题目 1. For each of the following s_n, determine s_n and its ordinary generating function.

1. $s_n := \sum_{k=0}^{n} k\binom{n}{k}$;

2. $s_n := \sum_{k=0}^{n} \binom{m+k}{m}$;

3. $s_n := \sum_{k=0}^{n} \binom{n-k}{k}$.

解答. 1.

Notice that $k\binom{n}{k} = n\binom{n-1}{k-1}$. Hence $s_n = n\sum_{k=0}^{n-1}\binom{n-1}{k-1} = n2^{n-1}$.

Let $F(x) := \sum_{n=0}^{\infty} s_n x^n = x\sum_{n=0}^{\infty} n\,(2x)^{n-1} = x\,D(\sum_{n=0}^{\infty}(2x)^n) = \frac{x}{(1-2x)^2}$.

2. Let $F_m(x) := \sum_{n=0}^{\infty} s_n x^n$. Then $F_m(x) = 1 + \sum_{n=1}^{\infty}\left(s_{n-1} + \binom{m+n}{m}\right)x^n = x\,F_m(x) + \sum_{n=0}^{\infty}\binom{m+n}{m}x^m$

Let $G_m(x) := \sum_{n=0}^{\infty}\binom{m+n}{n}x^n$.

Then $G_m(x) = \sum_{n=0}^{\infty}\binom{m+n}{n}x^n = 1 + \sum_{n=1}^{\infty}\binom{m+n-1}{n}x^n + \sum_{n=1}^{\infty}\binom{n+m-1}{n-1}x^n = x\,G_m(x) + \sum_{n=0}^{\infty}\binom{m-1+n}{n}x^n = x\,G_m(x) + G_{m-1}(x)$.

Since $G_0(x) = \frac{1}{1-x}$. $G_m(x) = \left(\frac{1}{1-x}\right)^{m+1}$. Hence $F_m(x) = \left(\frac{1}{1-x}\right)^{m+2}$.

Hence we get the recursion s. t. $F_m(x) = \frac{1}{1-2x}F_{m-1}(x)$. And we have the starting condition s. t. $F_0(x) = \left(\frac{1}{1-x}\right)^2$.

Therefore $F_m(x) = (\frac{1}{1-2x})^m \frac{1}{(1-x)^2}$.

When $n \geq 2$, $s_n = 1 + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+1}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+3}{2} + \binom{m+3}{3} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}$.

题目 2. Let S be a *colored set*, i.e., there exists a mapping $c: S \to C$, where the elements of C are called *colors*. A *rainbow subset* of a colored set is a subset in which every element has a distinct color.

Note that a rainbow subset does **not necessarily** contain all colors in C. Let S be a fixed colored set, where

$$C := [k] = \{1, 2, \ldots, k\}.$$

Let $n_i := |c^{-1}(i)|$, i.e., the number of elements of color i. Define f_m to be the number of rainbow subsets of S of size m.

1. Find the generating function of f_m. (You do not need to express it in a closed simple form, but simplify as much as possible.)

2. Suppose $n_1 = n_2 = \cdots = n_k = n$. Find f_m.

3. Suppose $n_1 = 1, n_2 = 2, \ldots, n_k = k$. Find f_m.

解答. 1.

By the question, each color has n_i choices,hence the mutiplication factor should be $(1 + n_i z_i)$. Hence the generating function should be $\prod_{i=1}^{k}(1 + n_i z_i)$. And We specilazation $z_i = z$. Hence the answer is $\prod_{i=1}^{k}(1 + n_i z)$.

2. Too silly. $\binom{m}{k}n^k$.

3. ## 同一行第一类斯特林数的计算

   类似第二类斯特林数，我们构造同行第一类斯特林数的生成函数，即

   $$F_n(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i$$

   根据递推公式，不难写出

   $$F_n(x) = (n-1)F_{n-1}(x) + xF_{n-1}(x)$$

   于是

   $$F_n(x) = \prod_{i=0}^{n-1}(x+i) = \frac{(x+n-1)!}{(x-1)!}$$

   ⟨hlink|斯特林数 - OI Wiki|⟩

⟨hlink| |https://oi-wiki.org/math/combinatorics/stirling/⟩

So the answer is the first stirling number.

题目 3. Let $R = \mathbb{C}[x]$, and $E = \mathrm{End}_{\mathbb{C}}(R)$. Consider the following two elements in E:

$$z\colon p \mapsto x\,p, \qquad D\colon p \mapsto \frac{d}{d\,x}\,p.$$

All the following equations are regarded as equalities in E.

1. Prove that $D\,z^k - z^k\,D = k\,z^{k-1}$. Here k is a natural number.
2. Find $a_{n,k}$ such that
$$(z\,D)^n = \sum_{k=0}^{n} a_{n,k}\,z^k\,D^k.$$
3. Find $b_{n,k}$ such that
$$z^n\,D^n = \sum_{k=0}^{n} b_{n,k}\,(z\,D)^k.$$
4. Find $c_{n,i,j}$ such that
$$(z + D)^n = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{n,i,j}\,z^i\,D^j.$$

解答. 1.

$\forall f \in \mathbb{C}[x]$, suppose $f = \sum_{n=0}^{\infty} a_n x^n$. $D\,z^k(f) = \sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1}$. $z^k D(f) = \sum_{n=0}^{\infty} n\,a_n x^{n+k-1}$.

2. $(z\,D)^{n+1} = z\,D(\sum_{k=0}^{n} a_{n,k} z^k D^k) = \sum_{k=0}^{n} a_{n,k} z(z^k D + k\,z^{k-1})D^k = \sum_{k=0}^{n} a_{n,k} z^{k+1} D^{k+1} + \sum_{k=0}^{n} a_{n,k} k\,z^k D^k$

Hence $a_{n+1} = a_{n,k-1} + k\,a_{n,k}$.

That is still the firsrt stirling number.

3. By the notes （you can see it in the following）. $(-1)^{n-k} S(n,k)$.

题目 4. Let $f(n)$ be the number of partitions if n such that for every k , k occurs in the partition at most k times. Let $g(n)$ be the number of partitions of n such that no part has the form $k(k+1)$ (i.e. no parts equal 2,6,12,...).Show that $f(n) = g(n)$.

解答. Just Let $F(x) := \sum_{n=0}^{\infty} f(x) x^n$ and $G(x) := \sum_{n=0}^{\infty} g(n) x^n$. By the question, $G(x) = \prod_{i \neq k(k+1), i \geq 1} \frac{1}{1 - x^i}$.

$F(x) = (1+x)(1+x^2+x^4)(1+x^3+x^6+x^9)\cdots = \prod_{i=1}^{\infty} \frac{1 - x^{i(i+1)}}{1 - x^i}$.

Problem 5

Let $f_k(m,n)$ denote the number of $m\times n$ $\{0,1\}$-matrices with exactly k ones such that **each row contains at least one 1**.
Let $g_k(m,n)$ denote the number of $m\times n$ $\{0,1\}$-matrices with exactly k ones such that **each row and each column contain at least one 1**.

(i) Find the generating function of $f_k(m,n)$ with respect to k, that is, find

$$\sum_{k\geq 0} f_k(m,n)z^k.$$

(ii) Find the generating function of $g_k(m,n)$ with respect to k, that is, find

$$\sum_{k\geq 0} g_k(m,n)z^k.$$

(iii) Prove that

$$\sum_{m,n,k\geq 0} g_k(m,n)z^k\frac{x^m}{m!}\frac{y^n}{n!}=e^{-x-y}\sum_{i,j\geq 0}(1+z)^{\mathrm{ij}}\frac{x^i}{i!}\frac{y^j}{j!}.$$

注记 1. Three polynomial bases

- **Ordinary powers**: x^n.
- **Falling factorials**:

$$x^{\underline{n}}=x(x-1)\cdots(x-n+1),x^{\underline{0}}=1.$$

- **Rising factorials**:

$$x^{\overline{n}}=x(x+1)\cdots(x+n-1),x^{\overline{0}}=1,$$

  with the basic relation $x^{\overline{n}}=(-1)^n(-x)^{\underline{n}}$.

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1) Ordinary powers ↔ Falling factorials (Stirling numbers)

- **Expand ordinary powers in the falling basis**

$$x^n=\sum_{k=0}^{n} S(n,k)\ x^{\underline{k}}$$

where $S(n,k)$ are the **Stirling numbers of the second kind** (partitions of an n-set into k nonempty blocks).

- **Expand falling factorials in ordinary powers**

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)\ x^k$$

where $s(n,k)$ are the **(signed) Stirling numbers of the first kind**, and $s(n,k) = (-1)^{\,n-k} \begin{bmatrix} n \\ k \end{bmatrix}$ with $\begin{bmatrix} n \\ k \end{bmatrix}$ the **unsigned** ones.

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2) Ordinary powers ↔ Rising factorials

- **Expand ordinary powers in the rising basis**

$$x^n = \sum_{k=0}^{n} (-1)^{\,n-k}\ S(n,k)\ x^{\overline{k}}$$

(obtained from $x^{\underline{k}} = (-1)^k (-x)^{\overline{k}}$).

- **Expand rising factorials in ordinary powers**

$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{\,n-k} s(n,k)$ are the **unsigned Stirling numbers of the first kind**.

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3) Falling factorials ↔ Rising factorials (Lah numbers)

The change of basis between the two factorial bases is governed by the **Lah numbers**

$$L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!} (1 \le k \le n).$$

- **Rising in terms of falling**

$$x^{\overline{n}} = \sum_{k=1}^{n} L(n,k)\ x^{\underline{k}}$$

- **Falling in terms of rising**

$$x^{\underline{n}} = \sum_{k=1}^{n} (-1)^{\,n-k}\ L(n,k)\ x^{\overline{k}}$$

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4) Quick table ($n \le 4$)

- $x^{\underline{1}} = x,\ x^{\underline{2}} = x^2 - x,\ x^{\underline{3}} = x^3 - 3x^2 + 2x,\ x^{\underline{4}} = x^4 - 6x^3 + 11x^2 - 6x.$
- $x^{\overline{1}} = x,\ x^{\overline{2}} = x^2 + x,\ x^{\overline{3}} = x^3 + 3x^2 + 2x,\ x^{\overline{4}} = x^4 + 6x^3 + 11x^2 + 6x.$

These identities give a complete and practical conversion kit among the three bases.