Econ 7040: Assignment #2
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Answer Key

1. The Euler equation implies

$$\frac{1}{C_t} = \alpha\beta E_t \frac{1}{C_{t+1}}(1-\tau_{t+1})A_{t+1}K_t^{\alpha-1} \tag{1}$$

Multiply both sides of (1) by K_t and perform the magic of adding 1 to both sides to yield

$$\frac{C_t + K_t}{C_t} = 1 + \alpha\beta E_t(1-\tau_{t+1})\frac{C_{t+1}+K_{t+1}}{C_{t+1}} \tag{2}$$

Let $z_t \equiv (C_t + K_t)/C_t$ and iterate forward on (2) to obtain

$$z_t = E_t \sum_{j=0}^{\infty} (\alpha\beta)^j d_j \equiv \varphi_t \tag{3}$$

$$d_j \equiv \prod_{i=0}^{j-1}(1-\tau_{t+i+1}), \quad d_0 = 1 \tag{4}$$

Use (3) in the aggregate resource constraint to solve for equilibrium C_t and K_t for $t = 0, 1, 2, \ldots$

$$C_t = \frac{1}{\varphi_t} A_t K_{t-1}^{\alpha} \tag{5}$$

$$K_t = \left(1 - \frac{1}{\varphi_t}\right) A_t K_{t-1}^{\alpha} \tag{6}$$

Notice that if the path of taxes is expected to rise, then d_j and φ_t fall, inducing the agent to substitute out of capital (saving) into consumption. Higher total factor productivity at t raises both consumption and capital today. Expected future TFP does not appear in the solution because the log preferences imply the income and substitution effects of anticipated TFP exactly cancel. ■

2. The deterministic steady state is given by

$$\bar{K} = \left(\frac{1}{\alpha\beta(1-\bar{\tau})\bar{A}}\right)^{1/(\alpha-1)}, \quad \bar{C} = \bar{A}\bar{K}^{\alpha} - \bar{K}, \quad \bar{Y} = \bar{A}\bar{K}^{\alpha}$$

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3. The log-linearized Euler equation and aggregate resource constraint are

$$c_t + (\alpha - 1)k_t = E_t\left[\frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_{t+1} - a_{t+1} + c_{t+1}\right] \tag{7}$$

$$\frac{\bar{C}}{\bar{K}}c_{t+1} + k_{t+1} = \alpha\frac{\bar{Y}}{\bar{K}}k_t + \frac{\bar{Y}}{\bar{K}}a_{t+1} \tag{8}$$

Define the forecast error as $\eta_{t+1} \equiv c_{t+1} - E_t c_{t+1}$ and use the AR(1) properties of the fundamental shocks to express

$$E_t a_{t+1} = \rho_a a_t + \varepsilon^a_{t+1} \tag{9}$$

$$E_t \hat{\tau}_{t+1} = \rho_\tau \hat{\tau}_t + \varepsilon^\tau_{t+1} \tag{10}$$

The system in the form $\Gamma_0 x_{t+1} = \Gamma_1 x_t + \Pi\eta_{t+1} + \Psi z_{t+1}$ with $x_t \equiv (c_t, k_t, a_t, \hat{\tau}_t)'$ and $z_t \equiv (\varepsilon^a_t, \varepsilon^\tau_t)'$ is

$$\Gamma_0 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1-\alpha\beta(1-\bar{\tau})}{\alpha\beta(1-\bar{\tau})} & 1 & -\frac{1}{\alpha\beta(1-\bar{\tau})} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_1 \equiv \begin{bmatrix} 1 & \alpha - 1 & \rho_a & -\frac{\rho_\tau\bar{\tau}}{1-\bar{\tau}} \\ 0 & \frac{\alpha}{\alpha\beta(1-\bar{\tau})} & 0 & 0 \\ 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & \rho_\tau \end{bmatrix},$$

$$\Pi \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Psi \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverting Γ_0 and premultiplying through by Γ_0^{-1} yields the system $x_{t+1} = Ax_t + B\xi_{t+1}$, where $\xi_{t+1} \equiv (\varepsilon^a_{t+1}, \varepsilon^\tau_{t+1}, \eta_{t+1})'$. Defining

$$\delta \equiv \alpha\beta(1-\bar{\tau}), \qquad \tau^* \equiv \frac{\bar{\tau}}{1-\bar{\tau}}$$

the new system matrices are

$$A \equiv \begin{bmatrix} 1 & \alpha - 1 & \rho_a & -\rho_\tau\tau^* \\ 1 - \frac{1}{\delta} & \frac{1}{\delta} + (\alpha - 1) & \rho_a & \rho_\tau\tau^*\left(\frac{1}{\delta} - 1\right) \\ 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & \rho_\tau \end{bmatrix}$$

$$B \equiv \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\delta} & 0 & 1-\frac{1}{\delta} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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4. The structure of the transition matrix, A, immediately reveals that the eigenvalues can be obtained as the eigenvalues of the 2×2 matrix

$$A_{[2,2]} \equiv \begin{bmatrix} 1 & \alpha - 1 \\ 1-\frac{1}{\delta} & \frac{1}{\delta} + (\alpha - 1) \end{bmatrix} \tag{11}$$

with the additional eigenvalues of ρ_a and ρ_τ. Now,

$$\mathrm{tr}(A_{[2,2]}) = \frac{1}{\delta} + \alpha, \qquad \det(A_{[2,2]}) = \frac{\alpha}{\delta}$$

so the eigenvalues are the roots of the polynomial

$$\begin{aligned} \mathcal{P}(\lambda) &= \lambda^2 - \mathrm{tr}(A_{[2,2]}) + \det(A_{[2,2]}) \\ &= \lambda^2 - \left(\frac{1}{\delta} + \alpha\right)\lambda + \frac{\alpha}{\delta} = 0 \end{aligned}$$

Factoring the polynomial yields the first two eigenvalues

$$0 < \lambda_1 = \alpha < 1, \qquad \lambda_2 = \frac{1}{\alpha\beta(1-\bar{\tau})} > 1$$

because $\bar{\tau} \in [0, 1)$. λ_2 is the unstable eigenvalue whose influence must be squashed for a solution to this growth model. Note that "squashing the explosive root" in this model is sufficient (but not necessary) to impose that the transversality condition, which is part of the necessary and sufficient conditions for an optimum, is satisfied. The transversality condition was imposed in part (a) to make the nonlinear difference equation converge. Transversality means that at an optimum, consumers cannot be made better off by permanently raising or lowering their consumption paths. The eigenvectors of the system are the W's that satisfy

$$A = W\Lambda W^{-1}$$

where

$$\Lambda \equiv \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \frac{1}{\delta} & 0 & 0 \\ 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & \rho_\tau \end{bmatrix}$$

It is straightforward to compute the eigenvectors to yield

$$W = \begin{bmatrix} 1 & -\frac{\delta(\alpha-1)}{\delta-1} & -\frac{\rho_a}{\alpha-\rho_a} & -\frac{\rho_\tau \tau^*(\alpha-\delta\rho_\tau)}{(\delta\rho_\tau-1)(\alpha-\rho_\tau)} \\ 1 & 1 & -\frac{\rho_a}{\alpha-\rho_a} & \frac{\rho_\tau^2 \tau^*(\delta-1)}{(\delta\rho_\tau-1)(\alpha-\rho_\tau)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W^{-1} = \begin{bmatrix} \frac{\delta-1}{\alpha\delta-1} & 1-\frac{\delta-1}{\alpha\delta-1} & \frac{\rho_a}{\alpha-\rho_a} & -\frac{\alpha\rho_\tau\tau^*(\delta-1)}{(\alpha-\rho_\tau)(\alpha\delta-1)} \\ -\left(\frac{\delta-1}{\alpha\delta-1}\right) & \frac{\delta-1}{\alpha\delta-1} & 0 & -\frac{\rho_\tau\tau^*(\delta-1)}{(\delta\rho_\tau-1)(\alpha\delta-1)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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5. The eigenvector associated with the unstable eigenvalue, $\lambda_2 = 1/\delta$, is given by the second row of W^{-1}:

$$W^{2\cdot} = \begin{pmatrix} -\left(\frac{\delta-1}{\alpha\delta-1}\right) & \frac{\delta-1}{\alpha\delta-1} & 0 & -\frac{\rho_\tau\tau^*(\delta-1)}{(\delta\rho_\tau-1)(\alpha\delta-1)} \end{pmatrix} \tag{12}$$

Stability condition (12) can be simplified as

$$W^{2\cdot} = \begin{pmatrix} -1 & 1 & 0 & -\frac{\rho_\tau\tau^*}{\delta\rho_\tau-1} \end{pmatrix} \tag{13}$$

We shall use (12) to produce the *stability condition* for the model. The stability condition implies

$$W^{2\cdot}x_t = 0, \qquad t = 0,1,2,\ldots \tag{14}$$

$$W^{2\cdot}B\xi_t = 0, \qquad t = 1,2,\ldots \tag{15}$$

Expression (14) is called the *stable manifold* of the model. It determines the linear combination of endogenous and exogenous variables that must hold in any equilibrium. For this model, the stability condition implies

$$W^{2\cdot}x_t = 0 \Rightarrow$$

$$-c_t + k_t = \frac{\rho_\tau\tau^*}{\delta\rho_\tau - 1}\hat{\tau}_t, \quad t = 0,1,2,\ldots \tag{16}$$

We use (15) to get the equilibrium mapping from the fundamental shocks—ε^a_{t+1} and ε^τ_{t+1}—to the forecast error, η_{t+1} as

$$W^{2\cdot}B\xi_{t+1} = 0 \Rightarrow$$

$$\eta_{t+1} = \varepsilon^a_{t+1} + \frac{\delta\rho_\tau\tau^*}{1-\delta\rho_\tau}\varepsilon^\tau_{t+1}, \quad t = 0,1,2,\ldots \tag{17}$$

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6. To obtain the equilibrium decision rules for c_t and k_t, combine (16) with (8), written as

$$\left(\frac{1-\delta}{\delta}\right)c_t + k_t = \frac{\alpha}{\delta}k_{t-1} + \frac{1}{\delta}a_t$$

to yield

$$k_t = \alpha k_{t-1} + a_t - \frac{(1-\delta)\rho_\tau\tau^*}{1-\delta\rho_\tau}\hat{\tau}_t, \quad t = 0,1,2,\ldots \tag{18}$$

$$c_t = \alpha k_{t-1} + a_t + \frac{\delta\rho_\tau\tau^*}{1-\delta\rho_\tau}\hat{\tau}_t, \quad t = 0,1,2,\ldots \tag{19}$$

Note that $\delta\rho_\tau < 1$ because $\lambda_2 = 1/\delta > 1$ and $\rho_\tau \in [0, 1)$ so a higher realization of $\hat{\tau}_t$ reduces capital and raises consumption.

From the decision rule for consumption, (19), we can compute the forecast error $\eta_{t+1} = c_{t+1} - E_t c_{t+1}$ to confirm that it is identical to the one obtained from the mapping in expression (17):

$$\begin{aligned}
\eta_{t+1} &= c_{t+1} - E_t c_{t+1} \\
&= a_{t+1} - \rho_a a_t + \frac{\delta\rho_\tau \tau^*}{1-\delta\rho_\tau}(\hat{\tau}_{t+1} - \rho_\tau \hat{\tau}_t) \\
&= \varepsilon^a_{t+1} + \frac{\delta\rho_\tau \tau^*}{1-\delta\rho_\tau}\varepsilon^\tau_{t+1}
\end{aligned}$$

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7. Decision rules (18) and (19) yield what `gensys` returns, where the transition matrix is G and the impact matrix is M:

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} c_{t-1} \\ k_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & \frac{\delta\rho_\tau\tau^*}{1-\delta\rho_\tau} \\ 1 & -\left(\frac{(1-\delta)\rho_\tau\tau^*}{1-\delta\rho_\tau}\right) \end{bmatrix} \begin{bmatrix} a_t \\ \hat{\tau}_t \end{bmatrix}$$

which holds for $t = 0, 1, 2, \ldots$.

Notice that if the tax-rate process is serially uncorrelated, $\rho_\tau = 0$, then the tax rate disappears from the equilibrium. The reason for this is that the tax is levied against output and the only endogenous variable output depends on is the existing capital stock. To see this in a different way, notice that only the expected tax rate, $E_t\hat{\tau}_{t+1}$, enters the Euler equation; surprise changes in taxes are lump sum and do not affect decisions. If output also depended on labor input, this result would be altered and both anticipated and unanticipated tax rates would affect the equilibrium.

Notice also that only the current realization of the technology shock affects the equilibrium. We saw this result in part (a): the functional form assumptions of the model drive this; different preferences, for example, would make the equilibrium depend on current and expected technology. ■

8. Assuming

$$\beta = 0.95, \quad \alpha = 0.33, \quad \bar{\tau} = 0.2, \quad \rho_a = \rho_\tau = 0.5, \quad \bar{A} = 1.0$$

we can plot the equilibrium responses to a unit one-time shock to productivity ($\varepsilon^a_t = 1$) and taxes ($\varepsilon^\tau_t = 1$):

Figure 1: Impulse responses to a productivity shock $\varepsilon_t^a = 1$

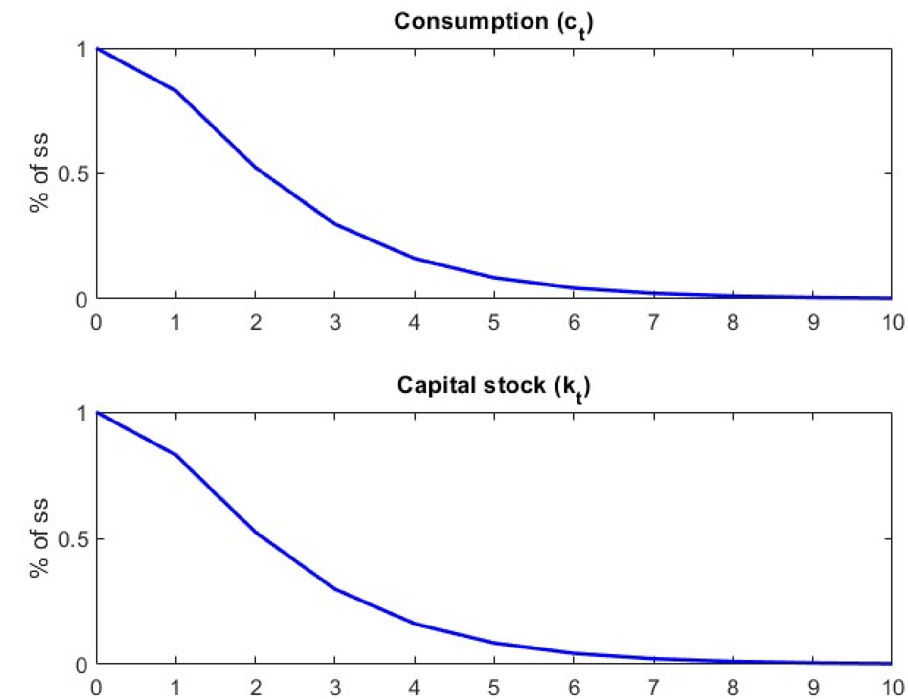

Figure 2: Impulse responses to a tax shock $\varepsilon_t^\tau = 1$