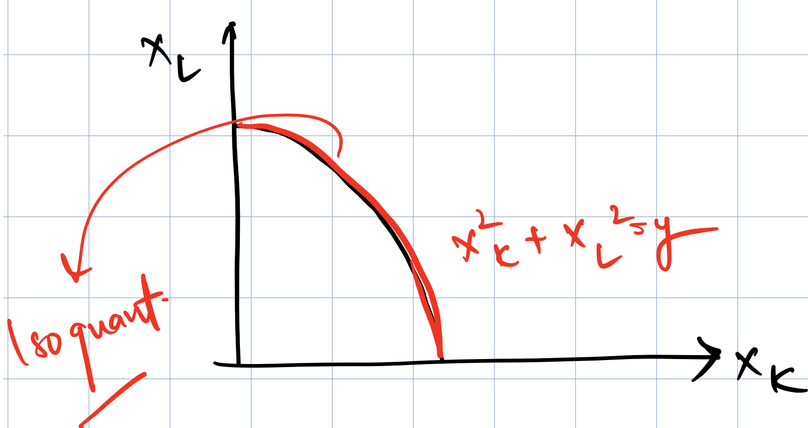
single output, fix y

$V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$ → Input requirement set.

$Q(y) = \{x \in \mathbb{R}^n : f(x) = y\}$ → Isoquant.

Example:

$f(x_K, x_L) = x_K^2 + x_L^2 = y$



This is all fine as the inputs here are considered positive as we are only working with inputs.

With a single output, we can define the marginal rate of technical substitution between two inputs

$$MRTS_{x_k, x_L} = \frac{\partial f / \partial x_k}{\partial f / \partial x_L}$$

$$\frac{\partial x_L}{\partial x_k} = -MRTS_{x_k, x_L} = \frac{-f_{x_k}}{f_{x_L}}$$

In the example

$$\frac{dx_L}{dx_k} = -\frac{\partial x_k}{\partial x_L} = -\frac{x_k}{x_L}$$

This is a funny example because the slope is increasing (MRTS is usually decreasing). This is because $V(y)$ is not convex.

Theorem: If Y is convex, then $V(y)$ is also convex.

$(P \Rightarrow Q \iff \text{not } Q \Rightarrow \text{not } P)$

<u>Proof</u> $(-x, y) \in Y$

$(-x', y) \in Y$

$$Y \text{ convex} \Rightarrow \underbrace{t(-x, y) + (1-t)(-x', y)}_{\text{linear combination}} \in Y$$

$\Rightarrow (-tx - (1-t)x', \; ty + (1-t)y) \in Y$

$\Rightarrow (-tx - (1-t)x', y) \in Y$

$\Rightarrow (-x'', y) \in Y$ where $x'' = tx + (1-t)x'$

$\Rightarrow x'' \in V(y)$

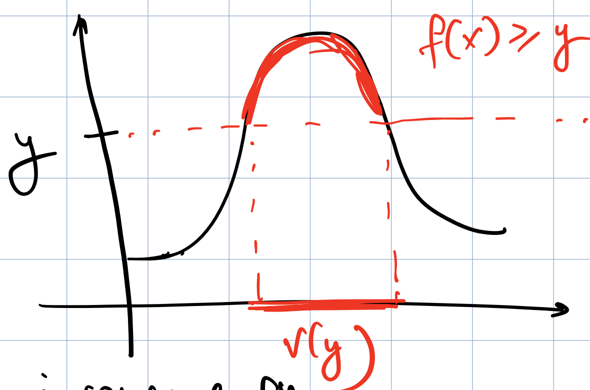
$\Rightarrow V(y)$ is <u>convex</u>.

Theorem: $V(y)$ is convex iff $f(x)$ is quasiconcave.

Proof: $V(y) = \{x \in \mathbb{R}^n : f(x) \geq y\}$ → upper contour set of f.

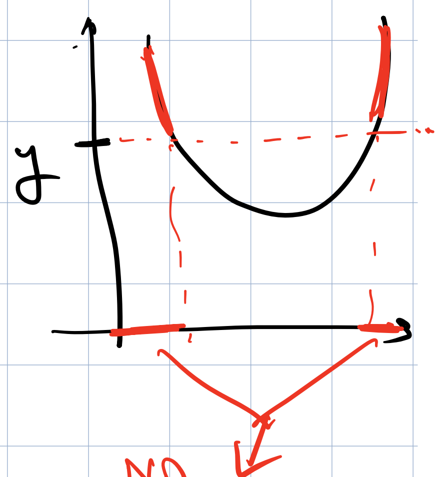
(This more of a def^n)

Definition of a quasi concave f^n that has convex upper contour sets.



Quasi concave f^n

Concave f^n are quasi concave f^n.



Not convex.
So this is not a quasi-concave f^n.

Homogeneity + Homotheticity

A function $f: \mathbb{R}^n \to \mathbb{R}$ is <u>homogeneous of degree k</u>

if $f(tx) = t^k f(x) \; \forall t = 0$

$k=0$: $f(tx) = f(x)$

$k=1$: $f(tx) = tf(x)$

Cobb-Douglas :

$$f(x_K, x_L) = x_K^{\alpha} x_L^{\beta} \qquad \alpha + \beta = 1.$$

$$f(tx_K, tx_L) = (tx_K)^{\alpha}(tx_L)^{\beta}$$

$$= t^{\alpha+\beta} x_K^{\alpha} x_L^{\beta}$$

when $\alpha + \beta = 1$.

$$= t\, x_K^{\alpha} x_L^{\beta} = tf(x)$$

What about MRTS?

$$\text{MRTS} = \frac{\alpha}{\beta}\frac{x_L}{x_K}$$

$$= \frac{\alpha}{\beta}\frac{(tx_L)}{(tx_K)} = \frac{\alpha}{\beta}\frac{x_L}{x_K}$$

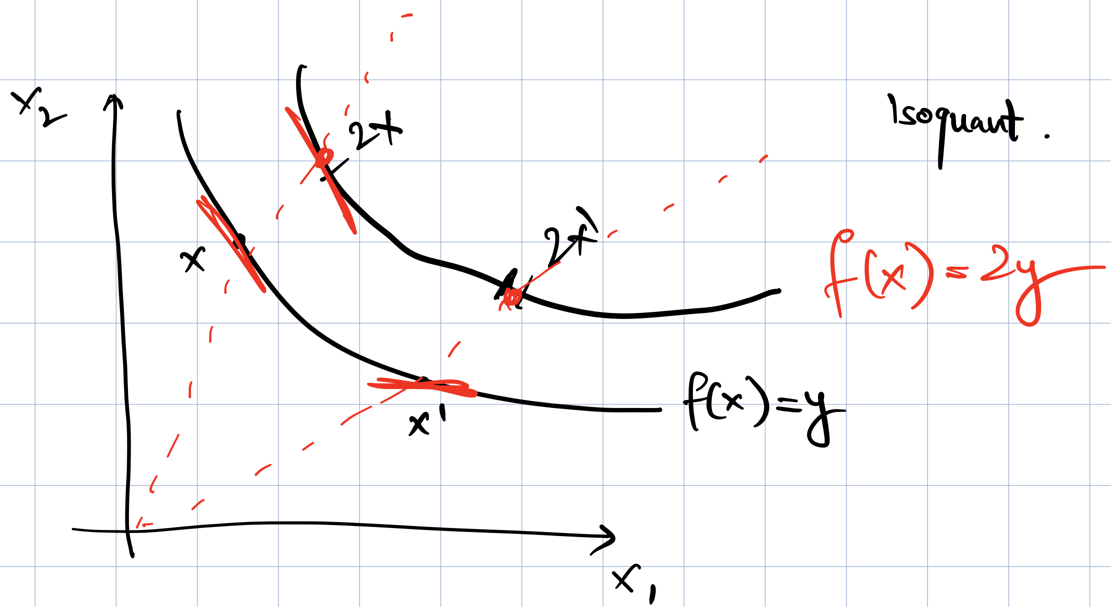
MRTS is homogeneous of degree 0.

Theorem: If f is hd$-k$, then the partial derivative f_{x_i} is hd$-(k-1)$.

Proof: $f(tx) = t^k f(x)$

$t f_{x_i}(tx) = t^k f_{x_i}(x)$

$f_{x_i}(tx) = t^{k-1} f_{x_i}(x)$



Say f is hd-1

$f(2x) = 2f(x) = 2y$

The slopes would be the same too.

$\Rightarrow$ Every isoquant is just a blown up version of the first one.

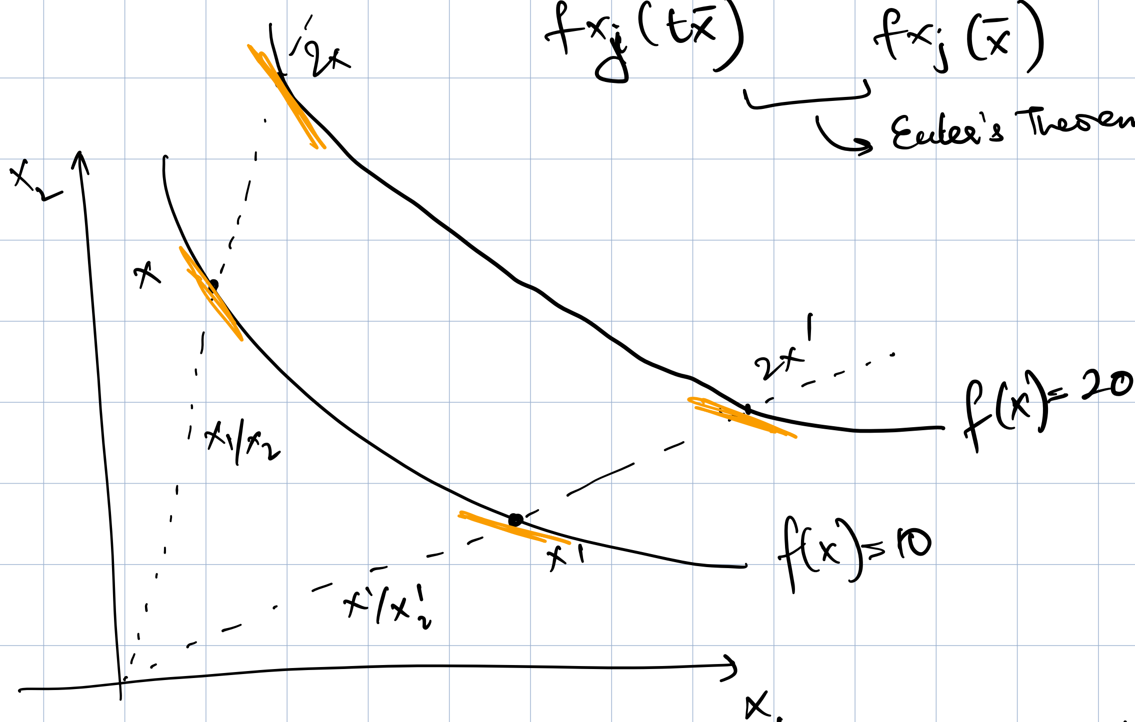
30 Aug 2023

If a production f^n $f(x)$ is homogeneous of degree 1 (hd-1), then the associated MRTS is hd-0.

Proof: $MRTS_{i,j}(\bar{x}) = \dfrac{f_{x_i}(\bar{x})}{f_{x_j}(\bar{x})}$

$$MRTS_{i,j}(t\bar{x}) = \frac{f_{x_i}(t\bar{x})}{f_{x_j}(t\bar{x})} = \frac{f_{x_i}(\bar{x})}{f_{x_j}(\bar{x})} = MRTS_{i,j}(\bar{x})$$

↳ Euler's Theorem



The slopes of the isoquants would be the same.

Correct later
factor ratio: x_2/x_1

Homothetic functions.

is a positive monotonic transformation of an hd-1 function.

i.e. $f: \mathbb{R}^n \to \mathbb{R}$ is homothetic if $\exists$ a

$g: \mathbb{R} \to \mathbb{R}$ strictly increasing and an

$h: \mathbb{R}^n \to \mathbb{R}$ that is hd-1 s.t. $f(x) = g(h(x))$

$$\text{MRTS of } f = \frac{f_{x_i}}{f_{x_j}} = \frac{g'(h(x))\, h_{x_i}}{g'(h(x))\, h_{x_j}} = \frac{h_{x_i}}{h_{x_j}}$$

↳ hd-0

The shape of the isoquant would be the same but the labelling could be different. Output could increase to 80 instead of 20 in the last example.

Returns to scale wrt a single output.

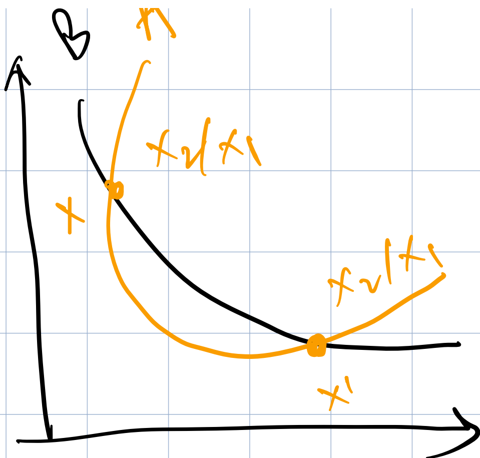
Production function f exhibits:

- decreasing returns to scale
  if $f(tx) \leq t f(x) \quad \forall t \geq 1$.
- increasing returns to scale
  if $f(tx) \geq t f(x) \quad \forall t \geq 1$
- constant returns to scale if *.
  $f(tx) = t f(x) \quad \forall t \geq 0$

Curvature of Isoquants.

① Elasticity of complementarity
(for "large" jumps)

$$\frac{1}{\sigma} = \frac{\dfrac{\Delta MRTS}{MRTS}}{\dfrac{\Delta (x_2/x_1)}{x_2/x_1}}$$



Denominator is the same for A, B in example.
Numerator much bigger for A.

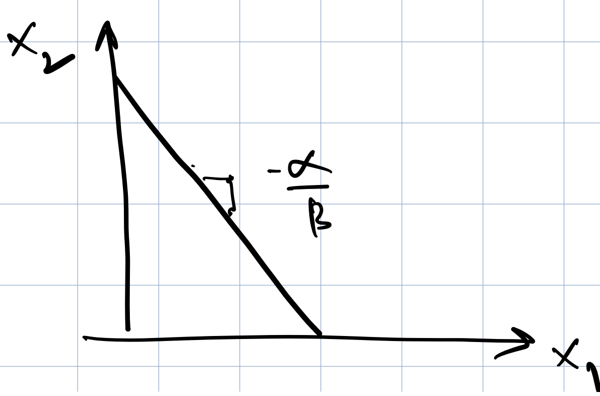
$$\frac{1}{\sigma_A} > \frac{1}{\sigma_B}$$

Defn through limits: —

$$\frac{1}{\sigma} = \frac{x_2/x_1}{MRTS} \cdot \frac{d\,MRTS}{d(x_2/x_1)}$$

Example

Perfect Substitution $f(x_1, x_2) = \alpha x_1 + \beta x_2$



$$\frac{1}{\sigma} = 0$$

The slope does not change.

Elasticity of substitution

$$\sigma = \frac{MRTS}{x_2/x_1} \frac{d(x_2/x_1)}{d\, MRTS}$$

For perfect substitutes, $\sigma = \infty$ (as $1/\sigma = 0$)

Less curvature → bigger elasticity of substitution

Perfect Complements

$f(x_1, x_2) = \min\{x_1, x_2\}$ $\quad \sigma = 0$

For any 2 quantities, y, x the <u>elasticity of y wrt x</u>

$$\varepsilon_{yx} = \frac{d \log(y)}{d \log(x)}$$

Way to see this:

$$d \log y = \frac{1}{y} dy \quad \text{(Total derivative)}$$

$$d \log x = \frac{1}{x} dx$$

$$= \frac{d \log(y)}{d \log(x)} = \frac{x}{y} \frac{dy}{dx}$$

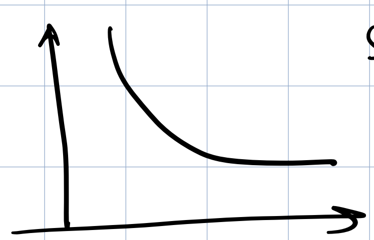
For σ:

$$y = x_2 / x_1$$

$$x = MRTS$$

(*) σ for cobb-douglas

<u>Cobb Douglas</u>:



Smoothly diminishing graphs

$$MRTS = \frac{\alpha}{\beta} \cdot \frac{x_L}{x_K}$$

$$\sigma = \frac{MRTS}{x_L / x_K} \cdot \frac{d(x_L / x_K)}{d(MRTS)}$$

Let $\theta = \frac{\alpha}{\beta} \frac{x_L}{x_K}$

or $\frac{x_L}{x_K} = \frac{\beta}{\alpha} \theta$

$= MRTS = \theta$ → constant.

$$\sigma = \frac{\theta}{\frac{\beta}{\alpha}\theta} \cdot \frac{d\left(\frac{\beta}{\alpha}\theta\right)}{d\theta}$$

$$= \frac{\beta/\alpha}{\beta/\alpha} = 1.$$

C-D, perf complements / substitutes are all special cases of ==constant elasticity of substitution (CES)== production.

$$f(x) = [\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}]^{1/\rho}$$ → row.

$\rho = 0, 1, \infty$ → leads to special cases.

Recall our assumption that firms are 1 price takers + 2 maximize profits

The profit maximization problem (PMP)

let p be a vector of n prices

$\max p \cdot y$ s.t. $y \in Y$, where $p \cdot y = \sum_{i=1}^{n} p_i y_i$ is dot product

Equivalently,

$$\max p \cdot y$$
$$\text{s.t. } T(y) \leq 0$$

$\pi : \mathbb{R}^n \to \mathbb{R}$ is also called the profit function.

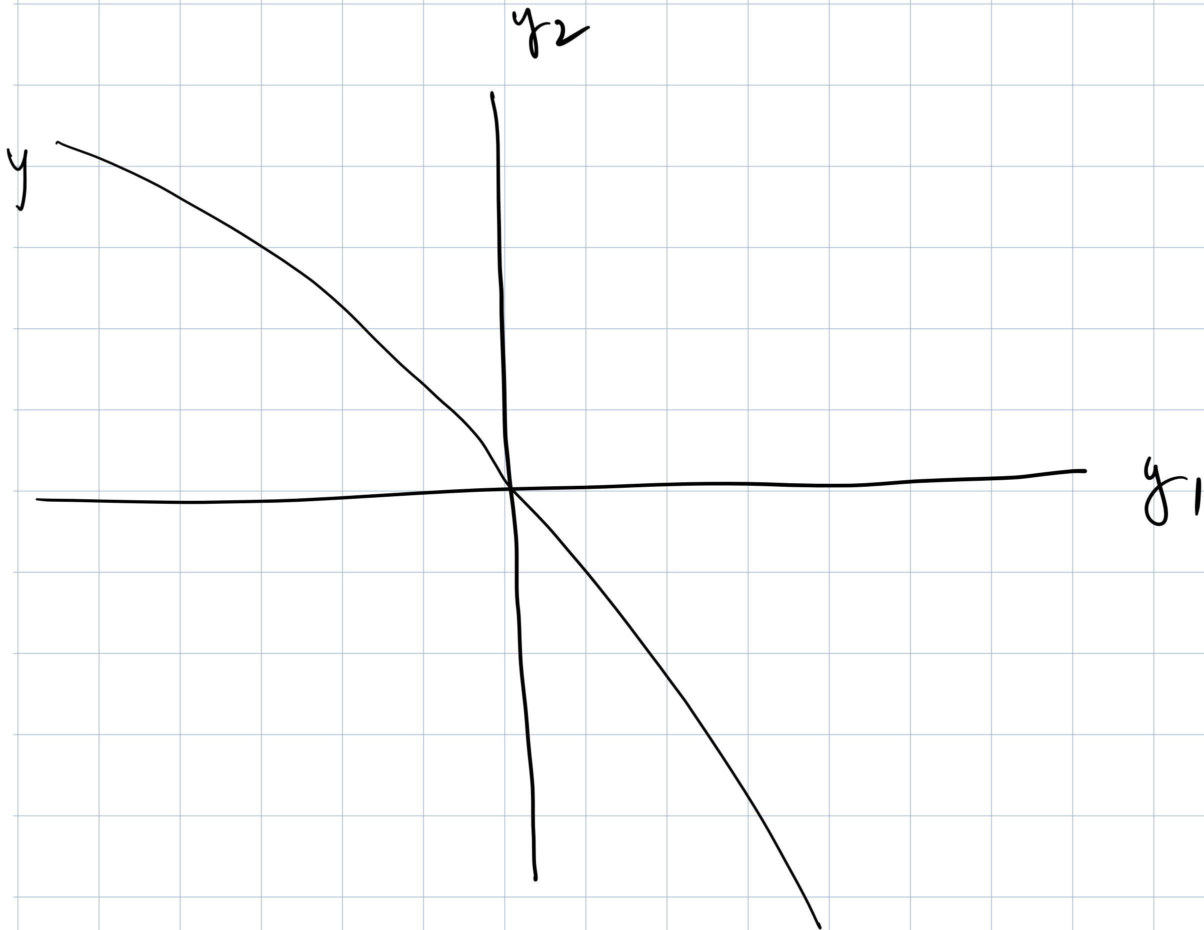
Example of a value function in constrained optimization.

$p \cdot y$ is called the objective function.

$y^*(p) = \{ y \in Y : p \cdot y = \pi(p) \}$ is called the supply correspondence.

(set valued mapping (function))

$y(p)$ could be multivalued (or $\{y(p)\} = \emptyset$)



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$\max \; p \cdot y$

s.t. $T(y) \leq C$

$\mathcal{L} = p \cdot y - \lambda T(y)$

KT conditions:

$$\frac{\partial \mathcal{L}}{\partial y_i} = 0 \Rightarrow p_i - \lambda \cdot \frac{\partial T}{\partial y_i} = 0$$

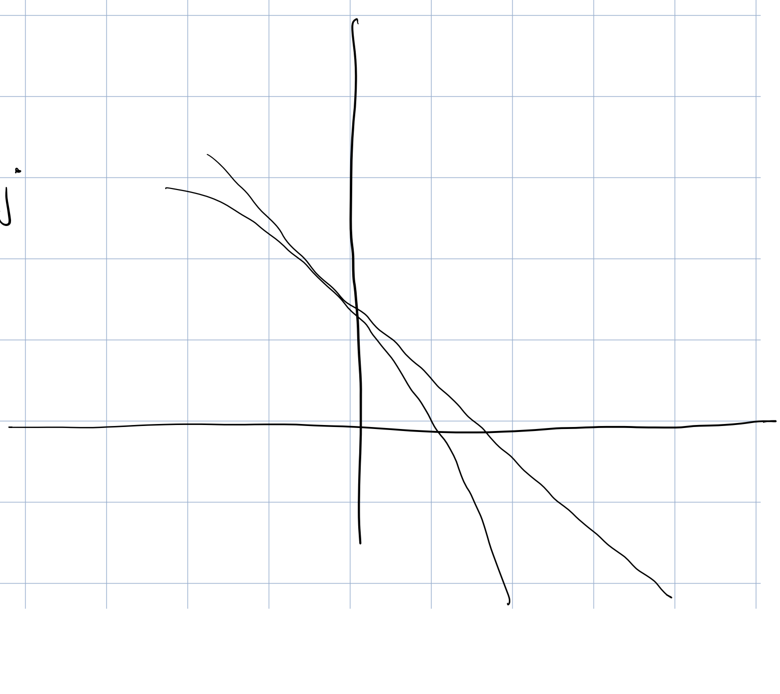
CSC : $\lambda T(y) = 0$

$\lambda \geq 0$

Implications of the FOCs:

$$\frac{p_i}{p_j} = \underbrace{\frac{\partial T / \partial y_i}{\partial T / \partial y_j}}_{MRT_{ij}} \quad \text{for any } i, j$$

Recall $\frac{dy_j}{dy_i} = -MRT_{ij}$



$$\frac{\partial T/\partial y_i}{p_i} = \frac{\partial T/\partial y_j}{p_j}$$: Marginal product per dollar is the same

* Single Output Case

Production function $f(x)$

$\max \; p f(x) - w \cdot x$ p: output ~~p~~price.

s.t. $x_i \geq 0$ w: (vector of) input prices

$\mathcal{L} = p f(x) - w \cdot x + \mu \cdot x$ μ_i is multiplier for $x_i \geq 0$ constraint.

KT conditions:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \Rightarrow p \frac{\partial f}{\partial x} - w_i + u_i = 0 \quad (*)$$

CSC : $u_i x_i = 0 \quad \forall i$

$\mu_i \geq 0 \quad \forall i$

Another way to write (*)

$$\underbrace{p \cdot \frac{\partial f}{\partial x_i}}_{\text{Marginal revenue of addition } x_i} \leq \underbrace{w_i}_{\text{Marginal cost of addition } x_i}$$ with equality if $x_i^* > 0$.

Divide i by j :

$$\frac{\partial f / \partial x_i}{\partial f / \partial x_j} = \frac{w_i}{w_j} \quad \text{for any 2 } x_i^*, x_j^* > 0$$

$$\frac{\partial f / \partial x_i}{w_i} = \frac{\partial f / \partial x_j}{w_j}$$

Notation for single output problems :

$x^*(p, w)$ factor demand correspondence (or function)

$y^*(p, w) = f(x^*(p, w))$: supply correspondence

$$\pi(p, w) = p f(x^*(p, w)) - w \cdot x^*(p, w) \quad \text{profit } f^n$$

$$= \max \; p f(x) - w \cdot x^*$$

$$x_i \geq 0$$

KT conditions are necessary but not (in general) sufficient. $\Rightarrow$ SOC

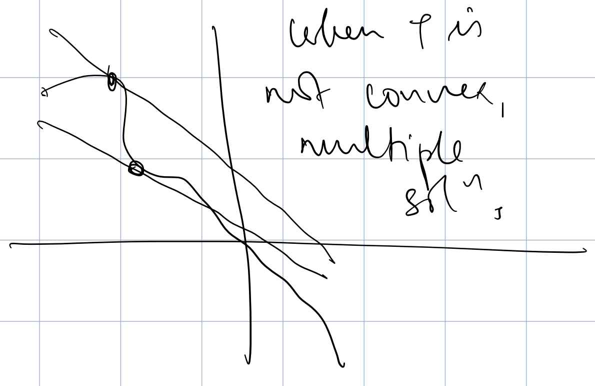
SOCs are also local. They don't tell you globals.

Going back to original formulation to make FOCs sufficient:

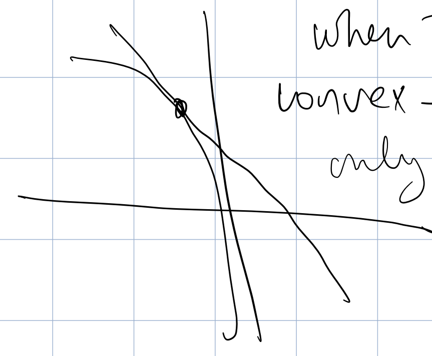
$\max p \cdot y$ s.t. $T(y) \leq 0$

Theorem: Assume that $T(y)$ is convex. If a point y^* satisfies the KT conditions for some $\lambda \geq 0$, then y^* is a solⁿ to the maximization problem.

[If your objective fⁿ ~~is~~ quasiconcave, if all the constraints are quasiconvex $\Rightarrow$ the KT conditions are sufficient.]



when T is not convex, multiple solⁿ,



when T is convex → there's only one KT solⁿ.

Single Output

$$\max \; pf(x) - w \cdot x$$
$$\text{s.t.} \quad x_i \geq 0$$

Theorem: Assume f is concave, then if x^* satisfies

$$p \cdot \frac{\partial f(x_i^*)}{\partial x_i} \leq w_i \;\; \forall i \quad \text{with equality if } x_i^* > 0$$

then x^* is a solution to the problem.

Caveats :-

- KT/FOCs are very useful, but be careful about following :-

1) Non differentiabilities
2) Non convexities
3) Corner solutions
4) Existence/Uniqueness issues

One input - one output

$$y = f(x) = \frac{1}{2}x$$

$$\max_{x \geq 0} \; pf(x) - wx$$

$$\mathcal{L} = p\left(\frac{1}{2}x\right) - wx + \mu x$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow \frac{1}{2}p - w + \mu = 0$$

CSC = $x\mu = 0$

$\mu \geq 0$

$x = 0$, $\mu = w - \frac{1}{2}p$ is a sol^n if

$$\mu = w - \frac{1}{2}p \geq 0$$

What if $w - \frac{1}{2}p < 0 \Rightarrow w < \frac{1}{2}p$.

The output price is too high that you keep producing $\Rightarrow$ no answer.

$$x(p,w) = \begin{cases} 0 & , p < 2w \\ (0, \infty) & , p = 2w \\ \phi & , p > 2w \end{cases}$$

$\phi \to$ infinity, whatever you want.

$$\pi(p,w) = \begin{cases} 0 & , p \leq 2w \\ +\infty & , p > 2w \end{cases}$$

Q/ $f(x_1, x_2) = \log(1 + x_1 + x_2)$

Output price = p

Input price : $w = (w_1, w_2)$

$$\max \; p \log(1 + x_1 + x_2) - w_1 x_1 - w_2 x_2$$

$$\mathcal{L} = p \log(1 + x_1 + x_2) - w_1 x_1 - w_2 x_2 + \mu_1 x_1 + \mu_2 x_2$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow \frac{p}{1 + x_1 + x_2} - w_1 + \mu_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow \frac{p}{1+x_1+x_2} - \omega_2 + \mu_2 = 0$$

CSC: $\mu_1 x_1 = 0$, $\mu_2 x_2 = 0$

$\mu_1, \mu_2 \geq 0$

Case 1: Both inputs are used

$x_1, x_2 > 0$

$\Rightarrow \mu_1 = \mu_2 = 0$

$$\omega_1 = \frac{p}{1+x_1+x_2} \qquad \omega_2 = \frac{p}{1+x_1+x_2}$$

$\Rightarrow \omega_1 = \omega_2 = \overline{\omega}$

$$\overline{\omega} = \frac{p}{1+x_1+x_2}$$

$$\Rightarrow 1+x_1+x_2 = \frac{p}{\overline{\omega}}$$

$$x_1+x_2 = \frac{p}{\overline{\omega}} - 1$$

Implicitly, we need $\frac{p}{w} > 1$

$\Rightarrow \quad p > \overline{w}$

Case 2 : One input (say x_i) is used

$x_2 = 0$

$\mu_1 = 0, \quad \mu_2 \geq 0$

$$\frac{p}{1+x_1} = w_1 \Rightarrow x_1^* = \frac{p}{w_1} - 1$$

$$\mu_2 = w_2 - \frac{p}{1+x_1} = w_2 - w_1 \geq 0$$

Case 3 Only x_2 used.

Same, switch x_1, x_2

$x_1 = 0 \qquad x_2 = \frac{p}{w_2} - 1$

$w_2 \leq w_1$

Factor Demands are (Assume $p \geq \min\{w_1, w_2\}$)

$$x_1(p,w) = \begin{cases} \frac{p}{w_1} - 1, & w_1 < w_2 \\ \alpha\left(\frac{p}{\overline{w}} - 1\right), & w_1 = w_2 = \overline{w} \\ 0, & w_1 > w_2 \end{cases}$$

for some $\alpha \in [0,1]$

$$x_2(p,w) = \begin{cases} 0, & w_1 < w_2 \\ (1-\alpha)\left(\frac{p}{\overline{w}} - 1\right), & w_1 = w_2 = \overline{w} \\ \frac{p}{w_2} - 1, & w_1 > w_2 \end{cases}$$

$$y(p,w) = f\left(x_1(p_1, w_1), x_2(p_2, w_2)\right)$$

$$= \begin{cases} 0, & p \leq \hat{w} \\ \log\left(\frac{p}{\hat{w}}\right), & p > \hat{w} \end{cases}$$

$$\hat{w} = \min \{ w_1, w_2 \}$$

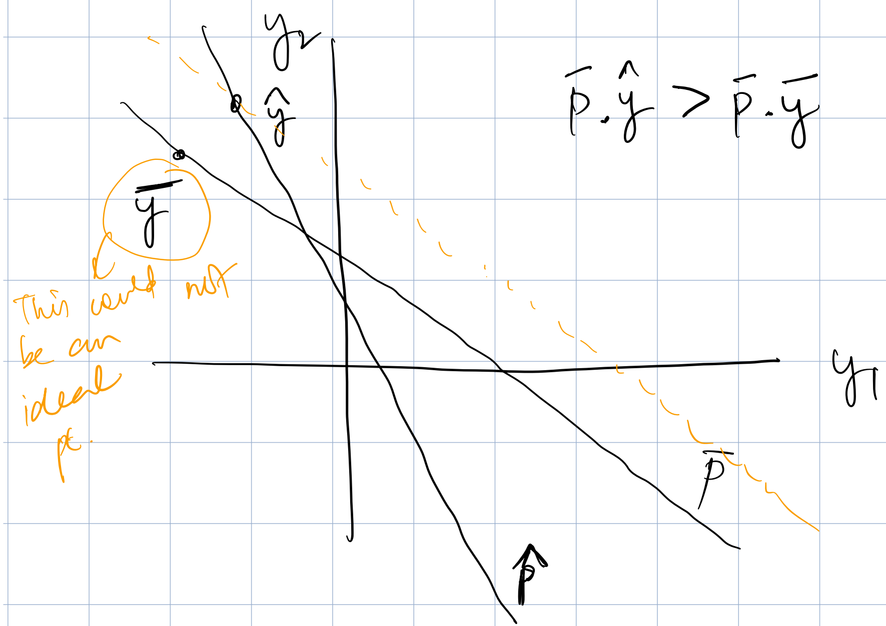
$$\pi(p, w) = \begin{cases} 0, & p \leq \hat{w} \\ p \log\left(\frac{p}{\hat{w}}\right) - p + \hat{w}, & p > \hat{w} \end{cases}$$

6 September

Rationalizability:

Q: Given some data $\{(p^1, y^1), \ldots, (p^T, y^T)\}$

Can we learn about the firm's technology?



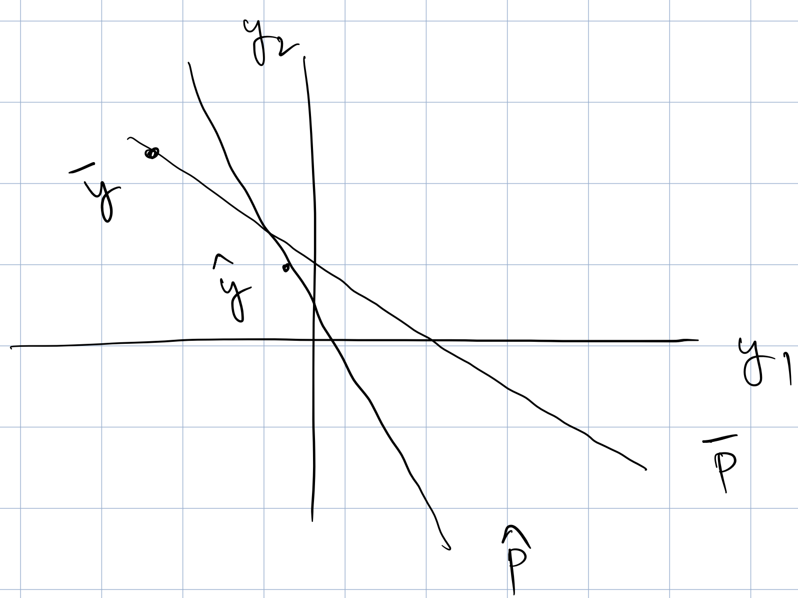
$\bar{p} \cdot \hat{y} > \bar{p} \cdot \bar{y}$

Weak Axiom of Profit Maximization

$p^t \cdot y^t \geq p^t \cdot y^s$

$p^1 \cdot y^1 \geq p^1 \cdot y^s \quad \forall \; s = 1, \ldots, T$

WAPM is a necessary condition for profit maximization.



What other production sets could "rationalize" the data?

A production set Y rationalize the data if
$y_t \in y^*(p^t) \quad \forall \; t = 1, \ldots, T$

$$y^*(p) = \arg\max_{y \in Y} \; p \cdot y$$

(optimal choices of the problem.)

Make some additional assumptions:

- data satisfies WAPM
- Y is convex & has free disposal

$Y^I = \text{conv}_{fd}(\{y^1, \dots y^T\})$

Does Y^I rationalize the data?

Claim: Yes.

Proof: Assume not. Then $\exists$ a point $y \in Y^I$ s.t. $p^t \cdot y > p^t \cdot y^t$ for some t.

By convexity & FO,

(Need to add notes from photo)

$Y^O = \{ y \in \mathbb{R}^n : p^t y \leq \pi(p^t), \forall t = 1, \ldots, T \}$

$\hat{y} \notin Y^O$ because $p^2 \cdot \hat{y} > p^2 \cdot y^2 = \pi(p^2)$

$\mathbb{R}^n \setminus Y^O$: bundles that yield higher profits than some observed choice.

$= \{ y \in \mathbb{R}^n : p^t \cdot y > \pi(p^t) \text{ for some } t \}$

Does Y^O rationalize the data?

Claim: Yes.

Proof: Let's assume not. Then, $\exists$ some $y \in Y^O$ s.t.

$p^t \cdot y > p^t \cdot y^t$

Directly contradicts the definition of Y^O.

Theorem: For any production set Y that rationalizes the data, we have

$Y^I \subseteq Y \subseteq Y^O$.

(Proof in the HW → draw picture to look for a contradiction)

What if we had infinite data?

Suppose we observe $y(p)$ for all p.
Let $\pi(p) = p \cdot y(p)$ be the (observed) profit function.

Theorem: Suppose Y is convex, closed & satisfies F.D. (free disposal)

Then, $Y = Y^0 = \{ y \in \mathbb{R}^n : p \cdot y(p) \leq \pi(p)$

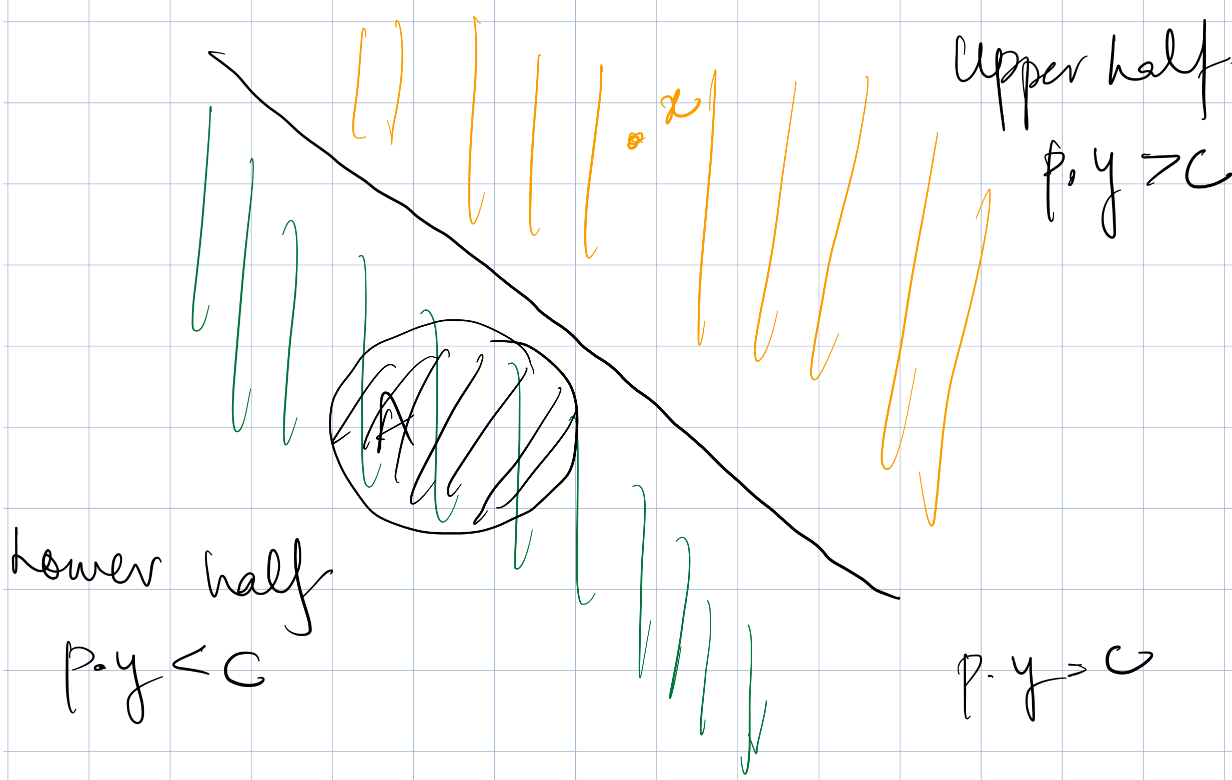
$\forall\, p \in \mathbb{R}^n_+ \}$

Detour: Separating Hyperplane Theorem

In $\mathbb{R}^n$, a <u>hyperplane</u> is a subspace of dim $(n-1)$.

Given $p \in \mathbb{R}^n$, $c \in \mathbb{R}$, the $(p-c)$-hyperplane is

$$H_{p,c} = \{ y \in \mathbb{R}^n : p \cdot y = c \}$$



Separating hyper plane Theorem :–

Suppose $A \subseteq \mathbb{R}^n$ is convex & closed, & $x \notin A$. Then, there is a $p \in \mathbb{R}^n$,

$p \neq 0$ & $c \in \mathbb{R}$ s.t.

$p \cdot x > c$ & $p \cdot y < c$ $\forall y \in A$

Back to the theorem :-

Theorem: Suppose Y is convex, closed & satisfies F.D. (free disposal)

Then, $Y = Y^0 = \{ y \in \mathbb{R}^n : p \cdot y(p) \leq \pi(p)$

$\forall p \in \mathbb{R}^n_+ \}$

Proof $Y = Y^0$

NTS (need to show)

① $Y \subseteq Y^0$ & ② $Y^0 \subseteq Y$

For ② : Take some $\hat{y} \in \mathbb{R}^n \setminus Y$

By SHT, $\exists$ some $p \in \mathbb{R}^n$ & c s.t.

$$p \cdot \hat{y} > c > p \cdot y \quad \forall y \in Y$$

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y \quad (*)$$

By free disposal, $p \geq 0$ if $p_i < 0$ for some i,
then $\max p \cdot y = \infty$
which contradicts $(*)$

Therefore, there exists $p \geq 0$ s.t.

$$p \cdot \hat{y} > \max_{y \in Y} p \cdot y = \pi(p)$$

i.e. $\hat{y} \notin Y^0$ or $Y^0 \subseteq Y$

25 Sep

$$C(w,y) = \min_{\substack{s.t.\\ f(x) \geq y}} w \cdot x$$

For now write as $c(y)$

Average Cost : $AC(y) = \dfrac{C(y)}{y}$

Marginal cost : $MC(y) = c'(y)$

Marginal - Average Relationship

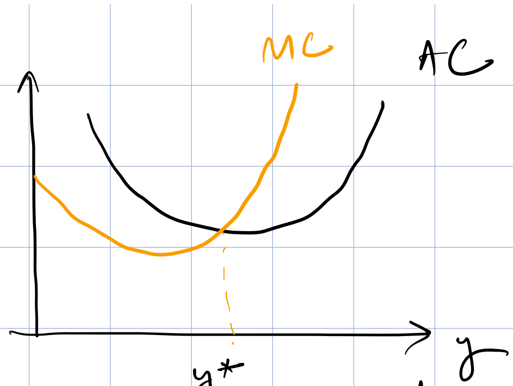
$$AC'(y) = \frac{c'(y)}{y} - \frac{c(y)}{y^2}$$

$$= \frac{c'(y)y - c(y)}{y^2} \qquad \text{Denom} \geq 0$$

AC incr $\iff$ $c'(y) > \dfrac{c(y)}{y}$ or $MC(y) > AC(y)$

AC decr $\iff$ $c'(y) < \dfrac{c(y)}{y}$ or $MC(y) < AC(y)$

$AC'(y) = 0 \iff MC(y) = AC(y)$



y^*_{min} → Minimum Efficient Scale (MES)

just a term → does not mean we have to produce at this point.

⊙ $\max_{y \geq 0} \; py - c(y)$

FONC: (First Order Necessary Condition)

$p \leq c'(y^*)$, with equality if $y^* > 0$

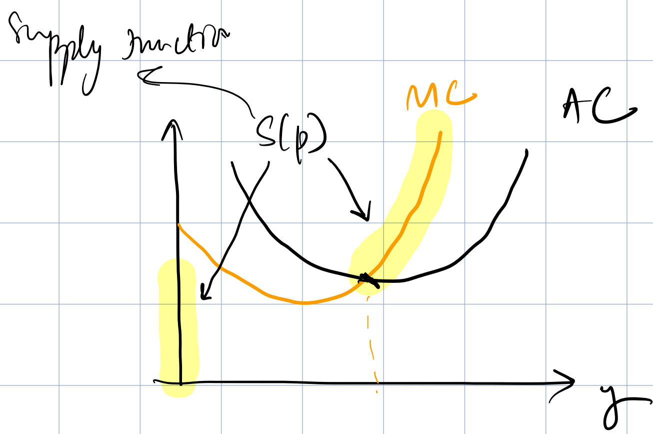
If $c(y)$ is convex, this is also sufficient.

price = MC if firm chooses to produce a strictly positive amount.

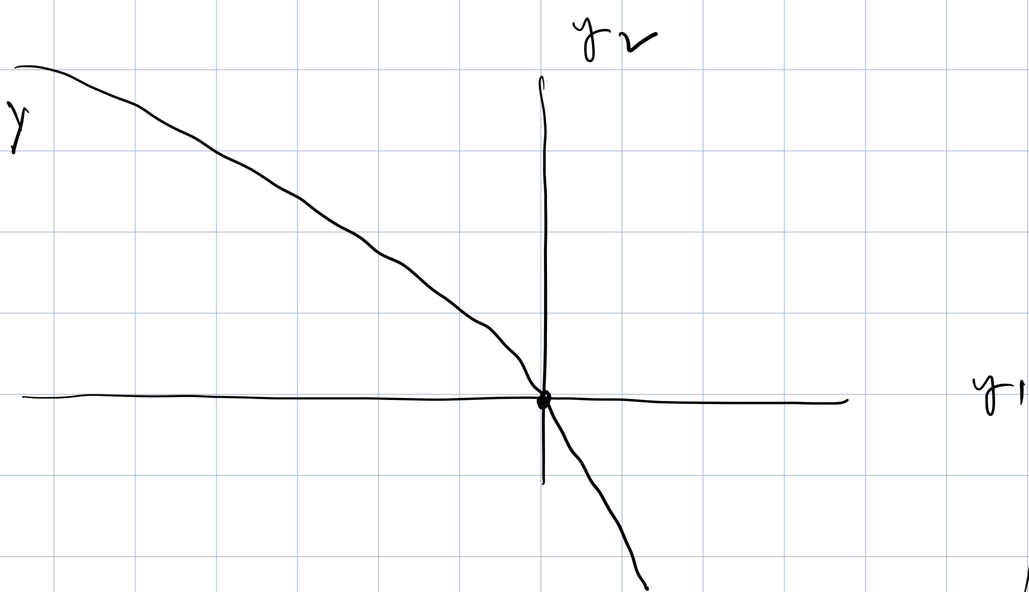
Shut Down Rule: Check that at y^* where $p = c'(y^*)$,

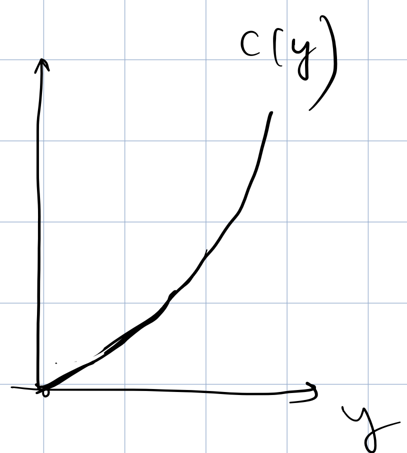
$py^* - c(y^*) \geq 0$

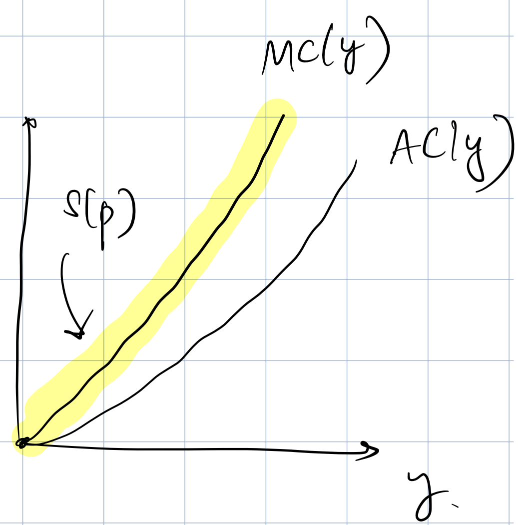
or $p \geq AC(y^*)$



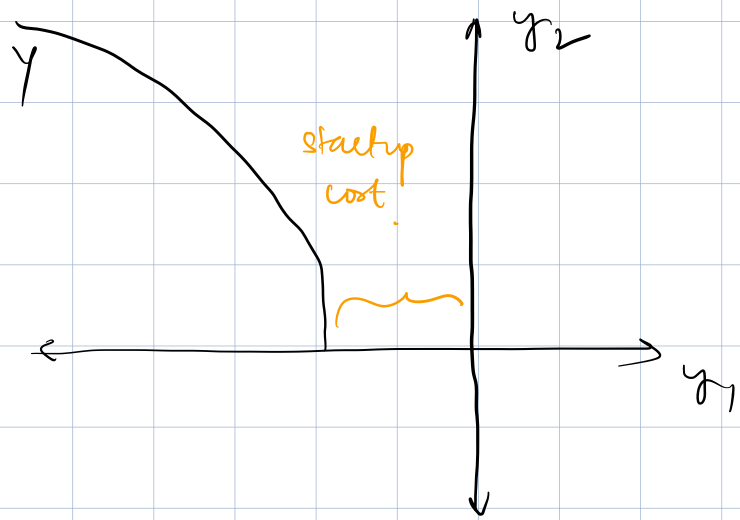
Only producing when P > AC, otherwise shut-down.

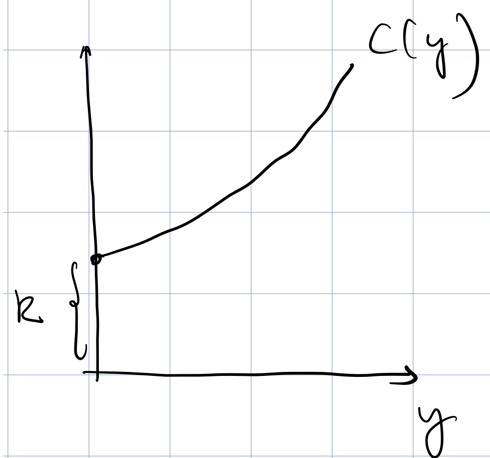


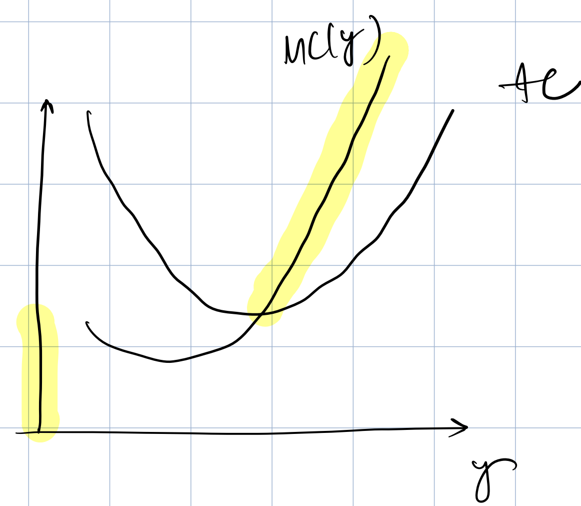




(Starts from origin as well)



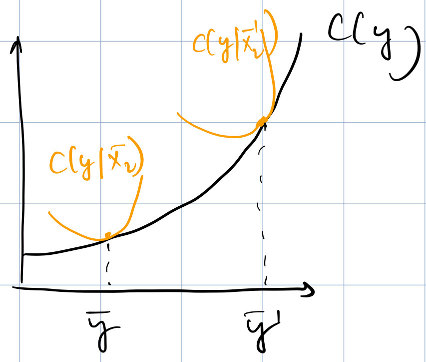




$$C(y) = \begin{cases} 0, & y = 0 \\ \underbrace{VC(y)}_{\text{variable cost.}} + k, & y > 0 \end{cases}$$

Long-run vs short-run costs

- 2 inputs (x_1, x_2)
- $C(y)$ be the long run cost function
- Let $\bar{x}_2$ be optimal x_2 when long run output is $\bar{y}$.
- $C(y \mid \bar{x}_2)$ short run cost, when x_2 fixed at $\bar{x}_2$.



The long run cost curve will be lower than the short run cost curves except at the optimal pts. → It's like a lower envelope.

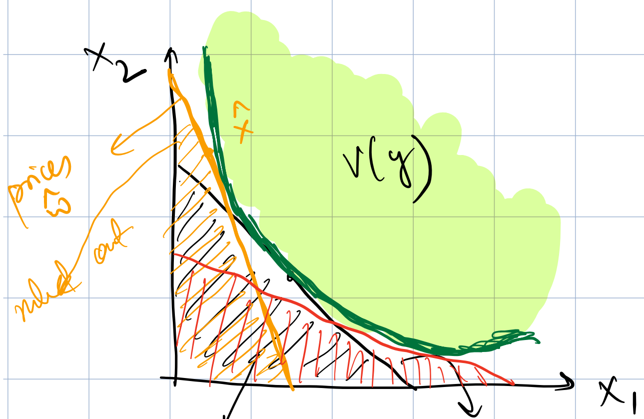
⊙ $f(x) \to c(w,y)$. Can we go from $c(w,y) \to f(x)$?
Yes - they are duals of each other.
- Observe $c(w,y)$ $\forall (w,y)$
- $V(y) = \{x.\ f(x) \geq y\}$

If we know $V(y)$ for all y, then we know $f(x)$
Can we recover $V(y)$?
Define an "outer bound"

$$V^0(y) = \{x : w \cdot x \geq c(w,y),\ \forall\ w \geq 0\}$$

At $\bar{w}_1$, cost is $c(\bar{w}, y) = \bar{c}$. Firm chose $\bar{x}$

$$\bar{w}_1 x_1 + \bar{w}_2 x_2 = \bar{c}$$



down here is cheaper but firm chose to not produce it because it does not produce enough.

As we draw more & more lines, we recover $V(y)$.

__Claim__: Suppose $V(y)$ is the input requirement set for a firm with a convex technology with free disposal. Then $V(y) = V^0(y)$

Proof: HW (use separating hyperplane theorem)

- Presupposes the function we start with is a "valid" cost function.
- Question - Given an arbitrary function $\phi(w, y)$ how do we know if ϕ is a valid cost function? That is, does:

$$\phi(w,y) = \min_{s.t.\ x \in V(y)} w \cdot x$$

where $V(y) = \{x \geq 0,\ w.x \geq \phi(w,y)\ \forall w\}$

- Showed before that if ϕ is a cost function, then:
  1. hd-1 in prices: $\phi(tw,y) = t\phi(w,y)$
  2. $\phi(w,y) \geq 0$ (positivity)
  3. $\phi(w',y) \geq \phi(w,y)$ for $w' \geq w$
  4. Concave in w

Theorem: Let $\phi(w,y)$ be a differentiable function satisfying 1–4, then $\phi(w,y)$ is the cost function for the technology

$$V(y) = \{x : w \cdot x \geq \phi(w,y),\ \forall w\}$$

<u>Proof</u> :- Define

$$x(w,y) = \left(\frac{\partial\phi(w,y)}{\partial w_1}, \ldots, \frac{\partial\phi(w,y)}{\partial w_n}\right)$$

By monotonicity, $x(w,y) \geq 0$

By hd-1, $\phi(w,y) = \sum_{i=1}^{n} w_i \frac{\partial\phi(w,y)}{\partial w_i} = w \cdot x(w,y)$

Euler's law

WTS for any $w \geq 0$,

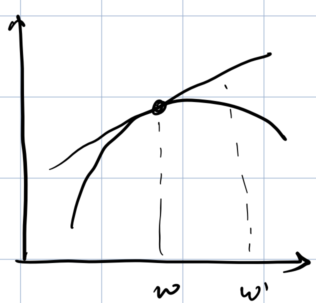
$$\phi(w,y) = w \cdot x(w,y) \leq w \cdot x \quad \forall x \in V(y)$$

This will imply:

$$\phi(w,y) = \min_{\text{st. } x \in V(y)} w \cdot x$$ (so long as $x(w,y)$ is feasible.)

To show: $x(w,y)$ is feasible

$$\phi(w',y) \leq \phi(w,y) + \underbrace{\nabla_w \phi(w,y)}_{x(w,y)}(w'-w) \quad \text{for any } w'.$$



$$\phi(w',y) \leq \cancel{\phi(w,y)} + w' \cdot x(w,y) \underbrace{- \cancel{w \cdot x(w,y)}}_{\phi(w,y)}$$

$$\phi(w',y) \leq w' \cdot x(w,y) \quad \text{for all } w'.$$

so, $x(w,y) \in V(y)$

$x(w,y)$ is feasible

Now notice that for any $x \in V(y)$

$$w \cdot x \geq \phi(w,y) = w \cdot x(w,y)$$

or $w \cdot x(w,y) \leq w \cdot x \quad \forall x \in V(y)$

i.e. $x(w,y) = \arg\min_{x \in V(y)} W \cdot x$

$$\phi(w,y) = w \cdot x(w,y)$$

Example :–

$$c(w, y) = y(w_1 + w_2)$$

$\Rightarrow$ satisfies the 4 criteria.

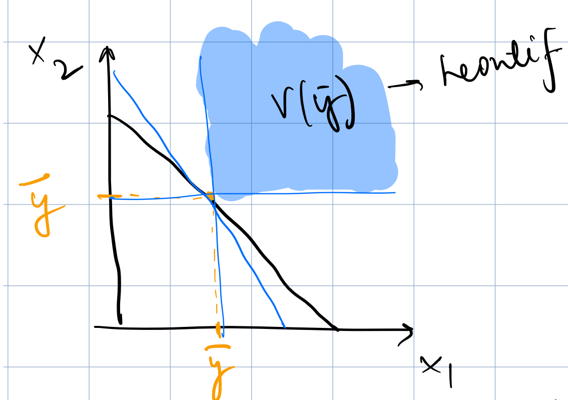
Fix an output $\bar{y}$. Graphically

Want points (x_1, x_2) s.t.

$$w \cdot x \geq c(w, \bar{y}) \quad \forall (w_1, w_2)$$

$$w_1 x_1 + w_2 x_2 \geq w_1 \bar{y} + w_2 \bar{y}$$

$$x_2 \geq \frac{w_1}{w_2}(\bar{y} - x_1) + \bar{y}$$



Algebrically :–

By Shephard's lemma:

$$\frac{\partial c}{\partial w_1} = x_1^*, \quad \frac{\partial c}{\partial w_2} = x_2^*$$

Here: $\bar{y} = x_1^*(w, y)$

$\bar{y} = x_2^*(w, y)$

$\Rightarrow$ Leontief production function

4th Oct

Monopoly

- Can "choose" output price (p)
- still face some constraints
  - Technological constraints (cost function $c(y)$)
  - consumer constraints (demand f^n $D(p)$)

Monopolist's problem: $\max\limits_{p,y} \; py - c(y)$

s.t. $D(p) = y$

$$\max_{p} \; p\,D(p) - c(D(p)) \qquad \text{(single variable)}$$

Another way uses inverse demand function

$$p(y) = D^{-1}(y) \quad \text{(How?)}$$

If $\boxed{\begin{aligned} D &= f(p) \\ \Rightarrow \; p &= f^{-1}(D) \end{aligned}}$

$$\max_{y \geq 0} \; p(y)\,y - c(y)$$

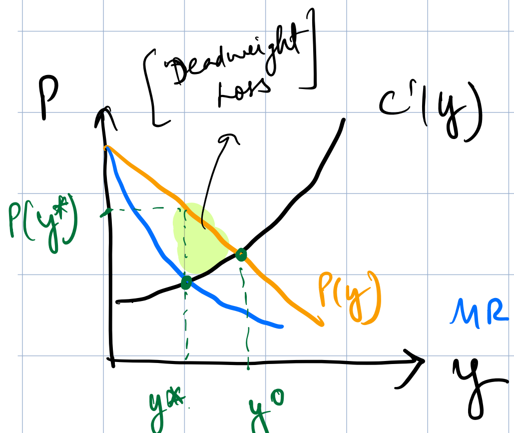
FOC : $\underbrace{p'(y^*)\,y^* + p(y^*)}_{\text{marginal revenue}} \leq \underbrace{c'(y^*)}_{\text{marginal cost}}$, with equality if $y^* > 0$

Assume $y^* > 0$, so $MR = MC$

$$p'(y^*) \cdot y^* + p(y^*) = c'(y^*)$$

Usual case: $p'(y) < 0$ (downward sloping demand)

$\Rightarrow p(y^*) > c'(y^*)$ ∴ Monopolist prices above marginal cost



This is a negative no.

$$MR(y) = \underbrace{p'(y)y}_{} + p(y)$$

y^o ↳ Socially optimal pt.

Q: How does monopolists price/output change as a parameter changes?

Tool here is the implicit function theorem: How does a soln fn change as a parameter changes.

Contrast with the envelope theorem which applies to the value function.

IFT "recipe":

- Start w/ FOC from max. problem
- Write solution function $x^*(\theta)$ implicitly as a f^n of parameter θ
- Differentiate FOC wrt to θ (chain rule)
- Solve resulting eq^n for $\frac{dx^*(\theta)}{d\theta}$

Back to monopolist

Costs: $c(y, \theta)$

FOC: $p'(y^*)y^* + p(y^*) - \frac{\partial c(y^*, \theta)}{\partial y} = 0$

Write y^* implicitly as a f^n of θ:

$p'(y^*(\theta))\,y^*(\theta) + p(y^*(\theta)) - \frac{\partial c(y^*(\theta), \theta)}{\partial}$

$$p''(y^*(\theta))\,y^*(\theta)\cdot\frac{dy^*(\theta)}{d\theta}+p'(y^*(\theta))\cdot\frac{dy^*(\theta)}{d\theta}+p'(y^*(\theta))\cdot\frac{dy^*(\theta)}{d\theta}$$

$$-\frac{\partial^2 c(y^*(\theta),\theta)}{\partial^2 y}\cdot\frac{dy^*}{d\theta}-\frac{\partial^2 c(y^*(\theta),\theta)}{\partial y\,\partial\theta}=0$$

Solve for $\frac{dy^*}{d\theta}$

$$\frac{dy^*}{d\theta}=\frac{\partial^2 c(y^*,\theta)/\partial y\,\partial\theta}{p''(y^*)\,y^*+2p'(y^*)-\frac{\partial^2 c(y^*,\theta)}{\partial y^2}}$$

<u>Example:</u> CES demand, constant marginal cost

$D(p)=Ap^{-\beta}$, $\beta>1$

$C(y)=\tilde{c}y$

Why is it called CES?

Calculate price elasticity of demand:

$$\sigma_{D,p}=\frac{d\log D(p)}{d\log p}$$

$\log D(p)=\log A-\beta\cdot\log p$

$\sigma_{D,p}=-\beta$

Inverse Demand

$$y = A p^{-\beta} \implies p(y) = \left(\frac{y}{A}\right)^{-1/\beta}$$

FOC: for monopolist:

$$p'(y)\,y + p(y) = c'(y)$$

$$p'(y) = \frac{-1}{\beta}\left(\frac{y}{A}\right)^{-1/\beta - 1} \cdot \left(\frac{1}{A}\right)$$

$$= \frac{-1}{\beta}\, p(y)\, \frac{1}{y}$$

FOC becomes:

$$\left(\frac{-1}{\beta}\, p(y)\, \frac{1}{y}\right) y + p(y) = \tilde{c}$$

$$p(y)\left[1 - \frac{1}{\beta}\right] = \tilde{c}$$

$$p(y) = \frac{\tilde{c}}{1 - 1/\beta}$$

Monopolist's price markup depends on β which is elasticity of dd.

$c \to$ here is $\tilde{c}$ (a constant)

What is $\frac{dy^*}{d\tilde{c}}$?

$y^*(\tilde{c})$ defined implicitly by

$$p(y^*(\tilde{c}))\left[1-\frac{1}{\beta}\right]-\tilde{c}=0$$

IFT:

$$p'(y^*(\tilde{c}))\left[1-\frac{1}{\beta}\right]\frac{dy^*}{d\tilde{c}}-1=0$$

$$\frac{dy^*}{d\tilde{c}}=\frac{1}{\underbrace{p'(y^*(\tilde{c}))}_{<0}\underbrace{\left[1-\frac{1}{\beta}\right]}_{>0}}<0$$

Consumer Theory

More complicated for few reasons:—

- Need to impose "rationality" axioms on preferences
- Can't use optimization techniques on preferences. Need to construct a utility f^n
- Consumer problem has prices in the constraint income / substitution effects.

Widely used in economics:

1. Normatively useful
2. Positive predictions
3. Widely applicable
4. Simple / sparse model

Two approaches:—

1. Preferences $\longrightarrow$ Choices
2. Choices $\longrightarrow$ Preferences

Starting with 2 :-

Let X: (abstract) set of alternatives
elements $x \in X$ are mutually exclusive

A choice structure $(\mathcal{B}, C(\cdot))$ is :-
- $\mathcal{B}$ is a family of nonempty subsets of X
- $C(\cdot)$ is a choice rule s.t. $C(B) \subseteq B \ \forall$
$B \in \mathcal{B}$, $C(B) \neq \phi$

$X = \{x, y, z\}$
$\mathcal{B} = \{\{x, y\}, \{x, z\}\}$

$C(\{x, y\}) = \{x\}$, $C(\{x, y, z\}) = \{x, z\}$
acceptable alternatives

What are some "reasonable" restrictions on behaviour?

- Weak Axiom of Revealed Preference (WARP)
If for some $B \in \mathcal{B}$ with $x, y \in B$, we have
$x \in C(B)$, then for any other $B' \in \mathcal{B}$ with

$x, y \in B'$, and $y \in C(B')$ we also have $x \in C(B')$.

$X = \{a, b, c\}$ $\quad$ $\mathcal{B} = \{\{a, b\}, \{a, b, c\}\}$

We observe $C(\{a, b, c\}) = \{b\}$
WARP implies that $C(\{a, b\}) = \{b\}$
Why? Assume $a \in C(\{a, b\})$, then since 4 was chosen when b was available, we must have $a \in C(\{a, b, c\})$. Contradiction.

$C(\{a, b\}) = \{b\}$
WARP implies $C(\{a, b, c\}) = \{b\}, \{c\}$, or $\{b, c\}$
(doesn't say anything about c).

Preference-based Approach :-

starting pt. of the model.

X set of alternatives

Primitive is a binary relation on X, $\succsim$

$x \succsim y$: "x is atleast as good as y"

$x \succ y$: $x \succsim y$ & $y \not\succsim x$ "x is strictly prefered to y".

$x \sim y$: $x \succsim y$ & $y \succsim x$ "x is indifferent to y"

A preference relation is rational if it is:-

1. Complete: $\forall\, x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both)
2. Transitive: $\forall\, x, y, z \in X$, if $x \succsim y$ & $y \succsim z$, then $x \succsim z$.

<u>Violation of transitivity</u> : Framing (Kahneman, Tversky)

Best buy to purchase a \$125 stereo & \$15 calculator

(A) Calculator on sale for \$5 off at a store 20 min away — Yes

(B) Stereo " " " " " " " " " — No

(C) Both are out of stock, but the store gives you \$5 coupon for either items. Does it matter which? (still travel to store B for purchase). — No

x = travel to store 2 for \$5 off calculator
y = \$5 off stereo

z = buy both items at store 1.

Responses imply: $x \succ z$ (A)
$z \succ y$ (B)
$x \sim y$ (C)

What are the links b/w choice & preference approach?

Q1 Does rationality imply WARP? $\boxed{\text{Yes}}$

Q2 Does WARP imply rationality? $\boxed{\text{No}}$

preferences

For Q1: Given $\succsim$, let $C(B, \succsim) = \{x \in B : x \succsim y \ \forall y \in B\}$

Theorem: Suppose $\succsim$ are rational. Then, the choice structure $(\mathcal{B}, C(B, \succsim))$ satisfies WARP.

<u>Proof</u>: Suppose for some $B \in \mathcal{B}$, we have $x, y \in B$ & $x \in C(B, \succsim)$. By def^n of C, $x \succsim y$.

Check WARP:

Suppose $x, y \in B'$, & $y \in C(B', \succsim)$. So, $y \succsim z \ \forall z \in B'$.

By transitivity, $x \succsim y \succsim z$, so $x \succsim z \ \forall z \in B'$.

Thus $x \in C(B', \succsim)$ and WARP holds.

Counterexample for Q 2:

$X = \{a, b, c\}$ $\quad$ $\mathcal{B} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$

Assume we observe :-

$C(\{a, b\}) = a$ $\quad$ $C(\{a, c\}) = c$ $\quad$ $C(\{b, c\}) = b$

Can check that $(\mathcal{B}, C(\,))$ satisfies WARP :

Observed choices imply :-

$$\left.\begin{array}{l} a > b \\ b > c \\ a > c \end{array}\right\} \text{violates transitivity}$$

Theorem (Arrow, 1959):

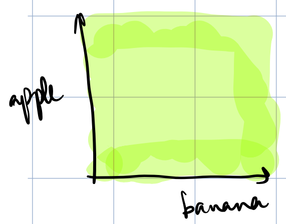
If $(\mathcal{B}, C(\,))$ is s.t. :

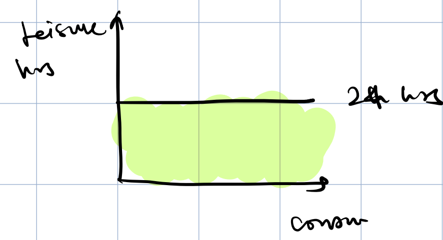
i) WARP is satisfied.

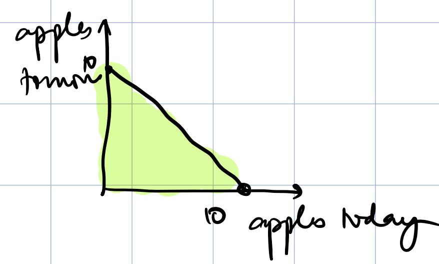
ii) $\mathcal{B}$ includes all subsets of X with upto 3 elements

then, there is a (unique) rational preference relation $\succsim$ that rationalizes $C(\cdot)$

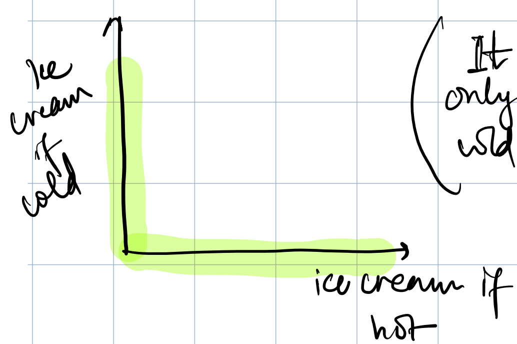
Restrict $X \subseteq \mathbb{R}^n_+$ → Consumption space.







State dependent cons^m



(It can only be hot or cold tomorrow)

Properties of preferences on $X \subseteq \mathbb{R}^n_+$

- Preference relation $\succsim$ is monotone if
  $y \gg x$ implies $y \succ x$
- $\succsim$ is strongly monotone if $y \geq x$ & $y \neq x$ implies $y \succ x$.

- $\succsim$ is <u>locally non satiated</u> if for every $x \in X$ &

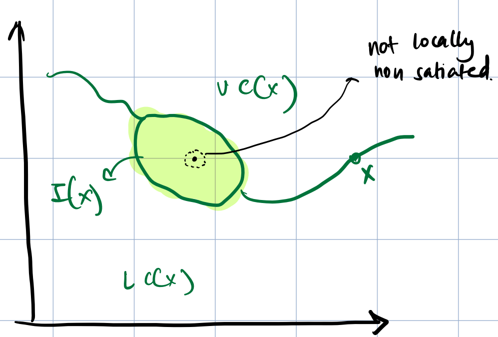
$\varepsilon > 0$, $\exists$ some $y \in X$ s.t. $\|y - x\| \leq \varepsilon$ & $y \succ x$.

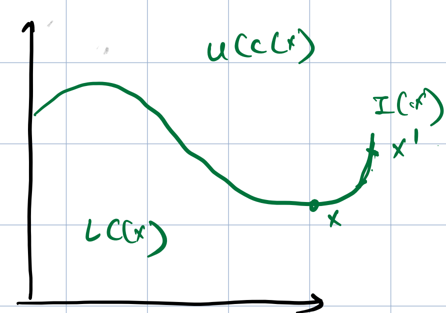
$uc(x) \rightarrow$ upper contour of x

$= \{ y \in X : y \succsim x \}$

$LC(x) = \{ y \in X : x \succsim y \}$ (lower contour set.)

$I(x) = \{ y \in X : x \sim y \}$ (Indifference set)

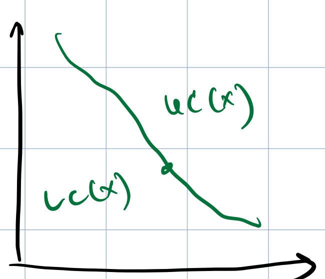




LNS, Not monotone.

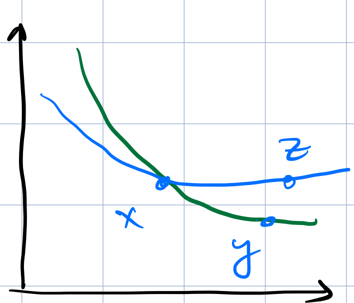
$x' \gg x$

but $x' \not\succ x$



Monotonicity $\Rightarrow$ downward sloping ICs.

IC's cannot cross



$x \sim z$

$x \sim y$.

$\Rightarrow z \sim y$ (transitivity)

But $z \succ y$ (monotonicity)

$\Rightarrow$ contradiction!