题目 1. For each of the following s_n, determine s_n and its ordinary generating function.

1. $s_n := \sum_{k=0}^{n} k\binom{n}{k}$;

2. $s_n := \sum_{k=0}^{n} \binom{m+k}{m}$;

3. $s_n := \sum_{k=0}^{n} \binom{n-k}{k}$.

解答. 1.

Notice that $k\binom{n}{k} = n\binom{n-1}{k-1}$. Hence $s_n = n\sum_{k=0}^{n-1}\binom{n-1}{k-1} = n2^{n-1}$.

Let $F(x) := \sum_{n=0}^{\infty} s_n x^n = x\sum_{n=0}^{\infty} n\,(2x)^{n-1} = x\,D(\sum_{n=0}^{\infty}(2x)^n) = \frac{x}{(1-2x)^2}$.

2. Let $F_m(x) := \sum_{n=0}^{\infty} s_n x^n$. Then $F_m(x) = 1 + \sum_{n=1}^{\infty}\left(s_{n-1} + \binom{m+n}{m}\right)x^n = x\,F_m(x) + \sum_{n=0}^{\infty}\binom{m+n}{m}x^m$

Let $G_m(x) := \sum_{n=0}^{\infty}\binom{m+n}{n}x^n$.

Then $G_m(x) = \sum_{n=0}^{\infty}\binom{m+n}{n}x^n = 1 + \sum_{n=1}^{\infty}\binom{m+n-1}{n}x^n + \sum_{n=1}^{\infty}\binom{n+m-1}{n-1}x^n = x\,G_m(x) + \sum_{n=0}^{\infty}\binom{m-1+n}{n}x^n = xG_m(x) + G_{m-1}(x)$.

Since $G_0(x) = \frac{1}{1-x}$. $G_m(x) = \left(\frac{1}{1-x}\right)^{m+1}$. Hence $F_m(x) = \left(\frac{1}{1-x}\right)^{m+2}$.

Hence we get the recursion s. t. $F_m(x) = \frac{1}{1-2x}F_{m-1}(x)$. And we have the starting condition s. t. $F_0(x) = \left(\frac{1}{1-x}\right)^2$.

Therefore $F_m(x) = (\frac{1}{1-2x})^m\frac{1}{(1-x)^2}$.

When $n \geq 2$, $s_n = 1 + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+1}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+3}{2} + \binom{m+3}{3} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}$.

题目 2. Let S be a *colored set*, i.e., there exists a mapping $c: S \to C$, where the elements of C are called *colors*. A *rainbow subset* of a colored set is a subset in which every element has a distinct color.

Note that a rainbow subset does **not necessarily** contain all colors in C. Let S be a fixed colored set, where

$$C := [k] = \{1, 2, \ldots, k\}.$$

Let $n_i := |c^{-1}(i)|$, i.e., the number of elements of color i. Define f_m to be the number of rainbow subsets of S of size m.

1. Find the generating function of f_m. (You do not need to express it in a closed simple form, but simplify as much as possible.)

2. Suppose $n_1 = n_2 = \cdots = n_k = n$. Find f_m.

3. Suppose $n_1 = 1, n_2 = 2, \ldots, n_k = k$. Find f_m.

解答. 1.

By the question, each color has n_i choices,hence the mutiplication factor should be $(1 + n_i z_i)$. Hence the generating function should be $\prod_{i=1}^{k}(1 + n_i z_i)$. And We specilazation $z_i = z$. Hence the answer is $\prod_{i=1}^{k}(1 + n_i z)$.

2. Too silly. $\binom{m}{k}n^k$.

3. 同一行第一类斯特林数的计算

类似第二类斯特林数，我们构造同行第一类斯特林数的生成函数，即

$F_n(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i$

根据递推公式，不难写出

$F_n(x) = (n-1)F_{n-1}(x) + xF_{n-1}(x)$

于是

$F_n(x) = \prod_{i=0}^{n-1}(x+i) = \frac{(x+n-1)!}{(x-1)!}$

⟨hlink|斯特林数 - OI Wiki|⟩

⟨hlink| |`https://oi-wiki.org/math/combinatorics/stirling/`⟩

So the answer is the first stirling number.

题目 3.

题目 4. Let $f(n)$ be the number of partitions if n such that for every k , k occurs in the partition at most k times. Let $g(n)$ be the number of partitions of n such that no part has the form $k(k+1)$ (i.e. no parts equal 2,6,12,$\ldots$).Show that $f(n)=g(n)$.

解答. Just Let $F(x):=\sum_{n=0}^{\infty} f(x)x^n$ and $G(x):=\sum_{n=0}^{\infty} g(n)x^n$. By the question, $G(x)=\prod_{i\neq k(k+1),i\geq 1}\frac{1}{1-x^i}$. $F(x)=(1+x)(1+x^2+x^4)(1+x^3+x^6+x^9)\cdots=\prod_{i=1}^{\infty}\frac{1-x^{i(i+1)}}{1-x^i}$.