**题目 1.** For each of the following $s_n$, determine $s_n$ and its ordinary generating function.

1. $s_n := \sum_{k=0}^{n} k\binom{n}{k}$;

2. $s_n := \sum_{k=0}^{n} \binom{m+k}{m}$;

3. $s_n := \sum_{k=0}^{n} \binom{n-k}{k}$.

**解答.** 1.

Notice that $k\binom{n}{k} = n\binom{n-1}{k-1}$. Hence $s_n = n\sum_{k=0}^{n-1}\binom{n-1}{k-1} = n2^{n-1}$.

Let $F(x) := \sum_{n=0}^{\infty} s_n x^n = x\sum_{n=0}^{\infty} n\,(2x)^{n-1} = x\,D(\sum_{n=0}^{\infty}(2x)^n) = \frac{x}{(1-2x)^2}$.

2. Let $F_m(x) := \sum_{n=0}^{\infty} s_n x^n$. Then $F_m(x) = 1 + \sum_{n=1}^{\infty}\left(s_{n-1} + \binom{m+n}{m}\right)x^n = x\,F_m(x) + \sum_{n=0}^{\infty}\binom{m+n}{m}x^m$

Let $G_m(x) := \sum_{n=0}^{\infty}\binom{m+n}{n}x^n$.

Then $G_m(x) = \sum_{n=0}^{\infty}\binom{m+n}{n}x^n = 1 + \sum_{n=1}^{\infty}\binom{m+n-1}{n}x^n + \sum_{n=1}^{\infty}\binom{n+m-1}{n-1}x^n = x\,G_m(x) + \sum_{n=0}^{\infty}\binom{m-1+n}{n}x^n = x\,G_m(x) + G_{m-1}(x)$.

Since $G_0(x) = \frac{1}{1-x}$. $G_m(x) = \left(\frac{1}{1-x}\right)^{m+1}$. Hence $F_m(x) = \left(\frac{1}{1-x}\right)^{m+2}$.

When $n \geq 2$, $s_n = 1 + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+1}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+3}{2} + \binom{m+3}{3} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}$.

3. We use the trick of swapping the summation order.

Let $F(x) := \sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n-k}{k}x^n = \sum_{k=0}^{\infty}\sum_{n\geq k}\binom{n-k}{k}x^n = \sum_{k=0}^{\infty}\frac{1}{k!}x^k\sum_{n\geq k}(n-k)(n-k-1)\cdots(n-k-k+1)x^{n-k} = \sum_{k=0}^{\infty}x^{2k}D^k\left(\sum_{n\geq k}x^{n-k}\right) = \sum_{k=0}^{\infty}\frac{x^{2k}}{(1-x)^{k+1}} = \frac{1}{1-x}\frac{1}{1-\frac{x^2}{1-x}} = \frac{1}{1-x-x^2}$.

so $s_n =$ the Fibonacci numbers.

**题目 2.** Let $S$ be a *colored set*, i.e., there exists a mapping $c: S \to C$, where the elements of $C$ are called *colors*. A *rainbow subset* of a colored set is a subset in which every element has a distinct color.

Note that a rainbow subset does **not necessarily** contain all colors in $C$. Let $S$ be a fixed colored set, where

$$C := [k] = \{1, 2, \ldots, k\}.$$

Let $n_i := |c^{-1}(i)|$, i.e., the number of elements of color $i$. Define $f_m$ to be the number of rainbow subsets of $S$ of size $m$.

1. Find the generating function of $f_m$. (You do not need to express it in a closed simple form, but simplify as much as possible.)

2. Suppose $n_1 = n_2 = \cdots = n_k = n$. Find $f_m$.

3. Suppose $n_1 = 1, n_2 = 2, \ldots, n_k = k$. Find $f_m$.

**解答.** 1.

By the question, each color has $n_i$ choices,hence the mutiplication factor should be $(1 + n_i z_i)$. Hence the generating function should be $\prod_{i=1}^{k}(1 + n_i z_i)$. And We specilazation $z_i = z$. Hence the answer is $\prod_{i=1}^{k}(1 + n_i z)$.

2. Too silly. $\binom{m}{k}n^k$.

3. 同一行第一类斯特林数的计算

类似第二类斯特林数，我们构造同行第一类斯特林数的生成函数，即

$F_n(x)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}x^i$

根据递推公式，不难写出

$F_n(x)=(n-1)F_{n-1}(x)+xF_{n-1}(x)$

于是

$F_n(x)=\prod_{i=0}^{n-1}(x+i)=\frac{(x+n-1)!}{(x-1)!}$

⟨hlink|斯特林数 - OI Wiki|⟩

⟨hlink| |https://oi-wiki.org/math/combinatorics/stirling/⟩

So the answer is the first stirling number.

**题目 3.** Let $R=\mathbb{C}[x]$, and $E=\mathrm{End}_{\mathbb{C}}(R)$. Consider the following two elements in $E$:

$$z\colon p\mapsto x\,p,\qquad D\colon p\mapsto\frac{d}{d\,x}\,p.$$

All the following equations are regarded as equalities in $E$.

1. Prove that $D\,z^k-z^k\,D=k\,z^{k-1}$. Here k is a natural number.
2. Find $a_{n,k}$ such that
$$(z\,D)^n=\sum_{k=0}^{n}a_{n,k}\,z^k\,D^k.$$
3. Find $b_{n,k}$ such that
$$z^n\,D^n=\sum_{k=0}^{n}b_{n,k}\,(z\,D)^k.$$
4. Find $c_{n,i,j}$ such that
$$(z+D)^n=\sum_{i=0}^{n}\sum_{j=0}^{n}c_{n,i,j}\,z^i\,D^j.$$

**解答.** 1.

$\forall f\in\mathbb{C}[x]$, suppose $f=\sum_{n=0}^{\infty}a_nx^n$. $D\,z^k(f)=\sum_{n=0}^{\infty}(n+k)a_nx^{n+k-1}$. $z^kD(f)=\sum_{n=0}^{\infty}n\,a_nx^{n+k-1}$.

2. $(z\,D)^{n+1}=z\,D(\sum_{k=0}^{n}a_{n,k}z^kD^k)=\sum_{k=0}^{n}a_{n,k}z(z^kD+k\,z^{k-1})D^k=\sum_{k=0}^{n}a_{n,k}z^{k+1}D^{k+1}+\sum_{k=0}^{n}a_{n,k}k\,z^kD^k$

Hence $a_{n+1}=a_{n,k-1}+k\,a_{n,k}$.

That is still the firsrt stirling number.

3. By the notes （you can see it in the following）. $(-1)^{n-k}S(n,k)$.

**题目 4.** Let $f(n)$ be the number of partitions if $n$ such that for every $k$ , $k$ occurs in the partition at most $k$ times. Let $g(n)$ be the number of partitions of n such that no part has the form $k(k+1)$ (i.e. no parts equal 2,6,12,…).Show that $f(n)=g(n)$.

**解答.** Just Let $F(x):=\sum_{n=0}^{\infty}f(x)x^n$ and $G(x):=\sum_{n=0}^{\infty}g(n)x^n$. By the question, $G(x)=\prod_{i\neq k(k+1),i\geq 1}\frac{1}{1-x^i}$.

$F(x)=(1+x)(1+x^2+x^4)(1+x^3+x^6+x^9)\cdots=\prod_{i=1}^{\infty}\frac{1-x^{i(i+1)}}{1-x^i}$.

## Problem 5

Let $f_k(m,n)$ denote the number of $m \times n$ $\{0,1\}$-matrices with exactly $k$ ones such that **each row contains at least one 1**.
Let $g_k(m,n)$ denote the number of $m \times n$ $\{0,1\}$-matrices with exactly $k$ ones such that **each row and each column contain at least one 1**.

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(i) Find the generating function of $f_k(m,n)$ with respect to $k$, that is, find

$$\sum_{k \geq 0} f_k(m,n) z^k.$$

(ii) Find the generating function of $g_k(m,n)$ with respect to $k$, that is, find

$$\sum_{k \geq 0} g_k(m,n) z^k.$$

(iii) Prove that

$$\sum_{m,n,k \geq 0} g_k(m,n) z^k \frac{x^m}{m!} \frac{y^n}{n!} = e^{-x-y} \sum_{i,j \geq 0} (1+z)^{\mathrm{ij}} \frac{x^i}{i!} \frac{y^j}{j!}.$$

**解答.** (i)

By the question, the generating fucntion should be written as $F_n = \prod_{i=1}^{m} \left( x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots \binom{n}{n} x^n \right) = ((1+x)^n - 1)^m$.

(ii)

We use the PIE

$$|\bigcup_{i=1}^{n} A_i| = \sum_{\emptyset \neq J \subseteq \{1,...,n\}} (-1)^{|J|+1} |\bigcap_{j \in J} A_j|$$

$$|\bigcap_{i=1}^{n} \overline{A_i}| = |S - \bigcup_{i=1}^{n} A_i| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

Set $S = \{m \times n \text{ matrices with at least one 1 in every row}\}$.

$E_i = \{E_i \in S : E_i \text{ has no 1 in the } i \text{ column}\}$.

Then we use the second formula.

So $\sum_{k \geq 0} g_k(m,n) z^k = F_n + \sum_{i=1}^{n} \binom{n}{i} (-1)^i F_{n-i}$.

**注记 1. Three polynomial bases**

- **Ordinary powers**: $x^n$.
- **Falling factorials**:

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1), x^{\underline{0}} = 1.$$

- **Rising factorials**:

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1), x^{\overline{0}} = 1,$$

with the basic relation $x^{\overline{n}} = (-1)^n(-x)^{\underline{n}}$.

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## 1) Ordinary powers ↔ Falling factorials (Stirling numbers)

- **Expand ordinary powers in the falling basis**

$$x^n = \sum_{k=0}^{n} S(n,k)\ x^{\underline{k}}$$

where $S(n,k)$ are the **Stirling numbers of the second kind** (partitions of an $n$-set into $k$ nonempty blocks).

- **Expand falling factorials in ordinary powers**

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)\ x^k$$

where $s(n,k)$ are the **(signed) Stirling numbers of the first kind**, and $s(n,k) = (-1)^{\,n-k} \begin{bmatrix} n \\ k \end{bmatrix}$ with $\begin{bmatrix} n \\ k \end{bmatrix}$ the **unsigned** ones.

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## 2) Ordinary powers ↔ Rising factorials

- **Expand ordinary powers in the rising basis**

$$x^n = \sum_{k=0}^{n} (-1)^{\,n-k}\ S(n,k)\ x^{\overline{k}}$$

(obtained from $x^{\underline{k}} = (-1)^k(-x)^{\overline{k}}$).

- **Expand rising factorials in ordinary powers**

$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{\,n-k} s(n,k)$ are the **unsigned Stirling numbers of the first kind**.

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## 3) Falling factorials ↔ Rising factorials (Lah numbers)

The change of basis between the two factorial bases is governed by the **Lah numbers**

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}(1 \le k \le n).$$

- **Rising in terms of falling**

$$x^{\overline{n}} = \sum_{k=1}^{n} L(n,k)\ x^{\underline{k}}$$

- **Falling in terms of rising**

$$x^{\underline{n}} = \sum_{k=1}^{n} (-1)^{\,n-k}\ L(n,k)\ x^{\overline{k}}$$

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## 4) Quick table ($n \le 4$)

- $x^{\underline{1}} = x,\ x^{\underline{2}} = x^2 - x,\ x^{\underline{3}} = x^3 - 3x^2 + 2x,\ x^{\underline{4}} = x^4 - 6x^3 + 11x^2 - 6x.$
- $x^{\overline{1}} = x,\ x^{\overline{2}} = x^2 + x,\ x^{\overline{3}} = x^3 + 3x^2 + 2x,\ x^{\overline{4}} = x^4 + 6x^3 + 11x^2 + 6x.$

These identities give a complete and practical conversion kit among the three bases.