6

Algebraic, D-Finite, and Noncommutative Generating Functions

6.1 Algebraic Generating Functions

In this chapter we will investigate two classes of generating functions called algebraic and D-finite generating functions. We will also briefly discuss the theory of noncommutative generating functions, especially their connection with rational and algebraic generating functions. The algebraic functions are a natural generalization of rational functions, while D-finite functions are a natural generalization of algebraic functions. Thus we have the hierarchy

$$\begin{array}{c} D\text{-finite} \\ | \\ \text{algebraic} \\ | \\ \text{rational} \end{array} \tag{6.1}$$

Various other classes could be added to the hierarchy, but the three classes of (6.1) seem the most useful for enumerative combinatorics.

6.1.1 Definition. Let K be a field. A formal power series $\eta \in K[[x]]$ is said to be *algebraic* if there exist polynomials $P_0(x), \dots, P_d(x) \in K[x]$, not all 0, such that

$$P_0(x) + P_1(x)\eta + \cdots + P_d(x)\eta^d = 0. \tag{6.2}$$

The smallest positive integer d for which (6.2) holds is called the *degree* of η.

Note that an algebraic series η has degree one if and only if η is rational. The set of all algebraic power series over K is denoted $K_{\text{alg}}[[x]]$.

6.1.2 Example. Let $\eta = \sum_{n \geq 0} \binom{2n}{n} x^n$. By Exercise 1.4(a) we have $(1-4x)\eta^2 - 1 = 0$. Hence η is algebraic of degree one or two. If K has characteristic 2 then

$\eta = 1$, which has degree one. Otherwise it is easy to see that η has degree two. Namely, if $\deg(\eta) = 1$ then $\eta = P(x)/Q(x)$ for some polynomials $P(x)$, $Q(x) \in K[x]$. Thus

$$(1 - 4x)P(x)^2 = Q(x)^2.$$

The degree (as a polynomial) of the left-hand side is odd while that of the right-hand side is even, a contradiction.

It is a great convenience in dealing with algebraic series to work over fields rather than over the rings $K[x]$ and $K[[x]]$. Thus we review some basic facts about the quotient fields of $K[x]$ and $K[[x]]$. The quotient field of $K[x]$ is just the field $K(x)$ of rational functions over K (since every rational function is by definition a quotient of polynomials). The ring $K[[x]]$ is almost a field; we simply need to invert x to obtain a field. (In other words, $K[[x]]$ is a local domain, and x generates the maximal ideal.) Thus the quotient field of $K[[x]]$ is given by

$$K((x)) = K[[x]][1/x].$$

Every element of $K((x))$ may be regarded as a Laurent series $\eta = \sum_{n \geq n_0} a_n x^n$ for some $n_0 \in \mathbb{Z}$ (depending on η). To see that such Laurent series indeed form a field, the only field axiom that offers any difficulty is the existence of multiplicative inverses. But if $\eta = \sum_{n \geq n_0} a_n x^n$ with $a_{n_0} \neq 0$, then $\eta = x^{n_0}\rho$ where ρ is an ordinary power series with nonzero constant term. Hence $\rho^{-1} \in K[[x]]$, so $\eta^{-1} = x^{-n_0}\rho^{-1} \in K((x))$, as desired.

Recall from any introductory algebra text that if D is an (integral) domain containing the field $K(x)$, then $\eta \in D$ is said to be *algebraic* over $K(x)$ if there exist elements $F_0(x), \ldots, F_d(x) \in K(x)$, not all 0, such that

$$F_0(x) + F_1(x)\eta + \cdots + F_d(x)\eta^d = 0. \tag{6.3}$$

The least such d is the *degree* of η over the field $K(x)$, denoted $\deg_{K(x)}(\eta)$. It is also the dimension of the field $K(x, \eta)$ (obtained by adjoining η to $K(x)$) as a vector space over $K(x)$. Equivalently, η is algebraic over $K(x)$ if and only if the $K(x)$-vector space spanned by $\{1, \eta, \eta^2, \ldots\}$ is finite-dimensional (in which case its dimension is $\deg_{K(x)}(\eta)$). Moreover (again from any introductory algebra text) the set of $\eta \in D$ that are algebraic over $K(x)$ form a subring of D containing $K(x)$ (and hence a $K(x)$-*subalgebra* of D, i.e., a subring of D that is also a vector space over $K(x)$).

Suppose (6.3) holds, and let $P(y) = F_0(x) + F_1(x)y + \cdots + F_d(x)y^d \in K(x)[y]$. Thus $P(y)$ is a polynomial in the indeterminate y with coefficients in the field $K(x)$. We then have $d = \deg_{K(x)}(\eta)$ if and only if the polynomial $P(y)$ is *irreducible*. If we normalize (6.3) by dividing by $F_d(x)$ so that $P(y)$ is monic, and if $d = \deg_{K(x)}(\eta)$, then the equation (6.3) is *unique*; for otherwise we could subtract two such equations and obtain one of smaller degree.

Note that we can multiply (6.3) by a common denominator of the F_i's and hence can assume that the F_i's are polynomials. Thus we see that $\eta \in K[[x]]$ is algebraic (as defined by Definition 6.1.1) if and only if it is algebraic over $K(x)$. The same is true for $\eta \in K((x))$. The set of all algebraic Laurent series over $K(x)$ is denoted $K_{\text{alg}}((x))$. Hence

$$K_{\text{alg}}[[x]] = K_{\text{alg}}((x)) \cap K[[x]]. \tag{6.4}$$

Again by standard algebraic arguments, $K_{\text{alg}}[[x]]$ is a subring of $K[[x]]$ containing $K(x) \cap K[[x]]$, the ring of rational functions P/Q with $P, Q \in K[x]$ and $Q(0) \neq 0$.

6.1.3 Example. Let us consider Example 6.1.2 from a somewhat more algebraic viewpoint. The series $\eta = \sum_{n\geq 0} \binom{2n}{n} x^n$ will have degree two if and only if $(1-4x)y^2 - 1$ is irreducible as a polynomial in y over $K(x)$. A quadratic polynomial $p = ay^2 + by + c$ is irreducible over a field F of characteristic not equal to two if and only if its discriminant $\text{disc}(p) = b^2 - 4ac$ is not a square in F. Now $\text{disc}((1-4x)y^2 - 1) = 4(1-4x)$, which is not a square, since its degree (as a polynomial in x) is odd. Hence $\deg_{K(x)}(\eta) = 2$ if $\text{char} K \neq 2$.

For most enumerative purposes involving algebraic series it suffices to work with Laurent series $y \in K((x))$. Note, however, that there exist elements η in some extension field of $K(x)$ that are algebraic over $K(x)$ but that cannot be represented as elements of $K((x))$. The simplest such η are defined by $\eta^N = x$ for $N \geq 2$. This suggests that we look at formal series $\eta = \sum_{n\geq n_0} a_n x^{n/N}$, where N is a positive integer depending on η. Such a series is called a *fractional (Laurent) series* (or *Puiseux series*). If we can take $n_0 = 0$ then we have a *fractional power series*. Let $K^{\text{fra}}((x))$ (respectively, $K^{\text{fra}}[[x]]$) denote the ring of all fractional Laurent series (respectively, fractional power series) over K. Thus $K^{\text{fra}}((x)) = K((x))[x^{1/2}, x^{1/3}, x^{1/4}, \ldots]$, i.e., every $\eta \in K^{\text{fra}}((x))$ can be written as a *polynomial* in $x^{1/2}, x^{1/3}, \ldots$ (and hence involving only finitely many of them) with coefficients in $K((x))$; and conversely every such polynomial is a fractional series. Thus for instance $\sum_{N\geq 1} x^{1/N}$ is not a fractional series in our sense of the term. It's easy to see that $K^{\text{fra}}[[x]]$ is a ring, and that $K^{\text{fra}}((x))$ is the quotient field of $K^{\text{fra}}[[x]]$. For instance, the product of $\sum_{m\geq m_0} a_m x^{m/M}$ and $\sum_{n\geq n_0} b_n x^{n/N}$ will be a series of the form $\sum_{n\geq m_0 N + n_0 M} c_n x^{n/MN}$.

For the remainder of this section we will develop some basic properties of fractional series and algebraic series. An understanding of these properties provides some interesting insight into the formal aspects of algebraic series, but such an understanding is not really necessary for solving enumerative problems. The reader may skip the remainder of this section with little loss of continuity. (The only real exception is the proof of Theorem 6.3.3.)

6.1.4 Proposition. *The field $K^{\text{fra}}((x))$ is an algebraic extension of $K((x))$, i.e., every $\eta \in K^{\text{fra}}((x))$ satisfies an equation*

$$P_0(x) + P_1(x)\eta + \cdots + P_d(x)\eta^d = 0,$$

where each $P_i(x) \in K((x))$ and not all $P_i(x) = 0$.

Proof. Let $\eta = \sum_{n\geq n_0} a_n x^{n/N} \in K^{\text{fra}}((x))$. There are then (unique) series $\eta_0, \eta_1, \ldots, \eta_{N-1} \in K((x))$ such that $\eta = \eta_0 + x^{1/N}\eta_1 + x^{2/N}\eta_2 + \cdots + x^{(N-1)/N}\eta_{N-1}$. The series (consisting of a single term each) $x^{1/N}, x^{2/N}, \ldots, x^{(N-1)/N}$ are clearly algebraic over $K((x))$. Since for any extension field E of any field F, the elements of E that are algebraic over F form a subfield of E containing F, we have that η is algebraic over $K((x))$, as desired. □

A considerably deeper result is the following.

6.1.5 Theorem. *Let K be an algebraically closed field of characteristic zero (e.g., $K = \mathbb{C}$). Then the field $K^{\text{fra}}((x))$ is algebraically closed (and hence by Proposition 6.1.4 is an algebraic closure of $K((x))$).*

Theorem 6.1.5 is known as *Puiseux's theorem.* We omit the rather lengthy proof, since we only use Theorem 6.1.5 here to prove the rather peripheral Theorem 6.3.3. For some references to proofs, see the Notes section of this chapter. Somewhat surprisingly, Theorem 6.1.5 is false for algebraically closed fields K of characteristic $p > 0$; see Exercise 6.4.

⋆ For the remainder of this section, unless explicitly stated otherwise, we assume that K is algebraically closed of characteristic zero. ⋆

Since $K(x)$ is a subfield of $K((x))$, it follows from Theorem 6.1.5 that $K^{\text{fra}}((x))$ contains the algebraic closure $\overline{K(x)}$ of $K(x)$. Thus any algebraic power series, as defined in Definition 6.1.1, can be represented as a fractional power series (and any algebraic Laurent series can be represented as a fractional Laurent series).

Suppose that $P(y) \in K((x))[y]$ is an *irreducible* polynomial in y of degree d over the field $K((x))$. The d fractional series $\eta_1, \ldots, \eta_d \in K^{\text{fra}}((x))$ satisfying $P(\eta_j) = 0$ are called *conjugates* of one another. The next result describes the relation between conjugate series.

6.1.6 Proposition. *With $P(y)$ as above, let $\eta = \sum_{n\geq n_0} a_n x^{n/N}$ satisfy $P(\eta) = 0$. Then the least possible value of N is equal to d, and the d conjugates to η are given by*

$$\eta_j = \sum_{n\geq n_0} a_n \zeta^{jn} x^{n/d}, \qquad 0 \leq j < d,$$

where ζ is a primitive d-th root of unity. (Note that ζ exists in K, since K is algebraically closed of characteristic zero.)

NOTE. The intuition behind Proposition 6.1.6 is the following. Let $t_j = \zeta^j x^{1/d}$. Then $t_j^d = x$. Hence each t_j is "just as good" a d-th root of x as $x^{1/d}$ itself and may be substituted for $x^{1/d}$ in η without affecting the validity of the equation $P(\eta) = 0$. A little field theory can make this argument rigorous, though we give a naive proof.

Sketch of the Proof. Let $\eta = \eta(x) = \sum_{n \geq n_0} a_n x^{n/N}$ with N minimal (given η) and let ξ be a primitive N-th root of unity. Let

$$P(y) = F_0(x) + F_1(x)y + \cdots + F_d(x)y^d,$$

so

$$P(\eta) = F_0(x) + F_1(x)\eta + \cdots + F_d(x)\eta^d = 0. \tag{6.5}$$

Now substitute $\xi^j x^{1/N}$ for $x^{1/N}$ in (6.5). Set $\eta_j^* = \sum_{n \geq n_0} a_n \xi^{jn} x^{n/N}$. Each $F_i(x)$ remains unchanged, so $P(\eta_j^*) = 0$, i.e., each η_j^* $(0 \leq j < N)$ is a conjugate of η. Hence $N \leq d$.

On the other hand, let $e_k = e_k(\eta_0^*, \ldots, \eta_{N-1}^*)$ be the k-th elementary symmetric function in $\eta_0^*, \ldots, \eta_{N-1}^*$, for $1 \leq k \leq N$. Each e_k, regarded as a Laurent series in $x^{1/N}$, is invariant under substituting $\xi x^{1/N}$ for $x^{1/N}$. Hence $e_k \in K((x))$, so

$$\prod_{j=0}^{N-1}(y - \eta_j^*) = \sum_{k=0}^{N}(-1)^k e_k y^{N-k} \in K((x))[y].$$

Since $P(y)$ is irreducible, there follows $N \geq d$. Hence $N = d$, and the proof follows. $\square$

The following corollary to Proposition 6.1.6 is sometimes regarded as part of Puiseux's theorem (Theorem 6.1.5).

6.1.7 Corollary. *Suppose that $\eta \in K^{\text{fra}}((x))$ is algebraic of degree d over $K(x)$. Then there exist positive integers $c_1, \ldots, c_r$ satisfying $c_1 + \cdots + c_r = d$, and fractional Laurent series*

$$\eta_j = \sum_{n \geq n_0} a_{j,n} x^{n/c_j}, \quad 1 \leq j \leq r,$$

such that the d conjugates of η (over $K(x)$) are given by

$$\sum_{n \geq n_0} a_{j,n} \zeta_{c_j}^{kn} x^{n/c_j},$$

where $1 \leq j \leq r$, $0 \leq k < c_j$, and ζ_{c_j} denotes a primitive c_j-th root of unity.

Proof. Immediate from Proposition 6.1.6, since conjugates over $K((x))$ remain conjugate over the subfield $K(x)$. $\square$

Now suppose that we are given a polynomial equation

$$P(y) = F_0(x) + F_1(x)y + \cdots + F_d(x)y^d = 0, \tag{6.6}$$

where $F_j(x) \in K((x))$. What information can we read off directly from (6.6) about the d roots $\eta_1, \ldots, \eta_d$ of $P(y)$? For instance, how many roots are ordinary power series (elements of $K[[x]]$)? In general there is no simple way to obtain such information, but there are some useful sufficient conditions for various properties of the η_j's.

6.1.8 Proposition. *Let*

$$P(y) = F_0(x) + F_1(x)y + \cdots + F_d(x)y^d,$$

where each $F_i(x) \in K[[x]]$ (or even $K^{\mathrm{fra}}[[x]]$) and for at least one j we have $F_j(0) \neq 0$. Let m be the number of solutions $y = \eta_i$ to $P(y) = 0$ that are fractional power series (i.e., no negative exponents). Then

$$m = \max\{j : F_j(0) \neq 0\}.$$

Proof. For any fractional series η, let

$$\nu(\eta) = \min\{a : [x^a]\eta \neq 0\},$$

and call $\nu(\eta)$ the *x-degree* of η (to distinguish it from the degree of η as an algebraic element over $K((x))$). Let $P(y) = F_d(x)(y-\eta_1)\cdots(y-\eta_d)$, where $\nu(\eta_i) = a_i < 0$ for $1 \leq i \leq d-m$, and $\nu(\eta_i) \geq 0$ for $d-m+1 \leq i \leq d$. Consider the elementary symmetric function

$$e_{d-m} = e_{d-m}(\eta_1, \ldots, \eta_d) = \sum_{i_1 < \cdots < i_{d-m}} \eta_{i_1} \cdots \eta_{i_{d-m}}.$$

Thus $[y^m]P(y) = (-1)^{d-m}F_d(x)e_{d-m}$. Now e_{d-m} has a term $\eta_1\eta_2\cdots\eta_{d-m}$ of x-degree $\sum_{i=1}^{d-m} a_i := A$, while all other terms have greater x-degree. Hence $\nu(e_{d-m}) = A$ exactly. Thus in order for $[y^m]P(y) \in K^{\mathrm{fra}}[[x]]$, we must have $\nu(F_d(x)) \geq -A$. Now every term $\eta_{i_1}\cdots\eta_{i_k}$ of every elementary symmetric function $e_k(\eta_1, \ldots, \eta_d)$ has x-degree at least A. Since some $F_j(0) \neq 0$, it follows that $\nu(F_d(x)) \leq -A$, so $\nu(F_d(x)) = -A$. Hence $F_m(0) \neq 0$. Moreover, if $k < d-m$ then every term of $e_k(\eta_1, \ldots, \eta_d)$ has x-degree strictly greater than A, so $F_d(x)e_k$ (which equals $(-1)^k[y^{d-k}]P(y)$) has strictly positive x-degree, i.e., $F_{d-k}(0) = 0$. Hence $m = \max\{j : F_j(0) \neq 0\}$, as desired. $\square$

If $P(y) = c_0 + c_1 y + \cdots + c_d y^d$ is a polynomial of degree d over a field F with roots $\alpha_1, \ldots, \alpha_d$ in some extension field, then define the *discriminant* of P by

$$\operatorname{disc}(P) = c_d^{2d-2} \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

It is well known that $\operatorname{disc}(P)$ can be expressed as a polynomial in the coefficients c_i of $P(y)$. If we set $\deg c_i = 1$, then $\operatorname{disc}(P)$ is homogeneous of degree $2d - 2$; while if we set $\deg c_i = d - i$, then $\operatorname{disc}(P)$ is homogeneous of degree $d(d-1)$. For instance,

$$\begin{aligned}
\operatorname{disc}(ay^2 + by + c) &= b^2 - 4ac \\
\operatorname{disc}(ay^3 + by^2 + cy + d) &= -27a^2d^2 + 18abcd - 4ac^3 - 4b^3d + b^2c^2 \\
\operatorname{disc}(ay^4 + by^3 + cy^2 + dy + e) &= 256a^3e^3 - 128a^2c^2e^2 + 144a^2cd^2e \\
&\quad - 192a^2bde^2 - 27a^2d^4 + 16ac^2e - 4ac^3d^2 \\
&\quad - 80abc^2de + 144ab^2ce^2 + 18abcd^3 \\
&\quad - 6ab^2d^2e - 4b^2c^3e + b^2c^2d^2 + 18b^3cde \\
&\quad - 27b^4e^2 - 4b^3d^3 \\
\operatorname{disc}(ay^d + by + c) &= (-1)^{\binom{d}{2}} a^{d-2} [d^d ac^{d-1} + (-1)^{d-1}(d-1)^{d-1} b^d].
\end{aligned} \tag{6.7}$$

For a proof of (6.7) see Exercise 6.8(a).

6.1.9 Proposition. *Let*

$$P(y) = F_0(x) + F_1(x)y + \cdots + F_d(x)y^d,$$

where each $F_j(x) \in K[[x]]$ and $F_d(0) \neq 0$ (so by Proposition 6.1.8, all the roots of $P(y)$ are fractional power series). Suppose that

$$\operatorname{disc}(P(y))|_{x=0} \neq 0. \tag{6.8}$$

Then every root $\eta_i \in K^{\text{fra}}[[x]]$ of $P(y)$ is an ordinary power series, i.e., $\eta_i \in K[[x]]$.

Proof. Suppose that $P(y)$ has a root $\eta_1 = \sum_{n\geq 0} a_n x^{n/N}$, for some $N > 1$, that is not a power series. If ζ is a primitive N-th root of unity, then by Proposition 6.1.6 (after factoring $P(y)$ into irreducibles) we see that another root of $P(y)$ is $\eta_2 = \sum_{n\geq 0} a_n \zeta^n x^{n/N}$ and that $\eta_1 \neq \eta_2$. Since $\eta_1(0) = \eta_2(0) = a_0$, the difference $\eta_1 - \eta_2$ vanishes at $x = 0$, and hence so does $\operatorname{disc}(P(y))$, contradicting (6.8). $\square$

6.1.10 Example. (a) Let $P(y) = y^2 - (x+1)$. Then $F_2(0) = 1 \neq 0$, $\operatorname{disc}(P(y)) = 4(x+1)$, and $\operatorname{disc}(P(y))|_{x=0} = 4 \neq 0$. Thus by Propositions 6.1.8 and 6.1.9, the equation $P(y) = 0$ has two power series solutions. Indeed, the solutions are given by

$$\pm(1+x)^{1/2} = \pm\sum_{n\geq 0}\binom{\frac{1}{2}}{n}x^n.$$

(b) Let $P(y) = y^2 - x$. Then $F_2(0) \neq 0$, so the two roots are fractional power series by Proposition 6.1.8. Since $\operatorname{disc}(P(y))|_{x=0} = 0$, we cannot tell just from $\operatorname{disc}(P(y))$ whether or not the two roots are in fact ordinary power series. Of course by inspection we see that the roots are $\pm x^{1/2}$, which are not ordinary power series.

(c) Let $P(y) = y^2 - x^2(x+1)$. Again $F_2(0) \neq 0$ and $\operatorname{disc}(P(y))|_{x=0} = 0$. This time, however, the roots *are* ordinary power series.

(d) Let $P(y) = xy^2 - y - 1$. By Proposition 6.1.8 exactly one of the roots is a fractional power series. Proposition 6.1.9 cannot be directly applied, since $F_2(0) = 0$. There are several ways, however, to see that both roots are ordinary Laurent series.

- Suppose that some root was not a Laurent series. Since $d = 2$, the roots η_1 and η_2 are of the form $\sum a_n x^{n/2}$ and $\sum(-1)^n a_n x^{n/2}$. Hence either both or none of the roots are fractional power series. Since we have seen that exactly one root is a fractional power series, it follows that both roots are ordinary Laurent series. More generally, if $r = 1$ (so $c_1 = d$) in Corollary 6.1.7, then either all or none of the η_j are fractional power series.
- Let $z = yx$. Then $xP(y) = z^2 - z - x$. Now Proposition 6.1.9 does apply, and we see that both roots ρ_1, ρ_2 of $z^2 - z - x$ are ordinary power series. Hence the roots $\eta_i = x^{-1}\rho_i$ of $P(y)$ are ordinary Laurent series.
- Since $P(y)$ is quadratic, we can just use the quadratic formula to obtain the two roots

$$\frac{1 \pm \sqrt{1+4x}}{2x} \tag{6.9}$$

where $\sqrt{1+4x} = \sum_{n\geq 0}\binom{1/2}{n}4^n x^n = 1 + 2x + \cdots$. Hence the plus sign in (6.9) produces a Laurent series η_1 with $\nu(\eta_1) = -1$, while the minus sign yields an ordinary power series η_2.

We mentioned earlier the standard result that the set $K_{\text{alg}}[[x]]$ of algebraic power series forms a subalgebra of $K[[x]]$. Thus if $u, v \in K_{\text{alg}}[[x]]$ and $\alpha, \beta \in K$, then $\alpha u + \beta v,\ uv \in K_{\text{alg}}[[x]]$. A further operation of combinatorial interest that can be performed on power series u and v is the *Hadamard product* $u * v$, defined in Section 4.2. Recall that if $u = \sum f(n)x^n$ and $v = \sum g(n)x^n$, then $u * v := \sum f(n)g(n)x^n$. What effect does this operation have on algebraicity? One can show that if u and v are algebraic, then $u * v$ need not be algebraic. An example (which takes some work to verify – see Exercise 6.3) is $u = v = (1 - 4x)^{-1/2} = \sum \binom{2n}{n} x^n$. The next result shows, however, that a weaker result is true.

6.1.11 Proposition. *Let K be a field of characteristic 0. If $u \in K[[x]]$ is algebraic and $v \in K[[x]]$ is rational, then $u * v$ is algebraic.*

Proof. Let $u = \sum f(n)x^n$, $v = \sum g(n)x^n$. By Theorem 4.1.1, there exist $\gamma_1, \ldots, \gamma_r \in K$ (since we are assuming K is algebraically closed, though we could just as easily let K be any field of characteristic 0 and work over an algebraic closure of K) and $P_1, \ldots, P_r \in K[x]$ such that

$$g(n) = \sum P_i(n)\gamma_i^n, \qquad n \gg 0. \tag{6.10}$$

Now changing finitely many coefficients of v has no effect on whether $u * v$ is algebraic [why?], so we can assume that (6.10) holds for *all* $n \geq 0$. Since linear combinations of algebraic functions are algebraic (because $K_{\text{alg}}[[x]]$ is a subalgebra of $K[[x]]$), it suffices to show that for $0 \neq \gamma \in K$, the two series $u_1 = \sum \gamma^n f(n)x^n$ and $u_2 = \sum nf(n)x^n$ are algebraic. Let $P(x, u) = 0$, where $0 \neq P(x, y) \in K[x, y]$. Then $P(\gamma x, y) \neq 0$ and $P(\gamma x, u_1) = 0$, so u_1 is algebraic. Now note that

$$0 = \frac{d}{dx} P(x, u) = \left.\frac{\partial P(x, y)}{\partial x}\right|_{y=u} + u' \left.\frac{\partial P(x, y)}{\partial y}\right|_{y=u} \tag{6.11}$$

If we assume that we have chosen $P(x, y)$ to be of minimal degree in y (so it is irreducible over $K(x)$), then $\partial P(x, y)/\partial y$ is a nonzero (since $\operatorname{char} K = 0$) polynomial in y of smaller degree than P, so $\left.\frac{\partial P(x,y)}{\partial y}\right|_{y=u} \neq 0$. It therefore follows from (6.11) that

$$u' = -\frac{\left.\frac{\partial P(x,y)}{\partial x}\right|_{y=u}}{\left.\frac{\partial P(x,y)}{\partial y}\right|_{y=u}} \in K(x, u). \tag{6.12}$$

In other words, u' is a rational function of x and u, so u' is algebraic. Hence $u_2 = xu'$ is algebraic, completing the proof. □

We conclude this section with a simple result on algebraic functions that we will need later (see the proof of Theorem 6.7.1). We will be considering Laurent series

$$y = f(x_1, \ldots, x_k) \in K(x)((x_1, \ldots, x_k)), \tag{6.13}$$

i.e., y is a Laurent series in $x_1, \ldots, x_k$ whose coefficients are rational functions (which we regard as Laurent series) in x over K. We will assume that $f(1, \ldots, 1)$ is a well-defined element of $K((x))$. More precisely, if $y = \sum_\alpha c_\alpha(x) x_1^{\alpha_1} \cdots x_k^{\alpha_k}$

then $\sum_\alpha c_\alpha(x)$ converges to an element of $K((x))$ in the topology of Section 1.1. For instance, if

$$f(x_1, x_2) = \sum_{i,j\geq 0} x^{i+j} x_1^i x_2^j,$$

then $f(1, 1)$ is the well-defined series $\sum_{n\geq 0}(n + 1)x^n$. On the other hand, if

$$f(x_1, x_2) = \sum_{i,j\geq 0} x^{i-j} x_1^i x_2^j,$$

then $f(1, 1)$ is undefined (e.g., the coefficient of x^0 is the meaningless expression $\sum_{i\geq 0} 1$).

6.1.12 Proposition. *Let $y = f(x_1, \ldots, x_k)$ be as in (6.13). Suppose that $f(1, \ldots, 1)$ is well defined and that y is algebraic over $K(x)(x_1, \ldots, x_k)$. Then $f(1, \ldots, 1)$ is algebraic over $K(x)$.*

Proof. Suppose

$$P_d(x_1, \ldots, x_k)y^d + \cdots + P_0(x_1, \ldots, x_k) = 0, \tag{6.14}$$

where $P_i \in K(x)[x_1, \ldots, x_k]$, the P_i's are relatively prime (as polynomials in $x_1, \ldots, x_k$), and $P_d \neq 0$. By clearing denominators we may assume in fact $P_i \in K[x][x_1, \ldots, x_k] = K[x, x_1, \ldots, x_k]$. We can't simply substitute $x_1 = \cdots = x_k = 1$ in (6.14), since we conceivably might have $P_i(1, \ldots, 1) = 0$ for all i. So instead first set $x_k = 1$. If $P_i(x_1, \ldots, x_{k-1}, 1) = 0$ then $P_i(x_1, \ldots, x_k)$ is divisible by $x_k - 1$. Since the P_i's are relatively prime, some $P_i(x_1, \ldots, x_{k-1}, 1) \neq 0$. Hence $f(x_1, \ldots, x_{k-1}, 1)$ is algebraic over $K(x)(x_1, \ldots, x_{k-1})$. The proof follows by induction on k (the case $k = 0$ being trivial). □

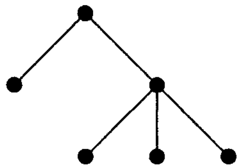
6.2 Examples of Algebraic Series

In this section we will consider some examples of algebraic series that arise in enumerative problems. For some further examples, see Sections 6.3 and 6.7.

We turn first to (unlabeled) plane trees and plane forests, as considered in Section 5.3. Let S be any subset of $\mathbb{P}$, and define a *plane S-tree* to be a plane tree for which any non-endpoint vertex has degree (number of successors) in S. For instance, a plane 2-tree (short for $\{2\}$-tree) is a plane binary tree. There are many interesting combinatorial structures that are "equivalent to" (can be put into a simple one-to-one correspondence with) plane trees. We collect some of the most important of these structures in the next result. We include an example of each structure with $S = \{2, 3\}$, $n = 6$, $m = 4$.

6.2.1 Proposition. *Let $S \subseteq \mathbb{P}$, $n \in \mathbb{P}$, $m \in \mathbb{P}$. There are "nice" bijections between the following sets:*

(i) *plane S-trees with n vertices and m endpoints:*



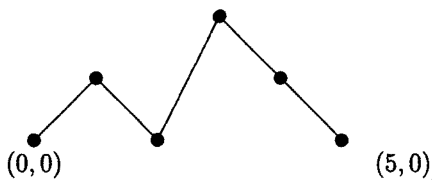
(ii) *sequences $i_1 i_2 \cdots i_{n-1}$, where each $i_j + 1 \in S$ or $i_j = -1$, such that there are a total of $m-1$ values of j for which $i_j = -1$, and such that $i_1 + i_2 + \cdots + i_j \geq 0$ for all j, and $i_1 + i_2 + \cdots + i_{n-1} = 0$:*

$$1, -1, 2, -1, -1;$$

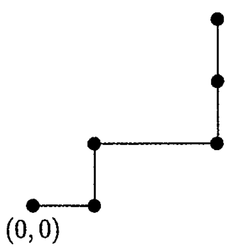
(iii) *parenthesizations (or* bracketings*) of a word of length m subject to $n - m$ k-ary operations, where $k \in S$:*

$$(x(xxx)) \qquad (\text{one 3-ary and one 2-ary operation});$$

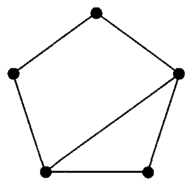
(iv) *paths P in the (x, y) plane from $(0, 0)$ to $(n-1, 0)$ using steps $(1, k)$, where $k + 1 \in S$ or $k = -1$, with a total of $m - 1$ steps of the form $(1, -1)$, such that P never passes below the x-axis:*



(v) *paths P in the (x, y) plane from $(0, 0)$ to $(m-1, m-1)$, using steps $(k, 0)$ or $(0, 1)$ with $k + 1 \in S$, with a total of $n - 1$ steps, such that P never passes above the line $x = y$:*



(vi) *dissections of a convex $(m + 1)$-gon C into $n - m$ regions, each a k-gon with $k - 1 \in S$, by drawing diagonals (necessarily $n - m - 1$ of them) that* *don't intersect in their interiors (when $k = 2$, we must draw two "curved" diagonals between the same pair of vertices of C):*



Proof. The bijection between (i) and (ii) was described in Section 5.3 (see Lemma 5.3.9 and the discussion preceding it). Recall that the bijection is obtained by doing a depth-first search through the plane tree τ and recording the integer $(\deg v) - 1$ whenever a vertex v is encountered for the first time. Here we should ignore the last vertex (which will be an endpoint), though in Section 5.3 it was included. For example, the tree of Figure 5-14 gives rise to the sequence

$$2, 0, 1, -1, -1, -1, 1, -1, 1, -1.$$

Now let τ be a plane tree. If τ is just a single vertex, then define the corresponding parenthesization $T_\tau = x$ (a single letter with no operation). Otherwise let the subtrees of the root of τ be $\tau_1, \ldots, \tau_j$ (in the given order), and define inductively $W_\tau = (W_{\tau_1} W_{\tau_2} \cdots W_{\tau_j})$. Clearly this construction sets up a bijection between (i) and (iii).

Now given a sequence $i_1 i_2 \cdots i_{n-1}$ enumerated by (ii), let P be the path from $(0, 0)$ to $(n - 1, 0)$ with successive steps $(1, i_1), (1, i_2), \ldots, (1, i_{n-1})$. This yields a bijection between (ii) and (iv).

The paths of type (v) are simply linear transformations of those of type (iv). More precisely, the sequence $i_1 i_2 \cdots i_{n-1}$ of (ii) corresponds to the path from $(0, 0)$ to $(m - 1, m - 1)$ whose j-th step is $(k, 0)$ if $i_j = k > 0$ and is $(0, 1)$ if $i_j = -1$.

Finally, consider a convex $(m + 1)$-gon C. Fix an edge e_0 of C, called the *root edge*. (The bijection to be described depends on the choice of e_0.) Given a dissection D of C as in (vi), define a plane tree $\tau = \tau(D, e_0)$ as follows. The vertices v_e of τ correspond to the edges e of D. The root vertex of τ is v_{e_0}. When we remove e_0 from D we obtain a sequence (in counterclockwise order) of edge-disjoint dissections $D_1, \ldots, D_k$ of polygons $C_1, \ldots, C_k$ (where $k + 1$ is the number of edges of the region R_0 of D that contains e_0). Let e_i be the edge of D_i (or C_i) that is also an edge of R_0. Define the subtrees $\tau_1, \ldots, \tau_k$ of the root v_{e_0} of τ by $\tau_i = \tau(D_i, e_i)$. This gives an inductive definition of τ and establishes a bijection between (vi) and (i). □

To understand this last bijection between (i) and (vi), "one picture is worth a thousand words." Figure 6-1 should make the bijection clear. The edges of D are solid, and the edges of τ are broken. The vertices of D are dots, and of τ are asterisks.

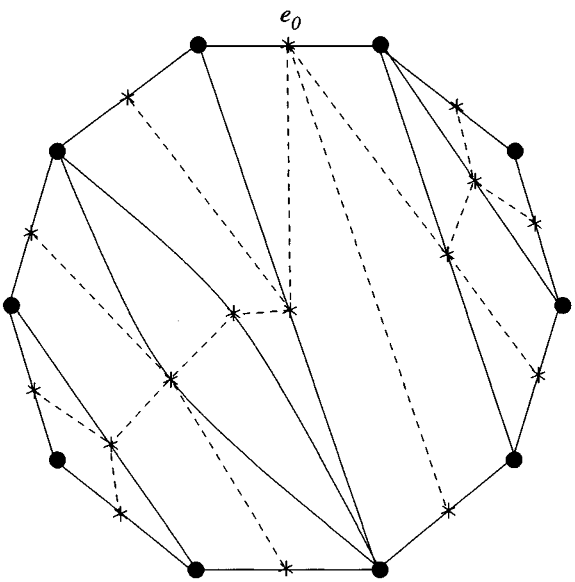


Figure 6-1. A plane tree obtained from a dissected polygon.

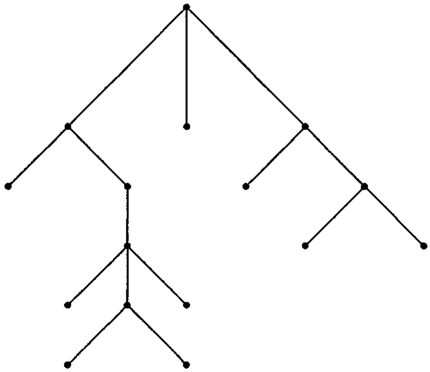


Figure 6-2. The tree of Figure 6-1.

The tree $\tau = \tau(D, e_0)$ is redrawn in Figure 6-2 for greater clarity. Sometimes it is clearer to regard the edges of τ as crossing the nonroot edges of D. Figure 6-3 shows Figure 6-2 redrawn in this way (with e_0 removed for even greater clarity).

One special case of Proposition 6.2.1 of particular interest occurs when S contains a single element $k \geq 2$. In this case the objects discussed in Proposition 6.2.1 exist only when

$$n = kj + 1, \qquad m = (k-1)j + 1$$

for some $j \geq 0$, or equivalently $(k-1)(n-1) = k(m-1)$. Let us write $T_S(m, n)$

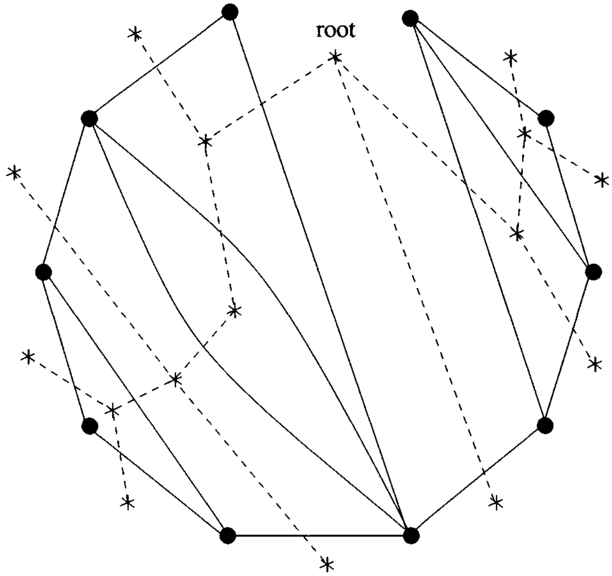


Figure 6-3. Figure 6-1 redrawn.

for the number of plane S-trees with n vertices and m endpoints, and $T_S(n)$ for the total number of plane S-trees with n vertices. We abbreviate $T_{\{k\}}$ by T_k. A special case of Theorem 5.3.10 is the following result.

6.2.2 Proposition. *We have*

$$T_k(m,n) = \begin{cases} \dfrac{1}{n}\dbinom{n}{j} & \text{if } n = kj+1 \text{ and } m = (k-1)j+1 \\ 0 & \text{otherwise.} \end{cases}$$

When $k = 2$, we recover the result (Example 5.3.12) that the number $T_2(n+1, 2n+1) = T_2(2n+1)$ of plane binary trees with $n+1$ endpoints (equivalently, $2n+1$ vertices) is the *Catalan number*

$$C_n := \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan numbers form one of the most ubiquitous and fascinating sequences of enumerative combinatorics. Proposition 6.2.1 yields a number of combinatorial interpretations of Catalan numbers, all closely connected (since there are simple bijections between the six classes). Let us reiterate the case $S = \{2\}$ of Proposition 6.2.1 directly in terms of Catalan numbers.

6.2.3 Corollary. *The Catalan number C_n counts the following:*

(i) *Plane binary trees with $n+1$ endpoints (or $2n+1$ vertices).*
(ii) *Sequences $i_1 i_2 \cdots i_{2n}$ of 1's and -1's with $i_1 + i_2 + \cdots + i_j \geq 0$ for all j and $i_1 + i_2 + \cdots + i_{2n} = 0$. Such sequences (as well as certain generalizations) are called* ballot sequences, *for the following reason. Suppose that an election is being held between two candidates A and B, and that a 1 (respectively, -1) indicates a vote for A (respectively, B). Then a ballot sequence corresponds to a sequence of $2n$ votes such that A never trails B and the election ends in a tie. (For a generalization of ballot sequences to any number of candidates, see Proposition 7.10.3.)*
(iii) *Ways to parenthesize a string of length $n+1$ subject to a nonassociative binary operation.*
(iv) *Paths P in the (x, y) plane from $(0, 0)$ to $(2n, 0)$, with steps $(1, 1)$ and $(1, -1)$, that never pass below the x-axis. Such paths are called* Dyck paths.
(v) *Paths P in the (x, y) plane from $(0, 0)$ to (n, n), with steps $(1, 0)$ and $(0, 1)$, that never pass above the line $y = x$.*
(vi) *Ways to dissect a convex $(n+2)$-gon into n triangles by drawing $n-1$ diagonals, no two of which intersect in their interior. Such dissections may be called* triangulations *of an $(n+2)$-gon (with no new vertices).*

A host of other appearances of Catalan numbers in enumerative combinatorics and in other areas of mathematics are given in Exercises 6.19–6.36. Some of the enumerative properties of Catalan numbers are quite surprising and subtle, and do not yield to "transparent" bijections such as those used to establish Corollary 6.2.3.

We are now ready to discuss the connection between plane S-trees and algebraic functions.

6.2.4 Proposition. *Let $S \subseteq \mathbb{P}$, and define*

$$u = u(t, x) = \sum_{n \geq 0} \sum_{m \geq 0} T_S(m, n) t^m x^n.$$

Then

$$u = tx + x \sum_{j \in S} u^j. \tag{6.15}$$

Proof. Note that u^j is the generating function for ordered j-tuples of plane S-trees, i.e.,

$$u^j = \sum_{n \geq 0} \sum_{m \geq 0} T_{S,j}(m, n) t^m x^n,$$

where $T_{S,j}(m, n)$ is the number of ordered j-tuples $(\tau_1, \ldots, \tau_j)$ of plane S-trees with m endpoints (where by definition an endpoint of $(\tau_1, \ldots, \tau_j)$ is an endpoint of some τ_i) and n vertices.

We obtain a plane S-tree τ from $(\tau_1, \ldots, \tau_j)$ for $j \in S$ by letting $\tau_1, \ldots, \tau_j$ be the subtrees of the root. The number $p(\tau)$ of vertices of τ satisfies $p(\tau) = 1+\sum p(\tau_i)$, while the number $q(\tau)$ of endpoints satisfies $q(\tau) = \sum q(\tau_i)$. Every plane S-tree τ with more than one point corresponds uniquely to such a j-tuple $(\tau_1, \ldots, \tau_j)$, so (6.15) follows. □

By specializing (6.15) in various ways we can obtain algebraic generating functions. In particular, if v enumerates plane S-trees by number of endpoints (where we must assume $1 \notin S$), then $v = u(t, 1)$ so

$$v = t + \sum_{j \in S} v^j. \tag{6.16}$$

Similarly if w enumerates plane S-trees by number of vertices, then $w = u(1, x)$ so

$$w = x + x \sum_{j \in S} w^j. \tag{6.17}$$

Clearly v and w will be algebraic for suitable choices of S. It can be shown that the following five conditions on S are equivalent when $\operatorname{char} K = 0$:

(i) v is algebraic (assuming $1 \notin S$, so v is defined).
(ii) w is algebraic.
(iii) $\sum_{j \in S} x^j$ is rational.
(iv) S differs by a finite set from a finite union of (infinite) arithmetic progressions in $\mathbb{P}$.
(v) The function $\chi_S : \mathbb{P} \to \{0, 1\}$ defined by $\chi_S(j) = 1$ if $j \in S$ and $\chi_S(j) = 0$ if $j \notin S$ is eventually periodic.

It's easy to see that

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftarrow \text{(iii)} \Leftarrow \text{(iv)} \Leftrightarrow \text{(v)}.$$

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) take considerably more work. (They can be deduced from Exercise 4.19(b), (c).)

Note that since (6.16) and (6.17) can be solved explicitly for t and x, respectively, we obtain explicit expressions for the compositional inverses $v^{\langle -1 \rangle}$ and $w^{\langle -1 \rangle}$:

$$v^{\langle -1 \rangle} = t - \sum_{j \in S} t^j, \qquad 1 \notin S$$

$$w^{\langle -1 \rangle} = \frac{x}{1 + \sum_{j \in S} x^j}.$$

6.2.5 Example. Let $S = 4\mathbb{P} \cup (5\mathbb{N}+1) \cup \{9, 11\} - \{1, 12, 16\}$. Then

$$\sum_{j \in S} x^j = \frac{x^4}{1-x^4} + \frac{x}{1-x^5} - \frac{x^{16}}{1-x^{20}} + x^9 + x^{11} - x - x^{12} - x^{16},$$

which is rational. Thus

$$v = x + \frac{v^4}{1-v^4} + \frac{v}{1-v^5} - \frac{v^{16}}{1-v^{20}} + v^9 + v^{11} - v - v^{12} - v^{16},$$

which clearly yields a polynomial equation (of degree 36) satisfied by v.

6.2.6 Example. Let $S = \{k\}$, $k \geq 2$. Combining Proposition 6.2.2 and equation (6.16) yields the following result: Let

$$\begin{aligned} v &= \sum_{n \geq 0} \frac{1}{kn+1} \binom{kn+1}{n} x^{(k-1)n+1} \\ &= \sum_{n \geq 0} \frac{1}{(k-1)n+1} \binom{kn}{n} x^{(k-1)n+1}. \end{aligned} \tag{6.18}$$

Then $v = x + v^k$, so $v = (x - x^k)^{\langle -1 \rangle}$. Of course we can obtain this result directly from the Lagrange inversion formula (Theorem 5.4.1). In fact, Proposition 6.2.2 is a special case of the basis for one of our combinatorial proofs of Lagrange inversion.

Suppose (continuing to assume $\operatorname{char} K = 0$) we let k be *any* element of $K - \{1\}$ (or an indeterminate over K), and define

$$\begin{aligned} y = y(x) &= \sum_{n \geq 0} \frac{1}{kn+1} \binom{kn+1}{n} x^n \\ &= \sum_{n \geq 0} \frac{1}{(k-1)n+1} \binom{kn}{n} x^n. \end{aligned}$$

If $k \in \mathbb{P}$ and v is as in (6.18), then $v = xy(x^{k-1})$. Hence $xy(x^{k-1}) = x + x^k y(x^{k-1})^k$, so

$$y = 1 + xy^k. \tag{6.19}$$

In fact, (6.19) remains valid for any $k \in K$. There are two ways to see this: (a) Lagrange inversion, and (b) equating coefficients of x^n on both sides of (6.19) yields a tentative polynomial identity involving k. Since we know it's true for all $k \in \mathbb{P} - \{1\}$, it follows that it's valid as a polynomial identity.

6.2.7 Example. An interesting series related to (6.18) is given by

$$z = \sum_{n \geq 0} \binom{kn}{n} x^n. \tag{6.20}$$

Here we may assume $k \in \mathbb{P}$, or even, as in the previous paragraph, $k \in K$. Thus, with v as in (6.18), we have $z(x^{k-1}) = v'$ (where $v' = dv/dx$). Differentiating $v = x + v^k$ yields $v' = 1 + kv'v^{k-1}$. Multiply by v to get $v'v = v + kv'(v - x)$. Solving for v yields

$$v = \frac{kv'x}{1 + (k-1)v'}.$$

Hence from $v = x + v^k$ we get

$$\frac{kv'x}{1 + (k-1)v'} = x + \left(\frac{kv'x}{1 + (k-1)v'} \right)^k,$$

so

$$\frac{v' - 1}{1 + (k-1)v'} = x^{k-1} \left(\frac{kv'}{1 + (k-1)v'} \right)^k.$$

Therefore

$$\frac{z - 1}{1 + (k-1)z} = x \left(\frac{kz}{1 + (k-1)z} \right)^k. \tag{6.21}$$

Hence if $k \in \mathbb{Q}$, then z is algebraic. For further information about the series z in the case $k \in \mathbb{P}$, see Exercise 6.13. Some series that seem closely related to (6.20), such as $\sum \binom{3n}{n,n,n} x^n$ and $\sum \binom{2n}{n}^2 x^n$, can be shown to be nonalgebraic. See Exercise 6.3.

6.2.8 Example. Some interesting enumerative problems correspond to the cases $S = \mathbb{P}$ and $S = \mathbb{P} - \{1\}$. For instance, if $S = \mathbb{P} - \{1\}$ then $v = u(t, 1)$ is the generating function for the total number of dissections of an $(n + 2)$-gon C by diagonals not intersecting in the interior of C (and only allowing straight diagonals, so no two-sided regions are formed). Similarly, if $S = \mathbb{P}$ then $w = u(1, x)$ is the generating function for the total number $T_{\mathbb{P}}(n)$ of plane trees with n vertices. From

(6.15), (6.16), and (6.17) we compute:

$$\begin{aligned}
S = \mathbb{P} - \{1\} \Rightarrow u &= \frac{1 + tx - \sqrt{1 - 2tx - 4tx^2 + t^2x^2}}{2(1+x)} \\
v &= \frac{1 + t - \sqrt{1 - 6t + t^2}}{4} \\
w &= \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2(1+x)} \\
&= \frac{1}{2}\left(1 - \sqrt{\frac{1-3x}{1+x}}\right)
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
S = \mathbb{P} \Rightarrow u &= \frac{1 - x + tx - \sqrt{1 - 2x - 2tx + x^2 - 2tx^2 + t^2x^2}}{2} \\
& v \text{ is undefined} \\
& w = \frac{1 - \sqrt{1-4x}}{2}.
\end{aligned}$$

From the simple form of w when $S = \mathbb{P}$ we deduce that $T_{\mathbb{P}}(n) = C_{n-1}$, another occurrence of Catalan numbers!

The problem of counting plane trees with no vertex of degree one by number of endpoints is equivalent to "Schröder's second problem" and is solved by the generating function (6.22). See the Notes for a surprising reference to this problem going back to the second century B.C. Now is a good time to give a general overview of Schröder's famous "*vier combinatorische Probleme*" (four combinatorial problems), since we have already given solutions to all four problems in Chapter 5 and the present chapter. Schröder was concerned with the enumeration of *bracketings*. He considered two classes of bracketings, viz., bracketings of words (or strings) and of sets, and two rules of combination (binary and arbitrary), giving four problems in all. A bracketing of a word $w = w_1 w_2 \cdots w_n$ is obtained by expressing w as a product of at least two nonempty words (unless w is a single letter), say $w = u_1 u_2 \cdots u_k$, and then inductively bracketing each u_i, continuing until only singleton words (letters) remain. Similarly, a bracketing of an n-set S is obtained by partitioning S into at least two (unless $\#S = 1$) nonempty pairwise disjoint subsets $S = T_1 \cup T_2 \cup \cdots \cup T_k$, and then inductively bracketing each T_i, continuing until only singletons remain. (The order of the T_i's is irrelevant.) A bracketing is *binary* if at each stage a non-singleton word or set is divided into exactly two parts; it is *arbitrary* if any number of parts (at least two) is allowed. Thus the four problems of Schröder are the following.

First Problem. Binary word bracketings. Clearly this coincides with binary parenthesizations of a string of length n, and was solved by Corollary 6.2.3(iii).

The generating function is given by

$$\begin{aligned}\sum_{n\ge 1} s_1(n)x^n &= \frac{1-\sqrt{1-4x}}{2}\\ &= x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + 429x^8 + \cdots,\end{aligned}$$

and

$$s_1(n) = C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1},$$

the $(n-1)$-st Catalan number. For a generalization, see Exercise 5.43.

Second Problem. Arbitrary word bracketings. This is equivalent to plane trees with no vertex of degree one and n endpoints, with generating function (6.22) given by

$$\begin{aligned}\sum_{n\ge 1} s_2(n)x^n &= \frac{1+x-\sqrt{1-6x+x^2}}{4}\\ &= x + x^2 + 3x^3 + 11x^4 + 45x^5 + 197x^6 + 903x^7 + 4279x^8 + \cdots.\end{aligned}$$

The numbers $r_n = 2s_2(n+1)$, $n \ge 1$ (with $r_0 = 1$), and $s_n = s_2(n+1)$, $n \ge 0$, are called *Schröder numbers*. Sometimes s_n is called a *little Schröder number* to distinguish it from r_n. Note that Proposition 6.2.1 yields several additional interpretations of Schröder numbers, e.g., s_n is the number of ways to dissect a convex $(n+2)$-gon with any number of diagonals that don't intesect in their interiors. For further information about Schröder numbers, see Exercise 6.39.

Third Problem. Binary set bracketings. This was the problem considered in Example 5.2.6 with exponential generating function

$$\begin{aligned}\sum_{n\ge 1} s_3(n)\frac{x^n}{n!} &= 1-\sqrt{1-2x}\\ &= x + \frac{x^2}{2!} + 3\frac{x^3}{3!} + 15\frac{x^4}{4!} + 105\frac{x^5}{5!} + 945\frac{x^6}{6!} + 10395\frac{x^7}{7!} + \cdots,\end{aligned}$$

and $s_3(n) = 1\cdot 3\cdot 5\cdots(2n-3)$.

Fourth Problem. Arbitrary set bracketings. This problem was considered in Example 5.2.5, with exponential generating function

$$\begin{aligned}\sum_{n\ge 1} s_4(n)\frac{x^n}{n!} &= (1+2x-e^x)^{\langle -1\rangle}\\ &= x + \frac{x^2}{2!} + 4\frac{x^3}{3!} + 26\frac{x^4}{4!} + 236\frac{x^5}{5!} + 2752\frac{x^6}{6!}\\ &\quad + 39208\frac{x^7}{7!} + 660032\frac{x^8}{8!} + \cdots.\end{aligned}$$

See also Exercises 5.26 and 5.40.

6.3 Diagonals

There is a useful general method for obtaining algebraic generating functions that includes Example 6.2.7 and related results.

6.3.1 Definition. Let

$$F(x_1, \ldots, x_k) = \sum f(n_1, \ldots, n_k) x_1^{n_1} \cdots x_k^{n_k} \in K[[x_1, \ldots, x_k]].$$

The *diagonal* of F, denoted $\operatorname{diag} F$, is the power series in a single variable x defined by

$$\operatorname{diag} F = (\operatorname{diag} F)(x) = \sum_n f(n, n, \ldots, n) x^n.$$

6.3.2 Example. Write s and t for x_1 and x_2, and let

$$F(s,t) = \frac{1}{1-s-t} = \sum_{i,j} \binom{i+j}{i} s^i t^j.$$

Then

$$\operatorname{diag} F = \sum_n \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

Example 6.3.2 is a prototype for the following general result. It will be our main instance of the use of Puiseux's theorem.

6.3.3 Theorem. *Suppose $F(s,t) \in K[[s,t]] \cap K(s,t)$, i.e., F is a power series in s and t that represents a rational function. Then* $\operatorname{diag} F$ *is algebraic.*

Proof. We may assume K is algebraically closed (since F is algebraic over K if and only if F is algebraic over any algebraic extension of K). We also assume that $\operatorname{char} K = 0$, though the theorem in fact holds for any K. (See Exercise 6.11(b) for a more general result when $\operatorname{char} K = p > 0$.) Let

$$G = G(x,s) = F(s, x/s) \in K[[s, x/s]]. \tag{6.23}$$

Thus G is a formal Laurent series in s and x, such that if a monomial $x^i s^j$ appears in G, then $i \geq 0$ and $j \geq -i$. Note that

$$\operatorname{diag} F = [s^0]G,$$

the constant term of G regarded as a Laurent series in s whose coefficients are power series in x.

Since F is rational, it follows that $G = P/Q$, where $P, Q \in K[s,x]$. Regarding G as a rational function of s whose coefficients lie in $K[x]$ (or in the field $K(x)$), we have a partial-fraction decomposition

$$G=\sum_{j=1}^{l}\frac{N_j(s)}{(s-\xi_j)^{e_j}},\tag{6.24}$$

where (i) $e_j\in\mathbb{P}$, (ii) $\xi_1,\dots,\xi_l$ are the distinct zeros of $Q(s)$, and (iii) $N_j(s)\in K(\xi_1,\dots,\xi_l)[s]$. Since the coefficients of $Q(s)$ are polynomials in x, the zeros $\xi_1,\dots,\xi_l$ are algebraic functions of x and hence by Puiseux's theorem (Theorem 6.1.5) we may assume $\xi_j\in K((x^{1/r}))$ for some $r\in\mathbb{P}$. Hence $N_j(s)\in K((x^{1/r}))[s]$.

Rename $\xi_1,\dots,\xi_l$ as $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n$ so that

$$\alpha_i\in x^{1/r}K\big[\big[x^{1/r}\big]\big],\qquad \beta_j\notin x^{1/r}K\big[\big[x^{1/r}\big]\big].$$

In other words, the Puiseux expansion of α_i has only positive exponents, while that of β_j has some nonpositive exponent. Equivalently, $\beta_j^{-1}\in K[[x^{1/r}]]$. Thus

$$\begin{aligned}G&=\sum_{i=1}^{m}\frac{p_i(s)}{(1-s^{-1}\alpha_i)^{c_i}}+\sum_{j=1}^{n}\frac{q_j(s)}{\left(1-s\beta_j^{-1}\right)^{d_j}}\\&=\sum_{i=1}^{m}p_i(s)\sum_{k\geq 0}\binom{c_i}{k}s^{-k}\alpha_i^k+\sum_{j=1}^{n}q_j(s)\sum_{k\geq 0}\binom{d_j}{k}s^k\beta_j^{-k}\end{aligned}\tag{6.25}$$

for some $c_i,d_j\in\mathbb{P}$ and $p_i(s),q_j(s)\in K((x^{1/r}))[s,s^{-1}]$. The expansion (6.25) for G satisfies

$$G\in K\big[\big[s^{1/r},(x/s)^{1/r}\big]\big]\big[s^{-1/r}\big]\supset K[[s,x/s]].$$

(This is the key point – we must expand each term of (6.24) so that each expansion lies in a ring R (independent of the term) that contains $K[[s,x/s]]$, the ring in which the expansion (6.23) of G lies. Thus all our operations are taking place inside a single ring and hence are well defined.) Since $G\in K[[s,x/s]]$, the expansion (6.25) agrees with the series $G=F(s,x/s)$.

Now take the coefficient of s^0 on both sides of (6.25). The left-hand side becomes $\operatorname{diag}F$. The right-hand side becomes a finite sum of terms $\gamma\alpha^a$ and $\delta\beta^b$, where $a,b\in\mathbb{Z}$ and α, β, γ, δ are algebraic. Hence $\operatorname{diag}F$ is algebraic, as was to be proved. □

6.3.4 Example. Consider the rational function $F(s,t)=(1-s-t)^{-1}$ of Example 6.3.2. Going through the previous proof yields

$$G=\frac{1}{1-s-\frac{x}{s}}=-\frac{s}{s^2-s+x}.$$

Now

$$s^2 - s + x = (s - \alpha)(s - \beta),$$

where

$$\alpha = \frac{1 - \sqrt{1 - 4x}}{2} = x + \cdots$$

$$\beta = \frac{1 + \sqrt{1 - 4x}}{2} = 1 + \cdots .$$

The partial-fraction expansion of G is

$$G = \frac{1}{\beta - \alpha}\left[\frac{\alpha}{s - \alpha} - \frac{\beta}{s - \beta}\right].$$

We must write this expression in a form so that its implied expansion as a fractional Laurent series (here just a Laurent series) agrees with the expansion

$$\begin{aligned} G = F(s, x/s) &= \sum_{i,j} \binom{i+j}{i} s^i \left(\frac{x}{s}\right)^j \\ &= \sum_{i,j} \binom{i+j}{i} x^j s^{i-j}. \end{aligned}$$

The difficulty is that a rational function such as $\alpha/(s - \alpha)$ has two expansions in powers of s, viz.,

$$\frac{\alpha/s}{1 - \frac{\alpha}{s}} = \sum_{n\geq 1} \alpha^n s^{-n}$$

$$-\frac{1}{1 - \frac{s}{\alpha}} = -\sum_{n\geq 0} \alpha^{-n} s^n.$$

The "correct" expansion will be the one that lies in the ring $K[[s^{1/r}, (x/s)^{1/r}]][s^{-1/r}]$ (here $r = 1$), viz., the first of the two possible expansions above (since $\alpha = x +$ higher order terms). Similarly we must regard $\beta/(s - \beta)$ as $-1/(1 - s\beta^{-1})$, so (using $\beta - \alpha = \sqrt{1 - 4x}$)

$$\begin{aligned} G &= \frac{1}{\sqrt{1 - 4x}}\left(\frac{\alpha/s}{1 - s^{-1}\alpha} + \frac{1}{1 - s\beta^{-1}}\right) \\ &= \frac{1}{\sqrt{1 - 4x}}\left(\sum_{n\geq 1} \alpha^n s^{-n} + \sum_{n\geq 0} \beta^{-n} s^n\right). \end{aligned}$$

We read off immediately that

$$\operatorname{diag} F = [s^0]G = \frac{1}{\sqrt{1-4x}}.$$

It is natural to ask whether Theorem 6.3.3 has a proof avoiding fractional series. There is an elegant proof based on contour integration (really a variant of the proof we have just given) that we sketch for the benefit of readers knowledgeable about complex analysis. (For another proof, see the end of Section 6.7.) The contour integration approach has the advantage, as exemplified in Example 6.3.5 below, that one can use the vast arsenal of residue calculus (in particular, techniques for computing residues) to carry out the necessary computations. We could develop an equivalent theory for dealing with partial fractions and fractional series arising in our first proof of Theorem 6.3.3, but for those who know residue calculus such a development is unnecessary.

We assume $K = \mathbb{C}$. The series expansion of $F(s,t) = \sum f(m,n)s^m t^n$ converges for sufficiently small $|s|$ and $|t|$ (since $F(s,t)$ is rational and we are assuming it has a power series expansion about $s = t = 0$). Thus $\operatorname{diag} F$ will converge for $|x|$ small. Fix such a small x. Then the series

$$F(s, x/s) = \sum f(m,n) s^{m-n} x^n,$$

regarded as a function of s, will converge in an annulus about $s = 0$ and hence on some circle $|s| = \rho > 0$. By Cauchy's integral theorem,

$$\operatorname{diag} F = [s^0]F(s, x/s) = \frac{1}{2\pi i}\int_{|s|=\rho} F(s, x/s)\frac{ds}{s}.$$

By the Residue Theorem,

$$\operatorname{diag} F = \sum_{s=s(x)} \operatorname{Res}_s F(s, x/s)\frac{1}{s},$$

where the sum ranges over all singularities of $\frac{1}{s}F(s, x/s)$ inside the circle $|s|=\rho$. (Such singularities are precisely the ones satisfying $\lim_{x\to 0} s(x) = 0$.) Since $F(s, x/s)$ is rational, all such singularities $s(x)$ are poles, and $s(x)$ is an algebraic function of x. The residue at a pole s belongs to the ring $\mathbb{C}(s, x)$, so the residues are algebraic. Hence $\operatorname{diag} F$ is algebraic.

6.3.5 Example. Let us apply the above proof to our canonical example $F(s,t) = (1 - s - t)^{-1}$. We have

$$\operatorname{diag} F = \frac{1}{2\pi i}\int \frac{ds}{s\left(1 - s - \frac{x}{s}\right)} = \frac{1}{2\pi i}\int \frac{ds}{-x + s - s^2}.$$

The poles are at $s = \frac{1}{2}(1 \pm \sqrt{1-4x})$. The only pole approaching 0 as $x \to 0$ is $s_0 = \frac{1}{2}(1 - \sqrt{1-4x})$. If $A(s)/B(s)$ has a simple pole at s_0 and $A(s_0) \neq 0$, then the residue at s_0 is $A(s_0)/B'(s_0)$. Hence

$$\operatorname{diag} F = \operatorname{Res}_{s_0} \frac{1}{-x+s-s^2} = \frac{1}{1-2s_0} = \frac{1}{\sqrt{1-4x}}.$$

6.3.6 Example. Let us consider a somewhat more complicated example than the preceding, where it is not feasible to find the poles explicitly. Consider the series $z = \sum_{n \geq 0} \binom{kn}{n} x^n$ of Example 6.2.7, where $k \in \mathbb{P}$. The contour integration argument given above for computing diagonals easily extends to give

$$\begin{aligned} z &= \frac{1}{2\pi i} \int_{s=|\rho|} \frac{ds}{s\left(1 - \frac{u}{s} - s^{k-1}\right)} \\ &= \frac{1}{2\pi i} \int_{s=|\rho|} \frac{ds}{-u+s-s^k} \end{aligned} \tag{6.26}$$

for a suitable $\rho > 0$, where $u = \sqrt[k-1]{x}$. Since the polynomial $-u+s-s^k$ has a zero of multiplicity one at $s=0$ when we set $x=0$, it follows that the integrand of (6.26) has a single (simple) pole $s_0 = s_0(x)$ satisfying $\lim_{x\to 0} s_0(x) = 0$. Thus

$$\begin{aligned} z &= \operatorname{Res}_{s=s_0} \frac{1}{-u+s-s^k} \\ &= \frac{1}{1-ks_0^{k-1}}. \end{aligned}$$

Hence $1/z = 1 - ks_0^{k-1}$, so $s_0/z = s_0 - ks_0^k = s_0 - k(s_0 - u)$. Solving for s_0 gives

$$s_0 = \frac{kuz}{1+(k-1)z},$$

so from $s_0^k - s_0 + u = 0$ we get

$$\left(\frac{kuz}{1+(k-1)z}\right)^k - \frac{kuz}{1+(k-1)z} + u = 0.$$

Dividing by u gives

$$x\left(\frac{kz}{1+(k-1)z}\right)^k - \frac{kz}{1+(k-1)z} + 1 = 0.$$

This equation is equivalent to (6.21) and simplifies to a polynomial equation satisfied by z.

In view of Theorem 6.3.3, it's natural to ask whether $\operatorname{diag} F$ is algebraic for a rational series in more than two variables. Unfortunately the answer in general is negative; we have mentioned earlier that $u = \sum \binom{3n}{n,n,n} x^n$ is not algebraic, yet $u = \operatorname{diag}(1 - x_1 - x_2 - x_3)^{-1}$. But diagonals of rational series in any number of variables do possess the desirable property of D-finiteness, to be discussed in the next section; see Exercise 6.61 for the connection between diagonals and D-finiteness.

We conclude this section with a class of enumerative problems that involve in a natural way rational functions of two variables. The general setting is that of *lattice paths in the plane*. For simplicity we will deal with paths whose steps come from $\mathbb{N} \times \mathbb{N}$, though many variations of this condition are possible. Given $S \subseteq \mathbb{N} \times \mathbb{N}$, an S-*path* of length l from $(0, 0)$ to (m, n) is a sequence $\sigma = (v_1, v_2, \ldots, v_l) \in S^l$ such that $v_1 + v_2 + \cdots + v_l = (m, n)$. Thus σ may be regarded as a *weak* (since $(0, 0) \in S$ is allowed) *composition* (or ordered partition) of (m, n) into l parts – the exact two-dimensional analogue of the one-dimensional situation discussed in Section 1.2. The set S is the set of *allowed steps*. Let $N_S(m, n; l)$ denote the number of S-paths of length l from $(0, 0)$ to (m, n). If $(0, 0) \notin S$ then let

$$N_S(m, n) = \sum_l N_S(m, n; l),$$

the total number of S-paths from $(0, 0)$ to (m, n). Define the generating function

$$F_S(s, t; z) = \sum_{m,n,l} N_S(m, n; l) s^m t^n z^l,$$

and if $(0, 0) \notin S$ then define

$$G_S(s, t) = F_S(s, t; 1) = \sum_{m,n} N_S(m, n) s^m t^n.$$

6.3.7 Proposition. *Let* $S \subseteq \mathbb{N} \times \mathbb{N}$. *Then*

$$F_S(s, t; z) = \frac{1}{1 - z \sum_{(i,j) \in S} s^i t^j}.$$

If moreover $(0, 0) \notin S$, *then*

$$G_S(s, t) = \frac{1}{1 - \sum_{(i,j) \in S} s^i t^j}.$$

Proof. It should be clear that for fixed l,

$$\sum_{m,n} N_S(m, n; l) s^m t^n = \left(\sum_{(i,j) \in S} s^i t^j \right)^l.$$

Now multiply by z^l and sum on $l \geq 0$. □

Under certain circumstances F_S and G_S will be rational functions of s and t, and their diagonals (or diagonals of closely related series) will be algebraic. Let us give some examples.

6.3.8 Example. Exercise 1.5(b) asked for the generating function $y = \sum f(n)x^n$, where $f(n)$ counts the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$. The solution $y = 1/\sqrt{1-6x+x^2}$ was obtained by an *ad hoc* argument, and the final answer appears rather mysterious. Using Proposition 6.3.7 and our techniques for computing diagonals, we have

$$\begin{aligned} y &= \operatorname{diag} \frac{1}{1-s-t-st} \\ &= [s^0] \frac{1}{1-s-\frac{x}{s}-s\frac{x}{s}} \\ &= [s^0] \frac{-s}{s^2+(x-1)s+x} \\ &= [s^0] \frac{1}{\beta-\alpha}\left(\frac{s}{s-\alpha}-\frac{s}{s-\beta}\right), \end{aligned}$$

where $\alpha = \frac{1}{2}(1-x-\sqrt{1-6x+x^2})$ and $\beta = \frac{1}{2}(1-x+\sqrt{1-6x+x^2})$. Thus

$$\begin{aligned} y &= [s^0] \frac{1}{\sqrt{1-6x+x^2}}\left(\frac{s\alpha^{-1}}{1-s\alpha^{-1}}+\frac{1}{1-s^{-1}\beta}\right) \\ &= \frac{1}{\sqrt{1-6x+x^2}} \\ &= 1+3x+13x^2+63x^3+321x^4+1683x^5+8989x^6+\cdots. \end{aligned} \tag{6.27}$$

The coefficient of $s^m t^n$ in the generating function $1/(1-s-t-st)$ is known as the *Delannoy number* $D(m, n)$, and $D(n, n)$ is called a *central Delannoy number*. Thus in particular

$$\sum_{n\geq 0} D(n, n)x^n = \frac{1}{\sqrt{1-6x+x^2}}.$$

We could also take into account the number of steps, i.e., now let

$$y = \sum_{n,l} N_S(n, n; l)x^n z^l,$$

where $S = \{(1, 0), (0, 1), (1, 1)\}$. Then one can compute that

$$\begin{aligned} y &= \operatorname{diag} \frac{1}{1-z(s+t+st)} \\ &= \frac{1}{\sqrt{1-x(2z+4z^2)+x^2z^2}}, \end{aligned} \tag{6.28}$$

where diag is computed with respect to the variables s and t only (not z). In particular, if $f(l)$ is the number of paths that end up on the line $y = x$ after l steps, then setting $x = 1$ in (6.28) yields

$$\begin{aligned}\sum_{l\geq 0} f(l)z^l &= \frac{1}{\sqrt{1-2z-3z^2}} \\ &= 1+z+3z^2+7z^3+19z^4+51z^5+141z^6 \\ &\quad +393z^7+1107z^8+3139z^9+8953z^{10}+\cdots. \qquad (6.29)\end{aligned}$$

For some more information on this generating function, see Exercise 6.42.

If we were even more ambitious, we could keep track of the number of steps of each type (i.e., $(1,0)$, $(0,1)$, $(1,1)$). Letting z_{ij} keep track of the number of times (i, j) is a step, it is clear that we want to compute

$$\begin{aligned}y &= \operatorname{diag}_{s,t} \frac{1}{1-z_{10}s-z_{01}t-z_{11}st} \\ &= \frac{1}{\sqrt{1-x(2z_{11}+4z_{10}z_{01})+x^2z_{11}^2}},\end{aligned}$$

where $\operatorname{diag}_{s,t}$ denotes the diagonal with respect to the variables s and t only. In this case we gain no further information than contained in (6.28), since knowing the destination (n, n) and the total number of steps determines the number of steps of each type, but for less "homogeneous" choices of S this is no longer the case.

6.3.9 Example. Set $S = \mathbb{N} \times \mathbb{N} - \{(0,0)\}$, so $N_S(n,n)$ is the total number of paths from $(0,0)$ to (n,n) with any allowed lattice steps that move closer to (n,n) (and never pass (n,n) and then backtrack). We have

$$\begin{aligned}y := \sum_{n\geq 0} N_S(n,n)x^n &= \operatorname{diag} \frac{1}{1-\left(\frac{1}{(1-s)(1-t)}-1\right)} \\ &= \operatorname{diag} \frac{(1-s)(1-t)}{1-2s-2t+2st}.\end{aligned}$$

By now it should be routine to compute that

$$y = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-12x+4x^2}}\right) \qquad (6.30)$$

$$= 1+3x+26x^2+252x^3+2568x^4+26928x^5+287648x^6+\cdots.$$

(The reader may find it instructive to carry out the computation in order to ascertain how the term $\frac{1}{2}$ arises in (6.30) when y is written as $\frac{1}{2}+\frac{1}{2\sqrt{1-12x+4x^2}}$.)

6.4 *D*-Finite Generating Functions

An important property of rational generating functions $\sum f(n)x^n$ is that the coefficients $f(n)$ satisfy a simple recurrence relation (Theorem 4.1.1(ii)). An analogous result holds for algebraic generating functions. However, the type of recurrence satisfied by the coefficients of an algebraic generating function also holds for the coefficients of more general series, whose basic properties we now discuss.

6.4.1 Proposition. *Let $u \in K[[x]]$. The following three conditions are equivalent:*

(i) *The vector space over $K(x)$ spanned by u and all its derivatives $u', u'', \ldots$ is finite-dimensional. In symbols,*

$$\dim_{K(x)}[K(x)u + K(x)u' + K(x)u'' + \cdots] < \infty.$$

(ii) *There exist polynomials $p_0(x), \ldots, p_d(x) \in K[x]$ with $p_d(x) \neq 0$, such that*

$$p_d(x)u^{(d)} + p_{d-1}(x)u^{(d-1)} + \cdots + p_1(x)u' + p_0(x)u = 0, \qquad (6.31)$$

where $u^{(j)} = d^j u/dx^j$.

(iii) *There exist polynomials $q_0(x), \ldots, q_m(x), q(x) \in K[x]$, with $q_m(x) \neq 0$, such that*

$$q_m(x)u^{(m)} + q_{m-1}(x)u^{(m-1)} + \cdots + q_1(x)u' + q_0(x)u = q(x). \qquad (6.32)$$

*If u satisfies any (and hence all) of the above three conditions, then we say that u is a D-*finite *(short for* differentiably finite*) power series.*

Proof. (i) $\Rightarrow$ (ii). Suppose

$$\dim_{K(x)}[K(x)u + K(x)u' + \cdots] = d.$$

Thus $u, u', \ldots, u^{(d)}$ are linearly dependent over $K(x)$. Write down a dependence relation and clear denominators so that the coefficients are all polynomials to get an equation of the form (6.31).

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Suppose that (6.32) holds and that the usual degree of $q(x)$ as a polynomial in x is t. Differentiate (6.32) $t + 1$ times to get an equation (6.31), with $d = m + t + 1$ and $p_d(x) = q_m(x) \neq 0$. Solving for $u^{(d)}$ shows that

$$u^{(d)} \in K(x)u + K(x)u' + \cdots + K(x)u^{(d-1)}. \qquad (6.33)$$

Differentiate (with respect to x) the equation expressing $u^{(d)}$ as a $K(x)$-linear combination of $u, u', \ldots, u^{(d-1)}$. We get

$$\begin{aligned} u^{(d+1)} &\in K(x)u + K(x)u' + \cdots + K(x)u^{(d)} \\ &= K(x)u + K(x)u' + \cdots + K(x)u^{(d-1)} \qquad \text{(by (6.33))}. \end{aligned}$$

Continuing in this way, we get

$$u^{(d+k)} \in K(x)u + K(x)u' + \cdots + K(x)u^{(d-1)}$$

for all $k \geq 0$, so (i) holds. □

6.4.2 Example. (a) $u = e^x$ is D-finite, since $u' = u$. Similarly, any linear combination of series of the form $x^m e^{ax}$ ($m \in \mathbb{N}$, $a \in K$) is D-finite, since such series satisfy a linear homogeneous differential equation with *constant* coefficients.
(b) $u = \sum_{n\geq 0} n!x^n$ is D-finite, since $(xu)' = \sum_{n\geq 0}(n+1)!x^n$, whence $1 + x(xu)' = u$. This equation simplifies to $x^2u' + (x-1)u = -1$.

We wish to characterize the coefficients of a D-finite power series u. To this end, define a function $f : \mathbb{N} \to K$ to be P*-recursive* (short for *polynomially recursive*) if there exist polynomials $P_0, \ldots, P_e \in K[n]$ with $P_e \neq 0$, such that

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \cdots + P_0(n)f(n) = 0, \qquad (6.34)$$

for all $n \in \mathbb{N}$. In other words, f satisfies a homogeneous linear recurrence of finite degree with polynomial coefficients.

6.4.3 Proposition. *Let* $u = \sum_{n\geq 0} f(n)x^n \in K[[x]]$. *Then* u *is* D*-finite if and only if* f *is* P*-recursive.*

Proof. Suppose u is D-finite, so that (6.31) holds (with $p_d(x) \neq 0$). Since

$$x^j u^{(i)} = \sum_{n\geq 0} (n+i-j)_i f(n+i-j)x^n,$$

when we equate coefficients of x^{n+k} in (6.31) for fixed k sufficiently large, we will obtain a recurrence of the form (6.34) for f. This recurrence will not collapse to $0 = 0$, because if $[x^j]p_d(x) \neq 0$, then $[n^d]P_{d-j+k}(n) \neq 0$.

Conversely, suppose that f satisfies (6.34) (with $P_e(n) \neq 0$). For fixed $i \in \mathbb{N}$, the polynomials $(n+i)_j$, $j \geq 0$, form a K-basis for the space $K[n]$ (since $\deg(n+i)_j = j$). Thus $P_i(n)$ is a K-linear combination of the polynomials $(n+i)_j$, so $\sum_{n\geq 0} P_i(n)f(n+i)x^n$ is a K-linear combination of series of the form $\sum_{n\geq 0}(n+i)_j f(n+i)x^n$. Now

$$\sum_{n\geq 0}(n+i)_j f(n+i)x^n = R_i(x) + x^{j-i}u^{(j)}$$

for some $R_i(x) \in x^{-1}K[x^{-1}]$ (i.e., $R_i(x)$ is a Laurent polynomial all of whose exponents are *negative*). For instance, $u' = \sum_{n\geq 0} nf(n)x^{n-1}$, so

$$\begin{aligned} x^{-1}u' &= \sum_{n\geq -1}(n+2)f(n+2)x^n \\ &= f(1)x^{-1} + \sum_{n\geq 0}(n+2)f(n+2)x^n. \end{aligned}$$

Hence multiplying (6.34) by x^n and summing on $n \geq 0$ yields

$$0 = \sum a_{ij}x^{j-i}u^{(j)} + R(x), \tag{6.35}$$

where the sum is finite, $a_{ij} \in K$, and $R(x) \in x^{-1}K[x^{-1}]$. One easily sees that not all $a_{ij} = 0$. Now multiply (6.35) by x^q for q sufficiently large to get an equation of the form (6.32). □

6.4.4 Example. (a) $f(n) = n!$ is clearly P-recursive, since $f(n+1) - (n+1)f(n) = 0$. Hence $u = \sum_{n\geq 0} n!x^n$ is D-finite. We don't need a trick to see this as in Example 6.4.2(b).
(b) $f(n) = \binom{2n}{n}$ is P-recursive, since

$$(n+1)f(n+1) - 2(2n+1)f(n) = 0.$$

Hence $u = \sum_{n\geq 0}\binom{2n}{n}x^n = 1/\sqrt{1-4x}$ is D-finite.

Soon we will give many examples of D-finite series and P-recursive functions (Theorems 6.4.6, 6.4.9, 6.4.10, 6.4.12 and Exercises 6.53–6.61). First we note a simple but useful result.

6.4.5 Proposition. *Suppose $f : \mathbb{N} \to K$ is P-recursive, and $g : \mathbb{N} \to K$ agrees with f for all n sufficiently large. Then g is P-recursive.*

Proof. Suppose $f(n) = g(n)$ for $n \geq n_0$ and f satisfies (6.34). Then

$$\left(\prod_{j=0}^{n_0-1}(n-j)\right)[P_e(n)g(n+e) + \cdots + P_0(n)g(n)] = 0,$$

so g is P-recursive. Alternatively, we could use

$$P_e(n+n_0)g(n+e+n_0)+P_{e-1}(n+n_0)g(n+e+n_0-1)$$
$$+\cdots+P_0(n+n_0)g(n+n_0)=0. \qquad \square$$

Our first main result on D-finite power series asserts that algebraic series are D-finite.

6.4.6 Theorem. *Let $u \in K[[x]]$ be algebraic of degree d. Then u is D-finite. More precisely, u satisfies an equation (6.31) of order d, or an equation (6.32) of order $m = d-1$. (For the least-order differential equation satisfied by u, see Exercise 6.62.)*

Proof. By (6.12) we have $u' \in K(x,u)$. Continually differentiating (6.12) with respect to x shows by induction that $u^{(k)} \in K(x,u)$ for all $k \geq 0$. But $\dim_{K(x)} K(x,u) = d$, so $u, u', \ldots, u^{(d)}$ are linearly dependent over $K(x)$, yielding an equation of the form (6.31). Similarly, $1, u, u', \ldots, u^{(d-1)}$ are linearly dependent over $K(x)$, yielding an equation (6.32) with $m \leq d-1$. $\square$

Proposition 6.4.3 and Theorem 6.4.6 together show that the coefficients of an algebraic power series u satisfy a simple recurrence (6.34). In particular, once this recurrence is found (which involves only a finite amount of computation), we have a method for rapidly computing the coefficients of u. Let us note that not all D-finite series are algebraic, e.g., e^x (Exercise 6.1).

6.4.7 Example. Let $F(x) \in K(x)$ with char$K = 0$ and $F(0) = 1$ (so $F(x)^{1/d}$ is defined formally as a power series for any $d \in \mathbb{P}$). Let $u = F(x)^{1/d}$ (with $u(0) = 1$), so $u^d = F(x)$. Then $du^{d-1}u' = F'(x)$, so multiplying by u yields

$$dF(x)u' = F'(x)u. \tag{6.36}$$

If we want polynomial coefficients in (6.36), suppose $F(x) = A(x)/B(x)$ where $A, B \in K[x]$. Then (6.36) becomes

$$dABu' = (A'B - AB')u. \tag{6.37}$$

For instance, if $B(x) = 1$, so $F(x) = A(x) = a_0 + a_1x + \cdots + a_rx^r$; and if $u = \sum f(n)x^n$, then equating coefficients of x^{n+r-1} in (6.37) yields

$$\sum_{j=0}^{r} a_{r-j}(dn + dj - r + j)f(n+j) = 0. \tag{6.38}$$

Similarly, if $A(x) = 1$, $B(x) = b_0 + b_1x + \cdots + b_rx^r$, and $u = \sum f(n)x^n$, then

$$\sum_{j=0}^{r} b_{r-j}(dn + dj + r - j)f(n+j) = 0. \tag{6.39}$$

6.4.8 Example. (a) In Exercise 1.5(b) and Example 6.3.8 we saw that the number $f(n)$ of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$ satisfied

$$u := \sum_{n\geq 0} f(n)x^n = \frac{1}{\sqrt{1-6x+x^2}}.$$

Putting $B(x) = 1 - 6x + x^2$ and $d = 2$ in (6.39) yields

$$(n+2)f(n+2) - 3(2n+3)f(n+1) + (n+1)f(n) = 0 \qquad (6.40)$$

for $n \geq 0$, with the initial conditions $f(0) = 1$, $f(1) = 3$. Though equation (6.40) is a simple recurrence, it is difficult to give a combinatorial proof.
(b) Here are the recurrences satisfied by the coefficients $f(n)$ of some of the algebraic functions $u = \sum f(n)x^n$ considered earlier in the chapter:

(i) $u = \frac{1+x-\sqrt{1-6x+x^2}}{4}$ (equation (6.22)):

$$(n+2)f(n+2) - 3(2n+1)f(n+1) + (n-1)f(n) = 0, \qquad n \geq 1.$$

(ii) $u = \frac{1}{\sqrt{1-2x-3x^2}}$ (equation (6.29)):

$$(n+2)f(n+2) - (2n+3)f(n+1) - 3(n+1)f(n) = 0, \qquad n \geq 0.$$

(iii) $u = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-12x+4x^2}}\right)$ (equation (6.30)):

$$(n+2)f(n+2) - 6(2n+3)f(n+1) + 4(n+1)f(n) = 0, \qquad n \geq 1.$$

Theorem 6.4.6 yields a large class of interesting D-finite series. But there are many simple series for which it is unclear, based on what we have presented up to now, whether they are D-finite, such as

$$\begin{aligned}
u_1 &= \sec x \\
u_2 &= \sqrt{\cos x} \\
u_3 &= e^{x/\sqrt{1-4x}} \\
u_4 &= e^{e^x - 1} \\
u_5 &= \frac{e^x}{\sqrt{1-4x}} \\
u_6 &= e^x + \sum_{n\geq 0} n!x^n \\
u_7 &= \sum_{n\geq 0} \left[1 + \binom{n}{1}^3 + \binom{n}{2}^3 + \cdots + \binom{n}{n}^3\right]^2 x^n.
\end{aligned}$$

Thus we need some further techniques for showing that power series are D-finite (or that their coefficients are P-recursive). It will follow from Theorem 6.4.10 that u_3 is D-finite, from Theorem 6.4.9 that u_5 and u_6 are D-finite, and from Theorem 6.4.12 that u_7 is D-finite. On the other hand, u_1, u_2, and u_4 are not D-finite. We will not develop here systematic methods for showing that power series are *not* D-finite, but see Exercises 6.59–6.60 for some tricks and results in that direction.

6.4.9 Theorem. *The set $\mathcal{D}$ of D-finite power series $u \in K[[x]]$ forms a subalgebra of $K[[x]]$. In other words, if $u, v \in \mathcal{D}$, and $\alpha, \beta \in K$, then $\alpha u + \beta v \in \mathcal{D}$ and $uv \in \mathcal{D}$.*

Proof. If $w \in K[[x]] \subset K((x))$, then let V_w denote the vector space over $K(x)$ spanned by $w, w', w'', \ldots$. Thus V_w is a subspace of $K((x))$. Let $u, v \in \mathcal{D}$ and $\alpha, \beta \in K$. Set $y = \alpha u + \beta v$. Then $y, y', y'', \ldots \in V_u + V_v$. Thus, taking dimensions over $K(x)$, we have

$$\dim V_y \leq \dim(V_u + V_v) \leq \dim V_u + \dim V_v < \infty.$$

Hence y is D-finite.

It remains to show that if $u, v \in \mathcal{D}$ then $uv \in \mathcal{D}$. We assume knowledge of the elementary properties of the tensor (or Kronecker) product of two vector spaces. Let $V = K((x))$, regarded as a vector space over $K(x)$. There is a unique linear transformation

$$\phi : V_u \otimes_{K(x)} V_v \to V$$

that satisfies $\phi(u^{(i)} \otimes v^{(j)}) = u^{(i)}v^{(j)}$ for all $i, j \geq 0$ (or $\phi(y \otimes z) = yz$ for all $y \in V_u$, $z \in V_v$). By Leibniz's rule for differentiating a product, we see that the image of ϕ contains V_{uv}. Hence

$$\dim V_{uv} \leq \dim(V_u \otimes V_v) = (\dim V_u)(\dim V_v) < \infty.$$

Thus $uv \in \mathcal{D}$, as desired. □

Our next result deals with the composition of D-finite series. In general, if $u, v \in \mathcal{D}$ with $u(0) = 0$, then $u(v(x))$ need not be D-finite, even if u is algebraic. For instance (Exercise 6.59) if $u = \sqrt{1+x} \in K_{\text{alg}}[[x]] \subset \mathcal{D}$ and $v = \log(1+x^2) - 1 \in \mathcal{D}$, then $u(v(x)) \notin \mathcal{D}$. Thus the next result may come as something of a surprise.

6.4.10 Theorem. *If $u \in \mathcal{D}$ and $v \in K_{\text{alg}}[[x]]$ with $v(0) = 0$, then $u(v(x)) \in \mathcal{D}$.*

Proof. Let $y = u(v(x))$ and $i \geq 0$. By iterating the chain rule and Leibniz's rule, we see that $y^{(i)}$ is a linear combination of $u(v(x)), u'(v(x)), u''(v(x)), \ldots$ with coefficients in $K[v', v'', v''', \ldots]$. Since v is algebraic, the proof of Theorem 6.4.6 shows that $v^{(i)} \in K(x, v)$. Hence

$$K[v', v'', v''', \ldots] \subset K(x, v).$$

Let V be the $K(x, v)$-vector space spanned by $u(v(x)), u'(v(x)), \ldots$. Since $u \in \mathcal{D}$, we have

$$\dim_{K(x)} \operatorname{span}_{K(x)}\{u(x), u'(x), \ldots\} < \infty.$$

Thus

$$\dim_{K(v)} \operatorname{span}_{K(v)}\{u(v(x)), u'(v(x)), \ldots\} < \infty,$$

so *a fortiori*

$$\dim_{K(x,v)} \operatorname{span}_{K(v)}\{u(v(x)), u'(v(x)), \ldots\} < \infty.$$

The above argument shows that

$$\dim_{K(x,v)} V < \infty \quad \text{and} \quad [K(x, v) : K(x)] < \infty,$$

where $[L : M]$ denotes the degree of the field L over the subfield M (i.e., $[L : M] = \dim_M L$). Hence

$$\dim_{K(x)} V = \left(\dim_{K(x,v)} V\right) \cdot [K(x, v) : K(x)] < \infty.$$

Since each $y^{(i)} \in V$, there follows $y \in \mathcal{D}$. $\square$

6.4.11 Example. Since $\sum_{n\geq 0} n!x^n \in \mathcal{D}$ (by Example 6.4.2(b)), $e^x \in \mathcal{D}$ (by Example 6.4.2(a)), and $x/\sqrt{1-4x}$ is algebraic, we conclude from Theorems 6.4.9 and 6.4.10 that

$$u := \left(\sum_{n\geq 0} n!x^n\right) e^{x/\sqrt{1-4x}} \in \mathcal{D}.$$

It would be a chore, however, to find a differential equation (6.31) or (6.32) satisfied by u, or a linear recurrence (6.34) satisfied by its coefficients.

For our final basic result on D-finite series, we consider the Hadamard product. We will show that if $u, v \in \mathcal{D}$ then $u * v \in \mathcal{D}$. Equivalently, if $f(n)$ and $g(n)$ are P-recursive, then so is $f(n)g(n)$. The proof will be quite similar to the proof that products of D-finite series are D-finite (see Theorem 6.4.9). We would like to work with the set $\mathcal{P}$ of all P-recursive functions $f : \mathbb{N} \to K$, regarded as a vector space over the field $K(n)$ of rational functions in the variable n. In order for $\mathcal{P}$ to be a $K(n)$-vector space, we need to have that if $f, g \in \mathcal{P}$ and $R(n) = \frac{A(n)}{B(n)} \in K(n)$ (with $A, B \in K[n]$), then $f + g \in \mathcal{P}$ and $Rf \in \mathcal{P}$. It follows immediately from Theorem 6.4.9 that $f + g \in \mathcal{P}$. If, on the other hand, f satisfies (6.34), then

$h := Rf$ satisfies

$$\frac{P_e(n)B(n+e)h(n+e)}{A(n+e)} + \frac{P_{e-1}(n)B(n+e-1)h(n+e-1)}{A(n+e-1)} + \cdots + \frac{P_0(n)B(n)h(n)}{A(n)} = 0.$$

Multiplying by $A(n)A(n+1)\cdots A(n+e)$ yields a nonzero linear recurrence with polynomial coefficients satisfied by h, so $h \in \mathcal{P}$. There is one technical flaw in this argument, however. (Can the reader find it without reading further?) The problem is that $B(n)$ may have zeros at certain $n_0 \in \mathbb{N}$, so $h(n)$ is undefined at $n = n_0$. However, since $B(n)$ can have only finitely many zeros, $h(n)$ is defined for all sufficiently large n. Thus we must deal not with $h(n)$ itself, but only with its behavior "in a neighborhood of ∞." (An alternative approach is to regard $\mathcal{P}$ as a $K[n]$-module, rather than a $K(n)$-module. However, we have tried to keep algebraic prerequisites at a minimum by working with vector spaces as much as possible.)

To make the above ideas precise, define a relation $\sim$ on functions $h_1, h_2 : \mathbb{N} \to K$ by the rule that $h_1 \sim h_2$ if $h_1(n) = h_2(n)$ for all sufficiently large n. Clearly $\sim$ is an equivalence relation; we call the equivalence classes *germs* (more precisely, germs at ∞ of functions $h : \mathbb{N} \to K$). Denote the class containing h by $[h]$. We define linear combinations (over K) and multiplication of germs by $\alpha_1[h_1] + \alpha_2[h_2] = [\alpha_1 h_1 + \alpha_2 h_2]$, $[h_1][h_2] = [h_1 h_2]$. Clearly these operations are well defined. Given $R(n) \in K(n)$ and $h : \mathbb{N} \to K$, we can also define a germ $[Rh]$ by requiring that $[Rh] = [h_1]$, where $h_1 : \mathbb{N} \to K$ is any function agreeing with Rh for those n for which $R(n)$ is defined. In this way the space $\mathcal{G}$ of germs acquires the structure of a vector space over the field $K(n)$. Finally note that if $h_1 \sim h_2$, then $h_1 \in \mathcal{P}$ if and only if $h_2 \in \mathcal{P}$ by Proposition 6.4.5. Hence we may speak of *P-recursive germs*. Thus a germ $[h]$ is P-recursive if and only if the $K(n)$-vector subspace

$$\mathcal{G}_h = \operatorname{span}_{K(n)}\{[h(n)], [h(n+1)], [h(n+2)], \ldots\}$$

of $\mathcal{G}$ is finite-dimensional. We are now ready to state and prove the desired result.

6.4.12 Theorem. *If $f, g : \mathbb{N} \to K$ are P-recursive, then so is the product fg. Equivalently, if $u, v \in \mathcal{D}$ then $u * v \in \mathcal{D}$.*

Proof. The above discussion shows that it suffices to prove that if $[f]$ and $[g]$ are P-recursive germs, then so is $[fg]$. There is a unique linear transformation

$$\phi : \mathcal{G}_f \otimes_{K(n)} \mathcal{G}_g \to \mathcal{G}$$

that satisfies

$$\phi([f(n+i)] \otimes [g(n+j)]) = [f(n+i)][g(n+j)] = [f(n+i)g(n+j)].$$

Clearly the image of ϕ contains $\mathcal{G}_{fg} = \mathrm{span}_K\{[f(n)g(n)], [f(n+1)g(n+1)], \ldots\}$. Hence (taking dimensions over $K(n)$),

$$\dim \mathcal{G}_{fg} \leq \dim(\mathcal{G}_f \otimes \mathcal{G}_g) = (\dim \mathcal{G}_f)(\dim \mathcal{G}_g) < \infty,$$

so fg is P-recursive. □

6.5 Noncommutative Generating Functions

A powerful tool for showing that power series $\eta \in K[[x]]$ or $\eta \in K[[x_1, \ldots, x_m]]$ are rational or algebraic is the theory of *noncommutative* formal series (in several variables). The connections with rational power series is more or less equivalent to the transfer-matrix method (Section 4.7) and so won't yield any really new (commutative) rational generating functions. Similarly, it is possible to develop an analogue of the transfer-matrix method for (commutative) algebraic generating functions, so that noncommutative series are not really needed. However, the noncommutative approach to both rational and algebraic commutative generating functions yields an elegant and natural conceptual framework which can greatly simplify complicated computations. We will give an overview of both rational and algebraic noncommutative series. In one nice application (Theorem 6.7.1) we will use the theory of both rational and algebraic series. We will adhere to standard terminology and notation in this area, though it will be slightly different from our previous terminology and notation involving commutative series.

Let K denote a fixed field. (Much of the theory can be developed over an arbitrary "semiring" (essentially a ring without additive inverses, such as $\mathbb{N}$), and that generalization has some interesting features, but for our purposes a field will suffice.) Let X be a set, called an *alphabet*, and let X^* be the free monoid generated by X, as defined in Section 4.7. Thus X^* consists of all finite strings (including the empty string 1) $w_1 \cdots w_n$ of letters $w_i \in X$. We write $|w| = n$ if $w = w_1 \cdots w_n \in X^*$, with each $w_i \in X$. Also define $X^+ = X^* \backslash 1$ $(= X^* - \{1\})$.

6.5.1 Definition. A *formal* (*power*) *series* in X (over K) is a function $S : X^* \to K$. We write $\langle S, w \rangle$ for $S(w)$ and then write

$$S = \sum_{w \in X^*} \langle S, w \rangle w.$$

The set of all formal series in X is denoted $K\langle\langle X \rangle\rangle$.

The set $K\langle\langle X \rangle\rangle$ has the obvious structure of a ring (or even a K-algebra) with identity 1. (We identify $1 \in X^*$ with $1 \in K$, and abbreviate the term $\alpha \cdot 1$ of the above series S as α.) Addition is componentwise, i.e.,

$$S + T = \sum_w (\langle S, w\rangle + \langle T, w\rangle)w,$$

while multiplication is given by the usual power series product, taking into account the noncommutativity of the variables. Thus

$$\begin{aligned}\left(\sum \langle S, u\rangle u\right)\left(\sum \langle T, v\rangle v\right) &= \sum_{u,v} \langle S, u\rangle\langle T, v\rangle uv \\ &= \sum_w \left(\sum_{uv=w} \langle S, u\rangle\langle T, v\rangle\right) w.\end{aligned}$$

Algebraically inclined readers can think of $K\langle\langle X\rangle\rangle$ as the completion of the monoid algebra of the free monoid X^* with respect to the ideal generated by X.

There is an obvious notion of *convergence* of a sequence $S_1, S_2, \ldots$ of formal series (and hence of a sum $\sum_{n\geq 0} T_n$) analogous to the commutative case (Section 1.1). Namely, we say that $S_1, S_2, \ldots$ converges to S if for all $w \in X^*$ the sequence $\langle S_1, w\rangle, \langle S_2, w\rangle, \ldots$ has only finitely many terms unequal to $\langle S, w\rangle$. Suppose now that $\langle S, 1\rangle = \alpha \neq 0$. Let

$$T = \frac{1}{\alpha}\sum_{n\geq 0}\left(1 - \frac{S}{\alpha}\right)^n.$$

This sum converges formally, and it is easy to check that $ST = TS = 1$. Hence $T = S^{-1}$ in $K\langle\langle X\rangle\rangle$. For instance,

$$\left(1 - \sum_{x\in X} x\right)^{-1} = \sum_{w\in X^*} w.$$

There are two subalgebras of $K\langle\langle X\rangle\rangle$ with which we will be concerned in this section.

6.5.2 Definition. (a) A (noncommutative) *polynomial* is a series $S \in K\langle\langle X\rangle\rangle$ that is a *finite* sum $\sum \langle S, w\rangle w$. The set of polynomials $S \in K\langle\langle X\rangle\rangle$ forms a subalgebra of $K\langle\langle X\rangle\rangle$ denoted $K\langle X\rangle$ (or sometimes $K_{\mathrm{pol}}\langle\langle X\rangle\rangle$).
(b) A (noncommutative) *rational* series is an element of the smallest subalgebra $K_{\mathrm{rat}}\langle\langle X\rangle\rangle$ of $K\langle\langle X\rangle\rangle$ containing $K\langle X\rangle$ (or equivalently, containing X) such that if $S \in K_{\mathrm{rat}}\langle\langle X\rangle\rangle$ and S^{-1} exists, then $S^{-1} \in K_{\mathrm{rat}}\langle\langle X\rangle\rangle$.

An example of a polynomial (when $K = \mathbb{Q}$, $X = \{x, y, z\}$) is $x^2z - xz^2 - 3x^5yxz^2 + \frac{2}{3}zyzy^2z^2$. An example of a rational series is

$$\begin{aligned}S = {}&[(1+x)^{-1} + y]^{-1}[x^2 - y^2xy(1 - xyz)^{-1} \\ &\times z(1 + xyxzx^2 + 2y^2z^3xyx)^{-1}zy^5z] + x.\end{aligned}$$

Note that notation such as $\frac{x}{1-y}$ is ambiguous; it could mean either $x(1-y)^{-1}$ or $(1-y)^{-1}x$. Note also that there does not exist a notion of "common denominator" for noncommutative series. For instance, there is no polynomial S satisfying

$$S[(1-x)^{-1}+(1-y)^{-1}] \in K\langle x, y\rangle,$$

or even polynomials S and T satisfying

$$S[(1-x)^{-1}+(1-y)^{-1}+(1-z)^{-1}]T \in K\langle x, y, z\rangle.$$

In particular, not every rational series is a quotient of two polynomials.

Let $\phi : K\langle\langle X\rangle\rangle \to K[[X]]$ be the continuous algebra homomorphism defined by $\phi(x) = x$ for all $x \in X$. Thus $\phi(S)$ is the "abelianization" of S, and the kernel of ϕ is the two-sided ideal of $K\langle\langle X\rangle\rangle$ generated by $\{xy - yx : x, y \in X\}$. Note that if $S \in K\langle X\rangle$ then $\phi(S) \in K[X]$. The converse is clearly false, e.g., if $S = \sum_{n\geq 0}(x^n y^n - y^n x^n)$ then $\phi(S) = 0 \in K[x, y]$, but $S \notin K\langle X\rangle$. Similarly if $S \in K_{\text{rat}}\langle\langle X\rangle\rangle$ then clearly $\phi(S) \in K_{\text{rat}}[[X]] = K[[X]] \cap K(X)$. Again the converse is false, but an example is not so obvious, since at this point we have no easy way to recognize when a series S is not rational (other than the condition that $\phi(S)$ is not rational). Exercise 6.65 will give us a method for showing that many series are not rational. For instance, $S = \sum_{n\geq 0} x^n y^n \notin K_{\text{rat}}\langle\langle x, y\rangle\rangle$, though $\phi(S) = 1/(1-xy) \in K_{\text{rat}}[[x, y]]$.

We next define a class of series of crucial importance in understanding rational series. We let $K^{n\times n}$ denote the monoid of all $n \times n$ matrices over K under the usual multiplication of matrices.

6.5.3 Definition. A series $S \in K\langle\langle X\rangle\rangle$ is *recognizable* if there exists a positive integer n and a homomorphism of monoids

$$\mu : X^* \to K^{n\times n},$$

as well as two matrices $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ (so λ is a row vector and γ a column vector) such that for all $w \in X^+$ we have

$$\langle S, w\rangle = \lambda \cdot \mu(w) \cdot \gamma. \tag{6.41}$$

Note. Equation (6.41) is only required to hold for $w \in X^+$, not $w \in X^*$. In other words, the property that a series S is recognizable does not depend on the constant term $\langle S, 1\rangle$ of S.

Note. If $\lambda \neq [0, 0, \ldots, 0]$ and $\gamma \neq [0, 0, \ldots, 0]^t$ (where ^{t} denotes transpose), then it is easy to see that we can find an invertible matrix $A \in K^{n\times n}$ such that $\lambda = [1, 0, \ldots, 0]A$ and $\gamma = A^{-1}[0, 0, \ldots, 1]^t$. If we define a new homomorphism

$\mu' : X^* \to K^{n\times n}$ by $\mu'(w) = A\mu(w)A^{-1}$, then

$$\langle S, w\rangle = \mu'(w)_{1n}, \tag{6.42}$$

the $(1, n)$ entry of $\mu'(w)$. Hence we may replace (6.41) by the stronger condition (6.42).

6.5.4 Example. Suppose that $X = \{x, y\}$ and that $\mu : X^* \to K^{2\times 2}$ is defined by

$$\mu(x) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} := a, \qquad \mu(y) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} := b.$$

The series B given by $\langle B, w\rangle = \mu(w)_{12}$ is recognizable by Definition 6.5.3. Let us see if we can obtain a formula for B. Define, for $w \in X^+$,

$$\mu(w) = \begin{bmatrix} A_w & B_w \\ C_w & D_w \end{bmatrix}, \tag{6.43}$$

and define series $A = \sum A_w w$, etc. Also set $\mu(1) = I$, the 2×2 identity matrix. Thus A_w is short for $\langle A, w\rangle$, etc., and the series B defined by (6.43) coincides with the definition of B preceding (6.43). Note that

$$\mu(wx) = \mu(w)a = \begin{bmatrix} A_w + 2B_w & B_w \\ * & * \end{bmatrix} \tag{6.44}$$

$$\mu(wy) = \mu(w)b = \begin{bmatrix} A_w & A_w + B_w \\ * & * \end{bmatrix}, \tag{6.45}$$

where $*$'s denote entries that turn out to be irrelevant. From (6.44) and (6.45) there follows

$$\begin{aligned} A_{wx} &= A_w + 2B_w, & B_{wx} &= B_w \\ A_{wy} &= A_w, & B_{wy} &= A_w + B_w. \end{aligned}$$

Hence

$$\begin{aligned} A &= 1 + \sum_w A_{wx} wx + \sum_w A_{wy} wy \\ &= 1 + \sum_w (A_w + 2B_w) wx + \sum_w A_w wy \\ &= 1 + A(x + y) + 2Bx. \end{aligned}$$

Similarly $B = Ay + B(x+y)$. Thus we obtain two linear equations in two unknowns A and B, viz.,

$$\begin{aligned} A(1-x-y) - 2Bx \qquad\qquad &= 1 \\ -Ay \qquad\qquad + \ B(1-x-y) &= 0. \end{aligned}$$

We now solve these equations, essentially by "noncommutative" Gaussian elimination. Since the unknowns A and B are only multiplied on the right and since the diagonal coefficients $1-x-y$ and $1-x-y$ are invertible, there will be no difficulty in carrying out the elimination. Multiply the first equation on the right by $(1-x-y)^{-1}y$ and add it to the second equation to get the following formula for B:

$$B = (1-x-y)^{-1}y[1-x-y-2x(1-x-y)^{-1}y]^{-1}. \qquad (6.46)$$

Note that

$$\begin{aligned} \phi(B) &= \frac{y}{(1-x-y)\left(1-x-y-\frac{2xy}{1-x-y}\right)} \\ &= \frac{y}{(1-x-y)^2-2xy}. \end{aligned}$$

We can also compute $\phi(B)$ directly from a and b using Theorem 4.7.2. We have

$$\begin{aligned} \phi(B) &= \left.\sum_{n\geq 0}(ax+by)^n\right|_{12} \\ &= (1-ax-by)^{-1}|_{12} \\ &= \left.\begin{bmatrix} 1-x-y & -y \\ -2x & 1-x-y \end{bmatrix}^{-1}\right|_{12}. \end{aligned}$$

Writing

$$c = \begin{bmatrix} 1-x-y & -y \\ -2x & 1-x-y \end{bmatrix},$$

there follows (using the notation of Theorem 4.7.2)

$$\begin{aligned} \phi(B) &= \frac{-\det[c:2,1]}{\det c} \\ &= \frac{y}{(1-x-y)^2-2xy}, \end{aligned}$$

as before.

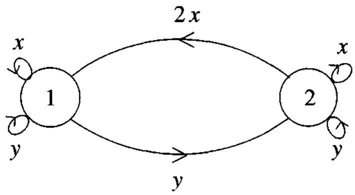


Figure 6-4. A directed graph Γ_μ.

There is an alternative graph-theoretic way to view recognizable series. Let $\mu : X^* \to K^{n\times n}$ be as in Definition 6.5.3, where we assume λ and γ are such that $\lambda \cdot \mu(w) \cdot \gamma = \mu(w)_{1n}$ as in (6.42). Define a directed graph Γ_μ with weighted edges as follows: Let $X = \{x_1, \dots, x_r\}$, and let Γ_μ have vertex set $V = V(\Gamma_\mu) = [n]$. For each triple $(i, j, k) \in [r]^3$, draw an edge e from i to j labeled $\omega(e) = \mu(x_k)_{ij} x_k$, where $\mu(x_k)_{ij}$ denotes the (i, j) entry of the matrix $\mu(x_k)$. Thus there are r edges from vertex i to vertex j (though edges labeled 0 may be suppressed). In the case of Example 6.5.4, the digraph Γ_μ is given by Figure 6-4.

Let P be a walk in Γ_μ of length m from i to j, say $i = i_0, e_1, i_1, e_2, \dots, i_{m-1}, e_m, i_m = j$. Define the *weight* of P by

$$\omega(P) = \omega(e_1)\omega(e_2)\cdots\omega(e_m) = \kappa(P) x_{i_1} \cdots x_{i_m},$$

a noncommutative monomial in the variables X, multiplied by some scalar $\kappa(P) \in K$. The scalar $\kappa(P)$ is just a term in the (i, j) entry of $\mu(x_{i_1}) \cdots \mu(x_{i_m})$, by definition of matrix multiplication. Hence the series S defined by $\langle S, w\rangle = \mu(w)_{1n}$ is also given by

$$S = \sum_P \omega(P), \tag{6.47}$$

summed over all walks from 1 to n.

6.5.5 Example. The walks from 1 to 2 of length at most two in Figure 6-4 are given by

$$1 \xrightarrow{y} 2$$

$$1 \xrightarrow{x} 1 \xrightarrow{y} 2$$

$$1 \xrightarrow{y} 1 \xrightarrow{y} 2$$

$$1 \xrightarrow{y} 2 \xrightarrow{x} 2$$

$$1 \xrightarrow{y} 2 \xrightarrow{y} 2$$

Hence the formal series B of Example 6.5.4 begins

$$B = y + xy + yx + 2y^2 + \cdots.$$

With a little practice one can see by inspection that the series B of (6.46) is also given by (6.47). The first factor $(1-x-y)^{-1}$ in (6.46) corresponds to the initial part of the walk P before it leaves vertex 1. We can walk along the loops at vertex 1 labeled x and y in any desired order. The factor y in (6.46) corresponds to the first step from vertex 1 to vertex 2. Now we are free to walk along the loops at 2 in any order (accounting for the terms $-x-y$ in the third factor of (6.46)), then to move back to 1 (the factor $2x$ of the term $2x(1-x-y)^{-1}y$ of the third factor in (6.46)), then to walk along the loops at 1 (the factor $(1-x-y)^{-1}$ of $2x(1-x-y)^{-1}y$), then to move back to 2 (the factor y of $2x(1-x-y)^{-1}y$), and then to iterate the procedure that begins at vertex 2.

The above discussion shows that the theory of recognizable series is essentially the same as the transfer-matrix method of Section 4.7, except that we must keep track of the actual walks (i.e., the order of their edges), and not just their unordered (commutative) weights. One might say that a recognizable series is simply the generating function for (weighted) walks in a digraph. One can also view a graph such as Figure 6-4 as a kind of finite-state machine (automaton) for producing the series S. We will not say more about this point of view here, though it can be a fruitful way of looking at recognizable series.

Before stating the main theorem on rational series, we need one simple lemma for ensuring that certain series are rational.

6.5.6 Lemma. *Suppose that $B_1, \ldots, B_n$ are formal series satisfying n linear equations of the form*

$$\begin{aligned}
B_1(1+c_{11}) + B_2c_{12} \quad &+ \cdots + B_nc_{1n} &&= d_1\\
B_1c_{21} \quad + B_2(1+c_{22}) &+ \cdots + B_nc_{2n} &&= d_2\\
& &&\ \vdots\\
B_1c_{n1} \quad + B_2c_{n2} \quad &+ \cdots + B_n(1+c_{nn}) &&= d_n,
\end{aligned}$$

where each c_{ij} is a rational series with zero constant term, and where each d_j is a rational series. Then $B_1, \ldots, B_n$ are rational series (and are the unique series satisfying the above system of linear equations).

Proof. Induction on n. When $n=1$ we have $B_1 = d_1(1+c_{11})^{-1}$, as desired. Now assume the result for $n-1$. Multiply the first equation on the right by $-(1+c_{11})^{-1}c_{j1}$ and add to the jth equation, for $2 \le j \le n$. We obtain a system of $n-1$ equations for $B_2, \ldots, B_n$ satisfying the hypotheses of the lemma, so $B_2, \ldots, B_n$ are rational (and unique). By symmetry (or by using the first equation to solve for B_1), we get that B_1 is also rational (and unique). □

We are now ready to state the main theorem on rational series.

6.5.7 Theorem (Fundamental theorem of rational formal series). *A formal series $S \in K\langle\langle X\rangle\rangle$ is rational if and only if it is recognizable.*

Proof (sketch). Assume S is recognizable. The proof that S is rational parallels the computation of Example 6.5.4. Let $\mu : X^+ \to K^{n\times n}$ be a homomorphism of monoids satisfying $\langle S, w\rangle = \mu(w)_{1n}$ for all $w \in X^+$. Set $\mu(1) = I$, the $n \times n$ identity matrix. Define series

$$A_{ij} = \sum_{w\in X^+} \mu(w)_{ij} w,$$

for $(i, j) \in [n] \times [n]$. If $x_k \in X$, then let $a^k = \mu(x_k)$ and $a^k_{ij} = \mu(x_k)_{ij}$, the (i, j) entry of the matrix a^k. Then $\mu(wx_k) = \mu(w)a^k$, so

$$\mu(wx_k)_{ij} = \sum_t \mu(w)_{it} a^k_{tj}.$$

Multiplying by wx_k and summing on w and k yields when $i = 1$ the equations

$$A_{1j} = \delta_{1j} + \sum_t \sum_k a^k_{tj} A_{1t} x_k,$$

or equivalently

$$\sum_t A_{1t}\left(\delta_{tj} - \sum_k a^k_{tj} x_k\right) = \delta_{1j}, \qquad 1 \le j \le n. \tag{6.48}$$

This is a system of n linear equations in the n unknowns $A_{11}, A_{12}, \dots, A_{1n}$, which has the form given by Lemma 6.5.6. Hence A_{1n} (as well as $A_{11}, A_{12}, \dots, A_{1,n-1}$) is rational, as desired.

To show conversely that rational series are recognizable, we must show the following four facts.

(i) If $x_i \in X$, then the series x_i is recognizable.
(ii) If S and T are recognizable and $\alpha, \beta \in K$, then $\alpha S + \beta T$ is recognizable.
(iii) If S and T are recognizable, then ST is recognizable.
(iv) If S is recognizable and $\langle S, 1\rangle \neq 0$, then S^{-1} is recognizable.

Fact (i) is trivial, while the remaining three facts are all proved by constructing appropriate homomorphisms $X^* \to K^{n\times n}$. The details are tedious and will not be given here. □

6.6 Algebraic Formal Series

In this section we consider an important generalization of the class of rational series. As before, we have a fixed (finite) alphabet X.

6.6.1 Definition. Let $Z = \{z_1, \dots, z_n\}$ be an alphabet disjoint from X. A *proper algebraic system* is a set of equations $z_i = p_i$, $1 \le i \le n$, where:

(a) $p_i \in K\langle X, Z\rangle$ (i.e., p_i is a polynomial in the alphabet $X \cup Z$);
(b) $\langle p_i, 1\rangle = 0$ and $\langle p_i, z_j\rangle = 0$ (i.e., p_i has no constant term and no terms $c_j z_j$, $0 \neq c_j \in K$).

We sometimes also call the n-tuple $(p_1, \ldots, p_n)$ a *proper algebraic system.*

A *solution* (sometimes called a *strong solution*) to a proper algebraic system $(p_1, \ldots, p_n)$ is an n-tuple $(R_1, \ldots, R_n) \in K\langle\langle X\rangle\rangle^n$ of formal series in X with zero constant term satisfying

$$R_i = p_i(X, Z)_{z_i=R_i}. \tag{6.49}$$

(The series $p_i(X, Z)_{z_i=R_i}$ will be formally well defined, since $\langle R_i, 1\rangle = 0$ by assumption.) Each R_i is called a *component* of the system $(p_1, \ldots, p_n)$.

6.6.2 Example. If $X = \{x, y\}$ and $Z = \{z\}$, then

$$z = xy + xzy$$

is a proper algebraic system with solution

$$R = \sum_{n\geq 1} x^n y^n. \tag{6.50}$$

Compare the system $z = xy + zxy$, with solution $R = \sum_{n\geq 1}(xy)^n = (1-xy)^{-1} - 1$, a rational series. It can be shown that the series (6.50) is not rational (see Exercise 6.65).

6.6.3 Proposition. *Every proper algebraic system $(p_1, \ldots, p_n)$ has a unique solution $R = (R_1, R_2, \ldots, R_n) \in K\langle\langle X\rangle\rangle^n$.*

Idea of Proof. The method of proof is "successive approximation." Define the first approximation $S^{(1)} = (S_1^{(1)}, \ldots, S_n^{(1)})$ to a solution by $S_i^{(1)} = p_i(X, Z) \in K\langle X, Z\rangle$. Now assuming that the k-th approximation $S^{(k)} = (S_1^{(k)}, \ldots, S_n^{(k)}) \in K\langle X, Z\rangle^n$ has been defined, let

$$S_i^{(k+1)} = p_i\big(X, S^{(k)}\big)$$

(i.e., substitute $S_j^{(k)}$ for Z_j in $p_i(X, Z)$). It is straightforward to verify from the definition of proper algebraic system that $\lim_{k\to\infty} S^{(k)}$ converges formally to a solution $R \in K\langle\langle X\rangle\rangle^n$, and that this solution must be unique.

As a simple example with just one equation, consider the proper algebraic system

$$z = x + xzy + yz.$$

The first approximation is

$$S^{(1)} = x + xzy + yz.$$

The second approximation is

$$\begin{aligned} S^{(2)} &= x + x(x + xzy + yz)y + y(x + xzy + yz) \\ &= x + x^2y + x^2zy^2 + xyzy + yx + yxzy + y^2z. \end{aligned}$$

The third approximation is

$$\begin{aligned} S^{(3)} &= x + xS^{(2)}y + yS^{(2)} \\ &= x + x^2y + x^3y^2 + xyxy + yx + yx^2y + y^2x + \text{ terms involving } z. \end{aligned}$$

This last approximation agrees with the solution R in all terms of degree at most three (and with some nonzero terms of higher degree).

NOTE (for logicians). Let $(p_1, \ldots, p_m)$ and $(p'_1, \ldots, p'_n)$ be proper algebraic systems with solutions $(R_1, \ldots, R_m)$ and $(R'_1, \ldots, R'_n)$, respectively. It is undecidable whether $R_1 = R'_1$, or in particular whether $R_1 = 0$. (One difficulty is that substituting $z_1 = 0$ in the system $z_1 = p_1, \ldots, z_m = p_m$ may yield a system which isn't proper.) On the other hand, it *is* decidable whether the abelianizations $\phi(R_1)$ and $\phi(R'_1)$ are equal.

6.6.4 Definition. (a) A series $S \in K\langle\langle X\rangle\rangle$ is *algebraic* if $S - \langle S, 1\rangle$ is a component of a proper algebraic system. The set of all algebraic series $S \in K\langle\langle X\rangle\rangle$ is denoted $K_{\text{alg}}\langle\langle X\rangle\rangle$.
(b) The *support* of a series $S = \sum \langle S, w\rangle w \in K\langle\langle X\rangle\rangle$ is defined by

$$\operatorname{supp}(S) = \{w \in X^* : \langle S, w\rangle \neq 0\}.$$

A *language* is a subset of X^*. A language L is said to be *rational* (respectively, *algebraic*) if it is the support of a rational (respectively, algebraic) series. A rational language is also called *regular*, and an algebraic language is also called *context-free*.

Our previous example (Example 6.6.2) of a proper algebraic system yields the algebraic series $\sum x^n y^n$ and $\sum (xy)^n$. Let us consider some examples of greater interest.

6.6.5 Example. Rational series are algebraic. The system (6.48) of linear equations is equivalent to a proper algebraic system in the unknowns $A_{11} - 1, A_{12}, \ldots, A_{1n}$.

6.6.6 Example. The *Dyck language* D is the subset of $\{x, y\}^*$ such that if x is replaced by a left parenthesis and y by a right parenthesis, then we obtain a sequence of properly nested parentheses. Equivalently, a word $w_1w_2\cdots w_m$ is in D (where $w_i = x$ or y) if for all $1 \leq j \leq m$ the number of x's among $w_1w_2\cdots w_j$ is at least as great as the number of y's among $w_1w_2\cdots w_j$, and the total number of x's is equal to the total number of y's (so m is even). An element w of D is called a *Dyck word*. The Dyck words of length six or less are given by

$$1 \quad xy \quad x^2y^2 \quad xyxy \quad x^3y^3 \quad x^2yxy^2 \quad x^2y^2xy \quad xyx^2y^2 \quad xyxyxy.$$

By Corollary 6.2.3(ii) or (iii), it follows that the number of words of length $2n$ in D is given by the Catalan number C_n. Now note the following key recursive property of a Dyck word w: If w is nonempty, then it begins with an x, followed by a Dyck word, then by a y (the right parenthesis matching the initial x), and finally by another Dyck word. Thus D is a solution to the system

$$z = 1 + xzyz,$$

and so $D^+ = D - 1$ is a solution to the proper algebraic system

$$\begin{aligned} z' &= x(z'+1)y(z'+1) \\ &= xy + xyz' + xz'y + xz'yz'. \end{aligned} \tag{6.51}$$

It follows from (6.51) that the Dyck language D is algebraic. It is perhaps the most important algebraic language for enumerative combinatorics. Many enumerative problems can be expressed in terms of the Dyck language, e.g., many of the parts of Exercise 6.19.

6.6.7 Example. Let $X = \{x_0, x_1, \ldots, x_m\}$. Define a weight $\omega : X \to \mathbb{Z}$ by $\omega(x_i) = i - 1$. Define a language $L \subset X^*$ by

$$\begin{aligned} L = \{x_{i_1}x_{i_2}\cdots x_{i_k} :\ & \omega(x_{i_1}) + \cdots + \omega(x_{i_j}) \geq 0 \text{ if } j < k, \\ & \text{and } \omega(x_{i_1}) + \cdots + \omega(x_{i_k}) = -1\}. \end{aligned}$$

Thus L consists of all words that encode plane trees of maximum degree m, as discussed in Section 5.3 (see Lemma 5.3.9). The language L is called the *Łukasiewicz language*, and its elements are *Łukasiewicz words*. Note that a Łukasiewicz word in the letters x_0 and x_2 is just a Dyck word with x replaced by x_2 and y by x_0, with an x_0 appended at the end. It is easily verified by inspection (essentially the definition of a plane tree) that L^+ is a solution (or component) of the proper algebraic system

$$z = x_0 + x_1z + x_2z^2 + \cdots + x_mz^m.$$

Hence the Łukasiewicz language is algebraic.

The next example will be of use to us later (Theorem 6.7.1) when we consider noncommutative diagonals. It is a good example of the nonobvious way in which auxiliary series may need to be introduced in order to obtain a proper algebraic system for which some given series S is a component.

6.6.8 Example. Let $X = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$, and let $\Delta \subset X^*$ be the set of those elements of X^* that reduce to the identity under the relations

$$x_iy_i = y_ix_i = 1, \qquad 1 \leq i \leq k.$$

In other words, if in the word $w \in X^*$ we replace y_i by x_i^{-1}, then we obtain the identity element of the free group generated by $x_1, \ldots, x_k$. Thus for example when $k = 3$ we have

$$x_1^2\, y_2^2\, x_2\, x_3\, y_3\, x_2\, y_1\, x_2\, y_2\, y_1 \in \Delta.$$

We claim that Δ is algebraic. If $t \in X$ let

$$G_t = \{w \in \Delta : w = tv,\ w \neq uu' \text{ for } u, u' \in \Delta^+\}.$$

Write

$$\bar{t} = \begin{cases} x_i & \text{if } t = y_i \\ y_i & \text{if } t = x_i. \end{cases}$$

Any word $w \in G_t$ must end in $\bar{t}$ [why?], so we can define a series (or language) B_t by $G_t = tB_t\bar{t}$. It is then not difficult to verify that

$$\begin{aligned} \Delta &= 1 + \Delta \sum_{t \in X} G_t \\ G_t &= tB_t\bar{t} \\ B_t &= 1 + B_t \sum_{\substack{q \in X \\ q \neq \bar{t}}} G_q. \end{aligned}$$

From these formulas we see that

$$\begin{aligned} \Delta^+ &= (\Delta^+ + 1) \sum_{t \in X} t\big(B_t^+ + 1\big)\bar{t} \\ B_t^+ &= \big(B_t^+ + 1\big) \sum_{\substack{q \in X \\ q \neq \bar{t}}} q\big(B_q^+ + 1\big)\bar{q}, \qquad t \in X. \end{aligned}$$

Hence $(\Delta^+, (B_t^+)_{t \in X})$ is a solution to a proper algebraic system, so Δ is algebraic.

We now want to relate algebraic formal series to *commutative* algebraic generating functions as discussed in Sections 6.1–6.3. We need a standard result in the theory of extension fields, which we simply state without proof.

6.6.9 Lemma. *Suppose that $\alpha_1, \ldots, \alpha_n$ belong to an extension field of a field K. Suppose also that there exist polynomials $f_1, \ldots, f_n \in K[X]$, where $X = (x_1, \ldots, x_n)$, satisfying:*

(i) $f_i(\alpha_1, \ldots, \alpha_n) = 0, \quad 1 \leq i \leq n,$
(ii) $\det(\partial f_i / \partial \alpha_j) \neq 0.$

Then each α_i is algebraic (in fact, separably algebraic, though separability is irrelevant here) over K.

Let us give a couple of examples showing the significance of the conditions (i) and (ii) above. If $f_1 = x_1 - x_2$ and $f_2 = (x_1 - x_2)^2$, then condition (i) is satisfied for *any* $\alpha_1 = \alpha_2$, but $\det(\partial f_i/\partial \alpha_j) = 0$. In fact, the Jacobian determinant $\det(\partial f_i/\partial x_j)$ is identically zero, since the polynomials f_1 and f_2 are algebraically dependent. Now let $f_1 = x_1$, $f_2 = x_1 x_2$, so f_1 and f_2 are algebraically independent and $\det(\partial f_i/\partial x_j) = x_1 \neq 0$. The solutions (α_1, α_2) to $f_1 = f_2 = 0$ are $(0, \alpha_2)$ for *any* α_2, so α_2 need not be algebraic over K. This does not contradict Lemma 6.6.9, since $\det(\partial f_i/\partial \alpha_j) = 0$.

6.6.10 Theorem. *Let $S \in K_{\mathrm{alg}}\langle\langle X\rangle\rangle$, where X is a finite alphabet. Then $\phi(S)$ is algebraic over the field $K(X)$ of rational functions in (the commuting variables) X.*

Proof. We can assume $\langle S, 1\rangle = 0$. Let $(p_1, \dots, p_n)$ be a proper algebraic system with solution $(S_1 = S, S_2, \dots, S_n)$. Let $\eta_i = \phi(S_i)$ and $\eta = (\eta_1, \dots, \eta_n)$. Let $Z = (z_1, \dots, z_n)$ and

$$f_i(Z) = z_i - p_i(X, Z).$$

Hence, regarding the p_i as *commutative* polynomials, the η_i satisfy

$$f_i(\eta) = \eta_i - p_i(X, \eta) = 0.$$

The Jacobian matrix at $Z = \eta$ is given by

$$J = \left(\frac{\partial f_i}{\partial \eta_j}\right) = I - \left(\frac{\partial p_i}{\partial \eta_j}\right).$$

By definition of proper algebraic system (and because $\langle S_i, 1\rangle = 0$), all entries $\partial p_i/\partial \eta_j$ belong to the maximal ideal $XK[[X]] = x_1 K[[X]] + \cdots + x_k K[[X]]$ of the ring $K[[X]]$. Hence $\det J = 1 + m$, where $m \in XK[[X]]$. Thus $\det J \neq 0$, so by Lemma 6.6.9 the η_i's are algebraic over $K(X)$. □

6.6.11 Example. Let D be the Dyck language of Example 6.6.6. We saw in that example that D is algebraic. If C_n is the number of Dyck words of length $2n$ (which we know is a Catalan number) then $\phi(D) = \sum_{n\geq 0} C_n x^n y^n$. Hence by Theorem 6.6.10 (substituting x for xy) we get that $\sum_{n\geq 0} C_n x^n$ is algebraic directly from the combinatorial structure of the Dyck language.

Our final topic in this section will be the Hadamard product. If $S = \sum \langle S, w\rangle w$ and $T = \sum \langle T, w\rangle w$ are two formal series (over the same alphabet X), then define, just as in the commutative case, the *Hadamard product*

$$S * T = \sum \langle S, w\rangle \langle T, w\rangle w.$$

(The notation $S \odot T$ is also used.) We will need the following result in the next section. It is the noncommutative analogue of Proposition 6.1.11. (For a related result, see Exercise 6.66).

6.6.12 Proposition. *If $S \in K_{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $T \in K_{\mathrm{alg}}\langle\langle X\rangle\rangle$, then $S * T \in K_{\mathrm{alg}}\langle\langle X\rangle\rangle$.*

Proof (sketch). Let $z_i = p_i$, $1 \le i \le n$, be a proper algebraic system with solution $z_1 = T$. Since S is rational, it is recognizable by Theorem 6.5.7. Hence for some $m \ge 1$ there is a monoid homomorphism $\mu : X^* \to K^{m\times m}$ such that

$$\langle S, w\rangle = \mu(w)_{1m} \qquad \text{for all } w \in X^+.$$

Introduce nm^2 new variables z_i^{jk}, $(i, j, k) \in [n] \times [m] \times [m]$. Let $Z' = \{z_i^{jk} : (i, j, k) \in [n] \times [m] \times [m]\}$, and let M_i be the $m \times m$ matrix with (j, k) entry given by $(M_i)_{jk} = z_i^{jk}$. For each $(i, j, k) \in [n] \times [m] \times [m]$, construct a polynomial $p_i^{jk} \in K\langle X \cup Z'\rangle$ as follows: Replace in p_i each letter z_t by the matrix M_t and each letter $x \in X$ by the matrix $\mu(x)x$ (scalar multiplication of $\mu(x)$ by x). Performing all matrix additions and multiplications involved in p_i transforms p_i into an $m \times m$ matrix. Let p_i^{jk} be the (j, k) entry of this matrix.

Now define a system $\mathcal{S}$ by

$$z_i^{jk} = p_i^{jk}, \qquad (i, j, k) \in [n] \times [m] \times [m].$$

Clearly $\mathcal{S}$ is proper. Moreover, it's not hard to check that if the original system $z_i = p_i$ has the solution $(z_1, \ldots, z_n) = (t_1 = T, t_2, \ldots, t_n)$, then for all $w \in X^*$ the solution $(z_i^{jk}) = (s_i^{jk})$ to $\mathcal{S}$ satisfies

$$\left\langle s_i^{jk}, w\right\rangle = \mu(w)_{jk}\langle t_i, w\rangle.$$

In particular,

$$\begin{aligned}\left\langle s_1^{1m}, w\right\rangle &= \mu(w)_{1m}\langle t_1, w\rangle\\ &= \langle S, w\rangle\langle T, w\rangle.\end{aligned}$$

Hence $S * T \in K_{\mathrm{alg}}\langle\langle X\rangle\rangle$. □

6.6.13 Example. Let $X = \{x_1, x_2\}$, and let

$$\mu(x_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} := a, \qquad \mu(x_2) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} := b.$$

It's easy to check that $(a^r b^r)_{12} = r$ for all $r \ge 1$, so the series S of the previous proposition satisfies $\langle S, x_1^r x_2^r\rangle = r$. Define T to be the solution to the proper algebraic system

$$z = x_1 z x_2 - x_1 x_2,$$

so $T = -\sum_{r\ge1} x_1^r x_2^r$ and $S * T = -\sum_{r\ge1} r x_1^r x_2^r$. The system $\mathcal{S}$ of the previous proof is defined by

$$\begin{bmatrix} z^{11} & z^{12} \\ z^{21} & z^{22} \end{bmatrix} = \begin{bmatrix} x_1 & x_1 \\ 0 & x_1 \end{bmatrix}\begin{bmatrix} z^{11} & z^{12} \\ z^{21} & z^{22} \end{bmatrix}\begin{bmatrix} -x_2 & 0 \\ 2x_2 & x_2 \end{bmatrix} - \begin{bmatrix} x_1 & x_1 \\ 0 & x_1 \end{bmatrix}\begin{bmatrix} -x_2 & 0 \\ 2x_2 & x_2 \end{bmatrix}.$$

In other words,

$$z^{11} = -x_1 z^{11} x_2 - x_1 z^{21} x_2 + 2x_1 z^{12} x_2 + 2x_1 z^{22} x_2 - x_1 x_2,$$

and similarly for z^{21}, z^{12}, z^{22}. The assertion of the previous proof is that the z^{12}-component is given by $s^{12} = -\sum_{r\geq 1} r x_1^r x_2^r$.

6.7 Noncommutative Diagonals

In this section we will use the theory of noncommutative formal series to show that certain (ordinary) Laurent series $\eta \in K((t))$ are algebraic. We will be dealing with series of the form

$$S(x_1, \ldots, x_k, y_1, \ldots, y_k) \in K(t)_{\mathrm{rat}}\langle\langle X, Y\rangle\rangle, \tag{6.52}$$

where $X = (x_1, \ldots, x_k)$, $Y = (y_1, \ldots, y_k)$. In other words, S is a rational series in X and Y with coefficients in the field $K(t)$ of rational functions in one variable t (commuting with X and Y). Since $K(t) \subset K((t))$ we can regard the coefficients of S as Laurent series in t. We will assume that S is such that the series $S(X, X^{-1}) := S(x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1})$ is well-defined, i.e., the coefficient of a Laurent monomial $x_{i_1}^{a_1} \cdots x_{i_j}^{a_j}$ $(a_1, \ldots, a_j \in \mathbb{Z})$ is a formal Laurent series $\eta \in K((t))$ in the variable t.

For instance (writing $x = x_1$ and $y = y_1$), if

$$S(x, y) = \sum_{n\geq 0} t^n (xy)^n = (1 - txy)^{-1},$$

then

$$S(x, x^{-1}) = \sum_{n\geq 0} t^n.$$

This is certainly a well-defined series; the coefficient of x^i for $i \neq 0$ is 0, while the coefficient of x^0 is the Laurent series (in fact, power series) $\sum_{n\geq 0} t^n$. For a more complicated example, let

$$\begin{aligned} S(X, Y) &= \sum_{n\geq 0} t^n (x_1 + \cdots + x_k + y_1 + \cdots + y_k)^n \\ &= [1 - t(x_1 + \cdots + x_k + y_1 + \cdots + y_k)]^{-1}. \end{aligned}$$

Fix a noncommutative Laurent monomial u in X, e.g., $u = x_1^{-2} x_2 x_3^{-1} x_1 x_3^2$. Then

$$[u]S(X, X^{-1}) = \sum_{n\geq 0} t^n [u]\big(x_1 + \cdots + x_k + x_1^{-1} + \cdots + x_k^{-1}\big)^n.$$

Since $[u](x_1 + \cdots + x_k + x_1^{-1} + \cdots + x_k^{-1})^n$ is simply an integer, we see that $[u]S(X, X^{-1})$ is a well-defined Laurent series for every u, so $S(X, X^{-1})$ is well defined. For an example of series $S(X, Y)$ for which $S(X, X^{-1})$ *isn't* well-defined, take for instance

$$S(x, y) = \sum_{n\geq 0} (xy)^n = (1 - xy)^{-1}.$$

Note that if $f(x, y)$ is a *commutative* power series, then

$$[x^0]f(x, tx^{-1}) = (\operatorname{diag} f)(t).$$

Hence, returning to the noncommutative case, we see that $[u]S(X, X^{-1})$ (where u is a Laurent monomial in $x_1, \ldots, x_k$) is a kind of "generalized noncommutative diagonal."

The main result of this section is given by the following theorem.

6.7.1 Theorem. *Let $S = S(X, Y)$ be given by (6.52). Suppose that $S(X, X^{-1})$ is well-defined formally, so for every Laurent monomial u in the variables X we have that $[u]S(X, X^{-1}) \in K((t))$. Then $[u]S(X, X^{-1}) \in K_{\text{alg}}((t))$, i.e., the Laurent series $[u]S(X, X^{-1})$ is algebraic over $K(t)$.*

Proof. Since

$$[u]S(X, X^{-1}) = [1]T(X, X^{-1})$$

where $T = u^{-1}S$, and since T is clearly rational when S is, we may assume $u = 1$. Let $\Delta = \Delta(X, Y)$ be as in Example 6.6.8. Thus the coefficient we want is the sum (which we are assuming exists formally) of all the coefficients of the Hadamard product $S * \Delta$, and hence also of the abelianization $\phi(S * \Delta)$. Since $S \in K(t)_{\text{rat}}\langle\langle X, Y\rangle\rangle$ and $\Delta \in K_{\text{alg}}\langle\langle X, Y\rangle\rangle \subset K(t)_{\text{alg}}\langle\langle X, Y\rangle\rangle$, it follows from Proposition 6.6.12 that $S * \Delta \in K(t)_{\text{alg}}\langle\langle X, Y\rangle\rangle$. Thus by Theorem 6.6.10, $\phi(S * \Delta)$ is algebraic over $K(t)((X, Y))$. By Proposition 6.1.12 it follows that $\phi(S*\Delta)_{x_i=y_i=1}$ is algebraic over $K(t)$, as was to be proved. □

6.7.2 Corollary. *Let P be a noncommutative Laurent polynomial over K in the variables $X = (x_1, \ldots, x_k)$. (Equivalently, $P \in K[F_k]$, the group algebra of the free group F_k generated by X.) Let u be a noncommutative Laurent monomial in X (i.e., $u \in F_k$). Then the power series*

$$y = \sum_{n\geq 0}([u]P^n)t^n \in K[[t]]$$

is algebraic (over $K(t)$).

Proof. We have

$$y = [u](1 - Pt)^{-1}.$$

The proof follows from Theorem 6.7.1. □

Note that the commutative analogue of Corollary 6.7.2 fails. For instance, if $P = x + x^{-1} + y + y^{-1} \in \mathbb{C}[x, x^{-1}, y, y^{-1}]$, then

$$\sum_{n\geq 0}([1]P^n)t^n = \sum_{m\geq 0}\binom{2m}{m}^2 t^{2m},$$

which according to Exercise 6.3 is not algebraic. Thus in the context of diagonals, we see that noncommutative series behave better than the commutative ones.

6.7.3 Example. An interesting special case of Corollary 6.7.2 is when $P = x_1 + x_2 + \cdots + x_k + x_1^{-1} + x_2^{-1} + \cdots + x_k^{-1}$ and $u = 1$. Then

$$y = \sum_{n \geq 0} [1]\big(x_1 + \cdots + x_k + x_1^{-1} + \cdots + x_k^{-1}\big)^n t^n$$
$$= 1 + 2kt^2 + (8k^2 - 2k)t^4 + (40k^3 - 24k^2 + 4k)t^6 + \cdots.$$

It would be extremely tedious to compute y explicitly using the method inherent in the proof of Theorem 6.7.1, but a direct combinatorial argument can be used to show that

$$y = \frac{2k-1}{k-1+k\sqrt{1-4(2k-1)t^2}}.$$

See Exercise 6.74.

As an application of Theorem 6.7.1, we give another proof that the diagonal of a (commutative) rational series in two variables is algebraic (Theorem 6.3.3).

Second Proof of Theorem 3.3. By Exercise 4.1(b), we can write $F(s,t) = P(s,t)/Q(s,t)$, where $P, Q \in K[s,t]$ and $Q(0,0) \neq 0$. Define a noncommutative series $\tilde{F}(\tilde{s},\tilde{t})$ by

$$\tilde{F}(\tilde{s},\tilde{t}) = P(\tilde{s}, x\tilde{t})Q(\tilde{s}, x\tilde{t})^{-1} \in K(x)_{\mathrm{rat}}\langle\langle \tilde{s},\tilde{t}\rangle\rangle.$$

The coefficient of $\tilde{s}^0$ in $\tilde{F}(\tilde{s},\tilde{s}^{-1})$ is the sum of the coefficients in $\tilde{F}(\tilde{s},\tilde{t})$ of monomials $w \in \{\tilde{s},\tilde{t}\}^*$ having equal total degree in $\tilde{s}$ and $\tilde{t}$. The contribution for $\deg \tilde{s} = \deg \tilde{t} = n$ is just $f(n,n)x^n$, where $F(s,t) = \sum f(i,j)s^i t^j$. Therefore the coefficient of $\tilde{s}^0$ in $\tilde{F}(\tilde{s},\tilde{s}^{-1})$ is just $\operatorname{diag} F$. Moreover, any coefficient $[\tilde{s}^i]\tilde{F}(\tilde{s},\tilde{s}^{-1})$ is well-defined, so the proof follows from Theorem 6.7.1. □

Notes

The theory of algebraic functions is a vast subject, but only a small part of it has been found to have direct relevance to enumerative combinatorics. It would be interesting to see whether some of the deeper aspects of algebraic functions, such as the Riemann–Roch theorem or the theory of abelian integrals, can be applied to enumerative combinatorics. The first result in our presentation that is not a simple consequence of an introductory algebra course is Puiseux's theorem (Theorem 6.1.5). First proved by V. Puiseux **[54]** in 1850, some expositions of the proof appear in [**12**, Ch. 4.6][**14**, pp. 373–396][**21**, Ch. III.6][**47**, Ch. V, Thm. 3.1][**74**, Ch. IV, Thm. 3.1]. A "modern" proof was given by P. M. Cohn **[17][18]**. For computational aspects see **[37]**.

Several interesting results concerning algebraic functions and related to enumeration appear in the paper **[40]** of R. Jungen. In particular, there is a proof of Proposition 6.1.11 and a determination of the asymptotic behavior of the coefficients of an algebraic power series. This latter result is a useful tool for showing that certain series are not algebraic (see Exercise 6.3). For a wealth of further information on the fascinating subject of discriminants, see **[28]**. Equation (6.7) is discussed further in Exercise 6.8(a).

∗ The enumeration of trees, parenthesizations (or bracketings), ballot sequences, lattice paths, and polygon dissections, and the close connections among them, as summarized by Proposition 6.2.1, goes back to Segner and Euler in 1760.* Segner [**67**] obtained a recurrence for the number of triangulations of a polygon (the problem discussed in our Corollary 6.2.3(vi)), and Euler [**23**] essentially solved this recurrence, though without details of a proof. (Euler published his result as an unsigned summary of the work of Segner, but it is evident that Euler is indeed the author.) The more general problem of computing the number of dissections of an n-gon by a fixed number m of its diagonals was posed by Pfaff to N. von Fuss, who generalized Segner's recurrence [**26**]. (For more information on this problem of Pfaff and Fuss, see Exercise 6.33(c).) In the period 1838–1839 four authors considered the Euler–Segner problem. The first was G. Lamé [**44**]. Lamé's proof of the Euler–Segner result was further developed and discussed by the Belgian mathematician Eugène Charles Catalan (1814–1894) [**10**], who wrote several other papers on this topic. The other two authors were O. Rodrigues [**56**][**57**] and J. Binet [**7**][**8**]. The term "Catalan number" arose from a citation by Netto [**1.14**, §122, §124], who attributed the problems of binary parenthesization and polygon triangulation to Catalan. A good historical discussion is given in [**9**]. An extensive bibliography (up to 1976) of Catalan numbers appears in [**30**]. The thesis [**42**] contains 31 combinatorial structures enumerated by Catalan numbers and 158 bijections among them. A very readable popular exposition of Catalan numbers appears in [**27**], while a recent survey aimed at a more mathematical audience is given by [**38**]. An earlier survey is [**2**]. For further information on Catalan numbers, see Exercises 6.19–6.36. For some interesting recent work on triangulations of polygons, see [**1**][**20**][**46**][**69**]. A little-known historical aspect of Catalan numbers is their independent discovery in China, beginning with Ming An-tu (1692?–1763?). He was a Mongolian mathematician who obtained several recurrences for Catalan numbers in the 1730s, though his work was not published until 1839. Ming and his successors, however, did not obtain combinatorial interpretations of Catalan numbers. For further information, see [**50**]. Recently there have been efforts to bring Catalan numbers into undergraduate and even secondary-school education; see for instance the papers [**15**][**39**][**41**][**73**].

The connection between bracketings and plane trees (Proposition 6.2.1(i) and (ii)) was known to Cayley [**11**]. The bijection with polygon dissections (Proposition 6.2.1(vi)) appears in [**5.22**] (with a sequel by Erdélyi and Etherington in [**5.20**]). Ballot sequences were first considered by J. Bertrand [**6**] in 1887. He sketched a proof by induction of a ballot theorem that includes Corollary 6.2.3(ii). A famous proof based on the "reflection principle" was given soon after by D. André. See Exercise 6.20 for further details. Additional information on ballot problems appears in Chapter 7 (see Proposition 7.10.3(c) and Corollary 7.21.6).

The formula (6.18) for the number of plane trees with $(k-1)n+1$ endpoints and every internal vertex of degree k is a special case of Theorem 5.3.10. See

*See the discussion below of Schröder's second problem for a remarkable earlier reference to a special bracketing problem.

the Notes to Chapter 5 for references. The power series solution to $y^5 + y = x$ (equivalent to (6.19) in the case $k = 5$) was obtained by Eisenstein **[22]** in his work on quintic equations. See **[51]** for an interesting historical discussion. The generating functions of Example 6.2.7, as well as some related ones, are considered by Pólya in **[53]**. Pólya mentions that Hurwitz posed the problem of showing that $\sum_n \binom{\beta n}{n} x^n$ is algebraic for $\beta \in \mathbb{Q}$. The "four combinatorial problems" (*vier combinatorische Probleme*) of Schröder appear in **[5.60]**. Much additional work related to Schröder's problems has been carried out. The first problem (equivalent to triangulations of a polygon) has already been discussed. The second problem (equivalent to arbitrary dissections of a polygon) is discussed further in Exercise 6.39. The term "Schröder number" seems to have been first used by Rogers **[58]**. The third and fourth problems are discussed in the Notes to Chapter 5. In 1994 D. Hough, while a graduate student at George Washington University, made a remarkable historical discovery related to the second problem of Schröder. He observed that the mysterious number 103,049 of Exercise 1.45 is just the tenth Schröder number $s_2(10)$. In other words, Hipparchus was aware of the Schröder numbers in the second century B.C. (Should they now be called Hipparchus numbers?) This discovery solves what is perhaps the oldest open problem related to combinatorics and shows that the ancient Greeks (or at least Hipparchus) were much more sophisticated in combinatorics than previously realized. The number 310,952 of Exercise 1.45 remains an enigma, though a possible interpretation of the nearby number 310,954 has been given by L. Habsieger, M. Kazarian, and S. Lando **[32]**. For further information related to Hough's discovery, see **[71]**.

The main result of Section 6.3, that the diagonal of a rational function of two variables is algebraic (Theorem 6.3.3), is due to H. Furstenburg **[25]**. His proof was based on contour integration as in the proof sketched preceding Example 6.3.5. (For further aspects of Furstenburg's paper, see Exercise 6.11. For a rigorous discussion of the contour integration technique for computing diagonals of power series in two variables, see **[36]**.) Our first proof of Theorem 6.3.3 follows Gessel [**29**, Thm. 6.1]. A proof based on noncommutative formal series, similar to the proof we give at the end of Section 6.7, was given by M. Fliess [**24**, Prop. 5]. For more information on the Delannoy numbers of Example 6.3.8, see [**2.3**, Exercise I.21].

D-finite power series and P-recursive functions were first systematically investigated in **[70]**, though much was known about them before **[70]** appeared. D-finite series are also called *holonomic* and are involved in much recent work dealing with algorithms for discovering and verifying combinatorial identities **[52][75]**. The basic connection between D-finite series and P-recursive functions (Proposition 6.4.1) is alluded to in [**40**, p. 299]. The earliest explicit statement of the fact that algebraic functions are D-finite (Theorem 6.4.6) of which we are aware is due to Comtet **[19]**. The extension of the theory of D-finite series to several variables is discussed in **[33][34][48][49]** and references given there. Additional references on D-finiteness may be found in Exercises 6.53–6.62.

The hierarchy rational $\Rightarrow$ algebraic $\Rightarrow$ D-finite can be further extended, though these extensions have not yet proved as useful in combinatorics as the original three classes. Two classes that may warrant further investigation are differentially finite algebraic series and constructible differentially finite algebraic series. For further information see [**3**][**4**][**59**][**60**].

The theory of rational formal noncommutative power series originated in the pioneering work of M. P. Schützenberger [**64**][**65**][**66**]. In particular, the Fundamental Theorem of Rational Formal Series (Theorem 6.5.7) appears in [**66**]. A similar theory of algebraic formal power series is due to Chomsky and Schützenberger [**13**]. For a comprehensive account of the theory of rational series, see [**5**]. Some good references for noncommutative series in general and their connections with languages and automata are [**16**][**43**][**55**][**61**][**62**][**63**][**68**]. These last seven references contain many additional historical remarks and references. For further information related to decidability aspects of noncommutative series, see e.g. [**43**, §8 and §16] [**61**, Ch. VIII][**63**, Chs. II.12 and IV.5]. An interesting survey of the connection between algebraic series and combinatorics appears in [**72**].

The Dyck language of Example 6.6.6 plays a fundamental role in a theorem of Chomsky and Schützenberger on the structure of arbitrary algebraic series. See [**63**, Thm. IV.4.5] for an exposition. Example 6.6.8 is due to Chomsky and Schützenberger [**13**] and is also discussed in [**35**, Prop. 3.2]. For a proof of Lemma 6.6.9, see [**45**, Chap. X, Prop. 8]. The result that the Hadamard product of a rational series and an algebraic series is algebraic (Proposition 6.6.12) is due to Schützenberger [**65**]. For further information on "closure properties" of formal series, see Exercise 6.71.

Our main result on noncommutative diagonals (Theorem 6.7.1) is a special case of a theorem of G. Jacob [**31**, Thm. 4]. We have closely followed Haiman [**35**] in our development of the theory of Section 6.7. For further information on Example 6.7.3, see Exercise 6.74.

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Exercises

6.1. [2+] Give a formal proof (no complex analysis, etc.) that e^x is not algebraic, i.e.,

$$\sum_{n\geq 0} \frac{x^n}{n!} \notin \mathbb{C}_{\text{alg}}[[x]].$$

* **6.2.** **a.** [3−] Suppose that $F(x) \in \mathbb{Q}[[x]]$ is algebraic. Show that there is an integer $m \geq 1$ such that $F(mx) \in \mathbb{Z}[[x]]$.

b. [1] Deduce that e^x is not algebraic.

6.3. [3+] Show that $y_1 = \sum_{n\geq 0} \binom{3n}{n,n,n} x^n$ and $y_2 = \sum_{n\geq 0} \binom{2n}{n}^2 x^n$ are not algebraic (over a field of characteristic zero). What about $\sum_{n\geq 0} \binom{2n}{n}^3 x^n$?

6.4. [3−] Show that Puiseux's theorem (Theorem 6.1.5) fails in characteristic $p > 0$.

6.5. [2] Let char $K = 0$ and

$$P(y) = F_d(x)y^d + \cdots + F_0(x) \in K[[x]][y],$$

with $F_d(0) \neq 0$. Suppose that $P(y)$ is irreducible, and let $c_1, \ldots, c_r$ be given by Corollary 6.1.7. Show that disc $P(y)$ is divisible by x^{d-r}.

6.6. [2+] Show that $\sum_{n\geq 0} f(n)x^n \in K[[x]]$ is algebraic of degree d if and only if $\sum_{n\geq 0}(\Delta^n f(0))x^n$ is algebraic of degree d. (See Section 1.4 for the definition of $\Delta^n f(0)$.)

6.7. [2−] If $u \in K[[x]]$ is algebraic with $u(0) = 0$ and $u'(0) \neq 0$, then is the compositional inverse $u^{\langle -1\rangle}$ algebraic?

6.8. **a.** [3−] Verify equation (6.7), i.e., show that

$$\operatorname{disc}(ay^d + by + c) = (-1)^{\binom{d}{2}} a^{d-2}[d^d ac^{d-1} + (-1)^{d-1}(d-1)^{d-1}b^d].$$

b. [3−] Find $\operatorname{disc}\left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1\right)$.

6.9. **a.** [2+] Let $F(x, y) = \sum_{m,n\geq 0} f(m, n)x^m y^n$, $G(x, y) = \sum_{m,n\geq 0} g(m, n) x^m y^n \in K[[x, y]]$, where K is any field. The Hadamard product is defined analogously to the univariate case to be

$$F * G = \sum_{m,n\geq 0} f(m, n)g(m, n)x^m y^n.$$

Show that if F and G are rational, then $F * G$ is algebraic (over $K(x, y)$).

b. [3−] For $k \geq 2$ define the power series

$$F_k(x_1, \ldots, x_k) = \sum_{n_1,\ldots,n_k\geq 0} \binom{n_1+n_2}{n_1}\binom{n_2+n_3}{n_2}\cdots\binom{n_k+n_1}{n_k} x_1^{n_1}\cdots x_k^{n_k}.$$

Show that F_k is algebraic (over $K(x_1, \ldots, x_k)$).

c. [2] Compute $F_2(x, y) = \sum_{m,n\geq 0}\binom{m+n}{m}^2 x^m y^n$ and $F_3(x, y, z)$ explicitly.

6.10. [2] Let $P(q) \in K[q, q^{-1}]$ be a Laurent polynomial over K, and fix an integer m. Define $f(n) = [q^m]P(q)^n$ for all $n \geq 0$. Show that $y = \sum_{n\geq 0} f(n)x^n$ is algebraic.

6.11. **a.** [3] Let K be a field of characteristic $p > 0$. Let $F, G \in K_{\text{alg}}[[x_1, \ldots, x_k]]$, i.e., F and G are algebraic power series over the field $K(x_1, \ldots, x_k)$. Show that the Hadamard product $F * G$ (defined for $k = 2$ in Exercise 6.9(a) and extended in the obvious way to arbitrary k) is also algebraic.

b. [2+] Deduce that if $F \in K_{\text{alg}}[[x_1, \ldots, x_k]]$ (with char $K = p > 0$), then diag F is algebraic.

6.12. [2+] Given power series $F(x) = F(x_1, \ldots, x_m) \in K[[x_1, \ldots, x_m]]$ and $G(y) = G(y_1, \ldots, y_n) \in K[[y_1, \ldots, y_n]]$, let F_k denote the part of F that is homogeneous of degree k, and similarly G_k, so $F = \sum F_k$ and $G = \sum G_k$. Define

$$(F \heartsuit G)(x, y) = \sum_{k\geq 0} F_k(x)G_k(y) \in K[[x, y]],$$

the "heartamard product" of F and G. Show that if F and G are rational, then so is $F \heartsuit G$. Moreover, if F is rational and G is algebraic, then $F \heartsuit G$ is algebraic.

6.13. **a.** [3−] Let $k \in \mathbb{P}$, and define $\eta = \sum_{n\geq 0} \binom{kn}{n} x^n$. Example 6.2.7 shows that η is a root of the polynomial

$$P(y) = k^k x y^k - (y-1)[(k-1)y+1]^{k-1}.$$

Find (as fractional series) the other $k-1$ roots of the polynomial $P(y)$. Deduce that $P(y)$ is irreducible (as a polynomial over $\mathbb{C}(x)$).

b. [3−] Find the discriminant of $P(y)$.

6.14. [3−] Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by

$$f(i-1, j) - 2f(i, j) + f(i+1, j-1) = 0 \tag{6.53}$$

for all $(i, j) \in \mathbb{N} \times \mathbb{N} - \{(0,0)\}$, with the initial conditions $f(0,0) = 1$ and $f(i, j) = 0$ if $i < 0$ or $j < 0$. Thus $f(i, 0) = 2^{-i}$, $f(0,1) = \frac{1}{4}$, $f(1,1) = \frac{1}{4}$, etc. Find the generating function $F(x, y) = \sum_{i,j\geq 0} f(i, j) x^i y^j$.

6.15. [2+] Let $f, g, h \in K[[x]]$ with $h(0) = 0$. Find a polynomial $P(f, g, h, x)$ so that

$$\operatorname{diag} \frac{1}{1 - sf(st) - tg(st) - h(st)} = \frac{1}{\sqrt{P}},$$

where diag is in the variable x.

6.16. [5−] Let $f(n)$ be the number of paths from $(0, 0)$ to (n, n) using the steps $(1, 0)$, $(0, 1)$, and $(1, 1)$; and let $g(n)$ be the number of paths from $(0, 0)$ to (n, n) using any elements of $\mathbb{N}^2 - \{(0,0)\}$ as steps. It is immediate from equations (6.27) and (6.30) that $g(n) = 2^{n-1} f(n)$, $n > 0$. Is there a combinatorial proof?

6.17. **a.** [2+] Let S be a subset of $\mathbb{N} \times \mathbb{N} - \{(0,0)\}$ such that (i) every element of S has the form (n, n), $(n+1, n)$, or $(n, n+1)$, and (ii) $(n, n+1) \in S$ if and only if $(n+1, n) \in S$. Let $g(n)$ be the number of paths from $(0, 0)$ to (n, n) using steps from S. Let $h(n)$ be the number of such paths that never go above the line $y = x$. (Let $g(0) = h(0) = 1$.) Define $G(x) = \sum_{n\geq 0} g(n) x^n$, $H(x) = \sum_{n\geq 0} h(n) x^n$, and $K(x) = \sum_{(n,n)\in S} x^n$. Show that

$$H(x) = \frac{2}{1 - K(x) + G(x)^{-1}}.$$

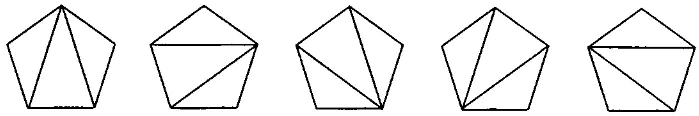
b. [2−] Compute $H(x)$ explicitly when $S = \{(0,1), (1,0), (1,1)\}$ and deduce that in this case $h(n)$ is the Schröder number r_n, thus confirming Exercise 6.39(j).

c. [3−] Give a *combinatorial* proof that when $S = \{(0,1), (1,0), (1,1)\}$ and $n \geq 2$, then $h(n)$ is twice the number of ways to dissect a convex $(n+2)$-gon with any number of diagonals that don't intersect in their interiors.

6.18. [3] Let S be a subset of $\mathbb{N} \times \mathbb{N} - \{(0,0)\}$ such that $\sum_{(m,n)\in S} x^m y^n$ is rational, e.g., S is finite or cofinite. Let $f(n)$ be the number of lattice paths from $(0, 0)$ to (n, n) with steps from S that never go above the line $y = x$. Show that $\sum_{n\geq 0} f(n) x^n$ is algebraic.

6.19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 66 sets S_i, (a) $\leq i \leq$ (nnn), given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

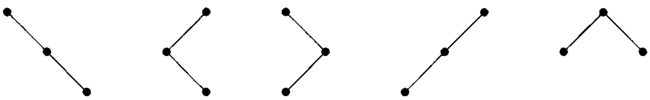
a. Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors:



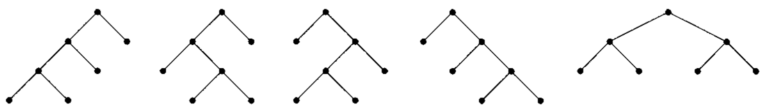
b. Binary parenthesizations of a string of $n+1$ letters:

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

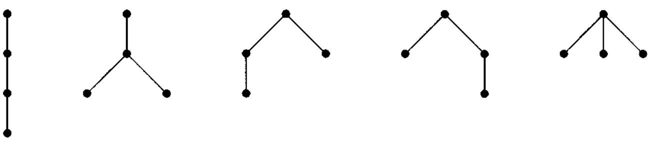
c. Binary trees with n vertices:



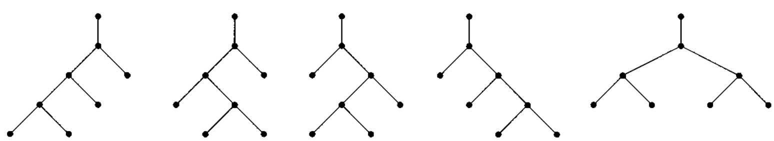
d. Plane binary trees with $2n+1$ vertices (or $n+1$ endpoints):



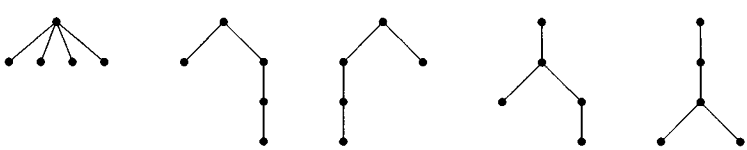
e. Plane trees with $n+1$ vertices:



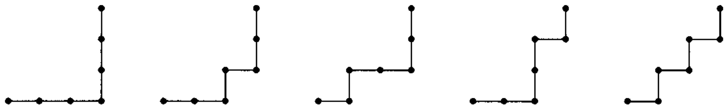
f. Planted (i.e., root has degree one) trivalent plane trees with $2n+2$ vertices:



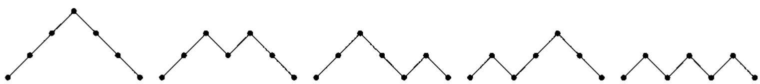
g. Plane trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length:



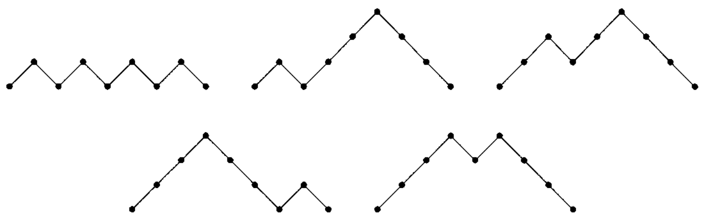
h. Lattice paths from $(0, 0)$ to (n, n) with steps $(0, 1)$ or $(1, 0)$, never rising above the line $y = x$:



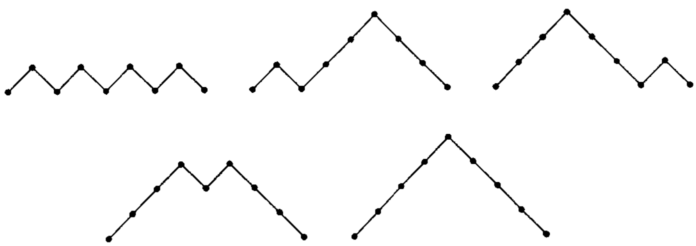
i. Dyck paths from $(0, 0)$ to $(2n, 0)$, i.e., lattice paths with steps $(1, 1)$ and $(1, -1)$, never falling below the x-axis:



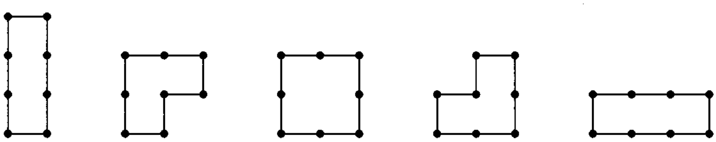
j. Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n+2, 0)$ such that any maximal sequence of consecutive steps $(1, -1)$ ending on the x-axis has odd length:



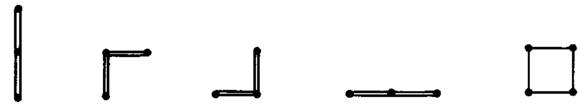
k. Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n + 2, 0)$ with no peaks at height two



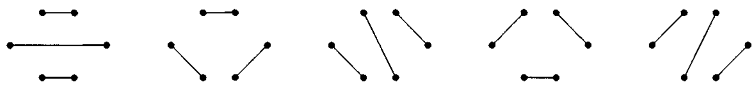
l. (Unordered) pairs of lattice paths with $n + 1$ steps each, starting at $(0, 0)$, using steps $(1, 0)$ or $(0, 1)$, ending at the same point, and only intersecting at the beginning and end:



m. (Unordered) pairs of lattice paths with $n - 1$ steps each, starting at $(0, 0)$, using steps $(1, 0)$ or $(0, 1)$, ending at the same point, such that one path never rises above the other path:



n. n nonintersecting chords joining $2n$ points on the circumference of a circle:



o. Ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points:

p. Ways of drawing in the plane $n + 1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right):

q. Ways of drawing in the plane $n + 1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right):

r. Sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below):

$$111--- \quad 11-1-- \quad 11--1- \quad 1-11-- \quad 1-1-1-$$

s. Sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$:

$$111 \quad 112 \quad 113 \quad 122 \quad 123$$

t. Sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq 2i$:

$$12 \quad 13 \quad 14 \quad 23 \quad 24$$

u. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$:

$$000 \quad 001 \quad 010 \quad 011 \quad 012$$

v. Sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative:

$$0,0 \quad 0,1 \quad 1,-1 \quad 1,0 \quad 1,1$$

w. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$:

$$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$$

x. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $0 \leq a_i \leq n-i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$, then $j - i > a_i - a_j$:

$$000 \quad 010 \quad 100 \quad 200 \quad 110$$

y. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$:

$$123 \quad 133 \quad 223 \quad 323 \quad 333$$

z. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r \leq j-1$:

$$111 \quad 112 \quad 113 \quad 121 \quad 123$$

aa. Equivalence classes B of words in the alphabet $[n-1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n-1]$ satisfying $|i-j| \geq 2$:

$$\{\emptyset\} \quad \{1\} \quad \{2\} \quad \{12\} \quad \{21\}$$

(For $n = 4$ a representative of each class is given by $\emptyset$, 1, 2, 3, 12, 21, 13, 23, 32, 123, 132, 213, 321, 2132.)

bb. Partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_1 \leq n-1$ (so the diagram of λ is contained in an $(n-1) \times (n-1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$:

$$(0,0) \quad (1,0) \quad (1,1) \quad (2,1) \quad (2,2)$$

cc. Permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$:

$$112233 \quad 112332 \quad 122331 \quad 123321 \quad 122133$$

dd. Permutations $a_1 a_2 \cdots a_{2n}$ of the set $[2n]$ such that: (i) $1, 3, \ldots, 2n-1$ appear in increasing order, (ii) $2, 4, \ldots, 2n$ appear in increasing order, and (iii) $2i - 1$ appears before $2i$, $1 \leq i \leq n$:

$$123456 \quad 123546 \quad 132456 \quad 132546 \quad 135246$$

ee. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations:

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

ff. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations):

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

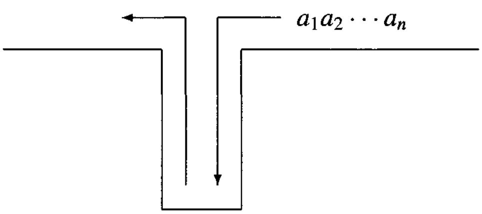
gg. Permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \ldots, 2n) \cdot w$ has $n + 1$ cycles:

$$(1,2,3,4,5,6)(1,2)(3,4)(5,6) = (1)(2,4,6)(3)(5)$$
$$(1,2,3,4,5,6)(1,2)(3,6)(4,5) = (1)(2,6)(3,5)(4)$$
$$(1,2,3,4,5,6)(1,4)(2,3)(5,6) = (1,3)(2)(4,6)(5)$$
$$(1,2,3,4,5,6)(1,6)(2,3)(4,5) = (1,3,5)(2)(4)(6)$$
$$(1,2,3,4,5,6)(1,6)(2,5)(3,4) = (1,5)(2,4)(3)(6)$$

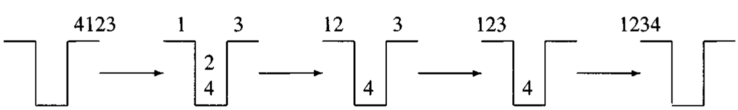
hh. Pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n + 1$ cycles, and $uv = (1, 2, \ldots, n)$:

$$(1)(2)(3) \cdot (1,2,3) \quad (1,2,3) \cdot (1)(2)(3) \quad (1,2)(3) \cdot (1,3)(2)$$
$$(1,3)(2) \cdot (1)(2,3) \quad (1)(2,3) \cdot (1,2)(3)$$

ii. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset) = \emptyset$. If $w = unv$ is a sequence of distinct integers with largest term n, then $S(w) = S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w) = w$:

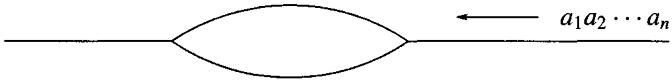


For example,



$$123 \quad 132 \quad 213 \quad 312 \quad 321$$

jj. Permutations $a_1a_2\cdots a_n$ of $[n]$ that can be put in increasing order on two parallel queues. Now the picture is



123 132 213 231 312

kk. Fixed-point-free involutions w of $[2n]$ such that if $i < j < k < l$ and $w(i) = k$, then $w(j) \neq l$ (in other words, 3412-avoiding fixed-point-free involutions):

(12)(34)(56) (14)(23)(56) (12)(36)(45) (16)(23)(45) (16)(25)(34)

ll. Cycles of length $2n + 1$ in $\mathfrak{S}_{2n+1}$ with descent set $\{n\}$:

2371456 2571346 3471256 3671245 5671234

mm. Baxter permutations (as defined in Exercise 6.55) of $[2n]$ or of $[2n + 1]$ that are reverse alternating (as defined at the end of Section 3.16) and whose inverses are reverse alternating:

132546 153426 354612 561324 563412

1325476 1327564 1534276 1735462 1756342

nn. Permutations w of $[n]$ such that if w has ℓ inversions then for all pairs of sequences $(a_1, a_2, \ldots, a_\ell), (b_1, b_2, \ldots, b_\ell) \in [n-1]^\ell$ satisfying

$$w = s_{a_1}s_{a_2}\cdots s_{a_\ell} = s_{b_1}s_{b_2}\cdots s_{b_\ell},$$

where s_j is the adjacent transposition $(j, j+1)$, we have that the ℓ-element *multisets* $\{a_1, a_2, \ldots, a_\ell\}$ and $\{b_1, b_2, \ldots, b_\ell\}$ are equal (thus, for example, $w = 321$ is not counted, since $w = s_1s_2s_1 = s_2s_1s_2$, and the multisets $\{1, 2, 1\}$ and $\{2, 1, 2\}$ are not equal):

123 132 213 231 312

oo. Permutations w of $[n]$ with the following property: Suppose that w has ℓ inversions, and let

$$R(w) = \left\{(a_1, \ldots, a_\ell) \in [n-1]^\ell : w = s_{a_1}s_{a_2}\cdots s_{a_\ell}\right\},$$

where s_j is as in (nn). Then

$$\sum_{(a_1,\ldots,a_\ell)\in R(w)} a_1a_2\cdots a_\ell = \ell!.$$

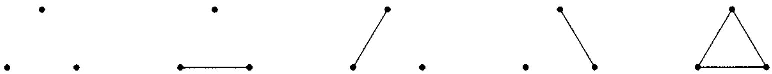
$$R(123) = \{\emptyset\}, \quad R(213) = \{(1)\}, \quad R(231) = \{(1, 2)\}$$

$$R(312) = \{(2, 1)\}, \quad R(321) = \{(1, 2, 1), (2, 1, 2)\}$$

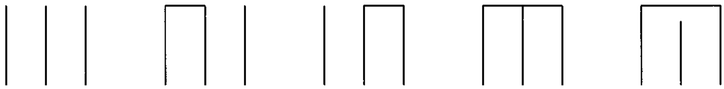
pp. Noncrossing partitions of $[n]$, i.e., partitions $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$:

123 12–3 13–2 23–1 1–2–3

qq. Partitions $\{B_1, \ldots, B_k\}$ of $[n]$ such that if the numbers $1, 2, \ldots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \ldots, B_k$ are pairwise disjoint:



rr. Noncrossing Murasaki diagrams with n vertical lines:



ss. Noncrossing partitions of some set $[k]$ with $n + 1$ blocks, such that any two elements of the same block differ by at least three:

1–2–3–4 14–2–3–5 15–2–3–4 25–1–3–4 16–25–3–4

tt. Noncrossing partitions of $[2n + 1]$ into $n + 1$ blocks, such that no block contains two consecutive integers:

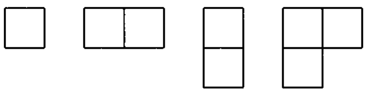
137–46–2–5 1357–2–4–6 157–24–3–6
17–246–3–5 17–26–35–4

uu. *Nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$:

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

vv. Young diagrams that fit in the shape $(n - 1, n - 2, \ldots, 1)$:

$\emptyset$ 

ww. Standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n-1)$):

123	124	125	134	135
456	356	346	256	246

or

123	124	125	134	135
45	35	34	25	24

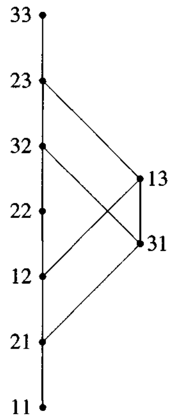


Figure 6-5. A poset with $C_4 = 14$ order ideals.

xx. Pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows:

$$(123, 123) \quad \begin{pmatrix} 12 & 12 \\ 3 & ,\ 3 \end{pmatrix} \quad \begin{pmatrix} 12 & 13 \\ 3 & ,\ 2 \end{pmatrix} \quad \begin{pmatrix} 13 & 12 \\ 2 & ,\ 3 \end{pmatrix} \quad \begin{pmatrix} 13 & 13 \\ 2 & ,\ 2 \end{pmatrix}$$

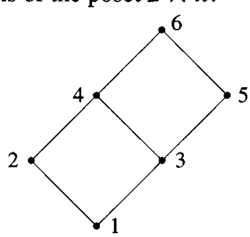
yy. Column-strict plane partitions of shape $(n-1, n-2, \ldots, 1)$, such that each entry in the i-th row is equal to $n-i$ or $n-i+1$:

$$\begin{matrix} 33 \\ 2 \end{matrix} \quad \begin{matrix} 33 \\ 1 \end{matrix} \quad \begin{matrix} 32 \\ 2 \end{matrix} \quad \begin{matrix} 32 \\ 1 \end{matrix} \quad \begin{matrix} 22 \\ 1 \end{matrix}$$

zz. Convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m), such that if $(i, j) \in S$ then $0 < i - j < n$:

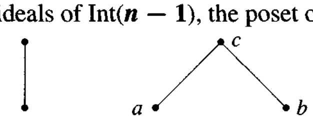
$$\emptyset \quad \{(1,0)\} \quad \{(2,0)\} \quad \{(1,0),(2,0)\} \quad \{(2,0),(2,1)\}$$

aaa. Linear extensions of the poset $\mathbf{2} \times \boldsymbol{n}$:



123456
123546
132456
132546
135246

bbb. Order ideals of $\mathrm{Int}(\boldsymbol{n} - \mathbf{1})$, the poset of intervals of the chain $\boldsymbol{n} - \mathbf{1}$:



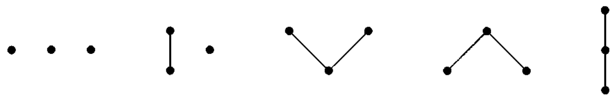
$\mathbf{2}$ $\qquad$ $\mathrm{Int}(\mathbf{2})$

$\emptyset,\ a,\ b,\ ab,\ abc$

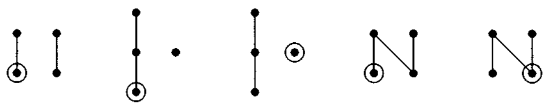
ccc. Order ideals of the poset A_n obtained from the poset $(\boldsymbol{n} - \mathbf{1}) \times (\boldsymbol{n} - \mathbf{1})$ by adding the relations $(i, j) < (j, i)$ if $i > j$ (see Figure 6-5 for the Hasse diagram of A_4):

$$\emptyset \quad \{11\} \quad \{11, 21\} \quad \{11, 21, 12\} \quad \{11, 21, 12, 22\}$$

ddd. Nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$:



eee. Nonisomorphic $(n+1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element:

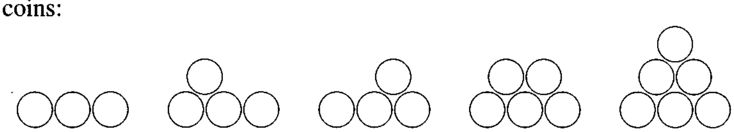


fff. Relations R on $[n]$ that are reflexive (iRi), symmetric $(iRj \Rightarrow jRi)$, and such that if $1 \le i < j < k \le n$ and iRk, then iRj and jRk (in the example below we write ij for the pair (i, j), and we omit the pairs ii):

$$\emptyset \quad \{12, 21\} \quad \{23, 32\} \quad \{12, 21, 23, 32\} \quad \{12, 21, 13, 31, 23, 32\}$$

ggg. Joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices:

hhh. Ways to stack coins in the plane, the bottom row consisting of n consecutive coins:



iii. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \ge 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors:

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

jjj. n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0:

$$000 \quad 013 \quad 022 \quad 112 \quad 233$$

kkk. n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \ge j$ and there is a lattice path from $(0, 0)$ to (i, j) with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$ that lies entirely inside S:

$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(2,1)\}$$

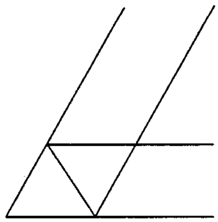
$$\{(0,0),(1,1),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$

lll. Regions into which the cone $x_1 \ge x_2 \ge \cdots \ge x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \le i < j \le n$ (the diagram below shows the

$$\begin{array}{ccccccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 1 & & 3 & & 2 & & 1 & & 5 & & 1 & & 2 & & 3 & & 1 & & 3 & & 2 & & 1 & & 5 & \\
& & 2 & & 5 & & 1 & & 4 & & 4 & & 1 & & 5 & & 2 & & 2 & & 5 & & 1 & & 4 & & \\
& & & 3 & & 2 & & 3 & & 3 & & 3 & & 2 & & 3 & & 3 & & 3 & & 2 & & 3 & & & \\
& & & & 1 & & 5 & & 2 & & 2 & & 5 & & 1 & & 4 & & 4 & & 1 & & 5 & & & & \\
& & & & & 2 & & 3 & & 1 & & 3 & & 2 & & 1 & & 5 & & 1 & & 2 & & & & & \\
& & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & &
\end{array}$$

Figure 6-6. The frieze pattern corresponding to the sequence $(1, 3, 2, 1, 5, 1, 2, 3)$.

situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$):



mmm. Positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (necessarily with $n + 1$ rows)

$$\begin{array}{cccccccccccc}
1 \quad 1 & 1 & \cdots & 1 & & 1 & & 1 & \cdots & 1 & & 1 \\
a_1 & a_2 & a_3 & \cdots & a_{n+2} & & a_1 & & a_2 & \cdots & a_{n-1} & \\
& b_1 & b_2 & b_3 & \cdots & b_{n+2} & b_1 & & \cdots & b_{n-2} & & \\
& & & & \vdots & & & & & & & \\
& & r_1 & r_2 & r_3 & \cdots & r_{n+2} & r_1 & & & & \\
& & & 1 & 1 & 1 & \cdots & 1 & & & &
\end{array} \tag{6.54}$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru + 1$ (an example of such an array for $(a_1, \ldots, a_8) = (1, 3, 2, 1, 5, 1, 2, 3)$ (necessarily unique) is given by Figure 6-6):

12213 22131 21312 13122 31221

nnn. n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

$$\begin{bmatrix}
a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & a_n
\end{bmatrix}$$

is positive definite with determinant one:

131 122 221 213 312

6.20. **a.** [2+] Let m, n be integers satisfying $1 \le n < m$. Show by a simple bijection that the number of lattice paths from $(1, 0)$ to (m, n) with steps $(0, 1)$ and $(1, 0)$ that intersect the line $y = x$ in at least one point is equal to the number of lattice paths from $(0, 1)$ to (m, n) with steps $(0, 1)$ and $(1, 0)$.

b. [2–] Deduce that the number of lattice paths from $(0, 0)$ to (m, n) with steps $(1, 0)$ and $(0, 1)$ that intersect the line $y = x$ only at $(0, 0)$ is given by $\frac{m-n}{m+n}\binom{m+n}{n}$.

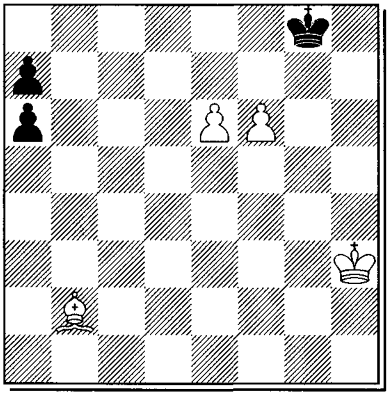
c. [1+] Show from (b) that the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$ that never rise above the line $y = x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (This gives a direct combinatorial proof of interpretation (h) of C_n in Exercise 6.19.)

6.21. **a.** [2+] Let X_n be the set of all $\binom{2n}{n}$ lattice paths from $(0, 0)$ to (n, n) with steps $(0, 1)$ and $(1, 0)$. Define the *excedance* (also spelled "exceedance") of a path $P \in X_n$ to be the number of i such that at least one point (i, i') of P lies above the line $y = x$ (i.e., $i' > i$). Show that the number of paths in X_n with excedance j is independent of j.

b. [1] Deduce that the number of $P \in X_n$ that never rise above the line $y = x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (a direct proof of interpretation (h) of C_n in Exercise 6.19). Compare with Example 5.3.11, which also gives a direct combinatorial interpretation of C_n when written in the form $\frac{1}{n+1}\binom{2n}{n}$ (as well as in the form $\frac{1}{2n+1}\binom{2n+1}{n}$).

6.22. [2+] Show (bijectively if possible) that the number of lattice paths from $(0, 0)$ to $(2n, 2n)$ with steps $(1, 0)$ and $(0, 1)$ that avoid the points $(2i - 1, 2i - 1)$, $1 \le i \le n$, is equal to the Catalan number C_{2n}.

6.23. [3–] Consider the following chess position:



Black is to make 19 consecutive moves, after which White checkmates Black in one move. Black may not move into check, and may not check White (except possibly on his last move). Black and White are *cooperating* to achieve the aim of checkmate. (In chess-problem parlance, this problem is called a *serieshelpmate in 19*.) How many different solutions are there?

6.24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, ...

6.25. [2]–[5] Show that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ has the algebraic interpretations given below:

a. number of two-sided ideals of the algebra of all $(n-1) \times (n-1)$ upper triangular matrices over a field,

b. dimension of the space of invariants of $\mathrm{SL}(2, \mathbb{C})$ acting on the $2n$-th tensor power $T^{2n}(V)$ of its "defining" two-dimensional representation V,

c. dimension of the irreducible representation of the symplectic group $\mathrm{Sp}(2(n-1), \mathbb{C})$ (or Lie algebra $\mathfrak{sp}(2(n-1), \mathbb{C})$) with highest weight λ_{n-1}, the $(n-1)$-st fundamental weight,

d. dimension of the primitive intersection homology (say with real coefficients) of the toric variety associated with a (rationally embedded) n-dimensional cube,

e. the generic number of $\mathrm{PGL}(2, \mathbb{C})$ equivalence classes of degree n rational maps with a fixed branch set,

f. number of translation conjugacy classes of degree $n+1$ monic polynomials in one complex variable, all of whose critical points are fixed,

g. dimension of the algebra (over a field K) with generators $\epsilon_1, \ldots, \epsilon_{n-1}$ and relations

$$\begin{aligned} \epsilon_i^2 &= \epsilon_i \\ \beta\epsilon_i\epsilon_j\epsilon_i &= \epsilon_i \qquad\qquad \text{if } |i-j| = 1 \\ \epsilon_i\epsilon_j &= \epsilon_j\epsilon_i \qquad\qquad \text{if } |i-j| \geq 2, \end{aligned}$$

where β is a nonzero element of K,

h. number of $\oplus$-sign types indexed by A_{n-1}^{+} (the set of positive roots of the root system A_{n-1}).

* **i.** Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)})$ for all $w \in \mathfrak{S}_n$. Let I be the ideal generated by all invariants of positive degree, i.e.,

$$I = \langle f \in A : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0 \rangle.$$

Then (conjecturally) C_n is the dimension of the subspace of A/I affording the sign representation, i.e.,

$$C_n = \dim\{f \in A/I : w \cdot f = (\operatorname{sgn} w) f \text{ for all } w \in \mathfrak{S}_n\}.$$

6.26. a. [3−] Let D be a Young diagram of a partition λ, as defined in Section 1.3. Given a square s of D let t be the lowest square in the same column as s, and let u be the rightmost square in the same row as s. Let $f(s)$ be the number of paths from t to u that stay within D, and such that each step is one square to the north or one square to the east. Insert the number $f(s)$ in square s, obtaining an array A. For instance, if $\lambda = (5, 4, 3, 3)$ then A is given by

16	7	2	1	1
6	3	1	1	
3	2	1		
1	1	1		

Let M be the largest square subarray (using consecutive rows and columns) of A containing the upper left-hand corner. Regard M as a matrix. For the above example we have

$$M = \begin{bmatrix} 16 & 7 & 2 \\ 6 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Show that $\det M = 1$.

b. [2] Find the unique sequence $a_0, a_1, \ldots$ of real numbers such that for all $n \geq 0$ we have

$$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1.$$

(When $n = 0$ the second matrix is empty and by convention has determinant one.)

6.27. a. [3−] Let V_n be a real vector space with basis $x_0, x_1, \ldots, x_n$ and scalar product defined by $\langle x_i, x_j \rangle = C_{i+j}$, the $(i+j)$-th Catalan number. It follows from Exercise 6.26(b) that this scalar product is positive definite, and therefore V has an orthonormal basis. Is there an orthonormal basis for V_n whose elements are *integral* linear combinations of the x_i's ?

b. [3−] Same as (a), except now $\langle x_i, x_j \rangle = C_{i+j+1}$.

* **c.** [5−] Investigate the same question for the matrices M of Exercise 6.26(a) (so $\langle x_i, x_j \rangle = M_{ij}$) when λ is self-conjugate (so M is symmetric).

6.28. a. [3−] Suppose that real numbers $x_1, x_2, \ldots, x_d$ are chosen uniformly and independently from the interval $[0, 1]$. Show that the probability that the sequence $x_1, x_2, \ldots, x_d$ is convex (i.e., $x_i \leq \frac{1}{2}(x_{i-1} + x_{i+1})$ for $2 \leq i \leq d-1$) is $C_{d-1}/(d-1)!^2$, where C_{d-1} denotes a Catalan number.

b. [3–] Let C_d denote the set of all points $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ such that $0 \leq x_i \leq 1$ and the sequence $x_1, x_2, \ldots, x_d$ is convex. It is easy to see that C_d is a d-dimensional convex polytope, called the *convexotope*. Show that the vertices of $\mathcal{C}_d$ consist of the points

$$\left(1, \frac{j-1}{j}, \frac{j-2}{j}, \ldots, \frac{1}{j}, 0, 0, \ldots, 0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\right) \tag{6.55}$$

(with at least one 0 coordinate), together with $(1, 1, \ldots, 1)$ (so $\binom{d+1}{2} + 1$ vertices in all). For instance, the vertices of $\mathcal{C}_3$ are $(0, 0, 0)$, $(0, 0, 1)$, $\left(0, \frac{1}{2}, 1\right)$, $(1, 0, 0)$, $\left(1, \frac{1}{2}, 0\right)$, $(1, 0, 1)$, $(1, 1, 1)$.

c. [3] Show that the Ehrhart quasi-polynomial $i(\mathcal{C}_d, n)$ of $\mathcal{C}_d$ (as defined in Section 4.6) is given by

$$\begin{aligned} y_d &:= \sum_{n \geq 0} i(\mathcal{C}_d, n) x^n \\ &= \frac{1}{1-x}\left(\sum_{r=1}^{d} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-r]!}\right. \\ &\quad \left. - \sum_{r=1}^{d-1} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-1-r]!}\right), \end{aligned} \tag{6.56}$$

where $[i] = 1 - x^i$, $[i]! = [1][2] \cdots [i]$, and $*$ denotes Hadamard product. For instance,

$$y_1 = \frac{1}{(1-x)^2}$$

$$y_2 = \frac{1+x}{(1-x)^3}$$

$$y_3 = \frac{1+2x+3x^2}{(1-x)^3(1-x^2)}$$

$$y_4 = \frac{1+3x+9x^2+12x^3+11x^4+3x^5+x^6}{(1-x)^2(1-x^2)^2(1-x^3)}$$

$$y_5 = \frac{1+4x+14x^2+34x^3+63x^4+80x^5+87x^6+68x^7+42x^8+20x^9+7x^{10}}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}.$$

Is there a simpler formula than (6.56) for $i(\mathcal{C}_d, n)$ or y_d?

6.29. [3] Suppose that $n+1$ points are chosen uniformly and independently from inside a square. Show that the probability that the points are in convex position (i.e., each point is a vertex of the convex hull of all the points) is $(C_n/n!)^2$.

6.30. [3−] Let f_n be the number of partial orderings of the set $[n]$ that contain no induced subposets isomorphic to $\mathbf{3}+\mathbf{1}$ or $\mathbf{2}+\mathbf{2}$. (This exercise is the labeled analogue of Exercise 6.19(ddd). As mentioned in the solution to this exercise, such posets are called *semiorders*.) Let $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ be the generating function for Catalan numbers. Show that

$$\sum_{n \geq 0} f_n \frac{x^n}{n!} = C(1-e^{-x}), \tag{6.57}$$

the composition of $C(x)$ with the series $1 - e^{-x} = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \cdots$.

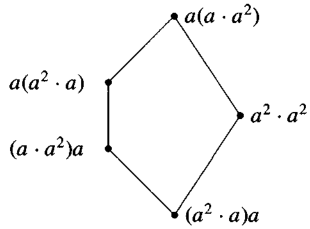


Figure 6-7. The Tamari lattice T_3.

6.31. **a.** [3−] Let $\mathcal{P}$ denote the convex hull in $\mathbb{R}^{d+1}$ of the origin together with all vectors $e_i - e_j$, where e_i is the i-th unit coordinate vector and $i < j$. Thus $\mathcal{P}$ is a d-dimensional convex polytope. Show that the relative volume of $\mathcal{P}$ (as defined in Section 4.6) is equal to $C_d/d!$, where C_d denotes a Catalan number.

b. [3] Let $i(\mathcal{P}, n)$ denote the Ehrhart polynomial of $\mathcal{P}$. Find a combinatorial interpretation of the coefficients of the i-Eulerian polynomial (in the terminology of Section 4.3)

$$(1-x)^{d+1} \sum_{n \geq 0} i(\mathcal{P}, n)x^n.$$

6.32. **a.** [3−] Define a partial order T_n on the set of all binary bracketings (parenthesizations) of a string of length $n+1$ as follows. We say that v covers u if u contains a subexpression $(xy)z$ (where x, y, z are bracketed strings) and v is obtained from u by replacing $(xy)z$ with $x(yz)$. For instance, $((a^2 \cdot a)a^2)(a^2 \cdot a^2)$ is covered by $((a \cdot a^2)a^2)(a^2 \cdot a^2)$, $(a^2(a \cdot a^2))(a^2 \cdot a^2)$, $((a^2 \cdot a)a^2)(a(a \cdot a^2))$, and $(a^2 \cdot a)(a^2(a^2 \cdot a^2))$. Figures 6-7 and 6-8 show the Hasse diagrams of T_3 and T_4. (In Figure 6-8, we have encoded the binary bracketing by a string of four +'s and four −'s, where a + stands for a left parenthesis and a − for the letter a, with the last a omitted.) Let U_n be the poset of all integer vectors $(a_1, a_2, \ldots, a_n)$ such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$ then $a_j \leq a_i$, ordered coordinatewise. Show that T_n and U_n are isomorphic posets.

b. [2] Deduce from (a) that T_n is a lattice (called the *Tamari lattice*).

6.33. Let C be a convex n-gon. Let $\mathcal{S}$ be the set of all sets of diagonals of C that do not intersect in the interior of C. Partially order the element of $\mathcal{S}$ by inclusion, and add a $\hat{1}$. Call the resulting poset A_n.

a. [3−] Show that A_n is a simplicial Eulerian lattice of rank $n-2$, as defined in Section 3.14.

b. [3] Show in fact that A_n is the lattice of faces of an $(n-3)$-dimensional convex polytope $\mathcal{Q}_n$.

c. [3−] Find the number $W_i = W_i(n)$ of elements of A_n of rank i. Equivalently, W_i is the number of ways to draw i diagonals of C that do not intersect in their interiors. Note that by Proposition 6.2.1, $W_i(n)$ is also the number of plane trees with $n+i$ vertices and $n-1$ endpoints such that no vertex has exactly one successor.

d. [3−] Define

$$\sum_{i=0}^{n-3} W_i(x-1)^{n-i-3} = \sum_{i=0}^{n-3} h_i x^{n-3-i}, \tag{6.58}$$

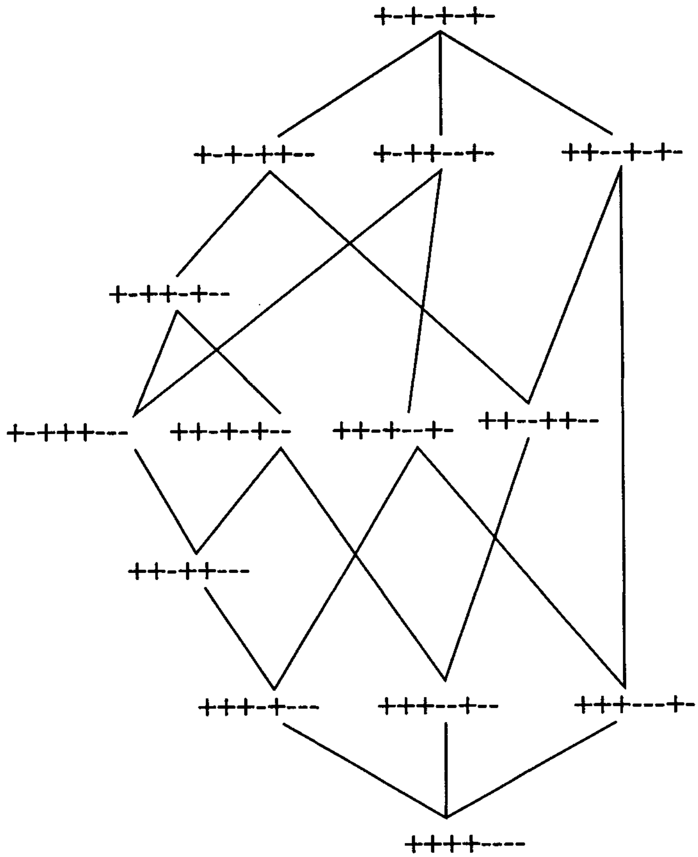


Figure 6-8. The Tamari lattice T_4.

as in equation (3.44). The vector $(h_0, \ldots, h_{n-3})$ is called the h-*vector* of A_n (or of the polytope $\mathcal{Q}_n$). Find an explicit formula for each h_i.

6.34. There are many possible q-analogues of Catalan numbers. In (a) we give what is perhaps the most natural "combinatorial" q-analogue, while in (b) we give the most natural "explicit formula" q-analogue. In (c) we give an interesting extension of (b), while (d) and (e) are concerned with another special case of (c).

a. [2+] Let

$$C_n(q) = \sum_P q^{A(P)},$$

where the sum is over all lattice paths P from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$, such that P never rises above the line $y = x$, and where $A(P)$ is the area under the path (and above the x-axis). Note that by Exercise 6.19(h), we have $C_n(1) = C_n$. (It is interesting to see what statistic corresponds to $A(P)$ for many of the other combinatorial interpretations of C_n given in Exercise 6.19.) For instance, $C_0(q) = C_1(q) = 1$, $C_2(q) = 1+q$, $C_3(q) = 1+q+2q^2+q^3$, $C_4(q) = 1+q+2q^2+3q^3+3q^4+3q^5+q^6$. Show that

$$C_{n+1}(q) = \sum_{i=0}^{n} C_i(q) C_{n-i}(q) q^{(i+1)(n-i)}.$$

Deduce that if $\tilde{C}_n(q) = q^{\binom{n}{2}} C_n(1/q)$, then the generating function

$$F(x) = \sum_{n \geq 0} \tilde{C}_n(q) x^n$$

satisfies

$$xF(x)F(qx) - F(x) + 1 = 0.$$

From this we get the continued fraction expansion

$$F(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{qx}{1 - \cfrac{q^2 x}{1 - \cdots}}}}. \tag{6.59}$$

b. [2+] Define

$$c_n(q) = \frac{1}{(n+1)} \binom{2n}{n}.$$

For instance, $c_0(q) = c_1(q) = 1$, $c_2(q) = 1 + q^2$, $c_3(q) = 1 + q^2 + q^3 + q^4 + q^6$, $c_4(q) = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$. Show that

$$q^n c_n(q) = \sum_w q^{\mathrm{maj}(w)},$$

where w ranges over all sequences $a_1 a_2 \cdots a_{2n}$ of n 1's and n -1's such that each partial sum is nonnegative, and where

$$\mathrm{maj}(w) = \sum_{\{i\,:\,a_i > a_{i+1}\}} i,$$

the major index of w.

c. [3−] Let t be a parameter, and define

$$c_n(t;q) = \frac{1}{(n+1)} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i+1} q^{i^2+it}.$$

Show that

$$c_n(t;q) = \sum_w q^{\mathrm{maj}(w)+(t-1)\mathrm{des}(w)},$$

where w ranges over the same set as in (b), and where

$$\mathrm{des}(w) = \#\{i : a_i > a_{i+1}\},$$

the number of descents of w. (Hence $c_n(1;q) = c_n(q)$.)

d. [3−] Show that

$$c_n(0;q) = \frac{1+q}{1+q^n} c_n(q).$$

For instance, $c_0(0;q) = c_1(0;q) = 1$, $c_2(0;q) = 1+q$, $c_3(0;q) = 1+q+q^2+q^3+q^4$, $c_4(0;q) = 1+q+q^2+2q^3+2q^4+2q^5+2q^6+q^7+q^8+q^9$.

e. [3+] Show that the coefficients of $c_n(0;q)$ are *unimodal*, i.e., if $c_n(0;q) = \sum b_i q^i$, then for some j we have $b_0 \le b_1 \le \cdots \le b_j \ge b_{j+1} \ge b_{j+2} \ge \cdots$. (In fact, we can take $j = \lfloor \frac{1}{2} \deg c_n(0;q) \rfloor = \lfloor \frac{1}{2}(n-1)^2 \rfloor$.)

6.35. Let Q_n be the poset of direct-sum decompositions of an n-dimensional vector space V_n over the field $\mathbb{F}_q$, as defined in Example 5.5.2(b). Let $\bar{Q}_n$ denote Q_n with a $\hat{0}$ adjoined, and let $\mu_n(q) = \mu_{\bar{Q}_n}(\hat{0}, \hat{1})$. Hence by (5.74) we have

$$-\sum_{n\ge 1} \mu_n(q) \frac{x^n}{q^{\binom{n}{2}}(\boldsymbol{n})!} = \log \sum_{n \ge 0} \frac{x^n}{q^{\binom{n}{2}}(\boldsymbol{n})!}.$$

a. [3−] Show that

$$\mu_n(q) = \frac{1}{n}(-1)^n (q-1)(q^2-1)\cdots(q^{n-1}-1)P_n(q),$$

where $P_n(q)$ is a polynomial in q of degree $\binom{n}{2}$ with nonnegative integral coefficients, satisfying $P_n(1) = \binom{2n-1}{n}$. For instance,

$$\begin{aligned}
P_1(q) &= 1 \\
P_2(q) &= 2+q \\
P_3(q) &= 3+3q+3q^2+q^3 \\
P_4(q) &= (2+2q^2+q^3)(2+2q+2q^2+q^3).
\end{aligned}$$

b. [3−] Show that

$$\exp \sum_{n\ge 1} q^{\binom{n}{2}} P_n(1/q) \frac{x^n}{n} = \sum_{n \ge 1} q^{\binom{n}{2}} C_n(1/q) x^n,$$

where $C_n(q)$ is the q-Catalan polynomial defined in Exercise 6.34(a).

6.36. a. [2+] The *Narayana numbers* $N(n,k)$ are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Let X_{nk} be the set of all sequences $w = w_1 w_2 \cdots w_{2n}$ of n 1's and n −1's with all partial sums nonnegative such that

$$k = \#\{j : w_j = 1, w_{j+1} = -1\}.$$

Give a combinatorial proof that $N(n,k) = \#X_{nk}$. Hence by Exercise 6.19(r), there follows

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

(It is interesting to find for each of the combinatorial interpretations of C_n given by Exercise 6.19 a corresponding decomposition into subsets counted by Narayana numbers.)

b. [2+] Let $F(x,t) = \sum_{n\geq 1}\sum_{k\geq 1} N(n,k)x^n t^k$. Using the combinatorial interpretation of $N(n,k)$ given in (a), show that

$$xF^2 + (xt + x - 1)F + xt = 0, \tag{6.60}$$

so

$$F(x,t) = \frac{1 - x - xt - \sqrt{(1-x-xt)^2 - 4x^2 t}}{2x}.$$

6.37. [2+] The *Motzkin numbers* M_n are defined by

$$\begin{aligned}\sum_{n\geq 0} M_n x^n &= \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}\\ &= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7\\ &\quad + 323x^8 + 835x^9 + 2188x^{10} + \cdots.\end{aligned}$$

Show that $M_n = \Delta^n C_1$ and $C_n = \Delta^{2n} M_0$, where C_n denotes a Catalan number.

6.38. [3−] Show that the Motzkin number M_n has the following combinatorial interpretations. (See Exercise 6.46(b) for an additional interpretation.)

a. Number of ways of drawing any number of nonintersecting chords among n points on a circle.

b. Number of walks on $\mathbb{N}$ with n steps, with steps -1, 0, or 1, starting and ending at 0.

c. Number of lattice paths from $(0,0)$ to (n,n), with steps $(0,2)$, $(2,0)$, and $(1,1)$, never rising above the line $y = x$.

d. Number of paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(1,1)$, and $(1,-1)$, never going below the x-axis. Such paths are called *Motzkin paths*.

e. Number of pairs $1 \leq a_1 < \cdots < a_k \leq n$ and $1 \leq b_1 < \cdots < b_k \leq n$ of integer sequences such that $a_i \leq b_i$ and every integer in the set $[n]$ appears at least once among the a_i's and b_i's.

f. Number of ballot sequences (as defined in Corollary 6.2.3(ii)) $(a_1, \ldots, a_{2n+2})$ such that we never have $(a_{i-1}, a_i, a_{i+1}) = (1, -1, 1)$.

g. Number of plane trees with $n/2$ edges, allowing "half edges" that have no successors and count as half an edge.

h. Number of plane trees with $n+1$ edges in which no vertex, the root excepted, has exactly one successor.

i. Number of plane trees with n edges in which every vertex has at most two successors.

j. Number of binary trees with $n-1$ edges such that no two consecutive edges slant to the right.

k. Number of plane trees with $n+1$ vertices such that every vertex of odd height (with the root having height 0) has at most one successor.

l. Number of noncrossing partitions $\pi = \{B_1, \ldots, B_k\}$ of $[n]$ (as defined in Exercise 3.68) such that if $B_i = \{b\}$ and $a < b < c$, then a and c appear in different blocks of π.

m. Number of noncrossing partitions π of $[n+1]$ such that no block of π contains two consecutive integers.

6.39. [3−] The Schröder numbers r_n and s_n were defined in Section 6.2. Show that they have the following combinatorial interpretations.

a. s_{n-1} is the total number of bracketings (parenthesizations) of a string of n letters.

b. s_{n-1} is the number of plane trees with no vertex of degree one and with n endpoints.

c. r_{n-1} is the number of plane trees with n vertices and with each endpoint colored red or blue.

d. s_n is the number of binary trees with n vertices and with each right edge colored either red or blue.

e. s_n is the number of lattice paths in the (x, y)-plane from $(0, 0)$ to the x-axis using steps $(1, k)$, where $k \in \mathbb{P}$ or $k = -1$, never passing below the x-axis, and with n steps of the form $(1, -1)$.

f. s_n is the number of lattice paths in the (x, y)-plane from $(0, 0)$ to (n, n) using steps $(k, 0)$ or $(0, 1)$ with $k \in \mathbb{P}$, and never passing above the line $y = x$.

g. r_{n-1} is the number of parallelogram polynominoes (defined in the solution to Exercise 6.19(l)) of perimeter $2n$ with each column colored either black or white.

h. s_n is the number of ways to draw any number of diagonals of a convex $(n+2)$-gon that do not intersect in their interiors

i. s_n is the number of sequences $i_1 i_2 \cdots i_k$, where $i_j \in \mathbb{P}$ or $i_j = -1$ (and k can be arbitrary), such that $n = \#\{j : i_j = -1\}$, $i_1 + i_2 + \cdots + i_j \geq 0$ for all j, and $i_1 + i_2 + \cdots + i_k = 0$.

j. r_n is the number of lattice paths from $(0, 0)$ to (n, n), with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$, that never rise above the line $y = x$.

k. r_{n-1} is the number of $n \times n$ permutation matrices P with the following property: We can eventually reach the all 1's matrix by starting with P and continually replacing a 0 by a 1 if that 0 has at least two adjacent 1's, where an entry a_{ij} is defined to be adjacent to $a_{i\pm 1, j}$ and $a_{i, j\pm 1}$.

l. Let $u = u_1 \cdots u_k \in \mathfrak{S}_k$. We say that a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ is *u-avoiding* if no subsequence $w_{a_1}, \ldots, w_{a_k}$ (with $a_1 < \cdots < a_k$) is in the same relative order as u, i.e., $u_i < u_j$ if and only if $w_{a_i} < w_{a_j}$. Let $\mathfrak{S}_n(u, v)$ denote the set of permutations $w \in \mathfrak{S}_n$ avoiding both the permutations $u, v \in \mathfrak{S}_4$. There is a group G of order 16 that acts on the set of pairs (u, v) of unequal elements of $\mathfrak{S}_4$ such that if (u, v) and (u', v') are in the same G-orbit (in which case we say that they are *equivalent*), then there is a simple bijection between $\mathfrak{S}_n(u, v)$ and $\mathfrak{S}_n(u', v')$ (for all n). Namely, identifying a permutation with the corresponding permutation matrix, the orbit of (u, v) is obtained by possibly interchanging u and v, and then doing a simultaneous dihedral symmetry of the square matrices u and v. There are then ten inequivalent pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ for which $\#\mathfrak{S}_n(u, v) = r_{n-1}$, namely, (1234, 1243), (1243, 1324), (1243, 1342), (1243, 2143), (1324, 1342), (1342, 1423), (1342, 1432), (1342, 2341), (1342, 3142), and (2413, 3142).

m. r_{n-1} is the number of permutations $w = w_1 w_2 \cdots w_n$ of $[n]$ with the following property: It is possible to insert the numbers $w_1, \ldots, w_n$ in order into a string, and to remove the numbers from the string in the order $1, 2, \ldots, n$. Each insertion must be at the beginning or end of the string. At any time we may remove the first (leftmost) element of the string. (*Example:* $w = 2413$. Insert 2, insert 4 at the right, insert 1 at the left, remove 1, remove 2, insert 3 at the left, remove 3, remove 4.)

n. r_n is the number of sequences of length $2n$ from the alphabet A, B, C such that: (i) for every $1 \le i < 2n$, the number of A's and B's among the first i terms is not less than the number of C's, (ii) the total number of A's and B's is n (and hence the also the total number of C's), and (iii) no two consecutive terms are of the form CB.

o. r_{n-1} is the number of noncrossing partitions (as defined in Exercise 3.68) of some set $[k]$ into n blocks, such that no block contains two consecutive integers.

p. s_n is the number of graphs G (without loops and multiple edges) on the vertex set $[n+2]$ with the following two properties: (α) All of the edges $\{1, n+2\}$ and $\{i, i+1\}$ are edges of G, and (β) G is *noncrossing*, i.e., there are not both edges $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. Note that an arbitrary noncrossing graph on $[n+2]$ can be obtained from those satisfying (α)–(β) by deleting any subset of the required edges in (α). Hence the total number of noncrossing graphs on $[n+2]$ is $2^{n+2} s_n$.

q. r_{n-1} is the number of reflexive and symmetric relations R on the set $[n]$ such that if iRj with $i < j$, then we never have uRv for $i \le u < j < v$.

r. r_{n-1} is the number of reflexive and symmetric relations R on the set $[n]$ such that if iRj with $i < j$, then we never have uRv for $i < u \le j < v$.

s. r_{n-1} is the number of ways to cover with disjoint dominos (or dimers) the set of squares consisting of $2i$ squares in the i-th row for $1 \le i \le n-1$, and with $2(n-1)$ squares in the n-th row, such that the row centers lie on a vertical line. See Figure 6-9 for the case $n = 4$.

6.40. [3−] Let a_n be the number of permutations $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ such that we never have $w_{i+1} = w_i \pm 1$, e.g., $a_4 = 2$, corresponding to 2413 and 3142.

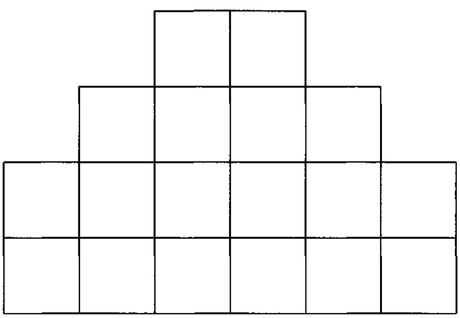


Figure 6-9. A board with $r_3 = 22$ domino tilings.

Equivalently, a_n is the number of ways to place n nonattacking kings on an $n \times n$ chessboard with one king in every row and column. Let

$$\begin{aligned} A(x) &= \sum_{n \geq 0} a_n x^n \\ &= 1 + x + 2x^4 + 14x^5 + 90x^6 + 646x^7 + 5242x^8 + \cdots. \end{aligned}$$

Show that $A(xR(x)) = \sum_{n \geq 0} n! x^n := E(x)$, where

$$R(x) = \sum_{n \geq 0} r_n x^n = \frac{1}{2x}(1 - x - \sqrt{1 - 6x + x^2}),$$

the generating function for Schröder numbers. Deduce that

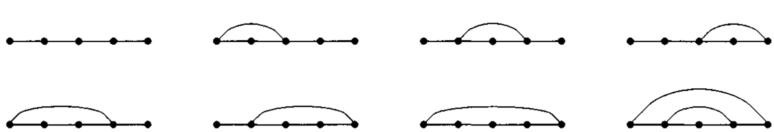
$$A(x) = E\left(\frac{x(1-x)}{1+x}\right).$$

6.41. [3] A permutation $w \in \mathfrak{S}_n$ is called *2-stack sortable* if $S^2(w) = w$, where S is the operator of Exercise 6.19(ii). Show that the number $S_2(n)$ of 2-stack sortable permutations in $\mathfrak{S}_n$ is given by

$$S_2(n) = \frac{2(3n)!}{(n+1)!\,(2n+1)!}.$$

6.42. [2] A king moves on the vertices of the infinite chessboard $\mathbb{Z} \times \mathbb{Z}$ by stepping from (i, j) to any of the eight surrounding vertices. Let $f(n)$ be the number of ways in which a king can walk from $(0, 0)$ to $(n, 0)$ in n steps. Find $F(x) = \sum_{n \geq 0} f(n) x^n$, and find a linear recurrence with polynomial coefficients satisfied by $f(n)$.

6.43. **a.** [2+] A *secondary structure* is a graph (without loops or multiple edges) on the vertex set $[n]$ such that (a) $\{i, i+1\}$ is an edge for all $1 \leq i \leq n-1$, (b) for all i, there is at most one j such that $\{i, j\}$ is an edge and $|j - i| \neq 1$, and (c) if $\{i, j\}$ and $\{k, l\}$ are edges with $i < k < j$, then $i \leq l \leq j$. (Equivalently, a secondary structure may be regarded as a 3412-avoiding involution (as in Exercise 6.19(kk)) such that no orbit consists of two consecutive integers.) Let $s(n)$ be the number of secondary structures with n vertices. For instance, $s(5) = 8$, given by



Let $S(x) = \sum_{n \geq 0} s(n) x^n = 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + 37x^7 + 82x^8 + 185x^9 + 423x^{10} + \cdots$. Show that

$$S(x) = \frac{x^2 - x + 1 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}.$$

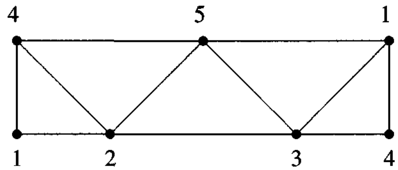


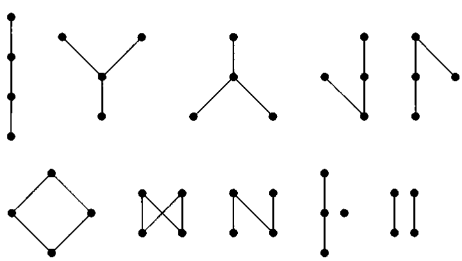
Figure 6-10. A Catalan triangulation of the Möbius band.

b. [3−] Show that $s(n)$ is the number of walks in n steps from $(0, 0)$ to the x-axis, with steps $(1, 0)$, $(0, 1)$, and $(0, -1)$, never passing below the x-axis, such that $(0, 1)$ is never followed directly by $(0, -1)$.

6.44. [3−] Define a *Catalan triangulation* of the Möbius band to be an abstract simplicial complex triangulating the Möbius band that uses no interior vertices, and has vertices labeled $1, 2, \ldots, n$ in order as one traverses the boundary. (If we replace the Möbius band by a disk, then we get the triangulations of Corollary 6.2.3(vi) or Exercise 6.19(a).) Figure 6-10 shows the smallest such triangulation, with five vertices (where we identify the vertical edges of the rectangle in opposite directions). Let $\mathrm{MB}(n)$ be the number of Catalan triangulations of the Möbius band with n vertices. Show that

$$\sum_{n\geq 0} \mathrm{MB}(n)x^n = \frac{x^2[(2-5x-4x^2)+(-2+x+2x^2)\sqrt{1-4x}]}{(1-4x)[1-4x+2x^2+(1-2x)\sqrt{1-4x}]}$$
$$= x^5 + 14x^6 + 113x^7 + 720x^8 + 4033x^9 + 20864x^{10} + \cdots.$$

6.45. [3−] Let $f(n)$ be the number of nonisomorphic n-element posets with no 3-element antichain. For instance, $f(4) = 10$, corresponding to



Let $F(x) = \sum_{n\geq 0} f(n)x^n = 1 + x + 2x^2 + 4x^3 + 10x^4 + 26x^5 + 75x^6 + 225x^7 + 711x^8 + 2311x^9 + 7725x^{10} + \cdots$. Show that

$$F(x) = \frac{4}{2-2x+\sqrt{1-4x}+\sqrt{1-4x^2}}.$$

6.46. **a.** [3+] Let $f(n)$ denote the number of subsets S of $\mathbb{N}\times\mathbb{N}$ of cardinality n with the following property: If $p \in S$ then there is a lattice path from $(0, 0)$

to p with steps (0, 1) and (1, 0), all of whose vertices lie in S. Show that

$$\sum_{n\geq 1} f(n)x^n = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-3x}}-1\right)$$
$$= x + 2x^2 + 5x^3 + 13x^4 + 35x^5 + 96x^6 + 267x^7 + 750x^8 + 2123x^9 + 6046x^{10} + \cdots.$$

b. [3+] Show that the number of such subsets contained in the first octant $0 \leq x \leq y$ is the Motzkin number M_{n-1} (defined in Exercise 6.37).

6.47. a. [3] Let P_n be the Bruhat order on the symmetric group $\mathfrak{S}_n$ as defined in Exercise 3.75(a). Show that the following two conditions on a permutation $w \in \mathfrak{S}_n$ are equivalent:

(i) The interval $[\hat{0}, w]$ of P_n is rank-symmetric, i.e., if ρ is the rank function of P_n (so $\rho(w)$ is the number of inversions of w), then

$$\#\{u \in [\hat{0}, w] : \rho(u) = i\} = \#\{u \in [\hat{0}, w] : \rho(w) - \rho(u) = i\},$$

for all $0 \leq i \leq \rho(w)$.

(ii) The permutation $w = w_1w_2\cdots w_n$ is 4231 and 3412-avoiding, i.e., there do not exist $a < b < c < d$ such that $w_d < w_b < w_c < w_a$ or $w_c < w_d < w_a < w_b$.

b. [3−] Call a permutation $w \in \mathfrak{S}_n$ *smooth* if it satisfies (i) (or (ii)) above. Let $f(n)$ be the number of smooth $w \in \mathfrak{S}_n$. Show that

$$\sum_{n\geq 0} f(n)x^n = \frac{1}{1 - x - \frac{x^2}{1-x}\left(\frac{2x}{1+x-(1-x)C(x)} - 1\right)}$$
$$= 1 + x + 2x^2 + 6x^3 + 22x^4 + 88x^5 + 366x^6 + 1552x^7 + 6652x^8 + 28696x^9 + \cdots,$$

where $C(x) = (1 - \sqrt{1-4x})/2x$ is the generating function for the Catalan numbers.

6.48. [3] Let $f(n)$ be the number of 1342-avoiding permutations $w = w_1w_2\cdots w_n$ in $\mathfrak{S}_n$, i.e., there do not exist $a < b < c < d$ such that $w_a < w_d < w_b < w_c$. Show that

$$\sum_{n\geq 0} f(n)x^n = \frac{32x}{1 + 20x - 8x^2 - (1-8x)^{3/2}}$$
$$= 1 + x + 2x^2 + 6x^3 + 23x^4 + 103x^5 + 512x^6 + 2740x^7 + 15485x^8 + \cdots.$$

6.49. a. [3−] Let B_n denote the board consisting of the following number of squares in each row (read top to bottom), with the centers of the rows lying on a

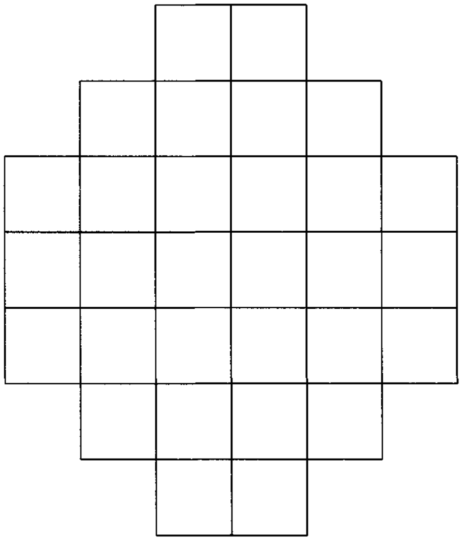


Figure 6-11. A board with $D(3, 3) = 63$ domino tilings.

vertical line: $2, 4, 6, \ldots, 2(n-1), 2n$ (three times), $2(n-1), \ldots, 6, 4, 2$. Figure 6-11 shows the board B_3. Let $f(n)$ be the number of ways to cover B_n with disjoint dominos (or dimers). (A domino consists of two squares with an edge in common.) Show that $f(n)$ is equal to the central Delannoy number $D(n, n)$ (as defined in Section 6.3).

b. [3−] What happens when there are only two rows of length $2n$?

6.50. [3] Let B denote the "chessboard" $\mathbb{N} \times \mathbb{N}$. A *position* consists of a finite subset S of B, whose elements we regard as *pebbles*. A *move* consists of replacing some pebble, say at cell (i, j), with two pebbles at cells $(i+1, j)$ and $(i, j+1)$, provided that each of these cells is not already occupied. A position S is *reachable* if there is some sequence of moves, beginning with a single pebble at the cell $(0, 0)$, that terminates in the position S. A subset T of B is *unavoidable* if every reachable set intersects T. A subset T of B is *minimally unavoidable* if T is unavoidable, but no proper subset of T is unavoidable. Let $u(n)$ be the number of n-element minimally unavoidable subsets of B. Show that

$$\begin{aligned}\sum_{n\geq 0} u(n)x^n &= x^3 \frac{(1-3x+x^2)\sqrt{1-4x} - 1 + 5x - x^2 - 6x^3}{1 - 7x + 14x^2 - 9x^3} \\ &= 4x^5 + 22x^6 + 98x^7 + 412x^8 + 1700x^9 \\ &\quad + 6974x^{10} + 28576x^{11} + \cdots .\end{aligned}$$

6.51. [3+] Let E_n denote the expected number of real eigenvalues of a random $n \times n$ real matrix whose entries are independent random variables from a standard (mean zero, variance one) normal distribution. Show that

$$\sum_{n\geq 0} E_n x^n = \frac{x(1-x+x\sqrt{2-2x})}{(1-x)^2(1+x)}.$$

6.52. [2] Let b_n be the number of ways of parenthesizing a string of n letters, subject to a *commutative* (but nonassociative) binary operation. Thus for instance $b_5 = 3$, corresponding to the parenthesizations

$$x^2 \cdot x^3 \qquad x \cdot (x \cdot x^3) \qquad x(x^2 \cdot x^2).$$

(Note that x^3 is unambiguous, since $x \cdot x^2 = x^2 \cdot x$.) Let

$$\begin{aligned} B(x) &= \sum_{n\geq 1} b_n x^n \\ &= x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 \\ &\quad + 23x^8 + 46x^9 + 98x^{10} + \cdots. \end{aligned}$$

Show that $B(x)$ satisfies the functional equation

$$B(x) = x + \tfrac{1}{2}B(x)^2 + \tfrac{1}{2}B(x^2). \tag{6.61}$$

6.53. [2−] Let $n \in \mathbb{P}$, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

6.54. **a.** [2] Fix $r \in \mathbb{P}$, and define

$$S_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r.$$

Show that the function $S_n^{(r)}$ is P-recursive (as a function of n). More generally, if $d \in \mathbb{P}$ is also fixed, let

$$S_n^{(r,d)} = \sum_{\substack{a_1+\cdots+a_d=n \\ a_i\in\mathbb{N}}} \binom{n}{a_1,\ldots,a_d}^r.$$

Then $S_n^{(r,d)}$ is P-recursive.

b. [3−] Show that

$$\begin{aligned} S_{n+1}^{(1)} - 2S_n^{(1)} &= 0 \\ (n+1)S_{n+1}^{(2)} - (4n+2)S_n^{(2)} &= 0 \\ (n+1)^2 S_{n+1}^{(3)} - (7n^2+7n+2)S_n^{(3)} - 8n^2 S_{n-1}^{(3)} &= 0 \\ (n+1)^3 S_{n+1}^{(4)} - 2(6n^3+9n^2+5n+1)S_n^{(4)} - (4n+1)(4n)(4n-1)S_{n-1}^{(4)} &= 0. \end{aligned}$$

c. [3] Show that in fact $S_n^{(r)}$ satisfies a homogeneous linear recurrence of order $\lfloor \frac{r+1}{2} \rfloor$ with polynomial coefficients.

6.55. **a.** [3] A *Baxter permutation* (originally called a *reduced* Baxter permutation) is a permutation $w \in \mathfrak{S}_n$ satisfying: if $w(r) = i$ and $w(s) = i + 1$, then there is a k_i between r and s (i.e., $r \leq k_i \leq s$ or $s \leq k_i \leq r$) such that $w(t) \leq i$ if t is between r and k_i, while $w(t) \geq i+1$ if $k_i + 1 \leq t \leq s$ or $s \leq t \leq k_i - 1$. For instance, all permutations $w \in \mathfrak{S}_4$ are Baxter permutations except 2413 and 3142. Let $B(n)$ denote the number of Baxter permutations in $\mathfrak{S}_n$. Show that

$$B(n) = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}. \tag{6.62}$$

b. [2+] Deduce that $B(n)$ is P-recursive.

c. [3−] Find a (nonzero) homogeneous linear recurrence with polynomial coefficients satisfied by $B(n)$.

6.56. **a.** [3] An *increasing subsequence* of length j of a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ is a sequence $w_{i_1} w_{i_2} \cdots w_{i_j}$ such that $i_1 < i_2 < \cdots < i_j$ and $w_{i_1} < w_{i_2} < \cdots < w_{i_j}$. Fix $k \in \mathbb{P}$. Let $A_k(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ that have no increasing subsequence of length k. Show that the function A_k is P-recursive.

b. [5] Show that the numbers $A_{k+1}(n)$ satisfy a recurrence of the form

$$\sum_{i=0}^{\lfloor k/2 \rfloor} p_i(n) A_{k+1}(n-i) = 0, \tag{6.63}$$

where $p_i(n)$ is a polynomial of degree at most k with the following additional properties:

(i)

$$p_0(n) = \prod_{i=1}^{\lfloor k/2 \rfloor} [n + i(k-i)]^2.$$

(ii) For $2 \leq i \leq \lfloor k/2 \rfloor + 1$ we have

$$p_i(n) = q_i(n) \prod_{j=1}^{i-1} (n-j)^2, \tag{6.64}$$

where $q_i(n)$ is a polynomial of degree at most $k - 2i$.

(iii) The polynomials $q_i(n)$ of (6.64) are such that the recurrence (6.63) is true with the unique initial condition $A_{k+1}(0) = 1$.

(iv) If $k = 2m + 1$ then the leading coefficient of $q_i(n)$ is the coefficient of z^i in the polynomial

$$\prod_{j=0}^{m} [1 - (2j+1)^2 z].$$

c. [5] Fix a permutation $u \in \mathfrak{S}_k$. Let $A_u(n)$ denote the number of u-avoiding permutations $w \in \mathfrak{S}_n$, as defined in Exercise 6.39(l). Is A_u P-recursive?

6.57. [3−] Let $f(n)$ for $n \in \mathbb{N}$ satisfy a homogeneous linear recurrence relation of order d with constant coefficients over $\mathbb{C}$, i.e.,

$$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0, \tag{6.65}$$

and suppose that f satisfies no such recurrence of smaller order. What is the smallest order of a (nonzero) homogeneous linear recurrence relation with *polynomial* coefficients satisfied by f? (The answer will depend on the recurrence (6.65), not just on d.) (*Example:* If $f(n) = n$, then f satisfies $f(n+2) - 2f(n+1) + f(n) = 0$ (constant coefficients) and $nf(n+1) - (n+1)f(n) = 0$ (polynomial coefficients).)

6.58. [2+] Consider the homogeneous linear equation (6.34), where $P_e(n) \neq 0$ and the ground field K has characteristic 0. Let $\mathcal{V}$ be the K-vector space of all solutions $f : \mathbb{N} \to K$ to (6.34). Show that

$$e \leq \dim \mathcal{V} \leq e + m, \tag{6.66}$$

where m is the number of distinct zeros of $P_e(n)$ that are nonnegative integers. Show that for fixed e and $P_e(n)$, any value of $\dim \mathcal{V}$ in the range given by (6.66) can occur.

6.59. [3−] Show by direct formal arguments that the series $\sec x$ and $\sqrt{\log(1+x^2)}$ are not D-finite. Hence the reciprocal and square root of a D-finite series need not be D-finite.

6.60. [3+] Let $y \in \mathbb{C}[[x]]$ be D-finite with $y(0) \neq 0$. Show that $1/y$ is D-finite if and only if y'/y is algebraic.

6.61. [3+] Let $F(x_1, \ldots, x_m) = \sum_{\alpha \in \mathbb{N}^m} f(\alpha) x^\alpha$ be a rational power series over the field K. Show that

$$\operatorname{diag}(F) := \sum_{n \geq 0} f(n, \ldots, n) t^n$$

is D-finite.

6.62. [3] Let $y \in \mathbb{C}_{\text{alg}}[[x]]$. Thus by Theorem 6.4.6, y satisfies a homogeneous linear differential equation with polynomial coefficients. Show that the least order of such an equation is equal to the dimension of the complex vector space V spanned by y and all its conjugates. (*Example:* Suppose $y^d = R(x) \in \mathbb{C}(x)$, where $y^d - R(x)$ is irreducible over $\mathbb{C}(x)$. The conjugates of y are given by ζy, where $\zeta^d = 1$. Hence y and all its conjugates span a one-dimensional vector space, and equation (6.37) gives the differential equation of order one satisfied by y.)

6.63. **a.** [5] Suppose that $y = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$ is D-finite. Define $\chi : \mathbb{C} \to \mathbb{Z}$ by

$$\chi(a) = \begin{cases} 1, & a \neq 0 \\ 0, & a = 0. \end{cases}$$

Is $\sum_{n \geq 0} \chi(a_n) x^n$ rational? This question is open even if y is just assumed to be algebraic; for the case when y is rational see Exercise 4.3.

b. [3] Show that (a) is false if y is assumed only to satisfy an *algebraic differential equation* (ADE), i.e., there is a nonzero polynomial $f(x_1, x_2, \ldots, x_{d+2}) \in \mathbb{C}[x_1, x_2, \ldots, x_{d+2}]$ such that $f(x, y, y', y'', \ldots, y^{(d)}) = 0$.

c. [5] Suppose that y satisfies an ADE and $y \notin \mathbb{C}[x]$. Can y be more than quadratically lacunary? In other words, if $y = \sum b_i x^{n_i}$, can one have $\lim_{i\to\infty} i^2/n_i = 0$?

6.64. **a.** [2] Let x and y be noncommuting indeterminates. Show that the following identity is valid, in the sense that the formal series represented by both sides are the same.

$$(1+x)(1-yx)^{-1}(1+y) = (1+y)(1-xy)^{-1}(1+x). \quad (6.67)$$

b. [3−] Is equation (6.67) valid in any associative algebra with identity for which both sides are defined?

c. [3] Even more generally, let R be the ring obtained from the noncommutative polynomial ring $K\langle x_1, \ldots, x_n\rangle$ by successively joining inverses of all elements whose series expansion has constant term 1. Let ω be the homomorphism from R to the ring $K\langle\langle x_1, \ldots, x_n\rangle\rangle$ that replaces a rational "function" by its series expansion. Is ω one-to-one?

6.65. **a.** [2+] Verify the statement preceding Definition 6.5.3 and in Example 6.6.2 that the series $\sum_{n\geq 1} x^n y^n$ is not rational.

b. [3−] More generally, suppose that $S \in K_{\mathrm{rat}}\langle\langle X\rangle\rangle$. Let supp$(S)$ be defined as in Definition 6.6.4. Show that there is a $p \in \mathbb{P}$ such that every $z \in \mathrm{supp}(S)$ of length at least p can be decomposed as $z = uvw$ where $v \neq 1$ and $\{uv^n w : n \geq 0\} \cap \mathrm{supp}(S)$ is infinite.

c. [3] Let $L \subseteq X^*$ be a language. Show that L is rational (as defined in Definition 6.6.4) if and only if there is a $p \in \mathbb{P}$ satisfying the following condition: Every word $x \in X^*$ of length p can be decomposed as $x = uvw$, where $v \neq 1$ and, for all $y \in X^*$ and all $n \in \mathbb{N}$, we have that $xy \in L$ if and only if $uv^n wy \in L$.

6.66. [2+] Let $S, T \in K\langle\langle X\rangle\rangle$ be rational series. Show that the Hadamard product $S * T$ (as defined in Section 6.6) is also rational.

6.67. [2] Let $b \geq 2$. If $n = i_k b^k + \cdots + i_1 b + i_0$ is the base b expansion of n (with $i_k > 0$), then associate with n the word $w(n) = x_{i_k} \cdots x_{i_1} x_{i_0}$ in the alphabet $X = \{x_0, \ldots, x_{b-1}\}$. Let $S = \sum_{n\geq 0} n w(n)$, the generating function for base b expansions of positive integers. For instance, if $b = 2$ then

$$S = x_1 + 2x_1x_0 + 3x_1^2 + 4x_1x_0^2 + 5x_1x_0x_1 + 6x_1^2x_0 + 7x_1^3 + \cdots.$$

Show that S is rational.

6.68. [3+] Let (W, S) be a Coxeter group, i.e., W is generated by $S = \{s_1, \ldots, s_n\}$ subject to relations $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$ for all $i < j$, where $m_{ij} \geq 2$ (possibly $m_{ij} = \infty$, meaning there is no relation $(s_i s_j)^{m_{ij}} = 1$). A word $x_{i_1} x_{i_2} \cdots x_{i_p}$ in the alphabet $X = \{x_1, \ldots, x_n\}$ is *reduced* if there is no relation $s_{i_1} s_{i_2} \cdots s_{i_p} = s_{j_1} s_{j_2} \cdots s_{j_q}$ in W with $q < p$. Show that the set of reduced words is a rational language, as defined in Definition 6.6.4.

6.69. [3−] Let A be a finite alphabet. Given $u, v \in A^*$, define $u \leq v$ if u is a subword of v. Write $u \otimes v$ for elements of the direct product $A^* \times A^*$. Show that the series $\sum_{u \leq v} u \otimes v$, and hence the language $\{u \otimes v \in A^* \times A^* : u \leq v\}$, is rational.

6.70. [1] Find explicitly the solution (R_1, R_2) to the proper algebraic system

$$\begin{aligned} z_1 &= y^2 z_1 - z_2^2 \\ z_2 &= z_2 x z_1 + z_1 z_2. \end{aligned}$$

6.71. [2] Let X be a finite alphabet. Show that the set $K_{\text{alg}}\langle\langle X\rangle\rangle$ of all algebraic series in X forms a subalgebra of $K\langle\langle X\rangle\rangle$. Moreover, if $u \in K_{\text{alg}}\langle\langle X\rangle\rangle$ and u^{-1} exists, then $u^{-1} \in K_{\text{alg}}\langle\langle X\rangle\rangle$.

6.72. **a.** [3−] Let $L \in K_{\text{alg}}\langle\langle X\rangle\rangle$ be an algebraic language, as defined in Definition 6.6.4. Show that there is a $p \in \mathbb{P}$ such that every $z \in L$ of length at least p can be decomposed as $z = uvwxy$ where $vx \neq 1$ and $\{uv^n wx^n y : n \geq 0\} \cap L$ is infinite.

b. [1+] Deduce from (a) that $\sum_{n \geq 0} x^n y^n z^n$ is not algebraic.

6.73. **a.** [2+] Let $S = \sum w$, summed over all words $w \in \{x, y\}^*$ with the same number of x's as y's. Thus

$$\phi(S) = \sum_{n \geq 1} \binom{2n}{n} x^n y^n = \frac{1}{\sqrt{1 - 4xy}},$$

where ϕ is the "abelianization" operator. Show that S is algebraic.

b. [5−] What is the least number of equations in a proper algebraic system for which $S - 1$ is a component?

6.74. [3−] Let $x_1, \ldots, x_k$ be noncommuting indeterminates. Let $f(n)$ be the constant term of the noncommutative Laurent polynomial $(x_1 + x_1^{-1} + \cdots + x_k + x_k^{-1})^n$. Show that

$$\sum_{n \geq 0} f(n) t^n = \frac{2k - 1}{k - 1 + k\sqrt{1 - 4(2k - 1)t^2}},$$

as stated in Example 6.7.3.

Solutions to Exercises

6.1. One way to proceed is as follows. Suppose that e^x is algebraic of degree d, and let

$$P_d(x) e^{dx} + P_{d-1}(x) e^{(d-1)x} + \cdots + P_0(x) = 0, \tag{6.68}$$

where $P_i(x) \in \mathbb{C}[x]$, and deg $P_d(x)$ is minimal. Differentiate (6.68) with respect to x and subtract equation (6.68) multiplied by d. We get the equation

$$(P_0' - dP_0) + [P_1' - (d-1)P_1] e^x + \cdots + (P_{d-1}' - P_{d-1}) e^{(d-1)x} + P_d' e^{dx} = 0,$$

which either has degree less than d or else contradicts the minimality of deg P_d. For a less straightforward but also less *ad hoc* solution, see Exercise 6.2.

6.2. **a.** This result is known as *Eisenstein's theorem.* For a proof and additional references, see G. Pólya and G. Szegö, *Problems and Theorems in Analysis,* vol. II, Springer-Verlag, New York/Berlin/Heidelberg, 1976 (Part 8, Ch. 3, §§2–3).

b. Easy from (a).

6.3. It follows from a theorem of R. Jungen [**40**] that if $y = \sum a_n x^n \in \mathbb{C}[[x]]$ is algebraic and $a_n \sim cn^r\alpha^n$ for some constants $0 \neq c \in \mathbb{C}$, $0 > r \in \mathbb{R}$, and $0 \neq \alpha \in \mathbb{C}$, then $r = s + \frac{1}{2}$ for some $s \in \mathbb{Z}$. Since by Stirling's formula $\binom{3n}{n,n,n} \sim cn^{-1}\alpha^n$ and $\binom{2n}{n}^2 \sim dn^{-1}\beta^n$, the proof follows.

For another proof that y_1 and y_2 aren't algebraic, we use the Gaussian hypergeometric series

$$F(a,b,c;x) = \sum_{n\geq 0} \frac{z^n}{n!} \prod_{i=0}^{n-1} \frac{(a+i)(b+i)}{c+i}.$$

It is easy to see that $y_1 = F\left(\frac{1}{3}, \frac{2}{3}, 1; 27x\right)$ and $y_2 = F\left(\frac{1}{2}, \frac{1}{2}, 1; 16x\right)$. It was determined by H. A. Schwarz, *J. Reine Angew. Math.* **75** (1873), 292–335; *Ges. Math. Abh. Bd. II*, Chelsea, New York, 1972, pp. 211–259, for exactly what parameters a, b, c is $F(a,b,c;x)$ algebraic. From this result it follows that y_1 and y_2 aren't algebraic. Alternatively, we could use a result of T. Schneider, *Einführung in die Tranzendenten Zahlen*, Springer, Berlin, 1957, which implies that $y_1(x)$ and $y_2(x)$ take nonalgebraic values (over $\mathbb{Q}$) when $x \neq 0$ is algebraic. Since it is easy to see that algebraic power series with rational coefficients satisfy a polynomial equation with rational coefficients, it follows that y_1 and y_2 aren't algebraic. For yet another proof that y_1 isn't algebraic (due to J. Rickard), see G. Almkvist, W. Dicks, and E. Formanek, *J. Algebra* **93** (1985), 189–214 (p. 209).

Finally, in the paper C. F. Woodcock and H. Sharif, *J. Algebra* **121** (1989), 364–369, appears a clever proof that $z := \sum_{n\geq 0} \binom{2n}{n}^t x^n$ isn't algebraic for any $t \in \mathbb{N}$, $t > 1$. (The asymptotic method of the first proof above works only for even t.) They use the fact that z is algebraic in characteristic $p > 0$ (for which they give a direct proof, though one could also use Exercise 6.11) and show that the degree of z over $\mathbb{F}_p(x)$ is unbounded as $p \to \infty$. From this it follows easily that z is not algebraic. Woodcock and Sharif show that some similar series are not algebraic by the same technique.

In general it seems difficult to determine whether some "naturally occurring" power series y is algebraic. If y is D-finite and one is given the linear differential equation $\mathcal{D}$ of least degree with polynomial coefficients that y satisfies, then a decision procedure for deciding whether y is algebraic (in which case all solutions to $\mathcal{D}$ are algebraic) was given by M. F. Singer, in *Proceedings of the 1979 Queens University Conference on Number Theory*, Queens Papers in Pure and Applied Mathematics **54**, pp. 378–420. (The problem had been almost completely solved by Boulanger and Painlevé in the nineteenth century, as discussed in the previous reference.) Singer has improved and generalized his result in several papers; for a brief discussion and references see his paper in *Computer Algebra and Differential Equations* (E. Tournier, ed.), Academic Press, New York, 1990, pp. 3–57.

* **6.4.** Let K be the algebraic closure of $\mathbb{F}_p$, the finite field of order p. Then the equation $y^p - y - x^{-1}$ has no solution that belongs to $K^{\text{fra}}((x))$, as may be seen by

considering the smallest integer $N > 0$ for which $y \in K((x))[x^{1/N}]$. This example is due to Chevalley [**12**, § IV.6]. Subsequently S. Abhyankar, *Proc. Amer. Math. Soc.* **7** (1956), 903–905, gave the factorization

$$y^p - y - x^{-1} = \prod_{i=0}^{p-1}\left(y - i - \sum_{j\geq 1} x^{-1/p^j}\right).$$

Using this example Men-Fon Huang, Ph.D. thesis, Purdue University, 1968, constructed a "generalized Puiseux field" $A(p)$ containing an algebraic closure of $K((x))$. Further properties of $A(p)$ were developed by S. Vaidya, *Illinois J. Math.* **41** (1997), 129–141. Another example of an equation with no solution in $K^{\mathrm{fra}}((x))$ is mentioned by Cohn [**17**, p. 198] and attributed to M. Ojanguren.

6.5. Let $\eta_1, \ldots, \eta_d$ be the roots of P, so by Proposition 6.1.8 we have $\eta_1, \ldots, \eta_d \in K^{\mathrm{fra}}[[x]]$. Suppose $\eta_j = \sum_{n\geq 0} a_n x^{n/N}$, where $\deg_{K((x))}\eta_j = N$ (so $N = c_s$ for some s). Let ζ be a primitive N-th root of unity, and set $\eta_j^{(k)} = \sum_{n\geq 0} a_n \zeta^{kn} x^{n/N}$ for $0 \leq k \leq N$. Thus disc $P(y)$ is divisible by $\prod_{0\leq a<b\leq N-1}\left(\eta_j^{(a)} - \eta_j^{(b)}\right)^2$. Each $\eta_j^{(a)} - \eta_j^{(b)}$ has zero constant term and hence is divisible by $x^{1/N}$, so the product is divisible by $(x^{1/N})^{2\binom{N}{2}} = x^{N-1}$. The stated result follows easily.

6.6. Follows easily from Exercise 1.37(a).

6.7. Yes. If u satisfies the polynomial equation $P(x, u) = 0$, then $u^{\langle -1\rangle}$ satisfies $P(u^{\langle -1\rangle}, x) = 0$.

6.8. **a.** The discriminant of a polynomial $F(y)$ is 0 if and only if $F(y)$ and $F'(y)$ have a common nonconstant factor. If $F(y) = ay^d + by + c$, then

$$dF(y) - yF'(y) = b(d-1)y + cd.$$

Assume $c \neq 0$. (Otherwise the problem is easy.) Then disc $F(y) = 0$ if and only if $F'(y)$ is divisible by $b(d-1)y + cd$, i.e., $F'(-cd/b(d-1)) = 0$. This leads to the condition

$$d^d ac^{d-1} + (-1)^{d-1}(d-1)^{d-1}b^d = 0,$$

and a simple normalization argument yields equation (6.7).

A somewhat different approach is the following. It is easily seen that for *any* polynomial $G(y) = ay^d + \cdots = a(y-y_1)\cdots(y-y_d)$, we have

$$\operatorname{disc} G = (-1)^{\binom{d}{2}} a^{d-2} G'(y_1)\cdots G'(y_d). \tag{6.69}$$

Letting $G(y) = F(y)$, we see that $F'(y_i) = ady_i^{d-1} + b$, so $y_i F'(y_i) = ady_i^d + by_i = d(-by_i - c) + by_i = b(1-d)y_i - cd$. Hence

$$\operatorname{disc} F = (-1)^{\binom{d}{2}} a^{2d-2} \prod_{i=1}^{d} \frac{b(1-d)y_i - cd}{y_i}$$

$$= (-1)^{\binom{d}{2}} a^{2d-2} \frac{b^d(d-1)^d a^{-1} F(cd/b(1-d))}{(-1)^d a^{-1} c},$$

which by routine manipulation becomes (6.7). This result appears in G. Salmon, *Lectures Introductory to the Modern Higher Algebra*, Dublin, 1885, reprinted by Chelsea, New York, 1969, and E. Netto, *Vorlesungen über Algebra*, Teubner-Verlag, Leipzig, 1896, though it may go back earlier. See also [**28**, Ch. 12, (1.38)].

b. Let

$$P = \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 = \frac{1}{n!}(x - x_1)\cdots(x - x_n).$$

Then $P = x^n/n! + P'$, so by (6.69) we have

$$\begin{aligned}\operatorname{disc} P &= (-1)^{\binom{n}{2}} n!^{-(n-2)} \prod_{i=1}^{n} \left(P(x_i) - \frac{x_i^n}{n!} \right) \\ &= (-1)^{\binom{n}{2}} n!^{-(n-2)} (-1)^n n!^{-n} \left(\prod_{i=1}^{n} x_i \right)^n \\ &= (-1)^{\binom{n-1}{2}} n!^{-2n+2} ((-1)^n n!)^n \\ &= (-1)^{\binom{n}{2}} n!^{-(n-2)}.\end{aligned}$$

This result follows from a more general result of D. Hilbert, *J. Reine Angew. Math.* **103** (1888), 337–345, though as in (a) it may have been known earlier. See also [**28**, Ch. 12, (1.42)].

6.9. a. Let $*_x$ denote Hadamard product with respect to x only, and let diag denote the diagonal (in the variable y) with respect to the variables y_1 and y_2. It is clear that

$$F * G = \operatorname{diag}(F(x, y_1) *_x F(x, y_2)). \tag{6.70}$$

Now $F(x, y_1)$ and $F(x, y_2)$ are both rational series in x over the field $K(y_1, y_2)$, so by Proposition 4.2.5 we have

$$F(x, y_1) *_x F(x, y_2) \in K(y_1, y_2)(x) = K(x)(y_1, y_2).$$

Then by Theorem 6.3.3, $\operatorname{diag}(F(x, y_1) *_x F(x, y_2))$ is algebraic over $K(x)(y) = K(x, y)$, as desired.

An explicit statement of the result of this exercise first appears in C. F. Woodcock and H. Sharif, *J. Algebra* **128** (1990), 517–527 (Thm. 5.1), who give a more complicated proof than ours. For the case when char $K > 0$, see Exercise 6.11.

b. Write $*_i$ for Hadamard product with respect to x_i, and $*_{ij}$ for Hadamard product with respect to x_i and x_j. Since $(1 - x - y)^{-1} = \sum \binom{m+n}{m} x^m y^n$, it is easy to see that

$$\begin{aligned}F_k = \Bigg[&\frac{1}{1 - x_1 - x_2} *_2 \frac{1}{1 - x_2 - x_3} *_3 \frac{1}{1 - x_3 - x_4} \\ &*_4 \cdots *_{k-1} \frac{1}{1 - x_{k-1} - x_k} \Bigg] *_{1k} \frac{1}{1 - x_1 - x_k}.\end{aligned} \tag{6.71}$$

By successive applications of Proposition 4.2.5, the expression within brackets is rational, so by (a) we have that F_k is algebraic.

c. By (6.70) we need to compute

$$\operatorname{diag}\left(\frac{1}{1-x-y_1} *_x \frac{1}{1-x-y_2}\right).$$

Now

$$\frac{1}{1-x-y_i} = \frac{1}{1-y_i}\cdot\frac{1}{1-\frac{x}{1-y_i}} = \frac{1}{1-y_i}\sum_{n\geq 0}(1-y_i)^{-n}x^n.$$

Hence

$$\frac{1}{1-x-y_1} * \frac{1}{1-x-y_2} = \frac{1}{(1-y_1)(1-y_2)}\sum_{n\geq 0}(1-y_1)^{-n}(1-y_2)^{-n}x^n$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-\frac{y_1+y_2-y_1y_2}{1-x}}.$$

Now apply Exercise 6.15 to obtain

$$F_2 = \frac{1}{\sqrt{(1-x+y)^2-4y}} = \frac{1}{\sqrt{(1-x-y)^2-4xy}}.$$

By similar reasoning we obtain

$$F_3 = \frac{1}{\sqrt{(1-x-y-z)^2-4xyz}}.$$

The results of (b) and (c) are due to L. Carlitz, *SIAM Review* **6** (1964), 20–30, essentially by the same methods as here. Carlitz did not appeal to (a) directly to show (b), but rather found an explicit (though based on a recurrence) formula for F_k. In particular, $F_k = P_k^{-1/2}$ where P_k is a polynomial in $x_1, \ldots, x_k$. (This result can also be obtained by a careful analysis of equation (6.71)). For further information and references related to the "cyclic binomial sums" of the type considered here, see V. Strehl, in *Séries Formelles et Combinatoire Algébrique* (P. Leroux and C. Reutenauer, eds.), Publ. du LACIM, vol. 11, Univ. du Québec à Montréal, 1992, pp. 363–377.

6.10. The series y is just the diagonal of

$$\frac{q^{-m}}{1-qP(q)t} = \sum_{n\geq 0} q^{n-m}P(q)^n t^n,$$

so the proof follows from Theorem 6.3.3. (Technically speaking, Theorem 6.3.3 does not apply because we are dealing with a Laurent series and not a power series, but the proof goes through in exactly the same way.) A somewhat more general result was proved by G. Pólya [**53**].

6.11. **a.** This result is due to H. Sharif and C. F. Woodcock, *J. London Math. Soc. (2)* **37** (1988), 395–403 (p. 401).

b. Let $G = 1/(1 - x_1 \cdots x_k)$. By (a), $F * G$ is algebraic over $K(x_1, \ldots, x_k)$. Now $F * G = (\text{diag } F)(x_1 \cdots x_k)$ (the diagonal of F in the variable $x_1 \cdots x_k$). From this it is not hard to deduce that $(\text{diag } F)(x)$ is algebraic over $K(x)$. See *ibid.*, Thm. 7.1, for the details.

This result was first proved by P. Deligne, *Invent. Math.* **76** (1983), 129–143, using sophisticated methods. Another elementary proof (in addition to the one of Sharif and Woodcock just sketched) appears in J. Denef and L. Lipshitz, *J. Number Theory* **26** (1987), 46–67 (Prop. 5.1). Earlier it was shown by H. Furstenberg [**25**] that the diagonal of a *rational* power series in several variables over a field of characteristic $p > 0$ is algebraic. An interesting survey of some aspects of power series in characteristic p (as well as some results valid in characteristic 0), including connections with finite automata, appears in L. Lipshitz and A. J. van der Poorten, in *Number Theory* (R. A. Mollin, ed.), de Gruyter, Berlin/New York, 1990, pp. 340–358.

Note that (a) follows from (b), since

$$(F * G)(x_1, \ldots, x_k) = \text{diag}_{y_1 z_1} \cdots \text{diag}_{y_k z_k} F(y_1, \ldots, y_k) G(z_1, \ldots, z_k),$$

where $\text{diag}_{y_i z_i}$ denotes the diagonal with respect to y_i and z_i in the variable x_i. Hence (a) and (b) are essentially equivalent (as noted by J.-P. Allouche, *Sém. Théor. Nombres Bordeaux (2)* **1** (1989), 163–187).

6.12. Let t be a new indeterminate, and write xt for the variables $x_1 t, \ldots, x_m t$, and similarly for yt. Let $*$ denote the usual Hadamard product with respect to the variable t. We may regard the series $F(xt)$ and $G(yt)$ as rational power series in t over the field $K(x, y)$. Moreover,

$$F(x) \heartsuit G(y) = F(xt) * G(yt)|_{t=1}.$$

Now consider Proposition 4.2.5, which is proved for the field $K = \mathbb{C}$. The proof actually shows that over any field K, the Hadamard product $A * B$ of rational power series A and B in one variable is rational over the algebraic closure of K. But since $A * B$ is defined over K, it is easy to see that $A * B$ is in fact rational over K. It therefore follows that

$$F(xt) * G(yt) \in K(x, y)(t),$$

and the proof follows. A similar argument applies to the case when F is rational and G is algebraic, using Proposition 6.1.11.

The product $\heartsuit$ was first defined (using a different notation) by S. Bochner and W. T. Martin, *Ann. Math.* **38** (1938), 293–302. The result of this exercise was stated without proof by M. P. Schützenberger [**65**, p. 885].

6.13. **a.** The other roots are given by

$$\theta_\zeta = -\frac{1}{k-1} \sum_{n \geq 0} \binom{kn/(k-1)}{n} \zeta^n x^{n/(k-1)}, \tag{6.72}$$

where $\zeta^{k-1} = 1$. By Corollary 6.1.7, it suffices to prove the assertion for

$\zeta = 1$. Let

$$\tau = \theta_1(x^{k-1}) = -\frac{1}{k-1}\sum_{n\geq 0}\binom{kn/(k-1)}{n}x^n.$$

We need to show that

$$k^k x^{k-1}\tau^k = (\tau - 1)[(k-1)\tau + 1]^k. \qquad (6.73)$$

Applying Example 6.2.7 to the series $-(k-1)\tau = (1-k)\tau$ (with k replaced by $k/(k-1)$) gives

$$\frac{(1-k)\tau - 1}{1 + \left(\frac{k}{k-1} - 1\right)(1-k)\tau} = x\left(\frac{\frac{k}{k-1}(1-k)\tau}{1 + \left(\frac{k}{k-1} - 1\right)(1-k)\tau}\right)^{\frac{k}{k-1}}$$

Simple algebraic manipulations show that this equation is equivalent to (6.73). Since the $k-1$ series (6.72) are all conjugate over $\mathbb{C}(x)$ by Corollary 6.1.7 (or Proposition 6.1.6), either $P(y)$ is irreducible or $y = \sum \binom{kn}{n}x^n$ is rational. It is easy to see [why?] that y can't be rational, so irreducibility follows.

b. *Answer:* $(-1)^{\binom{k}{2}}k^{k(k-1)}[k^k x - (k-1)^{k-1}]x^{k-2}$

6.14. The recurrence relation (6.53) and the values $f(i,0) = 2^{-i}$ imply that

$$(x^2 - 2x + y)F(x,y) = -2x + H(y)$$

for some $H(y) \in \mathbb{Q}[[y]]$. The left-hand side vanishes formally when $x = 1 - \sqrt{1-y}$, so hence also the right-hand side. Thus $H(y) = 2(1 - \sqrt{1-y})$, so

$$F(x,y) = \frac{2(1-\sqrt{1-y}) - 2x}{x^2 - 2x + y} = \frac{2}{1 - x + \sqrt{1-y}}.$$

This exercise is due to R. Pemantle. A slightly different result also due to Pemantle is cited in M. Larsen and R. Lyons, *J. Theor. Probability*, to appear.

6.15. Let

$$F(s,t) = [1 - sf(st) - tg(st) - h(st)]^{-1},$$

so

$$\begin{aligned}\operatorname{diag} F &= [u^0]\,\frac{1}{1 - uf(x) - \frac{x}{u}g(x) - h(x)}\\ &= [u^0]\,\frac{-u}{u^2 f - (h-1)u + xg}.\end{aligned}$$

The computation now parallels that of Examples 6.3.4, 6.3.8, and 6.3.9 (which the present exercise generalizes), yielding

$$\operatorname{diag} F = \frac{1}{\sqrt{(1-h)^2 - 4xfg}}.$$

A "naive" proof can also be given by generalizing the solution to Exercise 1.5(b).

6.17. **a.** Because of the conditions on S, no S-path can move from one side of the line $y = x$ to the other without actually landing on the line $y = x$. Let $\mathcal{P}_m$ be the set of all S-paths from (0, 0) to (m, m) that don't just consist of the single step (m, m) and that stay strictly below $y = x$ except at (0, 0) and (m, m). Every S-path from (0, 0) to (n, n) is a unique product (juxtaposition) of S-paths of the types (i) a single step from S of the form (j, j), (ii) elements of $\mathcal{P}_m$ for some m, and (iii) reflections about $y = x$ of elements of $\mathcal{P}_m$ for some m. Every S-path from (0, 0) to (n, n) that doesn't rise above $y = x$ is a unique product of S-paths of types (i) and (ii). Hence if $P(x) = \sum_{m\geq 1}(\#\mathcal{P}_m)x^m$, then

$$G = \frac{1}{1-K-2P}, \qquad H = \frac{1}{1-K-P}.$$

Eliminating P from these two equations gives $H = 2/(1-K+G^{-1})$, as desired.

b. We have $K(x) = x$, while from Exercise 1.5(b) or Example 6.3.8 we have $G(x) = 1/\sqrt{1-6x+x^2}$. Hence

$$H(x) = \frac{2}{1-x+\sqrt{1-6x+x^2}} = \frac{1-x-\sqrt{1-6x+x^2}}{2x},$$

so by the discussion of Schröder's second problem in Section 6.2 we have $h(n) = r_n$.

c. It was observed in the discussion of Schröder's second problem that $r_n/2$ $(= s_n)$ does indeed count dissections of the appropriate kind. However, this observation leads only to a generating function, not combinatorial, proof that $h(n) = r_n$. A combinatorial proof was subsequently given by L. W. Shapiro and R. A. Sulanke, Bijections for the Schröder numbers, preprint.

6.18. See I. M. Gessel, *J. Combinatorial Theory (A)* **28** (1980), 321–337 (Cor. 5.4).

6.19. It would require a treatise in itself to discuss thoroughly all the known interconnections among these problems. We will content ourselves with some brief hints and references that should serve as a means of further exposure to "Catalan disease" or "Catalania" (= Catalan mania). One interesting item omitted from the list because of its complicated description is the number of flexagons of order $n+1$; see C. O. Oakley and R. J. Wisner, *Amer. Math. Monthly* **64** (1957), 143–154 (esp. p. 152) for more information.

Parts a, b, c, h, i, r are covered by Corollary 6.2.3.

d. This is covered by Example 5.3.12. Alternatively, do a depth-first search, ignoring the root edge, recording 1 when a left edge is first encountered, and recording -1 when a right edge is first encountered. This gives a bijection with (r). Note also that when all endpoints are removed (together with the incident edges), we obtain the trees of (c).

e. This is covered by Example 6.2.8. For a bijection with (r), do a depth-first (preorder) search through the tree. When going "down" an edge (away from the root) record a 1, and when going up an edge record a -1. For further information and references, see D. A. Klarner, *J. Combinatorial Theory* **9** (1970), 401–411.

f. When the root is removed we obtain the trees of (d). See also Klarner, *op. cit.*

g. The bijection between parts (i) and (iv) of Proposition 6.2.1 gives a bijection between the present problem and (j). An elegant bijection with (e) was given by F. Bernhart (private communication, 1996).

j. Let $A(x) = x + x^3 + 2x^4 + 6x^5 + \cdots$ (respectively, $B(x) = x^2 + x^3 + 3x^4 + 8x^5 + \cdots$) be the generating function for Dyck paths from $(0, 0)$ to $(2n, 0)$ $(n > 0)$ such that the path only touches the x-axis at the beginning and end, and the number of steps $(1, -1)$ at the end of the path is odd (respectively, even). Let $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ be the generating function for all Dyck paths from $(0, 0)$ to $(2n, 0)$, so the coefficients are Catalan numbers by (i). It is easy to see that $A = x(1 + CB)$ and $B = xCA$. (Also $C = 1/(1 - A - B)$, though we don't need that fact here.) Solving for A gives $A = x/(1 - x^2C^2)$. The generating function we want is $1/(1 - A)$, which simplifies (using $1 + xC^2 = C$) to $1 + xC$, and the proof follows. This result is due to E. Deutsch (private communication, 1996).

k. This result is due to P. Peart and W. Woan, Dyck paths with no peaks at height 2, preprint. The authors give a generating function proof and a simple bijection with (i).

l. The region bounded by the two paths is called a *parallelogram polyomino*. It is an array of unit squares, say with k columns $C_1, \ldots, C_k$. Let a_i be the number of squares in column C_i, for $1 \leq i \leq k$, and let b_i be the number of rows in common to C_i and C_{i+1}, for $1 \leq i \leq k - 1$. Define a sequence σ of 1's and -1's as follows (where exponentiation denotes repetition):

$$\sigma = 1^{a_1}(-1)^{a_1-b_1+1}1^{a_2-b_1+1}(-1)^{a_2-b_2+1}1^{a_3-b_2+1}\cdots 1^{a_k-b_{k-1}+1}(-1)^{a_k}.$$

This sets up a bijection with (r). For the parallelogram polyomino of Figure 6-12 we have $(a_1, \ldots, a_7) = (3, 3, 4, 4, 2, 1, 2)$ and $(b_1, \ldots, b_6) =$

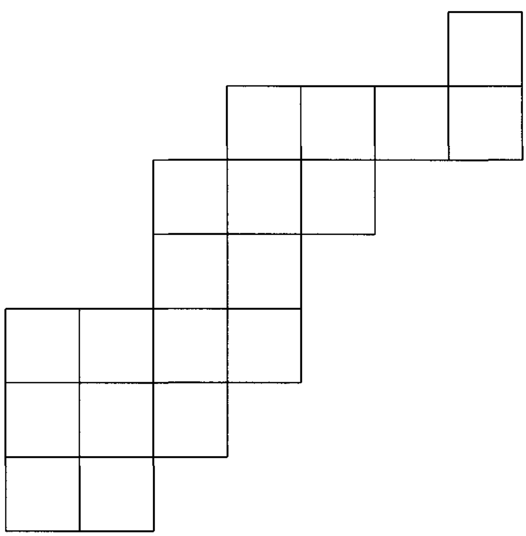


Figure 6-12. A parallelogram polyomino.

(3, 2, 3, 2, 1, 1). Hence (writing $-$ for -1)

$$\sigma = 111 - 1 - -111 - -11 - - - 1 - -1 - 11 - -.$$

The enumeration of parallelogram polyominoes is due to J. Levine, *Scripta Math.* **24** (1959), 335–338, and later G. Pólya, *J. Combinatorial Theory* **6** (1969) 102–105. See also L. W. Shapiro, *Discrete Math.* **14** (1976), 83–90; W.-J. Woan, L. W. Shapiro, and D. G. Rogers, *Amer. Math. Monthly* **104** (1997), 926–931; G. Louchard, *Random Structures and Algorithms* **11** (1997), 151–178; and R. A. Sulanke, *J. Difference Equations and Applications*, to appear. For more information on the fascinating topic of polyomino enumeration, see M.-P. Delest and G. Viennot, *Theoretical Computer Science* **34** (1984), 169–206, and X. G. Viennot, in *Séries Formelles et Combinatoire Algébrique* (P. Leroux and C. Reutenauer, eds.), Publications de Laboratoire de Combinatoire et d'Informatique Mathématique **11**, Université du Québec à Montréal, 1992, pp. 399–420.

m. Regarding a path as a sequence of steps, remove the first and last steps from the two paths in (l). This variation of (l) was suggested by L. W. Shapiro (private communication, 1998).

n. Fix a vertex v. Starting clockwise from v, at each vertex write 1 if encountering a chord for the first time and -1 otherwise. This gives a bijection with (r). This result is apparently due to A. Errera, *Mém. Acad. Roy. Belgique Coll. 8º (2)* **11** (1931), 26 pp. See also J. Riordan, *Math. Comp.* **29** (1975), 215–222, and S. Dulucq and J.-G. Penaud, *Discrete Math.* **17** (1993), 89–105.

o. Cut the circle in (n) between two fixed vertices and "straighten out."

p,q. These results are due to I. M. Gelfand, M. I. Graev, and A. Postnikov, in *The Arnold-Gelfand Mathematical Seminars*, Birkhäuser, Boston, 1997, pp. 205–221 (§6). For (p), note that there is always an arc from the leftmost to the rightmost vertex. When this arc is removed, we obtain two smaller trees satisfying the conditions of the problem. This leads to an easy bijection with (c). The trees of (p) are called *noncrossing alternating trees*.

An equivalent way of stating the above bijection is as follows. Let T be a noncrossing alternating tree on the vertex set $1, 2, \dots, n+1$ (in that order from left to right). Suppose that vertex i has r_i neighbors that are larger than i. Let u_i be the word in the alphabet $\{1, -1\}$ consisting of r_i 1's followed by a -1. Let $u(T) = u_1 u_2 \cdots u_{n+1}$. Then u is a bijection between the objects counted by (p) and (r). It was shown by M. Schlosser that exactly the same definition of u gives a bijection between (q) and (r)! The proof, however, is considerably more difficult than in the case of (p). (A more complicated bijection was given earlier by C. Krattenthaler.)

For further information on trees satisfying conditions (α), (β), and (δ) (called *alternating trees*), see Exercise 5.41.

s. Consider a lattice path P of the type (h). Let a_i be the area above the x-axis between $x = i-1$ and $x = i$, and below P. This sets up a bijection as desired.

t. Subtract $i-1$ from a_i and append a one at the beginning to get (s). This result is closely related to Exercise 6.25(c). If we replace the alphabet $1, 2, \dots, 2(n-1)$ with the alphabet $\overline{n-1}, n-1, \overline{n-2}, n-2, \dots, \overline{1}, 1$ (in that order) and write the new sequence $b_1, b_2, \dots, b_{n-1}$ in reverse order in a column, then we obtain the arrays of R. King, in *Lecture Notes in Physics*, vol. 50, Springer-Verlag, Berlin/Heidelberg/New York, 1975, pp. 490–499 (see also S. Sundaram, *J. Combinatorial Theory (A)* **53** (1990), 209–238 (Def. 1.1)) that index the weights of the $(n-1)$-st fundamental representation of $Sp(2(n-1), \mathbb{C})$.

u. Let $b_i = a_i - a_{i+1} + 1$. Replace a_i with one 1 followed by b_i -1's for $1 \le i \le n$ (with $a_{n+1} = 0$) to get (r).

v. Take the first differences of the sequences in (u).

w. Do a depth-first search through a plane tree with $n+1$ vertices as in (e). When a vertex is encountered for the first time, record one less than its number of successors, except that the last vertex is ignored. This gives a bijection with (e).

x. These sequences are just the inversion tables (as defined in Section 1.3) of the 321-avoiding permutations of (ee). For a proof see S. C. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (Thm. 2.1). (The previous reference deals with the *code* $c(w)$ of a permutation w rather than the inversion table $I(w)$. They are related by $c(w) = I(w^{-1})$. Since w is 321-avoiding if and only if w^{-1} is 321-avoiding, it makes no difference whether we work with the code or with the inversion table.)

y. If we replace a_i by $n - a_i$, then the resulting sequences are just the inversion tables of 213-avoiding permutations w (i.e., there does not exist $i < j < k$ such that $w_j < w_i < w_k$). Such permutations are in obvious bijection with the 312-avoiding permutations of (ff). For further aspects of this exercise, see Exercise 6.32.

z. Given a sequence $a_1, \ldots, a_n$ of the type being counted, define recursively a binary tree $T(a_1, \ldots, a_n)$ as follows. Set $T(\emptyset) = \emptyset$. If $n > 0$, then let the left subtree of the root of $T(a_1, \ldots, a_n)$ be $T(a_1, a_2, \ldots, a_{n-a_n})$ and the right subtree of the root be $T(a_{n-a_n+1}, a_{n-a_n+2}, \ldots, a_{n-1})$. This sets up a bijection with (c). Alternatively, the sequences $a_n - 1, a_{n-1} - 1, \ldots, a_1 - 1$ are just the inversion tables of the 312-avoiding permutations of (ff). Let us also note that the sequences $a_1, a_2, \ldots, a_n$ are precisely the sequences $\tau(u)$, $u \in \mathfrak{S}_n$, of Exercise 5.49(d).

aa. If $a = a_1 a_2 \cdots a_k$ is a word in the alphabet $[n-1]$, then let $w(a) = s_{a_1} s_{a_2} \cdots s_{a_k} \in \mathfrak{S}_n$, where s_i denotes the adjacent transposition $(i, i+1)$. Then $w(a) = w(b)$ if $a \sim b$; and the permutations $w(a)$, as a ranges over a set of representatives for the classes B being counted, are just those enumerated by (ee). This statement follows from S. C. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (Thm. 2.1).

bb. Regard a partition whose diagram fits in an $(n-1) \times (n-1)$ square as an order ideal of the poset $(\boldsymbol{n-1}) \times (\boldsymbol{n-1})$ in an obvious way. Then the partitions being counted correspond to the order ideals of (ccc). Bijections with other Catalan families were given by D. E. Knuth and A. Postnikov. Postnikov's bijection is the following. Let λ be a partition whose diagram is contained in an $(n-1) \times (n-1)$ square S. Let x be the lower right corner of the Durfee square of λ. Let L_1 be the lattice path from the upper right corner of S to x that follows the boundary of λ. Similarly let L_2 be the lattice path from the lower left corner of S to x that follows the boundary of λ. Reflect L_2 about the main diagonal of S. The paths L_1 and the reflection of L_2 form a pair of paths as in (m). Figure 6-13 illustrates this bijection for $n = 8$ and $\lambda = (5, 5, 3, 3, 3, 1)$. The path L_1 is shown in dark lines and L_2 and its reflection in dashed lines.

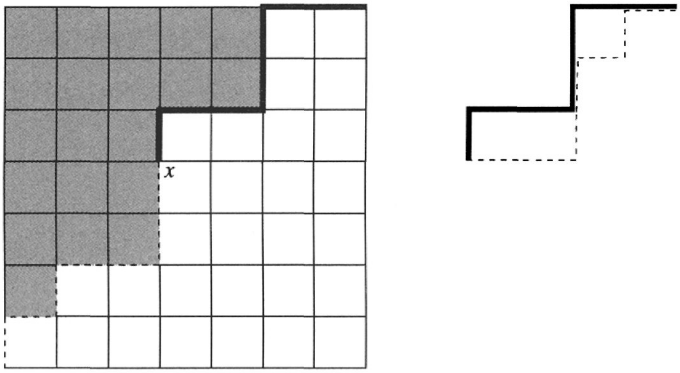


Figure 6-13. A bijection between (ccc) and (bb).

cc. Remove the first occurrence of each number. What remains is a permutation w of $[n]$ that uniquely determines the original sequence. These permutations are precisely the ones in (ff). There is also an obvious bijection between the sequences being counted and the nonintersecting arcs of (o).

dd. Replace each odd number by 1 and even number by -1 to get a bijection with (r).

ee. Corollary 7.23.11 shows that the RSK algorithm (Section 7.11) establishes a bijection with (xx). See also D. E. Knuth, *The Art of Computer Programming*, vol. 3, *Sorting and Searching*, Addison-Wesley, Reading, Massachusetts, 1973 (p. 64).

The earliest explicit enumeration of 321-avoiding permutations seems to be due to J. M. Hammersley, in *Proc. Sixth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, University of California Press, Berkeley/Los Angeles, 1972, pp. 345–394. In equation (15.9) he states the result, saying "and this can be proved in general." The first published proof is a combinatorial proof due to D. G. Rogers, *Discrete Math.* **22** (1978), 35–40. Another direct combinatorial proof, based on an idea of Goodman, de la Harpe, and Jones, appears in S. C. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (after the proof of Theorem 2.1). A sketch of this proof goes as follows. Given the 321-avoiding permutation $w = a_1a_2\cdots a_n$, define $c_i = \#\{j : j > i, w_j < w_i\}$. Let $\{j_1, \ldots, j_\ell\}_< = \{j : c_j > 0\}$. Define a lattice path from $(0, 0)$ to (n, n) as follows. Walk horizontally from $(0, 0)$ to $(c_{j_1} + j_1 - 1, 0)$, then vertically to $(c_{j_1} + j_1 - 1, j_1)$, then horizontally to $(c_{j_2} + j_2 - 1, j_1)$, then vertically to $(c_{j_2} + j_2 - 1, j_2)$, etc. The last part of the path is a vertical line from $(c_{j_\ell} + j_\ell - 1, j_{\ell-1})$ to $(c_{j_\ell} + j_\ell - 1, j_\ell)$, then (if needed) a horizontal line to (n, j_ℓ), and finally a vertical line to (n, n). This establishes a bijection with (h).

For an elegant bijection with (ff), see R. Simion and F. W. Schmidt, *Europ. J. Combinatorics* **6** (1985), 383–406 (Prop. 19). Two other bijections with (ff) appear in D. Richards, *Ars Combinatoria* **25** (1988), 83–86, and J. West, *Discrete Math.* **146** (1995) 247–262 (Thm. 2.8).

ff. There is an obvious bijection between 312-avoiding and 231-avoiding permutations, viz., $a_1a_2\cdots a_n \mapsto n+1-a_n,\ldots,n+1-a_2,n+1-a_1$. It is easily seen that the 231-avoiding permutations are the same as those of (ii), as first observed by D. E. Knuth [**5.41**, Exer. 2.2.1.5]. The enumeration *via* Catalan numbers appears in *ibid.*, Exer. 2.2.1.4. References to bijections with (ee) are given in the solution to (ee). For the problem of counting permutations in $\mathfrak{S}_n$ according to the number of subsequences with the pattern 132 (equivalently, 213, 231, or 312), see M. Bóna, in *Conference Proceedings*, vol. 1, *Formal Power Series and Algebraic Combinatorics, July 14–July 18, 1997,* Universität Wien, pp. 107–118.

gg. This result appears on p. 796 of D. M. Jackson, *Trans. Amer. Math. Soc.* **299** (1987), 785–801, but probably goes back much earlier. For a direct bijective proof, it is not hard to show that the involutions counted here are the same as those in (kk).

hh. A coding of planar maps due to R. Cori, *Astérisque* **27** (1975), 169 pp., when restricted to plane trees, sets up a bijection with (e).

ii. When an element a_i is put on the stack, record a 1. When it is taken off, record a -1. This sets up a bijection with (r). This result is due to D. E. Knuth [**5.41**, Exer. 2.2.1.4]. The permutations being counted are just the 231-avoiding permutations, which are in obvious bijection with the 312-avoiding permutations of (ff) (see Knuth, *ibid.*, Exer. 2.2.1.5).

jj. Same set as (ee), as first observed by R. Tarjan, *J. Assoc. Comput. Mach.* **19** (1972), 341–346 (the case $m = 2$ of Lemma 2). The concept of queue sorting is due to Knuth [**5.41**, Ch. 2.2.1].

kk. Obvious bijection with (o).

ll. See I. M. Gessel and C. Reutenauer, *J. Combinatorial Theory (A)* **64** (1993), 189–215 (Thm. 9.4 and discussion following).

mm. This result is due to O. Guibert and S. Linusson, in *Conference Proceedings*, vol. 2, *Formal Power Series and Algebraic Combinatorics, July 14 – July 18, 1997*, Universität Wien, pp. 243–252. Is there a nice formula for the number of alternating Baxter permutations of $[m]$?

nn. This is the same set as (ee). See Theorem 6.2.1 of the reference given in (ee) to S. C. Billey *et al.* For a generalization to other Coxeter groups, see J. R. Stembridge, *J. Alg. Combinatorics* **5** (1996), 353–385.

oo. These are just the 132-avoiding permutations $w_1\cdots w_n$ of $[n]$ (i.e., there does not exist $i<j<k$ such that $w_i<w_k<w_j$), which are in obvious bijection with the 312-avoiding permutations of (ff). This result is an immediate consequence of the following results: (i) I. G. Macdonald, *Notes on Schubert Polynomials*, Publications du LACIM, vol. 6, Univ. du Québec à Montréal, 1991, (4.7) and its converse stated on p. 46 (due to A. Lascoux and M. P. Schützenberger), (ii) *ibid.*, eqn. (6.11) (due to Macdonald), (iii) part (ff) of this exercise, and (iv) the easy characterization of dominant permutations (as defined in Macdonald, *ibid.*) as 132-avoiding permutations. For a simpler proof of the crucial (6.11) of Macdonald, see S. Fomin and R. Stanley, *Advances in Math.* **103** (1994), 196–207 (Lemma 2.3).

* **pp.** See Exercise 3.68(b). Noncrossing partitions first arose in the work of H. W. Becker, *Math. Mag.* **22** (1948–49), 23–26, in the form of *planar rhyme*

schemes, i.e., rhyme schemes with no crossings in the *Puttenham diagram*, defined by G. Puttenham, *The Arte of English Poesie*, London, 1589 (pp. 86–88). Further results on noncrossing partitions are given by H. Prodinger, *Discrete Math.* **46** (1983), 205–206; N. Dershowitz and S. Zaks, *Discrete Math.* **62** (1986), 215–218; R. Simion and D. Ullman, *Discrete Math.* **98** (1991), 193–206; P. H. Edelman and R. Simion, *Discrete Math.* **126** (1994), 107–119; R. Simion, *J. Combinatorial Theory (A)* **65** (1994), 270–301; R. Speicher, *Math. Ann.* **298** (1994), 611–628; A. Nica and R. Speicher, *J. Algebraic Combinatorics* **6** (1997), 141–160; R. Stanley, *Electron. J. Combinatorics* **4**, R20 (1997), 14 pp. See also Exercise 5.35.

qq. These partitions are clearly the same as the noncrossing partitions of (pp). This description of noncrossing partitions is due to R. Steinberg (private communication).

rr. Obvious bijection with (pp). (Vertical lines are in the same block if they are connected by a horizontal line.) As mentioned in the Notes to Chapter 1, Murasaki diagrams were used in *The Tale of Genji* to represent the 52 partitions of a five-element set. The noncrossing Murasaki diagrams correspond exactly to the noncrossing partitions. The statement that noncrossing Murasaki diagrams are enumerated by Catalan numbers seems first to have been observed by H. W. Gould, who pointed it out to M. Gardner, leading to its mention in [**27**]. Murasaki diagrams were not actually used by Lady Murasaki herself. It wasn't until the Wasan period of old Japanese mathematics, from the late 1600s well into the 1700s, that the Wasanists started attaching the Murasaki diagrams (which were actually incense diagrams) to illustrated editions of *The Tale of Genji*.

ss. This result was proved by M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 56), using generating function techniques.

tt. See R. C. Mullin and R. G. Stanton, *Pacific J. Math.* **40** (1972), 167–172 (p. 168). They set up a bijection with (e). They also show that $2n+1$ is the largest possible value of k for which there exists a noncrossing partition of $[k]$ with $n+1$ blocks such that no block contains two consecutive integers. A simple bijection with (a) was given by D. P. Roselle, *Utilitas Math.* **6** (1974), 91–93. The following bijection with (d) is due to A. Vetta (1997). Label the vertices $1, 2, \ldots, 2n+1$ of a tree in (d) in preorder. Define i and j to be in the same block of $\pi \in \Pi_{2n+1}$ if j is a right child of i.

uu. Let P_n denote the poset of intervals with at least two elements of the chain $\boldsymbol{n}$, ordered by inclusion. Let $\mathcal{A}_n$ denote the set of antichains of P_n. By the last paragraph of Section 3.1, $\#\mathcal{A}_n$ is equal to the number of order ideals of P_n. But P_n is isomorphic to the poset $\mathrm{Int}(\boldsymbol{n}-\boldsymbol{1})$ of *all* (nonempty) intervals of $\boldsymbol{n}-\boldsymbol{1}$, so by (bbb) we have $\#\mathcal{A}_n = C_n$. Given an antichain $A \in \mathcal{A}_n$, define a partition π of $[n]$ by the condition that i and j (with $i < j$) belong to the same block of π if $[i, j] \in A$ (and no other conditions not implied by these). This establishes a bijection between $\mathcal{A}_n$ and the nonnesting partitions of $[n]$. For a further result on nonnesting partitions, see the solution to Exercise 5.44. The present exercise was obtained in collaboration with A. Postnikov. The concept of nonnesting partitions for any reflection group (with the present case corresponding to the symmetric group $\mathfrak{S}_n$) is due to

Postnikov and is further developed in C. A. Athanasiadis, On noncrossing and nonnesting partitions for classical reflection groups, preprint, 1998.

vv. If $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \subseteq (n-1, n-2, \ldots, 1)$, then the sequences $(1, \lambda_{n-1} + 1, \ldots, \lambda_1 + 1)$ are in bijection with (s). Note also that the set of Young diagrams contained in $(n-1, n-2, \ldots, 1)$, ordered by inclusion (i.e., the interval $[\emptyset, (n-1, n-2, \ldots, 0)]$ in Young's lattice, as defined in Exercise 3.63), is isomorphic to $J(\mathrm{Int}(\boldsymbol{n}-\mathbf{1}))^*$, thereby setting up a bijection with (bbb).

ww. Given a standard Young tableau T of shape (n, n), define $a_1 a_2 \cdots a_{2n}$ by $a_i = 1$ if i appears in row 1 of T, while $a_i = -1$ if i appears in row 2. This sets up a bijection with (r). See also [**7.72**, p. 63] and our Proposition 7.10.3.

xx. See the solution to (ee) (first paragraph) for a bijection with 321-avoiding permutations. An elegant bijection with (ww) appears in [**2.15**, vol. 1, p. 131] (repeated in [**7.72**, p. 63] Namely, given a standard Young tableau T of shape (n, n), let P consist of the part of T containing the entries $1, 2, \ldots, n$; while Q consists of the complement in T of P, rotated 180°, with the entry i replaced by $2n + 1 - i$. See also Corollary 7.23.12.

yy. Let b_i be the number of entries in row i that are equal to $n - i + 1$ (so $b_n = 0$). The sequences $b_n + 1, b_{n-1} + 1, \ldots, b_1 + 1$ obtained in this way are in bijection with (s).

zz. This result is equivalent to Prop. 2.1 of S. C. Billey, W. Jockusch, and R. Stanley, *J. Algebraic Combinatorics* **2** (1993), 345–374. See also the last paragraph on p. 363 of this reference.

aaa. Obvious bijection with (ww). This interpretation of Catalan numbers appears in [**3.34**, p. 222]. Note also that if we label the elements of $\mathbf{2} \times \boldsymbol{n}$ analogously to what was illustrated for $n = 3$, then the linear extensions coincide with the permutations of (dd).

bbb. There is an obvious bijection with order ideals I of $\mathrm{Int}(\boldsymbol{n})$ that contain every one-element interval of $\boldsymbol{n}$. But the "upper boundary" of the Hasse diagram of I "looks like" the Dyck paths of (i). See [**3.34**, bottom of p. 222].

ccc. This result is equivalent to the $q = 1$ case of G. E. Andrews, *J. Statist. Plann. Inf.* **34** (1993), 19–22 (Corollary 1). For a more explicit statement and some generalizations, see R. G. Donnelly, Ph.D. thesis, University of North Carolina, 1997, and Symplectic and odd orthogonal analogues of $L(m, n)$, preprint. For a bijective proof, see the solution to (bb). A sequence of posets interpolating between the poset $\mathrm{Int}(\boldsymbol{n}-\mathbf{1})$ of (bbb) and A_{n-1}, and all having C_n order ideals, was given by D. E. Knuth (private communication, 9 December 1997).

ddd. Given a sequence $1 \le a_1 \le \cdots \le a_n$ of integers with $a_i \le i$, define a poset P on the set $\{x_1, \ldots, x_n\}$ by the condition that $x_i < x_j$ if and only if $j + a_{n+1-i} \ge n + 1$. (Equivalently, if Z is the matrix of the zeta function of P, then the 1's in $Z - I$ form the shape of the Young diagram of a partition, rotated 90° clockwise and justified into the upper right-hand corner.) This yields a bijection with (s). This result is due to R. L. Wine and J. E. Freund, *Ann. Math. Statist.* **28** (1957), 256–259. See also R. A. Dean and G. Keller, *Canad. J. Math.* **20** (1968), 535–554. Such posets are now called *semiorders*. For further information, see P. C. Fishburn, *Interval Orders and Interval Graphs*, Wiley-Interscience, New York, 1985, and W. T. Trotter, *Combinatorics and Partially Ordered Sets*, Johns Hopkins University Press, Baltimore/London, 1992 (Ch. 8). For the labeled version of this exercise, see Exercise 6.30.

eee. The lattice $J(P)$ of order ideals of the poset P has a natural planar Hasse diagram. There will be two elements covering $\hat{0}$, corresponding to the two minimal elements of P. Draw the Hasse diagram of $J(P)$ so that the rooted minimal element of P goes to the left of $\hat{0}$ (so the other minimal element goes to the right). The "outside boundary" of the Hasse diagram then "looks like" the pair of paths in (l) (rotated 45° counterclockwise).

fff. These relations are called *similarity relations*. See L. W. Shapiro, *Discrete Math.* **14** (1976), 83–90; V. Strehl, *Discrete Math.* **19** (1977), 99–101; D. G. Rogers, *J. Combinatorial Theory (A)* **23** (1977), 88–98; J. W. Moon, *Discrete Math.* **26** (1979), 251–260. Moon gives a bijection with (r). E. Deutsch (private communication) has pointed out an elegant bijection with (h), viz., the set enclosed by a path and its reflection in the diagonal *is* a similarity relation (as a subset of $[n] \times [n]$). The connectedness of the columns ensures the last requirement in the definition of a similarity relation.

ggg. A simple combinatorial proof was given by L. W. Shapiro, *J. Combinatorial Theory* **20** (1976), 375–376. Shapiro observes that this result is a combinatorial manifestation of the identity

$$\sum_{k\geq 0} \binom{n}{2k} 2^{n-2k} C_k = C_{n+1},$$

due to J. Touchard, in *Proc. Int. Math. Congress, Toronto (1924)*, vol. 1, 1928 (p. 465).

hhh. Obvious bijection with (bbb). This interpretation in terms of stacking coins is due to J. Propp. See A. M. Odlyzko and H. S. Wilf, *Amer. Math. Monthly* **95** (1988), 840–843 (Rmk. 1).

* **iii.** See J. H. van Lint, *Combinatorial Theory Seminar, Eindhoven University of Technology*, Lecture Notes in Mathematics, **382**, Springer-Verlag, Berlin/Heidelberg/New York, 1974 (pp. 22 and 26–27).

jjj. The total number of n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ is $\binom{2n}{n}$ (see Section 1.2). Call two such multisets M and N *equivalent* if for some $k \in \mathbb{Z}/(n+1)\mathbb{Z}$ we have $M = \{a_1, \ldots, a_n\}$ and $N = \{a_1+k, \ldots, a_n+k\}$. This defines an equivalence relation in which each equivalence class contains $n+1$ elements, exactly one of which has its elements summing to 0. Hence the number of multisets with elements summing to 0 (or to any other fixed element of $\mathbb{Z}/(n+1)\mathbb{Z}$) is $\frac{1}{n+1}\binom{2n}{n}$. This result appears in R. K. Guy, *Amer. Math. Monthly* **100** (1993), 287–289 (with a more complicated proof due to I. Gessel).

kkk. This result is implicit in the paper G. X. Viennot, *Astérisque* **121–122** (1985), 225–246. Specifically, the bijection used to prove (12), when restricted to Dyck words, gives the desired bijection. A simpler bijection follows from the work of J.-G. Penaud, in *Séminaire Lotharingien de Combinatoire*, 22^e Session, Université Louis Pasteur, Strasbourg, 1990, pp. 93–130 (Cor. IV-2-8). Yet another proof follows from more general results of J. Bétréma and J.-G. Penaud, *Theoret. Comput. Sci.* **117** (1993), 67–88. For some related problems, see Exercise 6.46.

* **lll.** Let I be an order ideal of the poset $\text{Int}(\boldsymbol{n}-\mathbf{1})$ defined in (bbb). Associate with I the set R_I of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $x_1 > \cdots > x_n$ and $x_i - x_j < 1$ if $[i, j-1] \in I$. This sets up a bijection between (bbb) and the regions R_I being counted. This result is implicit in R. Stanley, *Proc. Natl. Acad. Sci. U.S.A.* **93** (1996), 2620–2625 (§2), and also appears (as part of more general results) in C. A. Athanasiadis, Ph.D. thesis, Massachusetts Institute of Technology, 1996 (Cor. 7.1.3) and A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint (Prop. 7.2), available at http://front.math.ucdavis.edu/math.CO/9712213.

* **mmm.** Let P be a convex $(n+2)$-gon with vertices $v_1, v_2, \ldots, v_{n+2}$ in clockwise order. Let T be a triangulation of T as in (a), and let a_i be the number of triangles incident to v_i. Then the map $T \mapsto (a_1, \ldots, a_{n+2})$ establishes a bijection with (a). This remarkable result is due to J. H. Conway and H. S. M. Coxeter [**20**, problems (28) and (29)]. The arrays (6.54) are called *frieze patterns*.

nnn. See F. T. Leighton and M. Newman, *Proc. Amer. Math. Soc.* **79** (1980), 177–180, and L. W. Shapiro, *Proc. Amer. Math. Soc.* **90** (1984), 488–496.

6.20. **a.** Given a path P of the first type, let (i, i) be the first point on P that intersects $y = x$. Replace the portion of P from $(1, 0)$ to (i, i) by its reflection about $y = x$. This yields the desired bijection.

This argument is the famous "reflection principle" of D. André, *C. R. Acad. Sci. Paris* **105** (1887), 436–437. The application (b) below is also due to André. The importance of the reflection principle in combinatorics and probability theory was realized by W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, John Wiley and Sons, New York, 1950 (3rd edition, 1968). For a vast number of extensions and ramifications, see L. Takács, *Combinatorial Methods in the Theory of Stochastic Processes*, John Wiley and Sons, New York, 1967; T. V. Narayana, *Lattice Path Combinatorics with Statistical Applications*, Mathematical Expositions no. 23, University of Toronto Press, Toronto, 1979; and S. G. Mohanty, *Lattice Path Counting and Applications*, Academic Press, New York, 1979. For a profound generalization of the reflection principle based on the theory of Coxeter groups, as well as some additional references, see I. Gessel and D. Zeilberger, *Proc. Amer. Math. Soc.* **115** (1992), 27–31.

b. The first step in such a lattice path must be from $(0, 0)$ to $(1, 0)$. Hence we must subtract from the total number of paths from $(1, 0)$ to (m, n) the number that intersect $y = x$, so by (a) we get $\binom{m+n-1}{n} - \binom{m+n-1}{m} = \frac{m-n}{m+n}\binom{m+n}{n}$.

c. Move the path one unit to the right to obtain the case $m = n + 1$ of (b).

6.21. **a.** Given a path $P \in X_n$, define $c(P) = (c_0, c_1, \ldots, c_n)$, where c_i is the number of horizontal steps of P at height $y = i$. It is not difficult to verify that the cyclic permutations $C_j = (c_j, c_{j+1}, \ldots, c_n, c_1, \ldots, c_{j-1})$ of $c(P)$ are all distinct, and for each such there is a unique $P_j \in X_n$ with $c(P_j) = C_j$. Moreover, the number of excedances of the paths $P = P_0, P_1, \ldots, P_n$ are just the numbers $0, 1, \ldots, n$ in some order. From these observations the proof is immediate.

This result, known as the *Chung–Feller theorem*, is due to K. L. Chung and W. Feller, *Proc. Nat. Acad. Sci. U.S.A.* **35** (1949), 605–608. A refinement was given by T. V. Narayana, *Skand. Aktuarietidskr.* **1967** (1967), 23–30. For further information, see the books by Narayana (§I.2) and Mohanty (§3.3) mentioned in the solution to Exercise 6.20(a).

b. Immediate from (a).

6.22. This result was given by L. W. Shapiro, problem E2903, *Amer. Math. Monthly* **88** (1981), 619. An incorrect solution appeared in **90** (1983), 483–484. A correct but nonbijective solution was given by D. M. Bloom, **92** (1985), 430. The editors asked for a bijective proof in problem E3096, **92** (1985), 428, and such a proof was given by W. Nichols, **94** (1987), 465–466. Nichols's bijection is the following. Regard a lattice path as a sequence of E's (for the step $(1,0)$) and N's (for the step $(0,1)$). Given a path P of the type we are enumerating, define recursively a new path $\psi(P)$ as follows:

$$\psi(\emptyset)=\emptyset, \qquad \psi(DX)=D\psi(X), \qquad \psi(D'X)=E\psi(X)ND^*,$$

where (a) D is a path of positive length, with endpoints on the diagonal $x=y$ and all other points below the diagonal, (b) D' denotes the path obtained from D by interchanging E's and N's, and (c) $D=ED^*N$. Then ψ establishes a bijection between the paths we are enumerating and the paths of Exercise 6.19(h) with n replaced by $2n$. For an explicit description of ψ^{-1} and a proof that ψ is indeed a bijection, see the solution of Nichols cited above.

6.23. The Black pawn on a6 must promote to a knight and then move (in a unique way) to h7 in five additional moves. The Black pawn on a7 must also promote to a knight and then move (in a unique way) to f8 in four additional moves. White then plays Pf7 mate. The first move must be Pa5, after which the number of solutions is the same as if the pawn on a7 were on a6. Each pawn then makes nine moves (including moves after promotion). After the first move Pa5, denote a move by the pawn on a5 by $+1$ and a move by the pawn on a7 by -1. Since the pawn on a7 can never overtake the pawn on a5 (even after promotion), it follows that the number of solutions is just the number of sequences of nine 1's and nine -1's with all partial sums nonnegative. By Exercise 6.19(r), the number of solutions is therefore the Catalan number $C_9=4862$.

This problem is due to Kauko Väisänen, and appears in A. Puusa, *Queue Problems*, Finnish Chess Problem Society, Helsinki, 1992 (Problem 2). This booklet contains fifteen problems of a similar nature. See also Exercise 7.18. For more information on serieshelpmates in general, see A. Dickins, *A Guide to Fairy Chess*, Dover, New York, 1971, p. 10, and J. M. Rice and A. Dickins, *The Serieshelpmate*, second edition, Q Press, Kew Gardens, 1978.

6.24. These are just Catalan numbers! See for instance J. Gili, *Catalan Grammar*, Dolphin, Oxford, 1993, p. 39. A related question appears in *Amer. Math. Monthly* **103** (1996), 538 and 577.

6.25. **a.** Follows from Exercise 3.29(b) and 6.19(bbb). See L. W. Shapiro, *American Math. Monthly* **82** (1975), 634–637.

b. We assume knowledge of Chapter 7. It follows from the results of Appendix 2 of Chapter 7 that we want the coefficient of the trivial Schur function $s_\emptyset$ in the Schur function expansion of $(x_1+x_2)^{2n}$ in the ring $\Xi_2=\Lambda_2/(x_1x_2-1)$. Since $s_\emptyset=s_{(n,n)}$ in Ξ_2, the number we want is just $\langle s_1^{2n}, s_{(n,n)}\rangle = f^{(n,n)}$ (using Corollary 7.12.5), and the result follows from Exercise 6.19(ww).

c. See R. Stanley, *Ann. New York Acad. Sci.*, vol. 576, 1989, pp. 500–535 (Example 4 on p. 523).

d. See R. Stanley, in *Advanced Studies in Pure Math.*, vol. 11, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, 1987, pp. 187–213 (bottom of p. 194). A simpler proof follows from R. Stanley, *J. Amer. Math. Soc.* **5** (1992), 805–851 (Prop. 8.6). For a related result, see C. Chan, *SIAM J. Disc. Math.* **4** (1991), 568–574.

e. See L. R. Goldberg, *Adv. Math.* **85** (1991), 129–144 (Thm. 1.7).

f. See D. Tischler, *J. Complexity* **5** (1989), 438–456.

g. This algebra is the *Temperley–Lieb algebra* $A_{\beta,n}$ (over K), with many interesting combinatorial properties. For its basic structure see F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer-Verlag, New York, 1989, p. 33 and §2.8. For a direct connection with 321-avoiding permutations (defined in Exercise 6.19(ee)), see S. C. Billey, W. Jockusch, and R. Stanley, *J. Algebraic Combinatorics* **2** (1993), 345–374 (pp. 360–361).

h. See J.-Y. Shi, *Quart. J. Math.* **48** (1997), 93–105 (Thm. 3.2(a)).

i. This remarkable conjecture is a small part of a vast conjectured edifice due to M. Haiman, *J. Algebraic Combinatorics* **3** (1994), 17–76. See also A. M. Garsia and M. Haiman, *J. Algebraic Combinatorics* **5** (1996), 191–244; A. M. Garsia and M. Haiman, *Electron. J. Combinatorics* **3** (1996), no. 2, Paper 24; and M. Haiman, (t, q)-Catalan numbers and the Hilbert scheme, *Discrete Math.*, to appear.

6.26. a. This curious result can be proved by induction using suitable row and column operations. It arose from a problem posed by E. Berlekamp and was solved by L. Carlitz, D. P. Roselle, and R. A. Scoville, *J. Combinatorial Theory* **11** (1971), 258–271. A slightly different way of stating the result appears in [**3.34**, p. 223].

b. *Answer:* $a_n = C_n$, the nth Catalan number. One way (of many) to prove this result is to apply part (a) to the cases $\lambda = (2n+1, 2n, \ldots, 2, 1)$ and $\lambda = (2n, 2n-1, \ldots, 2, 1)$, and to use the interpretation of Catalan numbers given by Corollary 6.2.3(v). Related work appears in A. Kellogg (proposer), Problem 10585, *Amer. Math. Monthly* **104** (1997), 361, and C. Radoux, *Bull. Belgian Math. Soc. (Simon Stevin)* **4** (1997), 289–292.

6.27. a. The unique such basis $y_0, y_1, \ldots, y_n$, up to sign and order, is given by

$$y_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{i+j}{2i} x_i.$$

b. Now

$$y_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{i+j+1}{2i+1} x_i.$$

6.28. a. The problem of computing the probability of convexity was raised by J. van de Lune and solved by R. B. Eggleton and R. K. Guy, *Math. Mag.* **61** (1988), 211–219, by a clever integration argument. The proof of Eggleton and Guy can be "combinatorialized" so that integration is avoided. The decomposition of $\mathcal{C}_d$ given below in the solution to (c) also yields a proof. For a more general result, see P. Valtr, in *Intuitive Geometry (Budapest, 1995)*, Bolyai Soc. Math. Stud. **6**, János Bolyai Math Soc., Budapest, 1997, pp. 441–443.

b. Suppose that $x = (x_1, x_2, \ldots, x_d) \in \mathcal{C}_d$. We say that an index i is *slack* if $2 \le i \le d-1$ and $x_{i-1} + x_{i+1} > 2x_i$. If no index is slack, then either $x = (0, 0, \ldots, 0), x = (1, 1, \ldots, 1)$, or $x = \lambda(1, 1, \ldots, 1)+(1-\lambda)y$ for $y \in \mathcal{C}_d$ and sufficiently small $\lambda > 0$. Hence in this last case x is not a vertex. So suppose that x has a slack index. If for all slack indices i we have $x_i = 0$, then x is of the stated form (6.55). Otherwise, let i be a slack index such that $x_i > 0$. Let $j = i-p$ be the largest index such that $j < i$ and j is not slack. Similarly, let $k = i + q$ be the smallest index such that $k > i$ and k is not slack. Let

$$A(\epsilon) = \left(x_1, \ldots, x_j, x_{j+1} + \frac{\epsilon}{p}, x_{j+2} + \frac{2\epsilon}{p}, \ldots, \right.$$
$$\left. x_i + \epsilon, \ldots, x_{k-2} + \frac{2\epsilon}{q}, x_{k-1} + \frac{\epsilon}{q}, x_k, \ldots, x_n\right).$$

For small $\epsilon > 0$, both $A(\epsilon)$ and $A(-\epsilon)$ are in $\mathcal{C}_d$. Since $x = \frac{1}{2}\big[A(\epsilon) + \frac{1}{2}A(-\epsilon)\big]$, it follows that x is not a vertex. The main idea of this argument is due to A. Postnikov.

c. For $1 \le r \le s \le d$, let

$$F_{rs} = \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_r = x_{r+1} = \cdots = x_s = 0\}$$
$$F_r^- = \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_r = x_{r+1} = \cdots = x_d = 0\}$$
$$F_s^+ = \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_1 = x_2 = \cdots = x_s = 0\}.$$

Now F_r^- is a simplex with vertices $\left(1, \frac{k-1}{k}, \frac{k-2}{k}, \ldots, \frac{1}{k}, 0, 0, \ldots, 0\right)$ for $1 \le k \le r-1$, together with $(0, 0, \ldots, 0)$. These vertices have denominators (i.e., the smallest positive integer whose product with the vertex has integer coordinates) $1, 2, 3, \ldots, r-1, 1$, respectively. Hence

$$\sum_{n \ge 0} i(F_r^-, n)x^n = \frac{1}{[1][r-1]!}.$$

Similarly

$$\sum_{n \ge 0} i\left(F_s^+, n\right)x^n = \frac{1}{[1][d-s]!}.$$

Since $F_{rs} \cong F_r^- \times F_s^+$, we have $i(F_{rs}, n) = i(F_r^-, n)i(F_s^+, n)$ and

$$\sum_{n \ge 0} i(F_{rs}, n)x^n = \frac{1}{[1][r]!} * \frac{1}{[1][d-s]!}.$$

Let P be the poset of all F_{rs}'s, ordered by inclusion, and let μ denote the Möbius function of $P \cup \{\hat{1}\}$. Let $G = \bigcup_{r=1}^d F_{rr}$, a polyhedral complex in $\mathbb{R}^d$. By Möbius inversion we have

$$i(G, n) = -\sum_{F_{st} \in P} \mu(F_{st}, \hat{1})i(F_{st}, n).$$

But $F_{tu} \subseteq F_{rs}$ if and only if $t \le r \le s \le u$, from which it is immediate that

$$-\mu(F_{st}, \hat{1}) = \begin{cases} 1, & s = t \\ -1, & s = t-1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n\ge 0} i(G,n)x^n = \sum_{r=1}^{d} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-r]!}$$
$$- \sum_{r=1}^{d-1} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-1-r]!}.$$

Now the entire polytope C_d is just a cone over G with apex $(1, 1, \ldots, 1)$. From this it is not hard to deduce that

$$\sum_{n\ge 0} i(C_d, n)x^n = \frac{1}{1-x} \sum_{n\ge 0} i(G,n)x^n,$$

and the proof follows.

6.29. See P. Valtr, *Discrete Comput. Geom.* **13** (1995), 637–643. Valtr also shows in *Combinatorica* **16** (1996), 567–573, that if n points are chosen uniformly and independently from inside a triangle, then the probability that the points are in convex position is $\frac{2^n}{(2n)!}\binom{3n-3}{n-1,n-1,n-1}$.

6.30. Equation (6.57) is equivalent to

$$\sum_{n\ge 0} f_n \frac{1}{n!}[\log(1-x)^{-1}]^n x^n = C(x).$$

Hence by (5.25) we need to show that

$$n!\, C_n = \sum_{k=1}^{n} c(n,k) f_k,$$

where $c(n,k)$ is the number of permutations $w \in \mathfrak{S}_n$ with k cycles. Choose a permutation $w \in \mathfrak{S}_n$ with k cycles in $c(n,k)$ ways. Let the cycles of w be the elements of a semiorder P in f_k ways. For each cycle $(a_1, \ldots, a_i)$ of w, replace this element of P with an antichain whose elements are labeled $a_1, \ldots, a_i$. If $a = (a_1, \ldots, a_i)$ and $b = (b_1, \ldots, b_j)$ are two cycles of w, then define $a_r < b_s$ if and only if $a < b$ in P. In this way we get a poset $\rho(P, w)$ with vertices $1, 2, \ldots, n$. It is not hard to see that $\rho(P, w)$ is a semiorder, and that every isomorphism class of n-element semiorders occurs exactly $n!$ times among the posets $\rho(P, w)$. Since by Exercise 6.19(ddd) there are C_n nonisomorphic n-element semiorders, the proof follows.

This result was first proved by J. L. Chandon, J. Lemaire, and J. Pouget, *Math. Sci. Hum.* **62** (1978), 61–80, 83. For a more general situation in which the number A_n of unlabeled objects is related to the number B_n of labeled objects by $\sum B_n(x^n/n!) = \sum A_n(1-e^{-x})^n$, see R. Stanley, *Proc. Natl. Acad.*
* *Sci. U.S.A.* **93** (1996), 2620–2625 (Thm. 2.3) and A. Postnikov and R. Stanley,

Deformations of Coxeter hyperplane arrangements, preprint (§6), available at http://front.math.ucdavis.edu/math.CO/9712213.

6.31. **a.** (Sketch.) We will triangulate $\mathcal{P}$ into d-dimensional simplices σ, all containing 0. Thus each σ will have d vertices of the form $e_i - e_j$, where $i < j$. Given a graph G with d edges on the vertex set $[d+1]$, let σ_G be the convex hull of all vectors $e_i - e_j$ for which ij is an edge of G with $i < j$, and let $\tilde{\sigma}_G$ be the convex hull of σ_G and the origin. It is easy to see that $\tilde{\sigma}_G$ is a d-dimensional simplex if and only if G is a tree. Moreover, it can be shown that σ_G lies on the boundary of $\mathcal{P}$ (and hence can be part of a triangulation of the type we are looking for) if and only if G is an *alternating* tree, as defined in Exercise 5.41. We therefore want to choose a set $\mathcal{T}$ of alternating trees T on $[d+1]$ such that the $\tilde{\sigma}_T$'s are the facets of a triangulation of $\mathcal{P}$. One way to do this is to take $\mathbb{T}$ to consist of the *noncrossing* alternating trees on $[d+1]$, i.e., alternating trees such that if $i < j < k < l$, then not both ik and jl are edges. By Exercise 6.19(p) the number of such trees is C_d. (We can also take $\mathbb{T}$ to consist of alternating trees on $[d+1]$ such that if $i < j < k < l$ then not both il and jk are edges. By Exercise 6.19(q) the number of such trees is again C_d.) Moreover, it is easy to see that for any tree T on $[d+1]$ we have $V(\tilde{\sigma}_T) = 1/d!$, where V denotes relative volume. Hence $V(\mathcal{P}) = C_d/d!$. This result appears in I. M. Gelfand, M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, in *The Arnold-Gelfand Mathematical Seminars,* Birkhäuser, Boston, 1997, pp. 205–221 (Thm. 2.3(2)).

b. Order the $\binom{d+1}{2}$ edges ij, $1 \leq i < j \leq d+1$, lexicographically, e.g., $12 < 13 < 14 < 23 < 24 < 34$. Order the C_d noncrossing alternating trees $T_1, T_2, \ldots, T_{C_d}$ lexicographically by edge set, i.e., $T_i < T_j$ if for some k the first (in lexicographic order) k edges of T_i and T_j coincide, while the $(k+1)$-st edge of T_i precedes the $(k+1)$-st edge of T_j. For instance, when $d = 3$ the ordering on the noncrossing alternating trees (denoted by their set of edges) is

$$\{12, 13, 14\},\ \{12, 14, 34\},\ \{13, 14, 23\},\ \{14, 23, 24\},\ \{14, 23, 34\}.$$

One can check that $\tilde{\sigma}_{T_i}$ intersects $\tilde{\sigma}_{T_1} \cup \cdots \cup \tilde{\sigma}_{T_{i-1}}$ in a union of $j-1$ $(d-1)$-dimensional faces of $\tilde{\sigma}_{T_i}$, where j is the number of vertices of T_i that are less than all their neighbors. It is not hard to see that the number of noncrossing alternating trees on $[d+1]$ for which exactly j vertices are less that all their neighbors is just the Narayana number $N(d, j)$ of Exercise 6.36. It follows from the techniques of R. Stanley, *Annals of Discrete Math.* **6** (1980), 333–342 (especially Thm. 1.6), that

$$(1-x)^{d+1} \sum_{n \geq 0} i(\mathcal{P}, n) x^n = \sum_{j=1}^{d} N(d, j) x^{j-1}.$$

6.32. The Tamari lattice was first considered by D. Tamari, *Nieuw Arch. Wisk.* **10** (1962), 131–146, who proved it to be a lattice. A simpler proof of this result was given by S. Huang and D. Tamari, *J. Combinatorial Theory (A)* **13** (1972), 7–13. The proof sketched here follows J. M. Pallo, *Computer J.* **29** (1986), 171–175. For further properties of Tamari lattices and their generalizations, see P. H. Edelman and V. Reiner, *Mathematika* **43** (1996), 127–154; A. Björner and M. L. Wachs, *Trans. Amer. Math. Soc.* **349** (1997), 3945–3975 (§9); and the references given there.

6.33. a. Since $\mathcal{S}$ is a simplicial complex, A_n is a simplicial semilattice. It is easy to see that it is graded of rank $n-2$, the rank of an element being its cardinality (number of diagonals). To check the Eulerian property, it remains to show that $\mu(x, \hat{1}) = (-1)^{\ell(x,\hat{1})}$ for all $x \in A_n$. If $x \in A_n$ and $x \neq \hat{1}$, then x divides the polygon C into regions $C_1, \ldots, C_j$, where each C_i is a convex n_i-gon for some n_i. Let $\bar{A}_n = A_n - \{\hat{1}\}$. It follows that the interval $[x, \hat{1}]$ is isomorphic to the product $\bar{A}_{n_1} \times \cdots \times \bar{A}_{n_j}$, with a $\hat{1}$ adjoined. It follows from Exercise 5.61 (dualized) that it suffices to show that $\mu(\hat{0}, \hat{1}) = (-1)^{n-2}$. Equivalently (since we have shown that every proper interval is Eulerian), we need to show that A_n has as many elements of even rank as of odd rank. One way to proceed is as follows. For any subset B of A_n, let $\eta(B)$ denote the number of elements of B of even rank minus the number of odd rank. Label the vertices of C as $1, 2, \ldots, n$ in cyclic order. For $3 \leq i \leq n-1$, let $\mathcal{S}^*$ be the set of all elements of $\mathcal{S}$ for which either there is a diagonal from vertex 1 to some other vertex, or else such a diagonal can be adjoined without introducing any interior crossings. Given $S \in \mathcal{S}^*$, let i be the least vertex that is either connected to 1 by a diagonal or for which we can connect it to vertex 1 by a diagonal without introducing any interior crossing. We can pair S with the set S' obtained by deleting or adjoining the diagonal from 1 to i. This pairing (or involution) shows that $\eta(\mathcal{S}^*) = 0$. But $A_n - \mathcal{S}^*$ is just the interval $[T, \hat{1}]$, where T contains the single diagonal connecting 2 and n. By induction (as mentioned above) we have $\eta([T, \hat{1}]) = 0$, so in fact $\eta(A_n) = 0$.

b. See C. W. Lee, *Europ. J. Combinatorics* **10** (1989), 551–560. An independent proof was given by M. Haiman (unpublished). This polytope is called the *associahedron*. For a far-reaching generalization, see [**28**, Ch. 7] and the survey article C. W. Lee, in *DIMACS Series in Discrete Math. and Theo. Comput. Sci.* **4**, 1991, pp. 443–456.

c. Write

$$F(x, y) = x + \sum_{n \geq 2} \sum_{i=1}^{n-1} W_{i-1}(n+1) x^n y^i$$

$$= x + x^2 y + x^3(y + 2y^2) + x^4(y + 5y^2 + 5y^3) + \cdots.$$

By removing a fixed exterior edge from a dissected polygon and considering the edge-disjoint union of polygons thus formed, we get the functional equation

$$F = x + y \frac{F^2}{1-F}.$$

(Compare equation (6.15).) Hence by Exercise 5.59 we have

$$F = \sum_{m \geq 1} \frac{1}{m} [t^{m-1}] \left(x + y \frac{t^2}{1-t} \right)^m$$

$$= \sum_{m \geq 1} \frac{1}{m} [t^{m-1}] \sum_{n=0}^{m} \binom{m}{n} x^n \left(y \frac{t^2}{1-t} \right)^{m-n}.$$

From here it is a simple matter to obtain

$$F = x + \sum_{n\geq 2}\sum_{i=1}^{n-1} \frac{1}{n+i}\binom{n+i}{i}\binom{n-2}{i-1}x^n y^i,$$

whence

$$W_i(n) = \frac{1}{n+i}\binom{n+i}{i+1}\binom{n-3}{i}. \tag{6.74}$$

This formula goes back to T. P. Kirkman, *Phil. Trans. Royal Soc. London* **147** (1857), 217–272; E. Prouhet, *Nouvelles Annales Math.* **5** (1866), 384; and A. Cayley, *Proc. London Math. Soc. (1)* **22** (1890–1891), 237–262, who gave the first complete proof. For Cayley's proof see also [**28**, §7.3]. For a completely different proof, see Exercise 7.17. Another proof appears in D. Beckwith, *Amer. Math. Monthly* **105** (1998), 256–257.

* **d.** We have

$$h_i = \frac{1}{n-1}\binom{n-3}{i}\binom{n-1}{i+1}.$$

This result appears in C. W. Lee, *Europ. J. Combinatorics* **10** (1989), 551–560 (Thm. 6.3).

6.34. a.–d. See J. Fürlinger and J. Hofbauer, *J. Combinatorial Theory (A)* **40** (1985), 248–264, and the references given there. For (b) see also G. E. Andrews, *J. Statist. Plann. Inf.* **34** (1993), 19–22 (Cor. 1). The continued fraction (6.59) is of a type considered by Ramanujan. It is easy to show (see for instance [**1.1**, §7.1]) that

$$F(x) = \frac{\sum_{n\geq 0}(-1)^n q^{n^2}\frac{x^n}{(1-q)\cdots(1-q^n)}}{\sum_{n\geq 0}(-1)^n q^{n(n-1)}\frac{x^n}{(1-q)\cdots(1-q^n)}}.$$

e. See R. Stanley, *Ann. New York Acad. Sci.* **574** (1989), 500–535 (Example 4, p. 523). This result is closely related to Exercise 6.25(c).

6.35. These results are due to V. Welker, *J. Combinatorial Theory (B)* **63** (1995), 222–244 (§4).

6.36. a. There are $\binom{n}{k-1}\binom{n-1}{k-1}$ pairs of compositions $A : a_1 + \cdots + a_k = n+1$ and $B : b_1 + \cdots + b_k = n$ of $n+1$ and n into k parts. Construct from these compositions a circular sequence $w = w(A, B)$ consisting of a_1 1's, then b_1 -1's, then a_2 1's, then b_2 -1's, etc. Because n and $n+1$ are relatively prime, this circular sequence w could have arisen from exactly k pairs (A_i, B_i) of compositions of $n+1$ and n into k parts, viz., $A_i : a_i + a_{i+1} + \cdots + a_k + a_1 + \cdots + a_{i-1} = n+1$ and $B_i : b_i + b_{i+1} + \cdots + b_k + b_1 + \cdots + b_{i-1} = n$, $1 \leq i \leq k$. By the second proof of Theorem 5.3.10 (or more specifically, the paragraph following it), there is exactly one way to break w into a linear sequence $\overline{w}$ such that $\overline{w}$ begins with a 1, and when this initial 1 is removed every partial sum is nonnegative. Clearly there are exactly k 1's in $\overline{w}$ (with or without its initial 1 removed) followed by a -1. This sets up a bijection between the set of all "circular equivalence classes" $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ and X_{nk}. Hence

$$X_{nk} = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}.$$

b. For $k \geq 1$, let $X_k = X_{1k} \cup X_{2k} \cup \cdots$. Every $w \in X_k$ can be written uniquely in one of the forms (i) $1\ u\ {-1}\ v$, where $u \in X_j$ and $v \in X_{k-j}$ for some $1 \leq j \leq k-1$, (ii) $1\ {-1}\ u$, where $u \in X_{k-1}$, (iii) $1\ u\ {-1}$, where $u \in X_k$, and (iv) $1\ {-1}$ (when $k = 1$). Regarding X_k as a language as in Example 6.6.6, and replacing for notational comprehensibility 1 by α and -1 by β, conditions (i)–(iv) are equivalent to the equation

$$X_k = \sum_{j=1}^{k-1} \alpha X_j \beta X_{k-j} + \alpha\beta X_{k-1} + \alpha X_k \beta + \delta_{1k}\alpha\beta.$$

Thus if $y_k = \sum_{n\geq 1} N(n, k)x^n$, it follows that (setting $y_0 = 0$)

$$y_k = x\sum_{j=0}^{k} y_j y_{k-j} + xy_{k-1} + xy_k + \delta_{1k}x.$$

Since $F(x, t) = \sum_{k\geq 1} y_k t^k$, we get (6.60).

Narayana numbers were introduced by T. V. Narayana, *C. R. Acad. Sci. Paris* **240** (1955), 1188–1189, and considered further by him in *Sankhyā* **21** (1959), 91–98, and *Lattice Path Combinatorics with Statistical Applications*, Mathematical Expositions **23**, University of Toronto Press, Toronto, 1979 (§V.2). Further references include G. Kreweras and P. Moszkowski, *J. Statist. Plann. Inference* **14** (1986), 63–67; G. Kreweras and Y. Poupard, *Europ. J. Combinatorics* **7** (1986), 141–149; R. A. Sulanke, *Bull. Inst. Combin. Anal.* **7** (1993), 60–66; and R. A. Sulanke, *J. Statist. Plann. Inference* **34** (1993), 291–303.

6.37. Equivalent to Exercise 1.37(c). See also the nice survey R. Donaghey and L. W. Shapiro, *J. Combinatorial Theory (A)* **23** (1977), 291–301.

6.38. All these results except (f), (k), (l), and (m) appear in Donaghey and Shapiro, *loc. cit.* Donaghey and Shapiro give several additional interpretations of Motzkin numbers and state that they have found a total of about 40 interpretations. For (f), see M. S. Jiang, in *Combinatorics and Graph Theory (Hefei, 1992)*, World Scientific Publishing, River Edge, New Jersey, 1993, pp. 31–39. For (k), see A. Kuznetsov, I. Pak, and A. Postnikov, *J. Combinatorial Theory (A)* **76** (1996), 145–147. For (l), see M. Aigner, *Europ. J. Combinatorics* **19** (1998), 663–675. Aigner calls the partitions of (l) *strongly noncrossing*. Finally, for (m) see M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (pp. 55–56) (and compare Exercise 1.29). Klazar's paper contains a number of further enumeration problems related to the present one that lead to algebraic generating functions; see Exercise 6.19(ss), (tt) and Exercise 6.39(o) for three of them. The name "Motzkin number" arose from the paper T. Motzkin, *Bull. Amer. Math. Soc.* **54** (1948), 352–360.

6.39. **a.** This was the definition of Schröder numbers given in the discussion of Schröder's second problem in Section 6.2.

b,e,f,h,i. These follow from (a) using the bijections of Proposition 6.2.1.

c. See D. Gouyou-Beauchamps and D. Vanquelin, *RAIRO Inform. Théor. Appl.* **22** (1988), 361–388. This paper gives some other tree representations of Schröder numbers, as well as connections with Motzkin numbers and numerous references.

d. An easy consequence of the paper of Shapiro and Stephens cited below.

g. Due to R. A. Sulanke, *J. Difference Equations and Applications*, to appear. The objects counted by this exercise are called *zebras*. See also E. Pergola and R. A. Sulanke, *J. Integer Sequences* (electronic) **1** (1998), Article 98.1.7, available at http://www.research.att.com/~njas/sequences/JIS.

j,k. See L. W. Shapiro and A. B. Stephens, *SIAM J. Discrete Math.* **4** (1991), 275–280. For (j), see also Exercise 6.17(b).

l. L. W. Shapiro and S. Getu (unpublished) conjectured that the set $\mathfrak{S}_n(2413, 3142)$ and the set counted by (k) are identical (identifying a permutation matrix with the corresponding permutation). It was proved by J. West, *Discrete Math.* **146** (1995), 247–262, that $\#\mathfrak{S}_n(2413, 3142) = r_{n-1}$. Since it is easy to see that permutations counted by (k) are 2413-avoiding and 3142-avoiding, the conjecture of Shapiro and Getu follows from the fact that both sets have cardinality r_{n-1}. Presumably there is some direct proof that the set counted by (k) is identical to $\mathfrak{S}_n(2413, 3142)$.

West also showed in Theorem 5.2 of the above-mentioned paper that the sets $\mathfrak{S}_n(1342, 1324)$ and (m) are identical. The enumeration of $\mathfrak{S}_n(1342, 1432)$ was accomplished by S. Gire, Ph.D. thesis, Université Bordeaux, 1991. The remaining seven cases were enumerated by D. Kremer, Permutations with forbidden subsequences and a generalized Schröder number, preprint. Kremer also gives proofs of the three previously known cases. She proves all ten cases using the method of "generating trees" introduced by F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, *J. Combinatorial Theory (A)* **24** (1978), 382–394, and further developed by J. West, *Discrete Math.* **146** (1995), 247–262, and **157** (1996), 363–374. It has been verified by computer that there are no other pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ for which $\#\mathfrak{S}_n(u, v) = r_{n-1}$ for all n.

m. This is a result of Knuth [**5.41**, Exercises 2.2.1.10–2.2.1.11, pp. 239 and 533–534]; these permutations are now called *deque-sortable*. A combinatorial proof appears in D. G. Rogers and L. W. Shapiro, in Lecture Notes in Math. **884**, Springer-Verlag, Berlin, 1981, pp. 293–303. Some additional combinatorial interpretations of Schröder numbers and many additional references appear in the preceding reference. For q-analogues of Schröder numbers, see J. Bonin, L. W. Shapiro, and R. Simion, *J. Statist. Plann. Inference* **34** (1993), 35–55.

n. See D. G. Rogers and L. W. Shapiro, Lecture Notes in Mathematics **686**, Springer-Verlag, Berlin, 1978, pp. 267–276 (§5) for simple bijections with (a) and other "Schröder structures."

o. This result is due to R. C. Mullin and R. G. Stanton, *Pacific J. Math.* **40** (1972), 167–172 (§3), using the language of "Davenport–Schinzel sequences." It is also given by M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 55).

p. Remove a "root edge" from the polygon of (h) and "straighten out" to obtain a noncrossing graph of the type being counted.

q,r. These results (which despite their similarity are not trivially equivalent) appear in D. G. Rogers, Lecture Notes in Math. **622**, Springer-Verlag, Berlin, 1977, pp. 175–196 (equations (38) and (39)), and are further developed in D. G. Rogers, *Quart. J. Math. (Oxford) (2)* **31** (1980), 491–506. In particular, a bijective proof that (q) and (r) are equinumerous appears in §3 of this latter reference. It is also easy to see that (p) and (r) are virtually identical problems. A further reference is D. G. Rogers and L. W. Shapiro, Lecture Notes in Mathematics **686**, Springer-Verlag, Berlin, 1978, pp. 267–276.

s. See M. Ciucu, *J. Algebraic Combinatorics* **5** (1996), 87–103, Thm. 4.1.

6.40. Note that this exercise is the "opposite" of Exercise 6.39(k), i.e., here we are counting the permutation matrices P for which not even a single new 1 can be added (using the rules of Exercise 6.39(k)). The present exercise was solved by Shapiro and Stephens in §3 of the paper cited in the solution to Exercise 6.39(k). For a less elegant form of the answer and further references, see M. Abramson and M. O. J. Moser, *Ann. Math. Statist.* **38** (1967), 1245–1254.

6.41. This result was conjectured by J. West, Ph.D. thesis, M.I.T., 1990 (Conjecture 4.2.19), and first proved by D. Zeilberger, *Discrete Math.* **102** (1992), 85–93. For further proofs and related results, see M. Bóna, 2-stack sortable permutations with a given number of runs, MSRI Preprint 1997-055; M. Bousquet-Mélou, *Electron. J. Combinatorics* **5** (1998), R21, 12 pp.; S. Dulucq, S. Gire, and J. West, *Discrete Math.* **153** (1996), 85–103; I. P. Goulden and J. West, *J. Combinatorial Theory (A)* **75** (1996), 220–242; and J. West, *Theoret. Comput. Sci.* **117** (1993), 303–313.

6.42. It's easy to see that $f(n)$ is the number of lattice paths with n steps $(1,0)$, $(0,1)$, and $(1,1)$ that begin at $(0,0)$ and end on the line $y = x$. Hence by equation (6.29) we have $F(x) = 1/\sqrt{1-2x-3x^2}$. The linear recurrence is then given by Example 6.4.8(b)(ii).

It's also easy to see that $f(n) = [t^0](t^{-1}+1+t)^n$. For this reason $f(n)$ is called a *middle trinomial coefficient*. Middle trinomial coefficients were first considered by L. Euler, *Opuscula Analytica*, vol. 1, Petropolis, 1783, pp. 48–62, who obtained the generating function $1/\sqrt{1-2x-3x^2}$. See also Problem III.217 of [**5.53**, vol. I, pp. 147 and 349]. The interpretation of $f(n)$ in terms of chess is due to K. Fabel, Problem 1413, *Feenschach* **13** (October 1974), Heft 25, p. 382; solution, **13** (May–June–July, 1975), Heft 28, p. 91. Fabel gives the solution as

$$f(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{i!^2(n-2i)!},$$

but does not consider the generating function $F(x)$.

6.43. a. Define a secondary structure to be *irreducible* if either $n = 1$ or there is an edge from 1 to n. Let $t(n)$ be the number of irreducible secondary structures with n vertices, and set

$$T(x) = \sum_{n\geq 1} t(n)x^n = x + x^3 + x^4 + 2x^5 + 4x^6 + \cdots.$$

It is easy to see that $S = 1/(1 - T)$ and $T = x^2 S + x - x^2$. Eliminating T and solving for S yields the desired formula.

This result is due to P. R. Stein and M. S. Waterman, *Discrete Math.* **26** (1978), 261–272 (the case $m = 1$ of (10)). For further information and references, including connections with biological molecules such as RNA, see W. R. Schmitt and M. S. Waterman, *Discrete Applied Math.* **51** (1994), 317–323.

b. See A. Nkwanta, in *DIMACS Series in Discrete Math. and Theor. Comput. Sci.* **34**, 1997, pp. 137–147.

6.44. This result is due to P. H. Edelman and V. Reiner, *Graphs and Combinatorics* **13** (1997), 231–243 (Thm. 6.1).

6.45. See R. Stanley, solution to 6342, *American Math. Monthly* **90** (1983), 61–62. A labeled version of this result was given by S. Goodall, The number of labeled posets of width two, Mathematics Preprint Series LSE-MPS-46, London School of Economics, March 1993.

6.46. These remarkable results are due to G. Viennot and D. Gouyou-Beauchamps, *Advances in Appl. Math.* **9** (1988), 334–357. The subsets being enumerated are called *directed animals*. For a survey of related work, see G. Viennot, *Astérisque* **121–122** (1985), 225–246. See also the two other papers cited in the solution to Exercise 6.19(kkk), as well as the paper M. Bousquet-Mélou, *Discrete Math.* **180** (1998), 73–106. Let us also mention that Viennot and Goyou-Beauchamps show that $f(n)$ is the number of sequences of length $n - 1$ over the alphabet $\{-1, 0, 1\}$ with nonnegative partial sums. Moreover, M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 64) shows that $f(n)$ is the number of partitions of $[n + 1]$ such that no block contains two consecutive integers, and such that if $a < b < c < d$, a and d belong to the same block B_1, and b and c belong to the same block B_2, then $B_1 = B_2$.

6.47. **a.** There is a third condition equivalent to (i) and (ii) of (a) that motivated this work. Every permutation $w \in \mathfrak{S}_n$ indexes a closed Schubert cell $\bar{\Omega}_w$ in the complete flag variety $\mathrm{GL}(n, \mathbb{C})/B$. Then w is smooth if and only if the variety $\bar{\Omega}_w$ is smooth. The equivalence of this result to (i) and (ii) is implicit in K. M. Ryan, *Math. Ann.* **276** (1987), 205–244, and is based on earlier work of Lakshmibai, Seshadri, and Deodhar. An explicit statement that the smoothness of $\bar{\Omega}_w$ is equivalent to (ii) appears in V. Lakshmibai and B. Sandhya, *Proc. Indian Acad. Sci. (Math. Sci.)* **100** (1990), 45–52.

b. This generating function is due to M. Haiman (unpublished).

NOTE. It was shown by M. Bóna, *Electron. J. Combinatorics* **5**, R31 (1998), 12 pp., that there are four other inequivalent (in the sense of Exercise 6.39(l)) pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ such that the number of permutations in $\mathfrak{S}_n$ that avoid them is equal to $f(n)$, viz., (1324, 2413), (1342, 2314), (1342, 2431), and (1342, 3241). (The case (1342, 2431) is implicit in Z. Stankova, *Discrete Math.* **132** (1994), 291–316.)

6.48. This result is due to M. Bóna, *J. Combinatorial Theory (A)* **80** (1997), 257–272.

6.49. **a.** Given a domino tiling of B_n, we will define a path P from the center of the left-hand edge of the middle row to the center of the right-hand edge of the middle row. Namely, each step of the path is from the center of a domino

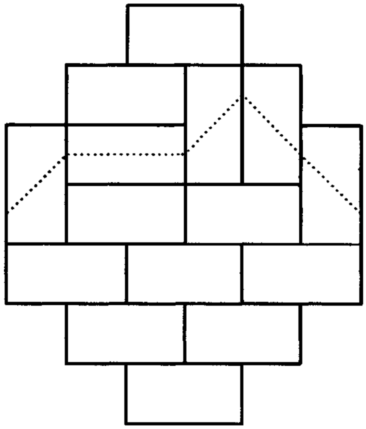


Figure 6-14. A path on the augmented Aztec diamond B_3.

edge (where we regard a domino as having six edges of unit length) to the center of another edge of the same domino D, such that the step is symmetric with respect to the center of D. One can check that for each tiling there is a unique such path P. Replace a horizontal step of P by $(1, 1)$, a northeast step by $(1, 0)$, and a southeast step by $(0, 1)$ (no other steps are possible), and we obtain a lattice path from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$, and conversely any such lattice path corresponds to a unique domino tiling of B_n. This establishes the desired bijection. For instance, Figure 6-14 shows a tiling of B_3 and the corresponding path P (as a dotted line). The steps in the lattice path from $(0, 0)$ to $(3, 3)$ are $(1, 0)$, $(1, 1)$, $(1, 0)$, $(0, 1)$, $(0, 1)$.

The board B_n is called the *augmented Aztec diamond*, and its number of domino tilings was computed by H. Sachs and H. Zernitz, *Discrete Appl. Math.* **51** (1994), 171–179. The proof sketched above is based on an explication of the proof of Sachs and Zernitz due to Dana Randall (unpublished).

b. The board is called an *Aztec diamond*, and the number of tilings is now $2^{\binom{n+1}{2}}$. (Note how much larger this number is than the solution $f(n)$ to (a).) Four proofs of this result appear in N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, *J. Alg. Combinatorics* **1** (1992), 111–132, 219–234. For some related work, see M. Ciucu, *J. Alg. Combinatorics* **5** (1996), 87–103, and H. Cohn, N. Elkies, and J. Propp, *Duke Math. J.* **85** (1996), 117–166. Domino tilings of the Aztec diamond and augmented Aztec diamond had actually been considered earlier by physicists, beginning with I. Carlsen, D. Grensing, and H.-Chr. Zapp, *Philos. Mag. A* **41** (1980), 777–781.

6.50. This result appears in F. R. K. Chung, R. L. Graham, J. Morrison, and A. M. Odlyzko, *Amer. Math. Monthly* **102** (1995), 113–123 (eqn. (11)). This paper contains a number of other interesting results related to pebbling.

6.51. This amazing result is due to A. Edelman, E. Kostlan, and M. Shub, *J. Amer. Math. Soc.* **7** (1994), 247–267 (Thm. 5.1).

6.52. Setting $b_k = 0$ for $k \notin \mathbb{P}$, it is easy to obtain the recurrence

$$b_n = \sum_{\substack{i+j=n \\ i<j}} b_i b_j + \left(\!\!\binom{b_{n/2}}{2}\!\!\right), \qquad n \geq 2,$$

from which (6.61) follows easily. This problem was considered by J. H. M. Wedderburn, *Ann. Math.* **24** (1922), 121–140, and I. M. H. Etherington, *Math. Gaz.* **21** (1937), 36–39, and is known as the *Wedderburn–Etherington commutative bracketing problem*. For further information and references, see [**2.3**, pp. 54–55] and H. W. Becker, *Amer. Math. Monthly* **56** (1949), 697–699.

6.53. We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)[f(n+1) - f(n)].$$

It follows that we can take $P(x) = x + 3$ and $Q(x) = x - 2$. This problem appeared as Problem B-1 in the Forty-Fifth (1984) William Lowell Putnam Competition. Of course the existence of *some* linear recurrence with polynomial coefficients satisfied by $f(n)$ follows from Theorem 6.4.9, since $n!$ is obviously P-recursive and $\sum_{n\geq 1} f(n)x^n = (1-x)^{-1} \sum_{n\geq 1} n!x^n$.

6.54. **a.** Let $y = \sum_{n\geq 0}(x^n/n!^r)$. Clearly $1/n!^r$ is P-recursive, so y is D-finite. Hence by Theorem 6.4.9, y^d is D-finite. But $y^d = \sum_{n\geq 0} S_n^{(r,d)}(x^n/n!^r)$, so $S_n^{(r,d)}/n!^r$ is P-recursive. Thus by Theorem 6.4.12 (or by a simple direct argument), $S_n^{(r,d)}$ is P-recursive. This argument appears in [**70**, Exam. 2.4] in the case $d = 2$.

b. The cases $r = 1$ and $r = 2$ are immediate from $S_n^{(1)} = 2^n$ and $S_n^{(2)} = \binom{2n}{n}$. The cases $r = 3$ and $r = 4$ are due to J. Franel, *L'Intermédiaire des Mathématiciens* **1** (1894), 45–47, and **2** (1895), 33–35. For $r = 5$ and $r = 6$, see M. A. Perlstadt, *J. Number Theory* **27** (1987), 304–309.

c. This result was conjectured by Franel in the 1895 reference above. An incomplete proof was given by T. W. Cusick, *J. Combinatorial Theory (A)* **52** (1989), 77–83. Cusick also gives a method for computing the recurrences that is simpler than Perlstadt's and that would allow the computation for some values of $r \geq 7$. The gap in Cusick's proof was pointed out by M. Stoll, who gives an elegant proof of his own in *Europ. J. Combinatorics* **18** (1997), 707–712. Franel made an additional conjecture about the form of the recurrences, but this is disproved by Perlstadt's computation. For analogous results concerning *alternating* sums of powers of binomial coefficients, see R. J. McIntosh, *J. Combinatorial Theory (A)* **63** (1993), 223–233.

6.55. **a.** See F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, *J. Combinatorial Theory (A)* **24** (1978), 382–394. For more information on this fascinating subject, see C. L. Mallows, *J. Combinatorial Theory (A)* **27** (1979), 394–396; R. Cori, S. Dulucq, and G. Viennot, *J. Combinatorial Theory (A)* **43** (1986), 1–22; and S. Dulucq and O. Guibert, *Discrete Math.* **157** (1996), 91–106. This last paper contains some additional references. See also Exercise 6.19(mm).

b. Let $f(k) = 1/(k-1)!k!(k+1)!$. Thus

$$B(n) = \binom{n+1}{1}^{-1}\binom{n+1}{2}^{-1}(n+1)!^3 \sum_{k=1}^{n} f(k)f(n+1-k).$$

Clearly $f(k)$ is P-recursive, so by Proposition 6.4.3 and Theorem 6.4.9 the function $\sum_{k=1}^{n} f(k)f(n+1-k)$ is P-recursive. It is then easy to see (e.g., from Theorem 6.4.12, though a simple direct argument can also be given) that $B(n)$ is P-recursive. This result can also be deduced from the general theory presented in [**75**].

c. According to Chung *et al.* (reference in (a)), P. S. Bruckman derived from (6.62) that

$$\begin{aligned}(n+1)(n+2)(n+3)(3n-2)B(n)\\ = 2(n+1)(9n^3+3n^2-4n+4)B(n-1)\\ + (3n-1)(n-2)(15n^2-5n-14)B(n-2)\\ + 8(3n+1)(n-2)^2(n-3)B(n-3),\end{aligned}$$

for $n \geq 4$.

6.56. a. Using the notation and techniques of Chapter 7 (see Corollary 7.23.12), we have

$$A_k(n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) < k}} (f^\lambda)^2,$$

where f^λ denotes the number of standard Young tableaux of shape λ. Using the hook-length formula for f^λ (Corollary 7.21.6), we get an explicit formula for $A_k(n)$ from which the techniques of [**75**] (especially §3.3) imply that A_k is P-recursive. Another approach is given by I. M. Gessel, *J. Combinatorial Theory (A)* **53** (1990), 257–285 (§7).

b. These conjectures are due to F. Bergeron, L. Favreau, and D. Krob, *Discrete Math.* **139** (1995), 463–468. They were obtained using the Maple package *gfun* and some related tools developed by S. Plouffe.

* **c.** Very little is known about the function A_v for arbitrary $v \in \mathfrak{S}_k$. For instance, it is conjectured that $\lim_{n\to\infty} A_v(n)^{1/n}$ exists (and is finite), but it is not even known whether $A_v(n) < c^n$ for some constant $c > 0$ depending on v. For further work on the function A_v, see Exercises 6.19(ee,ff), 6.39(l), 6.47, and 6.48

6.57. Let $Q(x) = 1+\alpha_1 x+\cdots+\alpha_d x^d = \prod_{i=1}^{k}(1-\gamma_i x)^{d_i}$, where the γ_i's are distinct nonzero complex numbers and $d_i > 0$. We claim that the answer is k, the number of *distinct* zeros of $Q(x)$. By Theorem 4.1.1 we have $f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n$ for some polynomials $0 \neq P_i(n) \in \mathbb{C}[n]$. Let E be the unit shift operator, given by $(Eg)(n) = g(n+1)$. Let

$$g(n) = [P_k(n)E - \gamma_k P_k(n+1)]f.$$

It's easy to see that $g(n) = \sum_{i=1}^{k-1} Q_i(n)\gamma_i^n$ for some polynomials $Q_i(n) \in \mathbb{C}[n]$. Hence by induction on k (the case $k=1$ being trivial), there is an operator $\Omega = H_{k-1}(n)E^{k-1}+\cdots+H_1(n)E+H_0(n)$, where $H_j(n) \in \mathbb{C}[n]$, satisfying

$\Omega g = 0$. Hence $\Omega \cdot [P_k(n)E - \gamma_k P_k(n+1)]f = 0$, a (nonzero) homogeneous linear recurrence of order at most k with polynomial coefficients satisfied by f.

It remains to show that f cannot satisfy a recurrence of order less than k. Suppose to the contrary that

$$T_{k-1}(n)f(n+k-1) + T_{k-2}f(n+k-2) + \cdots + T_0(n)f(n) = 0$$

for all $n \geq 0$, where $T_i(n) \in \mathbb{C}[n]$ and not all $T_i = 0$. Then

$$\begin{aligned} 0 &= \sum_{j=0}^{k-1} T_j(n) \sum_{i=1}^{k} P_i(n+j)\gamma_i^{n+j} \\ &= \sum_{i=1}^{k} \gamma_i^n \sum_{j=0}^{k-1} T_j(n)P_i(n+j)\gamma_i^j. \end{aligned}$$

Since the functions γ_i^n are linearly independent over $\mathbb{C}(n)$ (e.g., by Theorem 4.1.1), we have

$$\sum_{j=0}^{k-1} T_j(n)P_i(n+j)\gamma_i^j = 0, \qquad 1 \leq i \leq k.$$

Since the Vandermonde matrix $[\gamma_i^j]$ has nonzero determinant, we have $T_j(n)P_i(n + j) = 0$ for all $0 \leq j \leq k-1$, $1 \leq i \leq k$, and $n \geq 0$. Since each P_i and some T_j have only finitely many zeros, we have reached a contradiction.

6.58. Let r be the maximum nonnegative integer zero of $P_e(n)$ (where we set $r = -1$ if $P_e(n)$ has no such zero). Once we know $f(0), \ldots, f(r+e)$, then all other values of $f(n)$ are uniquely determined by (6.34). Hence $\mathcal{V}$ has the same dimension as the space of all $(r+e+1)$-tuples $(f(0), f(1), \ldots, f(r+e))$, where $f \in \mathcal{V}$. If we substitute $n = 0, 1, \ldots, r$ in (6.34), then we obtain $r+1$ homogeneous linear equations $E_0, E_1, \ldots, E_r$ in the $r+e+1$ unknowns $f(0), f(1), \ldots, f(r+e)$. If n is not a zero of $P_e(n)$, then the equation E_n involves $f(n+e)$ with a nonzero coefficient, while the equations $E_0, E_1, \ldots, E_{n-1}$ do not involve $f(n+e)$. Thus the system $(E_0, E_1, \ldots, E_n)$ has rank one more than the rank of the system $(E_0, E_1, \ldots, E_{n-1})$. It follows that the rank of the system $(E_0, E_1, \ldots, E_r)$ is at least $r + 1 - m$. On the other hand, the rank is at most the number $r + 1$ of equations. Since the dimension of the solution space is the number $r + e + 1$ of unknowns minus the rank, we get $e \leq \dim \mathcal{V} \leq e + m$, as desired. Given e and $P_e(n)$, it is easy to arrange for any value of $\dim \mathcal{V}$ in this range to occur. (In fact, since one can specify finitely many values of a polynomial arbitrarily, the system $(E_0, E_1, \ldots, E_r)$ can also be specified arbitrarily, except for being consistent with $P_e(n)$ having m zeros in $\mathbb{N}$, the largest being r.)

6.59. Let $u = \sec x$. Suppose that u satisfies the differential equation (6.31). Now $u' = u\sqrt{u^2-1}$, then $u'' = u^3 + u^2 - u$, and in general by induction it is easily seen that $u^{(2i+1)} = L_i(u)\sqrt{u^2-1}$ and $u^{(2i)} = M_i(u)$, where L_i and M_i are polynomials (with complex coefficients), both of degree $2i+1$. Making these substitutions into (6.31) yields a *nonzero* polynomial equation in x, u, and $\sqrt{u^2-1}$ satisfied by u. Hence u is algebraic, which is easily seen to be impossible (e.g., by Exercise 6.1). This argument appears in [**70**, Exam. 2.5]. An earlier proof was given by L. Carlitz, *J. Reine Angew. Math.* **214/215** (1964), 184–191

(Thm. 4). Another proof follows from the result within the theory of differential equations that an analytic D-finite series cannot have infinitely many poles. See [**70**, §4(a)] for a stronger result. Yet another way to see that $\sec x$ is not D-finite is to appeal to the difficult Exercise 6.60. An argument similar to the one given above can be applied to $\sqrt{\log(1+x^2)}$. See [**70**, Exam. 2.6] for further details.

6.60. Suppose that $u := y'/y$ is algebraic. Repeatedly differentiating the equation $y' = uy$ and using induction shows that $y^{(k)} = P_k(u, u', \ldots)y$, where P_k is a polynomial (over $\mathbb{C}$) in $u, u', \ldots$. Since u is algebraic, all of the series $P_k(u, u', \ldots)$ lie in the field $\mathbb{C}(x, u)$ (using (6.12)) and hence satisfy some linear dependence relation $\sum_{k=0}^{m} f_k(x)P_k = 0$, where $f_k(x) \in \mathbb{C}(x)$. Thus $\sum f_k(x)y^{(k)} = 0$, so y is D-finite.

The converse is considerably more difficult. It was first proved explicitly by W. A. Harris, Jr., and Y. Sibuya, *Advances in Math.* **58** (1985), 119–132, though it actually follows from an earlier result of S. Morrison appearing in P. Blum, *Amer. J. Math.* **94** (1972), 676–684 (Thm. 3). The result of Harris and Sibuya was successively generalized by W. A. Harris, Jr., and Y. Sibuya, *Proc. Amer. Math. Soc.* **97** (1986), 207–211, and S. Sperber, *Pacific J. Math.* **124** (1986), 249–256, culminating in M. F. Singer, *Trans. Amer. Math. Soc.* **295** (1986), 753–763.

6.61. This is a result of L. Lipshitz, *J. Algebra* **113** (1988), 373–378. Earlier proofs by I. Gessel, *Utilitas Math.* **19** (1981), 247–254, and D. Zeilberger, *J. Math. Anal. Appl.* **85** (1982), 114–145 (Thm. 11), were incomplete, though Zeilberger later completed his approach. A more general result was later proved by L. Lipshitz, *J. Algebra* **122** (1989), 353–373 (Thm. 2.7), using his generalization of D-finiteness to several variables.

6.62. Suppose that y satisfies an appropriate differential equation $\mathcal{E}$ of order e. Applying to $\mathcal{E}$ an element of the Galois group G of the irreducible polynomial P of which y is a root, we see that every conjugate of y also satisfies $\mathcal{E}$. It's easy to see that the space of fractional power series (or even fractional Laurent series) solutions to $\mathcal{E}$ has complex dimension at most e. Hence $e \geq \dim V$.

It remains to show that y satisfies *some* equation $\mathcal{E}$ whose degree is equal to $\dim V$. Let L be the splitting field of P. The derivation d/dx extends uniquely to L. G leaves V invariant, and so we have a representation $G \to \mathrm{GL}(V)$. For any $\mathbb{C}$-basis $z_1, \ldots, z_m$ of V, let

$$T(Y) = \frac{\mathrm{Wr}(Y, z_1, \ldots, z_m)}{\mathrm{Wr}(z_1, \ldots, z_m)},$$

where Y is an indeterminate and Wr denotes the Wronskian determinant. For any $\sigma \in G$, apply σ to the coefficients of $T(Y)$ and call the resulting differential polynomial $T^\sigma(Y)$. We then have

$$\begin{aligned} T^\sigma(Y) &= \frac{\mathrm{Wr}(Y, \sigma z_1, \ldots, \sigma z_m)}{\mathrm{Wr}(\sigma z_1, \ldots, \sigma z_m)} \\ &= \frac{\det([\sigma]) \cdot \mathrm{Wr}(Y, z_1, \ldots, z_m)}{\det([\sigma]) \cdot \mathrm{Wr}(z_1, \ldots, z_m)} \\ &= T(Y), \end{aligned}$$

where $[\sigma]$ is the matrix of σ with respect to the basis $z_1, \ldots, z_m$. Therefore the coefficients of $T(Y)$ are left invariant by G and so lie in $\mathbb{C}(x)$. $T(Y)$ is clearly nonzero (since $z_1, \ldots, z_m$ are linearly independent), vanishes at the roots of P, and has order equal to dim V, as desired.

The above argument is due to B. Dwork and later independently M. F. Singer (private communications).

6.63. b. It was shown by K. G. J. Jacobi, *J. Reine Angew. Math.* (= *Crelle's J.*) **36** (1847), 97–112, that the series $y = 1 + 2\sum_{n\geq 1} x^{n^2}$ satisfies the ADE

$$\left(y^2 z_3 - 15 y z_1 z_2 + 30 z_1^3\right)^2 + 32\left(y z_2 - 3 z_1^2\right)^3 - y^{10}\left(y z_2 - 3 z_1^2\right)^2 = 0,$$

where $z_1 = xy'$, $z_2 = xy' + x^2 y''$, $z_3 = xy' + 3x^2 y'' + x^3 y'''$.

c. The strongest related result known to date is due to L. Lipshitz and L. A. Rubel, *Amer. J. Math.* **108** (1986), 1193–1214 (Thm. 4.1). Their result shows in particular that y cannot have Hadamard gaps. In other words, if y satisfies an ADE, then there does not exist $r > 1$ such that $n_{i+1}/n_i > r$ for all i. It is open whether some series of the form $\sum_n b_n x^{n^3}$, with each $b_n \neq 0$, can satisfy an ADE. In fact, it does not seem to be known whether the series $\sum_n x^{n^3}$ satisfies an ADE. (It seems likely that $\sum_n b_n x^{n^3}$ does not satisfy an ADE, since otherwise there would be a completely unexpected result about representing integers as sums of cubes.)

6.64. a. The series represented by both sides is just the sum of all words $w \in \{x, y\}^*$ whose letters alternate (i.e., no two consecutive x's or y's).

b. Let $u = (1 - xy)^{-1}$ and $v = (1 - yx)^{-1}$. Note that $(1 - yx)y = y(1 - xy)$, and therefore $yu = vy$. Thus

$$\begin{aligned}(1 + x)v(1 + y) &= v + xv + yu + xyu \\ &= v + xv + yu + u - (1 - xy)u \\ &= u + v + xv + yu - 1.\end{aligned}$$

This last expression is symmetric with respect to the permutation (written in disjoint cycle form) $(x, y)(u, v)$, so an affirmative answer follows. This argument is due to S. Fomin (private communication), and the problem itself was motivated by the paper S. Fomin, *J. Combinatorial Theory (A)* **72** (1995), 277–292 (proof of Thm. 1.2).

c. An affirmative answer is due to D. Krob, in *Topics in Invariant Theory* (M.-P. Malliavin, ed.), Lecture Notes in Math. **1478**, Springer-Verlag, Berlin/Heidelberg/New York, 1991, pp. 215–243. A short discussion also appears in [**55**, §8].

6.65. a. First Solution. Suppose that $S = \sum_{n\geq 1} x^n y^n$ were rational. By Theorem 6.5.7, there exist $N \times N$ matrices A and B (for some N) such that $(A^i B^j)_{1N} = \delta_{ij}$ for all $i, j \geq 1$. Form the *commutative* generating function

$$\begin{aligned}F(s, t) &= \sum_{i,j\geq 1} (A^i B^j)_{1N} s^i t^j \\ &= [As(I - As)^{-1} Bt(I - Bt)^{-1}]_{1N}.\end{aligned}$$

By an argument as in Exercise 4.8(a), we see that $F(s, t)$ is a rational function of s and t with denominator $\det(I - As) \cdot \det(I - At)$. On the other hand,

$$F(s,t) = \sum_{i,j \geq 1} \delta_{ij} s^i t^j = \frac{st}{1 - st},$$

so that $1 - st$ is a factor of the denominator of $F(s,t)$, a contradiction (e.g., by the fact that $K[s,t]$ is a unique factorization domain).

Second Solution. Let A, B, and N be as above. By the Cayley–Hamilton theorem, we have $A^N = a_{N-1}A^{N-1} + \cdots + a_1 A + a_0 I$ for certain constants a_i. Hence

$$1 = (A^N B^N)_{1N} = \sum_{i=0}^{N-1} a_i (A^i B^N)_{1N} = 0,$$

a contradiction. This elegant argument is due to P. Hersh.

b. This is an example of an "iteration lemma" or "pumping lemma" for rational series, and is due to G. Jacob, *J. Algebra* **63** (1980), 389–412. For some stronger results and additional references, see C. Reutenauer, *Acta Inf.* **13** (1980), 189–197. See also [**5**, Ch. III, Thm. 4.1][**43**, Thm. 8.23].

c. This pumping lemma is a result of J. Jaffe, *SIGACT News* (1978), 48–49. See also [**62**, Thm. 3.14]. The paper of Jaffe mentions some further characterizations of rational languages, the earliest due to A. Nerode.

6.66. By Theorem 6.5.7, there are homomorphisms $\mu : X^+ \to K^{m\times m}$ and $\nu : X^+ \to K^{n\times n}$ such that $\langle S, w\rangle = \mu(w)_{1m}$ and $\langle T, w\rangle = \nu(w)_{1n}$ for all $w \in X^+$. Define $\mu \otimes \nu : X^+ \to K^{mn\times mn}$ by $(\mu \otimes \nu)(w) = \mu(w) \otimes \nu(w)$, where the latter $\otimes$ denotes tensor (Kronecker) product of matrices. Then $\mu \otimes \nu$ is a homomorphism of monoids satisfying $(\mu\otimes\nu)(w)_{1,mn} = \langle S, w\rangle \cdot \langle T, w\rangle$ for all $w \in X^+$, and the proof follows from Theorem 6.5.7. This result is due to M. P. Schützenberger [**65**, Property 2.2]. See also for instance [**63**, Thm. II.4.4]. (Schützenberger assumes only that K is a semiring, in which case the proof is considerably more difficult.)

6.67. Let $x = x_0 + x_1 + \cdots + x_{b-1}$ and $y = x_1 + 2x_2 + \cdots + (b-1)x_{b-1}$. Then one sees easily that

$$S = bSx + (x - x_0)(1 - x)^{-1}y + y,$$

whence

$$S = [(x - x_0)(1 - x)^{-1}y + y](1 - bx)^{-1}.$$

This result is a slight variation of [**5**, Exer. 4.4, p. 19].

6.68. A proof of this result is given in M. W. Davis and M. D. Shapiro, Coxeter groups are automatic, Ohio State University, 1991, preprint. However, the proof of a result called the parallel-wall property is incomplete. Subsequently B. Brink and R. B. Howlett, *Math. Ann.* **296** (1993), 179–190, asserted that their Thm. 2.8 implies the parallel-wall property, thereby completing the proof of Davis and Shapiro. Other proofs were given by H. Eriksson, Ph.D. thesis, Kungl. Tekniska Högskolan, 1994 (Thm. 73) and P. Headley, Ph.D. thesis, University of Michigan, 1994 (Thm. VII.12).

6.69. For this result and a number of related results, see A. Björner and C. Reutenauer, *Theoret. Comput. Sci.* **98** (1992), 53–63.

6.70. This is something of a trick question – clearly (0, 0) is a solution.

6.71. Let $\alpha, \beta \in K$. Let $s_1 = u$ be a component of a proper algebraic system $\mathcal{S}$ in variables $s_1, s_2, \ldots, s_j$, and let $t_1 = v$ be a component of a proper algebraic system $\mathcal{T}$ in variables $t_1, t_2, \ldots, t_k$. Then $z = \alpha u + \beta v$ is a component of the proper algebraic system consisting of $\mathcal{S}$, $\mathcal{T}$, and the equation $z = \alpha s_1 + \beta t_1$. Hence $\alpha u + \beta v$ is algebraic. A similar argument works for uv.

It remains to show that if u is algebraic and u^{-1} exists (i.e., $\langle u, 1\rangle \neq 0$), then u^{-1} is algebraic. Suppose $\langle u, 1\rangle = \alpha \neq 0$. Let $v = u^{-1}$, $u' = u - \alpha$, and $v' = v - \alpha^{-1}$, so $\langle u', 1\rangle = \langle v', 1\rangle = 0$. Then $v' = -\alpha^{-2}u' - \alpha^{-1}v'u'$, from which it is immediate that v' (and therefore v) is algebraic whenever u' (or u) is algebraic.

For additional "closure properties" of algebraic series, see [**63**, Ch. IV.3] and [**24**].

6.72. **a.** This pumping lemma for algebraic languages is due to Y. Bar-Hillel, *Z. Phonetik, Sprachwiss. u. Kommunicationsforschung* **14** (1961), 143–172 (Thm. 4.1); reprinted in Y. Bar-Hillel, *Language and Information*, Addison-Wesley, Reading, Massachusetts, 1964, p. 130. See also [**62**, Thm. 3.13]. For additional methods of showing that some languages are not algebraic, see [**63**, Exers. IV.2.4 and IV.5.8].

b. Easy.

6.73. **a.** Let D be the Dyck language of Example 6.6.6. Let $\bar{D}$ be the set of all words obtained from words in D by interchanging x's and y's. A simple combinatorial argument shows that $S = (1 - D^+ - \bar{D}^+)^{-1}$. Since D and $\bar{D}$ are algebraic, it follows immediately that S is algebraic. More explicitly, if $S' = S - 1$ then we have (using (6.51))

$$\begin{aligned} S' &= (S'+1)D^+ + (S'+1)\bar{D}^+ \\ D^+ &= x(D^+ + 1)y(D^+ + 1) \\ \bar{D}^+ &= y(\bar{D}^+ + 1)x(\bar{D}^+ + 1). \end{aligned}$$

b. Define a series Q by $Q = 1 + xQyQ + yQxQ$. Then Q is algebraic, and the *support* of Q is the language S, as noted by J. Berstel, *Transductions and Context-Free Languages*, Teubner, 1979 (Thm. II.3.7 on p. 41). However, this leaves open the question of whether $S - 1$ itself can be defined by a single equation (or even two equations).

6.74. Every word $w = w_1 w_2 \cdots w_n$ in the alphabet $\{x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}\}$ that reduces to 1 can be uniquely written as a product of irreducible such words. If $g(n)$ is the number of such irreducible words of length n and if $G(t) = \sum_{n\geq 1} g(n)t^n$, then it follows that $F = 1/(1-G)$. If an irreducible word u begins with a letter $y = x_i$ or x_i^{-1}, then it must end in y^{-1} [why?]. If $u = yvy^{-1}$, then v can be any word reducing to 1 whose irreducible components don't begin with y^{-1}. The generating function for irreducible words not beginning with y^{-1} is $\frac{2k-1}{k}G(t)$, so the generating function for sequences of such words is

$$\left(1 - \frac{2k-1}{k}G(t)\right)^{-1}.$$

Since the word u is two letters longer than v and there are $2k$ choices for y, there follows

$$G(t) = \frac{2kt^2}{1 - \frac{2k-1}{k} G(t)}.$$

From this it is easy to solve for $G(t)$ and then $F(t)$.

The argument given above is due to D. Grabiner. Another elegant solution by A. J. Schwenk appears in *Amer. Math. Monthly* **92** (1985), 670–671. The problem was formulated originally by M. Haiman and D. Richman for the case $k = 2$ in *Amer. Math. Monthly* **91** (1984), 259, though Schenk notes that his solution carries over to any k. The problem of Haiman and Richman provided the motivation for the paper [**35**], from which we obtained Theorem 6.7.1 and Corollary 6.7.2.

This problem was solved independently by physicists, in the context of random walks on the Bethe lattice. See A. Giacometti, *J. Phys. A: Math. Gen.* **28** (1995), L13–L17, and the references given there.