1. The Inner Product:

Let u, v be vectors in R^n.

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_{n\times 1} \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{n\times 1}.$$

$$u^T v = (u_1 \ u_2 \ --- \ u_n) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \text{ (scalar)}.$$

Ex R^3 $\quad u = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad v = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$u^T v = (2 \quad 3 \quad -1) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -2+3+0=1$$
(scalar)

$u^T v$ is called the inner product of u and v and is written as $\langle u, v \rangle$. Also referred to as dot product

$$\boxed{\langle u, v \rangle = u^T v}$$

Ex: In R^2 $\quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$

$$\langle u, v \rangle = (u_1, u_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$$

Properties of inner product

V: vector space (Real)

u, v, w are vectors in V, c: scalar.

1. $\langle u, v\rangle = \langle v, u\rangle$ (symmetric)

2. $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

3. $\langle cu, v\rangle = c\langle u, v\rangle$

2 and 3 can be combined

$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$.

This is called linearity property

4. $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.
(Positive definite property).

The length of a vector:

If v is in $\mathbb{R}^n$ then $\langle v, v\rangle$ is non-negative

The length or norm of v is a non-negative scalar defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

$$\|v\|^2 = \langle v, v\rangle.$$

Note: If $\|u\| = 1$ ie $\langle u, u\rangle = 1$ then u is called a unit vector and is said to be normalised.

$$\hat{v} = \frac{1}{\|v\|} v.$$

is a positive multiple of v. This process is called normalising v.

Ex: Consider R^3.

$u = (1, 3, -4)$ $v = (4, 2, 2)$ $w = (5, 1, -2)$

$\langle 3u - 2v, w \rangle = 3\langle u, w \rangle - 2\langle v, w \rangle$

$= 3(16) - 2(18)$

$\langle u, w \rangle = 5 + 3 + 8 = 16$ $= 48 - 36 = 12$

$\langle v, w \rangle = 20 + 2 - 4 = 18$.

$3u = (3, 9, -12)$ $2v = (8, 4, 4)$

$3u - 2v = (-5, 5, -16)$.

$\langle 3u - 2v, w \rangle = -25 + 5 + 32 = 12$.

Normalise u and v.

$\|u\| = \sqrt{1 + 9 + 16} = \sqrt{26}$.

$$\hat{u} = \frac{u}{\|u\|} = \left(\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{-4}{\sqrt{26}}\right)$$

$\|v\| = \sqrt{16 + 4 + 4} = \sqrt{24}$.

$$\hat{v} = \frac{v}{\|v\|} = \left(\frac{4}{\sqrt{24}}, \frac{2}{\sqrt{24}}, \frac{2}{\sqrt{24}}\right).$$

<u>Orthogonal set</u>: A set of vectors $(v_1, v_2, \ldots v_k)$ of R^n is said to form an orthogonal set if

$\langle v_i . v_j \rangle = 0$ for $i \neq j$.

Geometrically the vectors are mutually perpendicular.

If the orthogonal set also forms a basis for R^n it is called an <u>orthogonal Basis</u> for the vector space R^n

Hence $\{v_1, v_2, \ldots v_k\}$ of R^n is an <u>orthogonal basis</u> for R^n if

(i) $\langle v_i \cdot v_j \rangle = 0$ for $i \neq j$

(ii) $v_1, v_2, \ldots, v_k$ are linearly independent

(iii) Any $w \in R^n$, $w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$.

$$\langle w \cdot v_1 \rangle = c_1 \langle v_1 \cdot v_1 \rangle$$

$$c_1 = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} \qquad c_2 = \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle}$$

$$c_k = \frac{\langle w, v_k \rangle}{\langle v_k, v_k \rangle}$$

$$\therefore w = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \cdots$$

$$+ \frac{\langle w, v_k \rangle}{\langle v_k, v_k \rangle} v_k.$$

Problems

Determine which of the following sets are orthogonal.

1) $u = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ $v = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\langle u, v \rangle = -6 + 4 + 2 = 0$

$\langle u, w \rangle = -3 - 1 + 4 = 0$

$\langle v, w \rangle = 2 - 4 + 2 = 0$.

$\therefore \{u, v, w\}$ are orthogonal set of vectors

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2) $u = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ $v = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

$\langle u, v \rangle = -3 + 2 - 1 = -2 \neq 0$.

$\langle v, w \rangle = -2 - 4 + 4 = -2 \neq 0$

$\{u, v, w\}$ is not an orthogonal set of vectors

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3) S.T $v_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an orthogonal basis in R^2.

Given $w = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ find the coordinate vector $(w)_B$ of w w.r.t the basis

$\langle v_1, v_2 \rangle = 4 - 4 = 0$

$\{v_1, v_2\}$ is an orthogonal set.

$c_1 v_1 + c_2 v_2 = 0 \Rightarrow c_1 = c_2 = 0.$

$4c_1 + c_2 = 0.$

$-2c_1 + 2c_2 = 0.$

$$\begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 1 \\ 0 & 5/2 \end{pmatrix} \Rightarrow \begin{array}{l} 4c_1 + c_2 = 0 \\ \frac{5}{2} c_2 = 0 \end{array}$$

$\Rightarrow c_2 = c_1 = 0.$

$\therefore$ v_1, v_2 are linearly independent.

For $w = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ in $\mathbb{R}^2$.

$w = c_1 v_1 + c_2 v_2.$

$$\cancel{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \quad c_1 = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

$$= \frac{4 + 6}{16 + 4} = \frac{10}{20} = \frac{1}{2}.$$

$$c_2 = \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{1 - 6}{1 + 4} = -\frac{5}{5} = -1$$

$\therefore w = \frac{1}{2} v_1 - v_2.$

$$(w)_B = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}.$$

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Ex 2: $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ in R^3.

Show that $B = \{v_1, v_2, v_3\}$ form an orthogonal basis in R^3 and given $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. find the coordinate vector $\{w\}_B$ w.r.t. to the basis B.

$\langle v_1, v_2 \rangle = 1 + 0 - 1 = 0$

$\langle v_2, v_3 \rangle = 1 - 2 + 1 = 0$

$\langle v_3, v_1 \rangle = 1 + 0 - 1 = 0$.

Also v_1, v_2, v_3 are linearly independent

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_2 - c_3 &= 0 \\ -c_1 + c_2 + c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = c_3 = 0.$$

then

$$W = c_1 v_1 + c_2 v_2 + c_3 v_3.$$

$$c_1 = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{1 + 0 - 1}{1 + 1} = 0$$

$$c_2 = \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{1 + 2 + 1}{1 + 4 + 1} = \frac{4}{6} = \frac{2}{3}.$$

$$c_3 = \frac{\langle w, v_3 \rangle}{\langle v_3, v_3 \rangle} = \frac{1 - 1 + 1}{1 + 1 + 1} = \frac{1}{3}.$$

$\therefore \quad w = 0 v_1 + \frac{2}{3} v_2 + \frac{1}{3} v_3$

$$(w)_B = \begin{pmatrix} 0 \\ 2/3 \\ 1/3 \end{pmatrix}$$

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Ex: Find an orthogonal basis for the subspace W of R^3 given by $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$.

(* W is a plane through the origin in R^3)

$$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - 2z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ are a basis for W in R^3 but u and v are not orthogonal.

It is sufficient to find another non zero vector w that is orthogonal to either u or v.

let $w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \langle u, w \rangle = 0$

$\Rightarrow x + y = 0.$

Also $x - y + 2z = 0.$

$\Rightarrow x = -z, \quad y = z.$

$\therefore w = \begin{pmatrix} -z \\ z \\ z \end{pmatrix}$. For $z = 1$ $\quad w = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\{u, w\}$ is an orthogonal set in
and an orthogonal basis for W, since di

Defn: A set of vectors $S = \{v_1, v_2, \ldots v_k\}$ is an orthonormal set of vectors if

$$\langle v_i, v_j \rangle = 0 \quad \text{for } i \neq j$$

$$\|v_i\| = \langle v_i, v_j \rangle = 1 \quad \text{for } i = j.$$

An orthonormal basis for a subspace W of R^n is a basis of W that is an orthonormal set.

orthonormal $\langle v_i, v_j \rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$

Ex: S.T $S = \{v_1, v_2\}$ is an orthonormal set in R^3.

$$v_1 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}.$$

$$\langle v_1, v_2 \rangle = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$$

$$\langle v_1, v_1 \rangle = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\langle v_2, v_2 \rangle = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1.$$

Note: If $\{v_1, v_2, \ldots v_k\}$ be an orthonormal basis for a subspace W of R^n and let w be any vector in W, then

$$w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \cdots + \langle w, v$$

and this representation is unique.

Construct an orthonormal Basis for R^3.

① $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

② $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

$q_1 = \dfrac{v_1}{\|v_1\|}$ $q_2 = \dfrac{v_2}{\|v_2\|}$ $q_3 = \dfrac{v_3}{\|v_3\|}$

$\{q_1, q_2, q_3\}$ forms an orthonormal Basis for R^3

Ex: Let $v = (1, -2, 2, 0)$. Find a unit vector u in the same direction as v.

$$\|v\|^2 = \langle v . v \rangle = 1 + 4 + 4 = 9.$$

$$\|v\| = 3.$$

$$u = \frac{v}{\|v\|} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{pmatrix}$$

$$\|u\|^2 = 1$$

Ex 2: Let W be the subspace of $\mathbb{R}^2$ spanned by $x = \left(\frac{2}{3}, 1\right)$. Find a unit vector z that is a basis for W.

$$y = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\|y\|^2 = 2^2 + 3^2 = 13 \qquad \|y\| = \sqrt{13}$$

$$z = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Orthogonal matrices: An $(n \times n)$ matrix Q whose columns form an orthonormal set is called an orthogonal matrix.

Ex: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Q is an orthogonal matrix if and only if Q^TQ [...]
This is true if and only if Q is invertible a[...]

$Q^{-1} = Q^T$.

Determine whether the following matrices is orthogonal and hence find its inverse

(i) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ (iii) $\begin{pmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{pmatrix}$

Ex: Find the missing entries of Q to make Q an orthogonal matrix

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & - \\ 0 & 1/\sqrt{3} & - \\ -1/\sqrt{2} & 1/\sqrt{2} & - \end{pmatrix}$$

Ex: Determine whether the given orthogonal matrix represents a rotation or a reflection. If it is a rotation, give the angle of rotation, if it is a reflection, give the line of reflection

(a) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \rightarrow$ rotation $\theta = 45^0$

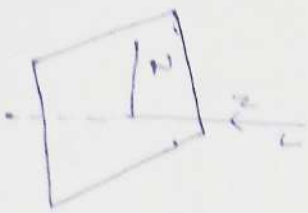
(b) $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

c) $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \rightarrow$ reflection $x = \sqrt{3}y$.

d) $\begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}$.

Orthogonal complement

Let W be a subspace of R^n. A vector v in R^n is orthogonal to W iff v is orthogonal to every vector in W.



The set of all vectors that are orthogonal to W is called orthogonal complement of W denoted by $W^{\perp}$ (W perp).

$$W^{\perp} = \{ v \text{ in } R^n ;\ v \cdot w = 0 \ \forall\ w \text{ in } W \}.$$

Note:

(i) A vector x is in $W^{\perp}$ if and only if x is orthogonal to every vector in a set that spans W

(ii) $W^{\perp}$ is a subspace of R^n

If $\{w_1, w_2, --- w_k\}$ span W then $z \perp W$

$\Rightarrow w_1 \cdot z = w_2 \cdot z = --- = w_k \cdot z = 0$. since any $w \in W$

If we have to find z. such that

$w_1 \cdot z = w_2 \cdot z = --- = w_k \cdot z = 0$ mean we have to find the null space of W.

Hence If A is an $(m \times n)$ matrix. The orthogonal complement of the row space of A is the null space of A. ie

$$\boxed{(\text{row } A)^{\perp} = \text{Null } A.}$$

Also ~~row A^T~~ row $A^T = \text{col } A$.

$$\therefore \boxed{(\text{Col } A)^{\perp} = \text{Null } A^T.}$$

orthogonal complement of column space of A is the null space of A^T.

Ex 1: Let W be the subspace spanned by the vectors $\{w_1, w_2\}$. Find a basis for $W^{\perp}$.

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad w_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 1 & -2 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_2} \begin{pmatrix} 2 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1/2 & -1 \\ 0 & 1 & -5/2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -5/2 \end{pmatrix}.$$

$$y_1 = -\frac{1}{4} y_3 \qquad \text{Basis} = \text{Span} \begin{pmatrix} -1/4 \\ 5/2 \\ 1 \end{pmatrix}$$

$$y_2 = \frac{5}{2} y_3.$$

$$= \left\{ \begin{pmatrix} -1 \\ 10 \\ 4 \end{pmatrix} \right\}$$

Ex 2: Let W be the subspace of R^5 spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of the $W^{\perp}$ of W.

Let $w = (x, y, z, s, t)$.

$$w \cdot u = 0 \Rightarrow x + 2y + 3z - s + 2t = 0$$

$$w \cdot v = 0 \Rightarrow 2x + 4y + 7z + 2s - t = 0.$$

$$\begin{pmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{pmatrix}$$

$$x + 2y + 3z - s + 2t = 0$$
$$z + 4s - 5t = 0$$

Let $t = a \quad s = b.$

$z = 5a - 4b.$

$y = c.$

$x = -2c - 3(5a - 4b) + b - 2a.$

$= -2c - 15a + 12b + b - 2a.$

$= \cancel{t + a} - 17a + 13b - 2c.$

$$\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = a \begin{pmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Basis} = \text{span}\left\{ \begin{pmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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Ex 3. Find the orthogonal complement of $W^\perp$
W and give a basis of $W^\perp$

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 2x - y = 0 \right\}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$v = \begin{pmatrix} a \\ b \end{pmatrix} \qquad v \cdot w = 0$

$\Rightarrow a + 2b = 0 \qquad b = t \Rightarrow a = -2t.$

$$\text{Basis of } W^\perp = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

Ex:

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x - y + 3z = 0 \right\}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x + 3z \\ z \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{w_1} + z \underbrace{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}_{w_2}.$$

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad w_1 \cdot v = 0 \Rightarrow a + 2b = 0.$

$\quad w_2 \cdot v = 0 \Rightarrow 3b + c = 0.$

let $c = t$ $\quad b = -t/3$ $\quad a = \frac{2}{3}t.$

Basis of $W^\perp = \text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\}.$

Ex 4: $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; x = t, \; y = -t, \; z = 3t \right\}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \to w_1$$

Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$w_1 \cdot v = 0 \Rightarrow a - b + 3c = 0.$

$c = t$ $\quad b = r.$ $\quad a = r - 3t$

Basis of $W^\perp = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

... for the row space and null space of A. Verify that every vector in row(A) is orthogonal to every vector in null(A).

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 7 & -14 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$Row(A) = span\{u_1, u_2\}$

$u_1 = (1, 0, 1) \qquad u_2 = (0, 1, -2).$

null (A) : $x - y + 3z = 0$

$y - 2z = 0$

$z = t \qquad y = 2t \qquad x = \cancel{2t-3t} = -t.$

$$null(A) = span\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}. \qquad v = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$v \cdot u_1 = -1 - 2 + 3 = 0$

$v \cdot u_2 = 0 + 2 - 2 = 0.$

To show $(row\, A)^{\perp} = null\, A$, it is enough to sh[ow th]at every u is orthogonal to v.

$$col(A) = span\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$v_1 = \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

Problems.

(1) Verify $(\text{Row } A)^{\perp} = \text{Null } A$ and $(\text{Col } A)^{\perp} = \text{Null } A^T$

a, $\begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 7 & -14 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\text{Row}(A) = \{(1, -1, 3)\ (0, 1, -2)\} = \{u_1, u_2\}$

Null (A): $x - y + 3z = 0$

$y - 2z = 0$

$z = t \quad y = 2t \quad x = 2t - 3t = -t$

$\text{Null}(A) = \text{span}\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\} = \text{span}(a_1)$.

To show that $(\text{Row})^{\perp} = \text{Null } A$, it is enough to show that u_i is orthogonal to a_1

$\langle u_1 . a_1 \rangle = u_1^T a_1 = -1 \; \cancel{\;} 2 + 3 = 0$

$\langle u_2, a_1 \rangle = 0 + 2 - 2 = 0$. $\therefore (\text{Row } A)^{\perp} = \text{Null } A$.

$\text{col}(A) : \left\{\begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \\ -1 \end{pmatrix}\right\} = \{v_1, v_2\}$

Null A^T:

$$A^T = \begin{pmatrix} 1 & 5 & 0 & -1 \\ -1 & 2 & 1 & -1 \\ 3 & 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 0 & -1 \\ 0 & 7 & 1 & -2 \\ 0 & -14 & -2 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 5 & 0 & -1 \\ 0 & 1 & 1/7 & -2/7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x + 5y - t = 0$$

$$y + \frac{1}{7} z - \frac{2}{7} t = 0$$

$$t_1 = k_1 \quad z = k_2 \quad y = -\frac{1}{7} k_2 + \frac{2}{7} k_1$$

$$x = -5y + t = \frac{5}{7} k_2 - \frac{10}{7} k_1 + k_1$$

$$x = \frac{5}{7} k_2 - \frac{3}{7} k_1$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \left\{ \begin{pmatrix} -3/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 5/7 \\ -1/7 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\text{Null } A^T = \text{span} \left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \\ 7 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 7 \\ 0 \end{pmatrix} \right\} = \{ b_1, b_2 \}$$

To show that (col A) = Null A^T it is enough to show that ie

$\langle v_1 . b_1 \rangle = 0 \Rightarrow -3 + 10 - 7 = 0$

$\langle v_2 , b_1 \rangle = 0 \Rightarrow$ ie $+3 + 4 - 7 = 0$

$\langle v_1 , b_2 \rangle = 5 - 5 = 0$

$\langle v_2 , b_2 \rangle = -5 - 2 + 7 = 0$

$\therefore (\text{col } A)^{\perp} =$ Nu

(2) $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Row A : $\{(1, 2, -1, 3)\ (0, 0, 3, -8)\} = \{u_1, u_2$

Null A : $x + 2y - z + 3t = 0$

$3z - 8t = 0$

$t = k_1 \quad z = 8/3\, k_1 \quad y = k_2 \quad x = -2k_2 + \frac{8}{3}k_1 - 3k$

$= -2k_2 - \frac{1}{3}k_1$

Null A $= \left\{ \begin{pmatrix} -1 \\ 0 \\ 8 \\ 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \{a_1, a_2\}$

$\langle u_1, a_1 \rangle = -1 - 8 + 9 = 0$

$\langle u_1, a_2 \rangle = -2 + 2 = 0$

$\langle u_2, a_1 \rangle = 24 - 24 = 0$

$\langle u_2, a_2 \rangle = 0 \qquad \therefore (\text{Row } A)^{\perp} = \text{Null } A$

Col A : $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right\} = \{v_1, v_2\}$

Null A^T : $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 1 & 3 \\ 3 & -2 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & -8 & -16 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$x + 2y + 3z = 0$$

$$y + 2z = 0$$

P $z = t \quad y = -2t \quad x = 4t - 3t = t$

Null $A^T : \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\} = \{ b_2 \}$

$\langle v_1, b_2 \rangle = 1 - 4 + 3 = 0$

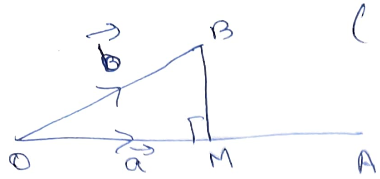
$\langle v_2, b_2 \rangle = -1 - 2 + 3 = 0$

$\therefore \quad (\text{col } A)^{\perp} = \text{Null } A^T$

—— x ——

Orthogonal Projection

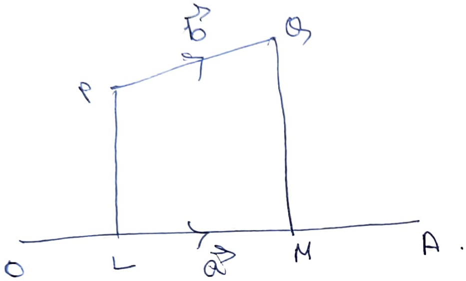
(Recaptulation of vector Algebra).



OM = component of $\vec{b}$ on $\vec{a}$

$\vec{OM}$ = component vector of $\vec{b}$ on $\vec{a}$.

$\vec{MB}$ = component vector $\vec{b} \perp \vec{a}$.



LM = component of $\vec{b}$ on $\vec{a}$.

= orthogonal projection of $\vec{b}$ on $\vec{a}$.

component of $\vec{b}$ on $\vec{a}$ = $\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$.

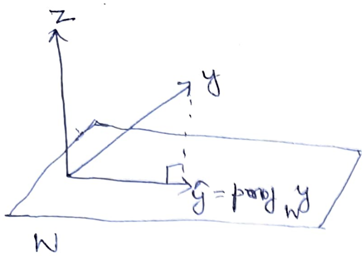
length or magnitude of projection of $\vec{b}$ on $\vec{a}$ = $\dfrac{|\vec{b}\cdot\vec{a}|}{|\vec{a}|}$

vector component of $\vec{b}$ on $\vec{a}$ = $\dfrac{(\vec{b}\cdot\vec{a})\,\vec{a}}{(\vec{a})^2}$

$\vec{b} \perp \vec{a}$.

($\perp$lar component $= \vec{b} - \dfrac{(\vec{b}\cdot\vec{a})\,\vec{a}}{(\vec{a})^2}$

Orthogonal Projection



Given a non-zero vector u in R^n, consider the problem of decomposing a vector y in R^n into sum of two vectors, one a multiple of u and the other orthogonal to u.

$$y = \hat{y} + z. \quad \text{——— (1)}$$

(* In physics, force is decomposed into two components one is the horizontal component and the other is the vertical component)

where $\hat{y} = \alpha u$ for some scalar α and z is some vector orthogonal to u.

$$\therefore \quad z = y - \alpha u$$

$y - \hat{y}$ is orthogonal to u if and only if

$$(y - \alpha u) \cdot u = 0$$

ie $\quad y \cdot u - \alpha (u \cdot u) = 0.$

$$\alpha = \frac{y \cdot u}{u \cdot u}.$$

Eqn (1) will be satisfied with z orthogonal to u if and only if $\alpha = \frac{\langle y \cdot u \rangle}{\langle u \cdot u \rangle}$ and $\hat{y} = \frac{\langle y \cdot u \rangle}{\langle u \; u \rangle} u$.

The vector $\hat{y}$ is called the ~~projection~~ <u>orthogonal projection</u> of y onto u. and the vector z is called the <u>component</u> of y orthogonal to u

Hence the projection of a vector v onto a non-zero vector u is

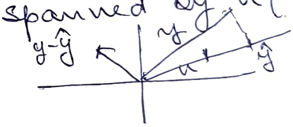
$$\boxed{\mathrm{proj}_u(v) = \frac{\langle v.u \rangle}{\langle u.u \rangle}\, u.}$$

Also $\mathrm{perp}_w v = v - \mathrm{proj}_u(v)$ is orthogonal to $\mathrm{proj}_u v$.

Hence we can decompose v as.

$$\boxed{v = \underset{(\hat{y})}{\mathrm{proj}_u(v)} + \underset{(z)}{\mathrm{perp}_w v}.}$$

If c is an nonzero scalar and if u is replaced by cu in the definition of $\hat{y}$ then the orthogonal projection of y onto cu is exactly same as the projection of y onto u. Hence this projection orthogonal projection of y onto u is determined by the subspace L spanned by u (the line through u and 0).



Definition: Let W be a subspace of R^n and let $\{u_1, u_2, \dots u_n\}$ be an orthogonal basis for W. For any vector v in R^n, the orthogonal projection of v onto W is defined as

$$\mathrm{proj}_W(v) = \frac{\langle u_1, v\rangle}{\langle u_1, u_1\rangle} u_1 + \frac{\langle u_2, v\rangle}{\langle u_2, u_2\rangle} u_2 + \dots \frac{\langle u_k, v\rangle}{\langle u_k, u_k\rangle} u_k$$

The complement of v orthogonal to W is the vector

$$\mathrm{perp}_W v = v - \mathrm{proj}_W(v)$$

——— x ———

Examples:

Let $y = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ $u = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Find the orthogonal

projection of y onto u.

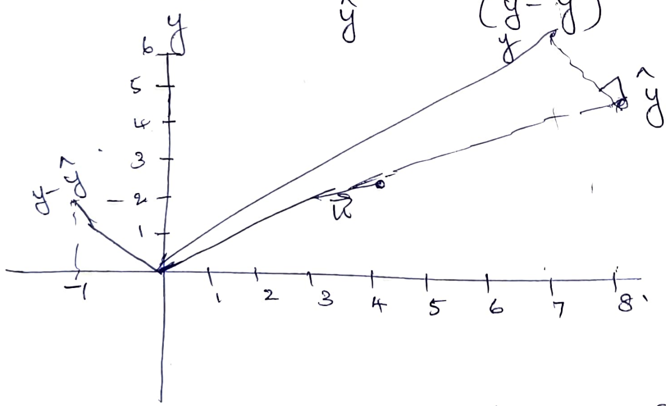
$$\hat{y} = \frac{\langle y \cdot u \rangle}{\langle u \cdot u \rangle} u = \frac{40}{20} u = 2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Component of y orthogonal to u.

$$\text{perp}_{w} y = y - \hat{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The sum of these two vectors is y.

i.e $\begin{pmatrix} 7 \\ 6 \end{pmatrix} = \underbrace{\begin{pmatrix} 8 \\ 4 \end{pmatrix}}_{\hat{y}} + \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{(y-\hat{y})}$



Note $\{ \hat{y} \cdot (y - \hat{y}) \}$ will be an orthogonal set

$$\hat{y} \cdot (y - \hat{y}) = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -8 + 8 = 0$$

——x——

Ex 2: Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$ and let $v = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. Find the orthogonal projection of v onto W and the component of v orthogonal to W.

Orthogonal Basis for W is

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

$\langle u_1, v \rangle = u_1^T v = 3 - 1 = 2.$

$\langle u_2, v \rangle = -3 - 1 + 2 = -2.$

$\langle \cancel{u_1, u_1} \rangle = u_1 \cdot u_1 = 2. \qquad u_2 \cdot u_2 = 3.$

$$\text{proj}_W v = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1} \right) u_1 + \left(\frac{u_2 \cdot v}{u_2 \cdot u_2} \right) u_2$$

$$= \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

$$\text{perp}_W v = v - \text{proj}_W v$$

$$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5/3 \\ 1/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -4/3 \\ 8/3 \end{pmatrix}.$$

~~pro~~ **Note:** $\text{proj}_W v$ is in W. $\text{perp}_W v$ is $\perp^{lar}$ to W.

normal to $W = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Find the orthogonal projection v onto the Subspace W Spanned by the vector u.

① $v = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$ $\quad u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\text{proj}_{u} v = \frac{\langle u . v \rangle}{\langle u . u \rangle} u = \left(\frac{7-4}{1+1} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$$

———— x ————

② $v = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $\quad u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\text{proj}_u v = \frac{\langle u_1 \, v \rangle}{\langle u_1 \, u_1 \rangle} u_1 + \frac{\langle u_2 \, v \rangle}{\langle u_2 \, u_2 \rangle} u_2$$

$$= \frac{3+1-2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{3-1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

———— x ————

The Orthogonal Decomposition Theorem:

Let W be a subspace of R^n and let v be a vector in R^n. Then there are unique vectors w in W and $w^{\perp}$ in $W^{\perp}$ such that

$$v = w + w^{\perp}$$

Prob:

Find the orthogonal decomposition of v with respect to W

(i) $v = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ $\quad W = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.

$$\text{proj}_W v = \frac{\langle w \cdot v \rangle}{\langle w, w \rangle} w = -\frac{4}{10} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2/5 \\ -6/5 \end{pmatrix}$$

$$W^T = v - \text{proj}_W v = \begin{pmatrix} 12/5 \\ -4/5 \end{pmatrix}$$

$$v = \text{proj}_W v + W^T$$

---

2) $v = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ $\quad W = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$$\text{proj}_W v = \frac{\langle w_1 \cdot v \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle w_2, v \rangle}{\langle w_2, w_2 \rangle} w_2.$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/2 \\ -2 \\ 7/2 \end{pmatrix}$$

$$w^T = v - \text{proj}_W v = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$v = \text{proj}_W v + w^T$$

The Gram-Schmidt Process

A simple method for constructing an orthogonal (or an orthonormal) basis for the subspace W of R^n

Let $\{x_1, x_2, \ldots x_n\}$ be a subspace of W of R^n and define the following

$v_1 = x_1 \qquad W_1 = \text{span}(x_1)$

Starting with x_1, we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1

$$
\begin{aligned}
V_2 &= \text{perp}_{x_1}(x_2) \\
&= x_2 - \text{proj}_{x_1}(x_2) \\
&= x_2 - \frac{(x_1 \cdot x_2)}{(x_1, x_1)} x_1 \\
&= x_2 - \frac{(v_1, x_2)}{(v_1, v_1)} v_1
\end{aligned}
$$

$$W_2 = \text{Span}\{x_1, x_2\}$$

$$V_3 = x_3 - \frac{\langle v_1, x_3 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle v_2, x_3 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$\boxed{W_3 = \text{Span}\{x_1, x_2, x_3\}}$$

$\vdots$

$$v_k = x_k - \frac{\langle v_1, x_k \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle v_2, x_k \rangle}{\langle v_2, v_2 \rangle} v_2 - \cdots - \frac{\langle v_{k-1}, x_k \rangle}{\langle v_{k-1}, v_{k-1} \rangle} v_{k-1}$$

$$W_k = \text{Span}\{x_1, x_2, \ldots x_k\}$$

Example

Apply the Gram-Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace R^4 spanned by

a) $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}$ $x_3 = \begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix}$

step 1

Set $v_1 = x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $W_1 = \text{span}(v_1)$.

Step 2: Compute the component of x_2 orthogonal to W_1

$$v_2 = \text{perp}_{W_1}(x_2) = x_2 - \frac{\langle v_1 \cdot x_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = x_2 - \frac{12}{4} v_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

$W_2 = \text{span}\{x_1, x_2\}$

Step 3: Compute the component of x_3 orthogonal to W_2

$$v_3 - \text{perp}_{W_2}(x_3) = x_3 - \frac{\langle v_1\ x_3 \rangle}{\langle v_1\ v_1 \rangle} v_1 - \frac{\langle v_2\ x_3 \rangle}{\langle v_2\ v_2 \rangle} v_2$$

$$= \left(\frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5}\right) = (-6, -17, -13, 14)$$

orthogonal basis $\{v_1, v_2, v_3\}$.

orthonormal basis: Normalise v_1, v_2, v_3

$$u_1 = \frac{v_1}{\|v_1\|^2} = \frac{1}{2}(1, 1, 1, 1)$$

$$u_2 = \frac{v_2}{\|v_2\|^2} = \frac{1}{\sqrt{10}}(-2, -1, 1, 2)$$

$$u_3 = \frac{v_3}{\|v_3\|^2} = \frac{1}{\sqrt{910}}(-6, -17, -13, 14)$$