Chern Class

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1 Via an Euler class

Theorem 1.1 (Thom isomorphism)**.** Let $\pi : E \to B$ be an oriented rank n real vector bundle. Then the cohomology group $H^i(E, E_0)$ is zero for $i < n$, and $H^n(E, E_0)$ contains one and only one cohomology class u whose restriction

$$u|_{(F,F_0)} \in H^n(F, F_0)$$

is equal to the preferred generator u_F for every fiber F. Furthermore the correspondence $y \mapsto y \smile u$ maps $H^k(E)$ isomorphically onto $H^{k+n}(E, E_0)$ for every integer k.

Definition 1.2. The inclusion $(E, \emptyset) \hookrightarrow (E, E_0)$ induce maps

$$H^r(E, E_0) \to H^r(E) \cong H^r(B).$$

The *Euler class* $e(E) \in H^n(B)$ is the image of u under the composition of these maps.

Note that since E and B are homotopic, $H^k(E) \cong H^k(B)$.

Proposition 1.3 (Gysin sequence)**.** Ther is a long exact sequence:

$$\cdots \to H^{k+n-1}(E_0) \to H^k(B) \xrightarrow{\smile e} H^{k+n}(B) \xrightarrow{\pi_0^*} H^{k+n}(E_0) \to \cdots .$$

Proof. By long exact sequence:

$$\cdots \to H^{r-1}(E_0) \to H^r(E, E_0) \to H^r(E) \to H^r(E_0) \to \cdots .$$

Then the following diagram commutes:

$$\begin{tikzcd}
H^{k+n}(E,E_0) \arrow[r] \arrow[d, "\cong"'] & H^{k+n}(E) \arrow[r] & H^{k+n}(E_0) \\
H^k(E) \arrow[d, "\cong"'] & & \\
H^k(B) \arrow[r, "\smile e(E)"] & H^{k+n}(B) \arrow[uu, "\pi^*"] \arrow[uur, "\pi_0^*"'] &
\end{tikzcd}$$

□

Corollary 1.4. Let $\pi : E \to B$ be a rank n complex vector bundle, there is an isomorphism:

$$H^k(B) \xrightarrow[\cong]{\pi_0^*} H^k(E_0),$$

for $k < 2n - 1$.

Proof. $H^{k-2n}(B) \cong H^{k-2n+1}(B) \cong 0$. Together with proposition 1.3. □

Definition 1.5. Let $\pi : E \to B$ be a rank n complex vector bundle over a paracompact space B. *Chern class* $c_k(E) \in H^{2k}(B_{\mathbb{R}})$ is given by

$$c_k(E) = \begin{cases} \pi_0^{*-1} c_k(E') & k < n \\ e(E_{\mathbb{R}}) & k = n \\ 0 & k > n \end{cases},$$

where $E' \to E_0$ is a vector bundle whose fiber on each point $v \in E_0$ is the quotient space $E/\mathbb{C}\{v\}$, whence E' is a rank $n-1$ complex vector bundle.

2 via Chern–Weil theory

2.1 Curvature from

Let $\pi : E \to M$ be a smooth rank n complex vector bundle over a smooth manifold M. Denote the space of smooth sections of E over M by $\Gamma(M, E)$.

Definition 2.1. We call any section of $\wedge^k T^*M \otimes E$ an *E-valued k-form* on M. The set of E-valued k-forms $\Gamma(M, \wedge^k T^*M \otimes E)$ is denoted by $\Omega^k(M; E)$

Definition 2.2. A *connection* on E is an $\mathbb{C}$-linear map

$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$$

such that the Leibniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

holds for all smooth functions f on M and all smooth sections s of E.

Let X be a tangent vector field on M, one can define a *covariant derivative along* X

$$\nabla_X : \Omega^0(M; E) \to \Omega^0(M; E),$$

by contracting X with the resulting covariant index in the connection: $\nabla_X(s) = (\nabla(s))(X)$.

Definition 2.3. Let $\varphi_U : U \times \mathbb{C}^n \to \pi^{-1}(U)$ be local trivializations of E. A set of local sections $\{s_1, s_2, \cdots, s_n\}$ is said to be basis of $\Gamma(U, \pi^{-1}(U))$ if for all point $p \in U, \{s_1(p), s_2(p), \cdots, s_n(p)\}$ is a basis for the fiber E_p. Such $\{s_1, s_2, \cdots, s_n\}$ is called *local frame* of E over U.

Definition 2.4. For the given connection ∇ and local frame $\{s_1, s_2, \cdots, s_n\}$ on U, we can write:

$$\nabla_X s_i = \sum_j (\omega_U)_{i,j}(X)s_j,$$

for all vector field X over U, where $\omega_U \in \Omega^1(M; \mathfrak{gl}(n; \mathbb{C}))$, called *connection* 1*-form* associated to the given local frame.

Definition 2.5. Let ∇ be the given connection on E, one can extend ∇ to a family of operators:

$$\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E),$$

by defining

$$\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s,$$

for all $\omega \in \Omega^k(M)$, $s \in \Omega^0(M; E)$

Definition 2.6. We call $R_\nabla = \nabla^2 : \Omega^0(M; E) \to \Omega^2(M; E)$ the *curvature tensor* of the connection ∇. In local frame:

$$\nabla^2 s_i(X, Y) = \sum_j (\Omega_U)_{i,j}(X, Y)s_j,$$

where $\Omega_U \in \Omega^2(M; \mathfrak{gl}(n; \mathbb{C}))$ is called *curvature 2-form*.

2.2 Invariant polynomial

Proposition 2.7. For the given vector space V, there is a bijection $\mathrm{Sym}^k V^*$ and homogeneous polynomial of degree k:

$$T \overset{\cong}{\mapsto} P_T := (v \mapsto T(v, \cdots, v)).$$

Proof. Just notice the *polarization formula*:

$$T(v_1, \cdots, v_k) = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k} P_T(t_1 v_1 + \cdots + t_k v_k).$$

□

In general, let $G = GL(n; \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n; \mathbb{C})$.

Definition 2.8. A symmetric k-tensor $T \in \mathrm{Sym}^k \mathfrak{g}^*$ is called *G-invariant* if

$$g \cdot T = T \quad \forall g \in G.$$

The set of all G-invariant symmetric k-tensor is denoted by $I^k(G)$.

Theorem 2.9 (Chevally restriction theorem)**.** Let G be a complex semi-simple Lie group, $\mathfrak{g}$ the corresponding Lie algebra, $\mathfrak{h}$ the Cartan subalgebra and W the Weyl group of $\mathfrak{g}$. Then G-invariant polynomial on $\mathfrak{g}$ is isomorphic to W-invariant polynomial on $\mathfrak{h}$.

2.3 Chern-Weil theory

Theorem 2.10. Let E be a vector bundle over M. Then

1. For any $T \in I^k(G)$ and any linear connection ∇ on E, $P_T(\Omega)$ is a closed $2k$-form.

2. The de Rham cohomology class $[P_T(\Omega)] \in H^{2k}_{dR}(M)$ is independent of the choices of ∇.

3. The Chern-Weil maps

$$\mathcal{CW} : (I^*(G), \circ) \to (H^*_{dR}(M), \wedge), \quad T \mapsto [P_T(\Omega)]$$

is a ring homomorphism, which is called *Chern-Weil homomorphism*.

Lemma 2.11 (Bianchi identity)**.**

$$\mathrm{d}\,\Omega = \Omega \wedge \omega - \omega \wedge \Omega = [\Omega, \omega].$$

Lemma 2.12. For all $T \in I^k(G)$ and $X, X_1, \cdots, X_k \in \mathfrak{g}$, we have

$$T([X, X_1], X_2, \cdots, X_k) + T(X_1, [X, X_2], \cdots, X_k) + T(X_1, X_2, \cdots, [X, X_k]) = 0.$$

proof of theorem 2.10. 1.

$$\begin{aligned}
\mathrm{d}\,P_T(\Omega) &= \mathrm{d}\,T(\Omega, \cdots, \Omega) \\
&= \sum T(\Omega, \cdots, \mathrm{d}\,\Omega, \cdots, \Omega) \\
&= \sum T(\Omega, \cdots, [\Omega, \omega], \cdots, \Omega) \\
&= 0.
\end{aligned}$$

2. Let ∇_1, ∇_2 be two different connection on E. The mapping $p_1 : M \times \mathbb{R} \to M$ induces two connection $\tilde{\nabla}_1, \tilde{\nabla}_2$ on $p_1^* E$.

$$\tilde{\nabla} = s\tilde{\nabla}_1 + (1 - s)\tilde{\nabla}_2$$

is also a connection on $p_1^* E$.
Consider $i_\epsilon : x \mapsto (x, \epsilon)$, then $i_0^* \tilde{\nabla} = \nabla_0, i_1^* \tilde{\nabla} = \nabla_1$. Hence

$$i_j^* \left[P_T \left(\tilde{\Omega} \right) \right] = [P_T(\Omega_j)], \quad j = 1, 2.$$

Since the mapping i_0 is homotopic to i_1, it follows that

$$[P_T(\Omega_0)] = [P_T(\Omega_1)].$$

3. For all $T \in I^k(G), S \in I^l(G), X_1, \cdots, X_{k+l} \in \Omega^*(M)$,

$$P_{T \circ S}(X_1, \cdots, X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} T(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \wedge S(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}).$$

$\square$

2.4 Chern Class

Definition 2.13. For the given G-invariant polynomial f of degree k, since the cohomology class $[f(\Omega)] \in H^{2k}_{dR}(M)$ does not depend on connections Ω, we will denote it by $f(E)$ and call it the *characteristic class* of E corresponding to f. The *total Chern class* $c(E)$ is defined by:

$$\left[\det\left(I - \frac{1}{2\pi i}\Omega\right)\right] = 1 + \sum_{k=1}^{n}\left(\frac{-1}{2\pi i}\right)^k \sigma_k(\Omega).$$

Remark 2.14.

$$H^k\left(\mathbb{CP}^n\right) = \begin{cases} \mathbb{R}, & \text{for } k \text{ even and } 0 \leq k \leq 2n \\ 0, & \text{otherwise} \end{cases}.$$

Moreover, $H^k\left(\mathbb{CP}^n\right) = \mathbb{R}[a]/\left(a^{n+1}\right)$, where a is the generator of $H^2\left(\mathbb{CP}^k;\mathbb{Z}\right)$.

Proof. Denote $U = \mathbb{CP}^n - \mathbb{CP}^{n-1} \cong \mathbb{C}^n$ and $V = \mathbb{CP}^n \backslash \{[0 : \cdots : 0 : 1]\}$. $U \cap V = \mathbb{C}^n - \{0\} \approx \mathbb{S}^{2n-1}$. Note that

$$([z_0 : \cdots : z_n], t) \mapsto [z_0 : \cdots : z_{n-1} : tz_n]$$

defines a homotopy between $\mathbb{CP}^{n-1}$ and V. Using Mayer–Vietoris sequence, the result is clear:

$$\cdots \to H^{k-1}\left(\mathbb{S}^{2n-1}\right) \to H^k\left(\mathbb{CP}^n\right) \to H^k\left(\mathbb{CP}^{n-1}\right) \to \cdots.$$

□

Example 2.15. Riemann sphere, which is isomorphic to complex projective line $\mathbb{CP}^1$, has Kahler metric:

$$\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\left(1+|z|^2\right)^2}$$

Hence there is a connection on tangent bundle $T\mathbb{CP}^1$ induced by the metric:

$$\begin{aligned} 2\frac{z\,\mathrm{d}\,\bar{z} + \bar{z}\,dz}{\left(1+|z|^2\right)^3} &= \mathrm{d}\,\frac{1}{\left(1+|z|^2\right)^2} \\ &= \mathrm{d}\left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right\rangle \\ &= \left\langle \nabla\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right\rangle + \left\langle \frac{\partial}{\partial z}, \nabla\frac{\partial}{\partial \bar{z}} \right\rangle \end{aligned}$$

$$\nabla\frac{\partial}{\partial z} = \frac{2z\,\mathrm{d}\bar{z}}{1+|z|^2} \otimes \frac{\partial}{\partial z}.$$

Then it has curvature form

$$\Omega = \frac{2\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2}.$$

To see the Chern class $c_1(T\mathbb{CP}^1) = [\frac{i}{2\pi}\mathrm{tr}\,\Omega] = [\frac{i}{2\pi}\Omega] \in H^2_{dR}\left(\mathbb{CP}^1\right)$ is non-zero,

$$\int c_1 = \frac{i}{\pi}\int \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2} = 2.$$

Proposition 2.16. The tirvial bundle $E = M \otimes \mathbb{C}^n$ has total Chern class $c(E) = 1$.

Proof. Sections s of E can be written by $s = (f_1, \cdots, f_n)$ and E has tirvial connection:

$$\nabla(f_1, \cdots, f_n) = (\mathrm{d}\, f_1, \cdots, \mathrm{d}\, f_n),$$

since

$$\begin{aligned}
\nabla f \cdot (f_1, \cdots, f_n) &= \nabla(f \cdot f_1, \cdots, f \cdot f_n) \\
&= (\mathrm{d}\, f \cdot f_1 + f \cdot \mathrm{d}\, f_1, \cdots, \mathrm{d}\, f \cdot f_n + f \cdot \mathrm{d}\, f_n) \\
&= \mathrm{d}\, f \otimes (f_1, \cdots, f_n) + f\nabla(f_1, \cdots, f_n).
\end{aligned}$$

Then

$$\begin{aligned}
\nabla^2(f_1, \cdots, f_n) &= \nabla(\mathrm{d}\, f_1, \cdots, \mathrm{d}\, f_n) \\
&= (\mathrm{d}^2\, f_1, \cdots, \mathrm{d}^2\, f_n) + (\mathrm{d}^2\, f_1, \cdots, \mathrm{d}^2\, f_n) \\
&= 0.
\end{aligned}$$

Hence the curvature 2-form $\Omega = 0$. □

3 Axiomatic definition

Definition 3.1. The Chern classes satisfy the following four axioms:

Axiom 1. $c_0(E) = 1$ for all E.

Axiom 2 (Naturality)**.** If $f : Y \to X$ is continuous and f^*E is the vector bundle pullback of E, then $(f^*E) = f^*c_k(E)$.

Axiom 3 (Whitney sum formula)**.** If $F \to X$ is another complex vector bundle, then the Chern classes of the direct sum $E \oplus F$ are given by

$$c(E \oplus F) = c(E) \smile c(F).$$

Axiom 4 (Normalization)**.** The total Chern class of the tautological line bundle over $\mathbb{CP}^k$ is $1 - a$, where a is Poincare-dual to the hyperplane $\mathbb{CP}^{k-1} \subseteq \mathbb{CP}^k$, i.e. the generator of $H^2\left(\mathbb{CP}^k; \mathbb{Z}\right)$

Theorem 3.2 (Poincare duality)**.** If a manifold M is orientable and compact, then

$$\begin{aligned} I_M : H^k(M) &\to \left(H^{n-k}(M)\right)^* \\ \omega &\mapsto \left(\eta \mapsto \int_M \omega \wedge \eta\right) \end{aligned}$$

is an isomorphism.

Remark 3.3. For the complex projective space $\mathbb{CP}^n$, the hyperplane bundle

$$\mathcal{O}_{\mathbb{CP}^n}(1) = \{(V, \lambda) \mid V \in \mathbb{CP}^n, \lambda \in V^*\}$$

has the property that there is a exact sequence:

$$0 \to \mathbb{CP}^n \otimes \mathbb{C} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \to T\mathbb{CP}^n \to 0,$$

since

$$\begin{aligned} T\mathbb{CP}^n \oplus (\mathbb{CP}^n \otimes \mathbb{C}) &= \mathrm{Hom}(\mathcal{O}(-1), \eta) \oplus \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(-1)) \\ &= \mathrm{Hom}(\mathcal{O}(-1), \mathbb{CP}^n \otimes \mathbb{C}) \\ &= \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}, \end{aligned}$$

where $\eta = \left\{(V, v) \mid V \in \mathbb{CP}^n, v \in V^\perp\right\}$. Note that $\phi \in \mathrm{Hom}(\mathcal{O}(-1), \eta)$ defines an element of $T_V\mathbb{CP}^n$ by

$$\mathrm{Id} \oplus \phi : V \to V \oplus V^\perp = \mathbb{C}^{n+1}.$$

Denote $a = c\left(\mathcal{O}_{\mathbb{CP}^n}(1)\right)$Note that $c\left(\mathbb{CP}^n \otimes \mathbb{C}\right) = 1$, whence

$$c\left(T\mathbb{CP}^n\right) = c\left(\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}\right) = (1 + a)^{n+1}.$$

Remark 3.4. In local affine coordinates, complex projective space $\mathbb{CP}^n$ has Fubini–Study metric:

$$ds^2 = \frac{(1 + z_i \bar{z}^i)\, dz_j\, d\bar{z}^j - \bar{z}^j z_i\, dz_j\, d\bar{z}^i}{(1 + z_i \bar{z}^i)^2}.$$

4 Application

4.1 Chern–Gauss–Bonnet theorem

Theorem 4.1 (Fundamental Theorem of Riemannian Geometry)**.** Let (M, g) be a Riemannian manifold (or pseudo-Riemannian manifold). Then there is a unique connection ∇ called *Levi-Civita connection* on the tangent bundle of M which satisfies the following conditions:

1. (compatible with the metric) for any vector fields X, Y, Z we have

$$\partial_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

   where $\partial_X \langle Y, Z \rangle$ denotes the derivative of the function $\langle Y, Z \rangle$ along vector field X.

2. (torsion free) for any vector fields X, Y,

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

   where $[X, Y]$ denotes the Lie bracket for vector fields X, Y.

Theorem 4.2 (Chern–Gauss–Bonnet)**.** For a closed even-dimensional Riemannian manifold M.

$$\chi(M) = \int_M e(\Omega),$$

where $\chi(M)$ denotes the Euler characteristic of M, and Ω is the associated curvature form of the Levi-Civita connection.

4.2 Chern polynomial

Definition 4.3. For a complex vector bundle E, the *Chern polynomial* of E is given by:

$$c_t(E) = 1 + c_1(E)t + \cdots + c_n(E)t^n.$$

Now, if $E = L_1 \oplus \cdots \oplus L_n$ is a direct sum of complex line bundles, then it follows from the Whitney sum formula (Axiom 3) that:

$$c_t(E) = (1 + a_1(E)t) \cdots (1 + a_n(E)t),$$

where $a_i(E) = c_1(L_i)$ are the first Chern classes.

The roots $a_i(E)$, called the *Chern roots* of E, determine the coefficients of the polynomial:

$$c_k(E) = \sigma_k(a_1(E), \ldots, a_n(E)).$$