Chordal Distance Graphs

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Let T be a tree and $d : V(T) \times V(T) \to \mathbb{N}_0$ the standard graph metric. Define $D_{\leq k} := D_{\leq k}(T)$ to be the graph with vertex set $V(T)$ and for each $u, v \in V(T)$, uv is an edge if and only if $d(u, v) \leq k$ in T. We similarly define $D_{=k}$ to be the k-distance graph. We show below that $D_{\leq k}$ is chordal.

1 Small Distance Graphs of Trees are Chordal.

Lemma 1. *Let T be a tree containing paths $P = (p_0, p_1, \ldots, p_{k-1})$ and $Q = (q_0, q_1, \ldots, q_{k-1})$ such that $p_{k-1} = q_0$ and $p_0 \neq q_{k-1}$. Then there is a unique vertex u satisfying $u \in P \cap Q$, and $N(u) \cap (P \triangle Q) \neq \emptyset$. That is, u intersects both P and Q and has a neighbour only in P and a neighbour only in Q.*

Proof. Since T is a tree and P and Q have a unique common end vertex, the set of common vertices between P and Q must induce a path ending at this end vertex, otherwise P and Q would induce a cycle. □

Lemma 2. *Suppose $D_{\leq k}$ has an induced cycle $C = (v_0, v_1, \ldots, v_{\ell-1}, v_0)$. For each $i \in [0, \ell-2]$ define P_i to be the unique (v_i, v_{i+1})-path in T. Define $u_{i,j}$ to be the unique intersecting vertex between paths P_i and P_j that has a neighbour only in P_i and a neighbour only in P_j. Then the subgraph of T induced by $\{P_i : i \in [0, \ell-2]\}$ contains a spinal path with vertex subsequence $(v_0, u_{0,1}, u_{1,2}, \ldots, u_{\ell-3,\ell-2}, v_{\ell-1})$, where v_0 and $v_{\ell-1}$ are the ends of the spine. Specifically, P_0 only intersects P_1 and $P_{\ell-2}$ only intersects $P_{\ell-3}$, and for each $i \in [1, \ell-3]$, P_i intersects exactly P_{i-1} and P_{i+1}.*

Proof. Since T is a tree, for each $i \in [0, \ell-2]$, P_i is the unique (v_i, v_{i+1})-path. Because each P_i has length at most k and C is an induced cycle in $D_{\leq k}$, it follows that $d(v_i, v_{i-2}) > k$. This means by Lemma 1 that for each $i \in [1, \ell-2]$, $u_{i-1,i}$ is the unique vertex intersecting P_{i-1} and P_i such that $N(u_{i-1,i}) \cap (P \triangle Q) \neq \emptyset$. Observe that $u_{i-1,i}, u_{i,i+1} \in P_i$ when $i \in [1, \ell-3]$, and also $u_{0,1} \in P_0$ and $u_{\ell-3,\ell-2} \in P_{\ell-2}$. So, P_0 intersects P_1, $P_{\ell-2}$ intersects $P_{\ell-3}$, and for each $i \in [1, \ell-3]$, P_i intersects both P_{i-1} and P_{i+1}.

It is sufficient to show that there are no other path intersections. Suppose otherwise that P_i intersects P_j for some $j \in [0, \ell-2] \setminus \{i-1, i+1\}$. Then if $j > i+1$, for each $r \in [i+1, j-1]$, P_r intersects P_{r+1}, so T contains two paths between v_{i+1} and v_j, namely

$$(P_{i+1}, P_{i+2}, \ldots, P_{j-1}), \text{ and } (v_j, \ldots, u_{j-1,j}, \ldots, u_{j,i}, \ldots, v_{i+1}).$$

This means that T has a cycle, a contradiction. The argument is analogous for $j < i-1$. □

Proposition 1.1. *The graph $D_{\leq k}$ is chordal.*

Proof. Let $C = (v_0, v_1, \ldots, v_{\ell-1}, v_0)$ be an induced cycle of $D_{\leq k}$ satisfying $\ell \geq 4$.

[***We know what to do when $\ell = 4$***]

Suppose $\ell \geq 5$. By Lemma 2, the subgraph of T induced by the consecutive (v_i, v_{i+1})-paths $P_0, P_1, \ldots, P_{\ell-2}$ satisfies the property that only consecutive paths intersect. Denote the intersecting vertex for P_{i-1} and P_i with the properties in Lemma 1 by $u_{i-1,i}$. For $i \in [1, \ell-1]$, let $a_i := d(v_i, u_{i-1,i})$, and for $i \in [2, \ell-2]$, $s_i := d(u_{i-2,i-1}, u_{i-1,i})$. For $i = 1$, define $s_1 := d(v_0, u_{0,1})$ and similarly for $i = \ell-1$, define $s_{\ell-1} := d(u_{\ell-3,\ell-2}, v_{\ell-1})$.

Note that $d(v_i, v_{i+1}) = a_{i-1} + s_i + a_i$ when $i \in [2, \ell-2]$, and $d(v_0, v_1) = s_1 + a_1$ and $d(v_{\ell-2}, v_{\ell-1}) = a_{\ell-2} + s_{\ell-1}$. Observe that the unique $(v_0, v_{\ell-1})$-path of length at most k is the path along the entire spine of the graph induced by $\{P_i : i \in [0, \ell-2]\}$, which contains the subsequence of vertices $(v_0, \ldots, u_{0,1}, \ldots, u_{1,2}, \ldots, u_{\ell-3,\ell-2}, \ldots, v_{\ell-1})$ and has length $\sum_{i=1}^{\ell-1} s_i$. Thus we have

$$d(v_0, v_{\ell-1}) = \sum_{i=1}^{\ell-1} s_i \leq k$$

But note that for all $j \in [2, \ell-2]$, $k < d(v_0, v_j) = a_j + \sum_{i=1}^{j} s_j$; and similarly, for $j \in [1, \ell-3]$, $k < d(v_j, v_{\ell-1}) = a_j + \sum_{i=j+1}^{\ell-1} s_j$. So, in particular, choose any $j' \in [2, \ell-3]$, which is nonempty because $\ell \geq 5$. Then

$$\begin{aligned} 2k &< d(v_0, v_{j'}) + d(v_{j'}, v_{\ell-1}) \\ &= 2a_{j'} + \sum_{i=1}^{\ell-1} s_i \\ &= 2a_{j'} + d(v_0, v_{\ell-1}) \leq 2a_{j'} + k. \end{aligned}$$

Thus we have $2a_{j'} > k$. However, since

$$\begin{aligned} k &< d(v_{j'-1}, v_{j'+1}) \\ &\leq a_{j'-1} + (k - a_{j'-1} - a_{j'}) + (k - a_{j'} - a_{j'+1}) + a_{j'+1} \\ &= 2k - 2a_{j'}, \end{aligned}$$

it follows that $2a_{j'} < k$, a contradiction. We have shown that $D_{\leq k}$ has no induced cycle of length at least 4, which means that it is chordal. $\square$