University of Victoria
Department of Mathematics and Statistics

Comprehensive Exam in Graph Theory
August 27, 2007
Examiners: Gary MacGillivray and Kieka Mynhardt

TIME: 3 HOURS. DO ANY 4 OF QUESTIONS 1-7 AND ANY 3 OF QUESTIONS 8-12.
This question paper has two pages. Write your answers in the booklets provided.

Part 1: Do any four of questions 1 - 7.

1. Prove that the number of labelled spanning trees of the complete graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$ is the multinomial coefficient

$$\binom{n-2}{d_1-1, d_2-1, \ldots, d_n-1}.$$

2. Describe Kruskal's algorithm and prove that it constructs a minimum weight spanning tree of the edge-weighted graph (G, w).

3. (a) State Hall's Theorem, Menger's Theorem, and the Max-Flow Min-Cut Theorem. Define all terms necessary to make your statements meaningful.

(b) Use the Max-Flow Min-Cut Theorem to prove Hall's Theorem.

4. (a) State and prove Brooks' Theorem.

(b) Use Brooks Theorem to show that if $\Delta = 3$ then the edge-chromatic number of G is at most 4.

5. (a) Prove that every tournament has a directed Hamilton path.

(b) Prove that a tournament T is strong if and only if it has a directed Hamilton cycle.

6. Let G be a connected graph with $n \geq 3$ vertices. Prove that there is a constant $c > 1$ such that if every pair of non-adjacent vertices u and v satisfy $d(u) + d(v) \geq n - c$, then G has a Hamilton path.

7. (a) Show that a plane graph G is 2-face-colourable if and only if G is eulerian.

(b) Show that a plane triangulation G is 3-vertex-colourable if and only if G is eulerian.

Part 2: Do any three of questions 8-12.

8. A graph is *minimally k-connected* if G is k-connected and, for every edge e of G, the vertex connectivity $\kappa(G-e) < \kappa(G)$. Prove that if G is minimally 2-connected, then G must have a vertex of degree 2.

9. Let H be a graph with p vertices, q edges, and automorphism group of size s. Prove that, for each $n < s^{1/p}2^{(q-1)/p}$, there exists a graph G on n vertices such that H is a subgraph of neither G nor of $\overline{G}$.

10. (a) Prove that, for $n \geq 1$, the edges of K_{2n} can be partitioned into Hamilton paths.

(b) Prove that, for $n \geq 0$, the edges of K_{2n+1} can be partitioned into Hamilton cycles.

(c) Prove that every cubic graph with no cut edges has a 2-factor.

11. Prove that if every vertex of G has even degree, then the edges of G can be oriented so that, at each vertex, the indegree equals the outdegree.

12. Suppose that a graph G has the following properties:

i) Any two non-adjacent vertices have the same degree;

ii) The degrees of any two adjacent vertices differ by at most one;

iii) Non-adjacency is a transitive relation on V.

(a) Prove the vertices of G can be partitioned into maximal independent sets. Assuming that there are m such sets, describe G.

(b) Suppose further that G has no complete subgraph on $r+1$ vertices, but the addition of any edge to G creates such a subgraph. Give an estimate of the maximum number of edges that G could have. (You need not be too careful in showing all of the details.)