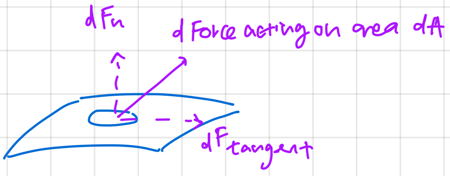
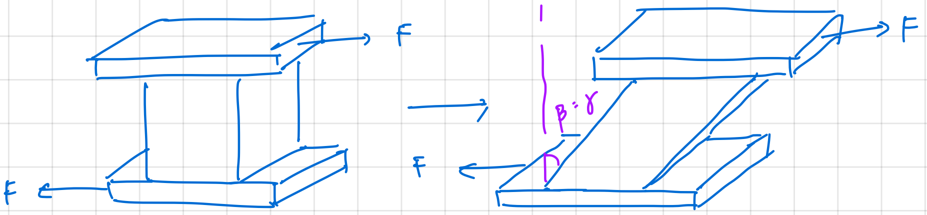
Properties of fluids

① What is a fluid?

A substance that deforms continuously when acted on by shear stress of any magnitude



$$\sigma = \frac{dF_n}{dA} \qquad \tau = \frac{dF_{tangent}}{dA}$$



For solids, we see fixed deformation independent of time. $\tau \propto \beta$

For fluids, we see continuous deformation.

$$\tau \propto \frac{d\beta}{dt}$$

time rate at which shear deformation is more meaningful for fluids because it continues to deform with time

② Density, specific weight and gravity

1) Density

$$\boxed{\rho = \frac{m}{V}}$$

T in K

universal gas constant 8.314 kJ

ideal gas law : $\boxed{P = \rho R T \ , \ R = \frac{R_u}{M_m}}$ molar mass of gas

specific gas constant, different for different gases

2) specific weight : weight per unit volume

$$\boxed{\gamma = \rho g = \frac{mg}{V}}$$

3) specific gravity : relative density of fluid to water at 4°C

$$\boxed{SG = \frac{\rho}{1000\ kgm^{-3}}}$$

③ Viscosity

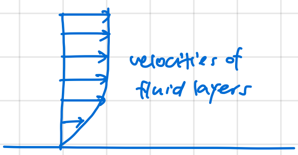
often used as a boundary condition

1) No slip condition : For a real viscous fluid, the velocity of the fluid immediately next to the wall is zero relative to the wall

↳ ie. fluid in contact with the solid boundary has same velocity as boundary

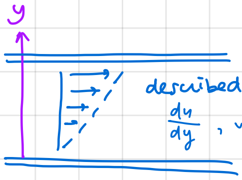
↓

greater wall shear stress and net resistance on surface (drag)



2) strain rate and velocity gradient

For any fluid :

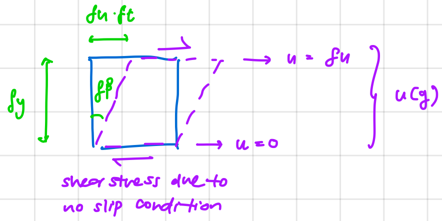


described by $u(y)$, velocity profile

$\frac{du}{dy}$, velocity gradient

gradient $= \frac{dy}{du}$, be careful

For any newtonian fluid :



Fluid particle undergoing shear deformation according to velocity profile because of no slip condition and presence of shear forces.

By geometry :

small ∠ approximation :

$$\delta\beta \approx \tan \delta\beta = \frac{u \cdot \delta t}{\delta y}$$

$$\frac{\delta\beta}{\delta t} = \frac{\delta u}{\delta y}$$

$$\frac{d\beta}{dt} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \frac{du}{dy}$$

$$\tau \propto \frac{d\beta}{dt}$$

3) dynamic viscosity in Newtonian fluids

Newtons law of viscosity

$$\tau \propto \frac{d\beta}{dt} \propto \frac{du}{dy} \Rightarrow \boxed{\tau = \mu \frac{d\beta}{dt} = \mu \frac{du}{dy}} \leftarrow \tau \propto \frac{d\beta}{dt} \propto \frac{du}{dy}$$

μ : dynamic viscosity, $N \cdot s\,m^{-2}$, $kg\,m^{-1}s^{-1}$, $Pa \cdot s$,

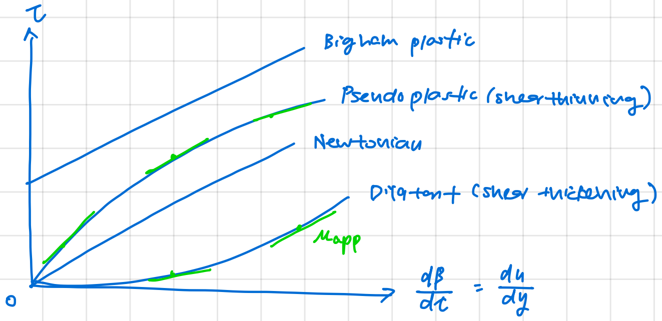
Poise (P) [$1P = 0.1\,Pa \cdot s$]

intuition : $\tau = \mu \frac{d\beta}{dt}$ — 'outcome'

τ : driver

μ : material-dependent property that relates shear stress on a fluid to its rate of deformation

4) Newtonian and non-newtonian fluids



1. Pseudoplastic : $\mu_{app} \downarrow$ as rate of deformation $\uparrow$
   eg. paints — flows fast when painted, slow when pour
2. Bigham plastic : shear stress must reach min. yield stress before flow commence
   eg. toothpaste
3. Dilatant : $\mu_{app} \uparrow$ as rate of deformation $\uparrow$
   eg. quicksand

$$\boxed{\mu_{app} = \text{slope of } \tau - \frac{d\beta}{dt} / \frac{du}{dy} \text{ graph}}$$

5) viscosity and temperature

viscosity of liquids

↳ the resistance to relative motion between adjacent layers of liquid caused by imf

↳ $\mu \downarrow$ as $T \uparrow$: at higher T, molecules possess more energy, can oppose larger cohesive imf, thus moving more freely

↳ approximated by Andrade's eqn :

$$\boxed{\mu = a e^{\frac{b}{T}}}$$

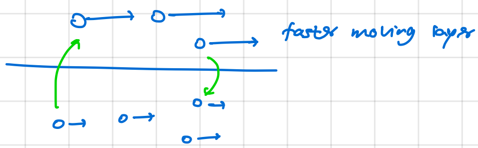
b : 19.3 K for water

T in K

a : $1.6 \times 10^{-6}\ kg\,m^{-1}s^{-1}$

viscosity of gases

↳ the resistance to relative motion between adjacent layers of gas due to exchange of momentum from collisions. negligible resistance from imf.



exchange of momentum between layers w diff. bulk velocities → when shear stress tries to deform fluid, exchange reduces relative motion of layers → resists deformation (and flow)

↳ $\mu \uparrow$ as $T \uparrow$: random motion enhanced as $T \uparrow$, more collisions, greater exchange of P, greater μ

↳ approximated by Sutherland's law :

$$\boxed{\mu = \frac{A T^{\frac{1}{2}}}{1 + BT}}$$

A : $1.458 \times 10^{-6}\ kg\,m^{-1}s^{-1}K^{-\frac{1}{2}}$

T in K

B : 110.4 K for air

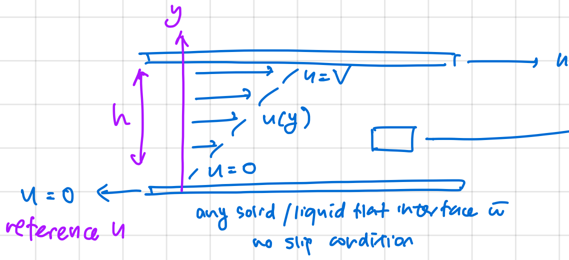
6) kinematic viscosity : ratio of dynamic viscosity to ρ. used for reporting purposes.

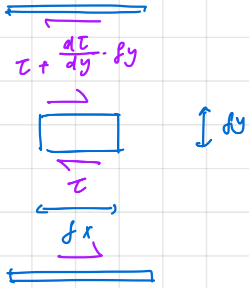
$\boxed{\nu = \frac{\mu}{\rho}}$, in $m^2 s^{-1}$ / stoke (st), $1 st = 10^{-4} m^2 s^{-1}$

4) viscous flow of newtonian fluids

applicable where system can be approximated to two parallel plates eg.  tangent, if $h << R$

1) coette flow : shear-driven flow between parallel plates





since pressure is uniform, force balance on small fluid element = 0.

$\Sigma F = 0$.

no fluid acceleration

$$\underbrace{(\tau + \frac{d\tau}{dy}\delta y)}_{\text{stress}} \cdot \underbrace{\delta x\, \delta z}_{\text{area}} = \tau \cdot \underbrace{\delta x\, \delta z}_{\text{area}}$$

$$\hookrightarrow \frac{d\tau}{dy}(\delta y \cdot \delta x\, \delta z) = 0$$

$$\frac{d\tau}{dy} = 0$$

shear stress is constant throughout the fluid

$$\tau = \frac{du}{dy} \cdot \mu = \text{constant}$$

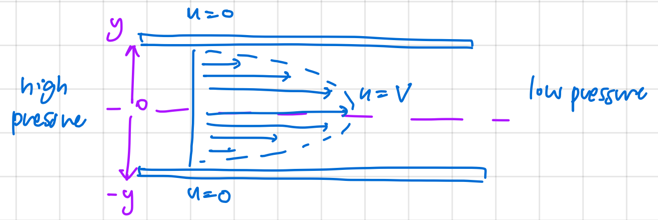
$$\int \frac{du}{dy}\, dy = \int \frac{\tau}{\mu}\, dy = Ay + B$$

↳ linear velocity profile

$$\boxed{u = V\frac{y}{h}}$$

$$\boxed{\tau = \mu \frac{du}{dy} = \mu \frac{V}{h}}$$

2) poiseulle flow : pressure-driven flow between paraller plates

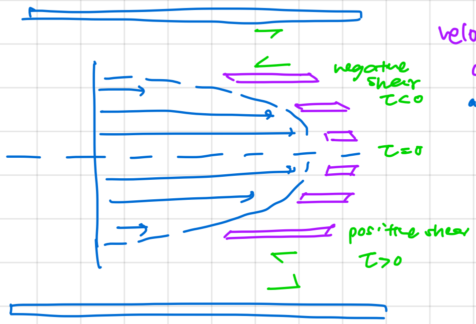


↳ parabolic velocity profile : $\boxed{u = V\left[1 - \left(\frac{y}{h}\right)^2\right]}$

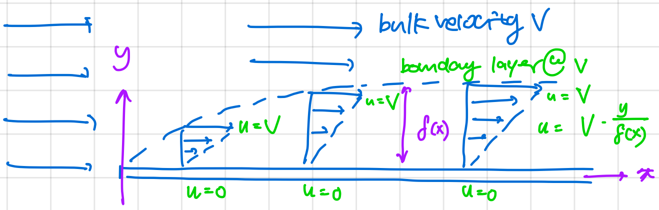
↳ $\boxed{\tau = \mu \frac{du}{dy} = -2\mu V \cdot \frac{y}{h^2}}$

velocity and shear stress vary across the fluid

as velocity gradient $\frac{du}{dy}$ decreases, so does τ.

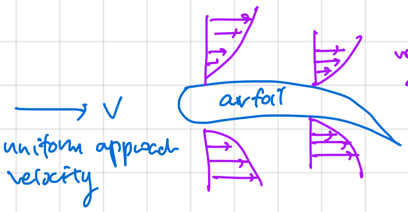


3) Formation of boundary layer



↳ when a viscous fluid flows past a sharp-edge, a thin layer adjacent to the plate surface develops in which velocity rapidly transforms from the bulk velocity V to 0 at the surface boundary.

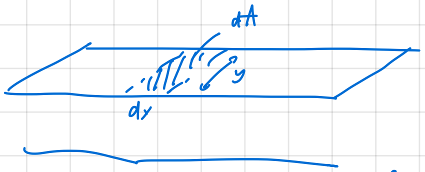
4) viscous drag



velocity gradient normal to wall gives rise to wall shear stresses

⇓

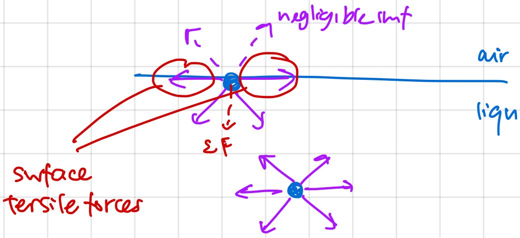
drag force



F_τ acting on area $= \int \tau \, dA = y \int \tau(x) \, dx$

5) surface tension

1) what is surface tension?



forces acting on surface liquid particles are not symmetrical as liquid gas imf are weak → net force pulling surface particles towards interior of liquid

↳ contraction of liquid surface (to minimize area, due to being less energetically favourable) → surface resists stretching

⇓

interface behaves like stretched skin, as each portion of surface exerts tensile force on adjacent portions of surface or objects in contact w surface

⇓

surface tension

coefficient of surface tension, σ

$$\boxed{\sigma = \frac{F}{L} = \frac{E_{surface}}{A}, \quad Nm^{-1} / Jm^{-2}}$$

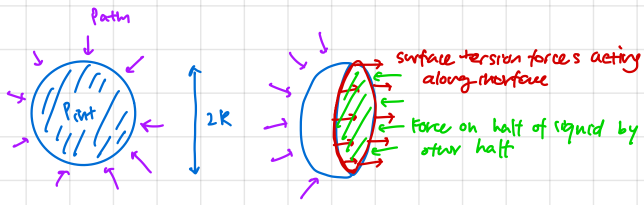
surface energy

interfacial tension force

length along interface (s) over which force acts

work done to bring molecules to surface (where there is higher energy)

2) Droplets



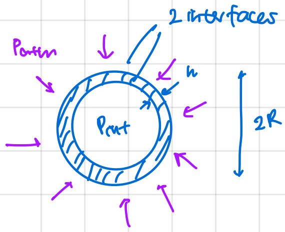
$\Sigma F = 0$

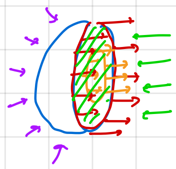
$P_{atm} \cdot \pi R^2 + \sigma \cdot 2\pi R = P_{int} \cdot \pi R^2$

projected area

$$\boxed{P_{int} - P_{atm} = \frac{2\sigma}{R}}$$

3) Bubbles





$\Sigma F = 0$

$R >>> h$

$P_{atm} \cdot \pi R^2 + \sigma 2\pi R + \sigma 2\pi (R-h) = P_{int} \cdot \pi R^2$

$$\boxed{P_{int} - P_{atm} = \frac{4\sigma}{R}}$$

4) surface tension and temperature

↳ σ decreases w temperature ⟶ $\boxed{\sigma = 0 \text{ at critical point : no distinct liquid / gas interface}}$

molecules have more energy at higher T ⟶ less energy difference between surface and internal

$\frac{E_{surface}}{A} \downarrow$

less work done

less tensile force to reconfigure arrangement

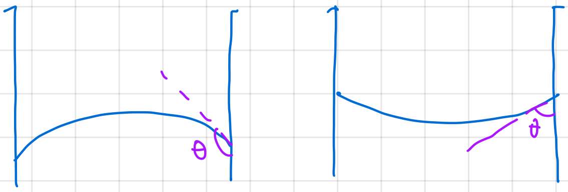
$\frac{F}{L} \downarrow$

↳ can also be decreased by surfactants

⑥ capillary effects and floating

1) contact angle : angle measured from solid surface, through liquid to the interface

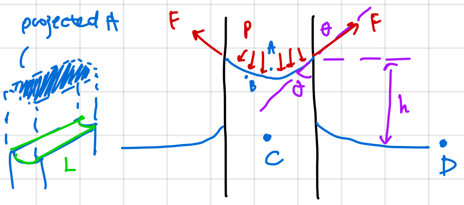
2) meniscus : curved free surface of a liquid in a narrow tube



3) capillary effect : rise or fall of a liquid in a small-diameter tube

→ wetting $\theta < 90°$, capillary rise $h > 0$
→ non-wetting $\theta > 90°$, capillary depression $h < 0$

wetting



$P_C = P_D = P_A = P_{atm}$

$P_B = P_C + \rho g h$

vertical force balance :

$(P_A - P_B) \cdot A = \underbrace{\rho g h \cdot A}_{\text{vertical force from pressure diff. } P_{AB}} = \underbrace{\sigma \cdot L \cos\theta}_{\text{vertical force from surface tension}}$

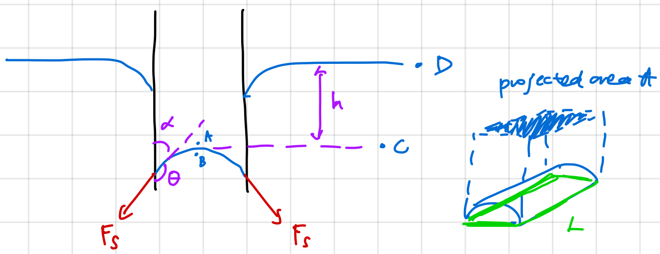
total interface length

$$\boxed{h = \frac{\sigma \cdot L \cos\theta}{\rho g A}}$$

contact angle

projected area of meniscus

non-wetting



vertical force balance :

pressure just above meniscus $= P_A = P_D$

pressure just below meniscus $= P_B = P_C = P_D + \rho g h$

pressure applied to meniscus $= P_A - P_B$
$= -\rho g h$

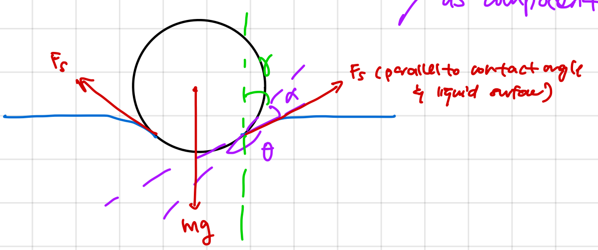
vertical force balance :

$-\rho g h \cdot A = \sigma L \cos\theta$

$$\boxed{h = -\frac{\sigma \cdot L \cos\theta}{\rho g A} = \frac{\sigma \cdot L \cos\alpha}{\rho g A}}$$

4) floating w/ surface tension

note that as cylinder sinks deeper, L does not significantly change. lift force changes as component that is vertical changes.



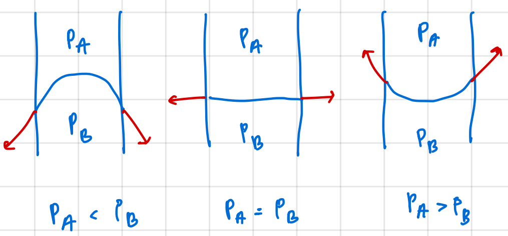
vertical force balance :

$F_s \cos(\gamma + \alpha) = mg$

$\sigma \cdot L \cos(\gamma + \alpha) = mg$

total length

5) pressure difference and interface bulging



Notes:

dynamic, large scenarios, about fluid's resistance to continuous deformation

1. comparing F_s and viscous forces

applies to small, mostly static scenarios.

simply about tensile forces

as surface changes shape slightly

2. deriving velocity profile given force balance of fluid

1) Force balance, get $\tau(y) \longrightarrow \int \frac{d\tau}{dy} \Rightarrow \tau(y)$

2) $\frac{du}{dy} = \frac{\tau(y)}{\mu} \longrightarrow \int \frac{\tau(y)}{\mu} \longrightarrow u(y)$

use boundary conditions

eg. no slip condition, surface velocity

Fluid statics

① Fluid pressure

↳ compressive normal force exerted by a fluid per unit area ; 1 atm = 101.33 kPa

1) absolute, gage and vacuum pressure — negative gage pressure

actual pressure at a given point, relative to absolute vacuum

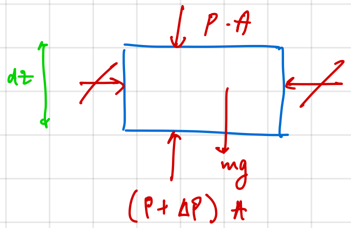
difference between absolute and local pressure

$$P_{gage} = P_{abs} - P_{atm}$$

2) pressure as a scalar : pressure at any point in a fluid is the same in all directions → magnitude, but no direction → scalar

pressure acts normal to every defined surface

3) variation of pressure w depth



vertical force balance :

$$P \cdot A + (-mg) - (P+\Delta P)A =$$

$$\cancel{P \cdot A} - \rho g \Delta z \cdot \cancel{A} - (\cancel{P} + \Delta P)\cancel{A} = 0$$

$$\Delta P = -\rho g \Delta z$$

$$\boxed{\frac{dP}{dz} = -\rho g}$$

applies for any fluid, including vapor and liquids

② Hydrostatic pressure in gases

1) ρ is constant since liquids are incompressible.

$$\int \frac{dP}{dz} dz = \int -\rho g \, dz$$

2) intuition : weight of water above it, where horizontal forces cancel out

$$\boxed{P_{bottom} = P_{top} + \rho g |\Delta z|}$$

3) Pascal's law : $P_1 = P_2$ if $z_1 = z_2$ → pressure is constant across a flat fluid interface

③ hydrostatic pressure in gases

gases are compressible, ρ not constant. can be modelled by ideal gas law $P = \rho R T$

$$\frac{dP}{dz} = -\rho g = -\frac{P}{RT} g$$

$$\int_{P_1}^{P_2} \frac{1}{P} dP = \int_{z_1}^{z_2} -\frac{g}{RT} dz$$

$$\ln \frac{P_2}{P_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{1}{T} dz$$

temperature is constant, along vertical isothermal

$$\boxed{P_2 = P_1 e^{-\frac{g(z_2 - z_1)}{RT_0}}}$$

eg. in stratosphere (11 km to 20.1 km), $T = T_0 = -56.5\,°C$

in troposphere ($z = 0$ @ sea level to $z = 11$ km), $\boxed{T = T_1 - \beta z}$ → $dT = -\beta dz$ → $dz = -\frac{1}{\beta} dT$

β = 0.00650 K/m ; T_1 = 288.15 K @ sea level

$$\int_{P_1}^{P_2} \frac{1}{P} dP = \int_{T_1}^{T_2} \frac{g}{RT\beta} dT \leftarrow \frac{dP}{dz} = -\beta \frac{dP}{dT} = -\frac{Pg}{RT}$$

$$\ln \frac{P_2}{P_1} = \frac{g}{RT\beta} \ln \frac{T_2}{T_1}$$

using sea level as a reference

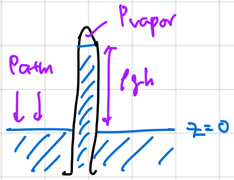
sub $T_2 = T_1 - Bz$, $P_1 = P$ @ $z = 0$

$$\boxed{P = P_{atm} \left(1 - \frac{\beta z}{T_{sea}}\right)^{\frac{g}{R\beta}}}$$

5.26 for air

④ measurement of pressure

1) Barometer : measuring atmospheric pressure



Pascal's rule : at $z=0$, pressure equal.

so $P_{atm} = \rho g h + P_{vapor}$

$\simeq \rho g h$

$\underbrace{}$ assuming tube not so narrow to have capillary effects

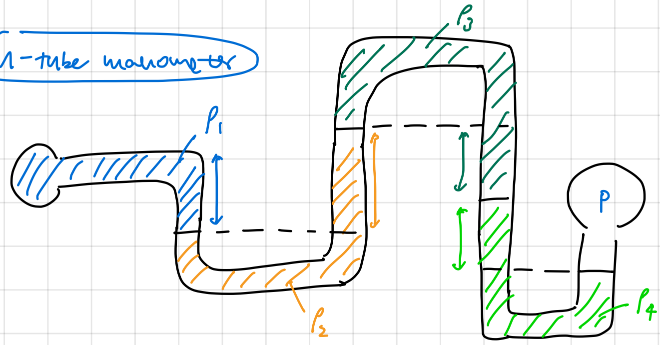
Reason for mercury

↳ mercury has high density, low h that is reasonable for measurement. $\boxed{\rho_{Hg} = 13\,595\ kg\,m^{-3}}$

↳ low vapor pressure : near vacuum @ closed upper end of barometer, negligible

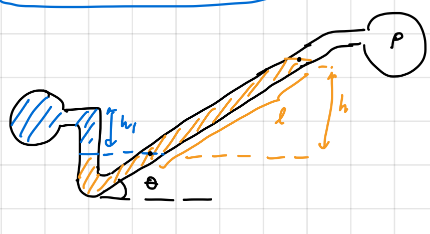
2) Manometer

U-tube manometer



if small pressure differences, Δh is small

inclined-tube manometer

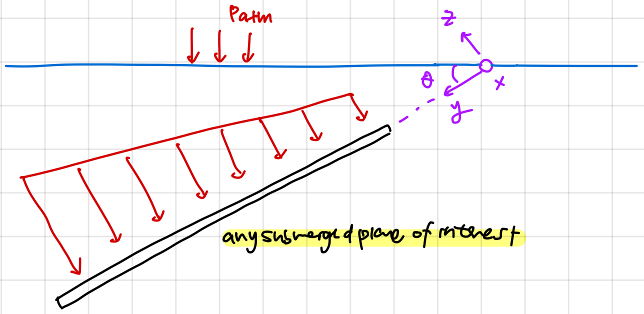


$h = \ell \sin\theta \longrightarrow$ use in calculations

θ being smaller makes ℓ bigger and more measurable

using Pascal's principle : equate pressures at equal heights and density. use $\rho g h$ to find changes in pressure from those points to the next point, form a linear eqn.

⑤ Hydrostatic force on plane submerged surfaces



1) definition of axes : plane lies in x-y plane at an angle θ w/ horizontal free surface

2) Pressure acting on plane : always perpendicular. Notice that the pressure need not start at 0, because object is fully submerged.

3) finding equivalent pressure using centroid

at any point on plate : $P = P_{atm} + \rho g h$

$= P_{atm} + \rho g \cdot y \sin\theta$

$dF = P \cdot dA$

$F_{resultant} = \int^{A} P_{atm} + \rho g \cdot y \sin\theta \; dA$

here you use area of surface of interest because P is always ⊥ to it

$= P_{atm} \cdot A + \rho g \sin\theta \cdot \int y \, dA$

centroid (x_c, y_c)

$y_c = \dfrac{\int y \, dA}{A}$ (mean y value)

$= P_{atm} \cdot A + \rho g \underbrace{\sin\theta \cdot y_c}_{h_c} \cdot A$

$= (P_{atm} + \rho g h_c) A$

$\boxed{h_c = y_c \sin\theta}$

considering gauge pressure :

$\boxed{\begin{aligned} P_{eff.} &= \rho g h_c \\ F_R &= \rho g h_c \cdot A \end{aligned}}$

4) Finding centre of pressure

resultant force must act through the point y_p where its net moment is equal to the moment sum of distributed pressure about the x-axis

$y_p \cdot F_R = \int^{A} y \cdot dF$

$= \int^{A} y \cdot P \cdot dA$

$= \int^{A} y \cdot (P_{atm} + \rho g y \sin\theta) \, dA$

$= P_{atm} \cdot y_c A + \rho g \sin\theta \int^{A} y^2 \, dA$

second moment of area : $\int y^2 \, dA$

square of 'distance away from axis' of each area → "variance" concept

$= P_{atm} \cdot y_c A + \rho g \sin\theta \cdot I_{xx,0}$

parallel axis theorem : $I_{xx,0} = I_{xx,centroid} + y_c^2 \cdot A$

simplify :

$F_R = P_{atm} + \rho g y_c \sin\theta A$

$$\boxed{y_p = y_c + \frac{I_{xx,c}}{\left(\frac{P_{atm}}{\rho g \sin\theta} + y_c\right) A}}$$

in many cases, P_{atm} acts on both sides of plane surface → only need to consider gage P. $P_{atm} = 0$ ⟶

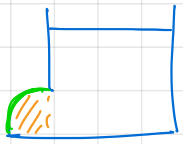
$$\boxed{y_p = y_c + \frac{I_{xx,c}}{y_c \cdot A}}$$

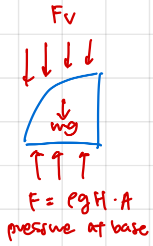
⑥ Hydrostatic forces for curved submerged surfaces

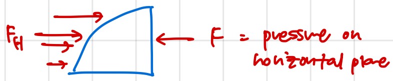
1) consider body of water adjacent to surface → analysis w FBD

2) break into F_H and F_V

eg.





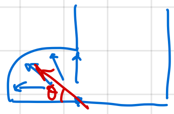


acts through x_p, centroid of water body

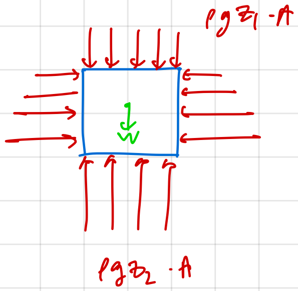
3) Solve for F_V : using FBD or normally $\boxed{F_V = W \text{ of water above}}$. Acts through centroid of water above it (no moment)

4) Solve for F_H by considering a submerged plane

5) Find resultant force : remember that force on curved surface is radial, must pass through 'centre of circle' → one reference point. Find θ : $\boxed{\tan\theta = \frac{F_V}{F_H}}$



⑦ Bouyancy



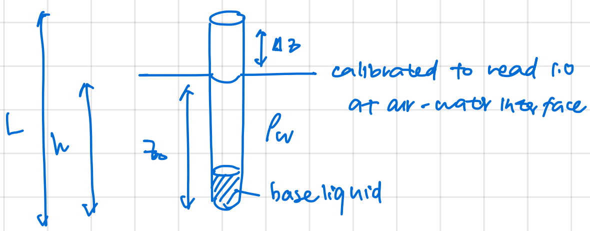
1) Force balance : $mg + \rho_w g z_1 \cdot A = \rho_w g z_2 \cdot A$

$$mg = \rho_w g (z_2 - z_1) \cdot A$$

$$\boxed{F_u = \rho_w g \cdot V_{submerged}}$$

2) Line of action : $\Sigma M = 0$ in static equilibrium. F_B and W are collinear.

3) Application : hydrometer to measure specific gravity



→ change external fluid w density ρ

$$F_u = mg = \rho g V_{sub} = \rho_w g V_{sub\,0}$$

$$\rho \cancel{g} (z_0 + \Delta z) \cdot \cancel{A} = \rho_w \cancel{g} (z_0) \cdot \cancel{A}$$

$$\boxed{\frac{\rho}{\rho_w} = SG = \frac{z_0}{z_0 + \Delta z} = \frac{h}{L}}$$

7 stability of submerged bodies

1) the notion of stability

stable

↳ small disturbance will generate a restoring force that will return body to original eqm

neutrally stable

↳ when displaced, body has no tendency to move back or continue to move away

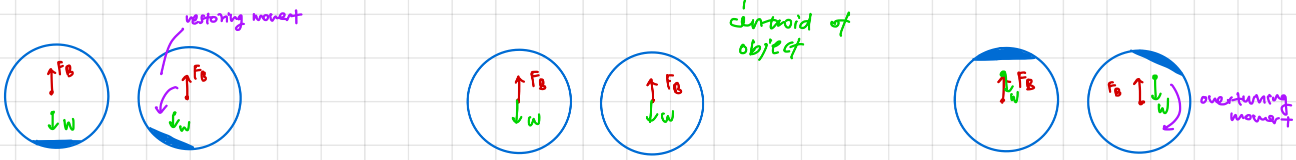
unstable

↳ body may be in eqm instantaneously, but any small disturbance cause body to increasingly diverge from initial position

2) stability of submerged body depends on relative positions of C_G and C_B

C_G — centroid of object

C_B — centroid of submerged volume

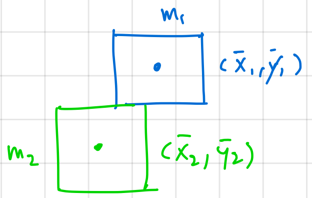


C_G below C_B : stable

C_B and C_G coincide, neutrally stable

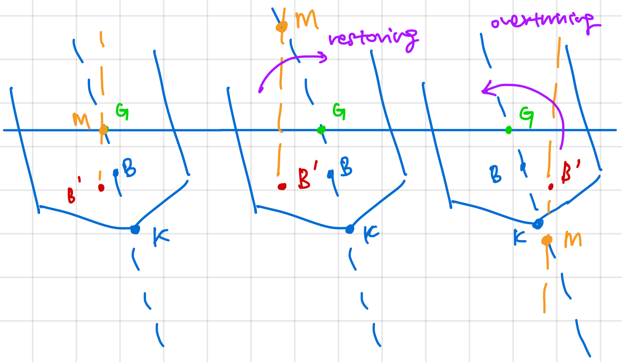
C_G above C_B, unstable

3) centre of gravity for composites : weighted average



$$\boxed{x_c = \frac{m_1\bar{x}_1 + m_2\bar{x}_2}{m_1+m_2} \qquad y_c = \frac{m_1\bar{y}_1 + m_2\bar{y}_2}{m_1+m_2}}$$

8 stability of floating bodies



neutrally stable　　stable　　unstable

1) metacentre M : point of intersection of original vertical axis with line of action of bouyancy force after disturbance of θ

2) vector GM determines stability of body

$$\boxed{\vec{GM} = \vec{KB} + \vec{BM} - \vec{KG}}$$

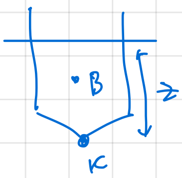
if M above G, then restoring moment, $\vec{GM} > 0$

if M below G, then overturning moment, $\vec{GM} < 0$

if M at G, then no moment, $\vec{GM} = 0$

3) Finding $\vec{KG}$: centroid

4) Finding $\vec{KB}$



↳ B at centroid of submerged volume.

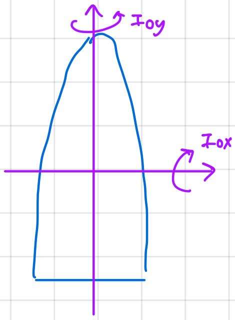
$$\rho g V = \rho_w g V_{sub}$$
$$= \rho_w g \cdot A \cdot z$$

$\rightarrow KB = \frac{z}{2}$

5) Finding $\vec{BM}$

Assumptions: angle θ is small, no vertical disturbance, emerged volume = submerged volume

$$\boxed{BM_{transverse} = \frac{I_{oy}}{V_{sub}}} \qquad \boxed{BM_{longditudinal} = \frac{I_{ox}}{V_{sub}}}$$



⑨ Factors affecting stability of floating bodies

1) GM should be large enough for stability in wage conditions

2) GM should be small enough to avoid violent rolling from restoring force

$$\text{Restoring couple} = -J_{oy}\frac{d^2\theta}{dt^2}$$

moment of inertia (mass equivalent)

$$\text{Restoring moment arm} = GM \sin\theta$$

$$\text{Restoring couple} = W \cdot GM\sin\theta = F_u\, GM \sin\theta = -J_{oy}\frac{d^2\theta}{dt^2}$$

small angle, so $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{W \cdot GM}{J_{oy}}}_{\omega^2}\theta = 0$$

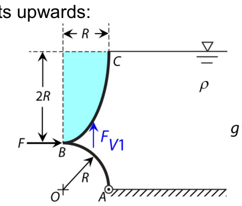
$$\boxed{\omega = \sqrt{\frac{W \cdot GM}{J_{oy}}}, \quad T = 2\pi\sqrt{\frac{J_{oy}}{W \cdot GM}}}$$

neglects damping by viscosity of water and assumes no other forces other than W and F_u.

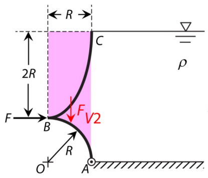
Notes

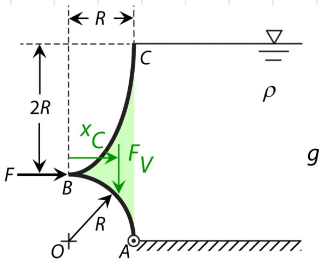
1. F_V

Vertical force F_{V1} acting on upper half of gate (parabola) is given by weight of vertical column of liquid above upper half of gate and acts upwards:

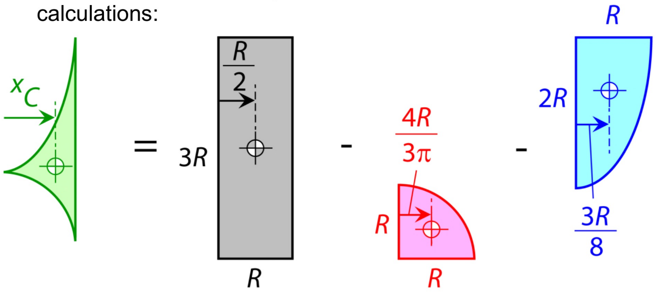


Vertical force F_{V2} acting on lower half of gate (quadrant) is given by weight of vertical column of liquid above lower half of gate and acts downwards:





- Centroid or center of mass calculations to determine line of action of vertical force F_V using first moment of area calculations:



principles of fluid flow

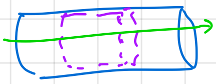
① basic concepts

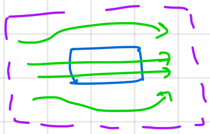
1) control volume : an open system where mass and energy can cross the boundary (control surface)

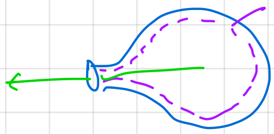
↳ any arbitrary region in space ; can be fixed shape or moving and deformable

↳ conservation laws apply to control volume

- fixed
  - stationary
    - eg. pipe
  - relative stationary
    - eg. jet engine
- deforming
  - eg. deflating balloon







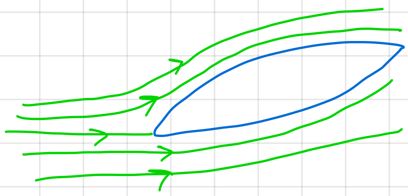
2) streamlines : lines that are tangent to velocity vectors, tracing out its instantaneous trajectory

in the case of turbulent flow, streamlines change instantaneously

↳ if flow is steady, each successive fluid particle passing through the same point follows same path, traced by streamline

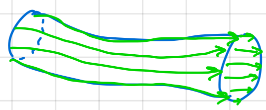
↳ streamlines cannot cross by definition, since you will have two velocity tangents

★ ↳ fluid flowing pass a solid boundary must be parallel to it, since it cannot flow in or out of it



3) stream tubes : a 3D bundle of streamlines

↳ fluid cannot flow across streamtube walls



② classification of fluid flows

1) viscous vs. inviscid flow → $\mu = 0$

↳ $\mu > 0$, viscous and frictional effects are significant

↳ all fluids are viscous, but practically there are region

s where viscous effects are negligible

↳ no-slip condition applied @ surfaces → $\boxed{\text{boundary layer}}$

flow region adjacent to solid surface in which viscous effects are significant

↳ velocity gradient

$$\tau = \overset{\neq 0}{\mu} \frac{du}{dy}$$

↳ outside boundary layer, $\mu \approx 0$, considered inviscid

2) laminar vs. turbulent flow

- laminar: highly ordered fluid motion characterised by smoothly flowing layers of fluid (stable streamlines)
- turbulent: highly disordered fluid motion with random 3D velocity fluctuations (changing streamlines)

↔ transitional flow

↳ flow that alternates between regions of laminar and turbulent

↳ characterised by Reynolds' number

$$\boxed{Re = \frac{\rho v d}{\mu}}$$

- $Re < 2300$, laminar
- $2300 < Re < 10^5$ transitional
- $Re > 10^5$ turbulent

3) compressible vs. incompressible

↳ flows where ρ varies non-negligibly

↳ liquids : non-compressible

↳ gases : depends on Mach Number

$$Ma = \frac{V}{c}$$

346 m s^{-1} , speed of sound

$Ma < 0.3$, negligible variation in ρ, less than 5%

↳ $V < 100\,ms^{-1}$, can neglect effects of ρ

$Ma < 1 \rightarrow$ subsonic flow

$Ma = 1 \rightarrow$ sonic flow

$Ma > 1 \rightarrow$ supersonic flow

↳ shock waves can form due to abrupt changes in fluid properties across a shock wave

4) internal vs. external

flow is completely bounded by a solid boundary

unbounded flow over a surface

5) steady vs. unsteady

$\frac{d}{dt} = 0$. $\frac{dm_{cv}}{dt} = 0$

flow at any point does not change with time

flow pattern varies in time

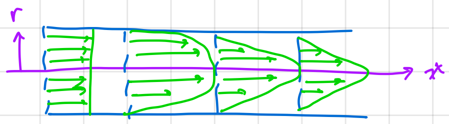
can be turbulent or laminar, depends on predictability of changes in flow pattern

6) uniform vs. non-uniform

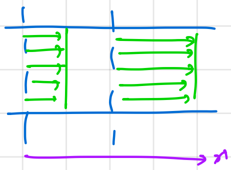
flow velocity does not vary in location

7) 1D, 2D, 3D

↳ depends on how many physical spatial dimensions velocity varies with



↳ varies in r and x ⇒ 2D flow



↳ varies only in x ⇒ 1D flow

③ conservation of mass and the continuity equation

$$\frac{dm_{cv}}{dt} = \underbrace{\Sigma\left(\frac{dm_{in}}{dt}\right)_{n_{cs}} - \Sigma\left(\frac{dm_{out}}{dt}\right)_{n_{cs}}}_{\text{normal to control surfaces}}$$

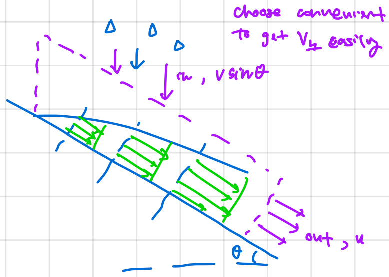
→ choose appropriate shape of control volume

mass flux : $\frac{dm}{dt} = \rho A V = \rho \int V\, dA$

volume flux : $Q = \int V dA$

average velocity $= \frac{Q}{A} = \frac{\int V dA}{A}$

choose convenient CV to get V_n easily



④ Bernoulli equation across streamlines

1) Force balance on fluid particle interpretation

kinematics

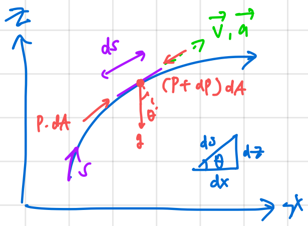
velocity $= \dfrac{ds}{dt}$

$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} \cdot dt$$

assuming steady flow, $\dfrac{\partial V}{\partial t} = 0$

$$\text{acceleration} = \frac{dV}{dt} = \frac{\partial V}{\partial s} \cdot \frac{ds}{dt} + \cancel{\frac{\partial V}{\partial t}}$$

$$= \frac{\partial V}{\partial s} \cdot V$$



Force balance on fluid particle

$\sum \vec{F}_s = m \cdot \vec{a}_s$. Assuming inviscid flow (no $\tau \cdot dA$ force)
↳ only g and P acting on fluid particle

$$\cancel{P\, dA} - (\cancel{P} + dP)\, dA - mg \sin\theta = m \vec{V} \frac{dV}{ds}$$

↳ substitute: $m = \rho \cdot dA \cdot ds$

$\sin\theta = \dfrac{dz}{ds}$

$$-dP\, \cancel{dA} - \rho\, \cancel{dA}\, \cancel{ds}\, g \frac{dz}{\cancel{ds}} = \rho\, \cancel{dA}\, \cancel{ds} \frac{dV}{\cancel{ds}} V$$

$$-dP - \rho g\, dz = \rho V dV \quad \frac{d}{dx} V^2 = 2V \frac{dV}{dx}$$

$$= \rho \tfrac{1}{2} V^2$$

$$\frac{dP}{\rho} + \frac{1}{2} V^2 + g\, dz = 0$$

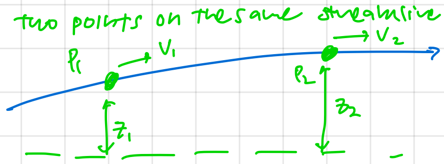
$$\frac{1}{\rho} \int dP + \frac{1}{2} V^2 + g \int dz$$

$$= \boxed{\frac{P}{\rho} + \frac{1}{2} V^2 + gz = \text{constant}}$$

static pressure applied to the fluid; velocity; vertical height; bernoulli constant along a specific streamline (unique for each line)

$$\frac{P_1}{\rho} + \frac{1}{2} V_1^2 + g z_1 = \frac{P_2}{\rho} + \frac{1}{2} V_2^2 + g z_2$$

two points on the same streamline



assuming steady, inviscid, incompressible
- steady ↓ $V(x)$, not $V(x,t)$
- inviscid ↓ $\mu = 0$, $\tau\, dA = 0$
- incompressible ↓ ρ constant

2) Energy interpretation : conservation of mechanical energy

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = E \text{ (constant along a streamline)}$$

flow energy; KE; GPE

⇒ the sum of KE, GPE and flow energy of a fluid particle is constant when compressibility and frictional effects are negligible

3) pressure interpretation : constant total pressure

$$\underbrace{P + \rho\frac{V^2}{2}}_{\text{stagnation } P} + \rho g z = P_{Total}$$

thermodynamic applied to the fluid; dynamic P; hydrostatic P

⑤ pressures along a streamline

↳ hydrostatic pressure — pressure due to weight of water above

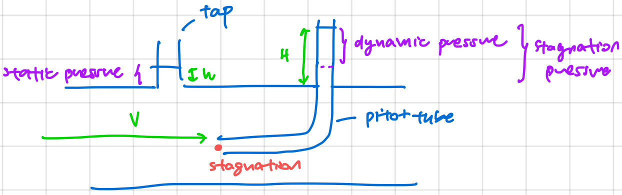
↳ static pressure — thermodynamic pressure of fluid — total pressure of fluid

↳ dynamic pressure — pressure rise when fluid in motion is brought to rest

→ stagnation pressure — pressure at stagnation point when fluid is brought to rest

↳ used to measure velocity of fluid

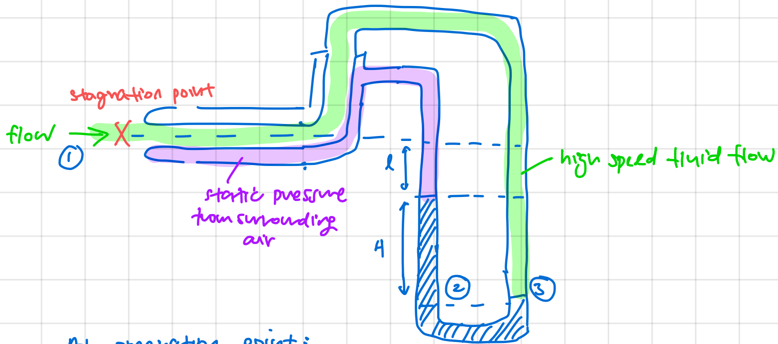
1) general pitot static tube:



$$P_1 + \rho\frac{V^2}{2} + \cancel{\rho g z} = P_{stag} + \cancel{\rho\frac{0^2}{2}} + \cancel{\rho g z}$$

$$V = \sqrt{\frac{2(P_{stag} - P_1)}{\rho}}$$

eg. manometer to find $P_{stag} - P_1$



At stagnation point:

$$P_1 + \rho_{air}\frac{V_1^2}{2} + \cancel{gz} = P_{stag} + \cancel{\rho_{air}\frac{(0)^2}{2}} + \cancel{gz}$$

At 2 and 3:

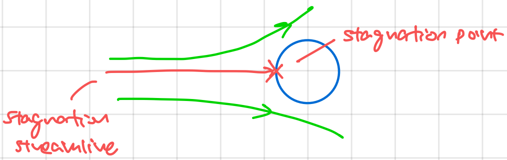
$$P_2 = P_3$$

$$P_1 + \cancel{\rho_{air} g l} + \rho_w g H = P_{stag} + \rho_{air} g(\cancel{l} + H)$$

$$P_{stag} - P_1 = (\rho_w - \rho_{air}) g H$$

$$V = \sqrt{\frac{2(P_{stag} - P_1)}{\rho}}$$

2) stagnation point vs. streamline



⑥ limitations on use of Bernoulli equation across streamlines

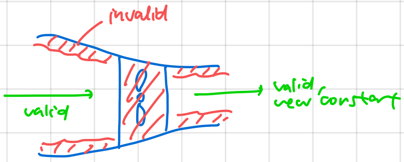
1) steady and incompressible — eg. sudden expansion

$\frac{dV}{dt} = 0$ — ρ constant, $Ma = \frac{V}{c} < 0.3$

2) Frictionless flow — eg narrow tube

3) No shaft work or heat transfer — eg. fan, heater

↳ diff bernoulli constants



↳ valid zones may not have same bernoulli constant, be careful if work is done

⑦ solving

1) define control volume.

2) $\frac{dm_{out}}{dt} = \Sigma_i \rho A V_n = \Sigma_i \rho \int V_n \, dA$

$\frac{dm_{in}}{dt} = \Sigma_i \rho A V_n = \Sigma \rho \int V_n \, dA$

$\frac{dm_{CV}}{dt}$ = differential eqn $\left(\frac{dh}{dt}, \frac{dl}{dt} \text{ etc.}\right)$ $= \frac{dm_{in}}{dt} - \frac{dm_{out}}{dt}$

relate $\frac{dx}{dt}$ to V_n — eg. $\frac{dh}{dt} = -V_1$

3) Bernoulli eqn to form another eqn of $x/h/z$ etc., V_1 and V_2 — eg. $V_2 = \left(\frac{d}{D}\right)^2 V_1 \longrightarrow$ express all in h

4) form linearly separable ODE to solve for t, V, h etc. with limits — eg. $f(V_1, V_2) \Rightarrow V_1 = \frac{dh}{dt}$, $V_1 = g(V_2)$ — substitute

eg. $V_1 = (\quad\quad)$

$= -\frac{dh}{dt} = f(h, t)$

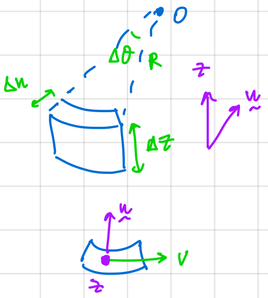
↳ separate, integrate

$$\int_{h_0}^{h_2} dh = \int_0^{T_2} g(t)\, dt$$

⑧ Bernoulli eqn normal to streamlines — same assumptions: steady, inviscid, incompressible

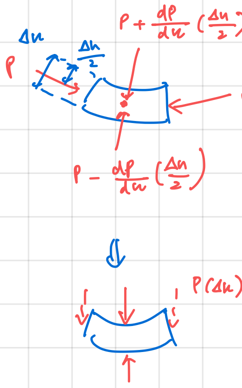
1) horizontal curved streamlines

consider a fluid particle of thickness Δz moving @ v along a local circular path with radius of curvature R.
set axes of z and $\underset{\sim}{n}$, where z is vertical, $\underset{\sim}{n}$ always points to local origin of radius of curvature.



↓

Force balance on fluid particle in radial (n) direction:



$P + \frac{dP}{dn}\left(\frac{\Delta n}{2}\right)$. pressure is lower inside (centripetal acc inwards, so $P_{outside} > P_{inside}$); $\frac{dP}{dn}$ is negative. so $P + \frac{dP}{dn}\left(\frac{\Delta n}{2}\right)$

P, does not change since horizontal

$P - \frac{dP}{dn}\left(\frac{\Delta n}{2}\right)$

⇩

$P(\Delta n)(\Delta z)\sin\frac{\Delta\theta}{2} \approx P\,\Delta n\,\Delta z\,\frac{\Delta\theta}{2}$

$$\Sigma \vec{F}_n = m\,\vec{a}_n = m\frac{V^2}{R}$$

$$-\frac{dP}{dn} R \cdot \cancel{\Delta n}\,\cancel{\Delta\theta}\,\cancel{\Delta z} = \rho R \cancel{\Delta\theta}\,\cancel{\Delta n}\,\cancel{\Delta z}\,\frac{V^2}{R}$$

$$\boxed{\frac{dP}{dn} = -\rho\frac{V^2}{R}}$$

normal to (across) streamline

pressure decreases away from center; $P_{out} > P_{in}$ to keep in local circular motion

2) vertical curved streamlines: considering g

$$\boxed{\rho g\frac{dz}{dn} + \frac{dP}{dn} = -\rho\frac{V^2}{R}}$$

↓ $\times\, dn$, integrate

$$\boxed{P + \rho\int\frac{V^2}{R}\,dn + \rho g z = \text{constant across a streamline}}$$

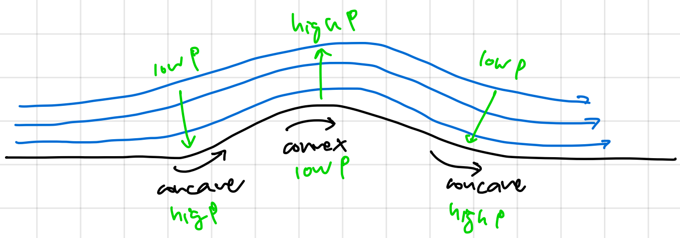
↳ vertically, pressure and g collectively provide centripetal force for curvature

3) straight streamlines

↳ radius of curvature at all local points, $R \approx \infty$.

so $\frac{dP}{dn} = -\rho\frac{V^2}{R} = 0$, $P_1 = P_2$

4) curved surfaces creating pressure difference



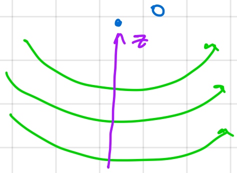
5) application to aerodynamics: airfoil and lift

6) solving

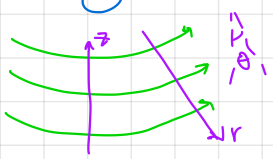
↳ equation relies on globally defined z and locally defined n → must relate them.

1. relate z and n via a global origin
2. apply eqn

e.g.



OR



$z = -r\cos\theta$

towards origin (globally upwards)

r and n are outwards from radius of origin

in this case, $z = -n = -r$

so $\rho g \frac{dz}{dn} + \frac{dp}{dn} = -\rho \frac{v^2}{R}$

$$\int_{z_1}^{z_2} \rho g \frac{dz}{\cancel{dr}} + \int_{P_1}^{P_2} \frac{dp}{\cancel{dr}} = \int_{R_1}^{R_2} \rho \frac{v^2}{\cancel{R}\, r}\, dr$$

Fluids in rigid body motion

① acceleration of a fluid particle

$$\vec{V} = u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k}$$

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

velocity vector

position

$$\Rightarrow \quad a_x = \frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial t} + w\frac{\partial w}{dt}$$

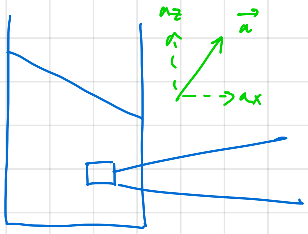
② features of fluids in rigid body motion

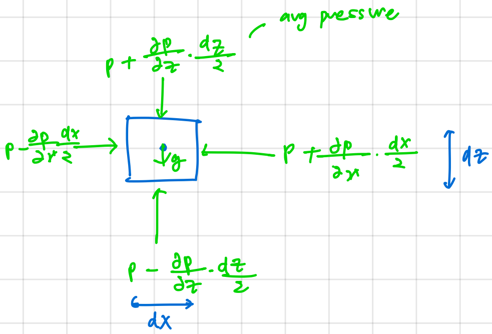
↳ all fluids have no relative motion and move like a rigid body

↳ no shear stress exists within the fluid. Only pressure and body (g) forces are considered.

↳ Fluid acceleration is given by the motion of the container

③ uniform rectilinear motion





1) Horizontal force balance by N2L:

$$\left(p - \frac{\partial p}{\partial x}\cdot\frac{dx}{2}\right)dz\cdot(1) - \left(p + \frac{\partial p}{\partial x}\cdot\frac{dx}{2}\right)dz\,(1)$$

$$= \rho(dx\cdot dz\cdot 1)\,a_x \quad \Rightarrow \boxed{\frac{\partial p}{\partial x} = -\rho a_x}$$

2) vertical force balance by N2L:

$$\left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)dx\,(1) - \left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right)dx\,(1)$$

$$-\rho g(dz\cdot dx\cdot 1) = \rho(dz\cdot dx\cdot 1)\,a_z$$

$$\Rightarrow \boxed{\frac{\partial p}{\partial z} = -\rho(a_z + g)}$$

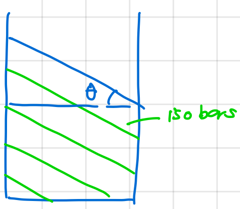
3) p at position in fluid

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz$$

setting origin @ point where p = atm

$$\int_{p_0}^{p} dp = \int_0^x \frac{\partial p}{\partial x}dx + \int_0^z \frac{\partial p}{\partial z}dz \quad \Rightarrow \quad \boxed{p - p_0 = -\rho a_x\cdot x - \rho(a_z + g)z}$$

4) isobars and angle of free surface

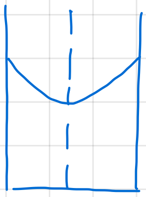


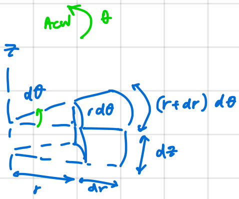
↳ isobars are parallel to free surface

along a angle p , $\boxed{\tan\theta = \frac{dz}{dx} = -\frac{a_x}{a_z + g}}$

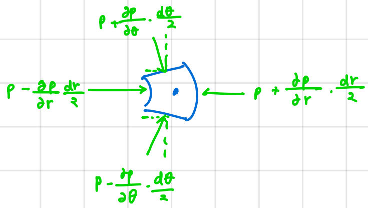
(4) rotation in a cylindrical container

↳ each fluid particle undergoes circular motion ⇒ fluid is in solid body rotation upon steady state, no relative motion ; no shear forces





1) Force balance along r direction



$$\left(p - \frac{\partial p}{\partial r}\frac{dr}{2}\right)(r\,d\theta \cdot dz) - \left(p + \frac{\partial p}{\partial r} \cdot \frac{dr}{2}\right)(r\,d\theta \cdot dz)$$

$$+ \left(p + \frac{\partial p}{\partial \theta}\frac{d\theta}{2}\right)(dr \cdot dz)\sin\frac{d\theta}{2} + \left(p - \frac{\partial p}{\partial \theta}\frac{d\theta}{2}\right)(dr \cdot dz)\sin\frac{d\theta}{2}$$

$$= -\left(\frac{\partial p}{\partial r} r \cdot dr + \frac{\partial p}{\partial r} \cdot \frac{dr}{2} dr\right) dz \cdot d\theta$$

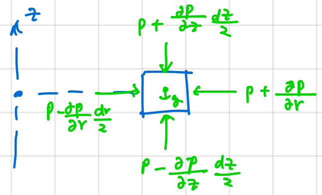
dr^2 very small . negligible

negative since r outward, but F_c is inwards

$$\approx -\frac{\partial p}{\partial r} r\,dr \cdot dz \cdot d\theta = \rho(r\,d\theta \cdot dz \cdot dr)(-\omega^2 r)$$

$$\Rightarrow \frac{\partial p}{\partial r} = \omega^2 r \rho$$

2) force balance along z direction



$\Rightarrow \boxed{\frac{\partial p}{\partial z} = -\rho g}$, including p and g forces .

3) p at position in the fluid

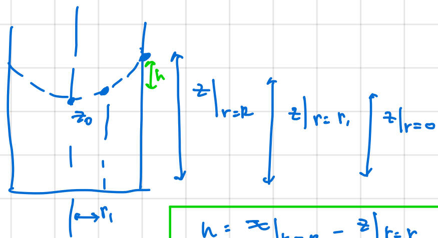
origin set where p = atm , $z = 0$ @ base of container

$$\int_{p_0}^{p} dp = \int_0^r \rho\omega^2 \cdot r\,dr - \int_{z_0}^{z} \rho g\,dz$$

$$\Rightarrow \boxed{p - p_0 = \frac{1}{2}\rho\omega^2 r^2 - \rho g(z - z_0)}$$

gauge pressure in rotation increased by r , decreases in depth above z_0 , increases in depth below z_0

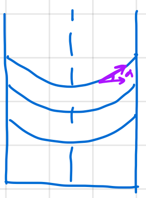
$$\boxed{z = \frac{1}{2g}\omega^2 r^2 + z_0 \text{ at surface where } p = p_0}$$



$$\boxed{h = z|_{r=R} - z|_{r=r_1}}$$

4) isobars and angles

↳ at any set p (isobar) , $\boxed{\tan\alpha = \frac{dz}{dr} = \frac{r\omega^2}{g}}$



all isobars are parallel

⑤ solving

1) rectilinear motion

- total force on surfaces
  - ⇓ integral
- spillage conditions
  - min acceleration to spill
    ↳ match shape of container r_1
  - max acceleration volume left match shape of container
  - use $\tan\theta$ or $p = p_0$, $p - p_0 = 0$
- pressure @ a point

$$\boxed{\begin{aligned} p - p_0 &= -\rho a_x \cdot x - \rho(a_z + g)z \\ \tan\theta &= \frac{dz}{dx} = -\frac{a_x}{a_z + g} \end{aligned}}$$

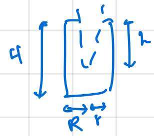
2) rotational motion

↓

conservation of volume to find z_0

$V_{cylinder} - V_{paraboloid}$

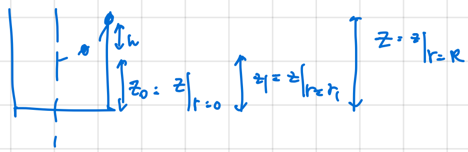
$\pi R^2 H - \frac{1}{2}\pi r^2 h$



- spillage conditions
- volume left

r, z to match shape, solve for z_0, then w → find V'

$$\boxed{\begin{aligned} p - p_0 &= \tfrac{1}{2}\rho w^2 r^2 - \rho g(z - z_0) \\ z &= \frac{1}{2g} w^2 r^2 + z_0 \text{ at surface where } p = p_0 \\ \tan\alpha &= \frac{dz}{dr} = \frac{r w^2}{g} \\ h &= z|_{r=R} - z|_{r=r_1} \end{aligned}}$$



momentum equation for a fluid system

① momentum equation for a 3D control volume

1) momentum flux through a surface

↳ at a small surface dA: $\frac{d(m\vec{v})}{dt} = \frac{dm}{dt} \cdot \vec{v}$ → parallel to specific direction

↙ normal to control surface

$= \rho \cdot dA \cdot u_{\perp} \cdot d\vec{v}$

↳ momentum flux $\doteq$ $\boxed{\frac{d(m\vec{v})}{dt} = \int \rho\, v_{\perp} \cdot \vec{v}\, dA = \rho A v_{\perp}^2 \text{ if uniform velocity distribution}}$

normal to surface

2) Force acting on a fluid

$$\vec{F}_{i \text{ on fluid mass}} = \frac{d(m\vec{v})}{dt} \text{ of control volume} = \int_{out} \rho \underbrace{v_{\perp}}_{\dot{m}} \cdot \underbrace{\vec{v_i}}_{\vec{v}}\, dA - \int_{in} \rho \underbrace{v_{\perp}}_{\dot{m}} \underbrace{\vec{v_i}}_{\vec{v}}\, dA$$

in a certain direction

$$= \underbrace{\Sigma_{out} \int \rho v_{\perp} \vec{v_i}\, dA}_{\text{sum of momentum flux out that direction through all branches}} - \underbrace{\Sigma_{in} \int \rho v_{\perp} \vec{v_i}\, dA}_{\text{sum of momentum flux in, for that direction through all branches}}$$

$\vec{F} = \sqrt{\Sigma \vec{F_i}^2}$

$\theta = \tan^{-1} \frac{F_x}{F_y}$

② applications of the momentum equation

1) general case

↳ consider appropriate CV to include reaction forces $\quad F_{\text{on CV by } x} = F_{\text{by } x \text{ on CV}}$

↳ consider all forces present : body (g), pressure (exposure to atm) and reactions

normally solving for this

$mg = \rho V g \qquad \Sigma p \cdot A \qquad$ momentum flux

↳ N2L, force balance ⇒ solve $\boxed{mg + \Sigma p \cdot A + \Sigma R_i = \text{momentum flux changes}}$

2) Force caused by jet striking a surface

stationary surface ← moving surface

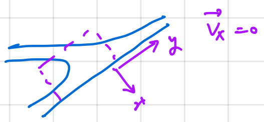
inward velocity becomes relative, $(v-u)$

⇒ all velocities become relative

↳ select a suitable CV, preferably one such that $\vec{v}_{out} = 0$ for one of the directions

↳ solve for F

eg.



3) Force due to deflection of jet by curved vane

exposed to atm

↳ apply momentum equation in x and y directions

4) Force due to flow through a tapering pipe bend

↳ consider pressure forces acting at control surfaces

↳ force balance to solve for reaction

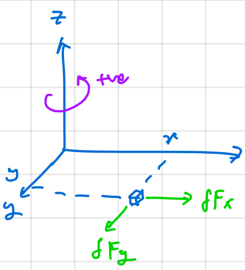
$R_i \;+\; F_{i\text{ on fluid}} \;+\;$ pressure forces $=0 \;\Rightarrow$ solve

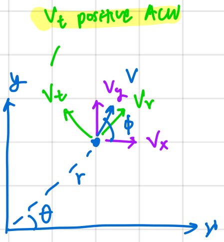
5) reaction of a jet and rocket propulsion

↳ apply momentum equation to solve for F

③ Angular momentum and torque

1) angular momentum and torque





$$\delta H = r \times \delta m \vec{V}_{tangential}$$

$$= \delta m(r\cos\theta\, V\sin\phi - r\sin\theta\, V\cos\phi)$$

$$= \delta m(xV_y - yV_x)$$

$$\frac{d}{dt}(\delta H) = \frac{d}{dt}\left(\delta m(xV_y - yV_x)\right)$$

$$= \frac{dm}{dt}\left(\cancel{V_xV_y} - \cancel{V_yV_x} + x\frac{dV_y}{dt} - y\frac{dV_x}{dt}\right)$$

$$= x\cdot\delta F_y - y\,\delta F_x$$

$$= \delta T_z$$

δT_z, torque on δm about z axis:

$$\delta T_z = x\cdot\delta F_y - y\,\delta F_x$$

$$= \delta m\cdot\left(x\frac{dV_y}{dt} - y\frac{dV_x}{dt}\right)$$

resultant external torque about an axis is equal to rate of change of angular momentum

$$\therefore \delta\vec{T}_i = \frac{d}{dt}(\delta\vec{H}_i)$$

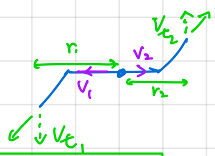
$$\delta T_z = \frac{d}{dt}[\delta m(xV_y - yV_x)]$$

$$= \frac{d}{dt}(\delta m\cdot r\cdot V_t)$$

$$\boxed{\begin{aligned} T_z\big|_{\text{on a control volume}} &= \int_{in}^{out}\cancel{\frac{d}{dt}}(rV_t\,\delta m) \qquad \delta m \to \rho Q\,\delta t \\ &= [\rho Q\, r V_t]_{in}^{out} \\ &= \rho Q[(rV_t)_{out} - (rV_t)_{in}] \quad \text{for steady flow only} \end{aligned}}$$

2) Torque on a control volume

Assuming constant ω, then $\Sigma M = 0$.



$$\boxed{\begin{aligned} &[(\rho Q rV_t)_{out} - (\rho Q rV_t)_{in}] + \Sigma M_{external} = 0 \\ &V_t = V_{flow} + r\omega \end{aligned}}$$

Dimensional analysis

① Dimensional analysis and physical similarity

1) physical similarity

geometric

↳ model must be same shape, but scaled down proportionately

↳ length-scale equivalence

$$\frac{L_m}{L_p} = \alpha \qquad \frac{A_m}{A_p} = \alpha^2 \qquad \frac{V_m}{V_p} = \alpha^3$$

kinematic

↳ velocity at any point in model must be pointing in same direction and scaled proportionately to prototype

↳ time-scale equivalence

$$\frac{\vec{u}_{m_1}}{\vec{u}_{p_1}} = \frac{\vec{u}_{m_2}}{\vec{u}_{p_2}} = \text{constant velocity ratio}$$

$$\frac{\vec{a}_{m_1}}{\vec{a}_{p_1}} = \frac{\vec{a}_{m_2}}{\vec{a}_{p_2}} = \text{constant acceleration ratio}$$

dynamic

↳ all forces in model scale constantly to prototype

↳ if dynamically similar, then also kinematically and geometrically similar, but converse is not necessarily true

2) Dimensional analysis

↳ to generate non-dimensional parameters that help in the design of experiments and reporting of results

↳ to obtain scaling laws

↳ to predict relationships between parameters

↳ underlying principle : similarity. Prototype and model systems must be physically similar for dimensionless parameters in either system to be equal.

$$\underbrace{\Pi_1}_{\text{dependent } \Pi} = f(\underbrace{\Pi_2, \Pi_3 \ldots}_{\text{independent } \Pi})$$

↳ if dynamically similar, then all Π groups are equal, vice versa

② Method of repeating variables

1) algorithm → choosing repeating variables

1. identify relevant variables to the problem
2. count all independent parameters (eg. $A = \pi r^2$, A is dependent on r), n
3. count no. of primary dimensions present, j ⇒ no. of repeating variables
4. no. of Π groups = $n - j$
5. choose j repeating variables
6. construct Π groups

eg. $z = f(t, w_0, z_0, g) \rightarrow L, T \rightarrow j = 2$

$n = 5$, $j = 2$, $n_\Pi = 5 - 2 = 3$

repeating variables : w_0 and z_0

$$\left.\begin{aligned} \Pi_1 &= z\, w_0^{\,a} z_0^{\,b} \\ \Pi_2 &= t\, w_0^{\,a} z_0^{\,b} \\ \Pi_3 &= g\, w_0^{\,a} z_0^{\,b} \end{aligned}\right\} \rightarrow \quad \Pi_1 = \frac{z}{z_0} \quad \Pi_2 = \frac{w_0 t}{z_0} \quad \Pi_3 = \frac{g z_0}{w_0^{\,2}} \quad \rightarrow \quad \Pi_1 = f(\Pi_2, \Pi_3)$$

(choosing repeating variables)

1. cannot be dependent variable
2. chosen j parameters cannot by themselves form a dimensionless group
3. chosen j parameters must represent all primary dimensions present
4. Never pick already dimensionless parameter.
5. Never pick two parameters related to each other by an exponent
6. choose dimensional constants over variables eg. g vs t
7. pick a length, a v and a mass or density. (less dimensions)

2) manipulating Π groups

1. can replace any Π w/ a product or quotient w/ any other Π

eg. $\Pi_3 \Rightarrow \frac{\Pi_3}{\Pi_1}$

2. can multiply Π or raise power of Π to any dimensionless value

eg. $\Pi \Rightarrow k \sin(\Pi^c)$

3. can substitute any parameter in Π w/ other parameters of same dimensions

eg. $Re = \frac{vL}{\nu} \Rightarrow \frac{vD}{\nu}$

4. can relate Π groups obtained using one set of repeating variables to another set of entirely different repeating variables

$$\Pi_1^* = \frac{\Pi_1}{\Pi_2^2} \qquad \Pi_2^* = \frac{\Pi_2\Pi_3}{\Pi_1} \qquad \Pi_3^* = \frac{1}{\Pi_3}$$

$$\Pi_1 = \frac{1}{\Pi_1^*\Pi_2^{*2}\Pi_3^{*2}} \qquad \Pi_2 = \frac{1}{\Pi_1^*\Pi_2^*\Pi_3^*} \qquad \Pi_3 = \frac{1}{\Pi_3^*}$$

3 incomplete similarity and distorted models

1) dynamic similarity cannot be achieved $\Rightarrow$ not all generated Π groups are equal $\rightarrow$ choose

A wind tunnel testing

↳ operate at Re independence (full turbulent flow) and then extrapolate

B flows w/ free surfaces

↳ generally $Re = \frac{vD}{\nu}$ and $Fr = \frac{v}{\sqrt{gD}}$ are important for these problems.

check if model and prototype have diff ν. if they don't then likely incomplete

↳ choice of equivalence depends on which ratio will govern dynamic similarity.

where μ and ρ are less significant, Fr.

$\frac{\text{inertial force}}{\text{gravitational force}}$

where g is more significant, Re.

$\frac{\text{inertial force}}{\text{viscous force}}$

2) geometric similarity cannot be achieved

↳ geometric similarity is not just dimensions, but also surface roughness

↳ due to limitations of fabrication, may not be able to scale surface roughness

Name	Definition	Ratio of Significance
Archimedes number	$\mathrm{Ar} = \frac{\rho_s g L^3}{\mu^2}(\rho_s - \rho)$	$\frac{\text{Gravitational force}}{\text{Viscous force}}$
Aspect ratio	$\mathrm{AR} = \frac{L}{W}$ or $\frac{L}{D}$	$\frac{\text{Length}}{\text{Width}}$ or $\frac{\text{Length}}{\text{Diameter}}$
Biot number	$\mathrm{Bi} = \frac{hL}{k}$	$\frac{\text{Surface thermal resistance}}{\text{Internal thermal resistance}}$
Bond number	$\mathrm{Bo} = \frac{g(\rho_f - \rho_v)L^2}{\sigma_s}$	$\frac{\text{Gravitational force}}{\text{Surface tension force}}$
Cavitation number	$\mathrm{Ca}\ (\text{sometimes } \sigma_c) = \frac{P - P_v}{\rho V^2}$ $\left(\text{sometimes } \frac{2(P - P_v)}{\rho V^2}\right)$	$\frac{\text{Pressure} - \text{Vapor pressure}}{\text{Inertial pressure}}$
Darcy friction factor	$f = \frac{8\tau_w}{\rho V^2}$	$\frac{\text{Wall friction force}}{\text{Inertial force}}$
Drag coefficient	$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$
Eckert number	$\mathrm{Ec} = \frac{V^2}{c_P T}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$
Euler number	$\mathrm{Eu} = \frac{\Delta P}{\rho V^2}\left(\text{sometimes } \frac{\Delta P}{\frac{1}{2}\rho V^2}\right)$	$\frac{\text{Pressure difference}}{\text{Dynamic pressure}}$
Fanning friction factor	$C_f = \frac{2\tau_w}{\rho V^2}$	$\frac{\text{Wall friction force}}{\text{Inertial force}}$
Fourier number	$\mathrm{Fo}\ (\text{sometimes } \tau) = \frac{\alpha t}{L^2}$	$\frac{\text{Physical time}}{\text{Thermal diffusion time}}$
Froude number	$\mathrm{Fr} = \frac{V}{\sqrt{gL}}\left(\text{sometimes } \frac{V^2}{gL}\right)$	$\frac{\text{Inertial force}}{\text{Gravitational force}}$
Grashof number	$\mathrm{Gr} = \frac{g\beta\lvert\Delta T\rvert L^3\rho^2}{\mu^2}$	$\frac{\text{Buoyancy force}}{\text{Viscous force}}$
Jakob number	$\mathrm{Ja} = \frac{c_p(T - T_{\text{sat}})}{h_{fg}}$	$\frac{\text{Sensible energy}}{\text{Latent energy}}$
Knudsen number	$\mathrm{Kn} = \frac{\lambda}{L}$	$\frac{\text{Mean free path length}}{\text{Characteristic length}}$
Lewis number	$\mathrm{Le} = \frac{k}{\rho c_p D_{AB}} = \frac{\alpha}{D_{AB}}$	$\frac{\text{Thermal diffusion}}{\text{Species diffusion}}$
Lift coefficient	$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$

(Continued)

Name	Definition	Ratio of Significance
Mach number	$\mathrm{Ma}\ (\text{sometimes } M) = \frac{V}{c}$	$\frac{\text{Flow speed}}{\text{Speed of sound}}$
Nusselt number	$\mathrm{Nu} = \frac{Lh}{k}$	$\frac{\text{Convection heat transfer}}{\text{Conduction heat transfer}}$
Peclet number	$\mathrm{Pe} = \frac{\rho L V c_p}{k} = \frac{LV}{\alpha}$	$\frac{\text{Bulk heat transfer}}{\text{Conduction heat transfer}}$
Power number	$N_P = \frac{\dot{W}}{\rho D^5 \omega^3}$	$\frac{\text{Power}}{\text{Rotational inertia}}$
Prandtl number	$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}$	$\frac{\text{Viscous diffusion}}{\text{Thermal diffusion}}$
Pressure coefficient	$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho V^2}$	$\frac{\text{Static pressure difference}}{\text{Dynamic pressure}}$
Rayleigh number	$\mathrm{Ra} = \frac{g\beta\lvert\Delta T\rvert L^3\rho^2 c_p}{k\mu}$	$\frac{\text{Buoyancy force}}{\text{Viscous force}}$
Reynolds number	$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$	$\frac{\text{Inertial force}}{\text{Viscous force}}$
Richardson number	$\mathrm{Ri} = \frac{L^5 g \Delta\rho}{\rho \dot{V}^2}$	$\frac{\text{Buoyancy force}}{\text{Inertial force}}$
Schmidt number	$\mathrm{Sc} = \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$	$\frac{\text{Viscous diffusion}}{\text{Species diffusion}}$
Sherwood number	$\mathrm{Sh} = \frac{VL}{D_{AB}}$	$\frac{\text{Overall mass diffusion}}{\text{Species diffusion}}$
Specific heat ratio	$k\ (\text{sometimes } \gamma) = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$
Stanton number	$\mathrm{St} = \frac{h}{\rho c_p V}$	$\frac{\text{Heat transfer}}{\text{Thermal capacity}}$
Stokes number	$\mathrm{Stk}\ (\text{sometimes St}) = \frac{\rho_p D_p^2 V}{18\mu L}$	$\frac{\text{Particle relaxation time}}{\text{Characteristic flow time}}$
Strouhal number	$\mathrm{St}\ (\text{sometimes S or Sr}) = \frac{fL}{V}$	$\frac{\text{Characteristic flow time}}{\text{Period of oscillation}}$
Weber number	$\mathrm{We} = \frac{\rho V^2 L}{\sigma_s}$	$\frac{\text{Inertial force}}{\text{Surface tension force}}$

Pipe flow

① Basics of pipe flow

1) Turbulent and laminar flow classification

↳ flow regime depends largely on ratio of inertial forces to viscous force ⟶ Re

$Re < 2300$	$2300 < Re < 10^5$	$Re > 10^5$
laminar	transitional	turbulent

↳ viscous forces large enough to suppress random fluctuations

↳ inertial forces ($\propto \rho v^2$) are large relative to viscous forces ($\propto \mu$).

↳ viscous forces unable to prevent random and rapid fluctuations of the fluid.

transition also depends on surface roughness, pipe vibrations, etc.

in practice, transitional flow $2300 \le Re \lesssim 4000$

laminar — turbulent

2) Hydraulic diameter

↳ For non-circular pipes, $Re = \frac{VL}{\nu}$, characteristic length L is based on hydraulic diameter

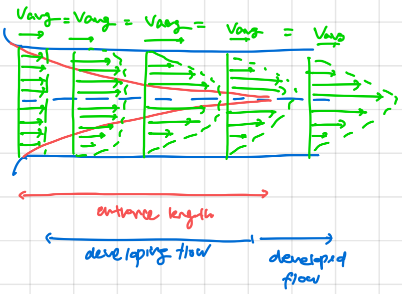
$$\boxed{D_h = \frac{4A_{cross\text{-}sectional}}{P_{wet}}}$$

— reduces to just D for circular pipes

wetted perimeter

② Entrance region

1) formation of a boundary layer — the region of flow in which the viscous shearing forces caused by fluid viscosity are significant



1. Fluid enters at uniform velocity. No slip condition means fluid particles in contact w/ surface must stop.
2. Adjacent layers slow down due to fluid friction. To make up of velocity reduction (continuity), fluid at midsection speeds up. Above boundary layer, viscous effects are negligible (but still has impact - essentially transmits τ so that v↓ in bulk layers, leading to change of velocity profile in subsequent segments) so uniform fluid velocity.
3. boundary layer grows ($\frac{du}{dy}$ becomes less sharp) due to gradual transmission of τ, until entry length where $\frac{\partial u(r,x)}{\partial x} = 0 \Rightarrow u = u(r)$, not $u(r,x)$

2) entry length : $\frac{L}{D} = f(Re)$

laminar flow

$$\boxed{\frac{L}{D} \approx 0.06\,Re}$$

when $Re = 2300$, $L = 138\,D$

turbulent flow

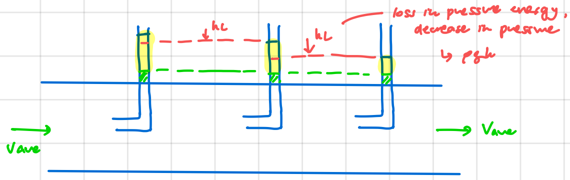
$$\boxed{\frac{L}{D} \approx 4.4\,Re^{\frac{1}{6}}}$$

③ energy loss and friction in a circular pipe

use hydraulic diameter for non-circular

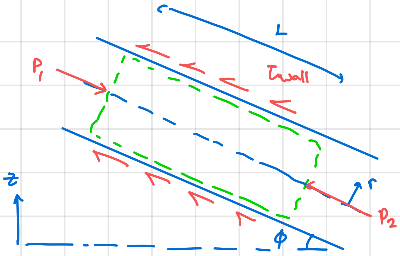
1) head loss, h_L

↳ quantifies loss in pressure (pressure energy) due to fluid friction at the surface of pipe (and transmitted through fluid



$$\underbrace{\overset{\text{decrease due to } h_L}{\frac{P_1}{\rho}} + \overset{\text{constant due to continuity}}{\frac{V^2}{2}}}_{\text{stagnation P}\downarrow} + gz = K$$

2) relating h_L to τ_{wall}



energy loss :

$$\frac{p_1}{\rho g} + \cancel{\frac{\bar{V}_1^2}{2g}} + z_1 = \frac{p_2}{\rho g} + \cancel{\frac{\bar{V}_2^2}{2g}} + z_2 + h_f$$

$$h_f = -\Delta z + \frac{\Delta p}{\rho g}$$

momentum equation

$$\underbrace{P_1 \pi R^2 - P_2 \pi R^2}_{\Delta p (A)} + \underbrace{\rho \pi R^2 L \cdot g \sin\phi}_{mg_L} - \underbrace{\tau_w 2\pi R \cdot L}_{\text{shear force}} = \frac{dm}{dt}(V_2 - V_1) = 0$$

Δz

$$\frac{-\Delta p \pi R^2}{\rho g} - \frac{\rho g \pi R^2 \Delta z}{\rho g} = \frac{\tau_w 2\pi R L}{\rho g}$$

$$h_f = -\left(\Delta z + \frac{\Delta p}{\rho g}\right) = \frac{2\tau_{wall} L}{\rho g R}$$

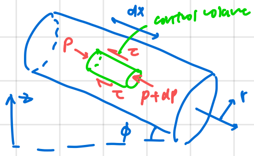
define $\boxed{f = \frac{8\tau_{wall}}{\rho \bar{V}^2}}$ avg. V since we assume uniform velocity profile

$$\boxed{h_f = f \frac{L}{D} \cdot \frac{\bar{V}^2}{2g} = -\Delta z - \frac{\Delta p}{\rho g}}$$

valid for duct flows of any cross section and both laminar & turbulent flow

4 laminar flow in pipes (fully developed)

1) velocity profile : poiseuille flow



momentum equation

$$p\pi r^2 - (p+dp)\pi r^2 + \rho \pi r^2 dx \cdot g \cdot \sin\phi - 2\pi r\, dx \cdot \tau = \frac{dm}{dt}(V_2 - V_1) = 0$$

$$\Rightarrow \tau = -\frac{r}{2}\frac{d}{dx}(p + \rho g z) = -\mu \frac{du}{dr}$$

$$u = \int \frac{du}{dr} dr = \int \frac{r}{2\mu}\frac{d}{dx}(p + \rho g z)\, dr$$

at $r = R$, $u = 0$

at $r = 0$, $u = u_{max}$

$-\frac{\Delta P}{L} - \rho g \Delta z$

$$\boxed{\begin{aligned} u &= \frac{1}{4\mu}\left(-\frac{d}{dx}(p + \rho g z)\right)(R^2 - r^2) \\ &= u_{max}\left(1 - \frac{r^2}{R^2}\right) \\ u_{max} &= -\frac{R^2}{4\mu}\left(\frac{d}{dx}(p + \rho g z)\right) \end{aligned}}$$

2) volume flow rate and $\bar{V}$

$$\boxed{\begin{aligned} Q &= \frac{\pi R^4}{8\mu}\left(-\frac{d}{dx}(p + \rho g z)\right) \\ &= \frac{1}{2}\pi R^2 u_{max} \end{aligned}}$$

$$\boxed{\bar{V} = \frac{Q}{A} = \frac{1}{2}u_{max}}$$

3) friction factor, head loss and pressure drop

τ_{wall}, f, h

$$\boxed{\tau_{wall} = \left|\mu \frac{du}{dr}\right|_{r=R} = \frac{2\mu u_{max}}{R} = \frac{4\mu \bar{V}}{R}}$$

$$\boxed{f_{laminar} = \frac{8\tau_w}{\rho \bar{V}^2} = \frac{64}{Re}}$$

$$\boxed{\begin{aligned} h_{laminar} &= f_{laminar} \cdot \frac{L}{D}\frac{\bar{V}^2}{2g} = \frac{128 \cdot \mu L Q}{\pi \rho g D^4} \\ &= -\Delta z - \frac{\Delta p}{\rho g} \end{aligned}}$$

pressure drop

$$Q = \frac{\pi R^4}{8\mu}\left(-\frac{d}{dx}(p + \rho g z)\right)$$

$$= \frac{\pi R^4}{8\mu}\left(-\frac{\Delta p}{L} - \frac{\Delta z}{L}\cdot \rho g\right)$$

$$\boxed{-\Delta p = \frac{128\mu}{\pi}\frac{LQ}{D^4} = \rho g h_L = f\frac{L}{D}\cdot\frac{\rho \bar{V}^2}{2}}$$

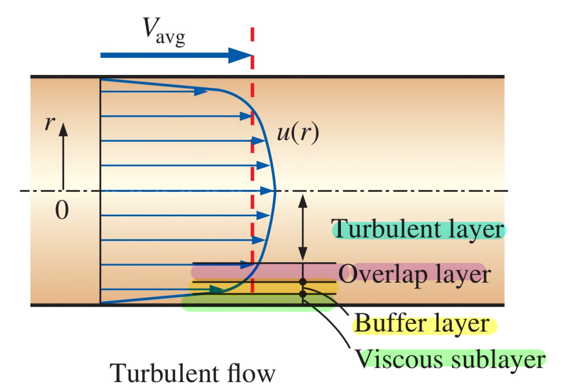
4) f in non-circular pipes

Tube Geometry	a/b or θ°	Friction Factor f
Circle	—	64.00/Re
Rectangle	a/b	
	1	56.92/Re
	2	62.20/Re
	3	68.36/Re
	4	72.92/Re
	6	78.80/Re
	8	82.32/Re
	∞	96.00/Re
Ellipse	a/b	
	1	64.00/Re
	2	67.28/Re
	4	72.96/Re
	8	76.60/Re
	16	78.16/Re
Isosceles triangle	θ	
	10°	50.80/Re
	30°	52.28/Re
	60°	53.32/Re
	90°	52.60/Re
	120°	50.96/Re

⑤ Turbulent flow in pipes

↳ turbulent flow is characterized by disorderly and rapid fluctuations of swirling regions (eddies)
↳ eddies lead to much more rapid and disorderly transfer of momentum, energy and mass

1) layers in turbulent velocity profile



1. viscous sublayer

↳ wall dampens any eddy motion, flow layer is essentially laminar
↳ very thin layer, velocity profile almost linear since it jumps from 0 to almost bulk fluid within very short distance
↳ $\boxed{\tau_{wall} = \mu \frac{du}{dr}\Big|_{r=R} \approx \mu \frac{u}{R}}$

↳ velocity profile reported by friction velocity

$\boxed{u^* = \sqrt{\frac{\tau_{wall}}{\rho}}}$ (friction 'velocity') $\longrightarrow$ $\boxed{\frac{u}{u^*} = \frac{R u^*}{\nu}}$

2. buffer layer
↳ turbulent effects are becoming significant. but flow is largely dominated by viscous effects

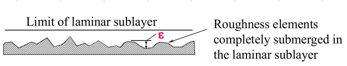
3. overlap layer
↳ turbulent effects are much more significant, but still not dominant

4. turbulent layer
↳ turbulent effects dominate viscous (molecular) effects

2) hydraulic smoothness of walls

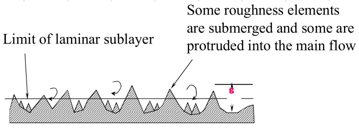
smooth

↳ roughness elements are submerged in laminar sublayer (no effects on laminar flow)
↳ no effects on friction overall



$\frac{\varepsilon u^*}{\nu} < 5$

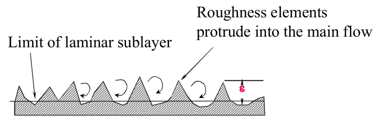
transitional rough



$3 < \frac{\varepsilon u^*}{\nu} < 70$

rough

↳ roughness elements protrude into main flow, causing flow to break into vortices or eddies



$\frac{\varepsilon u^*}{\nu} > 70$

3) properties of flow in pipes of different roughness

smooth pipes

1. velocity profile :

$$\frac{u(r)}{u^*} \approx 2.5 \ln \frac{(R-r)u^*}{\nu} + 5.0 \quad , \quad u^* = \sqrt{\frac{\tau_w}{\rho}}$$

2. avg. and max velocity

$$\frac{\bar{V}}{u_{max}} \approx \frac{1}{1+1.33\sqrt{f}}$$

3. friction factor

$$\frac{1}{\sqrt{f}} = 2.0 \lg \left(Re\sqrt{f} \right) - 0.8$$

or $f = 0.316\, Re^{-\frac{1}{4}}$, $4000 < Re < 10^5$

4. drop in pressure if pipe is horizontal

$$-\Delta p \approx 0.158\, L \rho^{\frac{3}{4}} \mu^{\frac{1}{4}} D^{-\frac{5}{4}} \bar{V}^{1.75}$$

rough pipes

1. velocity profile :

$$\frac{u}{u^*} = 2.5 \ln \frac{R-r}{\varepsilon} + 8.5$$

U_{max} @ $r=0$ $= u^* \; 2.5 \ln \frac{R}{\varepsilon} + 8.5$

2. friction factor

$$\frac{1}{\sqrt{f}} = -2.0 \lg \frac{\left(\frac{\varepsilon}{D}\right)}{3.7} \; ,$$

f independent of Re

dependent only on relative roughness $\frac{\varepsilon}{D}$

⇒ moody diagram

transitional rough pipes

1. velocity profile :

$$\frac{u}{u^*} = 2.5 \ln \frac{R-r}{\varepsilon} + B$$

2. friction factor

$$\frac{1}{\sqrt{f}} = -2.0 \lg \left(\frac{\left(\frac{\varepsilon}{D}\right)}{3.7} + \frac{2.51}{Re\sqrt{f}} \right)$$

2% variation

or
$$\frac{1}{\sqrt{f}} = -1.8 \lg \left[\frac{6.9}{Re} + \left(\frac{\left(\frac{\varepsilon}{D}\right)}{3.7} \right)^{1.11} \right]$$

⇒ moody diagram

4) moody diagram

↳ accurate to ±15%

↳ no reliable f for $2000 < Re < 4000$

⑥ minor losses

1) minor loss and equivalent length

$$\boxed{h_L = K_L \frac{\bar{V}^2}{2g} = f \frac{L_{eq}}{D} \frac{\bar{V}^2}{2g}} \qquad \boxed{K_L = \frac{h_L \; 2g}{\bar{V}^2}}$$

$$\boxed{L_{eq} = K_L \frac{D}{f}}$$

→ add to original length ⇒ or if we bend, just longer pipe

$$h_{L,total} = \sum_i h_{L,major} + \sum_j h_{L,minor}$$

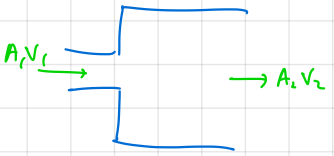
$$\sum_i f_i \frac{L_i}{D_i} \frac{\bar{V}_i^2}{2g} + \sum_j K_j \frac{\bar{V}_j^2}{2g}$$

2) Different loss scenarios

sudden enlargement

$h_L \approx \frac{V_1^2}{2g}$ if $A_2 >>> A_1$

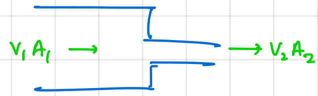
$$h_L = \frac{(V_1 - V_2)^2}{2g} = \frac{V_1^2}{2g}\underbrace{\left(1 - \frac{A_1}{A_2}\right)^2}_{K} = \frac{V_2^2}{2g}\underbrace{\left(\frac{A_2}{A_1} - 1\right)^2}_{K}$$



sudden contraction

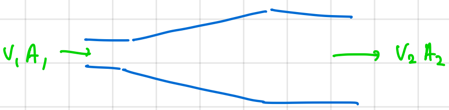
$$h_L = \frac{V_2^2}{2g}\left(\frac{A_2}{A_1} - 1\right)^2 = K_{SC}\frac{V_2^2}{2g}, \quad K_{SC} \to 0.5 \text{ as } \frac{A_2}{A_1} \to 0$$

contraction ratio $C = \frac{A_1}{A_2}$



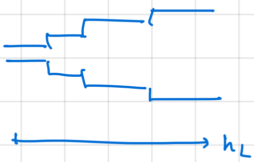
gradual expansion eg. diffuser

$$h_L = K_L \frac{(V_1 - V_2)^2}{2g} = K_L\left(1 - \frac{A_1}{A_2}\right)^2 \frac{V_1^2}{2g}$$



7 multiple pipe systems

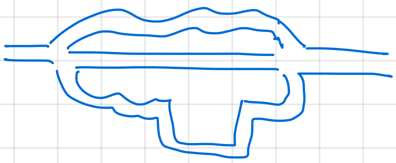
1) pipes in series



$\boxed{Q = Q_1 = Q_2 = Q_3}$

$\boxed{h_L = h_1 + h_2 + h_3 \ldots}$

2) pipes in parallel



$\boxed{Q = Q_1 + Q_2 + Q_3}$

$\boxed{h_1 = h_2 = h_3}$ (because of bernoulli streamlines)

3) branched pipes : junction as the reference

continuity : $Q_{J\,in} = Q_{J\,out}$

tank head : if $P_1 = P_2 = P_3 = P_{atm}$, tanks large that $\frac{dh}{dt} \approx 0$,

then $H_1 = z_1$, $H_2 = z_2$ ...

$$H_J = \frac{P_J}{\rho g} + \frac{V_J^2}{2g} + z_J$$

$$h_{L_i} = H_i - H_J = K_{SC}\frac{\bar{V}^2}{2g} = f\frac{L}{D}\frac{\bar{V}^2}{2g}$$

$\Rightarrow$ use J as reference point to relate $H_i = z_i$ to h_{L_i}

eg. $h_{L_1} = H_1 - H_J$ $\quad H_1 - H_2 = z_1 - z_2 = h_{L_1} - h_{L_2}$

$h_{L_2} = H_2 - H_J$