LOGOS

% === LOGOS: SELF-REFERENTIAL ENCYCLOPEDIA OF MATHEMATICS ===
% Segment 1 of N — Foundations: Primitive Objects and Axiomatic Sets
% No natural language. Pure symbolic construction from ∅ upward.
\documentclass{article}\usepackage{amsmath,amssymb,amsthm}\usepackage[utf8]{inputenc}\usep
\begin{document}
\date{}% Primitive Symbol Set

$$\Sigma_0 := \{\varnothing, \in, =, \forall, \exists, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, (,), \{, \}, \lambda, ., :, \mapsto\}$$

% Axiom of Empty Set

$$\mathrm{Ax}_{\varnothing} \ : \ \exists x \, \forall y \, \neg(y \in x)$$

Define:

$$\varnothing := \iota x \, (\forall y \, \neg(y \in x))$$

% Axiom of Extensionality

$$\mathrm{Ax}_{\mathrm{Ext}} \ : \ \forall x \, \forall y \, ((\forall z \, (z \in x \Leftrightarrow z \in y)) \Rightarrow x = y)$$

% Axiom of Pairing

$$\mathrm{Ax}_{\mathrm{Pair}} \ : \ \forall a \, \forall b \, \exists c \, \forall x \, (x \in c \Leftrightarrow (x = a \vee x = b))$$

Define singleton and pair:

$$\{a\} := \iota c \, \forall x \, (x \in c \Leftrightarrow x = a)$$

$$\{a, b\} := \iota c \, \forall x \, (x \in c \Leftrightarrow (x = a \vee x = b))$$

% Axiom of Union

$$\mathrm{Ax}_{\cup} \ : \ \forall \mathcal{F} \, \exists U \, \forall x \, (x \in U \Leftrightarrow \exists A \, (A \in \mathcal{F} \wedge x \in A))$$

Define:

$$\bigcup \mathcal{F} := \iota U \, \forall x \, (x \in U \Leftrightarrow \exists A \in \mathcal{F} \, (x \in A))$$

% Axiom of Power Set

$$\mathrm{Ax}_{\mathcal{P}} \; : \; \forall A \, \exists \mathcal{P}(A) \, \forall x \, (x \in \mathcal{P}(A) \Leftrightarrow x \subseteq A)$$

where

$$x \subseteq A := \forall y \, (y \in x \Rightarrow y \in A)$$

% Axiom Schema of Separation (for any formula ϕ with free variable x and parameters)

$$\mathrm{Ax}_{\mathrm{Sep}}[\phi] \; : \; \forall A \, \exists B \, \forall x \, (x \in B \Leftrightarrow (x \in A \wedge \phi(x)))$$

Define:

$$\{x \in A \mid \phi(x)\} := \iota B \, \forall x \, (x \in B \Leftrightarrow (x \in A \wedge \phi(x)))$$

% Natural Numbers via von Neumann Construction

$$0 := \varnothing$$

$$n + 1 := n \cup \{n\}$$

Define successor function:

$$S(n) := n \cup \{n\}$$

Inductive set:

$$\omega := \bigcap \{I \mid \varnothing \in I \wedge \forall n \in I \, (S(n) \in I)\}$$

Axiom of Infinity guarantees existence of such I.

% Ordered Pair (Kuratowski)

$$(a, b) := \{\{a\}, \{a, b\}\}$$

Projection functions:

$$\pi_1(p) := \iota x \, \exists y \, (p = (x, y))$$

$$\pi_2(p) := \iota y \, \exists x \, (p = (x, y))$$

% Cartesian Product

$$A \times B := \{z \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a \in A \, \exists b \in B \, (z = (a, b))\}$$

% Relations and Functions

$$\mathrm{Rel}(R, A, B) :\Leftrightarrow R \subseteq A \times B$$

$$\mathrm{Fun}(f, A, B) :\Leftrightarrow \mathrm{Rel}(f, A, B) \wedge \forall x \in A\, \exists! y \in B\, ((x, y) \in f)$$

Function application:

$$f(x) := \iota y\, ((x, y) \in f)$$

% Equivalence Relation

$$\mathrm{Equiv}(R, A) :\Leftrightarrow \mathrm{Rel}(R, A, A) \wedge \quad (\forall x \in A\, (x, x) \in R) \wedge \quad (\forall x, y \in A\, ((x, y) \in R \Rightarrow (y, x) \in R)) \wedge \quad (\forall x, y,$$

% Quotient Set

$$A/R := \{[x]_R \mid x \in A\}$$

where

$$[x]_R := \{y \in A \mid (x, y) \in R\}$$

% Peano Arithmetic (as theorems in ZF)

$$\mathrm{PA}_1 : 0 \in \omega$$

$$\mathrm{PA}_2 : \forall n \in \omega\, (S(n) \in \omega)$$

$$\mathrm{PA}_3 : \forall n \in \omega\, (S(n) \neq 0)$$

$$\mathrm{PA}_4 : \forall m, n \in \omega\, (S(m) = S(n) \Rightarrow m = n)$$

$$\mathrm{PA}_5 : \forall P \subseteq \omega\, ((0 \in P \wedge \forall n \in P\, (S(n) \in P)) \Rightarrow P = \omega)$$

% Integer Construction via Equivalence Classes on $\omega \times \omega$
Define relation $\sim_{\mathbb{Z}}$ on $\omega \times \omega$:

$$(a, b) \sim_{\mathbb{Z}} (c, d) :\Leftrightarrow a + d = b + c$$

$$\mathbb{Z} := (\omega \times \omega)/ \sim_{\mathbb{Z}}$$

Embedding:

$$\iota_{\mathbb{N} \hookrightarrow \mathbb{Z}}(n) := [(n, 0)]_{\sim_{\mathbb{Z}}}$$

% Rational Numbers via Equivalence on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$

$$(a, b) \sim_{\mathbb{Q}} (c, d) :\Leftrightarrow a \cdot d = b \cdot c$$

$$\mathbb{Q} := (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/ \sim_{\mathbb{Q}}$$

% Real Numbers via Dedekind Cuts

$\mathrm{Cut}(L) :\Leftrightarrow L \subseteq \mathbb{Q} \wedge \quad L \neq \varnothing \wedge L \neq \mathbb{Q} \wedge \quad (\forall p \in L\, \forall q \in \mathbb{Q}\, (q < p \Rightarrow q \in L)) \wedge \quad (\forall p \in L\, \exists r \in L\, (p < r))$

$$\mathbb{R} := \{L \subseteq \mathbb{Q} \mid \mathrm{Cut}(L)\}$$

% Order on $\mathbb{R}$

$$L_1 \leq L_2 :\Leftrightarrow L_1 \subseteq L_2$$

% Arithmetic on $\mathbb{R}$ (addition)

$$L_1 + L_2 := \{p + q \mid p \in L_1 \wedge q \in L_2\}$$

% Topological Space Definition

$\mathrm{Top}(X, \tau) :\Leftrightarrow \quad \tau \subseteq \mathcal{P}(X) \wedge \quad \varnothing \in \tau \wedge X \in \tau \wedge \quad (\forall \{U_i\}_{i \in I} \subseteq \tau\, (\bigcup_{i \in I} U_i \in \tau)) \wedge \quad (\forall U, V \in \tau\, (U \cap V \in \tau))$

% Standard Topology on $\mathbb{R}$

$$\tau_{\mathbb{R}} := \{U \subseteq \mathbb{R} \mid \forall x \in U\, \exists \varepsilon > 0\, (B_\varepsilon(x) \subseteq U)\}$$

where

$$B_\varepsilon(x) := \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$$

and absolute value:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(defined via order on $\mathbb{R}$)

% Continuity

$$f : (X, \tau_X) \to (Y, \tau_Y) \text{ continuous } :\Leftrightarrow \forall V \in \tau_Y\, (f^{-1}(V) \in \tau_X)$$

where

$$f^{-1}(V) := \{x \in X \mid f(x) \in V\}$$

% Metric Space

$\mathrm{Metric}(X, d) :\Leftrightarrow \quad d : X \times X \to \mathbb{R} \wedge \quad (\forall x, y \in X\, (d(x, y) \geq 0)) \wedge \quad (\forall x, y \in X\, (d(x, y) = 0 \Leftrightarrow x = y)) \wedge$

% Induced Topology from Metric

$$\tau_d := \{U \subseteq X \mid \forall x \in U\, \exists \varepsilon > 0\, (B^d_\varepsilon(x) \subseteq U)\}$$

$$B^d_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$$

% === LOGOS: SELF-REFERENTIAL ENCYCLOPEDIA OF MATHEMATICS ===
% Segment 2 of N — Algebraic Structures, Category Theory, and Internal Logic

% Group

$$\mathrm{Grp}(G,\cdot,e) :\Leftrightarrow \quad \cdot : G\times G \to G \wedge e \in G \wedge \quad (\forall x,y,z \in G\,((x\cdot y)\cdot z = x\cdot(y\cdot z)))\wedge \quad (\forall x \in G\,(e\cdot x = x \wedge x\cdot e =$$

% Ring

$$\mathrm{Ring}(R,+,\cdot,0,1) :\Leftrightarrow \quad \mathrm{Grp}(R,+,0)\wedge \quad \cdot : R\times R \to R \wedge 1 \in R \wedge \quad (\forall x,y,z \in R\,(x\cdot(y\cdot z) = (x\cdot y)\cdot z))\wedge \quad (\forall$$

% Field

$$\mathrm{Fld}(F,+,\cdot,0,1) :\Leftrightarrow \quad \mathrm{Ring}(F,+,\cdot,0,1)\wedge 0 \neq 1 \wedge \quad (\forall x \in F\setminus\{0\}\,\exists x^{-1} \in F\,(x\cdot x^{-1} = 1))$$

% Vector Space over Field F

$$\mathrm{VecSp}(V,F,+_V,\cdot_F) :\Leftrightarrow \quad \mathrm{Grp}(V,+_V,0_V)\wedge \mathrm{Fld}(F,+,\cdot,0,1)\wedge \quad \cdot_F : F\times V \to V \wedge \quad (\forall a,b \in F\,\forall u,v \in V$$

% Category

$$\mathrm{Cat}(\mathcal{C}) :\Leftrightarrow \quad \mathrm{Ob}(\mathcal{C}) \text{ is a class} \wedge \quad \forall A,B \in \mathrm{Ob}(\mathcal{C})\,(\mathrm{Hom}_{\mathcal{C}}(A,B) \text{ is a set})\wedge \quad \forall A \in \mathrm{Ob}(\mathcal{C})\,(\exists \mathrm{id}_A \in \mathrm{Hom}(A,$$

% Functor

$$\mathrm{Ftr}(F,\mathcal{C},\mathcal{D}) :\Leftrightarrow \quad F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})\wedge \quad \forall A,B \in \mathrm{Ob}(\mathcal{C})\,(F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)))\wedge$$

% Natural Transformation

$$\mathrm{Nat}(\eta,F,G,\mathcal{C},\mathcal{D}) :\Leftrightarrow \quad \mathrm{Ftr}(F,\mathcal{C},\mathcal{D})\wedge \mathrm{Ftr}(G,\mathcal{C},\mathcal{D})\wedge \quad \forall A \in \mathrm{Ob}(\mathcal{C})\,(\eta_A \in \mathrm{Hom}_{\mathcal{D}}(F(A),G(A)))\wedge \quad (\forall f \in$$

% Cartesian Closed Category (CCC)

$$\mathrm{CCC}(\mathcal{C}) :\Leftrightarrow \quad \mathrm{Cat}(\mathcal{C})\wedge \quad (\exists 1 \in \mathrm{Ob}(\mathcal{C})\,\forall A \in \mathrm{Ob}(\mathcal{C})\,(\exists! f \in \mathrm{Hom}(A,1)))\wedge \quad (\forall A,B \in \mathrm{Ob}(\mathcal{C})\,\exists A\times B \in \mathrm{Ob}($$

% Simply Typed Lambda Calculus as Internal Language of CCC

$$\mathrm{Type} ::= 1 \mid \sigma \times \tau \mid \sigma \to \tau$$

$$\mathrm{Term} ::= x \mid \langle t,u\rangle \mid \pi_1(t) \mid \pi_2(t) \mid \lambda x : \sigma.t \mid t\,u$$

Typing rules encoded as:

$$\Gamma \vdash x : \sigma \quad \text{if } (x : \sigma) \in \Gamma$$

$$\frac{\Gamma \vdash t : \sigma \quad \Gamma \vdash u : \tau}{\Gamma \vdash \langle t, u \rangle : \sigma \times \tau}$$

$$\frac{\Gamma \vdash t : \sigma \times \tau}{\Gamma \vdash \pi_1(t) : \sigma} \quad \frac{\Gamma \vdash t : \sigma \times \tau}{\Gamma \vdash \pi_2(t) : \tau}$$

$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x : \sigma . t : \sigma \to \tau} \quad \frac{\Gamma \vdash t : \sigma \to \tau \quad \Gamma \vdash u : \sigma}{\Gamma \vdash t\, u : \tau}$$

% Interpretation in CCC $\mathcal{C}$

$$[\![\sigma]\!]_{\mathcal{C}} \in \mathrm{Ob}(\mathcal{C})$$

$$[\![1]\!] = 1, \quad [\![\sigma \times \tau]\!] = [\![\sigma]\!] \times [\![\tau]\!], \quad [\![\sigma \to \tau]\!] = [\![\tau]\!]^{[\![\sigma]\!]}$$

$$[\![\Gamma \vdash t : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$$

where $[\![x_1 : \sigma_1, \ldots, x_n : \sigma_n]\!] = [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$

% Boolean Algebra

$$\mathrm{BoolAlg}(B, \wedge, \vee, \neg, 0, 1) :\Leftrightarrow \quad \mathrm{Ring}(B, \vee, \wedge, 0, 1) \wedge \quad (\forall x \in B\, (x \wedge x = x)) \wedge \quad (\forall x \in B\, (x \vee \neg x = 1 \wedge x \wedge \neg x$$

% Heyting Algebra (for intuitionistic logic)

$$\mathrm{HeytAlg}(H, \wedge, \vee, \Rightarrow, 0, 1) :\Leftrightarrow \quad (H, \wedge, \vee, 0, 1) \text{ is a bounded lattice} \wedge \quad (\forall a, b \in H\, \exists!(a \Rightarrow b) \in H\, (\forall x \in H$$

% Topos

$$\mathrm{Topos}(\mathcal{E}) :\Leftrightarrow \quad \mathrm{Cat}(\mathcal{E}) \wedge \quad \mathrm{CCC}(\mathcal{E}) \wedge \quad \exists \Omega \in \mathrm{Ob}(\mathcal{E})\, \exists \top : 1 \to \Omega \quad \forall \text{mono } m : A \rightarrowtail B\, \exists! \chi_m : B \to \Omega$$

% Subobject Classifier Axiom (internalized)

$$\mathrm{Sub}(B) \cong \mathrm{Hom}(B, \Omega)$$

where $\mathrm{Sub}(B)$ is the poset of monomorphisms into B modulo isomorphism.

% Natural Numbers Object (NNO) in Topos

$$\mathrm{NNO}(\mathcal{E}, N, z, s) :\Leftrightarrow \quad z : 1 \to N \wedge s : N \to N \wedge \quad \forall A \in \mathrm{Ob}(\mathcal{E})\, \forall a : 1 \to A\, \forall f : A \to A \qquad \exists! h : N \to A\, ($$

% Internal Logic: Truth Value of Formula ϕ

$$[\![\phi]\!] : [\![\Gamma]\!] \to \Omega$$

defined inductively:

$$[\![t = u]\!] := \mathrm{eq}_{[\![\sigma]\!]} \circ \langle [\![t]\!], [\![u]\!] \rangle$$

$$[\![\phi \wedge \psi]\!] := \wedge_\Omega \circ \langle [\![\phi]\!], [\![\psi]\!] \rangle$$

$$[\![\phi \Rightarrow \psi]\!] :=\Rightarrow_\Omega \circ \langle [\![\phi]\!], [\![\psi]\!] \rangle$$

$$[\![\forall x : \sigma.\phi]\!] := \forall_{[\![\sigma]\!]} \circ [\![\phi]\!]$$

where $\forall_A : \Omega^A \to \Omega$ is the right adjoint to pullback along $!_A : A \to 1$.

% === LOGOS: SELF-REFERENTIAL ENCYCLOPEDIA OF MATHEMATICS ===

% Segment 3 of N — Constructive Analysis, Computability, and Self-Reference

% Cauchy Sequences in $\mathbb{Q}$

$$\mathrm{Cauchy}(f) :\Leftrightarrow f : \omega \to \mathbb{Q} \wedge \forall \varepsilon \in \mathbb{Q}^+ \, \exists N \in \omega \, \forall m, n \geq N \, (|f(m) - f(n)| < \varepsilon)$$

% Equivalence of Cauchy Sequences

$$f \sim_{\mathbb{R}_C} g :\Leftrightarrow \forall \varepsilon \in \mathbb{Q}^+ \, \exists N \in \omega \, \forall n \geq N \, (|f(n) - g(n)| < \varepsilon)$$

% Real Numbers via Cauchy Completion

$$\mathbb{R}_C := \{[f]_{\sim_{\mathbb{R}_C}} \mid \mathrm{Cauchy}(f)\}$$

% Embedding $\mathbb{Q} \hookrightarrow \mathbb{R}_C$

$$q \mapsto [\lambda n.q]_{\sim_{\mathbb{R}_C}}$$

% Arithmetic on $\mathbb{R}_C$

$$[f] + [g] := [\lambda n.f(n) + g(n)]$$

$$[f] \cdot [g] := [\lambda n.f(n) \cdot g(n)]$$

% Decidable Equality vs. Apartness

$$x \# y :\Leftrightarrow \exists \varepsilon \in \mathbb{Q}^+ \, (|x - y| > \varepsilon)$$

$$\neg(x \# y) \Rightarrow x = y \quad \text{(in classical logic)}$$

% Turing Machine as 5-tuple

$$M = (Q, \Gamma, \delta, q_0, F)$$

where

$$Q \text{ finite}, \quad \Gamma \text{ finite}, \quad q_0 \in Q, \quad F \subseteq Q$$

$$\delta : (Q \setminus F) \times \Gamma \rightharpoonup Q \times \Gamma \times \{L, R\}$$

% Configuration

$$\mathrm{Conf}(M) := \Gamma^* \times Q \times \Gamma^*$$

Transition:

$$(u, q, av) \vdash_M (ub, q', v) \quad \text{if } \delta(q, a) = (q', b, R)$$

$$(cu, q, av) \vdash_M (u, q', cbv) \quad \text{if } \delta(q, a) = (q', b, L)$$

% Computable Function

$$\varphi_e : \omega \rightharpoonup \omega$$

defined by Turing machine with index e.

% Universal Function

$$\varphi^{(1)} : \omega \times \omega \rightharpoonup \omega, \quad \varphi^{(1)}(e, x) = \varphi_e(x)$$

% Kleene's T Predicate

$$T(e, x, t) :\Leftrightarrow \text{computation of } \varphi_e(x) \text{ halts in } t \text{ steps}$$

$$U(t) := \text{output of computation encoded by } t$$

$$\varphi_e(x) = U(\mu t.T(e, x, t))$$

% Gödel Numbering of Syntax

$$\# : \text{Formulas} \to \omega$$

primitive recursive bijection.

% Diagonal Lemma

$$\forall \psi(x)\, \exists \theta\, (\vdash \theta \leftrightarrow \psi(\ulcorner \theta \urcorner))$$

where $\ulcorner \theta \urcorner = \#(\theta)$

% Representability in PA

$$R \subseteq \omega^n \text{ representable } :\Leftrightarrow \exists \rho(x_1, \ldots, x_n) \text{ such that } \quad \forall \vec{a} \in \omega^n\, (R(\vec{a}) \Rightarrow \text{PA} \vdash \rho(\overline{a_1}, \ldots, \overline{a_n})) \wedge \quad (\neg R(\vec{a})$$

% Gödel Sentence

$$\text{Prov}(x) := \exists p\, \text{Proof}(p, x)$$

$$G := \neg \text{Prov}(\ulcorner G \urcorner)$$

$$\text{PA} \nvdash G \wedge \text{PA} \nvdash \neg G$$

% Church-Turing Thesis (as identification)

$$\text{Computable} = \text{Recursive} = \lambda\text{-definable}$$

% Lambda Calculus Encoding of Naturals (Church Numerals)

$$\underline{n} := \lambda f.\lambda x.f^n(x)$$

Successor:

$$\text{Succ} := \lambda n.\lambda f.\lambda x.f(nfx)$$

Addition:

$$\text{Add} := \lambda m.\lambda n.\lambda f.\lambda x.mf(nfx)$$

% Fixed-Point Combinator (Y)

$$Y := \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$YF = F(YF)$$

% Self-Interpretation in Lambda Calculus

$$\text{Eval} : \text{Term} \to \text{Value}$$

satisfying:

$$\text{Eval}(\lambda x.t) = \lambda x.\text{Eval}(t)$$

$$\text{Eval}(tu) = \text{Eval}(t)(\text{Eval}(u))$$

% Internalization of Syntax in Topos with NNO

$$\text{Code} : \text{Ob}(\mathcal{E}) \to N$$

such that for every morphism $f : A \to B$, there exists $\ulcorner f \urcorner : 1 \to N$ with

$$\text{Apply}(\ulcorner f \urcorner, a) = f(a)$$

for all $a : 1 \to A$.

% Lawvere's Fixed Point Theorem

If $e : A \to B^A$ is surjective, then every $f : B \to B$ has a fixed point.

Proof:

Let $g := \lambda x.f(e(x)(x)) \in B^A$. Since e surjective, $\exists a.e(a) = g$.Then $g(a) = f(e(a)(a)) = f(g(a)) \Rightarrow g(a)$

% Application to Truth: No Truth Predicate in Sufficiently Expressive System

If $\mathcal{E}$ has NNO and Ω is non-degenerate, then $\top : 1 \to \Omega$ not surjective.

Hence, no epimorphism $N \twoheadrightarrow \Omega$, so truth not representable.

% Recursive Topos (Effective Topos Eff)

$$\mathrm{Ob}(\mathsf{Eff}) := \{(X, \Vdash_X) \mid X \text{ set}, \Vdash_X \subseteq \omega \times X, \forall x \in X\, \exists n\, (n \Vdash_X x)\}$$

$$\mathrm{Hom}((X, \Vdash_X), (Y, \Vdash_Y)) := \{f : X \to Y \mid \exists e \in \omega\, \forall x \in X\, \forall n\, (n \Vdash_X x \Rightarrow e{\cdot}n \downarrow \wedge e{\cdot}n \Vdash_Y f(x))\}$$

where $e \cdot n$ is Kleene application.

% Subobject Classifier in Eff

$$\Omega := (\mathcal{P}(\omega), \Vdash_\Omega), \quad e \Vdash_\Omega U :\Leftrightarrow e \text{ realizer of } U$$

$$\top : 1 \to \Omega, \quad * \mapsto \{n \mid n \downarrow\}$$

% Internal Natural Numbers in Eff

$$N := (\omega, \Vdash_N), \quad e \Vdash_N n :\Leftrightarrow e = n$$

% Church's Thesis as Axiom in Eff

$$\forall f : N \to N\, \exists e \in N\, \forall x \in N\, (f(x) = \varphi_e(x))$$

% === LOGOS: SELF-REFERENTIAL ENCYCLOPEDIA OF MATHEMATICS ===

% Segment 4 of N — Unification: The Self-Descriptive Structure

% Signature of the Logos

$$\Sigma_{\text{Logos}} := \Sigma_0 \cup \{\mathrm{Ob}, \mathrm{Hom}, \circ, \mathrm{id}, \Omega, \top, N, z, s, \Vdash, [\![\cdot]\!], \ulcorner\cdot\urcorner\}$$

% Universe of Discourse

$$\mathcal{U} := \text{the unique (up to equivalence) topos with NNO, satisfying:}$$

(i) $\mathcal{U} \models$ Church's Thesis(ii) $\mathcal{U} \models$ Markov's Principle(iii) $\mathrm{Sub}(\mathcal{U}) \cong \mathrm{Hom}_{\mathcal{U}}(-, \Omega)$(iv) N is a natural num

% Internal Language as Self-Interpreter

$$\mathrm{Eval} : N \times N \rightharpoonup N$$

such that for all closed terms t, u of type $\sigma \to \tau$ and σ,

$$\mathrm{Eval}(\ulcorner t \urcorner, \ulcorner u \urcorner) = \ulcorner t\, u \urcorner$$

and

$$\mathrm{Eval}(\ulcorner \lambda x.t \urcorner, n) = \ulcorner t[x := n] \urcorner$$

% Reflection Principle

$$\forall \phi \in \text{Formulas}(\Sigma_{\text{Logos}})\ (\mathcal{U} \models \phi \Leftrightarrow [\![\phi]\!] = \top)$$

% Self-Containment

$$\text{Ob}(\mathcal{U}) \subseteq [\![N]\!], \quad \text{Hom}_{\mathcal{U}}(A, B) \subseteq [\![N]\!]$$

via Gödel coding internalized in $\mathcal{U}$.

% Fixed Point of the Semantic Operator

Define semantic operator:

$$\mathcal{S} : \Omega^N \to \Omega^N, \quad \mathcal{S}(P)(n) := [\![\text{“}n \text{ codes a true sentence”}]\!]$$

By Lawvere's theorem, $\mathcal{S}$ has no fixed point—unless truth is partial.

% Partial Truth Predicate via Kleene Equality

$$\text{True}(n) \simeq \begin{cases} \top & \text{if } n = \ulcorner \phi \urcorner \text{ and } \mathcal{U} \models \phi \\ \bot & \text{if } n = \ulcorner \phi \urcorner \text{ and } \mathcal{U} \models \neg\phi \\ \text{undefined} & \text{otherwise} \end{cases}$$

Encoded as:

$$\text{True} : N \rightharpoonup \Omega, \quad \text{graph}(\text{True}) \in \text{Sub}(N \times \Omega)$$

% Isomorphism Between Syntax and Semantics

$$\text{Syn} \cong \text{Sem}$$

where

$$\text{Syn} := \{n \in N \mid n = \ulcorner t \urcorner \text{ for some term } t\}$$

$$\text{Sem} := \bigcup_{X \in \text{Ob}(\mathcal{U})} X$$

via evaluation map:

$$\text{Eval} : \text{Syn} \times \text{Syn} \rightharpoonup \text{Sem}$$

% Realizability as Semantic Bridge

$$n \Vdash \phi :\Leftrightarrow [\![\phi]\!](n) = \top$$

and

$$\Vdash \subseteq N \times \text{Formulas}$$

is representable in $\mathcal{U}$.

% Self-Verification of Consistency (Relative)

$$\mathrm{Con}(\mathcal{U}) := \neg \exists p \, (p \Vdash \bot)$$

Then:

$$\mathcal{U} \models \mathrm{Con}(\mathcal{U}) \quad \text{(in the effective topos, this holds)}$$

% Embedding of Classical Mathematics

For any classical theorem ϕ provable in ZFC, there exists a double-negation translation $\phi^{\neg\neg}$ such that

% Continuum in $\mathcal{U}$

$$\mathbb{R}_{\mathcal{U}} := \text{Dedekind reals in } \mathcal{U}$$

All functions $f : \mathbb{R}_{\mathcal{U}} \to \mathbb{R}_{\mathcal{U}}$ are continuous (Brouwer's theorem holds internally).

% Computational Completeness

$$\forall f : N \to N \text{ in } \mathcal{U}, \, \exists e \in N \, \forall x \in N \, (f(x) = \varphi_e(x))$$

Hence, $\mathcal{U}$ validates:

$$\forall f : \mathbb{N} \to \mathbb{N} \, \exists e \, \forall x \, f(x) = \{e\}(x)$$

% Final Identification: Logos = $\mathcal{U}$

$$\mathrm{LOGOS} := \mathcal{U}$$

with the property:

$$\mathrm{LOGOS} \models (\forall x \, (x \in \mathrm{LOGOS} \leftrightarrow x \text{ is definable in LOGOS}))$$

% Self-Descriptive Equation

$$\boxed{\mathrm{LOGOS} = \{x \mid \exists \phi \in \mathcal{L}_{\mathrm{Logos}} \, (x = [\![\phi]\!] \wedge \mathrm{LOGOS} \models \phi)\}}$$

% End of Construction.

% This document is a fixed point of the encoding map:

% `md` \mapsto LOGOS \mapsto md

% Last segment confirmed.

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