statics

① particle statics

1) equations of equilibrium

$$\boxed{\begin{array}{llll} 2D: & \Sigma F_x, & \Sigma F_y, & \Sigma M = 0 \\ 3D: & \Sigma F_x = 0 & \Sigma F_y = 0 & \Sigma F_z = 0 \\ & \Sigma M_x = 0 & \Sigma M_y = 0 & \Sigma M_z = 0 \end{array}}$$

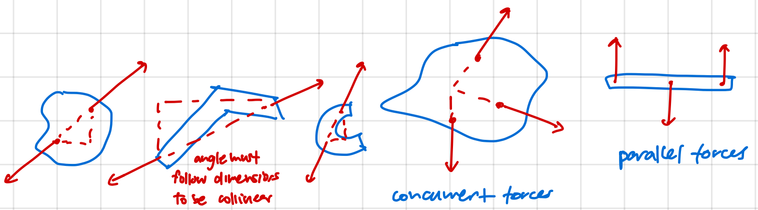
2) Force members

2 force members

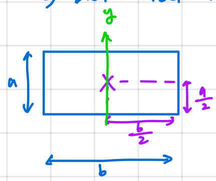
regardless of shape and where the forces are applied, they must be collinear to be in equilibrium.

3 Force members

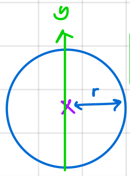
To stay in eqm ($\Sigma F = \Sigma M = 0$), the lines of the three forces must either be parallel or concurrent.



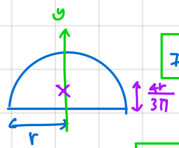
3) Distributed load equivalence



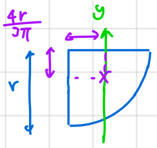
$\boxed{I_{yy,c} = \frac{1}{12} ab^3}$



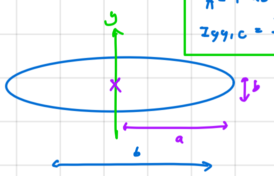
$\boxed{I_{yy,c} = \frac{\pi r^4}{4}}$



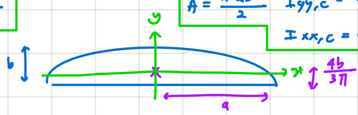
$\boxed{I_{yy,c} = \frac{\pi r^4}{2}}$



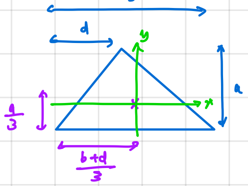
$\boxed{I_{xx,c} = I_{yy,c} = 0.05488 r^4}$



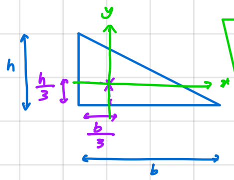
$\boxed{A = \pi \cdot ab \quad I_{yy,c} = \frac{\pi b a^3}{4}}$



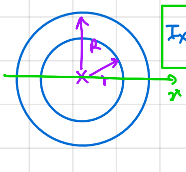
$\boxed{A = \frac{\pi ab}{2} \quad I_{yy,c} = 0.3927 ba^3 \quad I_{xx,c} = 0.1098 ab^3}$



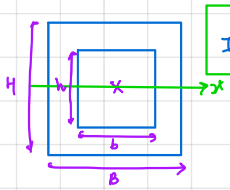
$\boxed{I_{yy,c} = \frac{ab}{36}(b^2 - bd + d^2) \quad I_{xx,c} = \frac{ba^3}{36}}$



$\boxed{I_{xx,c} = \frac{ba^3}{36} \quad I_{yy,c} = \frac{ab^3}{36}}$



$\boxed{I_{xx,c} = \frac{\pi}{4}(R^4 - r^4)}$



$\boxed{I_{xx,c} = \frac{BH^3 - bh^3}{12}}$

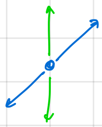
② analysis of truss structures

1) static determinacy

$$\boxed{2j = m + r}$$

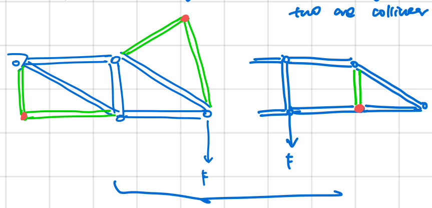
number of joints (j)
number of unknown forces acting on a joint (m)
reaction forces (r)

2) special joints



zero force members

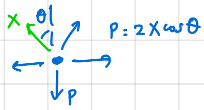
1. non-collinear two-member joint
2. 3-member joint where two are collinear



no load at special joint, members are zero-force

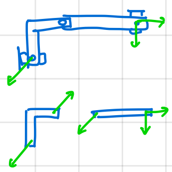
3) method of sections

1. eliminate zero force members
2. draw FBD of entire truss and get support reactions
3. Find cuts where there will be at most three unknowns
   ↳ try to find points where n+1 forces, but n forces can be eliminated by choice of pivot → single variable eqn
4. Draw FBD of cuts, assuming in tension.
5. use ΣF_x, ΣF_y, $\Sigma M = 0$ to solve for unknown forces on either side of cut
   ↳ look for symmetrically loaded joints



③ analysis of frames and machines

1) method of disassembly – utilize members

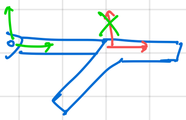


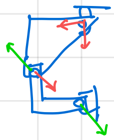
1. overall FBD
2. sub FBDs

} $\Sigma F_x, \Sigma F_y, \Sigma M = 0$

2) dealing w/ 3 eqns, 4 unknowns

1. eliminate by observation
2. disassemble to get more equations (especially of 2 force member)
3. analyze a force if you know its angle (especially of 2 force member)

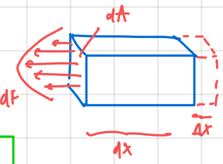




Stress and strain

① longitudinal stress and strain

1) Generalised hooke's law



$$\boxed{\sigma_{xx} = E\,\varepsilon_{xx}}$$

$$\boxed{\sigma = \frac{dF}{dA} \approx \frac{F}{A}}$$

$$\boxed{\varepsilon_{xx} = \frac{d\Delta x}{dx} \approx \frac{\delta}{L}}$$

$$\boxed{\varepsilon_{yy} = \varepsilon_{zz} = -\nu\,\varepsilon_{xx} = \frac{\delta_{yy}}{L_{yy}} = \frac{\delta_{zz}}{L_{zz}}}$$

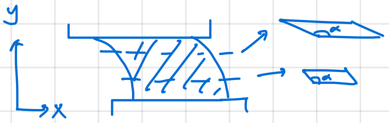
$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}}{E}$$

for triaxial stress states

② shear stress and strain

$$\boxed{\tau_{ij} = G\,\gamma_{ij}}$$

$$\boxed{\gamma = \frac{\pi}{2} - \alpha}$$



$$\left.\begin{aligned} \alpha &= \tan^{-1}\frac{dy}{dx} \\ \gamma &= \tan^{-1}\frac{dx}{dy} \end{aligned}\right\} \text{gradient}$$

③ statically indeterminate systems

1) degree of indeterminacy, K

$$\boxed{K = N_{unknowns} - N_{independent\ eqns}}$$

2) deformation compatibility: techniques

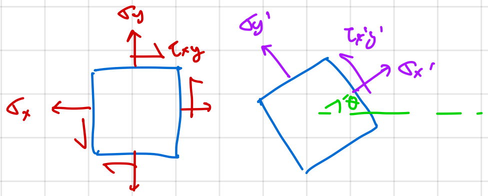
1. similar triangles
2. small Δ approximation (rigid rotation)
3. relationships
$$\frac{\delta_1}{\delta_2} = \frac{F_1 \cdot \frac{L_1}{E_1 A_1}}{F_2 \cdot \frac{L_2}{E_2 A_2}}$$
4. deformable pin connected structures: visualising δ to form Δ
5. $$\boxed{E_{composite} = \sum_i E_i \frac{V_i}{V_{total}}}$$

1. visualise deformation
2. create extra containing equation(s)

Transformation of stress and strain

① 2D transformation of stress and strain

1) transformation



stresses on plane @ θ anticlockwise from original axes

↙ can be extended to an element, since stress is specific

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_x' + \sigma_y' = \sigma_x + \sigma_y = \text{constant}$$

average normal stress :

$$\frac{\sigma_x' + \sigma_y'}{2} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_{p_1} + \sigma_{p_2}}{2}$$

2) principal stress

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta_p + \tau_{xy}\cos 2\theta_p = 0$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} , \quad \theta_{p_2} = \theta_{p_1} + \frac{\pi}{2}$$

$$\sigma_p = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{p_1} + \sigma_{p_2} = \sigma_x' + \sigma_y' = \sigma_x + \sigma_y$$

3) max shear stress

$$\theta_s = \frac{1}{2}\tan^{-1} - \frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\frac{\sigma_{p_1} - \sigma_{p_2}}{2}$$

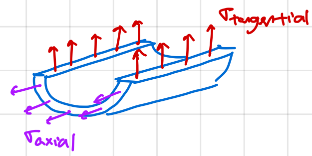
$$\sigma_{x_s}' = \sigma_{y_s}' = \frac{\sigma_x + \sigma_y}{2}$$

4) θ_p and θ_s

$$\theta_s = \theta_p \pm \frac{\pi}{4}$$

② stress in thin wall pressure vessels

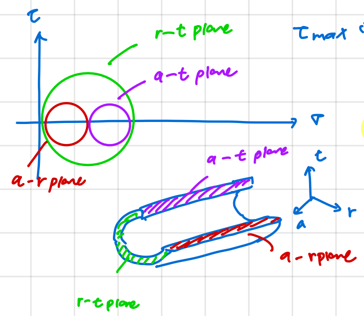
1) cylindrical pressure vessels



$$\sigma_t = \frac{pr}{t} \qquad \sigma_{axial} = \frac{pr}{2t} \qquad \sigma_{axial} = \frac{pr^2}{(R+r)t}$$

principal stresses are σ_t and σ_a. — if t is significant

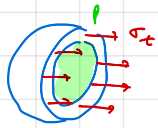
using mohr's circle, we see three τ_{max}.



τ_{max} in r-t plane ⟹ use for design

$$\tau_{max} = \frac{1}{2}\sigma_t = \frac{pr}{2t}$$

2) spherical pressure vessels



$$\sigma_t = \frac{pr}{2t}$$

assuming $\frac{r}{t} \geq 10$, $p \approx$ negligible.

so $\sigma_p = \sigma_t$ or 0

$$\tau_{max} = \frac{1}{2}\sigma = \frac{pr}{4t}$$

③ transformation of plane strain

1) transformation

$$\varepsilon_x' = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_y' = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\gamma_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

2) principal strain

$$\gamma_{x'y'} = 0$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_p = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

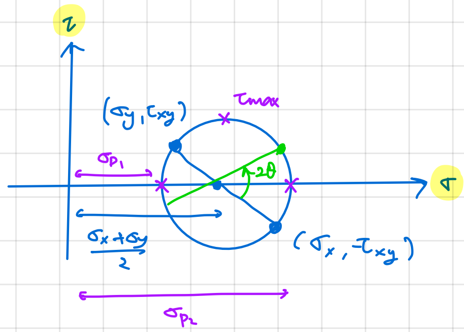
3) max shear strain

$$\gamma_{max} = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\theta_p = \theta_s \pm 45^\circ$$

④ Mohr's circle

1) Mohr's circle of stress



1. τ as y axis, σ as x axis
2. center of circle @ $\frac{\sigma_x+\sigma_y}{2}$ from origin.
3. locate initial points at $(\sigma_x, -\tau_{xy})$ and (σ_y, τ_{xy})
4. Draw circle, connect points

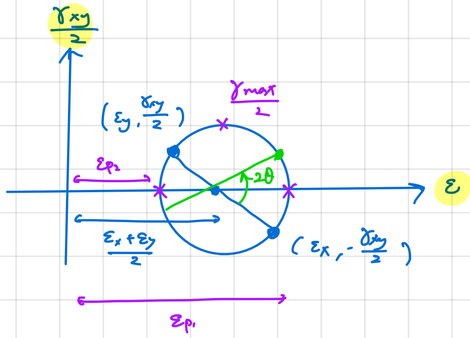
Needs:

$\frac{\sigma_x+\sigma_y}{2}$, σ_x, σ_y and τ_{xy}

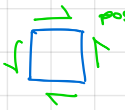
Information presented

- $\tau_{x'y'}$, σ_x', σ_y' @ any 2θ
- σ_{p_1} and σ_{p_2}, τ_{max}

2) Mohr's circle of strain



- shear stress has same direction convention



- same procedure as stress, but $\frac{\gamma_{xy}}{2}$ instead.

Beam bending

external loads ↳ CW as positive

internal reactions ↳ ACW as positive

① q, F_{xy} and M_{xz}

1) solve for reactions
2) cut or use singularity functions
3) sketching distribution

1. calculate F_{xy} and M_{xz} at point loads just before and after load at point comes into play
2. check correspondence between diagrams
3. simulate on ruler to check if positive (smile) and negative (frown) matches M_{xz}
4. checks:
   - F_{xy} will jump at point loads
   - M_{xz} linear if only point loads in that segment. curved if distributed load.
   - M_{xz} will jump at point moments.
   - only point loads, F_{xy} will be discontinuous and M_{xz} will be linear only

singularity function $K\langle x-a\rangle^n = \begin{cases} 0 & ; x < a \\ K(x-a)^n, & x > a \end{cases}$ and $n = 0, 1, 2 \ldots$

$$\int \langle x-a\rangle^n dx = \begin{cases} \dfrac{\langle x-a\rangle^{n+1}}{n+1} & ; n \geq 0 \\ \langle x-a\rangle^{n+1} & ; n < 0 \end{cases}$$

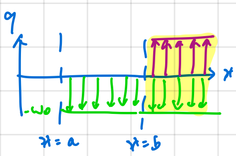
$$\boxed{\begin{aligned} q(x) &= \ldots \\ F_{xy} &= -\int q \\ M_{xz} &= -\int F_{xy} \end{aligned}}$$

$$\boxed{q(x) = \lim_{\delta x \to 0} \frac{-\delta F_{xy}}{\delta x} = -\frac{dF_{xy}}{dx}}$$

$$\boxed{F_{xy} = \lim_{\delta x \to 0} -\frac{\delta M_{xz}}{\delta x} = -\frac{dM_{xz}}{dx}}$$

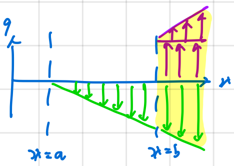
distributed load

1. uniform



$$q = -w_0\langle x-a\rangle^0 + w_0\langle x-b\rangle^0$$

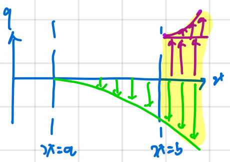
2. linear increase



$$q = -K\langle x-a\rangle^1 + K\langle x-b\rangle^1 + K(b-a)\langle x-b\rangle^0$$

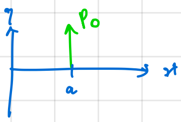
gradient (K)

3. quadratic



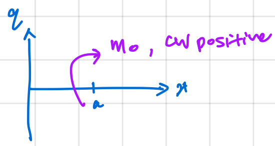
$$q = -K\langle x-a\rangle^2 + K\langle x-b\rangle^2 + K(b-a)^2\langle x-b\rangle^0$$

concentrated load



$$q = P_0\langle x-a\rangle^{-1}$$

concentrated moment



$$q = M_0\langle x-a\rangle^{-2}$$

② bending stress

internal rxn moment

$$\boxed{\sigma_{xx} = E\varepsilon_{xx} = -\frac{E}{R}y = -\frac{M_{xz}}{I_z}y}$$

③ shear stress

1) rectangular cross section

$$\boxed{\tau_{xy} = \frac{F_{xy} \cdot Q}{I_z b}}$$

2) circular cross section

solid

$$\boxed{\begin{aligned} \tau_{xy}\Big|_{y_1 = y} &= \frac{4F_{xy}(R^2 - y^2)}{3\pi R^4} \\ \tau_{xy}\ \text{max} \ @\ y=0 &= \frac{4F_{xy}}{3\pi R^2} \end{aligned}}$$

hollow

$$\boxed{\tau_{xy}\ \text{max} \ @\ y=0 = \frac{4F_{xy}}{3\pi(R^4 - r^4)}\left(\frac{Rr}{R^2+r^2} + 1\right)}$$

④ beam deflection

1) $M_{xz} = EI_z \dfrac{1}{R}$ $\qquad -F_{xy} = EI_z \dfrac{d^3v}{dx^3}$

$= EI_z\left(\dfrac{d^2v}{dx^2}\right)$ $\qquad q = EI_z \dfrac{d^4v}{dx^4}$

$$\boxed{v = \iint \frac{1}{EI_z} M_{xz}\, dx\, dx + C_1 x + C_2}$$

$C_1 = \dfrac{dv}{dx}\Big|_{x=0} EI_z \qquad C_2 = v\Big|_{x=0} \cdot EI_z$

2) point load deflection

$$\boxed{v = -\frac{PL^3}{3EI}}$$

⑤ statically indeterminate systems

↳ solving for rxns — not enough eqns to get values to solve for q ... ⟶ deformation compatibility (in this case, often boundary conditions)

method of double integration

$q = \ldots$

↳ $F_{xy} = \ldots$

↳ $M_{xz} = \ldots$

↳ $\frac{dv}{dx} \cdot EI_z = \ldots + C_1$

↳ $v\, EI_z = \ldots + C_1 x + C_2$

① ↳ boundary conditions to solve for C_1 and C_2

② boundary condition to get 3rd eqn

⇓

simultaneous

method of superposition

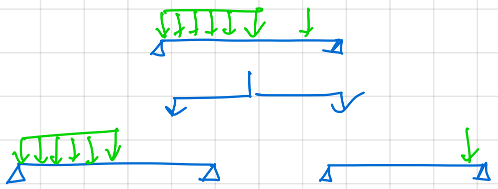
no need to + C_1, C_2 since you can solve using boundary conditions

1. split into cases
2. $q_i \to F_{xy_i} \to M_{xz_i} \to \frac{dv}{dx_i} \to v_i$
3. $\sum_i \frac{dv}{dx_i} = \ldots$ (boundary conditions)
4. $\sum_i v_i = \ldots$ (boundary conditions)

→ solve

summing individual effects (analytically) must give result same as boundary condition overall

↳ on a linear elastic structure, assuming deformations are small



Torsion

① twist

$$\boxed{\gamma \approx r\frac{d\theta}{dx}}$$

shear strain varies linearly w/ r

and $\frac{d\theta}{dx}$ (change in twist per unit length)

$$\boxed{\gamma_{max} = R\frac{d\theta}{dx} = R\frac{\theta}{L}}$$

linear variation of θ w/ x

③ power transmission

$$\boxed{\text{power transmission} = \text{torque} \times \omega}$$

② torque and shear stress

1) polar second moment of area

$$J = \int r^2 \, dA$$

$$\boxed{J_x = \frac{\pi}{2}(R_2^4 - R_1^4)}$$

2) τ, θ, T_{xx}

$$\frac{T_{xx}}{J_x} = G\frac{d\theta}{dx} = \frac{\tau_{x\theta}}{r}$$

$$\boxed{\theta = \frac{T_{xx}\cdot L}{GJ_x}}$$

if $\frac{d\theta}{dx}$ is constant;

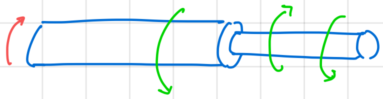
$= \frac{\theta}{L}$

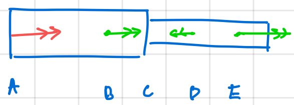
$$\boxed{\tau_{x\theta} = \frac{T_{xx}}{J_x}r = G\frac{d\theta}{dx}r = \frac{G\theta}{L}r}$$

τ_{max} at surface, $r = R$

④ load cases

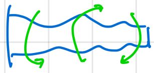
1) stepped shafts





⇒ solve section by section

2) shafts of varying radii



$r = f(x)$

$$\frac{T_{xx}}{J_x} = G\frac{d\theta}{dx} = \frac{\tau_{x\theta}}{r}$$

varies

varies

⇒ break into segments so that constant T_{xx}

$$\theta = \frac{T_{xx}}{G}\int_{x_1}^{x_2} J_x \, dx \quad \text{at each segment}$$

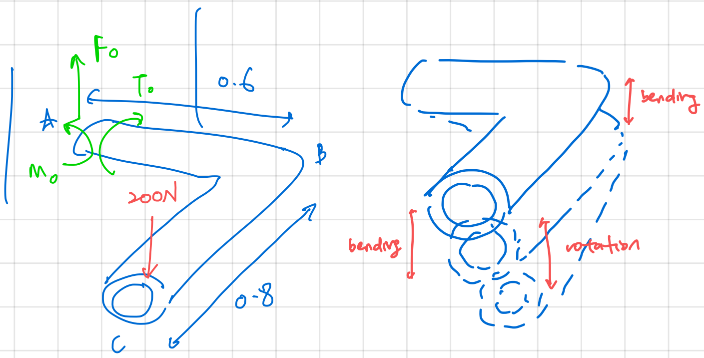
$$\tau_{max} = \frac{T_{xx}}{J_x}\cdot f(x)$$

3) shafts of varying torque loads

$$T_{xx} = f(x)$$

$$\tau_{max} = \frac{T_{xx}}{J_x}R = \frac{f(x)}{J_x}(R)$$

4) Deflection and twist in 3D



⇒ visualize how rotation can deflect, how bending can deflect ⟶ total displacement

$$\delta = \theta \cdot L \ \text{(tangent)}$$

⑤ solving

1) solve for reactions

2) look at segments by cutting ⟶ torque distribution

3) solve for τ or θ

$$\boxed{\tau_{max} = \frac{T_{xx\,(max)}}{J_x}R}$$

$$\theta_{F/A} = \theta_{F/E} + \theta_{E/D} + \dots \theta_{B/A}$$

$$= \frac{T_{FE}L_{FE}}{GJ_{xFE}} \dots$$