Cell Decomposition Compatibility of Plücker Embedding

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The decomposition of the Grassmannian into Schubert cells is compatible.

Example 1. [1, p. 124] The Grassmannian $G(2,4)$, has a decomposition into six Schubert cells $C_{(j_1,j_2)}$, with

$$(j_1, j_2) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

which are, respectively, of complex dimensions $0,1,2,2,3,4$, and correspond to 2×4-matrices in row echelon form. In terms of the Plücker embedding $G(2,4) \hookrightarrow \mathbb{P}^5$, if we write the defining equation for $G(2,4)$ in $\mathbb{P}^5$ as above, in the form

$$\Lambda^{(12)}\Lambda^{(34)} - \Lambda^{(13)}\Lambda^{(24)} + \Lambda^{(23)}\Lambda^{(14)} = 0$$

then the Schubert varieties $X_{(j_1,j_2)}$ given by the closures of the Schubert cells $C_{(j_1,j_2)}$, are given by

$$\begin{aligned}
X_{(1,2)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(13)} = \Lambda^{(14)} = \Lambda^{(23)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(1,3)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(14)} = \Lambda^{(23)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(1,4)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(23)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(2,3)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(14)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(2,4)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(34)} = 0 \right\}
\end{aligned}$$

with $X_{(3,4)} = G(2,4)$. The deformation described above then induces compatible noncommutative deformations on all the Schubert cells.

Proposition 2 (Plücker relation)**.** Let q be an integer $1 \le q \le k$[1]. Fix two length k sequences $I, J \subseteq \{1,2,\cdots,n\}$ and length q sequence $R \subseteq \{1,2,\cdots,n\}$;

$$\begin{aligned}
I &: i_1 < i_2 < \cdots < i_k; \\
J &: j_1 < j_2 < \cdots < j_k; \\
R &: r_1 < r_2 < \cdots < r_q.
\end{aligned}$$

If $S \subseteq \{1,2,\cdots,k\}$is a length q sequence, say $S : s_1 < s_2 < \cdots < s_q$; let T' be obtained from I by replacing $(i_{r_1}, i_{r_2}, \cdots, r_{r_q})$ with $(j_{s_1}, j_{s_2}, \cdots, j_{s_q})$ and likewise J' from J by replacing $(j_{s_1}, j_{s_2}, \cdots, j_{s_q})$ with $(i_{r_1}, i_{r_2}, \cdots, r_{r_q})$. Then we have the Plücker relation

$$\Lambda^I \Lambda^J = \sum_S \Lambda^{I'} \Lambda^{J'}, \tag{1}$$

where sum is quantified over all increasing length q sequences $S \subseteq \{1,2,\cdots,k\}$.

[1] when $q = k$, the relation is trivial.

1 $G(2, n)$

Take $G(2, n)$ for example, Schubert cells $C_{(j_1, j_2)}$ can be listed (in order) in the form

$$\begin{Bmatrix} (1,2) & (1,3) & (1,4) & \cdots & (1,n) \\ & (2,3) & (2,4) & \cdots & (2,n) \\ & & (3,4) & \cdots & (3,n) \\ & & & \ddots & \vdots \\ & & & & (n-1,n) \end{Bmatrix}.$$

Note that cells $C_{(j_1, j_2)}$:

$$C_{(j_1, j_2)} = \left\{ V \in G(2, n) \,\middle|\, \Lambda^{(j_1, j_2)} \neq 0 \text{ and } \Lambda^{(a,b)} = 0, \forall a > j_1 \text{ or } b > j_2 \right\}$$

has dimension $m - 3$.

In such order, $\mathbb{C}^{\binom{n}{2}}$ has cell

$$D_{(j_1, j_2)} = \left\{ v \in \mathbb{C}^{\binom{n}{2}} \,\middle|\, v_{(j_1, j_2)} \neq 0 \text{ and } v_{(a,b)} = 0, \forall a > j_1 \text{ or } a = j_1 \text{ and } b > j_2 \right\}.$$

It suffices to prove $\Lambda^{(a,b)} = 0$ for all $a < j_1$ and $b > j_2$: Since $a < j_1 < j_2 < b$ and

$$\begin{aligned} \Lambda^{(j_1, j_2)} \Lambda^{(a,b)} &= \Lambda^{(j_1, a)} \Lambda^{(j_2, b)} + \Lambda^{(j_1, b)} \Lambda^{(a, j_2)} \\ &= -\Lambda^{(a, j_1)} \underbrace{\Lambda^{(j_2, b)}}_{=0 \text{ since } j_2 > j_1 \text{ and } b > j_2} + \Lambda^{(a, j_2)} \underbrace{\Lambda^{(j_1, b)}}_{=0 \text{ since } b > j_2} \\ &= 0. \end{aligned}$$

By the fact that $\Lambda^{(j_1, j_2)} \neq 0$, we have $\Lambda^{(a,b)} = 0$.

2 $G(k, n)$

For $G(k, n)$, Schubert cell

$$C_{(j_1, j_2, \cdots, j_k)} = \left\{ V \in G(k, n) \,\middle|\, \Lambda^{(j_1, j_2, \cdots, j_k)} \neq 0 \text{ and } \Lambda^{(a_i)_i} = 0, \forall (a_i)_{1 \le i \le k}, \text{s.t. } a_h > j_h \text{ for some } h \right\},$$

and cell

$$D_{(j_1, j_2, \cdots, j_k)} = \\ \left\{ v \in \mathbb{C}^{\binom{n}{k}} \,\middle|\, v_{(j_1, j_2, \cdots, j_k)} \neq 0 \text{ and } v_{(a_i)_i} = 0, \forall (a_i)_{1 \le i \le k}, \text{s.t. } a_h > j_h \text{ and } a_l = j_l (\forall l < h) \text{ for some } h \right\}.$$

It suffices to prove for all $1 \le i \le n$,

$$\Lambda^{(a_1, a_2, \cdots, a_k)} = 0, \tag{2}$$

for all $(a_1, a_2, \cdots, a_k)$, s.t. $a_l \le j_l (\forall l < i)$ and $a_i > j_i$.

By induction, assume that equation 2 is true for $i = h - 1$ (when $i = 1$, it's clearly true).

Then let $q = 1, R = (h)$, equation 1 can be written as

$$\Lambda^{(j_1, j_2, \cdots, j_k)} \Lambda^{(a_1, a_2, \cdots, a_k)} = \sum_{l=1}^{k} \Lambda^{(j_1, j_2, \cdots, j_{h-1}, a_l, j_{h+1}, \cdots, j_k)} \Lambda^{(a_1, a_2, \cdots, j_h, \cdots, a_k)}$$

$$
\begin{aligned}
&= \sum_{l=1}^{h-1} \Lambda^{(j_1,j_2,\cdots,j_{h-1},a_l,j_{h+1},\cdots,j_k)} \underbrace{\Lambda^{(a_1,a_2,\cdots,a_{l-1},a_{l+1},\cdots,a_{h-1},\underbrace{j_h}_{(h-1)_{\text{th}}\text{ element}},a_h,\cdots,a_k)}}_{=0 \text{ since } a_{h-1}\leq j_{h-1}<j_h<a_h \text{ and } j_h>j_{h-1}} \\
&\quad + \sum_{l=h}^{k} \underbrace{\Lambda^{(j_1,j_2,\cdots,j_{h-1},a_l,j_{h+1},\cdots,j_k)}}_{=0 \text{ since } a_l\geq a_h>j_h \text{ and } j_{h+1}>j_h} \Lambda^{(a_1,a_2,\cdots,j_h,\cdots,a_k)} \\
&= 0.
\end{aligned}
$$

By the fact that $\Lambda^{(j_1,j_2,\cdots,j_k)} \neq 0$, we have $\Lambda^{(a_1,a_2,\cdots,a_k)} = 0$.

References

[1] M. Marcolli. *Noncommutative Cosmology*. World Scientific Publishing Company, 2017.