Find the derivation of the following formula:

$f(z) = e^{-\frac{z}{2}}$, where $z = g(y)$, $g(y) = y^T S^{-1} y$,

$y = h(x)$, $h(x) = x - \mu$

Solⁿ:

Given composite functions $f(z)$, $g(y)$, $h(x)$

So, According to chainrule:

$$\frac{df(z)}{dx} = \frac{df(z)}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Now,

$$\frac{df(z)}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = \frac{-1}{2} \cdot e^{-z/2}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right) = \frac{d}{dy}\left(y^T y . S^{-1}\right)$$

$$= S^{-1} \frac{d}{dy}\left(y^T y\right)$$

$$= 2y^T . S^{-1}$$

Metrics Derivation,

$\frac{d}{dy}(y^T y) = 2y^T$

and

S^{-1} = constant

$$\frac{dy}{dx} = \frac{d}{dx}(x-\mu) \quad \Big| \; \mu = \text{constant}$$

$$= 1$$

Now, after combining all

$$\frac{df(z)}{dx} = \frac{df(z)}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= \left\{ -\frac{1}{2} \cdot e^{\frac{-z}{2}} \right\} \cdot \left\{ 2 \cdot y^T S^{-1} \right\} \cdot 1$$

$$\frac{df(z)}{dx} = -e^{\frac{-z}{2}} \cdot y^T S^{-1}$$

$$\therefore \frac{df(z)}{dx} = -e^{\frac{-(y^T S^{-1} y)}{2}} \cdot (x-\mu)^T \cdot S^{-1}$$