MATRICES AND THE VERTICAL DIFFERENCE METHOD

DAVID GOMPRECHT

The Vertical Difference method leads to a system of equations which can be solved to find the derivative of a polynomial. As the degree of the polynomial increases so does the complexity of the system. Matrices are a great tool for helping us to handle these systems. Here's an example. Notice how the coefficients of $Q(x)$ are now denoted $q_0, q_1, \ldots, q_{n-2}$.

Example 1. Use the Vertical Difference Method to find the line tangent to $f(x) = 2x^3 + 3x$ when $x = 1$.

Let $v(x) = (x-1)^2 Q(x)$. Since $v(x) = f(x) - mx - b$ we have

$$\begin{aligned} f(x) &= v(x) + mx + b \\ &= (x-1)^2(q_1 x + q_0) + mx + b \\ &= (x^2 - 2x + 1)(q_1 x + q_0) + mx + b \\ &= q_1 x^3 + (q_0 - 2q_1)x^2 + (q_1 + m - 2q_o)x + q_0 + b \end{aligned}$$

Equating the coefficients we have:

$$\begin{array}{rl} 1 & 0 = b + q_0 \\ x & 3 = m - 2q_0 + q_1 \\ x^2 & 0 = q_0 - 2q_1 \\ x^3 & 2 = q_1 \end{array}$$

We might organize this a little differently to make it more clear:

$$\begin{array}{rrrrl} 1b+ & 0m+ & 1q_0+ & 0q_1 = & 0 \\ & 1m+ & -2q_0+ & 1q_1 = & 3 \\ & & 1q_0+ & -2q_1 = & 0 \\ & & & 1q_1 = & 2 \end{array}$$

Date: November 17, 2017.

When a system of linear equations is organized as above, with all variables lined up in columns on the left and the constants on the right, we say it is in *Standard Form*. (The variables in this system are $b,\ m,\ q_0,$ and q_1.)

When viewed in this form it is clear that the focus of our attention should be on the coefficients of the variables in this system. With this in mind we can create a *matrix equation* which represents this same system. The matrix on the left in this system is called the *Coefficient Matrix*.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}$$

This equation can be represented as follows:

$$M\overrightarrow{V} = \overrightarrow{F}.$$

Here M is the matrix of coefficients arising from the vertical difference method, $\overrightarrow{V}$ is the column vector whose entries are the variables $b, m, q_0, q_1, \ldots, q_{n-2}$ (if the degree of the polynomial is n), and $\overrightarrow{F}$ is the column vector whose entries are the coefficients of the polynomial $f(x)$. M depends only on $a \in \mathbb{R}$, and not on $f(x)$. $\overrightarrow{F}$ depends only on the polynomial $f(x)$, and not on $a \in \mathbb{R}$.

Matrices

A Matrix is an array of numbers organized into rows and columns. For example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

is a matrix with 2 rows and 3 columns, which we call a 2×3 matrix. We also say that its size is 2×3, or that it has dimensions 2×3.
In general, an $m \times n$ matrix has m rows and n columns.

Definition. A vector is a matrix with only one row or column. A $1 \times n$ matrix is called a **row vector** (because it consists of only one row) and an $m \times 1$ matrix is called a **column vector** (for a similar reason). We sometimes denote a row or column vector with an arrow: $\overrightarrow{v}$.

	Row Vector	Column Vector
An example:	$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

Before diving into general matrix multiplication we consider the *inner product*:

Given a $1 \times n$ row vector A with entries $a_1, a_2, a_3, \ldots a_n$ and an $n \times 1$ column vector B with entries $b_1, b_2, b_3, \ldots b_n$ we define the *inner product* by:

$$A \cdot B = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n$$

Definition. We can now use the inner product to define matrix multiplication in general. Given an $m \times k$ matrix A and a $k \times n$ matrix B, the product, $A \cdot B$ will be an $m \times n$ matrix whose entries are defined as follows:

Let A_i represent the i^{th} row of matrix A and let B_j represent the j^{th} column of matrix B. Then the entry in the i^{th} row and j^{th} column is the inner product of A_i and B_j. That is:

$(AB)_{i,j} = A_i \cdot B_j$.

Example 2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and Let $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Then $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

and $\quad BA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$

Notice that matrix multiplication does not commute. That is, order matters when multiplying matrices.

Example 3. $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (4)$

Example 4. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

The matrices (1), $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and so on, all have a special property. An $n \times n$ matrix with 1s along the diagonal from the upper left to the lower right and zeros elsewhere is the *identity element* for matrix multiplication of $n \times n$ matrices. Let us denote this matrix by I_n. If M is an $n \times n$ matrix, then $I_n M = M$ and $M I_n = M$.

Example 5. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

As long as the dimensions are such that matrix multiplication is defined, I_n behaves like a multiplicative identity. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

Matrices can also be added, subtracted, or multiplied by a scalar. (A scalar is just a real number.) For matrix addition or subtraction, the dimensions of the two matrices need to be the same. In this is true, then matrix addition or subtraction is defined adding or subtracting corresponding entries.

Example 6. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 9 \end{pmatrix}$ and

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

Scalar multiplication is always defined. The entries of a scalar times a matrix are the scalar times the entries of the matrix.

Example 7. $10 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}$

Solving Matrix Equations

One can compare the equation $M\overrightarrow{V} = \overrightarrow{F}$ to the simple algebraic equation $3x = 5$. One solves the latter by multiplying both sides by $\frac{1}{3}$, written suggestively as 3^{-1}:

$3x = 5$

$3^{-1} \cdot 3x = 3^{-1} \cdot 5$ (We multiply both sides of the equation by 3^{-1})

$1 \cdot x = 3^{-1} \cdot 5$ (Since $3^{-1} \cdot 3 = 1$)

$x = 3^{-1} \cdot 5$ (Since 1 is the multiplicative identity)

Suppose there were a matrix M^{-1} such that $M^{-1} \cdot M = I$, where I is the $n \times n$ identity matrix. Then we could solve the equation $M\overrightarrow{V} = \overrightarrow{F}$ as follows.

$M\overrightarrow{V} = \overrightarrow{F}$

$M^{-1}M\overrightarrow{V} = M^{-1}\overrightarrow{F}$ (We multiply both sides of the equation by M^{-1})

$I\overrightarrow{V} = M^{-1}\overrightarrow{F}$ (Since $M^{-1} \cdot M = I$)

$\overrightarrow{V} = M^{-1}\overrightarrow{F}$ (Since I is the multiplicative identity)

Definition. Suppose M and N are $n \times n$ matrices. If $MN = I$ and $NM = I$, then N is the inverse matrix of M (and conversely), and we write $N = M^{-1}$.

(It can be shown that if $MN = I$ then necessarily $NM = I$.)

Solving the system of equations arising from the vertical difference method is a very straightforward matter once the coefficient matrix M is inverted.

Matrix Inversion

In this section we show how a square matrix, if it is invertible, can be inverted. First, we need to define *Elementary Row Operations* (EROs), These are operations which change a system of equations without changing the set of solutions of the system of equations.

The first ERO is *switching rows*. For example,

$$\begin{array}{c} 6x+7y=4 \\ 7x+8y=3 \end{array} \quad \text{and} \quad \begin{array}{c} 7x+8y=3 \\ 6x+7y=4 \end{array}$$

have the same solution set.

The second ERO is *multiplying a scalar times a row*. For example,

$$\begin{array}{c} 6x+7y=4 \\ 7x+8y=3 \end{array} \quad \text{and} \quad \begin{array}{c} 60x+70y=40 \\ 7x+8y=3 \end{array}$$

have the same solution set.

The third ERO is *adding a scalar multiple of a row to another row*. In the following example, -1 times the first row is added to the second row. Make sure that you understand why the two systems have the same solution set.

$$\begin{array}{c} 6x+7y=4 \\ 7x+8y=3 \end{array} \quad \text{and} \quad \begin{array}{c} 6x+7y=4 \\ x+y=-1 \end{array}$$

have the same solution set.

The EROs are the operations one performs to solve a system of equations. This process is called *Gaussian Elimination*.

Here's an example of Gaussian elimination applied to the the linear system stemming from the vertical difference method applied to finding the line tangent to $f(x)=2x^3+3x$ at $x=1$.

Note: we'll refer to the equations as $E_1, E_2, E_3,$ and E_4 and the rows as $R_1, R_2, R_3,$ and R_4. Also, we keep track of the numbers on the right hand side of an equals sign with an *augmented matrix*.

Table 1. Gaussian Elimination

$\begin{aligned} 1b + 0m + 1q_0 + 0q_1 &= 0 \\ 1m + -2q_0 + 1q_1 &= 3 \\ 1q_0 + -2q_1 &= 0 \\ 1q_1 &= 2 \end{aligned}$	$\leftarrow$ Original System Original Matrix $\rightarrow$	$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
$\begin{aligned} 1b + 0m + 1q_0 + 0q_1 &= 0 \\ 1m + -2q_0 + 1q_1 &= 3 \\ 1q_0 + 0q_1 &= 4 \\ 1q_1 &= 2 \end{aligned}$	$\leftarrow$ Add $2E_4$ to E_3 Add $2R_4$ to R_3 $\rightarrow$	$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
$\begin{aligned} 1b + 0m + 1q_0 + 0q_1 &= 0 \\ 1m + -2q_0 + 0q_1 &= 1 \\ 1q_0 + 0q_1 &= 4 \\ 1q_1 &= 2 \end{aligned}$	$\leftarrow$ Add $-E_4$ to E_2 Add $-R_4$ to R_2 $\rightarrow$	$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
$\begin{aligned} 1b + 0m + 1q_0 + 0q_1 &= 0 \\ 1m + 0q_0 + 0q_1 &= 9 \\ 1q_0 + 0q_1 &= 4 \\ 1q_1 &= 2 \end{aligned}$	$\leftarrow$ Add $2E_3$ to E_2 Add $2R_3$ to R_2 $\rightarrow$	$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
$\begin{aligned} 1b + 0m + 0q_0 + 0q_1 &= -4 \\ 1m + 0q_0 + 0q_1 &= 9 \\ 1q_0 + 0q_1 &= 4 \\ 1q_1 &= 2 \end{aligned}$	$\leftarrow$ Add $-E_3$ to E_1 Add $-R_3$ to R_1 $\rightarrow$	$\left(\begin{array}{cccc\|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
$\begin{aligned} b &= -4 \\ m &= 9 \\ q_0 &= 4 \\ q_1 &= 2 \end{aligned}$	$\leftarrow$ Solution $\rightarrow$	$\left(\begin{array}{cccc\|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$

We will now use the EROs to solve a smaller system. We will write the system of equations along side the corresponding matrix equation, in both standard and augmented form.

$$\begin{aligned} 6x + 7y &= 1 \\ 7x + 8y &= 0 \end{aligned} \qquad \begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 6 & 7 & 1 \\ 7 & 8 & 0 \end{array}\right)$$

Add -1 times the top row to the bottom row:

$$\begin{aligned} 6x + 7y &= 1 \\ x + y &= -1 \end{aligned} \qquad \begin{pmatrix} 6 & 7 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 6 & 7 & 1 \\ 1 & 1 & -1 \end{array}\right)$$

Then add -7 times the second row to the first row.

$$\begin{aligned} -x \qquad &= 8 \\ x + y &= -1 \end{aligned} \qquad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix} \qquad \left(\begin{array}{cc|c} -1 & 0 & 8 \\ 1 & 1 & -1 \end{array}\right)$$

Then add 1 times the top row to the bottom row:

$$\begin{aligned} -x &= 8 \\ y &= 7 \end{aligned} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \qquad \left(\begin{array}{cc|c} -1 & 0 & 8 \\ 0 & 1 & 7 \end{array}\right)$$

Then multiply the top row by -1:

$$\begin{aligned} x &= -8 \\ y &= 7 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 7 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 7 \end{array}\right)$$

You should check that $x = -8$ and $y = 7$ is the solution to the system.

The following system, in which $\begin{pmatrix}1\\0\end{pmatrix}$ has been replaced by $\begin{pmatrix}0\\1\end{pmatrix}$, is solved by precisely the same EROs.

$$\begin{aligned}6x+7y&=0\\7x+8y&=1\end{aligned}\qquad \begin{pmatrix}6&7\\7&8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\qquad \left(\begin{array}{cc|c}6&7&0\\7&8&1\end{array}\right)$$

Add -1 times the top row to the bottom row:

$$\begin{aligned}6x+7y&=0\\x+y&=1\end{aligned}\qquad \begin{pmatrix}6&7\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\qquad \left(\begin{array}{cc|c}6&7&0\\1&1&1\end{array}\right)$$

Then add -7 times the second row to the first row.

$$\begin{aligned}-x\phantom{{}+y}&=-7\\x+y&=1\end{aligned}\qquad \begin{pmatrix}-1&0\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-7\\1\end{pmatrix}\qquad \left(\begin{array}{cc|c}-1&0&-7\\1&1&1\end{array}\right)$$

Then add 1 times the top row to the bottom row:

$$\begin{aligned}-x&=-7\\y&=-6\end{aligned}\qquad \begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-7\\-6\end{pmatrix}\qquad \left(\begin{array}{cc|c}-1&0&-7\\0&1&-6\end{array}\right)$$

Then multiply the top row by -1:

$$\begin{aligned}x&=7\\y&=-6\end{aligned}\qquad \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\-6\end{pmatrix}\qquad \left(\begin{array}{cc|c}1&0&7\\0&1&-6\end{array}\right)$$

You should check that $x=7$ and $y=-6$ is the solution to the system.

We now have two matrix equations:

$$\begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} -8 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 7 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The two column vectors can be put together to form a 2×2 matrix $\begin{pmatrix} -8 & 7 \\ 7 & -6 \end{pmatrix}$. One can then check that $\begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} -8 & 7 \\ 7 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

In other words, the inverse of the matrix $\begin{pmatrix} -8 & 7 \\ 7 & -6 \end{pmatrix}$ is the matrix $\begin{pmatrix} -8 & 7 \\ 7 & -6 \end{pmatrix}$.

Instead of solving two systems, one could solve both systems at once. To do this we form an augmented matrix with two columns appended. Here we go.

$$\left(\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 7 & 8 & 0 & 1 \end{array}\right)$$

Add -1 times the top row to the bottom row:

$$\left(\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{array}\right)$$

Then add -7 times the second row to the first row.

$$\left(\begin{array}{cc|cc} -1 & 0 & 8 & -7 \\ 1 & 1 & -1 & 1 \end{array}\right)$$

Then add 1 times the top row to the bottom row:

$$\left(\begin{array}{cc|cc} -1 & 0 & 8 & -7 \\ 0 & 1 & 7 & -6 \end{array}\right)$$

Then multiply the top row by -1:

$$\left(\begin{array}{cc|cc} 1 & 0 & -8 & 7 \\ 0 & 1 & 7 & -6 \end{array}\right).$$

The matrix $\begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix}$ has been inverted!

To recap, here are the steps for inverting the invertible square matrix M.

- Form the augmented matrix $\big(M \mid I\big)$.
- Perform EROs to turn M into I.
- These same EROs turn I in M^{-1}.

In other words, the EROs turn $\big(M \mid I\big)$ into $\big(I \mid M^{-1}\big)$.

As another example we'll invert $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ -4 & 0 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to 4R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 8 & 1 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{3}R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 8 & 1 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to -2R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 8 & 1 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to -8R_2 + R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 4 & -\frac{8}{3} & 1 \end{array}\right) \xrightarrow{R_1 \to R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -\frac{10}{3} & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 4 & -\frac{8}{3} & 1 \end{array}\right)$$

One can check that $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 5 & -\frac{10}{3} & 1 \\ 0 & \frac{1}{3} & 0 \\ 4 & -\frac{8}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Reduced Row Echelon Form

When we use Gaussian elimination to convert $\big(M \mid I\big)$ into $\big(I \mid M^{-1}\big)$ we are putting the $n \times 2n$ matrix $\big(M \mid I\big)$ in *row echelon form.* Even when a matrix M is not invertible, $\big(M \mid I\big)$ can still be put into row echelon form. M is invertible if and only if the left half of the resulting matrix is the identity matrix I, and in this case the right half of the resulting matrix is M^{-1}.

The first nonzero number in a given row of a matrix is known as the *pivot* of that row.

Definition. A matrix is in *row echelon form* if it satisfies the following conditions.

- All rows with at least one nonzero element lie above any rows consisting of only zeroes.
- The pivot of a nonzero row is always strictly to the right of the pivot of the row above it.

Notice that if a mtrix is in row echelon form then all entries below a leading coefficient (and in the same column) must be zero.

Example 8. The matrix below is in row echelon form.

$$\begin{bmatrix} 1 & 3 & 3 & 4 & 5 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition. A matrix is in *reduced row echelon form* if it is in row echelon form and every pivot is 1 and is the only nonzero entry in its column.

The reduced row echelon form of a matrix is unique and may be computed by Gaussian elimination.

Example 9. The matrix below is in reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To recap, one can invert an invertible matrix $n \times n$ M as follows.

(1) Form the augmented matrix $(M|I)$, where I is the $n \times n$ identity matrix.
(2) Use Gaussian elimination, i.e. elementary row operations, to put $(M|I)$ in reduced row echelon form.
(3) The resulting matrix will be $(I|M^{-1})$.

The augmented matrix $(M|I)$ can always be put in reduced row echelon form. If M is not invertible then the left half of the reduced row echelon form of $(M|I)$ will not be the $n \times n$ identity matrix. This is one way to determine whether a matrix is invertible.

Matrices that represent a system of equations arising from the vertical difference method will always be invertible.

Using Matrices to Differentiate

Armed with these new tools for dealing with systems, we can solve for the slope of the tangent line to the graph of a polynomial at a point.

We'll first use the vertical difference method to find the line tangent to $f(x) = 2x^3 + 3x$ when $x = 1$. Recall that the resulting system of linear equations, in matrix form , is as follows.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}$$

We now invert $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

So, $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

One can check: $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Coming back to the vertical difference method, we can multiply both sides of

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}$$

by the inverse of the coefficient matrix.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}$$

This then becomes

$$\begin{pmatrix} b \\ m \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 4 \\ 2 \end{pmatrix}$$

Therefore $m = 9$, $b = -4$, and the tangent line to $f(x) = 2x^3 + 3x$ at $(1, 5)$ is $y = 9x - 4$.

In this final example we will use matrices to differentiate $f(x) = 3x^4 - 4x^3 + x$.

$$\overrightarrow{F} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -4 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{V} = \begin{pmatrix} b \\ m \\ q_0 \\ q_1 \\ q_2 \end{pmatrix}.$$

$f(x) = (x-a)^2Q(x)+mx+b = (x^2-2ax+a^2)(q_2x^2+q_1x+q_0)+mx+b$
$= q_2x^4+(q_1-2aq_2)x^3+(q_0-2aq_1+a^2q_2)x^2+(a^2q_1-2aq_0+m)x+a^2q_0+b.$

Comparing coefficients of like terms lead to the system of equations $M\overrightarrow{V} = \overrightarrow{F}$, written below.

$$\begin{pmatrix} 1 & 0 & a^2 & 0 & 0 \\ 0 & 1 & -2a & a^2 & 0 \\ 0 & 0 & 1 & -2a & a^2 \\ 0 & 0 & 0 & 1 & -2a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \\ q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

Now we invert the coefficient matrix.

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & a^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2a & a^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2a & a^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2a & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccccc|ccccc} 1 & 0 & a^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2a & a^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2a & a^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & a^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2a & a^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2a & 0 & 0 & 0 & 1 & 0 & -a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccccc|ccccc} 1 & 0 & a^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2a & a^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & a^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2a & 0 & 0 & 0 & 1 & 0 & -a^2 & -2a^3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccccc|ccccc} 1 & 0 & a^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a & 3a^2 & 4a^3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -a^2 & -2a^3 & -3a^4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a & 3a^2 & 4a^3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

So, $\begin{pmatrix} b \\ m \\ q_0 \\ q_1 \\ q_2 \end{pmatrix} = \overrightarrow{V} = M^{-1}\overrightarrow{F} = \begin{pmatrix} 1 & 0 & -a^2 & -2a^3 & -3a^4 \\ 0 & 1 & 2a & 3a^2 & 4a^3 \\ 0 & 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -4 \\ 3 \end{pmatrix}$.

Therefore, $m = \begin{pmatrix} 0 & 1 & 2a & 3a^2 & 4a^3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -4 \\ 3 \end{pmatrix} = 1 - 12a^2 + 12a^3$, and so

$f'(x) = 12x^3 - 12x^2 + 1.$