Infinite period framework

① overview

↳ the two-period model allows us to understand the economy due to its simplicity. But now, we can relax some of its simplifying assumptions:

1. we now consider an infinite time horizon
2. we consider the effects of both asset price and dividend returns, since they can fluctuate with time.
3. we consider nominal prices more

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② consumer's ∞ budget constraints

1) savings in the form of stocks

↳ suppose instead of just a catch-all asset s, we model savings as an asset that has a price and dividend in some later period, a_t — real units of stock

price — nominal stock price S_{t+i}

dividend — nominal dividend return D_{t+i}

↳ suppose asset bought at time period t, a_t, has stock price S_i and dividend return D_i. Then if we hold it infinitely, in each period (that we can trade, and/or we receive a dividend) we get:

At period t ⟶ At period $t+i$

buy $\underbrace{a_t \cdot S_t}_{\text{nominal spending}}$ can sell $S_{t+i} \cdot a_t$, receive $D_{t+i} \cdot a_t$

2) period t budget constraint

inflow = outflow

$$\underbrace{\overbrace{P_t \cdot C_t}^{\text{consumption}} + \overbrace{S_t a_t}^{\text{new shares bought}}}_{\text{outflow}} = \underbrace{\overbrace{Y_t}^{\text{income}} + \overbrace{S_t a_{t-i}}^{\text{nominal value of tradeable shares owned @ start of period, tradeable } i \text{ periods after purchase}} + D_t a_{t-j}}_{\text{inflow}}$$

$D_t a_{t-j}$ — dividends from shares owned from j periods ago, paid only now

nominal interest rate observe that $S_t a_t \longrightarrow \underbrace{S_{t+i} a_t + D_{t+j} a_t}_{S_t a_t (1+i)}$

$$\boxed{1+i = \frac{\cancel{a_t}(S_{t+i} + D_{t+j})}{\cancel{a_t} \cdot S_t}}$$

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③ consumer optimisation

1) time-separable discounted utility

$$U(c_t, c_{t+1} \dots c_{t+\infty}) = u(c_t) + \beta\, u(c_{t+1}) \dots$$
$$= \sum_{i=0}^{\infty} \beta^i \cdot u(c_{t+i})$$

subjective discount factor

↳ as in intertemporal model, the discount factor introduced to represent the idea that utility further out in the future not as valuable as utility closer in time

↳ $\beta \in [0,1]$, and is a crude way of modelling impatience

↳ limited evidence on whether impatience "builds up" over time as we simply raise β^k, $k \to \infty$

2) sequential lagrangian formulation

↳ sequential formulation highlights the role of stock holdings between periods, and thus the interaction between asset prices and macroeconomic events

maximisation

$$\max. \sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$

$$\text{s.t.} \quad P_{t+i} C_{t+i} + S_{t+i} a_{t+i} = Y_{t+i} + a_{t+i-1}(S_{t+i} + D_{t+i}), \quad i \in [0, \infty)$$

$$\mathcal{L} = \sum_{i=0}^{\infty} \beta^i \cdot u(c_{t+i})$$
$$+ \sum_{i=0}^{\infty} \underbrace{\lambda_{t+i} \beta^i}_{\text{can envelope into } \lambda'_{t+i}} \left(Y_{t+i} + \underbrace{(S_{t+i} + D_{t+i}) a_{t+i-1}}_{\text{returns from previous period stock}} - P_{t+i} C_{t+i} - S_{t+i} a_{t+i} \right)$$

not strictly mathematically necessary, but helps w interpretation.

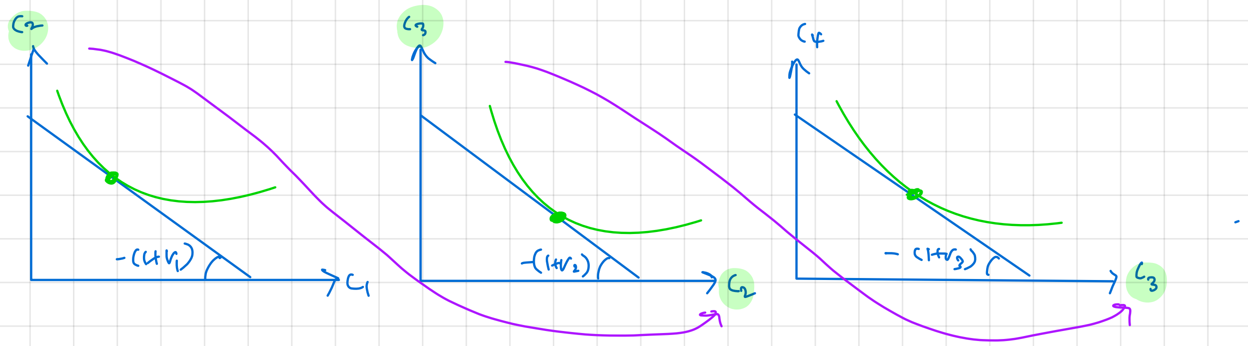
↳ choice variables : c_{t+i} , a_{t+i} , for all i , simultaneously

↳ compute FOCs wrt. $c_t, a_t, c_{t+1}, a_{t+1} \dots$

③ relationships in optimality

interpretation as overlapping 2-periods

↳ observation: for a given time period, a consumer's choices of c_t and a_t are only related to other periods by the returns from previously purchased assets - we can abstract to some i/r, r_t.



⇒ two period model as the starting point for modern macroeconomic theory

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λ_t as MU per dollar

$$\frac{\partial \mathcal{L}}{\partial c_{t+i}} = \cancel{\beta^i} \cdot u'(c_{t+i}) - \cancel{\beta^i} \lambda_{t+i} P_{t+i} = 0$$

appearance of λ_{t+i} used to relate to "relevant" constraints in this formulation, since each period has exactly 1 constraint

$$\lambda_{t+i} = \frac{u'(c_{t+i})}{P_{t+i}}$$

} λ_{t+i} works out to be MU/\$ for each period

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asset pricing

↳ simplifying assumption: assume that previously bought stock becomes tradeable and returns dividends in the same period. Then:

$$\mathcal{L} = \sum_{i=0}^{\infty} \beta^i \cdot u(c_{t+i})$$

$$+ \sum_{i=0}^{\infty} \lambda_{t+i} \beta^i \Big(Y_{t+i} + \underbrace{(S_{t+i} + D_{t+i})\, a_{t+i-j}}_{\text{returns from previous period stock}} - P_{t+i} C_{t+i} - S_{t+i} a_{t+i} \Big)$$

from some j periods ago

⇒ at any given time, optimization only relates period t (buy) and $t+j$ (returns), so simultaneous FOCs to solve for variables involving these two periods only need

$$\frac{\partial \mathcal{L}}{\partial a_t} = -\lambda_t S_t + \beta^j \lambda_{t+j} (S_{t+j} + D_{t+j}) = 0$$

$$S_t = \frac{\beta^j \lambda_{t+j}}{\lambda_t} (S_{t+j} + D_{t+j})$$

pricing kernel

price of financial asset = discounted value of future earnings, with influence of a pricing kernel

↳ observation : for a given time period, a consumer's choices of c_t and a_t are only related to other periods by the returns from previously purchased assets. i.e. when previous shares become tradeable and/or dividends come in. Equivalently, by when they receive the returns of what they buy now. Then :

↳ Also : recall λ_t as mu per dollar.

$$\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - \lambda_t P_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+j}} = \beta^j u'(c_{t+j}) - \beta^j \lambda_{t+j} P_{t+j} = 0$$

$$\frac{\lambda_{t+j}}{\lambda_t} = \left(\frac{u'(c_{t+j})}{P_{t+j}}\right) \Big/ \left(\frac{u'(c_t)}{P_t}\right) = \frac{u'(c_{t+j})}{u'(c_t)} \cdot \frac{P_t}{P_{t+j}}$$

$$\Rightarrow \quad S_t = \underbrace{\beta^j \left(\frac{u'(c_{t+j})}{u'(c_t)}\right)\left(\frac{P_t}{P_{t+j}}\right)}_{\text{pricing kernel}} (S_{t+j} + D_{t+j})$$

intuitive drivers

interpretation : stock price at a given period $t+i$, assuming all consumers are price-taking optimizers, depend on :

1. relative marginal utilities in later period to current period. If later more, makes more sense to save.

2. discount factor. offsets utility gain in future by how far away it is.

3. stock price and dividend in future, if higher, get more for saving

4. relative price levels in current to future period; adjusting nominal gains in real units

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$\boxed{\text{notion of steady state}}$ a situation in which all real variables stop fluctuating over time. (not necessarily so for nominal variables).

↳ think of it as long-run equilibrium, or "potential" performance of the economy.

↳ for real variable x, steady state when $x_t = x_{t+1} = x_{t+2} \ldots \bar{x}$

Steady state: impatience and real i/r

↳ observation: in an infinite period framework, we have:

$$s_t = \beta^j \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) \left(\frac{p_t}{p_{t+j}} \right) (s_{t+j} + D_{t+j})$$

rearranging, we get

$$\underbrace{\frac{S_{t+j} + D_{t+j}}{P_{t+j}} \Big/ \frac{s_t}{P_t}}_{\text{real returns of assets}} = \frac{u'(c_t)}{u'(c_{t+j})\,\beta^j}$$

real returns of assets $\Rightarrow 1 + r_t$

in steady state, $x_t = \bar{x}$. so:

$$1 + \bar{r} = \frac{\cancel{u'(\bar{c})}}{\cancel{u'(\bar{c})}\,\beta^j} = \frac{1}{\beta^j}$$

⇒ in the long run, the real interest rate — i.e. returns from loaning real resources to others, or borrowing from others — is fundamentally tied to the degree of impatience of consumers in the economy.

⇒ the most fundamental source of real interest rates, and why they cannot be negative, is because of human impatience, and the fact that will (on average) never view <u>future</u> consumption as <u>more</u> beneficial than current.

i.e. $\beta \in [0, 1]$.

Fundamentals of fiscal policy

① government budget constraint

budget constraint

↳ for simplicity, we consider that the government has the discretion to collect tax revenues T_i and spend G_i in each time period, as well as buy or sell assets to fund its expenditures.

↳ then we have it that in each period:

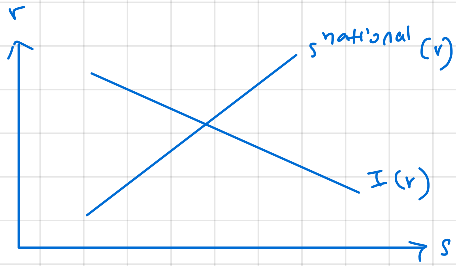
in this case, saving

$$\underbrace{g_t}_{\text{spending}} + \underbrace{b_t}_{\text{asset position}} = \underbrace{(1+r)\, b_{t-1}}_{\text{value of assets}} + \underbrace{t_t}_{\text{tax revenue}}, \text{ in real terms}$$

national savings

↳ idea: govt. and consumers both save (on aggregate), supplying loanable funds in the credit market. Then firms looking to invest in capital borrow these funds to grow their capital resources.

$$\boxed{\begin{aligned} s_t^{govt} &= t_t - g_t \\ s_t^{private} &= y - t_t - c_t \end{aligned}} \Bigg\} \; s_t^{national} = s_t^{govt} + s_t^{private}$$



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② Lump sum taxes

↳ questions: how does the govt. tax consumers, and how does that affect their optimal choices? ⟹ simplest model: lump sum taxation

1) lump sum taxation

↳ a tax whose incidence (amt. paid) is independent of the choices made

↳ then following our model w/ stocks, consumer budget:

$$\underbrace{\overbrace{p_t \cdot c_t}^{\text{consumption}} + \overbrace{s_t a_t}^{\text{new shares bought}}}_{\text{outflow}} + t_t = \underbrace{\overbrace{y_t}^{\text{income}} + \overbrace{s_t a_{t-j}}^{\text{nominal value of tradeable shares owned @ start of period, tradeable j periods after purchase}} + D_t a_{t-j}}_{\text{inflow}}$$

dividends from shares owned from j periods ago, paid only now

③ Ricardian equivalence

Theorem: for a given level of govt. expenditure for the first & second period, the exact timing of taxes has no impact on the real economy (eg. c^*, national savings)

i.e. the consumption choices and real i/r do not change when you change taxes in any period, so long as you fix G_t ⇒ because it will be financed through private savings from Δ disposable income

1) intuition

↳ individuals consume out of lifetime wealth

↳ so long as he knows govt. will spend G_1 now and G_2 later, he knows that a tax cut now means a tax raise later, and vice versa. So the change in disposable income from changes in taxes enters the credit market (eg. consumers save for future taxes), which are used to compensate for a govt.'s deficit / surplus based on the tax changes.

⇒ new govt. borrowing matches new consumer lending ⟶ competitive eqm., no change in c^* or r^*, only s^* and B^*

2) mathematically

1. consumer maximizes in infinite time horizon.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad P_t \cdot c_t + s_t a_t + t_t = Y_t + s_t a_{t-1} + D_t a_{t-1}$$

2. He chooses c_t, s_t at every time period, simultaneously
3. we model this in the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t \beta^t \left(P_t c_t + s_t a_t + t_t - Y_t - a_{t-1}(s_t + D_t) \right)$$

⇒ observe that FOCs wrt. s_t, a_t are independent of t_t

3) implications

↳ we can think of government expenditure & borrowing, like a consumer, as a "forced" manner of resource allocation across time for the whole of society

↳ The Ricardian equivalence tells us that there is no free lunch: government spending must be funded by taxation, and for a given degree of "forced" expenditure, consumers will be no different over their lifetimes because they will eventually have the same lifetime disposable income. We cannot create value out of thin air.

→ neither better nor worse

4) assumptions

1. changes in taxes are identical for every consumer.

↳ if change in tax is different for each consumer, then they face different impacts on their calculus to maximize life-time utility. consumption baskets will not be unchanged in general.

2. Debt is paid off eventually by the same consumers

↳ if across two periods, it is the same set of consumers, then the burden they accrue must be paid off by them. So they would not be better off.

↳ But if they are different groups, then you cannot conclude they will be the same — since their representative consumers are different!

☆ 3. Taxes are non-distortionary

↳ lump-sum taxes are non-distortionary since they do not influence per-unit consumption and choices

↳ unit taxes affect price signals and so distort consumption choices. If a distortionary tax is changed, then the decision landscape facing the consumer is different, and we may or may not be better off.

☆ 4. credit markets are perfect — no credit constraints that are binding

↳ The equivalence assumes that govt. surplus can be borrowed by consumers and vice versa and price signals are not distorted. And because credit flows smoothly as an intermediate, the final outcome is the same.

↳ in reality, diff. interest rates due to uncertainties and credit constraints restrict the flow of credit : so changes in taxes are not exactly offset by shifts in the credit market.

proof : by revealed preference

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④ Distortionary taxes and their effects

proportional taxes : taxes whose incidence depends on choices the agent makes.

i) consumption taxes and its implications

$$(1+\tau_1)c_1 + s_1 = y_1 + (1+r)s_0$$
$$(1+\tau_2)c_2 + s_2 = y_2 + (1+r)s_1$$

$$(1+\tau_1)c_1 + \frac{(1+\tau_2)c_2}{1+r} = y_1 + \frac{y_2}{1+r} + (1+r)s_0$$

$$\Rightarrow c_2 = -\left(\frac{1+\tau_1}{1+\tau_2}\right)(1+r)c_1 + \left(y_1 + \frac{y_2}{1+r}\right)\left(\frac{1+\tau_1}{1+\tau_2}\right)$$

budget line slope is $-\left(\frac{1+\tau_1}{1+\tau_2}\right)(1+r)$

$\Rightarrow$ distortionary effects on c_1^*, c_2^*

how does $\Delta \tau_t$ affect c_t^* and $s_t^{national}$?

1. consideration of infinite period as a series of two period optimisations

$$(1+\tau_1) c_1 + s_1 = y_1 + (1+r) s_0$$
$$(1+\tau_2) c_2 + s_2 = Y_2 + (1+r) s_1$$

$$(1+\tau_1) c_1 + \frac{(1+\tau_2) c_2}{1+r} = Y_1 + \frac{Y_2}{1+r} + (1+r) s_0$$

$$\Rightarrow \underbrace{c_2 = -\left(\frac{1+\tau_1}{1+\tau_2}\right)(1+r) c_1 + \left(y_1 + \frac{y_2}{1+r}\right)\left(\frac{1+\tau_1}{1+\tau_2}\right)}$$

budget line slope is $-\left(\frac{1+\tau_1}{1+\tau_2}\right)(1+r)$

$\Rightarrow$ distortionary effects on c_1^*, c_2^* since TOT is affected

2. consideration of income & substitution effects

2.1 suppose $\Delta \tau_1 > 0$. $\tau_1 \uparrow$.

2.2 Then by substitution effect, we know that $c_1^* \downarrow$, $c_2^* \uparrow$, $s_1 = Y_1 - c_1 - g_1 \uparrow$

2.3 income effect is ambiguous, depends on exact form of u.

2) asset and dividend taxes

↳ without distortionary taxes, we have it that

$$S_t = \beta^j \left(\frac{u'(c_{t+j})}{u'(c_t)}\right)\left(\frac{P_t}{P_{t+j}}\right)(S_{t+j} + D_{t+j})$$

↳ but suppose we consider a dividend tax, and a consumption tax. Then budget is

$$P_t C_t + P_t^b B_t + m_t + S_t a_t = Y_t + m_{t-1} + B_{t-1} + S_t a_{t-1} + D_t a_{t-1}$$

$P_t C_t \Downarrow P_t(1+\tau_t^c) c_t$

$D_t a_{t-1} \Downarrow (1-t_t^D) D_t a_{t-1}$

↳ optimising, we have that $\frac{\partial L}{\partial a_t} = 0$, $\frac{\partial L}{\partial c_t} = 0$, $\frac{\partial L}{\partial c_{t+1}} = 0$

$$\Rightarrow S_t = \frac{\beta u'(c_{t+1})}{u(c_t)} (S_{t+1} + (1-t_{t+1}^D) D_{t+1}) \left(\frac{P_t}{P_{t+1}}\right) \frac{1+\tau_t^c}{1+\tau_{t+1}^c}$$

how does τ_t^c affect S_t ?

↳ observe that τ_t^c directly affects S_t, but also affects c_t (and thus $\frac{u'(c_{t+1})}{u'(c_t)}$

↳ how? Let's consider ceteris paribus, an increase in τ_t^c. Then the substitution effect is directly in the equation — higher dividend tax, less demand, lower price.

↳ the income effect may appear ambiguous, but we can reason about it given what we know about c_t and r in general.

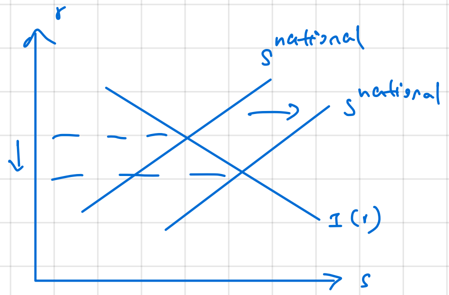
1. we know that an optimality condition is s.t. $\frac{u'(c_{t+1})}{u'(c_t)} = 1+r$

2. we also know that as $\tau_t^c \uparrow$, $c_t \downarrow$ by the substitution effect (which we assume will dominate). so $S_t \uparrow$.

2.1 in other words, $S_t^{national} = S_t^{private} + S_t^{govt} \uparrow$.

2.2 The increase in loanable funds supply implies $r \downarrow$.

3. So $\frac{u'(c_{t+1})}{u'(c_t)} = 1+r \downarrow$. so S_t will fall.



3) taxation on (general) interest returns

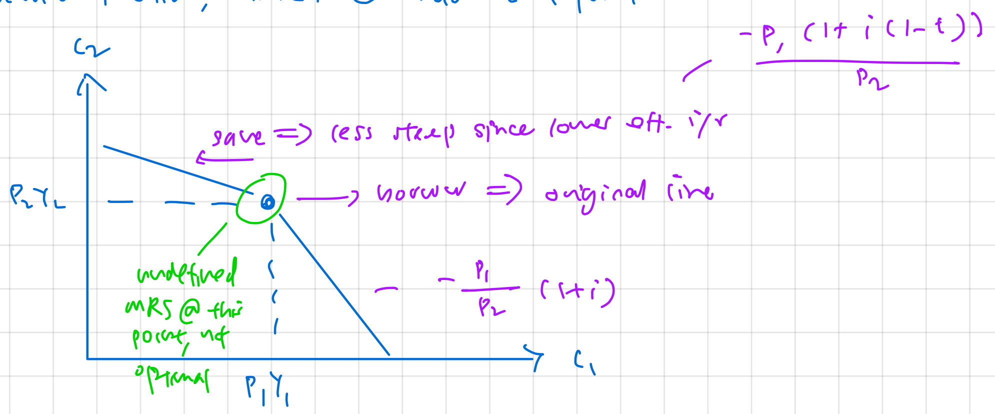
↳ usually, we have IBC where $P_1 c_1 + \frac{P_2 c_2}{1+i} = P_1 y_1 + \frac{P_2 y_2}{1+i}$

↳ if we tax on i/r iff. i/r returns > 0, we get a piecewise function!

⇒ effective $i = i(1-t)$

$$\begin{cases} P_1 c_1 + \frac{P_2 c_2}{1+i} = P_1 y_1 + \frac{P_2 y_2}{1+i} & \text{if } c_1 \geq y_1 \text{ (borrowed)} \\ P_1 c_1 + \frac{P_2 c_2}{1+i(1-t)} = P_1 y_1 + \frac{P_2 y_2}{1+i(1-t)} & \text{if } c_1 \leq y_1 \text{ (loaned, earned i/r)} \end{cases}$$

→ IBC is piecewise function, kinked @ endowment point



monetary and fiscal policy with infinite periods

① **money**

1) roles of money
- 1. medium of exchange : removes double coincidents of wants
- 2. unit of acount
- 3. store of value : purchasing power

2) the concept of money and price

people use money to transact because we don't have ideal "real units"

↳ in real life, a barter economy cannot work because of a double coincidence of wants
↳ money is the solution : a traded placeholder of value ie. we need money for all transactions

how much money do we pay for a unit of consumption ?
↳ money as a unit is a placeholder for some unit(s) of consumption.
↳ But how many units ? suppose there are more dollars in the economy than units of real consumption at equilibrium - what then ?

↓

the terms of trade between a dollar and a unit of consumption changes to reflect the dollars & units of consumption available

concept of price level : The TOT between a unit of money and a unit of consumption

3) the functional benefits of money : MIU model
↳ aside from the notion of transactable value, we need to consider that money and cash must be held to make transactions.

↳ but how should we model this functional role of money?

⇒ idea : that there is cash circulating in the economy, and different agents holding different amounts of it, and some want cash more than others, to make more / less transactions with its purchasing power

⇒ money-in-utility model : real purchasing power of money gives people utility

$$u_t = u\left(c_t, \frac{M^P_t}{P_t}\right)$$

② bonds as an asset

1) what is a bond? A debt obligation. If I buy a bond, I give my money to the seller in exchange for a fixed sum later on

2) returns from bonds

↳ in our intertemporal model, we consider that individuals "save" (i.e. loan for returns) by buying bonds (general loan) or equity (which return price & dividend)

↳ let's use a numeraire: we define P_t^b to be the price of 1 dollar of return in next period. Then in present value nominal terms:

$$\boxed{P_t^b = \frac{1}{1+i_t}}$$

3) why should we model bonds?

↳ in the simplest intertemporal model, due to impatience or govt. budget imbalance, individuals purchase equity to gain returns.

↳ we introduce a more "standard" notion of loans with bonds. Think of it as another alternative asset to holding cash.

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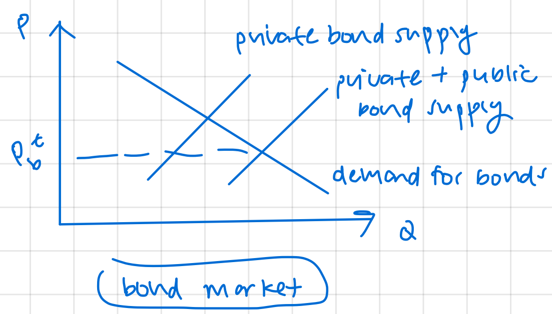
③ bonds and money supply

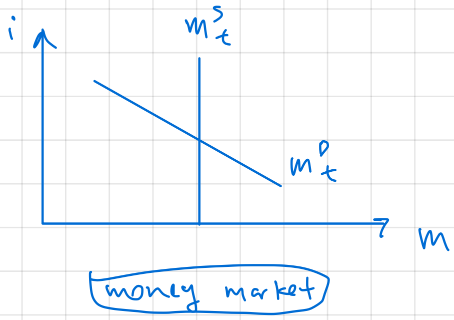
↳ observe that the central bank can print money at anytime — including in future.

↳ Also observe that all private agents can buy or sell bonds — including government bonds.

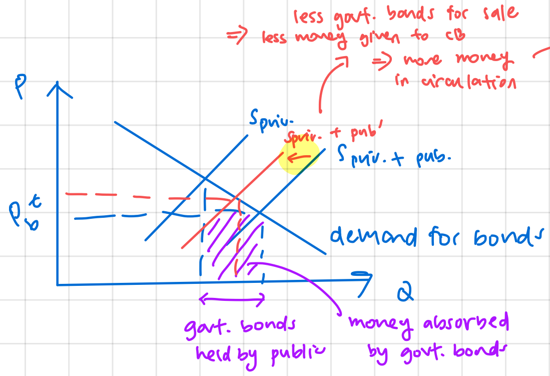
↳ Also observe that the act of buying a bond is also giving money away (in exchange for future monies)

⇒ by selling (taking cash) or buying (giving cash), the government can control money supply!

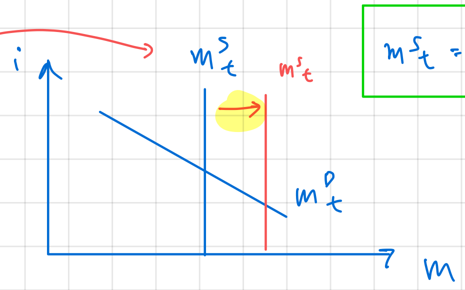




open market operations Thus, precisely by buying and selling bonds, the government can increase or absorb money in circulation.



bond market



money market

$$m^S_t = m^D_t = m_t$$

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④ consumer optimisation with bonds

1) budget constraint

$$P_t c_t + \underbrace{P_t^b B_t}_{\substack{\text{buy new}\\ \text{bonds at}\\ P_t^b < 1}} + \underbrace{m_t}_{\substack{\text{remaining}\\ \text{cash}}} + S_t a_t = Y_t + \underbrace{m_{t-1}}_{\substack{\text{previous}\\ \text{cash holdings}}} + \overbrace{B_{t-1}}^{\text{maturing bonds return 1 unit}} + S_t a_{t-1} + D_t a_{t-1}$$

2) optimisation and intuitive relationships

sequential lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t, \frac{m_t}{P_t}\right)$$
$$- \sum_{t=0}^{\infty} \lambda_t \beta^t \left(P_t c_t + P_t^b B_t + m_t + S_t a_t - Y_t - m_{t-1} - B_{t-1} - S_t a_{t-j} + D_t a_{t-j} \right)$$

⇒ choice variables of c_t, a_t, m_t, B_t, λ_t

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \frac{\partial u(c_t, \frac{m_t}{P_t})}{c_t} - \lambda_t P_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_t} = 0 \Rightarrow -\lambda_t S_t + \beta \lambda_{t+1} (S_{t+j} + D_{t+j}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0 \Rightarrow -\lambda_t P_t^b + \beta \lambda_{t+1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0 \Rightarrow \frac{\partial u(c_t, \frac{m_t}{P_t})}{m_t} - \lambda_t + \beta \lambda_{t+1} = 0 \Longleftrightarrow \frac{\partial u(c_t, \frac{m_t}{P_t})}{(P_t)\, \partial\left(\frac{m_t}{P_t}\right)} - \lambda_t + \beta \lambda_{t+1}$$

note chain rule!

no arbitrage condition

$$\frac{\partial L}{\partial B_t} = 0 \Rightarrow p^b_t = \frac{\beta \lambda_{t+1}}{\lambda_t}(1)$$

$$\frac{\partial L}{\partial a_t} = 0 \Rightarrow S_t = \frac{\beta \lambda_{t+1}}{\lambda_t}(S_{t+1} + D_{t+1})$$

pricing kernel as some driver relating nominal asset price to nominal returns.
$\Rightarrow$ observe that pricing kernel is the same

$$\frac{\beta \lambda_{t+1}}{\lambda_t} = p_t^b = \frac{S_t}{S_{t+1} + D_{t+1}} = \frac{1}{1+i} = \frac{1+r}{1+\pi} = \frac{P_t(1+r)}{P_{t+1}}$$

$\Rightarrow$ when consumer is optimising, $\frac{\text{price}}{\text{return}}$ is the same for all assets. no arbitrage!

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consumption – money optimality

intuitive derivation

↳ how much cash does an optimising rational consumer hold?

1. Let's think about the TOT between holding 1 unit of real purchasing power (P_t) now and a unit of consumption now, to determine the opportunity cost.

2. present value return of 1 unit of P.P. (P_t) in next period : (cash)

   hold (1) P_t $\Rightarrow$ in next period worth $\frac{P_t}{P_{t+1}}$ in real terms $\Rightarrow$ $\frac{P_t}{P_{t+1}} \cdot \frac{1}{1+r}$ in PV.

3. present value return of 1 unit of consumption in next period :

   1 unit of consumption = P_t $\Rightarrow$ can use to buy bond / stock (same return due to no arbitrage),

   $\Rightarrow \frac{P_t}{P^b_t}$ bonds $\Rightarrow \frac{P_t}{P^b_t} \frac{1}{P_{t+1}}$ in real terms in next period

   $\Rightarrow \frac{P_t}{P^b_t}\left(\frac{1}{P_{t+1}}\right)\left(\frac{1}{1+r}\right)$ in present value

4. so opportunity cost = total value forgone with choice

   = returns from alternative – returns from choice

$$= \frac{P_t}{P^b_t}\left(\frac{1}{P_{t+1}}\right)\left(\frac{1}{1+r}\right) - \frac{P_t}{P_{t+1}}\left(\frac{1}{1+r}\right)$$

$$= \frac{1}{(1+\pi)(1+r)}\left(\frac{1}{P^t_b} - 1\right)$$

$$= \frac{1}{i+1}\left((1+i) - 1\right) = 1 - \frac{1}{i+1} \quad //$$

5. utility maximisation when $MRS_{\frac{M_t}{P_t}, C_t}$ = opportunity cost

intuition

A rational, infinite period consumer chooses choice variables for all periods simultaneously, and because of the "infinite overflow", of two-periods, we compare the present value returns of any two variables to get their opportunity cost. Think of it as multiple simultaneous optimisations, so we convert everything to a "common currency" of present value returns.

mathematical derivation

$$\frac{\partial L}{\partial B_t} = 0 \Rightarrow -\lambda_t P_t^b + \beta \lambda_{t+1} = 0 \Rightarrow P_t^b = \beta \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1+i_t}$$

$$\frac{\partial L}{\partial M_t} = 0 \Rightarrow \frac{\frac{\partial u(c_t, \frac{M_t}{P_t})}{\partial (\frac{M_t}{P_t})}}{\lambda_t P_t} - 1 = -\beta \frac{\lambda_{t+1}}{\lambda_t} = -\frac{1}{1+i_t}$$

$$\frac{\partial L}{\partial c_t} = 0 \Rightarrow \frac{\partial u(c_t, \frac{M_t}{P_t})}{\partial c_t} = \lambda_t P_t$$

$$\frac{\frac{\partial u(c_t, \frac{M_t}{P_t})}{\partial (\frac{M_t}{P_t})}}{\frac{\partial u(c_t, \frac{M_t}{P_t})}{\partial c_t}} - 1 = -\frac{1}{1+i_t} \Rightarrow \boxed{MRS_{(\frac{M_t}{P_t}), c_t} = 1 - \frac{1}{1+i_t} = \frac{i_t}{1+i_t} = 1 - P_t^b}$$

---

money demand

↳ consumption-money optimality relates optimal choices of M_t to P_t, c_t and i_t, and we can use it to derive money demand.

$$MU_{(\frac{M_t}{P_t})} = MU_{c_t}\left(1 - \frac{1}{1+i_t}\right) \Rightarrow \text{make } M_t \text{ subject}$$

↳ intuitive interpretation : representative consumer's demand for cash in the current period is determined by current prices, how much he wants to optimally consume (c_t) and save (i_t)

Zero lower bound on interest rates

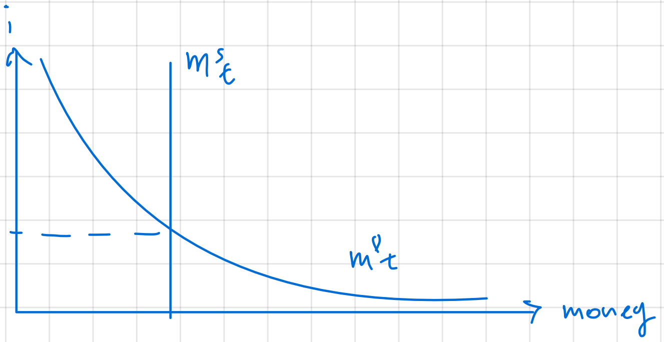
↳ observe that from the consumption-money optimality condition, if $i \le 0$, then $MRS_{\frac{M_t}{P_t}, C_t} \le 0$, which would imply either one of money or consumption had $MU \le 0$, violating most reasonable assumptions of utility.

⇒ $MU_{C_t} < 0$: then consumption gives negative utility

⇒ $MU_{\frac{M_t}{P_t}} \le 0$: convenience of purchasing power is negligible or negative

⇒ recall that an optimizing consumer consumes at opportunity cost. so a negative i_t reflects these behaviors of optimizing consumers!

⇒ Hence emerges the zero lower bound restriction on nominal interest rates



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⑤ monetary policy in the short run (short run) effects in a single period

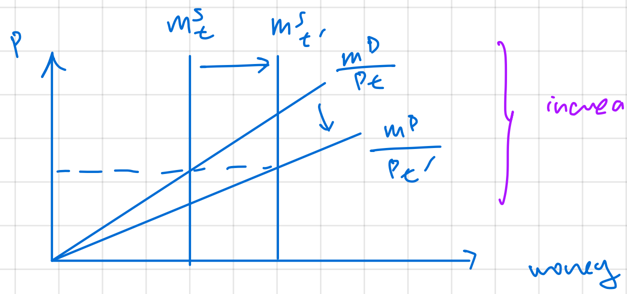
(monetary policy) central bank's intervention in the money market via money supply

(money neutrality) money is neutral if changes in the money supply at eqm. have no real effects on the economy

1) The real business cycle (RBC) view

↳ nominal prices adjust quickly
↳ changes in money supply simply change the ToT between 1 unit of consumption and 1 unit of money, since prices can change to reflect that

⇒ money (and mp) is neutral



increase in M^S ⇒ rise in prices

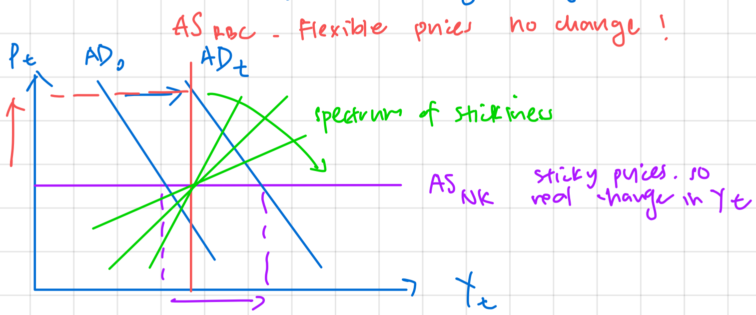
2) New keynsian model

↳ prices are sticky in the short run. with a constant ToT between real consumption and 1 unit of money, and changes in money, there are real effects in the short run

↳ c_t or i_t will adjust to reflect real changes

↳ money (and MP) are not neutral

3) implications of money neutrality on growth



period t goods market

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⑥ monetary policy in the long run — long run: consider effects across time periods

1) what are the real variables in the long run ?

↳ we consider growth of variables (nominal or real) as real variables

2) what is monetary policy in the long run?

↳ rate of growth of money supply, since that's what the central bank controls across time periods

3) is monetary policy neutral in the long run ?

1. money demand, $M^D_t = P_t f(c_t, i_t)$

2. Across two periods, $\frac{M^D_{t+1}}{M^D} = 1+\mu_t = \frac{P_{t+1}}{P_t} \frac{f(c_{t+1}, i_{t+1})}{f(c_t, i_t)}$ — rate of M^S growth

3. in long run steady state, $\frac{P_{t+1}}{P_t} = 1+\pi_t \longrightarrow 1+\bar{\pi}$. so $1+i_t = (1+\bar{r})(1+\bar{\pi})$ also goes to steady state.

$$\Rightarrow \quad 1+\mu_t = 1+\bar{\pi}_t \frac{\cancel{f(\bar{c}, \bar{i})}}{\cancel{f(\bar{c}, \bar{i})}} \quad \Rightarrow \quad \boxed{\mu = \bar{\pi}}$$

"monetarist" school of thought

4. in the long run, money growth = rate of inflation. so change in money only affects nominal (price) variables, and is thus neutral.

Fiscal and monetary policy

① policy authorities

1) fiscal authorities

- ↳ controls govt. spending g_t, collects taxes
- ↳ issues bonds and receives any left over "profit" from central bank, since it legally charters central bank

2) monetary authority (central bank)

- ↳ controls money supply by engaging in open market operations. Bonds sold are bought from govt.

② consolidated government flow budget

1) fiscal authority's flow budget

$$\underbrace{\underbrace{P_t g_t}_{\text{spending}} + \underbrace{B^F_{t-1}}_{\text{payment of borrowing (bonds) in previous period, to CB or public}}}_{\text{outflow}} = \underbrace{\overbrace{T_t}^{\text{taxes}} + \overbrace{P^b_t B^F_t}^{\text{new borrowing}} + \underbrace{RCB_t}_{\text{receipts turned over from CB. to fiscal authority}}}_{\text{inflow}}$$

2) monetary authority's flow budget

total borrowing, $B^F_t = B^m_t$ + bonds bought by public B_t

$$\underbrace{\underbrace{P^b_t B^m_t}_{\text{bonds bought in open mkt. operations (loans to public, ↑ MS)}} + \underbrace{RCB_t}_{\text{leftover receipts to fiscal authority}}}_{\text{outflow}} = \underbrace{\underbrace{B^m_{t-1}}_{\text{maturing bonds from previous period}} + \underbrace{(M_t - M_{t-1})}_{\text{printing of more money}}}_{\text{inflow}}$$

③ single period fiscal-monetary interactions

1) consolidated flow budget constraint

↳ to observe how two parts of govt. act as a single entity, we can combine their flow budget constraints

C.B. | fiscal authority

$$P_t^b \cdot B_t^m + RCB_t = B_{t-1}^m + M_t - M_{t-1} \qquad P_t g_t + B_{t-1}^F = P_t B_t^F + T_t + RIB_t$$

$$\underbrace{P_t g_t}_{\text{fiscal spending}} + \underbrace{B_{t-1}^F}_{\text{repayment of bonds issued (sold) in prev. period}} - \underbrace{B_{t-1}^m}_{\text{less the amt. central bank bought back}} = \underbrace{T_t}_{\text{tax}} + \underbrace{P_t^b (B_t^F - B_t^m)}_{\text{borrowing less internal transfer / borrowing}} + \underbrace{(M_t - M_{t-1})}_{\text{printing of money}}$$

↳ when we consolidate, let's consolidate $B_t^F - B_{t-1}^m = B_t$, bonds bought by private sector

$$\Rightarrow \boxed{\underbrace{P_t g_t + B_{t-1}}_{\text{outflow as a single govt. entity}} = \underbrace{T_t + P_t^b B_t + (M_t - M_{t-1})}_{\text{inflow as a single govt. entity}}}$$

2) The concept of seignorage revenue

seignorage revenue The real quantity of resources the govt. raises for itself by money creation

↳ intuition : printing money splits purchasing power over more units of money — of which the govt. holds the new monies! ⇒ printing more is essentially an "inflation tax", moving purchasing power to the govt.

$$\boxed{sr_t = \frac{M_t - M_{t-1}}{P_t}}$$

3) Active vs. passive policy

active policy: A policy authority is <u>active</u> if <u>every instrument</u> at its disposal can be freely chosen, without consideration to the consolidated GBC, and <u>passive</u> otherwise

policy pressure: observe that money cannot appear out of thin air. That is, the flow GBC must hold. If one authority is active, then the author is constrained, and must react accordingly with limited options to balance the budget.

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④ <u>implications of infinite period constraints</u>

↳ analysis of single period GBC reveals that the period-t choices of one authority <u>restrict</u> the choices of the other policy authority

↳ but when we consider an infinite period framework, we realize that the actions (and consequences) of a given period can flow to all future periods!

1) lifetime GBC

↳ sustainable govt. financing come from taxation & seignorage revenues.

↳ observe that debt to be repaid in a given period can be "rolled over", it can be paid for by borrowing more (since ∞ periods!)

$$P_t g_t + B_{t-1} = \underbrace{P_t^b B_t}_{\text{new borrowing}} + T_t + (M_t - M_{t-1})$$

$$\Leftrightarrow \underbrace{\frac{B_{t-1}}{P_t}}_{\substack{\text{debt repayments}\\ \text{in real terms}}} = \underbrace{(t_t - g_t)}_{\substack{\text{budget}\\ \text{surplus}}} + \underbrace{\left(\frac{M_t - M_{t-1}}{P_t}\right)}_{\substack{\text{seignorage}\\ \text{revenues}}} + P_t^b \underbrace{\frac{B_t}{P_t}}_{\substack{\text{new borrowing}\\ \text{in real terms}}}$$

↳ let's express current debt in terms of real resources in an ∞ period.

$$\frac{B_t}{P_{t+1}} = \frac{B_t}{P_t}\cdot\frac{P_t}{P_{t+1}} = \frac{b_t}{(1+\pi_{t+1})} = b_t\,\frac{1+r_t}{1+i_t} = (t_{t+1} - g_{t+1}) + sr_{t+1} + P^b_{t+1}\frac{B_{t+1}}{P_{t+1}}$$

$$\Rightarrow b_t = (t_{t+1} - g_{t+1})\frac{(1+i_t)}{1+r_t} + sr_{t+1}\frac{(1+i_t)}{1+r_t} + P^b_{t+1}\frac{B_{t+1}}{P_{t+1}}\left(\frac{1+i_t}{1+r_t}\right)$$

sub

$$\frac{B_{t-1}}{P_t} = (t_t - g_t) + \left(\frac{m_t - m_{t-1}}{P_t}\right) + \underbrace{P_t^b}_{\frac{1}{1+i_t}} \frac{B_t}{P_t}$$

$$= (t_t - g_t) + sr_t + \frac{(t_{t+1} - g_{t+1})}{1+r_t} + \frac{sr_{t+1}}{1+r_t} + \frac{P^b_{t+1} \, b_{t+1}}{1+r_t}$$

link to next period fiscal surplus & seignorage revenues

↳ if we recursively express b_i in terms of sr_{i+1} and $(t_{i+1} - g_{i+1})$, we get an infinite series.

$$\boxed{\frac{B_{t-1}}{P_t} = \underbrace{sr_t + \sum_{s=1}^{\infty} \frac{sr_{t+s}}{\prod_{x=1}^{s}(1+r_{t+x-1})}}_{\text{present value of } \infty \text{ seignorage revenues}} + \underbrace{(t_t - g_t) + \sum_{s=1}^{\infty} \frac{t_{t+s} - g_{t+s}}{\prod_{x=1}^{s}(1+r_{t+x-1})}}_{\text{present value of } \infty \text{ fiscal surpluses}}}$$

⇒ debt incurred today will be paid for by something in the next periods

2) steady state consolidated GBC

↳ in steady state, sr also tends to a constant.

⑤ Ricardian fiscal policy & price level

1) Ricardian fiscal policy

Definition: A Ricardian fiscal policy is in place if the fiscal authority sets its planned sequence of tax and spending to ensure that consolidated lifetime GBC is balanced, and non-Ricardian otherwise

↳ idea: Ricardian if all $(t_i - g_i)$ balance themselves out without need for help for sr_i

2) non-Ricardian fiscal policy

↳ suppose fiscal authority is active, and changes spending in a non-ricardian way — that is, future surpluses (in present value) cannot offset debts today (from deficit).

↳ but GBC must hold. How ?

monetary authority raises PV of lifetime sr

↓

at least one period's money creation increased. steady state / avg. π also increases

- at least one period has higher π
- steady state / avg. π is increased

⇒ monetary authority's response has implications on future inflation and was ultimately induced by fiscal policy

monetary authority helps a bit

↓

higher P_t, high inflation in future

monetary authority does nothing to prospective PV of sr

↓

govt. must somehow pay for current (one time) deficit

↓

inflation tax. Price level rises, and real value of debt falls

⇒ mechanism behind is that there are insufficient real resources going around to fund spending, but amt. of money is still being pumped in, so P_t rises to adjust for constant purchasing power

$P_t \uparrow$, $\frac{B_{t-1}}{P_t} \downarrow$

Fiscal theory of exchange rates

① exchange rates

$$E_t = {}^{\text{no. of \$B}}/_{\text{no. of \$A}}$$

1) definitions

nominal exchange rate: price of one currency in terms of another currency. Note that every exchange rate can be expressed in two directions

appreciation & depreciation: relative terms — speaks about the ratio, the change in exchange rate against another

real exchange rate:

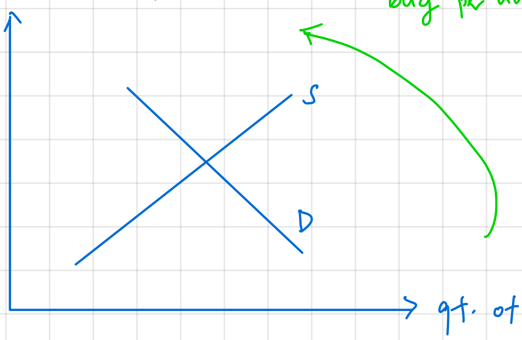
$$e_t = \frac{E_t P_t^*}{P_t}$$

converted real purchasing power of other

1 unit purchasing power of mine

2) demand and supply

P of B in units of A — how much B A can buy per unit



key notes: price is relative. qt. is in terms of what you have, price is how much of other you can buy per unit of what you have

② purchasing power parity condition

purchasing power parity: when prices of goods in any two different countries are converted into the same currency units, they are the same. i.e. No arbitrage by exchange rate

⇒ identical baskets of goods in different countries have the same prices when converted to a common currency

↳ mathematically, $P_t = E_t P_t$

↳ PPP is generally a long run condition

③ interest parity condition

interest parity condition: That there is no arbitrage of interest rates. That is, taking expected exchange rate into account, a nominal dollar in country A returns the same if it is converted to another currency, gains interest in country B, then converted back.

mathematically

$$1 + i_t = 1 + i_t^* \frac{(E^e_{t+1})}{E_t}$$

expected, because future unknown, investor makes decisions on expectation

convert back in next period

convert to B

④ Fiscal theory of exchange rates

foreign reserves a central bank's holding of foreign currencies for the purpose of govt. international transactions (most commonly e/r interventions)

floating / fixed e/r a nominal e/r is "floating" if it is determined solely by the forces of demand and supply. It is "fixed"/ pegged if it is determined through government intervention in exchange markets in order to fix the rate at some value.

1) fiscal theory of exchange rates & assumptions

assumptions

1. For simplicity, we assume consumption in every period is in "steady state".

$\Rightarrow$ $\boxed{\frac{M_t}{P_t} = \phi(\bar{c}, i_t)}$, so real money demand shifts only w/ i_t

2. purchasing power parity holds.

$e_t = 1$, so $\boxed{\frac{E_t P_t^*}{P_t} = 1 \iff E_t P_t^* = P_t}$

↳ if we set P_t^* as a numeraire, then $\boxed{E_t = P_t}$

3. interest rate parity holds : domestic & foreign financial markets are functioning well.

4. foreign real interest rate is fixed at r^*, and foreign inflation is 0.

then $\boxed{1 + i_t = (1 + i_t^*) \frac{E_{t+1}^e}{E_t} \iff (1 + r_t^*) \frac{E_{t+1}^e}{E_t}}$

modified GBC government borrowing only in foreign assets (foreign reserves). This is reasonably true for developing countries.

$$P_t G_t = T_t + (\underbrace{P_t^b B_t - B_{t-1}}_{\text{local assets}}) + (M_t - M_{t-1})$$

$$\Downarrow$$

$$\underbrace{P_t G_t}_{\text{spending}} = \underbrace{T_t}_{\text{tax}} + \underbrace{(M_t - M_{t-1})}_{\text{printing}} + \underbrace{E_t r^* B_{t-1}^F}_{\text{interest from foreign assets, converted}} + \underbrace{E_t (B_{t-1}^F - B_t^F)}_{\text{decrease in foreign reserves, converted}}$$

outflow (spending) inflow (remaining terms) we encapsulate spending − revenue, in real terms as DEF_t

$\Rightarrow \boxed{DEF_t = G_t - \frac{T_t}{P_t} - r^* B^F_{t-1}}$

P_t gone since $E_t = P_t$ by assumption. $\frac{E_t}{P_t} = 1$

then $\boxed{B_t^F - B^F_{t-1} = \frac{M_t - M_{t-1}}{P_t} - DEF_t}$

deficit is covered by drawdown from reserves, and then seignorage revenue

$\Leftrightarrow$ change in foreign reserves per period depend on surplus / deficit

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⑤ analysing fixed e/r policy in FTER

↳ nominal e/r pegged at some $\bar{E}$

case 1: pegged e/r, individuals expect it will remain in place

Case 1: Fixed exchange rate in place and individuals expect it will always remain in place

- ❑ $E_t = \bar{E}$ **always and** $E^e_{t+1} = \bar{E}$ **always (i.e.,** $\bar{E}$ **is a fixed, unchanging number)**

	Period 1	**Period 2**	**Period 3**
E_t	$E_1 = \bar{E}$	$E_2 = \bar{E}$	$E_3 = \bar{E}$
E^e_{t+1}	$E^e_2 = \bar{E}$	$E^e_3 = \bar{E}$	$E^e_4 = \bar{E}$
i_t **(using IRP condition)**	$1+i_1 = (1+r^*)\cancel{\frac{E^e_2}{E_1}}$ → $i_1 = r^*$	$1+i_2 = (1+r^*)\cancel{\frac{E^e_3}{E_2}}$ → $i_2 = r^*$	$1+i_3 = (1+r^*)\cancel{\frac{E^e_4}{E_3}}$ → $i_3 = r^*$
M_t **(using money demand function)**	$M_1 = P_1\phi(\bar{c}, i_1)$ $M_1 = E_1\phi(\bar{c}, r^*)$ (due to $P_1 = E_1$) → $M_1 = \bar{E}\phi(\bar{c}, r^*)$	$M_2 = P_2\phi(\bar{c}, i_2)$ $M_2 = E_2\phi(\bar{c}, r^*)$ (due to $P_2 = E_2$) → $M_2 = \bar{E}\phi(\bar{c}, r^*)$	$M_3 = P_3\phi(\bar{c}, i_3)$ $M_3 = E_3\phi(\bar{c}, r^*)$ (due to $P_3 = E_3$) → $M_3 = \bar{E}\phi(\bar{c}, r^*)$
SR_t	---	$SR_2 = \frac{M_2 - M_1}{P_2}$ $= 0$ (due to $M_2 = M_1$)	$SR_3 = \frac{M_3 - M_2}{P_3}$ $= 0$ (due to $M_3 = M_2$)

1. domestic nominal i/r equal to foreign real i/r.
   ↳ interest rate parity + no foreign inflation
2. domestic interest rate fixed at r^*_t
   ↳ since interest rate parity, $1+i_t = (1+r^*_t)\cancel{\frac{E_{t+1}}{E_t}}$

3. seignorage revenue = 0

↳ since $\bar{c}$ steady state, M_t only affected by i_t and P_t.

↳ by IRP and fixed e/r, i_t fixed at r^*_t, which we assume constant.

↳ by PPP, $e_t = 1$, so P_t also fixed. Then M_t is constant. $M_t - M_{t-1} = 0$

4. No inflation

↳ no seignorage/spending

↳ also, PPP + reliable peg, $P_t = \bar{P}$

5. if fiscal authority has deficits, drawdown on foreign reserves

case 2: one-time, unanticipated devaluation

Case 2: Unanticipated, one-time devaluation

- Government **unexpectedly** weakens the domestic currency to a new fixed rate and promises **(credibly)** to **never again** change the exchange rate – new rate is $E' > \bar{E}$ (i.e., E' is a fixed, unchanging *number*)

	Period 4	Period 5	Period 6
E_t	$E_4 = E' > \bar{E}$	$E_5 = E'$	$E_6 = E'$
E^e_{t+1}	$E^e_5 = E'$	$E^e_6 = E'$	$E^e_7 = E'$
i_t (using IRP condition)	$1+i_4 = (1+r^*)\frac{\cancel{E^e_5}}{\cancel{E_4}}$ $\longrightarrow i_4 = r^*$	$1+i_5 = (1+r^*)\frac{\cancel{E^e_6}}{\cancel{E_5}}$ $\longrightarrow i_5 = r^*$	$1+i_6 = (1+r^*)\frac{\cancel{E^e_7}}{\cancel{E_6}}$ $\longrightarrow i_6 = r^*$
M_t (using money demand function)	$M_4 = P_4\phi(\bar{c}, i_4)$ $M_4 = E_4\phi(\bar{c}, r^*)$ (due to $P_4 = E_4$) $\longrightarrow M_4 = E'\phi(\bar{c}, r^*)$	$M_5 = P_5\phi(\bar{c}, i_5)$ $M_5 = E_5\phi(\bar{c}, r^*)$ (due to $P_5 = E_5$) $\longrightarrow M_5 = E'\phi(\bar{c}, r^*)$	$M_6 = P_6\phi(\bar{c}, i_6)$ $M_6 = E_6\phi(\bar{c}, r^*)$ (due to $P_6 = E_6$) $\longrightarrow M_6 = E'\phi(\bar{c}, r^*)$
SR_t	$SR_4 = \frac{M_4 - M_3}{P_4}$ > 0 (due to $M_4 > M_3$)	$SR_5 = \frac{M_5 - M_4}{P_5}$ $= 0$ (due to $M_5 = M_4$)	$SR_6 = \frac{M_6 - M_5}{P_6}$ $= 0$ (due to $M_6 = M_5$)

1. devaluation allows for positive seignorage revenue

↳ in pegged case, since IRP and $\bar{c}$ and $\bar{E}$, money supply = money demand is necessarily constant

↳ with devaluation, $\bar{E}$ falls to $\bar{E}'$ (new peg). since IRP, $\bar{c}$ still, money supply necessarily increases

$\Rightarrow$ so $M_t - M_{t-1} > 0$

2. domestic i_t unchanged

↳ IRP implies $1+i_t = (1+r_t^*)\frac{E^e_{t+1}}{E_t}$

↳ since change was unexpected, and promise of new peg credible, then $\frac{E^e_{t+1}}{E_t}$ still at their respective per-period value, ratio is 1.

Case 3: anticipated fall in exchange rate

- **Case 3: Country is maintaining fixed exchange rate $\bar{E}$ *and* running fiscal deficit of *DEF* > 0 every period**
  - i.e., $\bar{E}$ and *DEF* are both fixed, unchanging *numbers*

	Period T-3	Period T-2	Period T-1	Period T
E_t	$E_{T-3} = \bar{E}$	$E_{T-2} = \bar{E}$	$E_{T-1} = \bar{E}$	
E^e_{t+1}	$E^e_{T-2} = \bar{E}$	$E^e_{T-1} = \bar{E}$	$E^e_T = (1+\mu)\bar{E}$	
i_t (using IRP condition)	$1+i_{T-3} = (1+r^*)\frac{\cancel{E^e_{T-2}}}{\cancel{E_{T-3}}}$ $\longrightarrow i_{T-3} = r^*$	$1+i_{T-2} = (1+r^*)\frac{\cancel{E^e_{T-1}}}{\cancel{E_{T-2}}}$ $\longrightarrow i_{T-2} = r^*$	$i_{T-1} > r^*$ (due to $\mu > 0$)	**WHAT HAPPENS HERE?**
M_t (using money demand function)	$M_{T-3} = P_{T-3}\phi(\bar{c}, i_{T-3})$ $M_{T-3} = E_{T-3}\phi(\bar{c}, r^*)$ $\longrightarrow M_{T-3} = \bar{E}\phi(\bar{c}, r^*)$	$M_{T-2} = P_{T-2}\phi(\bar{c}, i_{T-2})$ $M_{T-2} = E_{T-2}\phi(\bar{c}, r^*)$ $\longrightarrow M_{T-2} = \bar{E}\phi(\bar{c}, r^*)$	$M_{T-1} < M_{T-2}$	**Depends on how monetary policy is conducted post-collapse...**
SR_t	---	$SR_{T-2} = \frac{M_{T-2} - M_{T-3}}{P_{T-2}}$ $= 0$ (due to $M_{T-2} = M_{T-3}$)	$SR_{T-1} = \frac{M_{T-1} - M_{T-2}}{P_{T-1}}$ < 0 (due to $M_{T-1} < M_{T-2}$)	**Topic(s) for a course in international finance**
B^G_t (using GBC)	$B^G_{T-3} > 0$ (i.e., assume government has reserves)	$B^G_{T-2} = B^G_{T-3} + \cancel{SR_{T-2}} - DEF_{T-2}$ $\longrightarrow B^G_{T-2} < B^G_{T-3}$ **(Recall Key Result #3)**	$0 = B^G_{T-1} < B^G_{T-2}$	

1. anticipated devaluation (e/r ↑, i.e. need more to buy) by $(1+\mu)$ causes i_t to increase

↳ by IRP, $1+i_t = (1+r^*)\frac{\cancel{E^e_{t+1}}}{\cancel{E_t}}(1+\mu)$ intuitively because returns must be same in nominal terms (IRP)

2. domestic money supply falls, seignorage revenue negative

↳ recall $\bar{c}$, and by IRP, i_t falls. Then $P_t = \bar{E}_t$ by PPP. so money supply falls.

↳ Then SR_t must fall, worsening deficit.

↳ intuitively because e/r to fall, by IRP, people sell local currency to buy foreign, shrinking money supply

⑥ analysis of BOP crisis ū FTER

1) what might cause investors to expect changes in BOP ?

↳ recall that deficits during pegs cause drawdowns of foreign reserves
↳ recall also that if insufficient foreign reserves, govt. can devalue and use seignorage revenue to finance deficit

⇒ when deficit ū insufficient reserves expected, investors expect devaluation

2) phases of BOP collapse

pre-collapse phase govt. pegs currency, but runs a deficit and continues to draw down on foreign reserves. Analysis follows case 1, up to period T-2.

crisis in period T-1. investors expect that foreign reserves insufficient for expected deficits and expect devaluation to allow govt. to use seignorage revenues.

1. expected devaluation triggers individuals to buy foreign assets in foreign currency, to equalise by IRP. This simultaneously shrinks M_t and raises i_t.

2. seignorage revenue becomes negative, worsening deficit, with two sources of drain (DEF and s_r)

post-collapse no more foreign reserves ⟶ floating e/r

Production model of growth

① models of growth : motivation

- factually, we observe that different countries have different levels of income and rates of growth → why do these occur?
- we can build models based on our understanding, describe them mathematically and test their consistency with data

---

② production model

& constant RTS & diminishing MP

1) model overview
   - simplest representation of an economy : single, closed
   - one basket of good
   - production qt. determined by labour & capital inputs, via production function

2) production function $Y = f(K, L) = A K^{\alpha} L^{1-\alpha}$
   - productivity / technological level represented by TFP
   - cobb Douglas w/ constant RTS

increasing outputs w/ inputs

$$\frac{\partial Y}{\partial L}, \frac{\partial Y}{\partial K} > 0$$

diminishing marginal returns

$$\frac{\partial^2 Y}{\partial K^2}, \frac{\partial^2 Y}{\partial L^2} < 0$$

ln allows regression

$$\ln Y = \ln A + \alpha \ln K + (1-\alpha) \ln L$$

constant RTS

$$f(cK, cL) = cY$$

- by replication argument, double all inputs, naturally double all output

- rule of thumb, sum of exponents
  - $= 1 \Rightarrow$ constant RTS
  - $> 1 \Rightarrow$ increasing RTS
  - $< 1 \Rightarrow$ decreasing RTS

per capita / intensive form

$$Y/L = F(K/L, 1) \iff y = A k^{\alpha}$$

$$\frac{A K^{\alpha} L^{1-\alpha}}{L} = A \frac{K^{\alpha}}{L^{\alpha}} \frac{\cancel{L}}{\cancel{L}}$$

3) wages and interest
   - in this model, we assume that workers provide labour input at real wage w, and capital owners rent capital out at interest / rent rate r
   - assuming perfect competition, then we know optimality conditions for firms lie where $\boxed{MPL = w}$ and $\boxed{MPK = r}$
   - we assume fixed supply of labour & capital at $\bar{L}$ and $\bar{K}$ for simplicity

4) TFP

↳ a "multiplier" to both labour and capital

↳ reasons for difference

human capital

↳ stock of skills that individuals accumulate to make them more productive

technology

production techniques, tools etc.

institutions

factors that foster productivity
eg. rule of law, property rights

misallocation

eg. state run organisations, politically motivated use of resources

5) general equilibrium

$Y = A K^{\alpha} L^{1-\alpha}$ — production

$K = \bar{K}$ — capital supply

$MPK = r$ — capital demand @ π max.

$L = \bar{L}$ — labour supply

$MPL = w$ — labour demand @ π max.

6) key results

determinants of wage and rental

↳ in this simplified model, we observe that $w = MPL$, and $r = MPK$. That is, the equilibrium returns for working / renting are precisely proportional to how much output each unit can produce

cobb douglas & factor shares

↳ observe from $Y = AK^{\alpha}L^{1-\alpha}$, $w^* = MPL = AK^{\alpha}(1-\alpha)L^{-\alpha}$, $r^* = MPK = A\alpha K^{\alpha-1}L^{1-\alpha}$

↳ total wage paid out by firm $= \dfrac{w^* \cdot \bar{L}}{Y^*}$, total rent is $\dfrac{r^* \bar{K}}{Y^*}$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\frac{\cancel{A}\cancel{K^{\alpha}}(1-\alpha)\cancel{L^{-\alpha}}\cancel{L}}{\cancel{A}\cancel{K^{\alpha}}\cancel{L^{1-\alpha}}} \qquad\qquad \alpha$$

factor share is their exponent in cobb douglas

consistency w/ national income accounting

↳ firm : $\max_{K,L} \pi = F(K,L) - rK - wL$ when profit maximising

↳ factor shares of rK, wL imply that income = output = spending, a tenet of income accounting

TFP : wages, output, rental

↳ following from $MPL = w$, $MPK = r$ ⇒ A determines w and r

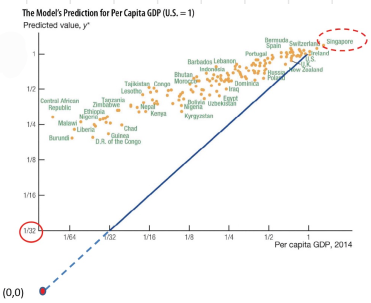
---

③ insights from data

different countries have different TFP

↳ if we purely assume that income purely decided by K and L, then we would observe that investment flow to poor countries w/ high unemployment, since room for arbitrage by low MPL, MPK from diminishing returns

↳ but we don't! ⇒ simple models w/ no differences in TFP are wrong



implicitly measuring TFP

↳ in the production model, TFP is an abstract term measuring "productivity" — we have no direct measures or concepts of it, unlike capital or labor

↳ we abstractly measure it as a "residual" → that is, assuming the model is correct, we infer A as a "residual", measuring the value so that the model fits exactly

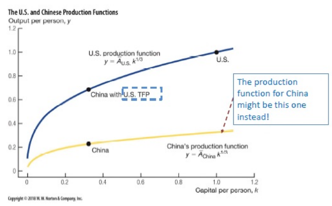
Measuring TFP So the Model Fits Exactly—1

TABLE 4.4
Measuring TFP So the Model Fits Exactly

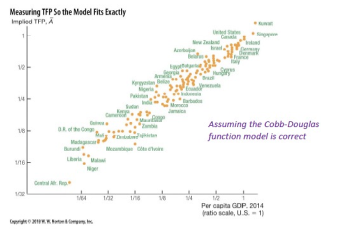
Country	Per capita GDP (y)	$\bar{k}^{1/3}$	Implied TFP ($\bar{A}$)
United States	1.000	1.000	1.000
Switzerland	1.147	1.123	1.022
United Kingdom	0.733	0.941	0.779
Japan	0.685	1.007	0.680
Italy	0.671	1.040	0.646
Spain	0.615	1.041	0.590
Brazil	0.336	0.771	0.436
South Africa	0.232	0.602	0.386
China	0.241	0.686	0.351
India	0.105	0.437	0.240
Burundi	0.016	0.192	0.085

Calculations are based on the equation $y = \bar{A}k^{1/3}$. Implied productivity $\bar{A}$ is calculated from data on y and k for the year 2010, so that this equation holds exactly as $\bar{A} = y/k^{1/3}$.

United States and Chinese Production Functions



Measuring TFP So the Model Fits Exactly—2



⑤ Implications & evaluation

1. in the absence of TFP, the model is incorrect. TFP is also abstract and not explained

2. model does not discuss growth

---

1. model is relatively consistent w/ factor shares, national income accounting, replication argument

Notes

1. showing RTS : derive for general case, and compare it to βY

   eg. βY vs. $F(\beta K, \beta L)$

   $> = <$

   conclude

2. per-capita graphs — recall that exponential → always intersect at $x = 1$

Solow model

① model formulation

key contribution : endogenising capital

↳ in the production model, capital and labor choices were exogenous.
↳ but intuitively we note that capital, in particular, can be accumulated, and depreciates with time.
↳ we also observe that the accumulation of capital could contribute to growth, that is, continual investment could be what drives long term growth.
↳ we endogenize capital accumulation

Assumptions

1. constant exogenous labor force $\bar{L}$, tech level / TFP $\bar{A}$, savings rate $\bar{s}$, depreciation rate $\bar{d}$
2. closed economy (no imports / exports)

formulation

we use it to produce more

1. $Y_t = \bar{A} K_t^{\beta} L_t^{1-\beta}$ — production function
2. $Y_t = C_t + I_t$ — resource constraint
3. $I_t = \bar{s} Y_t$ — investment → we save

we invest

4. $K_{t+1} = K_t + I_t - \bar{d} K_t$ — capital accumulation

↳ $0 \leq \bar{d} \leq 1$, usually $0.07 - 0.10$

$\Leftrightarrow \Delta K_{t+1} = K_{t+1} - K_t = I_t - \bar{d} K_t$

$\Leftrightarrow \Delta K_{t+1} = \bar{s} Y_t - \bar{d} K_t$

5. $L_t = \bar{L}$ — exogenous labor constraint

Y_t, K_t, L_t, C_t, I_t (and w, r) are endogenous, to be determined

——— optional ———

6. $w = MPL$
7. $r = MPK$

wage & interest constraints

main idea

1. At every time step, income Y_t is generated, paid out as wages.
2. Then some proportion of that is saved and used to invest in capital
   2.1 Due to depreciation from using the capital, actual change in capital is investment less depreciation.
   2.2 New capital then used to produce goods at next time step

② notion of a steady state and transition dynamics

1) why is there a steady state in the solow model?

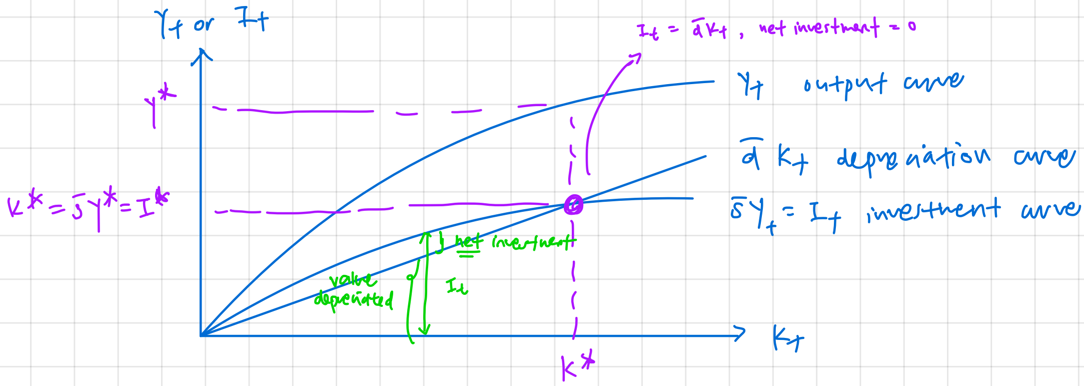
1. observe that in the model, depreciation is a constant factor that causes a constant diminishing of capital at every time step.

2. Also observe that the production function exhibits diminishing returns to capital.

3. Putting these two facts together, we have it that as we increase capital stock, production rises (and therefore investment). But because of diminishing returns, the amount by which production grows at the next time step gets smaller and smaller, and so does the growth in investment.

4. In contrast, a constant fraction of capital stock depreciates every period, so the amount of depreciation rises one-for-one with capital.

5. so there exists a point where the amount of investment the economy generates is equal to the amount of depreciation. Net investment is zero, and there is no capital-driven growth.



steady state condition:

$$\boxed{\bar{s}Y_t = I_t = \bar{d}K_t}$$

investment exactly offsets depreciation, no addtl. income from capital increase in future.

---

2) how does an economy evolve in relation to the steady state?

1. suppose the economy starts at K_0
2. case 1: $K_0 > K^*$.

   2.1 Then we see that $Y_t = \bar{A}^\alpha F(K_t, \bar{L}) > Y^*$, of which $\bar{s}Y_t$ is used to purchase new capital stock.

   2.2 Due to diminishing returns to capital, however, at K_0, depreciation $\bar{d}K_0$ exceeds investment. so $\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_0 < 0$ and capital stock falls.

   2.3 In the next period, therefore, $Y_t < Y_0$. Go to 1.

3. case 2: $K_0 < K^*$.

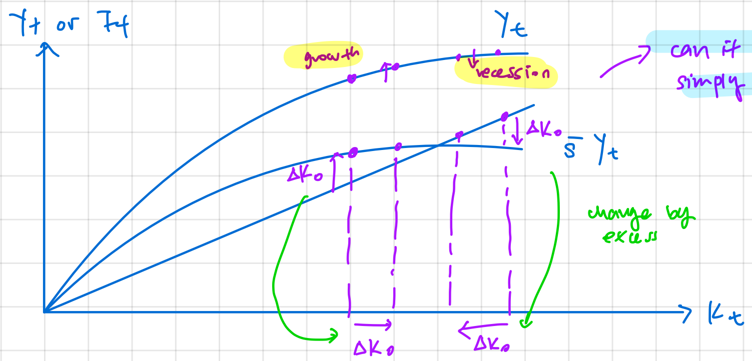
   3.1 Then $Y_0 < Y^*$, of which $\bar{s}Y_0$ is used to purchase new capital stock.

   3.2 Due to diminishing returns to capital, at $K_0 < K^*$, the returns to capital are greater than depreciation, and so there is an increase in capital stock.

2.3 In the next period, thus, output rises. Go to 1.

4. case 3: $K_0 = K^*$. Then the economy has arrived at the steady state.

$\Rightarrow$ observe the "recursive" and fluctuating behavior.



can it overshoot / undershoot, simply oscillate until steady state?

---

3) Solving for the steady state

1. in the steady state, investment = depreciation.

$$\boxed{I^* = \bar{s} Y^* = \bar{d} K^* \iff K^* = \frac{\bar{s} Y^*}{\bar{d}}}$$

2. substitute into production function. $Y^* = \bar{A}^{\alpha} K^{*\beta} \bar{L}^{1-\beta} = \bar{A}^{\alpha} \left(\frac{\bar{s} Y^*}{\bar{d}}\right)^{\beta} \bar{L}^{1-\beta}$

2.1 make Y^* subject.

$$Y^{*\,1-\beta} = \bar{A}^{\alpha} \left(\frac{\bar{s}}{\bar{d}}\right)^{\beta} \bar{L}^{1-\beta}$$

$$\boxed{Y^* = \bar{A}^{\frac{\alpha}{1-\beta}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\beta}{1-\beta}} \bar{L}}$$

increase, higher steady state income

increase, lower steady state income

3. per capita, we divide by $\bar{L}$.

$$\boxed{y^* = \bar{A}^{\frac{\alpha}{1-\beta}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\beta}{1-\beta}}}$$

③ insights of the model

↳ we develop these models of growth to understand why and how countries have different growth rates, income or why they even grow at all. How does the solow model fare here?

All things constant, there is no long-run growth

1. Transition dynamics predict that the economy will move towards the steady state, and that at the steady state, there is no growth all things constant.

⇒ So according to the model, capital accumulation / investment is not the engine of long run growth, but are beneficial in the short run.

↳ does not answer what is engine, but at least answers what is not

---

solow model attributes diff. income even more to TFP

1. we can compare the income per capita of two countries by way of a ratio.

$$\frac{y^*_{rich}}{y^*_{poor}} = \frac{A_{rich}^{\frac{\alpha_{rich}}{1-\beta_{rich}}} \left(\frac{\bar{s}_{rich}}{\bar{d}_{rich}}\right)^{\frac{\beta_{rich}}{1-\beta_{rich}}}}{A_{poor}^{\frac{\alpha_{poor}}{1-\beta_{poor}}} \left(\frac{\bar{s}_{poor}}{\bar{d}_{poor}}\right)^{\frac{\beta_{poor}}{1-\beta_{poor}}}}$$

2. For simplicity, suppose $\alpha = 1$, $\beta_{rich} = \beta_{poor}$.

2.1 then we see that among exogeneous factors, TFP is more important than $\frac{\bar{s}}{\bar{d}}$ in explaining differences in capital output.

---

factoring in population growth

1. growth in labour leads to economic growth

1.1 $Y_t = A_t^{\alpha} K_t^{\beta} L_t^{1-\beta} \iff \frac{dY_t}{dt} = \alpha A_t^{\alpha-1} K_t^{\beta} L_t^{1-\beta} \frac{dA_t}{dt}$ ↗ contribution by tech

$+ \beta K_t^{\beta-1} A_t^{\alpha} L_t^{1-\beta} \frac{dK_t}{dt}$ ↗ contribution by capital

$+ (1-\beta) L_t^{-\beta} A_t^{\alpha} K_t^{\beta} \frac{dL_t}{dt}$ ↗ contribution by labour

1.2 all else constant, if $\frac{dL_t}{dt} > 0$, $\frac{dY_t}{dt} > 0$. labour growth causes economic growth on aggregate.

2. growth in labour does not drive per-capita growth, but lessens it.

2.1 recall $y_t = A_t^{\alpha} \left(\frac{K_t}{L_t}\right)^{\beta}$

2.2 $\frac{dy_t}{dt} = \alpha A_t^{\alpha-1} \left(\frac{K_t}{L_t}\right)^{\beta} \frac{dA_t}{dt} + \beta \left(\frac{K_t}{L_t}\right)^{\beta-1} \left(\frac{\frac{dK_t}{dt} L_t - \frac{dL_t}{dt} K_t}{L_t^2}\right)$ by chain rule.

2.3 observe that all else constant, if $\frac{dL_t}{dt} < 0$, then $\frac{dy_t}{dt} < 0$.

3. From 2.2, observe that $\frac{dy_t}{dt} = \alpha A_t^{\alpha-1} (k_t)^{\beta} \frac{dA_t}{dt} + \beta k_t^{\beta-1} \left(\frac{dk_t}{dt} \right)$, k_t is capital / capita

3.1 So all else constant, $\frac{dy_t}{dt} = 0$ if $\frac{dk_t}{dt} = 0$

3.2 $$\Delta k_t = \underbrace{\frac{\bar{s} Y_t}{L}}_{\text{investment per capita}} - \underbrace{\frac{\bar{d} k_t}{L}}_{\text{depreciation loss per capita}} - \underbrace{\frac{\Delta L}{L} \frac{K_t}{L}}_{\text{redistribution loss}}$$

3.3 Then $\boxed{\Delta k_t = 0 \iff \bar{s} y_t = (\bar{d} + g_L) k_t}$

---

growth rate relative to steady state

0. Some useful results.

0.1 $$\boxed{\text{growth rate} = \frac{\Delta y_t}{y_t} = \frac{\frac{dy_t}{dt}}{y_t} = \frac{d \ln y_t}{dt} = g_{y_t}}$$

0.2 growth rate of exponential products is exponent-weighted sum of growth rates. Analogous for fractions.

$$\boxed{\begin{aligned} z_t = x_t^{\alpha} y_t^{\beta} &\iff \frac{d \ln z_t}{dt} = \alpha \frac{d \ln x_t}{dt} + \beta \frac{d \ln y_t}{dt} = \alpha g_{x_t} + \beta g_{y_t} \\ z_t = \frac{x_t^{\alpha}}{y_t^{\beta}} &\iff \frac{d \ln z_t}{dt} = \alpha \frac{d \ln x_t}{dt} - \beta \frac{d \ln y_t}{dt} = \alpha g_{x_t} - \beta g_{y_t} \end{aligned}}$$

1. we know that y_t and k_t are intimately tied in transition dynamics, k_t changes, which changes y_{t+1}, and so on. Express g_{y_t} in g_{k_t}

1.1 $Y_t = \bar{A}^{\alpha} K_t^{\beta} L_t^{1-\beta} \iff y_t = \bar{A}^{\alpha} \left(\frac{K_t}{L_t} \right)^{\beta}$

1.2 From 0.1 and 0.2, $g_{y_t} = \alpha g_{\bar{A}_t} + \beta (g_{K_t} - g_{L_t})$

1.3 $\Delta K_t = \bar{s} Y_t - \bar{d} K_t \iff \frac{\Delta K_t}{K_t} = \frac{\bar{s} Y_t}{K_t} - \bar{d}$

1.4 in the steady state, $\bar{s} Y^* = \bar{d} K^*$

1.5 Then

$$g_{y_t} = \alpha g_{\bar{A}_t} + \beta\left(\frac{\bar{s}Y_t}{K_t} - \bar{d} - g_{L_t}\right)$$

$$= \alpha g_{\bar{A}_t} + \beta\left(\bar{s}\left(\frac{Y_t}{K_t}\right)\left(\frac{Y^*}{K^*}\right)\left(\frac{K^*}{Y^*}\right) - \bar{s}\frac{Y^*}{K^*} - g_{L_t}\right)$$

$$= \alpha g_{A_t} + \bar{s}\frac{Y^*}{K^*}(\beta)\left(\frac{Y_t}{K_t}\cdot\frac{K^*}{Y^*} - 1\right) - \beta(g_{L_t})$$

$$= \alpha g_{A_t} + \bar{s}\frac{Y^*}{K^*}(\beta)\left(\frac{\cancel{A^*}K_t^{\beta}\cancel{L_t^{1-\beta}}}{K_t}\frac{K^*}{\cancel{A^*}K^{*\beta}\cancel{L_t^{1-\beta}}} - 1\right) - \beta(g_{L_t})$$

$$\boxed{g_{y_t} = \alpha g_{A_t} + \bar{s}\frac{Y^*}{K^*}(\beta)\left(\frac{K^{*\,1-\beta}}{K_t^{1-\beta}} - 1\right) - \beta(g_{L_t})}$$

Insights from growth rate / transition dynamics

1. Population growth dampens per capita GDP growth, weighted by β. Intuitively, this arises by way of how it reduces capital per capita

2. The relative ratio of $\frac{K^*}{K_t}$ determines direction of growth. Observe that $K^* > K_t$, $\frac{K^*}{K_t} > 1$, and vice versa, and for growth to be positive, $\left(\frac{K^*}{K_t}\right)^{1-\beta} > 1$ and vice versa.

   2.1 observe that analogously $\left(\frac{K^*}{K_t}\right)^{1-\beta}$ determines magnitude of growth rate.

3. technological growth boosts capita per GDP growth

---

④ exogeneous shocks

1. how change affects K_t, Y_t relative to K^*, Y^*, or K^* and Y^* itself
   1.1 what that means for amount of output produced and saved + curves
   1.2 How this compares to amount loss to depreciation + curves
   1.3 Effects on net investment and thus net change in capital stock

2. How this pattern continues in transition dynamics

3. go on until k at k^*, then investment is just able to cover for depreciation and steady state is reached.

if there is growth in labour, use per-capita analysis

transition dynamics

1. savings relative to depreciation : exceeds

1.1 net investment is positive, illustrated by distance ΔK_1. So there is capital accumulation.

1.2 the positive net investment increases production for the next period, creating more capital accumulation so long as savings exceed depreciation.

2. Savings is not enough to make up for depreciation

2.1 Net investment is negative, illustrated by distance ΔK.

2.2 In the next period, capital will fall from K_t to K_{t+1} by ΔK.

2.3 This decreases output further, and we continue to see capital depletion in capital (per worker) while savings are less than depreciation.

2.4 Due to diminishing returns to capital but fractional constant depreciation, the difference between $\bar{s}Y$ and $\bar{d}K$ approaches 0 as k approaches K^*.

a. Due to diminishing returns to capital but constant fractional depreciation, net investment approaches 0 as k_t approaches k^*.

b. At the steady state, $K_t = K^*$ and net investment exactly accounts for depreciation. So no more capital accumulation occurs and the economy has reached its new steady state at $Y = Y^*$.

---

technological shocks, change in $\bar{A}$

1. The change in $\bar{A}$ implies that productivity is higher, and for a given amount of capital and labor, the economy can produce more output

2. This is illustrated by an upward shift in the production & investment (per worker) curves from Y_t to Y_t' and I_t to I_t'

3. Since $\bar{a}$ is unchanged, we have a new steady state point at S_2, and the economy is now below the new steady state.

4. Savings exceed depreciation.

labour shocks, change in L

1. with an instaneous change in labour supply, for a given amount of capital, the economy is able to produce more output.

2. This implies for a constant fractional savings rate, more will be invested in capital.

3. These are illustrated by an upward shift in Y_t to Y_t' and K_t to K_t'

4. since $\bar{d}$ is unchanged, the new steady state is at S_2.

5. At S_1, the economy lies below the steady state, and net investment exceeds depreciation.

---

income/capital shocks, change in Y_t or K_t

1. With an instantaneous increase in Y_t/K_t the economy now finds itself above the steady state, though there is no change in the underlying productive capacity.

2. At this point Y_t', depreciation exceeds savings.

---

change in depreciation, $\bar{d}$

1. A change in depreciation rate implies that at a given time step, more capital is consumed in the production of goods.

2. This reduces net investment, and is illustrated by a pivot in the depreciation curve from D_1 to D_2 upwards.

3. This has the effect of moving the steady state down to S_2.

4. At original steady state S_1, thus, the economy lies above the steady state.

---

change in savings, $\bar{s}$

1. A change in savings rate implies that for a given level of output, more is reinvested into capital.

2. This is illustrated by an upward shift in I_1 to I_2, now closer to Y_t.

3. Since depreciation is $\bar{d}$, the new steady state lies at S_2, and at original state S_1, the economy is below S_2.

⑤ strengths & weaknesses of the Solow model

1) strengths

1. provides a theory that determines how rich a country is in the long run
2. transition dynamics helps explain different rates of growth, that countries farther from steady state grow faster

2) weaknesses

1. Does not explain TFP
2. Does not explain differences in $\bar{a}$ or $\bar{s}$
3. Does not provide explanation for sustained growth

---

⑥ Implications of Solow model on policy

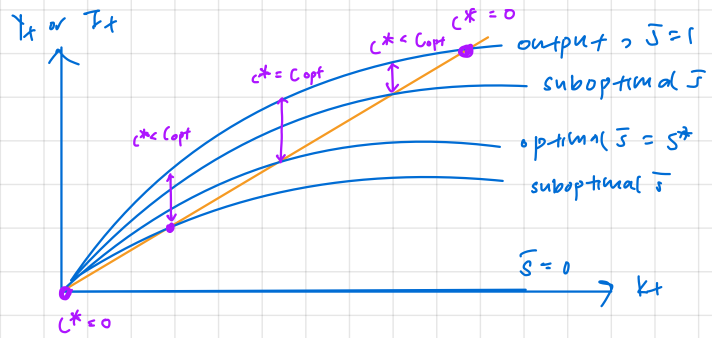
Steady state consumption vs. saving : golden rule of capital

1. If $\bar{s} = 0$, then long run output is $0 \to$ consumption $= 0$
2. If $\bar{s} = 1$, then long run output is Y^*, but all saved $\to$ consumption is 0

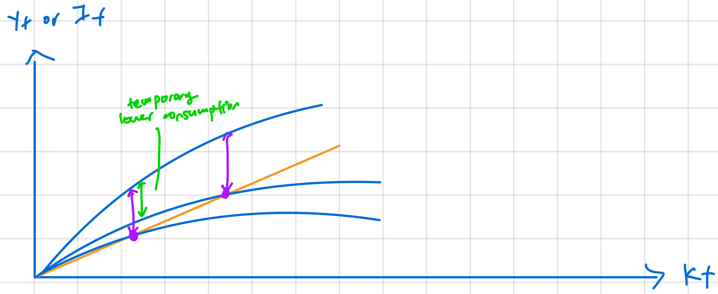
3. In between, $c = (1-\bar{s})Y^*$ at steady state

   3.1 The optimal $\bar{s} \to$ and optimal saving curve $\to$ is where c^* is maximum.

$\Rightarrow$ golden rule of capital : the level of capital per capita that yields highest level of consumption per capita



4. But, increasing $\bar{s}$ means lower short run consumption $\to$ politics



$\boxed{\text{Romer model}}$

① objects and ideas

1) objects & ideas

(objects) goods and items — raw materials in production

(ideas) Designs in making items — ways of arranging those raw materials

2) properties of ideas

(non-rivalry) one person's use does not reduce the usefulness of an idea to someone else

(excludable) Patent laws etc. can make ideas excludable

3) returns to ideas

↳ ideas, applied to production functions, lead to a function w/ increasing RTS

↳ with objects, must be constant RTS → $K^{\beta} L^{1-\beta}$

↳ suppose we include ideas A w/ some power α.

$$F(A,K,L) = A^{\alpha} K^{\beta} L^{1-\beta} \begin{cases} \text{if } \alpha >= 1 \longrightarrow \text{increasing RTS} \\ \text{if } \alpha <= 1 \longrightarrow \text{increasing RTS still} \end{cases}$$

$$\frac{A^{\alpha} K^{\beta} L^{1-\beta}}{(\gamma A)^{\alpha}(\gamma K)^{\beta}(\gamma L)^{1-\beta}} = \frac{1}{\gamma^{\alpha}\cdot\gamma} \Rightarrow \alpha \geq 0, \text{ then increasing RTS !}$$

4) implications on perfect competition

↳ an important note about ideas is the requirement for fixed investment

↳ increasing returns generated by non-rivalry

⇒ then to produce these increasing RTS effects, we require firms to R&D, but only charge $P = MC$ for Pareto optimality, which is not rational

⇒ to incentivise research, $P > MC$ (inefficient), welfare loss by patents (excludability)

⇒ we can correct for this using funding

$\underbrace{\hspace{10cm}}$

to enjoy increasing RTS from non-rivalry, we must permit imperfect competition

① the Romer model

1) formulation

$$Y_t = A_t L_{yt} \quad \text{output production function}$$

productivity of ideas

$$\Delta A_{t+1} = \bar{z} A_t L_{at} \quad \text{idea production function}$$

current stock of ideas

$$L_{yt} + L_{at} = \bar{L} \quad \text{resource constraint}$$

$$L_{at} = \bar{\ell} \bar{L} \quad \text{allocation of labour}$$

2) key relationships in the Romer model

the romer model produces long run growth

1. ideas are non-rival, and so do not have diminishing returns. $\alpha = 1$.

2. then growth rate of ideas $= \dfrac{\Delta A_{t+1}}{A_t} = \bar{z} L_{at} = \bar{z}\bar{\ell}\bar{L} = g_A$

   2.1 then stock of knowledge is iterative: $A_t = \bar{A}_0 (1+g_A)^t$

3. then GDP per capita $= \dfrac{Y_t}{\bar{L}} = \dfrac{A_t L_{yt}}{\bar{L}} = \bar{A}_0 (1+g_A)^t (1-\bar{\ell})$

balanced growth equation

↳ all endogeneous variables grow at constant rates. $(Y_t, A_t, L_{yt}, L_{at})$

1. $g_{L_{at}} = 0$

2. $g_{L_{yt}} = 0$

3. $g_{A_t} = \bar{z}\bar{\ell}\bar{L}$

4. $y_t = \bar{A}_0 (1+g_A)^t (1-\bar{\ell})$

   4.1 $\ln y_t = \ln A_0 (1+g_A)^t (1-\bar{\ell})$

   $$= \underbrace{\ln A_0 + \ln(1-\bar{\ell})}_{\text{intercept}} + \underbrace{t \ln(1+g_A)}_{\text{slope}}$$

   4.2 $\dfrac{d \ln y_t}{dt} = \ln(1+g_A) \approx g_A$

3) local vs. global knowledge

1. observe that $g_y = \bar{A}_0 (1+g_A)(1-\bar{\ell})$, and $g_A = \bar{z}\bar{\ell}\bar{L}$

1.1 That is, growth rates are driven by $\bar{\ell}$ (research allocation of labour), research productivity $\bar{z}$, labour force $\bar{L}$ and stock of ideas $\bar{A}_0$.

2. Were the Romer model to be applied only on a country-by-country basis, then larger countries eg. US would have much higher growth than small eg. Luxembourg

2.1 But this is not the case

3 Since knowledge is cross-border, the Romer model is better applied to the <u>world's</u> stock of ideas as opposed to a country-by-country basis

3.1 ie. $g_A = A_t^{world} \bar{z}^{country} \bar{L}^{country}$

3.2 Then $\frac{\Delta A_{t+1}^{world}}{A_t^{country}} = \bar{z} \frac{A_t^{world}}{A_t^{country}} \bar{L}^{country}$ → this ratio allows equilisation for labor shortfalls

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③ <u>experiments in the Romer model</u>

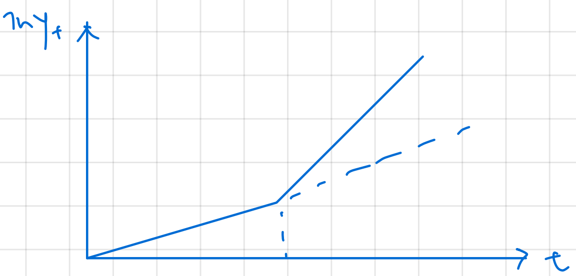
1. change in thing
2. growth (rate) effects
3. level (stock) effects

} graph

$\boxed{\text{changing } \bar{L}}$

1. recall that $g_A = \bar{z}\bar{\ell}\bar{L}$, and $g_y \approx g_A$

2. change in $\bar{L}$, immediate & permanent change in g_A and g_y, as illustrated by graph

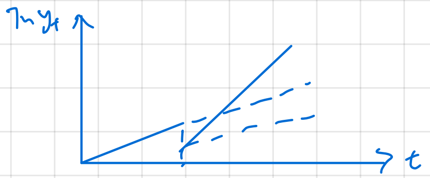


changing research share

1. recall that $Y_t = \bar{A}_0 (1+g_A)^t (1-\bar{\ell})$

2. Level effects. change in $\bar{\ell}$, immediate change in Y_t, illustrated by a break and shift in curve.

3. Growth effects. Recall $g_A = \bar{z}\bar{\ell}\bar{L}$. change in $\bar{\ell}$, immediate & permanent change in g_A and thus g_Y, illustrated by steeper/gentler curve.



what if ideas had diminishing returns?

1. $Y_t = A_t^{\alpha} L_{Y_t}$. $\Delta A_{t+1} = \bar{z} A_t L_{at}$.

   1.1 $g_A = \bar{z}\bar{\ell}\bar{L}$.

   1.2 $A_t = A_0 (1+g_A)^t$

   1.3 $Y_t = \left[A_0 (1+g_A)^t\right]^{\alpha} L_{yt}$, $\ln Y_t = \alpha \ln A_0 (1-\bar{\ell}) + \alpha t \ln (1+g_A)$

   1.4 $g_{Y_t} = \alpha \ln (1+g_A) \approx \alpha g_A$

2. $\alpha < 1$.

   2.1 Then if $\bar{z}, \bar{\ell} \uparrow$, $g_A \uparrow$, but $g_{Yt} \uparrow$ by a lesser extent.

3 converse is true if $\alpha > 1$.

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④ combining solow and romer models

1) formulation

1. $Y_t = A_t^{\alpha} K_t^{\beta} L_{Yt}^{1-\beta}$ production function w/ constant RTS to K, L
2. $\Delta K_{t+1} = \bar{s} Y_t - \bar{d} K_t$ capital accumulation
3. $\Delta A_{t+1} = \bar{z} A_t L_{at}$ idea production function
4. $L_{yt} + L_{at} = \bar{L}$ labour constraint
5. $L_{at} = \ell \bar{L}$ research labour allocation

2) key results

balanced growth path as steady state

1. growth due to ideas drives capital growth, even with diminishing returns to capital

1.1 suppose the economy starts at some steady state.

1.2 Now, because of growth of ideas, even with constant K and L, we have an increase in Y_t.

1.3 By capital accumulation, $\Delta K_{t+1} = \bar{s} Y_t - \bar{d} K_t$.

2. There is a steady state where $g_Y = g_K$.

2.1 steady state, where g_K is constant.

2.2 Then $g_K = \bar{s} \frac{Y_t}{K_t} - \bar{d}$ is constant. Then $\frac{Y_t}{K_t}$ is constant.

2.3 Then $\frac{d \ln \frac{Y_t}{K_t}}{dt}$ is constant, by monotonic transform.

2.4 then $g_Y - g_K = \frac{d}{dt}(\text{constant}) = 0$. Then $g_Y = g_K$

3. solving for balanced growth path conditions.

hold simult.

3.1 $\frac{dL_{at}}{dt} = \frac{dL_{yt}}{dt} = 0$

3.2 $g_A = \bar{z} \bar{\ell} \bar{L}$

3.3 $g_Y = g_K \iff \alpha g_A + \beta g_K = g_Y, \quad g_K = \bar{s} \frac{Y_t}{K_t} - \bar{d}$

capital as an amplifier

1. $g_Y^* = \alpha g_A + \beta g_K^* + \cancel{(1-\beta) g_{La}} = \alpha g_A + \beta g_Y^* \iff g_Y^* = \frac{\alpha}{1-\beta} g_A$

2. in the Romer model, $g_Y^* = \alpha g_A$. growth is higher in the combined model since ideas directly increase output, but higher income $\bar{\omega}$ a fixed savings rate also raises capital stock, further raising output

output per capita

1. growth in output per capita

1.1 recall $g_{Y^*} = \frac{\alpha}{1-\beta} g_A$

1.2 recall $g_y = g_Y - g_L$

1.3 Then $g_{y^*} = g_{Y^*} - \cancel{g_{L^*}}$

2. output per capita along balanced growth path

2.1 Recall $y^* = \frac{Y^*}{\bar{L}} = \frac{A_t^{*\alpha} K_t^{*\beta} L_{at}^{1-\beta}}{\bar{L}}$

$$= \frac{A_t^{\alpha} \left(\frac{K_t^*}{Y_t^*} Y_t^*\right)^{\beta} L_{at}^{1-\beta}}{\bar{L}}$$

2.2 Recall $g_K^* = g_Y^* = \bar{s}\frac{Y_t^*}{K_t^*} - \bar{d} \iff \frac{K_t^*}{Y_t^*} = \frac{\bar{s}}{g_{y^*} + \bar{d}}$

2.3 Recall $g_{y^*} = g_{Y^*}$ if $g_L = 0$

2.4 $y^* = A_t^{\alpha} \left(\frac{\bar{s}}{g_{y^*} + \bar{d}}\right)^{\beta} (Y_t^*)^{\beta} (1-\ell)^{1-\beta} \cdot \frac{\bar{L}^{1-\beta}}{\bar{L}}$

$$= A_t^{\alpha} \left(\frac{\bar{s}}{g_{y^*} + \bar{d}}\right)^{\beta} (1-\ell)^{1-\beta} y_t^{*\beta}$$

2.5 Then $\boxed{y^* = A_t^{\frac{\alpha}{1-\beta}} \left(\frac{\bar{s}}{g_{y^*} + \bar{d}}\right)^{\frac{\beta}{1-\beta}} (1-\ell)^{1-\beta}}$

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5 exogeneous shocks

1. growth effect
2. level effect
3. if level effects present

Transition dynamics

1. in every period, ideas production function means that $\Delta A_{t+1} = \bar{z} A_t L_{at}$

   1.1 recall $g_A = \bar{z}\bar{\ell}\bar{L}$, constant.

   1.2 Then output rises without change in capital, which due to savings, creates reinvestment in capital in the next period.

   1.3 For the current, new level of A_t, the new steady state capital K^* lies at $K_t + \Delta K_{t+1}$.

   1.4 This continuous, steady growth in A_t, Y_t and K_t lies along the balanced growth path.

2. Level effects occur, such that economy is above balanced growth path.

   2.1 savings from base income Y_t, where $Y_{t+1} = Y_t + \Delta Y_t$, exceeds depreciation.

   2.2 net investment is positive, illustrated by distance ΔK_1. So there is capital accumulation.

   2.3 the positive net investment increases production for the next period, creating excess capital accumulation so long as savings from output not from tech growth changes exceed depreciation

a. Due to diminishing returns to capital but constant fractional depreciation, net investment approaches ΔK_t^* as time goes to ∞.

b. At the steady state, $\Delta K_t = \Delta K^*$ and net investment arises purely from savings of output grown by technological growth. Then the economy is back on the steady state.

labour shocks, change in L

1. with an instaneous change in labour supply, for a given amount of capital, the economy is able to produce more output.

2. growth effects. $g_{Y*} = \frac{\alpha}{1-\beta} g_A$, $g_A = \bar{z}\bar{\lambda}\bar{L}$. so higher long run balanced growth. illustrated by different slope

3. level effects. instantaneous change in output. Transition dynamics, illustrated by offset then non-linear adjustment.

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income/capital shocks, change in Y_t or K_t

1. with an instantaneous increase in Y_t/K_t the economy now finds itself above the steady state, though there is no change in the underlying productive capacity.

2. growth effects. None. same slope.
3. Level effects. Illustrated by discontinuity. Transition dynamics.

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change in depreciation, $\bar{d}$

change in savings, $\bar{s}$

1. A change in depreciation rate implies that at a given time step, more capital is consumed in the production of goods.

2. growth effects. None.

3. level effects. None.

4. other effects. if shock, adjustment would be faster / slower.

Notes

1. compare graphs — intercept, slope, graph
   - before, after
   - level (stock), growth (rate) effects
   - capital accumulation effects