Lie Algebra is developed by **Marius Sophus Lie**

He was a Norwegian guy who was not as notable as Abel and mainly spent his life in Germany, which is disliked by our teacher Futorny.

Definition 1. An Algebra $\mathbb{A}$ is a system with a non-epmty set S wtih k operations σ_1,$\sigma_2, \ldots, \sigma_k$ s.t. all elements under operation are still in the set S.

Definition 2. A Lie Algebra Ł is a K- Vector Space with an opration [-,-]. Here, [-,-] is a bilinear operation s.t. $[x,x] = 0$ and $[x,[y,z]+[y,[x,z]+[z,[x,y]]=0], \forall x,y,z \in$ Ł

Warning. a) $[x,x] = 0 \;\; \forall x\in$ Ł

$\leftrightarrow$ b) $[x,y]+[y,x]=0 \;\forall\; x,y \in$ Ł. if and only if χ(K) $\neq 2$.

Reason: $[x+y,x+y]=0 \leftrightarrow [x,x]+[y,y]+[x,y]+[y,x]=0$.

Example 1. We Let $[x,y]=0, \forall x,y \in$ Ł. A trivial one.

Example 2. For vector space E^3, we let the cross-product be the bracket notion.

Example 3. We Let $[x,y]=x\cdot y - y\cdot x$. It is easy to verify that the Jaccobi identity is satisfied. Here $x,y \in A$. A is an associative Algebra. In this case, we denote by A^{lie}. Often(Maybe sometimes), $A=M_n(K)$, We denote $\text{End}(V)^{\text{lie}} := \mathfrak{gl}_n(K)$.

Definition 3. Given a subspace $\mathcal{H} \subset$ Ł, if the bracket operation [-,-] is closed under $\mathcal{H}$, Then $\mathcal{H}$ is a Lie Subalgebra.

Example 4. $\mathfrak{sl}_n(K) := \{\text{x} \in \mathfrak{gl}_n(\text{k}) : \text{Trace(x)} = 0\}$

Definition 4. A linear map d : A $\to$ A is called a derivation if d(ab) = ad(b) + bd(a) for all a,b $\in$ A. We denote the set of all derivations by Der(A).

Exercise 1. Show that for vector space $K[x]$, Der(A) $= \{h(x)\cdot\frac{d}{dx}|h(x)\in K[x]\}$

Solution. By the definition, $d(1)=d(1\times 1)=2d(1)$. Hence $d(1)=0$.

Let $d(x)=h(x)$.

By mathematics induction, we can say for $\deg(f) \leq n$, this holds.

Witout loss of generality, we can assume $f(x)=a_{n+1}x^{n+1}+a_nx^n+\ldots+a_1x$.

$d(f(x)) = d(x(a_{n+1}x^n + a_nx^{n-1}+1)) = h(x)((n+1)a_{n+1}x^{n-1}+(n-1)a_nx^{n-2}+\cdots a_1)+h(x)f(x) = h(x)\frac{df(x)}{dx}$.

Note 1. Let A be a commutative, associative algebra, Der(A) is closed under the bracket operation which can be noted as $[d_1,d_2]=d_1d_2-d_2d_1$.

Definition 5. Definition (Lie Ideal). I $\subset \mathfrak{g}$ is a Lie ideal if for all $x \in \mathfrak{g}$ and $a \in$ I, $[x, a] \in$ I.

Example 5. For vector space $M_n(K)^{\text{lie}}$, tI_n is an ideal, for you got an zero.