Proof of Forbenius's Formula

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To a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of d ($\sum_{i=1}^k \lambda_i = d$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$) is associated to a *Young diagram*

$$\ydiagram{3,2,2,1}$$

and also corresponding to an irreducible representation V_λ of $\mathfrak{S}_d$.

After numbering the box in Young diagram like

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 \end{ytableau}$$

we may define a subgroup of $\mathfrak{S}_d$:

$$P = P_\lambda = \{g \in \mathfrak{S}_d : g \text{ perserves each row}\}.$$

Note that

$$P_\lambda \cong \mathfrak{S}_\lambda \triangleq \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}.$$

In group algebra $\mathbb{C}\mathfrak{S}_d$, there is an element corresponding to such subgroup

$$a_\lambda = \sum_{g \in P} e_g.$$

Denote $C_{\mathbf{i}}$ the conjugacy class in $\mathfrak{S}_d$ determined by a sequence

$$\mathbf{i} = (i_1, \cdots, i_d) \quad \text{with} \sum \alpha i_\alpha = d,$$

where $C_{\mathbf{i}}$ consists of permutations having i_1 1-cycle, i_2 2-cycle, ..., and i_d d-cycle.

Define *power sums* $P_j(x), 1 \leq j \leq d$ and the *descriminant* $\Delta(x)$ by

$$P_j(x) = x_1^j + x_2^j + \cdots + x_k^j,$$
$$\Delta(x) = \prod_{i<j} (x_i - x_j).$$

For $f(x) = f(x_1, \cdots, x_k)$ a formal power series, $(l_1, \cdots, l_k)$ a k-tuple of non-negative integers, define

$$[f(x)]_{(l_1, \cdots, l_k)} = \text{ coeffcient of } x_1^{l_1} \cdot \ldots \cdot x_k^{l_k} \text{ in } f.$$

For given partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of d, set $l_1 = \lambda_1 + k - 1, l_2 = \lambda_2 + k - 2, \cdots, l_k = \lambda_k$, which is a strictly decreasing sequence of k non-negative integers.

To compute the character of representation V_λ,

Theorem 1 (Forbenius's Formula)**.**

$$\chi_\lambda(C_{\mathbf{i}}) = \left[\Delta(x)\cdot P^{(\mathbf{i})}\right]_{(l_1,\cdots,l_k)},$$

where

$$P^{(\mathbf{i})} = \prod_j P_j(x)^{i_j}.$$

Example 2. If $d=5, \lambda=(3,2)$, and $C_{\mathbf{i}}$ is the conjugacy class of $(123)(45)$, i.e. $\mathbf{i}=(0,1,1)$, then

$$\chi_{(3,2)}(C_{\mathbf{i}}) = \left[(x_1-x_2)\cdot(x_1^2+x_2^2)(x_1^3+x_2^3)\right]_{(4,2)} = 1.$$

Lemma 3. Let W be the trivial representation of H, which is a subgroup of G, then for conjugacy class C of G,

$$\chi_{\operatorname{Ind} W}(C) = \frac{[G:H]}{|C|}\cdot|C\cap H|$$

Proposition 4. Set U_λ the representation of $\mathfrak{S}_d$ induced from the trivial representation of $\mathfrak{S}_\lambda$. Let $\psi_\lambda = \chi_{U_\lambda}$ the character of U_λ, we have

$$\psi_\lambda(C_{\mathbf{i}}) = \left[P^{(\mathbf{i})}\right]_\lambda.$$

Lemma 5. For any symmetric polynomial P,

$$[P]_\lambda = \sum_\mu K_{\mu\lambda}\left[\Delta(x)\cdot P\right]_{(\mu_1+k-1,\mu_2+k-2,\cdots,\mu_k)},$$

where $K_{\mu\lambda}$ is the Kostka numbers.

Lemma 6.

$$K_{\lambda\lambda}=1, \qquad \text{and } K_{\mu\lambda}=0 \text{ for } \mu<\lambda.$$

Proposition 7.

$$\psi_\lambda(C_{\mathbf{i}}) = \sum_\mu K_{\mu\lambda}\omega_\mu(\mathbf{i}) = \omega_\lambda(\mathbf{i}) + \sum_{\mu>\lambda} K_{\mu\lambda}\omega_\mu(\mathbf{i}),$$

where

$$\omega_\mu(\mathbf{i}) = \left[\Delta\cdot P^{(\mathbf{i})}\right]_{(\mu_1+k-1,\mu_2+k-2,\cdots,\mu_k)}.$$

Lemma 8.

$$\frac{1}{d!}\sum_{\mathbf{i}}|C_{\mathbf{i}}|\omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i}) = \delta_{\lambda\mu}.$$

Proposition 9.

$$\chi_\lambda(C_{\mathbf{i}}) = \omega_\lambda(\mathbf{i}).$$

Proof of Lemma 3.

$$\begin{aligned}\chi_{\operatorname{Ind} W}(C) &= \frac{1}{|C|}\sum_{c\in C}\chi_{\operatorname{Ind} W}(c)\\ &= \frac{1}{|C|}\sum_{c\in C}\sum_{\substack{g\in G/H\\ gcg^{-1}\in H}}\chi_W(gcg^{-1})\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|C|}\sum_{c\in C}\frac{1}{|H|}\sum_{\substack{g\in G\\ gcg^{-1}\in H}}\chi_W(gcg^{-1})\\
&= \frac{1}{|C||H|}\#\{c\in C, g\in G \mid gcg^{-1}\in H\}\\
&= \frac{1}{|C||H|}\#\{c\in C, g\in G \mid gcg^{-1}\in C\cap H\}\\
&= \frac{|G|}{|H|}\frac{|C\cap H|}{|C|} = \frac{[G:H]}{|C|}\cdot|C\cap H|.
\end{aligned}
$$

□

Proof of Proposition 4.

$$
\begin{aligned}
|C_{\mathbf{i}}| &= \frac{d!}{1^{i_1}i_1!\cdot 2^{i_2}i_2!\cdot\ldots\cdot d^{i_d}i_d!};\\
|C_{\mathbf{i}}\cap\mathfrak{S}_\lambda| &= \sum_{\{r_{pq}\}}\prod_{p=1}^{k}\frac{\lambda_p!}{1^{r_{p1}}r_{p1}!\cdot 2^{r_{p2}}r_{p2}!\cdot\ldots\cdot d^{r_{pd}}r_{pd}!}
\end{aligned}
$$

(sum over $i_q = r_{1q}+\cdots+r_{kq}, \lambda_p = 1\cdot r_{p1} + 2\cdot r_{p2}+\cdots+d\cdot r_{pd}$);

By lemma 3,

$$
\begin{aligned}
\psi_\lambda(C_{\mathbf{i}}) &= \frac{[\mathfrak{S}_d:\mathfrak{S}_\lambda]}{|C_{\mathbf{i}}|}\cdot|C_{\mathbf{i}}\cap\mathfrak{S}_\lambda|\\
&= \frac{d!}{\lambda_1!\cdots\lambda_k!}\cdot\frac{1^{i_1}i_1!\cdot 2^{i_2}i_2!\cdot\ldots\cdot d^{i_d}i_d!}{d!}\cdot\sum_{\{r_{pq}\}}\prod_{p=1}^{k}\frac{\lambda_p!}{1^{r_{p1}}r_{p1}!\cdot 2^{r_{p2}}r_{p2}!\cdot\ldots\cdot d^{r_{pd}}r_{pd}!}\\
&= \sum_{\{r_{pq}\}}\frac{1^{i_1}i_1!\cdot 2^{i_2}i_2!\cdot\ldots\cdot d^{i_d}i_d!}{\lambda_1!\cdots\lambda_k!}\cdot\prod_{p=1}^{k}\frac{\lambda_p!}{1^{r_{p1}}r_{p1}!\cdot 2^{r_{p2}}r_{p2}!\cdot\ldots\cdot d^{r_{pd}}r_{pd}!}\\
&= \sum_{\{r_{pq}\}} i_1!\cdot i_2!\cdot\ldots\cdot i_d!\cdot\prod_{p=1}^{k}\frac{1}{r_{p1}!\cdot r_{p2}!\cdot\ldots\cdot r_{pd}!}\\
&= \sum_{\{r_{pq}\}}\prod_{q=1}^{d}\left(i_q!\prod_{p=1}^{k}\frac{1}{r_{pq}!}\right)\\
&= \sum_{\{r_{pq}\}}\prod_{q=1}^{d}\frac{i_q!}{r_{1q}!\cdot\ldots\cdot r_{kq}!}
\end{aligned}
$$

As

$$P^{(\mathbf{i})} = (x_1+\cdots+x_k)^{i_1}\cdot(x_1^2+\cdots+x_k^2)^{i_2}\cdot\ldots\cdot(x_1^d+\cdots+x_k^d)^{i_d},$$

$\frac{i_q!}{r_{1q}!\cdot\ldots\cdot r_{kq}!}$ equals to how many ways to gain $\prod_{p=1}^k x_p^{q\cdot r_{pq}}$ in $(x_1^q+\cdots+x_k^q)^{i_q}$, i.e.

$$\left[(x_1^q+\cdots+x_k^q)^{i_q}\right]_{(q\cdot r_{pq})_{p=1}^k},$$

and

$$\prod_{q=1}^{d}\prod_{p=1}^{k}x_p^{q\cdot r_{pq}} = \prod_{p=1}^{k}x_p^{\sum_{q=1}^d q\cdot r_{pq}} = \prod_{p=1}^{k}x_p^{\lambda_p} = X^\lambda.$$

Different $\{r_{pq}\}$ corresponding to different ways to split X^λ into $\{(x_1^q + \cdots + x_k^q)^{i_q}\}_{q=1}^d$. Therefore

$$\psi_\lambda(C_{\mathbf{i}}) = \sum_{\{r_{pq}\}} \prod_{q=1}^{d} \frac{i_q!}{r_{1q}! \cdot \ldots \cdot r_{kq}!} = \left[P^{(\mathbf{i})}\right]_\lambda .$$

□

Proof of Proposition 7. By proposition 4, lemma 5 and lemma 6,

$$\begin{aligned}
\psi_\lambda(C_{\mathbf{i}}) &= \left[P^{(\mathbf{i})}\right]_\lambda \\
&= \sum_\mu K_{\mu\lambda} \left[\Delta(x) \cdot P^{(\mathbf{i})}\right]_{(\mu_1+k-1, \mu_2+k-2, \cdots, \mu_k)} \\
&= \sum_\mu K_{\mu\lambda} \omega_\mu(\mathbf{i}) \\
&= \omega_\lambda(\mathbf{i}) + \sum_{\mu > \lambda} K_{\mu\lambda} \omega_\mu(\mathbf{i}),
\end{aligned}$$

□

Proof of Proposition 9. Note that

$$V_\lambda = \mathbb{C}\mathfrak{S}_d \cdot a_\lambda b_\lambda, \quad U_\lambda = \mathbb{C}\mathfrak{S}_d \cdot a_\lambda.$$

Hence $V_\lambda \subseteq U_\lambda$ and U_λ can then be decomposed with:

$$\psi_\lambda = \sum_\mu n_{\lambda\mu} \chi_\mu, \quad n_{\lambda\lambda} \geq 1, n_{\lambda\mu} \in \mathbb{N}. \tag{1}$$

ω_λ is a class function, then

$$\omega_\lambda = \sum_\mu m_{\lambda\mu} \chi_\mu, \quad m_{\lambda\mu} \in \mathbb{Z}.$$

By lemma 8, ω_λ are orthonormal. Hence

$$1 = (\omega_\lambda, \omega_\lambda) = \sum_\mu m_{\mu\lambda}^2,$$

so $\omega_\lambda = \pm\chi_\mu$ for somme μ.

Proof by induction: First, for $\lambda = (d)$, $\psi_\lambda = \omega_\lambda$. By equation 1, we have $\omega_\lambda = \chi_\lambda$.

Then, assume $\chi_\mu = \omega_\mu$ for all $\mu > \lambda$, we have

$$\psi_\lambda = \omega_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \chi_\mu.$$

By equation 1, $\omega_\lambda = \chi_\lambda$. □

Theorem 1 is equivalent to proposition 9. Therefore we have finished the proof of Forbenius's formula.

Lemma 5, 6 and 8 are results from symmetric polynomials which won't be proved here.