MIU model

① sequential Lagrangian

$$P_t c_t + \underbrace{P_t^b B_t}_{\text{buy new bonds at } P_t^b < 1} + \underbrace{m_t}_{\text{remaining cash}} + S_t a_t = Y_t + \underbrace{m_{t-1}}_{\text{previous cash holdings}} + \overbrace{B_{t-1}}^{\text{matching bonds return 1 unit}} + S_t a_{t-1} + D_t a_{t-1}$$

sequential lagrangian

$$\boxed{\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t u\left(c_t, \frac{m_t}{P_t}\right) \\ &\quad - \sum_{t=0}^{\infty} \lambda_t \beta^t \left(P_t c_t + P_t^b B_t + m_t + S_t a_t - Y_t - m_{t-1} - B_{t-1} - S_t a_{t-j} + D_t a_{t-j} \right) \end{aligned}}$$

$\Rightarrow$ choice variables of c_t, a_t, m_t, B_t, λ_t

$$\frac{\partial L}{\partial c_t} = 0 \Rightarrow \frac{\partial u(c_t, \frac{m_t}{P_t})}{c_t} - \lambda_t P_t = 0$$

$$\frac{\partial L}{\partial a_t} = 0 \Rightarrow -\lambda_t S_t + \beta \lambda_{t+1} (S_{t+j} + D_{t+j}) = 0$$

$$\frac{\partial L}{\partial B_t} = 0 \Rightarrow -\lambda_t P_t^b + \beta \lambda_{t+1} = 0$$

$$\frac{\partial L}{\partial m_t} = 0 \Rightarrow \frac{\partial u(c_t, \frac{m_t}{P_t})}{m_t} - \lambda_t + \beta \lambda_{t+1} = 0 \Leftrightarrow \frac{\partial u(c_t, \frac{m_t}{P_t})}{P_t \, \partial (\frac{m_t}{P_t})} - \lambda_t + \beta \lambda_{t+1}$$

note chain rule!

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② relationships

no arbitrage condition

$$\frac{\partial L}{\partial B_t} = 0 \Rightarrow P_t^b = \frac{\beta \lambda_{t+1}}{\lambda_t} \quad (1)$$

$$\frac{\partial L}{\partial a_t} = 0 \Rightarrow S_t = \frac{\beta \lambda_{t+1}}{\lambda_t} (S_{t+1} + D_{t+1})$$

pricing kernel as some driver relating nominal asset price to nominal returns.
$\Rightarrow$ observe that pricing kernel is the same

$$\frac{\beta \lambda_{t+1}}{\lambda_t} = P_t^b = \frac{S_t}{S_{t+1} + D_{t+1}} = \frac{1}{1+i} = \frac{1+r}{1+\pi} = \frac{P_t(1+r)}{P_{t+1}}$$

$\Rightarrow$ when consumer is optimising, $\frac{\text{price}}{\text{return}}$ is the same for all assets. No arbitrage!

consumption – money optimality

$$\frac{\partial L}{\partial B_t} = 0 \Rightarrow -\lambda_t P_t^b + \beta \lambda_{t+1} = 0 \Rightarrow P_t^b = \beta \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1+i_t}$$

$$\frac{\partial L}{\partial M_t} = 0 \Rightarrow \frac{\dfrac{\partial u\left(c_t, \frac{M_t}{P_t}\right)}{\partial \left(\frac{M_t}{P_t}\right)}}{\lambda_t P_t} - 1 = -\beta \frac{\lambda_{t+1}}{\lambda_t} = -\frac{1}{1+i_t}$$

$$\frac{\partial L}{\partial c_t} = 0 \Rightarrow \frac{\partial u\left(c_t, \frac{M_t}{P_t}\right)}{\partial c_t} = \lambda_t P_t$$

$$\frac{\dfrac{\partial u\left(c_t, \frac{M_t}{P_t}\right)}{\partial \left(\frac{M_t}{P_t}\right)}}{\dfrac{\partial u\left(c_t, \frac{M_t}{P_t}\right)}{\partial c_t}} - 1 = -\frac{1}{1+i_t} \Rightarrow \boxed{MRS_{\left(\frac{M_t}{P_t}\right), c_t} = 1 - \frac{1}{1+i_t} = \frac{i_t}{1+i_t} = 1 - P_t^b}$$

---

money demand

↳ consumption-money optimality relates optimal choices of M_t to P_t, c_t and i_t, and we can use it to derive money demand.

$$MU_{\left(\frac{M_t}{P_t}\right)} = MU_{c_t}\left(1 - \frac{1}{1+i_t}\right) \Rightarrow \text{make } M_t \text{ subject}$$

↳ intuitive interpretation : representative consumer's demand for cash in the current period is determined by current prices, how much he wants to optimally consume (c_t) and save (i_t)

---

zero lower bound on interest rates

↳ observe that from the consumption-money optimality condition, if $i \leq 0$, then $MRS_{\frac{M_t}{P_t}, c_t} \leq 0$, which would imply either one of money or consumption had $MU \leq 0$, violating most reasonable assumptions of utility.

⇒ $MU_{c_t} < 0$: then consumption gives negative utility

⇒ $MU_{\frac{M_t}{P_t}} \leq 0$: convenience of purchasing power is negligible or negative

→ recall that an optimizing consumer consumes at opportunity cost. so a negative i_t reflects these behaviours of optimizing consumers!

λ_t as mu per dollar

$$\frac{\partial \mathcal{L}}{\partial c_{t+i}} = \cancel{\beta^i} \cdot u'(c_{t+i}) - \cancel{\beta^i} \lambda_{t+i} P_{t+i} = 0$$

appearance of λ_{t+i} used to relate to "relevant" constraints in this formulation, since each period has exactly 1 constraint

$$\lambda_{t+i} = \frac{u'(c_{t+i})}{P_{t+i}}$$

λ_{t+i} works out to be mu/\$ for each period

---

asset pricing

↳ simplifying assumption: assume that previously bought stock becomes tradeable and returns dividends in the same period. Then:

$$\mathcal{L} = \sum_{i=0}^{\infty} \beta^i \cdot u(c_{t+i})$$

$$+ \sum_{i=0}^{\infty} \lambda_{t+i} \beta^i \left(Y_{t+i} + \underbrace{(S_{t+i} + D_{t+i})\, a_{t+i-j}}_{\text{returns from previous period stock}} - P_{t+i} c_{t+i} - S_{t+i} a_{t+i} \right)$$

from some j periods ago

⇒ at any given time, optimization only relates period t (buy) and $t+j$ (returns), so simultaneous FOCs to solve for variables involving these two periods only need

$$\frac{\partial \mathcal{L}}{\partial a_t} = -\lambda_t S_t + \beta^j \lambda_{t+j} (S_{t+j} + D_{t+j}) = 0$$

$$S_t = \frac{\beta^j \lambda_{t+j}}{\lambda_t} (S_{t+j} + D_{t+j})$$

pricing kernel

price of financial asset = discounted value of future earnings, with influence of a pricing kernel

↳ Also: recall λ_t as mu per dollar.

$$\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - \lambda_t P_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+j}} = \beta^j u'(c_{t+j}) - \beta^j \lambda_{t+j} P_{t+j} = 0$$

$$\frac{\lambda_{t+j}}{\lambda_t} = \left(\frac{u'(c_{t+j})}{P_{t+j}}\right) \Big/ \left(\frac{u'(c_t)}{P_t}\right) = \frac{u'(c_{t+j})}{u'(c_t)} \cdot \frac{P_t}{P_{t+j}}$$

$$\Rightarrow \quad S_t = \underbrace{\beta^j \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) \left(\frac{P_t}{P_{t+j}} \right)}_{\text{pricing kernel}} \left(S_{t+j} + D_{t+j} \right)$$

intuitive drivers

interpretation: stock price at a given period $t+i$, assuming all consumers are price-taking optimizers, depend on:

1. relative marginal utilities in later period to current period. If later more, makes more sense to save.

2. discount factor. offsets utility gain in future by how far away it is.

3. stock price and dividend in future, if higher, get more for saving

4. relative price levels in current to future period; adjusting nominal gains in real units

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Steady state: impatience and real i/r

↳ observation: in an infinite period framework, we have:

$$S_t = \beta^j \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) \left(\frac{P_t}{P_{t+j}} \right) \left(S_{t+j} + D_{t+j} \right)$$

rearranging, we get

$$\underbrace{\frac{\;\frac{S_{t+j} + D_{t+j}}{P_{t+j}}\;}{\frac{S_t}{P_t}}}_{\text{real returns of assets}} = \frac{u'(c_t)}{u'(c_{t+j})\, \beta^j}$$

real returns of assets $\Rightarrow 1 + r_t$

in steady state, $x_t = \bar{x}$. so:

$$1 + \bar{r} = \frac{\cancel{u'(\bar{c})}}{\cancel{u'(\bar{c})}\, \beta^j} = \frac{1}{\beta^j}$$

$\Rightarrow$ in the long run, the real interest rate — i.e. returns from loaning real resources to others, or borrowing from others — is fundamentally tied to the degree of impatience of consumers in the economy.

$\Rightarrow$ the most fundamental source of real interest rates, and why they cannot be negative, is because of human impatience, and the fact that will (on average) never view future consumption as more beneficial than current.

i.e. $\beta \in [0, 1]$.

fiscal policy

① budget & savings

budget constraint

$$\underbrace{g_t}_{\text{spending}} + \underbrace{b_t}_{\text{asset position}} = \underbrace{(1+r)\,b_{t-1}}_{\text{value of assets}} + \underbrace{t_t}_{\text{tax revenue}}, \text{ in real terms}$$

in this case, saving

national savings

$$\left.\begin{array}{l} s_t^{govt} = t_t - g_t \\ s_t^{private} = y - t_t - c_t \end{array}\right\} s_t^{national} = s_t^{govt} + s_t^{private}$$

---

② Taxation & ricardian equivalence

Theorem: for a given level of govt. expenditure for the first & second periods, the exact timing of taxes has no impact on the real economy (eg. c^*, national savings)

re-the consumption choices and real y/r do not change when you change taxes in any period, so long as you fix G ⇒ because it will be financed through private savings from Δ disposable income

mathematically

1. consumer maximizes in infinite time horizon.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad p_t \cdot c_t + s_t a_t + t_t = y_t + s_t a_{t-j} + D_t a_{t-j}$$

2. He choose c_t, s_t at every time period, simultaneously
3. we model this in the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t \beta^t \left(p_t c_t + s_t a_t + t_t - y_t - a_{t-1}(s_t + D_t) \right)$$

⇒ observe that FOCs wrt. c_t, a_t are independent of t_t

---

③ distortionary taxation

consumption taxes

$$\left.\begin{array}{l} (1+\tau_1)\,c_1 + s_1 = y_1 + (1+r)\,s_0 \\ (1+\tau_2)\,c_2 + s_2 = y_2 + (1+r)\,s_1 \end{array}\right\} (1+\tau_1)\,c_1 + \frac{(1+\tau_2)\,c_2}{1+r} = y_1 + \frac{y_2}{1+r} + (1+r)\,s_0$$

$$\Rightarrow \underbrace{c_2 = -\left(\frac{1+\tau_1}{1+\tau_2}\right)(1+r)\,c_1 + \left(y_1 + \frac{y_2}{1+r}\right)\left(\frac{1+\tau_1}{1+\tau_2}\right)}$$

budget line slope is $-\left(\frac{1+\tau_1}{1+\tau_2}\right)(1+r)$

⇒ distortionary effects on c_1^*, c_2^*

consumption / dividend taxes x S_t

↳ without distortionary taxes, we have it that

$$S_t = \beta^j \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) \left(\frac{P_t}{P_{t+j}} \right) (S_{t+j} + D_{t+j})$$

↳ but suppose we consider a dividend tax, <u>and</u> a consumption tax. Then budget is

$$P_t C_t + P_t^b B_t + M_t + S_t a_t = Y_t + M_{t-1} + B_{t-1} + S_t a_{t-1} + D_t a_{t-1}$$

$$P_t C_t \Downarrow P_t(1+\tau_t^c) c_t \qquad D_t a_{t-1} \Downarrow (1-t_t^D) D_t a_{t-1}$$

↳ optimising, we have that $\frac{\partial L}{\partial a_t} = 0$, $\frac{\partial L}{\partial c_t} = 0$, $\frac{\partial L}{\partial c_{t+1}} = 0$

$$\Rightarrow \boxed{S_t = \frac{\beta u'(c_{t+1})}{u(c_t)} (S_{t+1} + (1-t_{t+1}^D) D_{t+1}) \left(\frac{P_t}{P_{t+1}} \right) \frac{1+\tau_t^c}{1+\tau_{t+1}^c}}$$

---

taxes on real interest

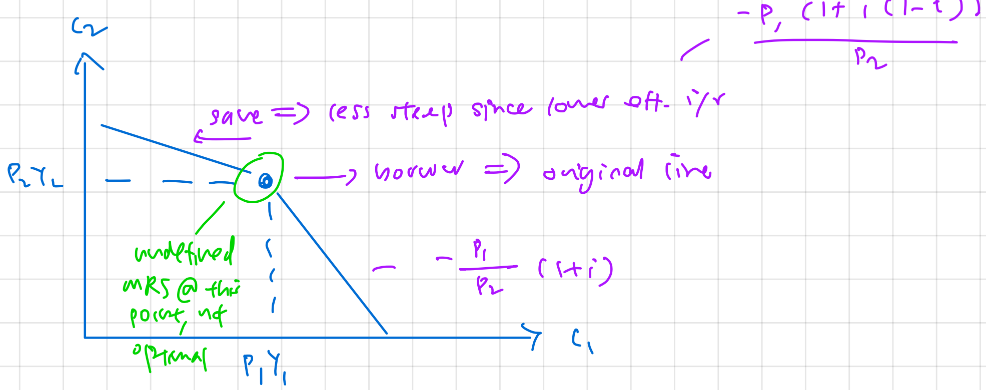
↳ usually, we have LBC where $P_1 c_1 + \frac{P_2 c_2}{1+i} = P_1 y_1 + \frac{P_2 y_2}{1+i}$

↳ if we tax on i/r iff. i/r returns >0, we get a piecewise function!

$\Rightarrow$ effective $i = i(1-t)$

$$\boxed{\begin{cases} P_1 c_1 + \frac{P_2 c_2}{1+i} = P_1 y_1 + \frac{P_2 y_2}{1+i} & \text{if } c_1 \geq y_1 \text{ (borrowed)} \\ P_1 c_1 + \frac{P_2 c_2}{1+i(1-t)} = P_1 y_1 + \frac{P_2 y_2}{1+i(1-t)} & \text{if } c_1 \leq y_1 \text{ (loaned, earned i/r)} \end{cases}}$$

→ LBC is piecewise function, kinked @ endowment point



monetary policy

bonds

$$\boxed{p_t^b = \frac{1}{1+i_t}}$$

1 nominal unit in next period

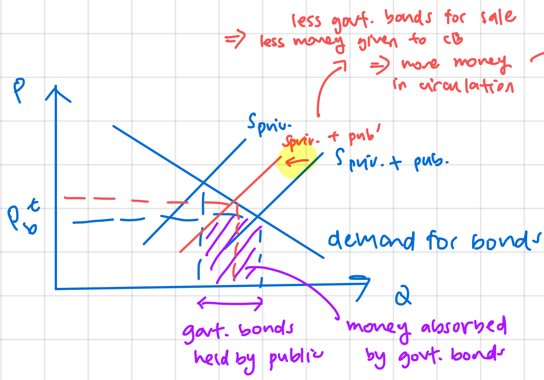
① open market operations

↳ observe that the central bank can print money at any time — including in future.

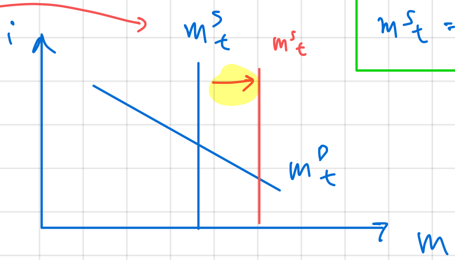
↳ Also observe that all private agents can buy or sell bonds — including government bonds.

↳ Also observe that the act of buying a bond is also giving money away (in exchange for future monies)

open market operations Thus, precisely by buying and selling bonds, the government can increase or absorb money in circulation.



bond market



money market

$$\boxed{m_t^S = m_t^D = m_t}$$

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② monetary policy in the short run

short run — effects in a single period

monetary policy central bank's intervention in the money market via money supply

money neutrality money is neutral if changes in the money supply at eqm. have no real effects on the economy

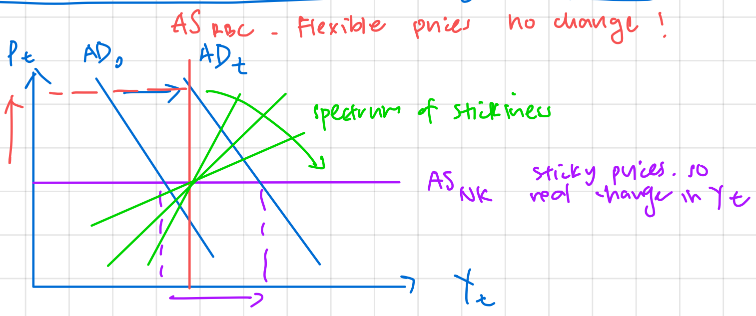
The real business cycle (RBC) view
↳ nominal prices adjust quickly

$\boxed{m_D \text{ function, } P_t \text{ changes}}$

New keynesian view
↳ nominal prices fixed.

$\frac{m_D}{P_t} = f(Y, i) \Rightarrow \boxed{Y_t, i_t \text{ change}}$

implications of money neutrality on growth



period$_t$ goods market

③ monetary policy in the long run

$\boxed{\text{long run}}$ consider effects across time periods

$\boxed{\text{is monetary policy neutral in the long run?}}$

1. money demand, $M^D_t = P_t f(c_t, i_t)$

2. Across two periods, $\frac{M^D_{t+1}}{M^D} = 1+\mu_t = \frac{P_{t+1}}{P_t} \frac{f(c_{t+1}, i_{t+1})}{f(c_t, i_t)}$

rate of M^S growth

3. in long run steady state, $\frac{P_{t+1}}{P_t} = 1+\pi_t \longrightarrow 1+\bar{\pi}$. so $1+i_t = (1+\bar{r})(1+\bar{\pi})$ also goes to steady state.

$$\Rightarrow \quad 1+\mu_t = 1+\bar{\pi}_t \cancel{\frac{f(\bar{c},\bar{i})}{f(\bar{c},\bar{i})}} \Rightarrow \boxed{\mu = \bar{\pi}}$$

"monetarist" school of thought

4. in the long run, money growth = rate of inflation. so change in money only affects nominal (price) variables, and is thus neutral.

$\boxed{\text{fiscal \& monetary policy}}$

① consolidated government flow budget

1) fiscal authority's flow budget

$$\underbrace{\underbrace{P_t g_t}_{\text{spending}} + \underbrace{B^F_{t-1}}_{\text{payment of borrowing (bonds) in previous period, to CB or public}}}_{\text{outflow}} = \underbrace{\overbrace{T_t}^{\text{taxes}} + \overbrace{P^b_t B^F_t}^{\text{new borrowing}} + \underbrace{RCB_t}_{\text{receipts turned over from CB. to fiscal authority}}}_{\text{inflow}}$$

2) monetary authority's flow budget

total borrowing, $B^F_t = B^m_t$ + bonds bought by public B_t

$$\underbrace{\underbrace{P^b_t B^m_t}_{\text{bonds bought in open mkt. operations (loans to public, } \uparrow \text{ MS)}} + \underbrace{RCB_t}_{\text{leftover receipts to fiscal authority}}}_{\text{outflow}} = \underbrace{\underbrace{B^m_{t-1}}_{\text{maturing bonds from previous period}} + \underbrace{(M_t - M_{t-1})}_{\text{printing of more money}}}_{\text{inflow}}$$

3) seignorage revenue $\boxed{sr_t = \frac{M_t - M_{t-1}}{P_t}}$

seignorage revenue : the real quantity of resources the govt. raises for itself by money creation

② policy interactions in the short run

(active policy) A policy authority is active if every instrument at its disposal can be freely chosen, without consideration to the consolidated GBC, and passive otherwise

(policy pressure) observe that money cannot appear out of thin air. That is, the flow GBC must hold. If one authority is active, then the author is constrained, and must react accordingly with limited options to balance the budget.

(consolidated budget) must hold in every period

$$B_t^F - B_{t-1}^m = B_t \text{ , bonds bought by private sector}$$

$$\Rightarrow \underbrace{P_t g_t + B_{t-1}}_{\text{outflow as a single govt. entity}} = \underbrace{T_t + p_t^b B_t + (M_t - M_{t-1})}_{\text{inflow as a single govt. entity}}$$

---

③ policy interactions in the long term

(pV infinite period budget)

$$\frac{B_{t-1}}{P_t} = \underbrace{sv_t + \sum_{s=1}^{\infty} \frac{sv_{t+s}}{\prod_{x=1}^{s}(1+r_{t+x-1})}}_{\text{present value of } \infty \text{ seignorage revenues}} + \underbrace{(t_t - g_t) + \sum_{s=1}^{\infty} \frac{t_{t+s} - g_{t+s}}{\prod_{x=1}^{s}(1+r_{t+x-1})}}_{\text{present value of } \infty \text{ fiscal surpluses}}$$

$\Rightarrow$ debt incurred today will be paid for by something in the next periods

1) Ricardian fiscal policy

Definition: A Ricardian fiscal policy is in place if the fiscal authority sets its planned sequence of tax and spending to ensure that consolidated lifetime GBC is balanced, and non-Ricardian otherwise

↳ idea: Ricardian if all $(t_i - g_i)$ balance themselves out without need for help for sr_i

2) non-Ricardian fiscal policy

↳ suppose fiscal authority is active, and changes spending in a non-ricardian way — that is, future surpluses (in present value) cannot offset debts today (from deficit).

↳ but GBC must hold. How?

monetary authority raises PV of lifetime sr

↓

at least one period's money creation increased. steady state / avg. π also increases

at least one period has higher π

steady state / avg. π is increased

⇒ monetary authority's response has implications on future inflation and was ultimately induced by fiscal policy

monetary authority helps a bit

↓

higher P_t, high inflation in future

monetary authority does nothing to prospective PV of sr

↓

govt. must somehow pay for current (one time) deficit

↓

inflation tax. Price level rises, and real value of debt falls

⇒ mechanism behind is that there are insufficient real resources going around to fund spending, but amt. of money is still being pumped in, so P_t rises to adjust for constant purchasing power

$P_t \uparrow$, $\frac{B_{t-1}}{P_t} \downarrow$

Fiscal theory of exchange rates

① fundementals

real exchange rate

converted real purchasing power of other

$$e_t = \frac{E_t P_t^*}{P_t}$$

1 unit purchasing power of mine

PPP

$$P_t = E_t P_t^*$$

IRP

expected, because future unknown, investor makes decisions on expectation

$$1 + i_t = 1 + i_t^* \frac{(E^e_{t+1})}{E_t}$$

convert back in next period

convert to B

② FTER

assumptions

1. PPP, P_t^* as numeraire. $P_t = E_t \cancel{P^*}$
2. IRP, no foreign inflation. $1 + i_t = 1 + \cancel{i_t^*}^{\,r_t^*} \frac{E^e_{t+1}}{E_t}$
3. consumption @ steady state. $\frac{M_t}{P_t} = \phi(\bar{c}, i_t)$

modified GBC

$$\frac{E_t}{P_t} = 1$$

$$\underbrace{B_t^F - B_{t-1}^F}_{\text{change in foreign assets}} = \frac{M_t - M_{t-1}}{P_t} - \underbrace{\left(\frac{G_t - T_t}{P_t} - r^* B_{t-1}^F\right)}_{\text{DEF: spend - tax - real returns from foreign assets}}$$

Production model of growth

① properties

1. increasing outputs $\bar{c}$ inputs
2. Diminishing marginal returns
3. constant RTs

general equilibrium

$y = Ak^\alpha$

$Y = AK^\alpha L^{1-\alpha}$ — production

$K = \bar{K}$ — capital supply

$mpK = r$ — capital demand @ π max.

$L = \bar{L}$ — labor supply

$MPL = w$ — labor demand @ π max.

② key results

Cobb Douglas & factor shares

1. $Y = AK^\alpha L^{1-\alpha}$
2. $w^* = mpL = AK^\alpha (1-\alpha) L^{-\alpha}$
3. $r^* = mpK = A\alpha K^{\alpha-1} L^{1-\alpha}$

(as fractions of Y)

total wages paid

$$\frac{w^* \bar{L}}{Y^*} = 1-\alpha$$

total rent paid

$$\frac{r^* \bar{K}}{Y} = \alpha$$

Solow model

① formulation

formulation

we use it to produce more

1. $Y_t = \bar{A}^{\alpha} K_t^{\beta} L_t^{1-\beta}$ — production function
2. $Y_t = C_t + I_t$ — resource constraint
3. $I_t = \bar{s} Y_t$ — investment → we save } we invest
4. $K_{t+1} = K_t + I_t - \bar{d} K_t$ — capital accumulation

$\Leftrightarrow \Delta K_{t+1} = K_{t+1} - K_t = I_t - \bar{d} K_t$

$\Leftrightarrow \Delta K_{t+1} = \bar{s} Y_t - \bar{d} K_t$

5. $L_t = \bar{L}$ — exogeneous labour constraint

——— optional ———

6. $w = MPL$
7. $r = MPK$

} wage & interest constraints

Y_t, K_t, L_t, C_t, I_t (and w, r) are endogeneous, to be determined

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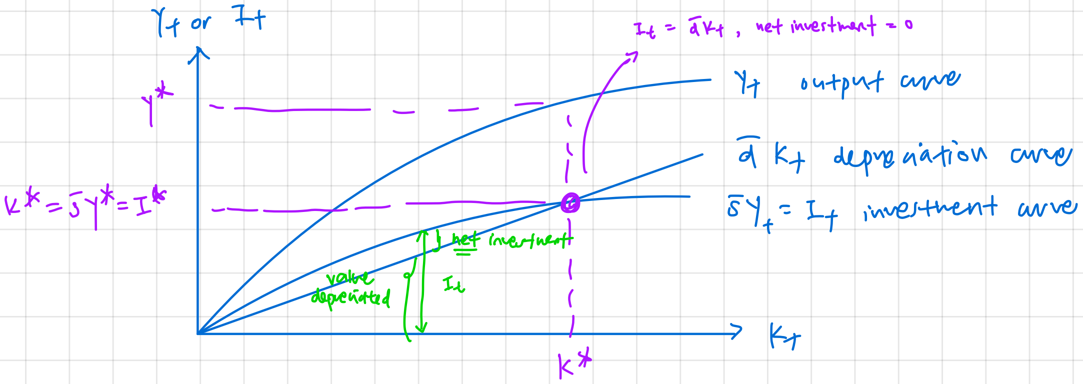
steady state condition



steady state condition :

$$\boxed{\bar{s} Y_t = I_t = \bar{d} K_t}$$

investment exactly offsets depreciation, no addtl. income from capital increase in future.

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output @ steady state

1. in the steady state, investment = depreciation.

$$\boxed{I^* = \bar{s} Y^* = \bar{d} K^* \Leftrightarrow K^* = \frac{\bar{s} Y^*}{\bar{d}}}$$

2. substitute into production function. $Y^* = \bar{A}^{\alpha} K^{*\beta} \bar{L}^{1-\beta} = \bar{A}^{\alpha} \left(\frac{\bar{s} Y^*}{\bar{d}}\right)^{\beta} \bar{L}^{1-\beta}$

2.1 make Y^* subject.

$$Y^{*\,1-\beta} = \bar{A}^{\alpha} \left(\frac{\bar{s}}{\bar{d}}\right)^{\beta} \bar{L}^{1-\beta}$$

$$\boxed{Y^* = \bar{A}^{\frac{\alpha}{1-\beta}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\beta}{1-\beta}} \bar{L}} \Leftrightarrow \boxed{y^* = \bar{A}^{\frac{\alpha}{1-\beta}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\beta}{1-\beta}}}$$

② key insights

growth relative to steady state

1. we know that y_t and k_t are intimately tied in transition dynamics, k_t changes, which changes y_{t+1}, and so on. Express g_{y_t} in g_{k_t}

1.1 $Y_t = \bar{A}^{\alpha} K_t^{\beta} L_t^{1-\beta} \iff y_t = \bar{A}^{\alpha}\left(\frac{K_t}{L_t}\right)^{\beta}$

1.2 From 0.1 and 0.2, $g_{y_t} = \alpha g_{\bar{A}_t} + \beta(g_{K_t} - g_{L_t})$

1.3 $\Delta K_t = \bar{s}Y_t - \bar{d}K_t \iff \frac{\Delta K_t}{K_t} = \frac{\bar{s}Y_t}{K_t} - \bar{d}$

1.4 in the steady state, $\bar{s}Y^* = \bar{d}K^*$

1.5 Then

$$g_{y_t} = \alpha g_{\bar{A}_t} + \beta\left(\frac{\bar{s}Y_t}{K_t} - \bar{d} - g_{L_t}\right)$$

$$= \alpha g_{\bar{A}_t} + \beta\left(\bar{s}\left(\frac{Y_t}{K_t}\right)\left(\frac{Y^*}{K^*}\right)\left(\frac{K^*}{Y^*}\right) - \bar{s}\frac{Y^*}{K^*} - g_{L_t}\right)$$

$$= \alpha g_{A_t} + \bar{s}\frac{Y^*}{K^*}(\beta)\left(\frac{Y_t}{K_t}\cdot\frac{K^*}{Y^*} - 1\right) - \beta(g_{L_t})$$

$$= \alpha g_{A_t} + \bar{s}\frac{Y^*}{K^*}(\beta)\left(\frac{\cancel{A^{\alpha}} K_t^{\beta} \cancel{L_t^{1-\beta}}}{K_t}\frac{K^*}{\cancel{A^{\alpha}} K^{*\beta} \cancel{L_t^{1-\beta}}} - 1\right) - \beta(g_{L_t})$$

$$\boxed{g_{y_t} = \alpha g_{A_t} + \bar{s}\frac{Y^*}{K^*}(\beta)\left(\frac{K^{*\,1-\beta}}{K_t^{\,1-\beta}} - 1\right) - \beta(g_{L_t})}$$

2. population growth dampens per capita GDP growth, weighted by β. Intuitively, this arises by way of how it reduces capital per capita

3. The relative ratio of $\frac{K^*}{K_t}$ determines direction of growth. Observe that $K^* > K_t$, $\frac{K^*}{K_t} > 1$, and vice versa, and for growth to be positive, $\left(\frac{K^*}{K_t}\right)^{1-\beta} > 1$ and vice versa.

3.1 observe that analogously $\left(\frac{K^*}{K_t}\right)^{1-\beta}$ determines magnitude of growth rate

existence of a steady state & long-run growth

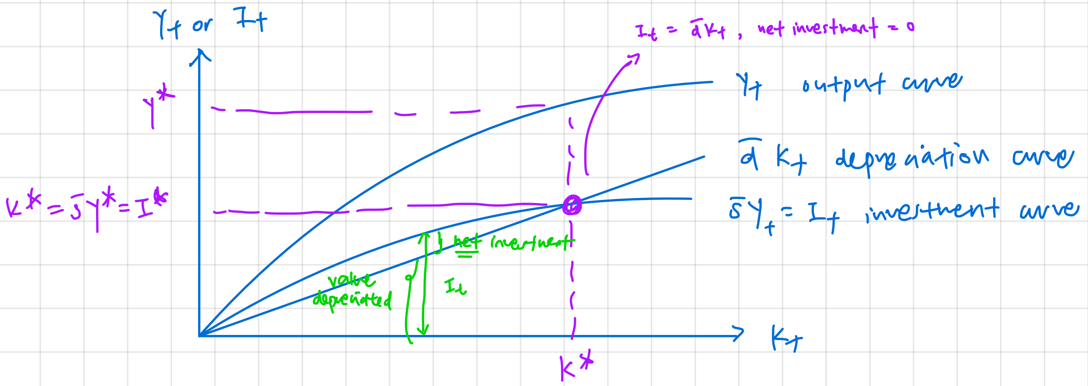
1. observe that in the model, depreciation is a constant factor that causes a constant diminishing of capital at every time step.

2. Also observe that the production function exhibits diminishing returns to capital.

3. Putting these two facts together, we have it that as we increase capital stock, production rises (and therefore investment). But because of diminishing returns, the amount by which production grows at the next time step gets smaller and smaller, and so does the growth in investment.

4. In contrast, a constant fraction of capital stock depreciates every period, so the amount of depreciation rises one-for-one with capital.

5. so there exists a point where the amount of investment the economy generates is equal to the amount of depreciation. Net investment is zero, and there is no capital-driven growth.



steady state condition :

$$\boxed{\bar{s}Y_t = I_t = \bar{d}K_t}$$

investment exactly offsets depreciation, no addtl. income than capital increase in future.

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③ exogeneous shocks

1. how change affects K_t, Y_t relative to K^*, Y^*, or K^* and Y^* itself
   - 1.1 what that means for amount of output produced and saved + curves
   - 1.2 How this compares to amount loss to depreciation + curves
   - 1.3 Effects on net investment and thus net change in capital stock

2. How this pattern continues w̄ transition dynamics

3. go on until K at K^*, then investment is just able to cover for depreciation and steady state is reached.

$\underbrace{\hspace{20em}}$

if there is growth in labour, use per-capita analysis

transition dynamics

1. savings relative to depreciation : exceeds

1.1 net investment is positive, illustrated by distance ΔK_1. So there is capital accumulation.

1.2 the positive net investment increases production for the next period, creating more capital accumulation so long as savings exceed depreciation.

2. Savings is not enough to make up for depreciation

2.1 Net investment is negative, illustrated by distance ΔK.

2.2 In the next period, capital will fall from K_t to K_{t+1} by ΔK.

2.3 This decreases output further, and we continue to see capital depletion in capital (per worker) while savings are less than depreciation.

2.4 Due to diminishing returns to capital but fractional constant depreciation, the difference between $\bar{s}Y$ and $\bar{d}K$ approaches 0 as k approaches K^*.

a. Due to diminishing returns to capital but constant fractional depreciation, net investment approaches 0 as k_t approaches k^*.

b. At the steady state, $K_t = K^*$ and net investment exactly accounts for depreciation. So no more capital accumulation occurs and the economy has reached its new steady state at $Y = Y^*$.

---

technological shocks, change in $\bar{A}$

1. The change in $\bar{A}$ implies that productivity is higher, and for a given amount of capital and labor, the economy can produce more output

2. This is illustrated by an upward shift in the production & investment (per worker) curves from Y_t to Y_t' and I_t to I_t'

3. Since $\bar{a}$ is unchanged, we have a new steady state point at S_2, and the economy is now below the new steady state.

4. Savings exceed depreciation.

Labour shocks, change in L

1. With an instaneous change in labour supply, for a given amount of capital, the economy is able to produce more output.

2. This implies for a constant fractional savings rate, more will be invested in capital.

3. These are illustrated by an upward shift in Y_t to Y_t' and K_t to K_t'

4. Since $\bar{d}$ is unchanged, the new steady state is at S_2.

5. At S_1, the economy lies below the steady state, and net investment exceeds depreciation.

---

income/capital shocks, change in Y_t or K_t

1. With an instantaneous increase in Y_t/K_t the economy now finds itself above the steady state, though there is no change in the underlying productive capacity.

2. At this point Y_t', depreciation exceeds savings.

---

change in depreciation, $\bar{d}$

1. A change in depreciation rate implies that at a given time step, more capital is consumed in the production of goods.

2. This reduces net investment, and is illustrated by a pivot in the depreciation curve from D_1 to D_2 upwards.

3. This has the effect of moving the steady state down to S_2.

4. At original steady state S_1, thus, the economy lies above the steady state.

---

change in savings, $\bar{s}$

1. A change in savings rate implies that for a given level of output, more is reinvested into capital.
2. This is illustrated by an upward shift in I_1 to I_2, now closer to Y_t.
3. Since depreciation is $\bar{d}$, the new steady state lies at S_2, and at original state S_1, the economy is below S_2.

Romer model

① formulation

$Y_t = A_t L_{yt}$ output production function

productivity of ideas

$\Delta A_{t+1} = \bar{z} A_t L_{at}$ idea production function

current stock of ideas

$L_{yt} + L_{at} = \bar{L}$ resource constraint

$L_{at} = \bar{\ell} \bar{L}$ allocation of labour

② α and long run growth

$g_y = \alpha g_A + \beta g_K + (1-\beta) g_L$

$g_A = \frac{\Delta A_{t+1}}{A_t} = \bar{z} \bar{\ell} \bar{L} A_t^{\alpha-1}$

$\frac{d g_A}{d A_t} = (\alpha-1) \bar{z} \bar{\ell} \bar{L} A_t^{\alpha-2}$

$\alpha \geq 1$

$\frac{d g_A}{d A_t} \geq 0$ if $\alpha \geq 1$.

As $A_t \to \infty$, $A_t \to \infty$

since $g_{A_0} > 0$

$\alpha < 1$

$\frac{d g_A}{d A_t} < 0$ if $\alpha < 1$.

as $A_t \to \infty$, $g_A \to 0$

balanced growth

all endogeneous variables grow at constant rates. (Y_t, A_t, L_{yt}, L_{at})

1. $g_{L_{at}} = 0$
2. $g_{L_{yt}} = 0$
3. $g_{A_t} = \bar{z} \bar{\ell} \bar{L}$
4. $y_t = \bar{A}_0 (1+g_A)^t (1-\bar{\ell})$

4.1 $\ln y_t = \ln A_0 (1+g_A)^t (1-\bar{\ell})$

$= \underbrace{\ln A_0 + \ln(1-\bar{\ell})}_{\text{intercept}} + \underbrace{t \ln(1+g_A)}_{\text{slope}}$

4.2 $\frac{d \ln y_t}{dt} = \ln(1+g_A) \approx g_A$

global knowledge

$g_A = A_t^{world} \bar{z}^{country} \bar{L}^{country}$

$\frac{\Delta A_{t+1}^{world}}{A_t^{country}} = \bar{z} \frac{A_t^{world}}{A_t^{country}} \bar{L}^{country}$ → this ratio allows equilisation for labor shortfalls

③ exogeneous shocks

changing research share

1. recall that $Y_t = \bar{A}_0 (1+g_A)^t (1-\bar{\ell})$

2. Level effects. change in $\bar{\ell}$, immediate change in Y_t, illustrated by a break and shift in curve.

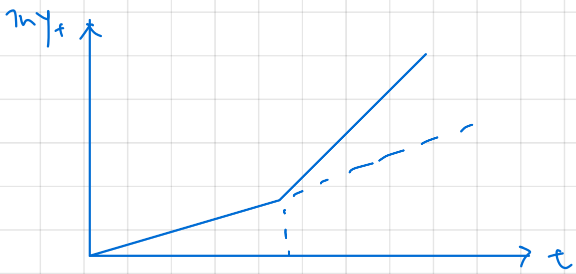
3. Growth effects. Recall $g_A = \bar{z} \bar{\ell} \bar{L}$. change in $\bar{\ell}$, immediate & permanent change in g_A and thus g_y, illustrated by steeper / gentler curve.

---

changing $\bar{L}$

1. recall that $g_A = \bar{z} \bar{\ell} \bar{L}$, and $g_y \approx g_A$

2. change in $\bar{L}$, immediate & permanent change in g_A and g_y, as illustrated by graph



$\boxed{\text{Solow \& Romer model}}$

① formulation

1. $Y_t = A_t^{\alpha} K_t^{\beta} L_{yt}^{1-\beta}$ production function w/ constant RTS to K, L
2. $\Delta K_{t+1} = \bar{s} Y_t - \bar{d} K_t$ capital accumulation
3. $\Delta A_{t+1} = \bar{z} A_t L_{at}$ idea production function
4. $L_{yt} + L_{at} = \bar{L}$ labour constraint
5. $L_{at} = \ell \bar{L}$ research labour allocation

---

② key results

output per capita

1. growth in output per capita

1.1 recall $g_{Y^*} = \frac{\alpha}{1-\beta} g_A$

1.2 recall $g_y = g_Y - g_L$

1.3 Then $g_{y^*} = g_{Y^*} - \cancel{g_{L^*}}$

2. output per capita along balanced growth path

2.1 Recall $y^* = \frac{Y^*}{\bar{L}} = \frac{A_t^{*\alpha} K_t^{*\beta} L_{at}^{1-\beta}}{\bar{L}}$

$$= \frac{A_t^{\alpha} \left(\frac{K_t^*}{Y_t^*} Y_t^*\right)^{\beta} L_{at}^{1-\beta}}{\bar{L}}$$

2.2 Recall $g_K^* = g_Y^* = \bar{s}\frac{Y_t^*}{K_t^*} - \bar{d} \iff \frac{K_t^*}{Y_t^*} = \frac{\bar{s}}{g_{Y^*} + \bar{d}}$

2.3 Recall $g_{y^*} = g_{Y^*}$ if $g_L = 0$

2.4 $y^* = \frac{A_t^{\alpha} \left(\frac{\bar{s}}{g_{y^*} + \bar{d}}\right)^{\beta} (Y_t^*)^{\beta} (1-\ell)^{1-\beta} \cdot \bar{L}^{1-\beta}}{\bar{L}}$

$$= A_t^{\alpha} \left(\frac{\bar{s}}{g_{y^*} + \bar{d}}\right)^{\beta} (1-\ell)^{1-\beta} y_t^{*\beta}$$

2.5 Then $\boxed{y^* = A_t^{\frac{\alpha}{1-\beta}} \left(\frac{\bar{s}}{g_{y^*} + \bar{d}}\right)^{\frac{\beta}{1-\beta}} (1-\ell)^{1-\beta}}$

Transition dynamics

1. in every period, ideas production function means that $\Delta A_{t+1} = \bar{z} A_t L_{at}$
   - 1.1 Recall $g_A = \bar{z}\bar{\ell}\bar{L}$, constant.
   - 1.2 Then output rises without change in capital, which due to savings, creates reinvestment in capital in the next period.
   - 1.3 For the current, new level of A_t, the new steady state capital K^* lies at $K_t + \Delta K_{t+1}$.
   - 1.4 This continuous, steady growth in A_t, Y_t and K_t lies along the balanced growth path.

2. Level effects occur, such that economy is above balanced growth path.
   - 2.1 Savings from base income Y_t, where $Y_{t+1} = Y_t + \Delta Y_t$, exceeds depreciation.
   - 2.2 Net investment is positive, illustrated by distance ΔK_1. So there is capital accumulation.
   - 2.3 The positive net investment increases production for the next period, creating excess capital accumulation so long as savings from output not from tech growth changes exceed depreciation.

a. Due to diminishing returns to capital but constant fractional depreciation, net investment approaches ΔK_t^* as time goes to ∞.

b. At the steady state, $\Delta K_t = \Delta K^*$ and net investment arises purely from savings of output grown by technological growth. Then the economy is back on the steady state.

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balanced growth path

1. $\boxed{g_{L_{at}} = 0}$
2. $\boxed{g_{L_{yt}} = 0}$
3. $\boxed{g_{A_t} = \dfrac{\Delta A_t}{A_t} = \bar{z}\bar{\ell}\bar{L}\, A_t^{\alpha-1}}$
4. steady state, $g_{Y^*} = g_{K^*}$

$$g_{K^*} = \text{constant} = \bar{s}\frac{Y_t^*}{K_t^*} - \bar{d}$$

$$\frac{d\ln(\text{constant} - \bar{d})}{dt} = \frac{d\ln\bar{s}}{dt} + \frac{d\ln Y_t^*}{dt} - \frac{d\ln K_t^*}{dt}$$

$$\boxed{0 = g_{Y^*} - g_{K^*}}$$

labour shocks, change in L

1. with an instaneous change in labour supply, for a given amount of capital, the economy is able to produce more output.

2. growth effects. $g_{Y^*} = \frac{\alpha}{1-\beta} g_A$, $g_A = \bar{z} \bar{\ell} \bar{L}$. so higher long run balanced growth. illustrated by different slope

3. level effects. Instantaneous change in output. Transition dynamics, illustrated by offset then non-linear adjustment.

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income/capital shocks, change in Y_t or K_t

1. with an instantaneous increase in Y_t/K_t the economy now finds itself above the steady state, though there is no change in the underlying productive capacity.

2. growth effects. None. same slope.
3. Level effects. Illustrated by discontinuity. Transition dynamics.

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change in depreciation, $\bar{d}$

change in savings, $\bar{s}$

1. A change in depreciation rate implies that at a given time step, more capital is consumed in the production of goods.

2. growth effects. None.

3. Level effects. None.

4. other effects. If shock, adjustment would be faster / slower.

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change in research share

1. level effects. more / less working in production.

2. Growth effects. $g_{y^*} = \alpha g_A + \beta g_{K^*} \iff \frac{\alpha}{1-\beta} g_A$, $\bar{\ell} \uparrow$, $g_A \uparrow$

3. transition dynamics due to level effects