CSC 418 Assignment 1 - Part A

Tanzeem Chowdhury
997574726

1) Spiral curve: $x(t) = t\sin(t)$
$y(t) = t\cos(t)$

a) Let $\bar{f}(t) = (x(t), y(t))$
$\vec{t}(t)$ be the Tangent Vector

$$\vec{t}(t) = \frac{d\bar{f}}{dt}(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)$$

$$x'(t) = t' \cdot \sin(t) + t \cdot \sin'(t)$$
$$= \sin(t) + t\cos(t)$$

$$y'(t) = t'\cos(t) + t \cdot \cos'(t)$$
$$= \cos(t) + t(-\sin(t))$$
$$= \cos(t) - t\sin(t)$$

$$\boxed{\vec{t}(t) = (\sin(t) + t\cos(t), \cos(t) - t\sin(t))}$$

Let $\vec{n}(t)$ be the Normal Vector
$\vec{n}(t)$ is perpendicular to $\vec{t}(t)$

$$\therefore \vec{n}(t) = \left(-\frac{dy}{dt}(t), \frac{dx}{dt}(t)\right)$$

$$\boxed{\begin{aligned}\vec{n}(t) &= (-[\cos(t) - t\sin(t)], \sin(t) + t\cos(t)) \\ &= (t\sin(t) - \cos(t), \sin(t) + t\cos(t))\end{aligned}}$$

1) b) $ax + by + c = 0$

$\Rightarrow ax + by = 0$, [In class, it was said to set $c=0$]

Convert to parametric form:

$ax + by = 0$

$\Rightarrow y = \frac{-ax}{b} = f(x)$

Let $x = t$

$y = f(t) = -\frac{a}{b}t$

x: $t\sin(t) = t$

$\Rightarrow \sin(t) = 1$

$\Rightarrow t = \frac{\pi}{2} + 2\pi n, \; n \in \mathbb{Z}$

since the curve is $t \in [0, 4\pi]$, we know at least two intersections happen, at $t = \frac{\pi}{2}$ and $t = \frac{5\pi}{2}$

$x(t) = t\sin(t)$ $\quad$ $y(t) = t\cos(t)$

$x(\pi/2) = \pi/2$ $\quad$ $y(\pi/2) = 0$

$x(5\pi/2) = 5\pi/2$ $\quad$ $y(5\pi/2) = 0$

Two points of intersection are $(\pi/2, 0)$ and $(\frac{5\pi}{2}, 0)$. Since both points have y as 0, this implies the line of intersection is $y=0$. We can find the remaining points by setting $y(t) = 0$

$y(t) = t\cos(t)$

$\Rightarrow 0 = t\cos(t)$

$\Rightarrow 0 = t, \quad 0 = \cos(t)$

$\Rightarrow t = \pi n - \frac{\pi}{2}, \; n \in \mathbb{Z}$

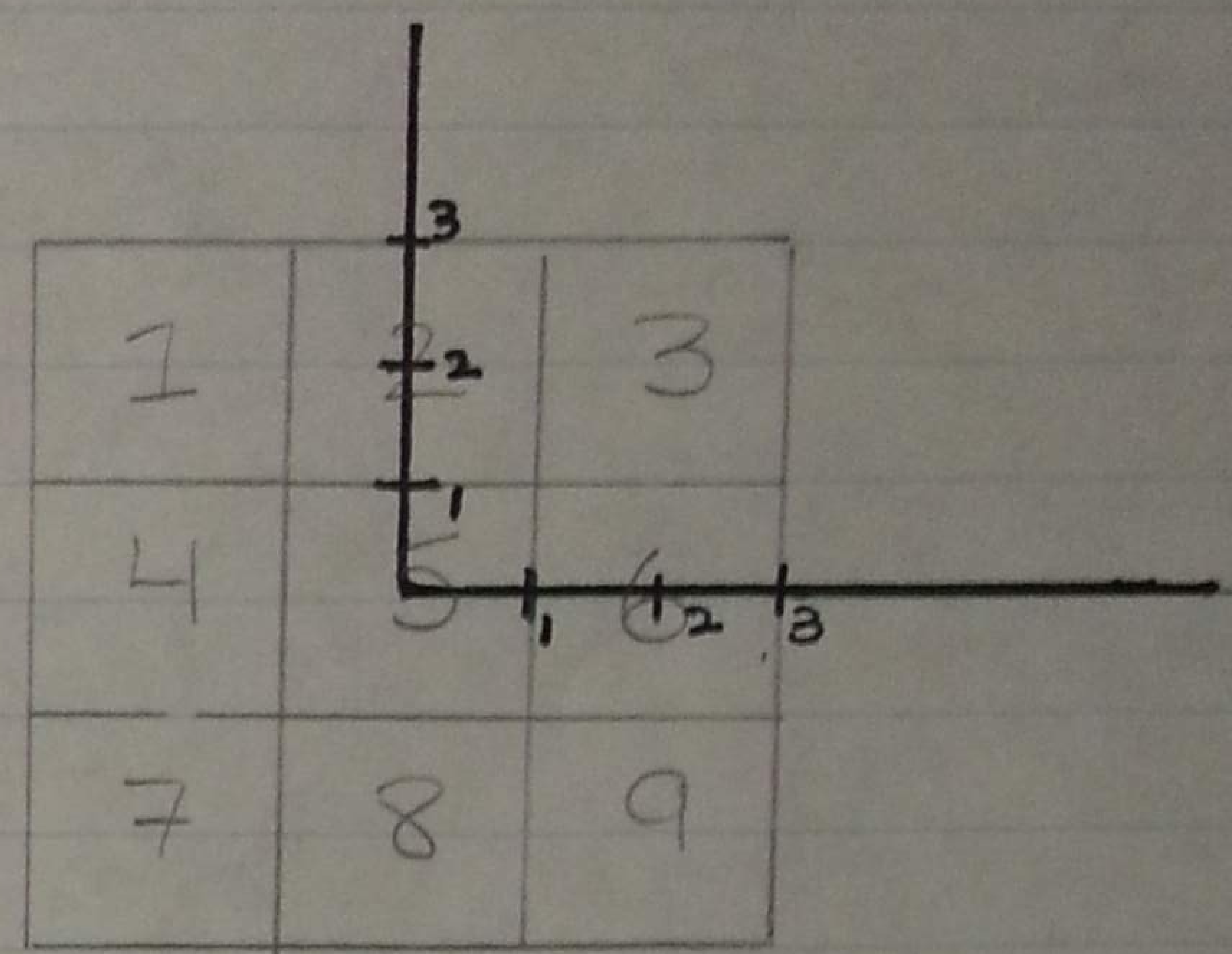
$\Rightarrow$ since t must be an $\in$ of $[0, 4\pi]$,

$t = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

We already found points at $t = \pi/2, 5\pi/2$, need to find $t = 0, 3\pi/2, 7\pi/2$ points.

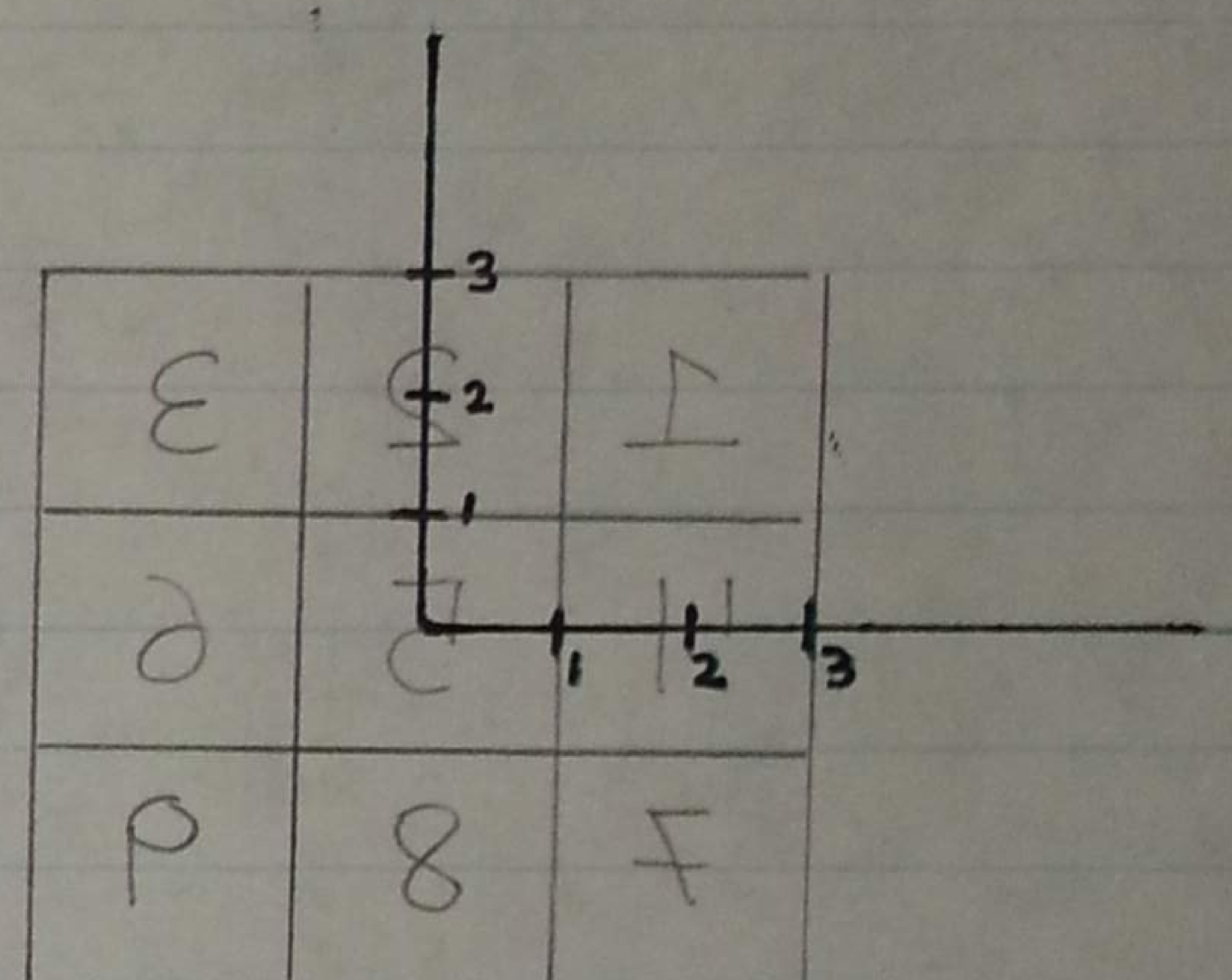
$\longrightarrow$

2) Starting shape:



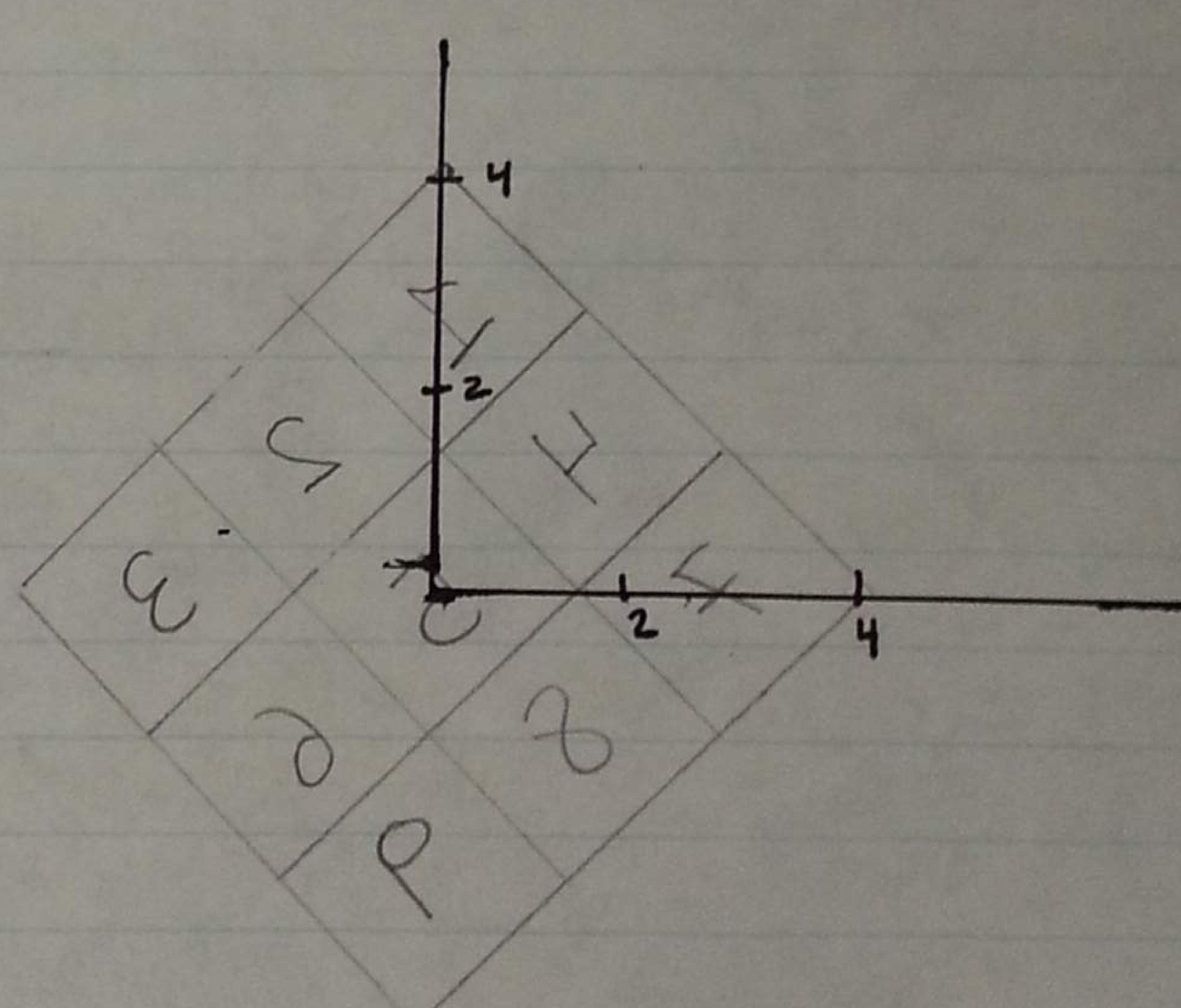
First, reflect about the y-axis

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Next, rotate 45 degrees ($\pi/4$) counter clockwise

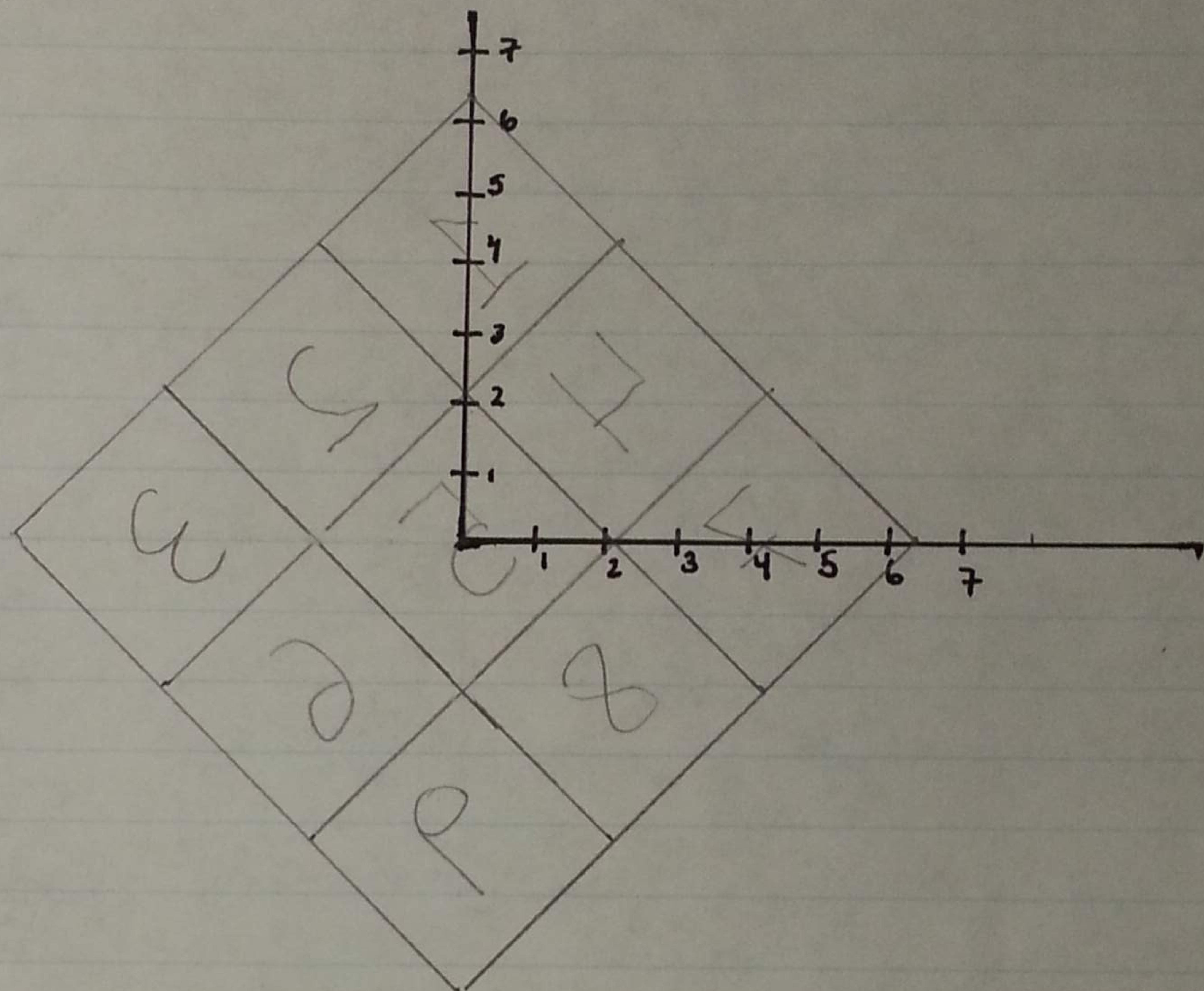
$$\begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$\longrightarrow$

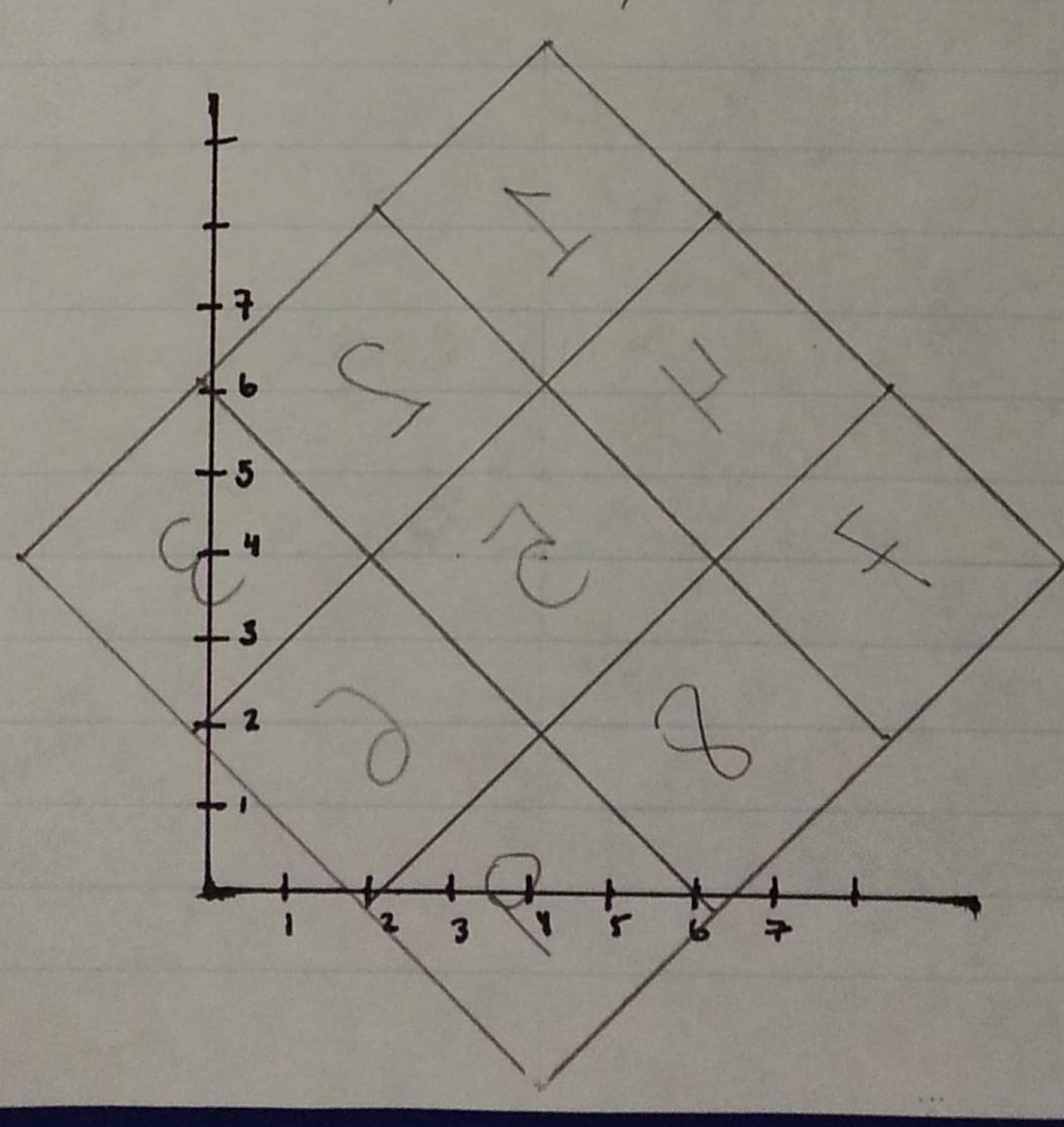
Next, the resulting image is 1.5 times larger, so scale by 1.5, both x and y. (1.5 was determined by measuring handout, 3cm → 4.5cm)

$$\begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Finally, the center of the image moves from (0,0) to (4,4). So we translate by 4 on x, and by 4 on y

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$



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All the individual transformations are represented as a matrix in the form of:

$$\begin{matrix} a & c & t_x \\ b & d & t_y \\ u & v & w \end{matrix}$$

$a; d$ - scale
b, c - skew
t_x, t_y - translate

$u, v, w = 0, 0, 1$

For a single matrix with all the transformations, we need to get the product of all the individual transformations, in reverse order

single matrix m = $\left(\begin{matrix}\text{Translate}\\\text{Matrix}\end{matrix}\right) \times \left(\begin{matrix}\text{Scale}\\\text{Matrix}\end{matrix}\right) \times \left(\begin{matrix}\text{Rotation}\\\text{Matrix}\end{matrix}\right) \times \left(\begin{matrix}\text{Reflection}\\\text{Matrix}\end{matrix}\right)$

$$m = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{m = \begin{bmatrix} -1.06066 & -1.06066 & 4 \\ -1.06066 & 1.06066 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

3) Homography that maps points:

(0,0), (0,1), (1,0), (1,1) TO

(4,2), (3.5,1), (3, 1.5), (3,1)

a) We can create a system of equations with the following matrice operations:

$$\begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & k & l \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ 1 \end{bmatrix}$$

i) $$\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$ax_k + by_k + c - u_k(hx_k + ky_k + 1) = 0$$
$$\Rightarrow (a)(0) + b(0) + c - 4(h(0) + k(0) + 1) = 0$$
$$\Rightarrow c - 4(1) = 0$$
$$\Rightarrow c = 4$$

$$dx_k + ey_k + f - v_k(hx_k + ky_k + 1) = 0$$
$$\Rightarrow d(0) + e(0) + f - 2(h(0) + k(0) + 1) = 0$$
$$\Rightarrow f - 2(1) = 0$$
$$\Rightarrow f = 2$$

Using a matrix calculater, and following what we did for (i), we get the following:

ii) $$\begin{bmatrix} 3.5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow e - k = -1$$
$$\Rightarrow b - 3.5k = -0.5$$

iii) $\begin{bmatrix} 3 \\ 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\Rightarrow a - 3h = -1$

$\Rightarrow d - 1.5h = -0.5$

iv) $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\Rightarrow a + b - 3h - 3k = -2$

$\Rightarrow d + e - h - k = -2$

From the 8 derived equations, we get

$a = 2$, $b = 3$, $c = 4$,

$d = 1$, $e = 0$, $f = 2$,

$h = 1$, $k = 1$

$$\boxed{H = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

b) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 11/4 \\ 1 \\ 1 \end{bmatrix}$

c) This homograph cannot be affine, because an affine transformation would have a bottom row of $[0 \; 0 \; 1]$, which H cannot be simplified to.

4) A rotation is represented by:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We want to prove that the product of 3 shears can achieve the above rotation.

Let a, b, c be the skews in 3 shear matrices

Show:

$$\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} ab+1 & a+(ab+1)c & 0 \\ b & ab+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow ab + 1 = \cos\theta$

$a + (ab+1)c = -\sin\theta$

$b = \sin\theta$

$bc + 1 = \cos\theta$

Substitute using $b = \sin\theta$

$\rightarrow a\sin\theta + 1 = \cos\theta$

$\rightarrow a + (a\sin\theta + 1)c = -\sin\theta$

$\rightarrow c\sin\theta + 1 = \cos\theta$

$\longrightarrow$

- $a\sin\theta + 1 = \cos\theta$

$$\Rightarrow a = \frac{\cos\theta - 1}{\sin\theta}$$

$$= -\left(\frac{1-\cos\theta}{\sin\theta}\right) = -\tan\left(\frac{\theta}{2}\right)$$

- $a\sin\theta + 1 = \cos\theta$

  $c\sin\theta + 1 = \cos\theta$

$$\Rightarrow a\sin\theta + 1 = c\sin\theta + 1$$

$$\Rightarrow a = c$$

$$\Rightarrow c = -\tan\left(\frac{\theta}{2}\right)$$

$$\boxed{\begin{bmatrix} 1 & -\tan(\frac{\theta}{2}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ \sin\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\tan(\frac{\theta}{2}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

b) Rotations can't be achieved when the angle is π. This would cause $-\tan(\frac{\theta}{2}) = -\tan(\frac{\pi}{2})$, which is infinity.