Chern Class

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All cohomology groups are with integer coefficients without notation.

1 Via an Euler class

Theorem 1.1 (Thom isomorphism)**.** Let $\pi : E \to B$ be an oriented rank n real vector bundle. Then the cohomology group $H^i(E, E_0)$ is zero for $i < n$, and $H^n(E, E_0)$ contains one and only one cohomology class u whose restriction

$$u|_{(F,F_0)} \in H^n(F, F_0)$$

is equal to the preferred generator u_F for every fiber F. Furthermore the correspondence $y \mapsto y \smile u$ maps $H^k(E)$ isomorphically onto $H^{k+n}(E, E_0)$ for every integer k.

Definition 1.2. The inclusion $(B, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E_0)$ induce maps

$$H^r(E, E_0) \to H^r(E) \to H^r(B).$$

The *Euler class* $e(E) \in H^n(B)$ is the image of u under the composition of these maps.

Note that since E and B are homotopic, $H^k(E) \cong H^k(B)$.

Proposition 1.3 (Gysin sequence)**.** Ther is a long exact sequence:

$$\cdots \to H^{k+n-1}(E_0) \to H^k(B) \xrightarrow{\smile e} H^{k+n}(B) \xrightarrow{\pi_0^*} H^{k+n}(E_0) \to \cdots .$$

Proof. By long exact sequence:

$$\cdots \to H^{r-1}(E_0) \to H^r(E, E_0) \to H^r(E) \to H^r(E_0) \to \cdots .$$

Then the following diagram commutes:

$$\begin{tikzcd}
H^{k+n}(E, E_0) \arrow[rrr] \arrow[dr, "\cong"'] & & & H^{k+n}(E) \arrow[r] & H^{k+n}(E_0) \\
& H^k(E) \arrow[dr, "\cong"'] & & & \\
& & H^k(B) \arrow[r, "\smile e(E)"] & H^{k+n}(B) \arrow[uu, "\pi^*"] \arrow[uur, "\pi_0^*"'] &
\end{tikzcd}$$

□

Corollary 1.4. Let $\pi : E \to B$ be a rank n complex vector bundle, there is an isomorphism:

$$H^k(B) \xrightarrow[\cong]{\pi_0^*} H^k(E_0),$$

for $k < 2n - 1$.

Proof. $H^{k-2n}(B) \cong H^{k-2n+1}(B) \cong 0$. Together with proposition 1.3. □

Definition 1.5. Let $\pi : E \to B$ be a rank n complex vector bundle over a paracompact space B. *Chern class* $c_k(E) \in H^{2k}(B_{\mathbb{R}})$ is given by

$$c_k(E) = \begin{cases} \pi_0^{*-1} c_k(E') & k < n \\ e(E_{\mathbb{R}}) & k = n \\ 0 & k > n \end{cases},$$

where $E' \to E_0$ is a vector bundle whose fiber on each point $v \in E_0$ is the quotient space $E/\mathbb{C}\{v\}$, whence E' is a rank $n-1$ complex vector bundle.

2 via Chern–Weil theory

2.1 Curvature from

Let $\pi : E \to M$ be a smooth rank n complex vector bundle over a differentiable manifold M. Denote the space of smooth sections of E over M by $\Gamma(M, E)$.

Definition 2.1. We call any section of $\wedge^k T^*M \otimes E$ an *E-valued k-form* on M. The set of E-valued k-forms $\Gamma(M, \wedge^k T^*M \otimes E)$ is denoted by $\Omega^k(M; E)$

Definition 2.2. A *connection* on E is an $\mathbb{C}$-linear map

$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$$

such that the Leibniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

holds for all smooth functions f on M and all smooth sections s of E.

Let X be a tangent vector field on M, one can define a *covariant derivative along X*

$$\nabla_X : \Omega^0(M; E) \to \Omega^0(M; E),$$

by contracting X with the resulting covariant index in the connection: $\nabla_X(s) = (\nabla(s))(X)$.

Definition 2.3. Let $\varphi_U : U \times \mathbb{C}^n \to \pi^{-1}(U)$ be local trivializations of E. A set of local sections $\{s_1, s_2, \cdots, s_n\}$ is said to be basis of $\Gamma(U, \pi^{-1}(U))$ if for all point $p \in U, \{s_1(p), s_2(p), \cdots, s_n(p)\}$ is a basis for the fiber E_p. Such $\{s_1, s_2, \cdots, s_n\}$ is called *local frame* of E over U.

Definition 2.4. For the given connection ∇ and local frame $\{s_1, s_2, \cdots, s_n\}$ on U, we can write:

$$\nabla_X s_i = \sum_j (\omega_U)_{i,j}(X) s_j,$$

for all vector field X over U, where $\omega_U \in \Omega^1(M; \mathfrak{gl}(n; \mathbb{C}))$, called *connection 1-form* associated to the given local frame.

Definition 2.5. Let ∇ be the given connection on E, one can extend ∇ to a family of operators:

$$\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E),$$

by defining

$$\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s,$$

for all $\omega \in \Omega^k(M), s \in \Omega^0(M; E)$

Definition 2.6. We call $R_\nabla = \nabla^2 : \Omega^0(M;E) \to \Omega^2(M;E)$ the *curvature tensor* of the connection ∇. In local frame:

$$\nabla^2 s_i(X,Y) = \sum_j (\Omega_U)_{i,j}(X,Y)s_j,$$

where $\Omega_U \in \Omega^2(M;\mathfrak{gl}(n;\mathbb{C}))$ is called *curvature 2-form.*

2.2 Invariant polynomial

Proposition 2.7. For the given vector space V, there is a bijection $\mathrm{Sym}^k V^*$ and homogeneous polynomial of degree k:

$$T \overset{\cong}{\mapsto} P_T := (v \mapsto T(v,\cdots,v)).$$

Proof. Just notice the *polarization formula*:

$$T(v_1,\cdots,v_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}P_T(t_1v_1+\cdots+t_kv_k).$$

$\square$

In general, let $G = GL(n;\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n;\mathbb{C})$.

Definition 2.8. A symmetric k-tensor $T \in \mathrm{Sym}^k\mathfrak{g}^*$ is called *G-invariant* if

$$g\cdot T = T \quad \forall g \in G.$$

The set of all G-invariant symmetric k-tensor is denoted by $I^k(G)$.

Theorem 2.9 (Chevally restriction theorem)**.** Let G be a complex semi-simple Lie group, $\mathfrak{g}$ the corresponding Lie algebra, $\mathfrak{h}$ the Cartan subalgebra and W the Weyl group of $\mathfrak{g}$. Then G-invariant polynomial on $\mathfrak{g}$ is isomorphic to W-invariant polynomial on $\mathfrak{h}$.

2.3 Chern-Weil theory

Theorem 2.10. Let E be a vector bundle over M. Then

1. For any $T \in I^k(G)$ and any linear connection ∇ on E, $P_T(R_\nabla)$ is a closed $2k$-form.
2. The de Rham cohomology class $[P_T(R_\nabla)] \in H^{2k}_{dR}(M)$ is independent of the choices of ∇.
3. The Chern-Weil maps

   $$\mathcal{CW} : (I^*(G),\circ) \to (H^*_{dR}(M),\wedge), \quad T \mapsto [P_T(R_\nabla)]$$

   is a ring homomorphism, which is called *Chern-Weil homomorphism.*

Definition 2.11. For the given G-invariant polynomial f of degree k, since the cohomology class $[f(\Omega)] \in H^{2k}_{dR}(M)$ does not depend on connections Ω, we will denote it by $f(E)$ and call it the *characteristic class* of E corresponding to f. The *total Chern class* $c(E)$ is defined by:

$$\left[\det\left(I - \frac{1}{2\pi i}\Omega\right)\right] = 1 + \sum_{k=1}^n \left(\frac{-1}{2\pi i}\right)^k \sigma_k(\Omega).$$