axioms

1. $0 \le P(A) \le 1$
2. $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$
3. $P(u) = 1$, $P(\phi) = 0$

joint probability $(x, y) \in D_x \times D_y$, $P(x, y) = P(X = x \wedge Y = y)$

conditional probability

1. $P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$ assuming $P(B) > 0$
2. $P(A) = P(A \mid B) P(B) + P(A \mid B') P(B')$

chain rule

$$P(E_1 \wedge E_2 \wedge E_3 \dots) = \prod_{i=1}^{n} P(E_i \mid E_1 \wedge E_2 \dots E_{i-1})$$

↳ derived from inductive application of Bayes' rule

independence A & B are independent $\iff P(A \wedge B) = P(A) \times P(B)$

$\iff P(A \mid B) = P(A)$ knowing B adds no information about A and vice versa

conditional independence

Given event B, event A is conditionally independent of event C if

$P(A \mid B, C) = P(A \mid B)$

⇒ A & C are related by a shared cause B

⇒ if we take it into account, they become unrelated

Bayesian networks

↳ suppose all n variables share a cause s. Then the full joint distribution

$P(x_1 \wedge x_2 \dots \wedge s) = P(x_1 \mid s) \cdot P(x_2 \mid s) \dots \cdot P(s)$ by chain rule.

↳ joint distributions represented by a graph

↳ vertices are random variables, an edge from $X \to Y$ implies X directly influences Y (some corr. assume X causes Y)

↳ Then conditional distribution for each node is $P(X \mid \text{Parents}(X))$

Structures in bayesian networks

independent causes

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

$$P(A \wedge B \wedge C) = P(A) \cdot P(B) \cdot P(C \mid A \wedge B)$$

conditionally independent effects

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

$$P(A \wedge B \wedge C) = P(A) \cdot P(C \mid A) \cdot P(B \mid A)$$

causal chain

$$A \rightarrow B \rightarrow C$$

$$P(A \wedge B \wedge C) = P(A) \cdot P(B \mid A) \cdot P(C \mid B)$$

Naive Bayes

↳ suppose we want to infer $P(\text{cause} \mid \underbrace{E_1 \wedge E_2 \ldots}_{\text{percepts}})$ i.e. given these observations, we want to find out likelihood that x caused it

↳ additionally suppose all $E_1 \ldots E_n$ are conditionally independent on cause.

1. $$P(\text{cause} \mid E_1 \wedge E_2 \ldots) = \frac{P(\text{cause}) \cdot P(E_1 \wedge E_2 \ldots \mid \text{cause})}{P(E_1 \wedge E_2 \ldots)}$$ by conditional prob.

$$= \frac{P(\text{cause})}{P(E_1 \wedge E_2 \ldots)} \prod_{i=1}^{n} P(E_i \mid \text{cause})$$

$$= \alpha \, P(\text{cause}) \prod_{i=1}^{n} P(E_i \mid \text{cause}), \quad \alpha = \frac{1}{P(E_1 \wedge E_2 \ldots)}$$

2. Now, to compare the probability of causes given observations, we can do it in relative terms of α!

That is, relative likelihood $$\boxed{\frac{P(X \mid E_1 \wedge E_2 \ldots)}{P(Y \mid E_1 \wedge E_2 \ldots)} = \frac{P(X) \prod_{i=1}^{n} P(E_i \mid X) \cancel{\alpha}}{P(Y) \prod_{i=1}^{n} P(E_i \mid Y) \cancel{\alpha}}}$$