Compact Lie Groups

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Pre-knowledge

Definition 1. Let B be connected space with base point $b_0 \in B$. The continuous map $\pi : E \to B$ is a *fiber bundle (locally trivial fibration)* with fiber F if it satisfies the following properties:

1. $\pi^{-1}(b_0) = F$.
2. π is surjective.
3. For every point $x \in B$ there is an open neighborhood $U_x \subset B$ and a "fiber preserving homeomorphism" $\Psi_{U_x} : \pi^{-1}(U_x) \to U_x \times F$, that is a homeomorphism making the following diagram commute:

$$\begin{tikzcd}
\pi^{-1}(U_x) \arrow[r, "\Psi_{U_x}"'] \arrow[d, "\pi"] & U_x \times F \arrow[dl, "proj_1"] \\
U_x &
\end{tikzcd}$$

Definition 2. A *covering space* is a fiber bundle s.t. the bundle projection π is a local homeomorphism. It follows that the fiber is a discrete space. If the fiber has exactly two elements, it's a *double cover*.

Definition 3. A *local section* of a fiber bundle is a continuous map $\sigma : U \to E$ where U is an open subset of B and

$$\pi(\sigma(x)) = x \quad \forall x \in U.$$

Definition 4. For a space X, define *n-th homotopy group* $\pi_n(X)$ to be the group of homotopy classes of maps $g : [0,1]^n \to X$ from the n-cube to X that take the boundary of the n-cube to b.

Basic Notions

Definition 5. A *Lie group* G is a group and a manifold so that

1. the *multiplication* map $\mu : G \times G \to G$ given by $\mu(g, g') = gg'$ is smooth and
2. the *inverse* map $\iota : G \to G$ by $\iota(g) = g^{-1}$ is smooth.

Definition 6. Let G be a matrix Lie group. The *Lie algebra* of G, denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for all real numbers t.

Definition 7. A *Lie subgroup* H of a Lie group G is the image in G of a Lie group H' under an injective immersive homomorphism $\varphi : H' \to G$ together with the Lie group structure on H making $\varphi : H' \to H$ a diffeomorphism.

Theorem 8. Let G be a Lie group and $H \subseteq G$ a subgroup (with no manifold assumption). Then H is a **regular** Lie subgroup if and only if H is **closed**.

Theorem 9. Let H be a closed subgroup of a Lie group G. Then there is a unique manifold structure on the quotient space G/H so the projection map $\pi : G \to G/H$ is smooth, and so there exist local smooth sections of G/H into G.

Definition 10. A *homomorphism* of Lie groups is a smooth homomorphism between two Lie groups.

Theorem 11. If G and G' are Lie groups and $\varphi : G \to G'$ is a homomorphism of Lie groups, then φ has constant rank and ker φ is a (closed) regular Lie subgroup of G of dimension $\dim G - \operatorname{rk}\varphi$ where $\operatorname{rk}\varphi$ is the rank of the differential of φ.

Proposition 12. The compact symplectic group

$$Sp(n) = \{g \in GL(n, \mathbb{H}) \mid g^\star g = I\}$$

is isomorphic to

$$Sp(n; \mathbb{C}) \cap U(2n),$$

where $Sp(n; \mathbb{C}) = \{g \in GL(2n; \mathbb{C}) \mid g^t \Omega g = \Omega\}$ is the symplectic group and

$$\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Remark. Consider $\mathbb{C}$-linear isomorphism

$$\begin{aligned} \vartheta : \mathbb{H}^n &\to \mathbb{C}^{2n} \\ a + jb &\mapsto (a, b), \end{aligned}$$

which induces a $\mathbb{C}$-linear isomorphism

$$\tilde{\vartheta} : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C}),$$

s.t.

$$\tilde{\vartheta} X = \vartheta \circ X \circ \vartheta^{-1} \quad (\forall X \in M_n(\mathbb{H})).$$

In fact,

$$\tilde{\vartheta}(A + jB) = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}.$$

Topology

Definition 13. If G is a Lie group, write G^0 for the connected component of G containing e.

Lemma 14. Let G be a Lie group. The connected component G^0 is a regular Lie subgroup of G. If G^1 is any connected component of G with $g_1 \in G^1$, then $G^1 = g_1 G^0$.

Lie Group	Name	Definition	C	SC	Lie Algebra
$O(n)$	orthogonal group	$\{g \in GL(n,\mathbb{R}) \mid g^t g = I\}$	$\mathbb{Z}_2$		$\mathfrak{so}(n) = \{X \in \mathfrak{o}(n) \mid X^t + X = 0, \operatorname{tr} X = 0\}$
$SO(n)$	special orthogonal group	$\{g \in O(n) \mid \det g = 1\}$	Y	$\mathbb{Z}$ $(n = 2)$ $\mathbb{Z}_2$ $(n > 2)$	$\mathfrak{so}(n)$
$U(n)$	unitary group	$\{g \in GL(n,\mathbb{C}) \mid g^\star g = I\}$	Y	$\mathbb{Z}$	$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^\star + X = 0\}$
$SU(n)$	special unitary group	$\{g \in U(n) \mid \det g = 1\}$	Y	Y	$\mathfrak{su}(n) = \{X \in \mathfrak{u}(n) \mid \operatorname{tr} X = 0\}$
$Sp(n)$	(compact) symplectic group	$\{g \in GL(n,\mathbb{H}) \mid g^\star g = I\}$	Y	Y	$\mathfrak{sp}(n) = \{X \in M_n(\mathbb{H}) \mid X^\star + X = 0\}$
$Spin(n)$	spin group		Y $(n > 1)$	Y $(n > 2)$	$\mathfrak{so}(n)$

Table 1: Table of Compact Classical Lie Groups [1]

Theorem 15. If G is a Lie group and H a connected Lie subgroup so that G/H is also connected, then G is connected.

Definition 16. Let be G a Lie group and M a manifold.

1. An *action* of G on M is a smooth map from $G \times M \to M$, denoted by $(g, m) \to g \cdot m$ for $g \in G$ and $m \in M$, so that:

    (a). $e \cdot m = m$, all $m \in M$ and

    (b). $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$ for all $g_1, g_2 \in G$ and $m \in M$.

2. The action is called *transitive* if for each $m, n \in M$, there is a $g \in G$, so $g \cdot m = n$.

3. The *stabilizer* of $m \in M$ is $G^m = \{g \in G \mid g \cdot m = m\}$.

Theorem 17. The compact classical groups, $SO(n), SU(n)$, and $Sp(n)$, are connected.

Remark.

$$\{1\} \to SO(n-1) \to SO(n) \to S^{n-1} \to \{1\},$$
$$\{1\} \to SU(n-1) \to SU(n) \to S^{2n-1} \to \{1\}$$
$$\{1\} \to Sp(n-1) \to Sp(n) \to S^{4n-1} \to \{1\}$$

Definition 18. The Lie group G is called *simply connected* if it's fundamental group $\pi_1(G)$ is trivial.

Lemma 19. If H is a discrete normal subgroup of a connected Lie group G, then H is contained in the center of G.

Theorem 20. Let G be a connected Lie group.

1. The connected simply connected cover $\widetilde{G}$ is a Lie group.

2. If π is the covering map and $\widetilde{Z} = \ker \pi$, then $\widetilde{Z}$ is a discrete central subgroup of $\widetilde{G}$

3. π induces a diffeomorphic isomorphism $G \cong \widetilde{G}/\widetilde{Z}$.

4. $\pi_1(G) \cong \widetilde{Z}$.

Lemma 21. $Sp(1)$ and $SU(2)$ are simply connected and isomorphic to each other. Either group is the simply connected cover of $SO(3)$, i.e., $SO(3)$ is isomorphic to $\mathrm{Sp}(1)/\{\pm 1\}$ or $SU(2)/\{\pm I\}$

Remark. Consider the 3-dimensional vector space

$$V = \left\{ \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_2 & x_1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\} = \mathfrak{su}(2),$$

with inner product

$$\langle X_1, X_2 \rangle = \frac{1}{2} \operatorname{trace}(X_1 X_2) = x_1 x_1' + x_2 x_2' + x_2 x_2'.$$

As vector space $V \cong \mathbb{R}^3$. Let $\Phi : SU(2) \to \operatorname{End}(V) \cong GL(3; \mathbb{R})$ by setting

$$\Phi_U(X) = UXU^{-1} \quad (\forall U \in SU(2)).$$

Since $\langle \Phi_U(X_1), \Phi_U(X_2) \rangle = \langle X_1, X_2 \rangle$, $\Phi_U \in SO(3)$.

Theorem 22. 1. $\pi_1(SO(2)) \cong \mathbb{Z}$ and $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$.

2. $SU(n)$ is simply connected for $n \geq 2$.

3. $\mathrm{Sp}(n)$ is simply connected for $n \geq 1$.

Remark. Use the long exact sequence of higher homotopy groups.

Clifford Algebras

Definition 23. The *Clifford algebra* is

$$\mathcal{C}(\mathbb{R}^n, Q) = \mathcal{T}_n(\mathbb{R})/\mathcal{I}$$

where $\mathcal{T}_n(\mathbb{R}) = \bigoplus_{k=0}^{\infty} \bigotimes^k \mathbb{R}^n$ and $\mathcal{I}$ is the ideal of $\mathcal{T}_n(\mathbb{R})$ generated by

$$\{(x \otimes x - Q(x)1) \mid x \in \mathbb{R}^n\},$$

and $Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form over $\mathbb{R}^n$.

Remark. To remove multiple copies of basis for $\mathbb{R}^n$,

$$\begin{aligned} x \otimes y + y \otimes x &= (x+y) \otimes (x+y) - x \otimes x - y \otimes y \\ &= Q(x+y) - Q(x) - Q(y). \end{aligned}$$

Remark. Let $Q(v) = -|v|^2$ $(\forall v \in \mathbb{R}^n)$, then $x \otimes y + y \otimes x = -2(x, y)$. For $\mathcal{C}_n(\mathbb{R}) := \mathcal{C}(\mathbb{R}^n, -|\cdot|^2)$

$$\mathcal{C}_0(\mathbb{R}) = \mathbb{R}, \quad \mathcal{C}_1(\mathbb{R}) = \mathbb{C}, \quad \mathcal{C}_2(\mathbb{R}) = \mathbb{H}.$$

Remark. If $Q = 0$ then the Clifford algebra is just the exterior algebra $\bigwedge \mathbb{R}^n$.

Proposition 24. There is a linear isomorphism $\Psi : \mathcal{C}_n(\mathbb{R}) \to \bigwedge \mathbb{R}^n$.

Remark.

$$\begin{aligned} \epsilon(x)(y) &= x \wedge y, \\ \iota(x)(y_1 \wedge \cdots y_k) &= \sum_{i=1}^{k} (-1)^{i+1} (x, y_i)\, y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_k, \end{aligned}$$

where $\hat{y}_i$ means to omit the term.

Let $L_x = \epsilon(x) + \iota(x)$, $\Phi : \mathcal{T}_n(\mathbb{R}) \to \mathrm{End}(\mathbb{R}^n)$ by setting $\Phi(x) = L_x$, which induces $\Phi : \mathcal{C}_n(\mathbb{R}) \to \mathrm{End}(\mathbb{R}^n)$ since $\Phi(\mathcal{I}) = 0$.

Let

$$\Psi(v) = \Phi(v)(1),$$

then

$$\Psi(x_1 \cdots x_k) = x_1 \wedge \cdots \wedge x_k + \text{ terms in } \bigoplus_{i \geq 1} \bigwedge^{k-2i} \mathbb{R}^n.$$

Remark. A basis of $\mathcal{C}_n(\mathbb{R})$ is then

$$\{1\} \cup \{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

where $\{e_1, e_2, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

$\mathrm{Spin}_n(\mathbb{R})$

Definition 25. 1. Let $\mathcal{C}_n^+(\mathbb{R})$ be the subalgebra of $\mathcal{C}_n(\mathbb{R})$ spanned by all products of an even number of elements of $\mathbb{R}^n$.

2. Let $\mathcal{C}_n^-(\mathbb{R})$ be the subspace of $\mathcal{C}_n(\mathbb{R})$ spanned by all products of an odd number of elements of $\mathbb{R}^n$ so $\mathcal{C}_n(\mathbb{R}) = \mathcal{C}_n^+(\mathbb{R}) \oplus \mathcal{C}_n^-(\mathbb{R})$ as a vector space.

3. Let the automorphism α, called the *main involution*, of $\mathcal{C}_n(\mathbb{R})$ act as multiplication by ± 1 on $\mathcal{C}_n^\pm(\mathbb{R})$

4. *Conjugation*, an anti-involution on $\mathcal{C}_n(\mathbb{R})$, is defined by

$$(x_1 x_2 \cdots x_k)^* = (-1)^k x_k \cdots x_2 x_1$$

for $x_i \in \mathbb{R}^n$.

Definition 26. 1. Let $\mathrm{Spin}_n(\mathbb{R}) = \{g \in \mathcal{C}_n^+(\mathbb{R}) \mid gg^* = 1 \text{ and } gxg^* \in \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n\}$.

2. Let $\mathrm{Pin}_n(\mathbb{R}) = \{g \in \mathcal{C}_n(\mathbb{R}) \mid gg^* = 1 \text{ and } \alpha(g)xg^* \in \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n\}$. Note $\mathrm{Spin}_n(\mathbb{R}) \subseteq \mathrm{Pin}_n(\mathbb{R})$.

3. For $g \in \mathrm{Pin}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$, define the homomorphism $\mathcal{A} : \mathrm{Pin}_n(\mathbb{R}) \to GL(n, \mathbb{R})$ by $(\mathcal{A}\})x = \alpha(g)xg^*$. Note $(\mathcal{A}g)x = gxg^*$ when $g \in \mathrm{Spin}_n(\mathbb{R})$.

Lemma 27. $\mathcal{A}$ is a covering map of $\mathrm{Pin}_n(\mathbb{R})$ onto $O(n)$ with $\ker \mathcal{A} = \{\pm 1\}$, so there is an exact sequence

$$\{1\} \to \{\pm 1\} \to \mathrm{Pin}_n(\mathbb{R}) \xrightarrow{\mathcal{A}} O(n) \to \{I\}$$

Remark. Outline of proof:

1. $\mathcal{A}g \in O(n)$.
2. $\mathcal{A}$ maps Pin_n onto $O(n)$.
3. $\ker \mathcal{A} = \{\pm 1\}$.
4. $\mathcal{A}$ is a covering map. (by theorem 11).

Lemma 28. $\mathrm{Pin}_n(\mathbb{R})$ and $\mathrm{Spin}_n(\mathbb{R})$ are compact I.ie groups with

$$\mathrm{Pin}_n(\mathbb{R}) = \left\{x_1 \cdots x_k \mid x_i \in S^{n-1} \text{ for } 1 \le k \le 2n\right\}$$
$$\mathrm{Spin}_n(\mathbb{R}) = \left\{x_1 x_2 \cdots x_{2k} \mid x_i \in S^{n-1} \text{ for } 2 \le 2k \le 2n\right\}$$

and $\mathrm{Spin}_n(\mathbb{R}) = \mathcal{A}^{-1}(SO(n))$.

Theorem 29. 1. $\mathrm{Pin}_n(\mathbb{R})$ has two connected $(n \ge 2)$ components with $\mathrm{Spin}_n(\mathbb{R}) = \mathrm{Pin}_n(\mathbb{R})^0$.

2. $\mathrm{Spin}_n(\mathbb{R})$ is the connected $(n \ge 2)$ simply connected $(n \ge 3)$ two-fold cover of $SO(n)$. The covering homomorphism is given by $\mathcal{A}$ with $\ker \mathcal{A} = \{\pm 1\}$, i.e., there is an exact sequence

$$\{1\} \to \{\pm 1\} \to \mathrm{Spin}_n(\mathbb{R}) \xrightarrow{\mathcal{A}} SO(n) \to \{I\}$$

Proposition 30. One can define the Lie algebra of $Spin_n(\mathbb{R})$ in terms of quadratic elements of the Clifford algebra, which is isomorphic to $\mathfrak{so}(n)$. [4]

Remark. $\mathfrak{so}(n)$ has a basis given by $L_{ij} = E_{ij} - E_{ji}$ $(\forall i < j)$. $\exp(tL_{ij})$ generates rotations in the $i - j$ plane.

$$[L_{ij}, L_{kl}] = \delta_{il} L_{kj} - \delta_{ik} L_{lj} + \delta_{jl} L_{ik} - \delta_{jk} L_{il}.$$

The generators of e_i of the Clifford algebra $C(n)$ satisfy the relations

$$\left[\frac{1}{2}e_i e_j, \frac{1}{2}e_k e_l\right] = \delta_{il}\left(\frac{1}{2}e_k e_j\right) - \delta_{ik}\left(\frac{1}{2}e_l e_j\right) + \delta_{jl}\left(\frac{1}{2}e_i e_k\right) - \delta_{jk}\left(\frac{1}{2}e_i e_l\right)$$

This shows that the vector space spanned by quadratic elements of $C(n)$ of the form $\frac{1}{2}e_i e_j$ $(i < j)$, together with the operation of taking commutators, is isomorphic to the Lie algebra $\mathfrak{so}(n)$.

Since $\left(\frac{1}{2}e_i e_j\right)^2 = -\frac{1}{4}$, the exponentials

$$e^{t\left(\frac{1}{2}e_i e_j\right)} = \cos\left(\frac{t}{2}\right) + e_i e_j \sin\left(\frac{t}{2}\right)$$

$\mathcal{A}e^{t\left(\frac{1}{2}e_i e_j\right)}$ also generates rotations in the $i - j$ plane.

As we go around this circle in $Spin_n(\mathbb{R})$ once, we go around the the circle of $SO(n)$ rotations in the $i - j$ plane twice. This is a reflection of the fact that $Spin_n(\mathbb{R})$ is a double-covering of the group $SO(n)$.

References

[1] Table of lie groups - wikipedia. `https://en.wikipedia.org/wiki/Table_of_Lie_groups`.

[2] Ralph Cohen. The topology of fiber bundles lecture notes. 1998.

[3] Mark R. Sepanski. *Compact Lie Groups*. Springer New York, 2007.

[4] Peter Woit. Clifford algebras and spin groups. `http://www.math.columbia.edu/~woit/LieGroups-2012/cliffalgsandspingroups.pdf`, 2012.