Cell Decomposition Compatibility of Plücker Embedding

Tao Huang
me@tau.ovh

The decomposition of the Grassmannian into Schubert cells is compatible.

Example 1. The Grassmannian $G(2,4)$, has a decomposition into six Schubert cells $C_{(j_1,j_2)}$, with

$$(j_1, j_2) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

which are, respectively, of complex dimensions $0,1,2,2,3,4$, and correspond to 2×4-matrices in row echelon form. In terms of the Plücker embedding $G(2,4) \hookrightarrow \mathbb{P}^5$, if we write the defining equation for $G(2,4)$ in $\mathbb{P}^5$ as above, in the form

$$\Lambda^{(12)}\Lambda^{(34)} - \Lambda^{(13)}\Lambda^{(24)} + \Lambda^{(23)}\Lambda^{(14)} = 0$$

then the Schubert varieties $X_{(j_1,j_2)}$ given by the closures of the Schubert cells $C_{(j_1,j_2)}$, are given by

$$\begin{aligned}
X_{(1,2)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(13)} = \Lambda^{(14)} = \Lambda^{(23)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(1,3)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(14)} = \Lambda^{(23)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(1,4)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(23)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(2,3)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(14)} = \Lambda^{(24)} = \Lambda^{(34)} = 0 \right\} \\
X_{(2,4)} &= \left\{ V \in G(2,4) \,\middle|\, \Lambda^{(34)} = 0 \right\}
\end{aligned}$$

with $X_{(3,4)} = G(2,4)$. The deformation described above then induces compatible noncommutative deformations on all the Schubert cells.

Proposition 2 (Plücker relation)**.** Let q be an integer $1 \le q \le k$[1]. Fix two length k sequences $I, J \subseteq \{1,2,\cdots,n\}$ and length q sequence $R \subseteq \{1,2,\cdots,n\}$;

$$\begin{aligned}
I &: i_1 < i_2 < \cdots < i_k; \\
J &: j_1 < j_2 < \cdots < j_k; \\
R &: r_1 < r_2 < \cdots < r_q.
\end{aligned}$$

If $S \subseteq \{1,2,\cdots,k\}$is a length q sequence, say $S: s_1 < s_2 < \cdots < s_q$; let T' be obtained from I by replacing $(i_{r_1}, i_{r_2}, \cdots, r_{r_q})$ with $(j_{s_1}, j_{s_2}, \cdots, j_{s_q})$ and likewise J' from J by replacing $(j_{s_1}, j_{s_2}, \cdots, j_{s_q})$ with $(i_{r_1}, i_{r_2}, \cdots, r_{r_q})$. Then we have the Plücker relation

$$\Lambda^I \Lambda^J = \sum_S \Lambda^{I'} \Lambda^{J'},$$

where sum is quantified over all increasing length q sequences $S \subseteq \{1,2,\cdots,k\}$.

[1]when $q = k$, the relation is trivial.

1 $G(2,n)$

Take $G(2,n)$ for example, Schubert cells $C_{(j_1,j_2)}$ can be listed (in order) in the form

$$\begin{Bmatrix} (1,2) & (1,3) & (1,4) & \cdots & (1,n) \\ & (2,3) & (2,4) & \cdots & (2,n) \\ & & (3,4) & \cdots & (3,n) \\ & & & \ddots & \vdots \\ & & & & (n-1,n) \end{Bmatrix}.$$

Note that cells $C_{(j_1,j_2)}$ has dimension $m-3$ and variety

$$X_{(j_1,j_2)} = \left\{ V \in G(2,n) \,\middle|\, \Lambda^{(a,b)} = 0, \forall a > j_1 \text{ or } b > j_2 \right\}$$

has dimension $\frac{(2j_2-j_1-1)j_1}{2}$.

In such order, $\mathbb{C}^{\binom{n}{2}}$ has variety

$$Y_{(j_1,j_2)} = \left\{ v \in \mathbb{C}^{\binom{n}{2}} \,\middle|\, v_{(a,b)} = 0, \forall a > j_1 \text{ or } a = j_1 \text{ and } b > j_2 \right\}.$$

It suffices to prove $\Lambda^{(a,b)} = 0$ for all $a \le j_1$ and $b > j_2$: Since $a \le j_1 < j_2 < b$ and

$$\begin{aligned} \Lambda^{(j_1,j_2)}\Lambda^{(a,b)} &= \Lambda^{(j_1,a)}\Lambda^{(j_2,b)} + \Lambda^{(j_1,b)}\Lambda^{(a,j_2)} \\ &= -\Lambda^{(a,j_1)}\Lambda^{(j_2,b)} + \Lambda^{(a,j_2)}\Lambda^{(j_1,b)} \\ &= -\Lambda^{(a,j_1)}0 + \Lambda^{(a,j_2)}0 \\ &= 0. \end{aligned}$$

Then $\Lambda^{(a,b)} = 0$.

2 $G(k,n)$

For $G(k,n)$, Schubert variety

$$X_{(j_1,j_2,\cdots,j_k)} = \left\{ V \in G(k,n) \,\middle|\, \Lambda^{(a_i)_i} = 0, \forall (a_i)_{i=1,2,\cdots,k}, \text{s.t. } a_i > j_i \text{ for some } i \right\},$$

and variety

$$Y_{(j_1,j_2,\cdots,j_k)} = \left\{ v \in \mathbb{C}^{\binom{n}{k}} \,\middle|\, v_{(a_i)_i} = 0, \forall (a_i)_{i=1,2,\cdots,k}, \text{s.t. } a_i > j_i \text{ and } a_l = j_l (\forall l < i) \text{ for some } i \right\}.$$

It suffices to prove $\Lambda^{(a_1,a_2,\cdots,a_k)} = 0$, for all $(a_1,a_2,\cdots,a_k)$, s.t. $a_l \le j_l (\forall l < i)$ and $a_i > j_i$ for some i.

By induction, assume such equation for $i' = i-1$ is true (when $i' = 1$, it's clearly true).

$$\begin{aligned} \Lambda^{(j_1,j_2,\cdots,j_k)}\Lambda^{(a_1,a_2,\cdots,a_k)} &= \sum_{l=1}^{k} \Lambda^{(j_1,j_2,\cdots,j_{i-1},a_l,j_{i+1},\cdots,j_k)}\Lambda^{(a_1,a_2,\cdots,j_i,\cdots,a_k)} \\ &= \sum_{l=1}^{i-1} \Lambda^{(j_1,j_2,\cdots,j_{i-1},a_l,j_{i+1},\cdots,j_k)} \underbrace{\Lambda^{(a_1,a_2,\cdots,a_{l-1},a_{l+1},\cdots,a_{i-1},\underbrace{j_i}_{(i-1)_{\text{th}}\text{ element}},a_i,\cdots,a_k)}}_{=0 \text{ since } a_{i-1} \le j_{i-1} < j_i < a_i \text{ and } j_i > j_{i-1}} \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{l=i}^{k} \underbrace{\Lambda^{(j_1, j_2, \cdots, j_{i-1}, a_l, j_{i+1}, \cdots, j_k)}}_{=0 \text{ since } a_l \geq a_i > j_l} \Lambda^{(a_1, a_2, \cdots, j_i, \cdots, a_k)} \\
&= 0.
\end{aligned}
$$