7

Symmetric Functions

7.1 Symmetric Functions in General

The theory of symmetric functions has many applications to enumerative combinatorics, as well as to such other branches of mathematics as group theory, Lie algebras, and algebraic geometry. Our aim in this chapter is to develop the basic combinatorial properties of symmetric functions; the connections with algebra will only be hinted at in Sections 7.18 and 7.24, Appendix 2, and in some exercises.

Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates, and let $n \in \mathbb{N}$. A *homogeneous symmetric function of degree n* over a commutative ring R (with identity) is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where (a) α ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n (of infinite length), (b) $c_\alpha \in R$, (c) x^α stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$, and (d) $f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation w of the positive integers $\mathbb{P}$. (A symmetric function of degree 0 is just an element of R.) Note that the term "symmetric function" is something of a misnomer; $f(x)$ is not regarded as a function but rather as a formal power series. Nevertheless, for historical reasons we adhere to the above terminology.

The set of all homogeneous symmetric functions of degree n over R is denoted Λ_R^n. Clearly if $f, g \in \Lambda_R^n$ and $a, b \in R$, then $af + bg \in \Lambda_R^n$; in other words, Λ_R^n is an R-module. For our purposes it will suffice to take $R = \mathbb{Q}$ (or sometimes $\mathbb{Q}$ with some indeterminates adjoined), so $\Lambda_{\mathbb{Q}}^n$ is a $\mathbb{Q}$-vector space. For the sake of convenience, then, and because some readers are doubtless more comfortable with vector spaces than with modules, we will henceforth work over $\mathbb{Q}$, though this is not the most general approach.

If $f \in \Lambda_{\mathbb{Q}}^m$ and $g \in \Lambda_{\mathbb{Q}}^n$, then it is clear that $fg \in \Lambda_{\mathbb{Q}}^{m+n}$ (where fg is a product of formal power series). Hence if we define

$$\Lambda_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}^0 \oplus \Lambda_{\mathbb{Q}}^1 \oplus \cdots \quad \text{(vector space direct sum)} \tag{7.1}$$

(so the elements of $\Lambda_{\mathbb{Q}}$ are power series $f = f_0 + f_1 + \cdots$ where $f_n \in \Lambda_{\mathbb{Q}}^n$ and all but finitely many $f_n = 0$), then $\Lambda_{\mathbb{Q}}$ has the structure of a $\mathbb{Q}$-algebra (i.e., a ring whose operations are compatible with the vector space structure), called the *algebra* (over $\mathbb{Q}$) *of symmetric functions*. Note that the algebra $\Lambda_{\mathbb{Q}}$ is commutative and has an identity element $1 \in \Lambda_{\mathbb{Q}}^0$. The decomposition (7.1) in fact gives $\Lambda_{\mathbb{Q}}$ the structure of a *graded algebra*, meaning that if $f \in \Lambda_{\mathbb{Q}}^m$ and $g \in \Lambda_{\mathbb{Q}}^n$, then $fg \in \Lambda_{\mathbb{Q}}^{m+n}$. From now on we suppress the subscript $\mathbb{Q}$ and write simply Λ^n and Λ for $\Lambda_{\mathbb{Q}}^n$ and $\Lambda_{\mathbb{Q}}$. Note, however, that in the outside literature Λ usually denotes $\Lambda_{\mathbb{Z}}$.

A central theme in the theory of symmetric functions is to describe various bases of the vector space Λ^n and the transition matrices between pairs of these bases. We will begin with four "simple" bases. In Sections 7.10–7.19 we consider a less obvious basis which is crucial for the deeper parts of the theory.

7.2 Partitions and Their Orderings

Recall from Section 1.3 that a *partition* λ of a nonnegative integer n is a sequence $(\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ satisfying $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum \lambda_i = n$. Any $\lambda_i = 0$ is considered irrelevant, and we identify λ with the *infinite* sequence $(\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$. We let $\text{Par}(n)$ denote the set of all partitions of n, with $\text{Par}(0)$ consisting of the empty partition $\emptyset$ (or the sequence $(0, 0, \ldots)$), and we let

$$\text{Par} := \bigcup_{n \geq 0} \text{Par}(n).$$

For instance (writing for example 4211 as short for $(4, 2, 1, 1, 0, \ldots)$),

$$\begin{aligned}
\text{Par}(1) &= \{1\}\\
\text{Par}(2) &= \{2, 11\}\\
\text{Par}(3) &= \{3, 21, 111\}\\
\text{Par}(4) &= \{4, 31, 22, 211, 1111\}\\
\text{Par}(5) &= \{5, 41, 32, 311, 221, 2111, 11111\}.
\end{aligned}$$

If $\lambda \in \text{Par}(n)$, then we also write $\lambda \vdash n$ or $|\lambda| = n$. The number of parts of λ (i.e., the number of nonzero λ_i) is the *length* of λ, denoted $\ell(\lambda)$. Write $m_i = m_i(\lambda)$ for the number of parts of λ that equal i, so in the notation of Section 1.3 we have $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$, which we sometimes abbreviate as $1^{m_1} 2^{m_2} \cdots$. Also recall from the discussion of entry 10 of the Twelvefold Way in Section 1.4 that the *conjugate partition* $\lambda' = (\lambda_1', \lambda_2', \ldots)$ of λ is defined by the condition that the Young (or Ferrers) diagram of λ' is the transpose of the Young diagram of λ; equivalently, $m_i(\lambda') = \lambda_i - \lambda_{i+1}$. Note that $\lambda_1' = \ell(\lambda)$ and $\lambda_1 = \ell(\lambda')$.

Three partial orderings on partitions play an important role in the theory of symmetric functions. We first define $\mu \subseteq \lambda$ for any $\mu, \lambda \in \text{Par}$ if $\mu_i \leq \lambda_i$ for all i. If we identify a partition with its (Young) diagram, then the partial order

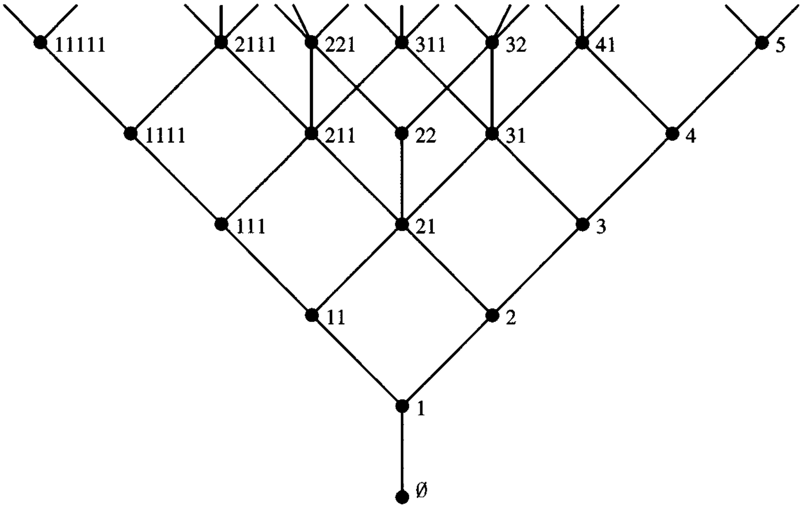


Figure 7-1. Young's lattice.

$\subseteq$ is given simply by containment of diagrams. The set Par with the partial order $\subseteq$ is called *Young's lattice Y* and (as mentioned in Exercise 3.63) is isomorphic to $J_f(\mathbb{N}^2)$. The rank of a partition λ in Young's lattice is equal to the sum $|\lambda|$ of its parts, so the rank-generating function by equation (1.30) is given by

$$F(Y, x) = \prod_{i \geq 1} (1 - x^i)^{-1}.$$

See Figure 7-1 for the first six levels of Young's lattice.

The second partial order is defined only on Par(n) for each $n \in \mathbb{N}$, and is called *dominance order* (also known as *majorization order* or the *natural order*), denoted $\leq$. Namely, if $|\mu| = |\lambda|$ then define $\mu \leq \lambda$ if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i \qquad \text{for all } i \geq 1.$$

For the reader's benefit we state the following basic facts about dominance order, though we have no need for them here.

- (Par(n), $\leq$) is a lattice.
- The map $\lambda \mapsto \lambda'$ is an anti-automorphism of (Par(n), $\leq$) (so Par(n) is self-dual under dominance order).
- (Par(n), $\leq$) is a chain if and only if $n \leq 5$.
- (Par(n), $\leq$) is graded if and only if $n \leq 6$.

- For $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$, let $\mathcal{P}_\lambda$ denote the convex hull in $\mathbb{R}^n$ of all permutations of the coordinates of λ. Then $(\mathrm{Par}(n), \leq)$ is isomorphic to the set of $\mathcal{P}_\lambda$'s ordered by inclusion.

For the Möbius function of $(\mathrm{Par}(n), \leq)$ see Exercise 3.55, and for some further properties see Exercise 7.2.

For our final partial order, also on $\mathrm{Par}(n)$, it suffices to take any linear order compatible with (i.e., a linear extension of) dominance order. The most convenient is *reverse lexicographic order*, denoted $\stackrel{R}{\leq}$. Given $|\lambda| = |\mu|$, define $\mu \stackrel{R}{\leq} \lambda$ if either $\mu = \lambda$, or else for some i,

$$\mu_1 = \lambda_1, \ldots, \quad \mu_i = \lambda_i, \quad \mu_{i+1} < \lambda_{i+1}.$$

For instance, the order $\stackrel{R}{>}$ on Par(6) is given by

$$6 \stackrel{R}{>} 51 \stackrel{R}{>} 42 \stackrel{R}{>} 411 \stackrel{R}{>} 33 \stackrel{R}{>} 321 \stackrel{R}{>} 3111 \stackrel{R}{>} 222 \stackrel{R}{>} 2211 \stackrel{R}{>} 21111 \stackrel{R}{>} 111111.$$

On the other hand, in dominance order the partitions 33 and 411, as well as 3111 and 222, are incomparable.

We would like to make one additional definition related to partitions. Define the *rank* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, denoted $\mathrm{rank}(\lambda)$, to be the largest integer i for which $\lambda_i \geq i$. Equivalently, $\mathrm{rank}(\lambda)$ is the length of the main diagonal of the diagram of λ or the side length of the Durfee square of λ (defined in the solution to Exercise 1.23(b)). Note that $\mathrm{rank}(\lambda) = \mathrm{rank}(\lambda')$.

7.3 Monomial Symmetric Functions

Given $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, define a symmetric function $m_\lambda(x) \in \Lambda^n$ by

$$m_\lambda = \sum_\alpha x^\alpha,$$

where the sum ranges over all *distinct* permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the entries of the vector $\lambda = (\lambda_1, \lambda_2, \ldots)$. For instance,

$$\begin{aligned} m_\emptyset &= 1 \\ m_1 &= \sum_i x_i \\ m_2 &= \sum_i x_i^2 \\ m_{11} &= \sum_{i<j} x_i x_j. \end{aligned}$$

We call m_λ a *monomial symmetric function*. Clearly if $f = \sum_\alpha c_\alpha x^\alpha \in \Lambda^n$ then $f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$. It follows that the set $\{m_\lambda : \lambda \vdash n\}$ is a (vector space) basis for

Λ^n, and hence that

$$\dim \Lambda^n = p(n),$$

the number of partitions of n. Moreover, the set $\{m_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ.

7.4 Elementary Symmetric Functions

Next we define the *elementary symmetric functions* e_λ for $\lambda \in$ Par by the formulas

$$\begin{aligned} e_n &= m_{1^n} = \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n}, \quad n \geq 1 \qquad (\text{with } e_0 = m_\emptyset = 1) \\ e_\lambda &= e_{\lambda_1}e_{\lambda_2}\cdots, \qquad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots). \end{aligned} \tag{7.2}$$

If $A = (a_{ij})_{i,j\geq 1}$ is an integer matrix with finitely many nonzero entries and with row and column sums

$$\begin{aligned} r_i &= \sum_j a_{ij} \\ c_j &= \sum_i a_{ij}, \end{aligned}$$

then define the *row-sum vector* $\mathrm{row}(A)$ and *column-sum vector* $\mathrm{col}(A)$ by

$$\begin{aligned} \mathrm{row}(A) &= (r_1, r_2, \ldots) \\ \mathrm{col}(A) &= (c_1, c_2, \ldots). \end{aligned}$$

Also define a (0, 1)-*matrix* to be a matrix whose entries are all 0 or 1.

7.4.1 Proposition. *Let $\lambda \vdash n$, and let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n. Then the coefficient $M_{\lambda\alpha}$ of x^α in e_λ, i.e.,*

$$e_\lambda = \sum_{\mu\vdash n} M_{\lambda\mu} m_\mu, \tag{7.3}$$

is equal to the number of (0, 1)*-matrices $A = (a_{ij})_{i,j\geq 1}$ satisfying* $\mathrm{row}(A) = \lambda$ *and* $\mathrm{col}(A) = \alpha$.

Proof. Consider the matrix

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

To obtain a term of e_λ, choose λ_1 entries from the first row, λ_2 from the second row, etc. Let the product of the chosen entries be x^α. If we convert the chosen entries to 1's and the other entries to 0's, then we obtain a matrix A with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$. Conversely, any such matrix corresponds to a term of e_λ, and the proof follows. □

7.4.2 Corollary. *Let $M_{\lambda\mu}$ be given by (7.3). Then $M_{\lambda\mu} = M_{\mu\lambda}$, i.e., the transition matrix between the bases* {$m_\lambda : \lambda \vdash n$} *and* {$e_\lambda : \lambda \vdash n$} *is a symmetric matrix.*

Proof. The (0, 1)-matrix A satisfies $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$ if and only if the transpose A^t satisfies $\mathrm{row}(A^t) = \mu$ and $\mathrm{col}(A^t) = \lambda$. □

It is easy to see that the coefficient $M_{\lambda\mu}$ of (7.3) has the following alternative combinatorial interpretation. We have n balls in all, with λ_i balls labeled i. We also have boxes labeled $1, 2, \ldots$. Then $M_{\lambda\mu}$ is the number of ways of placing the balls into the boxes so that: (a) no box contains more than one ball with the same label, and (b) box i contains exactly μ_i balls. The elegant combinatorial interpretations we have given of $M_{\lambda\mu}$ are our first hints of the combinatorial efficacy of the theory of symmetric functions.

In general, let $\{u_\lambda\}$ be a basis for Λ and let $f \in \Lambda$. If the expansion $f = \sum_\lambda a_\lambda u_\lambda$ of f in terms of the basis $\{u_\lambda\}$ satisfies $a_\lambda \geq 0$ for all λ, then we say that f is *u-positive*. If f is u-positive, then the coefficients a_λ often have a simple combinatorial or algebraic interpretation. (An example of an algebraic interpretation would be the dimension of a vector space.) For instance, it is obvious from the relevant definitions that e_λ is m-positive, and Proposition 7.4.1 gives a stronger result (viz., a combinatorial interpretation of the coefficients). Similarly to the definition of u-positivity, we also say that f is *u-integral* if the coefficients a_λ above are integers.

Proposition 7.4.1 has an equivalent formulation in terms of generating functions. The type of generating function that we will be considering throughout this chapter has the form $z = \sum_\lambda c_\lambda u_\lambda$, where λ ranges over Par, $\{u_\lambda\}$ is a $\mathbb{Q}$-basis for Λ (and usually $u_\lambda \in \Lambda^n$, where $\lambda \vdash n$), and c_λ belongs to some coefficient ring R (which for us will always be a $\mathbb{Q}$-algebra). We may think of z as belonging to the
* ring $\hat{\Lambda}_R = \hat{\Lambda} \otimes R$, where $\hat{\Lambda}_R$ denotes the completion of Λ_R with respect to the ideal $\Lambda_R^1 \oplus \Lambda_R^2 \cdots$. Readers unfamiliar with completion need not be concerned; the generating functions $\sum c_\lambda u_\lambda$ will always behave in a reasonable, intuitive way.

A frequently occurring class of generating functions has $R = \Lambda(y)$, i.e., symmetric functions in a new set of variables $y = (y_1, y_2, \ldots)$. For instance, the generating function $z = \sum_\lambda m_\lambda(x) e_\lambda(y)$ of the next proposition is of the form $\sum_\lambda c_\lambda u_\lambda$ where $u_\lambda = m_\lambda(x) \in \Lambda(x)$ and $c_\lambda = e_\lambda(y) \in \Lambda(y) = R$. In general, if a function c_λ indexed by $\lambda \in$ Par arises in an enumeration problem, then it is

* It follows from Theorem 7.4.4 below that the set $\{e_\lambda : \lambda \vdash n\}$ is indeed a basis.

natural to consider a generating function of the form $\sum c_\lambda u_\lambda$ for a suitable basis u_λ of Λ. The difficulty, of course, is deciding what basis is "suitable." We will see throughout this chapter various choices of u_λ, the most common being the Schur functions s_λ discussed in Sections 7.10–7.19.

Let us now give the promised reformulation of Proposition 7.4.1. The generating function that we consider, as well as a closely related one to be discussed later (see Proposition 7.5.3), plays an important role in the theory of symmetric functions.

7.4.3 Proposition. *We have*

$$\prod_{i,j}(1+x_iy_j) = \sum_{\lambda,\mu} M_{\lambda\mu} m_\lambda(x) m_\mu(y) \tag{7.4}$$

$$= \sum_{\lambda} m_\lambda(x) e_\lambda(y). \tag{7.5}$$

Here λ and μ range over Par. *(It suffices to take $|\lambda| = |\mu|$, since otherwise $M_{\lambda\mu} = 0$.)*

Proof. A monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots y_1^{\beta_1} y_2^{\beta_2} \cdots = x^\alpha y^\beta$ appearing in the expansion of $\prod(1 + x_iy_j)$ is obtained by choosing a (0, 1)-matrix $A = (a_{ij})$ with finitely many 1's, satisfying

$$\prod_{i,j}(x_iy_j)^{a_{ij}} = x^\alpha y^\beta.$$

But

$$\prod_{i,j}(x_iy_j)^{a_{ij}} = x^{\mathrm{row}(A)} y^{\mathrm{col}(A)},$$

so the coefficient of $x^\alpha y^\beta$ in the product $\prod(1 + x_iy_j)$ is the number of (0, 1)-matrices A satisfying $\mathrm{row}(A) = \alpha$ and $\mathrm{col}(A) = \beta$. Hence equation (7.4) follows. Equation (7.5) is then a consequence of (7.3). □

Note that Corollary 7.4.2 is immediate from (7.4), since the product $\prod(1+x_iy_j)$ is invariant under interchanging x_i and y_i for all i.

We now come to a basic result known as the "fundamental theorem of symmetric functions," though for us it will barely scratch the surface of this subject.

7.4.4 Theorem. *Let $\lambda, \mu \vdash n$. Then $M_{\lambda\mu} = 0$ unless $\mu \le \lambda'$, while $M_{\lambda\lambda'} = 1$. Hence the set $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ^n (so $\{e_\lambda : \lambda \in$ Par$\}$ is a basis for Λ). Equivalently, $e_1, e_2, \ldots$ are algebraically independent and generate Λ as a*

$\mathbb{Q}$-algebra, which we write as

$$\Lambda = \mathbb{Q}[e_1, e_2, \ldots].$$

Proof. Suppose that $M_{\lambda\mu} \neq 0$, so by Proposition 7.4.1 there is a (0, 1)-matrix A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$. Let A' be the matrix with $\text{row}(A') = \lambda$ and with its 1's left-justified, i.e., $A'_{ij} = 1$ precisely for $1 \leq j \leq \lambda_i$. For any i the number of 1's in the first i columns of A' clearly is not less than the number of 1's in the first i columns of A, so by definition of dominance order we have $\text{col}(A') \geq \text{col}(A) = \mu$. But $\text{col}(A') = \lambda'$, so $\lambda' \geq \mu$ as desired. Moreover, it is easy to see that A' is the *only* (0, 1)-matrix with $\text{row}(A') = \lambda$ and $\text{col}(A') = \lambda'$, so $M_{\lambda\lambda'} = 1$.

The previous argument shows the following: let $\lambda^1, \lambda^2, \ldots, \lambda^{p(n)}$ be an ordering of $\text{Par}(n)$ that is compatible with dominance order, and such that the "reverse conjugate" order $(\lambda^{p(n)})', \ldots, (\lambda^2)', (\lambda^1)'$ is also compatible with dominance order. (It is easily seen that such orders exist.) Then the matrix $(M_{\lambda\mu})$, with the row order $\lambda^1, \lambda^2, \ldots$ and column order $(\lambda^1)', (\lambda^2)', \ldots,$ is upper triangular with 1's on the main diagonal. Hence it is invertible, so $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ^n. (In fact, it is a basis for $\Lambda^n_{\mathbb{Z}}$ since the diagonal entries are actually 1's, and not merely nonzero.)

The set $\{e_\lambda : \lambda \in \text{Par}\}$ consists of all monomials $e_1^{a_1} e_2^{a_2} \cdots$ (where $a_i \in \mathbb{N}, \sum a_i < \infty$). Hence the linear independence of $\{e_\lambda : \lambda \in \text{Par}\}$ is equivalent to the algebraic independence of $e_1, e_2, \ldots,$ as desired. □

Figure 7-2 gives a short table of the coefficients $M_{\lambda\mu}$.

$$\begin{array}{lllllll}
e_1 & = m_1 \\
\\
e_{11} & = m_2 + 2m_{11} \\
e_2 & = \phantom{m_2 +{}} m_{11} \\
\\
e_{111} & = m_3 + 3m_{21} + 6m_{111} \\
e_{21} & = \phantom{m_3 +{}} m_{21} + 3m_{111} \\
e_3 & = \phantom{m_3 + m_{21} +{}} m_{111} \\
\\
e_{1111} & = m_4 + 4m_{31} + 6m_{22} + 12m_{211} + 24m_{1111} \\
e_{211} & = \phantom{m_4 +{}} m_{31} + 2m_{22} + 5m_{211} + 12m_{1111} \\
e_{22} & = \phantom{m_4 + m_{31} +{}} m_{22} + 2m_{211} + 6m_{1111} \\
e_{31} & = \phantom{m_4 + m_{31} + m_{22} +{}} m_{211} + 4m_{1111} \\
e_4 & = \phantom{m_4 + m_{31} + m_{22} + m_{211} +{}} m_{1111} \\
\\
e_{11111} & = m_5 + 5m_{41} + 10m_{32} + 20m_{311} + 30m_{221} + 60m_{2111} + 120m_{11111} \\
e_{2111} & = \phantom{m_5 +{}} m_{41} + 3m_{32} + 7m_{311} + 12m_{221} + 27m_{2111} + 60m_{11111} \\
e_{221} & = \phantom{m_5 + m_{41} +{}} m_{32} + 2m_{311} + 5m_{221} + 12m_{2111} + 30m_{11111} \\
e_{311} & = \phantom{m_5 + m_{41} + m_{32} +{}} m_{311} + 2m_{221} + 7m_{2111} + 20m_{11111} \\
e_{32} & = \phantom{m_5 + m_{41} + m_{32} + m_{311} +{}} m_{221} + 3m_{2111} + 10m_{11111} \\
e_{41} & = \phantom{m_5 + m_{41} + m_{32} + m_{311} + m_{221} +{}} m_{2111} + 5m_{11111} \\
e_5 & = \phantom{m_5 + m_{41} + m_{32} + m_{311} + m_{221} + m_{2111} +{}} m_{11111}
\end{array}$$

Figure 7-2. The coefficients $M_{\lambda\mu}$.

7.5 Complete Homogeneous Symmetric Functions

Define the *complete homogeneous symmetric functions* (or just *complete symmetric functions*) h_λ for $\lambda \in$ Par by the formulas

$$\begin{aligned} h_n &= \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}, \quad n \ge 1 \qquad (\text{with } h_0 = m_\emptyset = 1) \\ h_\lambda &= h_{\lambda_1} h_{\lambda_2} \cdots \qquad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots). \end{aligned} \tag{7.6}$$

Thus h_n is the sum of all monomials of degree n.

The symmetric functions h_λ are in many ways "dual" to the elementary symmetric functions e_μ. The underlying reason for this duality will be brought out by Theorem 7.6.1 and various subsequent developments. For now let us consider the "complete analogue" of Proposition 7.4.1.

7.5.1 Proposition. *Let $\lambda \vdash n$, and let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n. Then the coefficient $N_{\lambda\alpha}$ of x^α in h_λ, i.e.,*

$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu, \tag{7.7}$$

is equal to the number of $\mathbb{N}$-matrices $A = (a_{ij})_{i,j \ge 1}$ satisfying $\text{row}(A) = \lambda$ *and* $\text{col}(A) = \alpha$.

Proof. Analogous to the proof of Proposition 7.4.1. A term x^α from $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ is obtained by choosing a term $x_1^{a_{i1}} x_2^{a_{i2}} \cdots$ from each h_{λ_i} such that

$$\prod_i \left(x_1^{a_{i1}} x_2^{a_{i2}} \cdots \right) = x^\alpha.$$

But this is just the same as choosing (a_{ij}) to be an $\mathbb{N}$-matrix A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$, and the proof follows. □

7.5.2 Corollary. *Let $N_{\lambda\mu}$ be given by (7.7). Then $N_{\lambda\mu} = N_{\mu\lambda}$, i.e., the transition matrix between the bases* $\{m_\lambda : \lambda \vdash n\}$ and $\{h_\lambda : \lambda \vdash n\}$ is a symmetric matrix.*

Proof. Exactly analogous to the proof of Corollary 7.4.2. □

The coefficient $N_{\lambda\mu}$ of (7.7) has an alternative combinatorial interpretation in terms of balls into boxes, similar to the interpretation of $M_{\lambda\mu}$ in this way. We have

* It follows from Corollary 7.6.2 that the set $\{h_\lambda : \lambda \vdash n\}$ is indeed a basis.

n balls in all, with λ_i balls labeled i. We also have boxes labeled $1, 2, \ldots$. Then $N_{\lambda\mu}$ is the number of ways of placing the balls in the boxes so that box i contains exactly μ_i balls.

On the combinatorial level, the duality between e_λ and h_μ is manifested by (0, 1)-matrices *vs.* $\mathbb{N}$-matrices, or equivalently, balls into boxes (subject to certain conditions) not allowing repetitions or allowing repetitions. This situation is reminiscent of the reciprocity between $\binom{n}{k}$ and $\left(\!\binom{n}{k}\!\right) = (-1)^k\binom{-n}{k}$, or of the more general reciprocity theorems of Sections 4.5 and 4.6. Indeed, given a symmetric function $f(x)$ and $n \in \mathbb{N}$, let us write

$$f(1^n) = f(x_1 = x_2 = \cdots = x_n = 1,\ x_{n+1} = x_{n+2} = \cdots = 0). \tag{7.8}$$

Then

$$e_k(1^n) = \sum_{1 \le i_1 < \cdots < i_k \le n} 1 = \binom{n}{k}$$

$$h_k(1^n) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} 1 = \left(\!\binom{n}{k}\!\right).$$

We next give the generating function interpretation of Proposition 7.5.1. The proof is analogous to that of Proposition 7.4.3 and is omitted.

$$\begin{array}{lcl}
h_1 & = & m_1 \\
\\
h_{11} & = & 2m_{11} + m_2 \\
h_2 & = & m_{11} + m_2 \\
\\
h_{111} & = & 6m_{111} + 3m_{21} + m_3 \\
h_{21} & = & 3m_{111} + 2m_{21} + m_3 \\
h_3 & = & m_{111} + m_{21} + m_3 \\
\\
h_{1111} & = & 24m_{1111} + 12m_{211} + 6m_{22} + 4m_{31} + m_4 \\
h_{211} & = & 12m_{1111} + 7m_{211} + 4m_{22} + 3m_{31} + m_4 \\
h_{22} & = & 6m_{1111} + 4m_{211} + 3m_{22} + 2m_{31} + m_4 \\
h_{31} & = & 4m_{1111} + 3m_{211} + 2m_{22} + 2m_{31} + m_4 \\
h_4 & = & m_{1111} + m_{211} + m_{22} + m_{31} + m_4 \\
\\
h_{11111} & = & 120m_{11111} + 60m_{2111} + 30m_{221} + 20m_{311} + 10m_{32} + 5m_{41} + m_5 \\
h_{2111} & = & 60m_{11111} + 33m_{2111} + 18m_{221} + 13m_{311} + 7m_{32} + 4m_{41} + m_5 \\
h_{221} & = & 30m_{11111} + 18m_{2111} + 11m_{221} + 8m_{311} + 5m_{32} + 3m_{41} + m_5 \\
h_{311} & = & 20m_{11111} + 13m_{2111} + 8m_{221} + 7m_{311} + 4m_{32} + 3m_{41} + m_5 \\
h_{32} & = & 10m_{11111} + 7m_{2111} + 5m_{221} + 4m_{311} + 3m_{32} + 2m_{41} + m_5 \\
h_{41} & = & 5m_{11111} + 4m_{2111} + 3m_{221} + 3m_{311} + 2m_{32} + m_{41} + m_5 \\
h_5 & = & m_{11111} + m_{2111} + m_{221} + m_{311} + m_{32} + m_{41} + m_5
\end{array}$$

* **Figure 7-3.** The coefficients $N_{\lambda\mu}$.

7.5.3 Proposition. *We have*

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda,\mu} N_{\lambda\mu} m_\lambda(x) m_\mu(y) \tag{7.9}$$

$$= \sum_{\lambda} m_\lambda(x) h_\lambda(y), \tag{7.10}$$

where λ and μ range over Par *(and where it suffices to take $|\lambda| = |\mu|$).*

It is not as easy to argue as in the proof of Theorem 7.4.4 that the set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ^n (so $h_1, h_2, \ldots$ are algebraically independent), since the matrix $(N_{\lambda\mu})$ has no nice triangularity properties. But the linear independence of the h_λ's will be a trivial consequence of Theorem 7.6.1, so we save its "official" statement until then.

Figure 7-3 gives a short table of the coefficients $N_{\lambda\mu}$.

7.6 An Involution

Since $\Lambda = \mathbb{Q}[e_1, e_2, \ldots]$, an algebra endomorphism $f : \Lambda \to \Lambda$ is determined uniquely by its values $f(e_n)$, $n \geq 1$; and conversely any choice of $f(e_n) \in \Lambda$ determines an endomorphism f. Define an endomorphism $\omega : \Lambda \to \Lambda$ by $\omega(e_n) = h_n$, $n \geq 1$. Thus (since ω preserves multiplication) $\omega(e_\lambda) = h_\lambda$ for all partitions λ.

7.6.1 Theorem. *The endomorphism ω is an involution, i.e., $\omega^2 = 1$ (the identity automorphism), or equivalently $\omega(h_n) = e_n$. (Thus $\omega(h_\lambda) = e_\lambda$ for all partitions λ.)*

Proof. Consider the formal power series

$$H(t) := \sum_{n\geq 0} h_n t^n \in \Lambda[[t]]$$

$$E(t) := \sum_{n\geq 0} e_n t^n \in \Lambda[[t]].$$

We leave to the reader the easy verification of the identities

$$H(t) = \prod_n (1-x_n t)^{-1} \tag{7.11}$$

$$E(t) = \prod_n (1+x_n t). \tag{7.12}$$

Hence $H(t)E(-t) = 1$. Equating coefficients of t^n on both sides yields

$$0 = \sum_{i=0}^{n} (-1)^i e_i h_{n-i}, \qquad n \geq 1. \tag{7.13}$$

Conversely, if $\sum_{i=0}^{n}(-1)^i u_i h_{n-i} = 0$ for all $n \geq 1$, for certain $u_i \in \Lambda$ with $u_0 = 1$, then $u_i = e_i$. Now apply ω to (7.13) to obtain

$$0 = \sum_{i=0}^{n}(-1)^i h_i \omega(h_{n-i}) = (-1)^n \sum_{i=0}^{n}(-1)^i \omega(h_i) h_{n-i} = 0,$$

whence $\omega(h_i) = e_i$ as desired. □

The involution ω may be regarded as an algebraic elaboration of the reciprocity between sets and multisets expressed by the identity $\left(\!\binom{n}{k}\!\right) = (-1)^k \binom{-n}{k}$, as suggested in Section 7.5.

7.6.2 Corollary. *The set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ^n (so $\{h_\lambda : \lambda \in \text{Par}\}$ is a basis for Λ). Equivalently, $h_1, h_2, \ldots$ are algebraically independent and generate Λ as a $\mathbb{Q}$-algebra, which we write as*

$$\Lambda = \mathbb{Q}[h_1, h_2, \ldots].$$

Proof. Theorem 7.6.1 shows that the endomorphism $\omega : \Lambda \to \Lambda$ defined by $\omega(e_n) = h_n$ is invertible (since $\omega = \omega^{-1}$), and hence is an automorphism of Λ. The proof now follows from Theorem 7.4.4. □

7.7 Power Sum Symmetric Functions

We define a fourth set p_λ of symmetric functions indexed by $\lambda \in$ Par and called *power sum symmetric functions*, as follows:

$$p_n = m_n = \sum_i x_i^n, \quad n \geq 1 \qquad (\text{with } p_0 = m_\emptyset = 1)$$
$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \qquad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots).$$

7.7.1 Proposition. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, where $\ell = \ell(\lambda)$, and set*

$$p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu. \tag{7.14}$$

Let $k = \ell(\mu)$. Then $R_{\lambda\mu}$ is equal to the number of ordered partitions $\pi = (B_1, \ldots, B_k)$ of the set $[\ell]$ such that

$$\mu_j = \sum_{i \in B_j} \lambda_i, \qquad 1 \leq j \leq k. \tag{7.15}$$

Proof. $R_{\lambda\mu}$ is the coefficient of $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots$ in $p_\lambda = (\sum x_i^{\lambda_1})(\sum x_i^{\lambda_2}) \cdots$. To obtain the monomial x^μ in the expansion of this product, we choose a term $x_{i_j}^{\lambda_j}$ from each factor $\sum x_i^{\lambda_j}$ so that $\prod_j x_{i_j}^{\lambda_j} = x^\mu$. Define $B_r = \{j : i_j = r\}$. Then $(B_1, \ldots, B_k)$ will be an ordered partition of $[\ell]$ satisfying (7.15), and conversely every such ordered partition gives rise to a term x^μ. □

7.7.2 Corollary. *Let $R_{\lambda\mu}$ be as in (7.14). Then $R_{\lambda\mu} = 0$ unless $\lambda \leq \mu$, while*

$$R_{\lambda\lambda} = \prod_i m_i(\lambda)!, \tag{7.16}$$

where $\lambda = \langle 1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots \rangle$. Hence $\{p_\lambda : \lambda \vdash n\}$ is a basis for Λ^n (so $\{p_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ). Equivalently, $p_1, p_2, \ldots$ are algebraically independent and generate Λ as a $\mathbb{Q}$-algebra, i.e.,

$$\Lambda = \mathbb{Q}[p_1, p_2, \ldots].$$

Proof. If $R_{\lambda\mu} \neq 0$, then by Proposition 7.7.1 there is an ordered partition $\pi = (B_1, \ldots, B_k)$ of the set $[\ell] = [\ell(\lambda)]$ satisfying (7.15). Given $1 \leq r \leq \ell$, let $B_{i_1}, \ldots, B_{i_s}$ be the *distinct* blocks of π containing at least one of $1, 2, \ldots, r$. From (7.15) we have $\mu_{i_1} + \cdots + \mu_{i_s} \geq \lambda_1 + \cdots + \lambda_r$. But $\mu_1 + \cdots + \mu_r \geq \mu_{i_1} + \cdots + \mu_{i_s}$ since $r \geq s$ and $\mu_1 \geq \mu_2 \geq \cdots$. Hence $\mu \geq \lambda$, as desired.

If $\mu = \lambda$, then each block B_i is a singleton $\{j\}$, which we denote simply as j. $B_1, \ldots, B_{m_1}$ can be any ordering of $1, \ldots, m_1$. Then $B_{m_1+1}, \ldots, B_{m_1+m_2}$ can be any ordering of $m_1 + 1, \ldots, m_1 + m_2$, etc., giving a total of $R_{\lambda\lambda} = m_1! m_2! \cdots$ possibilities for π.

The fact that $\{p_\lambda : \lambda \vdash n\}$ is a basis for Λ (so $\Lambda = \mathbb{Q}[p_1, p_2, \ldots]$) follows by reasoning as in the proof of Theorem 7.4.4. □

NOTE. Because the diagonal elements $R_{\lambda\lambda}$ are not all ± 1, it follows that $\{p_\lambda : \lambda \vdash n\}$ is not a $\mathbb{Z}$-basis for $\Lambda^n_{\mathbb{Z}}$. Rather, the (additive) abelian subgroup P_n of $\Lambda^n_{\mathbb{Z}}$ generated by the p_λ's has index

$$\begin{aligned}\left[\Lambda^n_{\mathbb{Z}} : P_n\right] &= \det(R_{\lambda\mu})_{\lambda,\mu\vdash n} \\ &= \prod_{\lambda\vdash n} \prod_i m_i(\lambda)!.\end{aligned}$$

By Exercise 1.26, if follows that this index is also given by

$$\left[\Lambda^n_{\mathbb{Z}} : P_n\right] = \prod_{\lambda \vdash n} 1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots,$$

i.e., the product of all parts of all partitions of n.

We now consider the effect of the involution ω on p_λ. A generating function approach is most efficacious. For any partition $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$, define

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots. \tag{7.17}$$

For instance, $z_{442111} = 1^3 3!\, 2^1 1!\, 4^2 2! = 384$. If $w \in \mathfrak{S}_n$, then the *cycle type* $\rho(w)$ of w is the partition $\rho(w) = (\rho_1, \rho_2, \ldots) \vdash n$ such that the cycle lengths of w (in its factorization into disjoint cycles) are $\rho_1, \rho_2, \ldots$. Recall from Proposition 1.3.2 that the number of permutations $w \in \mathfrak{S}_n$ of a fixed cycle type $\rho = \langle 1^{m_1} 2^{m_2} \cdots \rangle$

is given by

$$\#\{w \in \mathfrak{S}_n : \rho(w) = \rho\} = \frac{n!}{1^{m_1} m_1!\, 2^{m_2} m_2! \cdots} = n!\, z_\rho^{-1}. \tag{7.18}$$

The set $\{v \in \mathfrak{S}_n : \rho(v) = \rho\}$ is just the conjugacy class in $\mathfrak{S}_n$ containing w. For any finite group G, the order $\#K_w$ of the conjugacy class K_w containing w is equal to the index $[G : C(w)]$ of the centralizer of w. Hence:

7.7.3 Proposition. *Let $\lambda \vdash n$. Then z_λ is equal to the number of permutations $v \in \mathfrak{S}_n$ that commute with a fixed w_λ of cycle type λ.*

For a bijective proof of Proposition 7.7.3, see Exercise 7.6.

For a partition $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ of n, define

$$\varepsilon_\lambda = (-1)^{m_2 + m_4 + \cdots} = (-1)^{n - \ell(\lambda)}. \tag{7.19}$$

Thus for $w \in \mathfrak{S}_n$, $\varepsilon_{\rho(w)}$ is $+1$ if w is an even permutation and -1 otherwise, so the map $\mathfrak{S}_n \to \{\pm 1\}$ defined by $w \mapsto \varepsilon_{\rho(w)}$ is the usual "sign homomorphism."

7.7.4 Proposition. *We have*

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \exp \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y) = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) \tag{7.20}$$

$$\prod_{i,j} (1 + x_i y_j) = \exp \sum_{n \geq 1} \frac{1}{n} (-1)^{n-1} p_n(x) p_n(y) = \sum_\lambda z_\lambda^{-1} \varepsilon_\lambda\, p_\lambda(x) p_\lambda(y). \tag{7.21}$$

Proof. We have

$$\begin{aligned}
\log \prod_{i,j} (1 - x_i y_j)^{-1} &= \sum_{i,j} \log(1 - x_i y_j)^{-1} \\
&= \sum_{i,j} \sum_{n \geq 1} \frac{1}{n} x_i^n y_j^n \\
&= \sum_{n \geq 1} \frac{1}{n} \left(\sum_i x_i^n \right) \left(\sum_j y_j^n \right) \\
&= \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y).
\end{aligned}$$

Thus the first equality of (7.20) is established. The second equality of (7.20) is a consequence of (and in fact is equivalent to) the permutation version of the exponential formula (Corollary 5.1.9). More specifically, in Corollary 5.1.9 set $f(n) = p_n(x)p_n(y)$ and $x = 1$. Then

$$\begin{aligned} h(n) &= \sum_{w \in \mathfrak{S}_n} p_{\rho(w)}(x) p_{\rho(w)}(y) \\ &= \sum_{\lambda \vdash n} c_\lambda p_\lambda(x) p_\lambda(y), \end{aligned}$$

where $c_\lambda = \#\{w \in \mathfrak{S}_n : \rho(w) = \lambda\}$. By equation (7.18) we have $c_\lambda = n! z_\lambda^{-1}$, and (7.20) follows from Corollary 5.1.9. The proof of (7.21) is entirely analogous. □

From the previous proposition it is easy to deduce the effect of ω on p_λ.

7.7.5 Proposition. *Let $\lambda \vdash n$. Then*

$$\omega p_\lambda = \varepsilon_\lambda p_\lambda.$$

In other words, p_λ is an eigenvector for ω corresponding to the eigenvalue ε_λ.

Proof. Regard ω as acting on symmetric functions in the variables $y = (y_1, y_2, \ldots)$; those in the variables x are regarded as scalars. Apply ω to (7.20). We obtain

$$\begin{aligned} \omega \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) &= \omega \prod_{i,j} (1 - x_i y_j)^{-1} \\ &= \sum_\nu m_\nu(x) \omega h_\nu(y) \qquad \text{(by (7.10))} \\ &= \sum_\nu m_\nu(x) e_\nu(y) \qquad \text{(by Theorem 7.6.1)} \\ &= \prod_{i,j} (1 + x_i y_j) \qquad \text{(by (7.5))} \\ &= \sum_\lambda z_\lambda^{-1} \varepsilon_\lambda \, p_\lambda(x) p_\lambda(y) \qquad \text{(by (7.21))}. \end{aligned}$$

Since the $p_\lambda(x)$'s are linearly independent, their coefficients in the first and last sums of the above chain of equalities must be the same. In other words, $\omega p_\lambda(y) = \varepsilon_\lambda p_\lambda(y)$, as desired. □

Note in particular that $\omega p_n = (-1)^{n-1} p_n$, or $\omega p_n(x) = -p_n(-x)$. Since ω is an automorphism, just the values of ωp_n in fact suffice to determine ωp_λ for any partition λ.

Consider ω restricted to the vector space (of dimension $p(n)$) Λ^n. Since the p_λ's for $\lambda \vdash n$ are linearly independent, it follows from Proposition 7.7.5 that the characteristic polynomial (normalized to be monic) of the linear transformation $\omega : \Lambda^n \to \Lambda^n$ is given by $(x-1)^{e(n)}(x+1)^{o(n)}$, where $e(n)$ (respectively, $o(n)$) is the number of partitions of n with an even number (respectively, odd number) of even parts. In other words, $e(n)$ is the number of *even conjugacy classes* (i.e., conjugacy classes contained in the alternating group) of $\mathfrak{S}_n$. By Exercise 1.1.9(b) we have

$$(x-1)^{e(n)}(x+1)^{o(n)} = (x^2-1)^{o(n)}(x-1)^{k(n)},$$

where $k(n)$ is the number of self-conjugate partitions of n. At the end of Section 7.14 we will see a simple reason for the factor $(x-1)^{k(n)}$ in terms of symmetric functions.

We can now ask how to express the symmetric functions m_λ, h_λ, and e_λ in terms of the p_μ's. Although combinatorial interpretations can be given to the coefficients in these expansions, they tend to be messy and not very useful. One special case, however, is of considerable importance.

7.7.6 Proposition. *We have*

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda \tag{7.22}$$

$$e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda. \tag{7.23}$$

Proof. Substituting $y = (t, 0, 0, \ldots)$ in (7.20) immediately yields (7.22). Equation (7.23) is similarly obtained from (7.21), or by applying ω to (7.22). $\square$

See Example 5.2.11 for a combinatorial proof of equation (7.22). Equation (7.23) can be given a similar proof.

7.8 Specializations

In many combinatorial problems involving symmetric functions f we only need partial information about f, such as a particular coefficient or value. In this section we give a brief overview of the most common specializations that arise in practice. (See Exercises 7.43 and 7.44 for two others.) Proofs for the most part are straightforward and will be omitted. First let us give a formal definition of the concept of specialization.

7.8.1 Definition. Let R be a commutative $\mathbb{Q}$-algebra with identity. A *specialization* of the ring Λ is a homomorphism $\varphi : \Lambda \to R$. (We always assume homomorphisms are unital, i.e., $\varphi(1) = 1$.)

The most obvious examples of specializations arise from substituting elements a_i of R for the variables x_i (provided of course this substitution is well-defined formally; it would make no sense, for instance, to set each $x_i = 1$ in $h_1(x) = x_1 + x_2 + \cdots$). We may then write

$$\varphi(f) = f(a_1, a_2, \ldots),$$

and we call φ the *substitution* of a_i for x_i.

Our first example may be called "reducing the number of variables" and is very common. Let Λ_n denote the set of all polynomials $f \in \mathbb{Q}[x_1, \ldots, x_n]$ in the variables $x_1, \ldots, x_n$ with rational coefficients which are invariant under any permutation of the variables. Thus f is just a symmetric function in the variables $x_1, \ldots, x_n$. Define $r_n : \Lambda \to \Lambda_n$ by $r_n(f) = f(x_1, \ldots, x_n, 0, 0, \ldots)$ (written $f(x_1, \ldots, x_n)$). The next proposition examines the behavior of the four bases m_λ, p_λ, h_λ, e_λ, as well as the involution ω, under r_n.

7.8.2 Proposition. *Let* Par_n *denote the set of all partitions* $\lambda \in \mathrm{Par}$ *of length at most n, i.e.,*

$$\mathrm{Par}_n = \{\lambda \in \mathrm{Par} : \ell(\lambda) \le n\}.$$

(a) *The sets* $\{r_n(m_\lambda) : \lambda \in \mathrm{Par}_n\}$, $\{r_n(p_\lambda) : \lambda' \in \mathrm{Par}_n\}$, $\{r_n(h_\lambda) : \lambda' \in \mathrm{Par}_n\}$, $\{r_n(e_\lambda) : \lambda' \in \mathrm{Par}_n\}$ *are all* $\mathbb{Q}$*-bases for* Λ_n*. Moreover, if* $\lambda \notin \mathrm{Par}_n$*, then* $r_n(m_\lambda) = r_n(e_{\lambda'}) = 0$.

(b) *For convenience identify an element* $f \in \Lambda$ *with its image* $r_n(f)$ *in* Λ_n*. Define a linear transformation* $\omega_n : \Lambda_n \to \Lambda_n$ *by* $\omega_n(e_\lambda) = h_\lambda$ *for* $\lambda' \in \mathrm{Par}_n$*. Then* ω_n *is an algebra automorphism and an involution, and* $\omega_n(p_\lambda) = \varepsilon_\lambda p_\lambda$ *for* $\lambda' \in \mathrm{Par}_n$.

Proof. Straightforward consequence of analogous properties for Λ and triangularity properties of the bases m_λ, p_λ, e_λ discussed previously. □

A little caution is needed when dealing with p_λ or h_λ in Λ_n when $\lambda' \notin \mathrm{Par}_n$. For instance, p_λ need not be zero nor an eigenvector of ω_n. For instance, when $n = 2$ we have $p_3 = \frac{1}{2}(3p_{21} - p_{111})$ and $\omega_2(p_3) = \frac{1}{2}(-3p_{21} - p_{111})$.

An important substitution $\mathrm{ps}_n : \Lambda \to \mathbb{Q}[q]$ is defined by

$$\mathrm{ps}_n(f) = f(1, q, q^2, \ldots, q^{n-1}),$$

and is called the *principal specialization* (of order n) of f. If we let $n \to \infty$ we obtain the limiting value

$$\mathrm{ps}(f) = f(1, q, q^2, \ldots) \in \mathbb{Q}[[q]], \tag{7.24}$$

called the *stable principal specialization* of f. (It is easily seen that $\lim_{n\to\infty} \mathrm{ps}_n(f)$ exists in the sense of Section 1.1.) A specialization $\mathrm{ps}_n^1 : \Lambda \to \mathbb{Q}$ of the principal specialization is obtained by letting $q = 1$, i.e.,

$$\mathrm{ps}_n^1(f) = f(\underbrace{1, 1, \ldots, 1}_{n\ 1\text{'s}}) = f(1^n),$$

in the notation of (7.8).

7.8.3 Proposition. *The following table summarizes the behavior of the bases* m_λ, p_λ, h_λ, e_λ *under* ps_n, ps, *and* ps_n^1*:*

basis b_λ	$\mathrm{ps}_n(b_\lambda)$	$\mathrm{ps}(b_\lambda)$	$\mathrm{ps}_n^1(b_\lambda)$
m_λ	messy	messy	$\binom{n}{\ell(\lambda)}\binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots}$
p_λ	$\prod_{i=1}^{\ell} \frac{1-q^{n\lambda_i}}{1-q^{\lambda_i}}$	$\prod_{i=1}^{\ell} \frac{1}{1-q^{\lambda_i}}$	$n^{\ell(\lambda)}$
e_λ	$\prod_i q^{\binom{\lambda_i}{2}}\binom{n}{\lambda_i}$	$\prod_i \frac{q^{\binom{\lambda_i}{2}}}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod_i \binom{n}{\lambda_i}$
h_λ	$\prod_i \binom{n+\lambda_i-1}{\lambda_i}$	$\prod_i \frac{1}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod_i \left(\!\binom{n}{\lambda_i}\!\right)$

Proof. All can be done by straightforward combinatorial or algebraic reasoning. As an example, we show how to obtain $\mathrm{ps}_n(h_\lambda)$.

Since $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ and ps_n is an algebra homomorphism, it suffices to compute $\mathrm{ps}_n(h_k)$. Since

$$h_k = \sum_{a_1+a_2+\cdots=k} x_1^{a_1} x_2^{a_2} \cdots,$$

we have

$$\mathrm{ps}_n(h_k) = \sum_{a_1+\cdots+a_n=k} q^{a_2+2a_3+\cdots+(n-1)a_n},$$

summed over all weak compositions of k into n parts. If we identify the sequence $(a_1, a_2, \ldots, a_n)$ with the partition $\lambda = \langle 1^{a_2}, 2^{a_3}, \ldots, (n-1)^{a_n}\rangle$, then we see that

$$\begin{aligned}\mathrm{ps}_n(h_k) &= \sum_{\substack{\ell(\lambda)\le k\\ \ell(\lambda')\le n-1}} q^{|\lambda|}\\ &= \binom{n+k-1}{k},\end{aligned}$$

by Proposition 1.3.19, and the proof follows. □

We now consider an important specialization that is not obtained simply by substituting for the variables x_i. We call this specialization the *exponential specialization* $\mathrm{ex} : \Lambda \to \mathbb{Q}[t]$ or $\mathrm{ex} : \hat{\Lambda} \to \mathbb{Q}[[t]]$, defined by

$$\mathrm{ex}(p_n) = t\delta_{1n}.$$

Note that since the p_n's are algebraically independent and generate Λ as a $\mathbb{Q}$-algebra, any homomorphism $\varphi : \Lambda \to R$ is determined by its values $\varphi(p_n)$. Here we are setting $\varphi(p_1) = t$ and $\varphi(p_n) = 0$ if $n > 1$. (If the domain of ex is taken to be $\hat{\Lambda}$, then we define ex to preserve infinite linear combinations, or equivalently, to be continuous in a suitable topology.)

7.8.4 Proposition. (a) *We have*

$$\mathrm{ex}(f) = \sum_{n\geq 0} [x_1x_2\cdots x_n]f\frac{t^n}{n!}, \tag{7.25}$$

for any $f \in \hat{\Lambda}$, where $[x_1x_2\cdots x_n]f$ denotes the coefficient of $x_1x_2\cdots x_n$ in f. Equivalently, if $f = \sum_\lambda c_\lambda m_\lambda$, then

$$\mathrm{ex}(f) = \sum_{n\geq 0} c_{1^n}\frac{t^n}{n!}.$$

(b) *We have*

$$\mathrm{ex}(m_\lambda) = \begin{cases} \dfrac{t^n}{n!} & \text{if } \lambda = \langle 1^n\rangle \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{ex}(p_\lambda) = \begin{cases} t^n & \text{if } \lambda = \langle 1^n\rangle \\ 0, & \text{otherwise} \end{cases} \tag{7.26}$$

$$\mathrm{ex}(h_\lambda) = \mathrm{ex}(e_\lambda) = \frac{t^{|\lambda|}}{\lambda_1!\lambda_2!\cdots}. \tag{7.27}$$

Proof. (a) Since the right-hand side of (7.25) is linear in f, we need only verify (7.25) for $f = p_\lambda$. This is a routine computation.
(b) Easy consequence of (a). □

7.8.5 Example. Let

$$F(x) = \prod_i (1 - x_i)^{-1} \cdot \prod_{i<j} (1 - x_ix_j)^{-1}.$$

(For the significance of this product, see Corollary 7.13.8.) In this example we will evaluate $\mathrm{ex}(F(x))$. We can save a little work by observing that

$$\begin{aligned}\mathrm{ex}\prod_i(1-x_i)^{-1} &= \mathrm{ex}\sum_{n\geq 0} h_n\\ &= \sum_{n\geq 0}\frac{t^n}{n!} = e^t,\end{aligned}$$

so (since ex is a homomorphism),

$$\mathrm{ex}(F(x)) = e^t\cdot \mathrm{ex}\prod_{i<j}(1-x_ix_j)^{-1}.$$

Now

$$\begin{aligned}\prod_{i<j}(1-x_ix_j)^{-1} &= \exp\sum_{i<j}\log(1-x_ix_j)^{-1}\\ &= \exp\sum_{i<j}\sum_{n\geq 1}\frac{(x_ix_j)^n}{n}\\ &= \exp\sum_{n\geq 1}\frac{1}{2n}\left(p_n^2-p_{2n}\right).\end{aligned}$$

(Do not confuse ex with the ordinary exponential function exp.) Hence

$$\begin{aligned}\mathrm{ex}\prod_{i<j}(1-x_ix_j)^{-1} &= \exp\sum_{n\geq 1}\frac{1}{2n}\mathrm{ex}\left(p_n^2-p_{2n}\right)\\ &= e^{t^2/2},\end{aligned}$$

by (7.26), so

$$\mathrm{ex}(F(x)) = e^{t+\frac{1}{2}t^2}.$$

We recognize from equation (5.32) that $e^{t+\frac{1}{2}t^2}$ is the exponential generating function for the number $e_2(n)$ of involutions in $\mathfrak{S}_n$. Indeed, it is easy to see directly from the definition of $F(x)$ that

$$[x_1\cdots x_n]F(x) = e_2(n).$$

In view of Proposition 7.8.4, it is natural to ask whether the specialization ex has a "natural" q-analogue ex_q. The definition that works best is given by

$$\mathrm{ex}_q(h_n) = \frac{t^n}{(\boldsymbol{n})!}.$$

By Proposition 7.8.3 we see that

$$\mathrm{ex}_q(f) = f((1-q)t,(1-q)qt,(1-q)q^2t,\ldots) \tag{7.28}$$

a minor variant of the stable principal specialization ps. In fact, if $f \in \Lambda^n$ then

$$\mathrm{ex}_q(f) = (1-q)^n t^n \,\mathrm{ps}(f). \tag{7.29}$$

Thus the exponential specialization ex is essentially a limiting case of ps, though we cannot simply set $q = 1$ in (7.29) to conclude that $\mathrm{ex}(f) = 0$ for all f! The substitution $q = 1$ is not a valid formal power series operation, since the operation of setting $q = 1$ in $\mathrm{ps}(f)$ is undefined (or is ∞, if one prefers).

7.9 A Scalar Product

Up to now we have been dealing with a graded algebra Λ with several distinguished bases. We now want to put on Λ the additional structure of a scalar product, i.e., a bilinear form $\Lambda \times \Lambda \to \mathbb{Q}$, which we will denote by $\langle\ ,\ \rangle$. If $\{u_i\}$ and $\{v_j\}$ are bases of a vector space V, then a scalar product on V is uniquely determined by specifying the values $\langle u_i, v_j\rangle$. In particular, we say that $\{u_i\}$ and $\{v_j\}$ are *dual bases* if $\langle u_i, v_j\rangle = \delta_{ij}$ (Kronecker delta) for all i and j. We now define a scalar product on Λ by requiring that $\{m_\lambda\}$ and $\{h_\mu\}$ be dual bases, i.e.,

$$\langle m_\lambda, h_\mu\rangle = \delta_{\lambda\mu}, \tag{7.30}$$

for all $\lambda, \mu \in \mathrm{Par}$. The motivation for this definition will become clear as we develop many desirable and useful properties. First notice that $\langle\ ,\ \rangle$ respects the grading of Λ, in the sense that if f and g are homogeneous then $\langle f, g\rangle = 0$ unless $\deg f = \deg g$.

We now give a series of results which elucidate the nature of the scalar product $\langle\ ,\ \rangle$.

7.9.1 Proposition. *The scalar product $\langle\ ,\ \rangle$ is symmetric, i.e., $\langle f, g\rangle = \langle g, f\rangle$ for all $f, g \in \Lambda$.*

Proof. The result is equivalent to Corollary 7.5.2. More specifically, it suffices by linearity to prove $\langle f, g\rangle = \langle g, f\rangle$ for some bases $\{f\}$ and $\{g\}$ of Λ. Take $\{f\} = \{g\} = \{h_\lambda\}$. Then

$$\langle h_\lambda, h_\mu\rangle = \left\langle \sum_\nu N_{\lambda\nu} m_\nu, h_\mu \right\rangle = N_{\lambda\mu}. \tag{7.31}$$

Since $N_{\lambda\mu} = N_{\mu\lambda}$ by Corollary 7.5.2, we have $\langle h_\lambda, h_\mu\rangle = \langle h_\mu, h_\lambda\rangle$, as desired. □

The following lemma is a basic tool for verifying orthogonality of certain classes of symmetric functions. Its proof is a straightforward exercise in linear algebra and can be omitted without significant loss of understanding.

7.9.2 Lemma. *Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be bases of Λ such that for all $\lambda \vdash n$ we have $u_\lambda, v_\lambda \in \Lambda^n$. Then $\{u_\lambda\}$ and $\{v_\lambda\}$ are dual bases if and only if*

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}.$$

Proof. Write $m_\lambda = \sum_\rho \zeta_{\lambda\rho} u_\rho$ and $h_\mu = \sum_\nu \eta_{\mu\nu} v_\nu$. Thus

$$\delta_{\lambda\mu} = \langle m_\lambda, h_\mu \rangle = \sum_{\rho,\nu} \zeta_{\lambda\rho} \eta_{\mu\nu} \langle u_\rho, v_\nu \rangle. \tag{7.32}$$

For each fixed $n \geq 0$, regard ζ and η as matrices indexed by $\mathrm{Par}(n)$, and let A be the matrix defined by $A_{\rho\nu} = \langle u_\rho, v_\nu \rangle$. Then (7.32) is equivalent to $I = \zeta A \eta^t$, where ^{t} denotes transpose and I the identity matrix. Therefore:

$$\begin{aligned} \{u_\lambda\} \text{ and } \{v_\mu\} \text{ are dual bases} &\iff A = I \\ &\iff I = \zeta \eta^t \\ &\iff I = \zeta^t \eta \\ &\iff \delta_{\rho\nu} = \sum_\lambda \zeta_{\lambda\rho} \eta_{\lambda\nu}. \end{aligned} \tag{7.33}$$

Now by Proposition 7.5.3 we have

$$\begin{aligned} \prod_{i,j} (1 - x_i y_j)^{-1} &= \sum_\lambda m_\lambda(x) h_\lambda(y) \\ &= \sum_\lambda \left(\sum_\rho \zeta_{\lambda\rho} u_\rho(x) \right) \left(\sum_\nu \eta_{\lambda\nu} v_\nu(y) \right) \\ &= \sum_{\rho,\nu} \left(\sum_\lambda \zeta_{\lambda\rho} \eta_{\lambda\nu} \right) u_\rho(x) v_\nu(y). \end{aligned}$$

Since the power series $u_\rho(x) v_\nu(y)$ are linearly independent over $\mathbb{Q}$, the proof follows from (7.33). □

7.9.3 Proposition. *We have*

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}. \tag{7.34}$$

Hence the p_λ's form an orthogonal basis of Λ. (They don't form an orthonormal basis, since $\langle p_\lambda, p_\lambda \rangle \neq 1$.)

Proof. By Proposition 7.7.4 and Lemma 7.9.2 we see that $\{p_\lambda\}$ and $\{p_\mu / z_\mu\}$ are dual bases, which is equivalent to (7.34). □

The length $\|p_\lambda\| = \langle p_\lambda, p_\lambda \rangle^{1/2} = z_\lambda^{1/2}$ is in general not rational. Thus the elements $p_\lambda / \|p_\lambda\|$ form an orthonormal basis of $\Lambda_{\mathbb{R}}$ but not of Λ (since they don't belong to Λ). It is natural to ask whether there is a "natural" orthonormal basis for Λ. Even better, is there an *integral* orthonormal basis for Λ, i.e., is there an orthonormal basis $\{b_\lambda\}$ for Λ such that each b_λ is an integer linear combination of m_μ's, and conversely each m_μ is an integer linear combination of b_λ's? Such a basis will thus be a basis for $\Lambda_{\mathbb{Z}}$ (as an abelian group). In Sections 7.10–7.17 we will construct such a basis (see Corollary 7.12.2) and derive many remarkable combinatorial properties that it possesses.

7.9.4 Corollary. *The scalar product $\langle\ ,\ \rangle$ is positive definite, i.e., $\langle f, f\rangle \geq 0$ for all $f \in \Lambda$, with equality if and only if $f = 0$.*

Proof. Write (uniquely) $f = \sum_\lambda c_\lambda p_\lambda$. Then

$$\langle f, f\rangle = \sum c_\lambda^2 z_\lambda.$$

The proof follows since each $z_\lambda > 0$. □

7.9.5 Proposition. *The involution ω is an isometry, i.e., $\langle \omega f, \omega g\rangle = \langle f, g\rangle$ for all $f, g \in \Lambda$.*

Proof. By the bilinearity of the scalar product, it suffices to take $f = p_\lambda$ and $g = p_\mu$. The result then follows from Propositions 7.7.5 and 7.9.3. □

7.10 The Combinatorial Definition of Schur Functions

The four bases m_λ, e_λ, h_λ, and p_λ of Λ discussed in the previous sections all have rather transparent definitions. In this section we consider a fifth basis, whose elements are denoted s_λ and are called *Schur functions*, and whose definition is considerably more subtle. In fact, there are many different (equivalent) ways in which we can define s_λ, viz., in terms of any of the four previous bases, or a "classical" definition involving quotients of determinants, or by abstract properties related to orthogonality and triangularity, or finally by sophisticated algebraic means. All these possible definitions will appear unmotivated to a neophyte. We choose to define s_λ in terms of the m_μ's because this approach is the most combinatorial, though other approaches have their own advantages. In the end, of course, all the approaches produce the same theory.

Much of the importance of Schur functions arises from their connections with such branches of mathematics as representation theory and algebraic geometry. We will discuss the connection with the representation theory of the symmetric group $\mathfrak{S}_n$ in Section 7.18 and with the general linear group $\mathrm{GL}(n, \mathbb{C})$ and related groups in Appendix 2. Another important application of Schur functions not developed here occurs in the Schubert calculus; the cohomology ring of the Grassmann variety $G_k(\mathbb{C}^n)$ can be described in a natural way in terms of Schur functions.

The fundamental combinatorial objects associated with Schur functions are semistandard tableaux. Let λ be a partition. A *semistandard (Young) tableau* (SSYT) of *shape* λ is an array $T = (T_{ij})$ of positive integers of shape λ (i.e., $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$) that is weakly increasing in every row and strictly increasing in every column. The *size* of an SSYT is its number of entries. An example of an SSYT of shape (6, 5, 3, 3) is given by

$$\begin{array}{llllll} 1&1&1&3&4&4 \\ 2&4&4&5&5 \\ 5&5&7 \\ 6&9&9. \end{array}$$

If T is an SSYT of shape λ then we write $\lambda = \mathrm{sh}(T)$. Hence the size of T is just $|\mathrm{sh}(T)|$. We may also think of an SSYT of shape λ as the Young diagram (as defined in Section 1.3) of λ whose boxes have been filled with positive integers (satisfying certain conditions). For instance, the above SSYT may be written

1	1	1	3	4	4
2	4	4	5	5	
5	5	7			
6	9	9			

We say that T has *type* $\alpha = (\alpha_1, \alpha_2, \ldots)$, denoted $\alpha = \mathrm{type}(T)$, if T has $\alpha_i = \alpha_i(T)$ parts equal to i. Thus the above SSYT has type (3, 1, 1, 4, 4, 1, 1, 0, 2). For any SSYT T of type α (or indeed for any multiset on $\mathbb{P}$ with possible additional structure), write

$$x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots.$$

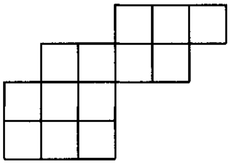
For our running example we have

$$x^T = x_1^3 x_2 x_3 x_4^4 x_5^4 x_6 x_7 x_9^2.$$

There is a generalization of SSYTs of shape λ that fits naturally into the theory of symmetric functions. If λ and μ are partitions with $\mu \subseteq \lambda$ (i.e., $\mu_i \leq \lambda_i$ for all i), then define a *semistandard tableau* of *(skew) shape* λ/μ to be an array $T = (T_{ij})$ of positive integers of shape λ/μ (i.e., $1 \leq i \leq \ell(\lambda)$, $\mu_i < j \leq \lambda_i$) that is weakly increasing in every row and strictly increasing in every column. An example of an SSYT of shape (6, 5, 3, 3)/(3, 1) is given by

$$\begin{array}{llllll} &&&3&4&4 \\ &1&4&7&7 \\ 2&2&6 \\ 3&8&8. \end{array}$$

We can similarly extend the definition of a Young diagram of shape λ to one of shape λ/μ. Thus the Young diagram of shape $(6, 5, 3, 3)/(3, 1)$ is given by



Thus an SSYT of shape λ/μ may be regarded as a Young diagram of shape λ/μ whose boxes have been filled with positive integers (satisfying certain conditions), just as for "ordinary shapes" λ. For instance, the above SSYT of shape $(6, 5, 3, 3)/(3, 1)$ may be written

			3	4	4
	1	4	7	7	
2	2	6			
3	8	8			

The definitions of type(T) and x^T carry over directly from SSYTs T of ordinary shape to those of skew shape.

We now come to the key definition of this entire chapter. As mentioned previously, this definition will appear entirely unmotivated until we proceed further.

7.10.1 Definition. Let λ/μ be a skew shape. The *skew Schur function* $s_{\lambda/\mu} = s_{\lambda/\mu}(x)$ of *shape* λ/μ in the variables $x = (x_1, x_2, \ldots)$ is the formal power series

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all SSYTs T of shape λ/μ. If $\mu = \emptyset$ so $\lambda/\mu = \lambda$, then we call $s_\lambda(x)$ the *Schur function* of shape λ.

For instance, the SSYTs of shape $(2, 1)$ with largest part at most three are given by

11	12	11	13	22	23	12	13
2	2	3	3	3	3	3	2

Hence

$$\begin{aligned} s_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3 \\ &= m_{21}(x_1, x_2, x_3) + 2m_{111}(x_1, x_2, x_3). \end{aligned}$$

Thus, since at most three distinct variables can occur in a term of s_{21}, we have $s_{21} = m_{21} + 2m_{111}$ (as elements of Λ, i.e., as symmetric functions in *infinitely* many variables). It is by no means obvious that $s_{\lambda/\mu}$ is in fact always a symmetric function.

7.10.2 Theorem. *For any skew shape λ/μ, the skew Schur function $s_{\lambda/\mu}$ is a symmetric function.*

Proof. It suffices to show [why?] that $s_{\lambda/\mu}$ is invariant under interchanging x_i and x_{i+1}. Suppose that $|\lambda/\mu| = n$ and that $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a weak composition of n. Let

$$\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots).$$

If $\mathcal{T}_{\lambda/\mu,\alpha}$ denotes the set of all SSYTs of shape λ/μ and type α, then we seek a bijection $\varphi : \mathcal{T}_{\lambda/\mu,\alpha} \to \mathcal{T}_{\lambda/\mu,\tilde{\alpha}}$.

Let $T \in \mathcal{T}_{\lambda/\mu,\alpha}$. Consider the parts of T equal to i or $i+1$. Some columns of T will contain no such parts, while some others will contain two such parts, viz., one i and one $i+1$. These columns we ignore. The remaining parts equal to i or $i+1$ occur once in each column, and consist of rows with a certain number r of i's followed by a certain number s of $i+1$'s. (Of course r and s depend on the row in question.) For example, a portion of T could look as follows:

$$\begin{array}{ccccc}
 & & & & i \\
i & i & \underbrace{i \quad i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1 \quad i+1}_{s=4} & i+1 \\
i+1 & i+1 & & &
\end{array}$$

In each such row convert the r i's and s $i+1$'s to s i's and r $i+1$'s:

$$\begin{array}{ccccc}
 & & & & i \\
i & i & \underbrace{i \quad i \quad i \quad i}_{s=4} & \underbrace{i+1 \quad i+1}_{r=2} & i+1 \\
i+1 & i+1 & & &
\end{array}$$

It's easy to see that the resulting array $\varphi(T)$ belongs to $\mathcal{T}_{\lambda/\mu,\tilde{\alpha}}$, and that φ establishes the desired bijection. $\square$

If $\lambda \vdash n$ and α is a weak composition of n, then let $K_{\lambda\alpha}$ denote the number of SSYTs of shape λ and type α. $K_{\lambda\alpha}$ is called a *Kostka number* and plays a prominent role in the theory of symmetric functions. By Definition 7.10.1 we have

$$s_\lambda = \sum_\alpha K_{\lambda\alpha} x^\alpha,$$

summed over all weak compositions α of n, so by Theorem 7.10.2 we have

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu. \tag{7.35}$$

More generally, we can define the *skew Kostka number* $K_{\lambda/\nu,\alpha}$ as the number of SSYTs of shape λ/ν and type α, so that if $|\lambda/\nu| = n$ then

$$s_{\lambda/\nu} = \sum_{\mu \vdash n} K_{\lambda/\nu,\mu} m_\mu. \tag{7.36}$$

No simple formula is known in general for $K_{\lambda/\nu,\mu}$, or even $K_{\lambda\mu}$, and it is unlikely that such a formula exists. For certain λ, ν, and μ a formula can be given, the most important being the case $\nu = \emptyset$ and $\mu = \langle 1^n \rangle$. While we will give this formula later (Corollary 7.21.6), let us here consider more closely the combinatorial significance of the number $K_{\lambda,1^n}$, also denoted f^λ. By definition, f^λ is the number of ways to insert the numbers $1, 2, \ldots, n$ into the shape $\lambda \vdash n$, each number appearing once, so that every row and column is increasing. Such an array is called a *standard Young tableau* (SYT) (or just *standard tableau*) of shape λ. For instance, the SYTs of shape $(3, 2)$ are

$$\begin{matrix}1\,2\,3 & 1\,2\,4 & 1\,2\,5 & 1\,3\,4 & 1\,3\,5\\ 4\,5 & 3\,5 & 3\,4 & 2\,5 & 2\,4\end{matrix},$$

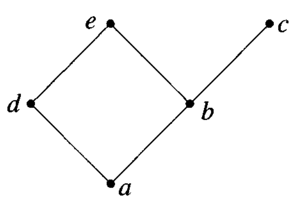
so $f^{(3,2)} = 5$. The number f^λ has several alternative combinatorial interpretations, as given by the following proposition.

7.10.3 Proposition. *Let $\lambda \in$ Par. Then the number f^λ counts the objects in items* (a)–(e) *below. We illustrate these objects with the case $\lambda = (3, 2)$.*

(a) Chains of partitions. *Saturated chains in the interval $[\emptyset, \lambda]$ of Young's lattice Y, or equivalently, sequences $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^n = \lambda$ of partitions (which we identify with their diagrams) such that λ^i is obtained from λ^{i-1} by adding a single square.*

$$\begin{aligned}
&\emptyset \subset 1 \subset 2 \ \subset 3 \ \subset 31 \subset 32\\
&\emptyset \subset 1 \subset 2 \ \subset 21 \subset 31 \subset 32\\
&\emptyset \subset 1 \subset 2 \ \subset 21 \subset 22 \subset 32\\
&\emptyset \subset 1 \subset 11 \subset 21 \subset 31 \subset 32\\
&\emptyset \subset 1 \subset 11 \subset 21 \subset 22 \subset 32
\end{aligned}$$

(b) Linear extensions. *Let P_λ be the poset whose elements are the squares of the diagram of λ, with t covering s if t lies directly to the right or directly below s (with no squares in between). Such posets are just the finite order ideals of $\mathbb{N} \times \mathbb{N}$. Then $f^\lambda = e(P_\lambda)$, the number of linear extensions of P_λ.*



abcde
abdce
abdec
adbce
adbec

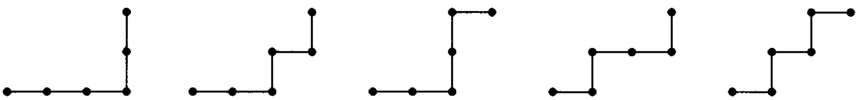
(c) Ballot sequences. *Ways in which n voters can vote sequentially in an election for candidates $A_1, A_2, \ldots$, so that for all i, A_i receives λ_i votes, and so that A_i never trails A_{i+1} in the voting. (We denote such a voting sequence as $a_1a_2\cdots a_n$, where the k-th voter votes for A_{a_k}.)*

11122 11212 11221 12112 12121

(d) Lattice permutations. *Sequences $a_1a_2\cdots a_n$ in which i occurs λ_i times, and such that in any left factor $a_1a_2\cdots a_j$, the number of i's is at least as great as the number of $i+1$'s (for all i). Such a sequence is called a* lattice permutation *(or* Yamanouchi word *or* ballot sequence*) of type λ.*

$$11122 \quad 11212 \quad 11221 \quad 12112 \quad 12121$$

(e) Lattice paths. *Lattice paths $0 = v_0, v_1, \ldots, v_n$ in $\mathbb{R}^\ell$ (where $\ell = \ell(\lambda)$) from the origin v_0 to $v_n = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, with each step a unit coordinate vector, and staying within the region (or cone) $x_1 \geq x_2 \geq \cdots \geq x_\ell \geq 0$.*



Proof. (a) Insert i into the square that was added to λ^{i-1} in order to obtain λ^i, to get an SYT of shape λ.
(b) The interval $[\emptyset, \lambda]$ in Y is just $J(P_\lambda)$, the lattice of order ideals of P_λ, so the equivalence between our interpretations (a) and (b) of f^λ is just a special case of the discussion following Proposition 3.5.2.
(c) If the k-th voter votes for A_i, then put k in the i-th row of the shape λ.
(d) Clearly the voting sequences in (c) are identical to the lattice permutations of (d).
(e) If $a_1a_2\cdots a_n$ is a lattice permutation as in (d), then let $v_i - v_{i-1}$ be the a_i-th unit coordinate vector (i.e., the vector with a one in position a_i and zeros elsewhere) to obtain a lattice path. Alternatively, the equivalence between (b) and (e) is a special case of the discussion preceding Example 3.5.3. □

All five of the above interpretations can be straightforwardly generalized to the skew case $f^{\lambda/\mu}$. We leave the details of this task to the interested reader.

There is a combinatorial object equivalent to an SSYT that is worth mentioning. A *Gelfand–Tsetlin pattern* (sometimes called just a *Gelfand pattern*), or *complete branching*, is a triangular array G of nonnegative integers, say

$$\begin{array}{ccccccccc} a_{11} & & a_{12} & & a_{13} & & \cdots & & a_{1n} \\ & a_{22} & & a_{23} & & \cdots & & a_{2n} & \\ & & a_{33} & & \cdots & & a_{3n} & & \\ & & & \ddots & & \iddots & & & \\ & & & & a_{nn} & & & & \end{array} \tag{7.37}$$

such that $a_{ij} \leq a_{i+1,j+1} \leq a_{i,j+1}$ when all three numbers are defined. In other words, the rows of G are weakly increasing, and $a_{i+1,j+1}$ lies weakly between its two neighbors above it. An example of a Gelfand–Tsetlin pattern is

$$\begin{array}{ccccccccc}
0 & & 2 & & 2 & & 3 & & 6 \\
& 0 & & 2 & & 2 & & 4 & \\
& & 1 & & 2 & & 4 & & \\
& & & 1 & & 3 & & & \\
& & & & 3. & & & &
\end{array}$$

Given the Gelfand–Tsetlin pattern G of equation (7.37), let λ^i be the i-th row of G in reverse order. Define a tableau $T = T(G)$ by inserting $n - i + 1$ into the squares of the skew shape λ^i/λ^{i+1}. For the example above, $T(G)$ is given by

$$\begin{array}{cccccc}
1 & 1 & 1 & 3 & 5 & 5 \\
2 & 3 & 5 & & & \\
3 & 4 & & & & \\
5 & 5. & & & &
\end{array}$$

We obtain an SSYT of shape λ^1 (the first row of G in reverse order) and largest part at most n. This correspondence between Gelfand–Tsetlin patterns with fixed first row α of length n and SSYT of shape α^r (the elements of α in reverse order) and largest part at most n is easily seen to be a bijection.

It is sometimes more convenient in dealing with Schur functions, Kostka numbers, etc., to work with arrays that are *decreasing* in rows and columns rather than with SSYT. Define a *reverse SSYT* or *column-strict plane partition* (sometimes abbreviated as *costripp*) of (skew) shape λ/μ to be an array of positive integers of shape λ/μ that is weakly decreasing in rows and strictly decreasing in columns. Define the *type* α of a reverse SSYT exactly as for ordinary SSYT. For instance, the array

$$\begin{array}{cccccc}
 & & & 6 & 5 & 5 \\
 & 8 & 5 & 2 & 2 & \\
7 & 7 & 3 & & & \\
6 & 1 & 1 & & &
\end{array}$$

is a reverse SSYT of shape $(6, 5, 3, 3)/(3, 1)$ and type $(2, 2, 1, 0, 3, 2, 2, 1)$.

Define $\hat{K}_{\lambda/\mu,\alpha}$ to be the number of reverse SSYTs of shape λ/μ and type α. The next proposition shows that for many purposes there is no significant difference between ordinary and reverse SSYTs.

7.10.4 Proposition. *Let λ/μ be a skew partition of n, and let α be a weak composition of n. Then $\hat{K}_{\lambda/\mu,\alpha} = K_{\lambda/\mu,\alpha}$.*

Proof. Suppose that T is a reverse SSYT of shape λ and type $\alpha = (\alpha_1, \alpha_2, \ldots)$. Let k denote the largest part of T. The transformation $T_{ij} \mapsto k + 1 - T_{ij}$ shows that $\hat{K}_{\lambda\alpha} = K_{\lambda\bar{\alpha}}$, where $\bar{\alpha} = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1, 0, 0, \ldots)$. But by Theorem 7.10.2 we have $K_{\lambda\bar{\alpha}} = K_{\lambda\alpha}$, and the proof is complete. □

We now wish to show that the Schur functions s_λ form a $\mathbb{Q}$-basis for Λ. The following proposition implies an even stronger result.

7.10.5 Proposition. *Suppose that μ and λ are partitions with $|\mu| = |\lambda|$ and $K_{\lambda\mu} \neq 0$. Then $\mu \leq \lambda$ (dominance order). Moreover, $K_{\lambda\lambda} = 1$.*

Proof. Suppose that $K_{\lambda\mu} \neq 0$. By definition, there exists an SSYT T of shape λ and type μ. Suppose that a part $T_{ij} = k$ appears below the k-th row (i.e., $i > k$). Then we have $1 \leq T_{1k} < T_{2k} < \cdots < T_{ik} = k$ for $i > k$, which is impossible. Hence the parts $1, 2, \ldots, k$ all appear in the first k rows, so $\mu_1 + \mu_2 + \cdots + \mu_k \leq \lambda_1 + \lambda_2 + \cdots + \lambda_k$, as desired. Moreover, if $\mu = \lambda$ then we must have $T_{ij} = i$ for all (i, j), so $K_{\lambda\lambda} = 1$. □

7.10.6 Corollary. *The Schur functions s_λ with $\lambda \in \mathrm{Par}(n)$ form a basis for Λ^n, so $\{s_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ. In fact, the transition matrix $(K_{\lambda\mu})$ which expresses the s_λ's in terms of the m_μ's, with respect to any linear ordering of $\mathrm{Par}(n)$ that extends dominance order, is lower triangular with 1's on the main diagonal.*

$$
\begin{aligned}
s_1 &= m_1 \\[1ex]
s_{11} &= m_{11} \\
s_2 &= m_{11} + m_2 \\[1ex]
s_{111} &= m_{111} \\
s_{21} &= 2m_{111} + m_{21} \\
s_3 &= m_{111} + m_{21} + m_3 \\[1ex]
s_{1111} &= m_{1111} \\
s_{211} &= 3m_{1111} + m_{211} \\
s_{22} &= 2m_{1111} + m_{211} + m_{22} \\
s_{31} &= 3m_{1111} + 2m_{211} + m_{22} + m_{31} \\
s_4 &= m_{1111} + m_{211} + m_{22} + m_{31} + m_4 \\[1ex]
s_{11111} &= m_{11111} \\
s_{2111} &= 4m_{11111} + m_{2111} \\
s_{221} &= 5m_{11111} + 2m_{2111} + m_{221} \\
s_{311} &= 6m_{11111} + 3m_{2111} + m_{221} + m_{311} \\
s_{32} &= 5m_{11111} + 3m_{2111} + 2m_{221} + m_{311} + m_{32} \\
s_{41} &= 4m_{11111} + 3m_{2111} + 2m_{221} + 2m_{311} + m_{32} + m_{41} \\
s_5 &= m_{11111} + m_{2111} + m_{221} + m_{311} + m_{32} + m_{41} + m_5
\end{aligned}
$$

Figure 7-4. The Kostka numbers $K_{\lambda\mu}$.

Proof. Proposition 7.10.5 is equivalent to the assertion about $(K_{\lambda\mu})$. Since a lower triangular matrix with 1's on the main diagonal is invertible, it follows that $\{s_\lambda : \lambda \in \mathrm{Par}(n)\}$ is a $\mathbb{Q}$-basis for Λ^n. □

Note that in fact $\{s_\lambda : \lambda \in \mathrm{Par}(n)\}$ is a $\mathbb{Z}$-basis for $\Lambda^n_{\mathbb{Z}}$, since each $K_{\lambda\lambda} = 1$, rather than just $K_{\lambda\lambda} \neq 0$.

In subsequent sections we will work out the basic theory of Schur functions, that is, the transition matrices between the s_λ's and the bases $m_\lambda, h_\lambda, e_\lambda, p_\lambda$, as well as connections with the scalar product $\langle\ ,\ \rangle$ and the automorphism ω. (We have already considered, essentially by definition, the transition matrix $(K_{\lambda\mu})$ from the m_λ's to the s_λ's, but we don't know yet what the inverse matrix looks like.) We will also give several enumerative applications of the theory of symmetric functions: the enumeration of plane partitions, some results on permutation statistics, and Pólya's theory of enumeration under group action.

Figure 7-4 gives a short table of the Kostka numbers $K_{\lambda\mu}$.

7.11 The RSK Algorithm

There is a remarkable combinatorial correspondence associated with the theory of symmetric functions, called the *RSK algorithm.* (For the meaning of the initials RSK, as well as for other names of the algorithm, see the Notes at the end of this chapter.) We will develop here only the most essential properties of the RSK algorithm, thereby allowing us to give combinatorial proofs of some fundamental properties of Schur functions. It is also possible to give purely algebraic proofs of these results, but of course in a text on enumerative combinatorics we prefer combinatorial proofs.

The basic operation of the RSK algorithm consists of the *row insertion* $P \leftarrow k$ of a positive integer k into a nonskew SSYT $P = (P_{ij})$. The operation $P \leftarrow k$ is defined as follows: Let r be the largest integer such that $P_{1,r-1} \leq k$. (If $P_{11} > k$ then let $r = 1$.) If P_{1r} doesn't exist (i.e., P has $r - 1$ columns), then simply place k at the end of the first row. The insertion process stops, and the resulting SSYT is $P \leftarrow k$. If, on the other hand, P has at least r columns, so that P_{1r} exists, then replace P_{1r} by k. The element then "bumps" $P_{1r} := k'$ into the second row, i.e., insert k' into the second row of P by the insertion rule just described. Continue until an element is inserted at the end of a row (possibly as the first element of a new row). The resulting array is $P \leftarrow k$.

7.11.1 Example. Let

$$
P = \begin{array}{lllllll}
1 & 1 & 2 & 4 & 5 & 5 & 6 \\
2 & 3 & 3 & 6 & 6 & 8 & \\
4 & 4 & 6 & 8 & & & \\
6 & 7 & & & & & \\
8 & 9. & & & & &
\end{array}
$$

Then $P \leftarrow 4$ is shown below, with the elements inserted into each row (either by bumping or by the final insertion in the fourth row) in boldface. Thus the 4 bumps the 5, the 5 bumps the 6, the 6 bumps the 8, and the 8 is inserted at the end of a row. The set of positions of these boldface elements is called the *insertion path* $I(P \leftarrow 4)$ of 4 (the number being inserted into P). Thus for this example we have $I(P \leftarrow 4) = \{(1,5),(2,4),(3,4),(4,3)\}$:

$$\begin{array}{l} 1\,1\,2\,4\,\mathbf{4}\,5\,6 \\ 2\,3\,3\,\mathbf{5}\,6\,8 \\ 4\,4\,6\,\mathbf{6} \\ 6\,7\,\mathbf{8} \\ 8\,9. \end{array}$$

There are two technical properties of insertion paths that are of great use in proving properties of the RSK algorithm.

7.11.2 Lemma. (a) *When we insert k into an SSYT P, then the insertion path moves to the left. More precisely, if $(r,s),(r+1,t) \in I(P \leftarrow k)$ then $t \le s$.*
(b) *Let P be an SSYT, and let $j \le k$. Then $I(P \leftarrow j)$ lies strictly to the left of $I((P \leftarrow j) \leftarrow k)$. More precisely, if $(r,s) \in I(P \leftarrow j)$ and $(r,t) \in I((P \leftarrow j) \leftarrow k)$, then $s < t$. Moreover, $I((P \leftarrow j) \leftarrow k)$ does not extend below the bottom of $I(P \leftarrow j)$. Equivalently,*

$$\#I((P \leftarrow j) \leftarrow k) \le \#I(P \leftarrow j).$$

Proof. (a) Suppose that $(r,s) \in I(P \leftarrow k)$. Now either $P_{r+1,s} > P_{rs}$ (since P is strictly increasing in columns) or else there is no $(r+1,s)$ entry of P. In the first case, P_{rs} cannot get bumped to the right of column s without violating the fact that the rows of $P \leftarrow k$ are weakly increasing, since P_{rs} would be to the right of $P_{r+1,s}$ on the same row. The second case is clearly impossible, since we would otherwise have a gap in row $r+1$. Hence (a) is proved.
(b) Since a number can only bump a strictly larger number, it follows that k is inserted in the first row of $P \leftarrow j$ strictly to the right of j. Since the first row of P is weakly increasing, j bumps an element no larger than the element k bumps. Hence by induction $I(P \leftarrow j)$ lies strictly to the left of $I((P \leftarrow j) \leftarrow k)$. The bottom element b of $I(P \leftarrow j)$ was inserted at the end of its row. By what was just proved, if $I((P \leftarrow j) \leftarrow k)$ has an element c in this row, then it lies to the right of b. Hence c was inserted at the end of the row, so the insertion procedure terminates. It follows that $I((P \leftarrow j) \leftarrow k)$ can never go below the bottom of $I(P \leftarrow j)$. □

7.11.3 Corollary. *If P is an SSYT and $k \ge 1$, then $P \leftarrow k$ is also an SSYT.*

Proof. It is clear that the rows of $P \leftarrow k$ are weakly increasing. Now a number a can only bump a larger number b. By Lemma 7.11.2(a), b does not move to the right when it is bumped. Hence b is inserted below a number that is strictly smaller than b, so $P \leftarrow k$ remains an SSYT. □

Now let $A = (a_{ij})_{i,j\geq 1}$ be an $\mathbb{N}$-matrix with finitely many nonzero entries. We will say that A is an $\mathbb{N}$-matrix of *finite support*. We can think of A as either an infinite matrix or as an $m \times n$ matrix when $a_{ij} = 0$ for $i > m$ and $j > n$. Associate with A a *generalized permutation* or *two-line array* w_A defined by

$$w_A = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_m \\ j_1 & j_2 & j_3 & \cdots & j_m \end{pmatrix}, \tag{7.38}$$

where (a) $i_1 \leq i_2 \leq \cdots \leq i_m$, (b) if $i_r = i_s$ and $r \leq s$, then $j_r \leq j_s$, and (c) for each pair (i, j), there are exactly a_{ij} values of r for which $(i_r, j_r) = (i, j)$. It is easily seen that A determines a unique two-line array w_A satisfying (a)–(c), and conversely any such array corresponds to a unique A. For instance, if

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \tag{7.39}$$

then the corresponding two-line array is

$$w_A = \begin{pmatrix} 1\,1\,1\,2\,2\,3\,3 \\ 1\,3\,3\,2\,2\,1\,2 \end{pmatrix}. \tag{7.40}$$

We now associate with A (or w_A) a pair (P, Q) of SSYTs of the same shape, as follows. Let w_A be given by (7.38). Begin with $(P(0), Q(0)) = (\emptyset, \emptyset)$ (where $\emptyset$ denotes the empty SSYT). If $t < m$ and $(P(t), Q(t))$ are defined, then let

(a) $P(t+1) = P(t) \leftarrow j_{t+1}$;
(b) $Q(t+1)$ be obtained from $Q(t)$ by inserting i_{t+1} (leaving all parts of $Q(t)$ unchanged) so that $P(t+1)$ and $Q(t+1)$ have the same shape.

The process ends at $(P(m), Q(m))$, and we define $(P, Q) = (P(m), Q(m))$. We denote this correspondence by $A \xrightarrow{\text{RSK}} (P, Q)$ and call it the *RSK algorithm*. We call P the *insertion tableau* and Q the *recording tableau* of A or of w_A.

7.11.4 Example. Let A and w_A be given by (7.39) and (7.40). The SSYTs $(P(1), Q(1)), \ldots, (P(7), Q(7)) = (P, Q)$ are as follows:

$P(i)$	$Q(i)$
1	1
1 3	1 1
1 3 3	1 1 1
1 2 3 3	1 1 1 2
1 2 2 3 3	1 1 1 2 2
1 1 2 2 3 3	1 1 1 2 2 3
1 1 2 2 2 3 3	1 1 1 3 2 2 3.

The main result on the RSK algorithm is the following.

7.11.5 Theorem. *The RSK algorithm is a bijection between* $\mathbb{N}$*-matrices* $A = (a_{ij})_{i,j\geq 1}$ *of finite support and ordered pairs* (P, Q) *of SSYT of the same shape. In this correspondence,*

$$j \text{ occurs in } P \text{ exactly } \sum_i a_{ij} \text{ times} \tag{7.41}$$

$$i \text{ occurs in } Q \text{ exactly } \sum_j a_{ij} \text{ times.} \tag{7.42}$$

(These last two conditions are equivalent to $\text{type}(P) = \text{col}(A), \text{type}(Q) = \text{row}(A)$.*)*

Proof. By Corollary 7.11.3, P is an SSYT. Clearly, by definition of the RSK algorithm P and Q have the same shape, and also (7.41) and (7.42) hold. Thus we must show the following: (a) Q is an SSYT, and (b) the RSK algorithm is a bijection, i.e., given (P, Q), one can uniquely recover A.

To prove (a), first note that since the elements of Q are inserted in weakly increasing order, it follows that the rows and columns of Q are weakly increasing. Thus we must show that the columns of Q are strictly increasing, i.e., no two equal elements of the top row of w_A can end up in the same column of Q. But if $i_k = i_{k+1}$ in the top row, then we must have $j_k \leq j_{k+1}$. Hence by Lemma 7.11.2(b), the insertion path of j_{k+1} will always lie strictly to the right of the path for j_k, and will never extend below the bottom of j_k's insertion path. It follows that the bottom elements of the two insertion paths lie in different columns, so the columns of Q are strictly increasing as desired.

The above argument establishes an important property of the RSK algorithm: *Equal elements of Q are inserted strictly left to right.*

It remains to show that the RSK algorithm is a bijection. Thus given $(P, Q) = (P(m), Q(m))$, let Q_{rs} be the rightmost occurrence of the largest entry of Q (where Q_{rs} is the element of Q in row r and column s). Since equal elements of Q are inserted left to right, it follows that $Q_{rs} = i_m$, $Q(m-1) = Q(m)\backslash Q_{rs}$ (i.e., $Q(m)$ with the element Q_{rs} deleted), and that P_{rs} was the last element of P to be bumped into place after inserting j_m into $P(m-1)$. But it is then easy to reverse the insertion procedure $P(m-1) \leftarrow j_m$. P_{rs} must have been bumped by the rightmost element $P_{r-1,t}$ of row $r-1$ of P that is smaller than P_{rs}. Hence remove P_{rs} from P, replace $P_{r-1,t}$ with P_{rs}, and continue by replacing the rightmost element of row $r-2$ of P that is smaller than $P_{r-1,t}$ with $P_{r-1,t}$, etc. Eventually some element j_m is removed from the first row of P. We have thus uniquely recovered (i_m, j_m) and $(P(m-1), Q(m-1))$. By iterating this procedure we recover the entire two-line array w_A. Hence the RSK algorithm is injective.

To show surjectivity, we need to show that applying the procedure of the previous paragraph to an arbitrary pair (P, Q) of SSYTs of the same shape always yields a valid two-line array

$$w_A = \begin{pmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix}.$$

Clearly $i_1 \leq i_2 \leq \cdots \leq i_m$, so we need to show that if $i_k = i_{k+1}$ then $j_k \leq j_{k+1}$. Let $i_k = Q_{rs}$ and $i_{k+1} = Q_{uv}$, so $r \geq u$ and $s < v$. When we begin to apply inverse bumping to P_{uv}, it occupies the end of its row (row u). Hence when we apply inverse bumping to P_{rs}, its "inverse insertion path" intersects row u strictly to the left of column v. Thus at row u the inverse insertion path of P_{rs} lies strictly to the left of that of P_{uv}. By a simple induction argument (essentially the "inverse" of Lemma 7.11.2(b)), the entire inverse insertion path of P_{rs} lies strictly to the left of that of P_{uv}. In particular, before removing i_{k+1} the two elements j_k and j_{k+1} appear in the first row with j_k to the left of j_{k+1}. Hence $j_k \leq j_{k+1}$ as desired, completing the proof. □

In Section 7.13 we will give an alternative "geometric" description of the RSK algorithm useful in proving some remarkable properties. This geometric description is only defined when the matrix A is a *permutation matrix*, i.e., an $n \times n$ $(0, 1)$-matrix with exactly one 1 in every row and column. In this case the top line of the two-line array is just $1\,2 \cdots n$, while the bottom line is a permutation w of $1, 2, \ldots, n$ that we can identify with A. When the RSK algorithm is applied to a permutation matrix A (or permutation $w \in \mathfrak{S}_n$), the resulting tableaux P, Q are just standard Young tableaux (of the same shape). Conversely, if P and Q are SYTs of the same shape, then the matrix A satisfying $A \xrightarrow{\text{RSK}} (P, Q)$ is a permutation matrix. Hence the RSK algorithm sets up a bijection between the symmetric group $\mathfrak{S}_n$ and pairs (P, Q) of SYTs of the same shape $\lambda \vdash n$. In particular, if f^λ denotes the number of SYTs of shape λ, then we have the fundamental identity

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!. \tag{7.43}$$

Although permutation matrices are very special cases of $\mathbb{N}$-matrices of finite support, in fact the RSK algorithm for arbitrary $\mathbb{N}$-matrices A can be reduced to the case of permutation matrices. Namely, given the two-line array w_A, say of length n, replace the first row by $1, 2, \ldots, n$. Suppose that the second row of w_A has c_i i's. Then replace the 1's in the second row from left-to-right with $1, 2, \ldots, c_1$, next the 2's from left-to-right with $c_1 + 1, c_1 + 2, \ldots, c_1 + c_2$, etc., until the second row becomes a permutation of $1, 2, \ldots, n$. Denote the resulting two-line array by $\tilde{w}_A$. For instance, if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix},$$

then

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 3 & 2 & 3 & 1 & 2 & 2 & 2 \end{pmatrix},$$

and w_A is replaced by

$$\tilde{w}_A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 8 & 4 & 9 & 3 & 5 & 6 & 7 \end{pmatrix}.$$

7.11.6 Lemma. *Let*

$$w_A = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

be a two-line array, and let

$$\tilde{w}_A = \begin{pmatrix} 1 & 2 & \cdots & n \\ \tilde{j}_1 & \tilde{j}_2 & \cdots & \tilde{j}_n \end{pmatrix}.$$

Suppose that $\tilde{w}_A \xrightarrow{\text{RSK}} (\tilde{P}, \tilde{Q})$. Let (P, Q) be the tableaux obtained from $\tilde{P}$ and $\tilde{Q}$ by replacing k in $\tilde{Q}$ by i_k, and $\tilde{j}_k$ in $\tilde{P}$ by j_k. Then $w_A \xrightarrow{\text{RSK}} (P, Q)$. In other words, the operation $w_A \mapsto \tilde{w}_A$ "commutes" with the RSK algorithm.

Proof. Suppose that when the number j is inserted into a row at some stage of the RSK algorithm, it occupies the k-th position in the row. If this number j were replaced by a larger number $j + \epsilon$, smaller than any element of the row which is greater than j, then $j + \epsilon$ would also be inserted at the k-th position. From this we see that the insertion procedure for elements $j_1 j_2 \cdots j_n$ exactly mimics that for $\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_n$, and the proof follows. □

The process of replacing w_A with $\tilde{w}_A$, P with $\tilde{P}$, etc., is called *standardization*. Compare the second proof of Proposition 1.3.17.

7.12 Some Consequences of the RSK Algorithm

The most important result concerning symmetric functions that follows directly from the RSK algorithm is the following, known as the *Cauchy identity*.

7.12.1 Theorem. *We have*

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y). \tag{7.44}$$

Proof. Write

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \prod_{i,j}\left[\sum_{a_{ij}\geq 0}(x_i y_j)^{a_{ij}}\right]. \tag{7.45}$$

A term $x^\alpha y^\beta$ in this expansion is obtained by choosing an $\mathbb{N}$-matrix $A^t = (a_{ij})^t$ (the transpose of A) of finite support with $\text{row}(A) = \alpha$ and $\text{col}(A) = \beta$. Hence the coefficient of $x^\alpha y^\beta$ in (7.45) is the number $N_{\alpha\beta}$ of $\mathbb{N}$-matrices A with $\text{row}(A) = \alpha$ and $\text{col}(A) = \beta$. (This statement is also equivalent to (7.9).) On the other hand, the coefficient of $x^\alpha y^\beta$ in $\sum_\lambda s_\lambda(x) s_\lambda(y)$ is the number of pairs (P, Q) of SSYT of the same shape λ such that $\text{type}(P) = \alpha$ and $\text{type}(Q) = \beta$. The RSK algorithm sets up a bijection between the matrices A and the tableau pairs (P, Q), so the proof follows. □

The Cauchy identity (7.44) has a number of immediate corollaries.

7.12.2 Corollary. *The Schur functions form an orthonormal basis for Λ, i.e., $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$.*

Proof. Combine Corollary 7.10.6 and Lemma 7.9.2. □

Linear-algebraic note. We say that the Schur functions form an *integral* orthonormal basis of Λ, since by Proposition 7.10.5 they actually generate $\Lambda_{\mathbb{Z}}$ as an abelian group. In general it is a subtle question whether a vector space with a distinguished basis (in our case the monomial symmetric functions) and a positive definite symmetric scalar product has an integral orthonormal basis. For our situation such a basis is equivalent to the existence of an *integral* matrix A such that $A^t A = N$, where N is the transition matrix from m_λ to h_λ. We then say N is *integrally equivalent to the identity*. The next result (which is nothing more than standard linear algebra) identifies A as the Kostka matrix K. Note that in general if an integral orthonormal basis exists, then it is unique up to sign and order. This is because the transition matrix between two such bases must be both integral and orthogonal. It is easy to see that the only integral orthogonal matrices are signed permutation matrices.

7.12.3 Corollary. *Fix partitions $\mu, \nu \vdash n$. Then*

$$\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu} = N_{\mu\nu} = \langle h_\mu, h_\nu \rangle,$$

where $K_{\lambda\mu}$ and $K_{\lambda\nu}$ denote Kostka numbers, and $N_{\mu\nu}$ is the number of $\mathbb{N}$-matrices A with $\mathrm{row}(A) = \mu$ and $\mathrm{col}(A) = \nu$.

Proof. Take the coefficient of $x^\mu y^\nu$ on both sides of (7.44). □

7.12.4 Corollary. *We have*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda. \tag{7.46}$$

In other words, if $M(u, v)$ denotes the transition matrix from the basis $\{v_\lambda\}$ to the basis $\{u_\lambda\}$ of Λ (so that $u_\lambda = \sum_\mu M(u, v)_{\lambda\mu} v_\mu$), then

$$M(h, s) = M(s, m)^t.$$

We give three proofs of this corollary, all essentially equivalent.

First Proof. Let $h_\mu = \sum_\lambda a_{\lambda\mu} s_\lambda$. By Corollary 7.12.2, we have $a_{\lambda\mu} = \langle h_\mu, s_\lambda \rangle$. Since $\langle h_\mu, m_\nu \rangle = \delta_{\mu\nu}$ by the definition (7.30) of the scalar product $\langle\ ,\ \rangle$, we have from (7.35) that $\langle h_\mu, s_\lambda \rangle = K_{\lambda\mu}$. □

Second Proof. Fix μ. Then

$$\begin{aligned} h_\mu &= \sum_A x^{\mathrm{col}(A)} \\ &= \sum_{(P,Q)} x^Q \qquad \text{by the RSK algorithm} \\ &= \sum_\lambda K_{\lambda\mu} \sum_Q x^Q \\ &= \sum_\lambda K_{\lambda\mu} s_\lambda, \end{aligned}$$

where (i) A ranges over all $\mathbb{N}$-matrices with $\mathrm{row}(A) = \mu$, (ii) (P, Q) ranges over all pairs of SSYT of the same shape with $\mathrm{type}(P) = \mu$, and (iii) Q ranges over all SSYT of shape λ. □

Third Proof. Take the coefficient of $m_\mu(x)$ on both sides of the identity

$$\sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

(The two sides are equal by (7.10) and (7.44).) □

The next corollary may be regarded as giving a generating function (with respect to the Schur functions s_λ) for the number f^λ of SYT of shape λ.

7.12.5 Corollary. We have

$$h_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda. \tag{7.47}$$

Proof. Take the coefficient of $x_1 x_2 \cdots x_n$ on both sides of (7.44). To obtain a bijective proof, consider the RSK algorithm $A \xrightarrow{\mathrm{RSK}} (P, Q)$ when $\mathrm{col}(A) = \langle 1^n \rangle$. □

Finally we come to an identity already given in (7.43) but worth repeating here.

7.12.6 Corollary. *We have*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

Proof. Regard (7.47) as being in the variables $x = (x_1, x_2, \ldots)$, and take the coefficient of $x_1 x_2 \cdots x_n$ on both sides. To obtain a bijective proof (as mentioned before equation (7.43)) consider the RSK algorithm applied to $n \times n$ permutation matrices. □

7.13 Symmetry of the RSK Algorithm

The RSK algorithm has a number of remarkable symmetry properties. We will discuss only the most important such property in this section.

7.13.1 Theorem. *Let A be an $\mathbb{N}$-matrix of finite support, and suppose that $A \xrightarrow{\mathrm{RSK}} (P, Q)$. Then $A^t \xrightarrow{\mathrm{RSK}} (Q, P)$, where ^{t} denotes transpose.*

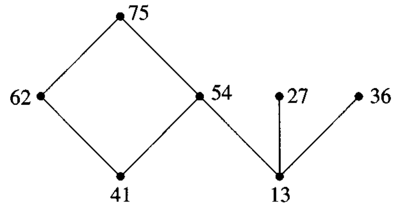
To prepare for the proof of this theorem, let $w_A = \binom{u}{v}$ be the two-line array associated to A. Hence $w_{A^t} = \binom{v}{u}_{\mathrm{sorted}}$, i.e., sort the columns of $\binom{v}{u}$ so that the columns are weakly increasing in lexicographic order. It follows from Lemma 7.11.6 that we may assume u and v have no repeated elements [why?].

Given

$$w_A = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

where the u_i's and the v_j's are distinct, define the *inversion poset* $I = I(A) = I\binom{u}{v}$ as follows. The vertices of I are the columns of $\binom{u}{v}$. For notational convenience, we often denote a column ${}^{a}_{b}$ as ab. Define $ab < cd$ in I if $a < c$ and $b < d$.

7.13.2 Example. Let $\binom{u}{v} = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 3\,7\,6\,1\,4\,2\,5 \end{pmatrix}$. Then I is given by



Note that the number of incomparable pairs in I is just the number of inversions of the permutation v, whence the terminology "inversion poset."

The following lemma is an immediate consequence of the definition of $I(A)$.

7.13.3 Lemma. *The map $\varphi : I(A) \to I(A^t)$ defined by $\varphi(ab) = ba$ is an isomorphism of posets.*

Now given the inversion poset $I = I(A)$, define I_1 to be the set of minimal elements of I, then I_2 to be the set of minimal elements of $I - I_1$, then I_3 to be the set of minimal elements of $I - I_1 - I_2$, etc. For the poset of Example 7.13.2 we have $I_1 = \{13, 41\}$, $I_2 = \{27, 36, 54, 62\}$, $I_3 = \{75\}$. Note that since I_i is an antichain of I, its elements can be labeled

$$(u_{i1}, v_{i1}),\ (u_{i2}, v_{i2}),\ \ldots,\ (u_{in_i},\ v_{in_i}), \tag{7.48}$$

where $n_i = \#I_i$, such that

$$\begin{aligned} u_{i1} &< u_{i2} < \cdots < u_{in_i} \\ v_{i1} &> v_{i2} > \cdots > v_{in_i}. \end{aligned} \tag{7.49}$$

7.13.4 Lemma. *Let $I_1, \ldots, I_d$ be the (nonempty) antichains defined above, labeled as in (7.49). Let $A \xrightarrow{\text{RSK}} (P, Q)$. Then the first row of P is $v_{1n_1} v_{2n_2} \cdots v_{dn_d}$, while the first row of Q is $u_{11} u_{21} \cdots u_{d1}$. Moreover, if $(u_k, v_k) \in I_i$, then v_k is inserted into the i-th column of the first row of the tableau $P(k-1)$ in the RSK algorithm.*

Proof. Induction on n, the case $n = 1$ being trivial. Assume the assertion for $n - 1$, and let

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}, \qquad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_{n-1} \\ v_1 & v_2 & \cdots & v_{n-1} \end{pmatrix}.$$

Let $(P(n-1), Q(n-1))$ be the tableaux obtained after inserting $v_1, \ldots, v_{n-1}$, and let the antichains $I'_i := I_i\binom{\tilde{u}}{\tilde{v}}$, $1 \le i \le e$ (where $e = d-1$ or $e = d$), be given by $(\tilde{u}_{i1}, \tilde{v}_{i1}), \ldots, (\tilde{u}_{im_i}, \tilde{v}_{im_i})$, where $\tilde{u}_{i1} < \cdots < \tilde{u}_{im_i}$ and $\tilde{v}_{i1} > \cdots > \tilde{v}_{im_i}$. By the induction hypothesis, the first row of $P(n-1)$ is $\tilde{v}_{1m_1}\tilde{v}_{2m_2}\cdots\tilde{v}_{em_e}$, while the first row of Q is $\tilde{u}_{11}\tilde{u}_{21}\cdots\tilde{u}_{e1}$. Now we insert v_n into $P(n-1)$. If $\tilde{v}_{im_i} > v_n$, then $I'_i \cup (u_n, v_n)$ is an antichain of $I\binom{u}{v}$. Hence $(u_n, v_n) \in I_i\binom{u}{v}$ if i is the *least* index for which $\tilde{v}_{im_i} > v_n$. If there is no such i, then (u_n, v_n) is the unique element of the antichain $I_d\binom{u}{v}$ of $I\binom{u}{v}$. These conditions mean that v_n is inserted into the i-th column of $P(n-1)$, as claimed. We start a new i-th column exactly when $v_n = v_{d1}$, in which case $u_n = u_{d1}$, so u_n is inserted into the i-th column of the first row of $Q(n-1)$, as desired. □

Proof of Theorem 7.13.1. If the antichain $I_i\binom{u}{v}$ is given by (7.48) such that (7.49) is satisfied, then by Lemma 7.13.3 the antichain $I_i\binom{v}{u}$ is just

$$(v_{im_i}, u_{im_i}), \ldots, (v_{i2}, u_{i2}), (v_{i1}, u_{i1}),$$

where

$$v_{im_i} < \cdots < v_{i2} < v_{i1}$$
$$u_{im_i} > \cdots > u_{i2} > u_{i1}.$$

Hence by Lemma 7.13.4, if $A^t \xrightarrow{\text{RSK}} (P', Q')$, then the first row of P' is $u_{11}u_{21}\cdots u_{d1}$, and the first row of Q' is $v_{im_1}v_{2m_2}\cdots v_{dm_d}$. Thus by Lemma 7.13.4, the first rows of P' and Q' agree with the first rows of Q and P, respectively.

When the RSK algorithm is applied to $\binom{u}{v}$, the element v_{ij}, $1 \le j < m_i$, gets bumped into the second row of P *before* the element v_{rs}, $1 \le s < m_r$, if and only if $u_{i,j+1} < u_{r,s+1}$. Let $\bar{P}$ and $\bar{Q}$ denote P and Q with their first rows removed. It follows that

$$\binom{a}{b} := \begin{pmatrix} u_{12}\cdots u_{1m_1} & u_{22}\cdots u_{2m_2} & \cdots u_{d2}\cdots u_{dm_d} \\ v_{11}\cdots v_{1,m_1-1} & v_{21}\cdots v_{2,m_2-1} & \cdots v_{d1}\cdots v_{d,m_d-1} \end{pmatrix}_{\text{sorted}}$$
$$\xrightarrow{\text{RSK}} (\bar{P}, \bar{Q}).$$

Similarly let $(\bar{P}', \bar{Q}')$ denote P' and Q' with their first rows removed. Applying the same argument to $\binom{v}{u}$ rather than $\binom{u}{v}$ yields

$$\binom{a'}{b'} := \begin{pmatrix} v_{1,m_1-1}\cdots v_{11} & v_{2,m_2-1}\cdots v_{21} & \cdots v_{d,m_d-1}\cdots v_{d1} \\ u_{1m_1}\cdots u_{12} & u_{2m_2}\cdots u_{22} & \cdots u_{dm_d}\cdots u_{d2} \end{pmatrix}_{\text{sorted}}$$
$$\xrightarrow{\text{RSK}} (\bar{P}', \bar{Q}').$$

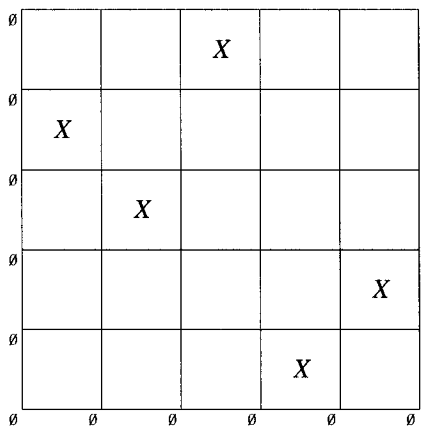
But $\binom{a}{b} = \binom{b'}{a'}_{\text{sorted}}$, so by induction on n (or on the number of rows) we have $(\bar{P}', \bar{Q}') = (\bar{Q}, \bar{P})$, and the proof follows. □

Second Proof (sketch). The above proof was somewhat mysterious and did not really "display" the symmetric nature of the RSK algorithm. We will describe an alternative "geometric" description of the RSK algorithm from which the symmetry property is obvious.

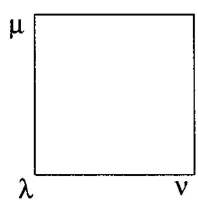
Given $w = w_1 \cdots w_n \in \mathfrak{S}_n$, construct an $n \times n$ square array with an X in the w_i-th square from the bottom of column i. For instance, if $w = 43512$ then we obtain

		X		
X				
	X			
				X
			X	
4	3	5	1	2

This is essentially the usual way of representing a permutation by a permutation matrix, except that we place the $(1, 1)$ entry at the bottom left instead of at the top left. We want to label each of the $(n+1)^2$ points that are corners of squares of our $n \times n$ array with a partition. We will write this partition just below and to the left of its corresponding point. Begin by labeling all points on the bottom row and left column with the empty partition $\emptyset$:



Suppose now that we have labeled all the corners of a square s except the upper right, say as follows:



Then label the upper right corner by the partition ρ defined by the following "local" rules (L1)–(L4):

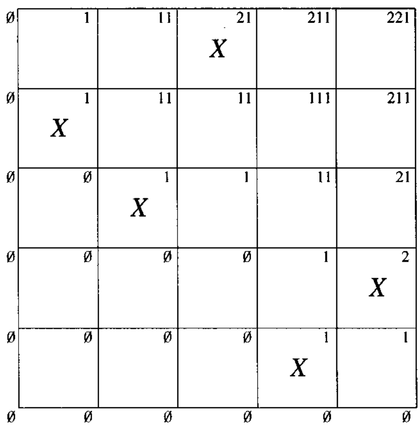
(L1) Suppose that the square s does not contain an X, and that $\lambda = \mu = \nu$. Then define $\rho = \lambda$.

(L2) Suppose that s does not contain an X, and that $\lambda \subset \mu = \nu$. This automatically implies that $|\mu/\lambda| = 1$, so μ is obtained from λ by adding 1 to some part λ_i. Let ρ be obtained from μ by adding 1 to μ_{i+1}.

(L3) Suppose that s does not contain an X and that $\mu \neq \nu$. Define $\rho = \mu \cup \nu$, i.e., $\rho_i = \max(\mu_i, \nu_i)$.

(L4) Suppose that s contains an X. This automatically implies that $\lambda = \mu = \nu$. Let ρ be obtained from λ by adding 1 to λ_1.

Using these rules, we can uniquely label every square corner, one step at a time. The resulting array is called the *growth diagram* $\mathcal{G}_w$ of w. For our example $w = 43512$, we get the growth diagram



(7.50)

It's easy to see that if a point p is labeled by λ, then the sum $|\lambda|$ of the parts of λ is equal to the number of X's in the quarter plane to the left and below p. In particular, if λ^i denotes the partition in row i (with the bottom row being row 0) and column n (the rightmost column), then $|\lambda^i| = i$. Moreover, it is immediate from the labeling procedure that $\emptyset = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^n$. Similarly, if μ^i denotes the partition in column i (with the leftmost column being 0) and row n (the top row), then $|\mu^i| = i$ and $\emptyset = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^n$.

The chains $\emptyset = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^n$ and $\emptyset = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^n$ correspond to standard tableaux P_w and Q_w, respectively (as explained in Proposition 7.10.3(a)). The main result concerning the geometric construction we have just described is the following.

7.13.5 Theorem. *The standard tableaux P_w and Q_w just described satisfy*

$$w \xrightarrow{\text{RSK}} (P_w, Q_w).$$

Proof (sketch). Let the partition appearing in row i and column j be $\nu(i, j)$. Thus for fixed j, we have

$$\emptyset = \nu(0, j) \subseteq \nu(1, j) \subseteq \cdots \subseteq \nu(n, j),$$

where $|\nu(i, j)/\nu(i-1, j)| = 0$ or 1. Let $T(i, j)$ be the tableau of shape $\nu(i, j)$ obtained by inserting k into the square $\nu(k, j)/\nu(k-1, j)$ when $0 \le k < i$ and $|\nu(k, j)/\nu(k-1, j)| = 1$. For the array (7.50) the tableaux $T(i, j)$ are given by

$$\begin{array}{ccccccccccc}
\emptyset & & 4 & & \begin{matrix}3\\4\end{matrix} & & \begin{matrix}35\\4\end{matrix} & & \begin{matrix}15\\3\\4\end{matrix} & & \begin{matrix}12\\35\\4\end{matrix} \\
 & & & & & X & & & & & \\
\emptyset & & 4 & & \begin{matrix}3\\4\end{matrix} & & \begin{matrix}3\\4\end{matrix} & & \begin{matrix}1\\3\\4\end{matrix} & & \begin{matrix}12\\3\\4\end{matrix} \\
 & X & & & & & & & & & \\
\emptyset & & \emptyset & & 3 & & 3 & & \begin{matrix}1\\3\end{matrix} & & \begin{matrix}12\\3\end{matrix} \\
 & & & X & & & & & & & \\
\emptyset & & \emptyset & & \emptyset & & \emptyset & & 1 & & 12 \\
 & & & & & & & & & X & \\
\emptyset & & \emptyset & & \emptyset & & \emptyset & & 1 & & 1 \\
 & & & & & & & X & & & \\
\emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset
\end{array}$$

We claim that the tableau $T(i, j)$ has the following alternative description: Let $(i_1, j_1), \ldots, (i_k, j_k)$ be the position of the x's to the left and below $T(i, j)$ (i.e., $i_r \le i$ and $j_r \le j$), labeled so that $j_1 < \cdots < j_k$. Then $T(i, j)$ is obtained by row inserting successively $i_1, i_2, \ldots, i_k$, beginning with an empty tableau. In symbols,

$$T(i, j) = ((\emptyset \leftarrow i_1) \leftarrow i_2) \leftarrow \cdots \leftarrow i_k.$$

The proof of the claim is by induction on $i + j$. The assertion is clearly true if $i = 0$ or $j = 0$, so that $T(i, j) = \emptyset$. If $i > 0$ and $j > 0$, then by the induction hypothesis we know that $T(i-1, j)$, $T(i, j-1)$, and $T(i-1, j-1)$ satisfy the desired conditions. One checks that $T(i, j)$ also satisfies these conditions by using the definition of $T(i, j)$ in terms of the local rules (L1)–(L4). There are thus four cases to check; we omit the rather straightforward details.

If we now take $i = n$, we see that

$$T(n, j) = ((\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots \leftarrow w_j, \tag{7.51}$$

where $w = w_1 w_2 \cdots w_n$. Thus $T(n, n)$ (which is the same as P_w) is indeed the insertion tableau of w, while Q_w (which is defined by the sequence $\nu(n,0) \subset \nu(n,1) \subset \cdots \subset \nu(n,n)$) is by (7.51) just the recording tableau. This completes the proof of Theorem 7.13.5. $\square$

It is now almost trivial to give our second proof of Theorem 7.13.1. If we transpose the growth diagram G_w (i.e., reflect about the diagonal from the lower left to the upper right corner) then the symmetry of the local rules (L1)–(L4) with respect to transposition shows that we get simply the growth diagram $G_{w^{-1}}$. Hence $P_w = Q_{w^{-1}}$ and $Q_w = P_{w^{-1}}$, and the proof follows from Theorem 7.13.5.

Growth diagrams and their variants are powerful tools for understanding the RSK algorithm and related algorithms. For further infomation, see the Notes to this chapter, as well as Exercise 7.28(a).

Let us now consider some corollaries of the symmetry property given by Theorem 7.13.1.

7.13.6 Corollary. *Let A be an $\mathbb{N}$-matrix of finite support, and let $A \xrightarrow{\text{RSK}} (P, Q)$. Then A is symmetric (i.e., $A = A^t$) if and only if $P = Q$.*

Proof. Immediate from the fact that $A^t \xrightarrow{\text{RSK}} (Q, P)$. $\square$

7.13.7 Corollary. *Let $A = A^t$ and $A \xrightarrow{\text{RSK}} (P, P)$, and let $\alpha = (\alpha_1, \alpha_2, \ldots)$, where $\alpha_i \in \mathbb{N}$ and $\sum \alpha_i < \infty$. Then the map $A \mapsto P$ establishes a bijection between symmetric $\mathbb{N}$-matrices with* $\mathrm{row}(A) = \alpha$ *and SSYTs of type α.*

Proof. Follows from Corollary 7.13.6 and Theorem 7.11.5. $\square$

7.13.8 Corollary. *We have*

$$\frac{1}{\prod_i (1 - x_i) \cdot \prod_{i<j} (1 - x_i x_j)} = \sum_{\lambda} s_\lambda(x), \tag{7.52}$$

summed over all $\lambda \in$ Par.

Proof. The coefficient of x^α on the left-hand side is the number of symmetric $\mathbb{N}$-matrices A with $\mathrm{row}(A) = \alpha$ [why?], while the coefficient of x^α on the right-hand side is the number of SSYTs of type α. Now apply Corollary 7.13.7. $\square$

7.13.9 Corollary. *We have*

$$\sum_{\lambda \vdash n} f^\lambda = \#\{w \in \mathfrak{S}_n : w^2 = 1\},$$

the number of involutions in $\mathfrak{S}_n$ (discussed in Example 5.2.10).

Proof. Let $w \in \mathfrak{S}_n$ and $w \xrightarrow{\text{RSK}} (P, Q)$, where P and Q are SYT of the same shape $\lambda \vdash n$. The permutation matrix corresponding to w is symmetric if and only if $w^2 = 1$. By Theorem 7.13.1 this is the case if and only if $P = Q$, and the proof follows.

Alternatively, take the coefficient of $x_1 \cdots x_n$ on both sides of (7.52). □

Corollary 7.13.9 asserts that the total number of SYT of size n is equal to the number of involutions in $\mathfrak{S}_n$. The RSK algorithm provides a bijective proof.

Note that if $t(n)$ denotes the coefficient of $x_1 \cdots x_n$ on the left-hand side of (7.52), then Example 7.8.5 shows directly that

$$\sum_{n\geq 0} t(n)\frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right),$$

in agreement with (5.32).

7.14 The Dual RSK Algorithm

There is a variation of the RSK algorithm that is related to the product $\prod(1 + x_i y_j)$ in the same way that the RSK algorithm itself is related to $\prod(1 - x_i y_j)^{-1}$. We call this variation the *dual RSK algorithm* and denote it by $A \xrightarrow{\text{RSK*}} (P, Q)$. The matrix A will now be a $(0, 1)$ matrix of finite support. Form the two-line array w_A just as before. The RSK* algorithm proceeds exactly like the RSK algorithm, except that an element i bumps the leftmost element $\geq i$, rather than the leftmost element $> i$. (In particular, RSK and RSK* agree for permutation matrices.) It follows that each row of P is *strictly* increasing.

7.14.1 Example. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$w_A = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 4 & 5 \\ 1 & 3 & 2 & 1 & 3 & 3 & 2 \end{pmatrix}.$$

The arrays $(P(1), Q(1)), \ldots, (P(7), Q(7))$, with $(P, Q) = (P(7), Q(7))$, obtained from RSK* are as follows:

$P(i)$	$Q(i)$
1	1
1 3	1 1
1 2 3	1 1 2
1 2 1 3	1 1 2 3
1 2 3 1 3	1 1 3 2 3
1 2 3 1 3 3	1 1 3 2 4 3
1 2 3 1 2 3 3	1 1 3 2 4 3 5

7.14.2 Theorem. *The RSK* algorithm is a bijection between* $(0, 1)$*-matrices* A *of finite support and pairs* (P, Q) *such that* P^t *(the transpose of* P*) and* Q *are SSYTs, with* $\mathrm{sh}(P) = \mathrm{sh}(Q)$*. Moreover,* $\mathrm{col}(A) = \mathrm{type}(P)$ *and* $\mathrm{row}(A) = \mathrm{type}(Q)$*.*

The proof of Theorem 7.14.2 is analogous to that of Theorem 7.11.5 and will be omitted.

Exactly as we obtained the Cauchy identity (7.44) from the ordinary RSK algorithm, we have the following result, known as the *dual Cauchy identity*.

7.14.3 Theorem. *We have*

$$\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y).$$

An important consequence of Theorem 7.14.3 is the evaluation of ωs_λ. First we need to see the effect of ω, acting on the y variables, on the product $\prod(1 + x_i y_j)$.

7.14.4 Lemma. *Let* ω_y *denote* ω *acting on the* y *variables only (so we regard the* x_i*'s as constants commuting with* ω*). Then*

$$\omega_y \prod(1 - x_i y_j)^{-1} = \prod(1 + x_i y_j).$$

Proof. We have

$$\begin{aligned}\omega_y \prod (1 - x_i y_j)^{-1} &= \omega_y \sum_\lambda m_\lambda(x) h_\lambda(y) \qquad \text{(by Proposition 7.5.3)} \\ &= \sum_\lambda m_\lambda(x) e_\lambda(y) \qquad \text{(by Theorem 7.6.1)} \\ &= \prod (1 + x_i y_j) \qquad \text{(by Proposition 7.4.3).}\end{aligned}$$

□

An alternative proof can be given by expanding the products $\prod(1 - x_i y_j)^{-1}$ and $\prod(1 + x_i y_j)$ in terms of the power sum symmetric functions (equations (7.20) and (7.21)) and applying Proposition 7.7.5.

7.14.5 Theorem. *For every* $\lambda \in \mathrm{Par}$ *we have*

$$\omega s_\lambda = s_{\lambda'}.$$

Proof. We have

$$\begin{aligned}\sum_\lambda s_\lambda(x) s_{\lambda'}(y) &= \prod (1 + x_i y_j) \qquad \text{(by Theorem 7.14.3)} \\ &= \omega_y \prod (1 - x_i y_j)^{-1} \qquad \text{(by Lemma 7.14.4)} \\ &= \omega_y \sum_\lambda s_\lambda(x) s_\lambda(y) \qquad \text{(by Theorem 7.12.1)} \\ &= \sum_\lambda s_\lambda(x) \omega_y \left(s_\lambda(y)\right).\end{aligned}$$

Take the coefficient of $s_\lambda(x)$ on both sides. Since the $s_\lambda(x)$'s are linearly independent, we obtain $s_{\lambda'}(y) = \omega_y\,(s_\lambda(y))$, or just $s_{\lambda'} = \omega s_\lambda$. □

Later (Theorem 7.15.6) we will extend Theorem 7.14.5 to skew Schur functions.

After Proposition 7.7.5 we mentioned that the characteristic polynomial of $\omega : \Lambda^n \to \Lambda^n$ is equal to $(x^2 - 1)^{o(n)}(x - 1)^{k(n)}$, where $o(n)$ is the number of odd conjugacy classes in $\mathfrak{S}_n$ and $k(n)$ is the number of self-conjugate partitions of n. In particular, the multiplicity of 1 as an eigenvalue exceeds the multiplicity of -1 by $k(n)$. This fact is also an immediate consequence of Theorem 7.14.5. For if $\lambda \neq \lambda'$ then ω transposes s_λ and $s_{\lambda'}$, accounting for one eigenvalue equal to 1 and one equal to -1. Left over are the $k(n)$ eigenvectors s_λ for which $\lambda = \lambda'$, with eigenvalue 1.

7.15 The Classical Definition of Schur Functions

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $w \in \mathfrak{S}_n$. As usual write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and define

$$w(x^\alpha) = x_1^{\alpha_{w(1)}} \cdots x_n^{\alpha_{w(n)}}$$

Now define

$$a_\alpha = a_\alpha(x_1, \ldots, x_n) = \sum_{w \in \mathfrak{S}_n} \varepsilon_w w(x^\alpha), \tag{7.53}$$

where

$$\varepsilon_w = \begin{cases} 1 & \text{if } w \text{ is an even permutation} \\ -1 & \text{if odd.} \end{cases}$$

(Thus $\varepsilon_w = \varepsilon_{\rho(w)}$, as defined in (7.19).) Note that the right-hand side of equation (7.53) is just the expansion of a determinant, namely,

$$a_\alpha = \det\left(x_i^{\alpha_j}\right)_{i,j=1}^n.$$

Note also that a_α is skew-symmetric, i.e., $w(a_\alpha) = \varepsilon_w a_\alpha$, so $a_\alpha = 0$ unless all the α_i's are distinct. Hence assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$, so $\alpha = \lambda + \delta$, where $\lambda \in \mathrm{Par}$, $\ell(\lambda) \leq n$, and $\delta = \delta_n = (n-1, n-2, \ldots, 0)$. Since $\alpha_j = \lambda_j + n - j$, we get

$$a_\alpha = a_{\lambda+\delta} = \det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n. \tag{7.54}$$

For instance,

$$a_{421} = a_{211+210} = \begin{vmatrix} x_1^4 & x_1^2 & x_1^1 \\ x_2^4 & x_2^2 & x_2^1 \\ x_3^4 & x_3^2 & x_3^1 \end{vmatrix}.$$

Note in particular that

$$a_\delta = \det\left(x_i^{n-j}\right) = \prod_{1 \leq i < j \leq n} (x_i - x_j), \tag{7.55}$$

the *Vandermonde determinant.*

If for some $i \neq j$ we put $x_i = x_j$ in a_α, then because a_α is skew-symmetric (or because the i-th row and j-th row of the determinant (7.54) become equal), we obtain 0. Hence a_α is divisible by $x_i - x_j$ and thus by a_δ (in the ring $\mathbb{Z}[x_1, \ldots, x_n]$).

Thus $a_\alpha/a_\delta \in \mathbb{Z}[x_1, \ldots, x_n]$. Moreover, since a_α and a_δ are skew-symmetric, the quotient is symmetric, and is clearly homogeneous of degree $|\alpha| - |\delta| = |\lambda|$. In other words, $a_\alpha/a_\delta \in \Lambda_n^{|\lambda|}$. (The quotient a_α/a_δ is called a *bialternant.*) It is therefore natural to ask for the relation between a_α/a_δ and the symmetric functions we have already considered. The answer is a fundamental result in the theory of symmetric functions.

7.15.1 Theorem. *We have*

$$a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_n).$$

Proof. There are many proofs of this result. We give one that can be extended to give an important result on skew Schur functions (Theorem 7.15.4).

Applying ω to (7.46) and replacing λ by λ' yields

$$e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda.$$

Since the matrix $(K_{\lambda'\mu})$ is invertible, it suffices to show that

$$e_\mu(x_1, \ldots, x_n) = \sum_\lambda K_{\lambda'\mu} \frac{a_{\lambda+\delta}}{a_\delta},$$

or equivalently (always working with n variables),

$$a_\delta e_\mu = \sum_\lambda K_{\lambda'\mu} a_{\lambda+\delta}. \tag{7.56}$$

Since both sides of (7.56) are skew-symmetric, it is enough to show that the coefficient of $x^{\lambda+\delta}$ in $a_\delta e_\mu$ is $K_{\lambda'\mu}$. We multiply a_δ by e_μ by successively multiplying by $e_{\mu_1}, e_{\mu_2}, \ldots$. Each partial product $a_\delta e_{\mu_1} \cdots e_{\mu_k}$ is skew-symmetric, so any term $x_1^{i_1} \cdots x_n^{i_n}$ appearing in $a_\delta e_{\mu_1} \cdots e_{\mu_k}$ has all exponents i_j *distinct*. When we multiply such a term $x_1^{i_1} \cdots x_n^{i_n}$ by a term $x_{m_1} \cdots x_{m_j}$ from $e_{\mu_{k+1}}$ (so $j = \mu_{k+1}$), either two exponents become equal or the exponents maintain their relative order. If two exponents become equal, then that term disappears from $a_\delta e_{\mu_1} \cdots e_{\mu_{k+1}}$. Hence to get the term $x^{\lambda+\delta}$, we must start with the term x^δ in a_δ and successively multiply by a term x^{α^1} of e_{μ_1}, then x^{α^2} of e_{μ_2}, etc., keeping the exponents strictly decreasing. The number of ways to do this is the coefficient of $x^{\lambda+\delta}$ in $a_\delta e_\mu$.

Given the terms $x^{\alpha^1}, x^{\alpha^2}, \ldots$ as above, define an SSYT $T = T(\alpha^1, \alpha^2, \ldots)$ as follows: Column j of T contains an i if the variable x_j occurs in x^{α^i} (i.e., the j-th coordinate of α^i is equal to 1). For example, suppose $n = 4$, $\lambda = 5332$, $\lambda' = 44311$, $\lambda + \delta = 8542$, $\mu = 3222211$, $x^{\alpha^1} = x_1x_2x_3$, $x^{\alpha^2} = x_1x_2$, $x^{\alpha^3} = x_3x_4$,

$x^{\alpha^4} = x_1x_2, x^{\alpha^5} = x_1x_4, x^{\alpha^6} = x_1, x^{\alpha^7} = x_3$. Then T is given by

$$\begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 5 \\ 4 & 4 & 7 & \\ 5 & & & \\ 6 & & & \end{array}$$

It is easy to see that the map $(\alpha^1, \alpha^2, \ldots) \mapsto T(\alpha^1, \alpha^2, \ldots)$ gives a bijection between ways of building up the term $x^{\lambda+\delta}$ from x^δ (according to the rules above) and SSYT of shape λ' and type μ, so the proof follows. □

From the combinatorial definition of Schur functions it is clear that $s_\lambda(x_1, \ldots, x_n) = 0$ if $\ell(\lambda) > n$. Since by Proposition 7.8.2(b) we have $\dim \Lambda_n = \#\{\lambda \in \mathrm{Par} : \ell(\lambda) \le n\}$, it follows that the set $\{s_\lambda(x_1, \ldots, x_n) : \ell(\lambda) \le n\}$ is a basis for Λ_n. (This also follows from a simple extension of the proof of Corollary 7.10.6.) We define on Λ_n a scalar product $\langle\ ,\ \rangle_n$ by requiring that $\{s_\lambda(x_1, \ldots, x_n)\}$ is an orthonormal basis. If $f, g \in \Lambda$, then we write $\langle f, g\rangle_n$ as short for $\langle f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)\rangle_n$. Thus

$$\langle f, g\rangle = \langle f, g\rangle_n,$$

provided that every monomial appearing in f involves at most n distinct variables, e.g., if $\deg f \le n$.

7.15.2 Corollary. *If $f \in \Lambda_n$, $\ell(\lambda) \le n$, and $\delta = (n-1, n-2, \ldots, 1, 0)$, then*

$$\langle f, s_\lambda\rangle_n = [x^{\lambda+\delta}]a_\delta f,$$

the coefficient of $x^{\lambda+\delta}$ in $a_\delta f$.

Proof. All functions will be in the variables $x_1, \ldots, x_n$. Let $f = \sum_{\ell(\lambda)\le n} c_\lambda s_\lambda$. Then by Theorem 7.15.1 we have

$$a_\delta f = \sum_{\ell(\lambda)\le n} c_\lambda a_{\lambda+\delta},$$

so

$$\langle f, s_\lambda\rangle_n = c_\lambda = [x^{\lambda+\delta}]a_\delta f.$$ □

For instance, we have

$$\left\langle a_\delta^{2k}, s_\lambda\right\rangle_n = [x^{\lambda+\delta}]a_\delta^{2k+1}, \tag{7.57}$$

for $\ell(\lambda) \le n$. It is an interesting problem (not completely solved) to compute the numbers (7.57); for further information on the case $k = 1$, see Exercise 7.37.

Let us now consider a "skew generalization" of Theorem 7.15.1. We continue to work in the n variables $x_1, \ldots, x_n$. For any $\lambda, \nu \in$ Par, $\ell(\lambda) \leq n$, $\ell(\nu) \leq n$, consider the expansion

$$s_\nu e_\mu = \sum_\lambda L^{\lambda'}_{\nu'\mu} s_\lambda,$$

or equivalently (multiplying by a_δ),

$$a_{\nu+\delta} e_\mu = \sum_\lambda L^{\lambda'}_{\nu'\mu} a_{\lambda+\delta}. \tag{7.58}$$

Arguing as in the proof of Theorem 7.15.1 shows that $L^{\lambda'}_{\nu'\mu}$ is equal to the number of ways to write

$$\lambda + \delta = \nu + \delta + \alpha^1 + \alpha^2 + \cdots + \alpha^k,$$

where $\ell(\mu) = k$, each α^i is a $(0, 1)$-vector with μ_i 1's, and each partial sum $\nu + \delta + \alpha^1 + \cdots + \alpha^i$ has strictly decreasing coordinates. Define a skew SSYT $T = T_{\lambda'/\nu'}(\alpha^1, \ldots, \alpha^k)$ of shape λ'/ν' and type μ by the condition that i appears in column j of T if the j-th coordinate of α^i is a 1. This establishes a bijection which shows that $L^{\lambda'}_{\nu'\mu}$ is equal to the skew Kostka number $K_{\lambda'/\nu',\mu}$, the number of skew SSYTs of shape λ'/ν' and type μ (see equation (7.36)). (If $\nu' \not\subseteq \lambda'$ then this number is 0.)

7.15.3 Corollary. *We have*

$$s_\nu e_\mu = \sum_\lambda K_{\lambda'/\nu',\mu} s_\lambda. \tag{7.59}$$

Proof. Divide (7.58) by a_δ, and let $n \to \infty$. □

It is now easy to establish a fundamental property of skew Schur functions.

7.15.4 Theorem. *For any $f \in \Lambda$, we have*

$$\langle f s_\nu, s_\lambda \rangle = \langle f, s_{\lambda/\nu} \rangle.$$

In other words, the two linear transformations $M_\nu : \Lambda \to \Lambda$ and $D_\nu : \Lambda \to \Lambda$ defined by $M_\nu f = s_\nu f$ and $D_\nu s_\lambda = s_{\lambda/\nu}$ are adjoint with respect to the scalar product $\langle\ ,\ \rangle$. In particular,

$$\langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\mu, s_{\lambda/\nu} \rangle. \tag{7.60}$$

Proof. Apply ω to (7.59) and replace ν by ν' and λ by λ'. We obtain

$$s_\nu h_\mu = \sum_\lambda K_{\lambda/\nu,\mu} s_\lambda.$$

Hence

$$\langle s_\nu h_\mu, s_\lambda \rangle = K_{\lambda/\nu,\mu} = \langle h_\mu, s_{\lambda/\nu} \rangle, \tag{7.61}$$

by (7.36) and the fact that $\langle h_\mu, m_\rho \rangle = \delta_{\mu\rho}$ by definition of $\langle\ ,\ \rangle$. But equation (7.61) is linear in h_μ, so since $\{h_\mu\}$ is a basis for Λ, the proof follows. $\square$

7.15.5 Example. We have $s_1 s_{31} = s_{41} + s_{32} + s_{311}$ and $s_1 s_{22} = s_{32} + s_{221}$. No other product $s_1 s_\mu$ involves s_{32}. It follows that $s_{32/1} = s_{22} + s_{31}$. For a generalization, see Corollary 7.15.9.

We can now give the generalization of Theorem 7.14.5 to skew Schur functions.

7.15.6 Theorem. *For any* $\lambda, \nu \in \mathrm{Par}$ *we have* $\omega s_{\lambda/\nu} = s_{\lambda'/\nu'}$.

Proof. By Proposition 7.9.5 and equation (7.60) we have

$$\langle \omega(s_\mu s_\nu), \omega s_\lambda \rangle = \langle \omega s_\mu, \omega s_{\lambda/\nu} \rangle.$$

Hence by Theorem 7.14.5 we get

$$\langle s_{\mu'} s_{\nu'}, s_{\lambda'} \rangle = \langle s_{\mu'}, \omega s_{\lambda/\nu} \rangle. \tag{7.62}$$

On the other hand, substituting λ', μ', ν' for λ, μ, ν respectively in (7.60) yields

$$\langle s_{\mu'} s_{\nu'}, s_{\lambda'} \rangle = \langle s_{\mu'}, s_{\lambda'/\nu'} \rangle. \tag{7.63}$$

From (7.62) and (7.63) there follows $\omega s_{\lambda/\nu} = s_{\lambda'/\nu'}$. $\square$

The integer $\langle s_\lambda, s_\mu s_\nu \rangle = \langle s_{\lambda/\nu}, s_\mu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle$ is denoted $c^\lambda_{\mu\nu}$ and is called a *Littlewood–Richardson coefficient*. Thus

$$\begin{aligned} s_\mu s_\nu &= \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \\ s_{\lambda/\nu} &= \sum_\mu c^\lambda_{\mu\nu} s_\mu \\ s_{\lambda/\mu} &= \sum_\nu c^\lambda_{\mu\nu} s_\nu. \end{aligned} \tag{7.64}$$

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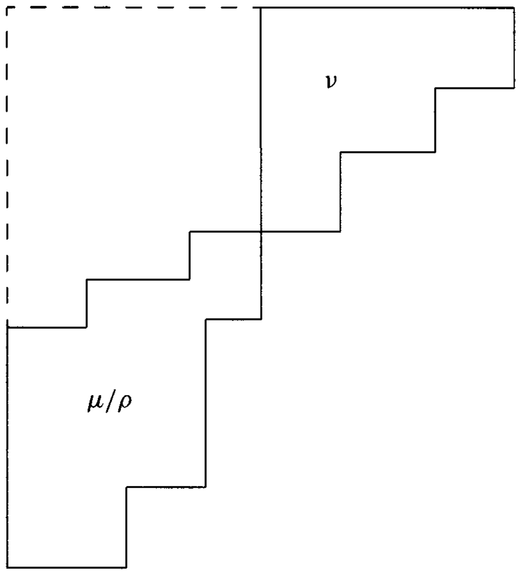


Figure 7-5. A skew shape.

Note that the seemingly more general $\langle s_{\lambda/\nu}, s_{\mu/\rho}\rangle$ is itself a Littlewood–Richardson coefficient, since $\langle s_{\lambda/\nu}, s_{\mu/\rho}\rangle = \langle s_\lambda, s_\nu s_{\mu/\rho}\rangle$ and $s_\nu s_{\mu/\rho}$ is just a skew Schur function, as Figure 7-5 (together with the combinatorial definition of Schur functions) makes evident. More generally, any product of skew Schur functions is a skew Schur function.

A central result in the theory of symmetric functions, called the *Littlewood–Richardson rule*, gives a combinatorial interpretation of the Littlewood–Richardson coefficient $c^\lambda_{\mu\nu}$. We will defer the statement and proof of the Littlewood–Richardson rule to Appendix 1 (Section A1.3). Here we consider the much easier special case when $\mu = (n)$, the partition with a single part equal to n. To state this result, known as *Pieri's rule*, define a *horizontal strip* to be a skew shape λ/ν with no two squares in the same column. Thus an SSYT of shape μ/ρ with largest part at most m may be regarded as a sequence $\rho = \mu^0 \subseteq \mu^1 \subseteq \cdots \subseteq \mu^m = \mu$ of partitions such that each skew shape μ^i/μ^{i-1} is a horizontal strip. (Simply insert i into each square of μ^i/μ^{i-1}.)

7.15.7 Theorem. *We have*

$$s_\nu s_n = \sum_\lambda s_\lambda, \tag{7.65}$$

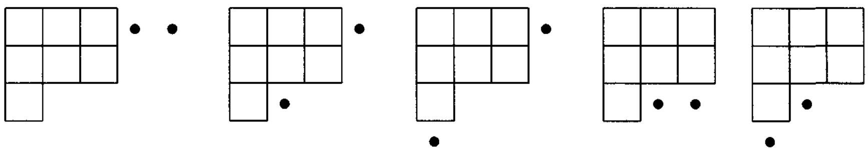
summed over all partitions λ such that λ/ν is a horizontal strip of size n.

Proof. We have $\langle s_\nu s_n, s_\lambda \rangle = \langle s_n, s_{\lambda/\nu} \rangle = \langle h_n, s_{\lambda/\nu} \rangle = K_{\lambda/\nu,n}$. Clearly by definition of $K_{\lambda/\nu,n}$ we have

$$K_{\lambda/\nu,n} = \begin{cases} 1 & \text{if } \lambda/\nu \text{ is a horizontal strip of size } n \\ 0 & \text{otherwise,} \end{cases}$$

and the proof follows. □

7.15.8 Example. Let $\nu = 331$ and $n = 2$. The ways of adding a horizontal strip of size 2 to the shape 331 are given by



Hence

$$s_{331} s_2 = s_{531} + s_{432} + s_{4311} + s_{333} + s_{3321}.$$

Note that by applying ω to (7.65) we get a dual version of Pieri's rule. Namely, defining a *vertical strip* in the obvious way, we have

$$s_\nu s_{1^n} = s_\nu e_n = \sum_\lambda s_\lambda,$$

summed over all partitions λ for which λ/ν is a vertical strip of size n.

We also have as an immediate consequence of (7.60) and Pieri's rule (Theorem 7.15.7) the following skew version of Pieri's rule.

7.15.9 Corollary. *We have*

$$s_{\lambda/n} = \sum_\nu s_\nu,$$

where ν ranges over all partitions $\nu \subseteq \lambda$ for which λ/ν is a horizontal strip of size n.

The proof we have given of Pieri's rule is rather indirect, but Pieri's rule is actually a simple combinatorial statement that deserves a direct bijective proof. Let $\mathcal{T}^\alpha_{\nu,n}$ be the set of all pairs (T, T') of SSYTs such that $\mathrm{sh}(T) = \nu$, $\mathrm{sh}(T') = (n)$, and $\mathrm{type}(T) + \mathrm{type}(T') = \alpha$. Similarly, let $\mathcal{T}^\alpha_\lambda$ be the set of all SSYTs T such that $\mathrm{sh}(T) = \lambda$ and $\mathrm{type}(T) = \alpha$. Pieri's rule asserts that

$$\#\mathcal{T}^\alpha_{\nu,n} = \#\left(\bigcup_\lambda \mathcal{T}^\alpha_\lambda\right),$$

where λ ranges over all partitions such that λ/ν is a horizontal strip of size n. Thus we seek a bijection

$$\varphi : \mathcal{T}^{\alpha}_{\nu,n} \to \bigcup_{\lambda} \mathcal{T}^{\alpha}_{\lambda}.$$

Let $T' = a_1 a_2 \cdots a_n$. It is not difficult to show (using Lemma 7.11.2) that φ is given just by iterated row insertion:

$$\varphi(T, T') = ((T \leftarrow a_1) \leftarrow a_2) \leftarrow \cdots \leftarrow a_n.$$

A further avatar of Theorem 7.15.4 is the following. Let $\Lambda(x)$ and $\Lambda(y)$ denote the rings of symmetric functions in the variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, respectively. Denote by $\Lambda(x) \otimes \Lambda(y)$ the ring of formal power series (over $\mathbb{Q}$) in x and y of bounded degree that are symmetric in the x variables and symmetric in the y variables. In other words, if $f(x_1, x_2, \ldots; y_1, y_2, \ldots) \in \Lambda(x) \otimes \Lambda(y)$ and if u and v are both permutations of $\mathbb{P}$, then

$$f(x_1, x_2, \ldots; y_1, y_2, \ldots) = f(x_{u(1)}, x_{u(2)}, \ldots; y_{v(1)}, y_{v(2)}, \ldots).$$

It is clear that if $\{b_\mu(x)\}$ is a basis for $\Lambda(x)$ and $\{c_\nu(y)\}$ for $\Lambda(y)$, then $\{b_\mu(x)c_\nu(y)\}$ is a basis for $\Lambda(x) \otimes \Lambda(y)$. The ring $\Lambda(x, y)$ of formal power series of bounded degree that are symmetric in the x and y variables *together* is a subalgebra of $\Lambda(x) \otimes \Lambda(y)$. Of course the containment is proper; for instance, if $f(x) \in \Lambda(x)$ and $\deg f > 0$, then $f(x) \in \Lambda(x) \otimes \Lambda(y)$ but $f(x) \notin \Lambda(x, y)$. If $\{b_\lambda(x)\}$ is a basis for $\Lambda(x)$ then $\{b_\lambda(x, y)\}$ is a basis for $\Lambda(x, y)$, where $b_\lambda(x, y)$ denotes the symmetric function b_λ in the variables $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$. It is now natural to ask how to expand $s_\lambda(x, y)$ in terms of the basis $\{s_\mu(x)s_\nu(y)\}$ of $\Lambda(x) \otimes \Lambda(y)$. Consider an ordered alphabet $A = \{1 < 2 < \cdots < 1' < 2' < \cdots\}$. If T is an SSYT of shape λ with respect to this alphabet, then define

$$(xy)^T = x_1^{\#(1)} x_2^{\#(2)} \cdots y_1^{\#(1')} y_2^{\#(2')} \cdots,$$

where $\#(a)$ denotes the number of occurrences of a in T. Thus from the combinatorial definition of s_λ (Definition 7.10.1), we have

$$s_\lambda(x, y) = \sum_T (xy)^T,$$

where T ranges over all SSYT of shape λ in the alphabet A. Now the part of T occupied by $1, 2, \ldots$ is just an SSYT of some shape $\mu \subseteq \lambda$, while the part of T occupied by $1', 2', \ldots$ is a skew SSYT of shape λ/μ. From this observation there follows

$$\begin{aligned} s_\lambda(x, y) &= \sum_{\mu \subseteq \lambda} s_\mu(x) s_{\lambda/\mu}(y) \\ &= \sum_{\mu \subseteq \lambda} s_\mu(x) \sum_\nu c^\lambda_{\mu\nu} s_\nu(y) \\ &= \sum_{\mu, \nu} c^\lambda_{\mu\nu} s_\mu(x) s_\nu(y), \end{aligned} \tag{7.66}$$

which gives us the desired expansion.

Note (for algebraists). Define $\Delta : \Lambda \to \Lambda(x) \otimes \Lambda(y)$ by $\Delta f = f(x, y)$. This operation makes the space Λ into a *coalgebra*, and together with the usual algebra structure on Λ it forms a *bialgebra*. If we take $1 \in \Lambda$ to be a unit and the map $f \mapsto f(0, 0, \ldots)$ to be a counit, then we get a *Hopf algebra*. Moreover, the scalar product on Λ is compatible with the bialgebra structure, in the sense that

$$\langle \Delta f, g(x)h(y) \rangle = \langle f, gh \rangle. \tag{7.67}$$

Here the first scalar product takes place in $\Lambda(x) \otimes \Lambda(y)$, where the elements $s_\mu(x)s_\nu(y)$ form an orthonormal basis. The second scalar product is just the usual one on Λ.

7.16 The Jacobi–Trudi Identity

In this section we will expand the Schur functions in terms of the complete symmetric functions. In effect we are computing the inverse to the Kostka matrix $(K_{\lambda\mu})$. Note that expanding Schur functions in terms of h_λ's is equivalent to expanding them in terms of e_λ's, for if $s_\lambda = \sum_\mu t_{\lambda\mu} h_\mu$, then applying ω yields $s_{\lambda'} = \sum_\mu t_{\lambda\mu} e_\mu$.

The main result of this section, known as the *Jacobi–Trudi identity*, expresses s_λ (in fact, $s_{\lambda/\mu}$) as a determinant whose entries are h_i's. Each term of the expansion of this determinant is thus of the form $\pm h_\nu$, so we get our desired expansion. The actual coefficient of h_ν must be obtained by collecting terms.

7.16.1 Theorem. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n) \subseteq \lambda$. Then*

$$s_{\lambda/\mu} = \det\big(h_{\lambda_i - \mu_j - i + j}\big)_{i,j=1}^{n}, \tag{7.68}$$

where we set $h_0 = 1$ and $h_k = 0$ for $k < 0$.

First Proof. Our first proof will be a direct application of Theorem 2.7.1, in which we evaluated a determinant combinatorially by constructing an involution that canceled out all unwanted terms. Indeed, the Jacobi–Trudi identity is perhaps the archetypal application of Theorem 2.7.1.

In Theorem 2.7.1, take $\alpha_j = \lambda_j + n - j$, $\beta_i = \mu_i + n - i$, $\gamma_i = \infty$ (more precisely, take $\gamma_i = N$ and let $N \to \infty$), and $\delta_j = 1$. The function $h(\alpha_j - \beta_i; \gamma_i, \delta_j)$ appearing in Theorem 2.7.1 is just the complete symmetric function $h_{\lambda_j - \mu_i - j + i}$. Thus the determinant appearing in Theorem 2.7.1 becomes (after interchanging the roles of i and j) the right-hand side of equation (7.68).

Therefore by Theorem 2.7.1 it remains to show that $B(\alpha, \beta, \gamma, \delta) = s_{\lambda/\mu}$. In other words, given a nonintersecting n-path $\mathbf{L}$ in $\mathcal{B}(\alpha, \beta, \gamma, \delta)$, we need to associate (in a bijective fashion) a skew SSYT T of shape λ/μ such that the weight of $\mathbf{L}$ is equal to $x^{\mathrm{type}(T)}$. Actually, we associate a *reverse* SSYT T, which by Proposition 7.10.4 does not make any difference. If the horizontal steps of the path from $(\mu_i + n - i, \infty)$ to $(\lambda_i + n - i, 1)$ occur at heights $a_1 \geq a_2 \geq \cdots \geq a_{\lambda_i - \mu_i}$, then let $a_1, a_2, \ldots, a_{\lambda_i - \mu_i}$ be the ith row of T. A little thought shows that this establishes the desired bijection. □

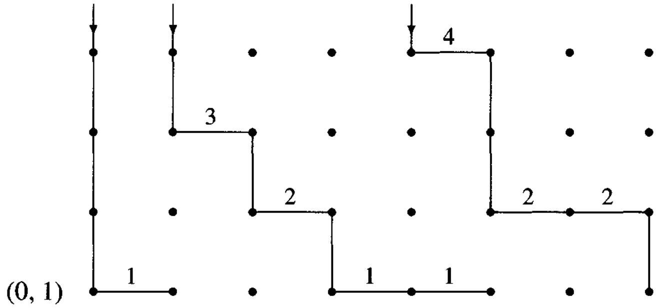


Figure 7-6. Nonintersecting lattice paths corresponding to an SSYT of shape 541/2.

As an example of the above bijection, take **L** as in Figure 7-6. Then

$$T = \begin{array}{ccccc} & & 4 & 2 & 2 \\ 3 & 2 & 1 & 1 & \\ 1 & & & & \end{array}$$

Second Proof. Though our first proof was a very elegant combinatorial argument, it is also worthwhile to give a purely algebraic proof. Let $c^{\lambda}_{\mu\nu} = \langle s_\lambda, s_\mu s_\nu \rangle$, so

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda, \qquad s_{\lambda/\mu} = \sum_\nu c^{\lambda}_{\mu\nu} s_\nu.$$

Then

$$\begin{aligned} \sum_\lambda s_{\lambda/\mu}(x) s_\lambda(y) &= \sum_{\lambda,\nu} c^{\lambda}_{\mu\nu} s_\nu(x) s_\lambda(y) \\ &= \sum_\nu s_\nu(x) s_\mu(y) s_\nu(y) \\ &= s_\mu(y) \sum_\nu h_\nu(x) m_\nu(y). \end{aligned}$$

Let $y = (y_1, \ldots, y_n)$. Multiplying by $a_\delta(y)$ gives

$$\begin{aligned} \sum_\lambda s_{\lambda/\mu}(x) a_{\lambda+\delta}(y) &= \left(\sum_\nu h_\nu(x) m_\nu(y) \right) a_{\mu+\delta}(y) \\ &= \left(\sum_{\alpha \in \mathbb{N}^n} h_\alpha(x) y^\alpha \right) \left(\sum_{w \in \mathfrak{S}_n} \varepsilon_w y^{w(\mu+\delta)} \right) \\ &= \sum_{w \in \mathfrak{S}_n} \sum_\alpha \varepsilon_w h_\alpha(x) y^{\alpha + w(\mu+\delta)} \end{aligned}$$

Now take the coefficient of $y^{\lambda+\delta}$ on both sides (so we are looking at terms where $\lambda+\delta=\alpha+w(\mu+\delta)$). We get

$$s_{\lambda/\mu}(x)=\sum_{w\in\mathfrak{S}_n}\varepsilon_w h_{\lambda+\delta-w(\mu+\delta)}(x) \tag{7.69}$$

$$=\det\big(h_{\lambda_i-\mu_j-i+j}(x)\big)_{i,j=1}^{n}. \qquad \square$$

If we substitute λ'/μ' for λ/μ in (7.68) and apply the automorphism ω, then we obtain the expansion of $s_{\lambda/\mu}$ in terms of the elementary symmetric functions, namely:

7.16.2 Corollary. *Let $\mu\subseteq\lambda$ with $\lambda_1\le n$. Then*

$$s_{\lambda/\mu}=\det\big(e_{\lambda'_i-\mu'_j-i+j}\big)_{i,j=1}^{n}. \tag{7.70}$$

Equation (7.70) is known as the *dual Jacobi–Trudi identity*.

Recall (Proposition 7.8.4) that the exponential specialization ex satisfies

$$\mathrm{ex}(f)=\sum_{n\ge 0}[x_1x_2\cdots x_n]f\,\frac{t^n}{n!}.$$

Let $\mathrm{ex}_1(f)=\mathrm{ex}(f)_{t=1}$, provided this number is defined. In particular, if $|\lambda/\mu|=N$ then

$$\mathrm{ex}_1(s_{\lambda/\mu})=\frac{f^{\lambda/\mu}}{N!},$$

where $f^{\lambda/\mu}$ is the number of SYT of shape λ/μ.

7.16.3 Corollary. *Let $|\lambda/\mu|=N$ and $\ell(\lambda)\le n$. Then*

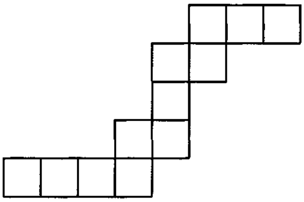
$$f^{\lambda/\mu}=N!\,\det\left(\frac{1}{(\lambda_i-\mu_j-i+j)!}\right)_{i,j=1}^{n}. \tag{7.71}$$

Proof. Apply ex_1 to the Jacobi–Trudi identity (equation (7.68)). Since $\mathrm{ex}_1(h_m)=1/m!$ by (7.27), the proof follows. $\square$

While it is certainly possible to prove Corollary 7.16.3 directly, our proof shows that it is just a specialization of the Jacobi–Trudi identity. When $\mu=\emptyset$ the determinant appearing in (7.71) can be explicitly evaluated (e.g., by induction and a clever use of row and column operations), thereby giving an explicit formula for f^λ. We will defer this formula to Corollary 7.21.6 and equation (7.113), where we give two less computational proofs.

7.17 The Murnaghan–Nakayama Rule

We have succeeded in expressing the Schur functions in terms of the bases m_λ, h_λ, and e_λ. In this section we consider the power sum symmetric functions p_λ. A skew shape λ/μ is *connected* if the interior of the diagram of λ/μ, regarded as a union of *solid* squares, is a connected (open) set. For instance, the shape $21/1$ is *not* connected. A *border strip* (or *rim hook* or *ribbon*) is a connected skew shape with no 2×2 square. An example of a border strip is $86554/5443$, whose diagram is



Given positive integers $a_1, \dots, a_k$, there is a unique border strip λ/μ (up to translation) with a_i squares in row i (i.e., $a_i = \lambda_i - \mu_i$). It follows that the number of border strips of size n (up to translation) is 2^{n-1}, the number of compositions of n. Define the *height* $\mathrm{ht}(B)$ of a border strip B to be one less than its number of rows. The next result shows the connection between border strips and symmetric functions.

7.17.1 Theorem. *For any $\mu \in$ Par and $r \in \mathbb{N}$ we have*

$$s_\mu p_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda, \tag{7.72}$$

summed over all partitions $\lambda \supseteq \mu$ for which λ/μ is a border strip of size r.

Proof. Let $\delta = (n-1, n-2, \dots, 0)$, and let all functions be in the variables $x_1, \dots, x_n$. In equation (7.53) let $\alpha = \mu + \delta$ and multiply by p_r. We get

$$a_{\mu+\delta} p_r = \sum_{j=1}^n a_{\mu+\delta+r\epsilon_j}, \tag{7.73}$$

where ϵ_j is the sequence with a 1 in the j-th place and 0 elsewhere. Arrange the sequence $\mu + \delta + r\epsilon_j$ in descending order. If it has two terms equal, then it will contribute nothing to (7.73). Otherwise there is some $p \le q$ for which

$$\mu_{p-1} + n - p + 1 > \mu_q + n - q + r > \mu_p + n - p,$$

in which case $a_{\mu+\delta+r\epsilon_j} = (-1)^{q-p} a_{\lambda+\delta}$, where λ is the partition

$$\lambda = (\mu_1, \dots, \mu_{p-1}, \mu_q + p - q + r, \mu_p + 1, \dots, \mu_{q-1} + 1, \mu_{q+1}, \dots, \mu_n).$$

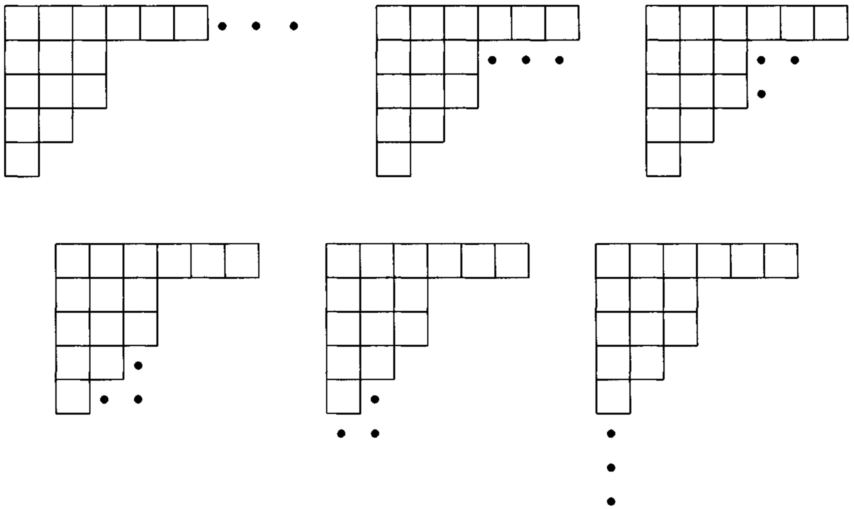


Figure 7-7. Border strips $\lambda/63321$ of size three.

Such partitions are precisely those for which λ/μ is a border strip B of size r, and $q - p$ is just $\mathrm{ht}(B)$. Hence

$$a_{\mu+\delta}p_r = \sum_{\lambda}(-1)^{\mathrm{ht}(\lambda/\mu)}a_{\lambda+\delta}.$$

Divide by a_δ and let $n \to \infty$ to obtain (7.72). □

7.17.2 Example. (a) Let $\mu = 63321$. The border strips of size 3 that can be added to μ are shown in Figure 7-7. Hence

$$s_{63321}p_3 = s_{93321} + s_{66321} - s_{65421} - s_{63333} - s_{633222} + s_{63321111}.$$

(b) Let $\delta = (n-1, n-2, \ldots, 0)$ as above. There are only *two* border strips of size 2 that can be added to δ, and we get

$$s_\delta p_2 = s_{n+1,n-2,n-3,\ldots,1} - s_{n-1,n-2,\ldots,2,1,1,1}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n. Define a *border-strip tableau* (or *rim-hook tableau*) of shape λ/μ (where $|\lambda/\mu| = n$) and type α to be an assignment of positive integers to the squares of λ/μ such that

(a) every row and column is weakly increasing,
(b) the integer i appears α_i times, and
(c) the set of squares occupied by i forms a border strip.

Equivalently, one may think of a border-strip tableau as a sequence $\mu = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^r \subseteq \lambda$ of partitions such that each skew shape λ^i/λ^{i+1} is a border-strip of size α_i (including the empty border-strip $\emptyset$ when $\alpha_i = 0$). For instance, the array

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 6 & 6 & 6 \\
1 & 2 & 2 & 5 & 6 & & \\
3 & 3 & 5 & 5 & 6 & & \\
3 & 5 & 5 & 6 & 6 & &
\end{array}$$

is a border-strip tableau of shape 7555 and type $(5, 2, 3, 0, 5, 7)$. (The border-strip outlines have been drawn in for the sake of clarity.) Define the *height* $\mathrm{ht}(T)$ of a border-strip tableau T to be

$$\mathrm{ht}(T) = \mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_k),$$

where $B_1, \ldots, B_k$ are the (nonempty) border strips appearing in T. For the example above we have $\mathrm{ht}(T) = 1 + 0 + 1 + 2 + 3 = 7$.

If we iterate Theorem 7.17.1, successively multiplying s_μ by $p_{\alpha_1}, p_{\alpha_2}, \ldots$, then we obtain immediately the following result.

7.17.3 Theorem. *We have*

$$s_\mu p_\alpha = \sum_\lambda \chi^{\lambda/\mu}(\alpha) s_\lambda, \tag{7.74}$$

where

$$\chi^{\lambda/\mu}(\alpha) = \sum_T (-1)^{\mathrm{ht}(T)}, \tag{7.75}$$

summed over all border-strip tableaux of shape λ/μ and type α.

Taking $\mu = \emptyset$ in Theorem 7.17.3 yields:

7.17.4 Corollary. *We have*

$$p_\alpha = \sum_\lambda \chi^\lambda(\alpha) s_\lambda, \tag{7.76}$$

where $\chi^\lambda(\alpha)$ is given by (7.75).

If we restrict ourselves to n variables where $n \geq \ell(\lambda)$ and apply Theorem 7.15.1, then equation (7.76) may be rewritten

$$p_\alpha a_\delta = \sum_\lambda \chi^\lambda(\alpha) a_{\lambda+\delta}.$$

Hence we obtain the following "formula" for $\chi^\lambda(\alpha)$:

$$\chi^\lambda(\alpha) = [x^{\lambda+\delta}] p_\alpha a_\delta. \tag{7.77}$$

It is easy to use equation (7.74) to express $s_{\lambda/\mu}$ in terms of the power sums. This result (at least in the case $\mu = \emptyset$) is known as the *Murnaghan–Nakayama rule*.

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7.17.5 Corollary. *We have*

$$s_{\lambda/\mu} = \sum_{\nu} z_\nu^{-1} \chi^{\lambda/\mu}(\nu) p_\nu, \tag{7.78}$$

where $\chi^{\lambda/\mu}(\nu)$ is given by (7.75).

Proof. We have from (7.74) that

$$\begin{aligned}\chi^{\lambda/\mu}(\nu) &= \langle s_\mu p_\nu, s_\lambda\rangle \\ &= \langle p_\nu, s_{\lambda/\mu}\rangle,\end{aligned}$$

and the proof follows from Proposition 7.9.3. □

The orthogonality properties of the bases $\{s_\lambda\}$ and $\{p_\lambda\}$ translate into orthogonality relations satisfied by the coefficients $\chi_\lambda(\mu)$.

7.17.6 Proposition. (a) *Fix μ, ν. Then*

$$\sum_\lambda \chi^\lambda(\mu)\chi^\lambda(\nu) = z_\mu \delta_{\mu\nu}.$$

(b) *Fix λ, μ. Then*

$$\sum_\nu z_\nu^{-1} \chi^\lambda(\nu)\chi^\mu(\nu) = \delta_{\lambda\mu}.$$

Proof. (a) Expand p_μ and p_ν by (7.76) and take $\langle p_\mu, p_\nu\rangle$.
(b) Expand s_λ and s_μ by (7.78) and take $\langle s_\lambda, s_\mu\rangle$. □

Proposition 7.17.6 is equivalent to the statement that the matrix $(\chi^\lambda(\mu) z_\mu^{-1/2})_{\lambda,\mu\vdash n}$ is an orthogonal matrix. This may be seen directly from the fact that this matrix is the transition matrix between the two orthonormal bases $\{s_\lambda\}$ and $\{p_\mu z_\mu^{-1/2}\}$.

A remarkable consequence of Corollary 7.17.4 is that the coefficients $\chi^\lambda(\alpha)$ do not depend on the order of the entries of α (since the same is true of the product $p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots$). This fact can be of great value in obtaining information about the numbers $\chi^\lambda(\alpha)$. As a sample application, we mention the following result.

7.17.7 Proposition. *Let δ be the "staircase shape" $\delta = (m-1, m-2, \ldots, 1)$. Then s_δ is a polynomial in the odd power sums $p_1, p_3, \ldots$.*

Proof. We need to show that $\chi^\delta(\nu) = 0$ if ν has an even part. Let α be an ordering of the parts of ν such that the *last* nonzero entry α_k of α is even. Thus the border-strip tableaux T in (7.75) have the property that the squares labeled k form a border strip δ/ν of size α_k. But every border strip δ/ν has odd size, so no such T exists. □

For the converse to Proposition 7.17.7, see Exercise 7.54.

NOTE (for algebraists). The coefficients $\chi^\lambda(\nu)$ for $\lambda, \nu \vdash n$ have a fundamental algebraic interpretation: They are the values of the irreducible (ordinary) characters of the symmetric group $\mathfrak{S}_n$. More precisely, the irreducible characters χ^λ of $\mathfrak{S}_n$ are indexed in a natural way by partitions $\lambda \vdash n$, and $\chi^\lambda(\nu)$ is the value of χ^λ at an element $w \in \mathfrak{S}_n$ of cycle type ν. Thus Proposition 7.17.6 is just the standard orthogonality relations satisfied by irreducible characters. Now it may be seen e.g. immediately from (7.75) that the degree (or dimension) of the character χ^λ is given by

$$\deg \chi^\lambda := \chi^\lambda(1^n) = f^\lambda. \tag{7.79}$$

Thus Corollary 7.12.6 agrees with the well-known result that for any finite group G,

$$\sum_{\chi \in \hat{G}} (\dim \chi)^2 = \#G, \tag{7.80}$$

where $\hat{G}$ is the set of irreducible characters of G. Moreover, Corollary 7.13.9 agrees with the less well-known result that

$$\sum_{\chi \in \hat{G}} \dim \chi = \#\{w \in G : w^2 = 1\}$$

if and only if every (ordinary) representation of G is equivalent to a real representation. For further information on the connections between symmetric functions and the characters of $\mathfrak{S}_n$, see the next section and many of the exercises for this chapter.

7.18 The Characters of the Symmetric Group

This section is not needed for the rest of the text (with a few minor exceptions) and assumes a basic knowledge of the representation theory of finite groups. Our goal will be to show that the functions χ^λ of the previous section (where $\chi^\lambda(\mu)$ is interpreted as $\chi^\lambda(w)$ when w is an element of $\mathfrak{S}_n$ of (cycle) type μ) are the irreducible characters of $\mathfrak{S}_n$.

Let CF^n denote the set of all class functions (i.e., functions constant on conjugacy classes) $f : \mathfrak{S}_n \to \mathbb{Q}$. Recall that CF^n has a natural scalar product defined by

$$\langle f, g \rangle = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) g(w).$$

Sometimes by abuse of notation we write $\langle \phi, \gamma \rangle$ instead of $\langle f, g \rangle$ when ϕ and γ are representations of $\mathfrak{S}_n$ with characters f and g.

NOTE. For general finite groups G, we can define $\mathrm{CF}(G)$ to be the set of all class functions $f : G \to \mathbb{C}$, and we can define the scalar product on $\mathrm{CF}(G)$ by

$$\langle f, g \rangle = \frac{1}{\#G} \sum_{w \in G} f(w) \bar{g}(w),$$

where $\bar{g}(w)$ denotes the complex conjugate of $g(w)$. Since all (complex) characters of $\mathfrak{S}_n$ turn out to be rational, it suffices to use the ground field $\mathbb{Q}$ instead of $\mathbb{C}$ when dealing with the characters of $\mathfrak{S}_n$.

Now let us recall some basic facts from the theory of permutation representations. If X is a finite set and G a finite group, then an *action* of G on X is a homomorphism $\varphi : G \to \mathfrak{S}_X$. If $s \in X$ and $w \in G$, then we write $w \cdot s$ for $\varphi(w)(s)$. The action of G on X extends to an action on $\mathbb{C}X$ (the complex vector space with basis X) by linearity. Hence φ can be regarded as a linear representation $\varphi : G \to \mathrm{GL}(\mathbb{C}X)$. The character of this representation is given by

$$\chi^{\varphi}(w) = \operatorname{tr} \varphi(w) = \#\mathrm{Fix}(w),$$

where $\mathrm{Fix}(w) = \{s \in X \,:\, w \cdot s = s\}$, the set of points fixed by w.

The action $\varphi : G \to \mathfrak{S}_X$ is *transitive* if for any $s, t \in X$ there is a $w \in G$ satisfying $w \cdot s = t$. If H is a subgroup of G, then G acts on the set G/H of left cosets of G by $w \cdot vH = wvH$. (We do not assume H is a normal subgroup, so G/H need not have the structure of a group.) Every transitive action of G is equivalent to an action on the left cosets of some subgroup H. Moreover, this action is equivalent to $\mathrm{ind}_H^G 1_H$, the *induction* from H to G of the trivial representation 1_H of H. We sometimes abbreviate this representation as 1_H^G. The well-known "Burnside's lemma" (see Lemma 7.24.5) is equivalent to the statement that

$$\left\langle 1_H^G, 1_G \right\rangle = \#\text{ of orbits of } G \text{ acting on } G/H. \tag{7.81}$$

Here $\langle 1_H^G, 1_G \rangle$ denotes the multiplicity of the trivial representation 1_G of G in 1_H^G, given more explicitly by

$$\left\langle 1_H^G, 1_G \right\rangle = \frac{1}{\#G} \sum_{w \in G} \#\mathrm{Fix}(w).$$

In the above sum $\mathrm{Fix}(w)$ refers to the action of G on the set G/H, so that $\mathrm{Fix}(w)$ is just the value of the character of this action on w.

Our present goal is to find "enough" subgroups H of $\mathfrak{S}_n$ so that we can obtain all the irreducible characters of $\mathfrak{S}_n$ as linear combinations of characters of the representations $1_H^{\mathfrak{S}_n}$. To this end, if $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{P}^\ell$ and $|\alpha| := \alpha_1 + \cdots + \alpha_\ell = n$, then define the *Young subgroup* $\mathfrak{S}_\alpha \subseteq \mathfrak{S}_n$ to be

$$\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_\ell},$$

where $\mathfrak{S}_{\alpha_1}$ permutes $1, 2, \ldots, \alpha_1$; $\mathfrak{S}_{\alpha_2}$ permutes $\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2$, etc. If α and β differ from each other only by a permutation of coordinates, then $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\beta$ are conjugate subgroups of $\mathfrak{S}_n$, and the representations $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}$ and $1_{\mathfrak{S}_\beta}^{\mathfrak{S}_n}$ are equivalent and hence have the same character. In particular, there is a unique $\lambda \vdash n$ for which $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\lambda$ are conjugate.

It is important to understand the combinatorial significance of the representations $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}$, so we will explain this topic in some detail. If we write a permutation w in the usual way as $w_1 w_2 \cdots w_n$ (so w_i is the *value* $w(i)$ of w at i), then it is easy to see that every left coset $v\mathfrak{S}_\alpha$ contains a unique permutation that is an α*-shuffle*, i.e., $1, 2, \dots, \alpha_1$ appear in increasing order, $\alpha_1 + 1, \alpha_1 + 2, \dots, \alpha_1 + \alpha_2$ appear in increasing order, etc. For instance, one of the left cosets of $\mathfrak{S}_{(2,2)}$ is given by $\{1324, 1423, 2314, 2413\}$, which contains the unique α-shuffle 1324. We can identify an α-shuffle with a permutation of the multiset $M_\alpha = \{a^{\alpha_1}, b^{\alpha_2}, \dots\}$, by replacing $1, 2, \dots, \alpha_1$ with a; $\alpha_1+1, \alpha_1+2, \dots, \alpha_1+\alpha_2$ with b; etc. $\mathfrak{S}_n$ then acts on a permutation π of M_α by permuting *positions*. For instance, $2431 \cdot baab = abab$, since the second element of $baab$ is moved to the first position, the fourth element to the second position, etc.

Alternatively, we can write a permutation $w \in \mathfrak{S}_n$ as the word $w^{-1}(1)w^{-1}(2)\cdots w^{-1}(n)$, so $w^{-1}(i)$ is the *position* of i in the word $w_1 w_2 \cdots w_n$. With this representation of permutations, every left coset of $\mathfrak{S}_\alpha$ contains a unique word $w' = w'_1 w'_2 \cdots w'_n$ such that $w'_1 < w'_2 < \cdots < w'_{\alpha_1}$, $w'_{\alpha_1+1} < w'_{\alpha_1+2} < \cdots < w'_{\alpha_1+\alpha_2}$, etc. Equivalently, the descent set $D(w')$ is contained in the set $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{\ell-1}\}$. We can also view these distinguished coset representatives as α*-flags*, i.e., chains $\emptyset = N_0 \subset N_1 \subset \cdots \subset N_\ell = [n]$ of subsets of $[n]$ such that $\#(N_i - N_{i-1}) = \alpha_i$, viz., $N_i = \{w'_1, w'_2, \dots, w'_{\alpha_1+\alpha_2+\cdots+\alpha_i}\}$. $\mathfrak{S}_n$ then acts on a flag F by permuting *elements*. For instance, if F is given by $\emptyset \subset 24 \subset 245 \subset 123456$ (so that $\alpha = (2, 1, 3)$) and if $w = 523614$, then $w \cdot F$ is given by $\emptyset \subset 26 \subset 126 \subset 123456$, since 2 and 6 are in the second and fourth position of w, 1 is in the fifth position of w, and 3, 4, 5 are in third, sixth, and fourth position of w.

Some special cases of the action of $\mathfrak{S}_n$ on α-flags should be noted. If $\alpha = (k, n-k)$, then an α-flag $\emptyset \subset N \subset [n]$ is equivalent to the k-subset N of $[n]$, and the action of $w \in \mathfrak{S}_n$ on F is equivalent to the "standard" action of $\mathfrak{S}_n$ on N that replaces $i \in N$ with $w^{-1}(i)$. We may write this equivalence as

$$\mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \cong \binom{[n]}{k}. \tag{7.82}$$

Similarly, if $\alpha = (1, 1, n-2)$ then we can identify the α-flag $\emptyset \subset \{a\} \subset \{a, b\} \subset [n]$ with the ordered pair (a, b) of distinct elements of $[n]$, so we can write

$$\mathfrak{S}_n/(\mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_{n-2}) \cong [n] \times [n] - \{(a, a) : a \in [n]\}. \tag{7.83}$$

Similar interpretations can be given for various other values of α.

Our main tool will be a linear transformation $\mathrm{ch}: \mathrm{CF}^n \to \Lambda^n$ called the *(Frobenius) characteristic map*. If $f \in \mathrm{CF}^n$, then define

$$\begin{aligned} \mathrm{ch}\, f &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) p_{\rho(w)} \\ &= \sum_\mu z_\mu^{-1} f(\mu) p_\mu, \end{aligned}$$

where $f(\mu)$ denotes $f(w)$ for any w of type $\rho(w)=\mu$. Equivalently, extending the ground field $\mathbb{Q}$ to the ring Λ and defining $\Psi(w)=p_{\rho(w)}$, we have

$$\operatorname{ch} f=\langle f,\Psi\rangle. \tag{7.84}$$

Note that if f_μ is the class function defined by

$$f_\mu(w)=\begin{cases}1 & \text{if } \rho(w)=\mu\\ 0 & \text{otherwise,}\end{cases}$$

then $\operatorname{ch} f_\mu = z_\mu^{-1}p_\mu$.

Note. Let $\varphi:\mathfrak{S}_n\to GL(V)$ be a representation of $\mathfrak{S}_n$ with character χ. Sometimes by abuse of notation we will write $\operatorname{ch}\varphi$ or $\operatorname{ch} V$ instead of $\operatorname{ch}\chi$. We also sometimes call the symmetric function $\operatorname{ch}\chi$ $(=\operatorname{ch}\varphi=\operatorname{ch} V)$ the *Frobenius image* of χ, φ, or V.

7.18.1 Proposition. *The linear transformation* ch *is an isometry, i.e.,*

$$\langle f,g\rangle_{\mathrm{CF}^n}=\langle \operatorname{ch} f,\operatorname{ch} g\rangle_{\Lambda^n}.$$

Proof. We have (using Proposition 7.9.3)

$$\begin{aligned}\langle \operatorname{ch} f,\operatorname{ch} g\rangle &= \left\langle \sum_\mu z_\lambda^{-1}f(\lambda)p_\mu,\ \sum_\mu z_\mu^{-1}g(\mu)p_\mu\right\rangle\\ &=\sum_\lambda z_\lambda^{-1}f(\lambda)g(\lambda)\\ &=\langle f,g\rangle.\end{aligned}$$ □

We now want to define a product on class functions that will correspond to the ordinary product of symmetric functions under the characteristic map ch. Let $f\in \mathrm{CF}^m$ and $g\in\mathrm{CF}^n$. Define the (pointwise) product $f\times g\in \mathrm{CF}(\mathfrak{S}_m\times\mathfrak{S}_n)$ by

$$(f\times g)(u,v)=f(u)g(v).$$

If f and g are characters of representations φ and ψ, then $f\times g$ is just the character of the tensor product representation $\varphi\otimes\psi$ of $\mathfrak{S}_m\times\mathfrak{S}_n$. Now define the *induction product* $f\circ g$ of f and g to be the induction of $f\times g$ to $\mathfrak{S}_{m+n}$, where as before $\mathfrak{S}_m$ permutes $1,2,\dots,m$ while $\mathfrak{S}_n$ permutes $m+1,m+2,\dots,m+n$. In symbols,

$$f\circ g=\operatorname{ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(f\times g).$$

Let $\mathrm{CF}=\mathrm{CF}^0\oplus\mathrm{CF}^1\oplus\cdots$, and extend the scalar product on CF^n to all of CF by setting $\langle f,g\rangle=0$ if $f\in\mathrm{CF}^m$, $g\in\mathrm{CF}^n$, and $m\neq n$. The induction product on characters extends to all of CF by (bi)linearity. It is not hard to check that this makes CF into an associative commutative graded $\mathbb{Q}$-algebra with identity $1 \in \mathrm{CF}^0$. Similarly we can extend the characteristic map ch to a linear transformation $\mathrm{ch}:\mathrm{CF} \to \Lambda$.

7.18.2 Proposition. *The characteristic map* $\mathrm{ch}:\mathrm{CF} \to \Lambda$ *is a bijective ring homomorphism, i.e.,* ch *is one-to-one and onto, and satisfies*

$$\mathrm{ch}(f \circ g) = (\mathrm{ch}\, f)(\mathrm{ch}\, g).$$

Proof. Let $\mathrm{res}_H^G f$ denote the restriction of the class function f on G to the subgroup H. We then have

$$\begin{aligned}
\mathrm{ch}(f \circ g) &= \mathrm{ch}\big(\mathrm{ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(f \times g)\big) \\
&= \big\langle \mathrm{ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(f \times g), \Psi \big\rangle \qquad \text{(by (7.84))} \\
&= \big\langle f \times g, \mathrm{res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} \Psi \big\rangle_{\mathfrak{S}_m \times \mathfrak{S}_n} \qquad \text{(by Frobenius reciprocity)} \\
&= \frac{1}{m!\,n!} \sum_{u \in \mathfrak{S}_m} \sum_{v \in \mathfrak{S}_n} f(u) g(v) \Psi(uv) \\
&= \frac{1}{m!\,n!} \sum_{u \in \mathfrak{S}_m} \sum_{v \in \mathfrak{S}_n} f(u) g(v) \Psi(u) \Psi(v) \\
&= \langle f, \Psi \rangle_{\mathfrak{S}_m} \langle g, \Psi \rangle_{\mathfrak{S}_n} \\
&= (\mathrm{ch}\, f)(\mathrm{ch}\, g).
\end{aligned}$$

Moreover, from the definition of ch and the fact that the power sums p_μ form a $\mathbb{Q}$-basis for Λ it follows that ch is bijective. □

We wish to apply Proposition 7.18.2 to evaluate the characteristic map at the character η^α of the representation $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}$ discussed above. First note that by equation (7.22) and the definition of ch we have

$$\mathrm{ch}\, 1_{\mathfrak{S}_n} = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda = h_n. \tag{7.85}$$

7.18.3 Corollary. *We have* $\mathrm{ch}\, 1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n} = h_\alpha$.

Proof. Since $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n} = 1_{\mathfrak{S}_{\alpha_1}}^{\mathfrak{S}_n} \circ 1_{\mathfrak{S}_{\alpha_2}}^{\mathfrak{S}_n} \circ \cdots \circ 1_{\mathfrak{S}_{\alpha_\ell}}^{\mathfrak{S}_n}$, the proof follows from Proposition 7.18.2 and equation (7.85). □

Now let R^n denote the set of all *virtual characters* of $\mathfrak{S}_n$, i.e., functions on $\mathfrak{S}_n$ that are the difference of two characters. Thus R^n is a lattice (discrete subgroup of maximum rank) in the vector space CF^n. The rank of R^n is $p(n)$, the number of partitions of n, and a basis consists of the irreducible characters of $\mathfrak{S}_n$. (Recall that in any finite group G, the number of linearly independent irreducible characters of G is equal to the number of conjugacy classes in G. For $\mathfrak{S}_n$ the number of conjugacy classes is $p(n)$.) This basis is the unique orthonormal basis for R^n up to sign and order, since the transition matrix between two such bases must be a integral orthogonal matrix and hence a signed permutation matrix. (See the note after the proof of Corollary 7.12.2 for similar reasoning.) Define $R = R^0 \oplus R^1 \oplus \cdots$.

7.18.4 Proposition. *The image of R under the characteristic map* ch *is* $\Lambda_{\mathbb{Z}}$. *Hence* $\mathrm{ch}: R \to \Lambda_{\mathbb{Z}}$ *is a ring isomorphism.*

Proof. It will suffice to find integer linear combinations of the characters η^α of the representations $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}$ that are the irreducible characters of $\mathfrak{S}_n$. The Jacobi–Trudi identity (Theorem 7.16.1) suggests that we define the (possibly virtual) characters $\psi^\lambda = \det\left(\eta^{\lambda_i - i + j}\right)$, where the product used in evaluating the determinant is the induction product. Then by the Jacobi–Trudi identity and Proposition 7.18.2 we have

$$\mathrm{ch}(\psi^\lambda) = s_\lambda. \tag{7.86}$$

Since ch is an isometry (Proposition 7.18.1) we get $\langle \psi^\lambda, \psi^\mu \rangle = \delta_{\lambda\mu}$. As pointed out above, this means that the class functions ψ^λ are, up to sign, the irreducible characters of $\mathfrak{S}_n$. Hence the ψ^λ for $\lambda \vdash n$ form a $\mathbb{Z}$-basis for R^n, and the image of R^n is the $\mathbb{Z}$-span of the s_λ's, which is just Λ^n as claimed. □

Finally we come to the main result of this section.

7.18.5 Theorem. *Regard the functions* χ^λ *(where* $\lambda \vdash n$*) of Section 7.17 as functions on* $\mathfrak{S}_n$ *given by* $\chi^\lambda(w) = \chi^\lambda(\mu)$, *where* w *has cycle type* μ. *Then the* χ^λ*'s are the irreducible characters of the symmetric group* $\mathfrak{S}_n$.

Proof. By the Murnaghan–Nakayama rule (Corollary 7.17.5), we have

$$\mathrm{ch}\,(\chi^\lambda) = \sum_\mu z_\mu^{-1} \chi^\lambda(\mu) p_\mu = s_\lambda.$$

Hence by equation (7.86), we get $\chi^\lambda = \psi^\lambda$. Since the ψ^λ's, up to sign, are the irreducible characters of $\mathfrak{S}_n$, it remains only to determine whether χ^λ or $-\chi^\lambda$ is a character. But $\chi^\lambda(1^n) = f^\lambda > 0$, so χ^λ is an irreducible character. □

Note. We have described a natural way to index the irreducible characters of $\mathfrak{S}_n$ by partitions of n, while the cycle type of a permutation defines a natural indexing of the conjugacy classes of $\mathfrak{S}_n$ by partitions of n. Hence we have a canonical bijection between the conjugacy classes and the irreducible characters of $\mathfrak{S}_n$. However, this bijection is essentially "accidental" and does not have any useful properties. For arbitrary finite groups there is in general no canonical bijection between irreducible characters and conjugacy classes.

7.18.6 Corollary. *Let $\mu \vdash m$, $\nu \vdash n$, $\lambda \vdash m+n$. Then the Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ of equation (7.64) is nonnegative*

Proof. By Propositions 7.18.1 and 7.18.2 we have

$$c^{\lambda}_{\mu\nu} = \langle s_\lambda, s_\mu s_\nu \rangle = \langle \chi^\lambda, \chi^\mu \circ \chi^\nu \rangle.$$

Since by Theorem 7.18.5 χ^μ and χ^ν are characters of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, respectively, it follows from the basic theory of induced characters that $\chi^\mu \circ \chi^\nu$ is a character of $\mathfrak{S}_{m+n}$. Hence $\langle \chi^\lambda, \chi^\mu \circ \chi^\nu \rangle \geq 0$. □

Combinatorial descriptions of the numbers $c^{\nu}_{\lambda\mu}$ are given in Appendix 1, Section A1.3. A primary use of Theorem 7.18.5 is the following. Let χ be any character (or even a virtual character) of $\mathfrak{S}_n$. Theorem 7.18.5 shows that the problem of decomposing χ into irreducibles is equivalent to expanding ch χ into Schur functions. Thus all the symmetric function machinery we have developed can be brought to bear on the problem of decomposing χ. An important example is given by the characters η^α of the representations $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}$. The result that expresses η^α in terms of irreducibles is known as *Young's rule*.

7.18.7 Proposition. *Let α be a composition of n and $\lambda \vdash n$. Then the multiplicity of the irreducible character χ^λ in the character η^α is just the Kostka number $K_{\lambda\alpha}$. In symbols,*

$$\langle \eta^\alpha, \chi^\lambda \rangle = K_{\lambda\alpha}.$$

Proof. By Corollary 7.18.3 we have ch $\eta^\alpha = h_\alpha$. The proof then follows from Corollary 7.12.4 and Theorem 7.18.5. □

7.18.8 Example. (a) Let χ denote the character of the action of $\mathfrak{S}_n$ on the k-subsets of $[n]$ (by permuting elements of $[n]$). By equation (7.82), we have ch $\chi = h_k h_{n-k}$. Assume without loss of generality that $k \leq n/2$ (since the action on k-element subsets is equivalent to the action on $(n-k)$-element subsets). The multiplicity of χ^λ in χ is the number of SSYTs of shape λ and type $1^{n-k}2^k$. There is one such SSYT if $\lambda = (n-m, m)$ with $0 \leq m \leq k$, and no SSYTs otherwise. Hence

$$\chi = \chi^{(n)} + \chi^{(n-1,1)} + \cdots + \chi^{(n-k,k)}.$$

Note the special case $k = 1$; this corresponds to the "defining representation" of $\mathfrak{S}_n$ (the action of $\mathfrak{S}_n$ on $[n]$), with character $\chi^{(n)} + \chi^{(n-1,1)}$.

(b) Let χ denote the character of the action of $\mathfrak{S}_n$ on ordered pairs (i, j) of distinct elements of $[n]$. By equation (7.83), we have $\operatorname{ch}\chi = h_1^2 h_{n-2}$. There are five kinds of SSYT of type $(n-2, 1, 1)$, viz.,

$$\begin{array}{lllll} 1\cdots 123 & 1\cdots 12 & 1\cdots 13 & 11\cdots 1 & 1\cdots 1 \\ & 3 & 2 & 23 & 2 \\ & & & & 3 \end{array}$$

Hence

$$\chi = \chi^{(n)} + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \chi^{(n-2,1,1)}.$$

(c) The *regular representation* of $\mathfrak{S}_n$ is defined to be the action of $\mathfrak{S}_n$ on itself by left multiplication. Hence it is given by $1_{\mathfrak{S}_1\times\cdots\times\mathfrak{S}_1}^{\mathfrak{S}_n}$, whose Frobenius image is h_1^n (by Corollary 7.18.3). By Corollary 7.12.5 we have $h_1^n = \sum_{\lambda\vdash n} f^\lambda s_\lambda$. Hence the multiplicity in the regular representation of the irreducible representation of $\mathfrak{S}_n$ whose character is χ^λ is just $f^\lambda = \chi^\lambda(1^n)$. Thus Corollary 7.12.5 is a symmetric function statement for $\mathfrak{S}_n$ of the fact that the multiplicity of an irreducible representation of a finite group in the regular representation is equal to its degree.

7.19 Quasisymmetric Functions

We have succeeded in expanding the Schur functions in terms of the four bases m_λ, h_λ, e_λ, and p_λ. We have also given a formula for s_λ (in n variables) as a quotient of determinants. There is one further expression for s_λ which has many combinatorial ramifications. We will write s_λ in terms of a basis not of the space Λ, but of a larger space $\mathcal{Q}$. The theory of P-partitions, as discussed in Section 4.5, will be generalized in order to obtain this expansion of s_λ (and more generally of $s_{\lambda/\mu}$).

A *quasisymmetric function* in the variables $x_1, x_2, \ldots$, say with rational coefficients, is a formal power series $f = f(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ of bounded degree such that for any $a_1, \ldots, a_k \in \mathbb{P}$ we have

$$\left[x_{i_1}^{a_1}\cdots x_{i_k}^{a_k}\right] f = \left[x_{j_1}^{a_1}\cdots x_{j_k}^{a_k}\right] f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Clearly every symmetric function is quasisymmetric, but not conversely. For instance, the series $\sum_{i<j} x_i^2 x_j$ is quasisymmetric but not symmetric.

Let $\mathcal{Q}^n$ denote the set of all homogeneous quasisymmetric functions of degree n. It is clear that $\mathcal{Q}^n$ is a vector space (over $\mathbb{Q}$). We will be indexing certain elements of $\mathcal{Q}^n$ by compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n. We will often be using the natural one-to-one correspondence between compositions α of n and subsets S of $[n-1]$. Thus we associate the set $S_\alpha = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\}$ with the composition α, and the composition $\operatorname{co}(S) = (s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_{k-1})$ with the set $S = \{s_1, s_2, \ldots, s_{k-1}\}_<$. It is clear that $\operatorname{co}(S_\alpha) = \alpha$ and $S_{\operatorname{co}(S)} = S$. We extend the definition of co to permutations $w \in \mathfrak{S}_n$ by defining $\operatorname{co}(w) = \operatorname{co}(D(w))$, where $D(w)$ denotes the descent set of w.

Let Comp(n) denote the set of compositions of n, so #Comp(n) $= 2^{n-1}$. Given $\alpha \in \mathrm{Comp}(n)$, define the *monomial quasisymmetric function* M_α by

$$M_\alpha = \sum_{i_1<\cdots<i_k} x_{i_1}^{\alpha_1}\cdots x_{i_k}^{\alpha_k}. \tag{7.87}$$

If $f \in \mathcal{Q}^n$ then it is clear that

$$f = \sum_{\alpha\in\mathrm{Comp}(n)} \left(\left[x_1^{\alpha_1}\cdots x_k^{\alpha_k}\right]f\right)M_\alpha. \tag{7.88}$$

From (7.88) it follows that the set $\{M_\alpha : \alpha \in \mathrm{Comp}(n)\}$ is a basis for $\mathcal{Q}^n$. In particular,

$$\dim \mathcal{Q}^n = 2^{n-1}.$$

It is an easy exercise to see that if $f \in \mathcal{Q}^m$ and $g \in \mathcal{Q}^n$, then $fg \in \mathcal{Q}^{m+n}$. (See Exercise 7.93 for a more precise result.) Hence if $\mathcal{Q} = \mathcal{Q}^0 \oplus \mathcal{Q}^1 \oplus \cdots$, then $\mathcal{Q}$ is a $\mathbb{Q}$-algebra, called the *algebra* (or *ring*) *of quasisymmetric functions* (over $\mathbb{Q}$).

We will now consider another important basis for $\mathcal{Q}^n$. Given $\alpha \in \mathrm{Comp}(n)$, define

$$L_\alpha = \sum_{\substack{i_1\leq\cdots\leq i_n\\ i_j<i_{j+1} \text{ if } j\in S_\alpha}} x_{i_1}x_{i_2}\cdots x_{i_n}. \tag{7.89}$$

It is clear that $L_\alpha \in \mathcal{Q}^n$. We call L_α a *fundamental quasisymmetric function.*

7.19.1 Proposition. *For $\alpha \in \mathrm{Comp}(n)$ we have*

$$L_\alpha = \sum_{S_\alpha\subseteq T\subseteq[n-1]} M_{\mathrm{co}(T)} \tag{7.90}$$

$$M_\alpha = \sum_{S_\alpha\subseteq T\subseteq[n-1]} (-1)^{\#(T-S_\alpha)} L_{\mathrm{co}(T)}. \tag{7.91}$$

Hence the set $\{L_\alpha : \alpha \in \mathrm{Comp}(n)\}$ is a basis for $\mathcal{Q}^n$.

Proof. Equation (7.90) is an immediate consequence of (7.89), on grouping the sequences $i_1 \leq \cdots \leq i_n$ according to whether $i_j < i_{j+1}$ or $i_j = i_{j+1}$ for each j. Equation (7.91) then follows from (7.90) by the Principle of Inclusion–Exclusion (Theorem 2.1.1). □

It is natural to ask under what conditions is a quasisymmetric function actually symmetric.

7.19.2 Proposition. *Let $f \in \mathcal{Q}^n$, say $f = \sum_{\alpha\in\mathrm{Comp}(n)} c_\alpha M_\alpha$. Then $f \in \Lambda^n$ if and only if for any two compositions α and β of n that have the same multiset of parts, we have $c_\alpha = c_\beta$.*

Proof. Let $\mathcal{R}^n$ be the subspace of $\mathcal{Q}^n$ consisting of all $f = \sum c_\alpha M_\alpha$ satisfying the conditions of the proposition. Given $\lambda \vdash n$, define $R_\lambda = \sum_\alpha M_\alpha$, summed over all distinct permutations of the parts of λ. It is clear that $\{R_\lambda : \lambda \vdash n\}$ is a basis for $\mathcal{R}^n$, so $\dim \mathcal{R}^n = p(n)$, the number of partitions of n. On the other hand, it is also evident that $R_\lambda = m_\lambda \in \Lambda^n$, so $\mathcal{R}^n \subseteq \Lambda^n$. Since $\dim \Lambda^n = p(n)$, the proof follows. □

Let us now consider the connection between quasisymmetric functions and the theory of P-partitions. If X is any finite set and $f : X \to \mathbb{P}$, then define

$$x^f = \prod_{t \in X} x_{f(t)} = \prod_{i \geq 1} x_i^{\# f^{-1}(i)}.$$

In Definition 4.5.1 we defined the notion of a π-compatible function $f : [n] \to C$, where $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ and C is a chain. It is more convenient here to work with the "reverse" notion. We say that f is *reverse π-compatible* if

$$\begin{aligned} &f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n) \\ &f(\pi_i) < f(\pi_{i+1}) \qquad \text{if } \pi_i > \pi_{i+1}. \end{aligned}$$

Clearly Lemma 4.5.1 carries over with "compatible" replaced with "reverse compatible." Thus every $f : [n] \to C$ is reverse π-compatible for a *unique* $\pi \in \mathfrak{S}_n$.

7.19.3 Lemma. *Let $\pi \in \mathfrak{S}_n$, and let S_π^r denote the set of all reverse π-compatible functions $f : [n] \to \mathbb{N}$. Then*

$$\sum_{f \in S_\pi^r} x^f = L_{\mathrm{co}(\pi)}(x).$$

Proof. Immediate from a comparison of the definition (7.89) of L_α and the definition of reverse π-compatible. □

Now let P be a finite poset of cardinality n. To be consistent with our treatment of SSYTs, we will deal with functions $\sigma : P \to \mathbb{P}$ rather than $\sigma : P \to \mathbb{N}$ as in Section 4.5. It is a trivial matter to modify the theory to allow also $\sigma(t) = 0$. A *reverse P-partition* is an order-preserving map $\sigma : P \to \mathbb{P}$ (and so is equivalent to an ordinary P^*-partition, where P^* denotes the dual of P). Let $\mathcal{A}^r(P)$ denote the set of reverse P-partitions. Define

$$K_P(x) = \sum_{\sigma \in \mathcal{A}^r(P)} x^\sigma. \tag{7.92}$$

Note that $K_P(x)$ is a quasisymmetric function that tells us for each weak composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n the number of reverse P-partitions with α_i parts equal to i. As in Section 4.5, regard P as a natural partial order on $[n]$, and let

$\mathcal{L}(P) \subseteq \mathfrak{S}_n$ be the Jordan–Hölder set of P. The fundamental decomposition $\mathcal{A}(P) = \dot{\bigcup}_{\pi\in\mathcal{L}(P)} S_\pi$ clearly works just as well in the reverse situation:

$$\mathcal{A}^r(P) = \dot{\bigcup_{\pi\in\mathcal{L}(P)}} S^r_\pi. \tag{7.93}$$

Hence, letting $J(P)$ denote the lattice of order ideals of P as in Chapter 3, we have

$$\begin{aligned} K_P(x) &= \sum_{\pi\in\mathcal{L}(P)} L_{\mathrm{co}(\pi)}(x) \qquad (7.94)\\ &= \sum_{S\subseteq[n-1]} \beta(J(P), S) L_{\alpha_S}(x), \end{aligned}$$

the latter equality by Theorem 3.12.1. This result shows that the expansion of a quasisymmetric function in terms of the L_α's can be of combinatorial significance, and that the coefficients may be regarded as an analogue of the numbers $\beta(J(P), S)$.

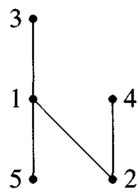
To apply the kind of reasoning of the previous paragraph to Schur functions, we need an extension of the theory of P-partitions to the case where P need not be a natural partial order. Define a *labeling* of P to be a bijection $\omega : P \to [n]$. (Do not confuse the labeling ω with the involution $\omega : \Lambda \to \Lambda$.) Alternatively, one could think of P as a partial ordering on $[n]$ by identifying $t \in P$ with $\omega(t)$. It is convenient, however, to work with labelings and so avoid having to deal with two different orderings on $[n]$.

A *reverse* (P, ω)*-partition* is a map $\sigma : P \to \mathbb{N}$ satisfying the two axioms:

(R1) If $s \le t$ in P then $\sigma(s) \le \sigma(t)$. In other words, σ is *order-preserving*.
(R2) If $s < t$ in P and $\omega(s) > \omega(t)$, then $\sigma(s) < \sigma(t)$.

Thus a reverse (P, ω)-partition is just a reverse P-partition with additional conditions specified by ω as to when *strict* inequality $\sigma(s) < \sigma(t)$ must occur. If, for instance, ω is order-preserving, then a reverse (P, ω)-partition is just a reverse P-partition. On the other hand, if ω is order-reversing, then a reverse (P, ω)-partition is just a strict reverse P-partition.

Let $\mathcal{L}(P, \omega)$ denote the set of all permutations $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ such that the map $w : P \to [n]$ defined by $w(\omega^{-1}(\pi_i)) = i$ is a linear extension of P. Thus $\mathcal{L}(P, \omega)$ may be regarded as the set of linear extensions of P, regarded as permutations of the labels of P. For instance, if (P, ω) is given by



then $\mathcal{L}(P, \omega) = \{52143, 52413, 25143, 25413, 24513, 52134, 25134\}$.

The following result is the "fundamental theorem of (reverse) (P, ω)-partitions." The proof is exactly like that of the special case Lemma 4.5.3 (in its "reverse form") and will be omitted.

7.19.4 Theorem. *Let $\mathcal{A}^r(P, \omega)$ denote the set of all reverse (P, ω)-partitions. Then*

$$\mathcal{A}^r(P, \omega) = \dot{\bigcup_{\pi \in \mathcal{L}(P,\omega)}} S_\pi^r \qquad \text{(disjoint union).}$$

In exact analogy to the definition (7.92) of $K_P(x)$, let

$$K_{P,\omega}(x) = \sum_{\sigma \in \mathcal{A}^r(P,\omega)} x^\sigma.$$

Note that $K_{P,\omega}(x)$ is quasisymmetric. Just as (7.94) follows from (7.93), we obtain:

7.19.5 Corollary. *We have*

$$K_{P,\omega} = \sum_{\pi \in \mathcal{L}(P,\omega)} L_{\mathrm{co}(\pi)}, \tag{7.95}$$

the expansion of $K_{P,\omega}$ in terms of the fundamental quasisymmetric functions.

We now want to apply Corollary 7.19.5 to the skew Schur functions $s_{\lambda/\mu}$, where $|\lambda/\mu| = n$. Define $P_{\lambda/\mu}$ to be the poset whose elements are the squares (i, j) of λ/μ, partially ordered componentwise. Thus $P_{\lambda/\mu}$ is a finite convex subset of $\mathbb{P} \times \mathbb{P}$, and every finite convex subset of $\mathbb{P} \times \mathbb{P}$ is equal to $P_{\lambda/\mu}$ for some λ/μ. (The case $\mu = \emptyset$ was discussed already in Proposition 7.10.3(b).) Define a labeling $\omega_{\lambda/\mu} : P_{\lambda/\mu} \to [n]$, called the *Schur labeling* of $P_{\lambda/\mu}$, as follows: The bottom square of the first column of $P_{\lambda/\mu}$ is labeled 1. The labeling then proceeds up the first column, then up the second column, etc. For instance, Figure 7-8 shows the Schur labeling of the skew shape $5331/2$, drawn both as a Young diagram and as a labeled poset.

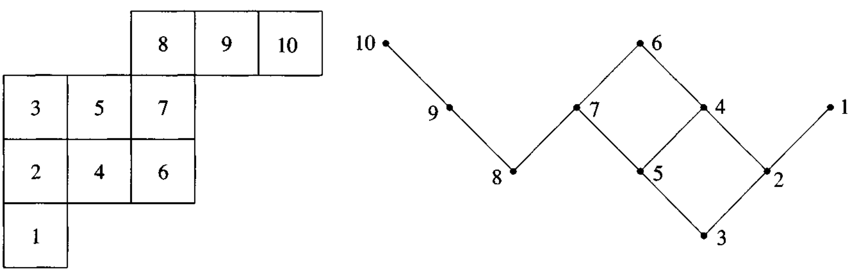


Figure 7-8. A Young diagram and the corresponding Schur labeled poset.

It is immediate from the definition of $K_{P,\omega}$ that a reverse $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$-partition is just an SSYT, so

$$K_{P_{\lambda/\mu},\omega_{\lambda/\mu}} = s_{\lambda/\mu}.$$

Hence as a special case of Corollary 7.19.5 we obtain the expansion of $s_{\lambda/\mu}$ in terms of fundamental quasisymmetric functions. Rather than describe this expansion directly in terms of $\mathcal{L}(P_{\lambda/\mu}, \omega_{\lambda/\mu})$ as in Corollary 7.19.5, we want a description in terms of SYTs of shape λ/μ.

A linear extension $w : P_{\lambda/\mu} \to [n]$ corresponds to an SYT T_w of shape λ/μ. Similarly w corresponds to a permutation $\pi_w \in \mathcal{L}(P_{\lambda/\mu}, \omega_{\lambda/\mu})$. Define a *descent* of an SYT T to be an integer i such that $i+1$ appears in a lower row of T than i, and define the *descent set* $D(T)$ to be the set of all descents of T. For instance, the SYT

$$\begin{array}{ccccc} & & & 2 & 8 \\ & 1 & 4 & 5 & 10 \\ 3 & 6 & 9 & & \\ 7 & & & & \end{array}$$

has descent set $\{2, 5, 6, 8\}$. We write $\mathrm{co}(T)$ for the composition $\mathrm{co}(D(T))$ of n associated with the descent set $D(T)$.

7.19.6 Lemma. *Let $w : P_{\lambda/\mu} \to [n]$ be a linear extension. Then $D(T_w) = D(\pi_w)$.*

Proof. Let $1 \le i \le n-1$. Let $s = (a, b)$ be the square of T_w containing i. The square $s' = (a', b')$ containing $i+1$ satisfies either (a) $a' \le a$ and $b' > b$, or (b) $a' > a$ and $b' \le b$. In the former case, $i \notin D(T_w)$ and $\omega(s') > \omega(s)$, where $\omega = \omega_{\lambda/\mu}$. Hence also $i \notin D(\pi_w)$. In the latter case, $i \in D(T_w)$ and $\omega(s') < \omega(s)$. Hence also $i \in D(\pi_w)$, and the proof follows. □

Combining Corollary 7.19.5 and Lemma 7.19.6 gives the main result of this section.

7.19.7 Theorem. *We have*

$$s_{\lambda/\mu} = \sum_T L_{\mathrm{co}(T)},$$

where T ranges over all SYTs of shape λ/μ.

7.19.8 Example. Let $\lambda/\mu = 32/1$. The five SYTs of shape λ/μ, with their descents shown in boldface, are:

$$\begin{array}{ccc} & \mathbf{1} & 4 \\ 2 & 3 & \end{array} \qquad \begin{array}{ccc} & 1 & \mathbf{2} \\ 3 & 4 & \end{array} \qquad \begin{array}{ccc} & \mathbf{2} & 4 \\ 1 & 3 & \end{array} \qquad \begin{array}{ccc} & 2 & \mathbf{3} \\ 1 & 4 & \end{array} \qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{3} \\ 2 & 4 & \end{array}$$

Hence

$$s_{32/1} = L_{13} + 2L_{22} + L_{31} + L_{121}.$$

As an illustration of the use of Theorem 7.19.7, we have the following somewhat surprising combinatorial result. See Exercise 7.90(b) for a more direct proof.

7.19.9 Proposition. *Let $|\lambda/\mu| = n$. For any $1 \le i \le n-1$, the number $d_i(\lambda/\mu)$ of SYTs T of shape λ/μ for which $i \in D(T)$ is independent of i.*

Proof. Define a linear transformation $\varphi_i : \mathcal{Q}^n \to \mathbb{Q}$ by

$$\varphi_i(L_\alpha) = \begin{cases} 1 & i \in S_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 7.19.7 we have $d_i(\lambda/\mu) = \varphi_i(s_{\lambda/\mu})$. Hence it suffices to prove that $\varphi_i(b_\nu)$ is independent of i for some basis $\{b_\nu\}$ of Λ^n, for then $\varphi_i(f)$ will be independent of i for all $f \in \Lambda^n$, including $f = s_{\lambda/\mu}$. We choose $b_\nu = m_\nu$.

By definition of M_α we have

$$m_\nu = \sum_\alpha M_\alpha,$$

where α ranges over all distinct permutations of the parts of ν. If $S \subseteq [n-1]$ and $\#S \le n-3$, then every $i \in [n-1]$ appears in the same number of sets T of even cardinality satisfying $S \subseteq T \subseteq [n-1]$ as of odd cardinality. It follows from (7.91) that $\varphi_i(M_\alpha) = 0$ if $\ell(\alpha) \le n-2$ (i.e., $\#S_\alpha \le n-3$), so $\varphi_i(m_\nu) = 0$ (independent of i) unless possibly $\nu = \langle 1^n \rangle$ or $\nu = \langle 21^{n-2} \rangle$.

If $\nu = \langle 1^n \rangle$, then $m_{1^n} = L_{1,1,\dots,1}$, so $\varphi_i(m_{1^n}) = 1$ (independent of i). If $\nu = \langle 21^{n-2} \rangle$, then by (7.91) we have

$$m_{21^{n-2}} = \sum_{j=1}^{n-1} (L_{1^{j-1},2,1^{n-j-1}} - L_{1,1,\dots,1}).$$

It follows that $\varphi_i(m_{21^{n-2}}) = -1$ (independent of i), and the proof follows. □

If we write $s_{\lambda/\mu} = \sum_\nu K_{\lambda/\mu,\nu} m_\nu$ and apply φ_i, then the above proof shows that

$$\begin{aligned} \varphi_i(s_{\lambda/\mu}) &= K_{\lambda/\mu,1^n} - K_{\lambda/\mu,21^{n-2}} \\ &= f^{\lambda/\mu} - K_{\lambda/\mu,21^{n-2}}. \end{aligned}$$

It is easy to see that when $\mu = \emptyset$ this quantity is equal to $f^{\lambda/11}$. Alternatively, it is clear that $\varphi_1(s_\lambda) = f^{\lambda/11}$, since if T is an SYT of shape λ and $1 \in D(T)$, then T has a 1 in the (1, 1) square and a 2 in the (2, 1) square. Hence, given that $\varphi_i(s_\lambda)$ is independent of i, there follows $\varphi_i(s_\lambda) = f^{\lambda/11}$ as before.

As another and more significant application of Theorem 7.19.7, we give a formula for the stable principal specialization $s_{\lambda/\mu}(1, q, q^2, \ldots)$.

7.19.10 Lemma. *Let* $\alpha \in \mathrm{Comp}(n)$. *Then*

$$L_\alpha(1, q, q^2, \ldots) = \frac{q^{e(\alpha)}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

where $e(\alpha) = \sum_{i \in S_\alpha}(n - i)$.

Proof. By (7.89) we have

$$L_\alpha(1, q, q^2, \ldots) = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} q^{i_1+i_2+\cdots+i_n-n}.$$

Define

$$r_j = i_j - 1 - \#\{m \in S_\alpha : m < j\}.$$

Then

$$L_\alpha(1, q, q^2, \ldots) = q^{i(\alpha)} \sum_{0 \le r_1 \le \cdots \le r_n} q^{r_1+r_2+\cdots+r_n},$$

where

$$i(\alpha) = \sum_{j=1}^{n} \#\{m \in S_\alpha : m < j\}.$$

It is easy to see that $i(\alpha) = e(\alpha)$, and the proof follows. □

For any SYT T define the *major index* * $\mathrm{maj}(T)$ by

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

7.19.11 Proposition. *Let* $|\lambda/\mu| = n$. *Then*

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \frac{\sum_T q^{\mathrm{maj}(T)}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

where T ranges over all SYTs of shape λ/μ.

Proof. Combining Theorem 7.19.7 and Lemma 7.19.10 yields

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \frac{\sum_T q^{\mathrm{comaj}(T)}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

* In Section 4.5 we used the notation $\iota(\pi)$ rather than $\mathrm{maj}(\pi)$ for the analogous concept of the major index (or greater index) of a permutation π.

where $\text{comaj}(T) = e(D(T)) = \sum_{i \in D(T)}(n-i)$, the *comajor index* of T. Let $s_{\lambda/\mu} = \sum_\alpha c_\alpha L_\alpha$. By Proposition 7.19.2 we see that $c_\alpha = c_{\alpha^*}$, where if $\alpha = (\alpha_1, \ldots, \alpha_k)$ then $\alpha^* = (\alpha_k, \ldots, \alpha_1)$. Hence

$$\sum_T q^{\text{comaj}(T)} = \sum_T q^{\text{maj}(T)},$$

summed over all SYT of shape λ/μ, and the proof follows. □

Suppose that λ/μ is a disjoint union of squares, e.g., 4321/321 is a disjoint union of four squares. Then the SYTs T of shape λ/μ correspond in a natural way to permutations $\pi \in \mathfrak{S}_n$ such that $D(T) = D(\pi)$. Clearly $s_{\lambda/\mu}(1, q, q^2, \ldots) = (1-q)^{-n}$, so Proposition 7.19.11 reduces to Corollary 4.5.9, viz.,

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

In Corollary 7.21.3 we will evaluate $s_\lambda(1, q, q^2, \ldots)$ explicitly, thereby yielding an explicit formula (Corollary 7.21.5) for $\sum_T q^{\text{maj}(T)}$, summed over all SYTs of shape λ. Another formula for $s_{\lambda/\mu}(1, q, q^2, \ldots)$, similar to Proposition 7.19.11 but with a different denominator, is given by Exercise 7.102(b).

As a variant of Proposition 7.19.11, we can find the *number* of descents of T, rather than the sum of the descents. Let $d(T) = \#D(T)$. For a power series $f(x_1, x_2, \ldots)$ write $f(1^m) = f(x_1 = \cdots = x_m = 1,\ x_{m+1} = x_{m+2} = \cdots = 0)$, as in equation (7.8). For $\alpha \in \text{Comp}(n)$ it is easy to see that

$$L_\alpha(1^m) = \left(\!\!\binom{m - \#S_\alpha}{n}\!\!\right) = \binom{m - \#S_\alpha + n - 1}{n}.$$

It now follows immediately from Theorem 7.19.7 (analogously to Theorem 4.5.14) that if $|\lambda/\mu| = n$ then

$$\sum_{m \geq 0} s_{\lambda/\mu}(1^m) t^m = \frac{\sum_T t^{d(T)+1}}{(1-t)^{n+1}}. \tag{7.96}$$

For instance, if λ is the "hook" $k1^{n-k}$, then all $\binom{n-1}{k-1}$ SYTs of shape λ have $n-k$ descents, from which there follows

$$s_{k1^{n-k}}(1^m) = \binom{n-1}{k-1}\binom{m+k-1}{n}.$$

Similarly we can take into account *both* $d(T)$ and $\text{maj}(T)$. We merely state the resulting formula, whose proof is analogous to that of Exercise 4.24(b).

7.19.12 Proposition. *We have*

$$s_{\lambda/\mu}(1, q, \ldots, q^{m-1}) = \sum_T \binom{m - d(T) + n - 1}{n} q^{\text{maj}(T)},$$

where T ranges over all SYTs of shape λ/μ.

An explicit formula for $s_\lambda(1, q, \ldots, q^{m-1})$ is given by Theorem 7.21.2.

7.20 Plane Partitions and the RSK Algorithm

We have now developed enough of the theory of symmetric functions that we can give a number of enumerative applications. This section and the next two will be devoted to a fascinating generalization of partitions of integers known as "plane partitions." A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition *of* n. When writing examples of plane partitions, the 0's (or all but finitely many 0's) are suppressed. Thus the plane partitions of integers $0 \leq n \leq 3$ are given by

$$\begin{array}{ccccccccccc} \emptyset & 1 & 2 & 11 & 1 & 3 & 21 & 111 & 11 & 2 & 1 \\ & & & & 1 & & & & 1 & 1 & 1 \\ & & & & & & & & & & 1. \end{array}$$

An ordinary partition $\lambda \vdash n$ may be regarded as a weakly decreasing *one-dimensional* array $(\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with finite support. Thus plane partitions are a natural generalization to two dimensions of ordinary partitions. It now seems obvious to define *r-dimensional partitions* for any $r \geq 1$. However, almost nothing significant is known for $r \geq 3$. Plane partitions have obvious similarities with semistandard tableaux. Indeed, a reverse SSYT is just a special kind of plane partition (with the irrelevant 0's removed), and in fact in our definition of reverse SSYT we mentioned the alternative term "column-strict plane partition." Because of the similarity between SSYTs and plane partitions, it is not surprising that symmetric functions play an important role in the enumeration of plane partitions.

A *part* of a plane partition $\pi = (\pi_{ij})$ is a positive entry $\pi_{ij} > 0$. The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the i-th row (so $\pi_{i\lambda_i} > 0$, $\pi_{i,\lambda_i+1} = 0$). We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda') = \lambda_1$. Write $\ell_1(\pi)$ for the number of rows and $\ell_2(\pi)$ for the number of columns of π. Finally, define the *trace* of π by the usual formula $\mathrm{tr}(\pi) = \sum \pi_{ii}$. For example, the plane partition

$$\begin{array}{cccccccc} 7 & 5 & 5 & 3 & 2 & 1 & 1 & 1 \\ 6 & 5 & 5 & 2 & 1 & 1 \\ 6 & 3 & 2 & 2 \end{array}$$

has shape $(8, 6, 4)$, 18 parts, 3 rows, 8 columns, and trace 14.

Let $\mathcal{P}(r, c)$ be the set of all plane partitions with at most r rows and at most c columns. For instance, if $\pi \in \mathcal{P}(1, c)$, then π is just an ordinary partition and $\mathrm{tr}(\pi)$ is the largest part of π. It is then clear by "inspection" (looking at the conjugate partition π' instead of π) that

$$\sum_{\pi \in \mathcal{P}(1,c)} q^{\operatorname{tr}(\pi)} x^{|\pi|} = \frac{1}{(1-qx)(1-qx^2)\cdots(1-qx^c)}. \tag{7.97}$$

The main result of this section is the following generalization of equation (7.97).

7.20.1 Theorem. *Fix $r, c \in \mathbb{P}$. Then*

$$\sum_{\pi \in \mathcal{P}(r,c)} q^{\operatorname{tr}(\pi)} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} (1 - qx^{i+j-1})^{-1}.$$

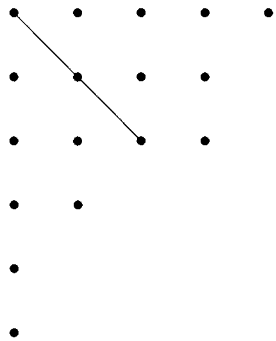
Proof. We will give an elegant bijective proof based on the RSK algorithm and a simple method of merging a pair of reverse SSYT of the same shape into a single plane partition. First we describe how to merge two partitions λ and μ with distinct parts and with the same number of parts into a single partition $\rho = \rho(\lambda, \mu)$. Draw the Ferrers diagram of λ but with each row indented one space to the right of the beginning of the previous row. Such a diagram is called the *shifted* Ferrers diagram of λ. For instance, if $\lambda = (5, 3, 2)$ then we get the shifted diagram

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \\
 & & \bullet & \bullet &
\end{array}$$

Do the same for μ, and then transpose the diagram. For instance, if $\mu = (6, 3, 1)$ then we get the transposed shifted diagram

$$\begin{array}{ccc}
\bullet & & \\
\bullet & \bullet & \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \\
\bullet & & \\
\bullet & &
\end{array}$$

Now merge the two diagrams into a single diagram by identifying their main diagonals. For λ and μ as above, we get the diagram (with the main diagonal drawn for clarity)



Define $\rho(\lambda, \mu)$ to be the partition for which this merged diagram is the Ferrers diagram. The above example shows that

$$\rho(532, 631) = 544211.$$

The map $(\lambda, \mu) \mapsto \rho(\lambda, \mu)$ is clearly a bijection between pairs of partitions (λ, μ) with k distinct parts, and partitions ρ of rank k (as defined in Section 7.2). Note that

$$|\rho| = |\lambda| + |\mu| - \ell(\lambda).$$

We now extend the above bijection to pairs (P, Q) of reverse SSYTs of the same shape. If λ^i denotes the i-th column of P and μ^i the i-th column of Q, then let $\pi(P, Q)$ be the array whose i-th column is $\rho(\lambda^i, \mu^i)$. For instance, if

$$P = \begin{array}{llll} 4 & 4 & 2 & 1 \\ 3 & 1 & 1 & \\ 2 & & & \end{array} \quad \text{and} \quad Q = \begin{array}{llll} 5 & 3 & 2 & 2 \\ 4 & 2 & 1 & \\ 1 & & & \end{array} ,$$

then

$$\pi(P, Q) = \begin{array}{llll} 4 & 4 & 2 & 1 \\ 4 & 2 & 2 & 1 \\ 4 & 2 & & \\ 2 & & & \\ 2 & & & \end{array} .$$

It is easy to see that $\pi(P, Q)$ is a plane partition. Replace each row of $\pi(P, Q)$ by its conjugate to obtain another plane partition $\pi'(P, Q)$. With $\pi(P, Q)$ as above we obtain

$$\pi'(P,Q) = \begin{matrix} 4\,3\,2\,2 \\ 4\,3\,1\,1 \\ 2\,2\,1\,1 \\ 1\,1 \\ 1\,1 \end{matrix}.$$

One can easily check that the map $(P, Q) \mapsto \pi'(P, Q)$ is a bijection between pairs (P, Q) of reverse SSYTs of the same shape and plane partitions π'. Write $\mathrm{diag}(\pi')$ for the main diagonal $(\pi'_{11}, \pi'_{22}, \ldots)$ of π', $\max(P)$ for the largest part P_{11} of the reverse SSYT P, etc. Recall that $\mathrm{sh}(P)$ denotes the shape of P, so $\mathrm{sh}(P) = \mathrm{sh}(Q)$, with P, Q as above. It is easy to see that

$$\begin{gathered} |\pi'| = |P| + |Q| - |\mathrm{sh}(P)| \\ \mathrm{diag}(\pi') = \mathrm{sh}(P) = \mathrm{sh}(Q), \qquad \text{so } \mathrm{tr}(\pi') = |\mathrm{sh}(P)| \\ \ell_1(\pi') = \max(Q) \\ \ell_2(\pi') = \max(P). \end{gathered} \tag{7.98}$$

Now let $A = (a_{ij})$ be an $\mathbb{N}$-matrix of finite support. We want to associate with A a pair of *reverse* SSYTs of the same shape. This can be done by an obvious variant of the RSK algorithm, where we reverse the roles of $\leq$ and $\geq$ in defining row insertion. Equivalently, if

$$w_A = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix}$$

is the two-line array associated with A, then apply the ordinary RSK algorithm to the two-line array,

$$\begin{pmatrix} -u_n & \cdots & -u_1 \\ -v_n & \cdots & -v_1 \end{pmatrix}$$

(whose entries are now *negative* integers) and then change the sign back to positive of all entries of the pair of SSYTs. We will obtain a pair (P, Q) of reverse SSYTs satisfying

$$\begin{aligned} |P| &= \sum_{i,j} j a_{ij} \\ |Q| &= \sum_{i,j} i a_{ij} \\ \max(P) &= \max\{j : a_{ij} \neq 0\} \\ \max(Q) &= \max\{i : a_{ij} \neq 0\} \\ |\mathrm{sh}(P)| &= |\mathrm{sh}(Q)| = \sum a_{ij}. \end{aligned} \tag{7.99}$$

It follows from the equations beginning with (7.98) and (7.99) that if $\mathcal{M}_{rc}$ is the set of all $r \times c$ $\mathbb{N}$-matrices, then

$$\begin{aligned}\sum_{\pi \in \mathcal{P}(r,c)} q^{\mathrm{tr}(\pi)} x^{|\pi|} &= \sum_{A=(a_{ij}) \in \mathcal{M}_{rc}} q^{\sum a_{ij}} x^{\sum (i+j)a_{ij} - \sum a_{ij}} \\ &= \prod_{i=1}^{r} \prod_{j=1}^{c} \left(\sum_{a_{ij} \geq 0} q^{a_{ij}} x^{(i+j-1)a_{ij}} \right) \\ &= \prod_{i=1}^{r} \prod_{j=1}^{c} (1 - qx^{i+j-1})^{-1}. \qquad \square\end{aligned}$$

Now let $\mathcal{P}(r)$ denote the set of all plane partitions with at most r rows. If we let $q = 1$ and $c \to \infty$ in Theorem 7.20.1, then we obtain the following elegant enumeration of the elements of $\mathcal{P}(r)$.

7.20.2 Corollary. *Fix $r \in \mathbb{P}$. Then*

$$\sum_{\pi \in \mathcal{P}(r)} x^{|\pi|} = \prod_{i \geq 1} (1 - x^i)^{-\min(i,r)}. \tag{7.100}$$

Proof. Theorem 7.20.1 yields

$$\sum_{\pi \in \mathcal{P}(r)} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j \geq 1} (1 - x^{i+j-1})^{-1},$$

which is easily seen to agree with the right-hand side of (7.100). $\square$

Finally, let $\mathcal{P}$ denote the set of all plane partitions, and let $r \to \infty$ in Corollary 7.20.2 to obtain the archetypal result in the theory of plane partitions:

7.20.3 Corollary. *We have*

$$\sum_{\pi \in \mathcal{P}} x^{|\pi|} = \prod_{i \geq 1} (1 - x^i)^{-i}.$$

A nice variation of Theorem 7.20.1 arises when we take into account the symmetry result Theorem 7.13.1 of the RSK algorithm. Define a plane partition $\sigma = (\sigma_{ij})$ to be *symmetric* if $\sigma_{ij} = \sigma_{ji}$ for all i, j. Let $\mathcal{S}(r)$ denote the set of all symmetric plane partitions with at most r rows (and therefore with at most r columns).

7.20.4 Theorem. *Fix $r \in \mathbb{P}$. Then*

$$\sum_{\sigma \in \mathcal{S}(r)} q^{\operatorname{tr}(\sigma)} x^{|\sigma|} = \prod_{i=1}^{r} (1 - qx^{2i-1})^{-1} \cdot \prod_{1 \le i < j \le r} \left(1 - q^2 x^{2(i+j-1)}\right)^{-1}.$$

First Proof. Let $\pi'(P, Q)$ be the plane partition described in the proof of Theorem 7.20.1. It is easy to see that π' is symmetric if and only if $P = Q$. Moreover, suppose that $A \xrightarrow{\mathrm{RSK}'} (P, Q)$, where RSK$'$ is the "reverse" RSK algorithm also described in the proof of Theorem 7.20.1 (so P and Q are reverse SSYTs). Theorem 7.13.1 holds equally well for RSK$'$, so $A \xrightarrow{\mathrm{RSK}'} (P, P)$ if and only if $A = A^t$. Let $\mathcal{M}'_r$ be the set of all $r \times r$ symmetric $\mathbb{N}$-matrices. We then obtain, exactly as in the proof of Theorem 7.20.1, that

$$\begin{aligned}
\sum_{\sigma \in \mathcal{S}(r)} q^{\operatorname{tr}(\sigma)} x^{|\sigma|} &= \sum_{A = (a_{ij}) \in \mathcal{M}'_r} q^{\sum a_{ij}} x^{\sum (i+j-1) a_{ij}} \\
&= \sum_A q^{\sum_{i \ge 1} a_{ii} + 2 \sum_{1 \le i < j \le r} a_{ij}} x^{\sum_{i \ge 1} (2i-1) a_{ii} + 2 \sum_{1 \le i < j \le r} (i+j-1) a_{ij}} \\
&= \prod_{i=1}^{r} \left(\sum_{a_{ii} \ge 0} q^{a_{ii}} x^{(2i-1) a_{ii}} \right) \cdot \prod_{1 \le i < j \le r} \left(\sum_{a_{ij} \ge 0} q^{2a_{ij}} x^{2(i+j-1) a_{ij}} \right) \\
&= \prod_{i=1}^{r} (1 - qx^{2i-1})^{-1} \cdot \prod_{1 \le i < j \le r} \left(1 - q^2 x^{2(i+j-1)}\right)^{-1}. \qquad \square
\end{aligned}$$

Second Proof. There is a clever alternative proof that avoids the use of Theorem 7.13.1. Suppose that π is a reverse SSYT with only odd parts. Thus each column of π is a partition into distinct odd parts. The solution to Exercise 1.22(d) gives a bijection between partitions of n with distinct odd parts and self-conjugate partitions of n. Apply this bijection to each column of π, and then take the conjugate of each row, producing a symmetric plane partition σ. An example is given by

$$\pi = \begin{matrix} 7\,5\,5\,5\,5\,1\,1 \\ 5\,3\,3\,1\,1 \\ 1\,1 \end{matrix} \quad \to \quad \begin{matrix} 4\,3\,3\,3\,3\,1\,1 \\ 4\,3\,3\,2\,2 \\ 3\,3\,2\,1\,1 \\ 2 \end{matrix} \quad \to \quad \begin{matrix} 7\,5\,5\,1 \\ 5\,5\,3\,1 \\ 5\,3\,2 \\ 1\,1 \end{matrix} = \sigma.$$

One can easily check that $|\pi| = |\sigma|$, $\operatorname{sh}(\pi) = \operatorname{diag}(\sigma)$, $|\operatorname{sh}(\pi)| = \operatorname{tr}(\sigma)$, and $\ell_1(\sigma) = \frac{1}{2}[1 + \max(\pi)]$. Hence from Corollary 7.13.8 there follows

$$\begin{aligned}
\sum_{\sigma \in \mathcal{S}(r)} q^{\operatorname{tr}(\sigma)} x^{|\sigma|} &= \prod_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} (1 - x_i)^{-1} \cdot \prod_{\substack{1 \le i < j \le 2r-1 \\ i,j \text{ odd}}} (1 - x_i x_j)^{-1} \Big|_{x_i = qx^i} \\
&= \prod_{i=1}^{r} (1 - qx^{2i-1})^{-1} \cdot \prod_{1 \le i < j \le r} \left(1 - q^2 x^{2(i+j-1)}\right)^{-1}. \qquad \square
\end{aligned}$$

Putting $q = 1$ and letting $r \to \infty$ yields the following elegant enumeration of all symmetric plane partitions of n.

7.20.5 Corollary. *Let S be the set of all symmetric plane partitions. Then*

$$\sum_{\sigma \in S} x^{|\sigma|} = \prod_{i \geq 1} \frac{1}{(1 - x^{2i-1})(1 - x^{2i})^{\lfloor i/2 \rfloor}}.$$

7.21 Plane Partitions with Bounded Part Size

Our main object in this section is to refine Theorem 7.20.1, in the case $q = 1$, by restricting the size of the largest part of the plane partition $\pi \in \mathcal{P}(r, c)$. Consider, for instance, the special case $r = 1$, so that π is just an ordinary partition $\lambda = (\lambda_1, \ldots, \lambda_c)$ with at most c parts. If we add the restriction $\lambda_1 \leq t$, then Proposition 1.3.19 tells us that

$$\sum_{\substack{\lambda = (\lambda_1, \ldots, \lambda_c) \\ \lambda_1 \leq t}} q^{|\lambda|} = \binom{c+t}{c},$$

a q-binomial coefficient. It is this result that we wish to generalize to plane partitions. We cannot expect a nice bijective proof like that of Theorem 7.20.1, because even in the case $r = 1$ the expansion of the numerator $(1 - q^{c+t}) \cdots (1 - q^{t+1})$ of the q-binomial coefficient $\binom{c+t}{c}$ has negative coefficients. A bijective proof would have to involve either an involution principle argument (or something similar), or else moving the numerator of $\binom{c+t}{c}$ over to the other side. While such proofs do exist, they lack the elegance of the proof of Theorem 7.20.1. The proof we give here will not be bijective, but will be a simple consequence of symmetric function theory.

To understand better the significance of the restrictions on the number of rows, the number of columns, and the largest part, we first discuss the notion of the diagram of a plane partition, generalizing the notion of the Young or Ferrers diagram of a partition. Formally, the *diagram* $D(\pi)$ (often identified with π) of a plane partition $\pi = (\pi_{ij})$ is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \{(i, j, k) \in \mathbb{P}^3 : 1 \leq k \leq \pi_{ij}\}.$$

Think of replacing the entry π_{ij} by a pillar of π_{ij} cubes (or dots). For instance, the (Ferrers) diagram of the plane partition

$$\pi = \begin{matrix} 4 & 2 & 1 \\ 3 & 1 & \\ 1 & & \\ 1 & & \end{matrix}$$

is given by Figure 7-9.

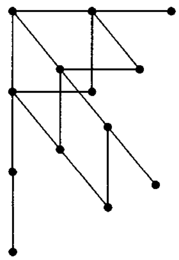


Figure 7-9. The diagram of a plane partition.

Any permutation w of the three coordinate axes transforms (the diagram of) a plane partition π of n into another plane partition $w(\pi)$ of n. Thus a plane partition has *six* "associates," called *aspects*, indexed by elements of $\mathfrak{S}_3$. Compare with the two "associates" λ and λ' of an ordinary partition λ. In terms of the plane partition $\pi = (\pi_{ij})$ itself, the six aspects are obtained as follows:

- leave π unchanged,
- conjugate every row of π,
- conjugate every column of π,
- transpose π,
- conjugate every row of π and then transpose,
- conjugate every column of π and then transpose.

The three statistics $\ell_1(\pi)$, $\ell_2(\pi)$, and $\max(\pi)$ are permuted among themselves when we take an aspect $w(\pi)$ of π. Thus for instance the number of plane partitions of n with at most r rows and at most c columns equals the number of plane partitions of n with at most c rows and with largest part at most r. Since we have enumerated plane partitions of n with at most r rows and at most c columns (Theorem 7.20.1 when $q = 1$), it now seems very natural to consider an additional restriction on the largest part.

Let $r, c, t \in \mathbb{P}$, and define the *box*

$$B(r,c,t) = \{(i,j,k) \in \mathbb{P}^3 : 1 \le i \le r,\ 1 \le j \le c,\ 1 \le k \le t\}.$$

Thus a plane partition π satisfies $\ell_1(\pi) \le r$, $\ell_2(\pi) \le c$, and $\max(\pi) \le t$ if and only if its diagram is contained in the box $B(r,c,t)$, which we write as $\pi \subseteq B(r,c,t)$. Our current goal, then, is to evaluate the generating function $\sum_{\pi \subseteq B(r,c,t)} q^{|\pi|}$.

As a preliminary step we will evaluate the principal specialization $s_\lambda(1, q, \ldots, q^{n-1})$. The most elegant formulation of this result involves two important statistics associated with the boxes of the Young diagram of a partition. Given a Young diagram λ (where we are identifying the diagram $\{(i, j) : 1 \le j \le \lambda_i\}$ with its shape) and a square $u = (i, j) \in \lambda$, define the *hook length $h(u)$ of λ at u* by

$$h(u) = \lambda_i + \lambda'_j - i - j + 1.$$

Equivalently, $h(u)$ is the number of squares directly to the right or directly below u, counting u itself once. For instance, the partition 4421 has hook lengths given by

7	5	3	2
6	4	2	1
3	1		
1			

Similarly define the *content* $c(u)$ of λ at $u = (i, j)$ by

$$c(u) = j - i.$$

For $\lambda = 4421$ the contents are given by

0	1	2	3
−1	0	1	2
−2	−1		
−3			

7.21.1 Lemma. *Let* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{Par}$ *and* $\mu_i = \lambda_i + n - i$. *Then*

$$\prod_{u\in\lambda}[h(u)] = \frac{\prod_{i\ge1}[\mu_i]!}{\prod_{1\le i<j\le n}[\mu_i - \mu_j]} \tag{7.101}$$

$$\prod_{u\in\lambda}[n + c(u)] = \prod_{i=1}^{n} \frac{[\mu_i]!}{[n-i]!}, \tag{7.102}$$

where $[k] = 1 - q^k$ *and* $[k]! = [1][2]\cdots[k]$.

Proof. Trusting that "one picture is worth a thousand words," we will illustrate the proofs with an example. Let $\lambda = 4421$. Add $n - i$ squares to the i-th row of the diagram of λ, obtaining the diagram of μ. In square (i, j) insert the number $\mu_i - j + 1$. Thus the multiset of inserted numbers is just $\bigcup_{i\ge1}\{1, 2, \ldots, \mu_i\}$, the exponents in the numerator of the right-hand side of equation (7.101) (when written as a product of factors $1 - q^k$). For each $1 \le i < j \le n$, write the number

$\mu_i - \mu_j$ in square $(i, \mu_j + 1)$ in boldface. We obtain the array

7	**6**	5	**4**	3	2	**1**
6	**5**	4	**3**	2	1	
3	**2**	1				
1						

A little thought shows that if we remove the columns $\mu_j + 1$ of bold numbers, we obtain just the diagram of λ with the hook length $h(u)$ in square u. This proves (7.101).

An analogous (but even simpler) argument works for equation (7.102). Here the relevant array is

1	**2**	**3**	4	5	6	7
	1	**2**	3	4	5	6
		1	2	3		
			1			

This completes the proof. $\square$

Given $\lambda \vdash n$, define

$$b(\lambda) = \sum (i-1)\lambda_i = \sum \binom{\lambda_i'}{2}. \tag{7.103}$$

Note that $b(\lambda)$ is the smallest possible sum of the entries of an SSYT (allowing 0 as a part) of shape λ, obtained uniquely by placing $i - 1$ in all the squares of the i-th row of λ. In particular, for $n \geq \ell(\lambda)$ we have $s_\lambda(1, q, \ldots, q^{n-1}) = q^{b(\lambda)} v_\lambda(q)$, where $v_\lambda(q)$ is a polynomial in q satisfying $v_\lambda(0) = 1$. (If $n < \ell(\lambda)$ then $s_\lambda(1, q, \ldots, q^{n-1}) = 0$.)

7.21.2 Theorem. *For any* $\lambda \in \mathrm{Par}$ *and* $n \in \mathbb{P}$ *we have*

$$s_\lambda(1, q, \ldots, q^{n-1}) = q^{b(\lambda)} \prod_{u \in \lambda} \frac{[n + c(u)]}{[h(u)]}.$$

Proof. If $n < \ell(\lambda)$ then both sides vanish, so assume $n \geq \ell(\lambda)$. By Theorem 7.15.1 (the bialternant formula for $s_\lambda(x_1, \ldots, x_n)$), we have

$$s_\lambda(1, q, \ldots, q^{n-1}) = \frac{\det\left(q^{(i-1)(\lambda_j + n - j)}\right)_{i,j=1}^n}{\det\left(q^{(i-1)(n-j)}\right)_{i,j=1}^n}. \tag{7.104}$$

The denominator is just a specialization of the Vandermonde determinant $a_\delta = \det(x_i^{n-j}) = \prod_{1 \leq i < j \leq n}(x_i - x_j)$, and so is equal to $\prod_{1 \leq i < j \leq n}(q^{i-1} - q^{j-1})$. But the numerator is also a specialization of a_δ, albeit somewhat disguised. Namely, let

$$a^*_\delta = \det\left(x_j^{i-1}\right)_{i,j=1}^n.$$

We have $a^*_\delta = (-1)^{\binom{n}{2}} a_\delta$ since the matrix $(x_j^{i-1})_{i,j=1}^n$ is obtained from that defining a_δ by transposing a_δ and then reversing the order of the rows. Thus the numerator of the right-hand side of (7.104) is just $a^*_\delta(q^{\mu_1}, q^{\mu_2}, \ldots, q^{\mu_n})$, where $\mu_j = \lambda_j + n - j$, so we get

$$s_\lambda(1, q, \ldots, q^{n-1}) = (-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} \frac{q^{\mu_i} - q^{\mu_j}}{q^{i-1} - q^{j-1}}. \tag{7.105}$$

By Lemma 7.21.1 there follows (using $\prod_{1 \le i < j \le n}[j - i] = \prod_{i=1}^n [n - i]!$)

$$\begin{aligned} s_\lambda(1, q, \ldots, q^{n-1}) &= \frac{q^{\sum_{i<j}\mu_j} \prod_{i<j}[\mu_i - \mu_j] \cdot \prod_{i \ge 1}[\mu_i]!}{q^{\sum_{i<j}(i-1)} \prod_{i<j}[j - i] \cdot \prod_{i \ge 1}[\mu_i]!} \\ &= q^{b(\lambda)} \prod_{u \in \lambda} \frac{[n + c(u)]}{[h(u)]}. \end{aligned}$$

□

Note that

$$s_\lambda(1, q, \ldots, q^{n-1}) = \sum_\pi q^{|\pi|},$$

where π ranges over all column-strict plane partitions (= reverse SSYTs) of shape λ and largest part at most $n - 1$, allowing 0 as a part. Hence Theorem 7.21.2 may be regarded as determining the generating function for this class of plane partitions, enumerated by the sum of their parts. If one prefers not to have 0 as a part, then the homogeneity of s_λ gives

$$\begin{aligned} q^{|\lambda|} s_\lambda(1, q, \ldots, q^{n-1}) &= s_\lambda(q, q^2, \ldots, q^n) \\ &= \sum_\pi q^{|\pi|}, \end{aligned}$$

where now π ranges over all column-strict plane partitions of shape λ and largest part at most n (with the usual condition that the parts are *positive* integers).

If we now let $n \to \infty$ in Theorem 7.21.2 then the numerator $\prod_{u \in \lambda}(1 - q^{n+c(u)})$ goes to 1, so we get a formula for the stable principal specialization $s_\lambda(1, q, q^2, \ldots)$.

7.21.3 Corollary. *For any* $\lambda \in \mathrm{Par}$ *we have*

$$s_\lambda(1, q, q^2, \ldots) = \frac{q^{b(\lambda)}}{\prod_{u \in \lambda}[h(u)]}.$$

Similarly, if we set $q = 1$ in Theorem 7.21.2, then we get (using the fact that $1 - q^k = (1-q)(1+q+\cdots+q^{k-1})$ and canceling the factors of $1-q$ from the numerator and denominator) the following result. (For its representation-theoretic significance, see Appendix 2, equation (A2.155) and the discussion following.)

7.21.4 Corollary. *For any* $\lambda \in \mathrm{Par}$ *and* $n \in \mathbb{P}$ *we have*

$$s_\lambda(1^n) = \prod_{u\in\lambda} \frac{n + c(u)}{h(u)}. \tag{7.106}$$

In particular, all the zeros of $s_\lambda(1^n)$ *(regarded as a polynomial in* n*) are integers.*

Corollary 7.21.3 has some interesting consequences. For instance, setting $\mu = \emptyset$ in Proposition 7.19.11 and comparing with Corollary 7.21.3 yields the following result.

7.21.5 Corollary. *For any* $\lambda \in \mathrm{Par}$ *we have*

$$\sum_T q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)}[n]!}{\prod_{u\in\lambda}[h(u)]},$$

where T *ranges over all SYTs of shape* λ.

From Corollary 7.21.5 we obtain the explicit formula for f^λ mentioned after Corollary 7.16.3 and thus also an enumeration of the combinatorial objects given in Proposition 7.10.3. This remarkable result is known as the *hook-length formula*.

7.21.6 Corollary. *Let* $\lambda \vdash n$. *Then*

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}.$$

Proof. Set $q = 1$ in Corollary 7.21.5. Alternatively, by equation (7.29) we can restate Corollary 7.21.3 for $\lambda \vdash n$ as

$$\mathrm{ex}_q(s_\lambda) = \frac{t^n q^{b(\lambda)}}{\prod_{u\in\lambda}\left(1+q+\cdots+q^{h(u)-1}\right)}.$$

Now let $q = 1$ and use the interpretation of ex given by Proposition 7.8.4(a) $\square$

We may regard both Corollaries 7.21.3 and 7.21.5 as q-analogues of the hook-length formula. Corollary 7.21.3 is the symmetric function q-analogue, while Corollary 7.21.5 is the combinatorial q-analogue.

Theorem 7.21.2 is a completely satisfactory generating function for column-strict plane partitions, but how is it related to ordinary plane partitions? The answer is that for *rectangular shapes* $\lambda = \langle c^r \rangle$ there is a simple bijection between column-strict plane partitions of shape $\langle c^r \rangle$ and ordinary plane partitions of shape $\langle c^r \rangle$, and this bijection has an easily computable effect on the largest part and the sum of the parts. These assertions will be explained in the proof of the following theorem, which is the main result of this section.

7.21.7 Theorem. *Fix r, c, t with $r \leq c$. Then*

$$\sum_{\pi \in B(r,c,t)} q^{|\pi|}$$

$$= \frac{[t+1][t+2]^2 \cdots [t+r]^r [t+r+1]^r \cdots [t+c]^r [t+c+1]^{r-1} \cdots [t+c+r-1]}{[1][2]^2 \cdots [r]^r [r+1]^r \cdots [c]^r [c+1]^{r-1} \cdots [c+r-1]}, \tag{7.107}$$

where $[i] = 1 - q^i$.

Proof. Let $\lambda = \langle c^r \rangle$, a rectangular shape with r rows and c columns. Note that the assumption $r \leq c$ entails no loss of generality since we can always replace λ with λ'. Let $\pi = (\pi_{ij})$ be a column-strict plane partition of shape λ, allowing 0 as a part. Define $\pi^* = (\pi^*_{ij})$ by $\pi^*_{ij} = \pi_{ij} - r + i$. We have simply applied to each column of π the usual method of converting a strictly decreasing sequence into a weakly decreasing one. For instance, if

$$\begin{matrix} & 6 & 6 & 4 & 4 & 4 & 3 \\ \pi = & 4 & 3 & 3 & 2 & 2 & 1 \\ & 2 & 2 & 1 & 1 & 0 & 0 \end{matrix}$$

then

$$\begin{matrix} & 4 & 4 & 2 & 2 & 2 & 1 \\ \pi^* = & 3 & 2 & 2 & 1 & 1 & 0. \\ & 2 & 2 & 1 & 1 & 0 & 0 \end{matrix}$$

It is clear that π^* is a plane partition satisfying

$$\ell_1(\pi^*) \leq r, \qquad \ell_2(\pi^*) \leq c,$$
$$\max(\pi^*) = \max(\pi) - r + 1, \qquad |\pi^*| = |\pi| - \binom{r}{2} c.$$

Moreover, given such a plane partition π^*, we can recover π by $\pi_{ij} = \pi^*_{ij} + r - i$. Hence we obtain from Theorem 7.21.2 that

$$\sum_{\pi \subseteq B(r,c,t)} q^{|\pi|} = q^{-\binom{r}{2}c} s_{\langle c^r \rangle}(1, q, \ldots, q^{t+r-1})$$
$$= q^{b(\langle c^r \rangle) - \binom{r}{2}c} \prod_{u \in \langle c^r \rangle} \frac{[t + r + c(u)]}{[h(u)]}. \tag{7.108}$$

Note that $b(\langle c^r \rangle) = \binom{r}{2}c$. Moreover, the multiset of hook lengths of $\langle c^r \rangle$ is $\{1, 2^2, 3^3, \ldots, r^r, (r+1)^r, \ldots, c^r, (c+1)^{r-1}, (c+2)^{r-2}, \ldots, c+r-1\}$, and the multiset of contents is obtained by subtracting r from the hook lengths. Substituting these values of $c(u)$ and $h(u)$ into (7.108) completes the proof. □

The reader can check that an alternative way of writing the generating function (7.107) that shows more clearly the symmetry between r, c, and t (but that has rct factors in the numerator and denominator rather than rc factors) is given by

$$\sum_{\pi \in B(r,c,t)} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \prod_{k=1}^{t} \frac{[i+j+k-1]}{[i+j+k-2]}. \tag{7.109}$$

Theorem 7.21.7 can be interpreted in terms of the theory of P-partitions developed in Section 4.5. A plane partition π satisfying $\ell_1(\pi) \le r$, $\ell_2(\pi) \le c$, and $\max(\pi) \le t$ may be regarded as an order-reversing map $\pi : \mathbf{r} \times \mathbf{c} \to [0, t]$. In other words, if P is the poset $\mathbf{r} \times \mathbf{c}$, then π is just a P-partition with largest part at most t. Thus in particular the number of such π is just $\Omega(P, t+1)$, where Ω denotes the order polynomial. Setting $q = 1$ and $t = m - 1$ in (7.107) yields

$$\begin{aligned}&\Omega(\mathbf{r} \times \mathbf{c}, m)\\ &\quad = \frac{m(m+1)^2 \cdots (m+r-1)^r (m+r)^r \cdots (m+c-1)^r (m+c)^{r-1} \cdots (m+c+r-2)}{1 \cdot 2^2 \cdots r^r (r+1)^r \cdots c^r (c+1)^{r-1} \cdots (c+r-1)}.\end{aligned}$$

In particular, all the zeros of $\Omega(\mathbf{r} \times \mathbf{c}, m)$ are nonpositive integers. Moreover, a simple extension of Proposition 3.5.1 (see Exercise 4.24(a)) shows that

$$\sum_{\pi \in B(r,c,t)} q^{|\pi|} = \sum_{I \in J(\mathbf{r} \times \mathbf{c} \times \mathbf{t})} q^{\#I}, \tag{7.110}$$

the rank-generating function of the distributive lattice $J(\mathbf{r} \times \mathbf{c} \times \mathbf{t})$. Thus equation (7.109) is equivalent to Exercises 3.27(b) and 4.25(f)(i). Setting $q = 1$ in (7.110) and (7.109) yields the elegant formula

$$\#J(\mathbf{r} \times \mathbf{c} \times \mathbf{t}) = \prod_{i=1}^{r} \prod_{j=1}^{c} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2}.$$

7.22 Reverse Plane Partitions and the Hillman–Grassl Correspondence

The proof of Theorem 7.21.7 involved a bijection between column-strict plane partitions of shape $\langle c^r \rangle$ and ordinary plane partitions whose shape is contained in $\langle c^r \rangle$. It is natural to ask whether we can do a similar bijection for any shape λ, thereby obtaining a formula for the generating function

$$\sum_{\substack{\mathrm{sh}(\pi)\subseteq\lambda \\ \max(\pi)\leq t}} q^{|\pi|}, \tag{7.111}$$

where π ranges over all plane partitions whose shape is contained in λ and with largest part $\leq t$. Unfortunately the bijection used in the proof of Theorem 7.21.7 does not carry over to nonrectangular shapes. For instance, if $\lambda = (2, 1)$ then we could associate with the ordinary plane partition (where 0 is allowed as a part)

$$\pi = \begin{matrix} a & b \\ c & \end{matrix}$$

the column-strict plane partition (also allowing 0 as a part)

$$\pi' = \begin{matrix} a+1 & b+1 \\ c & \end{matrix}.$$

But then there is no π corresponding to $\pi' = \begin{smallmatrix} 1 & 0 \\ 0 & \end{smallmatrix}$. Similarly, if we instead tried

$$\pi' = \begin{matrix} a+1 & b \\ c & \end{matrix},$$

then there is no π corresponding to $\pi' = \begin{smallmatrix} 1 & 1 \\ 0 & \end{smallmatrix}$ (since $\begin{smallmatrix} 0 & 1 \\ 0 & \end{smallmatrix}$ is not a plane partition). Indeed, the generating function (7.111), even in the case $t = \infty$, does not in general factor into a simple product (though there does exist a determinantal formula for (7.111)).

Although there is no such simple correspondence between column-strict plane partitions and plane partitions of a given shape as was used to prove Theorem 7.21.7, such a correspondence does exist in the *reverse* situation. This correspondence does not have a uniform effect on the largest part, but it is well behaved with respect to the sum of the parts. Thus we will get the generating function for reverse plane partitions of n of shape λ. It is easier to work with *weak* reverse plane partitions of shape λ, for which 0 can be a part. To get from a weak reverse plane partition of shape λ to a (nonweak) reverse plane partition λ^* of shape λ, simply add 1 to every entry of π. Note that $|\pi^*| = |\pi| + |\lambda|$. Similary define a *weak* SSYT to be an SSYT in which 0 is allowed to be a part.

7.22.1 Theorem. *Let* $\lambda \in$ Par. *Then*

$$\sum_{\pi} q^{|\pi|} = \frac{1}{\prod_{u\in\lambda}[h(u)]}, \tag{7.112}$$

where π ranges over all weak reverse plane partitions of shape λ. (If one does not want to allow 0 as a part, simply multiply by $q^{|\lambda|}$.)

Proof. Let $T = (a_{ij})$ be a weak SSYT of shape λ. Define $\pi = \pi(T)$ by $\pi_{ij} = a_{ij} - i + 1$. Then π is a weak reverse plane partition of shape λ satisfying

$$|\pi| = |T| - \sum (i-1)\lambda_i = |T| - b(\lambda).$$

This correspondence is easily seen to be a bijection. Hence if $\mathcal{R}_\lambda$ (respectively, $\mathcal{R}'_\lambda$) denotes the set of all weak reverse plane partitions (respectively, weak SSYTs) of shape λ, then

$$\begin{aligned}\sum_{\pi \in \mathcal{R}_\lambda} q^{|\pi|} &= q^{-b(\lambda)} \sum_{T \in \mathcal{R}'_\lambda} q^{|T|} \\ &= q^{-b(\lambda)} s_\lambda(1, q, q^2, \ldots),\end{aligned}$$

and the proof follows from Corollary 7.21.3. □

Theorem 7.22.1 is so elegant that we could ask for a simple bijective proof. Identifying λ with its diagram, we want a bijection between weak reverse plane partitions π of shape λ and functions $f : \lambda \to \mathbb{N}$, such that

$$|\pi| = \sum_{u \in \lambda} f(u) h(u).$$

We now describe such a bijection, known as the *Hillman–Grassl correspondence*.

We will successively define pairs $(\pi_0, f_0), (\pi_1, f_1), \ldots, (\pi_k, f_k)$, where each π_i is a weak reverse plane partition of shape λ and $f_i : \lambda \to \mathbb{N}$. We begin with $\pi_0 = \pi$ and $f_0(u) = 0$ for all $u \in \lambda$. We obtain π_{i+1} from π_i by decreasing $h(u_i)$ of the entries of π_i by 1 for a certain square $u_i \in \lambda$ (to be explained), and we define

$$f_{i+1}(v) = \begin{cases} f_i(v), & v \neq u_i \\ f_i(u_i) + 1, & v = u_i. \end{cases}$$

At the end π_k will have every entry equal to 0. Consequently,

$$|\pi| = \sum_{u \in \lambda} f_k(u) h(u),$$

and we define $f = f_k$.

It remains to describe the rule for obtaining π_{i+1} from π_i, and the corresponding choice of $u_i \in \lambda$. We will define a lattice path L in λ with steps N or E (i.e., one square up or one square to the right), beginning at the bottom of a column and ending at the end of a row of λ. The lattice path L begins at the location of the southwesternmost nonzero entry of π_i. If the path has reached the square (a, b), then move N if $(\pi_i)_{ab} = (\pi_i)_{a-1,b} > 0$; otherwise if $(\pi_i)_{ab} > 0$ move E. The lattice path terminates when no further move is possible. Define π_{i+1} to be the array obtained from π_i by subtracting 1 from every entry that lies in a square of the path L. If L begins in column b and ends in row a, then let $u_i = (a, b)$. It is easy to see that $\#L = h(u_i)$.

We illustrate this correspondence with a reverse plane partition π of shape (3, 3, 1). We indicate the lattice path L in boldface, and the function f_i by putting the value $f_i(u)$ in the square $u \in \lambda$:

π_i	f_i
0 1 3	0 0 0
2 4 4	0 0 0
3	0
0 1 3	0 0 0
2 4 4	0 0 0
2	1
0 1 **3**	0 0 0
1 3 3	1 0 0
1	1
0 1 **2**	1 0 0
0 **2 2**	1 0 0
0	1
0 **1 1**	1 1 0
0 **1** 1	1 0 0
0	1
0 0 0	1 2 0
0 0 **1**	1 0 0
0	1
0 0 0	1 2 0
0 0 0	1 0 1
0	1

We omit the proof that this correspondence is indeed a bijection, except for the hint that the square u_i is in the rightmost column among all the squares $u_1, \ldots, u_i$, and is in the highest row among the squares $u_1, \ldots, u_i$ in its column.

Now let P be an n-element poset, and let $\mathcal{A}(\mathcal{P})$ denote the set of P-partitions as in Section 4.5. Recall from Theorem 4.5.6 that

$$G_P(q) := \sum_{\sigma \in \mathcal{A}(\mathcal{P})} q^{|\sigma|} = \frac{W_P(q)}{(1-q)\cdots(1-q^n)},$$

where $W_P(1) = e(P)$, the number of linear extensions of P. If we set $P = P_\lambda^*$, the dual of the poset P_λ defined after Corollary 7.19.5, then the left-hand side of equation (7.112) is just $G_P(q)$. Thus we get

$$f^\lambda = e(P_\lambda) = e\big(P_\lambda^*\big) = \left.\frac{[n]!}{\prod_{u\in\lambda}[h(u)]}\right|_{q=1} = \frac{n!}{\prod_{u\in\lambda} h(u)} \qquad (7.113)$$

giving a proof of the hook-length formula (Corollary 7.21.6) avoiding the use of determinants.

7.23 Applications to Permutation Enumeration

We have already seen a number of connections between symmetric functions and permutations (such as the RSK algorithm applied to a permutation matrix, or the Jordan–Hölder set of the labeled poset $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$), so it is not too surprising that the theory of symmetric functions can be used to obtain some results related to permutation enumeration. Our first result is based on the RSK algorithm, and requires one additional fact about it.

7.23.1 Lemma. *Let $w \in \mathfrak{S}_n$ and $w \xrightarrow{\text{RSK}} (P, Q)$. Then $D(P) = D(w^{-1})$ and $D(Q) = D(w)$, where D denotes the descent set.*

Proof. Let $(P_0, Q_0), \ldots, (P_n, Q_n) = (P, Q)$ be the successive pairs of tableaux obtained in applying the RSK algorithm to w. Let $w = w_1 \cdots w_n$, and suppose that for some i we have $w_i < w_{i+1}$. As observed in the proof of Theorem 7.11.5, the insertion path of w_{i+1} lies to the right of that of w_i. Suppose that the shape of P_i is obtained from that of P_{i-1} by adjoining a square in the (a, b) position, so the (a, b) entry of Q is equal to i. When we insert w_{i+1} into P_i, if an element m is bumped into row a then it would occupy the $(a, b+1)$ position without bumping a further element. Thus $i + 1$ does not appear in Q in a lower row than i, so $i \notin D(Q)$.

Similarly if $w_i > w_{i+1}$ then the insertion path of w_{i+1} lies weakly to the left of that of w_i. Thus an element must be bumped into row a but not at the end, and hence must bump an element into row $a + 1$. This means $i \in D(Q)$, so $D(w) = D(Q)$ as claimed.

The symmetry property Theorem 7.13.1 of the RSK algorithm implies that $w^{-1} \xrightarrow{\text{RSK}} (Q, P)$, so by what was just proved we have $D(w^{-1}) = P$, completing the proof. □

Recall from Section 7.19 the definition $\mathrm{co}(S)$ of a set $S \subseteq [n-1]$, and the associated definitions of $\mathrm{co}(w)$ for $w \in \mathfrak{S}_n$ and $\mathrm{co}(T)$ when T is an SYT. Similarly define $\mathrm{co}'(w) = \mathrm{co}([n-1] - D(w))$. Note that $[n-1] - D(w)$ is just the ascent set $A(w)$ of w.

As a corollary to Lemma 7.23.1, we obtain the expansion of the Cauchy product $\prod(1 - x_i y_j)^{-1}$ in terms of quasisymmetric functions. (We regard n as fixed, so all our quasisymmetric functions are of degree n.) This result may be regarded as giving a generating function for the number of permutations $w \in \mathfrak{S}_n$ such that $D(w^{-1}) = S$ and $D(w) = T$, since this number is just the coefficient of $L_\alpha(x)L_\beta(y)$ in the expansion below, where $\alpha = \alpha_S$ and $\beta = \beta_T$.

7.23.2 Theorem. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\lambda \vdash n} s_\lambda(x)s_\lambda(y) = \sum_{w \in \mathfrak{S}_n} L_{\mathrm{co}(w^{-1})}(x)L_{\mathrm{co}(w)}(y) \tag{7.114}$$

$$\sum_{\lambda \vdash n} s_\lambda(x)s_{\lambda'}(y) = \sum_{w \in \mathfrak{S}_n} L_{\mathrm{co}(w^{-1})}(x)L_{\mathrm{co}'(w)}(y). \tag{7.115}$$

Proof. By the quasisymmetric expansion of s_λ (Theorem 7.19.7), we have

$$\begin{aligned}\sum_{\lambda \vdash n} s_\lambda(x)s_\lambda(y) &= \sum_{\lambda \vdash n}\left(\sum_{\mathrm{sh}(T)=\lambda} L_{\mathrm{co}(T)}(x)\right)\left(\sum_{\mathrm{sh}(T')=\lambda} L_{\mathrm{co}(T')}(y)\right)\\ &= \sum_{\lambda \vdash n}\sum_{\substack{\mathrm{sh}(T)=\lambda\\ \mathrm{sh}(T')=\lambda}} L_{\mathrm{co}(T)}(x)L_{\mathrm{co}(T')}(y),\end{aligned}$$

where $\mathrm{sh}(T) = \lambda$ signifies that the sum ranges over all SYT of shape λ (and similarly for $\mathrm{sh}(T') = \lambda$). If $w \in \mathfrak{S}_n$ satisfies $w \xrightarrow{\mathrm{RSK}} (T, T')$, then by Lemma 7.23.1 we have $D(w) = T'$ and $D(w^{-1}) = T$. Hence

$$\sum_{\lambda \vdash n}\sum_{\substack{\mathrm{sh}(T)=\lambda\\ \mathrm{sh}(T')=\lambda}} L_{\mathrm{co}(T)}(x)L_{\mathrm{co}(T')}(y) = \sum_{w \in \mathfrak{S}_n} L_{\mathrm{co}(w^{-1})}(x)L_{\mathrm{co}(w)}(y),$$

and (7.114) follows. The proof of (7.115) is analogous, using the dual RSK algorithm. Alternatively, apply the extension $\hat{\omega}$ of ω (acting on the y variables only) given by Exercise 7.94(a) to (7.114). $\square$

Although Theorem 7.23.2 may be regarded as "determining" the number of permutations $w \in \mathfrak{S}_n$ with $D(w^{-1}) = S$ and $D(w) = T$, a more useful or explicit expression would be desirable. Such an expression can be given in terms of skew Schur functions whose shape is a border strip. If $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathrm{Comp}(n)$, then let B_α denote the border strip with α_i squares in row $\ell - i + 1$. Regard B_α as a skew shape, so s_{B_α} is a skew Schur function.

7.23.3 Lemma. *The Jordan–Hölder set $\mathcal{L}(P_{B_\alpha}, \omega_{B_\alpha})$ (as defined in Section 7.19) consists of all permutations $w \in \mathfrak{S}_n$ satisfying $\mathrm{co}(w^{-1}) = \alpha$.*

Before proceeding to the proof, let us consider an example. Let $\alpha = (3, 1, 2, 3) \in \mathrm{Comp}(9)$. The corresponding Schur labeled border strip is

$$\begin{array}{cccccc} & & & 7 & 8 & 9\\ & & 5 & 6 & & \\ & & 4 & & & \\ 1 & 2 & 3 & & & \end{array}$$

A typical element of $\mathcal{L}(P,\omega)$ is $w = 578124963$. Then $w^{-1} = 459618237$, so $D(w^{-1}) = \{3,4,6\}$ and $\mathrm{co}(w^{-1}) = (3,1,2,3) = \alpha$.

Proof of Lemma 7.23.3. A permutation $w \in \mathfrak{S}_n$ belongs to $\mathcal{L}(P_{B_\alpha}, \omega_{B_\alpha})$ if and only if $i+1$ follows i in w (regarded as a word $w_1 \cdots w_n$) whenever i and $i+1$ are in the same row, and $i+1$ precedes i in w whenever i and $i+1$ are in the same column. Hence $i \in D(w^{-1})$ if and only if i and $i+1$ are in the same column, which is clearly equivalent to $i \in S_\alpha$. □

7.23.4 Corollary. *Let* $\alpha \in \mathrm{Comp}(n)$. *Then*

$$s_{B_\alpha} = \sum_{\substack{w \in \mathfrak{S}_n \\ \alpha = \mathrm{co}(w^{-1})}} L_{\mathrm{co}(w)}.$$

Proof. Immediate from Theorem 7.19.7 and Lemma 7.23.3. □

7.23.5 Corollary. *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(y) = \sum_{\alpha \in \mathrm{Comp}(n)} L_\alpha(x) s_{B_\alpha}(y). \tag{7.116}$$

Proof. By Corollary 7.23.4 we have

$$\begin{aligned}\sum_{\alpha \in \mathrm{Comp}(n)} L_\alpha(x) s_{B_\alpha}(y) &= \sum_{\alpha \in \mathrm{Comp}(n)} L_\alpha(x) \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{co}(w^{-1}) = \alpha}} L_{\mathrm{co}(w)}(y) \\ &= \sum_{w \in \mathfrak{S}_n} L_{\mathrm{co}(w^{-1})}(x) L_{\mathrm{co}(w)}(y),\end{aligned}$$

and the proof follows from Theorem 7.23.2. □

The next corollary gives a formula for the expansion of any symmetric function in terms of fundamental quasisymmetric functions.

7.23.6 Corollary. *For all* $f \in \Lambda^n$ *we have*

$$f = \sum_{\alpha \in \mathrm{Comp}(n)} \langle f, s_{B_\alpha} \rangle L_\alpha.$$

Proof. Take the scalar product of both sides of equation (7.116) with $s_\lambda(y)$ to obtain the desired result for $f = s_\lambda$. The general case follows by linearity. □

Next we come to an alternative expansion of $\sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(y)$ in terms of quasisymmetric functions.

7.23.7 Corollary. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(y) = \sum_{\alpha,\beta \in \mathrm{Comp}(n)} \langle s_{B_\alpha}, s_{B_\beta} \rangle L_\alpha(x) L_\beta(y).$$

Proof. By Corollary 7.23.6 we have

$$s_{B_\beta}(y) = \sum_{\alpha \in \mathrm{Comp}(n)} \langle s_{B_\alpha}, s_{B_\beta} \rangle L_\alpha.$$

Substitute into the right-hand side of equation (7.116) to complete the proof. □

Write B_S and B_T for the border strips B_{α_S} and B_{α_T}, where α_S and α_T are the compositions corresponding to S and T as defined in Section 7.19. Comparing Theorem 7.23.2 with Corollary 7.23.7 yields the following enumerative result.

7.23.8 Corollary. *Let $S, T \subseteq [n-1]$. Then the number of permutations $w \in \mathfrak{S}_n$ satisfying $D(w^{-1}) = S$ and $D(w) = T$ is equal to the scalar product $\langle s_{B_S}, s_{B_T} \rangle$.*

Theorem 7.23.2 can be specialized in several ways. For instance, we can obtain a generating function for the joint distribution of the statistics $\mathrm{maj}(w)$ and $\mathrm{maj}(w^{-1})$ for $w \in \mathfrak{S}_n$. Write $[k]_q = 1 - q^k$, $[k]_t = 1 - t^k$, $[k]!_q = [1]_q \cdots [k]_q$, and $[k]!_t = [1]_t \cdots [k]_t$. Recall the notation $b(\lambda) = \sum_i (i-1)\lambda_i$ from equation (7.103).

7.23.9 Corollary. *Let*

$$F_n(q,t) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w^{-1})} t^{\mathrm{maj}(w)}.$$

Then

$$F_n(q,t) = \sum_{\lambda \vdash n} \frac{q^{b(\lambda)}\, t^{b(\lambda)}\, [n]!_q\, [n]!_t}{\prod_{u \in \lambda} [h(u)]_q\, [h(u)]_t}, \tag{7.117}$$

and we have the "double Eulerian" generating function

$$\begin{aligned} \sum_{n \geq 0} F_n(q,t) \frac{z^n}{[n]!_q\, [n]!_t} &= \prod_{i,j \geq 0} \frac{1}{1 - q^i t^j z} \\ &= \exp \sum_{n \geq 1} \frac{1}{n} \frac{z^n}{(1-q^n)(1-t^n)}. \end{aligned}$$

Proof. Let $x_i = q^{i-1}$ and $y_i = t^{i-1}$ in equation (7.114). By Corollary 7.21.3, the left-hand side of (7.114) becomes

$$\sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots) s_\lambda(1, t, t^2, \ldots) = \sum_{\lambda \vdash n} \frac{q^{b(\lambda)}\, t^{b(\lambda)}}{\prod_{u \in \lambda} [h(u)]_q\, [h(u)]_t}.$$

On the other hand, by Lemma 7.19.10 the right-hand side of (7.114) becomes

$$\sum_{w\in\mathfrak{S}_n} L_{\mathrm{co}(w^{-1})}(1,q,q^2,\ldots)L_{\mathrm{co}(w)}(1,t,t^2,\ldots) = \sum_{w\in\mathfrak{S}_n} \frac{q^{\mathrm{comaj}(w^{-1})}t^{\mathrm{comaj}(w)}}{[n]!_q\,[n]!_t},$$

where $\mathrm{comaj}(w) = \sum_{i\in D(w)}(n-i)$. If $w = w_1\cdots w_n$ then define $w^* = n+1-w_n,\ldots,n+1-w_1$. The map $w\mapsto w^*$ from $\mathfrak{S}_n$ to itself is a bijection (in fact, involution), satisfying $\mathrm{comaj}(w) = \mathrm{maj}(w^*)$ and $(w^{-1})^* = (w^*)^{-1}$. Hence

$$\sum_{w\in\mathfrak{S}_n} q^{\mathrm{comaj}(w^{-1})}t^{\mathrm{comaj}(w)} = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w^{-1})}t^{\mathrm{maj}(w)},$$

so equation (7.117) follows. The remainder of the proof is an immediate consequence of Proposition 7.7.4 and Theorem 7.12.1, which assert that

$$\sum_{\lambda\in\mathrm{Par}} s_\lambda(x)s_\lambda(y) = \prod(1-x_iy_j)^{-1} = \exp\sum_{n\geq 1}\frac{1}{n}p_n(x)p_n(y). \qquad \square$$

It is not necessary to use the theory of symmetric functions to prove Corollary 7.23.9; see for instance Exercise 4.20 (in the case $m=2$).

Our second connection between symmetric functions (more accurately, the RSK algorithm) and permutation enumeration concerns increasing and decreasing subsequences of a permutation. If $w = w_1\cdots w_n\in\mathfrak{S}_n$, then let $v = w_{i_1}w_{i_2}\cdots w_{i_k}$ be a *subsequence* of w, i.e., $1\leq i_1<i_2<\cdots<i_k\leq n$. We say v is *increasing* if $w_{i_1}<w_{i_2}<\cdots<w_{i_k}$ and *decreasing* if $w_{i_1}>w_{i_2}>\cdots>w_{i_k}$. Write $\mathrm{is}(w)$ for the length (number of terms) of the longest increasing subsequence of w. Let $r_i(w)$ be the rightmost integer j in w such that the longest increasing subsequence of w whose last term is j has length i. If, for instance, $w = 725481963$, then $\mathrm{is}(w)=4$, $r_1(w)=1$, $r_2(w)=3$, $r_3(w)=6$, and $r_4(w)=9$, while $r_i(w)$ is undefined for $i>4$. Note that in general $1 = r_1(w) < r_2(w) < \cdots < r_{\mathrm{is}(w)}(w)$.

7.23.10 Proposition. *Let $w\in\mathfrak{S}_n$ and $m = \mathrm{is}(w)$. Suppose that $w \xrightarrow{\mathrm{RSK}} (P,Q)$. Then the first row of P is equal to $r_1(w), r_2(w), \ldots, r_m(w)$.*

Proof. The proposition is essentially a restatement of Lemma 7.13.4. An increasing subsequence $w_{i_1},\ldots,w_{i_k}$ of w is equivalent to a chain $(i_1,w_{i_1})<\cdots<(i_k,w_{i_k})$ in the inversion poset $I(w)$. If follows that the antichain I_j consists precisely of those pairs (i,w_i) for which the longest increasing subsequence of w ending at w_i has length j. The maximum value of i for such a pair is by definition u_{jn_j}, and the corresponding value of w_i is equal to v_{jn_j}. Hence $v_{jn_j} = r_j(w)$, so the proof follows from Lemma 7.13.4. $\square$

As an immediate corollary, we obtain a combinatorial interpretation of the length of the first row of P (or Q) when $w \xrightarrow{\mathrm{RSK}} (P,Q)$.

7.23.11 Corollary. *Suppose that $w \in \mathfrak{S}_n$ and $w \xrightarrow{\text{RSK}} (P, Q)$. Let* $\text{sh}(P) = \text{sh}(Q) = \lambda$. *Then* $\lambda_1 = \text{is}(w)$.

Corollary 7.23.11 can be used to obtain interesting enumerative results concerning the distribution of longest increasing subsequences. A quite general result is the following, though it is often difficult to extract further information from it. (See, for instance, Exercise 6.56 and Exercise 7.16.)

7.23.12 Corollary. *Let $g_p(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ for which* $\text{is}(w) = p$. *Then*

$$g_p(n) = \sum_{\substack{\lambda \vdash n \\ \lambda_1 = p}} (f^\lambda)^2.$$

Proof. There are $(f^\lambda)^2$ pairs (P, Q) of SYTs of shape λ. The proof now follows from Corollary 7.23.11. □

If $w \in \mathfrak{S}_n$, let us define the *shape* $\text{sh}(w)$ to be the shape of the SYT P or Q when $w \xrightarrow{\text{RSK}} (P, Q)$. If $\lambda = \text{sh}(w)$ then we have found a simple combinatorial interpretation of the largest part λ_1 of λ. It is natural to ask for a similar interpretation of the other parts λ_i of λ. For instance, it is tempting to conjecture that λ_2 is equal to the length of the longest possible increasing subsequence that can remain when an increasing subsequence of length λ_1 is removed from w. Unfortunately this conjecture is false. For instance, if $w = 247951368$ then $\text{sh}(w) = (5, 3, 1)$. There is a unique increasing subsequence of w of length 5, viz., 24568. When this is removed from w, we obtain the sequence 7913, which has no increasing subsequence of length 3. The correct result is given by the following fundamental theorem, whose proof is included in Appendix 1 (Theorem A1.1.1).

7.23.13 Theorem. *Let $w \in \mathfrak{S}_n$ and* $\text{sh}(w) = (\lambda_1, \lambda_2, \ldots)$. *Then for all $i \geq 1$, $\lambda_1 + \cdots + \lambda_i$ is equal to the length of the longest subsequence of w that can be written as a union of i increasing subsequences.*

For instance, let $w = 247951368$ as above. The subsequence 24791368 is the union of the two increasing subsequences 2479 and 1368. Hence $\lambda_1 + \lambda_2 \geq 8$. In fact w itself cannot be written as a union of two increasing subsequences, so actually $\lambda_1 + \lambda_2 = 8$.

Instead of increasing subsequences we can ask about both increasing and decreasing subsequences simultaneously. The key to this question is a further symmetry property of the RSK algorithm (in addition to Theorem 7.13.1). We outline one approach to this symmetry property here, while a different method of proof is given in Appendix 1 (Corollary A1.2.11). We have denoted the row insertion of the integer k into the SSYT T by $T \leftarrow k$. Assume that all entries of T are distinct and different from k, and let $k \rightarrow T$ denote the *column* insertion of k into T. This is defined exactly like row insertion, but with the roles of rows and columns interchanged. Equivalently, if ^{t} denotes transpose then

$$(k \rightarrow T) = (T^t \leftarrow k)^t.$$

We omit the proof, which consists of an elementary but tedious analysis of cases, of the following fundamental lemma. We are always assuming that our tableaux have distinct entries, and that this condition is maintained after inserting further elements.

7.23.14 Lemma. *If $i \neq j$ then*

$$j \rightarrow (T \leftarrow i) = (j \rightarrow T) \leftarrow i.$$

In other words, row insertion and column insertion commute with each other.

7.23.15 Lemma. *Let*

$$\begin{aligned} P(i_1, i_2, \ldots, i_n) &= ((i_1 \leftarrow i_2) \leftarrow i_3) \leftarrow \cdots \leftarrow i_n \\ \tilde{P}(i_1, i_2, \ldots, i_n) &= i_1 \rightarrow \cdots \rightarrow (i_{n-2} \rightarrow (i_{n-1} \rightarrow i_n)). \end{aligned}$$

Then

$$P(i_1, i_2, \ldots, i_n) = \tilde{P}(i_1, i_2, \ldots, i_n).$$

Proof. Induction on n. The assertion is clear for $n = 1$, since $P(i_1) = \tilde{P}(i_1) = i_1$. It is also easy to check directly the case $n = 2$. Now let $n \geq 2$ and assume the assertion for all $m \leq n$. We have

$$\begin{aligned} P(i_1, \ldots, i_{n+1}) &= P(i_1, \ldots, i_n) \leftarrow i_{n+1} && \text{(definition of } \leftarrow\text{)} \\ &= \tilde{P}(i_1, \ldots, i_n) \leftarrow i_{n+1} && \text{(induction hypothesis)} \\ &= [i_1 \rightarrow \tilde{P}(i_2, \ldots, i_n)] \leftarrow i_{n+1} && \text{(definition of } \tilde{P}\text{)} \\ &= i_1 \rightarrow [\tilde{P}(i_2, \ldots, i_n) \leftarrow i_{n+1}] && \text{(previous lemma)} \\ &= i_1 \rightarrow [P(i_2, \ldots, i_n) \leftarrow i_{n+1}] && \text{(induction hypothesis)} \\ &= i_1 \rightarrow P(i_2, \ldots, i_n, i_{n+1}) && \text{(definition of } \leftarrow\text{)} \\ &= i_1 \rightarrow \tilde{P}(i_2, \ldots, i_n, i_{n+1}) && \text{(induction hypothesis)} \\ &= \tilde{P}(i_1, \ldots, i_{n+1}) && \text{(definition of } \rightarrow\text{).} \end{aligned}$$

□

We now come to the new symmetry property of the RSK algorithm mentioned above.

7.23.16 Theorem. *Let $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ and $w \xrightarrow{\text{RSK}} (P, Q)$. Let $w^r = w_n \cdots w_2 w_1$, the word w written in reverse order. Suppose that $w^r \xrightarrow{\text{RSK}} (P^*, Q^*)$.*

Then $P^* = P^t$, the transpose of P. In particular, $\mathrm{sh}(w) = \mathrm{sh}(w^r)'$. *(The description of Q^* is more complicated and is discussed in Appendix 1, Section A1.2. The map $Q \mapsto Q^*$ is called the* Schützenberger involution.*)*

Proof. Using the notation of the previous lemma, we have $P(w_1, \ldots, w_n) = P$ and $\tilde{P}(w_1, \ldots, w_n)^t = P^*$. The proof follows from the previous lemma. $\square$

If we regard a permutation $w \in \mathfrak{S}_n$ as an $n \times n$ permutation matrix, then Theorem 7.13.1 tells us the effect on the RSK algorithm of reflecting w about the main diagonal, while Theorem 7.23.16 tells us the effect on the RSK algorithm of reflecting w about a horizontal line. These two reflections generate the entire eight-element dihedral group D_4 of symmetries of the square. Thus every "dihedral symmetry" of w has a predictable effect on the behavior of the RSK algorithm (when applied to a permutation).

Since a decreasing subsequence of w becomes an increasing subsequence (in reverse order) of w^r and vice versa, the following result is an immediate consequence of Theorem 7.23.13 and Theorem 7.23.16.

7.23.17 Theorem. *Let $w \in \mathfrak{S}_n$ and* $\mathrm{sh}(w) = \lambda$. *Then for all $i \geq 1$, $\lambda'_1 + \cdots + \lambda'_i$ is equal to the length of the longest subsequence of w that can be written as a union of i decreasing subsequences. In particular, λ'_1 is the length of the longest decreasing subsequence of w.*

Write $\mathrm{ds}(w)$ for the length of the longest decreasing subsequence of w. In the same way that Corollary 7.23.12 was obtained from Proposition 7.23.10, we deduce from both Proposition 7.23.10 and Theorem 7.23.17 the following result.

7.23.18 Corollary. *Let $g_{p,q}(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ satisfying* $\mathrm{is}(w) = p$ *and* $\mathrm{ds}(w) = q$. *Then*

$$g_{p,q}(n) = \sum_{\substack{\lambda \vdash n \\ \lambda_1 = p,\ \lambda'_1 = q}} (f^\lambda)^2.$$

7.23.19 Example. (a) If $w \in \mathfrak{S}_{pq+1}$ then either $\mathrm{is}(w) > p$ or $\mathrm{ds}(w) > q$, since no partition $\lambda \vdash pq + 1$ satisfies $\lambda_1 \leq p$ and $\lambda'_1 \leq q$.

(b) Exactly one partition $\lambda \vdash pq$ satisfies $\lambda_1 = p$ and $\lambda'_1 = q$, viz., $\lambda = \langle p^q \rangle$. Hence, assuming $p \leq q$ (which entails no real loss of generality, since $g_{p,q}(n) = g_{q,p}(n)$), we get from Corollary 7.23.18 and the hook-length formula (Corollary 7.21.6) that

$$\begin{aligned} g_{p,q}(pq) &= \left(f^{\langle p^q \rangle}\right)^2 \\ &= \left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)}\right)^2. \end{aligned}$$

(c) Let $p, q > n$. There are exactly $p(n)$ partitions (where $p(n)$ is the number of partitions of n) $\lambda \vdash p+q+n-1$ satisfying $\lambda_1 = p$ and $\lambda_1' = q$, viz., $\lambda = (p, 1+\mu_1, 1+\mu_2, \ldots, 1+\mu_{q-1})$ where $\mu \vdash n$. Write for short $\lambda = (p, 1+\mu)$. Then

$$g_{p,q}(p+q+n-1) = \sum_{\mu \vdash n} \left(f^{(p,1+\mu)}\right)^2.$$

We can combine information concerning descent sets together with increasing and decreasing subsequences. For instance, the following result should be apparent to any reader who has followed this section up to here.

7.23.20 Proposition. *Let $g_{p,q,S,T}(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ satisfying* $\mathrm{is}(w) = p$, $\mathrm{ds}(w) = q$, $D(w^{-1}) = S$, $D(w) = T$. *Then for fixed p, q, and n,*

$$\sum_{S,T} g_{p,q,S,T}(n) L_{\alpha_S}(x) L_{\alpha_T}(y) = \sum_{\substack{\lambda \vdash n \\ \lambda_1 = p,\ \lambda_1' = q}} s_\lambda(x) s_\lambda(y).$$

7.24 Enumeration under Group Action

The theory of enumeration under group action, or *Pólya theory*, is a standard topic within enumerative combinatorics which is usually presented without the use of symmetric functions. However, symmetric function theory does lead to a more natural development and is more convenient for certain extensions of the theory. Pólya theory is centered around a certain generating function $Z_G(x)$ for the cycle types of elements of a subgroup G of $\mathfrak{S}_S$, the symmetric group of all permutations of the finite set S. Actually, there is no need at first for G to be a subgroup, so we make the definition for any *subset* of $\mathfrak{S}_S$. (There are in fact interesting results for certain subsets that aren't subgroups; see for instance Exercise 7.111.)

7.24.1 Definition. Let K be a subset of the symmetric group $\mathfrak{S}_S$. Define the *augmented cycle indicator* $\tilde{Z}_K$ of K to be the symmetric function

$$\tilde{Z}_K = \sum_{w \in K} p_{\rho(w)},$$

where $\rho(w)$ denotes the cycle type of w as in Section 7.7. The *cycle indicator* Z_K of K is defined by

$$Z_K = \frac{1}{\#K} \tilde{Z}_K = \frac{1}{\#K} \sum_{w \in K} p_{\rho(w)}.$$

Thus $\tilde{Z}_K$ or Z_K is just a generating function for elements of K according to their cycle type. Note that if $n = \#S$, then $\tilde{Z}_K$ and Z_K are homogeneous of degree n, i.e., $\tilde{Z}_K, Z_K \in \Lambda^n$. In the traditional exposition of Pólya theory mentioned above, the power sum symmetric function p_i is replaced by an indeterminate t_i, and later one substitutes p_i or a specialization of p_i for t_i. (We have done this ourselves in Example 5.2.10.) This approach represents only a change in viewpoint, since the p_i's are algebraically independent. The cycle indicator, regarded as a polynomial in the indeterminates $t_1, t_2, \ldots$, is then also called the *cycle index polynomial* of K. The main result of Pólya theory expresses Z_G in terms of the monomial symmetric functions (i.e., gives a combinatorial interpretation of the scalar product $\langle Z_G, h_\lambda \rangle$) when G is a subgroup of $\mathfrak{S}_S$.

7.24.2 Example. (a) Let S be the set of vertices of a square, and let G be the dihedral group of all (Euclidean) symmetries of the square, acting on the vertex set S. The identity element has cycle indicator p_1^4. The rotations by 90° or 270° have indicator p_4. The rotation by 180° has indicator p_2^2. The horizontal and vertical reflections also have indicator p_2^2. Finally the two diagonal reflections have indicator $p_1^2 p_2$. Hence

$$Z_G = \tfrac{1}{8}\left(p_1^4 + 2p_1^2 p_2 + 3p_2^2 + 2p_4\right).$$

If instead we let G be the group of rotational symmetries of the square, then we would get

$$Z_G = \tfrac{1}{4}\left(p_1^4 + p_2^2 + 2p_4\right).$$

(b) Let V be a p-element vertex set, and let $S = \binom{V}{2}$. The symmetric group $\mathfrak{S}_V$ acts naturally on S, viz., if $w \in \mathfrak{S}_V$ and $\{s,t\} \in S$, then $w \cdot \{s,t\} = \{w \cdot s, w \cdot t\}$. Thus we have a subgroup $\mathfrak{S}_p \cong G \subset \mathfrak{S}_S \cong \mathfrak{S}_{\binom{p}{2}}$. For instance, when $p = 4$ we have

$$Z_G = \tfrac{1}{24}\left(p_1^6 + 9p_1^2 p_2^2 + 8p_3^2 + 6p_2 p_4\right).$$

(c) Let G be the group $\mathfrak{S}_S$ of *all* permutations of the n-element set S, so $G \cong \mathfrak{S}_n$. Let $\lambda \vdash n$. Recall (equation (7.18)) that $n! z_\lambda^{-1}$ is the number of permutations $w \in \mathfrak{S}_n$ of cycle type λ. Hence from equation (7.22) we get $Z_G = h_n$.

Now let $X = \{c_1, c_2, \ldots\}$ be a set of "colors," and let X^S denote the set of all functions $f : S \to X$. Think of f as a "coloring" of the set S, where the element $s \in S$ receives the color $f(s)$. Define the *weight* x^f of $f \in X^S$ by

$$x^f = \prod_{i \geq 1} x_i^{\#f^{-1}(c_i)}.$$

Thus x^f is a monomial of degree $n = \#S$ in the variables $x_1, x_2, \ldots$, which tells us for each i how many elements of S are colored c_i. There is a natural action of G on X^S, viz., if $w \in G$, $f \in X^S$, and $s \in S$, then

$$(w \cdot f)(s) = f(w \cdot s).$$

Let X^S/G denote the set of orbits of this action. In other words, define an equivalence relation $\sim$ on X^S by $f \sim g$ if there exists $w \in G$ such that $g = w \cdot f$. Then the elements of X^S/G are the equivalence classes with respect to $\sim$. Note that if $f \sim g$ then $x^f = x^g$. Hence if $\mathcal{O} \in X^S/G$, then we can define $x^{\mathcal{O}}$ to be x^f for any $f \in \mathcal{O}$. Equivalence classes $\mathcal{O}$ are called *patterns*. The *pattern inventory* (also known by various other names, such as *store enumerator* and *configuration counting series*) of G is the generating function

$$F_G(x) = \sum_{\mathcal{O} \in X^S/G} x^{\mathcal{O}}.$$

Thus the coefficient of a monomial x^α in $F_G(x)$ is the number of orbits $\mathcal{O} \in X^S/G$ of weight x^α. Since the elements of X are all "treated equally," it follows that $F_G(x)$ is a symmetric function. In fact, $F_G \in \Lambda^n$, since by definition F_G is homogeneous of degree n.

7.24.3 Example. (a) Traditionally the elements of S have some combinatorial or geometric structure. For instance, let S be the set of vertices of a square and G the group of dihedral symmetries as in Example 7.24.2(a). Two colorings of the vertices are equivalent if there is a symmetry of the square taking one coloring to the other. Let $\lambda \vdash 4$. The coefficient of m_λ in F_G is the number of inequivalent vertex colorings (or colorings "up to symmetry") using λ_i i's. Here are representatives (orbit members) of each of these inequivalent colorings (using the colors $1, 2, \ldots$):

$$\begin{array}{ccccccccc} 11 & 11 & 11 & 21 & 11 & 12 & 12 & 12 & 13 \\ 11 & 12 & 22 & 12 & 23 & 31 & 34 & 43 & 42 \end{array}$$

Hence

$$F_G = m_4 + m_{31} + 2m_{22} + 2m_{211} + 3m_{1111}.$$

If the group G were instead the group of rotational symmetries of the square, then we would get the additional inequivalent colorings

$$\begin{array}{cccc} 11 & 13 & 14 & 14 \\ 32 & 24 & 23 & 32 \end{array}.$$

Hence in this case

$$F_G = m_4 + m_{31} + 2m_{22} + 3m_{211} + 6m_{1111}.$$

(b) Let V, S, and G be as in Example 7.24.2(b), so $S = \binom{V}{2}$. Let $X = \mathbb{P}$. A function $f \in X^S$ may be regarded as a graph on the vertex set V, allowing multiple edges but not loops. Namely, if $f(\{s,t\}) = j$, then place $j-1$ (indistinguishable) edges between vertices s and t. Two graphs $f, g \in X^S$ are equivalent if and only if there is a permutation w of their vertex set such that w preserves edges, i.e., there are j edges between s and t if and only if there are j edges between $w \cdot s$ and $w \cdot t$. In other words, $f \sim g$ if and only if f and g are *isomorphic* (as graphs). Thus for $\alpha \in \mathrm{Comp}(\binom{p}{2})$ the coefficient of the monomial x^α in F_G is equal to the number of nonisomorphic loopless graphs with p vertices and α_j edges of multiplicity $j-1$. For the case $p = 4$ we have

$$\begin{aligned} F_G &= m_6 + m_{51} + 2m_{42} + 3m_{33} + 2m_{411} + 4m_{321} + 6m_{222} \\ &\quad + 5m_{3111} + 9m_{2211} + 15m_{21111} + 30m_{111111}. \end{aligned}$$

Moreover, $F_G(1^m)$ is equal to the total number of nonisomorphic loopless graphs with p vertices and all edges of multiplicity at most $m-1$. In particular, $F_G(1,1)$ is the number of nonisomorphic simple graphs (no loops or multiple edges) on p vertices. Note that the specialization $F_G(1^m)$ is obtained by expanding F_G in terms of the power sums p_i and setting each $p_i = m$. More generally, the coefficient of q^j in the polynomial $F_G(1, q, \dots, q^{m-1})$ is the number of nonisomorphic loopless graphs with p vertices, all edges of multiplicity at most $m-1$, and exactly j edges. Thus for instance when $p = 4$ we get

$$F_G(1,q) = 1 + q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6,$$

the generating function for nonisomorphic 4-vertex simple graphs by number of edges.

NOTE. Since traditionally the variables $t_1, t_2, \dots$ of Z_G and F_G correspond to the power sums $p_1, p_2, \dots$ instead of the *arguments* $x_1, x_2, \dots$ of the p_i's, what we write as $F_G(1, q, \dots, q^{m-1})$ is traditionally written

$$F_G\big(1 + q + \cdots + q^{m-1}, 1 + q^2 + \cdots + q^{2(m-1)}, 1 + q^3 + \cdots + q^{3(m-1)}, \dots\big).$$

(c) Let G be the group $\mathfrak{S}_S$ of *all* permutations of the n-element set S, as in Example 7.24.2(c). For simplicity take $S = [n]$, so $\mathfrak{S}_S = \mathfrak{S}_n$. Two colorings $f, g \in X^S$ are equivalent if and only if their coimages have the same type, i.e., the multisets $\{\#f^{-1}(c) : c \in X\}$ and $\{\#g^{-1}(c) : c \in X\}$ are the same. Hence the coefficient of every monomial x^α of degree n in $F_{\mathfrak{S}_n}$ is equal to 1, so

$$F_{\mathfrak{S}_n} = \sum_{\lambda \vdash n} m_\lambda = h_n.$$

We have defined two symmetric functions Z_G and F_G associated with the permutation group G. The cycle indicator Z_G is defined in terms of the power sum symmetric functions p_λ, while the pattern inventory is defined in terms of the monomial symmetric functions m_λ. The main result of Pólya theory is that these two symmetric functions are equal.

7.24.4 Theorem. *Let S be a finite set. For any subgroup G of $\mathfrak{S}_S$ we have $Z_G = F_G$.*

The proof is based on a simple but fundamental result on permutation groups known as *Burnside's lemma* (though see the Notes for the correct attribution).

7.24.5 Lemma. *Let Y be a finite set and G a subgroup of $\mathfrak{S}_Y$. For each $w \in G$ let*

$$\mathrm{Fix}(w) = \{y \in Y : w(y) = y\},$$

so $\#\mathrm{Fix}(w)$ is the number of cycles of length one in the permutation w. Let Y/G be the set of orbits of G. Then

$$\#(Y/G) = \frac{1}{\#G} \sum_{w \in G} \#\mathrm{Fix}(w).$$

In other words, the average number of elements of Y fixed by an element of G is equal to the number of orbits.

Proof. For $y \in Y$ let $G_y = \{w \in G : w(y) = y\}$, the *stabilizer* of y. Then

$$\begin{aligned}
\frac{1}{\#G} \sum_{w \in G} \mathrm{Fix}(w) &= \frac{1}{\#G} \sum_{w \in G} \sum_{\substack{y \in Y \\ w(y)=y}} 1 \\
&= \frac{1}{\#G} \sum_{y \in Y} \sum_{\substack{w \in G \\ w(y)=y}} 1 \\
&= \frac{1}{\#G} \sum_{y \in Y} \#G_y.
\end{aligned}$$

Let $Gy = \{w(y) : w \in G\}$, the orbit of G containing y. The multiset of elements $w(y)$, $w \in G$, contains every element in the orbit Gy the same number of times [why?], viz., $\#G/\#Gy$ times. Thus y occurs $\#G/\#Gy$ times among the elements $w(y)$, so

$$\frac{\#G}{\#Gy} = \#G_y.$$

Hence

$$\frac{1}{\#G}\sum_{w\in G}\#\mathrm{Fix}(w) = \frac{1}{\#G}\sum_{y\in Y}\frac{\#G}{\#Gy}$$
$$= \sum_{y\in Y}\frac{1}{\#Gy}.$$

For a fixed orbit $\mathcal{O} \in Y/G$, we have $Gy = \mathcal{O}$ if and only if $y \in \mathcal{O}$. Hence the term $1/\#\mathcal{O}$ appears $\#\mathcal{O}$ times in the last sum above, so the sum is equal to the number of orbits. □

Proof of Theorem 7.24.4. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n, and let $\mathcal{C}_\alpha$ denote the set of all "colorings" $f \in X^S$ with color c_j used α_j times. The set $\mathcal{C}_\alpha$ is invariant under the action of G on X^S. Let w_α denote the action of w on $\mathcal{C}_\alpha$. We want to apply Burnside's lemma (Lemma 7.24.5) to compute the number of orbits, so we need to find $\#\mathrm{Fix}(w_\alpha)$.

In order for $f \in \mathrm{Fix}(w_\alpha)$, we must color S so that (a) in any cycle of w, all the elements get the same color, and (b) the color c_j appears α_j times. It follows that

$$\#\mathrm{Fix}(w_\alpha) = [x^\alpha]\prod_j p_j^{m_j(w)} = [x^\alpha]p_{\rho(w)},$$

where $m_j(w)$ is the number of cycles of w of length j. Hence

$$p_{\rho(w)}(x) = \sum_\alpha \#\mathrm{Fix}(w_\alpha)x^\alpha.$$

Now sum over all $w \in G$ and divide by $\#G$. The left-hand side becomes Z_G, while by Burnside's lemma the right-hand side becomes F_G. □

When we put $x = 1^m$ in Theorem 7.24.4 we get the following result (which can also be obtained directly from Burnside's theorem).

7.24.6 Corollary. *Let $N_G(m)$ be the total number of inequivalent colorings of S from a set of m colors. Then*

$$N_G(m) = \frac{1}{\#G}\sum_{w\in G} m^{c(w)},$$

where $c(w)$ is the number of cycles of w.

Note (for algebraists). Let G be a subgroup of $\mathfrak{S}_S$ as above. Let $r = [\mathfrak{S}_S : G]$, the index of G in $\mathfrak{S}_S$. The group $\mathfrak{S}_S$ acts on the (right) cosets of G, defining a (transitive) permutation representation of $\mathfrak{S}_S$. Representing a permutation by the corresponding permutation matrix gives a linear representation $\sigma^G : \mathfrak{S}_S \to \mathrm{GL}(r, \mathbb{C})$. This linear representation is equivalent to the induced representation

$1_G^{\mathfrak{S}_S}$ (the induction of the trivial representation of G to $\mathfrak{S}_S$). Let $\chi^G : \mathfrak{S}_S \to \mathbb{C}$ denote the character of this representation, i.e., $\chi^G(w) = \operatorname{tr} \sigma^G(w)$. If $w \in \mathfrak{S}_S$ and $\rho(w) = \lambda$ (the cycle type of w), then

$$\chi^G(w) = \frac{\#\mathfrak{S}_S}{\#\mathfrak{S}^\lambda} \cdot \frac{\#(\mathfrak{S}^\lambda \cap G)}{\#G} = \frac{z_\lambda \cdot \#(\mathfrak{S}^\lambda \cap G)}{\#G}, \tag{7.118}$$

where $\mathfrak{S}^\lambda$ denotes the subset (conjugacy class) of $\mathfrak{S}_S$ of all permutations of cycle type λ. It follows from (7.118) that

$$\operatorname{ch}(\chi^G) = Z_G, \tag{7.119}$$

the cycle indicator of G, where ch is the characteristic map of Section 7.18. Hence Pólya theory is closely related to the interaction between the representation theory of the symmetric group and the theory of symmetric functions.

Write

$$\chi^G = \sum_{\lambda \vdash n} a_\lambda \chi^\lambda$$

as the decomposition of the character χ^G in terms of irreducible characters χ^λ. Hence a_λ is the multiplicity of χ^λ in χ^G, and so $a_\lambda \in \mathbb{N}$. Since the map ch is linear and $\operatorname{ch}(\chi^\lambda) = s_\lambda$, there follows from equation (7.119) that

$$Z_G = \sum_{\lambda \vdash n} a_\lambda s_\lambda. \tag{7.120}$$

Thus we have an algebraic interpretation of the expansion of Z_G in terms of Schur functions, showing in particular that the coefficients $a_\lambda = \langle Z_G, s_\lambda \rangle$ are nonnegative integers, or equivalently that Z_G is s-integral (which is trivial) and s-positive. No purely combinatorial or "formal" proof of the s-positivity of Z_G is known; all known proofs (which are essentially equivalent) use representation theory. (The fact that a_λ is an integer is immediate from Theorem 7.24.4.) There are a myriad of other "positivity theorems" in the theory of symmetric functions whose only known proofs use representation theory.

Notes

A good source of information for the early history of symmetric functions, such as the fundamental theorem of symmetric functions (Theorem 7.4.4) and the symmetry of the matrix $(M_{\lambda\mu})$ (Corollary 7.4.2), is the article **[154]** by Karl Theodor Vahlen. In particular, the first published work on symmetric functions is due to Albert Girard **[49]** in 1629, who gives an explicit formula expressing p_n in terms of the e_λ's (which we can obtain by equating coefficients of t^n in the formula $\sum_{n\geq 1} \frac{1}{n}(-1)^{n-1} p_n t^n = \log \sum_{k\geq 0} e_k t^k$). Other early researchers on symmetric functions include Gabriel Cramer, Francesco Faà di Bruno, Isaac Newton, and Edward Waring.

The earliest reference to Schur functions is the 1815 paper **[18]** of Augustin Louis Cauchy. (Cauchy's paper was submitted for publication in 1812.) He defines Schur functions (though of course not by that name) in the variables $x_1, \ldots, x_n$ as the bialternants of Theorem 7.15.1 and proves that they are indeed symmetric polynomials, and that (using our notation) $s_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ and (more trivially) $s_{1^n}(x_1, \ldots, x_n) = x_1 \cdots x_n$. The next paper of interest to us is that of Karl Gustav Jacob Jacobi **[64]** in 1841, in which he states without proof the Jacobi–Trudi identity (Theorem 7.16.1) in the case $\mu = \emptyset$, i.e., for ordinary Schur functions s_λ (defined as bialternants). In 1864 Jacobi's student Nicolò Trudi **[153]** gave a complete proof of the Jacobi–Trudi identity. The dual Jacobi–Trudi identity (Corollary 7.16.2) is due to Hans Eduard von Nägelsbach **[108]** in 1871, and was given a simpler proof in 1875 by Carl Franz Albert Kostka **[73]**. Our first proof of the Jacobi–Trudi formula (based on the theory of nonintersecting lattice paths) follows Ira Martin Gessel and Gérard Xavier Viennot **[47][2.5]**. See **[151**, Ch. 4.5] for an exposition. Our second proof comes from Ian G. Macdonald **[96**, Ch. I, (5.4)].

The expansion of $\prod(1 - x_i y_j)^{-1}$ in terms of Schur functions (Theorem 7.12.1) is universally attributed to Cauchy and is therefore called the "Cauchy identity." We have been unable, however, to find a clear statement of this identity in the work of Cauchy. On the other hand, the Cauchy identity is an almost trivial consequence of two results of Cauchy. The first result is the Binet–Cauchy formula for the determinant of the product of an $m \times n$ matrix and an $n \times m$ matrix. (For an interesting discussion of Cauchy's precise contribution to this formula, see **[106**, pp. 92–131]. It is at any rate clear from Cauchy's later work that he was adept at its use.) The second result of Cauchy **[19**, eqn. (10)] is the determinant evaluation (stated slightly differently)

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{n} = \frac{a_\delta(x) a_\delta(y)}{\prod(1 - x_i y_j)}, \tag{7.121}$$

where a_δ is the Vandermonde determinant (equation (7.55)). Applying the Binet–Cauchy formula to the product $A(x)A(y)^t$, where

$$A(z) = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots \\ 1 & z_2 & z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \\ 1 & z_n & z_n^2 & \cdots \end{bmatrix},$$

gives

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{n} = \sum_{\ell(\lambda) \le n} a_{\lambda+\delta}(x) a_{\lambda+\delta}(y).$$

Hence from (7.121) we have

$$\frac{1}{\prod_{i,j=1}^{n}(1-x_iy_j)} = \sum_{\ell(\lambda)\leq n} \frac{a_{\lambda+\delta}(x)}{a_\delta(x)} \cdot \frac{a_{\lambda+\delta}(y)}{a_\delta(y)},$$

and Cauchy's identity follows from Theorem 7.15.1 (which was Cauchy's definition of Schur functions).

Kostka **[74][75]** was the first person to consider the expansion of Schur functions into monomials, whence the term "Kostka number" for the numbers $K_{\lambda\mu}$ in the expansion $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ (equation (7.35)). (Foulkes [**36**, p. 85] suggested calling the matrix $(K_{\lambda\mu})$ the *Kostka matrix.*) In **[74]** Kostka gives a table of the Kostka numbers $K_{\lambda\mu}$ and the entries $(K^{-1})_{\lambda\mu}$ of the inverse Kostka matrix for $\lambda, \mu \vdash n \leq 8$. In **[75]** he extends the tables up to $n = 11$. Kostka asserts that the numbers in his tables also give the expansion of the h_μ's and m_μ's in terms of the s_λ's; this assertion is equivalent to the orthonormality of the s_λ's (Corollary 7.12.2).

Oscar Howard Mitchell **[103]** looked further at Kostka numbers in 1882. He showed that they were nonnegative without obtaining an explicit combinatorial interpretation of them, and he evaluated $s_\lambda(1^n)$ in the form obtained by letting $q \to 1$ in equation (7.105). A simpler proof of this evaluation of $s_\lambda(1^n)$ was later given by William Woolsey Johnson [**66**, §13]. Some efforts were subsequently made to find combinatorial interpretations of Kostka numbers in special cases, a typical example being Thomas Muir **[105]**. The first explicit statement of which we are aware that the Kostka number $K_{\lambda\mu}$ counts SSYTs of shape λ and type μ is due to Dudley Ernest Littlewood [**85**, Thm. VI] (see also [**88**, Ch. 10.1, IX]). The definition of the Gelfand–Tsetlin patterns of equation (7.37) was given by Israel M. Gelfand and Mikhail L. Tsetlin [**45**, (3)] in connection with the representation theory of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

Skew Schur functions were first investigated by Nägelsbach **[108]** and Alexander Craig Aitken **[1][2]**, in the form given by equation (7.68). Aitken proved what in our notation is the formula $\omega s_{\lambda/\nu} = s_{\lambda'/\nu'}$ (Theorem 7.15.6), and in a later paper **[3]** gave the combinatorial interpretation of $s_{\lambda/\mu}$ in terms of skew SSYTs. The connection between skew Schur functions and Littlewood–Richardson coefficients (Theorem 7.15.4) appears in [**88**, eqn. VIII, p. 110].

While the work described above was being carried out, a completely independent but ultimately equivalent avenue of research was being developed by a group of geometers. Certain enumerative questions involving intersections of subspaces of a vector space were reduced to algebraic computations formally the same as basic results in the theory of Schur functions. This general approach to enumerative geometry was first developed by Hermann Cäsar Hannibal Schubert (see **[68]** for references) and is now known as the *Schubert calculus.* It is not our intention to explain the Schubert calculus and its connections with symmetric functions here, but we will briefly mention the main highlights. The geometric result equivalent to Pieri's rule (Theorem 7.15.7) was given in 1893 by Mario Pieri **[118]**. A determinantal formula expressing a Schubert cycle in terms of special Schubert cycles was published in 1903 by Giovanni Zeno Giambelli [**48**]. This formula was formally the same as the Jacobi-Trudi identity (Theorem 7.16.1), thereby establishing the formal equivalence (though no one realized it yet) of the Schubert calculus with the algebra of Schur functions. The work of classical geometers such as Schubert, Pieri, and Giambelli on the Schubert calculus was vindicated rigorously by Charles Ehresmann [**28**], Bartel Leendert van der Waerden [**158**], William Vallance Douglas Hodge [**61**][**62**] *et al.* The formal equivalence between Schubert calculus and the algebra of Schur functions was first pointed out by Léonce Lesieur [**84**] in 1947. Conceptual explanations of this seeming "coincidence" were first given by Geoffrey Horrocks [**63**] and James B. Carrell [**17**]. More details of this history are discussed by William Fulton [**41**, pp. 278–279]. For three surveys of the Schubert calculus, the third one focusing on the connections with combinatorics, see [**68**][**69**][**148**].

The idea of unifying much of the theory of symmetric functions using linear algebra (scalar product, dual bases, involution, etc.) is due to Philip Hall in his important paper [**54**]. This paper was overlooked (undoubtedly because of its obscure place of publication) until an exposition (with most of the missing proofs filled in) was given by Stanley [**145**]. Stanley learned of Hall's paper from Robert James McEliece, who studied with Hall during the 1964–65 academic year in Cambridge (England). A later exposition of Hall's work was given by Macdonald [**94**].

The RSK algorithm (known by a variety of other names: either "correspondence" or "algorithm" in connection with some subset of the names Robinson, Schensted, and Knuth) was first described, in a rather vague form, by Gilbert de Beauregard Robinson [**131**, §5], as a tool in an attempted proof of the Littlewood–Richardson rule (Appendix 1, §A1.3). (See the Notes to Appendix 1 for the history of the Littlewood–Richardson rule.) The RSK algorithm was later rediscovered by Craige Eugene Schensted (see below), but no one actually analyzed Robinson's work until this was done by Marc A. A. van Leeuwen [**82**, §7]. It is interesting to note that Robinson says in a footnote on page 754 that "I am indebted for this association I to Mr. D. E. Littlewood." Van Leeuwen's analysis makes it clear that "association I" gives the recording tableau Q of the RSK algorithm $w \xrightarrow{\text{RSK}} (P, Q)$. Thus it might be correct to say that if $w \in \mathfrak{S}_n$ and $w \xrightarrow{\text{RSK}} (P, Q)$, then the definition of P is due to Robinson, while the definition of Q is due to Littlewood.

No further work related to Robinson's construction was done until Schensted published his seminal paper [**136**] in 1961. (For some information about the unusual
* life of Schensted, see [**7**].) Schensted's purpose was the enumeration of permutations in $\mathfrak{S}_n$ according to the length of their longest increasing and decreasing subsequences. For further information see the discussion of Section 7.23 below. According to Knuth [**71**, p. 726], the connection between the work of Robinson and that of Schensted was first pointed out by Marcel Paul Schützenberger, though as mentioned above the first person to describe this connection precisely was van Leeuwen.

Robinson states on page 755 of his paper [**131**] on the RSK algorithm that "it is not difficult to see" that if $w \xrightarrow{\text{RSK}} (P, Q)$, then $w^{-1} \xrightarrow{\text{RSK}} (Q, P)$. No indication of a proof of this fundamental result (our Theorem 7.13.1) is given. A proof was finally given by Schützenberger [**140**] in 1963. Schützenberger was the first to realize the great significance of Schensted's work for the theory of symmetric functions and the symmetric group. Our first proof of Theorem 7.13.1 follows Donald Ervin Knuth [**71**, §4][**72**, Ch. 5.1.4]. Corollary 7.13.8, which we derived from Theorem 7.13.1, was first proved by Issai Schur [**138**] (repeated in [**5.53**, VII.47]).

The theory of growth diagrams, which we used for our second proof of Theorem 7.13.1, was developed by Sergey V. Fomin [**31**][**32**][**33**][**34**]. Some further work was done by Thomas W. Roby [**132**][**133**]. Before Fomin a different "geometric" theory of the RSK algorithm had been developed by Viennot [**155**][**156**].

The extension of the RSK algorithm from permutations to arbitrary sequences of nonnegative integers (or from permutation matrices to $\mathbb{N}$-matrices of finite support) is due to Knuth [**71**]. Although, as pointed out by Lemma 7.11.6, this extension is actually equivalent to the original case, it is essential to use the more general form when dealing with symmetric functions. Thus for instance we obtained a direct bijective proof of the Cauchy identity (Theorem 7.12.1), as first done by Knuth [**71**, p. 726]. Knuth's paper deals with a number of further topics related to the RSK algorithm, in particular, a proof of the symmetry result Theorem 7.13.1 (working directly with $\mathbb{N}$-matrices of finite support and not reducing to the case of permutations), the definition and basic properties of the dual RSK algorithm of Section 7.14, and the definition and basic properties of Knuth equivalence, as discussed in Appendix 1. Some further variations of the RSK algorithm were given by William H. Burge [**13**] (see Exercises 7.28(c,e) and 7.29(a,b)). For good overviews of more recent work related to the RSK algorithm, see [**134**][**83**].

By this time the work described above had entered the general consciousness of algebraic and enumerative combinatorics, and the floodgates were opened. We will not attempt a survey of the enormous amount of more recent work done on symmetric functions, Young tableaux, the RSK algorithm, etc., but we will give some references to work discussed in the text. For further developments, see the end of these Notes, the Exercises to this chapter, and the Notes to Appendix 1.

Standard Young tableaux (SYT) were first enumerated by Percy Alexander MacMahon [**99**, p. 175] (see also [**101**, §103]). MacMahon formulated his result in terms of the ballot sequences or lattice permutations of Proposition 7.10.3(c,d), and stated the result not in terms of the product of hook lengths as in Corollary 7.21.6, but rather using the right-hand side of the case $q = 1$ of equation (7.101). The formulation in terms of hook lengths is due to James Sutherland Frame and appears first in the paper [**38**, Thm. 1] of Frame, Robinson, and Robert McDowell Thrall; hence it is sometimes called the "Frame–Robinson–Thrall hook-length formula." (The actual definition of standard Young tableaux is due to Alfred Young [**162**, p. 258].) Independently of MacMahon, Ferdinand Georg Frobenius [**5.27**, eqn. (6)] obtained the same formula for the degree of the irreducible character χ^λ

of $\mathfrak{S}_n$ as MacMahon obtained for the number of lattice permutations of type λ. Frobenius was apparently unaware of the combinatorial significance of deg χ^λ, but Young showed in [**162**, pp. 260–261] that deg χ^λ was the number of SYT of shape λ, thereby giving an independent proof of MacMahon's result. (Young also provided his own proof of MacMahon's result in [**162**, Thm. II].) A number of other proofs of the hook-length formula were subsequently found. Curtis Greene, Albert Nijenhuis, and Herbert Saul Wilf [**51**] gave an elegant probabilistic proof. The proof we gave in Section 7.22 based on the Hillman–Grassl correspondence appears in [**60**] and shows very clearly the role of hook lengths, though the proof is not completely bijective. A bijective version was later given by Christian Krattenthaler [**76**]. Completely bijective proofs of the hook-length formula were first given by Deborah Franzblau and Doron Zeilberger [**39**] and by Jeffrey Brian Remmel [**127**]. An exceptionally elegant bijective proof was later found by Jean-Christophe Novelli, Igor Pak, and Alexander V. Stoyanovskii [**113**].

For more information on the Hopf algebra approach to symmetric functions mentioned at the end of Section 7.15 see [**163**].

The determinantal formula (7.71) for $f^{\lambda/\mu}$ is due to Aitken [**3**, p. 310], who deduced it just as we have done from the Jacobi–Trudi identity for $s_{\lambda/\mu}$ (Theorem 7.16.1). Aitken's result was rediscovered by Walter Feit [**30**]. A generalization due to Germain Kreweras is given by Exercise 3.63.

The theory of representations of finite groups was developed by Frobenius; see [**24**][**56**][**57**][**58**][**59**] for an interesting discussion of this development. In particular, Frobenius computed the irreducible characters of $\mathfrak{S}_n$ (in the form given by Corollary 7.17.4 or equation (7.77)) in [**5.27**]. Much subsequent work on the representation theory of $\mathfrak{S}_n$ was done by Alfred Young; see Sagan [**135**] for a nice exposition of Young's work and its connection with symmetric functions. The Murnaghan–Nakayama rule (regarded as the formula (7.75) for $\chi^\lambda(\mu)$) is actually due to Littlewood and Archibald Read Richardson [**89**, §11]. Statements of this rule by Francis Dominic Murnaghan and Tadasi Nakayama are given in [**107**, (13)] and [**109**, §9].

The definition of quasisymmetric functions is due to Gessel, though they had appeared implicitly in earlier work. Gessel used quasisymmetric functions to prove such results as our Corollaries 7.23.6 and 7.23.8. The basic results on (P, ω)-partitions used here (Theorem 7.19.4 and Corollary 7.19.5) were given by Stanley [**3.28**, Ch. 2][**3.29**, Ch. 1], though not using the language of quasisymmetric functions. Proposition 7.19.11 seems to be due to George Lusztig (unpublished) and Stanley [**149**, Prop. 4.11]. Further work on quasisymmetric functions appears e.g. in [**26**][**102**] and the references given there.

Plane partitions were discovered by MacMahon in a series of papers which were not appreciated until much later. (See MacMahon's book [**101**, §§IX and X] for an exposition of his results.) MacMahon's first paper dealing with plane partitions was [**98**]. In Article 43 of this paper he gives the definition of a plane partition (though not yet with that name), and then goes on to discuss the six aspects of a plane partition. In Article 51 he conjectures that the generating function for plane partitions is the product

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-4}\cdots$$

(our Corollary 7.20.3). He also suggests (but doesn't call it a conjecture) that the generating function for r-dimensional partitions (whose diagram would be a finite order ideal of $\mathbb{N}^{r+1}$) is

$$\prod_{i\geq 1}\left(1-x^i\right)^{-\binom{i+r-2}{r-1}}. \tag{7.122}$$

MacMahon apparently never realized that this generating function, even for $r=3$, is incorrect, though he does mention in [**101**, vol. 2, footnote on p. 175] that an even stronger result is false. (When $r=3$, the smallest exponent n for which the coefficient of x^n in (7.122) fails to be the number of 3-dimensional partitions of n is $n=6$.) The incorrectness of (7.122) was first shown by Atkin *et al.* [**6**] and later by E. M. Wright [**160**]. V. S. Nanda [**110**][**111**] erroneously assumes (7.122) to be correct for $r=3$, stating in [**110**, p. 593] that

MacMahon has not given a rigorous derivation of the generating function for solid partitions. But a simple reasoning as in the case of plane partitions leads to the generating function

$$\frac{1}{(1)(2)^3(3)^6\cdots(s)^{\frac{1}{2}(s^2+s)}\cdots}$$

for solid partitions when there is no restriction on part magnitude.

Further computations by Knuth [**70**] show how useless it seems to write the generating function for 3-dimensional partitions in the form $\prod_{i\geq 1}(1-x^i)^{a_i}$. Returning to the paper [**98**] of MacMahon, in Article 52 he conjectures our Theorem 7.20.1, Corollary 7.20.2, and finally Theorem 7.21.7 (which includes all the previous conjectures as special cases). MacMahon goes on in Articles 56–62 to prove his conjecture in the case of plane partitions with at most 2 rows and c columns (the case $r=2$ of our Theorem 7.20.1), mentioning on page 662 that an independent solution was obtained by Andrew Russell Forsyth. (Though a publication reference is given to Forsyth's paper, apparently it never actually appeared.)

We will not attempt to describe MacMahon's subsequent work on plane partitions, except to say that the culmination of his work appears in [**101**, Art. 495], in which he proves his main conjecture from his first paper [**98**] on plane partitions, viz., our Theorem 7.21.7. MacMahon's proof is quite lengthy and indirect. We can regard a plane partition whose shape is contained in the partition $\lambda \vdash p$ as a P_λ-partition (in the sense of Section 4.5), where P_λ is the poset defined after Corollary 7.19.5. (We regard P_λ as a natural partial order on $[p]$, as in Section 4.5.) MacMahon anticipates the theory of P-partitions (as pointed out in the Notes to Chapter 4) by essentially establishing Exercise 4.24(b) for $P=P_\lambda$. He manages to convert this expression into a determinant, and then to evaluate the determinant when $\lambda=\langle c^r\rangle$.

Some interesting recent work on the shape of the diagram of a "typical" plane partition fitting in an $r \times c \times t$ box was done by Henry Cohn, Michael Larsen, and James Propp [**23**].

The theory of plane partitions stayed rather dormant until the early 1960s, when Basil Gordon and his students made a number of new contributions (but without using symmetric functions). For further discussion and references to the work of Gordon, see [**146**]. Also about this time Carlitz [**16**] gave a simpler proof of MacMahon's main result (our Theorem 7.21.7). (Limiting cases had earlier been given simpler proofs by Chaundy [**21**].) Then in 1972 Edward Anton Bender and Knuth [**8**], in an important paper, showed the connection between the theory of symmetric functions and the enumeration of plane partitions. They gave simple proofs, based on the RSK algorithm, of many of the results (and some generalizations) of Gordon *et al.*, as well as the first bijective proof (the same proof that we give) of our Theorem 7.20.1 in the case $q = 1$. The introduction of the variable q in Theorem 7.20.1 and related results to keep track of the trace of a plane partition is due to Stanley [**146**, Thm. 19.3][**147**].

The "hook-content formula" for $s_\lambda(1, q, \ldots, q^{n-1})$ (Theorem 7.21.2) was first stated explicitly by Stanley [**3.28**, Thm. V.2.3][**3.29**, Prop. 21.3][**146**, Thm. 15.3]. Earlier a less explicit statement (using the right-hand side of our equation (7.101) instead of the left-hand side) was given by Littlewood and Richardson [**90**, Thm. I] [**88**, I. on p. 124]. A bijective proof based on an involution principle argument was given by Remmel and Roger Whitney [**128**]. Krattenthaler [**77**] then gave a bijective proof not involving the involution principle, and generalized it in [**78**]. Finally Krattenthaler [**79**] gave a bijective proof of Theorem 7.21.2 analogous to the Novelli–Pak–Stoyanovskii proof [**113**] of the hook-length formula.

The generating function for symmetric plane partitions with at most r rows (the case $q = 1$ of Theorem 7.20.4) was conjectured by MacMahon [**100**, p. 153][**101**, Art. 520] and first proved by Gordon [**50**]. Bender and Knuth [**8**, pp. 42–43] give the same proof as ours based on the RSK algorithm. MacMahon actually makes a stronger conjecture, viz., an explicit formula for the generating function for symmetric plane partitions with at most r rows and largest part at most m (the symmetric analogue of our Theorem 7.21.7). This result proved to be considerably less tractable than the unrestricted (i.e., nonsymmetric) case; it was first proved by George E. Andrews [**4**][**5**]. A subsequent proof based on the Weyl character formula for type B_n was given by Macdonald [**92**, Exam. I.5.17, p. 52][**96**, Exam. I.5.17, pp. 84–85]. A somewhat different proof based on representation theory is due to Robert Alan Proctor [**121**, Prop. 7.2]. For further information related to the enumeration of symmetry classes of plane partitions, see the solution to Exercise 7.103(b).

The generating function (7.112) for reverse plane partitions of a fixed shape was first obtained by Stanley [**3.28**, Cor. V.2.6, p. 174][**146**, Prop. 18.3]; the proof is the same as our first proof of Theorem 7.22.1 (based on symmetric functions). The elegant bijective proof given after our first proof is due to Abraham P. Hillman and

Richard M. Grassl [**60**]. A different bijective proof was later given by Remmel and Whitney [**129**].

The fundamental result relating the RSK algorithm to descent sets (Lemma 7.23.1) is due to Schützenberger [**140**, Remarque 2] and was independently discovered later by Herbert Owen Foulkes [**37**, Thm. 8.1]. Foulkes anticipated Theorem 7.23.2 and its corollaries, but the first explicit statement of results of this nature is due to Gessel [**46**]. The basic connection between the RSK algorithm and increasing and decreasing subsequences (viz., that if $w \xrightarrow{\text{RSK}} (P, Q)$ where P and Q have shape λ, then $\text{is}(w) = \lambda_1$ and $\text{ds}(w) = \lambda'_1$) is the main result of Schensted [**136**]. Schensted's purpose in writing his paper was to obtain a formula for the number of $w \in \mathfrak{S}_n$ satisfying $\text{is}(w) = p$ and $\text{ds}(w) = q$, which he gave as his Theorem 7.3 (our Corollary 7.23.18). Theorems 7.23.13 and 7.23.17 are due to Greene; see Appendix 1 for further details. Example 7.23.19(a) is a famous result of Paul Erdős and George Szekeres [**29**, eqn. (8)] which was later given an elegant simple proof by Abraham Seidenberg [**141**]. Example 7.23.19(b) was posed as a problem by Stanley Rabinowitz [**122**] and solved by Stanley [**144**].

Burnside's lemma (Lemma 7.24.5) was actually first stated and proved by Frobenius [**40**, end of §4]. Frobenius in turn credits Cauchy [**20**, p. 286] for proving the lemma in the transitive case. Burnside, in the first edition of his book [**14**, §§118–119], attributes the lemma to Frobenius, but in the second edition [**15**] this citation is absent. For more on the history of Burnside's lemma, see [**112**] and [**161**]. Many authors (e.g., [**67**]) now call this result the Cauchy–Frobenius lemma. The cycle indicator $Z_G(x)$ (where G is a subgroup of $\mathfrak{S}_n$) was first considered by J. Howard Redfield [**124**], who called it the *group reduction function*, denoted $\text{Grf}(G)$. George Pólya [**119**] independently defined the cycle indicator, proved the fundamental Theorem 7.24.4, and gave numerous applications. For an English translation of Pólya's paper, see [**120**]. Much of Pólya's work was anticipated by Redfield. For interesting historical information about the work of Redfield and its relation to Pólya theory, see [**53**][**55**][**91**][**125**] (all in the same issue of *Journal of Graph Theory*). Subsequent to Pólya's work there have been a huge number of expositions, applications, and generalizations of Pólya theory. We mention here only the nice survey [**12**] by Nicolaas Govert de Bruijn and the paper [**123**] by Ronald C. Read, who was the first to consider the relevance of Schur functions to Pólya theory.

Appendix 1 has its own Notes, so now we discuss Appendix 2. The main result of Appendix 2, the classification of the rational representations of $\text{GL}(n, \mathbb{C})$ and the determination of their characters (Theorem A2.4), appears in the masterful doctoral dissertation of Schur [**137**]. He later gave a simpler proof [**139**] along the lines of the proof we have sketched. A slight refinement was given by Hermann Weyl [**159**, Thm. 4.4.C]. For a modern treatment of the work of Schur, see for instance [**96**, Ch. I, App. A]. Plethysm was introduced by D. E. Littlewood [**86**, p. 329] (see also [**88**, p. 289]). The term "plethysm" was suggested to Littlewood [**87**, p. 274] by M. L. Clark, after the Greek word *plethysmos* ($\pi\lambda\eta\theta\upsilon\sigma\mu\acute{o}\varsigma$) for "multiplication."

The connection between plethysm and wreath products (Theorem A2.8) is implicit in the determination of the characters of the wreath product $G\S\mathfrak{S}_n$ by Wilhelm Specht [**143**]. The special case when $\chi = 1_G^{\mathfrak{S}_k}$ and $\theta = 1_H^{\mathfrak{S}_n}$ for subgroups G and H of $\mathfrak{S}_k$ and $\mathfrak{S}_n$, respectively, is equivalent to Pólya's derivation of the cycle index of his so-called *Kranz group* (*Kranzgruppe*) [**119**, pp. 178–180]. Explicit statements of Theorem A2.8 (though not stated in the language of symmetric functions) may be found in [**93**, Rmk. 6.9][**65**, Ch. 5.4][**96**, App. A, (6.2)].

What should the reader who is interested in learning further about symmetric functions do next? An important topic, not treated at all here, is the myriad generalizations and variations of Schur functions. We present a short list, with some basic references (which are only provided as a means of entry into the subjects), of some of these generalizations.

- Hall–Littlewood symmetric functions [**96**, Ch. III]
- Shifted Schur functions (corresponding to shifted shapes), also called Schur P- and Q-functions [**52**][**96**, Ch. III.8]
- Super-Schur functions [**9**][**96**, Exam. I.3.23, pp. 58–60]
- Zonal symmetric functions [**96**, Ch. VII]
- Jack symmetric functions [**96**, Ch. VI.10][**150**]
- Macdonald symmetric functions [**43**][**96**, Ch. VI][**115**]
- Wreath-product Schur functions $s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(k)}}$ [**96**, Ch. I, App. B][**114**]
- Orthogonal and symplectic Schur functions [**152**]
- Flag Schur functions (or multi-Schur functions) [**157**][**95**, Ch. III][**126**]
- Factorial Schur functions and variations [**10**][**11**][**22**][**97**, §§4–6, 9][**104**]
- Shifted Schur functions (corresponding to "shifted" variables) [**116**][**117**]
- Noncommutative Schur functions of Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon [**44**][**80**][**27**][**81**]
- Noncommutative Schur functions of Fomin and Greene [**35**]
- Modular Schur functions (implicit in [**25**, §3.7])
- $\mathrm{GL}(n, \mathbb{F}_q)$-invariant Schur functions of Macdonald [**97**, §7]

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Chapter 7: Appendix 1

Knuth Equivalence, Jeu de Taquin, and the Littlewood–Richardson Rule

(by Sergey Fomin)

This Appendix is devoted to the study of several combinatorial constructions involving standard Young tableaux (SYTs) that lead to the proof of the Littlewood–Richardson rule, a combinatorial rule describing the coefficients in the Schur function expansion of an arbitrary skew Schur function (or in a product of two ordinary Schur functions).

Most of what follows can be straightforwardly generalized to semistandard Young tableaux (SSYTs). We do not do it here, in order to simplify the presentation.

A1.1 Knuth Equivalence and Greene's Theorem

The RSK algorithm $w \xrightarrow{\text{RSK}} (P, Q)$ associates to a permutation $w \in \mathfrak{S}_n$ a pair of SYTs: the *insertion tableau* P and the *recording tableau* Q; these tableaux have the same shape $\mathrm{sh}(w)$. In this section, we examine the following two questions:

- What are the conditions for two permutations to have the same shape $\mathrm{sh}(w)$?
- What are the conditions for two permutations to have the same insertion tableau P?

The first question has an answer involving a particular family of poset-theoretic invariants of permutations. The equivalence relation appearing in the second question can be described in terms of certain elementary transformations that change three consecutive entries of a permutation. We first state these two results, and devote the rest of this section to their proof.

For a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ and $k \in \mathbb{N}$, let $I_k = I_k(w)$ denote the maximal number of elements in a union of k increasing subsequences of w. Analogously, let D_l be the maximal size of a union of l decreasing subsequences of w. For example, for $w = 236145 \in \mathfrak{S}_6$, we have: $I_0 = 0$, $I_1 = 4$, $I_2 = I_3 = \cdots = 6$; $D_0 = 0$, $D_1 = 2$, $D_2 = 4$, $D_3 = 5$, $D_4 = D_5 = \cdots = 6$.

A1.1.1 Theorem (Greene's theorem). *Let $w \in \mathfrak{S}_n$ and* $\mathrm{sh}(w) = \lambda$. *Then, for any positive integer k and l,*

$$\begin{aligned} I_k(w) &= \lambda_1 + \cdots + \lambda_k , \\ D_l(w) &= \lambda_1' + \cdots + \lambda_l' . \end{aligned} \tag{A1.123}$$

(Note that Theorem A1.1.1 is a restatement of Theorems 7.23.13 and 7.23.17.)

To illustrate, take $w = 236145$. Then $w \xrightarrow{\mathrm{RSK}} (P, Q)$ with

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} \end{array}\,, \qquad Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}\,, \qquad \mathrm{sh}(w) = \lambda = (4, 2)\,.$$

To obtain the numbers I_k, we count boxes in the first several rows of the shape: $0, 4, 6, 6, \ldots$. Analogously, counting boxes in the first several columns of λ gives $0, 2, 4, 5, 6, 6, \ldots$, agreeing with our previous computations.

Theorem A1.1.1 implies that two permutations have the same shape $\mathrm{sh}(w)$ if and only if the values $I_1, I_2, \ldots$ (or $D_1, D_2, \ldots$) computed for these permutations are the same. Another direct implication of Theorem A1.1.1 is given below.

A1.1.2 Corollary. *For any permutation w, the sequences* $(I_1, I_2 - I_1, I_3 - I_2, \ldots)$ *and* $(D_1, D_2 - D_1, D_3 - D_2, \ldots)$ *define conjugate partitions.*

To formulate an answer to the second question posed at the beginning of this section, we will need the following definition.

A1.1.3 Definition. A *Knuth transformation* of a permutation is its transformation into another permutation that has one of the following forms:

$$\begin{array}{cccc} \cdots acb \cdots & \cdots cab \cdots & \cdots bac \cdots & \cdots bca \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \cdots cab \cdots & \cdots acb \cdots & \cdots bca \cdots & \cdots bac \cdots \end{array} \tag{A1.124}$$

where $a < b < c$ (all other entries remain intact). Thus each Knuth transformation switches two adjacent entries a and c provided an entry b satisfying $a < b < c$ is located next to a or c. Two permutations $u, v \in \mathfrak{S}_n$ are called *Knuth-equivalent* (denoted $u \overset{K}{\sim} v$) if one of them can be obtained from another by a sequence of Knuth transformations.

For example, the six permutations in (A1.125) below form a Knuth equivalence class; the ones that differ by a single Knuth transformation are connected by an edge.

$$\begin{tikzcd}[arrows=dash]
 & 51243 \arrow[r] \arrow[d] & 15243 \arrow[r] \arrow[d] & 12543 \\
54123 \arrow[r] & 51423 \arrow[r] & 15423 &
\end{tikzcd} \tag{A1.125}$$

A1.1.4 Theorem. *Permutations are Knuth-equivalent if and only if their insertion tableaux coincide.*

Permutations u and v are said to be *dual Knuth-equivalent* if $u^{-1} \overset{K}{\sim} v^{-1}$. For instance, 34521 is dual Knuth-equivalent to 12543. Recall the following symmetry of the RSK algorithm (see Theorem 7.13.1): the recording tableau for a permutation w is nothing but the insertion tableau for w^{-1}. Thus Theorem A1.1.4 implies that permutations have the same *recording* tableaux if and only if they are dual Knuth-equivalent.

Knuth equivalence classes can be given a more detailed description, which is provided in Theorem A1.1.6 below.

A1.1.5 Definition. Let T be a tableau. The *reading word* of T (denoted reading (T)) is the sequence of entries of T obtained by concatenating the rows of T bottom to top. For example, the tableau

$$\begin{array}{cccc}
 & & \boxed{1} & \boxed{2} \\
\boxed{3} & \boxed{5} & \boxed{6} & \boxed{8} \\
\boxed{4} & \boxed{7} & \boxed{9} &
\end{array}$$

has the reading word 479356812.

In what follows, we say that a tableau has a *straight shape* if its shape is a Young (or Ferrers) diagram. Observe that any straight-shape tableau T is uniquely reconstructed from its reading word. Indeed, to break a word $w = \text{reading}(T)$ into segments representing the rows of T, simply locate the descents of w.

A1.1.6 Theorem. *Each Knuth equivalence class contains exactly one reading word of a straight-shape SYT (call this tableau T), and consists of all permutations whose insertion tableau is T.*

For example, the only reading word in the Knuth equivalence class shown in (A1.125) is

$$54123 = \text{reading}(T)\,, \qquad T = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{4} & & \\ \boxed{5} & & \end{array}.$$

There are indeed six permutations with insertion tableau T (since $f^{\mathrm{sh}(T)} = 6$), and these are exactly the ones appearing in (A1.125).

Proofs of Theorems A1.1.1, A1.1.4, and A1.1.6

A1.1.7 Lemma. *For any k, the values $I_k(w)$ and $D_k(w)$ are invariant under Knuth transformations of a permutation w.*

Proof. It is enough to prove the invariance of the numbers I_k, since replacing a permutation $w = w_1 \cdots w_n$ by $w' = w_n \cdots w_1$ interchanges I_k and D_k, while clearly $u \overset{K}{\sim} v \Leftrightarrow u' \overset{K}{\sim} v'$. We need to show that I_k does not change under each of the two types of Knuth transformations:

$$u = \cdots acb \cdots \to v = \cdots cab \cdots, \qquad a < b < c,$$

and

$$u = \cdots bac \cdots \to v = \cdots bca \cdots, \qquad a < b < c$$

(cf. (A1.124)). Since these two cases are completely analogous, let us concentrate on the first one. Let $I_k(u) = m$. Obviously, $I_k(v) \leq m$. Moreover, the only situation where we may possibly have $I_k(v) < m$ is the following: *every* collection $\{\sigma_1, \ldots, \sigma_k\}$ of k disjoint increasing subsequences of u which jointly cover m elements has an element (say, σ_1) containing both a and c. Suppose this situation does indeed take place, and consider such a collection $\{\sigma_1, \ldots, \sigma_k\}$. If b does not belong to any σ_i, then simply replace c by b in σ_1, arriving at a contradiction with our assumption. We thus may assume that b belongs, say, to σ_2:

$$\sigma_1 = \left(u_{i_1} < \cdots < u_{i_s} < a < c < u_{i_{s+3}} < \cdots\right)$$

$$\sigma_2 = \left(u_{j_1} < \cdots < u_{j_t} < b < u_{j_{t+2}} < \cdots\right).$$

Then the increasing subsequences σ_1' and σ_2' defined by

$$\sigma_1' = (u_{i_1} < \cdots < u_{i_s} < a < b < u_{i_{t+2}} < \cdots)$$

$$\sigma_2' = (u_{j_1} < \cdots < u_{j_t} < c < u_{i_{s+3}} < \cdots)$$

will jointly cover the same elements of u as σ_1 and σ_2 do. The collection $\{\sigma_1', \sigma_2', \sigma_3, \ldots, \sigma_k\}$ will cover m elements, while not containing a subsequence to which both a and c belong. This contradicts our assumption, and the proof follows. □

We next show that the RSK insertion algorithm can be viewed as a sequence of Knuth transformations.

A1.1.8 Lemma. *Any permutation is Knuth-equivalent to the reading word of its insertion tableau.*

Proof. Recall from Section 7.11 that $P \leftarrow k$ denotes the result of inserting k into P. To prove the lemma, it suffices to show that, for any (straight-shape) SYT P and any positive integer k, we have

$$\mathrm{reading}(P) \cdot k \overset{K}{\sim} \mathrm{reading}(P \leftarrow k)\,, \tag{A1.126}$$

where $\cdot$ stands for concatenation. Because of the row-by-row nature of the RSK insertion algorithm, it is enough to check (A1.126) for a single-row tableau. This is a straightforward verification. □

A1.1.9 Corollary. *Let P be the insertion tableau for w. Then the permutations w and* reading(P) *have the same values of parameters I_k and D_k, for all k.*

Proof. Directly follows from Lemmas A1.1.7 and A1.1.8. □

A1.1.10 Lemma. *Let w be the reading word of a straight-shape SYT T. Then T is the insertion tableau for w.*

Proof. In the special case of a tableau word, the RSK insertion process is very simple: increasing segments of the word are consecutively placed atop each other, eventually forming the original tableau. □

Proof of Theorem A1.1.1. In view of Corollary A1.1.9 and Lemma A1.1.10, we may assume that w is a reading word of a straight-shape SYT T. To illustrate, let

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 8 \\ \hline 2 & 6 & 7 \\ \cline{1-3} 5 & 9 \\ \cline{1-2}\end{array}\quad ; \tag{A1.127}$$

then $w = 592671348$. Note that each row of T becomes an increasing subsequence in $w = \mathrm{reading}(T)$. Thus, for any k,

$$I_k(w) \geq \lambda_1 + \cdots + \lambda_k\,. \tag{A1.128}$$

Furthermore, the entries of each column of T form a decreasing subsequence in w. Therefore, for any l,

$$D_l(w) \geq \lambda'_1 + \cdots + \lambda'_l\,. \tag{A1.129}$$

Let us now consider a box lying at the border of the shape λ (such as the boxes containing the entries $5, 9, 6, 7, 4, 8$ in the example (A1.127)). Assume that this box is located in row k and column l. Then

$$(\lambda_1 + \cdots + \lambda_k) + (\lambda'_1 + \cdots + \lambda'_l) = n + kl\ . \tag{A1.130}$$

Combining (A1.128), (A1.129), and (A1.130), we obtain

$$I_k(w) + D_l(w) \geq n + kl\ .$$

On the other hand, an increasing and a decreasing subsequences may have at most one element in common. Hence

$$I_k(w) + D_l(w) \leq n + kl$$

for any k and l. Comparing this with the previous inequality, we conclude that $I_k(w) + D_l(w) = n + kl$, and moreover both (A1.128) and (A1.129) are actually *equalities* for the chosen values of k and l. Since every row and every column of λ contain at least one box that lies on the border, the identities (A1.123) hold for any k and l. □

A1.1.11 Corollary. *The shape* $\mathrm{sh}(w)$ *is invariant under Knuth transformations.*

Proof. In view of Theorem A1.1.1, the shape $\mathrm{sh}(w)$ is uniquely determined by the values $I_1(w), I_2(w), \ldots$, so the claim follows by Lemma A1.1.7. □

We will now show that a much stronger result holds.

A1.1.12 Corollary. *The insertion tableau P of a permutation w is invariant under Knuth transformations of w.*

Proof. For $k = 1, \ldots, n$, let $w_{(k)}$ denote the permutation in $\mathfrak{S}_k$ formed by the entries $1, \ldots, k$ of w. (For example, if $w = 236145$ then $w_{(4)} = 2314$.) Let us write $w \xrightarrow{\mathrm{RSK}} (P, Q)$ and $w_{(k)} \xrightarrow{\mathrm{RSK}} (P_{(k)}, Q_{(k)})$. Then $P_{(k)}$ is nothing but the tableau formed by the k smallest entries of P, since the larger entries do not interfere with the part of the insertion process that involves smaller entries. Now the crucial observation is the following: any Knuth transformation of w either does not change $w_{(k)}$ or else transforms the latter into a Knuth-equivalent permutation. By Corollary A1.1.11, this does not affect the shape $\mathrm{sh}(w_{(k)}) = \mathrm{sh}(P_{(k)})$. Since the tableau P can be viewed as a sequence of shapes

$$\emptyset \subset \mathrm{sh}(P_{(1)}) \subset \mathrm{sh}(P_{(2)}) \subset \cdots \subset \mathrm{sh}(P_{(n)}) = \mathrm{sh}(P)\ ,$$

and since all these shapes are unchanged by Knuth transformations, the proof follows. □

Proof of Theorems A1.1.4 and A1.1.6. It follows from Lemma A1.1.8 and Corollary A1.1.12 that two permutations are Knuth equivalent if and only if they have the same insertion tableau (whose reading word also belongs to the same equivalence class). Finally, two distinct reading words may not be Knuth-equivalent, since by Lemma A1.1.10 they have different insertion tableaux. □

A1.2 Jeu de Taquin

The constructions of the previous section are intimately related to the remarkable *jeu de taquin* equivalence relation among skew tableaux. In this section, we establish the fundamental properties of this equivalence, which will then be used in Section A1.3 to prove the main result – the Littlewood–Richardson rule.

Jeu de taquin, or the "teasing game," is a particular set of rules for transforming skew tableaux by viewing their entries as separate pieces that can be moved around on the "checkerboard" of the coordinate plane. These rules are designed so that the property of being a tableau was preserved. The concept of jeu de taquin is intuitively quite simple, and is probably best understood by looking at concrete examples. Still, we begin with a formal description.

A1.2.1 Definition. Let λ/μ be a skew shape. (In Figure A1-10, λ/μ is composed of the boxes made of solid lines segments.) Consider the boxes b that can be added to λ/μ, so that b shares at least one edge with λ/μ, and $\{b\} \cup \lambda/\mu$ is a valid skew shape. (In Figure A1-10, these boxes are made of dotted line segments.)

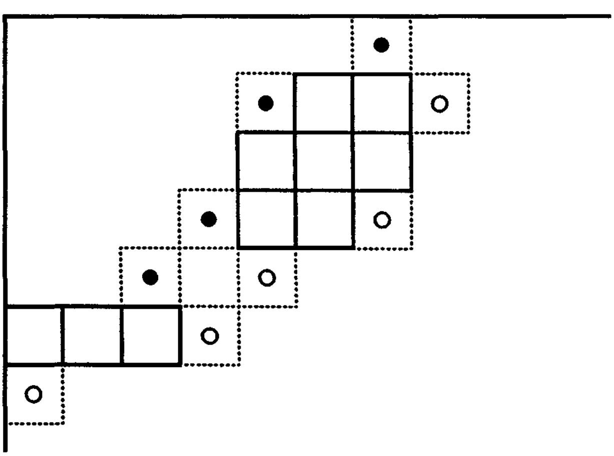


Figure A1-10. Adding boxes to a skew shape.

Two types of such boxes may occur, depending on the side of λ/μ that they are on. We mark by a bullet $\bullet$ the dashed boxes that share a lower or a right edge with λ/μ, while those that share an upper or a left edge are marked by a circle $\circ$.

Suppose we are given an SYT T of shape λ/μ. To each box b marked $\bullet$ or $\circ$, we will associate a transformation $\mathrm{jdt}_b(T)$ of T called a *jeu de taquin slide* of T into b. The definitions of the slides into inner boxes marked $\bullet$ and the outer boxes marked $\circ$ are completely analogous, so we will only discuss the first of these cases, and then provide examples illustrating both of them. Thus let us consider a box b_0 marked $\bullet$. There is at least one box b_1 in λ/μ that is adjacent to b_0 (i.e., such that b_0 and b_1 share an edge); if there are two such boxes, then let b_1 be the one with a smaller entry. Move the entry occupying b_1 into b_0. Then look at the tableau entries to the right and below b_1, and repeat the same procedure: if there is a unique such entry, then move it into b_1; if there are two to choose from, then move the smaller one. This will vacate some box b_2, and the process will continue until it reaches the outer boundary. The resulting tableau (indeed, it will be an SYT) is $\mathrm{jdt}_b(T)$, by definition.

For example, take

$$T = \begin{array}{cccc} a & \boxed{1} & \boxed{3} & \boxed{7} & \boxed{10} \\ \boxed{2} & \boxed{5} & \boxed{6} & \boxed{9} & \\ \boxed{4} & \boxed{8} & \boxed{11} & b & \end{array} \tag{A1.131}$$

(the boxes a and b are not included in T). Then

$$\mathrm{jdt}_a(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 6 & 7 & 10 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 4 & 8 & 11 \\ \cline{1-3} \end{array} \quad \text{and} \quad \mathrm{jdt}_b(T) = \begin{array}{cccc} & \boxed{1} & \boxed{3} & \boxed{7} & \boxed{10} \\ & \boxed{2} & \boxed{6} & \boxed{9} & \\ \boxed{4} & \boxed{5} & \boxed{8} & \boxed{11} & \end{array} \tag{A1.132}$$

A1.2.2 Definition. Tableaux T and T' are called *jeu de taquin equivalent* (denoted $T \overset{\mathrm{jdt}}{\sim} T'$) if one can be obtained from another by a sequence of jeu de taquin slides.

Note that $\overset{\mathrm{jdt}}{\sim}$ is a symmetric (and obviously transitive) relation, since any jeu de taquin slide can be reversed by performing a slide into the box that was vacated at the previous stage. For instance, in the example (A1.131) we have $\mathrm{jdt}_c(\mathrm{jdt}_a(T)) = T$, where c is the box occupied by 9 in T.

A1.2.3 Lemma. *Each jeu de taquin slide converts the reading word of a tableau into a Knuth-equivalent one:* $\mathrm{reading}(\mathrm{jdt}_b(T)) \overset{K}{\sim} \mathrm{reading}(T)$.

Proof. Let us verify that at every step of the sliding process, the reading word is transformed into a Knuth-equivalent one. The horizontal slides do not change the reading word at all. A vertical slide of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline a & \cdots & b & & c & \cdots & d & e & \cdots \\ \hline i & \cdots & j & k & l & \cdots & m \\ \cline{1-7}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline a & \cdots & b & k & c & \cdots & d & e & \cdots \\ \hline i & \cdots & j & & l & \cdots & m \\ \cline{1-7}\end{array}$$

replaces the segment

$$i \cdots jkl \cdots ma \cdots bc \cdots d$$

of the reading word by the segment

$$i \cdots jl \cdots ma \cdots bkc \cdots d.$$

To show that these two segments are Knuth-equivalent, we may use Theorem A1.1.4. Indeed, one easily checks that both segments have insertion tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\hline a & \cdots & b & k & c & \cdots & d \\ \hline i & \cdots & j & l & \cdots & m \\ \cline{1-6}\end{array},$$

and the lemma follows. □

The following result is sometimes called "the fundamental theorem of jeu de taquin."

A1.2.4 Theorem. *Each jeu de taquin equivalence class contains exactly one straight-shape tableau.*

Proof. If T is a tableau of a skew shape λ/μ, then performing consecutive slides into all boxes of μ (in any allowable order) will result in a straight-shape tableau, which is jeu de taquin equivalent to T. The uniqueness of such representative in a given equivalence class follows directly from Lemma A1.2.3 and the second statement of Theorem A1.1.4. □

We will use the notation jdt(T) to denote the unique straight-shape tableau in the jeu de taquin equivalence class of a given (skew) tableau T. Note that Lemma A1.2.3 implies that

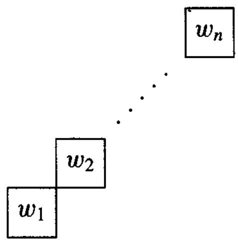
$$\text{reading}(\text{jdt}(T)) \overset{K}{\sim} \text{reading}(T)\,. \tag{A1.133}$$

A1.2.5 Corollary. *Two standard tableaux are jeu de taquin equivalent if and only if their reading words are Knuth-equivalent.*

Proof. Let T and T' be standard tableaux. Then

$$\begin{aligned} T \overset{\text{jdt}}{\sim} T' &\iff \text{jdt}(T) = \text{jdt}(T') && \text{(by Theorem A1.2.4)} \\ &\iff \text{reading}(\text{jdt}(T)) \overset{K}{\sim} \text{reading}(\text{jdt}(T')) && \text{(by Theorem A1.1.4)} \\ &\iff \text{reading}(T) \overset{K}{\sim} \text{reading}(T') && \text{(by (A1.133)).} \qquad \square \end{aligned}$$

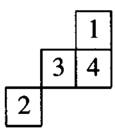
A1.2.6 Corollary. *For a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$, let T_w denote the skew tableau*



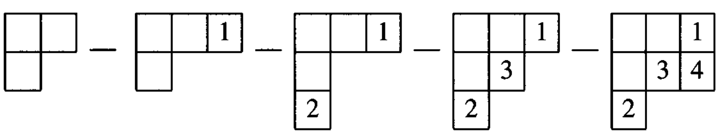
Then $\text{jdt}(T_w)$ *is the insertion tableau for* w.

Proof. By Lemma A1.2.3, $\text{reading}(\text{jdt}(T_w)) \overset{K}{\sim} \text{reading}(T_w) = w$, and the corollary follows from Theorem A1.1.4. $\square$

Similarly to the RSK algorithm, jeu de taquin can be described in terms of *growth diagrams* (cf. Section 7.13). This is best explained by an example. The tableau

$T =$  (A1.134)

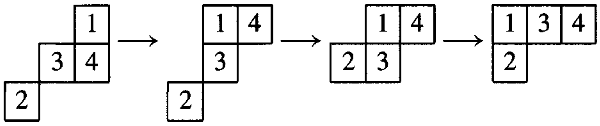
can be viewed as a sequence of shapes:



(disregard the entries, which are only shown to make the rules transparent), or as a sequence of partitions:

$$21 - 31 - 311 - 321 - 331 \,. \tag{A1.135}$$

Consider the sequence of jeu de taquin slides



Replace each of these tableaux by the corresponding sequence of partitions, place these sequences on top of each other and rotate the resulting table to obtain the growth diagram shown in Figure A1-11.

Its upper left row (or perhaps it should be called column) corresponds to the original tableau T (cf. (A1.134)–(A1.135)). The lower right row is the tableau jdt(T) obtained as a result of this sequence of slides, and the lower left row

$$\emptyset - 1 - 11 - 21$$

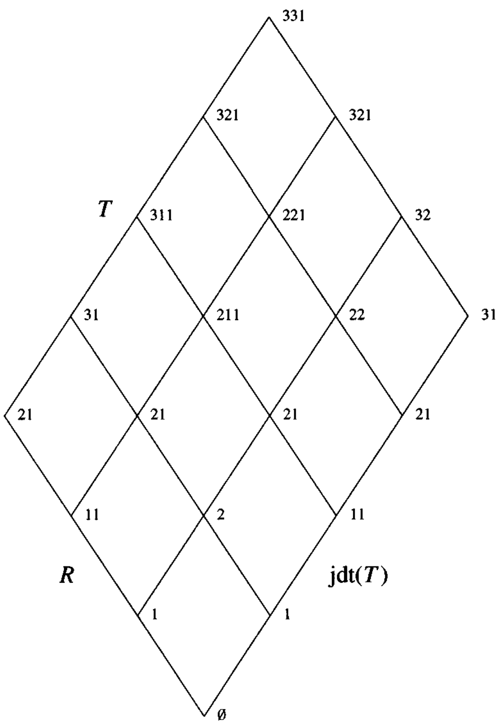


Figure A1-11. Growth diagram for jeu de taquin.

encodes the tableau

$$R = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,,$$

which records the order in which the slides were performed: we first made a slide into the box occupied by 3, then into the box occupied by 2, and finally, into the one occupied by 1.

Note that by virtue of Theorem A1.2.4, the resulting tableau jdt(T) does not depend on the order in which the slides were performed, i.e., it does not depend on the tableau R.

Growth diagrams for sequences of jeu de taquin slides can be described by very simple local rules. First of all, it is easy to verify that whenever shape λ covers μ in a growth diagram, λ can be obtained by from μ by adding a single box. Another property of jeu de taquin growth is stated below.

A1.2.7 Proposition. *Let*

$$\begin{tikzcd} & \rho \arrow[dl, dash] \arrow[dr, dash] & \\ \mu \arrow[dr, dash] & & \lambda \arrow[dl, dash] \\ & \nu & \end{tikzcd}$$

be a fragment of a jeu de taquin growth diagram. (Thus both μ and λ cover ν in the Young lattice, while ρ covers both μ and λ.) Then λ is uniquely determined from ν, μ, and ρ, according to the following rule:

- *if μ is the only shape of its size that contains ν and is contained in ρ, then $\lambda = \mu$;*
- *otherwise there is a unique such shape different from μ, and this is λ.* (A1.136)

In other words, $\mu \neq \lambda$ if and only if the interval $[\nu, \rho]$ in the Young lattice is isomorphic to a product of two 2-element chains.

Proof. Suppose we are given a tableau T of shape λ/μ and a box b such that $\mathrm{jdt}_b(T)$ is well-defined. Encode T as a sequence of shapes, and place these shapes on top of each other – and all together on top of the shape $\mu \setminus \{b\}$, as shown in Figure A1-12. We have to show that repeatedly applying the local rules (A1.136) will produce a tableau (encoded by the lower-right row in Figure A1-12) which is exactly $\mathrm{jdt}_b(T)$. Verification of this reformulation of the definition of jeu de taquin is straightforward, and is left to the reader. □

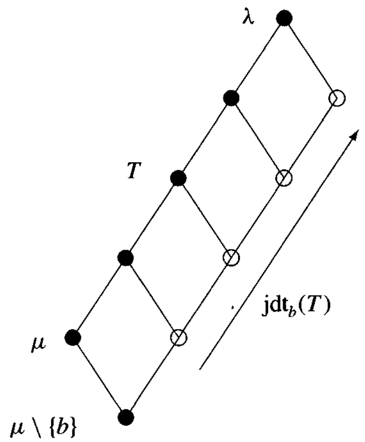


Figure A1-12. Jeu de taquin slide via local transformations.

The growth-diagram interpretation exhibits an important (and not obvious from the original description) *symmetry* of jeu de taquin, which will play a role in the next section. Notice that the rule (A1.136) is symmetric in μ and λ; in other words, λ is computed from ν, μ, and ρ in exactly the same way as μ is computed from ν, λ, and ρ. As a consequence, the "recording" tableau R in Figure A1-11 is equal to $\mathrm{jdt}(S)$, where S is the skew tableau encoded by the upper-right side of the growth diagram (cf. also Figure A1-14).

The Schützenberger Involution

This part of the appendix describes an involution on the set of SYTs of a given shape that is associated with the name of M. P. Schützenberger, and plays an important role in combinatorics, representation theory, and algebraic geometry. The material of this section is not used in the forthcoming proof of the Littlewood–Richardson rule.

A1.2.8 Definition. Let Q be an SYT of shape λ, and let b be its corner box occupied by entry 1. Define

$$\Delta(Q) = \mathrm{jdt}_b(\tilde{Q}) ,$$

where $\tilde{Q}$ is the skew SYT of shape $\lambda/(1)$ obtained from Q by removing the box b and subsequently decreasing all the remaining entries by 1. The *evacuation tableau* $\mathrm{evac}(Q)$ is by definition the SYT (of shape λ) that is encoded by the sequence of shapes

$$\emptyset\,,\ \Delta^{n-1}(Q)\,,\ \Delta^{n-2}(Q)\,,\ \ldots\,,\ \Delta^2(Q)\,,\ \Delta(Q)\,,\ Q\,. \qquad \text{(A1.137)}$$

The map $Q \mapsto \mathrm{evac}(Q)$ is called the *Schützenberger involution*. This terminology is justified by the following fact.

A1.2.9 Proposition. *The map $Q \mapsto$ evac(Q) is an involution.*

Before proving this proposition, let us illustrate Definition A1.2.8 by an example. Take

$$Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}\ . \qquad \text{(A1.138)}$$

Repeatedly applying the operator Δ, we obtain the tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline\end{array}\ ,\ \emptyset\,. \qquad \text{(A1.139)}$$

The sequence of their shapes (in the reverse order; cf. (A1.137)) encodes the tableau

$$\mathrm{evac}(Q) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}\ . \qquad \text{(A1.140)}$$

The reader is encouraged to verify that applying the same procedure to the tableau (A1.140) recovers (A1.138): $\mathrm{evac}(\mathrm{evac}(Q)) = Q$.

Proof. The involution property becomes less mysterious if one reformulates Definition A1.2.8 in terms of growth diagrams. This can be done as follows. The tableaux in (A1.137) can be viewed as sequences of shapes. Let us combine these sequences into a single triangular growth diagram, as shown in Figure A1-13. The rows of this diagram that go in the northeast direction correspond to the tableaux in (A1.137). The whole growth diagram can be reconstructed from its left side (which encodes the original tableau Q) using the local rule (A1.136), together with the fact that all the tableaux in the bottom row are obviously empty. Then

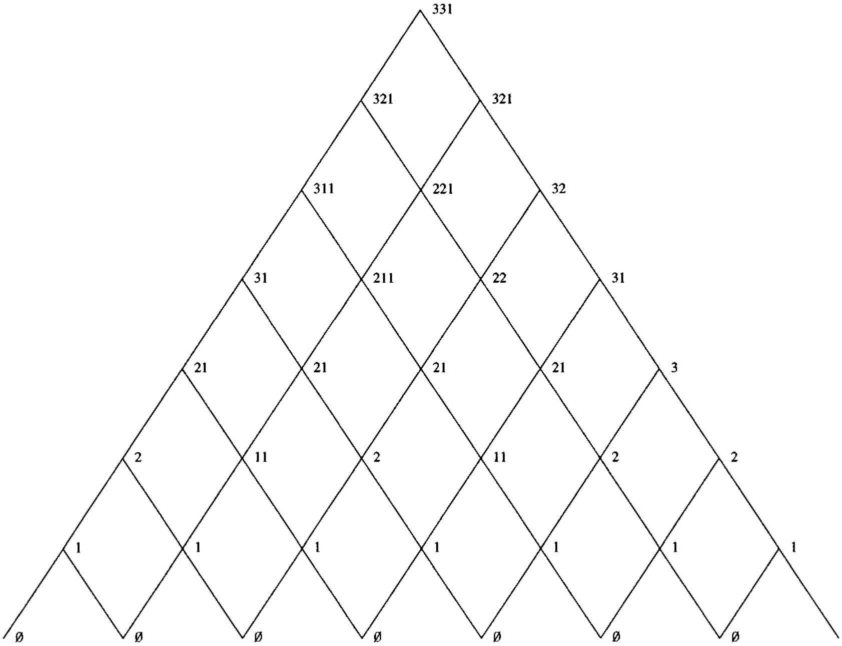


Figure A1-13. The Schützenberger involution.

the right side of the diagram is, by definition, the encoding of evac(Q). Since the rule (A1.136) is symmetric under interchanging λ and μ, applying the same procedure to the tableau evac(Q) would result in the mirror image of the same growth diagram, with its left and right sides interchanged. This proves that $Q \mapsto$ evac(Q) is an involution. □

The following theorem provides a direct interpretation for the Schützenberger involution in terms of the RSK algorithm; it also suggests another proof of Proposition A1.2.9.

For a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$, let $w^\sharp \in \mathfrak{S}_n$ be given by

$$w^\sharp = n+1-w_n \quad \cdots \quad n+1-w_2 \;\; n+1-w_1.$$

Equivalently, $w^\sharp = w_0 w w_0$, where w_0 denotes the permutation $n\ n-1\ \cdots\ 2\ 1$. For example, if $w = 3547126$, then $w^\sharp = 2671435$.

A1.2.10 Theorem. *If* $w \xrightarrow{\text{RSK}} (P, Q)$*, then* $w^\sharp \xrightarrow{\text{RSK}}$ (evac(P), evac(Q)).

To illustrate, let $w = 3547126$. Then

$$w \xrightarrow{\text{RSK}} (P,Q)\,, \qquad P = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}\,, \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}\,,$$

$$w^\sharp \xrightarrow{\text{RSK}} (P^\sharp, Q^\sharp)\,, \quad P^\sharp = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array} = \text{evac}(P)\,, \quad Q^\sharp = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array} = \text{evac}(Q)\,.$$

Proof. The nature of RSK insertion, Knuth equivalence, and jeu de taquin is such that these operations commute with removing the entries that are less than (or larger than) an arbitrary threshold value a. For instance, if we remove all entries of w that are larger than a (thus obtaining a permutation $w_{\le a} \in \mathfrak{S}_a$), then the insertion tableau $P_{\le a}$ of $w_{\le a}$ can be obtained from P by simply removing the boxes containing the entries $a+1, \dots, n$.

Less trivially, Corollary A1.2.6 implies that the insertion tableau for a permutation $w_{>a}$ is equal to $\text{jdt}(P_{>a})$, where $w_{>a}$ and $P_{>a}$ are obtained by removing the smallest a entries from w and P, respectively, and subtracting a from the remaining entries.

Let $w^\sharp \xrightarrow{\text{RSK}} (P^\sharp, Q^\sharp)$. By Theorem A1.1.1, the shape $\text{sh}(w^\sharp)$ of $P^\sharp$ can be described in terms of the parameters $I_k(w^\sharp)$ that count how many elements can be covered by a union of k increasing subsequences of $w^\sharp$. The argument used above shows that the shape of a partial tableau $P^\sharp_{\le j}$ has a similar description in terms of increasing subsequences of $w^\sharp$ with entries not exceeding j. Note that these subsequences correspond to increasing subsequences of w with entries $> n-j$. Therefore

$$\text{sh}\big(P^\sharp_{\le j}\big) = \text{sh}(w_{>n-j}) = \text{sh}(\text{jdt}(P_{>n-j})) = \text{sh}(\text{evac}(P)_{\le j}),$$

so $P^\sharp = \text{evac}(P)$.

By Theorem 7.13.1, the recording tableau for w coincides with the insertion tableau for w^{-1}. We already proved that as we pass from w to $w^\sharp$, the insertion tableau is replaced by its image under Schützenberger involution. Since $(w^{-1})^\sharp = (w^\sharp)^{-1}$, the same happens to the recording tableau. □

The following corollary of Theorem A1.2.10 is a reformulation of Theorem 7.23.16.

A1.2.11 Corollary. *Let* $w = w_1 \cdots w_n \xrightarrow{\text{RSK}} (P, Q)$. *Then*

$$w w_0 = w_n \cdots w_1 \xrightarrow{\text{RSK}} (P^t, \text{evac}(Q)^t).$$

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Proof. While replacing w by ww_0, we interchange increasing and decreasing subsequences, and each tableau $P_{\leq j}$ gets transposed. Hence the insertion tableau for ww_0 is P^t. As to the recording tableau, we have

$$\begin{array}{rrll} & w_0ww_0 \xrightarrow{\text{RSK}} & (\text{evac}(P), \text{evac}(Q)) & \text{(by Theorem A1.2.10)} \\ \Longrightarrow & w_0w^{-1}w_0 \xrightarrow{\text{RSK}} & (\text{evac}(Q), \text{evac}(P)) & \text{(by Theorem 7.13.1)} \\ \Longrightarrow & w_0w^{-1} \xrightarrow{\text{RSK}} & (\text{evac}(Q)^t, \cdots) & \text{(using what we just proved)} \\ \Longrightarrow & ww_0 \xrightarrow{\text{RSK}} & (\cdots, \text{evac}(Q)^t) & \text{(by Theorem 7.13.1),} \end{array}$$

as desired. □

A1.3 The Littlewood–Richardson Rule

The *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$ were defined in Section 7.15 (see (7.64)) as the structure constants for the multiplication in the basis of Schur functions:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \,, \tag{A1.141}$$

or as coefficients in the expansion of a skew Schur function in this basis:

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu \,. \tag{A1.142}$$

The celebrated Littlewood–Richardson rule is a combinatorial description of the coefficients $c^\lambda_{\mu\nu}$. In this section, we prove the rule in two different versions. Three more variations are then stated without proof.

A1.3.1 Theorem (the Littlewood–Richardson rule: jeu de taquin version). *Fix an SYT P of shape ν. The Littlewood–Richardson coefficient $c^\lambda_{\mu\nu}$ is equal to the number of SYT of shape λ/μ that are jeu de taquin equivalent to P.*

We will first illustrate Theorem A1.3.1 by an example, and then prove it.

A1.3.2 Example. Let $\lambda = (4,4,2,1)$, $\mu = (2,1)$, and $\nu = (4,3,1)$. Consider the tableau

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array} \tag{A1.143}$$

of shape ν. (According to Theorem A1.3.1, an SYT P of shape ν can be chosen arbitrarily. This special choice of P will later play a role in another version of the

Littlewood–Richardson rule.) There are exactly two SYTs T of shape λ/μ such that $\operatorname{jdt}(T) = P$, namely,

$$\begin{array}{cccc} & & 3 & 4 \\ & 2 & 6 & 7 \\ 1 & 8 & & \\ 5 & & & \end{array} \quad \text{and} \quad \begin{array}{cccc} & & 3 & 4 \\ & 2 & 6 & 7 \\ 1 & 5 & & \\ 8 & & & \end{array} . \tag{A1.144}$$

Hence $c^{\lambda}_{\mu\nu} = 2$.

Proof of Theorem A1.3.1. We may assume that $|\mu| + |\nu| = |\lambda|$, since otherwise the theorem simply tells that $0 = 0$. Let us then count the number of jeu de taquin growths of the form shown in Figure A1-14. (The shapes λ, μ, and ν, and the tableau P are fixed, while the tableaux R, S, and T are not.)

This number can be found in two different ways, which correspond to reconstructing the growth diagram from its left and right boundary, respectively, using the local rule (A1.136). First, we could count the SYTs T of shape λ/μ such that $P = \operatorname{jdt}(T)$. For each such T, there are f^{μ} possible choices for R. On the other hand, in order to define the values at the right boundary, we only need to pick an

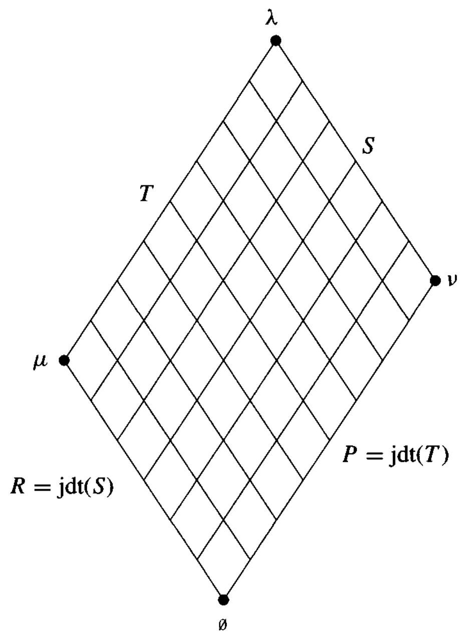

Figure A1-14. Counting jeu de taquin growths.

SYT S of shape λ/ν such that jdt(S) has shape μ. Comparing the two counts, we obtain

$$\begin{aligned}&\#\{\text{SYT } T \text{ of shape } \lambda/\mu \,:\, \text{jdt}(T) = P\} \cdot f^{\mu}\\ &= \#\{\text{SYT } S \text{ of shape } \lambda/\nu \,:\, \text{sh}(\text{jdt}(S)) = \mu\}\,.\end{aligned} \tag{A1.145}$$

This identity implies that the number

$$C^{\lambda}_{\mu\nu} = \#\{\text{SYTs } T \text{ of shape } \lambda/\mu \,:\, \text{jdt}(T) = P\} \tag{A1.146}$$

only depends on the shape ν of P, but not on P itself. (As an aside, notice that the right-hand side of (A1.145) equals $C^{\lambda}_{\nu\mu} \cdot f^{\mu}$, implying that $C^{\lambda}_{\mu\nu} = C^{\lambda}_{\nu\mu}$.)

To prove the theorem, we will need the expansion of a skew Schur function in terms of fundamental quasisymmetric functions that was given in Theorem 7.19.7:

$$s_{\lambda/\mu} = \sum_{T} L_{\text{co}(T)}, \tag{A1.147}$$

where the sum is over all SYTs T of shape λ/μ, and co(T) denotes the composition corresponding to the descent set $D(T)$ of T.

One easily checks that a jeu de taquin slide never changes the relative position of k and $k+1$, implying that the descent set of a skew tableau is invariant under jeu de taquin slides. Therefore the expansion (A1.147) can be rewritten as

$$s_{\lambda/\mu} = \sum_{P} \#\{T : \text{sh}(T) = \lambda/\mu\,,\ \text{jdt}(T) = P\} \cdot L_{\text{co}(P)},$$

where the sum is over all SYTs P of $n = |\lambda/\mu|$ boxes. In view of (A1.146), this can be further transformed into

$$s_{\lambda/\mu} = \sum_{\nu\vdash n} C^{\lambda}_{\mu\nu} \sum_{P} L_{\text{co}(P)},$$

where the internal sum is over all SYTs P of shape ν. Using (A1.147) for the shape ν, we obtain

$$s_{\lambda/\mu} = \sum_{\nu\vdash n} C^{\lambda}_{\mu\nu}\, s_{\nu}.$$

Comparing this with (A1.142), we conclude that $c^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu}$, which completes the proof of Theorem A1.3.1. □

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The Littlewood–Richardson coefficients $c^{\lambda}_{\mu\nu}$ are among the most important families of combinatorial numbers. They appear in the following contexts, among others:

- as coefficients in decompositions of tensor products of irreducible GL_n-modules;
- as coefficients in decompositions of skew Specht modules into irreducibles;
- as coefficients in decompositions of $\mathfrak{S}_n$-representations induced from Young subgroups;
- as intersection numbers in the Schubert calculus on a Grassmannian.

Note that Theorem A1.3.1 readily implies that the Littlewood–Richardson coefficients $c^{\lambda}_{\mu\nu}$ are *nonnegative* integers, a property that is hard to deduce directly from the definitions (A1.141)–(A1.142). Although nonnegativity immediately follows from each of the four interpretations of the $c^{\lambda}_{\mu\nu}$ listed in the previous paragraph, none of these interpretations provides by itself a combinatorial rule that can be used to compute the $c^{\lambda}_{\mu\nu}$. (Note that the third interpretation corresponds to Corollary 7.18.6. Moreover, the first interpretation is discussed in Appendix 2.)

There are many other ways to describe the Littlewood–Richardson coefficients as enumerative combinatorial constants. Once we know that $c^{\lambda}_{\mu\nu}$ is the cardinality of a certain set, then any bijection between this set and another family of combinatorial objects leads to a new description of $c^{\lambda}_{\mu\nu}$. Perhaps the most well-known of such reformulations is the one given in Theorem A1.3.3 below.

Recall from Section 7.10 (see Proposition 7.10.3(d)) that a *lattice permutation* (or Yamanouchi word, or ballot sequence) is a sequence $a_1a_2\cdots a_n$ such that in any initial factor $a_1a_2\cdots a_j$, the number of i's is at least as great as the number of $i+1$'s (for all i). We will also need the notion of a *reverse reading word* of a tableau, which is simply its reading word (cf. Definition A1.1.5) read backwards.

A1.3.3 Theorem (the Littlewood–Richardson rule). *The Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of semistandard Young tableaux of shape λ/μ and type ν whose reverse reading word is a lattice permutation.*

A1.3.4 Example. Semistandard Young tableaux with the lattice permutation property described in Theorem A1.3.3 are sometimes called *Littlewood–Richardson tableaux*, or *Littlewood–Richardson fillings* of the shape λ/μ. For the data in Example A1.3.2, there are two such tableaux (thus $c^{\lambda}_{\mu\nu}=2$):

$$\begin{array}{cccc} & & 1 & 1\\ & 1 & 2 & 2\\ 1 & 3 & & \\ 2 & & & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} & & 1 & 1\\ & 1 & 2 & 2\\ 1 & 2 & & \\ 3 & & & \end{array}\,. \tag{A1.148}$$

The corresponding reverse reading words 11221312 and 11221213 are indeed lattice permutations (of type ν). Note that the Littlewood–Richardson tableaux (A1.148) can be obtained from (A1.144) by replacing the entries 1, 2, 3, 4 by 1, the entries 5, 6, 7 by 2, and the entry 8 by 3.

We are going to deduce Theorem A1.3.3 from Theorem A1.3.1. This will require some preliminary work.

Through the end of this section, a partition $\nu = (\nu_1, \ldots, \nu_k)$ is assumed fixed, and we use the notation

$$N_0 = 0, \qquad N_1 = \nu_1, \qquad N_2 = \nu_1 + \nu_2, \qquad N_3 = \nu_1 + \nu_2 + \nu_3, \ldots .$$

Let P_ν denote the particular SYT of shape ν obtained by placing the entries $1, 2, \ldots, n$ in the boxes of ν row by row, beginning with the top row. For instance, if $\nu = (4, 3, 1)$, then P_ν is given by (A1.143). In general, the i-th row of P_ν will be

$$\boxed{N_{i-1}+1}\boxed{N_{i-1}+2}\boxed{\cdots}\boxed{\cdots}\boxed{\cdots}\boxed{N_i}\,, \tag{A1.149}$$

for $i = 1, \ldots, k$.

Let $\mathcal{L}_\nu$ denote the set of all Littlewood–Richardson tableaux of type ν and any shape whatsoever. The following construction will be needed in order to relate the concept of a Littlewood–Richardson tableau to jeu de taquin.

A1.3.5 Definition. Take any SSYT L of type ν (in particular, L could be a Littlewood–Richardson tableau in $\mathcal{L}_\nu$). For any i, the entries of L that are equal to i form a horizontal strip. Replace the 1's in L by $1, \ldots, N_1$, the 2's by $N_1 + 1, \ldots, N_2$, etc., so that the numbers increase left-to-right within each of these horizontal strips. Let us denote the resulting SYT by $\mathrm{st}(L)$ and call it the *standardization* of L. For example, applying this procedure to the Littlewood–Richardson tableaux (A1.148) would give the tableaux in (A1.144).

A1.3.6 Lemma. *A skew SYT T is a standardization of some Littlewood–Richardson tableau of type ν (i.e., $T \in \mathrm{st}(\mathcal{L}_\nu)$) if and only if the following condition holds, for $i = 1, \ldots, k-1$:*

the partial tableaux formed by the entries $N_{i-1}+1, \ldots, N_{i+1}$ of T and P_ν, respectively, are jeu de taquin equivalent. (A1.150)

Proof. First observe that an SYT L is a standardization of some SSYT of type ν (not necessarily a Littlewood–Richardson one) if and only if each of its partial tableaux formed by the entries $N_{i-1}+1, \ldots, N_i$ is jeu de taquin equivalent to the tableau (A1.149), i.e., to the ith row of P_ν. This condition is obviously satisfied whenever (A1.150) holds. A Littlewood–Richardson tableau L should also satisfy a lattice permutation condition, which is a certain restriction on the partial tableaux formed by the entries of L which are equal to i or $i+1$, for $i = 1, \ldots, k-1$. One easily checks that the standardization map translates this condition into (A1.150). □

A1.3.7 Lemma. *The set* $\mathrm{st}(\mathcal{L}_\nu)$ *of standardizations of Littlewood–Richardson tableaux of type* ν *coincides with the jeu de taquin equivalence class of* P_ν.

Proof. The condition (A1.150) is clearly invariant under jeu de taquin slides; hence the set $\mathrm{st}(\mathcal{L}_\nu)$ is a union of jeu de taquin equivalence classes. By Theorem A1.2.4, it is then enough to show that P_ν is the unique straight-shape tableau in $\mathrm{st}(\mathcal{L}_\nu)$.

Consider the SSYT L of type ν and shape ν obtained by placing i's in row i of ν, for every i. It is straightforward to check that L is the only straight-shape Littlewood–Richardson tableau of type ν. Since $\mathrm{st}(L) = P_\nu$, the lemma follows. □

Proof of Theorem A1.3.3. We need to show that $c^\lambda_{\mu\nu}$ equals the number of Littlewood–Richardson tableaux of shape λ/μ and type ν. This is done as follows:

$$\begin{aligned} c^\lambda_{\mu\nu} &= \#\{\text{SYT } T \,:\, \mathrm{sh}(T) = \lambda/\mu,\ T \overset{\text{jdt}}{\sim} P_\nu\} && \text{(by Theorem A1.3.1)} \\ &= \#\{T \in \mathrm{st}(\mathcal{L}_\nu) \,:\, \mathrm{sh}(T) = \lambda/\mu\} && \text{(by Lemma A1.3.7)} \\ &= \#\{L \in \mathcal{L}_\nu \,:\, \mathrm{sh}(L) = \lambda/\mu\} && \text{(since st is injective on } \mathcal{L}_\nu). \end{aligned}$$

□

Variations of the Littlewood–Richardson Rule

Note that a Littlewood–Richardson coefficient $c^\lambda_{\mu\nu}$ can be defined as a scalar product: $c^\lambda_{\mu\nu} = \langle s_{\lambda/\mu}, s_\nu\rangle$. The following two variations of the Littlewood–Richardson rule, stated without proof, provide combinatorial descriptions of more general *intertwining numbers* $\langle s_\theta, s_\sigma\rangle$, for arbitrary skew shapes θ and σ. (Recall from the discussion after equation (7.64) that $\langle s_\theta, s_\sigma\rangle$ is actually a special case of $\langle s_{\lambda/\mu}, s_\mu\rangle$. However, regarding $\langle s_\theta, s_\sigma\rangle$ in this way obscures the symmetry between θ and σ that appears in Theorems A1.3.8 and A1.3.9 below.)

We will say that a box a is located (weakly) northwest of box b if a occupies a row above b, or the same row as b, and also a column to the left of b, or the same column as b. In a similar fashion, we define what it means for one box to be (weakly) southwest of another.

A1.3.8 Theorem. *For a pair of skew shapes θ and σ, the intertwining number $\langle s_\theta, s_\sigma\rangle$ is equal to the number of bijective maps $f : \theta \to \sigma$ satisfying the following conditions:*

(i) *if box a is located (weakly) northwest of box b, then $f(a)$ is (weakly) southwest of $f(b)$;*
(ii) *if $f(a)$ is located (weakly) northwest of $f(b)$, then a is (weakly) southwest of b.*

(Note that condition (ii) is the same as (i) imposed on the inverse map f^{-1}.)

To illustrate, take the shapes $\theta = (4, 3, 1)$ and $\sigma = (4, 4, 2, 1)/(2, 1)$ (cf. Examples A1.3.2 and A1.3.4). Then there are two bijections $\theta \to \sigma$ satisfying conditions (i)–(ii) of Theorem A1.3.8, which are described by

$$\begin{array}{cccc} 1&2&3&4\\ 5&6&7& \\ 8&&& \end{array} \quad \longrightarrow \quad \begin{array}{cccc} &&8&7\\ &6&3&4\\ 5&2&& \\ 1&&& \end{array}$$

and

$$\begin{array}{cccc} 1&2&3&4\\ 5&6&7& \\ 8&&& \end{array} \quad \longrightarrow \quad \begin{array}{cccc} &&6&7\\ &8&3&4\\ 5&2&& \\ 1&&& \end{array}$$

(here each box on the left-hand side is mapped to the box with the same label on the right-hand side). Thus in this case $\langle s_\theta, s_\sigma\rangle = 2$.

A1.3.9 Theorem. *For a pair of skew shapes θ and σ, the intertwining number $\langle s_\theta, s_\sigma\rangle$ is equal to the number of pairs (P, Q) of standard Young tableaux of shapes θ and σ, respectively, such that the reverse reading words of P and Q are permutations inverse to each other.*

(In this theorem, reverse reading words could be replaced by ordinary reading words.)

For $\theta = (4, 3, 1)$ and $\sigma = (4, 4, 2, 1)/(2, 1)$, there are two pairs of tableaux (P, Q) satisfying the conditions of Theorem A1.3.9:

$$P = \begin{array}{cccc} 1&2&5&7\\ 3&4&8& \\ 6&&& \end{array}, \qquad Q = \begin{array}{cccc} &&3&4\\ &2&6&7\\ 1&8&& \\ 5&&& \end{array}$$

and

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\,, \qquad Q = \begin{array}{cccc} & & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \cline{2-4} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{7} \\ \cline{1-4} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{5} & & \\ \cline{1-2} \multicolumn{1}{|c|}{8} & & & \\ \cline{1-1} \end{array}\,,$$

with reverse reading words $w = 75218436$, $w^{-1} = 43762815$ and $w = 75216438$, $w^{-1} = 43762518$, respectively.

The last version of the Littlewood–Richardson rule that we are going to discuss exhibits certain symmetries of the coefficients $c^{\lambda}_{\mu\nu}$ that were hidden in the previous versions. Let λ, μ, and ν be partitions with at most r parts satisfying $|\lambda| = |\mu| + |\nu|$. Define the vectors $l = (l_1, \ldots, l_{r-1})$, $m = (m_1, \ldots, m_{r-1})$, and $n = (n_1, \ldots, n_{r-1})$ by

$$\begin{aligned} l_i &= \lambda_{r-i} - \lambda_{r-i+1} \,, \\ m_i &= \mu_i - \mu_{i+1} \,, \\ n_i &= \nu_i - \nu_{i+1} \,. \end{aligned} \tag{A1.151}$$

(It is possible to show that the Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ is in fact equal to the dimension of the space of SL_r-invariants in the tensor product of three irreducible SL_r-modules naturally associated to l, m, and n.)

The construction below is due to A. Berenstein and A. Zelevinsky, which explains our choice of terminology.

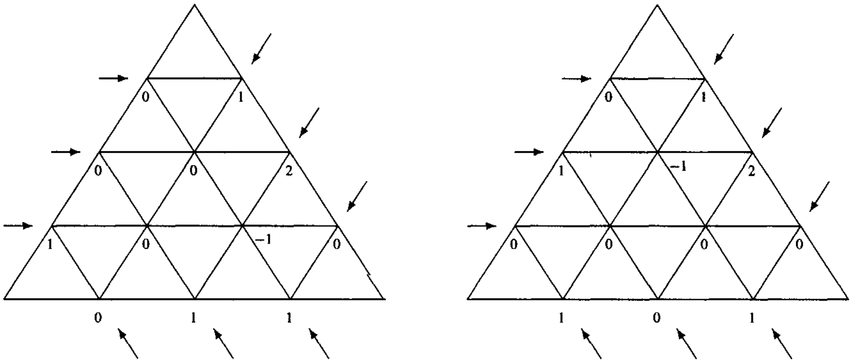
A1.3.10 Definition. Let $l = (l_1, \ldots, l_{r-1})$, $m = (m_1, \ldots, m_{r-1})$, and $n = (n_1, \ldots, n_{r-1})$ be vectors with nonnegative integer components. A *BZ pattern* of type (r, l, m, n) is a collection of integers $(y_{i,j,k})$ indexed by the set

$$\{(i, j, k) \in \mathbb{Z}^3 \; : \; 0 \le i, j, k < r \,, \; i + j + k = r\} \,;$$

and subject to certain linear equations and inequalities to be stated below. It is convenient to view $(y_{i,j,k})$ as a triangular array, as shown in Figure A1-15.

In order to form a BZ pattern, the integers $y_{i,j,k}$ should satisfy the following restrictions. First, the sums of entries along every line of the array that goes in one of the three distinguished directions (excluding the sides of the triangle) are prescribed:

- the sums in the horizontal rows are equal to $l_1, \ldots, l_{r-1}$, top down;
- the sums in the columns going northwest are equal to $m_1, \ldots, m_{r-1}$, left to right;
- the sums in the columns going southwest are equal to $n_1, \ldots, n_{r-1}$, right to left.



$l = (1, 2, 0), m = (1, 1, 0), n = (1, 2, 1)$

Figure A1-15. BZ-patterns for $r = 4$, $\lambda = (4, 4, 2, 1)$, $\mu = (2, 1, 0, 0)$, $\nu = (4, 3, 1, 0)$.

Second, in each of the $3r - 3$ sums above, the partial sum of several first entries, looking in the direction indicated by an arrow (see Figure A1-15), should be nonnegative. Figure A1-15 shows the two BZ patterns for the data from Examples A1.3.2 and A1.3.4.

A1.3.11 Theorem. *The Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of BZ patterns of type* (r, l, m, n), *where the vectors l, m, and n are defined by* (A1.151).

It is clear from this description that the Littlewood–Richardson coefficient in question is invariant under cyclic permutations of l, m, and n. It is possible to show that $c^{\lambda}_{\mu\nu}$ is in fact *symmetric* as a function of l, m, and n, and is also invariant under simultaneous rearrangement of the entries of each of the vectors l, m, and n in reverse order.

Acknowledgments. I am grateful to Curtis Greene and Andrei Zelevinsky for a number of valuable suggestions and corrections.

Notes

Theorem A1.1.1 was proved by C. Greene [**6**], generalizing C. E. Schensted's result [**7.136**]. Corollary A1.1.2 can be extended to arbitrary finite posets, as shown by C. Greene and D. J. Kleitman [**7**][**8**] (see also [**2**]).

Knuth equivalence and Theorems A1.1.4 and A1.1.6 are due to D. E. Knuth [**7.71**], who studied this equivalence in a more general setting, with permutations replaced by arbitrary words in the alphabet $\{1, \dots, n\}$. It is often useful to work in the *plactic monoid* [**12**], which is the quotient of the free monoid with generators $1, \ldots, n$ under the Knuth equivalence.

Jeu de taquin was invented by M. P. Schützenberger [**17**], as was the involution that bears his name [**7.140**]. Theorem A1.2.4 was proved by M. P. Schützenberger [**17**] and G. Thomas [**18**][**19**], and Theorem A1.2.10 by M. P. Schützenberger [**7.140**]. The growth-diagram interpretation of jeu de taquin was suggested in [**7.32**]. These constructions can be generalized to arbitrary finite posets (cf. [**16**]), although the analogue of Theorem A1.2.4 does not generally hold.

The Littlewood–Richardson rule was discovered by D. E. Littlewood and A. R. Richardson [**7.89**]. First complete proofs were given by M. P. Schützenberger [**17**] and G. Thomas [**18**][**20**]. An incomplete proof published by G. de B. Robinson [**7.131**] and reproduced by D. E. Littlewood [**7.88**, Ch. 6.3, Thm. V] was made precise by I. G. Macdonald [**7.92**, Ch. 1.9][**7.96**, Ch. 1.9] The proof given here is based on a combination of ideas taken from [**17**], [**7.32**], and [**9**].

Theorem A1.3.8 appeared in [**3**], and is a version of a result by A. Zelevinsky [**22**] (cf. also D. E. White [**21**]); the idea goes back to G. D. James and M. H. Peel [**10**]. Theorem A1.3.9 is a result of S. V. Kerov [**11**] and A. M. Garsia and J. B. Remmel [**5**]. Theorem A1.3.11 is due to A. D. Berenstein and A. Zelevinsky [**1**], who also gave reformulations exhibiting other symmetries of the Littlewood–Richardson coefficients.

Other versions of the Littlewood–Richardson rule, along with alternative proofs, can be found in [**3**][**7.35**] [**4**, Ch. 5.2–5.3] [**7.65**, 2.8.13] [**14**] [**15**, Thm. 4.9.4], among other sources. The history of the rule is presented in [**13**, pp. 3–7]. Generalizations and variations of the Littlewood–Richardson rule are numerous, and we do not attempt at reviewing them here.

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Chapter 7: Appendix 2

The Characters of GL $(n, \mathbb{C})$

In this appendix we state without proof the fundamental connection between Schur functions and the characters of the general linear group $\mathrm{GL}(n, \mathbb{C})$. By definition, $\mathrm{GL}(n, \mathbb{C})$ is the group of all invertible $n \times n$ matrices with complex entries (under the operation of matrix multiplication). If V is an n-dimensional complex vector space, then after choosing an ordered basis for V we can identify $\mathrm{GL}(n, \mathbb{C})$ with the group $\mathrm{GL}(V)$ of invertible linear transformations $A : V \to V$ (under the operation of composition of linear transformations).

A *linear representation* of $\mathrm{GL}(V)$ is a homomorphism $\varphi : \mathrm{GL}(V) \to \mathrm{GL}(W)$, where W is a complex vector space. From now on we assume that all representations are *finite-dimensional*, i.e., $\dim W < \infty$. We call $\dim W$ the *dimension* of the representation φ, denoted $\dim \varphi$. The representation φ is a *polynomial* (respectively, *rational*) representation if, after choosing ordered bases for V and W, the entries of $\varphi(A)$ are polynomials (respectively, rational functions) in the entries of $A \in \mathrm{GL}(n, \mathbb{C})$. It is clear that the notion of polynomial or rational representation is independent of the choice of ordered bases of V and W, since linear combinations of polynomials (respectively, rational functions) remain polynomials (respectively, rational functions). In general we do not distinguish between representations of $\mathrm{GL}(V)$ and the obvious analogous notion of a representation of $\mathrm{GL}(n, \mathbb{C})$. We say that the representation φ is *homogeneous of degree m* if $\varphi(\alpha A) = \alpha^m \varphi(A)$ for all $\alpha \in \mathbb{C}^* = \mathbb{C} - \{0\}$. If φ is a polynomial (or rational) representation, then this condition is equivalent to saying that each entry of $\varphi(A)$ is a homogeneous polynomial (or rational function) of degree m.

Note. Often the dimension of a representation φ is called its *degree*, so do not be confused by our different use of the term "degree."

A2.1 Example. Let $n = 2$, and define $\varphi : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{GL}(3, \mathbb{C})$ by

$$\varphi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}. \tag{A2.152}$$

One can directly check that φ is a group homomorphism. Since the entries of the matrix on the right-hand side of (A2.152) are homogeneous polynomials of degree two in a, b, c, d, it follows that φ is a homogeneous polynomial representation of dimension three and degree two.

A2.2 Example. Here are some simple examples of representations illustrating the terms defined above. In all these examples we take $A \in \mathrm{GL}(n, \mathbb{C})$.

- $\varphi(A) = 1 \in \mathbb{C}$ (the *trivial representation*). This is a homogeneous polynomial representation of dimension one and degree zero.
- $\varphi(A) = A$ (the *defining representation*). This is a homogeneous polynomial representation of dimension n and degree one.
- $\varphi(A) = (\det A)^m$, where $m \in \mathbb{Z}$. If $m \geq 0$, then this is a homogeneous polynomial representation of dimension one and degree mn. If $m < 0$ then φ is rational, but not polynomial. The degree remains mn.
- $\varphi(A) = |\det A|^{\sqrt{2}}$. Not a rational representation. It has dimension one and is not homogeneous. (The equation $\varphi(\alpha A) = \alpha^{n\sqrt{2}}\varphi(A)$ only holds when α is a positive real number.)
- $\varphi(A) = A^{-1}$. Not a representation.
- $\varphi(A) = (A^{-1})^t$, where ^{t} denotes transpose. A homogeneous rational (but not polynomial) representation of dimension n and degree -1. To see this, one needs the formula for the entries of the inverse of a matrix mentioned in the proof of Theorem 4.7.2.
- $\varphi(A) = (\det A)^m A$, where $m \in \mathbb{N}$. A homogeneous polynomial representation of dimension n and degree $mn + 1$.
- $\varphi(A) = \bar{A}$, where $\bar{\ }$ denotes complex conjugation. A nonrational representation of dimension n, and not homogeneous.
- $\varphi(A) = [\sigma(a_{ij})]$, where σ is a field automorphism of $\mathbb{C}$ which is not the identity or complex conjugation (so σ is necessarily discontinuous). This representation (of dimension n) is not only nonrational, but is not continuous.
- $\varphi(A) = \begin{bmatrix} 1 & \log|\det A| \\ 0 & 1 \end{bmatrix}$. A representation of dimension two that isn't homogeneous or rational, though it is continuous.

Note that many of the pathological examples given above disappear when we consider $\mathrm{SL}(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C}) : \det A = 1\}$ instead of $\mathrm{GL}(n, \mathbb{C})$. We will have more to say about $\mathrm{SL}(n, \mathbb{C})$ later.

Consider the representation φ of Example A2.1. One can check that if the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has eigenvalues θ_1 and θ_2, then $\varphi(A)$ has eigenvalues θ_1^2, $\theta_1\theta_2$, and θ_2^2. This computation illustrates the following fundamental result.

A2.3 Proposition. *If φ is a homogeneous rational representation of* $\mathrm{GL}(V)$ *of dimension N and degree m, then there exists a multiset $\mathcal{M}_\varphi$ of N Laurent monomials $x^a = x_1^{a_1} \ldots x_n^{a_n}$ of degree m (i.e., $\sum a_i = m$) with the following property. If $A \in \mathrm{GL}(V)$ has eigenvalues $\theta_1, \ldots, \theta_n$, then the eigenvalues of $\varphi(A)$ are given by θ^a, for all $x^a \in \mathcal{M}_\varphi$. Moreover, if φ is a polynomial representation, then the Laurent monomials $x^a \in \mathcal{M}_\varphi$ are actual monomials (no negative exponents).*

If φ is a rational representation of $\mathrm{GL}(n, \mathbb{C})$, then define its *character* $\operatorname{char}\varphi$ to be the Laurent polynomial

$$\operatorname{char}\varphi = (\operatorname{char}\varphi)(x) = \sum_{x^a \in \mathcal{M}_\varphi} x^a. \tag{A2.153}$$

Thus if $A \in \mathrm{GL}(n, \mathbb{C})$ has eigenvalues $\theta_1, \ldots, \theta_n$, then $(\operatorname{char}\varphi)(\theta) = \operatorname{tr}\varphi(A)$. Note that an immediate consequence of this definition is the fact that if $\varphi = \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_m$ (direct sum of rational representations), then

$$\operatorname{char}\varphi = \operatorname{char}\varphi_1 + \operatorname{char}\varphi_2 + \cdots + \operatorname{char}\varphi_m.$$

We are now ready to state the main theorem on rational and polynomial representations of $\mathrm{GL}(n, \mathbb{C})$.

A2.4 Theorem. (I) *Every rational representation $\varphi : \mathrm{GL}(V) \to \mathrm{GL}(W)$ is completely reducible, i.e., every $\mathrm{GL}(V)$-invariant subspace of W has a $\mathrm{GL}(V)$-invariant complement. Hence φ is a direct sum of irreducible representations.*
(II) *Let $\varphi : \mathrm{GL}(V) \to \mathrm{GL}(W)$ and $\varphi' : \mathrm{GL}(V) \to \mathrm{GL}(W')$ be rational representations of $\mathrm{GL}(V)$. Then φ and φ' are equivalent (i.e., there is a bijective linear transformation $\alpha : W \to W'$ such that $\alpha(\varphi(A)(v)) = \varphi'(A)(\alpha(v))$ for all $A \in \mathrm{GL}(V)$ and $v \in W$) if and only if $\operatorname{char}\varphi = \operatorname{char}\varphi'$.*
(III) *Irreducible rational representations φ of $\mathrm{GL}(V)$ are homogeneous, and $\operatorname{char}\varphi$ is a* symmetric *(homogeneous) Laurent polynomial in $x_1, \ldots, x_n$. Hence $\operatorname{char}\psi \in \Lambda_n^m$ if ψ is a homogeneous polynomial representation of degree m.*
(IV) (The main result.) *The irreducible polynomial representations φ^λ of $\mathrm{GL}(V)$ can be indexed by partitions λ of length at most n so that*

$$\operatorname{char}\varphi^\lambda = s_\lambda(x_1, \ldots, x_n).$$

(V) *Every irreducible rational representation φ of $\mathrm{GL}(V)$ is of the form $\varphi(A) = (\det A)^m \varphi'(A)$ for some $m \in \mathbb{Z}$ and some irreducible polynomial representation φ' of $\mathrm{GL}(V)$. The corresponding characters are hence related by*

$$\operatorname{char}\varphi = (x_1 \cdots x_n)^m \operatorname{char}\varphi'.$$

It follows from the above theorem that if φ is a polynomial representation of

GL(V), then the multiplicity of the irreducible character φ^λ in φ is given by

$$\langle \varphi, \varphi^\lambda \rangle = \langle \operatorname{char} \varphi, s_\lambda \rangle.$$

Let us consider some simple examples.

- If $\varphi(A) = 1$ (the trivial representation), then $\operatorname{char} \varphi = s_\emptyset = 1$. The representation is irreducible. (The main theorem is hardly needed for irreducibility – any linear representation of dimension one of any group is irreducible.)
- If $\varphi(A) = A$ (the defining representation), then $\operatorname{char} \varphi = x_1 + \cdots + x_n = s_1$ (understood to be in the variables $x_1, \ldots, x_n$). The representation is irreducible.
- If $\varphi(A) = (\det A)^m$ for $m \in \mathbb{Z}$, then $\operatorname{char} \varphi = (x_1 \cdots x_n)^m$. If $m \geq 0$, then $\operatorname{char} \varphi = s_\lambda$, where $\lambda = \langle m^n \rangle$. The representation is irreducible for any m.
- If $\varphi(A) = (A^{-1})^t$, then $\operatorname{char} \varphi = (x_1 \cdots x_n)^{-1} = (x_1 \cdots x_n)^{-1} s_\emptyset$. The representation is irreducible.

For a somewhat more substantial example, let End(V) denote the set of all linear transformations $X : V \to V$. Consider the action of GL(V) on End(V) given by $A \cdot X = AXA^{-1}$. This is called the *adjoint representation* of GL(V), denoted ad. Note that $\dim \mathrm{ad} = \dim \mathrm{End}(V) = n^2$. To compute char(ad), first choose an ordered basis for V, so we can identify End(V) with Mat(n), the ring of all $n \times n$ complex matrices. Let $A = \operatorname{diag}(\theta_1, \ldots, \theta_n)$, the diagonal matrix with diagonal entries $\theta_1, \ldots, \theta_n$. Let E_{ij} be the matrix in Mat(n) with a 1 in the (i, j)-position, and 0's elsewhere. Observe that

$$AE_{ij}A^{-1} = \theta_i \theta_j^{-1} E_{ij}.$$

Hence the E_{ij}'s are eigenvectors for ad(A), with eigenvalues $\theta_i \theta_j^{-1}$. We have found n^2 linearly independent eigenvectors, so

$$\operatorname{tr} \operatorname{ad}(A) = \sum_{i,j} \theta_i \theta_j^{-1},$$

where i, j range from 1 to n. It follows that

$$\begin{aligned} \operatorname{char}(\mathrm{ad}) &= \sum_{i,j} x_i x_j^{-1} \\ &= 1 + (x_1 \cdots x_n)^{-1} \left[(n-1)(x_1 \cdots x_n) + \sum_{i \neq j} \frac{x_i}{x_j} (x_1 \cdots x_n) \right] \\ &= s_\emptyset + (x_1 \cdots x_n)^{-1} s_{21^{n-2}}. \end{aligned} \tag{A2.154}$$

It follows that ad has two irreducible components, one being the trivial representation (with character $s_\emptyset$). In other words, the space Mat(n) contains a one-dimensional subspace invariant under the action of $\mathrm{GL}(n,\mathbb{C})$. This space consists of the scalar multiples of the identity matrix. The $\mathrm{GL}(n,\mathbb{C})$-invariant irreducible subspace complementary to these diagonal matrices consists of the matrices of trace 0.

The classical definition of Schur functions (Theorem 7.15.1) may be regarded as giving a formula for char φ^λ as a quotient $a_{\lambda+\delta}/a_\delta$ of two determinants. Readers familiar with the representation theory of semisimple Lie algebras or Lie groups will recognize this formula as the *Weyl character formula* for the group $\mathrm{GL}(n,\mathbb{C})$. The factorization $a_\delta = \prod_{i<j}(x_i - x_j)$ of the denominator is just the *Weyl denominator formula* for $\mathrm{GL}(n,\mathbb{C})$. Now note that for any polynomial (or rational) representation φ of $\mathrm{GL}(n,\mathbb{C})$, we have by the definition (A2.153) of char φ that

$$\dim\varphi = (\operatorname{char}\varphi)(1^n). \tag{A2.155}$$

Thus in particular $s_\lambda(1^n) = \dim\operatorname{char}\varphi^\lambda$. The formula for $s_\lambda(1^n)$ obtained by substituting $q = 1$ in equation (7.105) is equivalent to the *Weyl dimension formula* for $\mathrm{GL}(n,\mathbb{C})$. Corollary 7.21.4 is an alternative form of this formula.

Idea of Proof of Theorem A2.4. Although we will not prove Theorem A2.4 here, let us say a few words about the structure of the proof. Let V be an n-dimensional complex vector space. Then $\mathrm{GL}(V)$ acts diagonally on the k-th tensor power $V^{\otimes k}$, i.e,

$$A\cdot(v_1\otimes\cdots\otimes v_k) = (A\cdot v_1)\otimes\cdots\otimes(A\cdot v_k), \tag{A2.156}$$

and the symmetric group $\mathfrak{S}_k$ acts on $V^{\otimes k}$ by permuting tensor coordinates, i.e.,

$$w\cdot(v_1\otimes\cdots\otimes v_k) = v_{w^{-1}(1)}\otimes\cdots\otimes v_{w^{-1}(k)}. \tag{A2.157}$$

The actions of $\mathrm{GL}(V)$ and $\mathfrak{S}_k$ commute, so we have an action of $\mathfrak{S}_k\times\mathrm{GL}(V)$ on $V^{\otimes k}$. A crucial fact is that the actions of $\mathrm{GL}(V)$ and $\mathfrak{S}_k$ *centralize* each other, i.e., the (invertible) linear transformations $V^{\otimes k}\to V^{\otimes k}$ that commute with the $\mathfrak{S}_k$ action are just those given by (A2.156), while conversely the linear transformations that commute with the $\mathrm{GL}(V)$ action are those generated (as a $\mathbb{C}$-algebra) by (A2.157). From this it can be shown that $V^{\otimes k}$ decomposes into irreducible $\mathfrak{S}_k\times\mathrm{GL}(V)$-modules as follows:

$$V^{\otimes k} = \coprod_\lambda (M^\lambda\otimes F^\lambda), \tag{A2.158}$$

where $\coprod$ denotes direct sum. Here the M^λ's are nonisomorphic irreducible $\mathfrak{S}_k$ modules, the F^λ's are nonisomorphic irreducible $\mathrm{GL}(V)$ modules, and λ ranges over some index set. We know (Theorem 7.18.5) that the irreducible representations of $\mathfrak{S}_k$ are indexed by partitions λ of k, so we choose the indexing so that M^λ is the irreducible $\mathfrak{S}_k$-module corresponding to $\lambda\vdash k$ via Theorem 7.18.5. Thus we have constructed irreducible (or possibly 0) $\mathrm{GL}(V)$-modules F^λ. These modules afford polynomial representations φ^λ, and the nonzero ones are inequivalent. (The argument below shows that $F^\lambda \neq 0$ if and only if $\ell(\lambda) \leq n$.)

Next we compute the character of φ^λ. Let $w \times A$ be an element of $\mathfrak{S}_k \times \mathrm{GL}(V)$, and let $\mathrm{tr}(w \times A)$ denote the trace of $w \times A$ acting on $V^{\otimes k}$. Then by equation (A2.158) we have

$$\mathrm{tr}(w \times A) = \sum_\lambda \chi^\lambda(w) \cdot \mathrm{tr}(\varphi^\lambda(A)).$$

Let A have eigenvalues $\theta = (\theta_1, \ldots, \theta_n)$. A straightforward computation shows that $\mathrm{tr}(w \times A) = p_{\rho(w)}(\theta)$, so

$$p_{\rho(w)}(\theta) = \sum_\lambda \chi^\lambda(w)(\mathrm{char}\,\varphi^\lambda)(\theta).$$

But we know (Corollary 7.17.4) that

$$p_{\rho(w)} = \sum_\lambda \chi^\lambda(w) s_\lambda.$$

Since the χ^λ's are linearly independent, we conclude $\mathrm{char}\,\varphi^\lambda = s_\lambda$. A separate argument shows that there are no other irreducible polynomial characters, and the (sketched) proof is complete. □

The *special linear group* $\mathrm{SL}(n, \mathbb{C})$ is defined to be the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of the matrices in $\mathrm{GL}(n, \mathbb{C})$ of determinant one. It is sometimes more convenient to work with $\mathrm{SL}(n, \mathbb{C})$ rather than $\mathrm{GL}(n, \mathbb{C})$, so we will discuss the basics of the representation theory of $\mathrm{SL}(n, \mathbb{C})$. The main result is the following.

A2.5 Theorem. *Let $\lambda \in \mathrm{Par}$ with $\ell(\lambda) \leq n$. Then the restriction of φ^λ to $\mathrm{SL}(n, \mathbb{C})$ remains irreducible. The representations φ^λ for $\ell(\lambda) \leq n-1$ are all inequivalent. If $\ell(\lambda) = n$, then*

$$\varphi^\lambda = \varphi^{(\lambda_1-\lambda_n, \lambda_2-\lambda_n, \ldots, 0)},$$

as representations of $\mathrm{SL}(n, \mathbb{C})$. Every polynomial (or rational) representation of $\mathrm{SL}(n, \mathbb{C})$ is a direct sum of irreducible representations φ^λ for $\ell(\lambda) \leq n-1$.

If $\theta_1, \ldots, \theta_n$ are the eigenvalues of $A \in \mathrm{SL}(n, \mathbb{C})$, then $\theta_1 \cdots \theta_n = 1$. Hence it is natural to define the character $\mathrm{char}\,\varphi$ of a polynomial representation φ of $\mathrm{SL}(n, \mathbb{C})$ as lying in the ring $\Xi_n = \Lambda_n/(x_1 \cdots x_n - 1)$, the ring of symmetric functions in the variables $x_1, \ldots, x_n$, modulo the relation $x_1 \cdots x_n = 1$. A $\mathbb{C}$-basis for this ring consists of Schur functions s_λ with $\ell(\lambda) \leq n-1$, and two polynomial representations of $\mathrm{SL}(n, \mathbb{C})$ are equivalent if and only if they have the same character (regarded as lying in Ξ_n).

As an example of a computation of an $\mathrm{SL}(n, \mathbb{C})$ character, define the *adjoint representation* of $\mathrm{SL}(n, \mathbb{C})$ to be the action of $\mathrm{SL}(n, \mathbb{C})$ on the set $\mathrm{End}_0(V)$ of linear transformations $X : V \to V$ of trace 0 given by $A \cdot X = AXA^{-1}$. In other words, we restrict the adjoint representation of $\mathrm{GL}(n, \mathbb{C})$ to the subgroup $\mathrm{SL}(n, \mathbb{C})$ and its action to the subspace of $\mathrm{End}(V)$ consisting of linear transformations of trace 0 (the Lie algebra $\mathfrak{sl}(V)$). From equation (A2.154) we saw that $\mathrm{End}_0(V)$ was an irreducible subspace for the adjoint action of $\mathrm{GL}(n, \mathbb{C})$, with character $(x_1 \cdots x_n)^{-1} s_{21^{n-2}} \in \Lambda_n$. Hence the adjoint action of $\mathrm{SL}(n, \mathbb{C})$ is irreducible, with character $s_{21^{n-2}} \in \Xi_n$.

We conclude this appendix by discussing the connections between representation theory and two operations on symmetric functions. The first operation is the usual product fg. Let $\varphi : \mathrm{GL}(V) \to \mathrm{GL}(W)$ and $\varphi' : \mathrm{GL}(V) \to \mathrm{GL}(W')$ be two polynomial representations of $\mathrm{GL}(V)$. The tensor product representation

$$\varphi \otimes \varphi' : \mathrm{GL}(V) \to \mathrm{GL}(W \otimes W')$$

is defined by $A \cdot (w \otimes w') = (A \cdot w) \otimes (A \cdot w')$ (and extended to all of $W \otimes W'$ by bilinearity). If $B : W \to W$ and $B' : W' \to W'$ are linear transformations with eigenvalues $\theta_1, \ldots, \theta_N$ and $\theta'_1, \ldots, \theta'_{N'}$, respectively, then the eigenvalues of $B \otimes B'$ are the numbers $\theta_i \theta'_j$. It follows that

$$\mathrm{char}(\varphi \otimes \varphi') = \mathrm{char}\, \varphi \cdot \mathrm{char}\, \varphi'.$$

In particular if λ, μ, ν are partitions of length at most $n = \dim V$, then the multiplicity of φ^λ in $\varphi^\mu \otimes \varphi^\nu$ is just the Littlewood–Richardson coefficient

$$\langle s_\lambda, s_\mu s_\nu \rangle = c^\lambda_{\mu\nu}.$$

Hence the Littlewood–Richardson coefficients have a simple interpretation involving the representation theory of $\mathrm{GL}(n, \mathbb{C})$, showing in particular that they are nonnegative. Thus we have seen three fundamental ways to show that $c^\lambda_{\mu\nu} \geq 0$: (a) combinatorially, *via* the Littlewood–Richardson rule (Theorem A1.3.1), (b) algebraically, using the representation theory of the symmetric group (see Corollary 7.18.6), and (c) algebraically, using the representation theory of the general linear group (as done just above). The two methods (b) and (c) are in a sense "dual" to each other, the duality arising from the pairing (A2.158) of irreducible representations of $\mathfrak{S}_k$ and $\mathrm{GL}(V)$.

The second operation on symmetric functions we are considering here would appear at first sight rather unmotivated without understanding the connection with representation theory. Suppose that we have polynomial representations $\varphi : \mathrm{GL}(V) \to \mathrm{GL}(W)$ and $\psi : \mathrm{GL}(W) \to \mathrm{GL}(Y)$. Then the composition $\psi\varphi : \mathrm{GL}(V) \to \mathrm{GL}(Y)$ defines a polynomial representation of $\mathrm{GL}(V)$. We want to compute its character. Suppose that $A \in \mathrm{GL}(V)$ has eigenvalues $\theta_1, \ldots, \theta_n$. Then by Proposition A2.3 the eigenvalues of $\varphi(A)$ are the monomials θ^a for $x^a \in \mathcal{M}_\varphi$. Similarly, if B has eigenvalues $\zeta_1, \ldots, \zeta_N$, then the eigenvalues of $\psi(B)$ are the monomials ζ^b for $x^b \in \mathcal{M}_\psi$. Hence if we denote the monomials θ^a by $\theta^{a^1}, \ldots, \theta^{a^N}$ (in some order), then the eigenvalues of $\psi\varphi(A)$ are just the monomials

$$x^b\big|_{x_i=\theta^{a^i}}, \quad x^b \in \mathcal{M}_\psi.$$

Thus if $f = \sum_{i=1}^N \theta^{a^i} = \operatorname{char}\varphi$ and $g = \operatorname{char}\psi$, then we get

$$\operatorname{char}(\psi\varphi) = g\big(\theta^{a^1}, \ldots, \theta^{a^N}\big). \tag{A2.159}$$

This formula leads us to the following definition.

A2.6 Definition. Suppose that the symmetric function $f \in \Lambda$ is a sum of monomials, say, $f = \sum_{i\geq 1} x^{a^i}$. Given $g \in \Lambda$, define the *plethysm* $g[f]$ (sometimes denoted $f \circ g$) by

$$g[f] = g\big(x^{a^1}, x^{a^2}, \ldots\big).$$

(For the etymology of the term "plethysm," see the Notes.)

For instance, since $s_1 = x_1 + x_2 + \cdots$, we have $g[s_1] = g(x_1, x_2, \ldots) = g$. More generally, from $p_n = x_1^n + x_2^n + \cdots$ we have

$$f[p_n] = f\big(x_1^n, x_2^n, \ldots\big) = \sum_{i\geq 1} x^{a^i n} = p_n[f]. \tag{A2.160}$$

Clearly by definition of plethysm we have

$$(af + bg)[h] = af[h] + bg[h], \quad a, b \in \mathbb{Q} \tag{A2.161}$$

$$(fg)[h] = f[h] \cdot g[h]. \tag{A2.162}$$

We can use equation (A2.160) to define $p_n[f]$ for *any* $f \in \Lambda$, and then equations (A2.161) and (A2.162) allow us to define $g[f]$ for any $f, g \in \Lambda$. Specifically, if $g = \sum_\lambda c_\lambda p_\lambda$, then

$$g[f] = \sum_\lambda c_\lambda \prod_{i=1}^{\ell(\lambda)} f\big(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots\big).$$

For instance, using Proposition 7.7.6 and equation (7.19), we have

$$\begin{aligned} h_n[-s_1] &= \sum_{\lambda\vdash n} z_\lambda^{-1} p_\lambda[-s_1] \\ &= \sum_{\lambda\vdash n} z_\lambda^{-1} \prod_{i=1}^{\ell(\lambda)} \big(-s_1\big(x_1^{\lambda_i}\big)\big) \\ &= \sum_{\lambda\vdash n} z_\lambda^{-1} (-1)^{\ell(\lambda)} p_\lambda \\ &= (-1)^n e_n. \end{aligned}$$

Hence from equations (A2.161) and (A2.162) it follows that

$$f[-s_1] = (-1)^n \omega(f) \qquad \text{for any } f \in \Lambda^n. \tag{A2.163}$$

Certain symmetric function identities can often be recast in the language of plethysm. As just one example, Corollary 7.13.8 states that

$$\frac{1}{\prod_i (1-x_i) \cdot \prod_{i<j} (1 - x_i x_j)} = \sum_\lambda s_\lambda(x).$$

We leave for the reader to see that this identity is equivalent to the plethystic formula

$$\left(\sum_{n \geq 0} h_n\right)[e_1 + e_2] = \sum_\lambda s_\lambda. \tag{A2.164}$$

Returning to the representation-theoretic significance of plethysm, when we compare equation (A2.159) with the definition of plethysm (Definition A2.4), we see that

$$\operatorname{char}(\psi\varphi) = (\operatorname{char}\psi)[\operatorname{char}\varphi].$$

Here it is understood that the variables are restricted to $x_1, \ldots, x_n$ (where φ is a representation of $\mathrm{GL}(n, \mathbb{C})$). Many symmetric function identities thereby acquire a representation-theoretic significance. For instance, let $H(x) = h_0 + h_1 + \cdots$. Readers with a sufficient algebraic background will recognize $H(x_1, \ldots, x_n)$ as the character of $\mathrm{GL}(V)$, where $\dim V = n$, acting on the symmetric algebra $S(V)$. Similarly, $(e_1 + e_2)(x_1, \ldots, x_n)$ is the character of $\mathrm{GL}(V)$ acting on $V \oplus \Lambda^2(V)$, where Λ^2 denotes the second exterior power. Hence equation (A2.164) is equivalent to the assertion that the action of $\mathrm{GL}(V)$ on $S(V \oplus \Lambda^2(V))$ contains every irreducible polynomial representation of $\mathrm{GL}(V)$ exactly once (and contains no non-polynomial representation).

An important property of plethysm is given by the following result.

A2.7 Theorem. *Let f and g be $\mathbb{N}$-linear combinations of Schur functions. Then the plethysm $g[f]$ is also an $\mathbb{N}$-linear combination of Schur functions.*

Proof. Since f is an $\mathbb{N}$-linear combination of Schur functions, for any $m \in \mathbb{P}$ we have that $f(x_1, \ldots, x_m)$ is the character of a polynomial representation $\varphi : \mathrm{GL}(m, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ for n sufficiently large. Similarly, $g(x_1, \ldots, x_n)$ is the character of a polynomial representation $\psi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(Y)$. Hence the plethysm $g[f](x_1, \ldots, x_n)$ is the character of the composition $\psi\varphi$ and so is an $\mathbb{N}$-linear combination of Schur functions. Now let $n \to \infty$. □

No proof is known of Theorem A2.7 that doesn't use representation theory. In particular, a combinatorial rule for expanding the plethysm $s_n[s_m]$, analogous to the Littlewood–Richardson rule, is not known. Finding such a rule remains one of the outstanding open problems in the theory of symmetric functions.

Just as we defined in Section 7.18 an "induction product" of characters of symmetric groups to correspond to ordinary product of symmetric functions, we can ask whether there is some kind of product of $\mathfrak{S}_n$ characters that corresponds to plethysm. We will briefly sketch the answer to this question, assuming some knowledge of group theory. Suppose that $n = km$. We may regard the group $\mathfrak{S}_k^m$ as a (Young) subgroup of $\mathfrak{S}_n$ in a natural way. Let $N = N(\mathfrak{S}_k^m)$ denote the normalizer of $\mathfrak{S}_k^m$ in $\mathfrak{S}_n$. Thus N is isomorphic to the *wreath product* $\mathfrak{S}_k \S \mathfrak{S}_m$ (also denoted $\mathfrak{S}_k \wr \mathfrak{S}_m$, $\mathfrak{S}_k \sim \mathfrak{S}_m$, $\mathfrak{S}_k \operatorname{wr} \mathfrak{S}_m$, and $\mathfrak{S}_m[\mathfrak{S}_k]$). Given representations $\sigma : \mathfrak{S}_k \to \mathrm{GL}(V)$ and $\rho : \mathfrak{S}_m \to \mathrm{GL}(W)$, there is a natural (functorial) way to define a representation $\sigma \S \rho : N \to \mathrm{GL}(V^{\otimes m} \otimes W)$, as follows: we can regard in an obvious way an element of N as a pair (f, v), where $f : [m] \to \mathfrak{S}_k$ and $v \in \mathfrak{S}_m$. Then define

$$(f, v) \cdot (x_1 \otimes \cdots \otimes x_m \otimes y) = \left(f(1)x_{v^{-1}(1)}\right) \otimes \cdots \otimes \left(f(m)x_{v^{-1}(m)}\right) \otimes (v \cdot y).$$

If χ^α denotes the character of a representation α, then set $\chi^\sigma \S \chi^\rho = \chi^{\sigma \S \rho}$. We can now state the main result connecting plethysm with representations of $\mathfrak{S}_n$.

A2.8 Theorem. *Let χ be a character of $\mathfrak{S}_k$ and θ a character of $\mathfrak{S}_m$. Then*

$$\operatorname{ch} \operatorname{ind}_{N(\mathfrak{S}_k^m)}^{\mathfrak{S}_{km}} (\chi \S \theta) = (\operatorname{ch} \theta)[\operatorname{ch} \chi].$$

A2.9 Example. A *1-factor* (or *complete matching*) on $[2n]$ is a graph with the vertex set $[2n]$ and with n vertex-disjoint edges. Let O_n denote the set of all 1-factors on $[2n]$. The symmetric group $\mathfrak{S}_{2n}$ acts on O_n by permuting vertices. How does the character ψ^n of this action decompose into irreducibles? The action of $\mathfrak{S}_{2n}$ on O_n is transitive, and the stabilizer of the 1-factor with edges $\{1, 2\}$, $\{3, 4\}, \ldots, \{2n - 1, 2n\}$ is precisely the subgroup $N(\mathfrak{S}_2^n)$. Hence

$$\operatorname{ch} \psi^n = \operatorname{ch} 1_{N(\mathfrak{S}_2^n)}^{\mathfrak{S}_{2n}} = \left(\operatorname{ch} 1_{\mathfrak{S}_n}\right)\left[\operatorname{ch} 1_{\mathfrak{S}_2}\right] = h_n[h_2].$$

Now

$$\sum_{n \geq 0} h_n[h_2] = \prod_{i \leq j} (1 - x_i x_j)^{-1}.$$

Putting $q = 0$ in equation (7.202) shows that

$$\prod_{i \leq j} (1 - x_i x_j)^{-1} = \sum_{\mu \vdash n} s_{2\mu}.$$

Hence we get

$$\langle \psi^n, \chi^\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = 2\mu \text{ for some } \mu \vdash n \\ 0 & \text{otherwise.} \end{cases}$$

Exercises

7.1. [1] True or false? The Ferrers diagram of the partition (4, 4, 3) is given by

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

7.2. Let Par(n) denote the set of all partitions of n with the dominance ordering.

a. [2] Show that Par(n) is a lattice.

b. [2+] Show that Par(n) is self-dual.

c. [2+] Find the smallest value of n for which Par(n) is not graded.

d. [2+] Show that the maximum number of elements covered by an element of Par(n) is $\left\lfloor \frac{1}{2}(\sqrt{1+8n} - 3) \right\rfloor$.

e. [2+] Show that the shortest maximal chain in Par(n) has length $2n - 4$ for $n \geq 3$.

f. [3−] Show that the longest maximal chain in Par(n) has length

$$\tfrac{1}{3}m(m^2 + 3r - 1) \sim \tfrac{1}{3}(2n)^{3/2},$$

where $n = \binom{m+1}{2} + r,\ 0 \leq r \leq m$.

7.3. [2+] Expand the power series $\prod_{i \geq 1}(1 + x_i + x_i^2)$ in terms of the elementary symmetric functions.

7.4. [2+] Show that

$$h_r(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k^{n-1+r} \prod_{i \neq k} (x_k - x_i)^{-1}.$$

7.5. [2+] Prove the identity

$$\left(1 - \sum_{n \geq 1} p_n t^n\right)^{-1} = \frac{\sum_{n \geq 0} h_n t^n}{1 - \sum_{n \geq 1}(n-1)h_n t^n}. \tag{7.165}$$

7.6. [2+] Let $w \in \mathfrak{S}_n$ have cycle type λ. Give a direct bijective proof of Proposition 7.7.3, i.e., the number of elements $v \in \mathfrak{S}_n$ commuting with w is equal to $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$, where $m_i = m_i(\lambda)$.

7.7. [2+] Let Ω^n denote the subspace of Λ^n consisting of all $f \in \Lambda^n$ satisfying

$$f(x_1, -x_1, x_3, x_4, \ldots) = f(x_3, x_4, \ldots).$$

For instance, $m_1 = x_1 + x_2 + \cdots \in \Omega^1$. Find a "simple" basis for Ω^n. Express the dimension of Ω^n in terms of the number of partitions of n with a suitable restriction.

7.8. [2+] Let $f \in \Lambda^n$, and for any $g \in \Lambda^n$ define $g_k \in \Lambda^{nk}$ by

$$g_k(x_1, x_2, \dots) = g\big(x_1^k, x_2^k, \dots\big).$$

Show that

$$\omega f_k = (-1)^{n(k-1)}(\omega f)_k.$$

7.9. [2+] Let λ be a partition of n of length ℓ. Define the *forgotten symmetric function* f_λ by

$$f_\lambda = \varepsilon_\lambda\, \omega(m_\lambda),$$

where $\varepsilon_\lambda = (-1)^{n-\ell}$ as usual. (Sometimes f_λ is defined just as $\omega(m_\lambda)$.) Let $f_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$. Show that $a_{\lambda\mu}$ is equal to the number of distinct permutations $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$ of the sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_i : 1 \le i \le \ell\} \supseteq \{\mu_1 + \mu_2 + \cdots + \mu_j : 1 \le j \le \ell(\mu)\}.$$

For instance, if $\lambda = (3, 2, 1, 1)$ and $\mu = (5, 2)$, then $a_{\lambda\mu} = 5$, corresponding to (3, 2, 1, 1), (2, 3, 1, 1), (1, 1, 3, 2), (1, 3, 1, 2), and (3, 1, 1, 2).

7.10. [3−] Let $\lambda \in \mathrm{Par}$, and define the symmetric power series

$$A_\lambda(x) = \prod_\alpha \big(1 - x^\alpha\big)^{-1}$$
$$B_\lambda(x) = \prod_\alpha \big(1 + x^\alpha\big),$$

where α ranges over all *distinct* permutations of $(\lambda_1, \lambda_2, \dots)$. Find a formula for $\omega A_\lambda(x)$ and $\omega B_\lambda(x)$ in terms of $A_\mu(x)$'s and $B_\mu(x)$'s. For instance,

$$\omega A_1(x) = B_1(x)$$
$$\omega A_{11}(x) = A_2(x) A_{11}(x)$$
$$\omega A_2(x) = A_2(x)^{-1}.$$

In general, express the answer in terms of the coefficients $a_{\lambda\mu}$ defined in Exercise 7.9.

7.11. [2+] Let q be an indeterminate. Find the Schur function expansion of $\sum_{\mu \vdash n} q^{\ell(\mu)-1}\, m_\mu$.

7.12. [3−] Prove the converse to Proposition 7.10.5, i.e., if $\mu, \lambda \vdash n$ and $\mu \le \lambda$ (dominance order), then $K_{\lambda\mu} \ne 0$.

7.13. **a.** [3−] Let $\lambda \vdash n$ and $\mu \vdash n$, with $\lambda \ne (n)$. Suppose that $\lambda_1 \ne \mu_1$ and $\lambda'_1 \ne \mu'_1$. Show that $K_{\lambda\mu} = 1$ if and only if $\lambda = \langle (m+1)^m \rangle$ for some m (so $n = m(m+1)$), and $\mu = \langle m^{m+1} \rangle$. (Note that μ is assumed to be a *partition*, not just a composition.) Note that this result gives a complete characterization of when $K_{\lambda\mu} = 1$, since if $\lambda_1 = \mu_1$ then $K_{\lambda\mu} = K_{(\lambda_2, \lambda_3, \dots),(\mu_2, \mu_3, \dots)}$, while if $\lambda'_1 = \mu'_1 = \ell$ then $K_{\lambda\mu} = K_{(\lambda_1 - 1, \dots, \lambda_\ell - 1),(\mu_1 - 1, \dots, \mu_\ell - 1)}$.

b. [5−] Find a "reasonable" characterization of partitions $\lambda,\ \mu,\ \nu$ for which the Littlewood–Richardson coefficient $c^\lambda_{\mu\nu}$ is equal to one.

7.14. **a.** [2] Show that the number of SSYTs of type (r, r, r) (i.e., with r 1's, r 2's, and r 3's, and no other parts) is equal to

$$\frac{1}{16}[4r^3 + 18r^2 + 28r + 15 + (-1)^r].$$

b. [3+] Fix $n \geq 1$. Show that there are polynomials $P_n(r)$ of degree $\binom{n}{2}$ and $Q_n(r)$ of degree $\binom{2\lfloor (n-1)/2\rfloor}{2} - 1$ such that the number of SSYTs of type $(r, r, \dots, r)$ (k times) is given by $P_n(r) + (-1)^r Q_n(r)$. (The most difficult part is the degree of $Q_n(r)$.)

7.15. **a.** [2+] Let p be prime. Let $M_p(n)$ denote the number of partitions λ of n such that the number f^λ of SYTs of shape λ is relatively prime to p. Let

$$n = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots, \qquad 0 \leq \alpha_i \leq p - 1,$$

the base p expansion of n. Let $P(x) = \prod_{n\geq 1}(1 - x^n)^{-1}$. Show that

$$M_p(n) = \prod_{j\geq 0}\left(\text{coefficient of } x^{\alpha_j} \text{ in } P(x)^{p^j}\right).$$

b. [1+] Deduce from (a) that if $n = p^{k_1} + p^{k_2} + \cdots$ with $k_1 < k_2 < \cdots$, then $M_p(n) = p^{k_1+k_2+\cdots}$.

7.16. **a.** [3] Let $B_k = \sum_\lambda s_\lambda$, summed over all partitions with at most k parts. Let

$$c_i = \sum_{n=0}^{\infty} h_n h_{n+i}.$$

Show that

$$B_{2m} = \det(c_{i-j} + c_{i+j-1})_{1\leq i,j\leq m}$$

$$B_{2m+1} = h \cdot \det(c_{i-j} + c_{i+j})_{1\leq i,j\leq m},$$

where $h = \sum_{n\geq 0} h_n$.

b. [3−] Let $y_k(n)$ be the number of SYTs with n entries and at most k rows, and let C_n denote the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Deduce from (a) that

$$y_2(n) = \binom{n}{\lfloor n/2\rfloor}$$

$$y_3(n) = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}C_i$$

$$y_4(n) = C_{\lfloor (n+1)/2\rfloor}C_{\lceil (n+1)/2\rceil}$$

$$y_5(n) = 6\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}C_i\frac{(2i+2)!}{(i+2)!(i+3)!}.$$

c. [2]–[3+] Give combinatorial proofs of the above formulas for $y_2(n), \dots, y_5(n)$. Also give a simple symmetric function proof of the formula for $y_2(n)$, by considering the product $s_{\lfloor n/2\rfloor}s_{\lceil n/2\rceil}$.

d. [3] Let $R_k(x, y) = \sum_\lambda s_\lambda(x)s_\lambda(y)$, summed over all partitions with at most k parts. Let

$$A_i = \sum_{l=0}^{\infty} h_{l+i}(x)h_l(y).$$

Show that $R_k(x, y) = \det(A_{j-i})_{i,j=1}^{k}$.

e. [2+] Let $u_k(n)$ be the number of pairs of SYTs of the same shape with n entries each and at most k rows. Deduce from (c) that $u_2(n) = C_n$ and

$$u_3(n) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1}. \qquad (7.166)$$

f. [2], [5−] Give combinatorial proofs of the above formulas for $u_2(n)$ and $u_3(n)$.

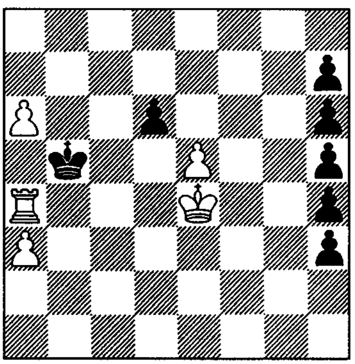
7.17. a. [3−] Let $W_i(n)$ be the number of ways to draw i diagonals in a convex n-gon such that no two diagonals intersect in their interiors. Give a bijective proof that $W_i(n)$ is the number of standard Young tableaux of shape $\langle (i+1)^2, 1^{n-i-3}\rangle$ (i.e., two parts equal to $i+1$ and $n-i-3$ parts equal to 1; when $i = 0$ there are $n-1$ parts equal to 1).

b. [2−] Deduce from (a) and the hook-length formula (Corollary 7.21.6) that

$$W_i(n) = \frac{1}{n+i}\binom{n+i}{i+1}\binom{n-3}{i},$$

in agreement with equation (6.74).

7.18. [3−] Solve the following chess problem:



Serieshelpmate in 25. How many solutions?

The definition of serieshelpmate is given in Exercise 6.23.

7.19. [3−] Let σ be a skew shape, regarded as a subset of $\mathbb{P}\times\mathbb{P}$. For instance, $32/1 = \{(1, 2), (1, 3), (2, 1), (2, 2)\}$. Let $p_\sigma(n)$ be the number of pairs (λ, μ) of partitions for which $\mu \subseteq \lambda$, $\mu \vdash n$, and λ/μ is a *translate* of σ. For instance, if

$\sigma = 32/1$ then $p_\sigma(6) = 3$, corresponding to $\lambda/\mu = 532/51,\ 43111/21111,\ 541/321$. Suppose that the smallest rectangle containing σ has r rows and s columns, and that t is the smallest integer for which $\sigma = \rho/\nu$ with $\nu \vdash t$. Show that

$$\sum_{n\geq 0} p_\sigma(n+t)q^n = \frac{[s-1]!\,[r-1]!}{[\infty]!\,[r+s-1]!},$$

where $[k]! = \prod_{i=1}^{k}(1-q^i)$.

7.20. **a.** [2] Let $\lambda \vdash n$. Show that $[\prod_i (m_i(\lambda)!)^{-1}]\langle p_1^n, h_\lambda\rangle$ is equal to the number of partitions of the set $[n]$ of type λ (i.e., with block sizes $\lambda_1, \lambda_2, \ldots$).

b. [2+] Let T be an SYT of shape $\lambda \vdash n$. For each entry of T not in the first column, let $f(i)$ be the number of entries j in the column immediately to the left of i and in a row not above i, for which $j < i$. Define $f(T) = \prod_i f(i)$, where i ranges over all entries of T not in the first column. For instance, if

$$T = \begin{array}{llll} 1 & 3 & 6 & 8 \\ 2 & 4 & 7 & \\ 5 & & & \end{array},$$

then $f(3) = 2$, $f(4) = 1$, $f(6) = 2$, $f(7) = 1$, $f(8) = 2$, and $f(T) = 8$. Show that

$$\sum_T f(T) = \left[\prod_i (m_i(\lambda)!)^{-1}\right]\langle p_1^n, h_\lambda\rangle,$$

where T ranges over all SYTs of shape λ.

c. [3−] Generalize (b) to SSYTs T of a given shape λ and type μ (so (b) is the special case $\mu = \langle 1^n\rangle$).

7.21. [3] Let $\lambda \vdash n$. An assignment $u \mapsto a_u$ of the distinct integers $1, 2, \ldots, n$ to the squares $u \in \lambda$ is a *balanced tableau* of shape λ if for each $u \in \lambda$ the number a_u is the k-th largest number in the hook of u, where k is the leg length (number of squares directly below u, counting u itself) of the hook of u. For instance, the balanced tableaux of shape (3, 2) are

$$\begin{array}{lll} 4 & 2 & 1 \\ 5 & 3 & \end{array}\quad \begin{array}{lll} 4 & 2 & 3 \\ 5 & 1 & \end{array}\quad \begin{array}{lll} 4 & 2 & 5 \\ 3 & 1 & \end{array}\quad \begin{array}{lll} 4 & 3 & 5 \\ 2 & 1 & \end{array}\quad \begin{array}{lll} 3 & 2 & 1 \\ 5 & 4 & \end{array}.$$

Let b^λ be the number of balanced tableaux of shape λ. Show that $b^\lambda = f^\lambda$, the number of SYTs of shape λ.

7.22. Let s_i denote the adjacent transposition $(i, i+1) \in \mathfrak{S}_n$, for $1 \leq i \leq n-1$. Let $w \in \mathfrak{S}_n$. It is easy to see that the smallest integer p for which w is a product of p adjacent transpositions is equal to $\ell(w)$, the number of inversions of w (defined in Section 1.3 and there denoted $i(w)$). A *reduced decomposition* of w is a sequence $a = (a_1, \ldots, a_p)$, where $p = \ell(w)$, such that $w = s_{a_1} \cdots s_{a_p}$. As usual, define the *descent set* $D(a) = \{i : 1 \leq i \leq p-1 \text{ and } a_i > a_{i+1}\}$; and write $\mathrm{co}(a)$ for the composition $\mathrm{co}(D(a)) \in \mathrm{Comp}(p)$, as defined in Section 7.19.

a. [1+] Let $R(w)$ denote the set of reduced decompositions of w, and $r(w) = \#R(w)$. Define the quasisymmetric function

$$F_w = \sum_{a \in R(w)} L_{\mathrm{co}(a)}.$$

Show that $r(w) = [x_1 x_2 \cdots x_p] F_w$. ($L_\alpha$ is defined in equation (7.89).)

b. [3] Show that $F_w \in \Lambda^p$, i.e., F_w is a homogeneous *symmetric* function of degree p. Hence if $F_w = \sum_{\lambda \vdash p} c_{w\lambda} s_\lambda$, then

$$r(w) = \sum_{\lambda \vdash p} c_{w\lambda} f^\lambda. \tag{7.167}$$

c. [3−] Define

$$r_i(w) = \#\{j \ : \ j < i \text{ and } a_j > a_i\}, \quad 1 \le i \le n$$
$$s_i(w) = \#\{j \ : \ j > i \text{ and } a_j < a_i\}, \quad 1 \le i \le n.$$

Thus $\sum_i r_i(w) = \sum_i s_i(w) = \ell(w)$. Let $\lambda(w)$ denote the partition obtained by arranging the numbers $r_1(w), \ldots, r_n(w)$ in descending order (and ignoring any 0's). Let $\mu(w)$ denote the *conjugate* to the partition $\mu'(w)$ obtained by arranging the numbers $s_1(w), \ldots, s_n(w)$ in descending order. Show that if $c_{w\nu} \neq 0$ then $\mu(w) \le \nu \le \lambda(w)$ (dominance order). Moreover, $c_{w,\mu(w)} = c_{w,\lambda(w)} = 1$. Hence F_w is a single Schur function s_ν (in which case $r(w) = f^\nu$) if and only if $\mu(w) = \lambda(w) = \nu$.

d. [3−] Show that $\lambda(w) = \mu(w)$ if and only if $w = w_1 \cdots w_n$ is 2143-*avoiding*, i.e., there do not exist $a < b < c < d$ such that $w_b < w_a < w_d < w_c$. 2143-avoiding permutations are also called *vexillary*, after the Latin word *vexillum* for "flag," because the Schubert polynomial indexed by a vexillary permutation is a flag Schur function; we will not define "Schubert polynomial" and "flag Schur function" here.

e. [3] Let $v(n)$ be the number of vexillary permutations in $\mathfrak{S}_n$. Show that

$$v(n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le 3}} (f^\lambda)^2.$$

A more explicit formula for $v(n)$ then follows from equation (7.166).

f. [2−] Let $w_0 = n, n-1, \ldots, 1$, the permutation in $\mathfrak{S}_n$ with the maximum number of inversions. Deduce from (c) that the number of reduced decompositions of w_0 is given by

$$r(w_0) = \frac{\binom{n}{2}!}{1^{n-1}\, 3^{n-2} \cdots (2n-3)^1} = \binom{n}{2}! \prod_{1 \le i < j \le n} \frac{2}{i+j-1}.$$

g. [3+] Let $p = \binom{n}{2}$. Show that

$$\sum_{(a_1, \ldots, a_p) \in R(w_0)} (x + a_1) \cdots (x + a_p) = \binom{n}{2}! \prod_{1 \le i < j \le n} \frac{2x + i + j - 1}{i + j - 1}. \tag{7.168}$$

Note that taking the coefficient of x^p on both sides gives (f). Moreover, setting $x = 0$ yields

$$\sum_{(a_1,\ldots,a_p)\in R(w_0)} a_1 \cdots a_p = \binom{n}{2}!. \tag{7.169}$$

h. [3] Show that $c_{w\lambda} \geq 0$ for all $w \in \mathfrak{S}_n$ and $\lambda \vdash \ell(w)$.

7.23. [3−] Let P be a finite graded poset of rank n. A *symmetric chain decomposition* of P is a partition of P into chains $x_i < x_{i+1} < \cdots < x_{n-i}$ such that x_j has rank j. Let M denote the (finite) multiset $\{1^{a_1}, 2^{a_2}, \ldots, k^{a_k}\}$, and let B_M denote the set of all submultisets of M, ordered by inclusion. (Thus B_M is just a product of chains of lengths $a_1, a_2, \ldots, a_k$.) Associate with each submultiset $N = \{1^{b_1}, \ldots, 1^{b_k}\} \in B_M$ the two-line array w_N whose first line consists of a_1 1's, then a_2 2's, etc., and whose second line has $a_i - b_i$ 0's followed by b_i 1's below the i's in the first line. Call two submultisets N and N' *equivalent* if, when the RSK algorithm is applied to w_N and $w_{N'}$, the same second tableau is obtained. This definition of equivalence partitions B_M into equivalence classes. Show that they form a symmetric chain decomposition of B_M.

7.24. **a.** [1] Let $U : \Lambda \to \Lambda$ and $D : \Lambda \to \Lambda$ be linear transformations defined by $U(f) = p_1 f$ and

$$D(f) = \frac{\partial}{\partial p_1} f,$$

where $\partial/\partial p_1$ is applied to f written as a polynomial in the p_i's. Show that $DU - UD = I$, the identity operator.

b. [1] Show that $DU^k = kU^{k-1} + U^k D$.

c. [2] Deduce from (a) and (b) that if $\ell \in \mathbb{N}$ then

$$(U + D)^\ell = \sum_{\substack{i+j\leq\ell \\ r:=(\ell-i-j)/2\in\mathbb{N}}} \frac{\ell!}{2^r r!\, i!\, j!} U^i D^j.$$

d. [2+] An *oscillating tableau* (or *up–down tableau*) of shape λ and length ℓ is a sequence $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^\ell = \lambda$ of partitions such that for all $1 \leq i \leq \ell - 1$, the diagram of λ^i is obtained from that of λ^{i-1} by either adding one square or removing one square. (If we add a square each time, then $\ell = |\lambda|$ and we have an SYT of shape λ.) Clearly if such an oscillating tableau exists, then $\ell = |\lambda| + 2r$ for some $r \in \mathbb{N}$. Deduce from (c) that the number $\tilde{f}^\lambda_\ell$ of oscillating tableaux of shape λ and length $\ell = |\lambda| + 2r$ is given by

$$\tilde{f}^\lambda_\ell = \frac{\ell!\, f^\lambda}{(\ell - 2r)!\, r!\, 2^r}.$$

e. [3−] Give a bijective proof of (d).

7.25. **a.** [3] Let $f_{2k}(n)$ be the number of ways to choose the diagram of a partition λ of n, then add or remove one square at a time for a total of $2k$ times, always keeping the diagram of a partition, and ending back at λ. Show that

$$\sum_{n\geq 0} f_{2k}(n)q^n = \frac{(2k)!}{2^k k!}\left(\frac{1+q}{1-q}\right)^k \prod_{i\geq 1}(1-q^i)^{-1}.$$

(Note that the case $n = 0$, obtained by setting $q = 0$, coincides with the case $\lambda = \emptyset$ of Exercise 7.24(d).)

b. [3−] Let $g_{2k}(n)$ be the number of ways to choose the diagram of a partition λ of n, then remove a square, then add a square, then remove a square, etc., for a total of k additions and k removals, always keeping the diagram of a partition, and ending back at λ. For instance, $g_0(n) = p(n)$, the number of partitions of n. Show that

$$g_{2k}(n) = \sum_{j=0}^{n}(p(j) - p(j-1))(n-j)^k.$$

7.26. [3+] Given $u = (i, j) \in \lambda \vdash n$, define the *arm length* $a(u) = \lambda_i - j$ and the *leg length* $\ell(u) = \lambda'_j - i$. Note that the hook length $h(u)$ satisfies $h(u) = a(u) + \ell(u) + 1$. Prove the identity

$$\sum_{\lambda\vdash n} \frac{\left(\sum_{(i,j)\in\lambda} t^{i-1}q^{j-1}\right)\prod_{(i,j)\in\lambda\setminus(1,1)}(1-t^{i-1}q^{j-1})}{\prod_{u\in\lambda}\left(1-t^{-\ell(u)}q^{1+a(u)}\right)\left(1-t^{1+\ell(u)}q^{-a(u)}\right)} = \frac{1}{(1-t)(1-q)}.$$

7.27. [3] Prove the following identities (combinatorially if possible). Here α and β are fixed partitions.

(a) $$\sum_{n\geq 0}\sum_{\substack{\mu\subseteq\lambda \\ |\lambda/\mu|=n}} f^{\lambda/\mu}\frac{q^{|\mu|}t^n}{n!} = \exp\left(\frac{t}{1-q}+\frac{t^2}{2(1-q^2)}\right)\cdot\prod_{i\geq 1}(1-q^i)^{-1}$$

(b) $$\sum_{n\geq 0}\sum_{\substack{\mu\subseteq\lambda \\ |\lambda/\mu|=n}} (f^{\lambda/\mu})^2\frac{q^{|\mu|}t^n}{n!} = \frac{1}{1-\frac{t}{1-q}}\prod_{i\geq 1}(1-q^i)^{-1}$$

(c) $$\sum_{\lambda} s_{\lambda/\alpha}(x)s_{\lambda/\beta}(y) = \left(\prod_{i,j}(1-x_iy_j)^{-1}\right)\sum_{\mu} s_{\beta/\mu}(x)s_{\alpha/\mu}(y)$$

(d) $$\sum_{\lambda} s_{\lambda/\alpha'}(x)s_{\lambda'/\beta}(y) = \left(\prod_{i,j}(1+x_iy_j)\right)\sum_{\mu} s_{\beta'/\mu}(x)s_{\alpha/\mu'}(y)$$

(e) $$\sum_{\lambda} s_{\lambda/\alpha}(x) = \left(\prod_{i}(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}\right)\sum_{\mu} s_{\alpha/\mu}(x)$$

(f) $$\sum_{\mu\subseteq\lambda} s_{\lambda/\mu}(x)s_{\lambda/\mu}(y)q^{|\mu|} = \prod_{n\geq 1}\left((1-q^n)\prod_{i,j}(1-x_iy_jq^{n-1})\right)^{-1}$$

(g) $$\sum_{\mu\subseteq\lambda} s_{\lambda/\mu}(x)s_{\lambda'/\mu'}(y)q^{|\mu|} = \prod_{n\geq 1}(1-q^n)^{-1}\prod_{i,j}(1+x_iy_jq^{n-1})$$

(h) $$\sum_{\mu\subseteq\lambda} s_{\lambda/\mu}(x)q^{|\mu|} = \prod_{n\geq 1}\left((1-q^n)\prod_{i<j}(1-x_ix_jq^{n-1})\prod_i(1-x_iq^{n-1})\right)^{-1}$$

7.28. a. [3−] Suppose that in the RSK algorithm $A \xrightarrow{\text{RSK}} (P, Q)$, the matrix A is symmetric (so $P = Q$). Show that $\text{tr}(A)$ is the number of columns of P of odd length.

b. [2−] Verify the identity

$$\prod_i (1-qx_i)^{-1}\cdot\prod_{i<j}(1-x_ix_j)^{-1} = \sum_\lambda q^{c(\lambda)}s_\lambda(x), \qquad (7.170)$$

where $c(\lambda)$ denotes the number of parts of λ' that are odd.

c. [1] Deduce that

$$\prod_{i<j}(1-x_ix_j)^{-1} = \sum_\mu s_{(2\mu)'}(x),$$

where $2\mu = (2\mu_1, 2\mu_2, \ldots)$.

d. [2−] Fix $k \geq 0$. Evaluate the sum $a(n,k) = \sum_\lambda f^\lambda$, where λ ranges over all partitions of n with k odd parts.

e. [2] What identity results when we apply ω to (7.170)?

7.29. a. [3−] Show that

$$\prod_i (1-x_i)\cdot\prod_{i<j}(1-x_ix_j) = \sum_\lambda (-1)^{\frac{1}{2}(|\lambda|+\text{rank}(\lambda))}s_\lambda, \qquad (7.171)$$

where λ ranges over all self-conjugate partitions.

b. [3−] Show that

$$\prod_i \left(1+x_i^2\right)\cdot\prod_{i<j}(1+x_ix_j) = \sum_\lambda s_\lambda,$$

where λ ranges over all partitions whose Frobenius notation (as defined in Exercise 7.39) has the form

$$\lambda = (\alpha_1+1 \cdots \alpha_r+1 \mid \alpha_1\cdots\alpha_r).$$

c. [3−] Show that

$$\prod_i (1+x_i)^{-1}\cdot\prod_{i\leq j}(1+x_ix_j)^{-1} = \sum_\lambda (-1)^{\frac{|\lambda|+o(\lambda)}{2}}s_\lambda,$$

where $o(\lambda)$ is the number of odd parts of λ, and λ runs over all partitions satisfying

$$\begin{aligned} \lambda_i - \lambda_{i+1} &\equiv 0, 1 \pmod 4), &\quad \lambda_i \text{ even} \\ \lambda_i - \lambda_{i+1} &\equiv 1, 2 \pmod 4), &\quad \lambda_i \text{ odd.} \end{aligned}$$

7.30. **a.** [2] Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be partitions of length at most n related by

$$\lambda_i + n - i = d(\mu_i + n - i), \qquad 1 \le i \le n,$$

for some fixed $d \in \mathbb{P}$. Show that

$$s_\lambda(x_1, \ldots, x_n) = s_\mu\left(x_1^d, \ldots, x_n^d\right) \prod_{1 \le i < j \le n} \frac{x_i^d - x_j^d}{x_i - x_j}.$$

b. [1+] Suppose that $\lambda = (d(n-1), d(n-2), \ldots, d, 0)$. Deduce from (a) that

$$s_\lambda(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} \left(x_i^{d-1} + x_i^{d-2}x_j + x_i^{d-3}x_j^2 + \cdots + x_j^{d-1}\right).$$

c. [3−] It follows from (b) that the number of SSYTs of shape $\lambda = (d(n-1), d(n-2), \ldots, d)$ and type $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (where $\alpha_1 + \cdots + \alpha_n = d\binom{n}{2}$) is equal to the number of ways of orienting the edges of the graph on the vertex set $\{1, 2, \ldots, n\}$ with $d - 1$ edges between any two distinct vertices, such that vertex i has outdegree α_i for $1 \le i \le n$. Give a direct combinatorial proof.

7.31. [3] Let p be a prime, and let A_p denote the matrix $[\zeta^{jk}]_{j,k=0}^{p-1}$, where $\zeta = e^{2\pi i/p}$. Show that every minor of A_p is nonzero. Equivalently, every square submatrix B of A_p is invertible. (Hint. Use Theorem 7.15.1.)

7.32. **a.** [2+] Let λ and μ be partitions of length at most n. Show that

$$s_\lambda(q^{\mu_1+n-1}, q^{\mu_2+n-2}, \ldots, q^{\mu_n}) = s_\mu(q^{\lambda_1+n-1}, q^{\lambda_2+n-2}, \ldots, q^{\lambda_n}) \times \prod_{1 \le i < j \le n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{\mu_i - \mu_j + j - i}}. \tag{7.172}$$

b. [2] Deduce from (a) that

$$s_\lambda(1, q, q^2, \ldots, q^{n-3}, q^{n-2}, q^n) = \frac{\sum_i q^{\lambda_i + n - i}}{1 + q + \cdots + q^{n-1}} s_\lambda(1, q, \ldots, q^{n-1})$$

$$s_\lambda(1, q^2, q^3, \ldots, q^{n-2}, q^{n-1}, q^{n+1}) = \frac{q^{\binom{n+1}{2}}\left(n - 1 + \sum_{i \ne j} q^{\lambda_i - \lambda_j + j - i}\right)}{(1 + q + \cdots + q^{n-2})(1 + q + \cdots + q^n)} s_\lambda(1, q, \ldots, q^{n-1}).$$

7.33. **a.** [2+] Let $\delta = (n-1, n-2, \ldots, 1)$. Let $t(n)$ denote the number of distinct monomials appearing in $s_\delta(x_1, \ldots, x_n)$, i.e., the number of sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ for which $K_{\delta\alpha} \ne 0$. For example, $t(3) = 7$. Show that $t(n)$ is equal to the number of labeled forests on n vertices, which by equation (5.42) and Proposition 5.3.2 is given by

$$\sum_{n \ge 0} t(n) \frac{x^n}{n!} = \exp \sum_{j \ge 1} j^{j-2} \frac{x^j}{j!}.$$

b. [3−] Let $k \in \mathbb{P}$. Generalize (a) to $s_{k\delta}(x_1, \ldots, x_n)$.
c. [5−] Can anything be said in general about the number of distinct monomials in $s_\lambda(x_1, \ldots, x_n)$ for arbitrary λ?

7.34. [3−] Let λ and μ be partitions of length at most n. Show that in the ring Λ_n (i.e., using only n variables), we have

$$s_\lambda s_\mu = \det\left(h_{\lambda_i+\mu_{n+1-j}-i+j}\right)_{i,j=1}^{n}.$$

7.35. a. [2] If R is a ring, then an additive group homomorphism $D : R \to R$ is called a *derivation* if $D(fg) = (Df)g + f(Dg)$ for all $f, g \in R$. Show that the linear transformation $\Lambda \to \Lambda$ defined by $D(s_\lambda) = s_{\lambda/1}$ is a derivation.
b. [2] Show that the bilinear operation $[f, g]$ on Λ given by $[s_\lambda, s_\mu] = s_{\lambda/1}s_\mu - s_\lambda s_{\mu/1}$ defines a Lie algebra structure on Λ. (In other words, verify the Jacobi identity $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$.)
c. [3−] Let $\rho_m = (m, m-1, \ldots, 1)$. Show that

$$\left[s_{\rho_{n+1}}, s_{\rho_{n-1}}\right] = s_{\rho_n}^2.$$

7.36. [2] Let $D_\mu : \Lambda \to \Lambda$ be the linear transformation given by $D_\mu(s_\lambda) = s_{\lambda/\mu}$. Show that $D_\mu D_\nu = D_\nu D_\mu$.

7.37. a. [2+] Let $a_\delta = \prod_{1\le i<j\le n}(x_i - x_j)$, as in equation (7.53). Write down a formula that expresses a_δ^2 in terms of the power sums $p_i(x_1, \ldots, x_n)$, $1 \le i \le 2n-2$. (You don't need to compute explicitly the coefficients in the expansion of a_δ^2 in terms of power sums; just some formula involving only power sums is wanted.)
b. [3−] Let $a_\delta^2 = \sum_{\lambda\vdash n(n-1)} c_\lambda s_\lambda(x_1, \ldots, x_n)$, where $c_\lambda \in \mathbb{Z}$. Show that if λ is the partition $\langle (n-1)^n \rangle$, then

$$c_\lambda = (-1)^{\binom{n}{2}} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

c. [3] More generally, if $\lambda = \langle (n+i-1)^{n-i}, (i-1)^i \rangle$, $1 \le i \le n$, then

$$c_\lambda = (-1)^{\frac{1}{2}(n-1)(n-2i)}[1 \cdot 3 \cdot 5 \cdots (2i-1)] \cdot [1 \cdot 3 \cdot 5 \cdots (2(n-i)-1)].$$

d. [3] Suppose that $\lambda = \mu + \langle (n-2)^n \rangle = (\mu_1 + n - 2, \ldots, \mu_n + n - 2)$, so $\mu \vdash n$. Show that

$$c_\lambda = (-1)^{\binom{n}{2}} f^\lambda \prod_{s\in\lambda}(1 - 2c(s)),$$

where $c(s)$ is the content of s, and f^λ is the number of SYTs of shape λ.
e. [3−] Show that if $\lambda \ne 2\delta = 2(n-1, n-2, \ldots, 1)$, then $c_\lambda \equiv 0 \pmod 3$.

7.38. a. [3] Fix $0 \le k \le \binom{n}{2}$, and let $\ell(w)$ denote the number of inversions of the permutation $w \in \mathfrak{S}_n$. Let λ and μ be partitions of length at most n, with $\mu \subseteq \lambda$. Define the symmetric function

$$t_{\lambda/\mu,k} = (-1)^k \sum_{\substack{w\in\mathfrak{S}_n \\ \ell(w)\ge k}} \varepsilon_w h_{\lambda+\delta-w(\mu+\delta)}.$$

Thus $t_{\lambda/\mu,k}$ is a "truncation" of the Jacobi–Trudi expansion (7.69) of $s_{\lambda/\mu}$. Show that $t_{\lambda/\mu,k}$ is s-positive.

b. [5−] Is there a "nice" combinatorial interpretation of the scalar product $\langle t_{\lambda/\mu,k}, s_\nu \rangle$?

7.39. [3−] Let λ be a partition of rank r. For $1 \le i \le r$ define $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$. The *Frobenius notation* for λ is the array

$$\lambda = (\alpha_1\, \alpha_2\, \cdots\, \alpha_r \mid \beta_1\, \beta_2\, \cdots\, \beta_r). \tag{7.173}$$

For instance,

$$(7, 7, 5, 4, 4, 2, 1, 1) = (6\,5\,2\,0 \mid 7\,4\,2\,1).$$

Note that if $a, b \in \mathbb{N}$, then $(a \mid b)$ is the Frobenius notation for the hook shape $\langle a+1, 1^b \rangle$. It is easy to see that any array (7.173) of integers satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_r \ge 0$ and $\beta_1 > \beta_2 > \cdots > \beta_r \ge 0$ is the Frobenius notation of a unique partition of rank r. Show that if $\lambda = (\alpha_1\, \cdots\, \alpha_r \mid \beta_1\, \cdots\, \beta_r)$, then

$$s_\lambda = \det\left(s_{(\alpha_i \mid \beta_j)}\right)_{i,j=1}^r.$$

7.40. [3−] Let u be a square of (the diagram of) the partition λ. Given $(i, j) \in \lambda$, let $B(i, j)$ be the border strip of λ whose top square is in row i of λ and whose bottom square is in column j of λ. Let r be the rank of λ. Show that

$$s_\lambda = \det\left(s_{B(i,j)}\right)_{i,j=1}^r.$$

7.41. [2+] Let $\ell(\lambda) \le m$ and $\lambda_1 \le n$. Define

$$\tilde{\lambda} = (n - \lambda_m, n - \lambda_{m-1}, \ldots, n - \lambda_1).$$

Give simple algebraic and combinatorial proofs that

$$(x_1 x_2 \cdots x_m)^n s_\lambda\left(x_1^{-1}, \ldots, x_m^{-1}\right) = s_{\tilde{\lambda}}(x_1, \ldots, x_m).$$

7.42. [2+] Show that

$$\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j) = \sum_\lambda s_\lambda(x) s_{\tilde{\lambda}'}(y),$$

summed over all partitions λ with $\ell(\lambda) \le m$ and $\lambda_1 \le n$, where $\tilde{\lambda}$ is defined in Exercise 7.41 above.

7.43. [3−] Let $\psi : \Lambda \to \mathbb{Q}[t]$ be the specialization (homomorphism) defined by $\psi(p_n) = 1 - (-t)^n$, $n > 0$. Show that

$$\psi(s_\lambda) = \begin{cases} t^k(1+t), & \lambda = \langle n-k, 1^k \rangle,\ 0 \le k \le n-1 \\ 0 & \text{otherwise.} \end{cases}$$

7.44. Define a specialization $\xi : \Lambda \to \mathbb{Q}[t]$ by

$$\xi(e_n) = \frac{(1-t)^{n-1}}{n!}, \qquad n > 0.$$

Let $\lambda \vdash n$ in (b)–(d) below.

a. [2+] Show that

$$\xi(h_n) = \frac{A_n(t)}{t \cdot n!},$$

where $A_n(t)$ denotes an Eulerian polynomial (Section 1.3).

b. [3−] Show that

$$\xi(p_\lambda) = \frac{z_\lambda}{n!} \sum_{\substack{w \in \mathfrak{S}_n \\ \rho(w)=\lambda}} t^{e(w)},$$

where $e(w) = \#\{i : w(i) > i\}$, the number of excedances of w.

c. [2+] Show that

$$\xi(s_\lambda) = \frac{1}{n!} \sum_{\substack{w \in \mathfrak{S}_n \\ c(w) \geq \operatorname{rank}(\lambda)}} \chi^\lambda(w) t^{e(w)},$$

where $c(w)$ is the number of cycles of w.

d. [2+] Show that

$$\xi(s_\lambda) = \frac{1}{n!} \sum_{\mu \vdash n} \binom{n}{\mu_1, \mu_2, \ldots} (K^{-1})_{\mu\lambda'} (1-t)^{n-\ell(\mu)},$$

where K^{-1} denotes the inverse of the Kostka matrix $(K_{\lambda\mu})$.

7.45. [3−] Suppose that $n = ab$, where $a, b \in \mathbb{P}$. If f is a symmetric function of degree n, then let $T_a(f)$ be the symmetric function obtained from f by expanding f in terms of monomials, dividing the exponents of these monomials by a, and then throwing away all terms whose exponents are not all integers. Thus $T_a(f)$ is a symmetric function of degree b. (For instance, $T_a(p_n) = p_b$, and $T_2(p_1^4) = m_2 + 6m_{11}$.) Show that if $\lambda \vdash n$, then $T_a(s_\lambda)$ is s-positive.

7.46. [3−] Recursively define symmetric functions q_n by

$$\sum_{\lambda \vdash n} q_\lambda = s_n,$$

where $q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots$. Show that for $n \geq 2$, the symmetric function $-q_n$ is s-positive.

7.47. Let G be a graph (without loops or multiple edges) on the d-element vertex set V. A *proper coloring* of G is a map $\kappa : V \to \mathbb{P}$ such that if $\{u, v\}$ is an edge of G, then $\kappa(u) \neq \kappa(v)$. Define $x^\kappa = \prod_i x_i^{\#\kappa^{-1}(i)}$, a monomial of degree d. Let $X_G = \sum_\kappa x^\kappa$, summed over all proper colorings of G. Thus the coefficient of $x_1^{a_1} x_2^{a_2} \cdots$ in X_G is the number of proper colorings of G such that a_i vertices are colored i for all $i \geq 1$. Clearly $X_G \in \Lambda^d$.

a. [1] Show that $X_G(1^n) = \chi_G(n)$, where χ_G denotes the chromatic polynomial of G (defined in Exercise 3.44). For this reason X_G is called the *chromatic symmetric function* of G.

b. [5] If T and T' are nonisomorphic trees, then is it true that $X_T \neq X_{T'}$?

c. [2−] A *stable partition* of G is a partition π of V such that every block B of π is *stable* (or *independent*), i.e., no two vertices of B are connected by an edge. Given a partition $\lambda = \langle 1^{r_1} 2^{r_2} \cdots \rangle$ of d, define the *augmented monomial symmetric function* $\tilde{m}_\lambda$ by $\tilde{m}_\lambda = r_1! \, r_2! \cdots m_\lambda$. Show that

$$X_G = \sum_{\lambda \vdash G} a_\lambda \tilde{m}_\lambda,$$

where a_λ is the number of stable partitions of G of type λ (i.e., with block sizes $\lambda_1, \lambda_2, \ldots$).

d. [2+] A *connected partition* of G is a partition π of V such that the restriction of G to every block of π is connected. Let L_G denote the lattice of all connected partitions of G ordered by refinement (as defined in Exercise 3.44). Show that

$$X_G = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}, \tag{7.174}$$

where μ denotes the Möbius function of L_G.

e. [2−] Deduce from equation (7.174) and Proposition 3.10.1 that ωX_G is *p-positive*.

f. [3−] Show that X_G is *L-positive*, i.e., a nonnegative linear combination of the fundamental quasisymmetric functions L_α (defined in equation (7.89)), where $\alpha \in \mathrm{Comp}(d)$.

g. [3] let $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$, and fix $k \in \mathbb{P}$. Show that the integer $\sum_{\substack{\lambda \vdash d \\ \ell(\lambda)=k}} c_\lambda$ is equal to the number $\mathrm{sink}(G, k)$ of acyclic orientations of G with exactly k sinks. (A *sink* is a vertex u with no edge $u \to v$. In particular, an isolated vertex of G is a sink in any acyclic orientation of G.)

h. [3] Let P be a d-element poset, and let $\mathrm{inc}(P)$ denote its *incomparability graph*, i.e., the vertices of $\mathrm{inc}(P)$ are the elements of P, with u and v connected by an edge if u and v are incomparable in P. A *P-tableau* of shape $\lambda \vdash d$ is a map $\tau : P \to \mathbb{P}$ satisfying: (i) If $\tau(u) = \tau(v)$ then $u \le v$ or $v \le u$ (in other words, τ is a proper coloring of $\mathrm{inc}(P)$), (ii) $\#\tau^{-1}(i) = \lambda_i$ for all i, and (iii) if $\tau^{-1}(i) = \{u_1, u_2, \ldots, u_{\lambda_i}\}$ with $u_1 < u_2 < \cdots < u_{\lambda_i}$ and $\tau^{-1}(i+1) = \{v_1, v_2, \ldots, v_{\lambda_{i+1}}\}$ with $v_1 < v_2 < \cdots < v_{\lambda_{i+1}}$, then for all i and all $1 \le j \le \lambda_{i+1}$ we require that $v_j \not< u_j$. Let f_P^λ denote the number of P-tableaux of shape λ. (Note that if P is a chain, then $f_P^\lambda = f^\lambda$, the number of SYTs of shape λ.) Define P to be $(\mathbf{3}+\mathbf{1})$*-free* if it contains no induced subposet isomorphic to $\mathbf{3}+\mathbf{1}$ (the disjoint union of a three-element chain and a one-element chain). Show that if P is $(\mathbf{3}+\mathbf{1})$-free, then

$$X_{\mathrm{inc}(P)} = \sum_{\lambda \vdash d} f_P^\lambda s_\lambda.$$

i. [2+] Let P be a $(\mathbf{3}+\mathbf{1})$-free poset, and let c_i denote the number of i-element chains in P (so in particular $c_0 = 1$). Deduce from (h) and Exercise 7.91(e) that every zero of the polynomial $C(t) = \sum c_i t^i$ is real.

j. [5] Suppose that P is a $(\mathbf{3}+\mathbf{1})$-free poset. Is it true that X_G is e-positive?

k. [3−] Let P_d be a d-element path. Show that

$$\sum_{d \ge 0} X_{P_d} \cdot t^d = \frac{\sum_{i \ge 0} e_i t^i}{1 - \sum_{i \ge 1}(i-1)e_i t^i}. \tag{7.175}$$

In particular, X_{P_d} is e-positive (a special case of (j)). Similarly, let C_d be a

d-vertex cycle. Show that

$$\sum_{d\geq 2} X_{C_d} \cdot t^d = \frac{\sum_{i\geq 0} i(i-1)e_i t^i}{1-\sum_{i\geq 1}(i-1)e_i t^i}.$$

l. [3−] Show that if the complement of G is triangle-free (equivalently, G contains no stable 3-element set of vertices), then X_G is e-positive.

m. [5] Suppose that G is *clawfree*, i.e., has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. Is it true that X_G is s-positive?

7.48. Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$. Define a formal power series F_P in the variables $x_1, x_2, \ldots$ by the formula

$$F_P = \sum_{\hat{0}=t_0\leq t_1\leq\cdots\leq t_{k-1}<t_k=\hat{1}} x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \cdots x_k^{\rho(t_{k-1},t_k)}, \qquad (7.176)$$

where $\rho(s,t)$ denotes the rank (length) of the interval $[s,t]$. (The sum ranges over all multichains from $\hat{0}$ to $\hat{1}$ of all possible lengths $k\geq 1$ such that $\hat{1}$ occurs with multiplicity one.)

a. [2] Note that F_P is a homogeneous quasisymmetric function of degree n. Show that

$$F_P = \sum_{\gamma\in\mathrm{Comp}(n)} \beta_P(S_\gamma) L_\gamma,$$

where (i) $\beta_P(S_\gamma)$ is the rank-selected Möbius invariant (now called the *flag h-vector*) of P, as defined in Section 3.12, (ii) S_γ is the subset of $[n-1]$ associated with γ, as defined in Section 7.19, and (iii) L_γ is given by (7.89).

b. [2+] Define

$$\bar{F}_P = \sum_{\hat{0}=t_0\leq t_1\leq\cdots\leq t_{k-1}<t_k=\hat{1}} \mu(t_0,t_1)\mu(t_1,t_2)\cdots\mu(t_{k-1},t_k)$$
$$\times x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \cdots x_k^{\rho(t_{k-1},t_k)},$$

where μ denotes the Möbius function of P. Show that

$$\bar{F}_P = (-1)^n \sum_{\gamma\in\mathrm{Comp}(n)} \beta_P(\bar{S}_\gamma) L_\gamma,$$

where $\bar{S}_\gamma = [n-1]-S_\gamma$. Deduce that if $F_P\in\Lambda^n$, then $\bar{F}_P = (-1)^n\omega F_P$.

c. [2+] Define P to be *locally rank-symmetric* if every interval of P is rank-symmetric, i.e., has the same number of elements of rank i as of corank i for all i. For instance, if P is *locally self-dual* (i.e., every interval is self-dual), then P is locally rank-symmetric. Show that if P is locally rank-symmetric, then $F_P\in\Lambda^n$.

d. [2] Let $P = (\boldsymbol{\mu}_1+\mathbf{1})\times\cdots\times(\boldsymbol{\mu}_\ell+\mathbf{1})$, a product of chains of cardinalities $\mu_1+1,\ldots,\mu_\ell+1$. Show that P is locally self-dual, and that $F_P = h_\mu$.

e. [3+] Let P be the lattice of subgroups of a finite abelian p-group G of type μ. Show that P is locally-self dual, and that

$$F_P = \sum_{\lambda\vdash n} \tilde{K}_{\lambda\mu}(p) s_\lambda,$$

where $\tilde{K}_{\lambda\mu}(p)$ is a polynomial in p with nonnegative integer coefficients satisfying $\tilde{K}_{\lambda\mu}(1) = K_{\lambda\mu}$ (a Kostka number). (The most difficult part is the nonnegativity of the coefficients of $\tilde{K}_{\lambda\mu}(p)$.)

f. [3−] Let $P = \mathrm{NC}_{n+1}$, the lattice of noncrossing partitions of $[n+1]$, as defined in Exercises 3.68 and 5.35. Show that NC_{n+1} is locally self-dual, and that

$$\begin{aligned} F_{\mathrm{NC}_{n+1}} &= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} \varepsilon_\lambda z_\lambda^{-1} p_\lambda \\ &= \sum_{\lambda \vdash n} \frac{1}{n+1} s_{\lambda'}(1^{n+1}) s_\lambda \\ &= \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{n+1}{\lambda_i} \right] m_\lambda \\ &= \sum_{\lambda \vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1(\lambda)!\cdots m_n(\lambda)!} e_\lambda \\ \omega F_{\mathrm{NC}_{n+1}} &= \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{\lambda_i + n}{\lambda_i} \right] m_\lambda . \end{aligned}$$

Here $m_i(\lambda)$ denotes the number of parts of λ equal to i. Show also that

$$\begin{aligned} F_{\mathrm{NC}_{n+1}} &= \frac{1}{n+1} [t^n] E(t)^{n+1} \\ \sum_{n \geq 0} F_{\mathrm{NC}_{n+1}} t^{n+1} &= (tH(-t))^{\langle -1 \rangle}, \end{aligned}$$

where $E(t) = \sum_{n \geq 0} e_n t^n$, $H(t) = \sum_{n \geq 0} h_n t^n$, and ${}^{\langle -1 \rangle}$ denotes compositional inverse with respect to the variable t.

g. [3−] Let $m, n \in \mathbb{N}$. Define the *shuffle poset* (or *poset of shuffles*) W_{mn} as follows. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ be two ordered alphabets. The elements of W_{mn} consist of all shuffles of subwords of the words $a_1 \cdots a_m$ and $b_1 \cdots b_n$, i.e, words whose restriction to the letters in A is a subword of $a_1 \cdots a_m$, and similarly for B. Some examples of elements of W_{mn} are $\emptyset$ (the empty word), $b_2 b_4 a_3 b_6 a_4 a_7 b_7$, and $a_4 a_5 a_8 b_1$. Define v to cover u in W_{mn} if v can be obtained from u either by deleting an element from A or inserting an element of B. Thus W_{mn} has minimum element $a_1 a_2 \cdots a_m$ and maximum $b_1 b_2 \cdots b_n$. Figure 7-16 shows the shuffle poset W_{21}, with $A = \{a, b\}$ and $B = \{x\}$. Show that W_{mn} is locally rank-symmetric (though not in general locally self-dual), and that

$$F_{W_{mn}} = \sum_{j \geq 0} \binom{m}{j} \binom{n}{j} e_1^{m+n-2j} e_2^j .$$

7.49. [3] Write F_n as short for the symmetric function $F_{\mathrm{NC}_{n+1}}$ of Exercise 7.48(f). Let $\psi : \Lambda \to \Lambda$ be the homomorphism defined by $\psi(h_n) = F_n$. Show that for

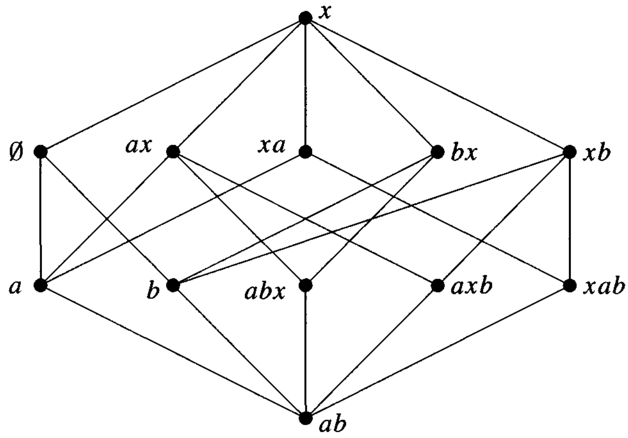


Figure 7-16. The shuffle poset W_{21}.

every skew shape λ/μ, the symmetric function $(-1)^{v(\lambda/\mu)}\psi(s_{\lambda/\mu})$ is s-positive, where $v(\lambda/\mu)$ is the number of nonzero entries below the main diagonal in the Jacobi–Trudi matrix for $s_{\lambda/\mu}$.

7.50. [2] Let $\lambda \vdash N$. Evaluate the sum

$$\frac{1}{N!}\sum_{w\in\mathfrak{S}_N}\chi^\lambda(w)n^{c(w)},$$

where $c(w)$ denotes the number of cycles of w. (Use the case $q = 1$ of Theorem 7.21.2.)

7.51. [2+] Show that if $\lambda \vdash N$ then

$$\binom{N}{2}\chi^\lambda(21^{N-2}) = f^\lambda\,(b(\lambda') - b(\lambda)),$$

where $b(\mu)$ is defined by equation (7.103).

7.52. [2+] Let λ be a partition of N of rank r. For $1 \le i \le r$, let $\mu_i = h(i,i)$, the hook length of λ at (i,i). Set $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, so μ is also a partition of N. Show that

$$\chi^\lambda(\mu) = (-1)^t,$$

where $t = \sum_{i=1}^r(\lambda_i' - i)$. Moreover, if $\chi^\lambda(\nu) \neq 0$, then show that $\nu \le \mu$ (dominance order).

7.53. [2+] Let λ be a partition of N of rank r. Show that

$$\sum_w \chi^\lambda(w) = f^\lambda\,(-1)^{t(\lambda)}\prod_{i=1}^r(\lambda_i - 1)!(\lambda_i' - 1)!,$$

where w ranges over all permutations in $\mathfrak{S}_N$ with exactly r cycles, and where $t(\lambda) = \sum_{i=1}^r(\lambda_i' - i)$.

7.54. [3−] Prove the converse to Proposition 7.17.7, i.e., if $\lambda \vdash n$ and $\chi^\lambda(\mu) = 0$ whenever μ has an even part, then $n = \binom{m}{2}$ and $\lambda = (m-1, m-2, \dots, 1)$ for some m.

7.55. **a.** [2+] Let $\rho^\lambda : \mathfrak{S}_n \to \mathrm{GL}(m, \mathbb{C})$ be an irreducible representation of $\mathfrak{S}_n$ with character χ^λ (so $m = f^\lambda$). Show that $\rho^\lambda(\mathfrak{S}_n) \subset \mathrm{SL}(m, \mathbb{C})$ if and only if

$$f^\lambda \left[\frac{b(\lambda') - b(\lambda)}{\binom{n}{2}} \right] \equiv f^\lambda \pmod 4, \tag{7.177}$$

where $b(\lambda)$ is defined by equation (7.103).

* **b.** [5−] Is there a simpler criterion? Is it possible to count the number of λ's satisfying (7.177)?

7.56. **a.** [2−] Given a skew shape θ, let θ^r denote the skew shape obtained by rotating θ 180°. For instance $(432/2)^r = 442/21$. Show that $s_\theta = s_{\theta^r}$.

b. [1+] Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a composition, and let $\tilde{\alpha} = (\alpha_k, \dots, \alpha_1)$. Show that $s_{B_\alpha} = s_{B_{\tilde{\alpha}}}$, where B_β denotes the border strip corresponding to β as defined in Section 7.23.

7.57. [2] Let $\lambda \vdash n$. How many border strips does λ have? In other words, how many partitions μ are there such that $\mu \subseteq \lambda$ and λ/μ is a (nonempty) border strip?

7.58. [2] Show that the number of odd hook lengths minus the number of even hook lengths of a partition λ is a triangular number.

7.59. This exercise deals with some basic combinatorial properties of border strips and hooks. Let λ be a partition, and let $p \in \mathbb{P}$. As noted in the solution to Exercise 7.57, there is a simple bijection between p-hooks (i.e., hooks of size p) of λ and border strips of λ of size p. Let D_λ denote the diagram of λ with its left-hand edge and upper edge extended to infinity, as shown in Figure 7-17 for $\lambda = (3,3,1)$. Put a 0 next to each vertical edge of the "lower envelope" of D_λ (whose definition should be clear from Figure 7-17), and a 1 next to each horizontal edge. If we read these numbers as we move north and east along the lower envelope, then we obtain an infinite binary sequence $C_\lambda = \cdots c_{-2}c_{-1}c_0c_1c_2\cdots$. For instance,

$$C_{331} = \cdots 0010110011 \cdots.$$

We regard a translate $\cdots b_{-1}b_0b_1\cdots$ of C_λ, where $b_i = c_{m+i}$ for some fixed $m \in \mathbb{Z}$, as being the same as C_λ. Thus the choice of which term of C_λ is labeled c_0 is arbitrary. Clearly the map $\lambda \mapsto C_\lambda$ is a one-to-one correspondence between partitions and infinite binary sequences beginning with infinitely many 0's and ending with infinitely many 1's. The size $|\lambda|$ of λ is equal to the number of pairs $i < j$ with $c_i = 1$ and $c_j = 0$.

a. [2−] Show that there is a (natural) one-to-one correpondence between the p-hooks of λ and integers i such that $c_i = 1$ and $c_{i+p} = 0$.

b. [2] Show that removing a border strip of size p from λ is equivalent to choosing i with $c_i = 1$ and $c_{i+p} = 0$, and then replacing c_i with 0 and c_{i+p} with 1.

c. [2] Let θ be a border strip of λ of size p, and let $\lambda\backslash\theta$ denote the partition obtained by removing θ from λ. Show that $\lambda\backslash\theta$ has exactly one less hook length divisible by p than λ.

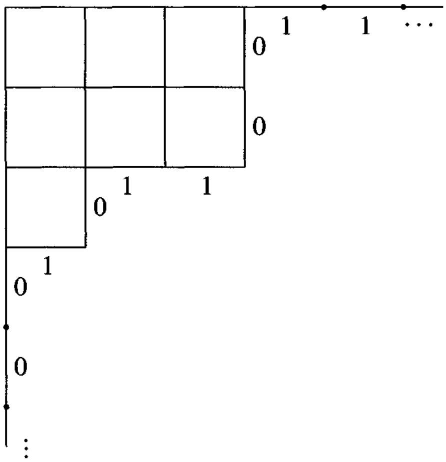


Figure 7-17. The coding C_λ of the partition $\lambda = (3, 3, 1)$.

d. [2+] Start with a partition λ, and continually remove border strips of size p until unable to do so. Show that the partition μ that remains is independent of the order in which the border strips are removed. The partition μ is called the *p-core* of λ, and a partition with no border strips (or hooks) of size p (equivalently, of size divisible by p) is called a *p-core*.

e. [2+] Let μ be a p-core. Let $Y_{p,\mu}$ be the set of all partitions whose p-core is μ. Define $\lambda \leq \nu$ in $Y_{p,\mu}$ if λ can be obtained from ν by removing border strips of size p. Show that $Y_{p,\mu} \cong Y^k$, where Y denotes Young's lattice. Deduce that if $f_\mu(n)$ is the number of partitions of n with p-core μ, then

$$\sum_{n\geq 0} f_\mu(n)x^n = x^{|\mu|}\prod_{i\geq 1}\left(1 - x^{pi}\right)^{-p}. \tag{7.178}$$

f. [2+] Let $n \in \mathbb{P}$. Show that the following three numbers are equal.
(i) The number of p-cores of size n.
(ii) The number of solutions $(x_1, \ldots, x_{p-1}) \in \mathbb{N}^{p-1}$ to the equation

$$\sum_{i=1}^{p-1}\left[ix_i + p\binom{x_i}{2}\right] - \binom{x_1 + \cdots + x_{p-1}}{2} = n.$$

(iii) The coefficient of x^n in

$$\prod_{i\geq 1}(1 - x^{pi})^p(1 - x^i)^{-1}.$$

g. [2] When $p = 2$, find all partitions in (f)(i) explicitly. What identity results from the equality of (i) and (iii)?

h. [2] Let $C_p(n)$ be the set of all $\lambda \vdash pn$ whose p-core is empty. Let f_p^λ be the number of border strip tableaux τ of shape λ such that all the border strips appearing in τ are of size p. Show that

$$\sum_{\lambda \in C_p(n)} \left(f_p^\lambda\right)^2 = p^n n!. \tag{7.179}$$

7.60. **a.** [2+] Let $\theta = \lambda/\mu$ be a border strip of size rs, where $r, s \in \mathbb{P}$. Show that there are partitions $\mu = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$ such that each skew shape μ^i/μ^{i-1} is a border strip of size s. Deduce in particular that if m is a hook length of λ and $k|m$, then k is also a hook length of λ.

b. [2+] Let $\lambda, \mu \vdash n$, with $\ell(\mu) = \ell$. Show that if $\chi^\lambda(\mu) \neq 0$, then the product $H_\lambda(q) := \prod_{u \in \lambda}(1 - q^{h(u)})$ is divisible (in $\mathbb{Z}[q]$) by $\prod_{i=1}^{\ell}(1 - q^{\mu_i})$. Here $h(u)$ denotes the hook length of λ at u.

7.61. [3−] Let $\lambda \vdash kn$. Show that

$$\left\langle h_n\left(x_1^k, x_2^k, \ldots\right), s_\lambda\right\rangle = 0 \text{ or } 1,$$

and give a rule for deciding which. In particular, show that this number is 0 unless λ has an empty k-core.

7.62. [2] Show that if $\lambda \vdash n$ and $\mu \vdash k \leq n$, then

$$\chi^\lambda(\mu 1^{n-k}) = \sum_{\nu \vdash k} f^{\lambda/\nu} \chi^\nu(\mu).$$

Here $\mu 1^{n-k}$ denotes the partition $\mu \cup \langle 1^{n-k} \rangle$.

7.63. **a.** [2+] For $\lambda \vdash n$ define

$$d_\lambda = \sum_{w \in \mathfrak{D}_n} \chi^\lambda(w),$$

where $\mathfrak{D}_n$ denotes the set of all derangements (permutations without fixed points) in $\mathfrak{S}_n$. Show that

$$\sum_{\lambda \vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (n)_k h_1^{n-k} h_k.$$

b. [2+] Deduce from (a) that for $1 \leq k \leq n$,

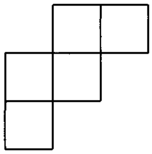
$$d_{\langle j, 1^{n-j} \rangle} = (-1)^{n-j} \binom{n}{j} D_j + (-1)^{n-1} \binom{n-1}{j},$$

where $D_j = \# \mathfrak{D}_j$ (discussed in Example 2.2.1).

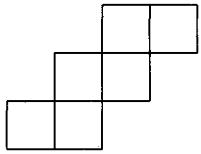
7.64. **a.** [2] For a skew shape λ/μ where $|\lambda/\mu| = n$, define the skew character $\chi^{\lambda/\mu}$ of $\mathfrak{S}_n$ by $\operatorname{ch} \chi^{\lambda/\mu} = s_{\lambda/\mu}$, so $\deg \chi^{\lambda/\mu} = f^{\lambda/\mu}$. Now fix n and set $m = \lfloor \frac{1}{2}(n+2) \rfloor$. Define the skew shape

$$\tau_n = \begin{cases} (m, m-1, \ldots, 1)/(m-2, m-3, \ldots, 1), & n \text{ odd} \\ (m, m-1, \ldots, 2)/(m-2, m-3, \ldots, 1), & n \text{ even.} \end{cases}$$

Thus τ_n is a "staircase border strip," e.g.,



τ_5



τ_6

Let $E_n = \deg \chi^{\tau_n}$. Show that E_n is an Euler number, as defined at the end of Chapter 3.16, so

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

b. [2+] Show that if $n = 2k+1$ and $\mu \vdash n$, then

$$\chi^{\tau_n}(\mu) = \begin{cases} 0 & \text{if } \mu \text{ has an even part} \\ (-1)^{k+r} E_{2r+1} & \text{if } \mu \text{ has } 2r+1 \text{ odd parts and no even parts.} \end{cases}$$

c. [2+] Show that if $n = 2k$ and $\mu \vdash n$, then

$$\chi^{\tau_n}(\mu) = (-1)^{k+r+e} E_{2r},$$

if μ has $2r$ odd parts and e even parts.

7.65. a. [2+] Let ψ_n be a character of $\mathfrak{S}_n$ for each $n \in \mathbb{P}$. Let us call the sequence $\psi_1, \psi_2, \ldots$ *elementary* if for all $w \in \mathfrak{S}_n$ we have that $\psi_n(w)$ is equal either to $\pm \deg \psi_m$ for some $m \leq n$ or to 0. For instance, the characters of the regular representations are elementary, as are the skew characters χ^{τ_n} of Exercise 7.64. Now define ψ_n by the condition that for $\lambda \vdash n$, $\langle \psi_n, \chi^\lambda \rangle$ is equal to the number of SYTs of shape λ whose largest descent has the same parity as n, where by convention every SYT has a descent at 0. For instance, $\psi_1 = 0$, $\psi_2 = \chi^2$, $\psi^3 = \chi^{21}$, $\psi^4 = \chi^4 + \chi^{31} + \chi^{22} + \chi^{211}$. Show that $\psi_1, \psi_2, \ldots$ is elementary, with $\deg \psi_n = D_n$, the number of derangements (permutations without fixed points) in $\mathfrak{S}_n$. Find $\psi_n(w)$ explicitly.

b. [5−] What other "interesting" elementary sequences are there? Can all elementary sequences be completely classified?

7.66. a. [3−] Let λ/μ be a skew shape. Define a *border strip decomposition* of λ/μ to be a partitioning of the squares of λ/μ into (nonempty) border strips. (We are not concerned with inserting the border strips in a particular order, as is the case for border strip tableaux.) For instance, Figure 7-18 shows a border strip decomposition of the shape 8877/211. Show that the number $d(\lambda/\mu)$ of border strip decompositions of λ/μ is a product of Fibonacci numbers. For instance, $d(8877/211) = 2 \cdot 3^2 \cdot 5 \cdot 8^2 \cdot 13 \cdot 21^2 \cdot 34$.

b. [3−] More generally, let

$$D_{\lambda/\mu}(q) = \sum_K q^{|\lambda/\mu| - \#K},$$

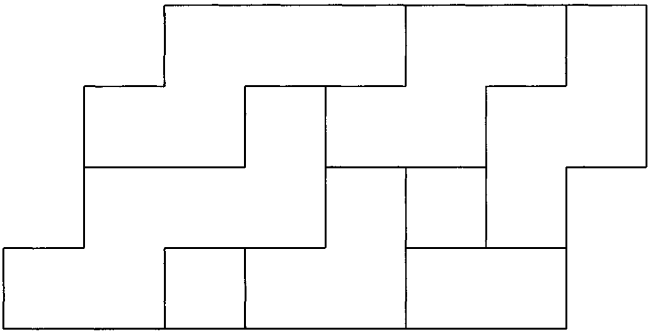


Figure 7-18. A border strip decomposition of the shape 8877/211.

where K ranges over all border strip decompositions of λ/μ and $\#K$ is the number of border strips appearing in K. Show that $D_{\lambda/\mu}(q)$ is a product of polynomials of the form $\sum_i \binom{m-i}{i} q^i$.

7.67. a. [2−] Let $0 \le s \le n-1$ and $\lambda \vdash n$. Show that if $w \in \mathfrak{S}_n$ is an n-cycle, then

$$\chi^\lambda(w) = \begin{cases} (-1)^s & \text{if } \lambda = \langle n-s, 1^s \rangle \\ 0 & \text{otherwise.} \end{cases}$$

b. [3] Let G be a finite group with conjugacy classes $C_1, \dots, C_t$. Fix w in some class C_k, and let $i_1, \dots, i_m \in [t]$. Let $\chi^1, \dots, \chi^t$ be the irreducible characters of G, and set $d_r = \deg \chi^r$. Write χ_i^r for the common value of χ^r at any $v \in C_i$. Show that the number of m-tuples $(u_1, \dots, u_m) \in G^m$ such that $u_j \in C_{i_j}$ and $u_1 \cdots u_m = w$ is equal to

$$\frac{\prod_{j=1}^m |C_{i_j}|}{|G|} \sum_{r=1}^t \frac{1}{d_r^{m-1}} \chi_{i_1}^r \cdots \chi_{i_m}^r \bar{\chi}_k^r. \tag{7.180}$$

c. [2] Fix $m \ge 1$. Use (a) and (b) to show that the number of m-tuples $(u_1, \dots, u_m)$ of n-cycles $u_i \in \mathfrak{S}_n$ satisfying $u_1 u_2 \cdots u_m = 1$ is equal to

$$\frac{(n-1)!^{m-1}}{n} \sum_{i=0}^{n-1} (-1)^{im} \binom{n-1}{i}^{-(m-2)}. \tag{7.181}$$

d. [2+] When $m = 3$, show that the above sum is equal to 0 if n is even, and to $2(n-1)!^2/(n+1)$ if n is odd.

7.68. a. [3−] Let G be a finite group of order g. Given $w \in G$, let $f(w)$ be the number of pairs $(u, v) \in G \times G$ satisfying $w = uvu^{-1}v^{-1}$ (the *commutator* of u and v). Thus f is a class function on G and hence a linear combination $\sum c_\chi \chi$ of irreducible characters χ of G. Show that the multiplicity c_χ of χ in f is equal to $g/\chi(1)$. Since $\chi(1) \mid g$, it follows that f is a character of G.

b. [5] Find an explicit G-module M whose character is f. This would provide a new proof of the basic result that $\chi(1)$ divides the order of G.

c. [2] Deduce from (a) that

$$\frac{1}{n!}\sum_{u,v\in\mathfrak{S}_n} p_{\rho(uvu^{-1}v^{-1})} = \sum_{\lambda\vdash n} H_\lambda s_\lambda, \tag{7.182}$$

where H_λ denotes the product of the hook lengths of λ.

d. [2+] Let n be an odd positive integer. Show that the number f_n of ways to write the n-cycle $(1,2,\ldots,n)\in\mathfrak{S}_n$ in the form $uvu^{-1}v^{-1}$ $(u,v\in\mathfrak{S}_n)$ is equal to $2n\cdot n!/(n+1)$.

e. [1+] Let $\kappa(w)$ denote the number of cycles of a permutation $w\in\mathfrak{S}_n$. Deduce from (c) that

$$\frac{1}{n!}\sum_{u,v\in\mathfrak{S}_n} q^{\kappa(uvu^{-1}v^{-1})} = \sum_{\lambda\vdash n}\prod_{t\in\lambda}[q+c(t)],$$

where $c(t)$ denotes the content of the square t.

f. [3−] Show that if u,v are chosen at random (uniformly, independently) from $\mathfrak{S}_n$, then the expected number E_n of cycles of $uvu^{-1}v^{-1}$ is

$$E_n = H_n + \frac{1}{n!}\left(\sum_{\substack{1\le i\le n\\ i\text{ odd}}}\frac{i!\,(n-i)!}{n-i+1} + \frac{(-1)^n}{2}\sum_{i=0}^{\lfloor (n-1)/2\rfloor} i!^2\,(n-1-2i)!\right), \tag{7.183}$$

where H_n denotes the harmonic series $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$. Note that H_n is the expected number of cycles of a random permutation in $\mathfrak{S}_n$, so the remaining terms in (7.183) are a "correction."

g. [3−] Fix $j\in\mathbb{P}$. Show that if u,v are chosen at random (uniformly, independently) from $\mathfrak{S}_n$, then the expected number e_{nj} of j-cycles of $uvu^{-1}v^{-1}$ is given by

$$e_{nj} = \frac{1}{j}\left(1+\frac{1}{\binom{n}{j}}\sum_{i=0}^{j-1}{}' \frac{(-1)^i}{\binom{j-1}{i}}\,\frac{n-j+i+1}{n-2j+i+1}\right),$$

where $\sum'$ indicates that we are to omit the term $i=2j-n-1$ when $2j>n$.

7.69. a. [2+] Expand $\frac{1}{n!}\sum_{w\in\mathfrak{S}_n} p_{\rho(w^2)}$ in terms of Schur functions.

b. [2] Show that the column sums of the character table of $\mathfrak{S}_n$ are nonnegative. These are the numbers $\sum_{\lambda\vdash n}\chi^\lambda(w)$, where $w\in\mathfrak{S}_n$ is fixed. (For the row sums of the character table of a finite group, see Exercise 7.71(b).)

c. [3] Let k be a positive integer. Show that $\sum_{w\in\mathfrak{S}_n} p_{\rho(w^k)}$ is a *nonnegative* integer linear combination of Schur functions. Equivalently, the function $r_k = r_{n,k} : \mathfrak{S}_n \to \mathbb{Z}$ defined by

$$r_k(w) = \#\{u\in\mathfrak{S}_n \,:\, u^k = w\}$$

is a character of $\mathfrak{S}_n$.

d. [3−] Let G be a finite group, and let $f_1, \ldots, f_m$ be class functions on G. Define a class function $F = F_{f_1,\ldots,f_m}$ by

$$F(w) = \sum_{u_1\cdots u_m = w} f_1(u_1)\cdots f_m(u_m).$$

Let χ be an irreducible character of G. Show that

$$\langle F, \chi\rangle = \left(\frac{|G|}{\chi(1)}\right)^{m-1} \langle f_1, \chi\rangle \cdots \langle f_m, \chi\rangle. \tag{7.184}$$

e. [2−] Show that equation (7.184) in the case $G = \mathfrak{S}_n$ and $m = 2$ is equivalent to the following result. Let $\tilde{s}_\lambda = H_\lambda s_\lambda$, called an *augmented Schur function*. Define a bilinear product $\square$ on Λ^n by

$$p_\lambda \square p_\mu = \frac{z_\lambda z_\mu}{n!} \sum_{\substack{\rho(u)=\lambda \\ \rho(v)=\mu}} p_{\rho(uv)} \qquad (\lambda, \mu \vdash n),$$

where the sum ranges over all $u, v \in \mathfrak{S}_n$ such that $\rho(u) = \lambda$ and $\rho(v) = \mu$. Then for $\lambda, \mu \vdash n$ we have

$$\tilde{s}_\lambda \square \tilde{s}_\mu = \delta_{\lambda\mu}\tilde{s}_\lambda, \tag{7.185}$$

i.e., the augmented Schur functions are orthogonal idempotents with respect to $\square$.

f. [1+] Let $(a_1, \ldots, a_m) \in \mathbb{Z}^m$, and define a class function $h = h_{a_1,\ldots,a_m}$ on $\mathfrak{S}_n$ by

$$h(w) = \#\{(u_1, \ldots, u_m) \in \mathfrak{S}_n^m \ : \ w = u_1^{a_1}\cdots u_m^{a_m}\}.$$

Show that h is a character of $\mathfrak{S}_n$.

g. [2] Let G be any finite group, and let $w \in G$. Find the number of pairs $(u, v) \in G \times G$ satisfying $w = uvu^2vuv$.

h. [2−] Fix $w \in \mathfrak{S}_n$. Show that the number $f(w)$ of solutions $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ to the equation $w = uvu^{-1}v^{-1}$ is equal to the number $g(w)$ of solutions $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ to the equation $w = u^2v^2$. Give an algebraic and a bijective proof.

i. [2+] Let $\gamma = \gamma(x_1, \ldots, x_r)$ be an element of the free group F_r on the generators $x_1, \ldots, x_r$. If G is a finite group and $w \in G$, let

$$f_{\gamma,G}(w) = \#\{(u_1, \ldots, u_r) \in G^r \ : \ \gamma(u_1, \ldots, u_r) = w\}.$$

Write $x = x_1$ and $y = x_2$, and let $k \in \mathbb{P}$. Show that for $\gamma = xy^kxy^{-k}$ and $\gamma = xy^kx^{-1}y^{-k}$, the class functions $f_{\gamma,\mathfrak{S}_n}$ are characters of $\mathfrak{S}_n$ (for all $n \in \mathbb{P}$).

j. [3] Preserve the notation of (i). Suppose that all characters of G are integer-valued, in which case we say that G is an *IC-group*. (Equivalently, if two elements of G generate the same cyclic subgroup, then they are conjugate. See e.g. [**142**, Chap. 13.1, Cor. 2].) Show that for all $\gamma \in F_r$ and all finite IC-groups G, the class function $f_{\gamma,G}$ is a difference of two characters of G.

Even more strongly, when $r = 1$ show that $f_{\gamma,G}$ is a difference of two characters for *any* finite group G. Morever, for each conjugacy class C_i in

G, define the class function g_i by

$$g_i(w) = \begin{cases} |G|/|C_i|, & w \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

If G is an IC-group, then it follows easily from the orthogonality properties of characters (see, e.g., [**142**, p. 20]) that g_i is a difference of characters. Show that if $r > 1$, then $f_{\gamma,G}$ is a $\mathbb{Z}$-linear combination of the g_i's. In particular, the symmetric function

$$\frac{1}{n!} \sum_{(u_1,\ldots,u_r)\in\mathfrak{S}_n^r} p_{\rho(\gamma(u_1,\ldots,u_r))}$$

is p-integral (and hence s-integral).

k. [5−] Preserve the notation of (i). For what γ is $f_{\gamma,\mathfrak{S}_n}$ a character of $\mathfrak{S}_n$ for all $n \geq 1$? For what γ is $f_{\gamma,G}$ a character of G for all finite groups G?

7.70. [3−] Let $x^{(1)}, \ldots, x^{(k)}$ be disjoint sets of variables, where $k \in \mathbb{N}$, and let H_λ denote the product of the hook lengths of λ. Show that

$$\sum_{\lambda\vdash n} H_\lambda^{k-2} s_\lambda\big(x^{(1)}\big)\cdots s_\lambda\big(x^{(k)}\big) = \frac{1}{n!} \sum_{\substack{w_1\cdots w_k=\mathrm{id} \\ \text{in } \mathfrak{S}_n}} p_{\rho(w_1)}\big(x^{(1)}\big)\cdots p_{\rho(w_k)}\big(x^{(k)}\big). \tag{7.186}$$

Note that the case $k = 2$ is just the Cauchy identity (in virtue of Proposition 7.7.4). What do the cases $k = 0$ and $k = 1$ say?

7.71. **a.** [2+] Show that the following two characters of a finite group G are the same:

(i) The character of the action of G on itself given by conjugation (in other words, the permutation representation $\rho : G \to \mathfrak{S}_G$ defined by $\rho(x)(y) = xyx^{-1}$, where $\mathfrak{S}_G$ is the group of permutations of G).

(ii) $\sum_\chi \chi\bar{\chi}$, where χ ranges over all irreducible characters of G.

b. [2+] Denote the above character by ψ_G, and let χ be an irreducible character of G. Show that

$$\langle \psi_G, \chi\rangle = \sum_K \chi(K),$$

where K ranges over all conjugacy classes of G and $\chi(K)$ denotes $\chi(w)$ for some $w \in K$. Thus $\langle \psi_G, \chi\rangle$ is the row sum of row χ of the character table of G. It is not *a priori* obvious that these row sums are nonnegative. (For the column sums of the character table of $\mathfrak{S}_n$, see Exercise 7.69(b).)

c. [2] Now let $G = \mathfrak{S}_n$, and write $\psi_n = \psi_G$. Show that $\operatorname{ch}\psi_n = \sum_{\lambda\vdash n} p_\lambda$, so $\sum_{n\geq 0} \operatorname{ch}\psi_n = \prod_{i\geq 1}(1-p_i)^{-1}$.

d. [3] Show that $\kappa_\lambda := \langle \psi_n, \chi^\lambda\rangle > 0$, with the sole exception $n = 2$, $\lambda = (1, 1)$.

e. [5−] Is there a "nice" combinatorial interpretation of the numbers κ_λ?

7.72. [3−] Let V be a vector space over a field K of characteristic 0 ($\mathbb{Q}$ will do) with basis $v_1, \ldots, v_n$. $\mathfrak{S}_n$ acts on V by permuting coordinates, i.e., $w \cdot v_i = v_{w^{-1}(i)}$.

Hence $\mathfrak{S}_n$ acts on the k-th exterior power $\Lambda^k V$ in a natural way, viz.,

$$w(v_i \wedge v_j \wedge \cdots) = v_{w^{-1}(i)} \wedge v_{w^{-1}(j)} \wedge \cdots.$$

Show that the character of this action is equal to $\chi^{\lambda^{k-1}} + \chi^{\lambda^k}$, where $\lambda^j = \langle n-j, 1^j \rangle$. (Set $\chi^{\lambda^{-1}} = 0$.)

7.73. [3−] As in Exercise 7.72, $\mathfrak{S}_n$ also acts on the polynomial ring $K[v_1, \ldots, v_n]$ (= the symmetric algebra $S(V^*)$) in a natural way, viz.,

$$w\left(v_1^{a_1} v_2^{a_2} \cdots\right) = v_{w^{-1}(1)}^{a_1} v_{w^{-1}(2)}^{a_2} \cdots.$$

Let ψ^k denote the character of this action on the forms (homogeneous polynomials) of degree k, so $\deg \psi^k = \binom{n+k-1}{k}$. Show that

$$\sum_{k \geq 0} (\operatorname{ch} \psi^k) q^k = \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots) s_\lambda,$$

where $s_\lambda(1, q, q^2, \ldots)$ is given explicitly by Corollary 7.21.3.

7.74. [3−] Let φ^λ be the irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ with character $s_\lambda(x_1, \ldots, x_n)$, as explained in Appendix 2. We may regard $\mathfrak{S}_n$ as the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of the $n \times n$ permutation matrices. Thus φ^λ restricts to a representation of $\mathfrak{S}_n$; let ξ^λ denote its character. Show that for $\mu \vdash n$,

$$\langle \operatorname{ch} \xi^\lambda, s_\mu \rangle = \langle s_\lambda, s_\mu[h] \rangle, \tag{7.187}$$

where $s_\mu[h]$ denotes the plethysm of s_μ with the symmetric function $h = h_0 + h_1 + h_2 + \cdots$. Note that Exercise 7.72 corresponds to the case $\lambda = \langle 1^k \rangle$, while Exercise 7.73 corresponds to $\lambda = (k)$.

7.75. **a.** [2+] Fix positive integers n and k. Let M denote the multiset $\{1^n, 2^n, \ldots, k^n\}$. The action of $\mathfrak{S}_k$ on $[k]$ induces an action of $\mathfrak{S}_k$ on the set $\binom{M}{j}$ of j-element submultisets of M. Let $v_j(\lambda)$ denote the multiplicity of the irreducible character χ^λ (where $\lambda \vdash k$) in the character of this action. Show that

$$\sum_j v_j(\lambda) q^j = s_\lambda(1, q, \ldots, q^n).$$

b. [3−] Let $\mathbb{Q}S$ denote the $\mathbb{Q}$-vector space with basis S. Define a linear map $U_j : \mathbb{Q}\binom{M}{j} \to \mathbb{Q}\binom{M}{j+1}$ by

$$U_j(X) = \sum_{Y \supset X} Y,$$

where $X \in \binom{M}{j}$ and Y ranges over all elements of $\binom{M}{j+1}$ that contain X. Show that U_j commutes with the action of $\mathfrak{S}_k$, and that if $j < kn/2$ then U_j is injective.

c. [2+] Deduce that if $s_\lambda(1, q, \ldots, q^n) = \sum_{j=0}^{kn} a_j q^j$, then $a_j = a_{kn-j}$ (this is easy to do directly) and $a_0 \leq a_1 \leq \cdots \leq a_{\lfloor kn/2 \rfloor}$. In other words, the polynomial $s_\lambda(1, q, \ldots, q^n)$ is *symmetric* and *unimodal*.

d. [2−] Show that the q-binomial coefficient $\binom{n}{k}$ is symmetric and unimodal.

7.76. **a.** [2] The *rank* of a finite group G acting transitively on a set T is defined to be the number of orbits of G acting in the obvious way on $T \times T$, i.e., $w \cdot (s, t) = (w \cdot s, w \cdot t)$. Thus G is doubly transitive if and only if rank $G = 2$. Let χ be the character of the action of G on T. Show that $\langle \chi, \chi \rangle = \operatorname{rank} G$.

b. [2] Find the rank of the natural action of $\mathfrak{S}_n$ on the set $\mathfrak{S}_n/\mathfrak{S}_\alpha$ of left cosets of the Young subgroup $\mathfrak{S}_\alpha$.

c. [2+] Give a direct bijective proof of (b).

7.77. **a.** [3−] If H and K are subgroups of a group G, then a *double coset* of (H, K) is a set $HwK = \{uwv | u \in H, v \in K\}$ for a fixed $w \in G$. The distinct double cosets of (H, K) partition G into pairwise disjoint nonempty subsets (not necessarily of the same cardinality). Show that when G is finite, the number of double cosets of (H, K) is given by

$$\left\langle \operatorname{ind}_H^G 1_H, \operatorname{ind}_K^G 1_K \right\rangle.$$

b. [2+] For any G (not necessarily finite), show that the number of double cosets of (H, H) is equal to the rank (as defined in Exercise 7.76) of G acting on G/H by left multiplication.

c. [2+] Let $G = \mathfrak{S}_n$, and let H and K be Young subgroups, say $H = \mathfrak{S}_\alpha$ and $K = \mathfrak{S}_\beta$. Interpret the number of double cosets of (H, K) in a simple combinatorial way, and give a combinatorial proof.

7.78. Let f and g be class functions (over $\mathbb{Z}$ or a field of characteristic 0) on a finite group G. Define the *Kronecker* (or *tensor*) product fg by $fg(w) = f(w)g(w)$, so fg is also a class function. Given (finite-dimensional) representations $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$, then define the tensor product representation $\varphi \otimes \rho : G \to \mathrm{GL}(V \otimes W)$ by

$$w \cdot (x \otimes y) = w \cdot x \otimes w \cdot y \qquad \text{(diagonal action).}$$

Let χ_φ and χ_ρ denote the characters of φ and ρ, respectively. Then the character $\chi_{\varphi \otimes \rho}$ of $\varphi \otimes \rho$ is just the Kronecker product $\chi_\varphi \chi_\rho$, so $\chi_\varphi \chi_\rho$ is a nonnegative integer linear combination of irreducible characters. In particular, for $G = \mathfrak{S}_n$ and $\lambda, \mu \vdash n$, we have

$$\chi^\lambda \chi^\mu = \sum_{\nu \vdash n} g_{\lambda\mu\nu} \chi^\nu, \tag{7.188}$$

for certain *nonnegative* integers $g_{\lambda\mu\nu}$. Define the *internal product* $s_\lambda * s_\mu$ by

$$s_\lambda * s_\mu = \sum_\nu g_{\lambda\mu\nu} s_\nu,$$

and extend to all of Λ by bilinearity. Clearly $*$ is associative and commutative, and $h_n * f = f$ for $f \in \Lambda^n$.

a. [2−] Show that $g_{\lambda\mu\nu}$ is invariant under permuting the indices λ, μ, ν.

b. [2−] Show that

$$s_\lambda * s_\mu = \operatorname{ch} \chi^\lambda \chi^\mu := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^\lambda(w) \chi^\mu(w) p_{\rho(w)}.$$

c. [2] Show that if $f \in \Lambda^n$ then $e_n * f = \omega f$.

d. [2] Show that $p_\lambda * p_\mu = z_\lambda \, p_\lambda \, \delta_{\lambda\mu}$.

e. [2] Let $\Lambda(x) \otimes \Lambda(y)$ be defined as at the end of Section 7.15, with the scalar product such that the elements $s_\mu(x)s_\nu(y)$ form an orthonormal basis. Let xy denote the set of variables $x_i y_j$. Show that for any $f, g, h \in \Lambda$ we have

$$\langle f(xy), g(x)h(y)\rangle = \langle f, g * h\rangle,$$

where the first scalar product takes place in $\Lambda(x) \otimes \Lambda(y)$ and the second in Λ. This gives a "basis-free" definition of the internal product, analogous to equation (7.67) for the ordinary product.

f. [2+] Show that

$$\prod_{i,j,k}(1 - x_i y_j z_k)^{-1} = \sum_{\lambda,\mu} s_\lambda * s_\mu(x)s_\lambda(y)s_\mu(z)$$
$$= \sum_{\lambda,\mu,\nu} g_{\lambda\mu\nu}s_\lambda(x)s_\mu(y)s_\nu(z).$$

g. [2+] More generally, show that if $x^{(1)}, \dots, x^{(k)}$ are disjoint sets of variables, then

$$\prod_{i_1,\dots,i_k}\left(1 - x^{(1)}_{i_1}\cdots x^{(k)}_{i_k}\right)^{-1}$$
$$= \sum_{\lambda^1,\dots,\lambda^k \vdash n}\left\langle 1_{\mathfrak{S}_n}, \chi^{\lambda^1}\cdots\chi^{\lambda^k}\right\rangle s_{\lambda^1}\left(x^{(1)}\right)\cdots s_{\lambda^k}\left(x^{(k)}\right).$$

7.79. **a.** [3−] Show that if $\langle s_\lambda, s_\mu * s_\nu\rangle \neq 0$, then $\ell(\lambda) \leq \ell(\mu)\ell(\nu)$.

b. [3−] Suppose that $\ell(\lambda) \leq ab$. Show that there exist partitions μ, ν satisfying $\ell(\mu) \leq a$, $\ell(\nu) \leq b$, and $\langle s_\lambda, s_\mu * s_\nu\rangle \neq 0$.

c. [3] Prove the following strengthening of (a) and (b): for fixed $\mu, \nu \vdash n$, we have

$$\max\{\ell(\lambda) : \langle s_\lambda, s_\mu * s_\nu\rangle \neq 0\} = |\mu \cap \nu'|,$$

where $\mu \cap \nu'$ is obtained by intersecting the diagrams of μ and ν'. Dually, we have

$$\max\{\lambda_1 : \langle s_\lambda, s_\mu * s_\nu\rangle \neq 0\} = |\mu \cap \nu|.$$

7.80. **a.** [3–] Let $\lambda, \mu, \nu \vdash n$ satisfy $s_\lambda * s_\mu = as_\nu$ for some $a \in \mathbb{P}$. Show that one of λ or μ is equal to (n) or $\langle 1^n\rangle$, and that $a = 1$.

b. [3] Let $\lambda, \mu, \nu, \sigma \vdash n$, where $\nu \neq \sigma$, satisfy $s_\lambda * s_\mu = as_\nu + bs_\sigma$ for some $a, b \in \mathbb{P}$. Show that one of λ or μ has nontrivial (i.e., not (n) or $\langle 1^n\rangle$) rectangular shape, and the other is equal to $(n-1, 1)$ or $\langle 2, 1^{n-2}\rangle$, and that $a = b = 1$.

7.81. [2+] Show that for $\lambda \vdash n$,

$$s_\lambda * s_{n-1,1} = s_1\, s_{\lambda/1} - s_\lambda.$$

7.82. **a.** [2] Show that

$$\sum_{\lambda\in\mathrm{Par}} s_\lambda * s_\lambda = \frac{1}{\prod_{i\geq 1}(1 - p_i)},$$

where $*$ denotes internal product.

b. [3−] Show that

$$\frac{\partial}{\partial p_1}\sum_{\ell(\lambda)\le 2} s_\lambda * s_\lambda = \left(\sum_{n\ge 0} s_n\right)\left(\sum_{\ell(\lambda)\le 3} s_\lambda\right).$$

7.83. **a.** [2+] Let χ and ψ be irreducible characters of a finite group G. Show that $\chi\psi$ is contained in the regular representation, i.e., $\langle\chi\psi,\phi\rangle \le \phi(1)$ for any irreducible character ϕ of G.

b. [1+] Deduce from (a) that if $\lambda,\mu,\nu\vdash n$, then $\langle s_\lambda * s_\mu, s_\nu\rangle \le f^\nu$ (the number of SYTs of shape ν).

7.84. **a.** [2+] Let $\lambda,\mu\vdash n$, with $\ell(\lambda)=\ell$. Show that

$$h_\lambda * s_\mu = \sum\prod_{i\ge 1} s_{\mu^i/\mu^{i-1}},$$

summed over all sequences $(\mu^0,\mu^1,\ldots,\mu^\ell)$ of partitions such that $\emptyset = \mu^0\subset\mu^1\subset\cdots\subset\mu^\ell=\mu$ and $|\mu^i/\mu^{i-1}|=\lambda_i$ for all $i\ge 1$.

b. [2+] Let $\lambda,\mu\vdash n$. Show that

$$h_\lambda * h_\mu = \sum_A \prod_{i,j=1}^{n} h_{a_{ij}},$$

where A ranges over all $n\times n$ $\mathbb{N}$-matrices (a_{ij}) with $\mathrm{row}(A)=\lambda$ and $\mathrm{col}(A)=\mu$.

7.85. [3] Fix $n\ge 1$. Given $\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbb{P}^k$ with $\alpha_1+\cdots+\alpha_k=n$, let $B(\alpha)$ denote the border strip whose ith row has α_i squares. Let $S=\{\alpha_1,\alpha_1+\alpha_2,\ldots,\alpha_1+\alpha_2+\cdots+\alpha_{k-1}\}\subseteq[n-1]$ and define $s_S=s_{B(\alpha)}$, the skew Schur function of shape $B(\alpha)$. Now let $S,T\subseteq[n-1]$ and $\lambda\vdash n$, and let $*$ denote internal product. Show that $\langle s_S * s_T, s_\lambda\rangle$ is equal to the number of triples $u,v,w\in\mathfrak{S}_n$ such that $uvw=1$, $D(u)=S$, $D(v)=T$, and if w is inserted into λ from right to left and from bottom to top, an SYT results. Note that the hook shapes $\langle n-k,1^k\rangle$ are border strips, so we have a combinatorial interpretation of the coefficients $g_{\lambda\mu\nu}$ when μ and ν are hooks.

Example. Let $n=3$ and $S=T=\{1\}$, so $s_S=s_T=s_{21}$. There are four triples $u,v,w\in\mathfrak{S}_3$ such that $uvw=1$ and $D(u)=D(v)=\{1\}$. In only one of these can we get an SYT by inserting w as required, viz., $u=312$, $v=213$, $w=321$. We can insert w exactly once into each $\lambda\vdash 3$, viz.,

$$\begin{array}{c}1\,2\,3\end{array}\qquad \begin{array}{l}1\,2\\3\end{array}\qquad \begin{array}{l}1\\2\\3\end{array}.$$

Hence $s_{21}*s_{21}=s_3+s_{21}+s_{111}$.

7.86. **a.** [3−] Given $\lambda,\mu\vdash n$, define

$$G_{\lambda\mu}(q)=s_\lambda * s_\mu(1,q,q^2,\ldots).$$

Show that $G_{\lambda\mu}(q)=P_{\lambda\mu}(q)H_\lambda(q)^{-1}$, where $P_{\lambda\mu}\in\mathbb{Z}[q]$ and $H_\lambda(q)$ is defined in Exercise 7.60(b).

b. [2+] Show that $P_{\lambda\mu}(1)=f^\mu$, the number of SYTs of shape μ.

c. [3] Show that if $\mu = \langle n-k, 1^k\rangle$, then $P_{\lambda\mu}(q)$ is the coefficient of t^k in the product

$$\prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (q^{i-1} + tq^{j-1}).$$

d. [5] Show that the coefficients of $P_{\lambda\mu}(q)$ are nonnegative. This has been checked for $n \leq 9$. Note that (c) shows that the coefficients of $P_{\lambda\mu}$ are indeed nonnegative when μ is a hook.

7.87. [3−] Let x, y, z be three sets of variables. Show that

$$\prod_{i,j}\prod_{r\geq 1}\prod_{a_1,\ldots,a_r}\left(1 - x_i y_j z_{a_1}\cdots z_{a_r}\right)^{-1}$$

$$= \left[\prod_{k\geq 1}(1 - p_k(z))\right] \sum_{\substack{\lambda,\mu,\nu \\ |\lambda|=|\mu|}} s_\lambda * s_\mu(z) s_{\lambda/\nu}(x) s_{\mu/\nu}(y).$$

Here $a_1, \ldots, a_r$ range independently over the positive integers and $p_k(z) = \sum z_i^k$.

7.88. a. [3−] Let C_n be the cyclic subgroup of $\mathfrak{S}_n$ of order n generated by an n-cycle w. Let χ be the character of C_n defined by $\chi(w) = e^{2\pi i/n}$. Let $\psi_m = \psi_{m,n}$ denote the induction of χ^m to $\mathfrak{S}_n$, for $m \in \mathbb{Z}$. Show that

$$\operatorname{ch}\psi_m = \frac{1}{n}\sum_{d|n}\frac{\phi(d)}{\phi(d/(m,d))}\mu(d/(m,d))p_d^{n/d}, \qquad (7.189)$$

where ϕ denotes Euler's totient function and (m, d) denotes the greatest common divisor of m and d.

b. [3−] Show that $\langle \psi_m, s_\lambda\rangle$ is equal to the number of SYTs T of shape λ satisfying $\operatorname{maj}(T) \equiv m \pmod n$.

c. [2−] Deduce from (a) and (b) that the number of SYTs T of shape λ satisfying $\operatorname{maj}(T) \equiv m \pmod n$ depends only on λ and $\gcd(m, n)$. Is there a bijective proof?

d. [3+] Let $y_k(\lambda)$ denote the number of SYTs T of shape λ satisfying $\operatorname{maj}(T) \equiv 1 \pmod k$. Show that if $\lambda \vdash n$ then

$$y_{n-1}(\lambda) \geq y_n(\lambda).$$

e. [2+] Let

$$J = \omega\sum_{n\geq 1}(-1)^{n-1}\operatorname{ch}(\psi_{1,n})$$

$$= \sum_{d\geq 1}\frac{\mu(d)}{d}\log(1 + p_d),$$

and let $H := 1 + h_1 + h_2 + \cdots$. Show that $J[H-1] = (H-1)[J] = h_1$, where brackets denote plethysm. In other words, J and $H - 1$ are plethystic inverses of one another.

7.89. **a.** [3−] Let $a < b < c < \cdots$ be an ordered alphabet. A *Lyndon word* is a word $w_1 w_2 \cdots w_n$ in the alphabet which is lexicographically strictly less than all its nonidentity cyclic shifts. Thus $aabcaabbc$ is not a Lyndon word, since its cyclic shift $aabbcaabc$ is lexicographically smaller; nor is $abab$ a Lyndon word, since it is equal to its cyclic shift of length two. Let $f(\alpha)$ be the number of Lyndon words with α_1 a's, α_2 b's, etc., where $\alpha = (\alpha_1, \alpha_2, \ldots)$. Define

$$L_n(x) = \sum_{\alpha} f(\alpha) x^{\alpha}, \tag{7.190}$$

where α ranges over all weak compositions of n. For instance, $L_3 = m_{21} + 2m_{111}$. Show that

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}, \tag{7.191}$$

where μ denotes the classical Möbius function of number theory.

b. [1+] Show that $L_n = \mathrm{ch}(\psi)$, where ψ denotes the character of $\mathfrak{S}_n$ obtained by inducing from a cyclic subgroup C_n generated by an n-cycle w to $\mathfrak{S}_n$ the character χ defined by $\chi(w) = e^{2\pi i/n}$. Deduce that $\langle L_n, s_\lambda \rangle \in \mathbb{N}$ for every $\lambda \vdash n$.

c. [1+] Even more strongly, show that $\langle L_n, s_\lambda \rangle$ is the number of standard Young tableaux T of shape λ satisfying $\mathrm{maj}(T) \equiv 1 \pmod{n}$.

d. [3−] Show that every word w in the letters $a, b, \ldots$ can be factored *uniquely* into a weakly decreasing (in lexicographic order) product of Lyndon words. For example, $bccbbcbaccaccabaabaa$ has the factorization $bcc \cdot bbc \cdot b \cdot acc \cdot acc \cdot ab \cdot aab \cdot a \cdot a$.

e. [2+] Given a word w as above, define its *Lyndon type* $\tau(w)$ to be the partition whose parts are the lengths of the Lyndon words in the factorization of w into a weakly decreasing product of Lyndon words. For instance, $\tau(dbca) = (2, 1, 1)$. Show that

$$\sum_{w} p_{\tau(w)} = n! h_n,$$

where w ranges over all permutations of n ordered letters. In other words, the distribution by Lyndon type of the permutations of an (ordered) n-set coincides with the distribution by cycle length.

f. [3−] Let M be a finite multiset on the set $\{a, b, \ldots\}$. Define

$$t_M = \sum_{w} p_{\tau(w)},$$

where w ranges over all permutations of M. Thus by (e), t_M is a multiset analogue of the cycle indicator of $\mathfrak{S}_n$. Define $L_{(i^m)}$ to be the plethysm $h_m[L_i]$ (as defined in Appendix 2). If $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$, then let $L_\lambda = L_{(1^{m_1})} L_{(2^{m_2})} \cdots$. If $M = \{a^{r_1}, b^{r_2}, \ldots\}$ and μ is the partition with parts $r_1, r_2, \ldots$, then show that $t_M(y)$ is the coefficient of $m_\mu(x)$ in $\sum_\lambda L_\lambda(x) p_\lambda(y)$.

g. [3−] Show that

$$\sum_{\lambda} L_\lambda(x) p_\lambda(y) = \sum_{\lambda} p_\lambda(x) L_\lambda(y).$$

h. [2+] Deduce that t_M is s-positive (and s-integral).

i. [5−] Is there a "nice" combinatorial interpretation of the coefficients $\langle t_M, s_\lambda \rangle$?

7.90. **a.** [2] Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a sequence of positive integers summing to n and let $|\lambda/\mu| = n$. Let $S = \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\cdots+\alpha_{k-1}\}$. Show that the number of (skew) SYTs τ of shape λ/μ satisfying $D(\tau) \subseteq S$ (where $D(\tau)$ denotes the descent set of τ) is equal to the Kostka number $K_{\lambda/\mu,\alpha}$. Give a simple bijective proof.

b. [2] Use (a) to give a simple direct proof of Proposition 7.19.9.

7.91. Let $F(t) = \sum_{j\geq 0} f_j t^j$ be a formal power series, where $f_0 = 1$. Expand the product $F(t_1)F(t_2)\cdots$ as a linear combination of Schur functions $s_\lambda(t_1, t_2, \dots)$. The coefficient of $s_\lambda(t_1, t_2, \dots)$ is called (in the terminology of D. E. Littlewood [**88**, pp. 99–100 and Chap. VII]) the *Schur function* (indexed by λ) *of the series* F, and we will denote it by s_λ^F. Equivalently, if R is a (commutative) ring containing $f_1, f_2, \dots$ and $\varphi : \Lambda \to R$ is the homomorphism defined by $\varphi(h_j) = f_j$, then $s_\lambda^F = \varphi(s_\lambda)$. Extend the definition of s_λ^F by defining $u^F = \varphi(u)$ for any $u \in \Lambda_R$.

a. [1] Show that if $F(t) = \prod_{i\geq 1}(1 - x_i t)^{-1}$, then $s_\lambda^F = s_\lambda(x)$. What if $F(t) = \prod_{i\geq 1}(1 + x_i t)$?

b. [1] Show that if $F(t) = \prod_{i=1}^n (1 - q^{i-1}t)^{-1}$, then

$$s_\lambda^F = q^{b(\lambda)} \prod_{u\in\lambda} \frac{1 - q^{n+c(u)}}{1 - q^{h(u)}},$$

where $b(\lambda)$, $c(u)$, and $h(u)$ have the same meaning as in Theorem 7.21.2.

c. [2+] Deduce from (b) that if

$$F(t) = \prod_{i\geq 0} \frac{1 - yq^i t}{1 - zq^i t},$$

then

$$s_\lambda^F = q^{b(\lambda)} \prod_{u\in\lambda} \frac{y - zq^{c(u)}}{1 - q^{h(u)}}. \tag{7.192}$$

d. [2−] Show that in general if $F(t) = \sum_{j\geq 0} f_j t^j$ and $\ell(\lambda) = \ell$, then

$$s_\lambda^F = \det\left(f_{\lambda_i - i + j}\right)_{i,j=1}^{\ell}.$$

e. [3+] Suppose that $F(t)$ is a nonconstant *polynomial* with complex coefficients (with $F(0) = 1$ as usual), so that s_λ^F is just a complex number. Show that the following four conditions are equivalent.

(i) Every zero of $F(t)$ is a negative real number.

(ii) For all partitions λ, s_λ^F is a nonnegative real number. Equivalently, when the product $F(t_1)F(t_2)\cdots$ is expanded as a linear combination of Schur functions $s_\lambda(t_1, t_2, \dots)$, all the coefficients are nonnegative real numbers. In other words, $F(t_1)F(t_2)\cdots$ is s-positive.

(iii) When the product $F(t_1)F(t_2)\cdots$ is expanded as a linear combination of elementary symmetric functions $e_\lambda(t_1, t_2, \dots)$, all the coefficients are nonnegative real numbers. In other words, $F(t_1)F(t_2)\cdots$ is e-positive. Equivalently, m_λ^F is a nonnegative real number for all partitions λ.

(iv) All coefficients of $F(t)$ are nonnegative real numbers, and the matrix $A = (p_{i+j}^F)_{i,j=0}^{n-1}$ is positive semidefinite. Here we set $p_0^F = \deg F$.

7.92. **a.** [3+] Let $A = (a_{ij})$ be an $n \times n$ real matrix such that every minor (= determinant of a square submatrix) is nonnegative. Define the symmetric function

$$F_A = \sum_{w \in \mathfrak{S}_n} a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} p_{\rho(w)}.$$

Show that F_A is s-positive.

b. [5] Show that F_A is h-positive.

7.93. [2+] Let $u = u_1 \cdots u_m \in \mathfrak{S}_m$ and $v = v_1 \cdots v_n \in \mathfrak{S}_{[m+1,m+n]}$. Let sh$(u, v)$ denote the set of *shuffles* of the words $u_1 \cdots u_m$ and $v_1 \cdots v_n$, i.e., sh(u, v) consists of all permutations $w_1 \cdots w_{m+n}$ of $[m+n]$ such that $u_1 \cdots u_m$ and $v_1 \cdots v_n$ are subsequences of w. Hence in particular $\#\mathrm{sh}(u, v) = \binom{m+n}{m}$. Let $\alpha = \mathrm{co}(u)$ and $\beta = \mathrm{co}(v)$, as defined at the beginning of Section 7.19. Show that

$$L_\alpha L_\beta = \sum_{w \in \mathrm{sh}(u,v)} L_{\mathrm{co}(w)}.$$

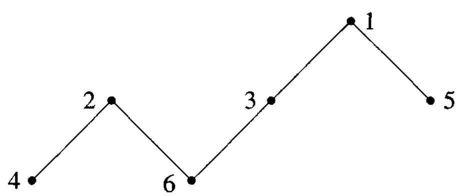
7.94. **a.** [2+] Let $\mathcal{Q}$ denote the ring of quasisymmetric functions (over $\mathbb{Q}$), and define a linear involution $\hat{\omega} : \mathcal{Q} \to \mathcal{Q}$ by $\hat{\omega}(L_\alpha) = L_{\bar{\alpha}}$, where if $\alpha \in \mathrm{Comp}(n)$ then $S_{\bar{\alpha}} = [n-1] - S_\alpha$. Show that $\hat{\omega}$ is an automorphism of $\mathcal{Q}$, and that $\hat{\omega}$ restricted to Λ coincides with the involution ω.

b. [3−] Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$. Let $f \in I(P)$ (the incidence algebra of P) satisfy $f(t, t) = 1$ for all $t \in P$. Define

$$F_f = \sum_{\hat{0}=t_0 \le t_1 \le \cdots \le t_{k-1} < t_k = \hat{1}} f(t_0, t_1) f(t_1, t_2) \cdots f(t_{k-1}, t_k) \times x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \cdots x_k^{\rho(t_{k-1},t_k)},$$

using the same notation as equation (7.176). Clearly $F_f \in \mathcal{Q}^n$. Show that $\hat{\omega}(F_f) = (-1)^n F_{f^{-1}}$. Note that Exercise 7.48(b) is the special case $f = \zeta$.

7.95. **a.** [2] Given $S \subseteq [n-1]$, let $\alpha = \mathrm{co}(S)$ be the corresponding composition of n, as defined at the beginning of Section 7.19. Let B_α be the border strip whose i-th row from the bottom has length α_i, and write P_S as short for the poset P_{B_α} (where $P_{\lambda/\mu}$ is defined after Corollary 7.19.5). Given $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$, let ω_w be the labeling of P_S obtained by inserting the numbers w_i into the squares of B_α from bottom to top and from left to right. For instance, if $\alpha = (2, 3, 1)$ (so $S = \{2, 5\}$) and $w = 426315$, then (P_S, ω_w) looks as follows:



Show that the Jordan–Hölder set $\mathcal{L}(P_S, \omega_w)$ consists of all permutations $v \in \mathfrak{S}_n$ such that $D(wv^{-1}) = S$.

b. [2+] Deduce from (a) the following statement. Given $w \in \mathfrak{S}_n$ and $S, T \subseteq [n-1]$, define

$$f(w, S, T) = \#\{(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n \,:\, uv = w,\ D(u) = S,\ D(v) = T\}.$$

Then $f(w, S, T) = f(w', S, T)$ whenever $D(w) = D(w')$.

7.96. [3] For $w \in \mathfrak{S}_n$ let $L_{\mathrm{co}(w)}$ denote the quasisymmetric function given by equation (7.89), where $\mathrm{co}(w)$ is defined at the beginning of Section 7.19. Define

$$T_n = \sum_{w \in \mathfrak{S}_n} L_{\mathrm{co}(w)} w,$$

regarded as an element of the group algebra $\mathcal{Q}\mathfrak{S}_n$ with coefficients in the ring $\mathcal{Q}$ of quasisymmetric functions. Thus T_n acts on $\mathcal{Q}\mathfrak{S}_n$ by left multiplication. Show that the eigenvalues of T_n are the power sums p_λ with multiplicity $n!/z_\lambda$, the number of permutations $w \in \mathfrak{S}_n$ of cycle type λ. What are the eigenvectors?

7.97. **a.** [2] Fix r, c, and $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash t$. Let $f(n)$ be the number of plane partitions $\pi = (\pi_{ij})$ of n with main diagonal $(\pi_{11}, \pi_{22}, \ldots) = \lambda$ and with at most r rows and at most c columns. Set $F(x) = \sum_{n \geq 0} f(n) x^n$, and show that

$$F(x) = x^{-t} s_\lambda(x, x^2, \ldots, x^r) s_\lambda(x, x^2, \ldots, x^c).$$

b. [2+] Show that if $g(n)$ is the number of *symmetric* plane partitions with main diagonal λ, then

$$\sum_{n \geq 0} g(n) x^n = s_\lambda(x, x^3, x^5, \ldots).$$

7.98. **a.** [2+] Given $\lambda \vdash n$ and $(i, j) \in \lambda$, define

$$w(i, j) = \prod_{(k,l)} x_{k-l},$$

where the x_m's are indeterminates and the product ranges over all squares $(k, l) \in \lambda$ in the hook of (i, j), i.e., such that either $k = i$ and $l \geq j$, or $l = j$ and $k \geq i$. If now $\pi = (\pi_{ij})$ is a reverse plane partition of shape λ (allowing 0 as a part), then define

$$W(\pi) = \prod_{(i,j) \in \lambda} x_{j-i}^{\pi_{ij}}.$$

Show that

$$\sum_\pi W(\pi) = \prod_{u \in \lambda} [1 - w(u)]^{-1},$$

where π ranges over all reverse plane partitions of shape λ (allowing 0 as a part).

b. [3−] State and prove an analogous result for *symmetric* reverse plane partitions of shape λ (where $\lambda = \lambda'$).

7.99. [2+] Let $K_t(n)$ be the number of plane partitions of n with trace t. Show that if $0 \le n \le t$, then $K_t(n+t)$ is equal to the coefficient of x^n in the expansion

$$\prod_{i\ge 1}(1-x^i)^{-i-1} = 1 + 2x + 6x^2 + 14x^3 + 33x^4 + 70x^5 + 149x^6 + \cdots.$$

7.100. **a.** [3−] Let A and B be two $\mathbb{N}$-matrices with the same support, i.e., $a_{ij} \neq 0$ if and only $b_{ij} \neq 0$. If $A \xrightarrow{\text{RSK}} (P, Q)$ and $B \xrightarrow{\text{RSK}} (P', Q')$, then show that P and P' have the same first columns, and that Q and Q' have the same first columns.

b. [2+] Let $t_\lambda(n)$ denote the number of plane partitions $\pi = (\pi_{ij})$ whose shape is contained in λ and that satisfy $n = \operatorname{tr}(\pi) := \pi_{11} + \pi_{22} + \cdots$. Show that $t_\lambda(n)$ is a polynomial function of n of degree $|\lambda| - 1$.

c. [2+] Show that if λ is an $a \times b$ rectangle (i.e., λ has a parts, all equal to b), then

$$t_\lambda(n) = \binom{ab+n-1}{ab-1}.$$

7.101. **a.** [3−] Let δ_n be the staircase shape $(n-1, n-2, \ldots, 1)$, and let $f_n(m)$ denote the number of plane partitions, allowing 0 as a part, of shape δ_n and with largest part at most m. For instance, it follows from Exercise 6.19(vv) that $f_n(1) = C_n$ (a Catalan number). Show that

$$f_n(m) = \prod_{i=1}^{n-1} \frac{m+i}{i}\left(\prod_{j=2}^{i} \frac{2m+i+j-1}{i+j-1}\right) = \prod_{1\le i<j\le n} \frac{2m+i+j-1}{i+j-1}. \tag{7.193}$$

* **b.** [3−] More generally, let $g_{Md\ell}(m)$ denote the number of plane partitions, allowing 0 as a part, of shape $\lambda = (M-d, M-2d, \ldots, M-\ell d)$. Show that

$$g_{Md\ell}(m) = \prod_{\substack{u=(i,j)\in\lambda \\ \ell+c(u)\le\lambda_i}} \frac{m+\ell+c(u)}{\ell+c(u)} \cdot \prod_{\substack{u=(i,j)\in\lambda \\ \ell+c(u)>\lambda_i}} \frac{(d+1)m+\ell+c(u)}{\ell+c(u)}, \tag{7.194}$$

where $c(u)$ denotes the content of the square u.

7.102. **a.** [2–] For $\lambda \in \text{Par}$, let n be large enough that $n + c(u) > 0$ for all $u \in \lambda$. (Specifically, $n \ge \ell(\lambda)$.) Define

$$t_{\lambda,n}(q) = s_\lambda(1, q, q^2, \ldots) \prod_{u\in\lambda} \left(1 - q^{n+c(u)}\right).$$

Show that $t_{\lambda,n}(q)$ is a polynomial in q with nonnegative integer coefficients.

b. [3−] Generalize (a) to skew shapes λ/μ. Here we define $c(u)$ for $u \in \lambda/\mu$ by restriction from λ. (For example, $21/1$ has contents 1 and -1, so we must take $n \ge 2$.) Thus if

$$t_{\lambda/\mu,n}(q) = s_{\lambda/\mu}(1, q, q^2, \ldots) \prod_{u\in\lambda/\mu} \left(1 - q^{n+c(u)}\right),$$

then show that $t_{\lambda/\mu,n}(q)$ is a polynomial in q with nonnegative integer coefficients. More precisely,

$$t_{\lambda/\mu,n} = \sum_T q^{|T|},$$

summed over all reverse SSYT $T = (T_{ij})$ (allowing 0 as a part) of shape λ/μ such that $T_{ij} \le n + \mu_i - i$. For instance, if $\lambda/\mu = 32/1$ and $n = 2$, then the tableaux enumerated by $t_{32/1,2}(q)$ are given by

*

$$\begin{matrix} 0\,0 \\ 0\,1 \end{matrix} \qquad \begin{matrix} 0\,0 \\ 1\,1 \end{matrix} \qquad \begin{matrix} 0\,0 \\ 0\,2 \end{matrix} \qquad \begin{matrix} 0\,0 \\ 1\,2 \end{matrix} \qquad \begin{matrix} 0\,0 \\ 2\,2 \end{matrix}.$$

Hence $t_{32/1,2}(q) = q + 2q^2 + q^3 + q^4$.

7.103. **a.** [3+] Let $A(r)$ be the number of plane partitions π with at most r rows such that π is symmetric and every row of π is a self-conjugate partition. (It follows that π has at most r columns and largest part at most r.) Such plane partitions are called *totally symmetric*. Show that

$$A(r) = \prod_{1\le i\le j\le k\le r} \frac{i+j+k-1}{i+j+k-2}.$$

b. [3+] Let $B(r)$ be the number of plane partitions as in (a) which are also self-complementary (as defined in Exercise 7.106(b)). Show that

$$B(r) = \frac{1!\,4!\,7!\,10!\cdots(3r-2)!}{r!\,(r+1)!\,(r+2)!\cdots(2r-1)!}.$$

c. [4−] A *monotone triangle* of order r is a Gelfand–Tsetlin pattern (as defined in Section 7.10) with first row $1, 2, \ldots, r$, for which every row is *strictly* increasing. Let $M(r)$ be the number of monotone triangles of order r. For instance, $M(3) = 7$, corresponding to

$$\begin{matrix} 1\,2\,3 \\ 1\,2 \\ 1 \end{matrix} \quad \begin{matrix} 1\,2\,3 \\ 1\,2 \\ 2 \end{matrix} \quad \begin{matrix} 1\,2\,3 \\ 1\,3 \\ 1 \end{matrix} \quad \begin{matrix} 1\,2\,3 \\ 1\,3 \\ 2 \end{matrix} \quad \begin{matrix} 1\,2\,3 \\ 1\,3 \\ 3 \end{matrix} \quad \begin{matrix} 1\,2\,3 \\ 2\,3 \\ 2 \end{matrix} \quad \begin{matrix} 1\,2\,3 \\ 2\,3 \\ 3 \end{matrix}.$$

Show that $M(r) = B(r)$.

d. [3] Let P be a poset with $\hat{1}$. The *MacNeille completion* $L(P)$ of P (mentioned in the solution to Exercise 3.12) is the meet semilattice of 2^P (the boolean algebra of all subsets of P) that is generated by the principal order ideals of P. Let P_n denote the Bruhat order of the symmetric group $\mathfrak{S}_n$, as defined in Exercise 3.75(a). Show that $\#L(P_n) = M(n)$. Figure 7-19 shows $L(P_4)$, with the elements of P_4 indicated by open circles.

7.104. [3+] Write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. Let $a(n)$ denote the number of plane partitions of n. Show that

$$a(n) \sim \zeta(3)^{1/36} 2^{-11/36} n^{-25/36} \exp(3\cdot 2^{-2/3}\zeta(3)^{1/3} n^{2/3} + 2C),$$

where ζ denotes the Riemann zeta function and

$$C = \int_0^\infty \frac{y\log y\,dy}{e^{2\pi y}-1} = -0.0827105718\cdots.$$

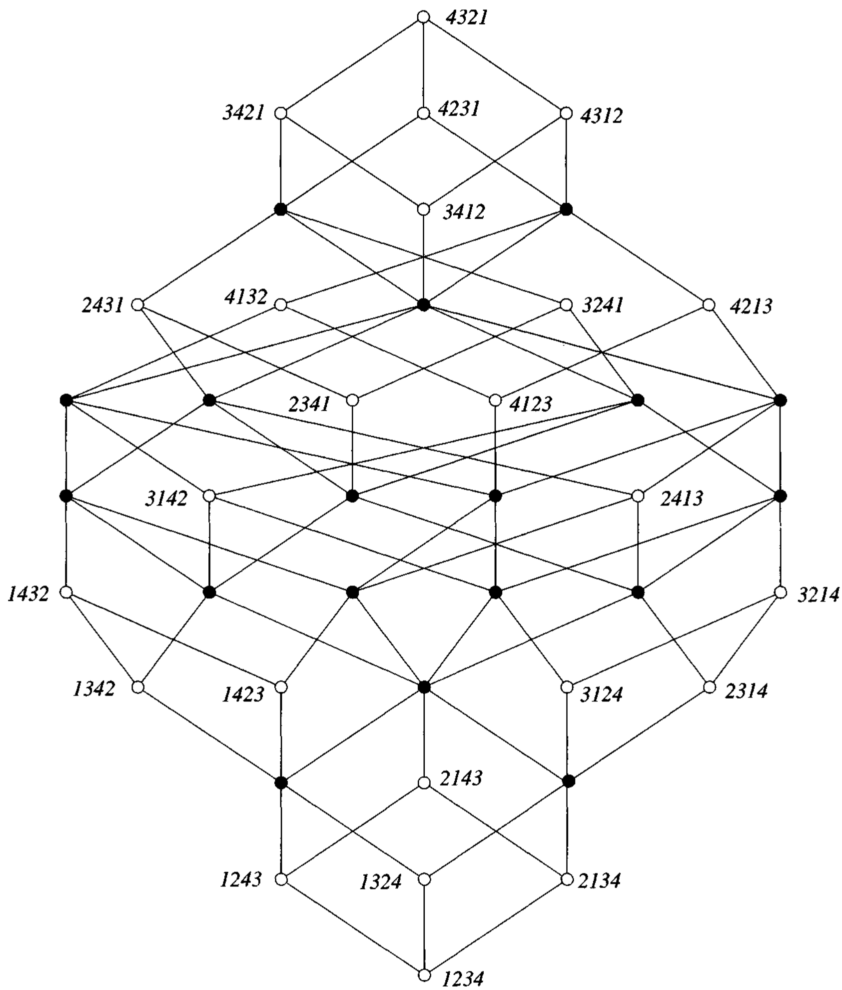


Figure 7-19. The MacNeille completion of the Bruhat order of $\mathfrak{S}_4$.

7.105. [3−] If the partitions λ and μ have the same multiset of hook lengths, does it follow that λ and μ are equal or conjugate?

7.106. **a.** [2] Let $\nu = \langle c^r \rangle$, the partition with r parts equal to c. Find the expansion of s_ν^2 in terms of Schur functions.

b. [3–] Fix r, c, and t. Let $\pi = (\pi_{ij})$ be a plane partition with at most r rows, at most c columns, and with largest part at most t. We say that π is *self-complementary*, or more precisely (r, c, t)*-self-complementary*, if $\pi_{ij} = t - \pi_{r-i,c-j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$. In other words, π is invariant under replacing each entry k by $t - k$ (where we regard π as being an $r \times c$ rectangular array) and rotating 180°. For example, the following plane partition is self-complementary for $(r, c, t) = (4, 5, 6)$:

$$\begin{bmatrix} 6 & 6 & 5 & 4 & 3 \\ 6 & 5 & 5 & 4 & 2 \\ 4 & 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \end{bmatrix}$$

Let $F(r, c, t)$ denote the number of plane partitions with at most r rows, at most c columns, and with largest part at most t. Let $G(r, c, t)$ denote the number of such plane partitions that are self-complementary. (It is easy to see that $G(r, c, t) \neq 0$ if and only if rct is even.) Show that

$$G(2r, 2c, 2t) = F(r, c, t)^2. \tag{7.195}$$

Find similar formulas for $G(2r, 2c, 2t+1)$ and $G(2r, 2c+1, 2t+1)$.

7.107. a. [2+] Let $\mu \in \mathrm{Par}$, and let A_μ be the infinite shape consisting of the quadrant $Q = \{(i, j) : i < 0,\ j > 0\}$ with the shape μ removed from the lower right-hand corner. Thus every square of A_μ has a finite hook and hence a hook length. For instance, when $\mu = (3, 1)$ we get the diagram

$$\begin{array}{c|c|c|c|c|c|c|}
 & & & & \vdots & & & \\
\hline
 & 10 & 9 & 8 & 6 & 5 & 3 \\
\hline
 & 9 & 8 & 7 & 5 & 4 & 2 \\
\hline
\cdots & 8 & 7 & 6 & 4 & 3 & 1 \\
\hline
 & 6 & 5 & 4 & 2 & 1 \\
\cline{1-6}
 & 3 & 2 & 1 \\
\cline{1-4}
\end{array}$$

Show that the multiset of hook lengths of A_μ is equal to the union of the multiset of hook lengths of Q (explicitly given by $\{1^1, 2^2, 3^3, \ldots\}$) and the multiset of hook lengths of μ.

b. [2+] Fix a plane partition μ, and let $a_\mu(n)$ be the number of skew plane partitions of n whose shape is λ/μ for some λ. For instance, $a_2(2) = 6$, corresponding to

$$\begin{array}{ccc} \cdot & \cdot & 2 \end{array} \qquad \begin{array}{cc} \cdot & \cdot \\ 2 & \end{array} \qquad \begin{array}{cccc} \cdot & \cdot & 1 & 1 \end{array} \qquad \begin{array}{ccc} \cdot & \cdot & 1 \\ 1 & & \end{array} \qquad \begin{array}{cc} \cdot & \cdot \\ 1 & 1 \end{array} \qquad \begin{array}{cc} \cdot & \cdot \\ 1 & \\ 1 & \end{array}$$

Show that

$$\sum_{n \geq 0} a_\mu(n) q^n = \left(\prod_{i \geq 1} (1 - q^i)^{-i} \right) \left(\prod_{u \in \mu} \left(1 - q^{h(u)}\right)^{-1} \right).$$

c. [3] It follows from (b), Corollary 7.20.3, and Theorem 7.22.1 that

$$a_\mu(n) = \sum_{k=0}^{n} a(k) b_\mu(n-k),$$

where $a(k)$ is the the number of plane partitions of k, and $b_\mu(n-k)$ is the number of reverse plane partitions of $n-k$ of shape μ. Find a bijective proof.

7.108. [2–] Let $p, q \geq 2$. Find explicitly the number $F(p,q)$ of $w \in \mathfrak{S}_{p+q}$ with longest increasing subsequence of length p and longest decreasing subsequence of length q.

7.109. [3] Let $E(n)$ denote the expected length of the longest increasing subsequence of $w \in \mathfrak{S}_n$. Equivalently,

$$E(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w).$$

a. [2–] Show that

$$E(n) = \sum_{\lambda \vdash n} \lambda_1 (f^\lambda)^2. \tag{7.196}$$

b. [3] Show that $\lim_{n\to\infty} E(n)/\sqrt{n}$ exists.

c. [2] Let α denote the limit in (b). Assuming that α does indeed exist, deduce from Example 7.23.19(a) that $\alpha \geq 1$.

d. [3–] Show that $\alpha \leq e$.

e. [3+] Write $\tilde{\lambda}^n = ((\tilde{\lambda}^n)_1, (\tilde{\lambda}^n)_2, \ldots)$ for some partition of n that maximizes f^λ (over all $\lambda \vdash n$). Identify $\tilde{\lambda}^n$ with the function from $\mathbb{R}_{>0}$ to $\mathbb{R}_{\geq 0}$ defined by

$$\tilde{\lambda}^n(x) = \frac{(\tilde{\lambda}^n)_i}{\sqrt{n}} \qquad \text{if } \frac{i-1}{\sqrt{n}} < x \leq \frac{i}{\sqrt{n}}.$$

Thus $\int_0^\infty \tilde{\lambda}^n(x)\,dx = 1$. Show that for weak convergence in a certain "reasonable" metric, we have

$$\lim_{n\to\infty} \tilde{\lambda}^n = f,$$

where $y = f(x)$ is defined parametrically by

$$x = y + 2\cos\theta, \quad y = \frac{2}{\pi}(\sin\theta - \theta\cos\theta), \qquad 0 \leq \theta \leq 2\pi,$$

and $f(x) = 0$ for $x > 2$. Thus f describes the "limiting shape" of the partitions that maximize f^λ.

f. [3–] Deduce from (e) that $\alpha \geq 2$.

g. [3] Use the RSK algorithm to show that $\alpha \leq 2$. Hence $\alpha = 2$.

7.110. [3–] Let $d(T)$ denote the number of descents of the SYT T. Define

$$Z = \sum_{\lambda \in \mathrm{Par}} \left(\sum_{\substack{T \text{ is an SYT} \\ \text{of shape } \lambda}} q^{d(T)} \right) s_\lambda.$$

Show that

$$Z = \frac{\sum_{n\geq 0} (1-q)^n s_n}{1 - q\sum_{n\geq 1}(1-q)^{n-1} s_n}$$

and

$$Z = \sum_{\lambda} z_\lambda^{-1} q^{-1}(1-q)^{n-\ell} A_\ell(q)\, p_\lambda,$$

where $n = |\lambda|$, $\ell = \ell(\lambda)$, and $A_\ell(q)$ denotes an Eulerian polynomial.

7.111. Let B be a subset (called a *board*) of $[n] \times [n]$. Let $X = X_B$ be the set of all permutations $w \in \mathfrak{S}_n$ satisfying $w(i) = j \Rightarrow (i, j) \in B$. Let $\tilde{Z}_X(x)$ denote the augmented cycle indicator of X, as defined in Definition 7.24.1.

a. [2−] Let $B = [n] \times [n]$. Show that $\tilde{Z}_X = n!h_n$.

b. [2+] Let $X = \{w \in \mathfrak{S}_n : w(n) \neq 1\}$. Express $\tilde{Z}_X$ in terms of the basis $\{h_\lambda\}$.

c. [3] Suppose that there are integers $a, b \geq 0$ such that $a + b \leq n$ and $(i, j) \in B$ whenever $i \leq n - a$ or $j > b$. Let $m = \min\{a, b\}$. Show that $\tilde{Z}_X$ is a nonnegative (integer) linear combination of the symmetric functions $h_j h_{n-j}$, $0 \leq j \leq m$. (Note that (b) corresponds to the case $a = b = 1$.)

d. [5] Let $B \subseteq [n] \times [n]$, and suppose that the set $\{(i, n+1-j) : (i, j) \in B\}$ is the diagram of a partition. Show that $\tilde{Z}_X$ is *h-positive*.

e. [3−] Let $w \in \mathfrak{S}_n$, and let B_w denote the $n \times n$ chessboard with w removed, i.e.,

$$B_w = \{(i, j) \in [n]^2 : w(i) \neq j\}.$$

Show that

$$\tilde{Z}_{B_w} = \sum_{\lambda \vdash n} (f^\lambda)^{-1} d_\lambda \chi^\lambda(w) s_\lambda,$$

where d_λ is defined in Exercise 7.63(a).

f. [3−] Let w be an n-cycle in (e). Show that

$$\tilde{Z}_{B_w} = \sum_{i=1}^{n} [nD_{i-1} + (-1)^i] s_{\langle i, 1^{n-i} \rangle},$$

where D_{i-1} denotes the number of derangements of $[i-1]$.

7.112. a. [2+] Define two sequences $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots b_n$ to be *equivalent* if one is a cyclic shift (conjugate) of the other. A *necklace* is an equivalence class of sequences. Show that the number $N(n, k)$ of necklaces of length n whose terms ("beads") belong to a k-element alphabet is given by

$$N(k, n) = \frac{1}{n} \sum_{d|n} \phi(d) k^{n/d}, \tag{7.197}$$

where ϕ denotes Euler's totient function.

b. [2+] Find a formula for the number of necklaces using n red beads and n blue beads (and no other beads).

7.113. [2+] Let $g_i(p)$ be the number of nonisomorphic graphs (without loops or multiple edges) with p vertices and i edges. Use Exercise 7.75(c) to show that the sequence $g_0(p), g_1(p), \ldots, g_{\binom{p}{2}}(p)$ is symmetric and unimodal.

Solutions to Exercises

7.1. True!
HINT: Where is the period at the end of the sentence?

7.2. See T. Brylawski, *Discrete Math.* **6** (1973), 201–219, and C. Greene and D. J. Kleitman, *Europ. J. Combinatorics* **7** (1986), 1–10. Note that Exercise 3.55 is concerned with the Möbius function of Par(n). (The answer to (c) is $n = 7$.)

7.3. Let ω be a primitive cube root of unity. Then

$$\begin{aligned}\prod_{i\geq 1}\left(1+x_i+x_i^2\right) &= \prod_{i\geq 1}(1-\omega x_i)(1-\omega^2 x_i)\\ &= \left(\sum_{n\geq 0}(-1)^n\omega^n e_n\right)\left(\sum_{n\geq 0}(-1)^n\omega^{2n} e_n\right)\\ &= \sum_n e_n^2 + \sum_{m<n}[(-1)^{m+n}\omega^{m+2n}+(-1)^{m+n}\omega^{2m+n}]e_m e_n\\ &= \sum_n e_n^2 + \sum_{m<n} c_{mn} e_m e_n,\end{aligned}$$

where

$$c_{mn} = \begin{cases} 2 & \text{if } m-n\equiv 0 \pmod 6\\ 1 & \text{if } m-n\equiv 1 \pmod 6\\ -1 & \text{if } m-n\equiv 2 \pmod 6\\ -2 & \text{if } m-n\equiv 3 \pmod 6\\ -1 & \text{if } m-n\equiv 4 \pmod 6\\ 1 & \text{if } m-n\equiv 5 \pmod 6). \end{cases}$$

This result is due to I. M. Gessel.

7.4. One of many ways to prove this formula (known to Jacobi) is to take the formula $s_\lambda = a_{\lambda+\delta}/a_\delta$ (Theorem 7.15.1), put $\lambda = (r)$, and expand the determinant $a_{(r)+\delta}$ by its last column. For further aspects, see R. A. Gustafson and S. C. Milne, *Advances in Math.* **48** (1983), 177–188.

7.5. By setting $y_1 = t$ and $y_2 = y_3 = \cdots = 0$ in (7.20) (or by reasoning directly from (7.11)), we get

$$\sum_{n\geq 1} p_n \frac{t^n}{n} = \log\sum_{n\geq 0} h_n t^n. \tag{7.198}$$

Differentiate with respect to t and multiply by t to get

$$\sum_{n\geq 1} p_n t^n = \frac{\sum_{n\geq 0} n h_n t^n}{\sum_{n\geq 0} h_n t^n}.$$

This is equivalent to the stated formula.

7.6. (Sketch.) Let $C_1, \ldots, C_j$ be the cycles of w of some fixed length i (so $j = m_i$). Choose a permutation $\pi \in \mathfrak{S}_j$ in $m_i!$ ways. Choose an element $a_k \in C_k$,

$1 \le k \le j$, in i^{m_i} ways. Do this for all i. Then there is a unique permutation v commuting with w such that if b_k is the least element of C_k, then $v(b_k) = a_{\pi(k)}$, and all v commuting with w are obtained in this way.

7.7. *Answer.* A basis consists of $\{p_\lambda : \lambda \vdash n,$ all parts $\lambda_i > 0$ are odd$\}$.

Proof. Clearly each such $p_\lambda \in \Omega^n$. Conversely, assume that $f \in \Omega^n$ but $f \notin \mathrm{span}_{\mathbb{Q}}\{p_\lambda : \lambda \vdash n\}$. We can assume that $f = \sum_\lambda c_\lambda p_\lambda$, where λ ranges over all partitions of n with at least one even part. Let μ be a *least* element in dominance order for which $c_\mu \neq 0$. Then the coefficient of $x_1^{\mu_1} x_3^{\mu_2} x_4^{\mu_3} \cdots$ in $f(x_1, -x_1, x_3, x_4, \ldots)$ is nonzero [why?], a contradiction. □

It follows that $\dim \Omega^n$ is the number of partitions of n into odd parts. By a famous theorem of Euler (see for instance item 10 of the Twelvefold Way in Section 1.4, as well as G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fourth edition, Oxford University Press, London, 1960 (Thm. 344), and [**1.1**, Cor. 1.2]), this is also the number of partitions of n into distinct parts. A natural basis for Ω^n (due to Schur) analogous to Schur functions and indexed by partitions of n into distinct parts appears in [**96**, Ch. III.8].

7.8. Let $f = \sum_{\lambda \vdash n} c_\lambda p_\lambda$. Then $\omega f = \sum_{\lambda \vdash n} c_\lambda \varepsilon_\lambda p_\lambda$, so

$$(\omega f)_k = \sum_{\lambda \vdash n} c_\lambda \varepsilon_\lambda p_\lambda\big(x_1^k, x_2^k, \ldots\big) = \sum_{\lambda \vdash n} c_\lambda \varepsilon_\lambda p_{k\lambda}.$$

On the other hand,

$$f_k = \sum_{\lambda \vdash n} c_\lambda p_\lambda\big(x_1^k, x_2^k, \ldots\big) = \sum_{\lambda \vdash n} c_\lambda p_{k\lambda},$$

so $\omega f_k = \sum_{\lambda \vdash n} c_\lambda \varepsilon_{k\lambda} p_{k\lambda}$. Now

$$\begin{aligned} \varepsilon_{k\lambda} &= (-1)^{|k\lambda| - \ell(k\lambda)} = (-1)^{k \cdot |\lambda| - \ell(\lambda)} \\ &= (-1)^{(k-1)n}(-1)^{n-\ell(\lambda)} = (-1)^{(k-1)n}\varepsilon_\lambda, \end{aligned}$$

and the proof follows.

7.9. (Sketch.) From $a_{\lambda\mu} = \langle f_\lambda, h_\mu \rangle$ we get $h_\mu = \sum_\lambda a_{\lambda\mu} e_\lambda$. From (7.13) or otherwise, one shows that

$$h_n = \sum_{\lambda \vdash n} \varepsilon_\lambda c_\lambda e_\lambda,$$

where c_λ is the number of distinct permutations of $(\lambda_1, \ldots, \lambda_\ell)$ (where $\ell = \ell(\lambda)$) and hence is just the multinomial coefficient

$$\binom{\ell}{m_1(\lambda), m_2(\lambda), \ldots} = \frac{\ell!}{m_1(\lambda)! m_2(\lambda)! \cdots}.$$

Since h_λ and e_λ are multiplicative, one can compute $a_{\lambda\mu}$ by expanding $h_{\mu_1} h_{\mu_2} \cdots$ in terms of the e_λ's.

Forgotten symmetric functions were first "remembered" by P. Doubilet, *Studies in Applied Math.* **51** (1972), 377–396. A different (less straightforward) proof from that given above appears in this reference.

7.10. We have

$$\begin{aligned}\log A_\lambda(x) &= \sum_\alpha \log(1-x^\alpha)^{-1}\\ &= \sum_\alpha \sum_{n\geq 1} \frac{x^{\alpha n}}{n}\\ &= \sum_{n\geq 1} \frac{1}{n} m_\lambda\big(x_1^n, x_2^n, \ldots\big)\\ &= \sum_{n\geq 1} \frac{1}{n} m_{n\lambda}.\end{aligned}$$

Suppose that $\lambda \vdash r$. Then [why?]

$$\omega m_\lambda\big(x_1^n, x_2^n, \ldots\big) = (-1)^{(n-1)r}\varepsilon_\lambda \sum_\mu a_{\lambda\mu} m_\mu\big(x_1^n, x_2^n, \ldots\big).$$

Hence

$$\begin{aligned}\omega \log A_\lambda(x) &= \log \omega A_\lambda(x)\\ &= \sum_{n\geq 1} \frac{1}{n}(-1)^{(n-1)r}\varepsilon_\lambda \sum_\mu a_{\lambda\mu} m_{n\mu}\\ &= (-1)^r \varepsilon_\lambda \sum_\mu \sum_{\beta\in \mathrm{Perm}(\mu)} \sum_{n\geq 1} \frac{(-1)^{nr}}{n} x^{\beta n},\end{aligned}$$

where $\mathrm{Perm}(\mu)$ denotes the set of distinct permutations of $(\mu_1, \mu_2, \ldots)$. We get

$$\log \omega A_\lambda(x) = (-1)^r \varepsilon_\lambda \sum_\mu a_{\lambda\mu} \sum_\beta \log[1-(-1)^r x^\beta]^{-1},$$

so

$$\omega A_\lambda(x) = \begin{cases} \prod_\mu A_\mu(x)^{\varepsilon_\lambda a_{\lambda\mu}}, & r \text{ even}\\ \prod_\mu B_\mu(x)^{\varepsilon_\lambda a_{\lambda\mu}}, & r \text{ odd.}\end{cases}$$

Similarly (or because the transition matrix $M(f, m)$ is an involution),

$$\omega B_\lambda(x) = \begin{cases} \prod_\mu B_\mu(x)^{\varepsilon_\lambda a_{\lambda\mu}}, & r \text{ even}\\ \prod_\mu A_\mu(x)^{\varepsilon_\lambda a_{\lambda\mu}}, & r \text{ odd.}\end{cases}$$

7.11. *Answer:* $\sum_{j=0}^{n-1}(q-1)^j s_{n-j,1^j}$. Once this answer is guessed, it can be verified as follows. We obtain an SSYT of shape $(n-j, 1^j)$ and type μ by choosing which parts of μ, excluding the part 1, go in the j squares in the first column below the first row. There are $\binom{j}{\ell(\mu)-1}$ such choices, so $K_{(n-j,1^j),\mu} = \binom{j}{\ell(\mu)-1}$. Hence the coefficient of m_μ in the claimed answer is given by

$$\sum_{j=0}^{n-1} \binom{j}{\ell(\mu)-1}(q-1)^j = q^{\ell(\mu)-1},$$

and the proof follows.

7.12. This result was conjectured by E. Snapper, *J. Algebra* **19** (1971), 520–535 (Conjecture 9.1), and proved independently by R. A. Liebler and M. R. Vitale, *J. Algebra* **25** (1973), 487–489, and T. Y. Lam, *J. Pure Appl. Algebra* **10** (1977), 81–94 (Thm. 1).

7.13. a. See A. D. Berenshtein (= Berenstein) and A. V. Zelevinskii (= Zelevinsky), *Funct. Analysis Appl.* **24** (1990), 259–269; Russian original, 1–13.

7.14. a. By Corollary 7.13.7, the number in question is the number $S_3(r)$ of 3×3 symmetric $\mathbb{N}$-matrices for which every row sum is r. The desired formula is now an easy consequence of the expression for $G_3(\lambda)$ following Proposition 4.6.21.

b. Now by Corollary 7.13.7, we are just counting the number $S_n(r)$ of $n \times n$ symmetric $\mathbb{N}$-matrices for which every row sum is r. Proposition 4.6.21 shows that $S_n(r)$ has the form $P_n(r) + (-1)^r Q_n(r)$. It is not difficult to find deg $P_n(r)$, e.g., by arguing as in the proof of Proposition 4.6.19 or by computing the maximum number of linearly independent $n \times n$ symmetric $\mathbb{N}$-matrices. The value of deg $Q_n(r)$ is mentioned in the Notes to Chapter 4 as a conjecture. This conjecture was proved by Rong Qing Jia, in *Formal Power Series and Algebraic Combinatorics, Proceedings of the Fifth Conference, Florence, Italy, June 21–25, 1993* (A. Barlotti, M. Delest, and R. Pinzani, eds.), Università di Firenze, pp. 292–300, using the theory of multivariate splines. For a related paper, see R. Q. Jia, *Trans. Amer. Math. Soc.* **340** (1993), 179–198.

7.15. See I. G. Macdonald, *Bull. London Math. Soc.* **3** (1971), 189–192. For the case $p = 2$, see also J. McKay, *J. Algebra* **20** (1972), 416–418.

7.16. a. This result was first stated explicitly by E. A. Bender and D. E. Knuth, *J. Combinatorial Theory (A)* **13** (1972), 40–54. An earlier Pfaffian expression for a generalization of B_k was given by B. Gordon and L. Houten, *J. Combinatorial Theory* **4** (1968), 81–99. Gordon, *J. Combinatorial Theory* **11** (1971), 157–168, simplified a special case, which was equivalent to a specialization of B_k, to a determinant. Bender and Knuth observed that Gordon's simplification applied to B_k itself. Further discussion appears in I. M. Gessel, *J. Combinatorial Theory (A)* **53** (1990), 257–285 (§6).

b.–c. By Pieri's rule (Thm. 7.15.7), we get

$$s_{\lfloor n/2 \rfloor} s_{\lceil n/2 \rceil} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq 2}} s_\lambda .$$

Now take the coefficient of $x_1 x_2 \cdots x_n$ on both sides, and the formula for $y_2(n)$ follows. This argument is due to A. Regev, *Advances in Math.* **41** (1981), 115–136. Regev gives a similar (though more complicated) argument for $k = 3$. (See Exercise 7.82(b).) Combinatorial proofs for $2 \leq k \leq 5$ are due to D. Gouyou-Beauchamps, *Europ. J. Combinatorics* **10** (1989), 69–82. Gessel, *loc. cit.*, Thm. 15, deduces the formulas for $y_k(n)$, $2 \leq k \leq 5$, from (a).

d.–e. See Gessel, *loc. cit.*, §7. In this reference Gessel gives a slightly more complicated formula than (7.166) for $u_3(n)$, but he subsequently found the simplification stated here.

f. The case $k = 2$ was done in Exercise 6.19(xx). Some formulas for $y_k(n)$ and $u_k(n)$ for large k are given by I. P. Goulden, *Canad. J. Math.* **42** (1990), 763–774. For some related work (using Corollary 7.23.12), see Exercise 6.56(c).

7.17. See R. Stanley, *J. Combinatorial Theory* **76** (1996), 169–172. For further information on $W_i(n)$, see Exercise 6.33(c).

7.18. Label the Black pawns $P_1, \ldots, P_5$ from the bottom up. When pawn P_i promotes to a rook, call that rook R_i. Black's 25 moves are shown in Figure 7-20 as the elements of a poset P. Black can play his moves in any order such that if $u < v$ in P, then move u must precede move v. Hence the number of solutions is the number $e(P)$ of linear extensions of P. This number is just $f^{(6,6,6,6)}$, the number of SYTs of shape (6, 6, 6, 6), and the hook-length formula (Corollary 7.21.6) yields the answer 140,229,804. This problem was composed by K. Väisänen and appears (Problem 7) in the booklet *Queue Problems* cited in the solution to Exercise 6.23.

7.19. Given $a, b \geq 0$, consider those λ of the form given by Figure 7-21, so $\sigma = \lambda/\mu$ for some μ. Let $p_{a,b,\sigma}(n)$ be the number of such μ satisfying $|\mu| = n$. Then [why?]

$$\sum_{n\geq 0} p_{a,b,\sigma}(n+t)q^n = \frac{q^{ab+ar+bs}}{[a]!\,[b]!},$$

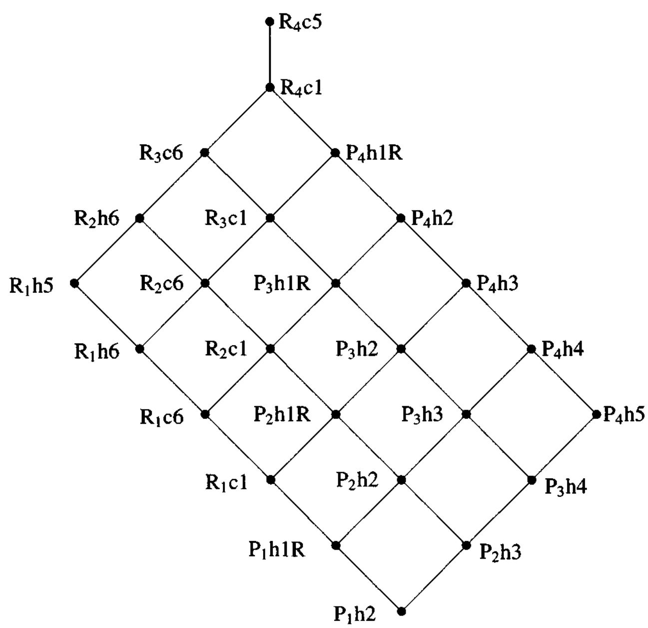


Figure 7-20. The solution poset to Exercise 7.18.

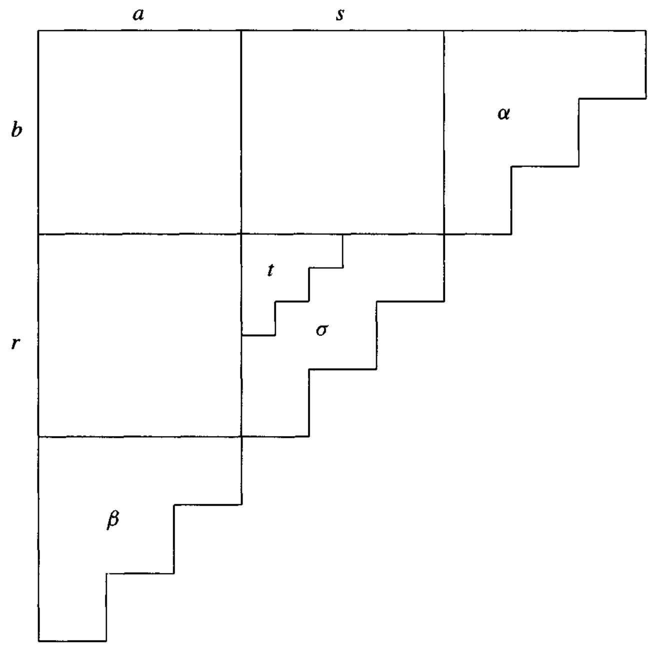


Figure 7-21. A skew shape λ/μ.

so

$$\sum_{n\geq 0} p_\sigma(n+t)q^n = \sum_{a,b\geq 0} \frac{q^{ab+ar+as}}{[a]!\,[b]!}.$$

Some simple manipulations show that the right-hand side is equal to the stated answer $[r-1]![s-1]!/[\infty]![r+s-1]!$.

7.20. a. In general, if $f \in \Lambda^n$ then $\langle p_1^n, f\rangle = [x_1\cdots x_n]f$, as follows e.g. from equation (7.25). Hence $\langle p_1^n, h_\lambda\rangle = [x_1\cdots x_n]h_\lambda = \binom{n}{\lambda_1,\lambda_2,\ldots}$. Since e.g. by Lemma 5.5.3 the number of partitions of $[n]$ of type λ is $\binom{n}{\lambda_1,\lambda_2,\ldots}\prod_i m_i(\lambda)^{-1}$, the result follows.

b. The following argument is due to Dale Worley. Let π be a partition of $[n]$ of type λ. Label the blocks $B_1, B_2, \ldots$ where $\#B_i = \lambda_i$, and if $\#B_i = \#B_j$ with $i < j$, then $\min B_i < \min B_j$. Insert the elements of B_i in increasing order into row i of λ. For instance, $B_1 = \{3,6,8\}$, $B_2 = \{5,7,9\}$, $B_3 = \{1,4\}$, $B_4 = \{2\}$ gives the array

$$\begin{array}{lll} 3 & 6 & 8 \\ 5 & 7 & 9 \\ 1 & 4 & \\ 2 & & \end{array}.$$

Sort each column into increasing order. For the above example, we get

$$\begin{array}{l} 1\,4\,8 \\ 2\,6\,9 \\ 3\,7 \\ 5 \end{array}.$$

The well-known "non-messing-up theorem" (see D. Gale and R. M. Karp, *J. Comput. System Sci.* **6** (1972), 103–115, for a more general result) states that the rows remain increasing, so an SYT T results. An easy combinatorial argument shows that the number of times a given SYT T occurs in this way is $f(T)$, and the proof follows.

c. Let T be an SSYT of shape λ and type μ. Let $T(s)$ denote the entry in the square $s \in \lambda$ of T. Call s *special* if $s = (i, j)$, $j > 1$, and $T(i, j-1) < T(i, j)$. If s is special, then define $f(s)$ exactly as in (b), i.e., $f(s)$ is the number of squares r in a column immediately to the left of s and in a row not above s, for which $T(r) < T(s)$. Now set

$$f(T) = \begin{cases} \prod_{\text{special } s} f(s) & \text{if } T \text{ has exactly } \ell(\mu) - \ell(\lambda) \text{ special squares} \\ 0 & \text{if } T \text{ has more than } \ell(\mu) - \ell(\lambda) \text{ special squares.} \end{cases}$$

(One can show that T always has at least $\ell(\mu) - \ell(\lambda)$ special squares.) Then

$$\sum_T f(T) = (m_i(\lambda)!)^{-1} \langle p_\mu, h_\lambda \rangle, \tag{7.199}$$

where T ranges over all SSYT of shape λ and type μ. The right-hand side of (7.199) is equal to the number of partitions of the multiset $\{1^{\mu_1}, 2^{\mu_2}, \ldots\}$ into *disjoint* blocks (where each block is a multiset) of sizes $\lambda_1, \lambda_2, \ldots$.

Example. Let $\lambda = (4, 2, 1)$ and $\mu = (2, 2, 1, 1, 1)$. There are five T with exactly $\ell(\lambda) - \ell(\mu) = 2$ special squares (whose entries are shown in boldface below), viz.,

$$\begin{array}{cccccc} & 11\mathbf{2}2 & 11\mathbf{2}2 & 11\mathbf{34} & 11\mathbf{35} & 11\mathbf{45} \\ T & 3\mathbf{4} & 3\mathbf{5} & 22 & 22 & 22 \\ & 5 & 4 & 5 & 4 & 3 \\ & & & & & \\ f(T) & 1 & 2 & 2 & 2 & 2 \end{array}.$$

Thus $\sum_T f(T) = 9$, corresponding to the nine partitions of the multiset $\{1, 1, 2, 2, 3, 4, 5\}$ given by $1122-34-5$, $1122-35-4$, $1122-45-3$, $1134-22-5$, $1135-22-4$, $1145-22-3$, $2234-11-5$, $2235-11-4$, $2245-11-3$.

The proof of (c) is analogous to that of (b).

NOTE. Parts (b) and (c) were originally proved algebraically. Define $P_\lambda(x; t)$ as in [**96**, Ch. 3.2], and write

$$P_\lambda(x; t) = \sum_\mu (1-t)^{\ell(\lambda)-\ell(\mu)} \alpha_{\lambda\mu}(t) m_\mu(x),$$

where $\alpha_{\lambda\mu} \in \mathbb{Z}[t]$. One interprets $\alpha_{\lambda\mu}(1)$ in two ways, using (4.4) on p. 224 and (5.11′) on p. 229 of [**96**] (note that (4.4) has the typographical error $Q_\lambda(x;t)$ instead of $Q_\lambda(y;t)$), and the proof follows. (The details are tedious.)

7.21. See P. H. Edelman and C. Greene, *Contemp. Math.* **34** (1984), 155–162 (Thm. 2), and *Advances in Math.* **63** (1987), 42–99 (Thm. 9.3). For a generalization of balanced tableaux, see S. Fomin, C. Greene, V. Reiner, and M. Shimozono, *Europ. J. Combinatorics* **18** (1997), 373–389.

7.22. **a.** For any $\alpha \in \mathrm{Comp}(p)$ we have $[x_1x_2\cdots x_p]L_\alpha = 1$, and the proof follows from the definition of F_w.

b. Define an algebra $\mathfrak{N}_n$ (over $\mathbb{Q}$, say), called the *nilCoxeter algebra* of the symmetric group $\mathfrak{S}_n$, as follows. $\mathfrak{N}_n$ has $n-1$ generators $u_1, \dots, u_{n-1}$, subject to the *nilCoxeter relations*

$$\begin{aligned} u_i^2 &= 0, \qquad 1 \le i \le n \\ u_iu_j &= u_ju_i \qquad \text{if } |i-j| \ge 2 \\ u_iu_{i+1}u_i &= u_{i+1}u_iu_{i+1}, \qquad 1 \le i \le n-2. \end{aligned}$$

If $(a_1, \dots, a_p) \in R(w)$, then identify the element $u_{a_1}\cdots u_{a_p}$ of $\mathfrak{N}_n$ with w. It is easy to see that this identification is well defined and that then $\mathfrak{S}_n$ is a $\mathbb{Q}$-basis for $\mathfrak{N}_n$. Write $\langle f, w\rangle$ for the coefficient of w when $f \in \mathfrak{N}_n$ is expanded in terms of the basis $\mathfrak{S}_n$.

Now let $\mathbf{x} = (x_1, x_2, \dots)$ and define $A(x) \in \mathfrak{N}_n \otimes_{\mathbb{Q}} \mathbb{Q}[x]$ and $G = G(\mathbf{x}) \in \mathfrak{N}_n \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbf{x}]$ by

$$\begin{aligned} A(x) &= (1+xu_{n-1})(1+xu_{n-2})\cdots(1+xu_1) \\ G &= A(x_1)A(x_2)\cdots. \end{aligned}$$

It is immediate from the definition of G that

$$G = \sum_{w\in\mathfrak{S}_n} F_{w^{-1}} \cdot w.$$

The crucial lemma, which has a simple proof by induction on n, asserts that

$$A(x)A(y) = A(y)A(x).$$

From this it follows that $F_w \in \Lambda^p$.

The result of this exercise was first given (with a more complicated proof) by R. Stanley, *Europ. J. Combinatorics* **5** (1984), 359–372 (Thm. 2.1). The proof sketched here appears in S. Fomin and R. Stanley, *Advances in Math.* **103** (1994), 196–207 (after Lemma 2.1).

c. Let $w = w_1w_2\cdots w_n$. Let a_w be the reduced decompostion of w obtained by starting with $12\cdots n$ and first moving w_n one step at a time to the last position, then w_{n-1} one step at a time to the next-to-last position, etc. For instance, if $w = 361524$ then $a_w = (4,5,2,3,4,3,1,2)$. One shows that $L_{\mathrm{co}(a_w)}$ contains the term $x_1^{\lambda_n(w)}x_2^{\lambda_{n-1}(w)}\cdots x_p^{\lambda_1(w)}$, that $L_{\mathrm{co}(a)}$ contains this term for no other $a \in R(w)$, and that no $L_{\mathrm{co}(a)}$ contains a term whose exponents, arranged in weakly decreasing order, are larger than $\lambda(w)$ in dominance order. Since $\sigma \le \rho$ whenever $K_{\sigma\rho} \ne 0$ and since $K_{\rho\rho} = 1$, we get that $\lambda \le \lambda(w)$ whenever $c_{w\lambda} \ne 0$ and that $c_{w,\lambda(w)} = 1$. The corresponding results for $\mu(w)$ are a consequence of the results for $\lambda(w)$ applied to $F_{w^{-1}}$, together with the fact that $\omega F_w = F_{w^{-1}}$. (It is easy to deduce that $\omega F_w = F_{w^{-1}}$ from Exercise 7.94(a).) For details, see R. Stanley, *ibid.* (Thm. 4.1).

d. Vexillary permutations (though not yet with that name) were introduced by A. Lascoux and M. P. Schützenberger, *C. R. Acad. Sci. Paris, Série I* **294** (1982), 447–450 (see Thm. 3.1), and were independently discovered by R. Stanley, *ibid.* (Cor. 4.2). In the paper A. Lascoux and M. P. Schützenberger, *Letters in Math. Physics* **10** (1985), 111–124, vexillary permutations are *defined* to be 2143-avoiding permutations (p. 115), and the equivalence with the definition we have given is proved as Lemma 2.3. See also (1.27) of I. G. Macdonald, *Notes on Schubert Polynomials*, Publications du LACIM **6**, Université du Québec à Montréal, 1991.

e. This result is due to J. West, Ph.D. thesis, Massachusetts Institute of Technology, 1990 (Cor. 3.1.7), and *Discrete Math.* **146** (1995), 247–262 (Cor. 3.5). West gives a bijection between 2143-avoiding permutations and 4321-avoiding permutations in $\mathfrak{S}_n$. The proof then follows from the case $p = 3$ of Corollary 7.23.12 (replacing λ with λ').

f. The permutation w_0 is vexillary, and one easily sees that $\lambda_{w_0} = \mu_{w_0} = (n-1, n-2, \dots, 1)$. Hence by (c) we get $r(w_0) = f^{n-1,n-2,\dots,1}$, and the proof follows from the hook-length formula (Corollary 7.21.6). This result is due to R. Stanley, *ibid.* (Cor. 4.3.).

g. Formula (7.168) is a result of S. Fomin and A. N. Kirillov, *J. Algebraic Combinatorics* **6** (1997), 311–319 (Thm. 1.1). Notice that by (7.193) the product on the right-hand side of (7.168) is exactly the number of plane partitions of staircase shape $(n-1, n-2, \dots, 1)$ with entries at most x. Formula (7.169) is due to Macdonald, *ibid.* (eqn. (6.11)). A simpler proof was given by S. Fomin and R. Stanley, *Advances in Math.* **103** (1994), 196–207 (Lemma 2.3). For a generalization, see Exercise 6.19(oo).

h. This result was first proved by P. H. Edelman and C. Greene, *Advances in Math.* **63** (1987), 42–99 (Cor. 8.4). For some subsequent proofs and related work, see W. Kraśkiewicz and P. Pragacz, Schubert functors and Schubert polynomials, preprint, October 1986, 22 pages; W. Kraśkiewicz and P. Pragacz, *C. R. Acad. Sci. Paris Ser. I Math.* **304** (1987), 209–211; W. Kraśkiewicz, *Europ. J. Combinatorics* **16** (1995), 293–313; S. Fomin and C. Greene, *Discrete Math.*, to appear (Thm. 1.2 and Example 2.2); V. Reiner and M. Shimozono, *J. Algebraic Combinatorics* **4** (1994), 331–351; anu V. Reiner and M. Shimozono, *J. Combinatorial Theory (A)* **82** (1998), 1–73.

7.23. This surprising connection between the RSK algorithm and symmetric chain decompositions is due to K. P. Vo, *SIAM J. Algebraic Discrete Methods* **2** (1981), 324–332. For the special case when P is a boolean algebra, see also D. Stanton and D. White, *Constructive Combinatorics*, Springer-Verlag, New York, 1986 (Thm. 7.5).

7.24. a.–b. These are simple properties of differentiation having nothing to do with symmetric functions *per se*.

c. Straightforward proof by induction on ℓ.

d. As special cases of Theorem 7.15.7 and Corollary 7.15.9, together with the fact that $(\partial/\partial p_1)s_\lambda = s_{\lambda/1}$ (see the solution to Exercise 7.35(a)), we have

$$Us_\mu = \sum_\nu s_\nu, \qquad Ds_\mu = \sum_\rho s_\rho,$$

where ν is obtained from μ by adding a box, and ρ is obtained from μ by deleting a box. It follows easily that

$$(U+D)^\ell\, 1 = \sum_\lambda \tilde{f}^\lambda_\ell s_\lambda.$$

Since $D1 = 0$, it follows from (c) that

$$\begin{aligned}(U+D)^\ell\, 1 &= \sum_{\substack{i\leq \ell \\ r:=(n-i)/2\in\mathbb{N}}} \frac{\ell!}{2^r r!\, i!} U^i 1 \\ &= \sum_{\substack{i\leq \ell \\ r:=(n-i)/2\in\mathbb{N}}} \frac{\ell!}{2^r r!\, i!} \left(\sum_{\lambda\vdash i} f^\lambda s_\lambda\right),\end{aligned}$$

and the proof follows.

The operators U and D are powerful tools for enumerating various kinds of sequences obtained by adding and removing single squares of diagrams of partitions. Exercises 7.25–7.27 give some further examples. From the viewpoint of partially ordered sets, the fundamental property $DU - UD = I$ holds because Young's lattice Y is a *1-differential poset*, i.e., Y is a locally finite poset with $\hat{0}$ such that (i) if $\lambda \in Y$ covers exactly k elements, then λ is covered by exactly $k+1$ elements (see Exercise 3.22), and (ii) if distinct elements $\lambda, \mu \in Y$ cover exactly k common elements, then they are covered by exactly k common elements. (Note that in fact $k = 0$ or 1 in (ii).) The general theory of differential posets is developed in R. Stanley, *J. Amer. Math. Soc.* **1** (1988), 919–961 (see the top of p. 940 for the present exercise), and R. Stanley, in *Invariant Theory and Tableaux* (D. Stanton, ed.), IMA Vols. in Math. Appl. **19**, Springer-Verlag, New York, 1990, pp. 145–165. A generalization was given by I. M. Gessel, *J. Statist. Plann. Inference* **34** (1993), 125–134. Further references related to differential posets are S. Fomin, *J. Algebraic Combinatorics* **3** (1994), 357–404, and **4** (1995), 5–45; S. Fomin, *J. Combinatorial Theory (A)* **72** (1995), 277–292; R. Kemp, in *Proc. Fifth Conf. on Formal Power Series and Algebraic Combinatorics 1993*, pp. 71–80; D. Kremer and K. M. O'Hara, *J. Combinatorial Theory (A)* **78** (1997), 268–279; T. W. Roby, Ph.D. thesis, Massachusetts Institute of Technology, 1991; T. W. Roby, Schensted correspondences for differential posets, preprint; and R. Stanley, *Europ. J. Combin.* **11** (1990), 181–188.

e. We want [why?] a bijection between the set $\mathcal{O}^\ell_\lambda$ of oscillating tableaux of length ℓ ending at λ and pairs (π, T), where π is a partition of some subset S (necessarily of even cardinality) of $[\ell]$ into blocks of size two, and T is an SYT of shape λ on the letters $[\ell] - S$. Given an oscillating tableau $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^\ell = \lambda$, we will recursively define a sequence

$(\pi_0, T_0), (\pi_1, T_1), \ldots, (\pi_\ell, T_\ell)$ with $(\pi_\ell, T_\ell) = (\pi, T)$. We leave to the reader the task of verifying that this construction gives a correct bijection. Let π_0 be the empty partition (of the empty set $\emptyset$), and let T_0 be the empty SYT (on the empty alphabet). If $\lambda^i \supset \lambda^{i-1}$, then $\pi_i = \pi_{i-1}$ and T_i is obtained from T_{i-1} by adding the entry i in the square λ^i/λ^{i-1}. If $\lambda^i \subset \lambda^{i-1}$, then let T_i be the unique SYT (on a suitable alphabet) of shape λ^i such that T_{i-1} is obtained from T_i by column-inserting some number j. In this case let π_i be obtained from π_{i-1} by adding the block $B_i = \{i, j\}$.

This bijection is due to S. Sundaram, *J. Combinatorial Theory (A)* **53** (1990), 209–238. For connections between oscillating tableaux and representation theory, see S. Sundaram, in *Invariant Theory and Tableaux* (D. Stanton, ed.), IMA Vols. Math. Appl. **19**, Springer-Verlag, New York, 1990, pp. 191–225. For an approach to oscillating tableaux based on the growth diagrams of Section 7.13, see T. W. Roby, Schensted correspondences for differential posets, preprint (§4.2). For the theory of *skew* oscillating tableaux, see S. Dulucq and B. E. Sagan, *Discrete Math.* **139** (1995), 129–142, and T. W. Roby, *Discrete Math.* **139** (1995), 481–485. As an example of the above bijection (given by Sundaram in the first reference above), let the oscillating tableau be ($\emptyset$, 1, 11, 21, 211, 111, 11, 21, 22, 221, 211). Then the pairs (B_i, T_i) (where B_i is the block added to π_{i-1} to obtain π_i) are given by

$$\begin{array}{cccccccccc}
1 & 1 & 1\,3 & 1\,3 & 1 & 1 & 1\,7 & 1\,7 & 1\,7 & 1\,7 \\
 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 8 \\
 & & & 4 & 4 & & & & 9 & 9 \\
 & & & & \{2,5\} & \{4,6\} & & & & \{3,10\}.
\end{array}$$

Hence

$$T = \begin{array}{l} 1\,7 \\ 8 \\ 9 \end{array}, \qquad \pi = \{\{2,5\},\{4,6\},\{3,10\}\}.$$

7.25. a. Let U and D be as in Exercise 7.24. It is easy to see that

$$f_{2k}(n) = \sum_{\lambda \vdash n} \langle (U+D)^{2k} s_\lambda, s_\lambda \rangle.$$

Now $U^i D^j(s_\lambda)$ is homogeneous of degree $n + i - j$. Hence setting

$$T = \sum_{i=0}^{k} \frac{(2k)!}{(k-i)!\, i!^2\, 2^{k-i}} U^i D^i,$$

we get by Exercise 7.24(c) that

$$\begin{aligned} f_{2k}(n) &= \sum_{\lambda \vdash n} \langle T s_\lambda, s_\lambda \rangle \\ &= \operatorname{tr}(T, \Lambda^n), \end{aligned}$$

where $\operatorname{tr}(T, \Lambda^n)$ denotes the trace of T acting on the space Λ^n. Note that

$$U^i D^i p_\mu = (m_1(\mu))_i\, p_\mu, \tag{7.200}$$

where $m_1(\mu)$ denotes the number of parts of μ equal to 1, and $(m_1(\mu))_i$ is the falling factorial. Since the trace of a linear transformation is the sum of its eigenvalues, we get

$$f_{2k}(n) = \sum_{\mu \vdash n} \sum_{i=0}^{k} \frac{(2k)!}{(k-i)!\, i!^2\, 2^{k-i}} m_1(\mu)_i$$
$$= \frac{(2k)!}{2^k\, k!} \sum_{\mu \vdash n} \sum_{i=0}^{k} \binom{m_1(\mu)}{i} \binom{k}{i} 2^i.$$

Now, writing $P(q) = \prod_{j \geq 0}(1-q^j)^{-1}$, we have [why?]

$$\sum_{n \geq 0} \left[\sum_{\mu \vdash n} \binom{m_1(\mu)}{i} \right] q^n = \frac{q^i}{(1-q)^i} P(q).$$

Hence

$$\sum_{n \geq 0} f_{2k}(n) q^n = \frac{(2k)!}{2^k k!} P(q) \sum_{i=0}^{k} \binom{k}{i} 2^i \frac{q^i}{(1-q)^i}$$
$$= \frac{(2k)!}{2^k k!} P(q) \left(1 + \frac{2q}{1-q}\right)^k$$
$$= \frac{(2k)!}{2^k k!} \left(\frac{1+q}{1-q}\right)^k P(q),$$

completing the proof. This result appears (with a different proof, in the context of differential posets) in R. Stanley, *J. Amer. Math. Soc.* **1** (1988), 919–961 (Cor. 3.14).

b. It is easy to see that

$$g_{2k}(n) = \sum_{\lambda \vdash n} \langle (UD)^k s_\lambda, s_\lambda \rangle$$
$$= \mathrm{tr}((UD)^k, \Lambda^n).$$

Hence if $\theta_1, \ldots, \theta_{p(n)}$ are the eigenvalues of UD acting on Λ^n, then

$$g_{2k}(n) = \theta_1^k + \cdots + \theta_{p(n)}^k.$$

It follows from the case $i = 1$ of (7.200) that the eigenvalues of UD are just the numbers $m_1(\mu)$, for $\mu \vdash n$. There are numerous ways to see that

$$\#\{\mu \vdash n : m_1(\mu) = n - j\} = p(j) - p(j-1),$$

and the proof follows. This result is related to R. Stanley, *J. Amer. Math. Soc.* **1** (1988), 919–961 (Thm. 4.1), and R. Stanley, in *Invariant Theory and Tableaux* (D. Stanton, ed.), IMA Vols. Math. Appl. **19**, Springer-Verlag, New York, 1990, pp. 145–165 (Prop. 2.9).

7.26. It is surprising that the only known proofs of this elementary identity involve either deep properties of Macdonald symmetric functions and q-Lagrange inversion (A. M. Garsia and M. Haiman, *J. Algebraic Combinatorics* **5** (1996), 191–244 (Thm. 2.10(a))) or of the Hilbert scheme of points in the plane (M. Haiman, (t, q)-Catalan numbers and the Hilbert scheme, *Discrete Math.*, to appear (the case $m = 0$ of (1.10)). Naturally a more elementary proof would be desirable.

7.27. Algebraic proofs of (c)–(h) appear in [**96**, Exams. I.5.26–I.5.28] and were discovered independently by various persons (Lascoux, Towber, Stanley, Zelevinsky). The identities (a) and (b) are easily deduced from (f) and (h) by considering the exponential specialization of Section 7.8. All the identities (a)–(h) can also be proved using the operators U and D of Exercise 7.24. See R. Stanley, *J. Amer. Math. Soc.* **1** (1988), 919–961 (Thms. 3.2 and 3.11) for two cases. Finally, bijective proofs of (a)–(h), based on a skew generalization of the RSK algorithm, were given by B. E. Sagan and R. Stanley, *J. Combinatorial Theory (A)* **55** (1990), 161–193 (Cors. 4.5, 4.2, 6.12, 7.6, 6.4, 6.7, 7.4, and 6.9, respectively). Related work appears in S. Fomin, *J. Algebraic Combinatorics* **3** (1994), 357–404, and **4** (1995), 5–45; and *J. Combinatorial Theory (A)* **72** (1995), 277–292.

There is a special case of (c) that is especially interesting. Let $\beta = \emptyset$ and take the coefficient of $x_1 \cdots x_n$ on both sides. If $\alpha \vdash m$ then we obtain

$$\sum_{\lambda \vdash n} f^{\lambda} f^{\lambda/\alpha} = (n)_m f^{\alpha}.$$

In particular, if $m = n - 1$ then

$$\sum_{\lambda} f^{\lambda} = (m+1) f^{\alpha}, \tag{7.201}$$

where the sum on the left ranges over all partitions λ covering α in Young's lattice Y. This result has numerous other proofs, including a simple combinatorial argument using only the fact that if a partition μ covers k elements in Y, then it is covered by $k + 1$ elements. Moreover, in terms of the character theory developed in Section 7.18, equation (7.201) asserts the obvious fact that

$$\dim \operatorname{ind}_{\mathfrak{S}_m}^{\mathfrak{S}_{m+1}} \chi^{\alpha} = (m+1) \dim \chi^{\alpha}.$$

7.28. **a.** As in the proof of Theorem 7.13.1, we may assume by Lemma 7.11.6 that if $\binom{u}{v}$ is the two-line array associated to A, then u and v have no repeated elements. The proof is by induction on the length n of u and v, the case $n = 0$ being trivial. Now (continuing the notation of the proof of Theorem 7.13.1) $\operatorname{tr}(A)$ is equal to the number of antichains $I_i\binom{u}{v}$ of odd cardinality [why?], and thus also equal to the number of antichains $I_i\binom{a}{b}$ of *even* cardinality. Thus by induction, $\operatorname{tr}(A)$ is the number of columns of $\bar{P}$ and $\bar{Q}$ of even length. Since the total number of columns of P and Q is the total number of antichains $I_i\binom{u}{v}$, the proof follows by induction. This result is due to M. P. Schützenberger [**140**, p. 127].

A proof can also be given based on the growth diagram of a permutation w used in the second proof of Theorem 7.13.1. Let $w \in \mathfrak{S}_n$ be an involution, so that the corresponding permutation matrix is symmetric. The

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entire growth diagram $\mathcal{G}_w$ will then be symmetric. Let $\nu(i, j)$ be the partition appearing in square corner (i, j) of $\mathcal{G}_w$ (where the bottom left corner is $(0, 0)$). We claim that for $0 \leq i \leq n$, the number of columns of (the diagram of) $\nu(i, i)$ is equal to the number of fixed points k of w satisfying $k \leq i$. The proof is by induction on i, the case $i = 0$ being trivial. Assume the assertion for i. Let s_{ab} denote the square in the a-th row (from the bottom) and b-th column (from the left) of $\mathcal{G}_w$. By the induction hypothesis, we are assuming that the number of columns of $\nu(i, i)$ of odd length is equal to the number of fixed points k of w satisfying $k \leq i$. Consider the use of the local rules (L1)–(L4) to define $\nu(i+1, i+1)$. By the symmetry of $\mathcal{G}_w$ we have $\nu(i, i+1) = \nu(i+1, i)$, so rule (L3) never occurs. If (L1) applies, then $i+1$ is not a fixed point of w and $\nu(i+1, i+1) = \nu(i, i)$, as desired. If (L2) applies, then $i+1$ is not a fixed point of w, and $\nu(i+1, i+1)$ is obtained from $\nu(i, i)$ by adding 1 to two consecutive parts. This does not affect the number of columns of odd length, as desired. Finally, if (L4) applies then $i+1$ is a fixed point of w and $\nu(i+1, i+1)$ is obtained from $\nu(i, i)$ by adding a square to the first row. This increases by one the number of columns of odd length, as desired. Hence the proof follows by induction.

b. The left-hand side of (7.170) is equal to $\sum_A q^{\mathrm{tr}(A)} x^{\mathrm{row}(A)}$, where A ranges over all symmetric $\mathbb{N}$-matrices of finite support. Now use (a) together with Corollary 7.13.7.

c. Put $q = 0$ in (7.170). This identity was first proved by D. E. Littlewood [**88**, (11.9;2)] using symmetric functions. A bijective proof based on a version of the RSK algorithm was given by W. H. Burge [**13**, §2].

d. Since $f^\lambda = f^{\lambda'}$, we see that $a(n, k) = \sum_\lambda f^\lambda$ where λ' has k odd parts. By letting A be a symmetric permutation matrix in (a) (or by considering the coefficient of $q^k x_1 \cdots x_n$ in (7.170)) we get that $a(n, k)$ is the number of permutations in $\mathfrak{S}_n$ of cycle type $\langle 2^j, 1^k \rangle$, where $2j + k = n$. Hence

$$a(n, k) = \frac{n!}{2^j\, j!\, k!}.$$

e. *Answer:*

$$\prod_i (1 + q x_i)\left(1 - x_i^2\right)^{-1} \cdot \prod_{i<j} (1 - x_i x_j)^{-1} = \sum_\lambda q^{o(\lambda)} s_\lambda(x), \qquad (7.202)$$

where $o(\lambda)$ denotes the number of odd parts of λ.

The case $q = 0$ appears in [**88**, (11.9;4)]. A combinatorial proof was given by Burge, *ibid.* (§3). For noncombinatorial "modern" proofs of (a)–(e), see [**96**, Exams. I.5.4–I.5.10, pp. 76–79].

7.29. a. This result was first proved by D. E. Littlewood [**88**, (11.9;5) on p. 238]. A combinatorial proof was given by W. H. Burge, *ibid.* (§6). For a proof based on Weyl's denominator formula for the root system C_n, see [**96**, Exams. 9(c), pp. 78–79].

There is an interesting connection between the result of this exercise and algebraic topology. Define the *matching complex* M_n to be the simplicial complex whose vertices are the two-element subsets of $[n]$, and whose faces consist of sets of pairwise disjoint vertices. The symmetric group $\mathfrak{S}_n$ acts naturally on M_n and hence also on its rational (reduced) homology $\tilde{H}_*(M_n;\ \mathbb{Q})$. We can therefore ask for the characteristic $\operatorname{ch} \tilde{H}_i(M_n; \mathbb{Q})$ of this action on the i-th homology group. Such a result was stated without proof (in a different form) by T. Józefiak and J. Weyman, *Math. Proc. Camb. Phil. Soc.* **103** (1988), 193–196 (p. 195). An explicit statement and proof was given by S. Bouc, *J. Algebra* **150** (1992), 158–186 (Proposition 4), and later independently by D. B. Karagueuzian, Ph.D. thesis, Stanford University, 1994. Namely,

$$\operatorname{ch} \tilde{H}_i(M_n; \mathbb{Q}) = \sum_{\lambda} s_\lambda, \tag{7.203}$$

where λ ranges over all self-conjugate partitions of n satisfying $i = \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lceil \frac{1}{2}\operatorname{rank}(\lambda) \right\rceil$. (In particular, the action of $\mathfrak{S}_n$ on the entire homology $\tilde{H}_*(M_n; \mathbb{Q})$ has the elegant characteristic $\sum_{\substack{\lambda \vdash n \\ \lambda = \lambda'}} s_\lambda$.) Now the Hopf trace formula (see S. Sundaram, *Contemp. Math.* **178** (1994), 277–309, for a discussion of this technique) shows that

$$\sum_i (-1)^i \operatorname{ch} C_i(M_n; \mathbb{Q}) = \sum_i (-1)^i \operatorname{ch} \tilde{H}_i(M_n; \mathbb{Q}),$$

where $C_i(M_n; \mathbb{Q})$ denotes the space of (oriented) rational i-chains of M_n. The left-hand (respectively, right-hand) side corresponds to the degree n part of the left-hand (respectively, right-hand) side of equation (7.171). Thus (7.171) is equivalent to the computation of the $\mathfrak{S}_n$-equivariant Euler characteristic of M_n, while (7.203) is a refinement that gives the actual homology.

For further information on matching complexes and related complexes (including *chessboard complexes*, which are the analogues of M_n for complete bipartite graphs), see A. Björner, L. Lovász, S. T. Vrećica, and R. Živaljević, *J. London Math. Soc.* **49** (1994), 25–39 (§4); P. F. Garst, Ph.D. thesis, University of Wisconsin–Madison, 1979, 130 pp.; V. Reiner and J. Roberts, Minimal resolutions and the homology of matching and chessboard complexes, preprint, July 1997; and G. M. Ziegler, *Israel J. Math.* **87** (1994), 97–110. The work of Józefiak and Weyman, of Bouc, and of Karagueuzian discussed above computes the homology of matching complexes over $\mathbb{Q}$ (with the additional structure of an $\mathfrak{S}_n$-action). It is also interesting to consider their homology over $\mathbb{Z}$. Computations of Bouc, *ibid.*
* (§3.3) and E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker, Complexes of not i-connected graphs, MSRI Preprint No. 1997-054 (Table 3), suggest that torsion only occurs for the prime 3, but this question remains open.

b. This was proved using symmetric functions by D. E. Littlewood [**88**, (11.9;3)], and by a variation of the RSK algorithm by W. H. Burge, *ibid.* (§5).

c. This result is due to T. Józefiak and J. Weyman, *Advances in Math.* **56** (1985), 1–8. For another proof and a number of related results, see A. Lascoux and P. Pragacz, *J. Phys. A* **21** (1988), 4105–4114. A further reference is J. B. Remmel and M. Yang, *SIAM J. Discrete Math.* **4** (1991), 253–274.

7.30. a. We have $\lambda + \delta = d(\mu + \delta)$. Hence by Theorem 7.15.1 we get

$$\begin{aligned} s_\lambda(x_1, \ldots, x_n) &= \frac{a_{\lambda+\delta}(x_1, \ldots, x_n)}{a_\delta(x_1, \ldots x_n)} \\ &= \frac{a_{\mu+\delta}\left(x_1^d, \ldots x_n^d\right)}{a_\delta\left(x_1^d, \ldots, x_n^d\right)} \cdot \frac{a_\delta\left(x_1^d, \ldots, x_n^d\right)}{a_\delta(x_1, \ldots, x_n)} \\ &= s_\mu\left(x_1^d, \ldots, x_n^d\right) \prod_{i<j} \frac{x_i^d - x_j^d}{x_i - x_j}. \end{aligned}$$

b. Put $\mu = \emptyset$ in (a).

c. See A.-A. A. Jucis (=Yutsis), *Mat. Zametki* **27** (1980), 353–359, 492; and T. S. Sundquist, Ph.D. thesis, University of Minnesota, 1992 (pp. 49–52).

7.31. Let $0 \le a_1 < a_2 < \cdots < a_n \le p-1$ and $0 \le b_1 < b_2 < \cdots < b_n \le p-1$. These two sequences define the submatrix $B = [\zeta^{a_j b_k}]_{j,k=1}^n$. Let $x = (x_1, \ldots, x_n)$ and define the matrix $B(x) = [x_j^{b_k}]_{j,k=1}^n$, so $B = B(\zeta^{a_1}, \ldots, \zeta^{a_n})$. Let $\lambda = (b_n - n + 1, b_{n-1} - n + 2, \ldots, b_1)$. By Theorem 7.15.1 we have

$$\det B(x) = \pm s_\lambda(x) \prod_{1 \le j < k \le n} (x_j - x_k).$$

Since $\prod_{1 \le j < k \le n} (\zeta^{a_j} - \zeta^{a_k}) \neq 0$, we need to show that $s_\lambda(\zeta^{a_1}, \ldots, \zeta^{a_n}) \neq 0$. Suppose the contrary. Then $q = \zeta$ is a zero of the integer polynomial $s_\lambda(q^{a_1}, \ldots, q^{a_n})$, so

$$s_\lambda(q^{a_1}, \ldots, q^{a_n}) = L(q)(1 + q + \cdots + q^{p-1})$$

for some $L(q) \in \mathbb{Z}[q]$. Putting $q = 1$ gives $s_\lambda(1^n) \equiv 0 \pmod{p}$. But by equation (7.105) we have

$$s_\lambda(1^n) = \prod_{1 \le j < k \le n} \frac{b_k - b_j}{k - j} \not\equiv 0 \pmod{p},$$

a contradiction.

This result was first proved (in a different way) by N. G. Čebotarev in 1948. For further proofs and references, see M. Newman, *Lin. Multilin. Algebra* **3** (1975/76), 259–262. The proof given here was found by R. Stanley in 1990.

7.32. a. Write the Schur functions appearing in (7.172) as quotients of determinants using Theorem 7.15.1. The numerators are transposes of each other, while the denominators can be evaluated from equation (7.105). This result, as well as the two examples in (b), is due to J. R. Stembridge (private communication). Stembridge's work was done in the more general context of characters of arbitrary complex semisimple Lie algebras.

b. Let $\mu = (1)$ and $\mu = \langle 21^{n-2} \rangle$, respectively, in (a).

7.33. a. It follows from Exercise 7.30(c) that $t(n)$ is the number of outdegree sequences $(\alpha_1, \ldots, \alpha_n)$ of a tournament on the vertex set $[n]$. Now use Exercise 4.32.

b. Let $t_k(n)$ be the number of distinct monomials appearing in $s_{k\delta}(x_1, \ldots, x_n)$. By a straightforward generalization of Exercise 7.30(c), $t_k(n)$ is the number of outdegree sequences of (loopless) directed graphs on n vertices with exactly k edges (ignoring direction) between any two distinct vertices. Applying Exercise 4.32(b) to the undirected graph Γ on n vertices with k edges between any two distinct vertices shows that $t_k(n)$ is the number of forests on n vertices whose edges are k-colored. Hence

$$\sum_{n\geq 0} t_k(n)\frac{x^n}{n!} = \exp \sum_{j\geq 1} j^{j-2}k^{j-1}\frac{x^j}{j!}.$$

7.34. See [**96**, Exams. 3.8, pp. 46–47]. This result is essentially due to Jacobi [**64**].

7.35. a. The easy way is to show that $D = \partial/\partial p_1$ (acting on polynomials in $p_1, p_2, \ldots$). See [**96**, Exam. I.5.3c, pp. 75–76]. One only needs to check that

$$\left\langle \frac{\partial}{\partial p_1} p_\lambda, p_\mu \right\rangle = \langle p_\lambda, p_1 p_\mu \rangle,$$

which is routine.

One can also give a direct combinatorial proof, based on the Littlewood–Richardson rule (Appendix 1, Section A1.3).

b. If D is *any* derivation on a ring R, then a simple formal computation shows that setting $[f, g] = (Df)g - f(Dg)$ defines a Lie algebra structure.

c. This identity first arose in the context of the Korteweg–deVries equation in M. Adler and J. Moser, *Commun. Math. Phys.* **61** (1978), 1–30. A combinatorial proof was given by I. P. Goulden, *Europ. J. Combinatorics* **9** (1988), 161–168. For additional information and references, see B. Leclerc, *Discrete Math.* **153** (1996), 213–227.

7.36. Use the fact that D_μ is adjoint to the homomorphism M_μ which multiplies by s_μ (Theorem 7.15.3). Since M_μ and M_ν commute, so do D_μ and D_ν.

7.37. a. We have $a_\delta = \det V_n$, where V_n denotes the Vandermonde matrix $(x_i^{n-j})_1^n$. Then $V_n^t V_n = (p_{2n-i-j})_1^n$, with the (temporary) convention $p_0 = n$. Take the determinant of both sides to get the expansion $a_\delta^2 = \det(p_{2n-i-j})$. This result is due to C. W. Borchardt, *Crelle's J.* **30** (1845), 38–45, and *J. de Math.* **12** (1845), 50–67.

b. In Exercise 7.42 set $m = n$ and $y_j = -qx_j$ to obtain

$$x_1 \cdots x_n (1-q)^n \prod_{i\neq j}(x_i - qx_j) = \sum_{\lambda\subseteq\langle n^n\rangle} (-q)^{\lambda} s_\lambda(x) s_{\tilde{\lambda}'}(x),$$

where $x = (x_1, \ldots, x_n)$. Now

$$\begin{aligned}
\left\langle s_\lambda s_{\tilde{\lambda}'}, s_{\langle n^n\rangle}\right\rangle_n &= \left\langle s_\lambda, s_{\langle n^n\rangle/\tilde{\lambda}'}\right\rangle_n \qquad \text{by (7.60)}\\
&= \langle s_\lambda, s_{\lambda'}\rangle_n \qquad \text{by Exercise 7.56(a)}\\
&= \begin{cases} 1, & \lambda = \lambda' \subseteq \langle n^n\rangle \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

(See D. E. Littlewood, *J. London Math. Soc.* **28** (1953), 494–500 (Thm. III), for a different argument.) Hence the coefficient of $s_{\langle n^n\rangle}$ in $x_1 \cdots x_n$

$(1-q)^n \prod_{i \neq j}(x_i - qx_j)$ is equal to

$$\sum_{\substack{\lambda \in \langle n^n \rangle \\ \lambda = \lambda'}} (-q)^{|\lambda|} = (1-q)(1-q^3)\cdots(1-q^{2n-1}) \qquad \text{[why?]}.$$

Divide by $(1-q)^n$ and set $q = 1$ to get that the coefficient of $s_{\langle n^n \rangle}$ in $x_1 \cdots x_n\, a_\delta^2$ is equal to $(-1)^{\binom{n}{2}} 1 \cdot 3 \cdots (2n-1)$. (The sign $(-1)^{\binom{n}{2}}$ arises because $a_\delta^2 = (-1)^{\binom{n}{2}} \prod_{i \neq j}(x_i - x_j)$.) This is equivalent [why?] to the statement that the coefficient of $s_{\langle (n-1)^n \rangle}$ in a_δ^2 is also $(-1)^{\binom{n}{2}} 1 \cdot 3 \cdots (2n-1)$.

c. This result follows from Theorem 11.4 or Example 11.6(a) of J. R. Stembridge, Ph.D. thesis, Massachusetts Institute of Technology, 1985.

d. This is the case $q = 1$ of J. R. Stembridge, *Trans. Amer. Math. Soc.* **299** (1987), 319–350 (Cor. 6.2). A formula for c_λ is unknown in general. A further reference is A. N. Kirillov, *Adv. Ser. Math. Phys.* **16** (1992), 545–579. For some of the sophisticated algebra related to the symmetric function a_δ^2, see P. Hanlon, *Advances in Math.* **84** (1990), 91–134, and B. Kostant, *Advances in Math.* **125** (1997), 275–350 (see especially §5).

e. Set $F(q) = \prod_{i \neq j}(x_i - qx_j)$. Let $\omega = e^{2\pi i/3}$. Since

$$(x_i - \omega x_j)(x_j - \omega x_i) = -\omega\left(x_i^2 + x_i x_j + x_j^2\right),$$

we have

$$F(\omega) = (-\omega)^{\binom{n}{2}} \prod_{i<j} \left(x_i^2 + x_i x_j + x_j^2\right)$$

$$= (-\omega)^{\binom{n}{2}} s_{2\delta}(x_1, \ldots, x_n),$$

using Exercise 7.30(b). By considering the largest exponent in dominance order of the monomials appearing in the expansion of $F(q)$, we see that

$$\langle F(q), s_{2\delta} \rangle = (-q)^{\binom{n}{2}}.$$

It follows that in the Schur function expansion of $F(q)$, all coefficients except that of $s_{2\delta}$ vanish at $q = \omega$. Hence (since these coefficients are polynomials with integer coefficients) they are divisible by $q^2 + q + 1$. Now put $q = 1$. This result was discovered empirically by J. Stembridge (private communication dated 13 May 1998) and proved by R. Stanley. For somewhat related results see [**96**, Ex. I.3.17, pp. 50–51].

7.38. a. For N sufficiently large (viz., $N \geq |\lambda/\mu|$), let V be a complex N-dimensional vector space, and let $F^{\lambda/\mu}$ denote a $\mathrm{GL}(V)$-module with character $s_{\lambda/\mu}(x_1, \ldots, x_N)$ (using the results and terminology of Appendix 2). For a (weak) composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, let S^α denote the $\mathrm{GL}(V)$-module with character h_α (so $S^\alpha = S^{\alpha_1}(V) \otimes S^{\alpha_2}(V) \otimes \cdots$, where $S^i(V)$ denotes the i-th symmetric power of V). For $0 \leq j \leq \binom{n}{2}$, define the $\mathrm{GL}(V)$-module

$$J^j = \coprod_{\substack{w \in \mathfrak{S}_n \\ \ell(w) = j}} S^{\lambda + \delta - w(\mu + \delta)},$$

where $\coprod$ denotes direct sum, and where δ is as in equation (7.69). The idea of the proof is to define an exact sequence (of GL(V)-modules)

$$0 \to J^{\binom{n}{2}} \to \cdots \to J^1 \to J^0 \to F^{\lambda/\mu} \to 0. \qquad (7.204)$$

The existence of such an exact sequence solves the problem at hand by an obvious extension of Exercise 2.3(b) from vector spaces to GL(V)-modules. The existence of the exact sequence (7.204) for the case $\mu = \emptyset$ was stated by A. Lascoux, Thèse, Université Paris VII, 1977, but without defining the maps. Actually, Lascoux considers the dual situation corresponding to Corollary 7.70, but the Schur positivity of $t_{\lambda/\mu,k}$ and that of $\omega(t_{\lambda/\mu,k})$ are equivalent. A rigorous treatment of Lascoux's work (for both the stated and the dual case) was subsequently given by K. Akin, *J. Algebra* **117** (1988), 494–503; in *Contemporary Math.* **88** (1989), pp. 209–217; and *J. Algebra* **152** (1992), 417–426. An independent treatment, for general λ/μ, was given by A. V. Zelevinskii (= Zelevinsky), *Functional Anal. Appl.* **21** (1987), 152–154.

7.39. This is a result of G. Z. Giambelli, *Atti Torino* **38** (1903), 823–844. See also [**96**, Exam. I.3.9, p. 47].

7.40. This is a result of A. Lascoux and P. Pragacz, *Europ. J. Combinatorics* **9** (1988), 561–574. See also [**96**, Exams. I.5.20–I.5.22, pp. 87–89].

7.41. *Algebraic proof.* By the classical definition of Schur functions (Theorem 7.15.1) we have

$$\begin{aligned}
(x_1\cdots x_m)^n s_\lambda\left(x_1^{-1},\ldots,x_m^{-1}\right) &= \frac{(x_1\cdots x_m)^n \det\left(x_i^{-(\lambda_j+m-j)}\right)_1^m}{\det\left(x_i^{-(m-j)}\right)_1^m}\\
&= \frac{\det\left(x_i^{m+n-1}x_i^{-(\lambda_j+m-j)}\right)}{\det\left(x_i^{m-1}x_i^{-(m-j)}\right)}\\
&= \frac{\det\left(x_i^{n-\lambda_j+j-1}\right)}{\det\left(x_i^{j-1}\right)}\\
&= s_{\tilde\lambda}(x_1,\ldots,x_m).
\end{aligned}$$

Combinatorial proof. Given an SSYT T of shape λ, let $\mu^1,\ldots,\mu^n$ be the columns of T (left to right). Let $\tilde\mu^i$ be the column whose entries are the complement in $[m]$ of the entries of μ^i, arranged in increasing order. Let $\tilde T$ have columns $\tilde\mu^n,\tilde\mu^{n-1},\ldots,\tilde\mu^1$. The map $T \mapsto T'$ yields the desired bijection. As an example, let $m = 6$, $n = 7$, and

$$T = \begin{array}{lllll} 1&1&2&3&4\\ 3&3&4&4&6\\ 4&5&5&&\\ 6&&&& \end{array}.$$

Then

$$T' = \begin{array}{l} 1\,1\,1\,1\,1\,2\,2 \\ 2\,2\,2\,2\,3\,4\,5 \\ 3\,3\,3\,5\,6\,6 \\ 4\,4\,5\,6 \\ 5\,5 \\ 6\,6 \end{array}$$

7.42. In the identity

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j) = \sum_{\lambda} s_\lambda(x)s_{\lambda'}(y),$$

(obtained by specializing Theorem 7.14.3), replace y_j by y_j^{-1}, multiply by $(y_1\cdots y_n)^m$, and use Exercise 7.41. See [**96**, Exam. I.4(5), p. 67].

7.43. Note that

$$\psi(p_n) = \omega_y p_n(x,y)\Big|_{\substack{x_1=1,\ y_1=t \\ x_i=y_i=0 \text{ if } i>0}},$$

where $x=(x_1,x_2,\ldots)$, $y=(y_1,y_2,\ldots)$, and ω_y denotes the involution ω acting on the y variables only. Hence (using equation (7.66)),

$$\begin{aligned}
\psi(s_\lambda) &= s_\lambda(x,y)\Big|_{\substack{x_1=1,\ y_1=t \\ x_i=y_i=0 \text{ if } i>0}} \\
&= \sum_{\mu\subseteq\lambda}\left(s_\mu(x)\Big|_{\substack{x_1=1 \\ x_i=0 \text{ if } i>0}}\right)\cdot\left(\omega_y s_{\lambda/\mu}\Big|_{\substack{y_1=t \\ y_i=0 \text{ if } i>0}}\right) \\
&= \sum_{\mu\subseteq\lambda} s_\mu(1)s_{\lambda'/\mu'}(t).
\end{aligned}$$

Now $s_\mu(1)=0$ unless μ consists of a single row, in which case $s_\mu(1)=1$. Similarly, $s_{\lambda'/\mu'}(t)=0$ unless λ/μ is a vertical strip, in which case $s_{\lambda'/\mu'}(t)=t^{|\lambda/\mu|}$. Thus if $s_\mu(1)s_{\lambda'/\mu'}(t)\neq 0$, then $\lambda=\langle n-k,1^k\rangle$ for some $0\le k\le n-1$, and either $\mu=(n-k)$ or $\mu=(n-k-1)$, in which case $s_\mu(1)s_{\lambda'/\mu'}(t)=t^{|\lambda/\mu|}$. From this the proof is immediate.

Suppose that $f=\sum_{\mu\vdash n} c_\mu s_\mu$. Applying ψ and dividing by $1+t$ yields

$$\frac{1}{1+t}\psi(f) = \sum_{k=0}^{n-1} c_{\langle n-k,1^k\rangle}t^k.$$

Hence this exercise can be a useful tool for evaluating hook coefficients of Schur function expansions. See Exercise 7.86(c) for an example.

7.44. This interesting specialization is due to F. Brenti, *Pacific J. Math.* **157** (1993), 1–28. Parts (a)–(d) appear as Theorem 4.1, Theorem 4.2, Proposition 4.5, and Proposition 4.8, respectively.

7.45. Let $s_\lambda = \sum_{\mu \vdash ab} K_{\lambda\mu} m_\mu$, so $K_{\lambda\mu}$ is a Kostka number. Thus

$$T_a(s_\lambda) = \sum_{\nu \vdash b} K_{\lambda, a\nu} m_\nu.$$

If follows that for each $\rho \vdash b$ we have

$$\begin{aligned}
\langle s_\rho, T_a(s_\lambda) \rangle &= \left\langle s_\rho, \sum_{\nu \vdash b} K_{\lambda, a\nu} m_\nu \right\rangle \\
&= \left\langle s_\rho, \sum_\nu \langle s_\lambda, h_{a\nu} \rangle\, m_\nu \right\rangle \\
&= \sum_\nu \langle s_\rho, m_\nu \rangle \cdot \langle s_\lambda, h_{a\nu} \rangle \\
&= \left\langle s_\lambda, \sum_\nu \langle s_\rho, m_\nu \rangle\, h_{a\nu} \right\rangle,
\end{aligned}$$

using the bilinearity of the scalar product together with Corollary 7.12.4 and the orthonormality of Schur functions (Corollary 7.12.2).

Consider the algebra endomorphism φ_a of the ring Λ defined by $\varphi_a(h_i) = h_{ai}$. If we apply φ_a to the Jacobi–Trudi identity defining s_ρ (Theorem 7.16.1), then we obtain the Jacobi–Trudi matrix for the skew Schur function of skew shape $[a\rho + (a-1)\delta]/(a-1)\delta$, where if $\ell(\rho) = \ell$ then $\delta = (\ell - 1, \ell - 2, \dots, 1, 0)$. Hence

$$\varphi_a(s_\rho) = s_{(a\rho + (a-1)\delta)/(a-1)\delta}.$$

Thus

$$\begin{aligned}
\sum_\nu \langle s_\rho, m_\nu \rangle h_{a\nu} &= \varphi_a\left(\sum_\nu \langle s_\rho, m_\nu \rangle h_\nu \right) \\
&= \varphi_a(s_\rho) \\
&= s_{(a\rho + (a-1)\delta)/(a-1)\delta}.
\end{aligned}$$

It follows that

$$\langle T_a(s_\lambda), s_\rho \rangle = \left\langle s_\lambda s_{(a-1)\delta}, s_{a\rho + (a-1)\delta} \right\rangle,$$

a Littlewood–Richardson coefficient. Such coefficients are always nonnegative (see Corollary 7.18.6 and Appendix 1, Section A1.3), and the proof follows.

This result is due independently to R. Stanley, *Electron. J. Combinatorics* **3**(2), R6 (1996), 22 pp. (Thm. 2.4), and to P. Littelmann, as a simple consequence of his path model theory developed in *Ann. Math.* **142** (1995), 499–525. More explicit statements appear in papers by Littelmann: *J. Amer. Math. Soc.* **11** (1998), 551–567 (§2); The path model, the quantum Frobenius map and Standard Monomial Theory, in *Algebraic Groups and Their Representations* (R. Carter and J. Saxl, eds.), Kluwer, Dordrecht, to appear.

7.46. This result was conjectured by C. Reutenauer, *Advances in Math.* **110** (1995), 234–246 (Conjecture 2), and proved independently by W. F. Doran, *J. Combinatorial Theory (A)* **74** (1996), 342–344, and by T. Scharf and J.-Y. Thibon, unpublished. Doran defines a symmetric function

$$f(n,k) = \sum_{\substack{\lambda \vdash n \\ \min\{\lambda_i : \lambda_i > 0\} \geq k}} q_\lambda$$

and shows that

$$-f(n,k) = s_{(n-1,1)} + \sum_{i=2}^{k} f(i,i)f(n-i,i).$$

It follows by induction that for $k \geq 2$ the symmetric function $-f(n,k)$ is a nonnegative linear combination of Schur functions. Since $q_n = f(n,n)$, the proof is complete.

7.47. **a.** The symmetric function X_G was first defined by R. Stanley, *Advances in Math.* **111** (1995), 166–194. (The reference in parentheses preceding the solutions to (b)–(g) and (j)–(k) below refers to the preceding reference.) The fact that $X_G(1^n) = \chi_G(n)$ is stated as Proposition 2.2, and is immediate from the definitions.

b. (p. 170) This question has been checked to be true for trees with at most nine vertices by T. Chow. Note that all trees with d vertices have the same chromatic polynomial, viz., $n(n-1)^{d-1}$.

c. (Proposition 2.4) The coefficient of a monomial $x_1^{\lambda_1}x_2^{\lambda_2}\cdots$ in X_G is equal to the number of ways to choose a stable partition π of G of type $\lambda = \langle 1^{r_1}2^{r_2}\cdots\rangle$, and then for each i to color some block of size λ_i with the color i. Once we choose π we have $r_1!\,r_2!\cdots$ ways to choose the coloring, and the proof follows.

d. (Theorem 2.6) The solution is analogous to the solution of Exercise 3.44. By a *coloring* of G, we mean *any* map $\kappa : V \to \mathbb{P}$. (Note that "coloring" in Exercise 3.44 is here called "proper coloring.") Given $\sigma \in L_G$, define $X_\sigma = \sum_\kappa x^\kappa$, summed over all colorings κ of G such that (i) if u and v are in the same block of σ then $\kappa(u) = \kappa(v)$, and (ii) if u and v are in different blocks and there is an edge with vertices u and v, then $\kappa(u) \neq \kappa(v)$. Given *any* $\kappa : V \to \mathbb{P}$, there is a unique $\sigma \in L_G$ such that κ indexes one of the terms appearing in the definition of X_σ. It follows that for any $\pi \in L_G$ we have

$$p_{\text{type}(\pi)} = \sum_{\sigma \geq \pi} X_\sigma.$$

By Möbius inversion (Proposition 3.72),

$$X_\pi = \sum_{\sigma \geq \pi} p_{\text{type}(\sigma)}\mu(\pi,\sigma).$$

But $X_{\hat{0}} = X_G$, and the proof follows.

e. (Corollary 2.7) Since for any $\lambda \vdash d$ we have $\varepsilon_\lambda = (-1)^{d-\ell(\lambda)}$ (see equation (7.19)), there follows $\varepsilon_{\text{type}(\pi)} = (-1)^{d-|\pi|}$. Now use Proposition 3.10.1, Proposition 7.7.5, and equation (7.174).

f. (Theorem 3.1 and equation (8)) Let P be a d-element poset. Fix an order-reversing bijection $\omega : P \to [d]$. Let

$$X_P = \sum_{\kappa} x^{\kappa},$$

summed over all *strict* order-preserving maps $\kappa : P \to \mathbb{P}$. By Corollary 7.19.5, we have

$$X_P = \sum_{\pi \in \mathcal{L}(P,\omega)} L_{\mathrm{co}(\pi)}, \tag{7.205}$$

so in particular X_P is L-positive. Now let $\mathfrak{o}$ be an acyclic orientation of G and κ a proper coloring. We say that κ is $\mathfrak{o}$-*compatible* if $\kappa(u) < \kappa(v)$ whenever (v, u) is an edge of $\mathfrak{o}$ (i.e., the edge uv of G is directed from v to u). It is easy to see that every proper coloring κ is compatible with exactly one acyclic orientation $\mathfrak{o}$. Hence if $X_{\mathfrak{o}} = \sum_{\kappa} x^{\kappa}$, summed over all $\mathfrak{o}$-compatible proper colorings κ, then $X_G = \sum_{\mathfrak{o}} X_{\mathfrak{o}}$, summed over all acyclic orientations of G. Let $\bar{\mathfrak{o}}$ denote the transitive and reflexive closure of $\mathfrak{o}$. Since $\mathfrak{o}$ is acyclic, $\bar{\mathfrak{o}}$ is a poset and $X_{\bar{\mathfrak{o}}} = X_{\mathfrak{o}}$. Since $X_{\bar{\mathfrak{o}}}$ is L-positive by (7.205), it follows that X_G is L-positive.

g. (Theorem 3.3) The idea of the proof is to define (using the notation of Section 7.19) a linear operator $\varphi : \mathcal{Q}^d \to \mathbb{Q}[t]$ by

$$\varphi(L_\alpha) = \begin{cases} t(t-1)^i, & \alpha = (i+1, 1, 1, \ldots, 1) \\ 0 & \text{otherwise.} \end{cases}$$

One then shows that $\varphi(e_\lambda) = t^{\ell(\lambda)}$ and

$$\varphi(X_G) = \sum_j \operatorname{sink}(G, j)\, t^j,$$

from which the proof follows. It would be interesting to have a more conceptual proof.

h. This beautiful result is due to V. N. Gasharov, *Discrete Math.* **157** (1996), 193–197, using an involution principle argument. For the case when P is a chain, a bijective proof follows immediately from the RSK algorithm. Thus it is natural to ask for a generalization of the RSK algorithm that proves the general case of Gasharov's result. When the poset P also contains no induced subposet isomorphic to the poset of Figure 7-22., such a generalization was given by T. S. Sundquist, D. G. Wagner, and J. West, *J. Combinatorial Theory (A)* **79** (1997), 36–52.

i. Let V be the vertex set of $G = \mathrm{inc}(P)$. If $\alpha : V \to \mathbb{N}$, then define G^α to be the graph obtained from G by replacing each vertex v of G by a clique (complete subgraph) $K_{\alpha(v)}$ of size $\alpha(v)$, and placing edges connecting every vertex of $K_{\alpha(v)}$ to every vertex of $K_{\alpha(u)}$ if uv is an edge of G. (If $\alpha(v) = 0$ then we are simply deleting the vertex v.) It follows from the definition of X_{G^α} that

$$\prod_i C(x_i) = \sum_{\alpha : V \to \mathbb{N}} X_{G^\alpha}.$$

It is easy to see that each G^α is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free

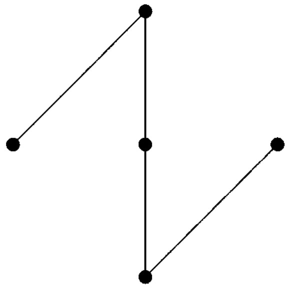

Figure 7-22. An obstruction to a generalized RSK algorithm.

poset. Hence by (h) the product $\prod_i C(x_i)$ is s-positive. The result now follows from Exercise 7.91(e). This argument appears in R. Stanley, Graph colorings and related symmetric functions: ideas and applications, *Discrete Math.*, to appear (Cor. 2.9). A different proof was given by M. Skandera in 1998.

j. (Conjecture 5.1) An equivalent conjecture appeared in R. Stanley and J. R. Stembridge, *J. Combinatorial Theory (A)* **62** (1993), 261–279 (Conjecture 5.5), in the context of "immanants of Jacobi–Trudi matrices." A special case of the conjecture asserts that for any fixed $k, d \geq 1$, the symmetric function $F_{k,d} = \sum x_{i_1} x_{i_2} \cdots x_{i_d}$, is e-positive, where the sum ranges over all sequences $i_1, i_2, \ldots, i_d$ such that any k consecutive terms are all distinct. Even the case $k = 3$ remains open. For the case $k = 2$, see equation (7.175).

k. (Propositions 5.3 and 5.4) Let $\lambda \vdash d$. The number b_λ of connected partitions of P_d (as defined in (d)) of type λ is just the number of distinct permutations of the parts of λ. Hence if $\lambda = \langle 1^{r_1} 2^{r_2} \cdots \rangle$, then $b_\lambda = \binom{\ell(\lambda)}{r_1, r_2, \ldots}$. Since P_d is a tree, the lattice L_{P_d} is just a boolean algebra, so by Example 3.8.3 we get $\mu(\hat{0}, \pi) = \varepsilon_{\mathrm{type}(\pi)}$. Hence from equation (7.174) there follows

$$X_{P_d} = \sum_{\lambda \vdash d} \varepsilon_\lambda \binom{\ell(\lambda)}{r_1, r_2, \ldots} p_\lambda,$$

so

$$\sum_{d \geq 0} X_{P_d} \cdot t^d = \frac{1}{1 - p_1 t + p_2 t^2 - p_3 t^3 + \cdots}.$$

The proof now follows by applying the involution ω to equation (7.165).

A second proof appears in the reference given in (a). The result (stated in a different form) seems first to have been proved by L. Carlitz, R. A. Scoville, and T. Vaughan, *Manuscripta Math.* **19** (1976), 211–243 (p. 242). A combinatorial proof was given by J. Dollhopf, I. P. Goulden, and C. Greene, in preparation. The generating function (7.175) (or its image under the involution ω) appears in (seemingly) completely unrelated contexts in R. Stanley, in *Graph Theory and Its Applications: East and West*, Ann.

New York Acad. Sci. **576**, 1989, pp. 500–535 (based on work of C. Procesi; for further references see J. R. Stembridge, *Advances in Math.* **106** (1994), 244–301 (p. 266)), and in [**6.28**, Thm. 14.2.4].

The formula for C_d can be deduced from that of P_d by using Corollary 4.7.3. See p. 190 of the reference in (a) for details.

l. In fact, X_G is a nonnegative linear combination of the symmetric functions $e_i e_{d-i}$. In the special case when the complement of G is bipartite, this result follows from R. Stanley and J. R. Stembridge, *ibid.*, Theorem 7.4.3, and was given a different proof in R. Stanley, *Advances in Math.* **111** (1995), 166–194 (Cor. 3.6). It was observed by T. Chow that this second proof only requires that the complement of G be triangle-free.

* **m.** This question is due to V. N. Gasharov and appears in R. Stanley, Graph colorings and related symmetric functions: ideas and applications, *Discrete Math.*, to appear (Conjecture 1.4). If this question has an affirmative answer, then the following conjecture (first mentioned as an open question by Y. O. Hamidoune, *J. Combinatorial Theory (B)* **50** (1990), 241–244 (p. 242)) would follow in the same manner as (i). Let G be a clawfree graph, and let c_i be the number of stable i-element subsets of the vertex set of G. Then every zero of the polynomial $\sum c_i t^i$ is real.

7.48. The theory of locally rank-symmetric posets developed in this exercise first appeared in R. Stanley, *Electron. J. Combinatorics* **3**, R6 (1996), 22 pp. The definition of F_P was suggested by R. Ehrenborg, *Advances in Math.* **119** (1996), 1–25 (Def. 4.1).

a. By considering the support of the multichain $\hat{0} = t_0 \leq t_1 \leq \cdots \leq t_{k-1} < t_k = \hat{1}$, we get

$$F_P = \sum_{\substack{S=\{m_1,\ldots,m_j\}_< \\ S\subseteq[n-1]}} \sum_{1\leq i_1<\cdots<i_{j+1}} x_{i_1}^{m_1} x_{i_2}^{m_2-m_1} \cdots x_{i_{j+1}}^{n-m_j}\, \alpha_P(S), \qquad (7.206)$$

where $\alpha_P(S)$ is defined in Section 3.12 (and is now called the *flag f-vector* of P). Since $\alpha_P(S) = \sum_{T\subseteq S} \beta_P(T)$ (equation (3.33)), we need to show that for each $T \subseteq [n-1]$,

$$\sum_{\substack{S\supseteq T \\ S=\{m_1,\ldots,m_j\}_<}} \sum_{1\leq i_1<\cdots<i_{j+1}} x_{i_1}^{m_1} x_{i_2}^{m_2-m_1} \cdots x_{i_{j+1}}^{n-m_j} = L_{\mathrm{co}(T)}.$$

But this is a routine verification, looking at all possible ways of choosing each symbol $\leq$ to be either $<$ or $=$ in the definition (7.89) of L_S. See R. Stanley, *ibid.* (Prop. 1.3).

b. This result is the special case $f = \zeta$ of Exercise 7.94(b). For a simplified version of the proof of Exercise 7.94(b) for the case at hand, see R. Simion and R. Stanley, *Discrete Math.* **204** (1999), 369–396 (Proposition 4.7.1).

c. It follows from (7.206) that $F_P \in \Lambda^n$ if and only if for all $S = \{m_1, \ldots, m_j\}_< \subseteq [n-1]$, we have that $\alpha_S(P)$ depends only on the multiset $\{m_1, m_2 - m_1, m_3 - m_2, \ldots, n - m_j\}$, not on the order of its elements R. Stanley, *Electron. J. Combinatorics*, **3**, R6 (1996), 22 pp., Cor. 1.2. Now use Exercise 3.65 (which applies to locally rank-symmetric posets, though it is stated only for locally self-dual posets).

d. *Ibid.* (Prop. 3.3).

e. *Ibid.* (Thm. 3.5). This result holds for a class of lattices, known as *q-semiprimary lattices of type* μ, more general than the subgroup lattices stated here. See the reference for further details. The polynomials $K_{\lambda\mu}(q) = q^{b(\lambda)}\tilde{K}_{\lambda\mu}(1/q)$ (where $b(\lambda) = \sum(i-1)\lambda_i$ as usual) are known as *Kostka polynomials* or *Kostka–Foulkes polynomials*. They occur in a variety of combinatorial and algebraic contexts and have many fascinating properties. For some further information on Kostka polynomials, see e.g. [**96**, Ch. 3.6], as well as A. M. Garsia and C. Procesi, *Advances in Math.* **94** (1992), 82–138; G.-N. Han, *Prépubl. Inst. Rech. Math. Av.* 1994/21, 71–85; and S. V. Kerov, A. N. Kirillov, and N. Yu. Reshetikhin, *Zap. Nauchn. Sem. Leningrad Otdel Mat. Inst. Steklov (LOMI)* **155** (1986), 50–64, 193.

f. See R. Stanley, *Electron. J. Combinatorics* **4**, R20 (1997), 14 pp. Some generalizations of NC_{n+1} related to root systems are discussed in §5 of that reference.

* g. See the reference cited in (b). Shuffle posets were first considered by C. Greene, *J. Combinatorial Theory (A)* **47** (1988), 191–206. A generalization was given by W. F. Doran, *J. Combinatorial Theory (A)* **66** (1994), 118–136.

7.49. This result was proved by C. Lenart, Lagrange inversion and Schur functions, preprint, 1998, by an intersecting lattice path argument.

7.50. In equation (7.78), set $x_1 = \cdots = x_n = 1$ and $x_{n+1} = x_{n+2} = \cdots = 0$ to get

$$s_\lambda(1^n) = \frac{1}{N!}\sum_{w\in\mathfrak{S}_N} \chi^\lambda(w)n^{c(w)}.$$

Now use Corollary 7.21.4 to get

$$\frac{1}{N!}\sum_{w\in\mathfrak{S}_N} \chi^\lambda(w)n^{c(w)} = \prod_{u\in\lambda}\frac{n+c(u)}{h(u)}. \tag{7.207}$$

(Of course a polynomial identity holding for all $n \in \mathbb{P}$ holds everywhere, i.e., when n is an indeterminate.)

7.51. Take the coefficient of n^{N-1} on both sides of equation (7.207), and the proof follows after some simple manipulation using the fact that $\sum_{u\in\lambda} c(u) = b(\lambda') - b(\lambda)$ (see [**96**, Exam. 3, p. 11]). This elegant proof is due to S. Sundaram and others. For other proofs, see [**96**, Exam. 7, pp. 117–118] and W. M. Benyon and G. Lusztig, *Math. Proc. Camb. Phil. Soc.* **84** (1978), 417–426 (pp. 419–420). One can also use Exercise 7.62 to get $\chi^\lambda(21^{n-2}) = f^{\lambda/2} - f^{\lambda/11}$. Clearly $f^{\lambda/2} + f^{\lambda/11} = f^\lambda$. To compute $f^{\lambda/2}$, see [**72**, Exer. 19, p. 70].

7.52. Define the *depth* $d(u)$ of a square $u = (i, j)$ of (the diagram of) λ to be the smallest integer $k > 0$ for which $u + (k, k) \notin \lambda$. The number of squares of depth k is just μ_k. Moreover, if we successively remove border strips from λ, then the k-th border strip removed contains no squares of depth greater than k. Hence the first k border strips removed contain a total of no more than $\mu_1 + \cdots + \mu_k$ squares. It is then immediate from the Murnaghan–Nakayama rule (Corollary 7.17.5) that if $\chi^\lambda(\nu) \neq 0$, then $\nu \leq \mu$. Now λ contains a unique border strip B_1 of size μ_1; $\lambda - B_1$ contains a unique border strip B_2 of size μ_2, etc. Since $\mathrm{ht}(B_i) = \lambda'_i - i$, we obtain from the Murnaghan–Nakayama rule the stated formula $\chi^\lambda(\mu) = (-1)^t$.

7.53. Consider equation (7.207). The coefficient of n^r on the left-hand side is equal to $1/N!$ times the sum $\sum_w \chi^\lambda(w)$ we are trying to evaluate. If now $\ell(\nu) > r$, then by Exercise 7.52 we have $\chi^\lambda(\nu) = 0$. Hence the left-hand side of (7.207) is divisible by n^r. There are r factors equal to n in the numerator of the right-hand side, coming from the r diagonal terms $u = (i, i)$. Hence the coefficient of n^r on the right-hand side is given by

$$\frac{\prod_{\substack{u\in\lambda\\ u\neq(i,i)}} c(u)}{\prod_{u\in\lambda} h(u)} = \frac{f^\lambda}{N!} \prod_{\substack{u\in\lambda\\ u\neq(i,i)}} c(u).$$

It is easy to see that this last product is just $(-1)^{t(\lambda)} \prod_{i=1}^r (\lambda_i - 1)!(\lambda_i' - 1)!$, and the proof follows.

7.54. Assume that $\chi^\lambda(\mu) = 0$ whenever some nonzero μ_i is even. Since $\chi^{\lambda'}(\mu) = \varepsilon_\lambda \chi^\lambda(\mu)$, it follows that $\lambda = \lambda'$. If $\lambda \neq (m, m-1, \dots, 1)$, then λ has a border strip of even length. Let B be a maximum-size border strip of even length. Since any even border strip of a self-conjugate partition can be extended either up to the first row or down to the first column and remain even, it follows that there exist exactly two maximum-size even border strips B, B', symmetrically placed on the boundary of λ, and $\text{ht}(B) \not\equiv \text{ht}(B') \pmod 2$. Let $|B| = 2r$. Then (by the Murnaghan–Nakayama rule) for all $\nu \vdash n - 2r$ we have

$$0 = \chi^\lambda(2r \cup \nu) = \pm[\chi^{\lambda-B}(\nu) - \chi^{\lambda-B'}(\nu)].$$

Let $\alpha = \lambda - B$, so $\alpha' = \lambda - B'$. Then for all ν we get that $\chi^\alpha(\nu) = \chi^{\alpha'}(\nu)$. Hence $\alpha = \alpha'$, which is impossible. This argument is due to to S. Sundaram (1984).

7.55. **a.** Since $\mathfrak{S}_n$ is generated by transpositions (even adjacent transpositions), we have $\rho^\lambda(\mathfrak{S}_n) \subset \mathrm{SL}(m, \mathbb{C})$ if and only if $\det \rho^\lambda(w) = 1$, where w has cycle type $\langle 21^{n-2}\rangle$. Now $\operatorname{tr} \rho^\lambda(w) = \chi^\lambda(21^{n-2})$, so $\rho^\lambda(w)$ has $\frac{1}{2}[f^\lambda + \chi^\lambda(21^{n-2})]$ eigenvalues equal to 1 and $\frac{1}{2}[f^\lambda - \chi^\lambda(21^{n-2})]$ eigenvalues equal to -1. Hence $\rho^\lambda(\mathfrak{S}_n) \subset \mathrm{SL}(m, \mathbb{C})$ if and only if $\frac{1}{2}[f^\lambda - \chi^\lambda(21^{n-2})]$ is even, and the proof follows from Exercise 7.51. This result is due to L. Solomon (unpublished).

7.56. **a.** If we rotate 180° an SSYT of shape θ, then we obtain a *reverse* SSYT of shape θ^r, and conversely. Now use Proposition 7.10.4.

b. Use (a) and the fact that $(B_\alpha)^r = B_{\tilde{\alpha}}$.

7.57. Let $u \in \lambda$. Let u_1 be the lowest square of λ in the same column as u, and let u_2 be the rightmost square of λ in the same row as u. Then there is a border strip B_u beginning at u_1 and ending at u_2, and this establishes a bijection between the squares and the border strips of λ. Hence the number of border strips is n. Note also that $\#B_u = h(u)$, the hook length at u.

7.58. Suppose λ has an even hook length. It is easy to see that λ then has a hook of length two. Remove it from the diagram of λ. The resulting diagram has one less even hook length and one less odd hook length than λ (see Exercise 7.59(c)), so the number of odd hook lengths minus the number of even hook lengths is unchanged. Continue removing hooks of length two until all hook lengths are odd. The resulting partition must then be of the form $(k, k-1, \ldots, 1)$ for some k. For references, see the solution to Exercise 7.59.

7.59. The code C_λ of the partition λ was first defined by S. Comét, *Numer. Math.* **1** (1959), 90–109, and was further developed by J. B. Olsson, *Math. Scand.* **61** (1987), 223–247. A technique equivalent to the code of a partition is the theory of bead configurations and abaci, developed in G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, 1981 (Ch. 2.7), based on work of T. Nakayama, H. K. Farahat, G. D. James, B. Wagner, G. de B. Robinson, and D. E. Littlewood.

a.–b. These are straightforward consequences of the relevant definitions.

c. This follows easily from (a) and (b).

d. Let C_λ^j be the subsequence $\cdots c_{j-2p}c_{j-p}c_jc_{j+p}c_{j+2p}\cdots$ of C_λ. The operation described in (b) is equivalent to choosing some $0 \le j < p$, and then replacing two consecutive terms 10 in C_λ^j with 01. From this it is clear that any order of performing such operations will result in the same final sequence C_λ^* for which no further operations are possible. If $C_\lambda^* = C_\mu$ then by (b) we have that μ is the (unique) p-core of λ. For a very general approach to uniqueness results such as this exercise or Exercise 3.9(a), see K. Eriksson, Ph.D. thesis, Kungl. Tekniska Högskolan, Stockholm, 1993, and *Discrete Math.* **153** (1996), 105–122.

e. If $p = 1$ then clearly $\mu = \emptyset$ and $Y_{1,\emptyset} = Y$. Suppose that $C_\lambda^j = C_{\lambda^j}$, where C_λ^j is defined in (d). Then (d) shows that the map $\lambda \mapsto (\lambda^0, \lambda^1, \ldots, \lambda^{p-1})$, where λ has fixed p-core μ, is a bijection between $Y_{p,\mu}$ and Y^k, with

$$|\lambda| = p(|\lambda^0| + \cdots + |\lambda^{p-1}|) + |\mu|. \tag{7.208}$$

Equation (7.178) is immediate from (7.208).

The sequence $(\lambda^0, \ldots, \lambda^{p-1})$ is known as the *p-quotient* of λ. The theory of p-cores and p-quotients was originally developed by T. Nakayama, *Japan. J. Math.* **17** (1940), 165–184, 411–423; an exposition appears in G. D James and A. Kerber, *ibid.* (Ch. 2.7). An explicit statement of the isomorphism between $Y_\emptyset$ and Y^k (which trivially generalizes to $\emptyset$ replaced with any p-core μ) was first given by S. Fomin and D. W. Stanton, Rim hook lattices, *St. Petersburg Math. J.* **9** (1998), to appear (Thm. 1.2).

f. Suppose that μ is a p-core. Let x_i be the number of squares in the first column of μ whose hook length is $\equiv i \pmod p$. Then $(x_1, \ldots, x_{p-1})$ satisfies the equation in (ii), and every solution $(x_1, \ldots, x_{p-1}) \in \mathbb{N}^{p-1}$ to (ii) is obtained exactly once in this way. Thus (i) = (ii).

Now let $g(n)$ be the number in (i). Sum equation (7.178) over all p-cores μ. Since $\sum_\mu f_\mu(n) = p(n)$ (the number of partitions of n), we get

$$\prod_{i\ge1}(1-x^i)^{-1} = \left(\sum_{n\ge0} g(n)x^n\right) \cdot \prod_{i\ge1}(1-x^{pi})^{-p},$$

as desired.

g. The only partitions with no even hook length are the "staircases" $(n, n-1, \ldots, 1)$. Hence we get

$$\prod_{i\geq 1} \frac{1-x^{2i}}{1-x^{2i-1}} = \sum_{n\geq 0} x^{\binom{n+1}{2}},$$

an identity due to C. F. Gauss (see e.g. [**1.1**, Cor. 2.10]).

h. The bijection $Y_\emptyset \xrightarrow{\cong} Y^k$ shows that the left-hand side of (7.179) is given by the sum $\sum_{t\in(Y^k)_n} e(t)^2$, where $(Y^k)_n$ is the set of elements of Y^k of rank n, and $e(t)$ is the number of saturated chains of Y^k between $\hat{0}$ and t. Hence

$$\begin{aligned}\sum_{\lambda\in C_p(n)} (f_p^\lambda)^2 &= \sum_{i_1+\cdots+i_p=n} \sum_{\lambda^1\vdash i_1} \cdots \sum_{\lambda^p\vdash i_p} \left[\binom{n}{i_1,\ldots,i_p} f^{\lambda^1}\cdots f^{\lambda^p}\right]^2 \\ &= \sum_{i_1+\cdots+i_p=n} \binom{n}{i_1,\ldots,i_p}^2 i_1!\cdots i_p! \\ &= p^n\, n! \qquad \text{[why?]}.\end{aligned}$$

This exercise gives a glimpse of a body of results concerned with hook lengths divisible by p. Some references not already mentioned include J. B. Olsson, *Math. Scand.* **38** (1976), 25–42; A. N. Kirillov, A. Lascoux, B. Leclerc, and J.-Y. Thibon, *C. R. Acad. Sci. Paris, Sér. I* **318** (1994), 395–400; D. W. Stanton and D. E. White, *J. Combinatorial Theory (A)* **40** (1985), 211–247; [**96**, Exams. I.1.8-I.1.11, pp. 12–16]. (Numerous other examples in [**96**] are related.)

7.60. a. Let the successive squares of θ, reading from left to right and bottom to top, be $u_1, u_2, \ldots, u_{rs}$. By induction on r it suffices to find a border strip λ/μ^{r-1} of λ contained in θ such that $|\lambda/\mu^{r-1}| = s$ and such that when we remove λ/μ^{r-1} from θ, the connected components thus formed (either one or two of them) will have a number of squares divisible by s.

Define $\theta_i = \{u_{is-s+1}, u_{is-s+2}, \ldots, u_{is}\}$, $1 \leq i \leq r$. Let j be the least positive integer for which u_{js+1} does not lie to the right of u_{js}. The integer j exists since u_{rs+1} is undefined and hence doesn't lie to the right of u_{rs}. Since j is minimal, u_{js-s+1} lies to the right of u_{js-s}. Hence θ_j is a border strip with the desired properties. This argument, due to A. M. Garsia and R. Stanley, appears in R. Stanley, *Linear and Multilinear Algebra* **16** (1984), 3–27 (Lemma 7.3). Since the size (number of squares) of a border strip of λ is a hook length of λ (see the solution to Exercise 7.57), the second assertion of the exercise follows. One could also solve this exercise using Exercise 7.59(a)–(b).

b. Regarding s as fixed, define for any integer k

$$k^* = \begin{cases} k/s & \text{if } s \mid k \\ 0 & \text{otherwise.}\end{cases}$$

Relabel the μ_i's so that $\mu_1, \ldots, \mu_j$ are not divisible by s and $\mu_{j+1}, \ldots, \mu_\ell$ are, where $\ell = \ell(\mu)$. If $\chi^\lambda(\mu) \neq 0$, then by the Murnaghan–Nakayama rule (Corollary 7.17.5) there exists a border strip tableau of shape λ and type μ. By (a), there exists a border strip tableau of shape λ and type $(\mu_1, \ldots, \mu_j, s, s, \ldots, s)$, where the number of s's is $m := \sum_i \mu_i^*$. It follows from Exercise 7.59(c) that to remove m successive border strips of size s from λ, we must have $m \leq h_s(\lambda)$, the number of hook lengths of λ divisible by s. Now the multiplicity of a primitive s-th root of unity ζ as a zero of $\prod \left(1 - q^{\mu_i}\right)$ is equal to $\#\{i : s \mid \mu_i\}$, while the multiplicity of ζ as a zero of $H_\lambda(q)$ is $h_s(\lambda)$. Clearly

$$\#\{i \ : \ s | \mu_i\} \leq m,$$

and the proof follows. This result first appeared in R. Stanley, *ibid.* (Cor. 7.5).

7.61. See, e.g., D. G. Duncan, *J. London Math. Soc.* **27** (1952), 235–236, or Y. M. Chen, A. M. Garsia, and J. B. Remmel, *Contemp. Math.* **34** (1984), 109–153.

In general, for any $f \in \Lambda$ we have $\langle f(x^k), s_\lambda \rangle = 0$ unless λ has an empty k-core. To see this, it suffices by linearity to assume $f = p_\mu$. Then

$$\langle p_\mu(x^k), s_\lambda \rangle = \langle p_{k\mu}, s_\lambda \rangle = \chi^\lambda(k\mu).$$

By the Murnaghan–Nakayama rule (Corollary 7.17.5), $\chi^\lambda(k\mu) = 0$ unless there exists a border strip tableau of shape λ and type $k\mu$. By Exercise 7.60(a), there then exists a border strip tableau of shape λ and type $\langle k^m \rangle$ (where $\lambda \vdash km$). Hence λ has an empty k-core.

7.62. Compute $\chi^\lambda(\mu 1^{n-k})$ by the Murnaghan–Nakayama rule (Corollary 7.17.5), choosing $\alpha = (\mu_1, \mu_2, \ldots, 1, 1, \ldots, 1)$.

7.63. **a.** It follows from the Murnaghan–Nakayama rule (Corollary 7.17.5) that

$$d_\lambda = n!\, s_\lambda|_{p_1=0,\ p_2=p_3=\cdots=1}. \tag{7.209}$$

From Proposition 7.7.4 and Theorem 7.12.1 we have

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \exp \sum_{n\geq 1} \frac{1}{n} p_n(x)p_n(y).$$

Set $p_1(y) = 0$ and $p_2(y) = p_3(y) = \cdots = 1$ to get

$$\begin{aligned} \sum_\lambda \frac{d_\lambda}{n!} s_\lambda &= \exp \sum_{n\geq 2} \frac{1}{n} p_n \\ &= e^{-h_1} \sum_{n\geq 0} h_n. \end{aligned}$$

It is now a simple matter to pick out the degree n terms to obtain the stated result.

b. It is easy to see from Pieri's rule (Theorem 7.15.7) that

$$\left\langle h_1^{n-k} h_k, s_{\langle j, 1^{n-j}\rangle} \right\rangle = \begin{cases} \binom{n-k}{j-k}, & k > 0 \\ \binom{n-1}{j-1}, & k = 0. \end{cases}$$

Hence from (a) we get

$$\begin{aligned}
d_{\langle j,1^{n-j}\rangle} &= \sum_{i=1}^{j}(-1)^{n-i}(n)_i\binom{n-i}{j-i} + (-1)^n\binom{n-1}{j-1}\\
&= (-1)^{n-j}\binom{n}{j}\left[\sum_{i=0}^{j}(-1)^{j-i}\frac{j!}{(j-i)!} - (-1)^j\right] + (-1)^n\binom{n-1}{j-1}\\
&= (-1)^{n-j}\binom{n}{j}D_j + (-1)^{n-1}\binom{n-1}{j},
\end{aligned}$$

using equation (2.11). This result was first obtained (stated slightly differently, and with a different proof) by S. Okazaki, Ph.D. thesis, Massachusetts Institute of Technology; 1992 (Cor. 1.3).

7.64. a. Just read an SYT of shape τ_n from right to left and from top to bottom, to obtain an alternating permutation of $[n]$ (as defined at the end of Section 3.16). This procedure establishes a bijection between SYT of shape τ_n and alternating permutations of $[n]$. Since E_n is the number of alternating permutations of $[n]$ (as shown at the end of Section 3.16), the result follows.

b. Apply the Murnaghan–Nakayama rule (Corollary 7.17.5) to the skew shape τ_n. When n is odd, τ_n has no even-length border strips, so assume μ has $2r+1$ odd parts. Consider a border strip tableau B of type μ, such as

$$\begin{array}{ccccccc}
 & & & & & 2 & 4\\
 & & & & 2 & 2 & \\
 & & & 1 & 5 & & \\
 & & 1 & 1 & & & \\
 & 1 & 1 & & & & \\
3 & 3 & & & & & \\
3 & & & & & &
\end{array}$$

so $\mu = (5, 3, 3, 1, 1)$. Reading the numbers from right to left and from top to bottom *without repetition* (e.g., here we get 4 2 5 1 3) gives an alternating permutation, and always $\mathrm{ht}(B) = k - r$. This gives the desired bijection.

c. Similar to (b), though a little more complicated.

Parts (b) and (c) are originally due to H. O. Foulkes, *Discrete Math.* **15** (1976), 311–324, who gave a more complicated proof.

Another approach was suggested by I. M. Gessel. Using e.g. the Jacobi–Trudi identity, one shows that

$$\sum_{n\geq 0}\left(\mathrm{ch}\,\chi^{\tau_n}\right)t^n = \frac{1}{\sum_{n\geq 0}(-1)^n h_{2n}t^{2n}} + \frac{\sum_{n\geq 0}h_{2n+1}t^{2n+1}}{\sum_{n\geq 0}(-1)^n h_{2n}t^{2n}}.$$

One can expand the right-hand side in terms of the p_λ's and compute coefficients explicitly.

7.65. a. First one shows that

$$\operatorname{ch}\psi_n = \sum_{k=0}^{n}(-1)^k h_1^{n-k} h_k,$$

after which it is easy to compute the character values $\psi_n(w)$. The symmetric function $\operatorname{ch}\psi_n$ was first considered by I. M. Gessel and C. Reutenauer, *J. Combinatorial Theory (A)* **64** (1993), 189–215 (Thm. 1, p. 206). Note that it follows from the equation $\deg \psi_n = D_n$ that D_n is equal to the number of $w \in \mathfrak{S}_n$ whose largest descent has the same parity as n. This fact was first shown by J. Désarménien, in *Actes 8ᵉ Séminaire Lotharingien*, Publ. 229/S08, IRMA, Strasbourg, 1984, pp. 11–16. A generalization was given by J. Désarménien and M. L. Wachs, in *Actes 19ᵉ Séminaire Lotharingien*, Publ. 361/S19, IRMA, Strasbourg, 1988, pp. 13–21.

7.66. a. We will illustrate the proof with the example $\lambda/\mu = 8877/211$. Consider the ten zigzag dashed paths in Figure 7-23.. Each such path consists of a number of horizontal or vertical steps (or edges) from the interior of a square to the interior of an adjacent square. Every border strip of a border-strip decomposition of λ/μ cannot contain three consecutive squares that a dashed path passes through. Equivalently, let S be the set of edges e of the dashed paths with the property that the two squares through which e passes belongs to the same border strip. Then S cannot contain two consecutive edges of any dashed path. Conversely, if we choose a subset S of the edges of the dashed paths such that S contains no two consecutive edges on any dashed path, then there is a unique border-strip decomposition of λ/μ with the following property: Let e be any edge of a dashed path, and let u and v be the two squares through which e passes. Then u and v belong to the same border strip if and only if $e \in S$.

It follows that $d(\lambda/\mu)$ is equal to the number of ways to choose S. The number of ways to choose a set of edges, no two consecutive, from a path

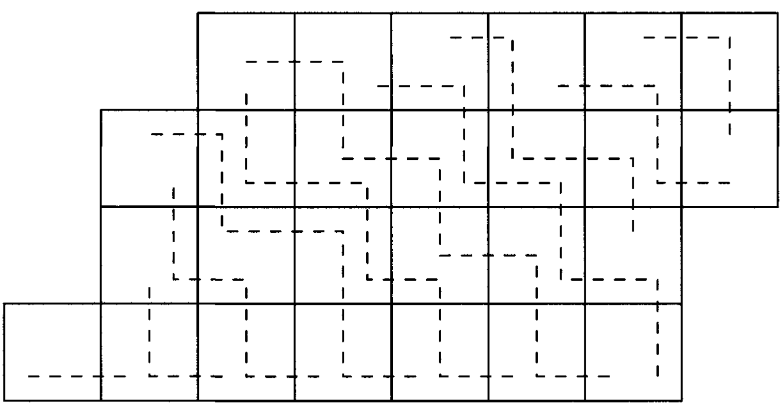

Figure 7-23. Dashed paths corresponding to the shape 8877/211.

of length n is the Fibonacci number F_{n+2} (see Exercise 1.14(a)), and the proof follows.

This exercise is actually a special case of Supplementary Exercise 3.14 from Volume 1 (second printing), which appeared in R. Stanley, Problem 10199, *Amer. Math. Monthly* **99** (1992), 162; solution by W. Y. C. Chen, **101** (1994), 278–279.

b. If P is a path of length $m-1$, then it follows from Exercise 1.13 that $\sum_T q^{\#T} = \sum_i \binom{m-i}{i} q^i$, where T ranges over all sets of edges, no two consecutive, of P. The proof is now a straightforward generalization of (a).

7.67. a. Immediate from the Murnaghan–Nakayama rule (Corollary 7.17.5).

b. Identify each C_i with the sum of its elements in the group algebra $\mathbb{C}G$. We use the standard result (e.g., [**15**, §229 and §236]) that the elements

$$E_r = \frac{d_r}{|G|} \sum_{j=1}^{t} \bar{\chi}_j^r C_j, \quad 1 \le r \le t, \tag{7.210}$$

form a complete set of orthogonal idempotents for the center of $\mathbb{C}G$. By the orthogonality of characters, inverting (7.210) yields

$$C_j = |C_j| \sum_{r=1}^{t} \frac{\chi_j^r}{d_r} E_r.$$

Since the E_i's are orthogonal idempotents (i.e., $E_r E_s = \delta_{rs} E_r$), it follows that

$$\begin{aligned} C_{i_1} \cdots C_{i_m} &= |C_{i_1}| \cdots |C_{i_m}| \sum_{r=1}^{t} \frac{\chi_{i_1}^r \cdots \chi_{i_m}^r}{d_r^m} E_r \\ &= \frac{|C_{i_1}| \cdots |C_{i_m}|}{|G|} \sum_{r=1}^{t} \frac{\chi_{i_1}^r \cdots \chi_{i_m}^r}{d_r^{m-1}} \sum_{k=1}^{t} \bar{\chi}_k^r C_k \\ &= \frac{|C_{i_1}| \cdots |C_{i_m}|}{|G|} \sum_{k=1}^{t} C_k \sum_{r=1}^{t} \frac{1}{d_r^{m-1}} \chi_{i_1}^r \cdots \chi_{i_m}^r \bar{\chi}_k^r. \end{aligned}$$

Expanding in terms of the basis G and taking the coefficient of w on both sides completes the argument. This result appears for instance in [**67**, Thm. 6.3.1] (in a somewhat more general form). Another reference is I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976; reprinted by Dover, New York, 1994 (Problem 3.9).

c. In (b) let $G = \mathfrak{S}_n$ and let $C_{i_1}, \ldots, C_{i_m}$ all be the conjugacy class consisting of the n-cycles. Let C_k be the conjugacy class consisting of the identity element. Then equation (7.180) reduces immediately to (7.181), using the fact that $\chi^{\langle n-s,1^s\rangle}(1^n) = f^{\langle n-s,1^s\rangle} = \binom{n-1}{s}$.

This argument appears in R. Stanley, *Discrete Math.* **37** (1981), 255–262. A survey of work related to the multiplication of conjugacy classes in $\mathfrak{S}_n$ appears in A. Goupil, *Contemp. Math.* **178** (1994), 129–143. For some recent work not mentioned in this survey, see I. P. Goulden, *Trans. Amer. Math. Soc.* **344** (1994), 421–440; I. P. Goulden and D. M. Jackson, *J. Algebra* **166** (1994), 364–378; D. M. Jackson, *Tran. Amer. Math. Soc.* **299** (1987), 785–801; D. M. Jackson and T. I. Visentin, *Trans. Amer. Math. Soc.* **322** (1990), 353–363, 365–376; S. V. Kerov, *C. R. Acad. Sci. Paris, Sér. I* **316** (1993), 303–308 (Prop. 2.2); D. Zagier, *Nieuw Arch. Wisk. (4)* **13** (1995), 489–495; and [**96**, Exams. I.7.24–I.7.25, pp. 131–134].

d. When n is even, the terms indexed by i and $n-1-i$ cancel, so the sum is 0. Alternatively, the product of three n-cycles is an odd permutation (when n is even) and hence cannot equal the identity permutation. When n is odd, the asserted result is equivalent to the identity

$$\sum_{k=0}^{r} \frac{(-1)^k}{\binom{r}{k}} = \frac{2(r+1)}{r+2}, \qquad r \text{ even}, \tag{7.211}$$

where we have set $r = n-1$. Recall the beta function integral

$$\int_0^1 t^k(1-t)^{r-k}dt = \frac{k!\,(r-k)!}{(r+1)!}.$$

Multiply by $(-1)^k$, sum on k from 0 to n, bring the sum inside the integral, evaluate the sum explicitly, and integrate to get the stated result. This argument was suggested by D. W. Stanton. Of course equation (7.211) is not new, and there are many other ways to prove it. An independent derivation of this exercise is due to A. D. Mednykh, *Comm. Alg.* **18** (1990), 1517–1533 (eqn. (32)).

7.68. a. Note that $uvu^{-1}v^{-1} = u(vu^{-1}v^{-1})$, a product of u and a conjugate of u^{-1}. Let C and $\bar{C}$ be the conjugacy classes of G containing u and u^{-1}, respectively (so $|C| = |\bar{C}|$). If y is a fixed conjugate of u^{-1}, then there are $|G|/|C|$ elements $v \in G$ satisfying $y = vu^{-1}v^{-1}$. Hence

$$f(w) = \sum_C \frac{|G|}{|C|}\#\{(u,y) \in C \times \bar{C} \,:\, w = uy\}, \tag{7.212}$$

where C ranges over the conjugacy classes of G. Let $\chi^1, \dots, \chi^t$ denote the irreducible characters of G, and let χ_C^r denote the value of χ^r at any element of C. By Exercise 7.67(b) we have

$$\#\{(u,y) \in C \times \bar{C} \,:\, w = uy\} = \frac{|C|^2}{|G|}\sum_{r=1}^{t} \frac{1}{d_r}\chi_C^r\,\bar{\chi}_C^r\,\bar{\chi}^r(w),$$

since $|C| = |\bar{C}|$ and $\chi^r(v^{-1}) = \bar{\chi}^r(v)$. Hence

$$f = \sum_C |C| \sum_{r=1}^{t} \frac{1}{d_r}\chi_C^r\,\bar{\chi}_C^r\,\bar{\chi}^r,$$

so

$$\langle f, \chi^r \rangle = \frac{1}{d_r} \sum_C |C|\, \bar{\chi}^r_C\, \chi^r_C$$

$$= \frac{|G|}{d_r},$$

by the orthogonality of characters, and the proof follows. This result was known to many researchers in finite groups. A closely related problem appears in the book of I. M. Isaacs cited above (Problem 3.10). It is also implicit in M. Leitz, *Arch. Math. (Basel)* **67** (1996), 275–280, and some of the references given there.

b. This problem has been looked at by a number of group theorists, such as J. L. Alperin, I. M. Isaacs, and L. Solomon.

c. For any class function F on $\mathfrak{S}_n$ we have

$$\operatorname{ch} F = \sum_{\lambda \vdash n} \langle F, \chi^\lambda \rangle\, s_\lambda.$$

Now let $F = f$ (as defined in (a)).

d. For any class function g on $\mathfrak{S}_n$ and any $w \in \mathfrak{S}_n$, we have $g(w) = \langle g, p_\lambda \rangle$. Hence by equation (7.182), we have

$$f_n = \sum_{\lambda \vdash n} H_\lambda \langle s_\lambda, p_\lambda \rangle.$$

By Exercise 7.67(a) there follows

$$f_n = \sum_{k=0}^{n-1} (-1)^k H_{\langle n-k, 1^k \rangle}$$

$$= \sum_{k=0}^{n-1} (-1)^k n(n-k-1)!\, k!.$$

Now use equation (7.211). This result is equivalent to equation (43) in A. D. Mednykh, *Comm. Alg.* **18** (1990), 1517–1533.

e. Put $x_1 = \cdots = x_q = 1$ and $x_i = 0$ for $i > q$ in (7.182) and use Corollary 7.21.4.

f. It follows from (e) that

$$E_n = \frac{1}{n!} \frac{d}{dq} \sum_{\lambda \vdash n} \prod_{t \in \lambda} [q + c(t)] \bigg|_{q=1}. \tag{7.213}$$

There are three cases: (i) No content of λ is equal to -1. Then $\lambda = (n)$, and the contribution of λ to E_n in equation (7.213) is H_n.

(ii) λ has exactly one content equal to -1. Then λ has the form $\langle a, b, 1^k \rangle$, where $a \geq b > 0$, $k \geq 0$, and $a + b + k = n$. In this case the contribution of λ to E_n is

$$\frac{1}{n!} \prod_{\substack{t \in \lambda \\ t \neq (2,1)}} [1 + c(t)] = (-1)^k a!\, (b-1)!\, k!. \tag{7.214}$$

(iii) λ has more than one content equal to -1. Then the contribution of λ to E_n is 0.

When we sum (7.214) over all (a, b, k) satisfying $a \geq b > 0, k \geq 0$, and $a + b + k = n$, then it is not hard to see (using equation (7.211)) that we get the right-hand side of (7.183) except for the term H_n, which already arose from $\lambda = (n)$.

g. Let Γ_j be the functional on symmetric functions defined by

$$\Gamma_j(f) = \frac{\partial}{\partial p_j} f \bigg|_{p_i=1},$$

where $g|_{p_i=1}$ indicates that we are to expand g as a polynomial in the p_i's and then set each $p_i = 1$. Thus from (7.182) we have

$$e_{nj} = \frac{1}{n!} \Gamma_j \left(\sum_{\lambda \vdash n} H_\lambda s_\lambda \right).$$

Let $m_j(\mu)$ denote the number of parts of μ equal to j, and note that

$$\begin{aligned} \Gamma_j(p_\mu) &= m_j(\mu) \\ &= \left\langle p_\mu, \sum_\lambda z_\lambda^{-1} m_j(\mu) p_\mu \right\rangle \\ &= \left\langle p_\mu, p_j \frac{\partial}{\partial p_j} h_n \right\rangle. \end{aligned}$$

But from $\sum_n h_n = \exp\left(\sum_n p_n/n\right)$ there follows

$$\frac{\partial}{\partial p_j} h_n = \frac{1}{j} h_{n-j}.$$

Hence from the linearity of Γ_j we get that for *any* $f \in \Lambda^n$,

$$\Gamma_j(f) = \left\langle f, \frac{1}{j} p_j h_{n-j} \right\rangle.$$

By Theorem 7.17.1 we have

$$p_j h_{n-j} = \sum_\rho (-1)^{\mathrm{ht}(\rho/(n-j))} s_\rho,$$

summed over all partitions $\rho \supseteq (n-j)$ for which $\rho/(n-j)$ is a border strip B of size j. For each $-1 \leq i \leq j-1$ there is exactly one such border strip B_i with $i+1$ squares in the first column, except that B_{2j-n-1} does not exist when $2j > n$. Write ρ^i for the partition ρ for which $\rho/(n-j) = B_i$. It follows that for $\lambda \vdash n$,

$$\Gamma_j(s_\lambda) = \begin{cases} \frac{1}{j}(-1)^{\mathrm{ht}(B_i)} & \text{if } \lambda = \rho^i \\ 0, & \text{otherwise.} \end{cases}$$

(This formula can also be obtained by showing that $\partial/\partial p_j$ is adjoint to multiplication by $\frac{1}{j}p_j$, and that $s_\lambda|_{p_i=1}=0$ unless $\lambda=(n)$.) In particular,

$$\begin{aligned} e_{nj} &= \frac{1}{n!}\Gamma_j\left(\sum_{\lambda\vdash n} H_\lambda s_\lambda\right) \\ &= \frac{1}{n!\,j}\sum_{i=-1}^{j-1}{}'(-1)^{\mathrm{ht}(B_i)}H_{\rho^i}. \end{aligned}$$

When $i=-1$ we have $\rho^{-1}=(n)$ and

$$\frac{1}{n!}(-1)^{\mathrm{ht}(B_{-1})}H_{\rho^{-1}}=1.$$

When $0\le i\le j-1$ (omitting $i=2j-n-1$ when $2j>n$) it is not hard to check (considering separately the cases $2j\le n+i$ and $2j>n+i+1$) that

$$\frac{1}{n!}(-1)^{\mathrm{ht}(B_i)}H_{\rho^i}=\frac{(-1)^i(n-j+i+1)}{\binom{n}{j}\binom{j-1}{i}(n-2j+i+1)},$$

and the proof follows. This result is due to R. Stanley and J. R. Stembridge (unpublished). For the problem of computing the expected number of j-cycles of more general expressions than $uvu^{-1}v^{-1}$, see A. Nica, *Random Structures and Algorithms* **5** (1994), 703–730.

7.69. a. The square of a cycle of odd length n is an n-cycle, while the square of a cycle of even length n is the product of two cycles of length $n/2$. It then follows from the exponential formula (Corollary 5.1.9) that

$$\begin{aligned} \sum_{n\ge 0}\frac{1}{n!}\sum_{w\in\mathfrak{S}_n}p_{\rho(w^2)} &= \exp\left(\sum_{\substack{n\ge 1\\ n\text{ odd}}}\frac{1}{n}p_n+\sum_{\substack{n\ge 1\\ n\text{ even}}}\frac{1}{n}p_{n/2}^2\right) \\ &= \exp\sum_{n\ge 1}\frac{1}{n}\left(\sum_i x_i^n+\sum_{i<j}(x_ix_j)^n\right) \\ &= \frac{1}{\prod_i(1-x_i)\cdot\prod_{i<j}(1-x_ix_j)} \\ &= \sum_{n\ge 0}\sum_{\lambda\vdash n}s_\lambda, \end{aligned}$$

the last step by Corollary 7.13.8, so we get

$$\frac{1}{n!}\sum_{w\in\mathfrak{S}_n}p_{\rho(w^2)}=\sum_{\lambda\vdash n}s_\lambda.$$

b. Let $f(w)=\sum_{\lambda\vdash n}\chi^\lambda(w)$, so $\mathrm{ch}\,f=\sum_\lambda s_\lambda$. We need to show that $\mathrm{ch}\,f$ is *p-positive*, i.e., a nonnegative linear combination of p_μ's. Now use (a). This argument shows in fact that

$$\sum_{\lambda\vdash n}\chi^\lambda(w)=\#\{u\in\mathfrak{S}_n : w=u^2\}.$$

See also [**96**, Exam. 11, p. 120]. More generally, it follows from the work of Frobenius and Schur (see Isaacs, *ibid.* (Ch. 4) for an exposition) that if G is a finite group for which every complex representation is equivalent to a real representation, then for any $w \in G$ we have

$$\sum_{\chi \in \hat{G}} \chi(w) = \#\{u \in G \,:\, w = u^2\},$$

where $\hat{G}$ denotes the set of irreducible characters of G.

c. This is a result of T. Scharf, *Bayreuther Math. Schr.*, No. 38 (1991), 99–207, and *J. Algebra* **139** (1991), 446–457. (See also [**67**, §6.2].) Scharf shows the following. Let $\mathrm{Par}_k(n)$ be the set of all partitions of n all of whose parts divide k. For each $\lambda \in \mathrm{Par}_k(n)$, choose an element $w_\lambda \in \mathfrak{S}_n$ of cycle type λ. Let $\zeta = e^{2\pi i/k}$. Define a one-dimensional character ψ^λ on the cyclic group generated by w_λ by

$$\psi^\lambda(w_\lambda) = \zeta^{k/\lambda_1 + \cdots + k/\lambda_\ell},$$

where $\ell = \ell(\lambda)$. This character extends naturally to a one-dimensional character ψ^λ of the centralizer $C(w_\lambda)$ of w_λ. Then

$$r_k = \sum_{\lambda \in \mathrm{Par}_k(n)} \mathrm{ind}_{C(w_\lambda)}^{\mathfrak{S}_n} \psi^\lambda.$$

Hence r_k is a character of $\mathfrak{S}_n$, and the proof follows.

A proof based more on the theory of symmetric functions was given by J.-Y. Thibon, *Bayreuther Math. Schrift.*, no. 40 (1992), 177–201 (Cor. 5.2). We give a somewhat simplified version of this proof (generalizing the argument in (a)) as follows. Let $\theta_{k,n} = \mathrm{ch}\, r_{n,k}$. Since the k-th power of an n-cycle is a product of (n,k) cycles of length $n/(n,k)$ (where (n,k) denotes the g.c.d. of n and k), it follows immediately from the exponential formula (Corollary 5.1.9) that

$$\sum_{n \geq 0} \theta_{k,n} = \exp \sum_{n \geq 1} \frac{1}{n} p_{n/(n,k)}^{(n,k)}. \tag{7.215}$$

Let L_d be the symmetric function of equation (7.191). A simple inclusion–exclusion argument shows that

$$\sum_{n \geq 1} \frac{1}{n} p_{n/(n,k)}^{(n,k)} = \sum_{d|k} \sum_{n \geq 1} \frac{1}{n} L_d(x^n), \tag{7.216}$$

where $L_d(x^n) = L_d(x_1^n, x_2^n, \ldots)$. Equation (7.215) is then equivalent to the plethystic formula

$$\sum_{n \geq 0} \theta_{k,n} = \sum_{n \geq 0} h_n \left[\sum_{d|k} L_d \right]. \tag{7.217}$$

Equivalently, if $h = \sum_{n \geq 0} h_n$, then

$$\sum_{n \geq 0} \theta_{k,n} = \prod_{d|k} h[L_d].$$

Now h_n is just the Schur function s_n, while L_d is a nonnegative (integer) linear combination of Schur functions by Exercise 7.89(b). Hence by Theorem A2.5 of Appendix 2, the right-hand side of (7.217) is also a nonnegative linear combination of Schur functions, and the proof follows. We don't know whether r_k (extended in an obvious way to any finite group) is a character of any finite group G for which every representation can be realized over $\mathbb{Z}$. (The quaternion group of order eight shows that it does not suffice just to assume that G is an IC-group, as defined in (j).)

d. It suffices by iteration to assume that $m = 2$ (though the general case can also be proved directly). Let $a = f_1$ and $b = f_2$. We follow the notation of Exercise 7.67(b). We have

$$h(w) = \sum_{i=1}^{t}\sum_{j=1}^{t} a_i b_j \,\#\{(u,v) \in G \times G \,:\, u \in C_i,\ v \in C_j,\ uv = w\}.$$

By Exercise 7.67(b), we have

$$\begin{aligned}\#\{(u,v) \in G \times G \,:\, u \in C_i,\ v \in C_j,\ uv = w\}\\ = \frac{|C_i|\cdot|C_j|}{|G|}\sum_{r=1}^{t}\frac{1}{d_r}\chi_i^r\,\chi_j^r\,\bar{\chi}^r(w).\end{aligned}$$

Hence

$$\begin{aligned}\langle h, \chi^r\rangle &= \frac{1}{|G|}\sum_{i=1}^{t}\sum_{j=1}^{t}|C_i|\cdot|C_j|\,a_i b_j\frac{1}{d_r}\chi_i^r\,\chi_j^r\\ &= \frac{|G|}{d_r}\left(\frac{1}{|G|}\sum_{i=1}^{t}|C_i|\,a_i\chi_i^r\right)\left(\frac{1}{|G|}\sum_{j=1}^{t}|C_j|\,b_j\chi_j^r\right)\\ &= \frac{|G|}{d_r}\langle f, \chi^r\rangle\cdot\langle g, \chi^r\rangle,\end{aligned}$$

and the proof is complete. It is possible to view this result as a special case of the theorem that the Fourier transform converts convolution to multiplication.

e. Define $\psi^\lambda \colon \mathfrak{S}_n \to \mathbb{Z}$ by $\psi^\lambda(w) = 1$ if $\rho(w) = \lambda$, and $\psi^\lambda(w) = 0$ otherwise. It is then easy to check that

$$\operatorname{ch} F_{\psi^\lambda,\psi^\mu} = (\operatorname{ch}\psi^\lambda)\Box\left(\operatorname{ch}\psi^\mu\right).$$

By bilinearity we get $\operatorname{ch} F_{f,g} = (\operatorname{ch} f)\Box(\operatorname{ch} g)$ for any class functions f, g on $\mathfrak{S}_n$. Now put $f = \chi^\lambda$, $g = \chi^\mu$, and use (7.184) to deduce (7.185). The steps can be reversed to deduce (7.184) from (7.185).

f. Letting r_k be as in (c), we have

$$h(w) = \prod_{u_1\cdots u_m = w} r_{a_1}(u_1)\cdots r_{a_m}(u_m).$$

The proof now follows from parts (c) and (f).

Some results closely related to (d) and (f) appear in A. Kerber and B. Wagner, *Arch. Math. (Basel)* **35** (1980), 252–262. Indeed, if one uses the fact (Isaacs, *ibid.*, Lemma 4.4, p. 49) that for any finite group G and any irreducible character χ we have

$$\langle r_k, \chi \rangle = \frac{1}{|G|} \sum_{w \in G} \chi(w^k),$$

then Satz 1 of Kerber and Wagner is equivalent to our (d) when $f_i = r_{a_i}$. Another related paper (though not as closely) is L. Solomon, *Arch. Math. (Basel)* **20** (1969), 241–247.

g. Let $a = uvu$ and $b = uv$. Then $u = b^{-1}a$ and $v = a^{-1}b^2$, so as (u, v) ranges over $G \times G$, so does (a, b). But $uvu^2vuv = ab^2$, which is clearly equidistributed over G. Thus we get

$$\#\{(u, v) \in G \times G \,:\, w = uvu^2vuv\} = |G|.$$

The main point is that the substitution $u = b^{-1}a$ and $v = a^{-1}b^2$ is invertible because the homomorphism $\varphi : F_2 \to F_2$ (where F_2 is the free group on generators x, y) defined by $\varphi(x) = y^{-1}x$ and $\varphi(y) = x^{-1}y^2$ is an automorphism of F_2. For the classification of automorphisms of free groups, see e.g. M. Hall, Jr., *The Theory of Groups*, Macmillan, New York, 1959 (Thm. 7.3.4).

h. We get from Exercise 7.68(a) and parts (a) and (d) of this exercise that $\langle f, \chi^\lambda \rangle = \langle g, \chi^\lambda \rangle = H_\lambda$ for all $\lambda \vdash n$, so the proof follows.

For a bijective proof, note that every element in $\mathfrak{S}_n$ is conjugate to its inverse. Hence we may replace the equation $w = u(vu^{-1}v^{-1})$ with $w = u(vuv^{-1})$ and maintain a simple bijection between the solutions to the two equations. Now let $a = uv$ and $v = b^{-1}$. Since $u = ab$ and $v = b^{-1}$, the elements a and b range over $\mathfrak{S}_n$ as u and v do. Hence we may replace the equation $w = uvuv^{-1}$ with $w = (ab)b^{-1}(ab)b = a^2b^2$, and the result follows. This argument is valid in any (finite) group for which every element is conjugate to its inverse, or equivalently (since $\chi(w^{-1}) = \bar{\chi}(w)$) for which every character is real, such as a (finite) Coxeter group. The case $w = 1$ was treated in Exercise 5.12 for any finite group G.

i. For $\gamma = xy^kxy^{-k}$ the second argument in (h) generalizes easily. Namely, let $a = xy^k$ and $b = y^{-1}$ to get $f_{\gamma, \mathfrak{S}_n} = f_{a^2b^{2k}, \mathfrak{S}_n}$, which is a character by (f). For $\gamma = xy^kx^{-1}y^{-k}$, note that as in (7.212) we have

$$f_{\gamma,G}(w) = \sum_C r_k(C) \frac{|G|}{|C|} \#\{(u, y) \in C \times \bar{C} \,:\, w = uy\},$$

where r_k is defined in (c). Reasoning exactly as in the solution to Exercise 7.68(a) shows that

$$\langle f, \chi \rangle = \frac{|G|}{\chi(1)} \langle r_k, \chi\bar{\chi} \rangle,$$

for every irreducible character χ of G. Since r_k is a character when $G = \mathfrak{S}_n$ by (c), it follows that f is also a character. Similar reasoning shows that if β is any word in the letters $x_1, \ldots, x_r$ and if x is a letter different from the

x_i's, then

$$\langle f_{x\beta x^{-1}\beta^{-1}}, \chi\rangle = \frac{|G|}{\chi(1)}\langle f_\beta, \chi\bar{\chi}\rangle$$
$$\langle f_{x\beta x^{-1}\beta}, \chi\rangle = \frac{|G|}{\chi(1)}\langle f_\beta, \chi^2\rangle.$$

j. When $r=1$ (so $f_{\gamma,G} = r_k$ for some $k \in G$), it follows from the work of Frobenius, *Sitz. Köngl. Preuß. Akad. Wissen. Berlin* (1907), 428–437; *Ges. Abh.*, vol. III, Art. 78, pp. 394–403, that r_k is a difference of two characters. A sketched proof appears in Isaacs, *ibid.* (Problem 4.7). When $r > 1$, then it suffices to show (since $f_{\gamma,G}$ is a class function) that for fixed i,

$$\#\{(u_1,\ldots,u_r)\in G^r \,:\, \gamma(u_1,\ldots,u_r)\in C_i\} \equiv 0 \pmod{|G|}.$$

But this is exactly the special case $m = 1$ of Theorem 1 of L. Solomon, *Arch. Math. (Basel)* **20** (1969), 241–247. For a closely related result, see I. M. Isaacs, *Canad. J. Math.* **22** (1970), 1040–1046 (Thm. B).

k. If $\gamma = x^2y^2x^2y^2$, $x^2y^3x^2y^{-3}$, or $x^2y^2x^2y^3$, then we don't know whether $f_{\gamma,\mathfrak{S}_n}$ is a character for all n. (The case $x^2y^2x^2y^2$ has been checked for $n \le 16$, and the other two for $n \le 7$.) On the other hand, if $\gamma = x\,y^{-1}x^2y$, $x^2y^3x^{-2}y^{-3}$, or $x^2y^3x^5y^4$, then $f_{\gamma,\mathfrak{S}_n}$ is *not* a character for all n. Note also that for a word like $\gamma = x_1^5\,x_2^3\,x_3^5\,x_4^4\,x_5^5\,x_6^3\,x_7^4\,x_8^5$ (where every exponent occurs an even number of times), it follows from (d) that $f_{\gamma,G}$ is a character for all finite groups G.

7.70. Let $\mu^1,\ldots,\mu^k \vdash n$. By Corollary 7.17.5, when the left-hand side of equation (7.186) is expanded in terms of power sums, the coefficient Q of $p_{\mu^1}(x^{(1)})\cdots p_{\mu^k}(x^{(k)})$ is given by

$$Q = \sum_{\lambda\vdash n} H_\lambda^{k-2}\prod_{i=1}^{k} z_{\mu^i}^{-1}\chi^\lambda(\mu^i).$$

Let C_μ denote the conjugacy class of $\mathfrak{S}_n$ consisting of permutations of cycle type μ. Since $|C_\mu| = n!/z_\mu$ (by equation (7.18)), $f^\lambda = n!/H_\lambda$ (by Corollary 7.21.6), and $\chi^\lambda(1^n) = f^\lambda$ (equation (7.79)), we have

$$Q = \left(\frac{1}{n!}\prod_{i=1}^{k}|C_{\mu^i}|\right)\sum_{\lambda\vdash n}\frac{1}{(f^\lambda)^{k-1}}\left(\prod_{i=1}^{k}\chi^\lambda(\mu^i)\right)\chi^\lambda(1^n).$$

Comparing with equation (7.180) (and using the fact that the χ^λ's are the irreducible characters of $\mathfrak{S}_n$ (Theorem 7.18.5) and that they are real), we see that

$$Q = \frac{1}{n!}\#\Big\{(w_1,\ldots,w_k)\in\mathfrak{S}_n^k \,:\, w_1\cdots w_k = \mathrm{id}\Big\},$$

as desired.

The case $k = 0$ is equivalent to Corollary 7.12.6, while the case $k=1$ is equivalent to Corollary 7.12.5.

Equation (7.186) first appeared in P. J. Hanlon, R. Stanley, and J. R. Stembridge, *Contemporary Math.* **138** (1992), 151–174 (Prop. 2.2), in connection with the distribution of the eigenvalues of the matrix $AUBU^*$, where A and B are fixed $n \times n$ Hermitian matrices, and U is a random $n \times n$ matrix whose entries are independent standard complex normal random variables.

7.71. **a.** By the orthogonality of characters,

$$\sum_{\chi} \chi\bar{\chi}(w) = \#C(w),$$

the order of the centralizer $C(w)$ of w (the number of $v \in G$ commuting with w). This is just the number of fixed points of the action of w on G by conjugation (the number of $v \in G$ such that $wvw^{-1} = v$). Since the number of fixed points of an element w acting on a set is its character value, (i) and (ii) agree.

b. Let $\hat{G}$ denote the set of irreducible characters of G. Then

$$\begin{aligned}
\langle \psi_G, \chi \rangle &= \frac{1}{\#G} \sum_{\theta \in \hat{G}} \sum_{w \in G} \theta(w)\bar{\theta}(w)\chi(w) \\
&= \sum_{w \in G} \chi(w) \left(\frac{1}{\#G} \sum_{\theta} \theta(w)\bar{\theta}(w) \right) \\
&= \sum_{w \in G} \chi(w)[G : C(w)]^{-1}.
\end{aligned}$$

But $[G : C(w)]$ is the number of conjugates of w. Thus the previous sum becomes $\sum_K \chi(K)$.

c. We have

$$\begin{aligned}
\operatorname{ch} \psi_n &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \psi_n(w) p_w \\
&= \sum_{w \in \mathfrak{S}_n} [\mathfrak{S}_n : C(w)]^{-1} p_w \\
&= \sum_{\lambda \vdash n} p_\lambda.
\end{aligned}$$

d. See A. Frumkin, *Israel J. Math. (1)* **55** (1986), 121–128; T. Scharf, *Arch. Math. (Basel)* **54** (1990), 427–429; and Y. Roichman, *Israel J. Math.* **97** (1997), 305–316.

e. For some related work, see H. Décoste, *Séries Indicatrices d'Espèces Pondérées et q-Analogues*, Publications du LACIM, vol. 2, Université du Québec à Montréal, 1989 (Example 3.7).

7.72. Let $\Lambda^k A$ denote the action of A on $\Lambda^k V$. If A has eigenvalues $\theta_1, \ldots, \theta_n$, then $\Lambda^k A$ has eigenvalues $\theta_{i_1} \cdots \theta_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$. Hence

$$\sum_{k=0}^{n} \left(\operatorname{tr} \Lambda^k A\right)(-q)^k = (1 - \theta_1 q) \cdots (1 - \theta_n q) = \det(I - qA). \qquad (7.218)$$

Now if $w \in \mathfrak{S}_n$ has cycle type $\mu = (\mu_1, \mu_2, \ldots)$ with $\ell(\mu) = \ell$, then $\det(I - qw) = \left(1 - q^{\mu_1}\right)\left(1 - q^{\mu_2}\right) \cdots \left(1 - q^{\mu_\ell}\right)$. Hence writing Ψ_k for the character of $\mathfrak{S}_n$ acting on $\Lambda^k V$, we have

$$\sum_{k=0}^{n} \Psi_k(w)(-q)^k = \left(1-q^{\mu_1}\right)\left(1-q^{\mu_2}\right)\cdots\left(1-q^{\mu_\ell}\right), \qquad (7.219)$$

so

$$\sum_{k=0}^{n} (\mathrm{ch}\,\Psi_k)(-q)^k = \frac{1}{n!}\sum_{\substack{w\in\mathfrak{S}_n\\ \rho(w)=\mu}} \left(1-q^{\mu_1}\right)\cdots\left(1-q^{\mu_\ell}\right) p_{\mu_1}\cdots p_{\mu_\ell}.$$

Note that since $p_j(-qx) = (-1)^j q^j p_j(x)$ and $\omega(p_j) = (-1)^{j-1} p_j$, we have

$$(1-q^j)p_j(x) = \omega_y p_k(x,y)|_{y=-qx},$$

where ω_y denotes ω acting on the y-variables only. Hence

$$\begin{aligned}
\sum_{k=0}^{n} (\mathrm{ch}\,\Psi_k)(-q)^k &= \omega_y\left[\frac{1}{n!}\sum_{w} p_{\rho(w)}(x,y)\right]_{y=-qx}\\
&= \omega_y h_n(x,y)|_{y=-qx} \qquad \text{(by Proposition 7.7.6)}\\
&= \omega_y \sum_{j=0}^{n} s_j(x)s_{n-j}(y)|_{y=-qx} \qquad \text{(by (7.66))}\\
&= \sum_{j=0}^{n} s_j(x)s_{1^{n-j}}(y)|_{y=-qx} \qquad \text{(by Theorem 7.14.5)}\\
&= \sum_{j=0}^{n} (-q)^{n-j} s_j(x)s_{1^{n-j}}(x).
\end{aligned}$$

It is a simple matter to multiply s_j by $s_{1^{n-j}}$ by Pieri's rule (Theorem 7.15.7) and collect terms to get

$$\sum_{k=0}^{n} (\mathrm{ch}\,\Psi_k)\,q^k = \sum_k (s_{\lambda^k} + s_{\lambda^{k-1}})\,q^k,$$

whence $\mathrm{ch}\,\Psi_k = s_{\lambda^k} + s_{\lambda^{k-1}}$ and $\Psi_k = \chi^{\lambda^k} + \chi^{\lambda^{k-1}}$.

Note that this result is equivalent to $\Lambda^k\chi^{(n-1,1)} = \chi^{(n-k,1^k)}$. The result of this exercise is due to A. C. Aitken, *Proc. Edinburgh Math. Soc. (2)* **7** (1946), 196–203.

7.73. The argument is similar to that of Exercise 7.72. Let $S^k A$ denote the action of A on $S^k V^*$, the space of homogeneous forms of degree k in the variables $v_1, \ldots, v_n$. Analogously to (7.218) and (7.219) we get

$$\sum_{k\geq 0} \left(\mathrm{tr}\, S^k A\right) q^k = \frac{1}{(1-\theta_1 q)\cdots(1-\theta_n q)} = \frac{1}{\det(I-qA)}$$

and

$$\sum_{k\geq 0} \psi^k(w) q^k = \frac{1}{\left(1-q^{\mu_1}\right)\left(1-q^{\mu_2}\right)\cdots\left(1-q^{\mu_\ell}\right)}.$$

Hence

$$\sum_{k\geq 0}\left(\operatorname{ch}\psi^k\right)q^k = \frac{1}{n!}\sum_{\substack{w\in\mathfrak{S}_n\\ \rho(w)=\mu}} \frac{p_{\mu_1}\cdots p_{\mu_\ell}}{\left(1-q^{\mu_1}\right)\cdots\left(1-q^{\mu_\ell}\right)}$$

$$= \frac{1}{n!}\sum_{\substack{w\in\mathfrak{S}_n\\ \rho(w)=\mu}} p_\mu(1,q,q^2,\ldots)p_\mu$$

$$= \sum_{\lambda\vdash n} s_\lambda(1,q,q^2,\ldots)s_\lambda,$$

by Proposition 7.7.4 and Theorem 7.12.1.

This result is due to A. C. Aitken, *Proc. Edinburgh Math. Soc. (2)* **5** (1937), 1–13 (Thm. 2). For a "modern" λ-ring proof of this exercise and the previous one (Exercise 7.72), see J.-Y. Thibon, *Bayreuth Math. Schr.*, No. 40 (1992), 177–201 (§3).

7.74. Let the permutation $w \in \mathfrak{S}_n$, regarded as an element of $\mathrm{GL}(n,\mathbb{C})$, have eigenvalues $\theta_1,\ldots,\theta_n$. Then $\xi^\lambda(w) = s_\lambda(\theta_1,\ldots,\theta_n)$. Let $\rho(w) = (\rho_1(w), \rho_2(w),\ldots)$. Since

$$\prod_{i=1}^{n}(1-\theta_i q) = \prod_j (1-q^j)^{\rho_j(w)},$$

we get by the Cauchy identity (Theorem 7.12.1) that

$$\sum_\lambda \xi^\lambda(w)s_\lambda(y) = \frac{1}{\prod_{i,j}\left(1-y_i^j\right)^{\rho_j(w)}}.$$

Taking the characteristic of both sides yields

$$\sum_\lambda (\operatorname{ch}\xi^\lambda)(x)s_\lambda(y) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\frac{1}{\prod_{i,j}\left(1-y_i^j\right)^{\rho_j(w)}}\cdot p_{\rho(w)}(x)$$

$$= \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} p_{\rho(w)}(x)p_{\rho(w)}[h](y)$$

$$= \sum_{\mu\vdash n} s_\mu(x)s_\mu[h](y),$$

and the proof follows.

Equation (7.187) appears in T. Scharf and J.-Y. Thibon, *Advances in Math.* **104** (1994), 30–58 (Thm. 5.1). The correspondence $s_\lambda \mapsto \operatorname{ch}\xi^\lambda$ is a special case of the operation of *inner plethysm*, defined as follows. Let σ : $\mathfrak{S}_n \to \mathrm{GL}(N,\mathbb{C})$ be any (finite-dimensional) representation of $\mathfrak{S}_n$, and let $\varphi : \mathrm{GL}(N,\mathbb{C}) \to \mathrm{GL}(M,\mathbb{C})$ be a polynomial representation of $\mathrm{GL}(N,\mathbb{C})$. The composition $\varphi\sigma$ is then a representation of $\mathfrak{S}_n$, and we define the inner plethysm $f \odot g$ of the symmetric functions $f = \operatorname{ch}\sigma$ and $g = \operatorname{char}\varphi$ by

$$f \odot g = \operatorname{ch}\varphi\sigma.$$

In particular,

$$\big(s_n + s_{(n-1,1)}\big) \odot s_\lambda = \operatorname{ch} \xi^\lambda,$$

since the character of the defining representation $\mathfrak{S}_n \to \mathrm{GL}(n, \mathbb{C})$ of $\mathfrak{S}_n$ is given by $\chi^n + \chi^{(n-1,1)}$. Inner plethysm was introduced by D. E. Littlewood,
* *Canad. J. Math.* **10** (1958), 1–16, 17–32. For further information and references, see T. Scharf and J.-Y. Thibon, *ibid.*

7.75. a. An orbit $\mathcal{O}_\mu$ of the action of $\mathfrak{S}_k$ on $\left(\binom{M}{j}\right)$ is specified by a partition $\mu = \langle 1^{m_1} 2^{m_2} \cdots n^{m_n}\rangle \vdash j$, where $\sum m_i = k$ and $\ell(\mu) = \sum m_i \leq k$. Here $\mathcal{O}_\mu$ consists of those submultisets $N \in \left(\binom{M}{j}\right)$ with m_i elements of multiplicity i. For instance, the orbit containing $\{1, 1, 1, 1, 3, 4, 4, 5, 5, 5, 5, 7, 8\}$ corresponds to the partition $\mu = \langle 1^3, 2^1, 4^2\rangle \vdash 13$. The characteristic $\operatorname{ch}(\mathcal{O}_\mu)$ of the action of $\mathfrak{S}_k$ on $\mathcal{O}_\mu$ is just $h_{m_1} h_{m_2} \cdots h_{m_n} h_{k-\ell(\mu)}$ [why?]. Hence (setting $r = k - \ell(\mu)$),

$$\sum_j \operatorname{ch}\left(\binom{M}{j}\right) q^j = \sum_{\substack{m_1,\ldots,m_n \geq 0 \\ r \geq 0}} h_{m_1} \cdots h_{m_n} h_r q^{\sum i m_i} \Bigg|_k ,$$

where $f|_k$ denotes the degree k part of the symmetric function f. We get

$$\begin{aligned}
\sum_j \operatorname{ch}\left(\binom{M}{j}\right) q^j &= (1 + h_1 + h_2 + \cdots) \prod_{i=1}^n (1 + q^i h_1 + q^{2i} h_2 + \cdots)|_k \\
&= \frac{1}{\prod (1 - x_i)(1 - q x_i) \cdots (1 - q^n x_i)} \Bigg|_k \\
&= \sum_{\lambda \vdash k} s_\lambda(1, q, \ldots, q^n) s_\lambda(x) \qquad \text{(by (7.44))},
\end{aligned}$$

and the proof follows.

b. It is easy to see that U_j commutes with the action of $\mathfrak{S}_k$. A proof of injectivity for $j < kn/2$ involving only elementary linear algebra is a special case of the argument given in §6 of R. A. Proctor, M. E. Saks, and D. G. Sturtevant, *Discrete Math.* **30** (1980), 173–180. For some related work see R. A. Proctor, *J. Combinatorial Theory (A)* **54** (1990), 235–247 (especially Cor. 1).

c. We omit the easy proof that $a_j = a_{kn-j}$. Let $j < \lfloor kn/2 \rfloor$. Since U_j commutes with the action of $\mathfrak{S}_k$ and is injective, it is an injective map of $\mathfrak{S}_k$-modules. Thus every irreducible representation of $\mathfrak{S}_k$ occurs at least as often in $\mathbb{Q}\left(\binom{M}{j+1}\right)$ as in $\mathbb{Q}\left(\binom{M}{j}\right)$, so by (a) we get $a_j \leq a_{j+1}$.

The unimodality of $s_\lambda(1, q, \ldots, q^n)$ was first proved (though not stated explicitly in terms of Schur functions) by E. B. Dynkin, *Dokl. Akad. Nauk SSSR (N.S.)* **71** (1950), 221–224, and *Amer. Math. Soc. Translations, Series 2* **6** (1957), 245–378 (p. 332) (translated from *Trudy Moskov. Mat. Obšč.* **1** (1957), 39–156), in the context of the representation theory of semisimple Lie groups. For a statement of Dynkin's result avoiding the language of representation theory, see R. Stanley, in *Young Day Proceedings* (T. V. Narayana, R. M. Mathsen, and J. G. Williams, eds.), Dekker, New

York/Basel, 1980, pp. 127–136. An elegant version of Dynkin's proof, in the special case we are considering here, is given in [**96**, Exam. I.8.4, pp. 137–138]. It is based on Theorem A2.5 of Appendix 2. A proof similar to the one given here (but using wreath products instead of multisets) appears in A. Kerber and K.-J. Thürlings, *Bayreuther Math. Schr.* **21** (1986), 156–278 (Satz 2.2). For further information see R. Stanley, *Ann. New York Acad. Sci.* **576**, 1989, pp. 500–535. No proof of the unimodality of $s_\lambda(1, q, \ldots, q^n)$ is known that does not involve representation theory (or even more sophisticated tools), but see (d) below for a special case for which simpler proofs are known.

d. Apply (c) to $s_{1^k}(1, q, \ldots, q^{n-1})$ or $s_k(1, q, \ldots, q^{n-k+1})$, where these specializations are evaluated in Proposition 7.8.3. This result goes back to J. J. Sylvester, *Phil. Mag.* **5** (1878), 178–188; in *Collected Math. Papers*, vol. 3, Chelsea, New York, 1973, pp. 117–126. A number of other proofs have subsequently been given, as discussed in R. Stanley, *ibid.* In particular, a long-sought-for combinatorial proof was found by K. M. O'Hara, *J. Combinatorial Theory (A)* **53** (1990), 29–52; an exposition was given by D. Zeilberger, *Amer. Math. Monthly* **96** (1989), 590–602.

7.76. a. Let f_w denote the number of fixed points of $w \in G$ acting on T, so $f_w = \chi(w)$. Thus

$$\langle \chi, \chi \rangle = \frac{1}{\#G} \sum_{w \in G} f_w^2.$$

On the other hand, the number of fixed points of w acting on $T \times T$ is just f_w^2. Thus by Burnside's lemma (Lemma 7.24.5), the above sum is the rank of G.

b. Let χ_α denote the character of this action, so by Corollary 7.18.3 we have $\mathrm{ch}\, \chi_\alpha = h_\alpha$. Since ch is an isometry (Proposition 7.18.1) the rank of $\mathfrak{S}_n$ acting on $\mathfrak{S}_n/\mathfrak{S}_\alpha$ is given by $\langle h_\alpha, h_\alpha \rangle$, which by equation (7.31) is the number of $\mathbb{N}$-matrices A with $\mathrm{row}(A) = \mathrm{col}(A) = \alpha$.

c. The left cosets of $\mathfrak{S}_\alpha$ are indexed in a natural way by permutations of the multiset $M_\alpha = \{1^{\alpha_1}, 2^{\alpha_2}, \ldots\}$, and the action of $\mathfrak{S}_n$ on $\mathfrak{S}_n/\mathfrak{S}_\alpha$ corresponds to the action of $\mathfrak{S}_n$ on M_α by permuting coordinates. Hence the action of $\mathfrak{S}_n$ on $T \times T$ is equivalent to the action of $\mathfrak{S}_n$ by column permutations on the set of $2 \times n$ matrices

$$B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix},$$

where each row is a permutation of M_α. Associate with B the matrix A whose (i, j) entry is the number of columns of B equal to $\substack{i \\ j}$. This establishes the desired bijection between the orbits of $\mathfrak{S}_n$ acting on $T \times T$ and the $\mathbb{N}$-matrices A with $\mathrm{row}(A) = \mathrm{col}(A) = \alpha$.

7.77. a. We have

$$\left\langle \mathrm{ind}_H^G 1_H, \mathrm{ind}_K^G 1_K \right\rangle = \left\langle \mathrm{ind}_H^G 1_H \otimes \mathrm{ind}_K^G 1_K, 1_G \right\rangle.$$

Now the representation $\mathrm{ind}_H^G 1_H \otimes \mathrm{ind}_K^G 1_K$ is a *permutation* representation, obtained by letting G act on pairs (aH, bK) of left cosets by $w \cdot (aH, bK) =$

(waH, wbK). Hence the multiplicity of the trivial representation is the number of orbits of this action. When are two elements in the same orbit? Every orbit contains an element of the form (H, bK), so we are asking for what $u, v \in G$ can we find $w \in G$ such that $w \cdot (H, uK) = (H, vK)$. Since $wH = H$ we have $w \in H$, so we want to know when we can find $h \in H$ with $huk = vk'$ for $k, k' \in K$. But clearly this conditions holds if and only if $HuK = HvK$, so the number of orbits is the number of double cosets as desired.

This result can be considerably strengthened, as part of a theory developed by G. Mackey. See for example C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Wiley (Interscience), New York, 1962, reprinted 1988 (§10C), and [**142**, §7.3].

b. The argument given in (a) shows that the number of double cosets of (H, H) is the number of orbits of G acting diagonally on $G/H \times G/H$. This latter number is just what is meant by the rank of G acting on G/H.

c. Since $\mathrm{ch}(\mathrm{ind}_{\mathfrak{S}_\gamma}^{\mathfrak{S}_n} 1_{\mathfrak{S}_\gamma}) = h_\gamma$ (by Corollary 7.18.3), the number of double cosets of (H, K) is by (a) and Proposition 7.18.1 given by $\langle h_\alpha, h_\beta \rangle$, the number of $\mathbb{N}$-matrices A with $\mathrm{row}(A) = \alpha$ and $\mathrm{col}(A) = \beta$ (by (7.31)). The solution to Exercise 7.76(c) generalizes straightforwardly to give a combinatorial proof of the present exercise.

7.78. a. Because all characters of $\mathfrak{S}_n$ are real, we have

$$\begin{aligned} g_{\lambda\mu\nu} &= \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^\lambda(w) \chi^\mu(w) \chi^\nu(w), \end{aligned}$$

which clearly has the desired symmetry.

b. We have

$$\begin{aligned} s_\lambda * s_\mu &= \sum_\nu g_{\lambda\mu\nu} s_\nu \\ &= \mathrm{ch} \sum_\nu g_{\lambda\mu\nu} \chi^\nu \\ &= \mathrm{ch}\, \chi^\lambda \chi^\mu. \end{aligned}$$

c. By linearity, we may take $f = s_\lambda$. Using (b) and the fact that $e_n = s_{1^n}$, we get

$$\begin{aligned} e_n * s_\lambda &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^{1^n}(w)\, \chi^\lambda(w) p_{\rho(w)} \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \varepsilon_w \chi^\lambda(w) p_{\rho(w)} \\ &= \omega \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^\lambda(w) p_{\rho(w)} \\ &= \omega\, s_\lambda. \end{aligned}$$

d. Let $g, h \in \Lambda^n$. We claim that

$$\langle g * h, p_\nu \rangle = \langle g, p_\nu \rangle \cdot \langle h, p_\nu \rangle. \tag{7.220}$$

By bilinearity, it suffices to take $g = s_\lambda$ and $h = s_\mu$. Equation (7.220) then follows from (7.78) and (b). Now let $g = p_\lambda$ and $h = p_\mu$.

e. Note that $p_\nu(xy) = p_\nu(x)p_\nu(y)$. Hence equation (7.220) is equivalent to the desired result when $f = p_\nu$, so the general case follows by linearity.

f. One way to set up this computation is as follows:

$$\begin{aligned}
\sum_{\lambda,\mu,\nu} g_{\lambda\mu\nu} s_\lambda(x)s_\mu(y)s_\nu(z) &= \sum_{\lambda,\mu,\nu} \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle s_\lambda(x)s_\mu(y)s_\nu(z) \\
&= \sum_{n\geq 0} \sum_{\lambda,\mu,\nu\vdash n} \left(\sum_{\rho\vdash n} z_\rho^{-1} \chi^\lambda(\rho)\chi^\mu(\rho)\chi^\nu(\rho) \right) \\
&\quad \times s_\lambda(x)s_\mu(y)s_\nu(z) \\
&= \sum_{n\geq 0} \sum_{\rho\vdash n} z_\rho^{-1} \left(\sum_\lambda \chi^\lambda(\rho) s_\lambda(x) \right) \\
&\quad \times \left(\sum_\mu \chi^\mu(\rho) s_\mu(x) \right) \left(\sum_\nu \chi^\nu(\rho) s_\nu(x) \right) \\
&= \sum_\rho z_\rho^{-1} p_\rho(x)p_\rho(y)p_\rho(z) \\
&= \prod_{i,j,k} (1 - x_i y_j z_k)^{-1},
\end{aligned}$$

by arguing as in the proof of Proposition 7.7.4.

g. Straightforward generalization of (f).

The internal product of symmetric functions was first defined by J. H. Redfield **[124]** (denoted something like $\mathcal{V}$), and later independently by D. E. Littlewood, *J. London Math. Soc.* **31** (1956), 89–93.

7.79. a. Let xy have the meaning of Exercise 7.78(e). By the Cauchy identity (Theorem 7.12.1) applied to the two sets of variables $x_i y_j$ and z_k, we have

$$\prod_{i,j,k} (1 - x_i y_j z_k)^{-1} = \sum_\lambda s_\lambda(xy) s_\lambda(z).$$

Comparing with Exercise 7.78(f), we get

$$\begin{aligned}
s_\lambda(xy) &= \sum_\nu s_\lambda * s_\nu(x) s_\nu(y) \\
&= \sum_{\mu,\nu} g_{\lambda\mu\nu} s_\mu(x) s_\nu(y).
\end{aligned} \tag{7.221}$$

Suppose that $\langle s_\lambda, s_\mu * s_\nu \rangle = g_{\lambda\mu\nu} \neq 0$. Let $a = \ell(\mu), b = \ell(\nu)$, and restrict the variables to $x = (x_1, \ldots, x_a)$ and $y = (y_1, \ldots, y_b)$. Then $s_\mu(x) \neq 0$ and $s_\nu(y) \neq 0$, so $s_\lambda(xy) \neq 0$. But $s_\lambda(xy)$ is a Schur function in the ab variables $x_i y_j$, so if $s_\lambda(xy) \neq 0$ then $\ell(\lambda) \leq ab$.

b. We can reverse the argument in (a). In equation (7.221) take $x = (x_1, \ldots, x_a)$ and $y = (y_1, \ldots, y_b)$. Since $\ell(\lambda) \leq ab$, we have $s_\lambda(xy) \neq 0$. Hence some term $g_{\lambda\mu\nu}s_\mu(x)s_\nu(y) \neq 0$, so $\ell(\mu) \leq a$ and $\ell(\nu) \leq b$. The results of this exercise (parts (a) and (b)) were first obtained (in another way) by A. Regev, *J. Algebra* **154** (1993), 125–140.

c. See Y. Dvir, *J. Algebra* **154** (1993), 125–140. For a continuation, see Y. Dvir, *Europ. J. Combinatorics* **15** (1994), 449–457.

7.80. These results are due to C. Bessenrodt and A. S. Kleshchev, On Kronecker products of complex representations of the symmetric and alternating groups, *Pacific J. Math.*, to appear.

7.81. Since $\chi^{n-1,1}(\lambda) = m_1(\lambda) - 1$ [why?], there follows for $f \in \Lambda^n$ the formula

$$f * s_{n-1,1} = p_1 \frac{\partial}{\partial p_1} f - f,$$

where we are regarding f as a polynomial in the power sums. Since

$$\frac{\partial}{\partial p_1} s_\lambda = s_{\lambda/1}$$

(see the solution to Exercise 7.35), the result follows.

7.82. **a.** Immediate consequence of Exercises 7.71(a)(ii) and 7.71(c).

b. This result is implicit in C. Procesi, *Advances in Math.* **19** (1976), 306–381, and *J. Algebra* **87** (1984), 342–359 (§2). An explicit statement and proof appears in A. Regev, *Linear and Multilinear Algebra* **21** (1987), 1–28 (Cor. 2.14), and 29–39 (Thm. 1).

7.83. **a.** We have

$$\langle \chi\psi, \phi \rangle = \frac{1}{\#G} \sum_{w \in G} \chi(w)\psi(w)\bar{\phi}(w) = \langle \chi\bar{\phi}, \bar{\psi} \rangle.$$

Hence by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\langle \chi\bar{\phi}, \bar{\psi} \rangle^2 &\leq \langle \chi\bar{\phi}, \chi\bar{\phi} \rangle \, \langle \bar{\psi}, \bar{\psi} \rangle \\
&= \frac{1}{\#G} \sum_{w \in G} |\chi\phi(w)|^2 \qquad \text{(since } \psi \text{ is irreducible)} \\
&\leq \frac{1}{\#G} \phi(1)^2 \sum_{w \in G} |\chi(w)|^2 \qquad \text{(since } |\phi(w)| \leq \phi(1)\text{)} \\
&= \phi(1)^2 \langle \chi, \chi \rangle \\
&= \phi(1)^2 \qquad \text{(since } \chi \text{ is irreducible).}
\end{aligned}$$

This result appears in I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976; reprinted by Dover, New York, 1994 (Problem 4.12).

b. Immediate from (a), equation (7.79), and the fact that

$$\langle s_\lambda * s_\mu, s_\nu \rangle = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle.$$

7.84. **a.** By Exercise 7.78(e) we have

$$\begin{aligned}\langle h_\lambda * s_\mu, s_\nu\rangle &= \langle h_\lambda * s_\nu, s_\mu\rangle \qquad \text{[why?]}\\ &= \langle h_\lambda(x)s_\nu(y), s_\mu(xy)\rangle.\end{aligned}$$

Ordering the variables xy as $x_1y_1 < x_1y_2 < \cdots < x_2y_1 < x_2y_2 < \cdots$, it follows from equation (7.66) that

$$\begin{aligned}s_\mu(xy) &= \sum_{\emptyset=\mu^0\subset\mu^1\subset\cdots\subset\mu^\ell=\mu}\left(\prod_{i\geq 1} x_i s_{\mu^i/\mu^{i-1}}(y)\right)\\ &= \sum_\lambda m_\lambda(x) \sum_{\substack{\emptyset=\mu^0\subset\mu^1\subset\cdots\subset\mu^\ell=\mu\\ |\mu^i/\mu^{i-1}|=\lambda_i}} \prod_{i\geq 1} s_{\mu^i/\mu^{i-1}}(y),\end{aligned}$$

and the proof follows easily. See [**96**, Exam. I.7.23(d), p. 130].

b. Let $\nu \vdash n$. By Exercise 7.78(e) we have

$$\langle m_\nu, h_\lambda * h_\mu\rangle = \langle m_\nu(xy), h_\lambda(x)h_\mu(y)\rangle.$$

Now

$$\begin{aligned}m_\nu(xy) &= \sum_A \prod_{i,j}(x_iy_j)^{a_{ij}}\\ &= \sum_A m_{\mathrm{row}(A)}(x)m_{\mathrm{col}(A)}(y),\end{aligned}$$

where A ranges over all $\mathbb{N}$-matrices (a_{ij}) such that the decreasing rearrangement of the a_{ij}'s is ν. The proof now follows easily for the duality between the bases $\{m_\lambda\}$ and $\{h_\lambda\}$ (equation (7.30)). See [**96**, Exam. I.7.23(e), p. 131].

* **7.85.** See [**46**, Corollary 15]. For some further evaluations of $g_{\lambda\mu\nu}$, see J. B. Remmel and T. Whitehead, *Bull. Belgian Math. Soc.* **1** (1994), 649–683; E. Vallejo, On the Kronecker product of irreducible characters of the symmetric group, preprint; and the references given there. One of the main open problems in the combinatorial representation theory of $\mathfrak{S}_n$ is to obtain a combinatorial interpretation of $g_{\lambda\mu\nu}$ in general.

7.86. **a.** By Exercise 7.78(b) we have

$$G_{\lambda\mu}(q) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\frac{\chi^\lambda(w)\chi^\mu(w)}{(1-q^{\rho_1})\cdots(1-q^{\rho_\ell})}, \tag{7.222}$$

where $\rho(w) = (\rho_1, \ldots, \rho_\ell)$ with $\rho_\ell > 0$. It follows from Exercise 7.60(b) that the denominators of the nonzero terms in the above sum are all divisors of $H_\lambda(q)$, and the proof follows. This result was first given in R. Stanley, *Linear and Multilinear Algebra* **16** (1984), 29–34 (Proposition 8.2(i)). For algebraic and geometric aspects of this exercise, see P. J. Hanlon, *Adv. in Math.* **56** (1985), 238–282; J. R. Stembridge, *J. Combin. Theory (A)* **46** (1987), 79–120; R. K. Brylinski, in Lecture Notes in Math. **1404**, Springer-Verlag, Berlin/New York, 1989, pp. 35–94; and R. K. Brylinski, *Advances in Math.* **100** (1993), 28–52.

b. Multiply (7.222) by $H_\lambda(q)$ and set $q = 1$. All terms vanish except the term indexed by $w = \mathrm{id}$, yielding

$$P_{\lambda\mu}(1) = \frac{1}{n!}\chi^\lambda(\mathrm{id})\chi^\mu(\mathrm{id})H_\lambda(1) = f^\mu.$$

c. Let xy have the meaning of Exercise 7.78(e). Let $\psi = \psi_t$ be the specialization of Exercise 7.43, acting on the y variables only; and let ps_n and ps be the principal specialization and stable principal specialization of Section 7.8, acting on the x variables only. By computing directly the case $f = p_k$, we see that for any $f \in \Lambda$ and $n \in \mathbb{P}$,

$$\mathrm{ps}\,\psi_{-q^N} f(xy) = \mathrm{ps}_N\, f(x). \tag{7.223}$$

Now by Exercise 7.78(f) we have

$$s_\lambda(xy) = \sum_\mu (s_\lambda * s_\mu)(x)s_\mu(y).$$

Apply the specialization ps_N. By equation (7.223) we get

$$s_\lambda(1, q, \ldots, q^{N-1}) = \sum_\mu (s_\lambda * s_\mu)(1, q, q^2, \ldots)\,\psi_{-q^N} s_\mu(y).$$

Using Theorem 7.21.2, Corollary 7.21.3, and Exercise 7.43, we have

$$q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{1 - q^{N+j-i}}{1 - q^{h(i,j)}} = \sum_{\mu=\langle n-k, 1^k\rangle} \frac{P_{\lambda\mu}(q)}{\prod_{(i,j)\in\lambda}\left(1 - q^{h(i,j)}\right)}(-q)^{Nk}(1 - q^N).$$

Multiply by $\prod(1 - q^{h(i,j)})$ and set $t = -q^N$. We obtain a polynomial identity in t valid for infinitely many values of t (viz., $t = -q^N$) and hence valid when t is an indeterminate. Therefore

$$q^{b(\lambda)} \prod_{(i,j)\in\lambda} (1 + tq^{j-i}) = \sum_{\mu=\langle n-k, 1^k\rangle} P_{\lambda\mu}(q)t^k(1 + t),$$

which is easily seen to be equivalent to the stated result.

This result is due to A. Lascoux (private communication). His proof uses the machinery of λ-rings. Since we have not introduced this machinery here, we have given a "naive" version of Lascoux's proof. However, the λ-ring approach does give a more natural and elegant proof. For more information on λ-rings, see D. Knutson, Lecture Notes in Math. **308**, Springer-Verlag, Berlin/Heidelberg/New York, 1973.

d. This deceptively simple statement follows from [**96**, Exam. VI.8.3, pp. 362–363] and a conjecture of I. G. Macdonald, in *Actes 20e Séminaire Lotharingien*, Publ. I.R.M.A. Strasbourg, 372/S-20, 1988, pp. 131–171 (§6), and [**96**, (8.18?), p. 355]. Part (b) suggests that the coefficients of $P_{\lambda\mu}(q)$ count some statistic on SYTs of shape μ, but such an interpretation remains open. See also A. N. Kirillov, *Adv. Ser. Math. Phys.* **16** (1992), 545–579.

7.87. See Theorem 5.1 of R. Stanley, *Linear and Multilinear Algebra* **16** (1984), 3–27.

7.88. a. It follows immediately from the standard formula for an induced character (e.g., Isaacs, *ibid.* (Def. 5.1) or [**142**, Prop. 20]) that

$$\operatorname{ch}\psi_m = \frac{1}{n}\sum_{d|n} p_d^{n/d}\left(\sum_{\zeta}\zeta^m\right),$$

where ζ ranges over all primitive d-th roots of unity. The result now follows from the well-known fact (see C. A. Nicol and H. S. Vandiver, *Proc. Nat. Acad. Sci.* **40** (1954), 825–835) that the above sum over ζ is equal to

$$\frac{\phi(d)}{\phi(d/(m,d))}\mu(d/(m,d)).$$

Equation (7.189) is due to H. O. Foulkes, in *Combinatorics* (D. J. A. Welsh and D. R. Woodall, eds.), The Institute for Mathematics and Its Applications, Southend-on-Sea, Essex, 1972, pp. 141–154 (Thm. 1).

b. (sketch) Let Ω_m be the operator on $\Lambda[[q]]$ defined by

$$\Omega_m f(q) = \frac{1}{n}\sum_{\zeta^n=1}\zeta^{-m}f(\zeta q),$$

regarding n as fixed. Thus Ω_m picks out from the power series $f(q)$ those terms whose exponents are congruent to m modulo n. Apply Ω_m to the identity

$$\sum_{\lambda\vdash n}\left(\sum_{\substack{T=\mathrm{SYT}\\ \text{of shape }\lambda}} q^{\mathrm{maj}(T)}\right)s_\lambda = \sum_\lambda z_\lambda^{-1}p_\lambda\frac{[n]!}{\prod[\lambda_i]},$$

where $[j] = 1-q^j$. The coefficient of s_λ on the left-hand side is equal to the number of SYT T of shape λ satisfying $\mathrm{maj}(T)\equiv m \pmod n$. The coefficient of p_λ on the right-hand side is given by

$$\begin{cases} 0, & \lambda\neq\langle d^{n/d}\rangle\\ \frac{1}{n}\sum_{\substack{\zeta=\text{primitive}\\ d\text{-th root of }1}}\zeta^m, & \lambda=\langle d^{n/d}\rangle,\end{cases}$$

and the proof follows from (a). This result was first proved independently by W. Kraśkiewicz and J. Weyman, Algebra of coinvariants and the action of Coxeter element, preprint, and by R. Stanley (unpublished). The proof by Stanley is the one given here. A similar proof appears in [**130**, Cor. 8.10]. Kraśkiewicz and Weyman extend the result to the Weyl groups of type B_n and D_n.

c. Regarding n as fixed, the expression (7.189) for ψ_m depends only on (m,n), and the proof follows from (b). A bijective proof is not known.

d. M. Kontsevich, in *The Gelfand Mathematical Seminars, 1990–1992* (L. Corwin *et al.*, eds.), Birkhäuser, Boston, 1993, pp. 173–187, mentions (p. 181) a certain representation of $\mathfrak{S}_n$ of dimension $(n-2)!$, described more explicitly (as an action on the multilinear part of the free Lie algebra on $n-1$ generators) by E. Getzler and M. M. Kapranov, in *Geometry, Topology, and Physics*, International Press, Cambridge, Massachusetts, 1995, pp. 167–201.

We will not give the definition here, but it follows from the definition that the characteristic W_n of this action is given by

$$W_n = p_1 L_{n-1} - L_n,$$

using the notation (7.191). It is easy to show from Exercise 7.89(c) that $\langle p_1 L_{n-1}, s_\lambda\rangle = y_{n-1}(\lambda)$, while Exercise 7.89(c) itself asserts that $\langle L_n, s_\lambda\rangle = y_n(\lambda)$, and the proof follows. No combinatorial proof is known.

The symmetric function W_n has subsequently appeared in a surprising number of disparate circumstances, and the $\mathfrak{S}_n$-module for which it is the characteristic is known as the *Whitehouse module*. Some references include
* E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker, Complexes of not i-connected graphs, MSRI preprint No. 1997-054, 31 pp.; P. Hanlon, *J. Combinatorial Theory (A)* **74** (1996), 301–320; P. Hanlon and R. Stanley, A q-deformation of a trivial symmetric group action, *Trans. Amer. Math. Soc.*, to appear; O. Mathieu, *Comm. Math. Phys.* **176** (1996), 467–474; C. A. Robinson, *Sonderforschungsbereich 343*, Universität Bielefeld, preprint 92-083, 1992; C. A. Robinson and S. Whitehouse, *J. Pure Appl. Algebra* **111** (1996), 245–253; S. Sundaram, Homotopy of non-modular partitions and the Whitehouse module, *J. Algebraic Combinatorics*, to appear; S. Sundaram, On the topology of two partition posets with forbidden block sizes, preprint, 1 May 1998; V. Turchin, Homology isomorphism of the complex of 2-connected graphs and the graph-complex of trees, preprint; S. Whitehouse, Ph.D. thesis, Warwick University, 1994; and S. Whitehouse, *J. Pure Appl. Algebra* **115** (1996), 309–321.

e. We have

$$\begin{aligned} H[J] &= \left(\sum_{i\geq 0} h_i\right)\left[\frac{\mu(d)}{d}\log(1+p_d)\right] \\ &= \left(\exp\sum_{k\geq 1}\frac{p_k}{k}\right)\left[\sum_{d\geq 1}\frac{\mu(d)}{d}\sum_{n\geq 1}(-1)^{n-1}\frac{p_d^n}{n}\right] \\ &= \exp\sum_{k\geq 1}\sum_{d\geq 1}\sum_{n\geq 1}\frac{1}{kdn}\mu(d)(-1)^{n-1}p_{kd}^n. \end{aligned}$$

Putting $N = kd$ gives

$$\begin{aligned} H[J] &= \exp\sum_{N\geq 1}\sum_{n\geq 1}\frac{1}{nN}(-1)^{n-1}p_N^n\sum_{d|N}\mu(d) \\ &= \exp\sum_{n\geq 1}(-1)^{n-1}\frac{p_1^n}{n} \\ &= \exp\log(1+p_1) \\ &= 1+p_1, \end{aligned}$$

whence $(H-1)[J] = p_1$. Since the invertible elements of $\hat{\Lambda}$ with respect to plethysm form a group, it follows also that $J[H-1] = p_1$, completing the proof.

The result of this exercise is due to C. C. Cadogan, *J. Combinatorial Theory (B)* **11** (1971), 193–200 (§3). For further aspects and a generalization, see A. R. Calderbank, P. Hanlon, and R. W. Robinson, *Proc. London Math. Soc. (3)* **53** (1986), 288–320.

7.89. a. This result can be proved by a straightforward use of the Principle of Inclusion–Exclusion. It is equivalent to enumerating primitive necklaces of length n by the number of occurrences of each color. In this form equation (7.191) appears in [**130**, Thm. 7.2].

b. This is the special case $m = 1$ of Exercise 7.88(a).

c. Let $m = 1$ in Exercise 7.88(b).

d. See e.g. [**4.21**, Thm. 5.1.5] or [**130**, (7.4.1)].

e. This result follows easily from Proposition 1.3.1.

f. This result is a consequence e.g. of I. M. Gessel and C. Reutenauer, *J. Combinatorial Theory (A)* **64** (1993), 189–215 (Thm. 3.6).

g. This key "reciprocity theorem" appears in T. Scharf and J.-Y. Thibon, *Advances in Math.* **104** (1994), 30–58 (Rmk. 3.11). A simpler proof was later given by I. M. Gessel (unpublished).

h. By (f) and (g) we have

$$\begin{aligned}\langle t_M, s_\mu\rangle &= \sum_\lambda \langle L_\lambda, h_\mu\rangle \cdot \langle p_\lambda, s_\mu\rangle\\ &= \sum_\lambda \langle p_\lambda, h_\mu\rangle \cdot \langle L_\lambda, s_\mu\rangle.\end{aligned}$$

But p_λ is clearly m-positive, so $\langle p_\lambda, h_\mu\rangle \geq 0$. Moreover, L_λ is s-positive by (b) and the fact (Appendix 2, Theorem A2.5) that the plethysm of s-positive symmetric functions remains s-positive. Hence $\langle L_\lambda, s_\mu\rangle \geq 0$, so $\langle t_M, s_\mu\rangle \geq 0$ as desired. This result was originally conjectured by R. Stanley and proved by Scharf and Thibon, *ibid.* (Example 3.15).

7.90. a. Given the SYT τ with $D(\tau) \subseteq S$, replace $1, 2, \ldots, \alpha_1$ in τ by 1's; replace $\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2$ by 2's, etc. This gives an SSYT of type $(\alpha_1, \ldots, \alpha_k)$, and the correspondence is a bijection. Compare the discussion preceding Lemma 7.11.6.

b. Let τ' be the transpose of τ, of shape λ'/μ'. The condition $i \in D(\tau)$ is equivalent to $i \notin D(\tau')$, i.e., $D(\tau') \subseteq \{1, \ldots, i-1, i+1, \ldots, n-1\}$. Thus by (a), the number of τ of shape λ/μ with $i \in D(\tau)$ is the Kostka number $K_{\lambda'/\mu',\alpha}$, where $\alpha = (1, 1, \ldots, 1, 2, 1, \ldots, 1)$. But $K_{\lambda'/\mu',\alpha}$ is independent of the order of the entries of α.

7.91. a. The first statement is immediate from the Cauchy identity (Theorem 7.12.1). Similarly, if $F(t) = \prod_{i\geq 1}(1 + x_i t)$, then it follows from the dual Cauchy identity (Theorem 7.14.3) that $s_\lambda^F = s_{\lambda'}(x)$.

b. Immediate consequence of (a) and Theorem 7.21.2.

c. Write $F_{y,z}(t)$ for $F(t)$. Note that s_λ^F is a polynomial $P_\lambda(y, z)$ in y and z. When $y = 1$ and $z = q^n$ then the problem reduces to (b). Since a polynomial in one variable is determined by any infinite set of its values, it follows that (7.192) holds for $F_{1,z}(t)$. Now for any $F(t)$ let $G(t) = F(yt)$. Clearly

$s_\lambda^G = y^{|\lambda|} s_\lambda^F$. Since $F_{y,z}(t) = F_{1,z/y}(yt)$, we get

$$\begin{aligned} s_\lambda^{F_{y,z}} &= y^{|\lambda|} s_\lambda^{F_{1,z/y}} \\ &= y^{|\lambda|} q^{b(\lambda)} \prod_{u\in\lambda} \frac{1 - \frac{z}{y} q^{c(u)}}{1 - q^{h(u)}} \\ &= q^{b(\lambda)} \sum_{u\in\lambda} \frac{y - zq^{c(u)}}{1 - q^{h(u)}}. \end{aligned}$$

This result is equivalent to that of D. E. Littlewood and A. R. Richardson, *Quart. J. Math. (Oxford)* **6** (1935), 184–198 (Thm. IX) (repeated in [**88**, II. on p. 125]), and also appears in [**96**, Exam. I.3, p. 45].

d. Apply the homomorphism φ to the Jacobi–Trudi identity (Theorem 7.16.1).

e. It is trivial that (iii) $\Rightarrow$ (ii), since e_λ is Schur-positive by the dual Jacobi–Trudi identity (Corollary 7.16.2).

Assume (i), so $F(t) = \prod_1^m (1 + \gamma_j t)$, where each $\gamma_j > 0$. Then

$$\prod_i F(t_i) = \prod_{j=1}^m \left(\sum_{n\geq 0} \gamma_j^n e_n(t) \right),$$

from which it follows that (i) $\Rightarrow$ (iii).

Assume (i). Clearly the coefficients of $F(t)$ are then nonnegative real numbers. Let the zeros of $F(t)$ be $\theta_1, \ldots, \theta_n$, and define the Vandermonde matrix $V = (\theta_j^i)_{i,j=0}^{n-1}$. Then V is a real matrix and $A = VV^t$, so A is semidefinite. Hence (i) $\Rightarrow$ (iv).

The difficult implications are (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i). The first of these implications is equivalent to a fundamental result of M. Aissen, I. J. Schoenberg, and A. Whitney, *J. Analyse Math.* **2** (1952), 93–103. This result states that if $a_0, a_1, \ldots, a_m \in \mathbb{R}$, then every zero of the polynomial $F(t) = a_0 + a_1 t + \cdots + a_m t^m$ is a nonpositive real number if and only if every minor of the (infinite) Toeplitz matrix $A = [a_{j-i}]_{i,j\geq 0}$ (where we set $a_k = 0$ if $k < 0$ or $k > m$) is nonnegative. To see the connection with the problem under consideration, suppose that $a_0 = 1$, so that $a_i = e_i(\gamma_1, \ldots, \gamma_m)$, where $-\gamma_1^{-1}, \ldots, -\gamma_m^{-1}$ are the zeros of $F(t)$. By the dual Jacobi–Trudi identity (Corollary 7.16.2), every minor of A is a skew Schur function $s_{\lambda/\mu}(\gamma_1, \ldots, \gamma_m)$. Since skew Schur functions are s-positive (by Corollary 7.18.6 or Theorem A1.3.1), it follows that if (ii) holds then every minor of A is nonnegative. Hence by the Aissen–Schoenberg–Whitney theorem and the fact that $F(0) = 1$, every zero of $F(t)$ is a negative real number.

The above formulation of the Aissen–Schoenberg–Whitney theorem in terms of symmetric functions seems first to have been stated explicitly in R. Stanley, Graph colorings and related symmetric functions: ideas and applications, *Discrete Math.,* to appear (Thm. 2.11). An extension to arbitrary
* Toeplitz matrices (not just those with finitely many nonzero diagonals) was

given by A. Edrei, *Canad. J. Math.* **5** (1953), 86–94, and *Trans. Amer. Math. Soc.* **74** (1953), 367–383, thereby proving a conjecture of Aissen, Schoenberg, and Whitney. The same result was rediscovered by E. Thoma, *Math. Zeitschrift* **85** (1964), 40–61, in the context of the characters of the infinite symmetric group. A matrix all of whose minors are nonnegative is called a *totally positive* (or sometimes *totally nonnegative*) *matrix*. Such matrices have been extensively investigated; see, e.g., S. Karlin, *Total Positivity*, vol. 1, Stanford University Press, Stanford, California, 1968; T. Ando, *Linear Algebra Appl.* **90** (1987), 165–219; J. R. Stembridge, *Bull. London Math. Soc.* **23** (1991), 422–428; B. Kostant, *J. Amer. Math. Soc.* **8** (1995), 181–186; F. Brenti, *J. Combinatorial Theory (A)* **71** (1995), 175–218; A. D. Berenstein, S. Fomin, and A. Zelevinsky, *Advances in Math.* **122** (1996), 49–149; A. Okounkov, *Zapiski Nauchnyh Seminarov POMI* **240** (1997), 167–230; and some of the papers in *Total Positivity and Its Applications (Jaca, 1994)*, Mathematics and Its Applications **359**, Kluwer, Dordrecht, 1996. For an interesting generalization of the Aissen–Schoenberg–Whitney–Edrei–Thoma theorem, see S. Kerov, A. Okounkov, and G. Olshanski, *Internat. Math. Res. Notices* (1998), no. 4, 173–179.

The equivalence of (i)–(iii) suggests that it might be possible to prove combinatorially that certain polynomials $F(t)$ have real zeros. Assuming that $F(0) = 1$, one wants to interpret combinatorially the coefficients of the product $F(t_1)F(t_2)\cdots$ when expanded in terms of Schur functions or elementary symmetric functions, thereby showing that they are nonnegative. For an example of such an argument, see Exercise 7.47(i).

The implication (iv) $\Rightarrow$ (i) is a consequence of the work of A. Hurwitz, E. J. Routh, J. C. F. Sturm, and others on the zeros of polynomials. It seems first to have been explicitly stated by F. R. Gantmacher, *The Theory of Matrices*, vol. 2, Chelsea, New York, 1959 (Cor. on p. 203). Since a real symmetric matrix $(a_{ij})_{i,j=1}^{n}$ is semidefinite if and only if the leading principal minors $\det(a_{ij})_{i,j=1}^{k}$ are nonnegative, condition (iv) yields $n-1$ inequalities (in addition to the nonnegativity of the coefficients) on the coefficients of $F(t)$ that are necessary and sufficient for every zero of $F(t)$ to be a negative real number. (There are $n-1$ rather than n inequalities because $a_{11} = p_0^F = \deg F \geq 0$.)

7.92. **a.** See J. R. Stembridge, *Bull. London Math. Soc.* **23** (1991), 422–428. A different proof was later given by B. Kostant, *J. Amer. Math. Soc.* **8** (1995), 181–186. Note that the matrices A of this exercise are the totally positive matrices discussed in the solution to Exercise 7.91(e).

b. See J. R. Stembridge, *Canad. J. Math.* **44** (1992), 1079–1099 (Conjecture 2.1). Exercise 7.111(d) is a special case. An even stronger conjecture involving Kazhdan–Lusztig theory appears in M. Haiman, *J. Amer. Math. Soc.* **6** (1993), 569–595 (Conjecture 2.1).

7.93. Let (P, ω) be the labeled poset that is a disjoint union of chains $t_1 < \cdots < t_m$ and $t_1' < \cdots < t_n'$ with $\omega(t_i) = u_i$ and $\omega(t_j') = v_j$. It is immediate from the definition of a reverse (P, ω)-partition and from the definition (7.95) of $K_{P,\omega}$

that

$$K_{P,\omega} = L_{\text{co}(u)} L_{\text{co}(v)}.$$

On the other hand, we have $\mathcal{L}(P, \omega) = \text{sh}(u, v)$, and the proof follows from Corollary 7.19.5.

7.94. a. Preserve the notation of Exercise 7.93. By that exercise, we have

$$\hat{\omega}(L_\alpha L_\beta) = \sum_{w \in \text{sh}(u,v)} L_{\overline{\text{co}(w)}}.$$

On the other hand we have

$$\begin{aligned} \hat{\omega}(L_\alpha)\hat{\omega}(L_\beta) &= L_{\bar{\alpha}} L_{\bar{\beta}} \\ &= L_{\bar{\beta}} L_{\bar{\alpha}} \\ &= \sum_{w \in \text{sh}(\bar{v}, \bar{u})} L_{\text{co}(w)}, \end{aligned}$$

where $\bar{v} \in \mathfrak{S}_n$ and $\bar{u} \in \mathfrak{S}_{[n+1,n+m]}$ satisfy $\text{co}(\bar{v}) = \bar{\beta}$ and $\text{co}(\bar{u}) = \bar{\alpha}$. But then the natural bijection $\varphi : \text{sh}(u, v) \to \text{sh}(\bar{v}, \bar{u})$ satisfies $\overline{\text{co}(w)} = \text{co}(\varphi(w))$ for $w \in \text{sh}(u, v)$. Hence $\hat{\omega}(L_\alpha L_\beta) = \hat{\omega}(L_\alpha)\hat{\omega}(L_\beta)$. Since the L_α's form a basis for $\mathcal{Q}$ and $\hat{\omega}$ is linear, it follows that $\hat{\omega}$ is an endomorphism of $\mathcal{Q}$. Since $\hat{\omega}$ is an involution, it is in fact an automorphism. Now let $\alpha = (1, 1, \ldots, 1) \in \text{Comp}(n)$, so $\bar{\alpha} = (n)$. Then $L_\alpha = e_n$ and $L_{\bar{\alpha}} = h_n$. Hence $\hat{\omega}(e_n) = h_n$. Since $\hat{\omega}$ is an automorphism we have $\hat{\omega}(e_\lambda) = h_\lambda$ for all $\lambda \in \text{Par}$, so $\hat{\omega}|_\Lambda = \omega$.

b. *First Proof.* If $C : \hat{0} = t_0 < t_1 < \cdots < t_k = \hat{1}$ is a chain of P, then write

$$f(C) = f(t_0, t_1) f(t_1, t_2) \cdots f(t_{k-1}, t_k).$$

Also for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \text{Comp}(n)$ let C_α denote the set of all chains $\hat{0} = t_0 < t_1 < \cdots < t_k = \hat{1}$ of P for which $\rho(t_i) - \rho(t_{i-1}) = \alpha_i$. Hence

$$F_{f^{-1}} = \sum_{\alpha \in \text{Comp}(n)} \sum_{C \in C_\alpha} f^{-1}(C) M_\alpha,$$

where M_α is given by equation (7.87). Now let $f = 1 + g$, so $f^{-1} = 1 - g + g^2 - \cdots$. Then

$$\begin{aligned} f^{-1}(C) &= (1 - g + g^2 - \cdots)(t_0, t_1) \cdots (1 - g + g^2 - \cdots)(t_{k-1}, t_k) \\ &= \sum_{D \succeq C} (-1)^{\ell(D)} f(D), \end{aligned}$$

where $D \succeq C$ indicates that D is a chain refining C, and where $\ell(D)$ denotes the length of D. Hence

$$F_{f^{-1}} = \sum_{\alpha \in \text{Comp}(n)} \sum_{C \in C_\alpha} \sum_{D \succeq C} (-1)^{\ell(D)} f(D) M_\alpha.$$

Now sum first over D. If $D \in C_\beta$, then α satisfies $S_\alpha \subseteq S_\beta$. Moreover,

$\ell(D) = \#S_\beta$, so we get

$$F_{f^{-1}} = \sum_{\beta\in\mathrm{Comp}(n)} \sum_{D\in C_\beta} (-1)^{\#S_\beta} f(D) \sum_{\alpha\,:\,S_\alpha\subseteq S_\beta} M_\alpha$$

$$= \sum_{\beta\in\mathrm{Comp}(n)} \sum_{D\in C_\beta} (-1)^{\#S_\beta} f(D)$$

$$\times \sum_{\alpha\,:\,S_\alpha\subseteq S_\beta} \sum_{S_\alpha\subseteq T\subseteq[n-1]} (-1)^{\#(T-S_\alpha)} L_{\mathrm{co}(T)} \qquad \text{(by (7.91))}.$$

Let $\bar{T} = [n-1] - T$. By (a) we get

$$\hat{\omega} F_{f^{-1}} = \sum_{\beta\in\mathrm{Comp}(n)} \sum_{D\in C_\beta} (-1)^{\#S_\beta} f(D) \sum_{\alpha\,:\,S_\alpha\subseteq S_\beta} \sum_{S_\alpha\subseteq T\subseteq[n-1]} (-1)^{\#(T-S_\alpha)} L_{\mathrm{co}(\bar{T})}$$

$$= \sum_{\beta\in\mathrm{Comp}(n)} \sum_{D\in C_\beta} (-1)^{\#S_\beta} f(D) \sum_{T\subseteq[n-1]} L_{\mathrm{co}(\bar{T})} \sum_{\alpha\,:\,S_\alpha\subseteq S_\beta\cap T} (-1)^{\#(T-S_\alpha)}.$$

But

$$\sum_{\alpha\,:\,S_\alpha\subseteq S_\beta\cap T} (-1)^{\#(T-S_\alpha)} = \begin{cases} (-1)^{\#T} & \text{if } S_\beta\cap T=\emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Since $S_\beta \cap T = \emptyset$ if and only if $S_\beta \subseteq \bar{T}$, we get

$$\hat{\omega} F_{f^{-1}} = \sum_{\beta\in\mathrm{Comp}(n)} \sum_{D\in C_\beta} (-1)^{\#S_\beta} f(D) \sum_{S_\beta\subseteq\bar{T}\subseteq[n-1]} (-1)^{n-\#\bar{T}} L_{\mathrm{co}(\bar{T})}$$

$$= (-1)^n \sum_{\beta\in\mathrm{Comp}(n)} \sum_{D\in C_\beta} f(D) M_{\mathrm{co}(\beta)} \qquad \text{(by (7.91))}$$

$$= (-1)^n F_f,$$

completing the proof.

Second Proof (Sketch). Let $m \in \mathbb{P}$. We see immediately from the relevant definitions that

$$F_{f^m}(x) = F_f(mx), \tag{7.224}$$

where mx denotes the multiset of variables consisting of m x_1's, m x_2's, etc., in that order. If for any $G \in \mathcal{Q}^n$ we expand $g(mx)$ in terms of some basis for $\mathcal{Q}^n$, then the coefficients will be polynomials in m. One can show (using the basis $\{L_\alpha\}$) that setting $m = -1$ yields $(-1)^n\hat{\omega}(G)$. (Compare equation (A2.163) of Appendix 2.) Similarly if we expand the left-hand side of (7.224) in terms of some basis for $\mathcal{Q}^n$, then the coefficients will be polynomials in m, and setting $m = -1$ yields $F_{f^{-1}}(x)$. Hence the proof follows by setting $m = -1$ in (7.224).

7.95. a. This is a straightforward generalization of Lemma 7.23.3. Regarding P_S as the border strip B_α, we have that a permutation $v \in \mathfrak{S}_n$ belongs to

$\mathcal{L}(P_S, \omega_w)$ if and only if w_{i+1} follows w_i in v whenever w_i is to the left of w_{i+1} in the same row, and w_{i+1} precedes w_i in v whenever w_i is below w_{i+1} in the same column. This condition is easily seen to be equivalent to $D(wv^{-1}) = S$.

b. The set $\mathcal{A}^r(P_S, \omega_w)$ of reverse (P_S, ω_w)-partitions depends on S and on the set of those pairs $(u, v) \in P_S \times P_S$ for which $u < v$ and $\omega_w(u) > \omega_w(v)$. This set of pairs in turn depends only on $D(w)$. Hence by Corollary 7.19.5 the multiset $M = \{D(v) : v \in \mathcal{L}(P_S, \omega_w)\}$ depends only on S and $D(w)$. By (a), we have $M = \{D(v) : D(wv^{-1}) = S\}$. Letting $u = wv^{-1}$, we get that the number of u, v for which $D(u) = S$, $D(v) = T$, and $uv = w$ depends only on u, v, and $D(w)$, as was to be proved.

This result may be formulated algebraically as follows. In the group algebra $\mathbb{Q}\mathfrak{S}_n$ of $\mathfrak{S}_n$, define for each $S \subseteq [n-1]$

$$B_S = \sum_{w\,:\,D(w)=S} w.$$

Then the algebra $\mathcal{D}_n$ generated by the B_S's is equal (as a set) to their linear span. In other words, $\dim \mathcal{D}_n = 2^{n-1}$. The algebra $\mathcal{D}_n$ is known as the *descent algebra* of $\mathfrak{S}_n$ and has many remarkable properties. It was first defined (for any finite Coxeter group) by L. Solomon, *J. Algebra* **41** (1976), 255–268. For a connection between descent algebras and quasisymmetric functions, see C. Malvenuto and C. Reutenauer, *J. Algebra* **177** (1995), 967–982. The proof that $\dim \mathcal{D}_n = 2^{n-1}$ given here is due to I. M. Gessel. A good reference to the descent algebra is [**130**, Ch. 9].

7.96. This result is implicit in A. M. Garsia and C. Reutenauer, *Advances in Math.* **77** (1989), 189–262 (Thm. 4.4).

7.97. a. It follows from the four equations beginning with (7.98) that

$$\begin{aligned} F(x) &= \sum_{P,Q} x^{|P|+|Q|-|\lambda|} \\ &= x^{-t}\left(\sum_P x^{|P|}\right)\left(\sum_Q x^{|Q|}\right), \end{aligned}$$

where P ranges over all reverse SSYT of shape λ and largest part at most c, while Q ranges over all reverse SSYT of shape λ and largest part at most r. The sum over P is thus just $s_\lambda(x, x^2, \ldots, x^r)$, while the sum over Q is $s_\lambda(x, x^2, \ldots, x^c)$, and the proof follows.

b. The proof is parallel to that of (a), using the correspondence $\pi \mapsto \sigma$ defined in the second proof of Theorem 7.20.4.

7.98. a. This formula can be proved by generalizing the proof of Theorem 7.22.1 (the Hillman–Grassl algorithm). See E. R. Gansner, *J. Combinatorial Theory (A)* **30** (1981), 71–89 (Thm. 5.1).

b. See the above reference (Thm. 6.1).

7.99. From the proof of Theorem 7.20.1 we have

$$\sum_{t,n\geq 0} K_t(n) q^t x^n = \sum_{a_{ij}} q^{\sum a_{ij}} x^{\sum (i+j-1)a_{ij}},$$

where each a_{ij} for $i, j \geq 1$ ranges over the set $\mathbb{N}$. Putting q/x for q gives

$$\sum_{t,n\geq 0} K_t(n+t)q^t x^n = \sum_{a_{ij}} q^{\sum a_{ij}} x^{\sum (i+j-2)a_{ij}}.$$

The condition that $n \leq t$ becomes

$$\sum (i+j-2)a_{ij} \leq \sum a_{ij},$$

or equivalently,

$$a_{11} \geq \sum{}' (i+j-3)a_{ij},$$

where $\sum'$ indicates that the term $(i, j) = (1, 1)$ is missing. Hence

$$\begin{aligned}
\sum_{0\leq n\leq t} K_t(n+t)q^t x^n &= \sum_{a_{11}\geq \sum'(i+j-3)a_{ij}} q^{\sum a_{ij}} x^{\sum (i+j-2)a_{ij}} \\
&= \frac{1}{1-q} \sum_{a_{ij}\,:\,(i,j)\neq(1,1)} q^{\sum (i+j-2)a_{ij}} x^{\sum (i+j-2)a_{ij}} \\
&= \frac{1}{1-q} \prod_{i,j}{}' \left(\sum_{a_{ij}\geq 0} (qx)^{(i+j-2)a_{ij}} \right) \\
&= \frac{1}{1-q} \prod_{i,j}{}' \frac{1}{1-(qx)^{i+j-2}} \\
&= \frac{1}{1-q} \prod_{k\geq 1} \frac{1}{[1-(qx)^k]^{k+1}}.
\end{aligned}$$

From this the desired conclusion is immediate.

This result was first proved (by a more complicated method) in R. Stanley, *J. Combinatorial Theory (A)* **14** (1973), 53–64 (Cor. 5.3(v)).

The result of this exercise is equivalent to the formula

$$K_t(n+t) = \sum_{k=0}^{n} p(k)a(n-k), \qquad 0 \leq n \leq t,$$

where $p(k)$ denotes the number of partitions of k and $a(n-k)$ the number of plane partitions of $n-k$. Is there a direct bijective proof?

7.100. **a.** Let $A \xrightarrow{\text{RSK}} (P, Q)$. Proposition 7.23.10 tells us the first row of P. Theorem 7.23.16 then allows us to describe the first column of P directly in terms of A. Using Theorem 7.13.1 we can then describe the first column of Q also in terms of A. These are all the ingredients necessary for the proof, though we omit the details.

b. The result of (a) applies equally well to the "reverse" version of the RSK algorithm used in the proof of Theorem 7.20.1. Hence the merged plane partitions $\pi(P, Q)$ and $\pi(P', Q')$ (as defined in the proof of Theorem 7.20.1) will have the same first column, so the row conjugates $\pi'(P, Q)$ and

$\pi'(P', Q')$ will have the same shape. In other words, the shape $\mathrm{sh}(\pi')$ depends only on the support $\mathrm{supp}(A)$. Since under the correspondence $A \mapsto \pi'$ we have $\mathrm{tr}(\pi') = \sum a_{ij}$, it follows that there is a collection $\mathcal{S}$ of finite subsets of $\mathbb{P} \times \mathbb{P}$ such that the following condition holds: $\mathrm{sh}(\pi') \subseteq \lambda$ and $\mathrm{tr}(\pi') = n$ if and only if $\mathrm{supp}(A) \in \mathcal{S}$ and $\sum a_{ij} = n$. If $S \subset \mathbb{P}^2$ with $\#S = k$, then the number of $\mathbb{N}$-matrices of support S and element sum n is just $\binom{n-1}{k-1}$, the number of compositions of n into k parts. Thus we get

$$t_\lambda(n) = \sum_{S \in \mathcal{S}} \binom{n-1}{\#S - 1},$$

which is a polynomial in n. Given that $t_\lambda(n)$ is a polynomial, there are several ways to see that its degree is $|\lambda| - 1$. For instance, Theorem 4.5.4 (or the more general Corollary 7.19.5) allows the generating function $\sum_n t_\lambda(n)x^n$ to be expressed as a sum of f^λ rational functions of the form $x^a/(1-x^{b_1})\cdots(1-x^{b_m})$ where $\lambda \vdash m$, from which it is immediate that $\deg t_\lambda = |\lambda| - 1$. See E. R. Gansner, *Illinois J. Math.* **25** (1981), 533–554.

c. The condition that π' fits in an $a \times b$ rectangle is equivalent (by the proof of Theorem 7.20.1) to $\max\{j : (i, j) \in \mathrm{supp}(A)\} \le b$ and $\max\{i : (i, j) \in \mathrm{supp}(A)\} \le a$. Hence $t_{\langle b^a \rangle}(n)$ is equal to the number of $a \times b$ $\mathbb{N}$-matrices whose entries sum to n, which is just $\binom{ab+n-1}{ab-1}$.

7.101. Equation (7.193) was stated without proof (with a misprint) by R. Stanley, in *Combinatoire et Représentation du Groupe Symétrique (Strasbourg 1976)*, Lecture Notes in Math. **579**, Springer-Verlag, Berlin/Heidelberg/New York, 1977, pp. 217–251 (Thm. 4.3(b)). The first proof was given by R. A. Proctor, in *Lie Algebras and Related Topics* (D. J. Britten, F. W. Lemire, and R. V. Moody, eds.), CMS Conf. Proc. **5**, American Mathematical Society, Providence, 1986, pp. 357–360, and *Invent. Math.* **92** (1988), 307–332 (Cor. 4.1). Proctor actually proves the case $d = 1$ of equation (7.194), and later states (immediately after Cor. 4.1) equation (7.194) in its full generality. Proctor's proof is based on representation theory; the number $f_n(m)$ is in fact the dimension of the irreducible representation of the symplectic group $\mathrm{Sp}(2(n-1), \mathbb{C})$ (or Lie algebra $\mathfrak{sp}(2(n-1), \mathbb{C})$) with highest weight $m\lambda_{n-1}$, where λ_{n-1} denotes the $(n-1)$-st fundamental weight. (See also Exercise 6.25(c).) Proctor proves the more general case $d = 1$ of (7.194) also using representation theory, but when $M - \ell$ is even this involves the construction of a non-semisimple analogue $\mathfrak{sp}(2n+1, \mathbb{C})$ of the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$. Proctor's unpublished proof of the general case of (7.194) uses entirely different techniques, viz., the evaluation of the $q = 1$ case of MacMahon's determinantal expression (P. A. MacMahon, *Phil. Trans. Roy. Soc. London (A)* **211** (1911), 345–373 (p. 367); *Collected Works*, vol. 1 (G. E. Andrews, ed.), MIT Press, Cambridge, Massachusetts, 1978, pp. 1406–1434 (p. 1428)) for the polynomial $\sum_\pi q^{|\pi|}$, summed over all plane partitions, allowing 0 as a part, of an arbitrary shape μ. Subsequently a much more general determinant evaluation was given by C. Krattenthaler, *Manuscripta Math.* **69** (1990), 173–202, and in *Number-Theoretic Analysis* (H. Hlawka and R. F. Tichy, eds.), Lecture Notes in Math. **1452**, Springer-Verlag, Berlin/Heidelberg/New York, 1990, pp. 121–131 (though in the special case of (7.194) Krattenthaler does not state the result in the elegant form we have given, due to Proctor). For some further applications of representation theory to the enumeration of plane partitions, see R. A. Proctor, *Europ. J. Combinatorics* **11** (1990), 289–300.

7.102. **a.** By Theorem 7.21.2 and Corollary 7.21.3 we have

$$t_\lambda(q) = s_\lambda(1, q, \ldots, q^n),$$

which clearly has the desired properties.

b. By the Jacobi–Trudi identity (Theorem 7.16.1) and the specialization $h_i(1, q, \ldots) = 1/[i]!$ (where $[i]! = (1-q)(1-q^2)\cdots(1-q^i)$), we have

$$t_{\lambda/\mu,n}(q) = \prod_{u\in\lambda/\mu}\left(1 - q^{n+c(u)}\right)\cdot\det\left(\frac{1}{[\lambda_i - \mu_j - i + j]!}\right)$$

$$= \prod_{u\in\lambda/\mu}\left(1 - q^{n+c(u)}\right)\cdot\sum_{w\in\mathfrak{S}_n}\varepsilon_w\prod_i\frac{1}{\left[\lambda_i - \mu_{w(i)} - i + w(i)\right]!}.$$

It is not hard to see that

$$\prod_{u\in\lambda/\mu}\left(1 - q^{n+c(u)}\right)\cdot\prod_i\frac{1}{[\lambda_i - \mu_j - i + j]!} = \prod_i\binom{n + \lambda_i - i}{\lambda_i - \mu_j - i + j}.$$

It follows that

$$t_{\lambda/\mu,n} = \det\left(\binom{n + \lambda_i - i}{\lambda_i - \mu_j - i + j}\right). \tag{7.225}$$

The proof now follows from a straightforward application of Theorem 2.7.1. (In fact, equation (7.225) is a specialization of a result known as the "Jacobi–Trudi identity for flag Schur functions," due to I. M. Gessel and appearing in M. L. Wachs, *J. Combinatorial Theory (A)* **40** (1985), 276–289 (Thm. 3.5).) The proof we have just given is due to H. L. Wolfgang (private communication, 13 November 1996). A version of the proof, based on the theory of Schubert polynomials, appears in S. C. Billey, W. Jockusch, and R. Stanley,
* *J. Algebraic Combinatorics* **2** (1993), 345–375 (Thm. 3.1). Is there a "nice" bijective proof?

7.103. **a.** This result was conjectured by I. G. Macdonald and proved by J. R. Stembridge, *Advances in Math.* **111** (1995), 227–243.

b. This result was conjectured by D. P. Robbins and R. Stanley, and proved by G. E. Andrews, *J. Combinatorial Theory (A)* **66** (1994), 28–39.

Both (a) and (b) (as well as Theorem 7.20.4 and Exercise 7.106(b)) are part of the subject of the enumeration of symmetry classes of plane partitions. For an overview of this subject (written when most of the current theorems were conjectures), see R. Stanley, *J. Combinatorial Theory (A)* **43** (1986), 103–113; Erratum, **44** (1987), 310. All ten symmetry classes discussed in this paper have now been enumerated, though the q-enumeration of totally symmetric plane partitions (i.e., the plane partitions of (a)) remains open. For a recent paper with further references, see G. Kuperberg, *J. Combinatorial*

Theory (A) **75** (1996), 295–315. For an entertaining account of the numbers $B(r)$ of (b), see D. P. Robbins, *Math. Intelligencer* **13** (1991), 12–19.

c. This intriguing and surprisingly difficult result was conjectured by W. H. Mills, D. P. Robbins, and H. Rumsey, Jr., *Invent. Math.* **66** (1982), 73–87 (Conjecture 1), and *J. Combinatorial Theory (A)* **34** (1983), 340–359 (Conjecture 1). It was first proved by D. Zeilberger, *Electron. J. Combinatorics* **3** (1996), R13, 84 pp.; also published in *The Foata Festschrift* (J. Désarménien, A. Kerber, and V. Strehl, eds.), Imprimerie Louis-Jean, Gap, France, 1996, pp. 289–372. A simpler proof was later given by G. Kuperberg, *Int. Math. Res. Notices (3)*, 1996, 139–150. For more information concerning this result, see the paper of Robbins cited in (b). For a textbook devoted to symmetry classes of plane partitions, monotone triangles, and related topics, see D. M. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press and Mathematical Association of America, to appear.

d. If $w = w_1 \cdots w_n \in \mathfrak{S}_n$, then associate with w the monotone triangle $\mathrm{mt}(w) = (a_{ij}(w))_{1\le i\le j\le n}$ whose i-th row consists of the numbers $w_1, w_2, \ldots, w_{n-i+1}$ in increasing order. It was shown by C. Ehresmann, *Ann. Math.* **35** (1934), 396–443 (§20) that the set $\{\mathrm{mt}(w) : w \in \mathfrak{S}_n\}$, ordered componentwise, is isomorphic to P_n. Given triangular arrays $a = (a_{ij})$ and $b = (b_{ij})$, define the *meet* $a \wedge b$ to be the triangular array $(\min\{a_{ij}, b_{ij}\})$. It is not difficult to check that the set of all arrays obtained by repeatedly taking meets of the triangles $\mathrm{mt}(w)$, $w \in \mathfrak{S}_n$, coincides with the set of monotone triangles. Hence $L(P_n)$ is a completion of P_n, and it is not hard to show that it is in fact the MacNeille completion. The surprising formula $\#L(P_n) = M(n)$ is due to A. Lascoux and M. P. Schützenberger, *Electron. J. Combinatorics* **3** (1996), R27, 35 pp.; also published in *The Foata Festschrift* (J. Désarménien, A. Kerber, and V. Strehl, eds.), Imprimerie Louis-Jean, Gap, France, 1996, pp. 653–685. Note also the unexpected fact that $L(P_n)$ is a distributive lattice.

7.104. This is a result of E. M. Wright, *Quart. J. Math. Oxford (2)* **2** (1931), 177–189.

7.105. Yes if $n < 14$, but no for $n = 14$, an example being $\lambda = (5, 5, 2, 1, 1)$ and $\mu = (4, 4, 3, 1, 1, 1)$. These results are due to L. A. Shepp, private communication, 1975.

7.106. a. It is a straightforward application of the Littlewood–Richardson rule (Appendix 1, Section A1.3) that

$$s_\nu^2 = \sum_{\lambda\subseteq\langle c^r\rangle} s_{(c+\lambda_1, c+\lambda_2,\ldots,c+\lambda_r, c-\lambda_r, c-\lambda_{r-1},\ldots,c-\lambda_1)}. \tag{7.226}$$

A direct bijective proof using jeu de taquin (essentially a proof of the Littlewood–Richardson rule in the special case s_ν^2) can also be given; see R. Stanley, *J. Combinatorial Theory (A)* **43** (1986), 103–113; Erratum, **44** (1987), 310.

b. Let π be a $(2r, 2c, 2t)$-self-complementary plane partition. Add $2r - i$ to every entry of the i-th row of π to obtain a reverse column-strict plane partition σ of shape $\langle (2c)^{2r}\rangle$, allowing 0 as a part, with largest part at most $2r + 2t - 1$.

Let τ be the subarray of σ consisting of all entries less than $r+t$. Then τ is a (rotated) SSYT (allowing 0 as a part) with largest part at most $r+t-1$ and whose shape is one of the partitions μ such that s_μ is a term of the sum in equation (7.226). Conversely, given such an SSYT, we can reverse the steps to obtain a $(2r, 2c, 2t)$-self-complementary plane partition. It follows that

$$G(2r, 2c, 2t) = s_{\langle c^r\rangle}(1^{r+t})^2.$$

But $s_{\langle c^r\rangle}(1^{r+t}) = F(r, c, t)$, since an SSYT with $\leq r$ rows, $\leq c$ columns, and largest part $\leq r+t-1$ (allowing 0 as a part) can be converted to a plane partition with $\leq r$ rows, $\leq c$ columns, and largest part $\leq r+t$ by subtracting $r-i$ from the entries in the i-th row and rotating 180°. Hence we get equation (7.195).

In a similar manner we obtain

$$G(2r+1, 2c, 2t) = F(r, c, t)F(r+1, c, t)$$
$$G(2r+1, 2c+1, t) = F(r+1, c, t)F(r, c+1, t).$$

These results appear in R. Stanley, *ibid.* (eqns. (3a)–(3c)).

7.107. a. Let $\mu^{[n]}$ denote the partition whose diagram is an $n\times n$ square (with n sufficiently large) with the shape μ removed from the bottom right-hand corner. By Exercise 7.41 we have

$$s_{\mu^{[n]}}(x_1, \ldots, x_n) = (x_1\cdots x_n)^n s_\mu\left(x_1^{-1}, \ldots, x_n^{-1}\right).$$

Put $x_i = q^{i-1}$. By Theorem 7.21.2 we get

$$q^k \prod_{u\in\mu^{[n]}} \frac{1-q^{n+c(u)}}{1-q^{h(u)}} = \prod_{v\in\mu} \frac{1-q^{-n-c(v)}}{1-q^{-h(v)}}$$

for some $k \in \mathbb{Z}$. The right-hand side is equal to $q^m \prod_{v\in\mu}(1-q^{n+c(v)})/(1-q^{h(v)})$ for some $m\in\mathbb{Z}$, so

$$q^{k-m} \prod_{u\in\mu^{[n]}} \frac{1-q^{n+c(u)}}{1-q^{h(u)}} = \prod_{v\in\mu} \frac{1-q^{n+c(v)}}{1-q^{h(v)}}.$$

Putting $q=0$ shows that $k-m=0$. Now as $n\to\infty$, it is easily seen that

$$\prod_{u\in\mu^{[n]}}\left(1-q^{n+c(u)}\right) \to \prod_{i\geq 1}(1-q^i)^i$$
$$\prod_{u\in\mu^{[n]}}\left(1-q^{h(u)}\right) \to \prod_{u\in A_\mu}\left(1-q^{h(u)}\right)$$
$$\prod_{v\in\mu}\left(1-q^{n+c(v)}\right) \to 1,$$

and the proof follows.

A bijective proof of this exercise was first given by D. E. White (private communication). We sketch another such proof (found in collaboration with C. Bessenrodt) using the binary sequence coding C_μ of μ explained in Exercise 7.59. By considering the sequences C_μ and C_{A_μ} together with Exercise 7.59(a), one sees that we need to prove bijectively the following.

Lemma. *Let C be a binary sequence $\cdots c_{-1}c_0c_1\cdots$ beginning with infinitely many 0's and ending with infinitely many 1's. For each $p \geq 1$, let $r_p(C)$ be the number of integers i such that $c_i = 0$ and $c_{i+p} = 1$, and let $s_p(C)$ be the number of integers i such that $c_i = 1$ and $c_{i+p} = 0$. Then $r_p(C) = p + s_p(C)$.*

Proof. First note that the case $p = 1$ is easy to prove bijectively. But when we apply the case $p = 1$ to each of the subsequences $C^j = \{c_{pi+j}\}_{i\in\mathbb{Z}}$, where $0 \leq j < p$, then we obtain the stated result. □

C. Bessenrodt has observed that the present exercise is equivalent to the statement that for each $p \geq 1$, the number of ways to add a border strip of size p to μ is exactly p more than the number of border strips of μ of size p. Note that the case $p = 1$ is the familiar fact (see Exercise 3.22(c)) that in Young's lattice the element μ is covered by one more element than it covers. The general case then follows from the isomorphism $Y_{p,\emptyset} \xrightarrow{\cong} Y^p$
* of Exercise 7.59(e). For related work see C. Bessenrodt, On hooks of Young diagrams, preprint.

b. A weak *reverse* plane partition of shape $\mu^{[n]}$, rotated 180°, is just a skew plane partition of shape $\langle n^n\rangle/\mu$. Hence by Theorem 7.22.1,

$$\sum_\pi q^{|\pi|} = \frac{1}{\prod_{u\in\langle n^n\rangle/\mu}[h(u)]}.$$

Now let $n \to \infty$ and use (a).

c. Such a proof was given by K. Kadell, *J. Combinatorial Theory (A)* **77** (1997), 110–133 (§6).

7.108. The only partition of $p+q$ with largest part p and with q parts is $\langle p, 2, 1^{q-2}\rangle$. Hence by Theorem 7.23.13, we have

$$\begin{aligned} F(p,q) &= \left(f^{\langle p,2,1^{q-2}\rangle}\right)^2 \\ &= \left(\frac{(p+q)!}{pq(p+q-1)(p-2)!(q-2)!}\right)^2. \end{aligned}$$

7.109. a. Immediate from Corollary 7.23.12.

b. This result was first shown by J. M. Hammersley, in *Proc. Sixth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, University of California Press, Berkeley/Los Angeles, 1972, pp. 345–394 (Thm. 4), using subadditive ergodic theory.

c. If $w = w_1w_2\cdots w_n \in \mathfrak{S}_n$, then let $w^r = w_n\cdots w_2w_1$. It follows from Example 7.23.19(a) that $\mathrm{is}(w)\cdot\mathrm{is}(w^r) \geq n$. Hence (using the arithmetic–

geometric-mean inequality),

$$\begin{aligned} E(n) &= \frac{1}{n!}\frac{1}{2}\sum_{w\in\mathfrak{S}_n}[\mathrm{is}(w)+\mathrm{is}(w^r)] \\ &\geq \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\sqrt{\mathrm{is}(w)\cdot\mathrm{is}(w^r)} \\ &\geq \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\sqrt{n} \\ &= \sqrt{n}, \end{aligned}$$

and the proof follows. By a more sophisticated argument Hammersley, *ibid.* (p. 360), shows that $\alpha \geq \pi/2 = 1.57\cdots$, and explains how this bound can be improved to $\sqrt{8/\pi} = 1.59\cdots$ (also done independently by D. H. Blackwell).

d. The number of subsequences of $w \in \mathfrak{S}_n$ of length k is $\binom{n}{k}$, and the probability that a specified one of them is increasing is $1/k!$. Hence the probability that $\mathrm{is}(w) \geq k$ cannot exceed $\frac{1}{k!}\binom{n}{k}$, and the proof can be completed by a judicious use of Stirling's formula. This argument is due to Hammersley, *ibid.* (Thm. 6).

e. This result was proved independently by B. F. Logan and L. A. Shepp, *Advances in Math.* **26** (1977), 206–222, and by A. M. Vershik and S. V. Kerov, *Dokl. Akad. Nauk SSSR* **233** (1977), 1024–1027, English translation in *Soviet Math. Dokl.* **18** (1977), 527–531.

f. Roughly speaking, most of the contribution to the sum on the right-hand side of (7.196) comes from terms indexed by λ "near" $\tilde{\lambda}^n$. Moreover, since $\sum_{\lambda\vdash n}(f^\lambda)^2 = n!$ and the number of terms of this sum is small compared with $n!$, we see that $(f^{\tilde{\lambda}^n})^2$ is "near" $n!$. Thus the largest part $(\tilde{\lambda}^n)_1$ of $\tilde{\lambda}^n$ is "near" $\alpha\sqrt{n}$. Since $\lim_{x\to 0} f(x) = 2$, it follows that $(\tilde{\lambda}^n)_1$ is asymptotically at least as large as $2\sqrt{n}$, so $\alpha \geq 2$. For rigorous treatments of this argument, see the two papers cited in (e) above.

g. See A. M. Vershik and S. V. Kerov, *ibid.*

Much further work has been subsequently done on the problems of estimating $E(n)$ and describing $\tilde{\lambda}^n$, and the closely related problem of finding the "typical" shape of a permutation $w \in \mathfrak{S}_n$ (i.e., the shape of the SYT P or Q obtained from w *via* the RSK algorithm). See for instance S. V. Kerov and A. M. Vershik, *SIAM J. Alg. Disc. Meth.* **7** (1986), 116–124; J. M. Steele, in *Discrete Probability and Applications (Minneapolis, MN, 1993)*, IMA Vol. Math. Appl. **72**, Springer, New York, 1995, pp. 111–131; D. Aldous and P. Diaconis, *Probab. Theory Related Fields* **103** (1995), 199–213; J. H. Kim, *J. Combinatorial Theory (A)* **76** (1996), 148–155; T. Seppäläinen, *Electron. J. Probab.* **1** (1996), no. 5, 51 pp.; B. Bollobás and S. Janson, in *Combinatorics, Geometry and Probability (Cambridge, 1993)*, Cambridge University Press, Cambridge, 1997, pp. 121–128; and J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of a random permutation, preprint dated May 11, 1998. In this last paper the following remarkable result is proved. Let $u(x)$ be the (unique) solution of the Painlevé II equation

$$u_{xx} = 2u^3 + xu, \quad \text{and } u \sim \mathrm{Ai}(x) \text{ as } x \to \infty,$$

where $\mathrm{Ai}(x)$ is the Airy function. Let L_n denote the length of the longest increasing subsequence of a random permutation $w \in \mathfrak{S}_n$. Then

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \le t\right) = \exp\left(-\int_t^\infty (x-t)u(x)^2 dx\right).$$

In particular, the variance of L_n is an explicit constant times $n^{1/3}$.

7.110. From equation (7.96) there follows

$$\begin{aligned}
&\frac{1-q}{q}\sum_{m\ge 0} s_\lambda(\underbrace{1-q,\dots,1-q}_{m})q^m \\
&\quad = \frac{\sum_{\mathrm{sh}\,T=\lambda} q^{d(T)+1}}{(1-q)^{|\lambda|+1}} \\
\implies \quad Z &= \sum_\lambda \left(\sum_{\mathrm{sh}\,T=\lambda} q^{d(T)}\right) s_\lambda \\
&= \frac{1-q}{q}\sum_\lambda s_\lambda \sum_{m\ge 0} s_\lambda(1-q,\dots,1-q)q^m \\
&= \frac{1-q}{q}\sum_{m\ge 0} q^m \sum_\lambda s_\lambda \cdot s_\lambda(1-q,\dots,1-q) \\
&= \frac{1-q}{q}\sum_{m\ge 0} q^m \prod_i \frac{1}{[1-x_i(1-q)]^m},
\end{aligned}$$

the last step by the Cauchy identity (Theorem 7.12.1). Note that

$$\prod_i \frac{1}{1-x_i(1-q)} = \sum_{n\ge 0}(1-q)^n s_n.$$

Hence

$$Z = \frac{1-q}{q}\,\frac{1}{1-\frac{1}{\sum_{n\ge 0}(1-q)^n s_n}} = \frac{\sum_{n\ge 0}(1-q)^n s_n}{1-q\sum_{n\ge 1}(1-q)^{n-1}s_n}. \qquad (7.227)$$

Now note that from the generating function for the Eulerian polynomials given in Exercise 3.81(c) we get

$$\begin{aligned}
1+\frac{1}{q}\sum_{\ell\ge 1} A_\ell(q)\frac{x^\ell}{\ell!} &= \frac{(1-q)e^{(1-q)x}}{1-qe^{(1-q)x}} \\
&= \frac{e^{(1-q)x}}{1-q\sum_{\ell\ge 1}(1-q)^{\ell-1}\frac{x^\ell}{\ell!}}.
\end{aligned}$$

Letting $H(z) = \sum_{n\geq 0} s_n z^n$, the right-hand side of (7.227) can be written as

$$Z = \frac{(1-q)H(1-q)}{1-qH(1-q)}.$$

By equation (7.198) we have

$$H(z) = \exp\left(p_1 z + p_2 \frac{z^2}{2} + \cdots\right),$$

so we get

$$Z = \frac{1}{q}\sum_{\ell\geq 1} A_\ell(q) \frac{\left[p_1(1-q) + \frac{1}{2}p_2(1-q)^2 + \cdots\right]^\ell}{\ell!}.$$

Expanding by the multinomial theorem shows that the coefficient of p_λ is

$$z_\lambda^{-1} q^{-1}(1-q)^{n-\ell} A_\ell(q) p_\lambda,$$

as desired. This argument was done in collaboration with I. M. Gessel.

7.111. a. We have $X = \mathfrak{S}_n$. Hence the formula $Z_X = n! h_n$ is equivalent to equation (7.22).

b. Let $Y_i = \{w \in \mathfrak{S}_n : w(n) = i\}$. Clearly $\tilde{Z}_{Y_1} = \tilde{Z}_{Y_2} = \cdots = \tilde{Z}_{Y_{n-1}}$, while by (a) we have $\tilde{Z}_{Y_n} = (n-1)!\, h_{n-1}h_1$. But

$$n!\, h_n = \tilde{Z}_{[n]\times[n]} = \tilde{Z}_{Y_1} + \cdots + \tilde{Z}_{Y_n},$$

so

$$\begin{aligned} \tilde{Z}_X &= \tilde{Z}_{[n]\times[n]} - \tilde{Z}_{Y_1} \\ &= n!\, h_n - \frac{1}{n-1}[n!\, h_n - (n-1)!\, h_{n-1}h_1] \\ &= n(n-2)(n-2)!\, h_n + (n-2)!\, h_{n-1}h_1. \end{aligned}$$

c. The h-positivity of $\tilde{Z}_X$ is equivalent to Exercise 7.47(l) in the case when the complement of G is bipartite. See the solution to that exercise for references. It follows from this solution that moreover the only h_λ's appearing in $\tilde{Z}_X$ are of the form $h_j h_{n-j}$.

d. This result is equivalent to Exercise 7.47(j) in the special case that P is also $(\mathbf{2}+\mathbf{2})$-free. (It is also a special case of Exercise 7.92(a).) See R. Stanley and J. R. Stembridge, *J. Combinatorial Theory (A)* **62** (1993), 261–279 (§5). The weaker result that $\tilde{Z}_X$ is s-positive follows from Exercise 7.47(h) and also from Exercise 7.92(a).

e. In equation (7.186) put $k=3$, $x^{(1)}=x$, $x^{(2)}=y$, $p_1(x^{(3)})=0$, $p_2(x^{(3)}) = p_3(x^{(3)}) = \cdots = 1$, $w_1 = v$, and $w_2 = w$. Using equation (7.209) we get

$$\sum_{\lambda\vdash n} H_\lambda \frac{d_\lambda}{n!} s_\lambda(x) s_\lambda(y) = \frac{1}{n!}\sum_{vw\in\mathfrak{D}_n} p_{\rho(v)}(x) p_{\rho(w)}(y).$$

Hence (since $\langle p_\mu, p_\nu\rangle = z_\mu \delta_{\mu\nu}$),

$$\tilde{Z}_{B_w}(x) = \left\langle \frac{1}{n!}\sum_{\lambda\vdash n} H_\lambda d_\lambda s_\lambda(x)s_\lambda(y),\ p_{\rho(w)}(y)\right\rangle_y,$$

where $\langle\,,\,\rangle_y$ indicates that we are taking the scalar product with respect to the y variables only. Since $p_\alpha = \sum_\lambda \chi^\lambda(\alpha)s_\lambda$ (Corollary 7.17.4) and $f^\lambda = n!/H_\lambda$ (Corollary 7.21.6), we get

$$\tilde{Z}_{B_w} = \sum_{\lambda\vdash n}(f^\lambda)^{-1}d_\lambda\chi^\lambda(w)s_\lambda,$$

as desired. This formula is a result of S. Okazaki, Ph.D. thesis, Massachusetts Institute of Technology, 1992 (Thm. 1.2).

f. Follows easily from (e) and Exercise 7.63(b). See Okazaki, *ibid.* (Thm. 1.6). Note that this result implies that $\tilde{Z}_{B_w}$ is s-positive when w is an n-cycle. In general, $\tilde{Z}_{B_w}$ need not be s-positive, e.g., if $w = \mathrm{id} \in \mathfrak{S}_2$, then $\tilde{Z}_{B_w} = s_2 - s_{11}$.

7.112. a. Let G be the subgroup of $\mathfrak{S}_n$ generated by the n-cycle $(1, 2, \ldots, n)$. A necklace with beads from an alphabet A is just an orbit of the action of G on $A^{[n]}$, the set of functions $[n] \to A$. Hence by Corollary 7.24.6, we have

$$N(k,n) = \frac{1}{n}\sum_{w\in G} k^{c(w)}.$$

Since G has $\phi(d)$ elements of cycle type $\langle d^{n/d}\rangle$, the proof follows. The enumeration (7.197) of necklaces is due to P. A. MacMahon, *Proc. London Math. Soc.* **23** (1892), 305–313 (p. 308); in *Collected Papers* (G. E. Andrews, ed.), MIT Press, Cambridge, 1978, pp. 468–476, and is a precursor of Pólya's theory of enumeration under group action. MacMahon mentions that the enumeration of necklaces according to the number of beads of each color (and hence including (b) as a special case) had earlier been done by M. E. Jablonski and M. Moreau, independently.

b. By Theorem 7.24.4, we want the coefficient of $x_1^n x_2^n$ in

$$Z_G(p_1, p_2, \ldots) = \frac{1}{2n}\sum_{d|2n}\phi(d)p_d^{2n/d}.$$

It is easy to see that this coefficient is just

$$\frac{1}{2n}\sum_{d|n}\phi(d)\binom{2n/d}{n/d}.$$

7.113. Let V be a p-element set, and let G be the group of permutations of $S = \binom{V}{2}$ induced by permutations of V, as in Example 7.24.2(b). By Example 7.24.3(b) we have

$$\sum_{i=0}^{\binom{p}{2}} g_i(p)q^i = Z_G(1,q).$$

By equation (7.120) we have

$$Z_G(1,q) = \sum_{\lambda \vdash \binom{p}{2}} a_\lambda s_\lambda(1,q),$$

where $a_\lambda \in \mathbb{N}$. By Exercise 7.75(c) each polynomial $s_\lambda(1,q)$ is symmetric and unimodal, with center of symmetry $\frac{1}{2}\binom{p}{2}$. Hence the same is true of $Z_G(1,q)$, and the proof follows.

This result (in a more general form that can be proved in the same way as above) is due to D. Livingstone and A. Wagner, *Math. Z.* **90** (1965), 393–403. For further references and ramifications, see R. Stanley, *Ann. New York Acad. Sci.* **576** (1989), 500–535 (esp. Thm. 10) and *Discrete Appl. Math.* **34** (1991), 241–277 (§3).

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