3

Partially Ordered Sets

3.1 Basic Concepts

The theory of partially ordered sets (or *posets*) plays an important unifying role in enumerative combinatorics. In particular, the theory of Möbius inversion on a partially ordered set is a far-reaching generalization of the Principle of Inclusion-Exclusion, and the theory of binomial posets provides a unified setting for various classes of generating functions. These two topics will be among the highlights of this chapter, though many other interesting uses of partially ordered sets will also be given.

To get a glimpse of the potential scope of the theory of partially ordered sets as it relates to the Principle of Inclusion-Exclusion, consider the following example. Suppose we have four finite sets A, B, C, D such that

$$D = A \cap B = A \cap C = B \cap C = A \cap B \cap C.$$

It follows from the Principle of Inclusion-Exclusion that

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \\ &= |A| + |B| + |C| - 2|D|. \end{aligned} \tag{3.1}$$

The relations $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ collapsed the general seven-term expression for $|A \cup B \cup C|$ into a four-term expression, since the collection of intersections of A, B, C has only four distinct members. What is the significance of the coefficient -2 in equation (3.1)? Can we compute such coefficients efficiently for more complicated sets of equalities among intersections of sets $A_1, \dots, A_n$? It is clear that the coefficient -2 depends only on the *partial order relation* among the distinct intersections A, B, C, D of the sets A, B, C – that is, on the fact that $D \subseteq A$, $D \subseteq B$, $D \subseteq C$ (where we continue to assume that $D = A \cap B = A \cap C = B \cap C = A \cap B \cap C$). In fact, we shall see that -2 is a certain value of the Möbius function of this partial order (with an additional element corresponding to the empty intersection adjoined). Hence, Möbius inversion results in a simplification

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of Inclusion-Exclusion under appropriate circumstances. However, we shall also see that the applications of Möbius inversion are much further-reaching than as a generalization of Inclusion-Exclusion.

Before plunging headlong into the theory of incidence algebras and Möbius functions, it is worthwhile to develop some feeling for the structure of finite partially ordered sets. Hence in the first five sections of this chapter, we collect together some of the basic definitions and results on the subject, though strictly speaking most of them are not needed in order to understand the theory of Möbius inversion.

A *partially ordered set* P (or *poset*, for short) is a set (which by abuse of notation we also call P), together with a binary relation denoted $\leq$ (or $\leq_P$ when there is a possibility of confusion), satisfying the following three axioms:

1. For all $t \in P$, $t \leq t$ (*reflexivity*).
2. If $s \leq t$ and $t \leq s$, then $s = t$ (*antisymmetry*).
3. If $s \leq t$ and $t \leq u$, then $s \leq u$ (*transitivity*).

We use the obvious notation $t \geq s$ to mean $s \leq t$, $s < t$ to mean $s \leq t$ and $s \neq t$, and $t > s$ to mean $s < t$. We say that two elements s and t of P are *comparable* if $s \leq t$ or $t \leq s$; otherwise, s and t are *incomparable**, denoted $s \parallel t$.

Before giving a rather lengthy list of definitions associated with posets, let us first look at some examples of posets of combinatorial interest that will later be considered in more detail.

3.1.1 Example. a. Let $n \in \mathbb{P}$. The set $[n]$ with its usual order forms an n-element poset with the special property that any two elements are comparable. This poset is denoted $\boldsymbol{n}$. Of course $\boldsymbol{n}$ and $[n]$ coincide as sets, but we use the notation $\boldsymbol{n}$ to emphasize the order structure.

b. Let $n \in \mathbb{N}$. We can make the set $2^{[n]}$ of all subsets of $[n]$ into a poset B_n by defining $S \leq T$ in B_n if $S \subseteq T$ as sets. One says that B_n consists of the subsets of $[n]$ "ordered by inclusion."

c. Let $n \in \mathbb{P}$. The set of all positive integer divisors of n can be made into a poset D_n in a "natural" way by defining $i \leq j$ in D_n if j is divisible by i (denoted $i|j$).

d. Let $n \in \mathbb{P}$. We can make the set Π_n of all partitions of $[n]$ into a poset (also denoted Π_n) by defining $\pi \leq \sigma$ in Π_n if every block of π is contained in a block of σ. For instance, if $n = 9$ and if π has blocks 137, 2, 46, 58, 9, and σ has blocks 13467, 2589, then $\pi \leq \sigma$. We then say that π is a *refinement* of σ and that Π_n consists of the partitions of $[n]$ "ordered by refinement."

e. In general, any collection of sets can be ordered by inclusion to form a poset. Some cases will be of special combinatorial interest. For instance, let $B_n(q)$ consist of all subspaces of the n-dimensional vector space $\mathbb{F}_q^n$, ordered by inclusion. We will see that $B_n(q)$ is a nicely behaved q-analogue of the poset B_n defined in (b).

* "Comparable" and "incomparable" are accented on the syllable "com."

We now list a number of basic definitions and results connected with partially ordered sets. Some readers may wish to skip directly to Section 3.6, and to consult the intervening material only when necessary.

Two posets P and Q are *isomorphic*, denoted $P \cong Q$, if there exists an *order-preserving bijection* $\phi\colon P \to Q$ whose inverse is order-preserving; that is,

$$s \le t \text{ in } P \iff \phi(s) \le \phi(t) \text{ in } Q.$$

For example, if B_S denotes the poset of all subsets of the set S ordered by inclusion, then $B_S \cong B_T$ whenever $\#S = \#T$.

Some care has to be taken in defining the notion of "subposet." By a *weak subposet* of P, we mean a subset Q of the elements of P and a partial ordering of Q such that if $s \le t$ in Q, then $s \le t$ in P. If Q is a weak subposet of P with $P = Q$ as sets, then we call P a *refinement* of Q. By an *induced subposet* of P, we mean a subset Q of P and a partial ordering of Q such that for $s,t, \in Q$ we have $s \le t$ in Q if and only if $s \le t$ in P. We then say the subset Q of P has the *induced order*. Thus, the finite poset P has exactly $2^{\#P}$ induced subposets. By a *subposet* of P, we will always mean an *induced* subposet. A special type of subposet of P is the (closed) *interval* $[s,t] = \{u \in P : s \le u \le t\}$, defined whenever $s \le t$. (Thus, the empty set is *not* regarded as a closed interval.) The interval $[s,s]$ consists of the single point s. We similarly define the *open* interval $(s,t) = \{u \in P : s < u < t\}$, so $(s,s) = \emptyset$. If every interval of P is finite, then P is called a *locally finite* poset. We also define a subposet Q of P to be *convex* if $t \in Q$ whenever $s < t < u$ in P and $s,u \in Q$. Thus, an interval is convex.

If $s,t \in P$, then we say that t *covers* s or s is *covered by* t, denoted $s \lessdot t$ or $t \gtrdot s$, if $s < t$ and no element $u \in P$ satisfies $s < u < t$. Thus t covers s if and only if $s < t$ and $[s,t] = \{s,t\}$. A locally finite poset P is completely determined by its cover relations. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $s < t$ then t is drawn "above" s (i.e., with a higher vertical coordinate). Figure 3.1 shows the Hasse diagrams of all posets (up to isomorphism) with at most four elements. Some care must be taken in "recognizing" posets from their Hasse diagrams. For instance,

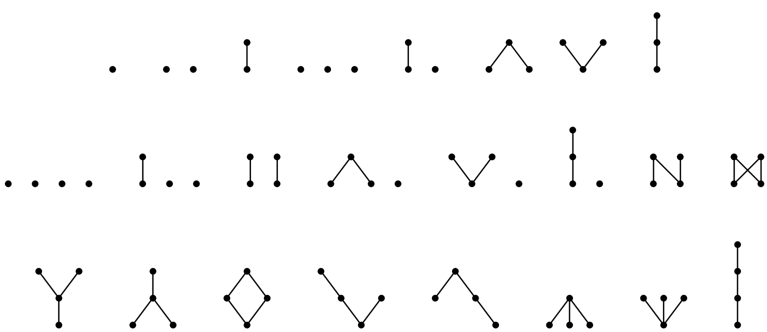

Figure 3.1 The posets with at most four elements.

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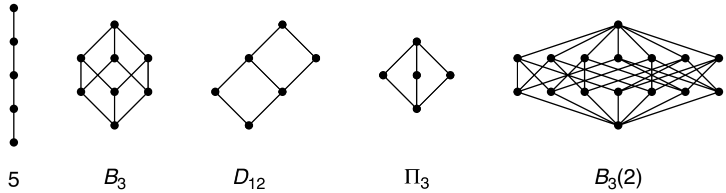


Figure 3.2 Some examples of posets.

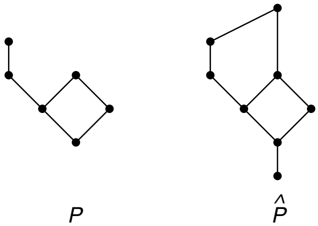
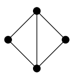


Figure 3.3 Adjoining a $\hat{0}$ and $\hat{1}$.

the graph  is a perfectly valid Hasse diagram, yet appears to be missing from Figure 3.1. We trust the reader will resolve this anomaly. Similarly, why does the graph  not appear above? Figure 3.2 illustrates the Hasse diagrams of some of the posets considered in Example 3.1.1.

We say that P *has a* $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $t \geq \hat{0}$ for all $t \in P$. Similarly, P *has a* $\hat{1}$ if there exists $\hat{1} \in P$ such that $t \leq \hat{1}$ for all $t \in P$. We denote by $\widehat{P}$ the poset obtained from P by adjoining a $\hat{0}$ and $\hat{1}$ (in spite of a $\hat{0}$ or $\hat{1}$ that P may already possess). See Figure 3.3 for an example.

A *chain* (or *totally ordered set* or *linearly ordered set*) is a poset in which any two elements are comparable. Thus, the poset $\boldsymbol{n}$ of Example 3.1.1(a) is a chain. A subset C of a poset P is called a *chain* if C is a chain when regarded as a subposet of P. The chain C of P is called *maximal* if it is not contained in a larger chain of P. The chain C of P is called *saturated* (or *unrefinable*) if there does not exist $u \in P - C$ such that $s < u < t$ for some $s,t \in C$ and such that $C \cup \{u\}$ is a chain. Thus, maximal chains are saturated, but not conversely. In a locally finite poset, a chain $t_0 < t_1 < \cdots < t_n$ is saturated if and only if $t_{i-1} \lessdot t_i$ for $1 \leq i \leq n$. The *length* $\ell(C)$ of a finite chain is defined by $\ell(C) = \#C - 1$. The length (or *rank*) of a finite poset P is

$$\ell(P) := \max\{\ell(C) : C \text{ is a chain of } P\}.$$

The length of an interval $[s,t]$ is denoted $\ell(s,t)$. If every maximal chain of P has the same length n, then we say that P is *graded of rank* n. In this case there is a unique *rank function* $\rho\colon P \to \{0,1,\ldots,n\}$ such that $\rho(s) = 0$ if s is a minimal element of P, and $\rho(t) = \rho(s) + 1$ if $t \gtrdot s$ in P. If $s \leq t$, then we also write $\rho(s,t) = \rho(t) - \rho(s) = \ell(s,t)$. If $\rho(s) = i$, then we say that s has *rank* i. If P is

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graded of rank n and has p_i elements of rank i, then the polynomial

$$F(P,x) = \sum_{i=0}^{n} p_i x^i$$

is called the *rank-generating function* of P. For instance, all the posets $\boldsymbol{n}$, B_n, D_n, Π_n, and $B_n(q)$ are graded. The reader can check the entries of the following table (some of which will be discussed in more detail later).

Poset P	Rank of $t \in P$	Rank of P
$\boldsymbol{n}$	$t-1$	$n-1$
B_n	$\operatorname{card} t$	n
D_n	number of prime divisors of t (counting multiplicity)	number of prime divisors of n (counting multiplicity)
Π_n	$n-\lvert t\rvert$	$n-1$
$B_n(q)$	$\dim t$	n

The rank-generating functions of these posets are as follows. For D_n, let $n = p_1^{a_1} \cdots p_k^{a_k}$ be the prime power factorization of n. We write, for example, $\boldsymbol{(n)}_x$ for the q-analogue $\boldsymbol{(n)}$ of n in the variable x, so

$$\boldsymbol{(n)}_x = \frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1}.$$

$$\begin{aligned}
F(\boldsymbol{n},x) &= \boldsymbol{(n)}_x,\\
F(B_n,x) &= (1+x)^n,\\
F(D_n,x) &= \boldsymbol{(a_1+1)}_x \cdots \boldsymbol{(a_k+1)}_x,\\
F(\Pi_n,x) &= \sum_{i=0}^{n-1} S(n,n-i)x^i,\\
F(B_n(q),x) &= \sum_{i=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{i}} x^i.
\end{aligned}$$

We can extend the definition of a graded poset in an obvious way to certain infinite posets. Namely, we say that P is *graded* if it can be written $P = P_0 \cup P_1 \cup \cdots$ such that every maximal chain has the form $t_0 \lessdot t_1 \lessdot \cdots$, where $t_i \in P_i$. We then have a rank function $\rho\colon P \to \mathbb{N}$ just as in the finite case. If each P_i is finite then we also have a rank-generating function $F(P,q)$ as before, though now it may be a power series rather than a polynomial.

A *multichain* of the poset P is a chain with repeated elements; that is, a multiset whose underlying set is a chain of P. A *multichain of length* n may be regarded as a sequence $t_0 \le t_1 \le \cdots \le t_n$ of elements of P.

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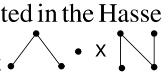
An *antichain* (or *Sperner family* or *clutter*) is a subset A of a poset P such that any two distinct elements of A are incomparable. An *order ideal* (or *semi-ideal* or *down-set* or *decreasing subset*) of P is a subset I of P such that if $t \in I$ and $s \leq t$, then $s \in I$. Similarly, a *dual order ideal* (or *up-set* or *increasing subset* or *filter*) is a subset I of P such that if $t \in I$ and $s \geq t$, then $s \in I$. When P is finite, there is a one-to-one correspondence between antichains A of P and order ideals I. Namely, A is the set of maximal elements of I, while

$$I = \{s \in P : s \leq t \text{ for some } t \in A\}. \tag{3.2}$$

The set of all order ideals of P, ordered by inclusion, forms a poset denoted $J(P)$. In Section 3.4 we shall investigate $J(P)$ in greater detail. If I and A are related as in equation (3.2), then we say that A *generates* I. If $A = \{t_1, \ldots, t_k\}$, then we write $I = \langle t_1, \ldots, t_k \rangle$ for the order ideal generated by A. The order ideal $\langle t \rangle$ is the *principal order ideal* generated by t, denoted Λ_t. Similarly V_t denotes the principal dual order ideal generated by t, that is, $V_t = \{s \in P : s \geq t\}$.

3.2 New Posets from Old

Various operations can be performed on one or more posets. If P and Q are posets on *disjoint* sets, then the *disjoint union* (or *direct sum*) of P and Q is the poset $P + Q$ on the union $P \cup Q$ such that $s \leq t$ in $P + Q$ if either (a) $s, t \in P$ and $s \leq t$ in P, or (b) $s, t \in Q$ and $s \leq t$ in Q. A poset that is not a disjoint union of two nonempty posets is said to be *connected*. The disjoint union of P with itself n times is denoted nP; hence, an n-element antichain is isomorphic to $n\mathbf{1}$. If P and Q are on disjoint sets as previously, then the *ordinal sum* of P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that $s \leq t$ in $P \oplus Q$ if (a) $s, t \in P$ and $s \leq t$ in P, or (b) $s, t \in Q$ and $s \leq t$ in Q, or (c) $s \in P$ and $t \in Q$. Hence, an n-element chain is given by $\boldsymbol{n} = \mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$ (n times). Of the 16 four-element posets, exactly one of them cannot be built up from the poset $\mathbf{1}$ using the operations of disjoint union and ordinal sum. Posets that *can* be built up in this way are called *series-parallel posets*. (See Exercises 3.14, 3.15(c), and 5.39 for further information on such posets.)

If P and Q are posets, then the *direct* (or *cartesian*) *product* of P and Q is the poset $P \times Q$ on the set $\{(s,t) : s \in P \text{ and } t \in Q\}$ such that $(s,t) \leq (s',t')$ in $P \times Q$ if $s \leq s'$ in P and $t \leq t'$ in Q. The direct product of P with itself n times is denoted P^n. To draw the Hasse diagram of $P \times Q$ (when P and Q are finite), draw the Hasse diagram of P, replace each element t of P by a copy Q_t of Q, and connect corresponding elements of Q_s and Q_t (with respect to some isomorphism $Q_s \cong Q_t$) if s and t are connected in the Hasse diagram of P. For instance, the Hasse diagram of the direct product  is drawn as indicated in Figure 3.4.

It is clear from the definition of the direct product that $P \times Q$ and $Q \times P$ are isomorphic. However, the Hasse diagrams obtained by interchanging P and Q

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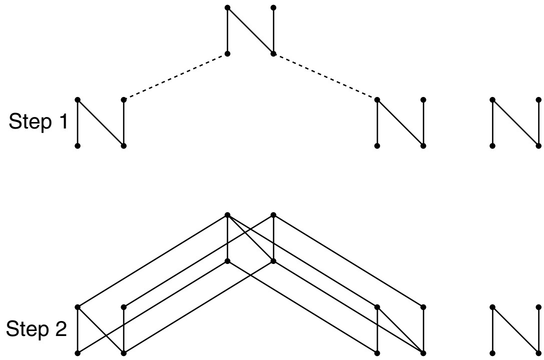

Figure 3.4 Drawing a direct product of posets.

in the above procedure in general look completely different, although they are of course isomorphic. If P and Q are graded with rank-generating functions $F(P,x)$ and $F(Q,x)$, then it is easily seen that $P \times Q$ is graded and

$$F(P \times Q, x) = F(P,x)F(Q,x). \tag{3.3}$$

A further operation on posets is the *ordinal product* $P \otimes Q$. This is the partial ordering on $\{(s,t) : s \in P \text{ and } t \in Q\}$ obtained by setting $(s,t) \leq (s',t')$ if (a) $s = s'$ and $t \leq t'$, or (b) $s < s'$. To draw the Hasse diagram of $P \otimes Q$ (when P and Q are finite), draw the Hasse diagram of P, replace each element t of P by a copy Q_t of Q, and then connect every maximal element of Q_s with every minimal element of Q_t whenever t covers s in P. If P and Q are graded and Q has rank r, then the analogue of equation (3.3) for ordinal products becomes

$$F(P \otimes Q, x) = F(P, x^{r+1})F(Q,x).$$

Note that in general $P \otimes Q$ and $Q \otimes P$ do not have the same rank-generating function, so in particular they are not isomorphic.

A further operation that we wish to consider is the *dual* of a poset P. This is the poset P^* on the same set as P, but such that $s \leq t$ in P^* if and only if $t \leq s$ in P. If P and P^* are isomorphic, then P is called *self-dual*. Of the 16 four-element posets, 8 are self-dual.

If P and Q are posets, then Q^P denotes the set of all order-preserving maps $f\colon P \to Q$; that is, $s \leq t$ in P implies $f(s) \leq f(t)$ in Q. We give Q^P the structure of a poset by defining $f \leq g$ if $f(t) \leq g(t)$ for all $t \in P$. It is an elementary exercise to check the validity of the following rules of *cardinal arithmetic* (for posets).

a. $+$ and $\times$ are associative and commutative.
b. $P \times (Q + R) \cong (P \times Q) + (P \times R)$.
c. $R^{P+Q} \cong R^P \times R^Q$.
d. $(R^P)^Q \cong R^{P \times Q}$.

3.3 Lattices

We now turn to a brief survey of an important class of posets known as *lattices*. If s and t belong to a poset P, then an *upper bound* of s and t is an element $u \in P$ satisfying $u \geq s$ and $u \geq t$. A *least upper bound* (or *join* or *supremum*) of s and t is an upper bound u of s and t such that every upper bound v of s and t satisfies $v \geq u$. If a least upper bound of s and t exists, then it is clearly unique and is denoted $s \vee t$ (read "s join t" or "s sup t"). Dually, one can define the *greatest lower bound* (or *meet* or *infimum*) $s \wedge t$ (read "s meet t" or "s inf t"), when it exists. A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound. One can also define a lattice axiomatically in terms of the operations $\vee$ and $\wedge$, but for combinatorial purposes this is not necessary. The reader should check, however, that in a lattice L:

a. The operations $\vee$ and $\wedge$ are associative, commutative, and idempotent (i.e., $t \wedge t = t \vee t = t$);
b. $s \wedge (s \vee t) = s = s \vee (s \wedge t)$ (absorption laws);
c. $s \wedge t = s \Leftrightarrow s \vee t = t \Leftrightarrow s \leq t$.

Clearly all finite lattices have a $\hat{0}$ and $\hat{1}$. If L and M are lattices, then so are L^*, $L \times M$, and $L \oplus M$. However, $L + M$ will never be a lattice unless one of L or M is empty, but $\widehat{L + M}$ (i.e., $L + M$ with an $\hat{0}$ and $\hat{1}$ adjoined) is always a lattice. Figure 3.5 shows the Hasse diagrams of all lattices with at most six elements.

In checking whether a (finite) poset is a lattice, it is sometimes easy to see that meets, say, exist, but the existence of joins is not so clear. Thus, the criterion of the next proposition can be useful. If every pair of elements of a poset P has a

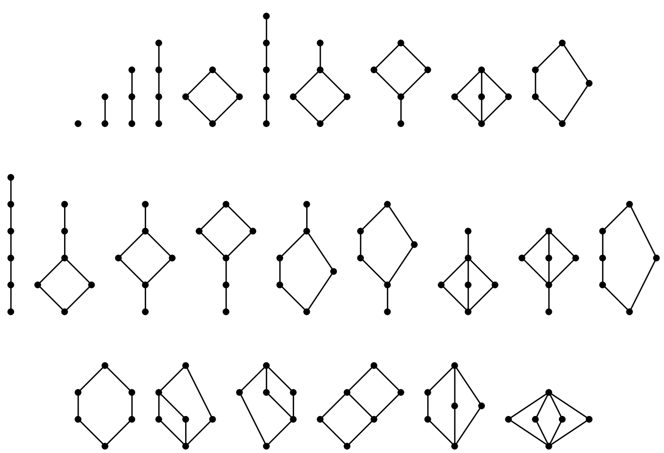

Figure 3.5 The lattices with at most six elements.

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meet (respectively, join), then we say that P is a *meet-semilattice* (respectively, *join-semilattice*).

3.3.1 Proposition. *Let P be a finite meet-semilattice with $\hat{1}$. Then P is a lattice. (Of course, dually a finite join-semilattice with $\hat{0}$ is a lattice.)*

Proof. If $s,t \in P$, then the set $S = \{u \in P : u \geq s \text{ and } u \geq t\}$ is finite (since P is finite) and nonempty (since $\hat{1} \in P$). Clearly by induction, the meet of finitely many elements of a meet-semilattice exists. Hence, we have $s \vee t = \bigwedge_{u \in S} u$. □

Proposition 3.3.1 fails for infinite lattices because an *arbitrary* subset of L need not have a meet or join. (See Exercise 3.26.) If in fact every subset of L does have a meet and join, then L is called a *complete lattice*. Clearly, a complete lattice has a $\hat{0}$ and $\hat{1}$.

We now consider one of the types of lattices of most interest to combinatorics.

3.3.2 Proposition. *Let L be a finite lattice. The following two condtions are equivalent.*

i. L is graded, and the rank function ρ of L satisfies

$$\rho(s) + \rho(t) \geq \rho(s \wedge t) + \rho(s \vee t)$$

for all $s,t \in L$.

ii. If s and t both cover $s \wedge t$, then $s \vee t$ covers both s and t.

Proof. (i)⇒(ii) Suppose that s and t cover $s \wedge t$. Then $\rho(s) = \rho(t) = \rho(s \wedge t) + 1$ and $\rho(s \vee t) > \rho(s) = \rho(t)$. Hence by (i), $\rho(s \vee t) = \rho(s) + 1 = \rho(t) + 1$, so $s \vee t$ covers both s and t.

(ii)⇒(i) Suppose that L is not graded, and let $[u,v]$ be an interval of L of minimal length that is not graded. Then there are elements s_1, s_2 of $[u,v]$ that cover u and such that all maximal chains of each interval $[s_i,v]$ have the same length ℓ_i, where $\ell_1 \neq \ell_2$. By (ii), there are saturated chains in $[s_i,v]$ of the form $s_i < s_1 \vee s_2 < t_1 < t_2 < \cdots < t_k = v$, contradicting $\ell_1 \neq \ell_2$. Hence, L is graded.

Now suppose that there is a pair $s,t \in L$ with

$$\rho(s) + \rho(t) < \rho(s \wedge t) + \rho(s \vee t), \tag{3.4}$$

and choose such a pair with $\ell(s \wedge t, s \vee t)$ minimal, and then with $\rho(s) + \rho(t)$ minimal. By (ii), we cannot have both s and t covering $s \wedge t$. Thus, assume that $s \wedge t < s' < s$, say. By the minimality of $\ell(s \wedge t, s \vee t)$ and $\rho(s) + \rho(t)$, we have

$$\rho(s') + \rho(t) \geq \rho(s' \wedge t) + \rho(s' \vee t). \tag{3.5}$$

Now $s' \wedge t = s \wedge t$, so equations (3.4) and (3.5) imply

$$\rho(s) + \rho(s' \vee t) < \rho(s') + \rho(s \vee t).$$

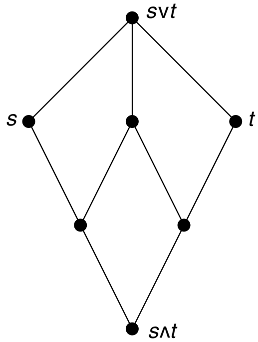

Figure 3.6 A semimodular but nonmodular lattice.

Clearly, $s \wedge (s' \vee t) \geq s'$ and $s \vee (s' \vee t) = s \vee t$. Hence, setting $S = s$ and $T = s' \vee t$, we have found a pair $S, T \in L$ satisfying $\rho(S) + \rho(T) < \rho(S \wedge T) + \rho(S \vee T)$ and $\ell(S \wedge T, S \vee T) < \ell(s \wedge t, s \vee t)$, a contradiction. This completes the proof. $\square$

A finite lattice satisfying either of the (equivalent) conditions of the previous proposition is called a *finite upper semimodular lattice*, or a just a *finite semimodular lattice*. The reader may check that of the 15 lattices with six elements, exactly 8 are semimodular.

A finite lattice L whose dual L^* is semimodular is called *lower semimodular*. A finite lattice that is both upper and lower semimodular is called a *modular lattice*. By Proposition 3.3.2, a finite lattice L is modular if and only if it is graded, and its rank function ρ satisfies

$$\rho(s) + \rho(t) = \rho(s \wedge t) + \rho(s \vee t) \quad \text{for all } s, t \in L. \tag{3.6}$$

For instance, the lattice $B_n(q)$ of subspaces (ordered by inclusion) of an n-dimensional vector space over the field $\mathbb{F}_q$ is modular, since the rank of a subspace is just its dimension, and equation (3.6) is then familiar from linear algebra. Every semimodular lattice with at most six elements is modular. There is a unique seven-element non-modular, semimodular lattice, which is shown in Figure 3.6. This lattice is not modular since $s \vee t$ covers s and t, but s and t don't cover $s \wedge t$. It can be shown that a finite lattice L is modular if and only if for all $s, t, u \in L$ such that $s \leq u$, we have

$$s \vee (t \wedge u) = (s \vee t) \wedge u. \tag{3.7}$$

This allows the concept of modularity to be extended to nonfinite lattices, though we will only be concerned with the finite case. Equation (3.7) also shows immediately that a sublattice of a modular lattice is modular. (A subset M of a lattice L is a *sublattice* if it is closed under the operations of $\wedge$ and $\vee$ in L.)

A lattice L with $\hat{0}$ and $\hat{1}$ is *complemented* if for all $s \in L$ there is a $t \in L$ such that $s \wedge t = \hat{0}$ and $s \vee t = \hat{1}$. If for all $s \in L$ the complement t is unique, then L is *uniquely complemented*. If every interval $[s, t]$ of L is itself complemented, then L is *relatively complemented*. An *atom* of a finite lattice L is an element covering

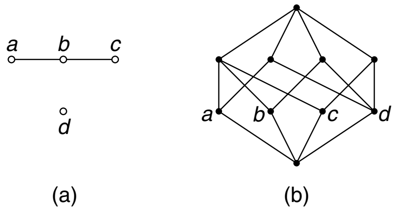

Figure 3.7 A subset S of the affine plane and its corresponding geometric lattice $L(S)$.

$\hat{0}$, and L is said to be *atomic* (or a *point lattice*) if every element of L is a join of atoms. (We always regard $\hat{0}$ as the join of an empty set of atoms.) Dually, a *coatom* is an element that $\hat{1}$ covers, and a *coatomic* lattice is defined in the obvious way. Another simple result of lattice theory, whose proof we omit, is the following.

3.3.3 Proposition. *Let L be a finite semimodular lattice. The following two conditions are equivalent.*

i. L is relatively complemented.
ii. L is atomic.

A finite semimodular lattice satisfying either of the two (equivalent) conditions (i) or (ii) is called a *finite geometric lattice*. A basic example is the following. Take any finite set S of points in some affine space (respectively, vector space) V over a field K (or even over a division ring). Then the subsets of S of the form $S \cap W$, where W is an affine subspace (respectively, linear subspace) of V, ordered by inclusion, form a geometric lattice $L(S)$. For instance, taking $S \subset \mathbb{R}^2$ (regarded as an affine space) to be as in Figure 3.7(a), then the elements of $L(S)$ consist of $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,d\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,c,d\}$. For this example, $L(S)$ is in fact modular and is shown in Figure 3.7(b).

NOTE. A geometric lattice is intimately related to the subject of *matroid theory*. A (finite) *matroid* may be defined as a pair $(S,\mathcal{I})$, where S is a finite set and $\mathcal{I}$ is a collection of subsets of S satisfying the two conditions:

- If $F \in \mathcal{I}$ and $G \subseteq F$, then $G \in \mathcal{I}$. In other words, $\mathcal{I}$ is an order ideal of the boolean algebra B_S of all subsets of S (defined in Section 3.4).
- For any $T \subseteq S$, let $\mathcal{I}_T$ be the restriction of $\mathcal{I}$ to T, that is, $\mathcal{I}_T = \{F \in \mathcal{I} : F \subseteq T\}$. Then all maximal (under inclusion) elements of $\mathcal{I}_T$ have the same number of elements.

(There are several equivalent definitions of a matroid.) The elements of $\mathcal{I}$ are called *independent* sets. They are an abstraction of linear independent sets of a vector space or affinely independent subsets of an affine space. Indeed, if S is a finite subset of a vector space (respectively, affine subset of an affine space) and $\mathcal{I}$ is the collection of linearly independent (respectively, affinely independent) subsets of S, then $(S,\mathcal{I})$ is a matroid. A matroid is *simple* if every two-element subset of $\mathcal{I}$ is independent. Every matroid can be "simplified" (converted to a simple

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matroid) by removing all elements of S not contained in any independent set and by identifying any two points that are not independent. It is not hard to see that matroids on a set S are in bijection with geometric lattices L whose set of atoms is S, where a set $T \subseteq S$ is independent if and only if its join in L has rank $\#T$.

The reader may wish to verify the (partly redundant) entries of the following table concerning the posets of Example 3.1.1.

Poset P	Properties that P possesses	Properties that P lacks (n large)
$\boldsymbol{n}$	modular lattice	complemented, atomic, coatomic, geometric
B_n	modular lattice, relatively complemented, uniquely complemented, atomic, coatomic, geometric	
D_n	modular lattice	complemented, atomic coatomic, geometric (unless n is squarefree, in which case $D_n \cong B_k$)
Π_n	geometric lattice	modular
$B_n(q)$	modular lattice, complemented, atomic, coatomic, geometric	uniquely complemented

3.4 Distributive Lattices

The most important class of lattices from the combinatorial point of view are the *distributive* lattices. These are defined by the distributive laws

$$\begin{aligned} s \vee (t \wedge u) &= (s \vee t) \wedge (s \vee u), \\ s \wedge (t \vee u) &= (s \wedge t) \vee (s \wedge u). \end{aligned} \tag{3.8}$$

(One can prove that either of these laws implies the other.) If we assume $s \le u$ in the first law, then we obtain equation (3.7), since $s \vee u = u$. Hence, every distributive lattice is modular. The lattices $\boldsymbol{n}$, B_n, and D_n of Example 3.1.1 are distributive, while Π_n ($n \ge 3$) and $B_n(q)$ ($n \ge 2$) are not distributive. Further examples of distributive lattices are the lattices $J(P)$ of order ideals of the poset P. The lattice operations $\wedge$ and $\vee$ on order ideals are just ordinary intersection and union (as subsets of P). Since the union and intersection of order ideals is again an order ideal, it follows from the well-known distributivity of set union and intersection over one another that $J(P)$ is indeed a distributive lattice. The *fundamental theorem for finite distributive lattices* (FTFDL) states that the converse is true when P is finite.

3.4.1 Theorem (FTFDL). *Let L be a finite distributive lattice. Then there is a unique (up to isomorphism) poset P for which $L \cong J(P)$.*

Remark. For combinatorial purposes, it would in fact be best to *define* a finite distributive lattice as any poset of the form $J(P)$, P finite. However, to avoid conflict with established practices we have given the usual definition.

To prove Theorem 3.4.1, we first need to produce a candidate P and then show that indeed $L \cong J(P)$. Toward this end, define an element s of a lattice L to be *join-irreducible* if $s \neq \hat{0}$ and one cannot write $s = t \vee u$ where $t < s$ and $u < s$. (*Meet-irreducible* is defined dually.) In a finite lattice, an element is join-irreducible if and only if it covers exactly one element. An order ideal I of the finite poset P is join-irreducible in $J(P)$ if and only if it is a principal order ideal of P. Hence, there is a one-to-one correspondence between the join-irreducibles Λ_s of $J(P)$ and the elements s of P. Since $\Lambda_s \subseteq \Lambda_t$ if and only if $s \leq t$, we obtain the following result.

3.4.2 Proposition. *The set of join-irreducibles of $J(P)$, considered as an (induced) subposet of $J(P)$, is isomorphic to P. Hence, $J(P) \cong J(Q)$ if and only if $P \cong Q$.*

Proof of Theorem 3.4.1. Because of Proposition 3.4.2, it suffices to show that if P is the subposet of join-irreducibles of L, then $L \cong J(P)$. Given $t \in L$, let $I_t = \{s \in P : s \leq t\}$. Clearly $I_t \in J(P)$, so the mapping $t \mapsto I_t$ defines an order-preserving (in fact, meet-preserving) map $L \xrightarrow{\phi} J(P)$ whose inverse is order-preserving on $\phi(L)$. Moreover, ϕ is injective since $J(P)$ is a lattice. Hence, we need to show that ϕ is surjective. Let $I \in J(P)$ and $t = \bigvee\{s : s \in I\}$. We need to show that $I = I_t$. Clearly, $I \subseteq I_t$. Suppose that $u \in I_t$. Now

$$\bigvee\{s : s \in I\} = \bigvee\{s : s \in I_t\}. \tag{3.9}$$

Apply $\wedge u$ to equation (3.9). By distributivity, we get

$$\bigvee\{s \wedge u : s \in I\} = \bigvee\{s \wedge u : s \in I_t\}. \tag{3.10}$$

The right-hand side is just u, since one term is u and all others are $\leq u$. Since u is join-irreducible (being by definition an element of P), it follows from equation (3.10) that some $t \in I$ satisfies $t \wedge u = u$, that is, $u \leq t$. Since I is an order ideal we have $u \in I$, so $I_t \subseteq I$. Hence, $I = I_t$, and the proof is complete. □

In certain combinatorial problems, infinite distributive lattices of a special type occur naturally. Thus we define a *finitary* distributive lattice to be a locally finite distributive lattice L with $\hat{0}$. It follows that L has a unique rank function $\rho\colon L \to \mathbb{N}$ given by letting $\rho(t)$ be the length of any saturated chain from $\hat{0}$ to t. If L has finitely many elements p_i of any given rank $i \in \mathbb{N}$, then we can define the *rank-generating function* $F(L,x)$ by

$$F(L,x) = \sum_{t \in L} x^{\rho(t)} = \sum_{i \geq 0} p_i x^i.$$

In this case, of course, $F(L,x)$ need not be a polynomial but in general is a formal power series. We leave to the reader to check that FTFDL carries over to finitary distributive lattices as follows.

3.4.3 Proposition. *Let P be a poset for which every principal order ideal is finite. Then the poset $J_f(P)$ of* finite *order ideals of P, ordered by inclusion, is a finitary distributive lattice. Conversely, if L is a finitary distributive lattice and P is its subposet of join-irreducibles, then every principal order ideal of P is finite and $L \cong J_f(P)$.*

3.4.4 Example. (a) If P is an infinite antichain, then $J_f(P)$ has infinitely many elements on each level, so $F(J_f(P),x)$ is undefined.

(b) Let $P = \mathbb{N} \times \mathbb{N}$. Then $J_f(P)$ is a very interesting distributive lattice known as *Young's lattice*, denoted Y. It is not hard to see that

$$F(Y,x) = \sum_{i \geq 0} p(i)x^i = \frac{1}{\prod_{n \geq 1}(1-x^n)},$$

where $p(i)$ denotes the number of partitions of i (Sections 1.7 and 1.8). In fact, Y is isomorphic to the poset of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of all integers $n \geq 0$, ordered componentwise (or by containment of Young diagrams). For further information on Young's lattice, see Exercise 3.149, Section 3.21, and various places in Chapter 7.

We now turn to an investigation of the combinatorial properties of $J(P)$ (where P is finite) and of the relationship between P and $J(P)$. If I is an order ideal of P, then the elements of $J(P)$ that cover I are just the order ideals $I \cup \{t\}$, where t is a minimal element of $P - I$. From this observation we conclude the following result.

3.4.5 Proposition. *If P is an n-element poset, then $J(P)$ is graded of rank n. Moreover, the rank $\rho(I)$ of $I \in J(P)$ is just the cardinality $\#I$ of I, regarded as an order ideal of P.*

It follows from Propositions 3.4.2, 3.4.5, and FTFDL that there is a bijection between (nonisomorphic) posets P of cardinality n and (nonisomorphic) distributive lattices of rank n. This bijection sends P to $J(P)$, and the inverse sends $J(P)$ to its poset of join-irreducibles. In particular, the number of nonisomorphic posets of cardinality n equals the number of nonisomorphic distributive lattices of rank n.

If $P = \boldsymbol{n}$, an n-element chain, then $J(P) \cong \boldsymbol{n+1}$. At the other extreme, if $P = n\mathbf{1}$, an n-element antichain, then any subset of P is an order ideal, and $J(P)$ is just the set of subsets of P, ordered by inclusion. Hence $J(n\mathbf{1})$ is isomorphic to the poset B_n of Example 3.1.1(b), and we simply write $B_n = J(n\mathbf{1})$. We call B_n a *boolean algebra* of rank n. (The usual definition of a boolean algebra gives it more structure than merely that of a distributive lattice, but for our purposes we simply regard B_n as a certain distributive lattice.) It is clear from FTFDL (or otherwise) that the following conditions on a finite distributive lattice L are equivalent.

a. L is a boolean algebra.
b. L is complemented.

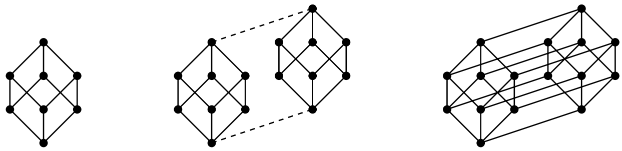

Figure 3.8 Drawing B_4 from B_3.

c. L is relatively complemented.
d. L is atomic.
e. $\hat{1}$ is a join of atoms of L.
f. L is a geometric lattice.
g. Every join-irreducible of L covers $\hat{0}$.
h. If L has n join-irreducibles (equivalently, $\operatorname{rank}(L) = n$), then L has at least (equivalently, exactly) 2^n elements.
i. The rank-generating function of L is $(1+x)^n$ for some $n \in \mathbb{N}$.

Given an order ideal I of P, define a map $f_I\colon P \to \mathbf{2}$ by

$$f_I(t) = \begin{cases} 1, & t \in I \\ 2, & t \notin I. \end{cases}$$

Clearly, f is order-preserving, that is, $f \in \mathbf{2}^P$. Then $f_I \le f_{I'}$ in $\mathbf{2}^P$ if and only if $I \supseteq I'$. Hence, $J(P)^* \cong \mathbf{2}^P$. Note also that $J(P^*) \cong J(P)^*$ and $J(P+Q) \cong J(P) \times J(Q)$. In particular, $B_n = J(n\mathbf{1}) \cong J(\mathbf{1})^n \cong \mathbf{2}^n$. This observation gives an efficient method for drawing B_n using the method of the previous section for drawing products. For instance, the Hasse diagram of B_3 is given by the first diagram in Figure 3.8. The other two diagrams show how to obtain the Hasse diagram of B_4.

If $I \le I'$ in the distributive lattice $J(P)$, then the interval $[I, I']$ is isomorphic to $J(I' - I)$, where $I' - I$ is regarded as an (induced) subposet of P. In particular, $[I, I']$ is a distributive lattice. (More generally, any sublattice of a distributive lattice is distributive, an immediate consequence of the definition (3.8) of a distributive lattice.) It follows that there is a one-to-one correspondence between intervals $[I, I']$ of $J(P)$ isomorphic to B_k ($k \ge 1$) such that no interval $[K, I']$ with $K < I$ is a boolean algebra, and k-element antichains of P. Equivalently, k-element antichains in P correspond to elements of $J(P)$ that cover exactly k elements.

We can use these ideas to describe a method for drawing the Hasse diagram of $J(P)$, given P. Let I be the set of minimal elements of P, say of cardinality m. To begin with, draw $B_m \cong J(I)$. Now choose a minimal element of $P - I$, say t. Adjoin a join-irreducible to $J(I)$ covering the order ideal $\Lambda_t - \{t\}$. The set of joins of elements covering $\Lambda_t - \{t\}$ must form a boolean algebra, so draw in any new joins necessary to achieve this. Now there may be elements covering $\Lambda_t - \{t\}$ whose

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covers don't yet have joins. Draw these in to form boolean algebras. Continue until all sets of elements covering a particular element have joins. This yields the distributive lattice $J(I \cup \{t\})$. Now choose a minimal element u of $P - I - \{t\}$ and adjoin a join-irreducible to $J(I \cup \{t\})$ covering the order ideal $\lambda_u - \{u\}$. "Fill in" the covers as before. This yields $J(I \cup \{t, u\})$. Continue until reaching $J(P)$. The actual process is easier to carry out than describe. Let us illustrate with P given by Figure 3.9(a). We will denote subsets of P such as $\{a, b, d\}$ as abd. First, draw $B_3 = J(abc)$ as in Figure 3.9(b). Adjoin the order ideal $\Lambda_d = abd$ above ab (and label it d) (Figure 3.9(c)). Fill in the joins of the elements covering ab (Figure 3.9(d)). Adjoin bce above bc (Figure 3.9(e)). Fill in joins of elements covering bc (Figure 3.9(f)). Fill in joins of elements covering abc (Figure 3.9(g)). Adjoin cf above c (Figure 3.9(h)). Fill in joins of elements covering c. These joins (including the empty join c) form a rank three boolean algebra. The elements c, ac, bc, cf, and abc are already there, so we need the three additional elements acf, bcf, and $abcf$ (Figure 3.9(i)). Now fill in joins of elements covering bc (Figure 3.9(j)). Finally, fill in joins of elements covering abc (Figure 3.9(k)). With a little practice, this procedure yields a fairly efficient method for computing the rank-generating function $F(J(P), x)$ by hand. For the present example, we see that

$$F(J(P), x) = 1 + 3x + 4x^2 + 5x^3 + 4x^4 + 3x^5 + x^6.$$

For further information about "zigzag" posets (or fences) as in Figure 3.9, see Exercise 3.66.

3.5 Chains in Distributive Lattices

We have seen that many combinatorial properties of the finite poset P have simple interpretations in terms of $J(P)$. For instance, the number of k-element order ideals of P equals the number of elements of $J(P)$ of rank k, and the number of k-element antichains of P equals the number of elements of $J(P)$ that cover exactly k elements. We wish to discuss one further example of this nature.

3.5.1 Proposition. *Let P be a finite poset and $m \in \mathbb{N}$. The following quantities are equal:*

a. The number of order-preserving maps $\sigma : P \to \boldsymbol{m}$,
b. The number of multichains $\hat{0} = I_0 \le I_1 \le \cdots \le I_m = \hat{1}$ of length m in $J(P)$,
c. The cardinality of $J(P \times \boldsymbol{m-1})$.

Proof. Given $\sigma : P \to \boldsymbol{m}$, define $I_j = \sigma^{-1}(\boldsymbol{j})$. Given $\hat{0} = I_0 \le I_1 \le \cdots \le I_m = \hat{1}$, define the order ideal I of $P \times \boldsymbol{m-1}$ by $I = \{(t, j) \in P \times \boldsymbol{m-1} : t \in I_{m-j}\}$. Given the order ideal I of $P \times \boldsymbol{m-1}$, define $\sigma : P \to \boldsymbol{m}$ by $\sigma(t) = \min\{m - j : (t, j) \in I\}$ if $(t, j) \in I$ for some j, and otherwise $\sigma(t) = m$. These constructions define the desired bijections. $\square$

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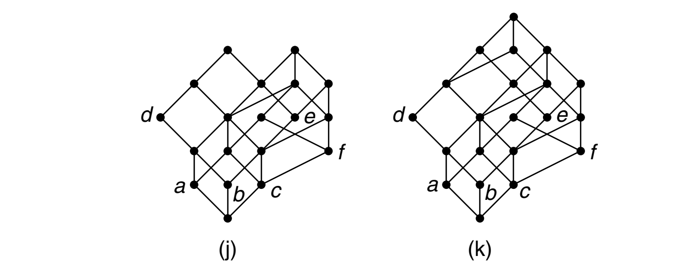

Figure 3.9 Drawing $J(P)$.

Note that the equivalence of (a) and (c) also follows from the computation

$$\boldsymbol{m}^P \cong (\boldsymbol{2}^{\boldsymbol{m-1}})^P \cong \boldsymbol{2}^{\boldsymbol{m-1}\times P}.$$

As a modification of the preceding proposition, we have the following result.

3.5.2 Proposition. *Preserve the notation of Proposition 3.5.1. The following quantities are equal:*

a. The number of surjective *order-preserving maps* $\sigma: P \to \boldsymbol{m}$,
b. The number of chains $\hat{0} = I_0 < I_1 < \cdots < I_m = \hat{1}$ *of length* m *in* $J(P)$.

Proof. Analogous to the proof of Proposition 3.5.1. □

One special case of Proposition 3.5.2 is of particular interest. If $\#P = p$, then an order-preserving bijection $\sigma: P \to \boldsymbol{p}$ is called a *linear extension* or *topological*

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sorting of P. The number of linear extensions of P is denoted $e(P)$ and is probably the single most useful number for measuring the "complexity" of P. It follows from Proposition 3.5.2 that $e(P)$ is also equal to the number of maximal chains of $J(P)$.

We may identify a linear extension $\sigma\colon P \to \boldsymbol{p}$ with the permutation $\sigma^{-1}(1),\ldots,\sigma^{-1}(p)$ of the elements of P. Similarly we may identify a maximal chain of $J(P)$ with a certain type of lattice path in Euclidean space, as follows. Let $C_1,\ldots,C_k$ be a partition of P into chains. (It is a consequence of a well-known theorem of Dilworth that the smallest possible value of k is equal to the cardinality of the largest antichain of P. See Exercise 3.77(d).) Define a map $\delta\colon J(P) \to \mathbb{N}^k$ by

$$\delta(I) = (\#(I \cap C_1), \#(I \cap C_2), \ldots, \#(I \cap C_k)).$$

If we give $\mathbb{N}^k$ the obvious product order, then δ is an injective lattice homomorphism that is cover-preserving (and therefore rank-preserving). Thus in particular, $J(P)$ is isomorphic to a sublattice of $\mathbb{N}^k$. If we choose each $\#C_i = 1$, then we get a rank-preserving injective lattice homomorphism $J(P) \to B_p$, where $\#P = p$. Given $\delta\colon P \to \mathbb{N}^k$ as above, define $\Gamma_\delta = \bigcup_T \operatorname{conv}(\delta(T))$, where conv denotes convex hull in $\mathbb{R}^k$ and T ranges over all intervals of $J(P)$ that are isomorphic to boolean algebras. (The set $\operatorname{conv}(\delta(T))$ is just a cube whose dimension is the length of the interval T.) Thus, Γ_δ is a compact polyhedral subset of $\mathbb{R}^k$, which is independent of δ (up to geometric congruence). It is then clear that the number of maximal chains in $J(P)$ is equal to the number of lattice paths in Γ_δ from the origin $(0,0,\ldots,0) = \delta(\hat{0})$ to $\delta(\hat{1})$, with unit steps in the directions of the coordinate axes. In other words, $e(P)$ is equal to the number of ways of writing $\delta(\hat{1}) = v_1 + v_2 + \cdots + v_p$, where each v_i is a unit coordinate vector in $\mathbb{R}^k$ and where $v_1 + v_2 + \cdots + v_i \in \Gamma_\delta$ for all i. The enumeration of lattice paths is an extensively developed subject which we encountered in various places in Chapter 1 and in Section 2.7, and which is further developed in Chapter 6. The point here is that certain lattice path problems are equivalent to determining $e(P)$ for some P. Thus, they are also equivalent to the problem of counting certain types of permutations.

3.5.3 Example. Let P be given by Figure 3.10(a). Take $C_1 = \{a,c\}$, $C_2 = \{b,d,e\}$. Then $J(P)$ has the embedding δ into $\mathbb{N}^2$ given by Figure 3.10(b). To get the polyhedral set Γ_δ, we simply "fill in" the squares in Figure 3.10(b), yielding the polyhedral set of Figure 3.10(c). There are nine lattice paths of the required type from $(0,0)$ to $(2,3)$ in Γ_δ, that is, $e(P) = 9$. The corresponding nine permutations of P are $abcde$, $bacde$, $abdce$, $badce$, $bdace$, $abdec$, $badec$, $bdaec$, $bdeac$.

3.5.4 Example. Let P be a disjoint union $C_1 + C_2$ of chains C_1 and C_2 of cardinalities m and n. Then Γ_δ is an $m \times n$ rectangle with vertices $(0,0)$, $(m,0)$, $(0,n)$, (m,n). As noted in Proposition 1.2.1, the number of lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$ is just $\binom{m+n}{m} = e(C_1 + C_2)$. A linear extension $\sigma : P \to \boldsymbol{m+n}$ is completely determined by the image $\sigma(C_1)$, which can be any m-element subset of $\boldsymbol{m+n}$. Thus once again, we obtain $e(C_1 + C_2) = \binom{m+n}{m}$. More

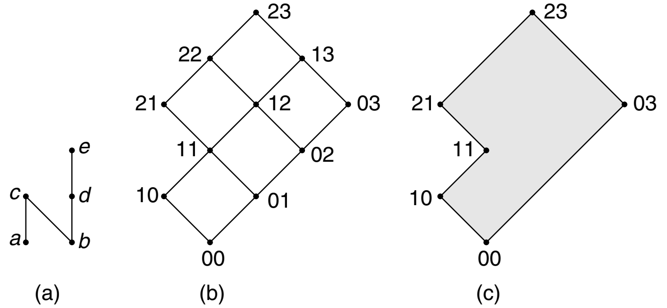

Figure 3.10 A polyhedral set associated with a finite distributive lattice.

generally, if $P = P_1 + P_2 + \cdots + P_k$ and $n_i = \#P_i$, then

$$e(P) = \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} e(P_1)e(P_2)\cdots e(P_k).$$

3.5.5 Example. Let $P = \mathbf{2} \times \boldsymbol{n}$, and take $C_1 = \{(2, j) : j \in \boldsymbol{n}\}, C_2 = \{(1, j) : j \in \boldsymbol{n}\}$. Then $\delta(J(P)) = \{(i, j) \in \mathbb{N}^2 : 0 \le i \le j \le n\}$. For example, the embedded poset $\delta(J(\mathbf{2} \times \mathbf{3}))$ is shown in Figure 3.11. Hence, $e(P)$ is equal to the number of lattice paths from (0,0) to (n,n), with steps $(1,0)$ and $(0,1)$, that never fall below (or by symmetry, that never rise above) the main diagonal $x = y$ of the (x, y)-plane. These lattice paths arose in the enumeration of 321-avoiding permutations in Section 1.5, where it was mentioned that they are counted by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. It follows that $e(\mathbf{2} \times \boldsymbol{n}) = C_n$. By the definition of $e(P)$, we see that this number is also equal to the number of $2 \times n$ matrices with entries the distinct integers $1, 2, \ldots, 2n$, such that every row and column is increasing. For instance, $e(\mathbf{2} \times \mathbf{3}) = 5$, corresponding to the matrices

123	124	125	134	135
456	356	346	256	246.

Such matrices are examples of *standard Young tableaux* (SYT), discussed extensively in Chapter 7.

We have now seen two ways of looking at the numbers $e(P)$: as counting certain order-preserving maps (or permutations) and as counting certain chains (or lattice paths). There is yet another way of viewing $e(P)$—as satisfying a certain *recurrence*. Regard e as a function on $J(P)$, that is, if $I \in J(P)$ then $e(I)$ is the number of linear extensions of I (regarded as a subposet of P). Thus, $e(I)$ is also the number of saturated chains from $\hat{0}$ to I in $J(P)$. From this observation it is clear that

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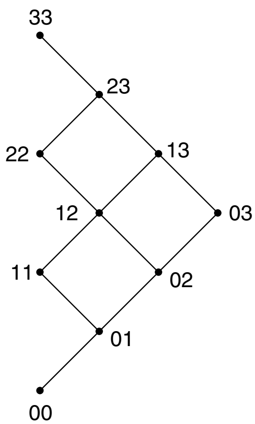

Figure 3.11 The distributive lattice $J(\mathbf{2} \times \mathbf{3})$.

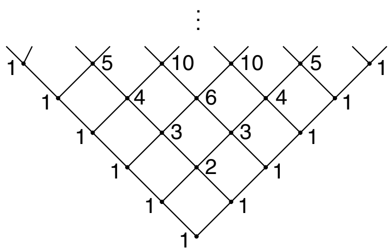

Figure 3.12 The distributive lattice $J_f(\mathbb{N}+\mathbb{N})$.

$$e(I) = \sum_{I' \lessdot I} e(I'), \tag{3.11}$$

where I' ranges over all elements of $J(P)$ that I covers. In other words, if we label the element $I \in J(P)$ by $e(I)$, then $e(I)$ is the sum of those $e(I')$ that lie "just below" I. This recurrence is analogous to the definition of Pascal's triangle, where each entry is the sum of the two "just above." Indeed, if we take P to be the infinite poset $\mathbb{N}+\mathbb{N}$ and let $J_f(P)$ be the lattice of finite order ideals of P, then $J_f(P) \cong \mathbb{N} \times \mathbb{N}$, and labeling the element $I \in J_f(P)$ by $e(I)$ yields precisly Pascal's triangle (though upside-down from the usual convention in writing it). Each finite order ideal I of $\mathbb{N}+\mathbb{N}$ has the form $\boldsymbol{m}+\boldsymbol{n}$ for some $m,n \in \mathbb{N}$, and from Example 3.5.4 we indeed have $e(\boldsymbol{m}+\boldsymbol{n}) = \binom{m+n}{m}$, the number of maximal chains in $\boldsymbol{m} \times \boldsymbol{n}$. See Figure 3.12.

Because of the previous example, we define a *generalized Pascal triangle* to be a finitary distributive lattice $L = J_f(P)$, together with the function $e\colon L \to \mathbb{P}$. The entries $e(I)$ of a generalized Pascal triangle thus have three properties in common with the usual Pascal triangle: (a) They count certain types of permutations, (b) they count certain types of lattice paths, and (c) they satisfy a simple recurrence.

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3.6 Incidence Algebras

Let P be a locally finite poset, and let $\mathrm{Int}(P)$ denote the set of (closed) intervals of P. (Recall that the empty set is not an interval.) Let K be a field. If $f : \mathrm{Int}(P) \to K$, then we write $f(x,y)$ for $f([x,y])$.

3.6.1 Definition. The *incidence algebra* $I(P,K)$ (denoted $I(P)$ for short) of P over K is the K-algebra of all functions

$$f : \mathrm{Int}(P) \to K$$

(with the usual structure of a vector space over K), where multiplication (or *convolution*) is defined by

$$fg(s,u) = \sum_{s \le t \le u} f(s,t)g(t,u).$$

This sum is finite (and hence fg is well-defined), since P is locally finite. It is easy to see that $I(P,K)$ is an associative algebra with (two-sided) identity, denoted δ or 1, defined by

$$\delta(s,t) = \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

One can think of $I(P,K)$ as consisting of all infinite linear combinations of symbols $[s,t]$, where $[s,t] \in \mathrm{Int}(P)$. Convolution is defined uniquely by requiring that

$$[s,t] \cdot [u,v] = \begin{cases} [s,v], & \text{if } t = u, \\ 0, & \text{if } t \neq u, \end{cases}$$

and then extending to all of $I(P,K)$ by bilinearity (allowing infinite linear combinations of the $[s,t]$'s). The element $f \in I(P,K)$ is identified with the expression

$$f = \sum_{[s,t] \in \mathrm{Int}(P)} f(s,t)[s,t].$$

If P if finite, then label the elements of P by $t_1, \dots, t_p$ where $t_i < t_j \Rightarrow i < j$. (The number of such labelings is $e(P)$, the number of linear extensions of P.) Then $I(P,K)$ is isomorphic to the algebra of all upper triangular matrices $M = (m_{ij})$ over K, where $1 \le i, j \le p$, such that $m_{ij} = 0$ if $t_i \not\le t_j$. (*Proof.* Identify m_{ij} with $f(t_i, t_j)$.) For instance, if P is given by Figure 3.13, then $I(P)$ is isomorphic to the algebra of all matrices of the form

$$\begin{bmatrix} * & 0 & * & 0 & * \\ 0 & * & * & * & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

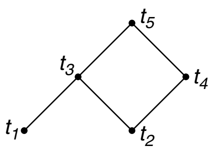

Figure 3.13 A five-element poset.

3.6.2 Proposition. *Let $f \in I(P)$. The following conditions are equivalent:*

a. f has a left inverse.
b. f has a right inverse.
c. f has a two-sided inverse (which is necessarily the unique left and right inverse).
d. $f(t,t) \neq 0$ for all $t \in P$.

Moreover, if f^{-1} exists, then $f^{-1}(s,u)$ depends only on the poset $[s,u]$.

Proof. The statement that $fg = \delta$ is equivalent to

$$f(s,s)g(s,s) = 1 \text{ for all } s \in P \tag{3.12}$$

and

$$g(s,u) = -f(s,s)^{-1} \sum_{s<t\leq u} f(s,t)g(t,u), \quad \text{for all } s < u \text{ in } P. \tag{3.13}$$

It follows that f has a right inverse g if and only if $f(s,s) \neq 0$ for all $s \in P$, and in that case $f^{-1}(s,u)$ depends only on $[s,u]$. Now the same reasoning applied to $hf = \delta$ shows that f has a left inverse h if and only if $f(s,s) \neq 0$ for all $s \in P$; that is, if and only if f has a right inverse. But from $fg = \delta$ and $hf = \delta$, we have that $g = h$, and the proof follows. □

NOTE. The fact that a right-inverse of f is a two-sided inverse also follows from general algebraic reasoning. Namely, the restriction of f to $\mathrm{Int}([s,u])$ satisfies a polynomial equation with nonzero constant term. An example of such an equation is the characteristic equation

$$\prod_{t\in[s,u]} (f - f(t,t)) = 0. \tag{3.14}$$

Hence, a right inverse of f is a polynomial in f and therefore commutes with f.

Let us now survey some useful functions in $I(P)$. The *zeta function* ζ is defined by

$$\zeta(t,u) = 1, \quad \text{for all } t \leq u \text{ in } P.$$

Thus,

$$\zeta^2(s,u) = \sum_{s\leq t\leq u} 1 = \#[s,u].$$

More generally, if $k \in \mathbb{P}$, then

$$\zeta^k(s,u) = \sum_{s=s_0 \le s_1 \le \cdots \le s_k = u} 1,$$

the number of multichains of length k from s to u. Similarly,

$$(\zeta - 1)(s,u) = \begin{cases} 1, & \text{if } s < u, \\ 0, & \text{if } s = u. \end{cases}$$

Hence, if $k \in \mathbb{P}$, then $(\zeta - 1)^k(s,u)$ is the number of chains $s = s_0 < s_1 < \cdots < s_k = u$ of length k from s to u. By Propositions 3.5.1 and 3.5.2, we have additional interpretations of $\zeta^k(s,u)$ and $(\zeta-1)^k(s,u)$ when P is a distributive lattice.

Now consider the element $2 - \zeta \in I(P)$. Thus,

$$(2-\zeta)(s,t) = \begin{cases} 1, & \text{if } s = t, \\ -1, & \text{if } s < t. \end{cases}$$

By Proposition 3.6.2, $2-\zeta$ is invertible. We claim that $(2-\zeta)^{-1}(s,t)$ is equal to the *total* number of chains $s = s_0 < s_1 < \cdots < s_k = t$ from s to t. We sketch two justifications of this fact.

First Justification. Let ℓ be the length of the longest chain in the interval $[s,t]$. Then $(\zeta-1)^{\ell+1}(u,v) = 0$ for all $s \le u \le v \le t$. Thus, for $s \le u \le v \le t$, we have

$$\begin{aligned}
&(2-\zeta)[1 + (\zeta-1) + (\zeta-1)^2 + \cdots + (\zeta-1)^\ell](u,v) \\
&= [1-(\zeta-1)][1 + (\zeta-1) + \cdots + (\zeta-1)^\ell](u,v) \\
&= [1-(\zeta-1)^{\ell+1}](u,v) = \delta(u,v).
\end{aligned}$$

Hence, $(2-\zeta)^{-1} = 1 + (\zeta-1) + \cdots + (\zeta-1)^\ell$ when restricted to $\mathrm{Int}([s,t])$. But by the definition of ℓ, it is clear that $[1 + (\zeta-1) + \cdots + (\zeta-1)^\ell](s,t)$ is the total number of chains from s to t, as desired. □

Second Justification. Our second justification is essentially equivalent to the first one, but it uses a little topology to avoid having to restrict our attention to an interval. The topological approach can be used to perform without effort many similar kinds of computations in $I(P)$. We define a topology on $I(P)$ (analogous to the topology on $\mathbb{C}[[x]]$ defined in Section 1.1) by saying that a sequence $f_1, f_2, \ldots$ of functions converges to f if for all $s \le t$, there exists $n_0 = n_0(s,t) \in \mathbb{P}$ such that $f_n(s,t) = f(s,t)$ for all $n \ge n_0$. With this topology, the following computation is valid (because the infinite series converges):

$$(2-\zeta)^{-1} = (1-(\zeta-1))^{-1} = \sum_{k \ge 0} (\zeta-1)^k,$$

so

$$\begin{aligned}(2-\zeta)^{-1}(s,t) &= \sum_{k\geq 0}(\zeta-1)^k(s,t)\\ &= \sum_{k\geq 0}(\text{number of chains of length } k \text{ from } s \text{ to } t)\\ &= \text{total number of chains from } s \text{ to } t.\end{aligned}$$

□

Similarly to the above interpretation of $(2-\zeta)^{-1}$, we leave to the reader to verify that $(1-\eta)^{-1}(s,t)$ is equal to the total number of maximal chains in the interval $[s,t]$, where η is defined by

$$\eta(s,t) = \begin{cases} 1, & \text{if } t \text{ covers } s,\\ 0, & \text{otherwise.}\end{cases}$$

3.7 The Möbius Inversion Formula

It follows from Proposition 3.6.2 that the zeta function ζ of a locally finite poset is invertible; its inverse is called the *Möbius function* of P and is denoted μ (or μ_P if there is possible ambiguity). One can define μ recursively without reference to the incidence algebra. Namely, the relation $\mu\zeta = \delta$ is equivalent to

$$\begin{aligned}\mu(s,s) &= 1, \quad \text{for all } s \in P,\\ \mu(s,u) &= -\sum_{s\leq t<u}\mu(s,t), \quad \text{for all } s<u \text{ in } P.\end{aligned} \tag{3.15}$$

3.7.1 Proposition (Möbius inversion formula). *Let P be a poset for which every principal order ideal Λ_t is finite. Let $f,g\colon P\to K$, where K is a field. Then*

$$g(t) = \sum_{s\leq t} f(s), \quad \text{for all } t\in P, \tag{3.16}$$

if and only if

$$f(t) = \sum_{s\leq t} g(s)\mu(s,t), \quad \text{for all } t\in P. \tag{3.17}$$

Proof. The set K^P of all functions $P\to K$ forms a vector space on which $I(P,K)$ acts (on the right) as an algebra of linear transformations by

$$(f\xi)(t) = \sum_{s\leq t} f(s)\xi(s,t),$$

where $f\in K^P$, $\xi\in I(P,K)$. The Möbius inversion formula is then nothing but the statement

$$f\zeta = g \iff f = g\mu.$$

□

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Note. It is also easy to give a naive computational proof of Proposition 3.7.1. Assuming (3.16), we have (for fixed $t \in P$)

$$\begin{aligned}\sum_{s\le t} g(s)\mu(s,t) &= \sum_{s\le t}\mu(s,t)\sum_{u\le s} f(u)\\ &= \sum_{u\le t} f(u)\sum_{u\le s\le t}\mu(s,t)\\ &= \sum_{u\le t} f(u)\delta(u,t)\\ &= f(t),\end{aligned}$$

which is (3.17). A completely analogous argument shows that (3.16) follows from (3.17).

A dual formulation of the Möbius inversion formula is sometimes convenient.

3.7.2 Proposition (Möbius inversion formula, dual form). *Let P be a poset for which every principal dual order ideal V_t is finite. Let $f,g\colon P \to K$. Then*

$$g(s) = \sum_{t\ge s} f(t), \text{ for all } s \in P,$$

if and only if

$$f(s) = \sum_{t\ge s}\mu(s,t)g(t), \text{ for all } s \in P.$$

Proof. Exactly as above, except now $I(P,K)$ acts on the *left* by

$$(\xi f)(s) = \sum_{t\ge s}\xi(s,t)f(t).$$

□

As in the Principle of Inclusion-Exclusion, the purely abstract statement of the Möbius inversion formula as given here is just a trivial observation in linear algebra. What is important are the applications of the Möbius inversion formula. First, we show the Möbius inversion formula does indeed explain formulas such as equation (3.1).

Given n finite sets $S_1,\dots,S_n$, let P be the poset of all their intersections ordered by inclusion, including the empty intersection $S_1\cup\cdots\cup S_n = \hat{1}$. If $T \in P$, then let $f(T)$ be the number of elements of T that belong to no $T' < T$ in P, and let $g(T) = \#T$. We want an expression for $\#(S_1\cup\cdots\cup S_n) = \sum_{T\le\hat{1}} f(T) = g(\hat{1})$. Now $g(T) = \sum_{T'\le T} f(T')$, so by Möbius inversion on P we have

$$0 = f(\hat{1}) = \sum_{T\in P} g(T)\mu(T,\hat{1}) \Rightarrow g(\hat{1}) = -\sum_{T<\hat{1}} \#T\cdot\mu(T,\hat{1}),$$

as desired. In the example given by equation (3.1), P is given by Figure 3.14. Indeed, $\mu(A,\hat{1}) = \mu(B,\hat{1}) = \mu(C,\hat{1}) = -1$ and $\mu(D,\hat{1}) = 2$, so (3.1) follows.

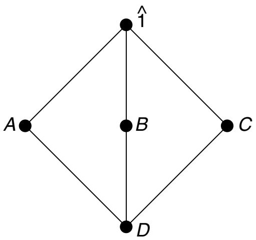

Figure 3.14 The poset related to equation (3.1).

3.8 Techniques for Computing Möbius Functions

In order for the Möbius inversion formula to be of any value, it is necessary to be able to compute the Möbius function of posets P of interest. We begin with a simple example that can be done by brute force.

3.8.1 Example. Let P be the chain $\mathbb{N}$. It follows directly from equation (3.15) that

$$\mu(i,j) = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i+1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

The Möbius inversion formula takes the form

$$g(n) = \sum_{i=0}^{n} f(i) \text{ for all } n > 0$$

if and only if

$$f(0) = g(0), \text{ and } f(n) = g(n) - g(n-1) \text{ for all } n > 0.$$

In other words, the operations Σ and Δ (with Σ suitably initialized) are inverses of one another, the finite difference analogue of the "fundamental theorem of calculus."

Since only in rare cases can Möbius functions be computed by inspection as in Example 3.8.1, we need general techniques for their evaluation. We begin with the simplest result of this nature.

3.8.2 Proposition (the product theorem). *Let P and Q be locally finite posets, and let $P \times Q$ be their direct product. If $(s,t) \le (s',t')$ in $P \times Q$ then*

$$\mu_{P\times Q}((s,t),(s',t')) = \mu_P(s,s')\mu_Q(t,t').$$

Proof. Let $(s,t) \le (s',t')$. We have

$$\sum_{(s,t)\le(u,v)\le(s',t')} \mu_P(s,u)\mu_Q(t,v) = \left(\sum_{s\le u\le s'} \mu_P(s,u)\right)\left(\sum_{t\le v\le t'} \mu_Q(t,v)\right)$$
$$= \delta_{ss'}\delta_{tt'} = \delta_{(s,t),(s',t')}.$$

Comparing with equation (3.15), which determines μ uniquely, completes the proof. $\square$

For readers familiar with tensor products, we mention a more conceptual way of proving the previous proposition. Namely, one easily sees that

$$I(P \times Q, K) \cong I(P,K) \otimes_K I(Q,K)$$

and $\zeta_{P\times Q} = \zeta_P \otimes \zeta_Q$. Taking inverses gives $\mu_{P\times Q} = \mu_P \otimes \mu_Q$.

3.8.3 Example. Let $P = B_n$, the boolean algebra of rank n. Now $B_n \cong \mathbf{2}^n$, and the Möbius function of the chain $\mathbf{2} = \{1,2\}$ is given by $\mu(1,1) = \mu(2,2) = 1$, $\mu(1,2) = -1$. Hence if we identify B_n with the set of all subsets of an n-set X, then we conclude from the product theorem that

$$\mu(T,S) = (-1)^{\#(S-T)}.$$

Since $\#(S-T)$ is the length $\ell(T,S)$ of the interval $[T,S]$, in purely order-theoretic terms, we have

$$\mu(T,S) = (-1)^{\ell(T,S)}. \tag{3.18}$$

The Möbius inversion formula for B_n becomes the following statement. Let $f,g\colon B_n \to K$; then

$$g(S) = \sum_{T\subseteq S} f(T), \ \text{ for all } S \subseteq X,$$

if and only if

$$f(S) = \sum_{T\subseteq S} (-1)^{\#(S-T)} g(T), \ \text{ for all } S \subseteq X.$$

This is just equation (2.8). Hence, we can say that "Möbius inversion on a boolean algebra is equivalent to the Principle of Inclusion-Exclusion." Note that equation (2.8), together with the Möbius inversion formula (Proposition 3.7.1), actually *proves* (3.18), so now we have two proofs of this result.

3.8.4 Example. Let $n_1,\dots,n_k$ be nonnegative integers, and let $P = (\boldsymbol{n_1+1}) \times (\boldsymbol{n_2+1}) \times \cdots \times (\boldsymbol{n_k+1})$, a product of chains of lengths $n_1,\dots,n_k$. Note that P is isomorphic to the distributive lattice $J(\boldsymbol{n_1} + \boldsymbol{n_2} + \cdots + \boldsymbol{n_k})$. Identify P with the set of all k-tuples $(a_1,a_2,\dots,a_k) \in \mathbb{N}^k$ with $0 \le a_i \le n_i$, ordered componentwise. If $a_i \le b_i$ for all i, then the interval $[(a_1,\dots,a_k),(b_1,\dots,b_k)]$ in P is isomorphic to $(\boldsymbol{b_1 - a_1 + 1}) \times \cdots \times (\boldsymbol{b_k - a_k + 1})$. Hence by Example 3.8.1 and Proposition 3.8.2, we have

$$\mu((a_1,\dots,a_k),(b_1,\dots,b_k)) = \begin{cases} (-1)^{\sum(b_i - a_i)}, & \text{if each } b_i - a_i = 0 \text{ or } 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

Equivalently,

$$\mu(s,t) = \begin{cases} (-1)^{\ell(s,t)}, & \text{if } [s,t] \text{ is a boolean algebra,} \\ 0, & \text{otherwise.} \end{cases}$$

(See Example 3.9.6 for a mild generalization.)

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There are two further ways of interest to interpret the lattice $P = (\boldsymbol{n_1+1}) \times \cdots \times (\boldsymbol{n_k+1})$. First, P is isomorphic to the poset of submultisets of the multiset $\{x_1^{n_1}, \ldots, x_k^{n_k}\}$, ordered by inclusion. Second, if N is a positive integer of the form $p_1^{n_1} \cdots p_k^{n_k}$, where the p_i's are distinct primes, then P is isomorphic to the poset D_N defined in Example 3.1.1(c) of positive integral divisors of N, ordered by divisibility (i.e., $r \leq s$ in D_N if $r|s$). In this latter context, equation (3.19) takes the form

$$\mu(r,s) = \begin{cases} (-1)^t, & \text{if } s/r \text{ is a product of } t \text{ distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\mu(r,s)$ is just the classical number-theoretic Möbius function $\mu(s/r)$. The Möbius inversion formula becomes the classical one, namely,

$$g(n) = \sum_{d|n} f(d), \quad \text{for all } n|N,$$

if and only if

$$f(n) = \sum_{d|n} g(d)\mu(n/d), \quad \text{for all } n|N.$$

This example explains the termimology "Möbius function of a poset."

Rather than restricting ourselves to the divisors of a fixed integer N, it is natural to consider the poset P of *all* positive integers, ordered by divisibility. Since any interval $[r,s]$ of this poset appears as an interval in the lattice of divisors of s (or of any N for which $s|N$), the Möbius function remains $\mu(r,s) = \mu(s/r)$. Abstractly, the poset P is isomorphic to the finitary distributive lattice

$$J_f(\mathbb{P}+\mathbb{P}+\mathbb{P}+\cdots) = J_f\left(\sum_{n\geq 1}\mathbb{P}\right) \cong \prod_{n\geq 1}\mathbb{N}, \tag{3.20}$$

where the product $\prod_{n\geq 1}\mathbb{N}$ is the *restricted direct product*; that is, only finitely many components of an element of the product are nonzero. Alternatively, P can be identified with the lattice of all finite multisets of the set $\mathbb{P}$ (or any countably infinite set).

We now come to a very important way of computing Möbius functions.

3.8.5 Proposition (Philip Hall's theorem). *Let P be a finite poset, and let $\widehat{P}$ denote P with a $\hat{0}$ and $\hat{1}$ adjoined. Let c_i be the number of chains $\hat{0} = t_0 < t_1 < \cdots < t_i = \hat{1}$ of length i between $\hat{0}$ and $\hat{1}$. (Thus, $c_0 = 0$ and $c_1 = 1$.) Then*

$$\mu_{\widehat{P}}(\hat{0},\hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots. \tag{3.21}$$

Proof. We have

$$\begin{aligned}\mu_{\widehat{P}}(\hat{0},\hat{1}) &= (1+(\zeta-1))^{-1}(\hat{0},\hat{1})\\ &= (1-(\zeta-1)+(\zeta-1)^2-\cdots)(\hat{0},\hat{1})\\ &= \delta(\hat{0},\hat{1})-(\zeta-1)(\hat{0},\hat{1})+(\zeta-1)^2(\hat{0},\hat{1})-\cdots\\ &= c_0-c_1+c_2-\cdots.\end{aligned}$$

□

The significance of Proposition 3.8.5 is that it shows that $\mu(\hat{0},\hat{1})$ (and thus $\mu(s,t)$ for any interval $[s,t]$) can be interpreted as an Euler characteristic, and therefore links the Möbius function of P with the powerful machinery of algebraic topology. To see the connection, recall that an (abstract) *simplicial complex* on a vertex set V is a collection Δ of subsets of V satisfying:

a. If $t \in V$ then $\{t\} \in \Delta$,
b. if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Thus, Δ is just an order ideal of the boolean algebra B_V that contains all one-element subsets of V. An element $F \in \Delta$ is called a *face* of Δ, and the *dimension* of F is defined to be $\#F-1$. In particular, the empty set $\emptyset$ is always a face of Δ (provided $\Delta \neq \emptyset$), of dimension -1. Also define the *dimension* of Δ by

$$\dim \Delta = \max_{F\in\Delta}(\dim F).$$

If Δ is finite, then let f_i denote the number of i-dimensional faces of Δ. Define the *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ by

$$\widetilde{\chi}(\Delta) = \sum_i (-1)^i f_i = -f_{-1}+f_0-f_1+\cdots. \tag{3.22}$$

Note that $f_{-1}=1$ unless $\Delta=\emptyset$. The simplicial complexes $\Delta_1=\emptyset$ and $\Delta_2=\{\emptyset\}$ are not the same; in particular, $\widetilde{\chi}(\Delta_1)=0$ and $\widetilde{\chi}(\Delta_2)=-1$.

NOTE. The reduced Euler characteristic $\widetilde{\chi}(\Delta)$ is related to the ordinary Euler characteristic $\chi(\Delta)$ by $\widetilde{\chi}(\Delta)=\chi(\Delta)-1$ (if $\Delta\neq\emptyset$). Thus, in computing $\chi(\Delta)$, the empty set is not considered as a face, whereas for $\widetilde{\chi}(\Delta)$ we do regard it as a face.

Now if P is any poset, then define a simplicial complex $\Delta(P)$ as follows: The vertices of $\Delta(P)$ are the elements of P, and the faces of $\Delta(P)$ are the chains of P. The simplicial complex $\Delta(P)$ is called the *order complex* of P. We then conclude from equations (3.21) and (3.22) the following result.

3.8.6 Proposition (Proposition 3.8.5, restated). *Let P be a finite poset. Then*

$$\mu_{\widehat{P}}(\hat{0},\hat{1}) = \widetilde{\chi}(\Delta(P)).$$

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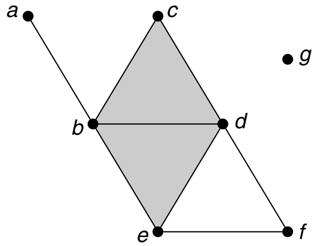

Figure 3.15 A geometric realization of a simplicial complex.

Proposition 3.8.5 gives an expression for $\mu(\hat{0},\hat{1})$ that is self-dual (i.e., remains unchanged if P is replaced by P^*). Thus, we see that in any locally finite poset P,

$$\mu_P(s,t) = \mu_{P^*}(t,s).$$

(One can also prove this fact using $\mu\zeta = \zeta\mu$.)

Let us recall that in topology one associates a topological space $|\Delta|$, called the *geometric realization* of Δ, with a simplicial complex Δ. (One also says that Δ is a *triangulation* of the space $|\Delta|$.) Informally, place the vertices of Δ in sufficiently general position (e.g., linearly independent) in some Euclidean space. Then

$$|\Delta| = \bigcup_{F\in\Delta} \operatorname{conv}(F),$$

where conv denotes convex hull. For instance, if the maximal faces of Δ are (abbreviating $\{a,b\}$ as ab, etc.) ab, bcd, bde, df, ef, g, then the geometric realization $|\Delta|$ is shown in Figure 3.15.

The reduced Euler characteristic $\widetilde{\chi}(X)$ of the space $X = |\Delta|$ is defined by

$$\widetilde{\chi}(X) = \sum_i (-1)^i \operatorname{rank} \widetilde{H}_i(X;\mathbb{Z}),$$

where $\widetilde{H}_i(X;\mathbb{Z})$ denotes the ith reduced homology group of X. One then has from elementary algebraic topology that

$$\widetilde{\chi}(X) = \widetilde{\chi}(\Delta), \tag{3.23}$$

so that $\mu_{\widehat{P}}(\hat{0},\hat{1})$ depends only on the topological space $|\Delta(P)|$ of $\Delta(P)$. For instance, if $\Delta(P)$ is a triangulation of an n-dimensional sphere, then $\mu_{\widehat{P}}(\hat{0},\hat{1}) = (-1)^n$.

3.8.7 Example (for readers familiar with some topology). A *finite regular cell complex* Γ is a finite set of nonempty pairwise-disjoint open cells $\sigma_i \subset \mathbb{R}^N$ such that

a. $(\bar{\sigma}_i, \bar{\sigma}_i - \sigma_i) \approx (\mathbb{B}^n, \mathbb{S}^{n-1})$, for some $n = n(i)$,
b. each $\bar{\sigma}_i - \sigma_i$ is a union of σ_j's.

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Here $\bar{\sigma}_i$ denotes the closure of σ_i (in the usual topology on $\mathbb{R}^N$), $\approx$ denotes homeomorphism, $\mathbb{B}^n$ is the unit ball $\{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_1^2+\cdots+x_n^2\le 1\}$, and $\mathbb{S}^{n-1}$ is the unit sphere $\{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_1^2+\cdots+x_n^2=1\}$. Note that a cell σ_i may consist of a single point, corresponding to the case $n=0$. Also, define the *underlying space* of Γ to be the topological space $|\Gamma|=\bigcup\sigma_i\subset\mathbb{R}^N$. Given a finite regular cell complex Γ, define its (first) *barycentric subdivision* $\mathrm{sd}(\Gamma)$ to be the abstract simplicial complex whose vertices consist of the closed cells $\bar{\sigma}_i$ of Γ, and whose faces consist of those sets $\{\bar{\sigma}_{i_1},\ldots,\bar{\sigma}_{i_k}\}$ of vertices forming a *flag* $\bar{\sigma}_{i_1}\subset\bar{\sigma}_{i_2}\subset\cdots\subset\bar{\sigma}_{i_k}$. The crucial property of a regular cell complex to concern us here is that the geometric realization $|\mathrm{sd}(\Gamma)|$ of the simplicial complex $\mathrm{sd}(\Gamma)$ is homeomorphic to the underlying space $|\Gamma|$ of the cell complex Γ:

$$|\mathrm{sd}(\Gamma)|\approx|\Gamma|. \tag{3.24}$$

Now given a finite regular cell complex Γ, let $P(\Gamma)$ be the poset of cells of Γ, ordered by defining $\sigma_i\le\sigma_j$ if $\bar{\sigma}_i\subseteq\bar{\sigma}_j$. It follows from the definition of $\mathrm{sd}(\Gamma)$ that $\Delta(P(\Gamma))=\mathrm{sd}(\Gamma)$. From Proposition 3.8.6 and equations (3.23) and (3.24), we conclude the following.

3.8.8 Proposition. *Let Γ be a finite regular cell complex, and let $P=P(\Gamma)$. Then*

$$\mu_{\widehat{P}}(\hat{0},\hat{1})=\widetilde{\chi}(|\Gamma|), \tag{3.25}$$

where $\widetilde{\chi}(|\Gamma|)$ is the reduced Euler characteristic of the topological space $|\Gamma|$.

Propositions 3.8.6 and 3.8.8 deal with the topological significance of the integer $\mu_{\widehat{P}}(\hat{0},\hat{1})$. We are also interested in other values $\mu_{\widehat{P}}(s,t)$, so we briefly discuss this point. Let Δ be any finite simplicial complex, and let $F\in\Delta$. The *link* of F is the subcomplex of Δ defined by

$$\mathrm{lk}\,F=\{G\in\Delta : G\cap F=\emptyset \text{ and } G\cup F\in\Delta\}. \tag{3.26}$$

If P is a finite poset and $s<t$ in P, then choose saturated chains $s_1\lessdot s_2\lessdot\cdots\lessdot s_j=s$ and $t=t_1\lessdot t_2\lessdot\cdots\lessdot t_k$ in P such that s_1 is a minimal element and t_k is a maximal element of P. Let $F=\{s_1,\ldots,s_j,t_1,\ldots,t_k\}\in\Delta(P)$. Then $\mathrm{lk}\,F$ is just the order complex of the open interval $(s,t)=\{u\in P : s<u<t\}$, so by Proposition 3.8.6 we have

$$\mu(s,t)=\widetilde{\chi}(\mathrm{lk}\,F). \tag{3.27}$$

Now suppose that Δ is an abstract simplicial complex that triangulates a manifold M, with or without boundary. (In other words, $|\Delta|\approx M$.) Let $\emptyset\ne F\in\Delta$. It is well known from algebraic topology that $\mathrm{lk}\,F$ has the same homology groups as a sphere or ball of dimension equal to $\dim(\mathrm{lk}\,F)=\max_{G\in\mathrm{lk}\,F}(\dim G)$. Moreover, $\mathrm{lk}\,F$ will have the homology groups of a ball precisely when F lies on the boundary $\partial\Delta$ of Δ. Equivalently, F is contained in some face F' such that $\dim F'=\dim\Delta-1$ and

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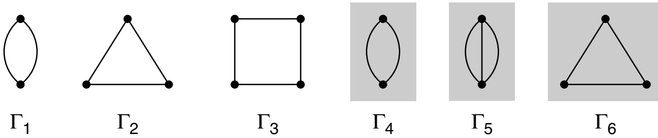

Figure 3.16 Some regular cell complexes.

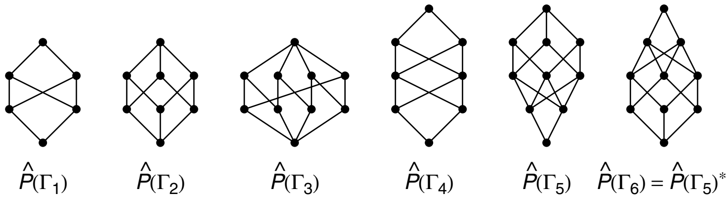

Figure 3.17 The face posets of the regular cell complexes of Figure 3.16.

F' is contained in a unique maximal face of Δ. (Somewhat surprisingly, lk F need not be simply connected and $|\text{lk}\, F|$ need not be a manifold!) Since $\widetilde{\chi}(\mathbb{S}^n) = (-1)^n$ and $\widetilde{\chi}(\mathbb{B}^n) = 0$, we deduce from equations (3.25) and (3.27) the following result.

3.8.9 Proposition. *Let Γ be a finite regular cell complex. Suppose that $|\Gamma|$ is a manifold, with or without boundary. Let $P = P(\Gamma)$. Then*

$$\mu_{\widehat{P}}(s,t) = \begin{cases} 0, & \text{if } s \neq \hat{0},\ t = \hat{1}, \text{ and the cell } s \text{ lies on the} \\ & \text{boundary of } |\Gamma|, \\ \widetilde{\chi}(|\Gamma|), & \text{if } (s,t) = (\hat{0},\hat{1}), \\ (-1)^{\ell(s,t)}, & \text{otherwise.} \end{cases}$$

Motivated by Proposition 3.8.9, we define a finite graded poset P with $\hat{0}$ and $\hat{1}$ to be *semi-Eulerian* if $\mu_P(s,t) = (-1)^{\ell(s,t)}$ whenever $(s,t) \neq (\hat{0},\hat{1})$, and to be *Eulerian* if in addition $\mu_P(\hat{0},\hat{1}) = (-1)^{\ell(\hat{0},\hat{1})}$. Thus, Proposition 3.8.9 implies that if $|\Gamma|$ is a manifold (without boundary), then $\widehat{P}(\Gamma)$ is semi-Eulerian. Moreover, if $|\Gamma|$ is a sphere, then $\widehat{P}(\Gamma)$ is Eulerian. By Example 3.8.3, boolean algebras B_n are Eulerian; indeed, $B_n = \widehat{P}(\Gamma)$, where Γ is the boundary complex of an $(n-1)$-simplex. Hence, $|\Delta(B_n)| \approx \mathbb{S}^{n-2}$, a vast topological strengthening of the mere computation of the Möbius function of B_n. Some interesting properties of Eulerian posets appear in Sections 3.16 and 3.17.

3.8.10 Example. a. The diagrams of Figure 3.16 represent finite regular cell complexes Γ such that $|\Gamma| \cong \mathbb{S}^1$ or $|\Gamma| \cong \mathbb{S}^2$. (Shaded regions represent 2-cells.) The corresponding Eulerian posets $\widehat{P}(\Gamma)$ are shown in Figure 3.17. Note that $\widehat{P}(\Gamma_2)$ and $\widehat{P}(\Gamma_3)$ are lattices. This is because in Γ_2 and Γ_3, any intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ is some $\bar{\sigma}_k$.

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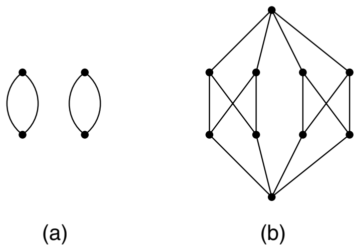

Figure 3.18 A nonspherical regular cell complex and its Eulerian face poset.

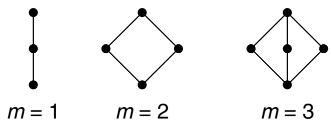

Figure 3.19 Face posets of some 0-dimensional manifolds.

b. The diagram  represents a certain cell complex Γ that is *not* regular, since for the unique 1-cell σ we do not have $\bar{\sigma} - \sigma \approx \mathbb{S}^0$. (The sphere $\mathbb{S}^0$ consists of two points, while $\bar{\sigma} - \sigma$ is just a single point.) The corresponding poset $P = P(\Gamma)$ is the two-element chain, and $|\Delta(P)|$ is not homeomorphic to $|\Gamma|$. (We have $|\Gamma| \approx \mathbb{S}^1$ while $|\Delta(P)| \approx \mathbb{B}^1$.) Note that $\widehat{P}$ is not Eulerian even though $|\Gamma|$ is a sphere.
c. Let Γ be given by Figure 3.18(a). Then $|\Gamma|$ is a manifold without boundary with the same Euler characteristic as $\mathbb{S}^1$ (namely, 0), though $|\Gamma| \not\approx \mathbb{S}^1$. Hence, $\widehat{P}(\Gamma)$ is Eulerian even though Γ does not have the same homology groups as a sphere. See Figure 3.18(b).
d. If Γ is a disjoint union of m points then $|\Gamma|$ is a manifold with Euler characteristic m. Hence, $\widehat{P}(\Gamma)$ is semi-Eulerian, but not Eulerian if $m \neq 2$. See Figure 3.19.

For our final excursion into topology, let P be a finite graded poset with $\hat{0}$ and $\hat{1}$. We say that the Möbius function of P *alternates in sign* if

$$(-1)^{\ell(s,t)}\mu(s,t) \geq 0, \text{ for all } s \leq t \text{ in } P.$$

A finite poset P is said to be *Cohen-Macaulay* over an abelian group A if for every $s < t$ in $\widehat{P}$, the order complex $\Delta(s,t)$ of the open interval (s,t) satisfies

$$\widetilde{H}_i(\Delta(s,t);A) = 0, \text{ if } i < \dim \Delta(s,t). \tag{3.28}$$

Here $\widetilde{H}_i(\Delta(s,t);A)$ denotes reduced simplicial homology with coefficients in A. It follows from standard topological arguments that if P is Cohen-Macaulay over some group A, then P is Cohen-Macaulay over $\mathbb{Q}$. Hence, we may as well take $A = \mathbb{Q}$ to get the widest class of posets. One can easily show that a Cohen-Macaulay poset is graded. If equation (3.28) holds (say with $A = \mathbb{Q}$) and if $d = \dim \Delta(s,t)$,

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then equation (3.27) implies that

$$\mu_P(s,t) = \widetilde{\chi}(\Delta(s,t)) = (-1)^d \dim_{\mathbb{Q}} \widetilde{H}_d(\Delta(s,t);\mathbb{Q}) \geq 0.$$

Since $d = \ell(s,t) - 2$, we conclude that

$$(-1)^{\ell(s,t)}\mu_P(s,t) = \dim_{\mathbb{Q}} \widetilde{H}_d(\Delta(s,t);\mathbb{Q}) \geq 0.$$

We have therefore proved the following result.

3.8.11 Proposition. *If P is Cohen-Macaulay, then the Möbius function of P alternates in sign.*

Examples of Cohen-Macaulay posets include those of the form $P(\Gamma)$, where Γ is a finite regular cell complex such that $|\Gamma|$ is a manifold of dimension d, with or without boundary, satisfying $\widetilde{H}_i(\Gamma;\mathbb{Q}) = 0$ if $i < d$. It can be shown that for any finite regular cell complex Γ, the question of whether $P(\Gamma)$ is Cohen-Macaulay depends only on the space $|\Gamma|$. It can also be shown that if $\widehat{P}$ is a finite semimodular lattice, then P is Cohen-Macaulay. Though we will not prove this fact here, we will later (Proposition 3.10.1) prove the weaker assertion that the Möbius function of a finite semimodular lattice alternates in sign.

3.9 Lattices and Their Möbius Functions

There are special methods for computing the Möbius function of a lattice that are inapplicable to general posets. We will develop these results in a unified way using the theory of Möbius algebras. While the applications to Möbius functions can also be proved without recourse to Möbius algebras, we prefer the convenience and elegance of the algebraic viewpoint.

3.9.1 Definition. Let L be a lattice and K a field. The *Möbius algebra* $A(L,K)$ is the semigroup algebra of L with the meet operation, over K. In other words, $A(L,K)$ is the vector space over K with basis L, with (bilinear) multiplication defined by $s \cdot t = s \wedge t$ for all $s,t \in L$.

The Möbius algebra $A(L,K)$ is commutative and has a vector space basis consisting of idempotents, namely, the elements of L. It follows from general ring-theoretic considerations (Wedderburn theory or otherwise) that when L is finite we have $A(L,K) \cong K^{\#L}$. We wish to make this isomorphism more explicit. To do so, define for $t \in L$ the element $\delta_t \in A(L,K)$ by

$$\delta_t = \sum_{s \leq t} \mu(s,t)s.$$

Hence by the Möbius inversion formula,

$$t = \sum_{s \leq t} \delta_s. \tag{3.29}$$

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The number of δ_t's is equal to $\#L = \dim_K A(L,K)$, and equation (3.29) shows that they span $A(L,K)$. Hence, the δ_t's form a K-basis for $A(L,K)$.

3.9.2 Theorem. *Let L be a finite lattice and let $A'(L,K)$ be the abstract algebra $\bigoplus_{t\in L} K_t$, where each $K_t \cong K$. Denote by δ'_t the identity element of K_t, so $\delta'_s\delta'_t = \delta_{st}\delta'_t$ (where δ_{st} denotes the Kronecker delta). Define a linear transformation $\theta\colon A(L,K) \to A'(L,K)$ by setting $\theta(\delta_t) = \delta'_t$ and extending by linearity. Then θ is an isomorphism of algebras.*

Proof. If $t \in L$, then let $t' = \sum_{s\le t} \delta'_s \in A'(L,K)$. Since θ is clearly a vector space isomorphism, we need only show that $s't' = (s\wedge t)'$. Now

$$s't' = \left(\sum_{u\le s}\delta'_u\right)\left(\sum_{v\le t}\delta'_v\right) = \sum_{\substack{u\le s\\ v\le t}} \delta_{uv}\delta'_u$$

$$= \sum_{w\le s\wedge t} \delta'_w = (s\wedge t)'. \qquad \square$$

3.9.3 Corollary (Weisner's theorem). *Let L be a finite lattice with at least two elements, and let $\hat{1} \ne a \in L$. Then*

$$\sum_{t\colon t\wedge a=\hat{0}} \mu(t,\hat{1}) = 0.$$

Proof. In the Möbius algebra $A(L,K)$ we have

$$a\delta_{\hat{1}} = \left(\sum_{b\le a}\delta_b\right)\delta_{\hat{1}} = 0, \text{ if } a \ne \hat{1}. \tag{3.30}$$

On the other hand,

$$a\delta_{\hat{1}} = a\sum_{t\in L}\mu(t,\hat{1})t = \sum_{t\in L}\mu(t,\hat{1})(a\wedge t). \tag{3.31}$$

Writing $a\delta_{\hat{1}} = \sum_{t\in L} c_t\cdot t$, we conclude from equation (3.30) that $c_{\hat{0}} = 0$ and from (3.31) that $c_{\hat{0}} = \sum_{t\colon t\wedge a=\hat{0}} \mu(t,\hat{1})$. $\square$

Looking at the defining recurrence (3.15) for the Möbius function, we see that Corollary 3.9.3 gives a similar recurrence, but in general with many fewer terms. Some applications of Corollary 3.9.3 will be given soon. First, we give some other consequences of Theorem 3.9.2. The next result is known as the *Crosscut Theorem*.

3.9.4 Corollary (Crosscut Theorem). *Let L be a finite lattice, and let X be a subset of L such that (a) $\hat{1} \notin X$, and (b) if $s \in L$ and $s \ne \hat{1}$, then $s \le t$ for some $t \in X$. Then*

$$\mu(\hat{0},\hat{1}) = \sum_k (-1)^k N_k, \tag{3.32}$$

where N_k is the number of k-subsets of X whose meet is $\hat{0}$.

Proof. For any $t \in L$, we have in $A(L,K)$ that

$$\hat{1} - t = \sum_{s \le \hat{1}} \delta_s - \sum_{s \le t} \delta_s = \sum_{s \not\le t} \delta_s.$$

Hence by Theorem 3.9.2,

$$\prod_{t \in X} (\hat{1} - t) = \sum_{s} \delta_s,$$

where s ranges over all elements of L satisfying $s \not\le t$ for all $t \in X$. By hypothesis, the only such element is $\hat{1}$. Hence,

$$\prod_{t \in X} (\hat{1} - t) = \delta_{\hat{1}}.$$

If we now expand both sides as linear combinations of elements of L and equate coefficients of $\hat{0}$, the result follows. □

NOTE (for topologists). Let Γ be the set of all subsets of X (as in the previous corollary) whose meet is *not* $\hat{0}$. Then Γ is a simplicial complex, and equation (3.32) asserts that $\widetilde{\chi}(\Gamma) = \mu(\hat{0}, \hat{1})$. Let $P' = P - \{\hat{0}, \hat{1}\}$. Comparing with Proposition 3.8.6, which asserts that $\mu(\hat{0}, \hat{1}) = \widetilde{\chi}(\Delta(P'))$, suggests that the two simplicial complexes Γ and $\Delta(P')$ might have deeper topological similarities than merely having the same Euler characteristic. Indeed, it can be shown that Γ and $\Delta(P')$ are homotopy equivalent, a good example of combinatorial reasoning leading to a stronger topological result.

It is clear that a subset X of L satisfies conditions (a) and (b) of Corollary 3.9.4 if and only if X contains the set A^* of all coatoms (= elements covered by $\hat{1}$) of L. To make the numbers N_k as small as possible, we should take $X = A^*$. Note that if $\hat{0}$ is not the meet of all the coatoms of L, then each $N_k = 0$. Hence, we obtain the following corollary.

3.9.5 Corollary. *If L is a finite lattice for which $\hat{0}$ is not a meet of coatoms, then $\mu(\hat{0}, \hat{1}) = 0$. Dually, if $\hat{1}$ is not a join of atoms, then again $\mu(\hat{0}, \hat{1}) = 0$.*

3.9.6 Example. Let $L = J(P)$ be a finite distributive lattice. The interval $[I, I']$ of L is a boolean algebra if and only if $I' - I$ is an antichain of P. More generally, the join of all atoms of the interval $[I, I']$ (regarded as a sublattice of L) is the order ideal $I \cup M$, where M is the set of minimal elements of the subposet $I' - I$ of P. Hence, I' is a join of atoms of $[I, I']$ if and only if $[I, I']$ is a boolean algebra. From Example 3.8.3 and Corollary 3.9.5, we obtain the Möbius function of L, namely,

$$\mu(I, I') = \begin{cases} (-1)^{\ell(I,I')} = (-1)^{\#(I'-I)}, & \text{if } [I, I'] \text{ is a boolean algebra (i.e., if} \\ & \quad I' - I \text{ is an antichain of } P), \\ 0, & \text{otherwise.} \end{cases}$$

3.10 The Möbius Function of a Semimodular Lattice

We wish to apply the dualized form of Corollary 3.9.3 to a finite semimodular lattice L of rank n with rank function ρ. Pick a to be an atom of L. Suppose $a \vee t = \hat{1}$. If also $a \leq t$, then $t = \hat{1}$. Hence, either $t \wedge a = \hat{0}$ or $t = \hat{1}$. Now from the definition of semimodularity, we have $\rho(t) + \rho(a) \geq \rho(t \wedge a) + \rho(t \vee a)$, so either $t = \hat{1}$ or $\rho(t) + 1 \geq 0 + n$. Hence, either $t = \hat{1}$, or t is a coatom. From Corollary 3.9.3 (dualized) there follows

$$\mu(\hat{0}, \hat{1}) = - \sum_{\substack{\text{coatoms } t \\ \text{such that} \\ t \not\geq a}} \mu(\hat{0}, t). \tag{3.33}$$

Since every interval of a semimodular lattice is again semimodular (e.g., by Proposition 3.3.2), we conclude from equation (3.33) and induction on n the following result, mentioned at the end of Section 3.8.

3.10.1 Proposition. *The Möbius function of a finite semimodular lattice alternates in sign.*

Since $(-1)^{\ell(s,t)}\mu(s,t)$ is a nonnegative integer for any $s \leq t$ in a finite semimodular lattice L, we can ask whether this integer actually counts something associated with the structure of L. This question will be answered in Section 3.14.

We now turn to two of the most important examples of semimodular lattices.

3.10.2 Example. Let q be a prime power, and let $V_n = \mathbb{F}_q^n$, an n-dimensional vector space over the finite field $\mathbb{F}_q$. (Any n-dimensional vector space over $\mathbb{F}_q$ will do, but for definiteness we choose $\mathbb{F}_q^n$.) Let $B_n(q)$ denote the poset of all subspaces of V_n, ordered by inclusion, as defined in Example 3.1.1(e). We observed in Section 3.3 that $B_n(q)$ is a graded lattice of rank n, where the rank $\rho(W)$ of a subspace is just its dimension. We also mentioned that since any two subspaces W, W' of V satisfy the "modular equality"

$$\dim W + \dim W' = \dim(W \cap W') + \dim(W \cup W'),$$

it follows from equation (3.6) that $B_n(q)$ is in fact a *modular lattice*. Since every subspace of $B_n(q)$ is the span of its one-dimensional subspaces, $B_n(q)$ is also a *geometric lattice*. The interval $[W, W']$ of $B_n(q)$ is isomorphic to the lattice of subspaces of the quotient space W'/W, so $[W, W'] \cong B_m(q)$, where $m = \ell(W, W') = \dim W' - \dim W$. Hence, $\mu(W, W')$ depends only on the integer $\ell = \ell(W, W')$, so we write $\mu_\ell = \mu(W, W')$. It is now an easy task to compute μ_ℓ using equation (3.33). Let a be an element of $B_n(q)$ of rank 1 (i.e., an atom). Now $B_n(q)$ has a total of $\binom{\boldsymbol{n}}{\boldsymbol{n-1}} = q^{n-1} + q^{n-2} + \cdots + 1$ coatoms, of which $\binom{\boldsymbol{n-1}}{\boldsymbol{n-2}} = q^{n-2} + q^{n-3} + \cdots + 1$ lie above a. Hence, there are q^{n-1} coatoms t satisfying $t \not\geq a$, so from equation (3.33) we have

$$\mu_n = -q^{n-1}\mu_{n-1}.$$

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Together with the initial condition $\mu_0 = 1$, there follows

$$\mu_n = (-1)^n q^{\binom{n}{2}}. \tag{3.34}$$

3.10.3 Example. We give one simple example of the use of equation (3.34). We wish to count the number of spanning subsets of V_n. For the purpose of this example, we say that the empty set $\emptyset$ spans no space, while the subset $\{0\}$ spans the zero-dimensional subspace $\{0\}$.* If $W \in B_n(q)$, then let $f(W)$ be the number of subsets of V_n whose span is W, and let $g(W)$ be the number whose span is contained in W. Hence, $g(W) = 2^{q^{\dim W}} - 1$, since $\emptyset$ has no span. Clearly,

$$g(W) = \sum_{T \le W} f(T),$$

so by Möbius inversion in $B_n(q)$,

$$f(W) = \sum_{T \le W} g(T)\mu(T,W).$$

Putting $W = V_n$, there follows

$$\begin{aligned} f(V_n) &= \sum_{T \in B_n(q)} g(T)\mu(T,V_n) \\ &= \sum_{k=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-1)^{n-k} q^{\binom{n-k}{2}} \left(2^{q^k} - 1\right). \end{aligned}$$

3.10.4 Example. Let Π_S denote the set of all partitions of the finite set S, and write Π_n for $\Pi_{[n]}$. As in Example 3.1.1(d), we partially order Π_S by *refinement*; that is, define $\pi \le \sigma$ if every block of π is contained in a block of σ. For instance, Π_1, Π_2, and Π_3 are shown in Figure 3.20. It is easy to check that Π_n is graded of rank $n-1$. The rank $\rho(\pi)$ of $\pi \in \Pi_n$ is equal to $n - (\text{number of blocks of } \pi) = n - \#\pi$. Hence, the rank-generating function of Π_n is given by

$$f(\Pi_n, x) = \sum_{k=0}^{n-1} S(n,n-k)x^k, \tag{3.35}$$

where $S(n,n-k)$ is a Stirling number of the second kind. If $\pi,\sigma \in \Pi_n$, then $\pi \wedge \sigma$ has as blocks the nonempty sets $B \cap C$, where $B \in \pi$ and $C \in \sigma$. Hence, Π_n is a meet-semilattice. Since the partition of $[n]$ with a single block $[n]$ is a $\hat{1}$ for Π_n, it follows from Proposition 3.3.1 that Π_n is a *lattice*.

Suppose that $\pi = \{B_1,\dots,B_k\}$. Then the interval $[\pi,\hat{1}]$ is isomomorphic in an obvious way to Π_π, the lattice of partitions of the set $\{B_1,\dots,B_k\}$. Hence, $[\pi,\hat{1}] \cong \Pi_k$. Now it is easy to see that in Π_k, the join of any two distinct atoms has rank two. Morever, any $\pi \in \Pi_n$ is the join of those atoms $\{B_1,\dots,B_{n-1}\}$ such

* The standard convention is that the empty set spans $\{0\}$. If we wish to retain this convention, then we need to enlarge $B_n(q)$ by adding $\emptyset$ below $\{0\}$.

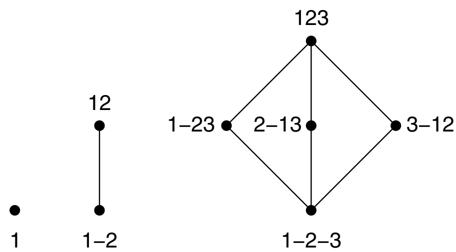

Figure 3.20 Small partition lattices.

that $\#B_1 = 2$ (so $\#B_i = 1$ for $2 \le i \le n-1$) and B_1 is a subset of some block of π. Hence Π_n is a *geometric lattice*.

The previous paragraph determined the structure of $[\pi, \hat{1}]$. Let us now consider the structure of any interval $[\sigma, \pi]$. Suppose that $\pi = \{B_1, \dots, B_k\}$ and that B_i is partitioned into λ_i blocks in σ. We leave to the reader the easy argument that

$$[\sigma,\pi] \cong \Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_k}.$$

In particular, $[\hat{0}, \pi] \cong \Pi_1^{a_1} \times \cdots \times \Pi_n^{a_n}$ if π has a_i blocks of size i.

NOTE. In analogy to the type of a permutation, we define the *type* of $\pi \in \Pi_n$ by $\text{type}(\pi) = (a_1, \dots, a_n)$ if π has a_i blocks of size i, for $1 \le i \le n$. It is easy to prove in analogy with Proposition 1.3.2 that the number of partitions $\pi \in \Pi_n$ of type $(a_1, \dots, a_n)$ is given by

$$\#\{\pi \in \Pi_n : \text{type}(\pi) = (a_1, \dots, a_n)\} = \frac{n!}{1!^{a_1} a_1! \, 2!^{a_2} a_2! \cdots n!^{a_n} a_n!}. \tag{3.36}$$

As an example of the structure of $[\sigma, \pi]$, let $\sigma = $ 1-2-3-45-67-890 and $\pi = $ 14567-2890-3. Then

$$[\sigma,\pi] \cong \Pi_{\{1,45,67\}} \times \Pi_{\{2,890\}} \times \Pi_{\{3\}} \cong \Pi_3 \times \Pi_2 \times \Pi_1.$$

Now set $\mu_n = \mu(\hat{0}, \hat{1})$, where μ is the Möbius function of Π_n. If $[\sigma, \pi] = \Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_k}$, then by Proposition 3.8.2 we have $\mu(\sigma, \pi) = \mu_{\lambda_1} \times \mu_{\lambda_2} \times \cdots \times \mu_{\lambda_k}$. Hence to determine μ completely, it suffices to compute μ_n. Although Π_n is geometric so that equation (3.33) applies, it is easier to appeal directly to Corollary 3.9.3. Pick a to be the partition with the two blocks $\{1, 2, \dots, n-1\}$ and $\{n\}$. An element t of Π_n satisfies $t \wedge a = \hat{0}$ if and only if $t = \hat{0}$ or t is an atom whose unique two-element block has the form $\{i, n\}$ for some $i \in [n-1]$. The interval $[t, \hat{1}]$ is isomorphic to Π_{n-1}, so from Corollary 3.9.3 we have $\mu_n = -(n-1)\mu_{n-1}$. Since $\mu_0 = 1$, we conclude

$$\mu_n = (-1)^{n-1}(n-1)!. \tag{3.37}$$

There are many other ways to prove this important result, some of which we shall consider later. Let us simply point out here the more general result (which is proved

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in Example 3.11.11)

$$\sum_{\pi \in \Pi_n} \mu(\hat{0},\pi)x^{\#\pi} = (x)_n = x(x-1)\cdots(x-n+1). \tag{3.38}$$

To get equation (3.37), equate coefficients of x.

Equation (3.38) can be put in the following more general context. Let P be a finite graded poset with $\hat{0}$, say of rank n. Define the *characteristic polynomial* $\chi_P(x)$ of P by

$$\begin{aligned}\chi_P(x) &= \sum_{t \in P} \mu(\hat{0},t)x^{n-\rho(t)} \\ &= \sum_{k=0}^{n} w_k x^{n-k}, \text{ say.}\end{aligned} \tag{3.39}$$

The coefficient w_k is called the kth *Whitney number of P of the first kind*:

$$w_k = \sum_{\substack{t \in P \\ \rho(t)=k}} \mu(\hat{0},t).$$

In this context, the number of elements of P of rank k is denoted W_k and is called the kth *Whitney number of P of the second kind*. Thus the rank-generating function $F(P,x)$ of P is given by

$$\begin{aligned}F(P,x) &= \sum_{t \in P} x^{\rho(t)} \\ &= \sum_{k=0}^{n} W_k x^k.\end{aligned}$$

It follows from equation (3.38) that

$$\chi_{\Pi_n}(x) = (x-1)(x-2)\cdots(x-n+1),$$

since Π_n has rank $n-1$ and $\#\pi = n - \rho(\pi)$. Hence from Proposition 1.3.7, we have $w_k = s(n,n-k)$, a Stirling number of the first kind. Moreover, equation (3.35) yields $W_k = S(n,n-k)$ for the lattice Π_n. For a poset-theoretic reason for the inverse relationship between $S(n,k)$ and $s(n,k)$ given by Proposition 1.9.1(a), see Exercise 3.130(a).

3.11 Hyperplane Arrangements

3.11.1 Basic Definitions

In this section, we give an interesting geometric application of Möbius functions which has a vast number of further applications and extensions. The basic geometric concept to concern us will be a (finite) *hyperplane arrangement* (or just

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arrangement for short), that is, a finite set $\mathcal{A}$ of affine hyperplanes in a finite-dimensional vector space $V \cong K^n$, where K is a field. To make sure that the definition of a hyperplane arrangement is clear, we define a *linear hyperplane* to be an $(n-1)$-dimensional subspace H of V, that is,

$$H = \{v \in V : \alpha \cdot v = 0\},$$

where α is a fixed nonzero vector in V and $\alpha \cdot v$ is the usual dot product (after identifying V with K^n):

$$(\alpha_1, \ldots, \alpha_n) \cdot (v_1, \ldots, v_n) = \sum \alpha_i v_i. \tag{3.40}$$

An *affine hyperplane* is a translate J of a linear hyperplane, that is,

$$J = \{v \in V : \alpha \cdot v = a\},$$

where α is a fixed nonzero vector in V and $a \in K$. The vector α is the *normal* to J, unique up to multiplication by a nonzero scalar.

Let $\mathcal{A}$ be an arrangement in the vector space V. The *dimension* $\dim(\mathcal{A})$ of $\mathcal{A}$ is defined to be $\dim(V)$ $(= n)$, while the *rank* $\operatorname{rank}(\mathcal{A})$ of $\mathcal{A}$ is the dimension of the space spanned by the normals to the hyperplanes in $\mathcal{A}$. We say that $\mathcal{A}$ is *essential* if $\operatorname{rank}(\mathcal{A}) = \dim(\mathcal{A})$. Suppose that $\operatorname{rank}(\mathcal{A}) = r$, and take $V = K^n$. Let Y be a complementary space in K^n to the subspace X spanned by the normals to hyperplanes in $\mathcal{A}$. Define

$$W = \{v \in V : v \cdot y = 0, \text{for all } y \in Y\}.$$

If $K = \mathbb{R}$, then we can simply take $W = X$. (More generally, if $\operatorname{char}(K) = 0$, then we can take $W = X$ provided that we modify the definition of the scalar product (3.40).) By elementary linear algebra, we have

$$\operatorname{codim}_W(H \cap W) = 1 \tag{3.41}$$

for all $H \in \mathcal{A}$. In other words, $H \cap W$ is a hyperplane of W, so the set $\mathcal{A}_W := \{H \cap W : H \in \mathcal{A}\}$ is an essential arrangement in W. Moreover, the arrangements $\mathcal{A}$ and $\mathcal{A}_W$ are "essentially the same," meaning in particular that they have the same intersection poset (as defined below in Subsection 3.11.2). Let us call $\mathcal{A}_W$ the *essentialization* of $\mathcal{A}$, denoted $\operatorname{ess}(\mathcal{A})$. When $K = \mathbb{R}$ and we take $W = X$, then the arrangement $\mathcal{A}$ is obtained from $\mathcal{A}_W$ by "stretching" the hyperplane $H \cap W \in \mathcal{A}_W$ orthogonally to W. Thus, if $W^\perp$ denotes the orthogonal complement to W in V, then $H' \in \mathcal{A}_W$ if and only if $H' \oplus W^\perp \in \mathcal{A}$. Note that in characteristic p this type of reasoning fails since the orthogonal complement of a subspace W can intersect W in a subspace of dimension greater than 0.

3.11.1 Example. Let $\mathcal{A}$ consist of the lines $x = a_1, \ldots, x = a_k$ in K^2 (with coordinates x and y). Then we can take W to be the x-axis, and $\operatorname{ess}(\mathcal{A})$ consists of the points $x = a_1, \ldots, x = a_k$ in K.

3.11.2 The Intersection Poset and Characteristic Polynomial

Let $\mathcal{A}$ be an arrangement in a vector space V, and let $L(\mathcal{A})$ be the set of all *nonempty* intersections of hyperplanes in $\mathcal{A}$, including V itself as the intersection over the empty set. Define $s \leq t$ in $L(\mathcal{A})$ if $s \supseteq t$ (as subsets of V). In other words, $L(\mathcal{A})$ is partially ordered by *reverse* inclusion. The vector space V is the $\hat{0}$ element of $L(\mathcal{A})$. We call $L(\mathcal{A})$ the *intersection poset* of $\mathcal{A}$. It is the fundamental combinatorial object associated with an arrangement.

An arrangement $\mathcal{A}$ is called *central* if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$. We can translate all the hyperplanes in a central arrangement by a fixed vector so that $\mathbf{0} \in \bigcap_{H\in\mathcal{A}} H$, where $\mathbf{0}$ denotes the origin of V. Thus, each hyperplane $H \in \mathcal{A}$ is a *linear* hyperplane and can therefore be identified with a point f_H in the dual space V^* (namely, if H is defined by $\alpha \cdot v = 0$, then H corresponds to the linear functional $f(v) = \alpha \cdot v$). Let L be the geometric lattice, as defined after Proposition 3.3.3, consisting of all intersections of $\{f_H : H \in \mathcal{A}\}$ with linear subspaces of V^*, ordered by inclusion. It is straightforward to see that $L \cong L(\mathcal{A})$. We have therefore proved the following result.

3.11.2 Proposition. *Let $\mathcal{A}$ be a (finite) hyperplane arrangement in a vector space V. If $\mathcal{A}$ is central, then $L(\mathcal{A})$ is a geometric lattice. For any $\mathcal{A}$, every interval of $L(\mathcal{A})$ is a geometric lattice.*

We now define the *characteristic polynomial* $\chi_{\mathcal{A}}(x)$ of the arrangement $\mathcal{A}$ by

$$\chi_{\mathcal{A}}(x) = \sum_{t\in L(\mathcal{A})} \mu(\hat{0},t)x^{\dim(t)}.$$

Compare this definition with that of the characteristic polynomial $\chi_{L(\mathcal{A})}$ of the poset $L(\mathcal{A})$ itself (equation (3.39)). If $\mathcal{A}$ is essential, then $\chi_{\mathcal{A}}(x) = \chi_{L(\mathcal{A})}(x)$; in general,

$$\chi_{\mathcal{A}}(x) = x^{n-r}\chi_{L(\mathcal{A})}(x),$$

where $n = \dim(\mathcal{A})$ and $r = \operatorname{rank}(\mathcal{A})$. Since every interval of $L(\mathcal{A})$ is a geometric lattice, it follows from Proposition 3.10.1 that the coefficients of $\chi_{\mathcal{A}}(x)$ alternate in sign. More precisely, we have

$$\chi_{\mathcal{A}}(x) = x^n - a_1x^{n-1} + \cdots + (-1)^{n-r}a_{n-r}x^{n-r},$$

where each $a_i > 0$. Note that $a_1 = \#\mathcal{A}$, the number of hyperplanes in $\mathcal{A}$.

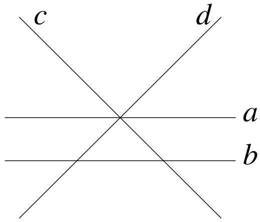
We now use the Crosscut Theorem (Corollary 3.9.4) to give a formula (Proposition 3.11.3) for the characteristic polynomial $\chi_{\mathcal{A}}(x)$. Next we employ this formula for $\chi_{\mathcal{A}}(x)$ to give a recurrence (Proposition 3.11.5) for $\chi_{\mathcal{A}}(x)$. We then use this recurrence to give a formula (Theorem 3.11.7) for the number of regions and number of (relatively) bounded regions of a real arrangement.

Extending slightly the definition of a central arrangement, call any subset $\mathcal{B}$ of $\mathcal{A}$ *central* if $\bigcap_{H\in\mathcal{B}} H \neq \emptyset$.

3.11.3 Proposition. *Let $\mathcal{A}$ be an arrangement in an n-dimensional vector space. Then*

$$\chi_{\mathcal{A}}(x) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \mathcal{B}\,\text{central}}} (-1)^{\#\mathcal{B}} x^{n-\operatorname{rank}(\mathcal{B})}. \tag{3.42}$$

3.11.4 Example. Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^2$ shown below.



The following table shows all central subsets $\mathcal{B}$ of $\mathcal{A}$ and the values of $\#\mathcal{B}$ and $\operatorname{rank}(\mathcal{B})$.

$\mathcal{B}$	$\#\mathcal{B}$	$\operatorname{rank}(\mathcal{B})$
$\emptyset$	0	0
a	1	1
b	1	1
c	1	1
d	1	1
ac	2	2
ad	2	2
bc	2	2
bd	2	2
cd	2	2
acd	3	2

It follows that $\chi_{\mathcal{A}}(x) = x^2 - 4x + (5-1) = x^2 - 4x + 4$.

Proof of Proposition 3.11.3. Let $t \in L(\mathcal{A})$. Let

$$\Lambda_t = \{s \in L(\mathcal{A}) \,:\, s \le t\},$$

the principal order ideal generated by t. Define

$$\mathcal{A}_t = \{H \in \mathcal{A} \,:\, H \le t \text{ (i.e., } t \subseteq H)\}. \tag{3.43}$$

By the Crosscut Theorem (Corollary 3.9.4), we have

$$\mu(\hat{0},t) = \sum_k (-1)^k N_k(t),$$

where $N_k(t)$ is the number of k-subsets of $\mathcal{A}_t$ with join t. In other words,

$$\mu(\hat{0},t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}_t \\ t=\bigcap_{H\in\mathcal{B}} H}} (-1)^{\#\mathcal{B}}.$$

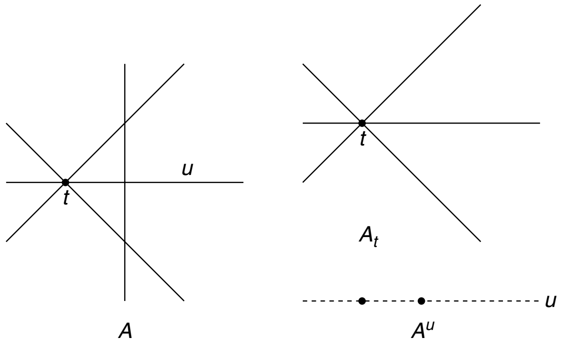


Figure 3.21 An illustration of the definitions of $\mathcal{A}_t$ and $\mathcal{A}^K$.

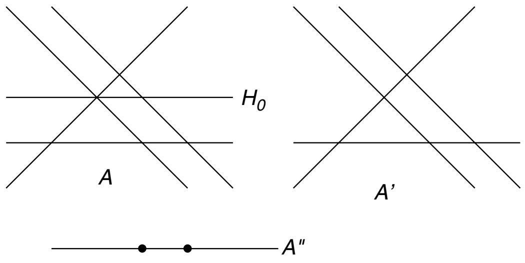


Figure 3.22 A triple of arrangements.

Note that $t = \bigcap_{H\in\mathcal{B}} H$ implies that $\operatorname{rank}(\mathcal{B}) = n - \dim t$. Now multiply both sides by $x^{\dim(t)}$ and sum over t to obtain equation (3.42). □

The characteristic polynomial $\chi_{\mathcal{A}}(x)$ satisfies a fundamental recurrence, which we now describe. Let $\mathcal{A}$ be an arrangement in the vector space V. A *subarrangement* of $\mathcal{A}$ is a subset $\mathcal{B} \subseteq \mathcal{A}$. Thus, $\mathcal{B}$ is also an arrangement in V. If $t \in L(\mathcal{A})$, then let $\mathcal{A}_t$ be the subarrangement of equation (3.43). Also define an arrangement $\mathcal{A}^t$ in the affine subspace $t \in L(\mathcal{A})$ by

$$\mathcal{A}^t = \{t \cap H \neq \emptyset : H \in \mathcal{A} - \mathcal{A}_t\}. \tag{3.44}$$

Note that if $t \in L(\mathcal{A})$, then

$$\begin{aligned} L(\mathcal{A}_t) &\cong \Lambda_t := \{s \in L(\mathcal{A}) : s \leq t\}, \\ L(\mathcal{A}^t) &\cong V_t := \{s \in L(\mathcal{A}) : s \geq t\}. \end{aligned} \tag{3.45}$$

Figure 3.21 shows an arrangement $\mathcal{A}$, two elements $t, u \in L(\mathcal{A})$, and the arrangements $\mathcal{A}_t$ and $\mathcal{A}^u$.

Choose $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a *triple* of arrangements with *distinguished hyperplane* H_0. An example is shown in Figure 3.22.

3.11.5 Proposition (Deletion-Restriction). *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple of real arrangements. Then*

$$\chi_{\mathcal{A}}(x) = \chi_{\mathcal{A}'}(x) - \chi_{\mathcal{A}''}(x).$$

Proof. Let $H_0 \in \mathcal{A}$ be the hyperplane defining the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. Split the sum on the right-hand side of (3.42) into two sums, depending on whether $H_0 \notin \mathcal{B}$ or $H_0 \in \mathcal{B}$. In the former case, we get

$$\sum_{\substack{H_0 \notin \mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} x^{n-\mathrm{rank}(\mathcal{B})} = \chi_{\mathcal{A}'}(x).$$

In the latter case, set $\mathcal{B}_1 = (\mathcal{B} - \{H_0\})^{H_0}$, a central arrangement in $H_0 \cong K^{n-1}$ and a subarrangement of $\mathcal{A}^{H_0} = \mathcal{A}''$. Suppose that S is a set of $r \geq 1$ hyperplanes in $\mathcal{A}$ that all have the same intersection with H_0. Then

$$\sum_{\emptyset \neq T \subseteq S} (-1)^{\#T} = -1,$$

the same result we would get if $r = 1$. Since $\#\mathcal{B}_1 = \#\mathcal{B} - 1$ and $\mathrm{rank}(\mathcal{B}_1) = \mathrm{rank}(\mathcal{B}) - 1$, we get

$$\begin{aligned}\sum_{\substack{H_0 \in \mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} x^{n-\mathrm{rank}(\mathcal{B})} &= \sum_{\mathcal{B}_1 \in \mathcal{A}''} (-1)^{\#\mathcal{B}_1+1} x^{(n-1)-\mathrm{rank}(\mathcal{B}_1)} \\ &= -\chi_{\mathcal{A}''}(x),\end{aligned}$$

and the proof follows. □

3.11.3 Regions

Hyperplane arrangements have special combinatorial properties when $K = \mathbb{R}$, which we assume for the remainder of this subsection. A *region* of an arrangement $\mathcal{A}$ (defined over $\mathbb{R}$) is a connected component of the complement X of the hyperplanes:

$$X = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H.$$

Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$, and let

$$r(\mathcal{A}) = \#\mathcal{R}(\mathcal{A}),$$

the number of regions. For instance, the arrangement $\mathcal{A}$ of Figure 3.23 has $r(\mathcal{A}) = 14$.

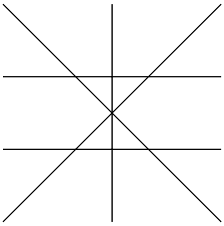


Figure 3.23 An arrangement with 14 regions and four bounded regions.

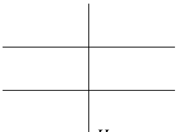
It is a simple exercise to show that every region $R \in \mathcal{R}(\mathcal{A})$ is open and convex (continuing to assume $K = \mathbb{R}$), and hence homeomorphic to the interior of an n-dimensional ball $\mathbb{B}^n$. Note that if W is the subspace of V spanned by the normals to the hyperplanes in $\mathcal{A}$, then the map $R \mapsto R \cap W$ is a bijection between $\mathcal{R}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_W)$. We say that a region $R \in \mathcal{R}(\mathcal{A})$ is *relatively bounded* if $R \cap W$ is bounded. If $\mathcal{A}$ is essential, then relatively bounded is the same as bounded. We write $b(\mathcal{A})$ for the number of relatively bounded regions of $\mathcal{A}$. For instance, in Example 3.11.1, take $K = \mathbb{R}$ and $a_1 < a_2 < \cdots < a_k$. Then the relatively bounded regions are the regions $a_i < x < a_{i+1}$, $1 \leq i \leq k-1$. In $\mathrm{ess}(\mathcal{A})$, they become the (bounded) open intervals (a_i, a_{i+1}). There are also two regions of $\mathcal{A}$ that are not relatively bounded, namely, $x < a_1$ and $x > a_k$. As another example, the arrangement of Figure 3.23 is essential and has four bounded regions.

3.11.6 Lemma. *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of real arrangements with distinguished hyperplane H_0. Then*

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}''),$$

$$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}''), & \text{if } \mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}'), \\ 0, & \text{if } \mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}') + 1. \end{cases}$$

NOTE. If $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}')$, then also $\mathrm{rank}(\mathcal{A}) = 1 + \mathrm{rank}(\mathcal{A}'')$. The following figure illustrates the situation when $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}') + 1$.



Proof. Note that $r(\mathcal{A})$ equals $r(\mathcal{A}')$ plus the number of regions of $\mathcal{A}'$ cut into two regions by H_0. Let R' be such a region of $\mathcal{A}'$. Then $R' \cap H_0 \in \mathcal{R}(\mathcal{A}'')$. Conversely, if $R'' \in \mathcal{R}(\mathcal{A}'')$, then points near R'' on either side of H_0 belong to the same region $R' \in \mathcal{R}(\mathcal{A}')$, since any $H \in \mathcal{R}(\mathcal{A}')$ separating them would intersect R''. Thus, R' is cut in two by H_0. We have established a bijection between regions of $\mathcal{A}'$ cut into two by H_0 and regions of $\mathcal{A}''$, establishing the first recurrence.

The second recurrence is proved analogously; the details are omitted. □

We come to one of the central theorems in the subject of hyperplane arrangements.

3.11.7 Theorem. *Let $\mathcal{A}$ be an arrangement in an n-dimensional real vector space. Then*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1), \tag{3.46}$$

$$b(\mathcal{A}) = (-1)^{\mathrm{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1). \tag{3.47}$$

Proof. Equation (3.46) holds for $\mathcal{A} = \emptyset$, since $r(\emptyset) = 1$ and $\chi_{\emptyset}(x) = x^n$. By Lemma 3.11.6 and Proposition 3.11.5, both $r(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(-1)$ satisfy the same recurrence, so the proof of (3.46) follows.

Now consider equation (3.47). Again it holds for $\mathcal{A} = \emptyset$ since $b(\emptyset) = 1$. (Recall that $b(\mathcal{A})$ is the number of *relatively* bounded regions. When $\mathcal{A} = \emptyset$, the entire ambient space $\mathbb{R}^n$ is relatively bounded.) Now

$$\chi_{\mathcal{A}}(1) = \chi_{\mathcal{A}'}(1) - \chi_{\mathcal{A}''}(1).$$

Let $d(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1)$. If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A}'') + 1$, then $d(\mathcal{A}) = d(\mathcal{A}') + d(\mathcal{A}'')$. If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') + 1$ then $b(\mathcal{A}) = 0$ [why?] and $L(\mathcal{A}') \cong L(\mathcal{A}'')$ [why?]. Thus, $d(\mathcal{A}) = 0$. Hence in all cases, $b(\mathcal{A})$ and $d(\mathcal{A})$ satisfy the same recurrence, so $b(\mathcal{A}) = d(\mathcal{A})$. □

As an application of Theorem 3.11.7, we compute the number of regions of an arrangement whose hyperplanes are in *general position*, that is,

$$\{H_1, \ldots, H_p\} \subseteq \mathcal{A},\ p \le n \Rightarrow \dim(H_1 \cap \cdots \cap H_p) = n - p,$$
$$\{H_1, \ldots, H_p\} \subseteq \mathcal{A},\ p > n \Rightarrow H_1 \cap \cdots \cap H_p = \emptyset.$$

For instance, if $n = 2$ then a set of lines is in general position if and only if no two are parallel and no three meet at a point.

3.11.8 Proposition (general position). *Let $\mathcal{A}$ be an n-dimensional arrangement of m hyperplanes in general position. Then*

$$\chi_{\mathcal{A}}(x) = x^n - m x^{n-1} + \binom{m}{2} x^{n-2} - \cdots + (-1)^n \binom{m}{n}.$$

In particular, if $\mathcal{A}$ is a real arrangement, then

$$r(\mathcal{A}) = 1 + m + \binom{m}{2} + \cdots + \binom{m}{n},$$

$$\begin{aligned} b(\mathcal{A}) &= (-1)^n \left(1 - m + \binom{m}{2} - \cdots + (-1)^n \binom{m}{n} \right) \\ &= \binom{m-1}{n}. \end{aligned}$$

Proof. Every $\mathcal{B} \subseteq \mathcal{A}$ with $\#\mathcal{B} \le n$ defines an element $x_{\mathcal{B}} = \bigcap_{H \in \mathcal{B}} H$ of $L(\mathcal{A})$. Hence, $L(\mathcal{A})$ is a *truncated boolean algebra*:

$$L(\mathcal{A}) \cong \{S \subseteq [m] : \#S \le n\},$$

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ordered by inclusion. If $t \in L(\mathcal{A})$ and $\operatorname{rank}(t) = k$, then $[\hat{0}, t] \cong B_k$, a boolean algebra of rank k. By equation (3.18), there follows $\mu(\hat{0}, t) = (-1)^k$. Hence,

$$\begin{aligned}\chi_{\mathcal{A}}(x) &= \sum_{\substack{S \subseteq [m] \\ \#S \le n}} (-1)^{\#S} x^{n-\#S} \\ &= x^n - m x^{n-1} + \cdots + (-1)^n \binom{m}{n}.\end{aligned}$$

□

3.11.4 The Finite Field Method

In this subsection we will describe a method based on finite fields for computing the characteristic polynomial of an arrangement defined over $\mathbb{Q}$. We will then give two examples; further examples may be found in Exercise 3.115.

Suppose that the arrangement $\mathcal{A}$ is defined over $\mathbb{Q}$. By multiplying each hyperplane equation by a suitable integer, we may assume $\mathcal{A}$ is defined over $\mathbb{Z}$. In that case, we can take coefficients modulo a prime p and get an arrangement $\mathcal{A}_q$ defined over the finite field $\mathbb{F}_q$, where $q = p^r$. We say that $\mathcal{A}$ has *good reduction* mod p (or over $\mathbb{F}_q$) if $L(\mathcal{A}) \cong L(\mathcal{A}_q)$.

For instance, let $\mathcal{A}$ be the affine arrangement in $\mathbb{Q}^1 = \mathbb{Q}$ consisting of the points 0 and 10. Then $L(\mathcal{A})$ contains three elements, namely, $\mathbb{Q}$, $\{0\}$, and $\{10\}$. If $p \neq 2, 5$ then 0 and 10 remain distinct, so $\mathcal{A}$ has good reduction. On the other hand, if $p = 2$ or $p = 5$ then $0 = 10$ in $\mathbb{F}_p$, so $L(\mathcal{A}_p)$ contains just two elements. Hence, $\mathcal{A}$ has bad reduction when $p = 2, 5$.

3.11.9 Proposition. *Let $\mathcal{A}$ be an arrangement defined over $\mathbb{Z}$. Then $\mathcal{A}$ has good reduction for all but finitely many primes p.*

Proof. Let $H_1, \dots, H_j$ be affine hyperplanes, where H_i is given by the equation $\alpha_i \cdot x = a_i$ ($\alpha_i \in \mathbb{Z}^n, a_i \in \mathbb{Z}$). By linear algebra, we have $H_1 \cap \cdots \cap H_j \neq \emptyset$ if and only if

$$\operatorname{rank} \begin{bmatrix} \alpha_1 & a_1 \\ \vdots & \vdots \\ \alpha_j & a_j \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \end{bmatrix}. \tag{3.48}$$

Moreover, if (3.48) holds, then

$$\dim(H_1 \cap \cdots \cap H_j) = n - \operatorname{rank} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \end{bmatrix}.$$

Now for any $r \times s$ matrix A, we have $\operatorname{rank}(A) \ge t$ if and only if some $t \times t$ submatrix B satisfies $\det(B) \neq 0$. It follows that $L(\mathcal{A}) \not\cong L(\mathcal{A}_p)$ if and only if at least one member S of a certain finite collection $\mathcal{S}$ of subsets of integer matrices B satisfies the following condition:

$$(\forall B \in S)\ \det(B) \neq 0 \text{ but } \det(B) \equiv 0 \,(\mathrm{mod}\ p).$$

This can only happen for finitely many p, namely, for certain B we must have $p \mid \det(B)$, so $L(\mathcal{A}) \cong L(\mathcal{A}_p)$ for p sufficiently large. □

The main result of this subsection is the following. Like many fundamental results in combinatorics, the proof is easy but the applicability very broad.

3.11.10 Theorem. *Let $\mathcal{A}$ be an arrangement in $\mathbb{Q}^n$, and suppose that $L(\mathcal{A}) \cong L(\mathcal{A}_q)$ for some prime power q. Then*

$$\begin{aligned}\chi_{\mathcal{A}}(q) &= \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right)\\ &= q^n - \# \bigcup_{H \in \mathcal{A}_q} H.\end{aligned}$$

Proof. Let $t \in L(\mathcal{A}_q)$ so $\#t = q^{\dim(t)}$. Here $\dim(t)$ can be computed either over $\mathbb{Q}$ or $\mathbb{F}_q$. Define two functions $f, g : L(\mathcal{A}_q) \to \mathbb{Z}$ by

$$f(t) = \#t$$

$$g(t) = \#\left(t - \bigcup_{u > t} u\right).$$

In particular,

$$g(\hat{0}) = g(\mathbb{F}_q^n) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right).$$

Clearly,

$$f(t) = \sum_{u \geq t} g(u).$$

Let μ denote the Möbius function of $L(\mathcal{A}) \cong L(\mathcal{A}_q)$. By the Möbius inversion formula (Proposition 3.7.1),

$$\begin{aligned}g(t) &= \sum_{u \geq t} \mu(t,u) f(u)\\ &= \sum_{u \geq t} \mu(t,u) q^{\dim(u)}.\end{aligned}$$

Put $t = \hat{0}$ to get

$$g(\hat{0}) = \sum_{u} \mu(\hat{0},u) q^{\dim(u)} = \chi_{\mathcal{A}}(q).$$

□

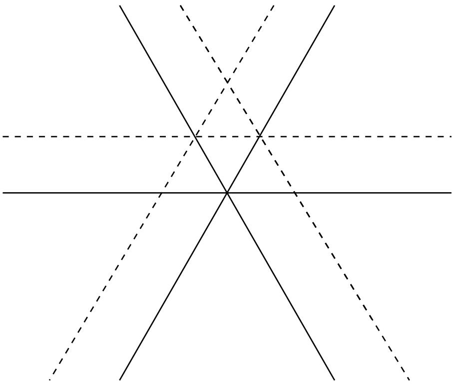

Figure 3.24 The Shi arrangement $\mathcal{S}_3$ in $\ker(x_1+x_2+x_3)$.

3.11.11 Example. The *braid arrangement* $\mathcal{B}_n$ of rank $n-1$ is the arrangement in K^n with hyperplanes $x_i - x_j = 0$ for $1 \le i < j \le n$. The characteristic polynomial of $\mathcal{B}_n$ is particularly easy to compute by the finite field method. Namely, for a large prime p (actually, any prime) $\chi_{\mathcal{B}_n}(p)$ is equal to the number of vectors $(x_1,\dots,x_n) \in \mathbb{F}_p^n$ such that $x_i \neq x_j$ for all $i < j$. There are p choices for x_1, then $p-1$ choices for x_2, etc., giving $\chi_{\mathcal{B}_n}(p) = p(p-1)\cdots(p-n+1) = (p)_n$. Hence,

$$\chi_{\mathcal{B}_n}(x) = (x)_n. \tag{3.49}$$

In fact, it is not hard to see that $L_{\mathcal{B}_n} \cong \Pi_n$, the lattice of partitions of the set $[n]$. (See Exercise 3.108(b).) Thus in particular, we have proved equation (3.38).

3.11.12 Example. In this example, we consider a modification (or deformation) of the braid arrangement called the *Shi arrangement* and denoted $\mathcal{S}_n$. It consists of the hyperplanes

$$x_i - x_j = 0, 1, \quad 1 \le i < j \le n.$$

Thus, $\mathcal{S}_n$ has $n(n-1)$ hyperplanes and $\operatorname{rank}(\mathcal{S}_n) = n-1$. Figure 3.24 shows the Shi arrangement $\mathcal{S}_3$ in $\ker(x_1+x_2+x_3) \cong \mathbb{R}^2$ (i.e., the space $\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1+x_2+x_3 = 0\}$).

3.11.13 Theorem. *The characteristic polynomial of $\mathcal{S}_n$ is given by*

$$\chi_{\mathcal{S}_n}(x) = x(x-n)^{n-1}.$$

Proof. Let p be a large prime. By Theorem 3.11.10, we have

$$\chi_{\mathcal{S}_n}(p) = \#\{(\alpha_1,\dots,\alpha_n) \in \mathbb{F}_p^n : i < j \Rightarrow \alpha_i \neq \alpha_j \text{ and } \alpha_i \neq \alpha_j + 1\}.$$

Choose a weak ordered partition $\pi = (B_1,\dots,B_{p-n})$ of $[n]$ into $p-n$ blocks (i.e., $\bigcup B_i = [n]$ and $B_i \cap B_j = \emptyset$ if $i \neq j$, such that $1 \in B_1$). ("Weak" means that we allow $B_i = \emptyset$.) For $2 \le i \le n$ there are $p-n$ choices for j such that $i \in B_j$, so $(p-n)^{n-1}$ choices in all. We will illustrate the following argument with the example $p = 11$, $n = 6$, and

$$\pi = (\{1,4\},\{5\},\emptyset,\{2,3,6\},\emptyset). \tag{3.50}$$

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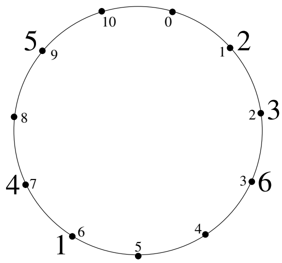
Arrange the elements of $\mathbb{F}_p$ clockwise on a circle. Place $1,2,\ldots,n$ on some n of these points as follows. Place elements of B_1 consecutively (clockwise) in increasing order with 1 placed at some element $\alpha_1 \in \mathbb{F}_p$. Skip a space and place the elements of B_2 consecutively in increasing order. Skip another space and place the elements of B_3 consecutively in increasing order, and so on. For our example, (3.50), say $\alpha_1 = 6$. We then get the following placement of $1,2,\ldots,6$ on $\mathbb{F}_{11}$.



Let α_i be the position (element of $\mathbb{F}_p$) at which i was placed. For our example, we have

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) = (6,1,2,7,9,3).$$

It is easily verified that we have defined a bijection from the $(p-n)^{n-1}$ weak ordered partitions $\pi = (B_1,\ldots,B_{p-n})$ of $[n]$ into $p-n$ blocks such that $1 \in B_1$, together with the choice of $\alpha_1 \in \mathbb{F}_p$, to the set $\mathbb{F}_p^n - \cup_{H\in(\mathcal{S}_n)_p} H$. There are $(p-n)^{n-1}$ choices for π and p choices for α_1, so it follows from Theorem 3.11.10 that $\chi_{\mathcal{S}_n}(p) = p(p-n)^{n-1}$. Hence, $\chi_{\mathcal{S}_n}(x) = x(x-n)^{n-1}$. □

We obtain the following corollary immediately from Theorem 3.11.7.

3.11.14 Corollary. *We have* $r(\mathcal{S}_n) = (n+1)^{n-1}$ *and* $b(\mathcal{S}_n) = (n-1)^{n-1}$.

Note. Since $r(\mathcal{S}_n)$ and $b(\mathcal{S}_n)$ have such simple formulas, it is natural to ask for a direct bijective proof of Corollary 3.11.14. A number of such proofs are known; a sketch that $r(\mathcal{S}_n) = (n+1)^{n-1}$ is given in Exercise 3.111.

3.12 Zeta Polynomials

Let P be a finite poset. If $n \geq 2$, then define $Z(P,n)$ to be the number of multichains $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$ in P. We call $Z(P,n)$ (regarded as a function of n) the *zeta polynomial* of P. First we justify this nomenclature and collect together some elementary properties of $Z(P,n)$.

3.12.1 Proposition. *a. Let b_i be the number of chains $t_1 < t_2 < \cdots < t_{i-1}$ in P. Then $b_{i+2} = \Delta^i Z(P,2)$, $i \geq 0$, where Δ is the finite difference operator. In other words,*

$$Z(P,n) = \sum_{i\geq 2} b_i \binom{n-2}{i-2}. \tag{3.51}$$

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In particular, $Z(P,n)$ is a polynomial function of n whose degree d is equal to the length of the longest chain of P, and whose leading coefficient is $b_{d+2}/d!$. Moreover, $Z(P,2) = \#P$ (as is clear from the definition of $Z(P,n)$).

b. *Since $Z(P,n)$ is a polynomial for all integers $n \geq 2$, we can define it for all $n \in \mathbb{Z}$ (or even all $n \in \mathbb{C}$). Then*

$$Z(P,1) = \chi(\Delta(P)) = 1 + \mu_{\widehat{P}}(\hat{0},\hat{1}),$$

where $\Delta(P)$ denotes the order complex of P.

c. *If P has a $\hat{0}$ and $\hat{1}$, then $Z(P,n) = \zeta^n(\hat{0},\hat{1})$ for all $n \in \mathbb{Z}$ (explaining the term* zeta polynomial*). In particular,*

$$Z(P,-1) = \mu(\hat{0},\hat{1}), \;\; Z(P,0) = 0 \text{ (if } \hat{0} \neq \hat{1}), \text{ and } Z(P,1) = 1.$$

Proof.

a. The number of $(n-1)$-element multichains with support $t_1 < t_2 < \cdots < t_{i-1}$ is $\left(\!\binom{i-1}{n-1-(i-1)}\!\right) = \binom{n-2}{i-2}$, from which equation (3.51) follows. The additional information about $Z(P,n)$ can be read off from (3.51).

b. Putting $n = 1$ in (3.51) yields

$$Z(P,1) = \sum_{i \geq 2} b_i \binom{-1}{i-2} = \sum_{i \geq 2} (-1)^i b_i.$$

Now use Proposition 3.8.5.

c. If P has a $\hat{0}$ and $\hat{1}$, then the number of multichains $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$ is the same as the number of multichains $\hat{0} = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq t_n = \hat{1}$, which is $\zeta^n(\hat{0},\hat{1})$ for $n \geq 2$. There are several ways to see that $Z(P,n)$, as defined by (3.51) for all $n \geq 2$, is equal to $\zeta^n(\hat{0},\hat{1})$ for all $n \in \mathbb{Z}$. For instance, it follows from equation (3.14) that $\Delta^{d+1}\zeta^k|_{k=0} = 0$ (as linear transformations) [why?]. Multiplying by ζ^n gives $\Delta^{d+1}\zeta^n = 0$ for any $n \in \mathbb{Z}$. Hence by Proposition 1.9.2, $\zeta^n(\hat{0},\hat{1})$ is a polynomial function for all $n \in \mathbb{Z}$ and, thus, must agree with (3.51) for all $n \in \mathbb{Z}$. □

If $m \in \mathbb{P}$, then let $\Omega_P(m)$ denote the number of order-preserving maps $\sigma : P \to \boldsymbol{m}$. It follows from Proposition 3.5.1 that $\Omega_P(m) = Z(J(P),m)$. Hence, $\Omega_P(m)$ is a polynomial function of m of degree $p = \#P$ and leading coefficient $e(P)/p!$. (This can easily be seen by a more direct argument.) We call $\Omega_P(m)$ the *order polynomial* of P. Thus, the order polynomial of P is the zeta polynomial of $J(P)$. For further information on order polynomials in a more general setting of labeled posets, see Section 3.15.3.

3.12.2 Example. Let $P = B_d$, the boolean algebra of rank d. Then $Z(B_d,n)$ for $n \geq 1$ is equal to the number of multichains $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S$ of subsets of a d-set S. For any $s \in S$, we can pick arbitrarily the least positive integer $i \in [n]$ for which $s \in S_i$. Hence, $Z(B_d,n) = n^d$. (We can also see this from $Z(B_d,n) = \Omega_{d\mathbf{1}}(n)$, since *any* map $\sigma: d\mathbf{1} \to \boldsymbol{n}$ is order-preserving.) Putting $n = -1$ yields $\mu_{B_d}(\hat{0}, \hat{1}) = (-1)^d$, a third proof of equation (3.18). This computation of $\mu(\hat{0}, \hat{1})$ is an interesting example of a "semicombinatorial" proof. We evaluate $Z(B_d, n)$ combinatorially for $n \geq 1$ and then substitute $n = -1$. Many other theorems involving Möbius functions of posets P can be proved in such a fashion, by proving combinatorially for $n \geq 1$ an appropriate result for $Z(P, n)$ and then letting $n = -1$.

3.13 Rank Selection

Let P be a finite graded poset of rank n, with rank function $\rho\colon P \to [0,n]$. If $S \subseteq [0,n]$ then define the subposet

$$P_S = \{t \in P : \rho(t) \in S\},$$

called the *S-rank-selected subposet* of P. For instance, $P_\emptyset = \emptyset$ and $P_{[0,n]} = P$. Now define $\alpha_P(S)$ (or simply $\alpha(S)$) to be the number of maximal chains of P_S. For instance, $\alpha(i)$ (short for $\alpha(\{i\})$) is just the number of elements of P of rank i. The function $\alpha_P : 2^{[0,n]} \to \mathbb{Z}$ is called the *flag f-vector* of P. Also define $\beta_P(S) = \beta(S)$ by

$$\beta(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha(T). \tag{3.52}$$

Equivalently, by the Principle of Inclusion-Exclusion,

$$\alpha(S) = \sum_{T \subseteq S} \beta(T). \tag{3.53}$$

The function β_P is called the *flag h-vector* of P.

NOTE. The reason for the terminology "flag f-vector" and "flag h-vector" is the following. Let Δ be a finite $(d-1)$-dimensional simplicial complex with f_i i-dimensional faces. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of Δ. Define integers $h_0, \ldots, h_d$ by the condition

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

(Recall that $f_{-1} = 1$ unless $\Delta = \emptyset$.) The vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is called the *h-vector* of Δ and is often more convenient to work with than the f-vector. It is easy to check that for a finite graded poset P with order complex $\Delta = \Delta(P)$, we have

$$f_i(\Delta) = \sum_{\#S = i+1} \alpha_P(S),$$

$$h_i(\Delta) = \sum_{\#S = i} \beta_P(S).$$

Thus α_P and β_P extend in a natural way the counting of faces by dimension (or cardinality) to the counting of *flags* (or chains) of P (which are just faces of $\Delta(P)$) by the ranks of the elements of the flags.

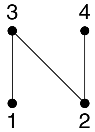

Figure 3.25 A naturally labeled poset.

If μ_S denotes the Möbius function of the poset $\widehat{P_S} = P_S \cup \{\hat{0}, \hat{1}\}$, then it follows from Proposition 3.8.5 that

$$\beta_P(S) = (-1)^{\#S-1}\mu_S(\hat{0}, \hat{1}). \tag{3.54}$$

For this reason, the function β_P is also called the *rank-selected Möbius invariant* of P.

Suppose that P has a $\hat{0}$ and $\hat{1}$. It is then easily seen that

$$\alpha_P(S) = \alpha_P(S \cap [n-1]),$$
$$\beta_P(S) = 0, \text{ if } S \not\subseteq [n-1] \text{ (i.e., if } 0 \in S \text{ or } n \in S).$$

Hence, we lose nothing by restricting our attention to $S \subseteq [n-1]$. For this reason, if we know in advance that P has a $\hat{0}$ and $\hat{1}$ (e.g., if P is a lattice), then we will only consider $S \subseteq [n-1]$.

Equations (3.53) and (3.54) suggest a combinatorial method for interpreting the Möbius function of P. The numbers $\alpha(S)$ have a combinatorial definition. If we can define numbers $\gamma(S) \geq 0$ so that there is a combinatorial proof that $\alpha(S) = \sum_{T \subseteq S} \gamma(T)$, then it follows that $\gamma(S) = \beta(S)$ so $\mu_S(\hat{0}, \hat{1}) = (-1)^{\#S-1}\gamma(S)$. We cannot expect to define $\gamma(S)$ for any P since in general we need not have $\beta(S) \geq 0$. However, there are large classes of posets P for which $\gamma(S)$ can indeed be defined in a nice combinatorial manner. To introduce the reader to this subject, we will consider two special cases here, while the next section is concerned with a more general result of this nature.

Let $L = J(P)$ be a finite distributive lattice of rank n (so $\#P = n$). Let $\omega\colon P \to [n]$ be an order-preserving bijection (i.e., a linear extension of P). In the present context we call ω a *natural labeling* of P. We may identify a linear extension $\sigma: P \to [n]$ of P with a permutation $\omega(\sigma^{-1}(1)), \ldots, \omega(\sigma^{-1}(n))$ of the set $[n]$ of *labels* of P. (Compare Section 3.5, where we identified a linear extension with a permutation of the *elements* of P.) The set of all $e(P)$ permutations of $[n]$ obtained in this way is denoted $\mathcal{L}(P,\omega)$ and is called the *Jordan-Hölder set* of P. For instance, if (P,ω) is given by Figure 3.25, then $\mathcal{L}(P,\omega)$ consists of the five permutations 1234, 2134, 1243, 2143, 2413.

3.13.1 Theorem. *Let $L = J(P)$ as above, and let $S \subseteq [n-1]$. Then $\beta_L(S)$ is equal to the number of permutations $w \in \mathcal{L}(P,\omega)$ with descent set S.*

Proof. Let $S = \{a_1, a_2, \ldots, a_k\}_<$. It follows by definition that $\alpha_L(S)$ is equal to the number of chains $I_1 \subset I_2 \subset \cdots \subset I_k$ of order ideals of P such that $\#I_i = a_i$.

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Given such a chain of order ideals, define a permutation $w \in \mathcal{L}(P,\omega)$ as follows: First, arrange the labels of the elements of I_1 in increasing order. To the right of these arrange the labels of the elements of $I_2 - I_1$ in increasing order. Continue until at the end we have the labels of the elements of $P - I_k$ is increasing order. This establishes a bijection between maximal chains of L_S and permutations $w \in \mathcal{L}(P,\omega)$ whose descent set is *contained in* S. Hence if $\gamma_L(S)$ denotes the number of $w \in \mathcal{L}(P,\omega)$ whose descent set *equals* S, then

$$\alpha_L(S) = \sum_{T \subseteq S} \gamma_L(T),$$

and the proof follows. □

3.13.2 Corollary. *Let $L = B_n$, the boolean algebra of rank n, and let $S \subseteq [n-1]$. Then $\beta_L(S)$ is equal to the total number of permutations of $[n]$ with descent set S. Thus, $\beta_L(S) = \beta_n(S)$ as defined in Example 2.2.4.*

Just as Example 2.2.5 is a q-generalization of Example 2.2.4, so we can generalize the previous corollary.

3.13.3 Theorem. *Let $L = B_n(q)$, the lattice of subspaces of the vector space $\mathbb{F}_q^n$. Let $S \subseteq [n-1]$. Then*

$$\beta_L(S) = \sum_{w} q^{\mathrm{inv}(w)},$$

where the sum is over all permutations $w \in \mathfrak{S}_n$ with descent set S, and where $\mathrm{inv}(w)$ *is the number of inversions of w.*

Proof. Let $S = \{a_1, a_2, \ldots, a_k\}_<$. Then

$$\begin{aligned} \alpha_L(S) &= \binom{\mathbf{n}}{\mathbf{a_1}}\binom{\mathbf{n-a_1}}{\mathbf{a_2-a_1}}\binom{\mathbf{n-a_2}}{\mathbf{a_3-a_2}}\cdots\binom{\mathbf{n-a_k}}{\mathbf{n-a_k}} \\ &= \binom{\mathbf{n}}{\mathbf{a_1, a_2-a_1, \ldots, n-a_k}}. \end{aligned}$$

The proof now follows by comparing equation (2.20) from Chapter 2 with equation (3.53). □

3.14 R-Labelings

In this section we give a wide class $\mathcal{A}$ of posets P for which the flag h-vector $\beta_P(S)$ has a direct combinatorial interpretation (and is therefore nonnegative). If $P \in \mathcal{A}$, then every interval of P will also belong to $\mathcal{A}$, so in particular the Möbius function of P alternates in sign.

Let $\mathcal{H}(P)$ denote the set of pairs (s,t) of elements of P for which t covers s. We may think of elements of $\mathcal{H}(P)$ as edges of the Hasse diagram of P.

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3.14.1 Definition. Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$. A function $\lambda\colon \mathcal{H}(P) \to \mathbb{Z}$ is called an *R-labeling* of P if, for every interval $[s,t]$ of P, there is a unique saturated chain $s = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_\ell = t$ satisfying

$$\lambda(t_0,t_1) \le \lambda(t_1,t_2) \le \cdots \le \lambda(t_{\ell-1},t_\ell). \tag{3.55}$$

A poset P possessing an R-labeling λ is called *R-labelable* or an *R-poset*, and the chain $s = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_\ell = t$ satisfying equation (3.55) is called the *increasing chain* from s to t.

Note that if $Q = [s,t]$ is an interval of P, then the restriction of λ to $\mathcal{H}(Q)$ is an R-labeling of $\mathcal{H}(Q)$. Hence Q is also an R-poset, so any property satisfied by all R-posets P is also satisfied by any interval of P.

3.14.2 Theorem. *Let P be an R-poset of rank n. Let λ be an R-labeling of P, and let $S \subseteq [n-1]$. Then $\beta_P(S)$ is equal to the number of maximal chains $\mathfrak{m}\colon \hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n = \hat{1}$ of P for which the sequence*

$$\lambda(\mathfrak{m}) := (\lambda(t_0,t_1), \lambda(t_1,t_2), \ldots, \lambda(t_{n-1},t_n))$$

has descent set S; that is, for which

$$D(\lambda(\mathfrak{m})) := \{i : \lambda(t_{i-1},t_i) > \lambda(t_i,t_{i+1})\} = S.$$

Proof. Let $\mathfrak{c}\colon \hat{0} < u_1 < \cdots < u_s < \hat{1}$ be a maximal chain in $\widehat{P_S}$. We claim there is a unique maximal chain $\mathfrak{m}$ of P containing $\mathfrak{c}$ and satisfying $D(\lambda(\mathfrak{m})) \subseteq S$. Let $\mathfrak{m}\colon \hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n = \hat{1}$ be such a maximal chain (if one exists), and let $S = \{a_1, \ldots, a_s\}_<$. Thus, $t_{a_i} = u_i$. Since $\lambda(t_{a_{i-1}}, t_{a_{i-1}+1}) \le \lambda(t_{a_{i-1}+1}, t_{a_{i-1}+2}) \le \cdots \le \lambda(t_{a_i-1}, t_{a_i})$ for $1 \le i \le s+1$ (where we set $a_0 = \hat{0}$, $a_{s+1} = \hat{1}$), we must take $t_{a_{i-1}}, t_{a_{i-1}+1}, \ldots, t_{a_i}$ to be the unique increasing chain of the interval $[u_{i-1}, u_i] = [t_{a_{i-1}}, t_{a_i}]$. Thus, M exists and is unique, as claimed.

It follows that the number $\alpha'_P(S)$ of maximal chains $\mathfrak{m}$ of P satisfying $D(\lambda(\mathfrak{m})) \subseteq S$ is just the number of maximal chains of P_S; that is, $\alpha'_P(S) = \alpha_P(S)$. If $\beta'_P(S)$ denotes the number of maximal chains $\mathfrak{m}$ of P satisfying $D(\lambda(\mathfrak{m})) = S$, then clearly

$$\alpha'_P(S) = \sum_{T \subseteq S} \beta'_P(T).$$

Hence from equation (3.53) we conclude $\beta'_P(S) = \beta_P(S)$. $\square$

3.14.3 Example. We now consider some examples of R-posets. Let P be a natural partial order on $[n]$, as in Theorem 3.13.1. Let $(I, I') \in \mathcal{H}(J(P))$, so I and I' are order ideals of P with $I \subset I'$ and $\#(I' - I) = 1$. Define $\lambda(I, I')$ to be the unique element of $I' - I$. For any interval $[K, K']$ of $J(P)$ there is a unique increasing chain $K = K_0 < K_1 < \cdots < K_\ell = K'$ defined by letting the sole element of $K_i - K_{i-1}$ be the least integer (in the usual linear order on $[n]$) contained in $K' - K_{i-1}$. Hence λ is an R-labeling, and indeed Theorems 3.13.1 and 3.14.2 coincide. We next mention without proof two generalizations of this example.

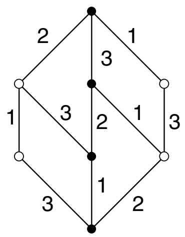

Figure 3.26 A supersolvable lattice.

3.14.4 Example. A finite lattice L is *supersolvable* if it possesses a maximal chain $\mathfrak{c}$, called an *M-chain*, such that the sublattice of L generated by $\mathfrak{c}$ and any other chain of L is distributive. Example of supersolvable lattices include modular lattices, the partition lattice Π_n, and the lattice of subgroups of a finite supersolvable group. For modular lattices, any maximal chain is an M-chain. For the lattice Π_n, a chain $\hat{0} = \pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_{n-1} = \hat{1}$ is an M chain if and only if each partition π_i $(1 \le i \le n-1)$ has exactly one block B_i with more than one element (so $B_1 \subset B_2 \subset \cdots \subset B_{n-1} = [n]$). The number of M-chains of Π_n is $n!/2$, $n \ge 2$. For the lattice L of subgroups of a supersolvable group G, an M-chain is given by a *normal series* $\{1\} = G_0 \lessdot G_1 \lessdot \cdots \lessdot G_n = G$; that is, each G_i is a normal subgroup of G, and each G_{i+1}/G_i is cyclic of prime order. (There may be other M-chains.)

If L is supersolvable with M-chain $\mathfrak{c}$: $\hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n = \hat{1}$, then an R-labeling $\lambda\colon \mathcal{H}(P) \to \mathbb{Z}$ is given by

$$\lambda(s,t) = \min\{i : s \vee t_i = t \vee t_i\}. \tag{3.56}$$

If we restrict λ to the (distributive) lattice L' of L generated by $\mathfrak{c}$ and some other chain, then we obtain an R-labeling of L' that coincides with Example 3.14.3. Figure 3.26 shows a (nonsemimodular) supersolvable lattice L with an M-chain denoted by solid dots, and the corresponding R-labeling λ. There are five maximal chains, with labels 312, 132, 123, 213, 231 and corresponding descent sets $\{1\}$, $\{2\}$, $\emptyset$, $\{1\}$, $\{2\}$. Hence $\beta(\emptyset) = 1$, $\beta(1) = \beta(2) = 2$, $\beta(1,2) = 0$. Note that all maximal chain labels are permutations of $[3]$; for the significance of this fact see Exercise 3.125.

3.14.5 Example. Let L be a finite (upper) semimodular lattice. Let P be the subposet of join-irreducibles of L. Let $\omega\colon P \to [k]$ be an order-preserving bijection (so $\#P = k$), and write $t_i = \omega^{-1}(i)$. Define for $(s,t) \in \mathcal{H}(L)$,

$$\lambda(s,t) = \min\{i : s \vee t_i = t\}. \tag{3.57}$$

Then λ is an R-labeling, and hence semimodular lattices are R-posets. Figure 3.27 shows on the left a semimodular lattice L with the elements t_i denoted by i, and on the right the corresponding R-labeling λ. There are seven maximal chains, with labels 123, 132, 213, 231, 312, 321, 341, and corresponding descent sets $\emptyset$, $\{2\}$,$\{1\}$,$\{2\}$, $\{1\}$, $\{1,2\}$, $\{2\}$. Hence, $\beta(\emptyset) = 1$, $\beta(1) = 2$, $\beta(2) = 3$, $\beta(1,2) = 1$.

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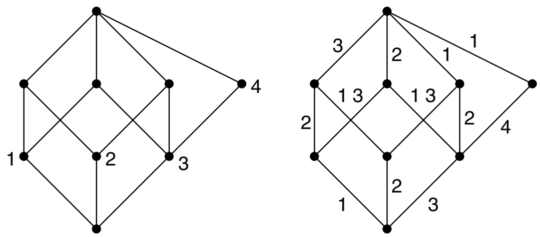

Figure 3.27 A semimodular lattice.

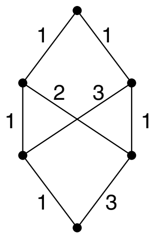

Figure 3.28 A nonlattice with an R-labeling.

Examples 3.14.4 and 3.14.5 both have the property that we can label certain elements of L as t_i (or just i) and then define λ by the similar formulas (3.56) and (3.57). Many additional R-lattices have this property, though not all of them do. Of course, equations (3.56) and (3.57) are meaningless for posets that are not lattices. Figure 3.28 illustrates a poset P that is not a lattice, together with an R-labeling λ.

3.15 (P,ω)-Partitions

3.15.1 The Main Generating Function

A (P,ω)-partition is a kind of interpolation between partitions and compositions. The poset P specifies inequalities among the parts, and the labeling ω specifies which of these inequalities are strict. There is a close connection with descent sets of permutations and the related statistics maj (the major index) and des (the number of descents).

Let P be a finite poset of cardinality p. Let $\omega\colon P \to [p]$ be a bijection, called a *labeling* of P.

3.15.1 Definition. A (P,ω)*-partition* is a map $\sigma\colon P \to \mathbb{N}$ satisfying the conditions:

- If $s \leq t$ in P, then $\sigma(s) \geq \sigma(t)$. In other words, σ is *order-reversing*.
- If $s < t$ and $\omega(s) > \omega(t)$, then $\sigma(s) > \sigma(t)$.

If $\sum_{t\in P}\sigma(t) = n$, then we say that σ is a (P,ω)-partition *of* n.

If ω is *natural* (i.e., $s < t \Rightarrow \omega(s) < \omega(t)$), then a (P,ω)-partition is just an order-reversing map $\sigma\colon P \to \mathbb{N}$. We then call σ simply a P*-partition*. Similarly,

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if ω is *dual natural* (i.e., $s < t \Rightarrow \omega(s) > \omega(t)$), then a (P,ω)-partition is a *strict* order-reversing map $\sigma: P \to \mathbb{N}$ (i.e., $s < t \Rightarrow \sigma(s) > \sigma(t)$). We then call σ a *strict* *P-partition*.

Let $P = \{t_1, \dots, t_p\}$. The fundamental generating function associated with (P,ω)-partitions is defined by

$$F_{P,\omega} = F_{P,\omega}(x_1, \dots, x_p) = \sum_\sigma x_1^{\sigma(t_1)} \cdots x_p^{\sigma(t_p)},$$

where σ ranges over all (P,ω)-partitions $\sigma: P \to \mathbb{N}$. If ω is natural, then we write simply F_P for $F_{P,\omega}$. The generating function $F_{P,\omega}$ essentially lists all (P,ω)-partitions and contains all possible information about them. Indeed, it is easy to recover the labeled poset (P,ω) if $F_{P,\omega}$ is known.

3.15.2 Example. (a) Suppose that P is a naturally labeled p-element chain $t_1 < \cdots < t_p$. Then

$$\begin{aligned} F_P &= \sum_{a_1 \geq a_2 \geq \cdots \geq a_p \geq 0} x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p} \\ &= \frac{1}{(1-x_1)(1-x_1x_2)\cdots(1-x_1x_2\cdots x_p)}. \end{aligned}$$

(b) Suppose that P is a dual naturally labeled p-element chain $t_1 < \cdots < t_p$. Then

$$\begin{aligned} F_{P,\omega} &= \sum_{a_1 > a_2 > \cdots > a_p \geq 0} x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p} \\ &= \frac{x_1^{p-1} x_2^{p-2} \cdots x_{p-1}}{(1-x_1)(1-x_1x_2)\cdots(1-x_1x_2\cdots x_p)}. \end{aligned}$$

(c) If P is a p-element antichain, then all labelings are natural. We get

$$\begin{aligned} F_P &= \sum_{a_1, \dots, a_p \geq 0} x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p} \\ &= \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_p)}. \end{aligned}$$

(d) Suppose that P has a minimal element t_1 and two elements t_2, t_3 covering t_1, with the labeling $\omega(t_1) = 2$, $\omega(t_2) = 1$, $\omega(t_3) = 3$. Then

$$F_{P,\omega} = \sum_{b < a \geq c} x_1^a x_2^b x_3^c.$$

Let $\mathcal{L}(P,\omega)$ denote the set of linear extensions of P, regarded as permutations of the *labels* $\omega(t)$, as done in Section 3.13 for natural labelings. Thus, $\mathcal{L}(P,\omega) \subseteq \mathfrak{S}_p$. Again following Section 3.13, we call $\mathcal{L}(P,\omega)$ the *Jordan-Hölder set* of the labeled

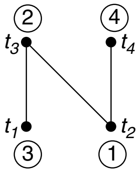

Figure 3.29 A labeled poset.

poset (P,ω). For instance, if (P,ω) is given by Figure 3.29 (with the labels circled) then

$$\mathcal{L}(P,\omega) = \{3124, 3142, 1324, 1342, 1432\}.$$

Write $\mathcal{A}(P,\omega)$ for the set of all (P,ω)-partitions $\sigma\colon P \to \mathbb{N}$, and let $\sigma \in \mathcal{A}(P,\omega)$. Define $\sigma'\colon [p] \to \mathbb{N}$ by

$$\sigma'(i) = \sigma(\omega^{-1}(i)).$$

In other words, σ and σ' are essentially the same function, but the argument of σ is an *element* $t \in P$, while the argument of σ' is the *label* $\omega(t)$ of that element. This distinction is particularly important in Subsection 3.15.3 (reciprocity), where we deal with two different labelings of the same poset. We know from Lemma 1.4.11 that there is a unique permutation $w \in \mathfrak{S}_p$ for which σ' is w-compatible. For any $w \in \mathfrak{S}_p$, we write S_w for the set of all functions $\sigma\colon P \to \mathbb{N}$ for which σ' is w-compatible. We come to the fundamental lemma on (P,ω)-partitions.

3.15.3 Lemma. *A function $\sigma\colon P \to \mathbb{N}$ is a (P,ω)-partition if and only if σ' is w-compatible with some (necessarily unique) $w \in \mathcal{L}(P,\omega)$. Equivalently, we have the disjoint union*

$$\mathcal{A}(P,\omega) = \dot{\bigcup}_{w\in\mathcal{L}(P,\omega)} S_w.$$

Proof. We first show that if $\sigma \in \mathcal{A}(P,\omega) \cap S_w$, then $w \in \mathcal{L}(P,\omega)$. Let $w = w_1w_2\cdots w_p$. Suppose that $i < j$ and $\omega(s) = w_i$, $\omega(t) = w_j$. We need to show that we cannot have $s > t$. If $\sigma(s) = \sigma(t)$, then by definition of w-compatibility we have $w_i < w_{i+1} < \cdots < w_j$. Hence, by definition of (P,ω)-partition, we cannot have $s > t$. If instead $\sigma(s) > \sigma(t)$, then again by definition of (P,ω)-partition we cannot have $s > t$, so $w \in \mathcal{L}(P,\omega)$.

It remains to show that if $w \in \mathcal{L}(P,\omega)$ and σ' is w-compatible, then $\sigma \in \mathcal{A}(P,\omega)$. Clearly, σ is order-reversing, so we need to show that if $s < t$ and $\omega(s) = w_i > w_j = \omega(t)$, then $\sigma(s) > \sigma(t)$. Since $w_i > w_j$, somewhere in w between w_i and w_j is a descent $w_k > w_{k+1}$. Thus,

$$\sigma(s) = \sigma(w_i) \geq \sigma(w_{i+1}) \geq \cdots \geq \sigma(w_k) > \sigma(w_{k+1}) \geq \cdots \geq \sigma(w_j) = \sigma(t),$$

and the proof follows. $\square$

Given $w \in \mathfrak{S}_p$, let

$$F_w = \sum_{\sigma\in S_w} x_1^{\sigma(t_1)} \cdots x_p^{\sigma(t_p)}, \tag{3.58}$$

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be the generating function for all functions $\sigma\colon P\to\mathbb{N}$ for which σ' is w-compatible. The next result is a straightforward extension of Lemma 1.4.12. Write $w'_i = j$ if $w_i = \omega(t_j)$.

3.15.4 Lemma. *Let $w = w_1\cdots w_p \in \mathfrak{S}_p$. Then*

$$F_w = \frac{\prod_{j\in D(w)} x_{w'_1}x_{w'_2}\cdots x_{w'_j}}{\prod_{i=1}^{p}\left(1 - x_{w'_1}x_{w'_2}\cdots x_{w'_i}\right)}. \tag{3.59}$$

Proof. Let $\sigma\in S_w$. Define numbers c_i, $1\le i\le p$, by

$$c_i = \begin{cases} \sigma'(w_i) - \sigma'(w_{i+1}), & \text{if } i\notin D(w),\\ \sigma'(w_i) - \sigma'(w_{i+1}) - 1, & \text{if } i\in D(w), \end{cases} \tag{3.60}$$

where we set $\sigma'(w_{p+1}) = 0$. Note that $c_i\ge 0$ and that any choice of $c_1,c_2,\dots,c_p\in\mathbb{N}$ defines a unique function $\sigma\in S_w$ satisfying equation (3.60). Then

$$x_1^{\sigma(t_1)}\cdots x_p^{\sigma(t_p)} = \prod_{i=1}^{p}\left(x_{w'_1}x_{w'_2}\cdots x_{w'_i}\right)^{c_i}\cdot\prod_{j\in D(w)} x_{w'_1}x_{w'_2}\cdots x_{w'_j}.$$

This sets up a one-to-one correspondence between the terms in the left- and right-hand sides of equation (3.59), so the proof follows. □

Combining Lemmas 3.15.3 and 3.15.4, we obtain the main theorem on the generating function $F_{P,\omega}$.

3.15.5 Theorem. *Let (P,ω) be a labeled p-element poset. Then*

$$F_{P,\omega}(x_1,\dots,x_p) = \sum_{w\in\mathcal{L}(P,\omega)} \frac{\prod_{j\in D(w)} x_{w'_1}x_{w'_2}\cdots x_{w'_j}}{\prod_{i=1}^{p}\left(1 - x_{w'_1}x_{w'_2}\cdots x_{w'_i}\right)}. \tag{3.61}$$

3.15.6 Example. Let (P,ω) be given by Figure 3.29. Then Lemma 3.15.3 says that every (P,ω)-partition $\sigma\colon P\to\mathbb{N}$ satisfies exactly one of the conditions

$$\begin{array}{ccccccc}
\sigma'(3) & > & \sigma'(1) & \ge & \sigma'(2) & \ge & \sigma'(4),\\
\sigma'(1) & \ge & \sigma'(3) & > & \sigma'(2) & \ge & \sigma'(4),\\
\sigma'(3) & > & \sigma'(1) & \ge & \sigma'(4) & > & \sigma'(2),\\
\sigma'(1) & \ge & \sigma'(3) & \ge & \sigma'(4) & > & \sigma'(2),\\
\sigma'(1) & \ge & \sigma'(4) & > & \sigma'(3) & > & \sigma'(2).
\end{array}$$

It follows that

$$\begin{aligned}F_{P,\omega}(x_1,x_2,x_3,x_4) &= \frac{x_1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}\\ &\quad+\frac{x_1x_2}{(1-x_2)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}\\ &\quad+\frac{x_1^2x_2x_4}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)}\\ &\quad+\frac{x_1x_2x_4}{(1-x_2)(1-x_1x_2)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)}\\ &\quad+\frac{x_1x_2^2x_4^2}{(1-x_2)(1-x_2x_4)(1-x_1x_2x_4)(1-x_1x_2x_3x_4)}.\end{aligned}$$

This example illustrates the underlying combinatorial meaning behind the efficacy of the fundamental Lemma 3.15.3—it allows the set $\mathcal{A}(P,\omega)$ of all (P,ω)-partitions to be partitioned into finitely many (namely, $e(P)$) "simple" subsets, each of which can be handled separately.

3.15.2 Specializations

We now turn to two basic specializations of the generating function $F_{P,\omega}$. Let $a(n)$ denote the number of (P,ω)-partitions of n. Define the generating function

$$G_{P,\omega}(x) = \sum_{n\geq 0} a(n)x^n. \tag{3.62}$$

Clearly, $G_{P,\omega}(x) = F_{P,\omega}(x,x,\dots,x)$. Moreover, $\prod_{j\in D(w)} x^j = x^{\mathrm{maj}(w)}$. Hence from Theorems 3.15.5, we obtain the following result.

3.15.7 Theorem. *The generating function $G_{P,\omega}(x)$ has the form*

$$G_{P,\omega}(x) = \frac{W_{P,\omega}(x)}{(1-x)(1-x^2)\cdots(1-x^p)}, \tag{3.63}$$

where $W_{P,\omega}(x)$ is a polynomial given by

$$W_{P,\omega}(x) = \sum_{w\in\mathcal{L}(P,\omega)} x^{\mathrm{maj}(w)}. \tag{3.64}$$

If we take P to be the antichain $p\mathbf{1}$ (with any labeling), then clearly $G_P(x) = (1-x)^{-p}$. Comparing with equation (3.64) yields

$$\sum_{w\in\mathfrak{S}_p} x^{\mathrm{maj}(w)} = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{p-1}),$$

which is the same (up to a change in notation) as equation (1.42).

Recall from Section 3.12 that we defined the order polynomial $\Omega_P(m)$ to be the number of order-preserving maps $\sigma : P \to \boldsymbol{m}$. By replacing $\sigma(t)$ with $m+1-\sigma(t)$, we see that $\Omega_P(m)$ is also the number of order-*reversing* maps $P \to \boldsymbol{m}$ (i.e., the number of P-partitions $P \to \boldsymbol{m}$). We can therefore extend the definition to labeled posets by defining the (P,ω)-*order polynomial* $\Omega_{P,\omega}(m)$ for $m \in \mathbb{P}$ to be the number of (P,ω)-partitions $\sigma : P \to \boldsymbol{m}$. Let $e_{P,\omega}(s)$ be the number of *surjective* (P,ω)-partitions $P \to \boldsymbol{s}$. Note that for any ω we have $e_{P,\omega}(p) = e(P)$. By first choosing the image $\sigma(P)$ of the (P,ω)-partition $\sigma : P \to \boldsymbol{m}$, it is clear that

$$\Omega_{P,\omega}(m) = \sum_{s=1}^{p} e_{P,\omega}(s)\binom{m}{s}.$$

It follows that $\Omega_{P,\omega}(m)$ is a polynomial in m of degree p and leading coefficient $e(P)/p!$.

Now define

$$H_{P,\omega}(x) = \sum_{m \geq 0} \Omega_{P,\omega}(m) x^m.$$

The fundamental property of order polynomials is the following.

3.15.8 Theorem. *We have*

$$H_{P,\omega}(x) = \frac{\sum_{w \in \mathcal{L}(P,\omega)} x^{1+\mathrm{des}(w)}}{(1-x)^{p+1}}. \tag{3.65}$$

Proof. Immediate from equation (1.46) and Lemma 3.15.3. □

In analogy to equation (1.36), we write

$$A_{P,\omega}(x) = \sum_{w \in \mathcal{L}(P,\omega)} x^{1+\mathrm{des}(w)},$$

called the (P,ω)-*Eulerian polynomial*. As usual, when ω is natural, we just write $A_P(x)$ and call it the P-*Eulerian polynomial*. Note that Proposition 1.4.4 corresponds to the case $P = p\mathbf{1}$, when the order polynomial is just the Eulerian polynomial $A_p(x)$. Note also that if we take coefficients of x^m in equation (3.65) (or by equation (1.45)), then we obtain

$$\Omega_{P,\omega}(m) = \sum_{w \in \mathcal{L}(P,\omega)} \left(\!\!\binom{m-\mathrm{des}(w)}{p}\!\!\right).$$

3.15.3 Reciprocity

With a labeling ω of P, we can associate a certain "dual" labeling $\overline{\omega}$. The connection between ω and $\overline{\omega}$ will lead to a generalization of the reciprocity formula $\left(\!\binom{n}{k}\!\right) = (-1)^k\binom{-n}{k}$ of equation (1.21).

Let ω be a labeling of the p-element poset P. Define the *complementary labeling* $\overline{\omega}$ by

$$\overline{\omega}(t) = p + 1 - \omega(t).$$

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For instance, if ω is natural so that a (P,ω)-partition is just a P-partition, then a $(P,\overline{\omega})$-partition is a strict P-partition. If $w = w_1 w_2 \cdots w_p \in \mathfrak{S}_p$, then let $\overline{w} = p+1-w_1, p+1-w_2, \ldots, p+1-w_p \in \mathfrak{S}_p$. Note that $D(\overline{w}) = [p-1] - D(w)$.

3.15.9 Lemma. *Let F_w be as in equation (3.58). Then as rational functions,*

$$x_1 x_2 \cdots x_p F_{\overline{w}}(x_1,\ldots,x_p) = (-1)^p F_w\left(\frac{1}{x_1},\ldots,\frac{1}{x_p}\right).$$

Proof. Let $w = w_1 \cdots w_p$ and $w'_i = j$ if $w(i) = \omega(t_j)$ as before. We have by Lemma 3.15.4 that

$$F_w\left(\frac{1}{x_1},\ldots,\frac{1}{x_p}\right) = \frac{\prod_{j\in D(w)}\left(x_{w'_1}x_{w'_2}\cdots x_{w'_j}\right)^{-1}}{\prod_{i=1}^p\left(1-\left(x_{w'_1}x_{w'_2}\cdots x_{w'_i}\right)^{-1}\right)}$$

$$= (-1)^p \frac{x_{w'_1}^p x_{w'_2}^{p-1}\cdots x_{w'_p}\prod_{j\in D(w)}\left(x_{w'_1}x_{w'_2}\cdots x_{w'_j}\right)^{-1}}{\prod_{i=1}^p\left(1-x_{w'_1}x_{w'_2}\cdots x_{w'_i}\right)}. \tag{3.66}$$

But

$$\left(\prod_{j\in D(w)} x_{w'_1}x_{w'_2}\cdots x_{w'_j}\right)\left(\prod_{k\in D(\overline{w})} x_{w'_1}x_{w'_2}\cdots x_{w'_k}\right) = \prod_{j=1}^{p-1} x_{w'_1}x_{w'_2}\cdots x_{w'_j}$$
$$= x_{w'_1}^{p-1}x_{w'_2}^{p-2}\cdots x_{w'_{p-1}}.$$

The proof now follows upon comparing equation (3.66) with Lemma 3.15.4 for $\overline{w}$. □

3.15.10 Theorem (the reciprocity theorem for (P,ω)-partitions). *The rational functions $F_{P,\omega}(x_1,\ldots,x_p)$ and $F_{P,\overline{\omega}}(x_1,\ldots,x_p)$ are related by*

$$x_1 x_2 \cdots x_p F_{P,\overline{\omega}}(x_1,\ldots,x_p) = (-1)^p F_{P,\omega}\left(\frac{1}{x_1},\ldots,\frac{1}{x_p}\right).$$

Proof. Immediate from Theorem 3.15.5 and Lemma 3.15.9. □

The power series and polynomials $G_{P,\omega}(x)$, $W_{P,\omega}(x)$, $H_{P,\omega}(x)$, and $A_{P,\omega}(x)$ are well-behaved with respect to reciprocity. It is immediate from the preceding discussion that

$$\begin{aligned}
x^p G_{P,\overline{\omega}}(x) &= (-1)^p G_{P,\omega}(1/x),\\
W_{P,\overline{\omega}}(x) &= x^{\binom{p}{2}} W_{P,\omega}(1/x),\\
H_{P,\overline{\omega}}(x) &= (-1)^{p+1} H_{P,\omega}(1/x),\\
A_{P,\overline{\omega}}(x) &= x^{p+1} A_{P,\omega}(1/x).
\end{aligned} \tag{3.67}$$

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There is also an elegant reciprocity result for the order polynomial $\Omega_{P,\omega}(m)$ itself (and not just its generating function $H_{P,\omega}(x)$). We first need the following lemma. It is a special case of Proposition 4.2.3 and Corollary 4.3.1, where more conceptual proofs are given than the naive argument below.

3.15.11 Lemma. *Let $f(m)$ be a polynomial over a field K of characteristic 0, with* $\deg f \le p$. *Let $H(x) = \sum_{m\ge 0} f(m)x^m$. Then there is a polynomial $P(x) \in K[x]$, with* $\deg P \le p$*, such that*

$$H(x) = \frac{P(x)}{(1-x)^{p+1}}. \tag{3.68}$$

Moreover, as rational functions we have

$$\sum_{m\ge 1} f(-m)x^m = -H(1/x). \tag{3.69}$$

Proof. By linearity it suffices to prove the result for some basis of the space of polynomials $f(m)$ of degree at most p. Choose the basis $\binom{m+i}{p}$, $0 \le i \le p$. Let

$$\begin{aligned} H_i(x) &= \sum_{m\ge 0} \binom{m+i}{p} x^m \\ &= \frac{x^{p-i}}{(1-x)^{p+1}}, \end{aligned}$$

establishing equation (3.68). Now

$$\begin{aligned} -H_i(1/x) &= \frac{-x^{-p+i}}{(1-1/x)^{p+1}} \\ &= \frac{(-1)^p x^{i+1}}{(1-x)^{p+1}} \\ &= (-1)^p \sum_{m\ge 1} \binom{m+p-i-1}{p} x^m \\ &= \sum_{m\ge 1} \binom{-m+i}{p} x^m, \end{aligned}$$

and the proof of equation (3.69) follows. □

3.15.12 Corollary (the reciprocity theorem for order polynomials). *The polynomials $\Omega_{P,\overline{\omega}}(m)$ and $\Omega_{P,\omega}(m)$ are related by*

$$\Omega_{P,\overline{\omega}}(m) = (-1)^p \Omega_{P,\omega}(-m).$$

Proof. Immediate from equation (3.67) and Lemma 3.15.11. □

3.15.4 Natural Labelings

When ω is a natural labeling many properties of P dealing with the length of chains are closely connected with the generating functions we have been considering. Recall that we suppress the labeling ω from our notation when ω is natural, so for instance we write $\mathcal{L}(P)$ for $\mathcal{L}(P,\omega)$ when ω is natural. We also use an overline to denote that a labeling is dual natural; for instance, $\overline{G}_P(x)$ denotes $G_{P,\omega}(x)$ for ω dual natural. To begin, if $t \in P$, then define $\delta(t)$ to be the length ℓ of the longest chain $t = t_0 < t_1 < \cdots < t_\ell$ of P whose first element is t. Also define

$$\delta(P) = \sum_{t\in P} \delta(t).$$

3.15.13 Corollary. *Let $p = \#P$. Then the degree of the polynomial $W_P(x)$ is $\binom{p}{2} - \delta(P)$. Moreover, $W_P(x)$ is a monic polynomial. (See Corollary 4.2.4(ii) for the significance of these results.)*

Proof. By equation (3.64) we need to show that

$$\max_{w\in\mathcal{L}(P)} \operatorname{maj}(w) = \binom{p}{2} - \delta(P),$$

and that there is a unique w achieving this maximum. Let $w = a_1a_2\cdots a_p \in \mathcal{L}(P)$, and suppose that the longest chain of P has length ℓ. Given $0 \le i \le \ell$, let j_i be the largest integer for which $\delta(a_{j_i}) = i$. Clearly $j_1 > j_2 > \cdots > j_\ell$. Now for each $1 \le i \le \ell$, there is some element a_{k_i} of P satisfying $a_{j_i} < a_{k_i}$ in P (and thus also $a_{j_i} < a_{k_i}$ in $\mathbb{Z}$) and $\delta(a_{k_i}) = \delta(a_{j_i}) - 1$. It follows that $j_i < k_i \le j_{i-1}$. Hence, somewhere in w between positions j_i and j_{i-1} there is a pair $a_r < a_{r+1}$ in $\mathbb{Z}$, so

$$\operatorname{maj}(w) \le \binom{p}{2} - \sum_{i=1}^{\ell} j_i.$$

If δ_i denotes the number of elements t of P satisfying $\delta(t) = i$, then by definition $j_i \ge \delta_i + \delta_{i+1} + \cdots + \delta_\ell$. Hence,

$$\begin{aligned} \operatorname{maj}(w) &\le \binom{p}{2} - \sum_{i=1}^{\ell} (\delta_i + \delta_{i+1} + \cdots + \delta_\ell) \\ &= \binom{p}{2} - \sum_{i=1}^{\ell} i\delta_i \\ &= \binom{p}{2} - \sum_{t\in P} \delta(t). \end{aligned}$$

If equality holds, then the last δ_0 elements t of w satisfy $\delta(t) = 0$, the next δ_1 elements t from the right satisfy $\delta(t) = 1$, and so on. Moreover, the last δ_0 elements must be arranged in decreasing order as elements of $\mathbb{Z}$, the next δ_1 elements also in decreasing order, etc. Hence there is a unique w for which equality hold. $\square$

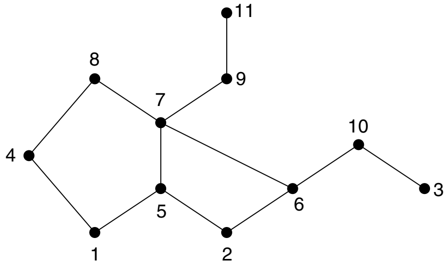

Figure 3.30 A naturally labeled poset P with $\delta(P) = 19$.

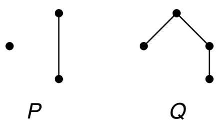

Figure 3.31 Nongraded posets satisfying the δ-chain condition.

3.15.14 Example. Let P be the naturally labeled poset shown in Figure 3.30. Then the unique $w \in \mathcal{L}(P)$ satisfying $\mathrm{maj}(w) = \binom{p}{2} - \delta(P)$ is given by

$$w = 2,1,6,5,7,9,4,3,11,10,8,$$

so $\mathrm{maj}(w) = 36$ and $\delta(P) = 19$.

For our next result concerning the polynomial $W_P(x)$, let $\mathcal{A}(P)$ (respectively, $\overline{\mathcal{A}}(P)$) denote the set of all P-partitions (respectively, strict P-partitions). Define a map (denoted $'$) $\mathcal{A}(P) \to \overline{\mathcal{A}}(P)$ by the formula

$$\sigma'(t) = \sigma(t) + \delta(t), \ \ t \in P. \tag{3.70}$$

Clearly, this correspondence is injective.

We say that P satisfies the δ-*chain condition* if for all $t \in P$, all maximal chains of the principal dual order ideal $V_t = \{u \in P : u \geq t\}$ have the same length. If P has a $\hat{0}$, then this is equivalent to saying that P is graded. Note, however, that the posets P and Q of Figure 3.31 satisfy the δ-chain condition but are not graded.

3.15.15 Lemma. *The injection $\sigma \mapsto \sigma'$ is a bijection from $\mathcal{A}(P)$ to $\overline{\mathcal{A}}(P)$ if and only if P satisfies the δ-chain condition.*

Proof. The "if" part is easy to see. To prove the "only if" part, we need to show that if P fails to satisfy the δ-chain condition, then there is a $\tau \in \overline{\mathcal{A}}(P)$ such that $\tau - \delta \notin \mathcal{A}(P)$. Assume that P does not satisfy the δ-chain condition. Then there exist two elements t_0, t_1 of P such that t_1 covers t_0 and $\delta(t_0) > \delta(t_1) + 1$. Define τ by

$$\tau(t) = \begin{cases} \delta(t), & \text{if } t \geq t_0 \text{ and } t \neq t_1 \text{ (in } P\text{)}, \\ \delta(t) + 1, & \text{if } t \not\geq t_0 \text{ or } t = t_1 \text{ (in } P\text{)}. \end{cases}$$

It is easily seen that $\tau \in \overline{\mathcal{A}}(P)$, but

$$\tau(t_0) - \delta(t_0) = 0 < 1 = \tau(t_1) - \delta(t_1).$$

Since $t_0 < t_1$, $\tau - \delta \notin \mathcal{A}(P)$. $\square$

3.15.16 Theorem. *Let P be a p-element poset. Then P satisfies the δ-chain condition if and only if*

$$x^{\binom{p}{2}-\delta(P)} W_P(1/x) = W_P(x). \tag{3.71}$$

(Since $\deg W_P(x) = \binom{p}{2} - \delta(P)$*, equation* (3.71) *simply says that the coefficients of $W_P(x)$ read the same backwards as forwards.)*

Proof. Let $\sigma \in \mathcal{A}(P)$ with $|\sigma| = n$. Then the strict P-partition σ' defined by (3.70) satisfies $|\sigma'| = n + \delta(P)$. Hence from Lemma 3.15.15, it follows that P satisfies the δ-chain condition if an only if $a(n) = \overline{a}(n + \delta(P))$ for all $n \geq 0$. In terms of generating functions, this condition becomes $x^{\delta(P)} G_P(x) = \overline{G}_P(x)$. The proof now follows from Theorem 3.15.7. $\square$

Theorem 3.15.16 has an analogue for order polynomials. Recall that P is *graded* if all maximal chains of P have the same length. We say that P satisfies the λ-*chain condition* if every element of P is contained in a chain of maximum length. Clearly, a graded poset satisfies the λ-chain condition. The converse is false, as shown by Exercise 3.7(a).

Let $\mathcal{A}_m(P)$ (respectively, $\overline{\mathcal{A}}_m(P)$) denote the set of all order-reversing maps (respectively, strict order-reversing maps) $\sigma : P \to \boldsymbol{m}$. The next result is the analogue of Lemma 3.15.15 for graded posets and for the λ-chain condition.

3.15.17 Lemma. *Let P be a finite poset with longest chain of length ℓ. For each $i \in \mathbb{P}$, define an injection $\theta_i : \mathcal{A}_i(P) \to \overline{\mathcal{A}}_{\ell+i}(P)$ by $\theta_i(\sigma) = \sigma + \delta$.*

a. The map θ_1 is a bijection (i.e, $\#\overline{\mathcal{A}}_{\ell+1}(P) = 1$) if and only if P satisfies the λ-chain condition.

b. The maps θ_1 and θ_2 are both bijections if and only if P is graded. In this case θ_i is a bijection for all $i \in \mathbb{P}$.

Proof. **a.** The "if" part is clear. To prove the converse, define $\delta^*(t)$ for $t \in P$ to be the length k of the longest chain $t_0 < t_1 < \cdots < t_k = t$ in P with top t. Thus, $\delta(t) + \delta^*(t)$ is the length of the longest chain of P containing t, and $\delta(t) + \delta^*(t) = \ell$ for all $t \in P$ if and only if P satisfies the λ-chain condition. Define $\sigma, \tau \in \overline{\mathcal{A}}_{\ell+1}(P)$ by $\sigma(t) = 1 + \delta(t)$ and $\tau(t) = \ell - \delta^*(t) + 1$. Then $\sigma \neq \tau$ if (and only if) P fails to satisfy the λ-chain condition, so in this case θ_1 is not a bijection.

b. Again the "if" part is clear. To prove the converse, assume that P is not graded. If P does not satisfy the λ-chain condition, then by (a) θ_1 is not a bijection. Hence, assume that P satisfies the λ-chain condition. Let $t_0 < t_1 < \cdots < t_m$ be a maximal chain of P with $m < \ell$. Let k be the greatest integer, $0 \leq k \leq m$, such that $\delta(t_k) > m - k$. Since P satisfies the λ-chain condition and t_0 is a minimal element of P, $\delta(t_0) = \ell > m$; so k always exists. Furthermore, $k \neq m$ since t_m is a maximal

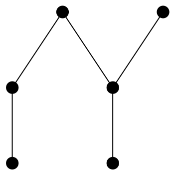

Figure 3.32 A graded poset.

element of P. Define a map $\sigma : P \to [\ell+2]$ as follows:

$$\sigma(t) = \begin{cases} 1+\delta(t), & \text{if } t \not\leq t_{k+1}, \\ 1+\max(\delta(t), \delta(t_{k+1})+\lambda(t,t_{k+1})+1), & \text{if } t \leq t_{k+1}, \end{cases}$$

where $\lambda(t,t_{k+1})$ denotes the length of the longest chain in the interval $[t,t_{k+1}]$. It is not hard to see that $\sigma \in \overline{\mathcal{A}}_{\ell+2}(P)$. Moreover,

$$\sigma(t_k) - \delta(t_k) = 1, \quad \sigma(t_{k+1}) - \delta(t_{k+1}) = 2,$$

so $\sigma - \delta \notin \mathcal{A}(P)$. Hence, θ_2 is not a bijection, and the proof is complete. $\square$

3.15.18 Corollary. *Let P be a p-element poset with longest chain of length ℓ. Then*

$$\Omega_P(-1) = \Omega_P(-2) = \cdots = \Omega_P(-\ell) = 0.$$

Moreover:

a. P satisfies the λ-chain condition if and only if $\Omega_P(-\ell-1) = (-1)^p$.
b. The following three conditions are equivalent:
 i. P is graded.
 ii. $\Omega_P(-\ell-1) = (-1)^p$ and $\Omega_P(-\ell-2) = (-1)^p\Omega_P(2)$.
 iii. $\Omega_P(-\ell-m) = (-1)^p\Omega_P(m)$ for all $m \in \mathbb{Z}$.

The following example illustrates the computational use of Corollary 3.15.18.

3.15.19 Example. Let P be given by Figure 3.32. Thus, $\Omega_P(m)$ is a polynomial of degree 6, and by the preceding corollary $\Omega_P(0) = \Omega_P(-1) = \Omega_P(-2) = 0$, $\Omega_P(1) = \Omega_P(-3) = 1$, $\Omega_P(2) = \Omega_P(-4)$. Thus as soon as we compute $\Omega_P(2)$, we know seven values of $\Omega_P(m)$, which suffice to determine $\Omega_P(m)$ completely. In fact, $\Omega_P(2) = 14$, from which we compute

$$\sum_{m\geq 0} \Omega_P(m)x^m = \frac{x+7x^2+7x^3+x^4}{(1-x)^7}$$

and

$$\begin{aligned} \Omega_P(m) &= \frac{1}{180}(4m^6+24m^5+55m^4+60m^3+31m^2+6m) \\ &= \frac{1}{180}m(m+1)^2(m+2)(2m+1)(2m+3). \end{aligned}$$

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3.16 Eulerian Posets

Let us recall the definition of an Eulerian poset following Proposition 3.8.9: A finite graded poset P with $\hat{0}$ and $\hat{1}$ is *Eulerian* if $\mu_P(s,t) = (-1)^{\ell(s,t)}$ for all $s \le t$ in P. Eulerian posets enjoy many remarkable properties concerned with the enumeration of chains. In this section, we will consider several duality properties of Eulerian posets, while the next section deals with a generalization of the cd-index.

3.16.1 Proposition. *Let P be an Eulerian poset of rank n. Then $Z(P,-m) = (-1)^n Z(P,m)$.*

Proof. By Proposition 3.12.1(c) we have

$$\begin{aligned} Z(P,-m) &= \mu^m(\hat{0},\hat{1}) \\ &= \sum \mu(t_0,t_1)\mu(t_1,t_2)\cdots\mu(t_{m-1},t_m), \end{aligned}$$

summed over all multichains $\hat{0} = t_0 \le t_1 \le \cdots \le t_m = \hat{1}$. Since P is Eulerian, $\mu(t_{i-1},t_i) = (-1)^{\ell(t_{i-1},t_i)}$. Hence, $\mu(t_0,t_1)\mu(t_1,t_2)\cdots\mu(t_{m-1},t_m) = (-1)^n$, so $Z(P,-m) = (-1)^n \zeta^m(\hat{0},\hat{1}) = (-1)^n Z(P,m)$. □

Define a finite poset P with $\hat{0}$ to be *simplicial* if each interval $[\hat{0},t]$ is isomorphic to a boolean algebra.

3.16.2 Proposition. *Let P be simplicial. Then $Z(P,m) = \sum_{i\ge 0} W_i(m-1)^i$, where*

$$W_i = \#\{t \in P \,:\, [\hat{0},t] \cong B_i\}.$$

In particular, if P is graded then $Z(P,q+1)$ is the rank-generating function of P.

Proof. Let $t \in P$, and let $Z_t(P,m)$ denote the number of multichains $t_1 \le t_2 \le \cdots \le t_{m-1} = t$ in P. By Example 3.12.2, $Z_t(P,m) = (m-1)^i$ where $[\hat{0},t] \cong B_i$. But $Z(P,m) = \sum_{t\in P} Z_t(P,m)$, and the proof follows. □

Now suppose that P is Eulerian and $P' := P - \{\hat{1}\}$ is simplicial. By considering multichains in P that do not contain $\hat{1}$, we see that

$$Z(P',m+1) = Z(P,m+1) - Z(P,m) = \Delta Z(P,m).$$

Hence by Proposition 3.16.2,

$$\Delta Z(P,m) = \sum_{i=0}^{n-1} W_i m^i, \tag{3.72}$$

where P has W_i elements of rank i (and $n = \operatorname{rank}(P)$ as usual). On the other hand, by Proposition 3.16.1 we have $Z(P,-m) = (-1)^n Z(P,m)$, so $\Delta Z(P,-m) = (-1)^{n-1}\Delta Z(P,m-1)$. Combining with equation (3.72) yields

$$\sum_{i=0}^{n-1} W_i(m-1)^i = \sum_{i=0}^{n-1} (-1)^{n-1-i} W_i m^i. \tag{3.73}$$

Equation (3.73) imposes certain linear relations on the W_i's, known as the *Dehn-Sommerville equations*. In general, there will be $\lfloor n/2 \rfloor$ independent equations (in addition to $W_0 = 1$). We list below these equations for $2 \le n \le 6$, where we have set $W_0 = 1$.

$$\begin{aligned}
n=2: &\qquad & W_1 &= 2,\\
n=3: && W_1 - W_2 &= 0,\\
n=4: && W_1 - W_2 + W_3 &= 2,\\
&& 2W_2 - 3W_3 &= 0,\\
n=5: && W_1 - W_2 + W_3 - W_4 &= 0,\\
&& W_3 - 2W_4 &= 0,\\
n=6: && W_1 - W_2 + W_3 - W_4 + W_5 &= 0,\\
&& 2W_2 - 3W_3 + 4W_4 - 5W_5 &= 0,\\
&& 2W_4 - 5W_5 &= 0.
\end{aligned}$$

A more elegant way of stating these equations will be discussed in conjunction with Theorem 3.16.9.

A fundamental example of an Eulerian lattice L for which $L - \{\hat{1}\}$ is simplicial is the lattice of faces of a triangulation Δ of a sphere, with a $\hat{1}$ adjoined. In this case W_i is just the number of $(i-1)$-dimensional faces of Δ.

Let us point out that although we have derived equation (3.73) as a special case of Proposition 3.16.1, one can also deduce Proposition 3.16.1 from (3.73). Namely, given an Eulerian poset P, apply (3.73) to the poset of chains of P with a $\hat{1}$ adjoined. The resulting formula is formally equivalent to Proposition 3.16.1.

Next we turn to a duality theorem for the numbers $\beta_P(S)$ when P is Eulerian.

3.16.3 Lemma. *Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let $t \in P - \{\hat{0},\hat{1}\}$. Then*

$$\mu_{P-t}(\hat{0},\hat{1}) = \mu_P(\hat{0},\hat{1}) - \mu_P(\hat{0},t)\mu_P(t,\hat{1}).$$

Proof. This result is a simple consequence of Proposition 3.8.5. □

3.16.4 Lemma. *Let P be as above, and let Q be any subposet of P containing $\hat{0}$ and $\hat{1}$. Then*

$$\mu_Q(\hat{0},\hat{1}) = \sum (-1)^k \mu_P(\hat{0},t_1)\mu_P(t_1,t_2)\cdots\mu_P(t_k,\hat{1}),$$

where the sum ranges over all chains $\hat{0} < t_1 < \cdots < t_k < \hat{1}$ in P such that $t_i \notin Q$ for all i. (The chain $\hat{0} < \hat{1}$ contributes $\mu(\hat{0},\hat{1})$ to the sum.)

Proof. Iterate Lemma 3.16.3 by successively removing elements of $P - Q$ from P. □

3.16.5 Proposition. *Let P be Eulerian of rank n, and let Q be any subposet of P containing $\hat{0}$ and $\hat{1}$. Set $\overline{Q} = (P - Q) \cup \{\hat{0}, \hat{1}\}$. Then*

$$\mu_Q(\hat{0}, \hat{1}) = (-1)^{n-1} \mu_{\overline{Q}}(\hat{0}, \hat{1}).$$

Proof. Since P is Eulerian, we have

$$\mu_P(\hat{0}, t_1)\mu_P(t_1, t_2) \cdots \mu_P(t_k, \hat{1}) = (-1)^n$$

for all chains $\hat{0} < t_1 < \cdots < t_k < \hat{1}$ in P. Hence from Lemma 3.16.4, we have $\mu_Q(\hat{0}, \hat{1}) = \sum (-1)^{k+n}$, where the sum ranges over all chains $\hat{0} < t_1 < \cdots < t_k < \hat{1}$ in $\overline{Q}$. The proof follows from Proposition 3.8.5. □

3.16.6 Corollary. *Let P be Eulerian of rank n, let $S \subseteq [n-1]$, and set $\overline{S} = [n-1] - S$. Then $\beta_P(S) = \beta_P(\overline{S})$.*

Proof. Apply Proposition 3.16.5 to the case $Q = P_S \cup \{\hat{0}, \hat{1}\}$ and use equation (3.54). □

Topological digression

Proposition 3.16.5 provides an instructive example of the usefulness of interpreting the Möbius function as a (reduced) Euler characteristic and then considering the actual homology groups. In general, we expect that if we suitably strengthen the hypotheses to take into account the homology groups, then the conclusion will be similarly strengthened. Indeed, suppose that instead of merely requiring that $\mu_P(s,t) = (-1)^{\ell(s,t)}$, we assume that

$$\widetilde{H}_i(\Delta(s,t); K) = \begin{cases} 0, & i \neq \ell(s,t) - 2, \\ K, & i = \ell(s,t) - 2, \end{cases}$$

where K is a field (or any coefficient group), and $\Delta(s,t)$ denotes the order complex (as defined in Section 3.8) of the open interval (s,t). Equivalently, P is Eulerian and Cohen-Macaulay over K. (We then say that P is a *Gorenstein* poset* over K. The asterisk is part of the notation, not a footnote indicator.) Let Q, $\overline{Q}$ be as in Proposition 3.16.5, and set $Q' = Q = \{\hat{0}, \hat{1}\}$, $\overline{Q}' = \overline{Q} - \{\hat{0}, \hat{1}\}$. The *Alexander duality theorem* for simplicial complexes asserts in the present context that

$$\widetilde{H}_i(\Delta(Q'); K) \cong \widetilde{H}^{n-i-3}(\Delta(\overline{Q}'); K).$$

(When K is a field there is a (noncanonical) isomorphism $\widetilde{H}^j(\Delta; K) \cong \widetilde{H}_j(\Delta; K)$.) In particular, $\widetilde{\chi}(\Delta(Q')) = (-1)^{n-1} \widetilde{\chi}(\Delta(\overline{Q}'))$, which is equivalent to Proposition 3.16.5 (by Proposition 3.8.6). Hence, Proposition 3.16.5 may be regarded as the "Möbius-theoretic analogue" of the Alexander duality theorem.

Finally, we come to a remarkable "master duality theorem" for Eulerian posets P. We will associate with P two polynomials $f(P,x)$ and $g(P,x)$ defined next. Define $\widetilde{P}$ to be the set of all intervals $[\hat{0}, t]$ of P, ordered by inclusion. Clearly, the map $P \to \widetilde{P}$ defined by $t \mapsto [\hat{0}, t]$ is an isomorphism of posets. The polynomials f and g are defined inductively as follows.

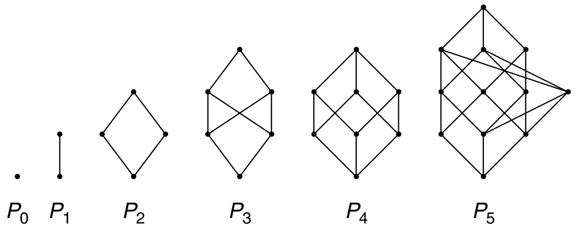

Figure 3.33 Some Eulerian posets.

1.
$$f(\mathbf{1},x) = g(\mathbf{1},x) = 1. \tag{3.74}$$

2. If $n+1 = \text{rank}\, P > 0$, then $f(P,x)$ has degree n, say $f(P,x) = h_0 + h_1 x + \cdots + h_n x^n$. Then define
$$g(P,x) = h_0 + (h_1 - h_0)x + (h_2 - h_1)x^2 + \cdots + (h_m - h_{m-1})x^m, \tag{3.75}$$
where $m = \lfloor n/2 \rfloor$.
3. If $n+1 = \text{rank}\, P > 0$, then define
$$f(P,x) = \sum_{\substack{Q \in \widetilde{P} \\ Q \neq P}} g(Q,x)(x-1)^{n-\rho(Q)}. \tag{3.76}$$

We call $f(P,x)$ the *toric h-polynomial* of P, and we call $g(P,x)$ the *toric g-polynomial* of P. The sequence $(h_0,\ldots,h_n)$ of coefficients of $f(P,x)$ is called the *toric h-vector* of P. The *toric g-vector* is defined similarly.

3.16.7 Example. Consider the six Eulerian posets of Figure 3.33. Write f_i and g_i for $f(P_i,x)$ and $g(P_i,x)$, respectively. We compute recursively that

$$\begin{aligned}
&f_0 = g_0 = 1,\\
&f_1 = g_0 = 1, \;\; g_1 = 1,\\
&f_2 = 2g_1 + g_0(x-1) = 1+x, \;\; g_2 = 1,\\
&f_3 = 2g_2 + 2g_1(x-1) + (x-1)^2 = 1 + x^2, \;\; g_3 = 1 - x,\\
&f_4 = 3g_2 + 3g_1(x-1) + (x-1)^2 = 1 + x + x^2, \;\; g_4 = 1,\\
&f_5 = 2g_4 + g_3 + 4g_2(x-1) + 3g_1(x-1)^2 + (x-1)^3 = 1 + x^3,\\
&\qquad g_5 = 1 - x.
\end{aligned}$$

3.16.8 Example. Write $f_n = f(B_n,x)$ and $g_n = g(B_n,x)$, where B_n is a boolean algebra of rank n. A simple computation yields

$$\begin{aligned}
&f_0 = 1, \;\; g_0 = 1, \;\; f_1 = 1, \;\; g_1 = 1, \;\; f_2 = 1+x, \;\; g_2 = 1,\\
&f_3 = 1 + x + x^2, \;\; g_3 = 1, \;\; f_4 = 1 + x + x^2 + x^3, \;\; g_4 = 1.
\end{aligned}$$

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This computation suggests that $f_n = 1 + x + \cdots + x^{n-1}$ $(n > 0)$ and $g_n = 1$. Clearly, equations (3.74) and (3.75) hold; we need only to check (3.76). The recurrence (3.76) reduces to

$$f_{n+1} = \sum_{k=0}^{n} g_k \binom{n+1}{k} (x-1)^{n-k}.$$

Substituting $g_k = 1$ yields

$$\begin{aligned} f_{n+1} &= \sum_{k=0}^{n} \binom{n+1}{k} (x-1)^{n-k} \\ &= (x-1)^{-1}\left[((x-1)+1)^{n+1} - 1\right], \text{ by the binomial theorem} \\ &= 1 + x + \cdots + x^n. \end{aligned}$$

Hence, we have shown

$$\begin{aligned} f(B_n, x) &= 1 + x + \cdots + x^{n-1}, \ \ n \geq 1, \\ g(B_n, x) &= 1, \ \ n \geq 0. \end{aligned}$$

Now suppose that P is Eulerian of rank $n+1$ and $P - \{\hat{1}\}$ is simplicial. Since $g(B_n, x) = 1$ we get from equation (3.76) that

$$\begin{aligned} f(P, x) &= \sum_{Q \neq P} (x-1)^{n-\rho(Q)} \\ &= \sum_{i=0}^{n} W_i (x-1)^{n-i}, \end{aligned} \tag{3.77}$$

where P has W_i elements of rank i.

We come to the main result of this section.

3.16.9 Theorem. *Let P be Eulerian of rank $n+1$. Then $f(P, x) = x^n f(P, 1/x)$. Equivalently, if $f(P, x) = \sum_{i=0}^{n} h_i x^{n-i}$, then $h_i = h_{n-i}$.*

Proof. We write $f(P)$ for $f(P, x)$, $g(P)$ for $g(P, x)$, and so on. Set $y = x - 1$. Multiply equation (3.76) by y and add $g(P)$ to obtain

$$\begin{aligned} g(P) + y f(P) &= \sum_{Q \in \widetilde{P}} g(Q) y^{\rho(P) - \rho(Q)} \\ \Rightarrow y^{-\rho(P)}(g(P) + y f(P)) &= \sum_{Q} g(Q) y^{-\rho(Q)}. \end{aligned}$$

By Möbius inversion, we obtain

$$g(P) y^{-\rho(P)} = \sum_{Q} (g(Q) + y f(Q)) y^{-\rho(Q)} \mu_{\widetilde{P}}(Q, P).$$

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Since $\widetilde{P}$ is Eulerian, we get $\mu_{\widetilde{P}}(Q,P) = (-1)^{\ell(Q,P)}$, so

$$g(P) = \sum_{Q} (g(Q) + yf(Q))(-y)^{\ell(Q,P)}. \tag{3.78}$$

Let $f(Q) = a_0 + a_1 x + \cdots + a_r x^r$, where $\rho(Q) = r+1$. Then

$$g(Q) + yf(Q) = (a_s - a_{s+1})x^{s+1} + (a_{s+1} - a_{s+2})x^{s+2} + \cdots,$$

where $s = \lfloor r/2 \rfloor$. By induction on $\rho(Q)$ we may assume that $a_i = a_{r-i}$ for $r < n$. In this case,

$$\begin{aligned} g(Q) + yf(Q) &= \begin{cases} (a_s - a_{s+1})x^{s+1} + (a_{s-1} - a_{s-2})x^{s+2} + \cdots, & r \text{ even} \\ (a_s - a_{s+1})x^{s+2} + (a_{s-1} - a_{s-2})x^{s+3} + \cdots, & r \text{ odd} \end{cases} \\ &= x^{\rho(Q)} g(Q, 1/x). \end{aligned} \tag{3.79}$$

Now subtract $yf(P) + g(P)$ from both sides of equation (3.78) and use (3.79) to obtain

$$\begin{aligned} -yf(P) &= \sum_{Q < \hat{1}} x^{\rho(Q)} g(Q, 1/x)(-y)^{\ell(Q,P)} \\ \Rightarrow f(P) &= \sum_{Q < \hat{1}} x^{\rho(Q)} g(Q, 1/x)(-y)^{\ell(Q,P)-1} \\ &= x^n f(P, 1/x), \text{ by equation (3.76),} \end{aligned}$$

and the proof is complete. □

Equation (3.77) gives a direct combinatorial interpretation of the polynomial $f(P,x)$ provided $P - \{\hat{1}\}$ is simplicial, and in this case Theorem 3.16.9 is equivalent to equation (3.73). In general, however, $f(P,x)$ seems to be an exceedingly subtle invariant of P. See Exercises 176–177 and 179 for further information.

3.17 The *cd*-Index of an Eulerian Poset

In Corollary 3.16.6 we showed that $\beta_P(S) = \beta_P(\overline{S})$ for any Eulerian poset P of rank n, where $S \subseteq [n-1]$ and $\overline{S} = [n-1] - S$. We can ask whether the flag h-vector β_P satisfies additional relations, valid for all Eulerian posets of rank n. For instance, writing $\beta(i)$ as short for $\beta(\{i\})$, we always have the relation

$$\beta_P(1) - \beta_P(2) + \cdots + (-1)^n \beta_P(n-1) = \begin{cases} 1, & n \text{ even,} \\ 0, & n \text{ odd,} \end{cases}$$

an immediate consequence of the Eulerian property. However, there can be additional relations, such as

$$\beta_P(1,2) - \beta_P(1,3) + \beta_P(2,3) = 1$$

when $n = 5$.

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In this section, we will find all the linear relations satisfied by the flag h-vector (or equivalently, flag f-vector, since the two flag vectors are linearly related) of an Eulerian poset of rank n. This information is best codified by a certain noncommutative polynomial $\Phi_P(c,d)$, called the cd-index of P. When $P = B_n$, the cd-index $\Phi_{B_n}(c,d)$ coincides with the cd-index $\Phi_n(c,d)$ of $\mathfrak{S}_n$ defined in Section 1.6. The equality of $\Phi_n(c,d)$ and $\Phi_{B_n}(c,d)$ is a consequence of Corollary 3.13.2, which states that $\beta_{B_n}(S)$ is the number $\beta_n(S)$ of permutations $w \in \mathfrak{S}_n$ with descent set S.

Recall from equation (1.60) that we defined the *characteristic monomial* u_S of $S \subseteq [n-1]$ by

$$u_S = e_1 e_2 \cdots e_{n-1},$$

where

$$e_i = \begin{cases} a, & \text{if } i \notin S, \\ b, & \text{if } i \in S. \end{cases}$$

Given any graded poset P of rank n with $\hat{0}$ and $\hat{1}$, define the noncommutative polynomial $\Psi_P(a,b)$, called the *ab-index* of P, by

$$\Psi_P(a,b) = \sum_{S \subseteq [n-1]} \beta_P(S) u_S.$$

Thus, $\Psi_P(a,b)$ is a noncommutative generating function for the flag h-vector β_P. Note that it is an immediate consequence of the definition (3.52) (or equivalently, equation (3.53)) that

$$\Psi_P(a+b,b) = \sum_{S \subseteq [n-1]} \alpha_P(S) u_S.$$

The main result of this section is the following.

3.17.1 Theorem. *Let P be an Eulerian poset of rank n. Then there exists a polynomial $\Phi_P(c,d)$ in the noncommutative variables c and d such that*

$$\Psi_P(a,b) = \Phi_P(a+b, ab+ba).$$

The polynomial $\Phi_P(c,d)$ is called the *cd-index* of P. For instance, let P be the face poset of the regular cell complex of Figure 3.34 (a decomposition of the 2-sphere). Then

$$\Psi_P(a+b,b) = aaa + 4baa + 7aba + 5aab + 14bba + 14bab + 14abb + 28bbb,$$

whence

$$\begin{aligned} \Psi_P(a,b) &= aaa + 3baa + 6aba + 4aab + 4bba + 6bab + 3abb + bbb \\ &= (a+b)^3 + 3(a+b)(ab+ba) + 2(ab+ba)(a+b). \end{aligned}$$

It follows that $\Phi_P(c,d) = c^3 + 3cd + 2dc$.

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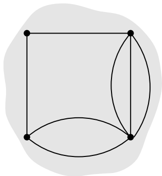

Figure 3.34 A regular cell complex with four vertices, seven edges, and five faces.

Proof of Theorem 3.17.1. The proof is by induction on $\operatorname{rank}(P)$. The result is clearly true for $\operatorname{rank}(P) = 1$, where $\Phi_P(c,d) = 1$. Assume for posets of rank less than n, and let $\operatorname{rank}(P) = n$. We have

$$\begin{aligned}\Psi_P(a+b,b) &= \sum_{S\subseteq[n-1]} \alpha_P(S) u_S \\ &= \sum_{\hat{0}=t_0<t_1<\cdots<t_k=\hat{1}} a^{\rho(t_0,t_1)-1} b a^{\rho(t_1,t_2)-1} b \cdots a^{\rho(t_{k-1},t_k)-1}.\end{aligned}$$

Hence,

$$\Psi_P(a,b) = \sum_{\hat{0}=t_0<t_1<\cdots<t_k=\hat{1}} (a-b)^{\rho(t_0,t_1)-1} b (a-b)^{\rho(t_1,t_2)-1} b \cdots (a-b)^{\rho(t_{k-1},t_k)-1}. \tag{3.80}$$

This formula remains true when P is replaced by any interval $[s,u]$ of P (since intervals of Eulerian posets are Eulerian). Write $\Psi_Q = \Psi_Q(a,b)$. Let (s,u) denote as usual the open interval $\{v \in P : s < v < u\}$, and similarly for the half-open interval $(s,u] = \{v \in P : s < v \leq u\}$. Replacing P by $[s,u]$ in equation (3.80) and breaking up the right-hand side according to the value of $t = t_{k-1}$ gives

$$\Psi_{[s,u]} = (a-b)^{\rho(s,u)-1} + \sum_{t\in(s,u)} \Psi_{[s,t]} b (a-b)^{\rho(t,u)-1}, \tag{3.81}$$

where the first term on the right-hand side comes from the chain $s = t_0 < t_1 = u$. Multiply on the right by $a-b$ and add $\Psi_{[s,u]} b$ to both sides:

$$\Psi_{[s,u]} a = (a-b)^{\rho(s,u)} + \sum_{t\in(s,u]} \Psi_{[s,t]} b (a-b)^{\rho(t,u)}. \tag{3.82}$$

Let $I(P)$ denote the incidence algebra of P over the noncommutative polynomial ring $\mathbb{Q}\langle a,b\rangle$. Define $f,g,h \in I(P)$ by

$$f(s,t) = \begin{cases} \Psi_{[s,t]} a, & s < t, \\ 1, & s = t, \end{cases}$$

$$g(s,t) = \begin{cases} \Psi_{[s,t]} b, & s < t, \\ 1, & s = t, \end{cases}$$

$$h(s,t) = (a-b)^{\rho(s,t)}.$$

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Thus, equation (3.82) is equivalent to $f = gh$.

Note that h^{-1} exists (since $h(t,t) = 1$). Since P is Eulerian and h has the multiplicative property $h(s,t)h(t,u) = h(s,u)$, we have

$$h^{-1}(s,t) = \mu(s,t)(a-b)^{\rho(s,t)} = (-1)^{\rho(s,t)}(a-b)^{\rho(s,t)}.$$

Therefore, the relation $g = fh^{-1}$ can be rewritten

$$\Psi_{[s,u]}b = (-1)^{\rho(s,u)}(a-b)^{\rho(s,u)} + \sum_{t\in(s,u]} \Psi_{[s,t]}a(-1)^{\rho(t,u)}(a-b)^{\rho(t,u)}.$$

Move $\Psi_{[s,u]}a$ to the left-hand side and cancel the factor $b-a$:

$$\Psi_{[s,u]} = -(-1)^{\rho(s,u)}(a-b)^{\rho(s,u)-1} - \sum_{t\in(s,u)} \Psi_{[s,t]}a(-1)^{\rho(t,u)}(a-b)^{\rho(t,u)-1}. \tag{3.83}$$

Add equations (3.81) and (3.83). We obtain an expression for $2\Psi_{[s,u]}$ as a function of $\Psi_{[s,t]}$ for $s<t<u$ and of $a+b=c$ and $(a-b)^{2m} = (c^2-2d)^m$. By the induction hypothesis, $\Psi_{[s,t]}$ is a polynomial in c and d, so the proof follows by induction. □

It is easy to recover the result $\beta_P(S) = \beta_P(\overline{S})$ (Corollary 3.16.6) from Theorem 3.17.1. For this result is equivalent to $\Psi_P(a,b) = \Psi_P(b,a)$, which is an immediate consequence of $\Psi_P(a,b) = \Phi_P(a+b, ab+ba)$.

If P is an Eulerian poset of rank n, then $\Phi_P(c,d)$ is a homogeneous polynomial in c,d of degree $n-1$, where we define $\deg(c) = 1$ and $\deg(d) = 2$. The number of monomials of degree $n-1$ in c and d is the number of compositions of $n-1$ into parts equal to 1 and 2, which by Exercise 1.35(c) is the Fibonacci number F_n. The coefficients of Φ_P are linear combinations of the 2^{n-1} numbers $\beta_P(S)$, $S \subseteq [n-1]$, where the coefficients in the linear combination depend only on n, not on P. The coefficients of Φ_P are also linear combinations of the 2^{n-1} numbers $\alpha_P(S)$, $S \subseteq [n-1]$, since the $\beta_P(S)$'s are linear combinations of the $\alpha_P(T)$'s (equation (3.52)). Let $\mathcal{F}_n$ denote the linear span of all flag h-vectors β_P (or flag f-vectors α_P) of Eulerian posets of rank n in the 2^{n-1}-dimensional vector space of all functions $2^{[n-1]} \to \mathbb{R}$. This argument shows that

$$\dim \mathcal{F}_n \le F_n. \tag{3.84}$$

We can also ask for the dimension of the *affine* subspace spanned by flag h-vectors of Eulerian posets of rank n. There is at least one additional affine relation, namely, $\beta(\emptyset) = 1$. Equivalently, the coefficient of c^{n-1} in $\Phi_P(c,d)$ is 1. Let $\mathcal{G}_n$ denote the affine span of all flag h-vectors of Eulerian posets of rank n. Thus,

$$\dim \mathcal{G}_n \le F_n - 1. \tag{3.85}$$

We now show that the bounds in equations (3.84) and (3.85) are tight.

First, we need to define an operation on posets P and Q with $\hat{0}$ and $\hat{1}$. The *join* $P * Q$ of P and Q is the poset

$$P * Q = (P - \{\hat{1}\}) \oplus (Q - \{\hat{0}\}), \tag{3.86}$$

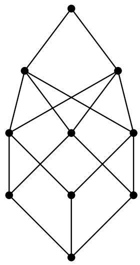

Figure 3.35 The join $B_3 * B_2$.

where $\oplus$ denotes ordinal sum. For instance, Figure 3.35 shows the join of the boolean algebras B_3 and B_2. It is easy to see (Exercise 3.191) that the join of Eulerian posets is Eulerian. Moreover, in this case we have

$$\Phi_{P*Q}(c,d) = \Phi_P(c,d)\Phi_Q(c,d). \tag{3.87}$$

Let Q_m denote the face lattice of a polygon with m vertices, so Q_m is Eulerian with rank-generating function $F(Q_m,q) = 1 + mq + mq^2 + q^3$. Moreover,

$$\Phi_{Q_m}(c,d) = c^2 + (m-2)d.$$

3.17.2 Theorem. *We have* $\dim \mathcal{F}_n = F_n$ *and* $\dim \mathcal{G}_n = F_n - 1$.

Proof. By the previous discussion, we need to show that F_n and $F_n - 1$ are lower bounds for the dimensions of $\mathcal{F}$ and $\mathcal{G}$. Given a cd-monomial $\rho = e_1e_2\cdots e_r$, let $P_\rho = Q_1 * Q_2 * \cdots * Q_r$, where

$$Q_i = \begin{cases} B_2, & e_i = c, \\ Q_m, & e_i = d. \end{cases}$$

By equations (3.86) and (3.87), P_ρ is Eulerian, and $\Phi_{P_\rho}(c,d) = h_1h_2\cdots h_r$, where

$$h_i = \begin{cases} c, & e_i = c, \\ c^2 + (m-2)d, & e_i = d. \end{cases}$$

For instance, if $\rho = dc^2d^2$, then

$$\begin{aligned} \Phi_{P_\rho}(c,d) &= (c^2 + (m-2)d)c^2(c^2 + (m-2)d)^2 \\ &= m^3dc^2d^2 + O(m^2). \end{aligned}$$

In general, $\Phi_{P_\rho}(c,d) = m^k\rho + O(m^{k-1})$, where k is the number of d's in ρ. Hence, unless $\rho = c^n$, we can make the coefficient of ρ arbitrarily large compared to the other coefficients, so it is affinely (and hence linearly) independent from the other coefficients. Hence, $\dim \mathcal{G}_n = F_n - 1$. Since all the coefficients other than that of c^n are affinely independent, while the coefficient of c^n is 1, it follows

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that the coefficient of c^n is linear independent from the other coefficients. Hence, $\dim \mathcal{F}_n = F_n$. □

3.18 Binomial Posets and Generating Functions

We have encountered many examples of generating functions so far, primarily of the form $\sum_{n\geq 0} f(n)x^n$ or $\sum_{n\geq 0} f(n)x^n/n!$. Why are these types so ubiquitous, and why do generating functions such as $\sum_{n\geq 0} f(n)x^n/(1+n^2)$ never seem to occur? Are there additional classes of generating functions besides the previous two that are useful in combinatorics? The theory of binomial posets seeks to answer these questions. It allows a unified treatment of many of the different types of generating functions that occur in combinatorics. This section and the next will be devoted to this topic. Most of the material in subsequent chapters of this book will be devoted to more sophisticated aspects of generating functions that are not really appropriate to the theory of binomial posets. We should mention that there are several alternative approaches to unifying the theory of generating functions. We have chosen binomial posets for two reasons: (a) We have already developed much of the relevant background information concerning posets, and (b) of all the existing theories, binomial posets give the most explicit combinatorial interpretation of the numbers $B(n)$ appearing in generating functions of the form $\sum_{n\geq 0} f(n)x^n/B(n)$. (Do not confuse these $B(n)$'s with the Bell numbers.)

Let us first consider some of the kinds of generating functions $F(x) \in \mathbb{C}[[x]]$ that have actually arisen in combinatorics. These generating functions should be regarded as "representing" the function $f\colon \mathbb{N} \to \mathbb{C}$ by the power series $F(x) = \sum_{n\geq 0} f(n)x^n/B(n)$, where the $B(n)$'s are certain complex numbers (which turn out in the theory of binomial posets always to be positive integers). The field $\mathbb{C}$ can be replaced with any field K of characteristic 0 throughout.

3.18.1 Example. a. (ordinary generating functions). These are generating functions of the form $F(x) = \sum_{n\geq 0} f(n)x^n$. (More precisely, we say that F is the *ordinary generating function* of f.) Of course, we have seen many examples of such generating functions, such as

$$\sum_{n\geq 0} \binom{t}{n} x^n = (1+x)^t,$$

$$\sum_{n\geq 0} \left(\!\binom{t}{n}\!\right) x^n = (1-x)^{-t},$$

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} (1-x^i)^{-1}.$$

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b. (exponential generating functions). Here $F(x) = \sum_{n\geq 0} f(n)x^n/n!$. Again, we have many examples, such as

$$\sum_{n\geq 0} B(n)\frac{x^n}{n!} = e^{e^x-1},$$

$$\sum_{n\geq 0} D(n)\frac{x^n}{n!} = \frac{e^{-x}}{1-x}.$$

c. (Eulerian generating functions). Let q be a fixed positive integer (almost always taken in practice to be a prime power corresponding to the field $\mathbb{F}_q$). Sometimes it is advantageous to regard q as an indeterminate, rather than an integer. The corresponding generating function is

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{(\boldsymbol{n})!},$$

where $(\boldsymbol{n})! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ is in Section 1.3. Note that $(\boldsymbol{n})!$ reduces to $n!$ upon setting $q=1$. As discussed in Section 1.8, sometimes in the literature one sees the denominator replaced with $[n]! = (1-q)(1-q^2)\cdots(1-q^n)$; this amounts to the transformation $x \mapsto x/(1-q)$. We will see that our choice of denominator is the natural one insofar as binomial posets are concerned. One immediate advantage is that an Eulerian generating function reduces to an exponential generating function upon setting $q=1$. An example of an Eulerian generating function is

$$\sum_{n\geq 0} f(n)\frac{x^n}{(\boldsymbol{n})!} = \left(\sum_{n\geq 0}\frac{x^n}{(\boldsymbol{n})!}\right)^2,$$

where $f(n)$ is the total number of subspaces of $\mathbb{F}_q^n$, that is, $f(n) = \sum_{k=0}^{n}\binom{\boldsymbol{n}}{\boldsymbol{k}}$.

d. (doubly exponential generating functions). These have the form $F(x) = \sum_{n\geq 0} f(n)x^n/n!^2$. For instance, if $f(n)$ is the number of $n\times n$ matrices of nonnegative integers such that every row and column sum equals two, then $F(x) = e^{x/2}(1-x)^{-1/2}$ (see Corollary 5.5.11). Sometimes one has occasion to deal with the more general *r-exponential generating function* $F(x) = \sum_{n\geq 0} f(n)x^n/n!^r$, where r is any positive integer.

e. (chromatic generating functions). Fix $q \in \mathbb{P}$. Then

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{q^{\binom{n}{2}}n!}.$$

Sometimes one sees $q^{\binom{n}{2}}$ replaced with $q^{n^2/2}$, amounting to the transformation $x \to xq^{-1/2}$. An example is

$$\sum_{n\geq 0} f(n)\frac{x^n}{2^{\binom{n}{2}}n!} = \left(\sum_{n\geq 0}(-1)^n\frac{x^n}{2^{\binom{n}{2}}n!}\right)^{-1}, \tag{3.88}$$

where $f(n)$ is the number of *acyclic digraphs* on n vertices; that is, the number of subsets of $[n]\times[n]$ not containing a sequence of elements $(i_0,i_1),(i_1,i_2),(i_2,i_3),\dots,(i_{j-1},i_j),(i_j,i_0)$. For instance, $f(3)=25$, corresponding to the empty set, the six 1-subsets $\{(i,j) : i\neq j\}$, the twelve 2-subsets $\{(i,j),(k,\ell) : i\neq j, k\neq \ell, (i,j)\neq(\ell,k)\}$, and the six 3-subsets obtained from $\{(1,2),(2,3),(1,3)\}$ by permuting 1,2,3. See the solution to Exercise 3.200.

The basic concept that will be used to unify these three above examples follows.

3.18.2 Definition. A poset P is called a *binomial poset* if it satisfies the three conditions:

a. P is locally finite with $\hat{0}$ and contains an infinite chain.
b. Every interval $[s,t]$ of P is graded. If $\ell(s,t)=n$, then we call $[s,t]$ an *n-interval*.
c. For all $n\in\mathbb{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains. We call $B(n)$ the *factorial function* of P.

NOTE. Condition (a) is basically a matter of convenience, and several alternative conditions are possible.

Note that from the definition of binomial poset we have $B(0)=B(1)=1$, $B(2)=\#[s,t]-2$, where $[s,t]$ is any 2-interval, and $B(0)\leq B(1)\leq B(2)\leq\cdots$.

3.18.3 Example. The posets below are all binomial posets.

a. Let $P=\mathbb{N}$ with the usual linear order. Then $B(n)=1$.
b. Let P be the lattice of all finite subsets of $\mathbb{P}$ (or any infinite set), ordered by inclusion. Then P is a distributive lattice and $B(n)=n!$. We will denote this poset as $\mathbb{B}$.
c. Let P be the lattice of all finite-dimensional subspaces of a vector space of infinite dimension over $\mathbb{F}_q$, ordered by inclusion. Then $B(n)=\boldsymbol{(n)!}$. We denote this poset by $\mathbb{B}(q)$.
d. Let P be the set of all ordered pairs (S,T) of finite subsets S,T of $\mathbb{P}$ satisfying $\#S=\#T$, ordered componentwise (i.e., $(S,T)\leq(S',T')$ if $S\subseteq S'$ and $T\subseteq T'$). Then $B(n)=n!^2$. This poset will be denoted $\mathbb{B}_2$. More generally, let $P_1,\dots,P_k$ be binomial posets with factorial functions $B_1,\dots,B_k$. Let P be the subposet of $P_1\times\cdots\times P_k$ consisting of all k-tuples $(t_1,\dots,t_k)$ such that $\ell(\hat{0},t_1)=\cdots=\ell(\hat{0},t_k)$. Then P is binomial with factorial function $B(n)=B_1(n)\cdots B_k(n)$. We write $P=P_1*\cdots*P_k$, the *Segre product* of $P_1,\dots,P_k$. Thus $\mathbb{B}_2=\mathbb{B}*\mathbb{B}$. More generally, we set $\mathbb{B}_r=\mathbb{B}*\cdots*\mathbb{B}$ (r times).

e. Let V be an infinite vertex set, let $q \in \mathbb{P}$ be fixed, and let P be the set of all pairs (G,σ), where G is a function from all 2-sets $\{u,v\} \in \binom{V}{2}$ into $\{0,1,\dots,q-1\}$ such that all but finitely many values of G are 0 (think of G as a graph with finitely many edges labeled $1,2,\dots,q-1$), and where $\sigma : V \to \{0,1\}$ is a map satisfying the two conditions:
   1. If $G(\{u,v\}) \neq 0$ then $\sigma(u) \neq \sigma(v)$, and
   2. $\sum_{v \in V} \sigma(v) < \infty$.

   If $(G,\sigma),(H,\tau) \in P$, then define $(G,\sigma) \leq (H,\tau)$ if
   1. $\sigma(v) \leq \tau(v)$ for all $v \in V$, and
   2. If $\sigma(u) = \tau(u)$ and $\sigma(v) = \tau(v)$, then $G(\{u,v\}) = H(\{u,v\})$.

   Then P is a binomial poset with $B(n) = n!q^{\binom{n}{2}}$. We leave to the reader the task of finding a binomial poset Q with factorial function $B(n) = q^{\binom{n}{2}}$ such that $P = Q * \mathbb{B}$, where $\mathbb{B}$ is the binomial poset of Example 3.18.3(b).

f. Let P be a binomial poset with factorial function $B(n)$, and let $k \in \mathbb{P}$. Define the rank-selected subposet (called the *k-th Veronese subposet*)
$$P^{(k)} = \{t \in P \,:\, \ell(\hat{0},t) \text{ is divisible by } k\}.$$
Then $P^{(k)}$ is binomial with factorial function
$$B_k(n) = B(nk)/B(k)^n.$$

Observe that the numbers $B(n)$ considered in Example 3.18.3(a)–(e) appear in the power series generating functions of Example 3.18.1. If we can somehow associate a binomial poset with generating functions of the form $\sum f(n)x^n/B(n)$, the we will have "explained" the form of the generating functions of Example 3.18.3. We also will have provided some justification of the heuristic principle that ordinary generating functions are associated with the nonnegative integers, exponential generating functions with sets, Eulerian generating functions with vector spaces over $\mathbb{F}_q$, and so on.

To begin our study of binomial posets P, choose $i,n \in \mathbb{N}$ and let $\binom{n}{i}_P$ denote the number of elements u of rank i in an n-interval $[s,t]$. Note that since $B(i)B(n-i)$ maximal chains of $[s,t]$ pass through a given element u of rank i, we have
$$\binom{n}{i}_P = \frac{B(n)}{B(i)B(n-i)}, \tag{3.89}$$
so $\binom{n}{i}_P$ depends only on n and i, not on the choice of the n-interval $[s,t]$. When $P = \mathbb{B}$ as in Example 3.18.3(b), then $B(n) = n!$ and $\binom{n}{i}_P = \binom{n}{i}$, explaining our terms "binomial poset" and "factorial function." The analogy with factorials is strengthened further by observing that
$$B(n) = A(n)A(n-1)\cdots A(1),$$
where $A(i) = \binom{i}{1}_P$, the number of atoms in an i-interval.

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We can now state the main result concerning binomial posets. Let P be a binomial poset with factorial function $B(n)$ and incidence algebra $I(P)$ over $\mathbb{C}$. Define

$$R(P) = \{f \in I(P)\colon\ f(s,t) = f(s',t') \text{ if } \ell(s,t) = \ell(s',t')\}.$$

If $f \in R(P)$, then write $f(n)$ for $f(s,t)$ when $\ell(s,t) = n$. Clearly $R(P)$ is a vector subspace of $I(P)$.

3.18.4 Theorem. *The space $R(P)$ is a subalgebra of $I(P)$, and we have an algebra isomorphism $\phi\colon\ R(P) \to \mathbb{C}[[x]]$ given by*

$$\phi(f) = \sum_{n\geq 0} f(n)\frac{x^n}{B(n)}.$$

The subalgebra $R(P)$ is called the reduced incidence algebra *of $I(P)$.*

Proof. Let $f,g \in R(P)$. We need to show that $fg \in R(P)$. By definition of $\binom{n}{i}_P$ we have for an n-interval $[s,t]$

$$\begin{aligned} fg(s,t) &= \sum_{u\in[s,t]} f(s,u)g(u,t) \\ &= \sum_{i=0}^{n} \binom{n}{i}_P f(i)g(n-i). \end{aligned} \tag{3.90}$$

Hence, $fg(s,t)$ depends only on $\ell(s,t)$, so $R(P)$ is a subalgebra of $I(P)$. Moreover, the right-hand side of equation (3.90) is just the coefficient of $x^n/B(n)$ in $\phi(f)\phi(g)$, so the proof follows. □

Let us note a useful property of the algebra $R(P)$ that follows directly from Theorem 3.18.4 (and that can also be easily proved without recourse to Theorem 3.18.4).

3.18.5 Proposition. *Let P be a binomial poset and $f \in R(P)$. Suppose that f^{-1} exists in $I(P)$, i.e., $f(t,t) \neq 0$ for all $t \in P$. Then $f^{-1} \in R(P)$.*

Proof. The constant term of the power series $F = \phi(f)$ is equal to $f(t,t) \neq 0$ for any $t \in P$, so F^{-1} exists in $\mathbb{C}[[x]]$. Let $g = \phi^{-1}(F^{-1})$. Since $FF^{-1} = 1$ in $\mathbb{C}[[x]]$, we have $fg = 1$ in $I(P)$. Hence, $f^{-1} = g \in R(P)$. □

We now turn to some examples of the unifying power of binomial posets. We make no attempt to be systematic or as general as possible, but simply try to convey some of the flavor of the subject.

3.18.6 Example. Let $f(n)$ be the cardinality of an n-interval $[s,t]$ of P, that is, $f(n) = \sum_{i=0}^{n} \binom{n}{i}_P$. Clearly by definition the zeta function ζ is in $R(P)$ and

$\phi(\zeta) = \sum_{n\geq 0} x^n/B(n)$. Since $R(P)$ is a subalgebra of $I(P)$, we have $\zeta^2 \in R(P)$. Since $\zeta^2(s,t) = \#[s,t]$, it follows that

$$\sum_{n\geq 0} f(n)\frac{x^n}{B(n)} = \left(\sum_{n\geq 0}\frac{x^n}{B(n)}\right)^2.$$

Thus from Example 3.18.3(a), we have the the cardinality of a chain of length n (or the number of integers in the interval $[0,n]$) satisfies

$$\sum_{n\geq 0} f(n)x^n = \left(\sum_{n\geq 0} x^n\right)^2 = \frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n,$$

whence $f(n) = n+1$ (not exactly the deepest result in enumerative combinatorics). Similarly from Example 3.18.3(b), the number $f(n)$ of subsets of an n-set satisfies

$$\sum_{n\geq 0} f(n)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\frac{x^n}{n!}\right)^2 = e^{2x} = \sum_{n\geq 0} 2^n\frac{x^n}{n!},$$

whence $f(n) = 2^n$. The analogous formula for Eulerian generating functions was given in Example 3.18.1(c).

3.18.7 Example. If $\mu(n)$ denotes the Möbius function $\mu(s,t)$ of an n-interval $[s,t]$ of P (which depends only on n, by Proposition 3.18.5), then from Theorem 3.18.4 we have

$$\sum_{n\geq 0}\mu(n)\frac{x^n}{B(n)} = \left(\sum_{n\geq 0}\frac{x^n}{B(n)}\right)^{-1}. \tag{3.91}$$

Thus with P as in Example 3.18.3(a),

$$\sum_{n\geq 0}\mu(n)x^n = \left(\sum_{n\geq 0} x^n\right)^{-1} = 1-x,$$

agreeing, of course, with Example 3.8.1. Similarly for Example 3.18.3(b),

$$\sum_{n\geq 0}\mu(n)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\frac{x^n}{n!}\right)^{-1} = e^{-x} = \sum_{n\geq 0}(-1)^n\frac{x^n}{n!},$$

giving yet another determination of the Möbius function of a boolean algebra. Thus, formally the Principle of Inclusion-Exclusion is equivalent to the identity $(e^x)^{-1} = e^{-x}$.

3.18.8 Example. The previous two examples can be generalized as follows. Let $Z_n(\lambda)$ denote the zeta polynomial (in the variable λ) of an n-interval $[s,t]$ of P.

Then since $Z_n(\lambda) = \zeta^\lambda(s,t)$, we have

$$\sum_{n\geq 0} Z_n(\lambda)\frac{x^n}{B(n)} = \left(\sum_{n\geq 0}\frac{x^n}{B(n)}\right)^\lambda .$$

This formula is valid for any complex number (or indeterminate) λ.

3.18.9 Example. As a variant of the previous example, fix $k \in \mathbb{P}$ and let $c_k(n)$ denote the number of chains $s = s_0 < s_1 < \cdots < s_k = t$ of length k between s and t in an n-interval $[s,t]$. Since $c_k(n) = (\zeta - 1)^k[s,t]$, we have

$$\sum_{n\geq 0} c_k(n)\frac{x^n}{B(n)} = \left(\sum_{n\geq 1}\frac{x^n}{B(n)}\right)^k .$$

The case $P = \mathbb{B}$ is particularly interesting. Here $c_k(n)$ is the number of chains $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = [n]$, or alternatively the number of ordered partitions $(S_1, S_2 - S_1, S_3 - S_2, \ldots, [n] - S_{k-1})$ of $[n]$ into k (nonempty) blocks. Since there are $k!$ ways of ordering a partition with k blocks, we have $c_k(n) = k!S(n,k)$. Hence,

$$\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(e^x - 1\right)^k .$$

Thus, the theory of binomial posets "explains" the simple form of the generating function from equation (1.94b).

3.18.10 Example. Let $c(n)$ be the *total* number of chains from s to t in the n-interval $[s,t]$; that is, $c(n) = \sum_k c_k(n)$. We have seen (Section 3.6) that $c(n) = (2-\zeta)^{-1}(s,t)$. Hence,

$$\sum_{n\geq 0} c(n)\frac{x^n}{B(n)} = \left(2 - \sum_{n\geq 0}\frac{x^n}{B(n)}\right)^{-1} .$$

For instance, if $P = \mathbb{N}$, then

$$\sum_{n\geq 0} c(n)x^n = \left(2 - \frac{1}{1-x}\right)^{-1} = 1 + \sum_{n\geq 1} 2^{n-1}x^n .$$

Thus, $c(n) = 2^{n-1}$, $n \geq 1$. Indeed, in the n-interval $[0,n]$, a chain $0 = t_0 < t_1 < \cdots < t_k = n$ can be identified with the composition $(t_1, t_2 - t_1, \ldots, n - t_{k-1})$, so we recover the result in Section 1.1 that there are 2^{n-1} compositions of n. If instead $P = \mathbb{B}$, then

$$\sum_{n\geq 0} c(n)\frac{x^n}{n!} = \frac{1}{2-e^x} .$$

As seen from Example 3.18.9, $c(n)$ is the total number of ordered partitions of the set $[n]$; that is, $c(n) = \sum_k k!S(n,k)$. One sometimes calls an ordered partition of a set S a *preferential arrangement*, since it corresponds to ranking the elements of S in linear order where ties are allowed.

3.18.11 Example. Let $f(n)$ be the total number of chains $s = s_0 < s_1 < \cdots < s_k = t$ in an n-interval $[s,t]$ of P such that $\ell(s_{i-1}, s_i) \geq 2$ for all $1 \leq i \leq k$, where k is allowed to vary. By now it should be obvious to the reader that

$$\begin{aligned} \sum_{n\geq 0} f(n)\frac{x^n}{B(n)} &= \sum_{k\geq 0}\left(\sum_{n\geq 0}\frac{x^n}{B(n)} - 1 - x\right)^k \\ &= \left(1 - \sum_{n\geq 2}\frac{x^n}{B(n)}\right)^{-1}. \end{aligned} \tag{3.92}$$

For instance, when $P = \mathbb{N}$, we are enumerating subsets of $[0,n]$ that contain 0 and n, and that don't contain two consecutive integers. Equivalently, we are counting compositions $(t_1 - t_0, t_2 - t_1, \ldots, n - t_{k-1})$ of n with no part equal to 1. From equation (3.92), we have

$$\begin{aligned} \sum_{n\geq 0} f(n)x^n &= \left(1 - \frac{x^2}{1-x}\right)^{-1} \\ &= \frac{1-x}{1-x-x^2} = 1 + \sum_{n\geq 2} F_{n-1}x^n, \end{aligned}$$

where F_{n-1} denotes a Fibonacci number, in agreement with Exercise 1.35(b). Similarly when $P = \mathbb{B}$, we get $(2 + x - e^x)^{-1}$ as the exponential generating function for the number of ordered partitions of an n-set with no singleton blocks.

3.19 An Application to Permutation Enumeration

In Section 3.13, we related Möbius functions to the counting of permutations with certain properties. Using the theory of binomial posets, we can obtain generating functions for counting some of these permutations.

Throughout this section P denotes a binomial poset with factorial function $B(n)$. Let $S \subseteq \mathbb{P}$. If $[s,t]$ is an n-interval of P, then denote by $[s,t]_S$ the S-rank selected subposet of $[s,t]$ with s and t adjoined; that is,

$$[s,t]_S = \{u \in [s,t] : u = s, u = t, \text{or } \ell(s,u) \in S\}. \tag{3.93}$$

Let μ_S denote the Möbius function of $[s,t]_S$, and set $\mu_S(n) = \mu_S(s,t)$. (It is easy to see that $\mu_S(n)$ depends only on n, not on the choice of the n-interval $[s,t]$.)

3.19.1 Lemma. *We have*

$$-\sum_{n\geq 1}\mu_S(n)\frac{x^n}{B(n)} = \left[\sum_{n\geq 1}\frac{x^n}{B(n)}\right]\left[1 + \sum_{n\in S}\mu_S(n)\frac{x^n}{B(n)}\right]. \tag{3.94}$$

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Proof. Define a function $\chi : \mathbb{N} \to \{0,1\}$ by $\chi(n) = 1$ if $n = 0$ or $n \in S$, and $\chi(n) = 0$ otherwise. Then the defining recurrence (3.15) for Möbius functions yields $\mu_S(0) = 1$ and

$$\mu_S(n) = -\sum_{i=0}^{n-1} \binom{n}{i}_P \mu_S(i)\chi(i), \ \ n \geq 1,$$

where $\binom{n}{i}_P = B(n)/B(i)B(n-i)$ as usual. Hence,

$$-\mu_S(n)(1-\chi(n)) = \sum_{i=0}^{n} \binom{n}{i}_P \mu_S(i)\chi(i), \ \ n \geq 1,$$

which translates into the generating function identity

$$-\sum_{n\geq 0} \mu_S(n)\frac{x^n}{B(n)} + \sum_{n\geq 0} \mu_S(n)\chi(n)\frac{x^n}{B(n)}$$
$$= \left[\sum_{n\geq 0} \frac{x^n}{B(n)}\right]\left[\sum_{n\geq 0} \mu_S(n)\chi(n)\frac{x^n}{B(n)}\right] - 1.$$

This formula is clearly equivalent to equation (3.94). □

We now consider a set S for which the power series $1 + \sum_{n\in S} \mu_S(n)x^n/B(n)$ can be explicitly evaluated.

3.19.2 Lemma. *Let $k \in \mathbb{P}$, and let $S = k\mathbb{P} = \{kn : n \in \mathbb{P}\}$. Then*

$$1 + \sum_{n\in S} \mu_S(n)\frac{x^n}{B(n)} = \left[\sum_{n\geq 0} \frac{x^{kn}}{B(kn)}\right]^{-1}. \tag{3.95}$$

Proof. Let $P^{(k)}$ be the binomial poset of Example 3.18.3(f), with factorial function $B_k(n) = B(kn)/B(k)^n$. If $\mu^{(k)}$ is the Möbius function of $P^{(k)}$, then it follow from equation (3.91) that

$$\sum_{n\geq 0} \mu^{(k)}(n)\frac{x^n}{B_k(n)} = \left[\sum_{n\geq 0} \frac{x^n}{B_k(n)}\right]^{-1}. \tag{3.96}$$

But $\mu^{(k)}(n) = \mu_S(kn)$. Putting $B_k(n) = B(kn)/B(k)^n$ in equation (3.96), we obtain

$$\sum_{n\geq 0} \mu_S(kn)\frac{(B(k)x)^n}{B(kn)} = \left[\sum_{n\geq 0} \frac{(B(k)x)^n}{B(kn)}\right]^{-1}.$$

If we put x^k for $B(k)x$, then we get equation (3.95). □

Combining Lemmas 3.19.1 and 3.19.2 we obtain:

3.19.3 Corollary. *Let $k \in \mathbb{P}$ and $S = k\mathbb{P}$. Then*

$$-\sum_{n\geq 1}\mu_S(n)\frac{x^n}{B(n)} = \left[\sum_{n\geq 1}\frac{x^n}{B(n)}\right]\left[\sum_{n\geq 0}\frac{x^{kn}}{B(kn)}\right]^{-1}.$$

Now specialize to the case $P = \mathbb{B}(q)$ of Example 3.18.3(c). For any $S \subseteq \mathbb{P}$, it follows from Theorem 3.13.3 that

$$(-1)^{\#(S\cap[n-1])-1}\mu_S(n) = \sum_w q^{\mathrm{inv}(w)},$$

where the sum is over all permutations $w \in \mathfrak{S}_n$ with descent set S. If $S = k\mathbb{P}$, then $\#(S\cap[n-1]) = \lfloor (n-1)/k \rfloor$. Hence, we conclude:

3.19.4 Proposition. *Let $k \in \mathbb{P}$, and let $f_{n,k}(q) = \sum_w q^{\mathrm{inv}(w)}$, where the sum is over all permutations $w = a_1a_2\cdots a_n \in \mathfrak{S}_n$ such that $a_i > a_{i+1}$ if and only if $k|i$. Then*

$$\sum_{n\geq 1}(-1)^{\lfloor (n-1)/k\rfloor} f_{n,k}(q)\frac{x^n}{(\boldsymbol{n})!} = \left[\sum_{n\geq 1}\frac{x^n}{(\boldsymbol{n})!}\right]\left[\sum_{n\geq 0}\frac{x^{kn}}{(\boldsymbol{kn})!}\right]^{-1}. \tag{3.97}$$

Although Proposition 3.19.4 can be proved without the use of binomial posets, our approach yields additional insight as to why equation (3.97) has such a simple form. In particular, the simple denominator $\sum_{n\geq 0} x^{kn}/(\boldsymbol{kn})!$ arises from dealing with the Möbius function of the poset $P^{(k)}$ where $P = \mathbb{B}(q)$.

We can eliminate the unsightly factor $(-1)^{\lfloor (n-1)/k\rfloor}$ in equation (3.97) by treating each congruence class of n modulo k separately. Fix $1 \leq j \leq k$, substitute $x^k \to -x^k$, and extract from (3.97) only those terms whose exponent is congruent to j modulo k to obtain the elegant formula

$$\sum_{\substack{m\geq 0\\ n=mk+j}} f_{n,k}(q)\frac{x^n}{(\boldsymbol{n})!} = \left[\sum_{\substack{m\geq 0\\ n=mk+j}}(-1)^m\frac{x^n}{(\boldsymbol{n})!}\right]\left[\sum_{n\geq 0}(-1)^n\frac{x^{nk}}{(\boldsymbol{nk})!}\right]^{-1} \tag{3.98}$$

In particular, when $j = k$, we can add 1 to both sides of equation (3.98) to obtain

$$\sum_{m\geq 0} f_{mk,k}(q)\frac{x^{mk}}{(\boldsymbol{mk})!} = \left[\sum_{n\geq 0}(-1)^n\frac{x^{nk}}{(\boldsymbol{nk})!}\right]^{-1}. \tag{3.99}$$

Equation (3.99) is also a direct consequence of Lemma 3.19.2.

One special case of equation (3.98) deserves special mention. Recall (Sections 1.4 and 1.6) that a permutation $a_1a_2\cdots a_n \in \mathfrak{S}_n$ is *alternating* if $a_1 > a_2 < a_3 > \cdots$. It is clear from the definition of $f_{n,k}(q)$ that $f_{n,2}(1)$ is the number E_n of alternating permutations in $\mathfrak{S}_n$. Substituting $k = 2$, $q = 1$, and $j = 1, k$ in equation (3.98) recovers Proposition 1.6.1, namely,

$$\sum_{n\geq 0} E_n\frac{x^n}{n!} = \sec x + \tan x. \tag{3.100}$$

Thus, we have a poset-theoretic explanation for the remarkable elegance of equation (3.100). Moreover, equation (1.59) and Exercise 1.147 can be proved in exactly the same way using $f_{n,k}(1)$.

3.20 Promotion and Evacuation

This section is independent of the rest of this book (except for a few exercises) and can be omitted without loss of continuity. Promotion and evacuation are certain bijections on the set of linear extensions of a finite poset P. They have some remarkable properties and arise in a variety of unexpected situations. See for instance Section A1.2 of Appendix 1 of Chapter 7, as well as Exercises 3.79–3.80.

Let $\mathcal{L}(P)$ denote the set of linear extensions of P. For now we regard a linear extension as an order-preserving bijection $f\colon P \to \boldsymbol{p}$, where $\#P = p$. Think of the element $t \in P$ as being labeled by $f(t)$. Now define a bijection $\partial\colon \mathcal{L}(P) \to \mathcal{L}(P)$, as follows. Remove the label 1 from P. Let $t_1 \in P$ satisfy $f(t_1) = 1$. Among the elements of P covering t_1, let t_2 be the one with the smallest label $f(t_2)$. Remove this label from t_2 and place it at t_1. (Think of "sliding" the label $f(t_2)$ down from t_2 to t_1.) Now among the elements of P covering t_2, let t_3 be the one with the smallest label $f(t_3)$. Slide the label from t_3 to t_2. Continue this process until eventually reaching a maximal element t_k of P. After we slide $f(t_k)$ to t_{k-1}, label t_k with $p+1$. Now subtract 1 from every label. We obtain a new linear extension $f\partial \in \mathcal{L}(P)$, called the *promotion* of f. Note that we let ∂ operate on the *right*. Note also that $t_1 \lessdot t_2 \lessdot \cdots \lessdot t_k$ is a maximal chain of P, called the *promotion chain* of f. Figure 3.36(a) shows a poset P and a linear extension f. The promotion chain is indicated by circled dots and arrows. Figure 3.36(b) shows the labeling after the sliding operations and the labeling of the last element of the promotion chain by $p+1 = 10$. Figure 3.36(c) shows the linear extension $f\partial$ obtained by subtracting 1 from the labels in Figure 3.36(b).

It should be obvious that $\partial\colon \mathcal{L}(P) \to \mathcal{L}(P)$ is a bijection. In fact, let ∂^* denote *dual promotion*, (i.e., we remove the *largest* label p from some element $u_1 \in P$, then slide the *largest* label of an element covered by u_1 up to u_1, etc.). After reaching a minimal element u_k, we label it by 0 and then add 1 to each label, obtaining $f\partial^*$. It is easy to check that

$$\partial^{-1} = \partial^*.$$

We next define a variant of promotion called *evacuation*. The evacuation of a linear extension $f \in \mathcal{L}(P)$ is denoted $f\epsilon$ and is another linear extension of P. First,

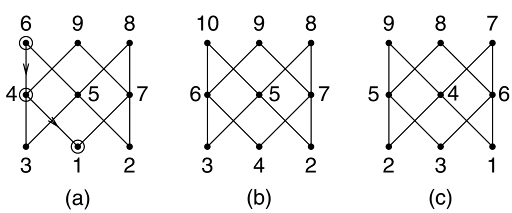

Figure 3.36 The promotion operator ∂ applied to a linear extension.

compute $f\partial$. Then "freeze" the label p into place and apply ∂ to what remains. In other words, let P_1 consist of those elements of P labelled $1, 2, \ldots, p-1$ by $f\partial$, and apply ∂ to the restriction of $f\partial$ to P_1. Then freeze the label $p-1$ and apply ∂ to the $p-2$ elements that remain. Continue in this way until every element has been frozen. Let $f\epsilon$ be the linear extension, called the *evacuation* of f, defined by the frozen labels.

Figure 3.37 illustrates the evacuation of a linear extension f. The promotion paths are shown by arrows, and the frozen elements are circled. For ease of understanding we don't subtract 1 from the unfrozen labels since they all eventually disappear. The labels are always frozen in descending order $p, p-1, \ldots, 1$. Figure 3.38 shows the evacuation of $f\epsilon$, where f is the linear extension of Figure 3.37. Note that (seemingly) miraculously, we have $f\epsilon^2 = f$. This example illustrates a fundamental property of evacuation given by Theorem 3.20.1(a).

We can define *dual evacuation* ϵ^* analogously to dual promotion (i.e., evacuate from the top of P rather than from the bottom). In symbols, if $f \in \mathcal{L}(P)$, then define $f^* \in \mathcal{L}(P^*)$ by $f^*(t) = p + 1 - f(t)$. Then ϵ^* is given by

$$f\epsilon^* = (f^*\epsilon)^*.$$

We can now state the main result of this section.

3.20.1 Theorem. *Let P be a p-element poset. Then the operators ϵ, ϵ^*, and ∂ satisfy the following properties.*

(a) Evacuation is an involution, that is, $\epsilon^2 = 1$ (the identity operator).
(b) $\partial^p = \epsilon\epsilon^*$.
(c) $\partial\epsilon = \epsilon\partial^{-1}$.

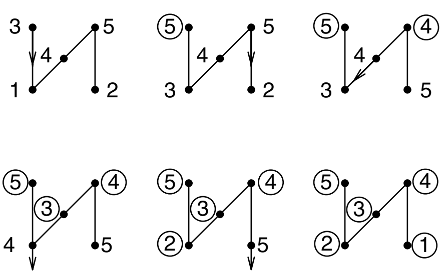

Figure 3.37 The evacuation of a linear extension f.

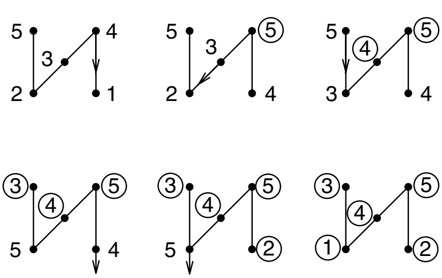

Figure 3.38 The linear extension $f\epsilon^2$.

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Theorem 3.20.1 can be interpreted algebraically as follows. The bijections ϵ and ϵ^* generate a subgroup of the symmetric group $\mathfrak{S}_{\mathcal{L}(P)}$ on all the linear extensions of P. Since ϵ and (by duality) ϵ^* are involutions, the group they generate is a dihedral group (possibly degenerate, i.e., isomorphic to $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$) of order 1 or $2m$ for some $m\geq 1$. If ϵ and ϵ^* are not both trivial, so they generate a group of order $2m$, then m is the order of ∂^p. In general the value of m, or more generally the cycle structure of ∂^p, is mysterious. For a few cases in which further information is known, see Exercise 3.80.

The main idea for proving Theorem 3.20.1 is to write linear extensions as *words* rather than functions and then to describe the actions of ∂ and ϵ on these words. The proof then becomes a routine algebraic computation. Let us first develop the necessary algebra in a more abstract context.

Let G be the group generated by elements $\tau_1,\dots,\tau_{p-1}$ satisfying

$$\begin{aligned} \tau_i^2 &= 1, \quad 1\leq i\leq p-1 \\ \tau_i\tau_j &= \tau_j\tau_i, \quad \text{if } |i-j|>1. \end{aligned} \tag{3.101}$$

Some readers will recognize these relations as a subset of the Coxeter relations defining the symmetric group $\mathfrak{S}_p$. Define the following elements of G for $1\leq j\leq p-1$:

$$\begin{aligned} \delta_j &= \tau_1\tau_2\cdots\tau_j, \\ \gamma_j &= \delta_j\delta_{j-1}\cdots\delta_1, \\ \gamma_j^* &= \tau_j\tau_{j-1}\cdots\tau_1\cdot\tau_j\tau_{j-1}\cdots\tau_2\cdots\tau_j\tau_{j-1}\cdot\tau_j. \end{aligned}$$

3.20.2 Lemma. *In the group G, we have the following identities for $1\leq j\leq p-1$:*

(a) $\gamma_j^2=(\gamma_j^*)^2=1$.
(b) $\delta_j^{j+1}=\gamma_j\gamma_j^*$.
(c) $\delta_j\gamma_j=\gamma_j\delta_j^{-1}$.

Proof. (a) Induction on j. For $j=1$, we need to show that $\tau_1^2=1$, which is given. Now assume for $j-1$. Then

$$\gamma_j^2=\tau_1\tau_2\cdots\tau_j\cdot\tau_1\cdots\tau_{j-1}\cdots\tau_1\tau_2\tau_3\cdot\tau_1\tau_2\cdot\tau_1\cdot\tau_1\tau_2\cdots\tau_j\cdot\tau_1\cdots\tau_{j-1}$$
$$\cdots\tau_1\tau_2\tau_3\cdot\tau_1\tau_2\cdot\tau_1.$$

We can cancel the two middle τ_1's since they appear consecutively. We can then cancel the two middle τ_2's since they are now consecutive. We can then move one of the middle τ_3's past a τ_1 so that the two middle τ_3's are consecutive and can be canceled. Now the two middle τ_4's can be moved to be consecutive and then canceled. Continuing in this way, we can cancel the two middle τ_i's for all $1\leq i\leq j$. When this cancellation is done, what remains is the element γ_{j-1}, which is 1 by induction.

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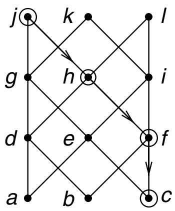

Figure 3.39 The promotion chain of the linear extension $cabdfeghjilk$.

(b,c) Analogous to (a). Details are omitted. □

Proof of Theorem 3.20.1. A glance at Theorem 3.20.1 and Lemma 3.20.2 makes it obvious that they should be connected. To see this connection, regard the linear extension $f \in \mathcal{L}(P)$ as the word (or permutation of P) $f^{-1}(1),\ldots,f^{-1}(p)$. For $1 \le i \le p-1$ define operators $\tau_i : \mathcal{L}(P) \to \mathcal{L}(P)$ by

$$(u_1u_2\cdots u_p)\tau_i = \begin{cases} u_1u_2\cdots u_p, & \text{if } u_i \text{ and } u_{i+1} \text{ are comparable in } P, \\ u_1u_2\cdots u_{i+1}u_i\cdots u_p, & \text{if } u_i \parallel u_{i+1}. \end{cases} \tag{3.102}$$

Clearly, τ_i is a bijection, and the τ_i's satisfy the relations (3.101). By Lemma 3.20.2, the proof of Theorem 3.20.1 follows from showing that

$$\partial = \delta_{p-1} := \tau_1\tau_2\cdots\tau_{p-1}.$$

Note that if $f = u_1u_2\cdots u_p$, then $f\delta_{p-1}$ is obtained as follows. Let j be the least integer such that $j > 1$ and $u_1 < u_j$. Since f is a linear extension, the elements $u_2, u_3, \ldots, u_{j-1}$ are incomparable with u_1. Move u_1 so it is between u_{j-1} and u_j. (Equivalently, cyclically shift the sequence $u_1u_2\cdots u_{j-1}$ one unit to the left.) Now let k be the least integer such that $k > j$ and $u_j < u_k$. Move u_j so it is between u_{k-1} and u_k. Continue in this way reaching the end. For example, let z be the linear extension $cabdfeghjilk$ of the poset in Figure 3.39 (which also shows the evacuation chain for this linear extension). (We denote the linear extension for this one example by z instead of f since we are denoting one of the elements of P by f.) We factor z from left to right into the longest factors for which the first element of each factor is incomparable with the other elements of the factor:

$$z = (cabd)(feg)(h)(jilk).$$

Cyclically shift each factor one unit to the left to obtain $z\delta_{p-1}$:

$$z\delta_{p-1} = (abdc)(egf)(h)(ilkj) = abdcegfhkilj.$$

Now consider the process of promoting the linear extension f of the previous paragraph, given as a function by $f(u_i) = i$ and as a word by $u_1u_2\cdots u_p$. The elements $u_2,\ldots,u_{j-1}$ are incomparable with u_1 and thus will have their labels reduced by 1 after promotion. The label j of u_j (the least element in the linear extension f greater than u_1) will slide down to u_1 and be reduced to $j-1$. Hence,

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$f\partial = u_2u_3\cdots u_{j-1}u_1\cdots$. Exactly analogous reasoning applies to the next step of the promotion process, when we slide the label k of u_k down to u_j. Hence,

$$f\partial = u_2u_3\cdots u_{j-1}u_1\cdot u_{j+1}u_{j+2}\cdots u_{k-1}u_j\cdots.$$

Continuing in this manner shows that $z\delta = z\partial$, completing the proof of Theorem 3.20.1. □

3.21 Differential Posets

Differential posets form a class of posets with many explicit enumerative properties that can be proved by linear algebraic techniques. Their combinatorics is closely connected with the combinatorics of the relation $DU - UD = I$.

3.21.1 Definition. Let $r \in \mathbb{P}$. An *r-differential poset* is a poset P satisfying the following three axioms.

(D1) P has a $\hat{0}$ and is locally finite and graded (i.e., every interval $[\hat{0},t]$ is graded).
(D2) If $t \in P$ covers exactly k elements, then t is covered by exactly $k+r$ elements.
(D3) Let $s,t \in P$, $s \neq t$. If exactly j elements are covered by both s and t, then exactly j elements cover both s and t.

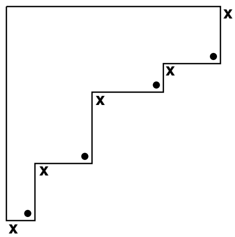
Note.

(a) It is easy to see that in axiom (D3) we must have either $j = 0$ or $j = 1$. For suppose $j > 1$ elements $u_1, u_2, \ldots$ are covered by both s and t, where the rank m of s and t is minimal with respect to this property. Then u_1 and u_2 are covered by both s and t, so by (D3) u_1 and u_2 cover at least two elements, contradicting the minimality of m.
(b) Suppose that L is an r-differential lattice. Then by Proposition 3.3.2, axiom (D3) is equivalent to the statement that L is modular.

Let us first give some examples of differential posets.

3.21.2 Example. 1. It is easy to see that if P is r-differential and Q is s-differential, then $P \times Q$ is $(r+s)$-differential.

2. Young's lattice $Y = J_f(\mathbb{N}\times\mathbb{N})$, defined in Example 3.4.4(b), is a 1-differential poset. Axiom (D1) of Definition 3.21.1 is clear, while (D3) follows from the distributivity (and hence modularity) of L. For (D2), note that the positions where we can add a square to the Young diagram of a partition λ (marked $\times$ in the diagram at the top of the next page) alternate along the boundary with the positions where we can remove a square (marked $\bullet$), with $\times$ at the beginning and end.
3. Let P be r-differential "up to rank n." This means that (a) P is graded of rank n with $\hat{0}$, (b) P satisfies axiom (D2) of Definition 3.21.1 if $\mathrm{rank}(t) < n$, and (c) P satisfies axiom (D3) of Definition 3.21.1 if $\mathrm{rank}(s) = \mathrm{rank}(t) < n$. Define $\Omega_r P$, the *reflection extension* of P, as follows. Let P_i denote the set of elements of P



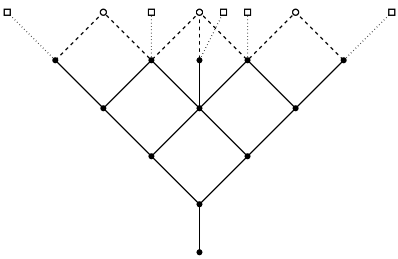


Figure 3.40 The Reflection-Extension construction.

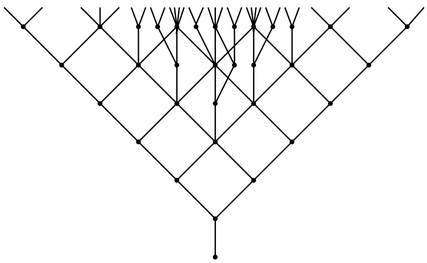


Figure 3.41 The Fibonacci differential poset Z_1.

of rank i, and suppose that $P_{n-1} = \{s_1, \ldots, s_k\}$. First, place $t_1, \ldots, t_k \in P_{n+1}$ with $t_i \gtrdot u \in P_n$ if and only if $s_i \lessdot u$ (the "reflection of P_{n-1} through P_n"). Then for each $u \in P_n$, adjoin r new elements covering u. Figure 3.40 gives an example for $r = 1$ and $n = 4$. The open circles and dashed lines show the reflection of rank 3 to rank 5, while the open squares and dotted lines indicate the extension of each element of rank 4.

It is easy to see that $\Omega_r P$ is r-differential up to rank $n+1$, and that

$$p_{n+1} = r p_n + p_{n-1}, \tag{3.103}$$

where $p_i = \#P_i$. In particular, if we infinitely iterate the operation Ω_r on P, then we obtain an r-differential poset $\Omega_r^\infty P$.

4. Let P be the one-element poset, and write $Z_r = \Omega_r^\infty P$. It is easy to check (e.g., by induction) that Z_r is a (modular) lattice, called the *r-Fibonacci* differential poset. If $r = 1$ then it follows from equation (3.103) that $\#(Z_1)_n = F_{n+1}$, a Fibonacci number. See Figure 3.41 for Z_1 up to rank 6.

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We now turn to the connection between the axioms (D1)–(D3) of a differential poset and linear algebra. Let K be a field and P any poset. Let KP be the vector space with basis P (i.e., the set of all formal linear combinations of elements of P with only finitely many nonzero coefficients). Let

$$\widehat{KP} = \left\{ \sum_{t \in P} c_t t \, : \, c_t \in K \right\},$$

the vector space of all *infinite* linear combinations of elements of P. A linear transformation $\varphi \colon \widehat{KP} \to \widehat{KP}$ is *continuous* if it preserves *infinite* linear combinations, that is,

$$\varphi\left(\sum c_t t\right) = \sum c_t \varphi(t), \tag{3.104}$$

so in particular the right-hand side of equation (3.104) must be well defined. For instance, if P is infinite and $u \in P$, then a map $\varphi \colon \widehat{KP} \to \widehat{KP}$ satisfying $\varphi(t) = u$ for all $t \in P$ cannot be extended to a continuous linear transformation since $\varphi\left(\sum_{t \in P} t\right) = \sum_{t \in P} u$, which is not defined.

Now assume that P satisfies axiom (D1). Define continuous linear transformations $U \colon \widehat{KP} \to \widehat{KP}$ and $D \colon \widehat{KP} \to \widehat{KP}$ by

$$U(s) = \sum_{t \gtrdot s} t, \text{ for all } s \in P,$$

$$D(s) = \sum_{t \lessdot s} t, \text{ for all } s \in P.$$

3.21.3 Proposition. *Let P satisfy axiom (D1). Then P is r-differential if and only if*

$$DU - UD = rI,$$

where I is the identity linear transformation on $\widehat{KP}$.

Proof. The proof is of the type "follow your nose." By the definition of the product of linear transformations, axiom (D2) is equivalent to the statement that the coefficient of t (when we expand as an infinite linear combination of elements of P) in $(DU - UD)t$ is r. If $s \neq t$, then axiom (D3) is equivalent to the statement that the coefficient of s in $(DU - UD)t$ is 0. $\square$

If $X \subseteq P$, then write $\boldsymbol{X} = \sum_{t \in X} t \in \widehat{KP}$. Define a pairing $\widehat{KP} \times KP \to K$ by

$$\langle s, t \rangle = \delta_{st}, \quad s, t \in P.$$

Thus,

$$\left\langle \sum_{t \in P} a_t t, \sum_{t \in P} b_t t \right\rangle = \sum a_t b_t,$$

a finite sum since $\sum_{t \in P} b_t t \in KP$. If $f \in \widehat{KP}$ and $t \in P$, then $\langle f, t \rangle$ is just the coefficient of t in f.

3.21.4 Proposition. *Let P be an r-differential poset. Then*

$$D\boldsymbol{P} = (U+r)\boldsymbol{P}.$$

Proof. Equivalent to axiom (D2). In more detail, for $t \in P$ let

$$C^-(t) = \{s \in P : s \lessdot t\}.$$

Then $\langle (U+r)\boldsymbol{P}, t\rangle = r + \#C^-(t)$, and the proof follows. □

3.21.5 Example. Let $t \in P_n$ (the set of elements of P of rank n), and let $e(t)$ denote the number of saturated chains $\hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n = t$ from $\hat{0}$ to t. Then

$$e(t) = \langle U^n\hat{0}, t\rangle.$$

Similarly [why?],

$$\sum_{t\in P_n} e(t) = \langle U^n\hat{0}, \boldsymbol{P_n}\rangle,$$

$$\sum_{t\in P_n} e(t)^2 = \langle D^nU^n\hat{0}, \hat{0}\rangle.$$

Let us now consider some relations between U and D that are formal consequences of $DU - UD = rI$. The basic goal is to "push" to the right the D's in an expression (power series) involving U and D by using $DU = UD + rI$. A useful way to understand such results is the following. We can "represent" $U = z$ (i.e., multiplication by the indeterminate z) and $D = r\frac{d}{dz}$ since $(r\frac{d}{dz})z - z(r\frac{d}{dz}) = r$ as operators. Thus, familiar identities involving differentiation can be transferred to identities involving U and D. For the algebraically minded, a more precise statement is that we have an isomorphism

$$K\langle\langle U, D\rangle\rangle/(DU - UD - r) \cong K\left\langle\left\langle z, r\frac{d}{dz}\right\rangle\right\rangle,$$

where $K\langle\langle\cdot\rangle\rangle$ denotes noncommutative formal power series.

As an example, let $f(U)$ be any power series in U. Since $r\frac{d}{dz}f(z)g(z) = rf'(z)g(z) + f(z)r\frac{d}{dz}g(z)$, we have

$$Df(U) = rf'(U) + f(U)D.$$

This identity can be verified directly by proving by induction on n that $DU^n = rnU^{n-1} + U^nD$ and using linearity and continuity.

3.21.6 Theorem. *(a) Suppose that $DU - UD = rI$. Then*

$$e^{(U+D)x} = e^{\frac{1}{2}rx^2 + Ux}e^{Dx}. \tag{3.105}$$

(b) Let $f(U)$ be a power series in y whose coefficients are polynomials (independent of another variable x) in U. Thus, the action of $f(U)$ on $\widehat{KP}$ is well defined. Then

$$e^{Dx}f(U) = f(U+rx)e^{Dx}.$$

In particular, setting $f(U) = e^{Uy}$ and then $x = y$, we get

$$e^{Dx}e^{Ux} = e^{rx^2+Ux}e^{Dx}. \tag{3.106}$$

NOTE. Care must be taken in interpreting Theorem 3.21.6 since U and D don't commute. For instance, equation (3.105) asserts that

$$\sum_{n\geq 0}(U+D)^n\frac{x^n}{n!} = \left(\sum_{n\geq 0}\frac{r^n x^{2n}}{2^n n!}\right)\left(\sum_{n\geq 0}U^n\frac{x^n}{n!}\right)\left(\sum_{n\geq 0}D^n\frac{x^n}{n!}\right).$$

Thus, when we equate coefficients of $x^3/3!$ on both sides, we obtain

$$(U+D)^3 = 3!\left[\left(\frac{U^3}{6}+\frac{U^2D}{2}+\frac{UD^2}{2}+\frac{D^3}{6}\right)+\frac{r}{2}(U+D)\right].$$

Proof of Theorem 3.21.6. Both equations (3.105) and (3.106) can be proved straightforwardly by verifying them for the coefficient of x^n using induction on n. We give other proofs based on the representation $U = z$, $D = r\frac{d}{dz}$ discussed above.

(a) Let $H(x) = e^{(D+U)x} = \sum_{n\geq 0}(D+U)^n\frac{x^n}{n!}$. Then $H(x)$ is uniquely determined by

$$(D+U)H(x) = \frac{d}{dx}H(x), \quad H(0) = 1.$$

Let

$$J(x) = e^{\frac{1}{2}rx^2+Ux}e^{Dx}.$$

Clearly $J(0) = 1$. Regarding $D = r\frac{d}{dU}$, we have

$$(D+U)J(x) = (rxe^{\frac{1}{2}rx^2+Ux}e^{Dx} + e^{\frac{1}{2}rx^2+Ux}De^{Dx}) + UJ(x),$$

$$\begin{aligned}\frac{d}{dx}J(x) &= (rx+U)J(x) + e^{\frac{1}{2}rx^2+Ux}De^{Dx}\\ &= (D+U)J(x).\end{aligned}$$

Hence, $H(x) = J(x)$, proving (3.105).

(b) The Taylor series expansion of $f(z+rx)$ at z is given by

$$\begin{aligned}f(z+rx) &= \sum_{n\geq 0}\left(r\frac{d}{dz}\right)^n f(z)\frac{x^n}{n!}\\ &= e^{x\left(r\frac{d}{dz}\right)}f(z),\end{aligned}$$

and the proof follows from the representation $D = r\frac{d}{dU}$.

□

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We are now ready to give some enumerative applications. A *Hasse walk* of length ℓ from s to t in a poset P is a sequence

$$s = t_0, t_1, \dots, t_\ell = t, \;\; t_i \in P,$$

such that either $t_{i-1} \lessdot t_i$ or $t_{i-1} \gtrdot t_i$ for $1 \le i \le \ell$. Note that in a graded poset, all closed Hasse walks (i.e., those with $s = t$) have even length.

3.21.7 Theorem. *Let P be an r-differential poset, and let κ_ℓ be the number of Hasse walks of length ℓ from $\hat{0}$ to $\hat{0}$, so $\kappa_\ell = 0$ if ℓ is odd. Then*

$$\kappa_{2n} = (2n-1)!!\, r^n = 1 \cdot 3 \cdot 5 \cdots (2n-1) r^n.$$

Proof. Note that $\kappa_{2n} = \langle (U+D)^{2n}\hat{0}, \hat{0}\rangle$. Hence using equation (3.105) and $D^n\hat{0} = 0$ for $n \ge 1$, we get

$$\begin{aligned}
\sum_{n \ge 0} \kappa_{2n} \frac{x^{2n}}{(2n)!} &= \left\langle \sum_{n \ge 0} (U+D)^n \frac{x^n}{n!} \hat{0}, \hat{0} \right\rangle \\
&= \left\langle e^{(U+D)x} \hat{0}, \hat{0} \right\rangle \\
&= \left\langle e^{\frac{1}{2}rx^2 + Ux} e^{Dx} \hat{0}, \hat{0} \right\rangle \\
&= \left\langle e^{\frac{1}{2}rx^2 + Ux} \hat{0}, \hat{0} \right\rangle \\
&= e^{\frac{1}{2}rx^2} \\
&= \sum_{n \ge 0} r^n (2n-1)!! \frac{x^{2n}}{(2n)!},
\end{aligned}$$

and the proof follows from equating coefficients of $x^{2n}/(2n)!$. □

3.21.8 Theorem. *For any r-differential poset P we have*

$$\sum_{t \in P_n} e(t)^2 = r^n n!.$$

Proof. Completely analogous to the previous proof, using equation (3.106) and

$$\sum_{n \ge 0} \left(\sum_{t \in P_n} e(t)^2 \right) \frac{x^{2n}}{n!^2} = \left\langle e^{Dx} e^{Ux} \hat{0}, \hat{0} \right\rangle.$$

□

We next give some enumerative applications of the identity $D\boldsymbol{P} = (U + r)\boldsymbol{P}$ (Proposition 3.21.4).

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3.21.9 Theorem. *Let P be an r-differential poset. Then*

$$e^{Dx}\boldsymbol{P} = e^{rx+\frac{1}{2}rx^2+Ux}\boldsymbol{P}, \tag{3.107}$$

$$e^{(U+D)x}\boldsymbol{P} = e^{rx+rx^2+2Ux}\boldsymbol{P}, \tag{3.108}$$

$$e^{Dx}e^{Ux}\boldsymbol{P} = e^{rx+\frac{3}{2}rx^2+2Ux}\boldsymbol{P}. \tag{3.109}$$

Proof. Let $H(x) = e^{Dx}$. Then $H(x)\boldsymbol{P}$ is uniquely determined by the conditions

$$DH(x)\boldsymbol{P} = \frac{d}{dx}H(x)\boldsymbol{P}, \quad H(0)\boldsymbol{P} = \boldsymbol{P}.$$

Let $L(x) = e^{rx+\frac{1}{2}rx^2+Ux}\boldsymbol{P}$. Then

$$\begin{aligned} DL(x)\boldsymbol{P} &= (rxL(x) + L(x)D)\boldsymbol{P} \quad (\text{since } Df(U) = rf'(U) + f(U)D) \\ &= (rxL(x) + L(x)(U+r))\boldsymbol{P} \\ &= (rx + U + r)L(x)\boldsymbol{P} \\ &= \frac{d}{dx}L(x)\boldsymbol{P}. \end{aligned}$$

Clearly, $L(0)\boldsymbol{P} = \boldsymbol{P}$, so $L(x) = H(x)$, proving (3.107).

Now

$$\begin{aligned} e^{(U+D)x}\boldsymbol{P} &= e^{\frac{1}{2}rx^2+Ux}e^{Dx}\boldsymbol{P} \\ &= e^{\frac{1}{2}rx^2+Ux}e^{rx+\frac{1}{2}rx^2+Ux}\boldsymbol{P} \\ &= e^{rx+rx^2+2Ux}\boldsymbol{P}, \end{aligned}$$

and

$$\begin{aligned} e^{Dx}e^{Ux}\boldsymbol{P} &= e^{rx^2+Ux}e^{Dx}\boldsymbol{P} \\ &= e^{rx^2+Ux}e^{rx+\frac{1}{2}rx^2+Ux}\boldsymbol{P} \\ &= e^{rx+\frac{3}{2}rx^2+2Ux}\boldsymbol{P}, \end{aligned}$$

completing the proof. □

Write $\alpha(0 \to n) = \sum_{t\in P_n} e(t)$, and let δ_n be the number of Hasse walks of length n from $\hat{0}$ (with any ending element).

3.21.10 Theorem. *For any r-differential poset P we have*

$$\sum_{n\geq 0} \alpha(0 \to n)\frac{x^n}{n!} = e^{rx+\frac{1}{2}rx^2}$$

$$\sum_{n\geq 0} \delta_n \frac{x^n}{n!} = e^{rx+rx^2}.$$

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NOTE. It follows from Theorem 1.3.3 or from Section 5.1 that Theorem 3.21.10 can be restated in the form

$$\alpha(0 \to n) = \sum_{\substack{w \in \mathfrak{S}_n \\ w^2=1}} r^{c(w)}$$

$$\sum_{n \geq 0} \delta_n = \sum_{\substack{w \in \mathfrak{S}_n \\ w^2=1}} r^{c(w)} 2^{c_2(w)},$$

where $c(w)$ denotes the number of cycles of w and $c_2(w)$ the number of 2-cycles. For instance, $\alpha(0 \to 3) = r^3 + 3r^2$, and $\delta_3 = r^3 + 6r^2$. Note that in particular if $r = 1$, then $\alpha(0 \to n)$ is just the number of involutions in $\mathfrak{S}_n$. For the case where P is Young's lattice Y, this result is equivalent to Corollary 7.13.9.

Proof of Theorem 3.21.10. Clearly,

$$\alpha(0 \to n) = \langle D^n \boldsymbol{P}, \hat{0} \rangle, \quad \delta_n = \langle (U+D)^n \boldsymbol{P}, \hat{0} \rangle.$$

Now for any $f(U)$ we have $\langle f(U)\boldsymbol{P}, \hat{0} \rangle = f(0)$, so

$$\begin{aligned} \sum_{n \geq 0} \alpha(0 \to n) \frac{x^n}{n!} &= \sum_{n \geq 0} \langle D^n \boldsymbol{P}, \hat{0} \rangle \frac{x^n}{n!} \\ &= \langle e^{Dx} \boldsymbol{P}, \hat{0} \rangle \\ &= \langle e^{rx + \frac{1}{2}rx^2 + Ux} \boldsymbol{P}, \hat{0} \rangle \\ &= e^{rx + \frac{1}{2}rx^2}. \end{aligned}$$

An analogous argument works for $\sum_{n \geq 0} \delta_n x^n / n!$. $\square$

Let us now generalize this result for $\alpha(0 \to n)$ by considering increasing Hasse walks from rank n to rank $n+k$. More precisely, let $\alpha(n \to n+k)$ be the number of such walks, that is,

$$\alpha(n \to n+k) = \#\{t_n \lessdot t_{n+1} \lessdot \cdots \lessdot t_{n+k} : \rho(t_i) = i\},$$

where ρ denotes the rank function of P. In particular, $\alpha(n \to n) = \#P_n$. This special case suggests that the rank-generating function

$$F(P,q) = \sum_{t \in P} q^{\rho(t)} = \sum_{n \geq 0} (\#P_n) q^n$$

will be relevant, so let us note that

$$F(Y^r, q) = \prod_{i \geq 1} \frac{1}{(1-q^i)^r},$$

$$F(Z_r, q) = \frac{1}{1 - rq - q^2}.$$

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3.21.11 Theorem. *For any r-differential poset P we have*

$$\sum_{n\geq 0}\sum_{k\geq 0}\alpha(n\to n+k)q^n\frac{x^k}{k!} = F(P,q)\exp\left(\frac{rx}{1-q}+\frac{rx^2}{2(1-q^2)}\right).$$

NOTE. Taking the coefficient of $x^k/k!$ for $0\leq k\leq 2$ yields

$$\sum_{n\geq 0}\alpha(n\to n)q^n = F(P,q),$$

$$\sum_{n\geq 0}\alpha(n\to n+1)q^n = \frac{r}{1-q}F(P,q),$$

$$\sum_{n\geq 0}\alpha(n\to n+2)q^n = \frac{r(r+1)+r(r-1)q}{(1-q)^2(1-q^2)}F(P,q).$$

In general, it is immediate from Theorem 3.21.11 that for fixed k we have

$$\sum_{n\geq 0}\alpha(n\to n+k)q^n = A_k(q)F(P,q),$$

where $A_k(q)$ is a rational function of q (and r) satisfying

$$\sum_{k\geq 0}A_k(q)\frac{x^k}{k!} = \exp\left(\frac{rx}{1-q}+\frac{rx^2}{2(1-q^2)}\right). \tag{3.110}$$

Proof of Theorem 3.21.11. Let $\gamma\colon \widehat{KP}\to K[[q]]$ be the continuous linear transformation defined by $\gamma(t)=q^{\rho(t)}$ for all $t\in P$, so $\gamma(\boldsymbol{P})=F(P,q)$. Now

$$\begin{aligned}\gamma(e^{Dx}\boldsymbol{P}) &= \sum_{k\geq 0}\gamma(D^k\boldsymbol{P})\frac{x^k}{k!}\\ &= \sum_{n\geq 0}\sum_{k\geq 0}\alpha(n\to n+k)q^n\frac{x^k}{k!} := G(q,x).\end{aligned}$$

But also

$$\begin{aligned}\gamma(e^{Dx}\boldsymbol{P}) &= e^{rx+\frac{1}{2}rx^2}\gamma(e^{Ux}\boldsymbol{P})\\ &= e^{rx+\frac{1}{2}rx^2}\sum_{n\geq 0}\sum_{k\geq 0}\alpha(n-k\to n)q^n\frac{x^k}{k!}\\ &= e^{rx+\frac{1}{2}rx^2}\sum_{n\geq 0}\sum_{k\geq 0}\alpha(n-k\to n)q^{n-k}\frac{(qx)^k}{k!}\\ &= e^{rx+\frac{1}{2}rx^2}G(q,qx).\end{aligned}$$

Hence, we have shown that

$$G(q,x)=e^{rx+\frac{1}{2}rx^2}G(q,qx). \tag{3.111}$$

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Moreover, it is clear that

$$G(q,0) = F(P,q). \tag{3.112}$$

We claim that equations (3.111) and (3.112) determine $G(q,x)$ uniquely. For if $G(q,x) = \sum a_k(q)x^k$, then

$$\sum a_k(q)x^k = e^{rx+\frac{1}{2}rx^2} \sum a_k(q)(qx)^k.$$

We can equate coefficients of x^k and solve for $a_k(q)$ in terms of $a_0(q), a_1(q), \dots, a_{k-1}(q)$, with $a_0(q) = F(P,q)$, proving the claim.

Now note that $F(P,q)\exp\left(\frac{rx}{1-q} + \frac{rx^2}{2(1-q^2)}\right)$ satisfies equations (3.111) and (3.112), completing the proof. □

A further aspect of differential posets is the computation of eigenvalues and eigenvectors of certain linear transformations. We illustrate this technique here by computing the eigenvalues of the adjacency matrix of the graph obtained by restricting the Hasse diagram of a differential poset to two consecutive levels. In general, if G is a finite graph, say with no multiple edges, then the *adjacency matrix* of G is the (symmetric) matrix $\boldsymbol{A}$, say over $\mathbb{C}$, with rows and columns indexed by the vertices of G (in some order), with

$$\boldsymbol{A}_{uv} = \begin{cases} 1, & \text{if } uv \text{ is an edge of } G, \\ 0, & \text{otherwise.} \end{cases}$$

Let P be an r-differential poset, and let $P_{j-1,j}$ denote the restriction of P to $P_{j-1} \cup P_j$. Identify $P_{j-1,j}$ with its Hasse diagram, regarded as an undirected (bipartite) graph. Let $\boldsymbol{A}$ denote the adjacency matrix of $P_{j-1,j}$. We are interested in computing the eigenvalues (or characteristic polynomial) of $\boldsymbol{A}$. By definition of matrix multiplication, the matrix entry $(\boldsymbol{A}^n)_{uv}$ is the number of walks of length n from u to v. On the other hand, $(\boldsymbol{A}^n)_{uv}$ is closely related to the eigenvalues of $\boldsymbol{A}$, as discussed in Section 4.7. This suggests that differential poset techniques might be useful in computing the eigenvalues.

3.21.12 Theorem. *Let $p_i = \#P_i$. Then the eigenvalues of $\boldsymbol{A}$ (over $\mathbb{C}$) are as follows:*

- *0 with multiplicity $p_j - p_{j-1}$,*
- *$\pm\sqrt{rs}$ with multiplicity $p_{j-s} - p_{j-s-1}$, $1 \le s \le j$.*

NOTE. The total number of eigenvalues is

$$p_j - p_{j-1} + 2\sum_{s=1}^{j}(p_{j-s} - p_{j-s-1}) = p_{j-1} + p_j,$$

the number of elements of the poset $P_{j-1,j}$.

Proof of Theorem 3.21.12. For any set S write $\mathbb{C}S$ for the complex vector space with basis S. Since $\mathbb{C}P_{j-1,j} = \mathbb{C}P_{j-1} \oplus \mathbb{C}P_j$, any $v \in \mathbb{C}P_{j-1,j}$ can be uniquely written

$$v = v_{j-1} + v_j, \ \ v_i \in \mathbb{C}P_i.$$

Then $\boldsymbol{A}$ acts on $\mathbb{C}P_{j-1,j}$ by

$$\boldsymbol{A}(v) = D(v_j) + U(v_{j-1}).$$

Write U_i for the restriction of U to $\mathbb{C}P_i$, and similarly for D_i and I_i. Thus, the identity $DU - UD = rI$ takes the form

$$D_{i+1}U_i - U_{i-1}D_i = rI_i.$$

Now U_{i-1} and D_i are adjoint linear transformations with respect to the bases P_{i-1} and P_i (in other words, their matrices are transposes of one another). Thus, by standard results in linear algebra, the linear transformation $U_{i-1}D_i$ is (positive) semidefinite and hence has nonnegative real eigenvalues. Now

$$D_{i+1}U_i = U_{i-1}D_i + rI_i.$$

The eigenvalues of $U_{i-1}D_i + rI_i$ are obtained by adding r to the eigenvalues of the semidefinite transformation $U_{i-1}D_i$. Hence, $D_{i+1}U_i$ has positive eigenvalues and is therefore invertible. In particular, U_i is injective, so its adjoint D_{i+1} is surjective. Therefore,

$$\begin{aligned}\dim(\ker D_i) &= \dim \mathbb{C}P_i - \dim \mathbb{C}P_{i-1} \\ &= p_i - p_{i-1}.\end{aligned}$$

Case 1. Let $v \in \ker(D_j)$, so $v \in \mathbb{C}P_j$, i.e., $v = v_j$. Hence $\boldsymbol{A}v = Dv = 0$, so $\ker(D_j)$ is an eigenspace of $\boldsymbol{A}$ with eigenvalue 0. Thus 0 is an eigenvalue of $\boldsymbol{A}$ with multiplicity at least $p_j - p_{j-1}$.

Case 2. Let $w \in \ker(D_s)$ for some $0 \leq s \leq j-1$. Let

$$w^* = \underbrace{\sqrt{r(j-s)}U^{j-1-s}(w)}_{w^*_{j-1}} + \underbrace{U^{j-s}(w)}_{w^*_j} \in \mathbb{C}P_{j-1,j}.$$

We can choose either sign for the square root. Then

$$\begin{aligned}\boldsymbol{A}(w^*) &= U(w^*_{j-1}) + D(w^*_j) \\ &= \sqrt{r(j-s)}U^{j-s}(w) + DU^{j-s}(w) \\ &= \sqrt{r(j-s)}U^{j-s}(w) + U^{j-s}\underbrace{D(w)}_{0} + r(j-s)U^{j-s-1}(w) \\ &= \sqrt{r(j-s)}U^{j-s}(w) + r(j-s)U^{j-s-1}(w) \\ &= \sqrt{r(j-s)}w^*.\end{aligned}$$

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If $w_1, \ldots, w_t$ is a basis for $\ker(D_s)$ then $w_1^*, \ldots, w_t^*$ are linearly independent (since U is injective). Hence, $\pm\sqrt{r(j-s)}$ is an eigenvalue of $\boldsymbol{A}$ with multiplicity at least $t = \dim(\ker D_s) = p_s - p_{s-1}$. We have found a total of

$$p_j - p_{j-1} + 2\sum_{s=0}^{j-1}(p_s - p_{s-1}) = p_{s-1} + p_s$$

eigenvalues, so we have them all. □

3.21.13 Corollary. *Fix $j \geq 1$. The number of closed walks of length $2m > 0$ in $P_{j-1,j}$ beginning and ending at some $t \in P_j$ is given by*

$$\sum_{s=1}^{j}(p_{j-s} - p_{j-s-1})(rs)^m.$$

Proof. By the definition of matrix multiplication, the total number of closed walks of length $2m$ in $P_{j-1,j}$ is equal to $\operatorname{tr}\boldsymbol{A}^{2m} = \sum \theta_i^{2m}$, where the θ_i's are the eigenvalues of $\boldsymbol{A}$. (See Theorem 4.7.1.) Exactly half these walks start at P_j. By Theorem 3.21.12 we have

$$\begin{aligned}\frac{1}{2}\operatorname{tr} A^{2m} &= \frac{1}{2}\sum_{s=1}^{j}(p_{j-s} - p_{j-s-1})\left((\sqrt{rs})^{2m} + (-\sqrt{rs})^{2m}\right)\\ &= \frac{1}{2}\sum_{s=1}^{j}(p_{j-s} - p_{j-s-1})2(rs)^m,\end{aligned}$$

and the proof follows. □

Notes

The subject of partially ordered sets and lattices has its origins in the work of G. Boole, C. S. Peirce, E. Schröder, and R. Dedekind during the nineteenth century. However, it was not until the work of Garrett Birkhoff in the 1930s that the development of poset theory and lattice theory as subjects in their own right really began. In particular, the appearance in 1940 of the first edition of Birkhoff's famous book [3.13] played a seminal role in the development of the subject. It is interesting to note that the three successive editions of this book used the terms "partly ordered set," "partially ordered set," and "poset," respectively. More explicit references to the development of posets and lattices can be found in [3.13]. Another important impetus to lattice theory was the work of John von Neumann on continuous geometries, also in the 1930s. For two surveys of this work see Birkhoff [3.12] and Halperin [3.40].

A bibliography of around 1400 items dealing with posets (but not lattices!) appears in Rival [3.57]. This latter reference contains many valuable surveys of the status of poset theory up to 1982. In particular, we mention the survey [3.35] of C. Greene on Möbius functions. An extensive bibliography of lattice theory appears in Grätzer [3.33].

Matroid theory was mentioned at the end of Section 3.3. Some books on this subject are by Oxley [3.54], Welsh [3.89], and White (ed.) [3.90][3.91][3.92].

The idea of incidence algebras can be traced back to Dedekind and E. T. Bell, whereas the Möbius inversion formula for posets is essentially due to L. Weisner in 1935. It was rediscovered shortly thereafter by P. Hall, and stated in its full generality by M. Ward in 1939. Hall proved the basic Proposition 3.8.5 (therefore known as "Philip Hall's theorem"), and Weisner, the equally important Corollary 3.9.3 ("Weisner's theorem"). However, it was not until 1964 that the seminal paper [3.58] of G.-C. Rota that began the systematic development of posets and lattices within combinatorics appeared. Reference to earlier work in this area cited earlier appear in [3.58]. Much additional material on incidence algebras appears in the book of E. Spiegel and C. O'Donnell [3.66].

We now turn to more specific citations, beginning with Section 3.4. Theorem 3.4.1 (the fundamental theorem for finite distributive lattices) was proved by Birkhoff [3.11, Thm. 17.3]. Generalizations to arbitrary distributive lattices were given by M. H. Stone [3.86, Thm. 4] and H. A. Priestley [3.55][3.56]. A nice survey is given by Davey and Priestley [3.24]. The connection between chains in distributive lattices $J(P)$ and order-preserving maps $\sigma : P \to \mathbb{N}$ (Section 3.5) was first explicitly observed by Stanley in [3.67] and [3.68]. The notion of a "generalized Pascal triangle" appears in Stanley [3.73].

The development of a homology theory for posets was considered by Deheuvels, Dowker, Farmer, Nöbeling, Okamoto, and others (see Farmer [3.27] for references), but the combinatorial ramifications of such a theory, including the connection with Möbius functions, was not perceived until Rota [3.58, pp. 355–356]. Some early work along these lines was done by Farmer, Folkman, Lakser, Mather, and others (see Walker [3.87][3.88] for references). In particular, Folkman proved a result equivalent to the statement that geometric lattices are Cohen-Macaulay. The systematic development of the relationship between combinatorial and topological properties of posets was begun by K. Baclawski and A. Björner and continued by J. Walker, followed by many others. A nice survey of topological combinatorics up to 1995 was given by Björner [3.18], while Kozlov [3.48] has written an extensive text. The connection between regular cell complexes and posets is discussed by Björner [3.15]. Cohen-Macaulay complexes were discovered independently by Baclawski [3.3] and Stanley [3.75, §8]. A survey of Cohen-Macaulay posets, including their connection with commutative algebra, appears in Björner–Garsia–Stanley [3.16]. For further information on the subject of "combinatorial commutative algebra" see the books by Stanley [3.78] and by E. Miller and B. Sturmfels [3.52]. The statement preceding Proposition 3.8.9 that $\operatorname{lk} F$ need not be simply connected and $|\operatorname{lk} F|$ need not be a manifold when $|\Delta|$ is a manifold is a consequence of a deep result of R. D. Edwards. See for instance [3.23, II.12].

The Möbius algebra of a poset P (generalizing our definition in Section 3.10 when P is a lattice) was introduced by L. Solomon [3.65] and first systematically investigated by C. Greene [3.34], who showed how it could be used to derive many apparently unrelated properties of Möbius functions.

Proposition 3.10.1 (stated for geometric lattices) is due to Rota [3.58, Thm. 4, p. 357]. The formula (3.34) for the Möbius function of $B_n(q)$ is due to P. Hall [3.39, (2.7)], while the generalization (3.38) appeared in Stanley [3.74, Thm. 3.1] (with $r = 1$). The formula (3.37) for Π_n is due independently to Schützenberger and to Frucht and Rota (see [3.58, p. 359]).

Theorem 3.11.7 is perhaps the first significant result in the theory of hyperplane arrangements. It was obtained by T. Zaslavsky [3.95][3.96] in 1975, though some special cases were known earlier. In particular, Proposition 3.11.8 goes back to L. Schläfli [3.59] (written in 1850–1852 and published in 1901).

Proposition 3.11.3, known as *Whitney's theorem*, was proved by H. Whitney [3.93, §6] for graphs (equivalent to graphical arrangements). Whitney considered this formula further in [3.94]. Many aspects of hyperplane arrangments are best understood *via* matroid theory, as explained for example in Stanley [3.83]. The finite field method had its origins in the work of Crapo and Rota [3.22, §16] but was not applied systematically to computing characteristic polynomials until the work of C. A. Athanasiadis [3.1][3.2]. The theory of hyperplane arrangements has developed into a highly sophisticated subject with deep connections with topology, algebraic geometry, and so on. For a good overview, see the text of P. Orlik and H. Terao [3.53]. For an introduction to the combinatorial aspects of hyperplane arrangements, see the lecture notes of Stanley [3.83]. The Shi arrangement was first defined by Jian-Yi Shi (时俭益) [3.60][3.61], who computed the number of regions in connection with determining the left cells of the affine Weyl group of type $\widetilde{A}_{n-1}$. The characteristic polynomial was first computed by P. Headley [3.41].

Zeta polynomials were introduced by Stanley [3.72, §3] and further developed by P. Edelman [3.26].

The idea of rank-selected subposets and the corresponding functions $\alpha_P(S)$ and $\beta_P(S)$ was considered for successively more general classes of posets by Stanley in [3.68, Ch. II][3.69][3.71], finally culminating in [3.76, §5]. Theorem 3.13.1 appeared (in a somewhat more general form) in [3.68, Thm. 9.1], while Theorem 3.13.3 appeared in [3.74, Thm. 3.1] (with $r = 1$).

R-labelings had a development parallel to that of rank-selection. The concept was successively generalized in [3.68][3.69][3.71], culminating this time in Björner [3.14] (from which the term "R-labeling" is taken) and Björner and Wachs [3.17]. Example 3.14.4 comes from [3.69], while Example 3.14.5 is found in [3.71]. A more stringent type of labeling than R-labeling, originally called *L-labeling* and now called *EL-labeling*, was introduced by Björner [3.14] and generalized to *CL-labeling* by Björner and Wachs [3.17]. (The definition of CL-labeling implicitly generalizes the notion of R-labeling to what logically should be called "CR-labeling.") A poset with a CL-labeling (originally, just with an EL-labeling) is called *lexicographically shellable*. While R-labelings are used (as in Section 3.14) to compute Euler characteristics (i.e., Möbius functions), CL-labelings allow one to compute the actual homology groups. From the many important examples, beginning with [3.17][3.19], of posets that can be proved to have a CL-labeling but not an EL-labeling, it seems clear that CL-labeling is the right level of generality for this subject. Note, however, that an even more general concept is due to D. Kozlov [3.47]. We have treated only R-labelings here for ease of presentation and because we are focusing on enumeration, not topology.

The theory of (P,ω)-partitions was foreshadowed by the work of MacMahon (see, for example, [1.55, §§439, 441]) and more explicitly Knuth [3.46], but the first general development appeared in Stanley [3.67][3.68]. Our treatment closely follows [3.68].

Eulerian posets were first explicitly defined in Stanley [3.77, p. 136], though they had certainly been considered earlier. A survey was given by Stanley [3.82]. In particular, Proposition 3.16.1 appears in [3.72, Prop. 3.3] (though stated less generally), while our approach to the Dehn-Sommerville equations (Theorem 3.16.9 in the case when $P - \{\hat{1}\}$ is Eulerian) appears in [3.72, p. 204]. Classically the Dehn-Sommerville equations were stated for face lattices of simplicial convex polytopes or triangulations of spheres (see [3.37, Ch. 9.8]); Klee [3.45] gives a treatment equivalent in generality to ours. A good general reference on polytopes is the book of Ziegler [3.97].

Lemma 3.16.3 and its generalization Lemma 3.16.4 are due independently to Baclawski [3.4, Lem. 4.6] and Stečkin [3.85]. A more general formula is given by Björner and Walker [3.20]. Proposition 3.16.5 and Corollary 3.16.6 appear in [3.77, Prop. 2.2]. Theorem 3.16.9 has an interesting history. It first arose when P is the lattice of faces of a rational convex polytope $\mathcal{P}$ as a byproduct of the computation of the intersection homology $IH(X(\mathcal{P});\mathbb{C})$ of the toric variety $X(\mathcal{P})$ associated with $\mathcal{P}$. Specifically, setting $\beta_i = \dim IH_i(X(\mathcal{P});\mathbb{C})$ one has

$$\sum_{i\geq 0} \beta_i q^i = f(P,q^2).$$

But intersection homology satisfies Poincaré duality, which implies $\beta_i = \beta_{2n-i}$. For references and further information, see Exercise 3.179. It was then natural to ask for a more elementary proof in the greatest possible generality, from which Theorem 3.16.9 arose. For further developments in this area, see Exercise 3.179.

The cd-index arose from the work of M. Bayer and L. Billera [3.5] on flag f-vectors of Eulerian posets. J. Fine (unpublished) observed that the linear relations obtained by Bayer and Billera are equivalent to the existence of the cd-index. Bayer and A. Klapper [3.6] wrote up the details of Fine's argument and developed some further properties of the cd-index. Stanley [1.70] gave the proof of the existence of the cd-index appearing here as the proof of Theorem 3.17.1 (with a slight improvement due to G. Hetyei; see L. Billera and R. Ehrenborg, [3.10, p. 427]) and gave some additional results. An important breakthrough was later given by K. Karu (Exercise 3.192).

The theory of binomial posets was developed by Doubilet, Rota, and Stanley [3.25, §8]. Virtually all the material of Section 3.18 (some of it in a more general form) can be found in this reference, with the exception of chromatic generating functions [3.70]. The generating function $(2-e^x)^{-1}$ of Example 3.18.10 was first considered by A. Cayley [3.21] in connection with his investigation of trees. See also O. A. Gross [3.36]. The application of binomial posets to permutation enumeration (Section 3.19) was developed by Stanley [3.74].

Among the many alternative theories to binomial posets for unifying various aspects of enumerative combinatorics and generating functions, we mention the theory of prefabs of Bender and Goldman [3.7], dissects of Henle [3.42], linked sets of Gessel [3.31], and species of Joyal [3.44]. The most powerful of these theories is perhaps that of species, which is based on category theory. An exposition is given by F. Bergeron, G. Labelle, and P. Leroux [3.8]. We should also mention the book of Goulden and Jackson [3.32], which gives a fairly unified treatment of a large part of enumerative combinatorics related to the counting of sequences and paths.

Evacuation (Section 3.20) first arose in the theory of the RSK algorithm. See pages 425–429 of Chapter 7, Appendix I, for this connection. Evacuation was described by M.-P. Schützenberger [3.62] in a direct way not involving the RSK algorithm. In two follow-up papers [3.63][3.64], Schützenberger extended the definition of evacuation to linear extensions of any finite poset and developed the connection with promotion. Schützenberger's work was simplified by Haiman [3.38] and Malvenuto and Reutenauer [3.51]. A survey of promotion and evacuation with many additional references was given by Stanley [3.84].

Differential posets were discovered independently by S. Fomin [3.29][3.30] and Stanley [3.80] [3.81]. Fomin's work goes back to his M.S. thesis [3.28] and is done in the more general context of "dual graded graphs" (essentially where the U and D operators act on different posets). Our exposition follows [3.80], where many further results may be found. Theorem 3.21.6(b) is based on a suggestion of Yan Zhang. Generalizations of differential posets in addition to dual graded graphs include sequentially differential posets [3.81, §2], weighted differential posets (an example appearing in [3.81, §3]), down-up algebras [3.9], signed differential posets [3.49], quantized dual graded graphs [3.50], and the updown categories of M. E. Hoffman [3.43].

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Exercises for Chapter 3

1. [3] What is the connection between a partially ordered set and itinerant salespersons who take revenge on customers who don't pay their bills?

2. **a.** [1+] A *preposet* (or *quasi-ordered set*) is a set P with a binary relation $\leq$ satisfying reflexivity and transitivity (but not necessary antisymmetry). Given a preposet P and $s,t \in P$, define $s \sim t$ if $s \leq t$ and $t \leq s$. Show that $\sim$ is an equivalence relation.

b. [1+] Let $\widetilde{P}$ denote the set of equivalence classes under $\sim$. If $S,T \in \widetilde{P}$, then define $S \leq T$ if there is an $s \in S$ and $t \in T$ for which $s \leq t$ in P. Show that this definition of $\leq$ makes $\widetilde{P}$ into a poset.

c. [2–] Let Q be a poset and $f\colon P \to Q$ order-preserving. Show that there is a unique order-preserving map $g\colon \widetilde{P} \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[rr] \arrow[dr, "f"'] & & \widetilde{P} \arrow[dl, "g"] \\ & Q & \end{tikzcd}$$

Here the map $P \to \widetilde{P}$ is the canonical map taking t into the equivalence class containing t.

3. **a.** [1+] Let P be a finite preposet (as defined in Exercise 3.2). Define a subset U of P to be *open* if U is an order ideal (defined in an obvious way for preposets) of P. Show that P becomes a finite topological space, denoted P_{top}.

b. [2–] Given a finite topological space X, show that there is a unique preposet P (up to isomorphism) for which $P_{\text{top}} = X$. Hence, the correspondence $P \to P_{\text{top}}$ is a bijection between finite preposets and finite topologies.

c. [2–] Show that the preposet P is a poset if and only if P_{top} is a T_0-space (i.e., distinct points have distinct sets of neighborhoods).

d. [2–] Show that a map $f\colon P \to Q$ of preposets is order-preserving if and only if f is continuous when regarded as a map $P_{\text{top}} \to Q_{\text{top}}$.

4. [2–] Let P be a poset. Show that there exists a collection $\mathcal{S}$ of sets such that if we partially order $\mathcal{S}$ by defining $S \leq T$ if $S \subseteq T$, then $\mathcal{S} \cong P$.

5. **a.** [2] Draw diagrams of the 63 five-element posets (up to isomorphism), 318 six-element posets, and 2045 seven-element posets (straightforward, but time consuming). For readers with a lot of spare time on their hands, continue with eight-element posets, nine-element posets, and so on, obtaining the numbers 16999, 183231, 2567284, 46749427, 1104891746, 33823827452, 1338193159771, 68275077901156, and 4483130665195087.

b. [5] Let $f(n)$ be the number of nonisomorphic n-element posets. Find a "reasonable" formula for $f(n)$ (probably impossible, and similarly for the case of *labeled posets*, i.e., posets on the vertex set $[n]$).

c. [5] With $f(n)$ as previously, let $\mathcal{P}$ denote the statement that infinitely many values of $f(n)$ are palindromes when written in base 10. Show that $\mathcal{P}$ cannot be proved or disproved in Zermelo–Fraenkel set theory.

d. [3] Show that

$$\log f(n) \sim (n^2/4)\log 2.$$

e. [3+] Improve (d) by showing

$$f(n) \sim C2^{n^2/4+3n/2}e^n n^{-n-1},$$

where C is a constant given by

$$C = \frac{2}{\pi}\sum_{i\geq 0} 2^{-i(i+1)} \approx 0.80587793 \ \ (n \text{ even}),$$

and similarly for n odd.

6. **a.** [2] Let P be a finite poset and $f : P \to P$ an order-preserving bijection. Show that f is an automorphism of P (i.e., f^{-1} is order-preserving).

b. [2] Show that (a) fails for infinite P.

7. **a.** [1+] Give an example of a finite poset P such that if ℓ is the length of the longest chain of P, then every $t \in P$ is contained in a chain of length ℓ, yet P has a maximal chain of length less than ℓ.

b. [2] Let P be a finite poset with no isolated points and with longest chain of length ℓ. Assume that for every t covering s in P there exists a chain of length ℓ containing both s and t. Show that every maximal chain of P has length ℓ.

8. [3–] Find a finite poset P for which there is a bijection $f : P \to P$ such that $s \leq t$ if and only if $f(s) \geq f(t)$ (i.e., P is self-dual), but for which there is *no* such bijection f satisfying $f(f(t)) = t$ for all $t \in P$.

9. [2–] True or false: The number of nonisomorphic 8-element posets that are not self-dual is 16507.

10. **a.** [2–]* If P is a poset, then let $\mathrm{Int}(P)$ denote the poset of (nonempty) intervals of P, ordered by inclusion. Show that for any posets A and B, we have $\mathrm{Int}(A \times B) \cong \mathrm{Int}(A \times B^*)$.

b. [2+] Let P and Q be posets. If P has a $\hat{0}$ and $\mathrm{Int}(P) \cong \mathrm{Int}(Q)$, show that $P \cong A \times B$ and $Q \cong A \times B^*$ for some posets A and B.

c. [3] Find finite posets P, Q such that $\mathrm{Int}(P) \cong \mathrm{Int}(Q)$, yet the conclusion of (b) fails.

11. **a.** [2] Let A be the set of all isomorphism classes of finite posets. Let $[P]$ denote the class of the poset P. Then A has defined on it the operations $+$ and $\cdot$ given by $[P] + [Q] = [P + Q]$ and $P \cdot Q = [P \times Q]$. Show that these operations make A into a commutative semiring (i.e., A satisfies all the axioms of a commutative ring except the existence of additive inverses).

b. [3–] We can formally adjoin additive inverses to A in an obvious way to obtain a ring B (exactly the same way as one obtains $\mathbb{Z}$ from $\mathbb{N}$). Define a poset to be *irreducible* if it cannot be written in a nontrivial way as a direct product. Show that B is just the polynomial ring $\mathbb{Z}[[P_1], [P_2], \dots]$ where the $[P_i]$'s are the classes

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of irreducible connected finite posets with more than one element. (The additive identity of B is given by the class of the empty poset, and the multiplicative identity by the class of the one-element poset.)

c. [3–] Find irreducible finite posets P_i satisfying $P_1 \times P_2 \cong P_3 \times P_4$, yet $P_1 \not\cong P_3$ and $P_1 \not\cong P_4$. Why does this not contradict the known fact that $\mathbb{Z}[x_1, x_2, \dots]$ is a unique factorization domain?

12. [2+] True or false: If every chain and every antichain of a poset P is finite, then P is finite.

13. **a.** [3] Let P be a poset for which every antichain is finite. Show that every antichain of $J_f(P)$ is finite.

b. [2] Show that if every antichain of P is finite, it need not be the case that every antichain of $J(P)$ is finite.

14. A finite poset P is a *series-parallel poset* if it can be built up from a one-element poset using the operations of disjoint union and ordinal sum. There is a unique four-element poset (up to isomorphism) that is not series-parallel, namely, the zigzag poset Z_4 of Exercise 3.66.

a. [2+]* Show that a finite poset P is series-parallel if and only if it contains no induced subposet isomorphic to Z_4. Such posets are sometimes called *N-free posets*.

b. [2+]* Let P_w be the inversion poset of the permutation $w \in \mathfrak{S}_n$, as defined in the solution to Exercise 3.20. Show that P_w is N-free if and only if w is 3142-avoiding and 2413-avoiding. Such permutations are also called *separable*. Note. The number of separable permutations in $\mathfrak{S}_n$ is the Schröder number r_{n-1}, as defined in Vol. II, Section 6.2.

15. An *interval order* is a poset P isomorphic to a set of closed intervals of $\mathbb{R}$, with $[a,b] < [c,d]$ if $b < c$.

a. [3–] Show that a finite poset P is an interval order if and only if it is $(\mathbf{2}+\mathbf{2})$-free (i.e., has no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$).

b. [3–] A poset P is a *semiorder* (or *unit interval order*) if it is an interval order corresponding to a set of intervals all of length one. Show that an interval order P is a semiorder if and only if P is $(\mathbf{3}+\mathbf{1})$-free. (For the enumeration of labeled and unlabeled semiorders, see Exercises 6.30 and 6.19(ddd), Vol. II, respectively.)

c. [3] Let $t(n)$ be the number of nonisomorphic interval orders with n elements. Show that

$$\sum_{n\geq 0} t(n)x^n = \sum_{n\geq 0}\prod_{k=1}^{n}(1-(1-x)^k)$$
$$= 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + \cdots.$$

d. [3] Let $u(n)$ be the number of labeled n-element interval orders (i.e., interval orders on the set $[n]$). Show that

$$\sum_{n\geq 0} u(n)\frac{x^n}{n!} = \sum_{n\geq 0}\prod_{k=1}^{n}(1-e^{-kx})$$
$$= 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 207\frac{x^4}{4!} + 3451\frac{x^5}{5!} + 81663\frac{x^6}{6!} + \cdots.$$

e. [2+] Show that the number of nonisomorphic n-element interval orders that are also series-parallel posets (defined in Exercise 3.14) is the Catalan number C_n.

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f. [2]* Let $\ell_1,\ldots,\ell_n$ be positive real numbers. Let $g(\ell_1,\ldots,\ell_n)$ be the number of interval orders P that can be formed from intervals $I_1,\ldots,I_n$, where I_i has length ℓ_i, such that the element t of P corresponding to I_i is labeled i. For instance, if $n=4$ and P is isomorphic to $\mathbf{3}+\mathbf{1}$, then $\mathbf{3}$ can be labeled a,b,c from bottom to top and $\mathbf{1}$ labeled d if and only if $\ell_d \geq \ell_b$ (12 labelings in all if l_a,l_b,l_c,l_d are distinct). Show that $g(\ell_1,\ldots,\ell_n)$ is equal to the number of regions of the real hyperplane arrangement

$$x_i - x_j = \ell_i, \quad i \neq j$$

($n(n-1)$ hyperplanes in all).

g. [3] Suppose that $\ell_1,\ldots,\ell_n$ in (f) are linearly independent over $\mathbb{Q}$. Define a power series

$$y = 1 + x + 5\frac{x^2}{2!} + 46\frac{x^3}{3!} + 631\frac{x^4}{4!} + 9655\frac{x^5}{5!} + \cdots$$

by the equation

$$1 = y(2-e^{xy}),$$

or equivalently

$$y - 1 = \left(\frac{1}{1+x}\log\frac{1+2x}{1+x}\right)^{\langle -1\rangle},$$

where $^{\langle -1\rangle}$ denotes compositional inverse. Let

$$\begin{aligned} z &= \sum_{n\geq 0} g(\ell_1,\ldots,\ell_n)\frac{x^n}{n!} \\ &= 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + 2831\frac{x^5}{5!} + 53703\frac{x^6}{6!} + \cdots. \end{aligned}$$

Show that z is the unique power series satisfying

$$\frac{z'}{z} = y^2, \quad z(0) = 1.$$

Note that it is by no means a priori obvious that $g(\ell_1,\ldots,\ell_n)$ is independent of $\ell_1,\ldots,\ell_n$ (provided they are linearly independent over $\mathbb{Q}$).

16. **a.** [3] Let $f(n)$ be the number of graded $(\mathbf{3}+\mathbf{1})$-free partial orderings of an n-element set. Set

$$G(x) = \sum_{m,n\geq 0} 2^{mn}\frac{x^{m+n}}{m!\,n!}.$$

Show that

$$\begin{aligned} \sum_{n\geq 0} f(n)\frac{x^n}{n!} &= \frac{e^{2x}(2e^x-3) + e^x(e^x-2)^2 G(x)}{e^x(1+2e^x) + (e^{2x}-2e^x-1)G(x)} \\ &= 1 + x + 3\frac{x^2}{2!} + 13\frac{x^3}{3!} + 111\frac{x^4}{4!} + 1381\frac{x^5}{5!} + 22383\frac{x^6}{6!} + \cdots. \end{aligned}$$

b. [5] What can be said about the *total* number of $(\mathbf{3}+\mathbf{1})$-free posets on an n-element set?

17. **a.** [3–] Let $\mathcal{S}$ be a collection of finite posets, all of whose automorphism groups are trivial. Let $\mathcal{T}$ be the set of all nonisomorphic posets that can be obtained by replacing each element t of some $P \in \mathcal{S}$ with a finite nonempty antichain A_t. (Thus if t

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covers s in P, then each $t' \in A_t$ covers each $s' \in A_s$.) Let $f(n)$ be the number of nonisomorphic n-element posets in $\mathcal{T}$. Let $g(n)$ be the number of posets on the set $[n]$ that are isomorphic to some poset in $\mathcal{T}$. Set

$$F(x) = \sum_{n\geq 0} f(n)x^n, \quad G(x) = \sum_{n\geq 0} g(n)\frac{x^n}{n!}.$$

Show that $G(x) = F(1-e^{-x})$.

b. [2] What are $F(x)$ and $G(x)$ when $\mathcal{S} = \{\mathbf{1},\mathbf{2},\ldots\}$, where $\boldsymbol{i}$ denotes an i-element chain?

c. [2+] Show that we can take $\mathcal{S}$ to consist of all interval orders (respectively, all semiorders) with no nontrivial automorphisms. Then $\mathcal{T}$ consists of all nonisomorphic interval orders (respectively, semiorders). Formulas for $F(x)$ and $G(x)$ appear in Exercises 3.15(c,d): in Vol. II, Exercises 6.30, and 6.19(ddd).

d. [3–]* Show that the number of nonisomorphic n-element graded semiorders is $1+F_{2n-2}$, where F_{2n-2} is a Fibonacci number.

18. [3] Let P be a finite $(\mathbf{3}+\mathbf{1})$-free poset. Let c_i denote the number of i-element chains of P (with $c_0 = 1$). Show that all the zeros of the polynomial $C(P,x) = \sum_i c_i x^i$ are real.

19. **a.** [3–] An element t of a finite poset P is called *irreducible* if t covers exactly one element or is covered by exactly one element of P. A subposet Q of P is called a *core* of P, written $Q = \operatorname{core} P$, if

i. one can write $P = Q \cup \{t_1,\ldots,t_k\}$ such that t_i is an irreducible element of $Q \cup \{t_1,\ldots,t_i\}$ for $1 \leq i \leq k$, and

ii. Q has no irreducible elements.

Show that any two cores of P are isomorphic (though they need not be equal). Hence, the notation $\operatorname{core} P$ determines a unique poset up to isomorphism.

b. [1+]* If P has a $\hat{0}$ or $\hat{1}$, then show that $\operatorname{core} P$ consists of a single element.

c. [3–] Show that $\#(\operatorname{core} P) = 1$ if and only if the poset P^P of order-preserving maps $f\colon P \to P$ is connected. (Such posets are called *dismantlable*.)

d. [5–] Is it possible to enumerate nonisomorphic n-element dismantlable posets or dismantlable posets on an n-element set?

20. [2+]

a. Let $d \in \mathbb{P}$. Show that the following two conditions on a finite poset P are equivalent:

i. P is the intersection of d linear orderings of $[n]$, where $\#P = n$.

ii. P is isomorphic to a subposet of $\mathbb{N}^d$.

b. Moreover, show that when $d = 2$ the two conditions are also equivalent to:

iii. There exists a poset Q on $[n]$ such that $s < t$ or $s > t$ in Q if and only if s and t are incomparable in P.

21. [3+] A finite poset $P = \{t_1,\ldots,t_n\}$ is a *sphere order* if for some $d \geq 1$ there exist $(d-1)$-dimensional spheres $S_1,\ldots,S_n$ in $\mathbb{R}^d$ such that S_i is inside S_j if and only if $t_i < t_j$. *Prove or disprove:* Every finite poset is a sphere order.

22. [2+] Let P be a poset with elements $t_1,\ldots,t_p$, which we regard as indeterminates. Define a $p \times p$ matrix A by

$$A_{ij} = \begin{cases} 0, & \text{if } t_i < t_j, \\ 1, & \text{otherwise.} \end{cases}$$

Define the diagonal matrix $D = \operatorname{diag}(t_1,\ldots,t_p)$, and let I denote the $p \times p$ identity matrix. Show that

$$\det(I + DA) = \sum_C \prod_{t_i \in C} t_i,$$

where C ranges over all chains in P.

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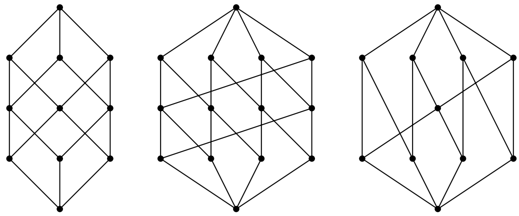

Figure 3.42 Which are lattices?

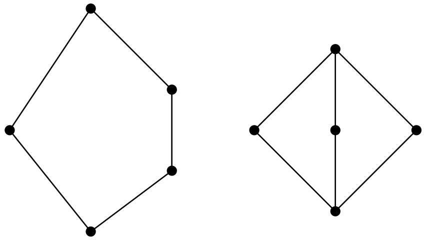

Figure 3.43 Obstructions to distributivity and modularity.

23. [2+] Show that the boolean algebra $B_{\mathbb{P}}$ of *all* subsets of $\mathbb{P}$, ordered by inclusion, contains both countable and uncountable maximal chains.

24. [2+] Let $n \geq 5$. Show that up to isomorphism there is one n-element poset with 2^n order ideals, one with $(3/4)2^n$ order ideals, two with $(5/8)2^n$ order ideals, three with $(9/16)2^n$, and two with $(17/32)2^n$. Show also that all other n-element posets have at most $(33/64)2^n$ order ideals.

25. [2–] Which of the posets of Figure 3.42 are lattices?

26. [2] Give an example of a meet-semilattice with $\hat{1}$ (necessary infinite) that is not a lattice.

27. [3–] Let L be a finite lattice, and define the subposet $\mathrm{Irr}(L)$ of irreducibles of L by

$$\mathrm{Irr}(L) = \{x \in L : x \text{ is join-irreducible or meet-irreducible (or both)}\}.$$

Show that L can be uniquely recovered from the poset $\mathrm{Irr}(L)$.

28. [3–] Give an example of a finite atomic and coatomic lattice that is not complemented.

29. [5–] A finite lattice L has n join-irreducibles. What is the most number $f(n)$ of meet-irreducible elements L can have?

30. **a.** [2+] Show that a lattice is distributive if and only if it does not contain a sublattice isomorphic to either of the two lattices of Figure 3.43.
b. [2+] Show that a lattice is modular if and only if it does not contain a sublattice isomorphic to the first lattice of Figure 3.43.

31. **a.** [2+]* A poset is called *locally connected* if every nonempty open interval (s,t) is either an antichain or is connected. Show that a finite locally connected poset with $\hat{0}$ and $\hat{1}$ is graded.
b. [3] Let L be a finite locally connected lattice for which every interval of rank 3 is a distributive lattice. Show that L is a distributive lattice.
c. [2–] Deduce from (b) that if L is a finite locally connected lattice for which every interval of rank 3 is a product of chains, then L is a product of chains.
d. [2–] Deduce from (b) that if L is a finite locally connected lattice for which every interval of rank 3 is a boolean algebra, then L is a boolean algebra.

32. **a.** [3–] For a finite graded poset P with $\hat{0}$ and with rank function ρ, let $f(P)$ be the largest integer d such that there exists a partition of $P - \{\hat{0}\}$ into (pairwise disjoint) closed intervals $[s,t]$ satisfying $\rho(t) \geq d$. Find $f(B_n)$. For instance, $f(B_3) = 2$, corresponding to the partition

$$\pi = \{[1,12],[2,23],[3,13],[123,123]\}.$$

b. [3–] Show that $f(\boldsymbol{k}^n) = (k-1)f(B_n)$.
c. [2] Let $a \leq b$. Show that $f(\boldsymbol{a} \times \boldsymbol{b}) = b$.
d. [2+] Let $a \leq b \leq c$. Show that $f(\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}) = \max\{a+b, c\}$.
e. [3–] Let $a \leq b \leq c \leq d$. Show that

$$f(\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} \times \boldsymbol{d}) = \max\{d, \min\{b+d, a+b+c\}\}.$$

f. [3] Find $f(\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} \times \boldsymbol{d} \times \boldsymbol{e})$.

33. [2+] Characterize all positive integers n for which there exists a connected poset with exactly n chains (including the empty chain). The empty poset is not considered to be connected.

34. [2]* Find all nonisomorphic posets P such that

$$F(J(P),x) = (1+x)(1+x^2)(1+x+x^2).$$

35. **a.** [2] Let $f_k(n)$ be the number of nonisomorphic n-element posets P such that if $1 \leq i \leq n-1$, then P has exactly k order ideals of cardinality i. Show that $f_2(n) = 2^{n-3}$, $n \geq 3$.
b. [2+] Let $g(n)$ be the number of those posets enumerated by $f_3(n)$ with the additional property that the only 3-element antichains of P consist of the three minimal elements and three maximal elements of P. Show that $g(n) = 2^{n-7}$, $n \geq 7$.
c. [3] Show that

$$\begin{aligned}\sum_{n\geq 0} f_3(n)x^n &= \frac{x^3 - x^4 - x^5 - xg(x) - x^2 g(x)}{1-2x-3x^2}\\ &= x^3 + x^5 + x^6 + 3x^7 + 6x^8 + 16x^9 + 39x^{10} + \cdots,\end{aligned}$$

where

$$g(x) = \frac{x^3 - 2x^4 - x^5 - x^6}{1-2x-2x^2+2x^4+3x^5}.$$

d. [1+] Find $f_k(n)$ for $k > 3$.

36. **a.** [2] Let L be a finite semimodular lattice. Let L' be the subposet of L consisting of elements of L that are joins of atoms of L (including $\hat{0}$ as the empty join). Show that L' is a geometric lattice.
b. [3–] Is L' a sublattice of L?

37. **a.** [3–] Let W be a subspace of the vector space K^n, where K is a field of characteristic 0. The *support* of a vector $v = (v_1,\ldots,v_n) \in K^n$ is given by $\mathrm{supp}(v) = \{i : v_i \neq 0\}$. Let L denote the set of supports of all vectors in W, ordered by reverse inclusion. Show that L is a geometric lattice.
b. [2+] An *isthmus* of a graph H is an edge e of H whose removal disconnects the component to which e belongs. Let G be a finite graph, allowing loops and multiple

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edges. Let D_G be the set of all spanning subgraphs of G that do not have an isthmus, ordered by reverse edge inclusion. Use (a) to show that D_G is a geometric lattice.

38. [2+] Let $k \in \mathbb{N}$. In a finite distributive lattice L, let P_k be the subposet of elements that cover k elements, and let R_k be the subposet of elements that are covered by k elements. Show that $P_k \cong R_k$, and describe in terms of the structure of L an explicit isomorphism $\phi\colon P_k \to R_k$.

39. [2+]* Find all finitary distributive lattices L (up to isomorphism) such that $L \cong V_t$ for all $t \in L$. If we only require that L is a locally finite distributive lattice with $\hat{0}$, are there other examples?

40. **a.** [3–] Let L be a finite distributive lattice of length kr that contains k join-irreducibles of rank i for $1 \leq i \leq r$ (and therefore no other join-irreducibles). What is the most number of elements that L can have? Show that the lattice L achieving this number of elements is unique (up to isomorphism).

b. [2+] Let L be a finitary distributive lattice with exactly two join-irreducible elements at each rank $n \in \mathbb{P}$, and let L_i denote the set of elements of L at rank i. Show that $\#L_i \leq F_{i+2}$ (a Fibonacci number), with equality for all i if and only if $L \cong J_f(P+\mathbf{1})$, where P is the poset of Exercise 3.62(b).

c. [5–] Suppose that L is a finitary distributive lattice with an infinite antichain $t_1, t_2, \dots$ such that t_i has rank i. Does it follow that $\#L_i \geq F_{i+2}$?

41. **a.** [2] Let L be a finite lattice. Given $f\colon L \to \mathbb{N}$, choose s,t incomparable in L such that $f(s) > 0$ and $f(t) > 0$. Define $\Gamma f\colon L \to \mathbb{N}$ by

$$\begin{aligned}
\Gamma f(s) &= f(s) - 1,\\
\Gamma f(t) &= f(t) - 1,\\
\Gamma f(s \wedge t) &= f(s \wedge t) + 1,\\
\Gamma f(s \vee t) &= f(s \vee t) + 1,\\
\Gamma f(u) &= f(u), \text{ otherwise.}
\end{aligned}$$

Show that for some $n > 0$ we have $\Gamma^n f = \Gamma^{n+1} f$ (i.e., $\Gamma^n f$ is supported on a chain).

b. [3–] Show the limiting function $\tilde{f} = \Gamma^n f$ (where $\Gamma^n f$ is supported on a chain) does not depend on the way in which we choose the pairs u, v (though the number of steps n may depend on these choices) if and only if L is distributive.

c. [2+] Show from (b) that $\tilde{f}$ has the following description: It is the unique function $\tilde{f}\colon L \to \mathbb{N}$ supported on a chain, such that for all join-irreducibles $t \in L$ and for $t = \hat{0}$, we have

$$\sum_{s \geq t} f(s) = \sum_{s \geq t} \tilde{f}(s).$$

d. [2] For $t \in L$ let $g(t) = \sum_{s \geq t} f(s)$. Order the join-irreducibles $t_1, \dots, t_m$ of L such that $g(t_1) \geq g(t_2) \geq \cdots \geq g(t_m)$, and let $u_i = t_1 \vee t_2 \vee \cdots \vee t_i$. Deduce from (c) that $\tilde{f}(u_i) = g(t_i) - g(t_{i+1})$ (where we set $g(t_0) = g(\hat{0})$ and $g(t_{m+1}) = 0$), and that $\tilde{f}(u) = 0$ for all other $u \in L$.

e. [2+] Let $L = B_n$, the boolean algebra of subsets of $[n]$. Define $f : B_n \to \mathbb{N}$ by $f(S) = \#\{w \in \mathfrak{S}_{n+1} : \mathrm{Exc}(w) = S\}$, where

$$\mathrm{Exc}(w) = \{i : w(i) > i\},$$

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the *excedance set* of w. Show that for all $0 \le i \le n$, we have

$$\tilde{f}(\{n-i+1, n-i+2, \dots, n\}) = n!,$$

so all other $\tilde{f}(S) = 0$ (since $\tilde{f}$ is supported on a chain).

42. **a.** [2–]* Regard Young's lattice Y as the lattice of all partitions of all integers $n \ge 0$, ordered componentwise. Let Z be the subposet of Y consisting of all partitions with odd parts. Show that Z is a sublattice of Y.

b. [2]* Since sublattices of distributive lattices are distributive, it follows that Z is a finitary distributive lattice. (This fact is also easy to see directly.) For what poset P do we have $Z \cong J_f(P)$?

43. [3–] Let P be the poset with elements s_i and t_i for $i \ge 1$, and cover relations

$$s_1 \lessdot s_2 \lessdot \cdots, \quad t_1 \lessdot t_2 \lessdot \cdots, \quad s_{2i} \lessdot t_i \text{ for } i \ge 1.$$

Find a nice product formula for the rank-generating function $F_{J_f(J_f(P))}(q)$.

44. **a.** [3–]* Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. Let $P_w = \{(i, a_i) : i \in [n]\}$, regarded as a subposet of $\mathbb{P} \times \mathbb{P}$. In other words, define $(i, a_i) \le (k, a_k)$ if $i \le k$ and $a_i \le a_k$. Let $j(P)$ denote the number of order ideals of the poset P. Show that

$$\sum_{w \in \mathfrak{S}_n} j(P_w) = \sum_{i=0}^{n} \frac{n!}{i!} \binom{n}{i}.$$

b. [3]* Let w be as in (a), and let $Q_w = \{(i, j) : 1 \le i < j \le n,\ a_i < a_j\}$. Partially order Q_w by $(i, j) \le (r, s)$ if $r \le i < j \le s$. Show that

$$\sum_{w \in \mathfrak{S}_n} j(Q_w) = (n+1)^{n-1}.$$

45. **a.** [2]* Let $L_k(n)$ denote the number of k-element order ideals of the boolean algebra B_n. Show that for fixed k, $L_k(n)$ is a polynomial function of n of degree $k-1$ and leading coefficient $1/(k-1)!$. Moreover, the differences $\Delta^i L_k(0)$ are all nonnegative integers.

b. [3–]* Show that

$$\begin{aligned}
L_0(n) = L_1(n) &= 1, \\
L_2(n) &= \binom{n}{1}, \\
L_3(n) &= \binom{n}{2}, \\
L_4(n) &= \binom{n}{2} + \binom{n}{3}, \\
L_5(n) &= 3\binom{n}{3} + \binom{n}{4}, \\
L_6(n) &= 3\binom{n}{3} + 6\binom{n}{4} + \binom{n}{5}, \\
L_7(n) &= \binom{n}{3} + 15\binom{n}{4} + 10\binom{n}{5} + \binom{n}{6}, \\
L_8(n) &= \binom{n}{3} + 20\binom{n}{4} + 45\binom{n}{5} + 15\binom{n}{6} + \binom{n}{7},
\end{aligned}$$

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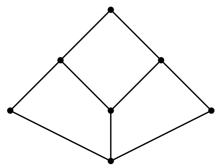

Figure 3.44 A meet-distributive lattice that is not distributive.

$$L_9(n) = 19\binom{n}{4} + 120\binom{n}{5} + 105\binom{n}{6} + 21\binom{n}{7} + \binom{n}{8},$$

$$L_{10}(n) = 18\binom{n}{4} + 220\binom{n}{5} + 455\binom{n}{6} + 210\binom{n}{7} + 28\binom{n}{8} + \binom{n}{9},$$

$$L_{11}(n) = 13\binom{n}{4} + 322\binom{n}{5} + 1385\binom{n}{6} + 1330\binom{n}{7} + 378\binom{n}{8} + 36\binom{n}{9} + \binom{n}{10}.$$

NOTE. It was conjectured that $L_k(n)$ has only real zeros. This conjecture fails, however, for $k = 11$.

46. **a.** [2]* Let $f(n)$ be the number of sublattices of rank n of the boolean algebra B_n. Show that $f(n)$ is also the number of partial orders P on $[n]$.

b. [2+]* Let $g(n)$ be the number of sublattices of B_n that contain $\emptyset$ and $[n]$ (the $\hat{0}$ and $\hat{1}$ of B_n). Write

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!},$$

$$G(x) = \sum_{n\geq 0} g(n)\frac{x^n}{n!}.$$

Show that $G(x) = F(e^x - 1)$.

c. [2]* Let $h(n)$ be the number of nonempty sublattices of B_n. Write

$$H(x) = \sum_{n\geq 0} h(n)\frac{x^n}{n!}.$$

Using (b), show that $H(x) = e^{2x}G(x)$.

47. A finite meet-semilattice is *meet-distributive* if for any interval $[s,t]$ of L such that s is the meet of the elements of $[s,t]$ covered by t, we have that $[s,t]$ is a boolean algebra. For example, distributive lattices are meet-distributive, while the lattice of Figure 3.44 is meet-distributive but not distributive.

a. [2-]* Show that a meet-distributive lattice is lower semimodular and hence graded.

b. [2] Let L be a meet-distributive meet-semilattice, and let $f_k = f_k(L)$ be the number of intervals of L isomorphic to the boolean algebra B_k. Also let $g_k = g_k(L)$ denote the number of elements of L that cover exactly k elements. Show that

$$\sum_{k\geq 0} g_k(1+x)^k = \sum_{k\geq 0} f_k x^k.$$

c. [1] Deduce from (b) that

$$\sum_{k\geq 0} (-1)^k f_k = 1. \tag{3.113}$$

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d. [2+] Let $L = J(\boldsymbol{m} \times \boldsymbol{n})$ in (a). Explicitly compute f_k and g_k.
e. [3–] Given $m \le n$, let Q_{mn} be the subposet of $\mathbb{P} \times \mathbb{P}$ defined by

$$Q_{mn} = \{(i, j) \in P \times P : 1 \le i \le j \le m + n - i,\ 1 \le i \le m\},$$

and set $P_{mn} = \boldsymbol{m} \times \boldsymbol{n}$. Show that P_{mn} and Q_{mn} have the same zeta polynomial.
f. [3+] Show that P_{mn} and Q_{mn} have the same order polynomial.
g. [3–] Show that $J(P_{mn})$ and $J(Q_{mn})$ have the same values of f_k and g_k.

48. [2+] Let L be a meet-distributive lattice, as defined in Exercise 3.47, and let $t \in L$. Show that the number of join-irreducibles s of L satisfying $s \le t$ is equal to the rank $\rho(t)$ of t.

49. [2] Let L_p denote the set of all natural partial orders P of $[p]$ (that is, $i <_P j \Rightarrow i <_{\mathbb{Z}} j$), ordered by refinement. The bottom element is an antichain, and the top element is the chain $\boldsymbol{p}$. Figure 3.6 shows a poset isomorphic to L_3. Show that L_p is meet-distributive of rank $\binom{p}{2}$.

50. [2+] Let L be a finitary distributive lattice with finitely many elements of each rank. Let $u(i, j)$ be the number of elements of L of rank i that cover exactly j elements, and let $v(i, j)$ be the number of elements of rank i that are covered by exactly j elements. Show that for all $i \le j \le 0$,

$$\sum_{k \ge 0} u(i,k) \binom{k}{j} = \sum_{k \ge 0} v(i - j,k) \binom{k}{j}. \tag{3.114}$$

(Each sum has finitely many nonzero terms.)

51. Let $f : \mathbb{N} \to \mathbb{N}$. A finitary distributive lattice L is said to have the *cover function* f if whenever $t \in L$ covers i elements, then t is covered by $f(i)$ elements.
a. [2+] Show that there is at most one (up to isomorphism) finitary distributive lattice with a given cover function f.
b. [2+] Show that if L is a *finite* distributive lattice with a cover function f, then L is a boolean algebra.
c. [2+] Let $k \in \mathbb{P}$. Show that there exist finitary distributive lattices with cover functions $f(n) = k$ and $f(n) = n + k$.
d. [2+] Let $a, k \in \mathbb{P}$ with $a \ge 2$. Show that there does not exist a finitary distributive lattice L with cover function $f(n) = an + k$.
e. [3] Show in fact that $f(n)$ is the cover function of a finitary distributive lattice L if and only if it belongs to one of the following seven classes. (Omitted values of f have no effect on L.)

- Let $k \ge 1$. Then $f(n) = k$ for $0 \le n \le k$.
- Let $k \ge 1$. Then $f(n) = n + k$.
- Let $k \ge 2$. Then $f(0) = 1$, and $f(n) = k$ for $1 \le n \le k$.
- $f(0) = 2$, and $f(n) = n + 1$ for $n \ge 1$.
- Let $k \ge 0$. Then $f(n) = k - n$ for $0 \le n \le k$.
- Let $k \ge 2$. Then $f(n) = k - n$ for $0 \le n < k$, and $f(k) = k$.
- $f(0) = 2$, $f(1) = f(2) = 1$.

52. [2+]* What is the maximum possible value of $e(P)$ for a connected n-element poset P?

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53. [2]* Let P be a finite n-element poset. Simplify the two sums

$$f(P) = \sum_{I \in J(P)} e(I)e(\bar{I}),$$

$$g(P) = \sum_{I \in J(P)} \binom{n}{\#I} e(I)e(\bar{I}),$$

where $\bar{I}$ denotes the complement $P - I$ of the order ideal I.

54. [2+]* Let P be a finite poset. Simplify the sum

$$f(P) = \sum_{t_1 < \cdots < t_n} \frac{1}{(\#V_{t_1} - 1) \cdots (\#V_{t_{n-1}} - 1)},$$

where the sum ranges over all nonempty chains of P for which t_n is a maximal element of P. Generalize.

55. **a.** [3] Generalize Corollary 1.6.5 as follows. Let T be a tree on the vertex set $[n]$. Given an orientation $\mathfrak{o}$ of the edges of T, let $P(T,\mathfrak{o})$ be the reflexive and transitive closure of $\mathfrak{o}$, so $P(T,\mathfrak{o})$ is a poset. Clearly for fixed T, exactly two of these posets (one the dual of the other) have no 3-element chains. Let us call the corresponding orientations *bipartite*. Show that for fixed T, the number $e(P(T,\mathfrak{o}))$ of linear extensions of $(P,\mathfrak{o})$ is maximized when $\mathfrak{o}$ is bipartite. (Corollary 1.6.5 is the case when T is a path.)

b. [5–] Does (a) continue to hold when T is replaced with any finite bipartite graph?

56. [3–] Let P be a finite poset, and let $f(P)$ denote the number of ways to partition the elements of P into (nonempty) disjoint saturated chains. For instance, $f(\boldsymbol{n}) = 2^{n-1}$. Suppose that every element of P covers at most two elements and is covered by at most two elements. Show that $f(P)$ is a product of Fibonacci and Lucas numbers. In particular, compute $f(\boldsymbol{m} \times \boldsymbol{n})$.

57. **a.** [2]* Let P be an n-element poset. If $t \in P$, then set $\lambda_t = \#\{s \in P : s \leq t\}$. Show that

$$e(P) \geq \frac{n!}{\prod_{t \in P} \lambda_t}. \tag{3.115}$$

b. [2+]* Show that equality holds in equation (3.115) if and only if every component of P is a rooted tree (where the root as usual is the maximum element of the tree).

58. [3–]* Let P be a finite poset. Let A be an antichain of P which intersects every maximal chain. Show that

$$e(P) = \sum_{t \in A} e(P - t).$$

Try to give an elegant bijective proof.

59. [2+] Let P be a finite p-element poset. Choose two incomparable elements $s,t \in P$. Define $P_{s<t}$ to be the poset obtained from P by adjoining the relation $s < t$ (and all those implied by transitivity). Similarly define $P_{s>t}$. Define $P_{s=t}$ to be the poset obtained from P by identifying s and t. Hence $\#P_{s=t} = p - 1$. Write formally

$$P \to P_{s<t} + P_{s>t} + P_{s=t}.$$

Now choose two incomparable elements (if they exist) of each summand and apply the same decomposition to them. Continue until P is formally written as a linear

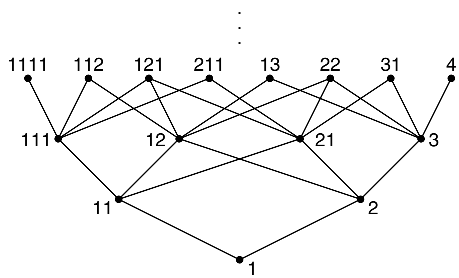

Figure 3.45 The composition poset $\mathcal{C}$.

combination of chains:

$$P \to \sum_{i=1}^{p} a_i \boldsymbol{i}.$$

Show that the numbers a_i are independent of the way in which the decomposition was obtained, and find a combinatorial interpretation of a_i.

60. Let P be a p-element poset. A bijection $f\colon P \to [p]$ is called a *dropless labeling* if we never have $s < t$ and $f(s) = f(t) + 1$.

a. [1]* Show that every linear extension of P is a dropless labeling.

b. [3–] Let G be an (undirected) graph, say with no loops or multiple edges. An *acyclic orientation* of G is an assignment of a direction $u \to v$ or $v \to u$ to each edge uv of G so that no directed cycles $u_1 \to u_2 \to \cdots \to u_k \to u_1$ result. Show that the number of dropless labelings of P is equal to the number of acyclic orientations of the incomparability graph $\mathrm{inc}(P)$. (For further information on the number of acyclic orientations of a graph, see Exercise 3.109.)

c. [2+]* Give a bijective proof that the number of dropless labelings of P is equal to the number of bijections $g\colon P \to P$ such that we never have $g(t) < t$. *Hint.* Use Proposition 1.3.1(a).

61. [2+] Let $\mathcal{C}$ be the set of all compositions of all positive integers. Define a partial ordering on $\mathcal{C}$ by letting τ cover $\sigma = (\sigma_1, \dots, \sigma_k)$ if τ can be obtained from σ either by adding 1 to a part, or adding 1 to a part and then splitting this part into two parts. More precisely, for some j we have either

$$\tau = (\sigma_1, \dots, \sigma_{j-1}, \sigma_j + 1, \sigma_{j+1}, \dots, \sigma_k)$$

or

$$\tau = (\sigma_1, \dots, \sigma_{j-1}, h, \sigma_j + 1 - h, \sigma_{j+1}, \dots, \sigma_k)$$

for some $1 \le h \le \sigma_j$. See Figure 3.45. For each $\sigma \in \mathcal{C}$, find in terms of a "familiar" number the number of saturated chains from the composition 1 (the bottom element of $\mathcal{C}$) to σ. What is the total number of saturated chains from 1 to some composition of n?

62. **a.** [2]* Let P_n be the poset with elements s_i, t_i for $i \in [n]$, and cover relations $s_1 < s_2 < \cdots < s_n$ and $t_i > s_i$ for all $i \in [n]$. For example, P_3 has the Hasse diagram of Figure 3.46.

Find a "nice" expression for the rank-generating function $F(J(P_n), x)$.

b. [2–]* Let $P = \lim_{n\to\infty} P_n$. Find the rank-generating function $F(J_f(P), x)$.

c. [2]* Find a simple formula for $e(P_n)$.

d. [2]* Let $\Omega_{P_n}(m)$ denote the order polynomial of P_n (naturally labeled). For $m \in \mathbb{P}$ express $\Omega_{P_n}(m)$ in terms of Stirling numbers of the second kind.

e. [2+]* For $m \in \mathbb{P}$ express $\Omega_{P_n}(-m)$ in terms of Stirling numbers of the first kind.

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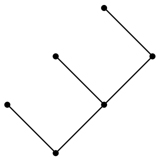

Figure 3.46 The poset P_3 of Exercise 3.62.

f. [2+] Let P be as in (b). The generating function $U_{P,m}(x)$ of Exercise 3.171 is still well-defined although P is infinite, namely, $U_{P,m}(x)) = \sum_\sigma x^{|\sigma|}$, where σ ranges over all order-reversing maps $\sigma\colon P \to [0,m]$ such that $|\sigma| := \sum_{t\in P}\sigma(t) < \infty$. Such a map σ is called a *protruded partition* of $n = |\sigma|$ (with largest part at most m). Thus, we can regard a protruded partition of n as a pair (λ,μ), where $\lambda = (\lambda_1,\lambda_2,\dots)$ is a partition, $\mu = (\mu_1,\mu_2,\dots)$ is a sequence of nonnegative integers satisfying $\mu_i \le \lambda_i$, and $\sum(\lambda_i+\mu_i) = n$. For instance, there are six protruded partitions of 3, given by

$$(3,0),\ (21,00),\ (111,000),\ (2,1),\ (11,10),\ (11,01).$$

Show that

$$U_{P,m}(x) = \prod_{i=1}^{m}(1-x^i-x^{i+1}-\cdots-x^{2i})^{-1}.$$

g. [2+] Show that

$$\sum_{m\ge 0} U_{P_n}(x)q^n = P(q,x)\sum_{j\ge 0}\frac{x^{j(j+1)}q^j}{[j]!(1-x-x^2)(1-x-x^3)\cdots(1-x-x^{j+1})},$$

where $[j]! = (1-x)(1-x^2)\cdots(1-x^j)$ and $P(q,x) = 1/(1-q)(1-qx)(1-qx^2)\cdots$.

63. **a.** [2]* Let P be a p-element poset, with every maximal chain of length ℓ. Let e_s (respectively, $\bar{e}_S$) denote the number of surjective (respectively, strict surjective) order-preserving maps $f\colon P \to \boldsymbol{s}$. (The order-preserving map f is *strict* if $s < t$ in P implies $f(s) < f(t)$.) Use Corollary 3.15.18 to show that

(i) $2e_{p-1} = (p+\ell-1)e(P)$.
(ii) $2\bar{e}_{p-1} = (p-\ell-1)e(P)$.
(iii) $\sum_{s=1}^{p} e_s = 2^\ell\sum_{s=1}^{p}\bar{e}_s$.

b. [1+]* With P as in (a). show that if $p \equiv \ell\,(\mathrm{mod}\,2)$, then $e(P)$ is even.

c. [2]* With P as in (a), suppose that $\ell = p-4$. Let $j(P)$ denote the number of order ideals of P. Show that $e(P) = 2(j(P)-p)$.

64. **a.** [2+]* Let $\varphi\colon \mathbb{Q}[n] \to \mathbb{Q}[x]$ be the $\mathbb{Q}$-linear function on polynomials with rational coefficients that takes n^k to $\sum_j c_j(k)x^j$, where $c_j(k)$ is the number of ordered partitions of $[k]$ into j blocks (entry 3 of the Twelvefold Way or Example 3.18.9). Let (P,ω) be a labeled poset. Show that

$$\varphi\Omega_{P,\omega}(n) = \sum_j a_j(P,\omega)x^j,$$

where $a_j(P,\omega)$ is the number of chains $\emptyset = I_0 < I_1 < \cdots < I_j = P$ in $J(P)$ for which the restriction of ω to every set $I_i - I_{i-1}$ is order-preserving.

b. [1+]* Let $c(k)$ denote the total number of ordered set partitions of $[k]$. Deduce from (a) that when we substitute $c(k)$ for n^k in $\Omega_P(n)$ (so P is naturally labeled), then we obtain the total number of chains from $\hat{0}$ to $\hat{1}$ in $J(P)$.

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c. [2+]* Now let $\sigma : \mathbb{Q}[n] \to \mathbb{Q}[x]$ be defined by $\sigma(n^k) = (-1)^k \sum_j c_j(k) x^j$. Let $\#P = p$. Show that

$$\sigma \Omega_{P,\omega}(n) = (-1)^p \sum_j b_j(P,\omega) x^j,$$

where $b_j(P,\omega)$ is the number of chains $\emptyset = I_0 < I_1 < \cdots < I_j = P$ in $J(P)$ for which the restriction of ω to every set $I_i - I_{i-1}$ is order-reversing.

d. [1+]* Deduce from (c) that when we substitute $(-1)^k c(k)$ for n^k in $\Omega(P,n)$, then we obtain $(-1)^p$ times the number of chains $\emptyset = I_0 < I_1 < \cdots < I_j = P$ in $J(P)$ for which every interval $I_i - I_{i-1}$ is a boolean algebra.

65. [2] Let $n \in \mathbb{P}$ and $r,s,t \in \mathbb{N}$. Let $P(r,s,2t,n)$ be the poset with elements x_i $(1 \le i \le n)$, y_{ij} $(1 \le i \le r, 1 \le j \le n)$, z_{ij} $(1 \le i \le s, 1 \le j \le n)$, and a_{ijk} $(1 \le j < k \le n, 1 \le i \le 2t)$, and cover relations

$$x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n,$$

$$y_{1j} \lessdot y_{2j} \lessdot \cdots \lessdot y_{rj} \lessdot x_j, \;\; 1 \le j \le n,$$

$$x_j \lessdot z_{1j} \lessdot z_{2j} \lessdot \cdots \lessdot z_{sj}. \;\; 1 \le j \le n,$$

$$x_j \lessdot a_{1jk} \lessdot a_{2jk} \lessdot \cdots \lessdot a_{2t,j,k} \lessdot x_k, \;\; 1 \le j < k \le n.$$

Use Exercise 1.11(b) to show that

$$e(P,r,s,2t,n) = \frac{[(r+s+1)n + tn(n-1)]!}{n! r!^n s!^n t!^n (2t)!^{\binom{n}{2}}} \cdot \prod_{j=1}^{n} \frac{(r+(j-1)t)!(s+(j-1)t)!(jt)!}{(r+s+1+(n+j-2)t)!}. \tag{3.116}$$

Figure 3.47 shows the poset $p = P(2,1,2,3)$ for which $e(P) = 4725864 = 2^3 \cdot 3^5 \cdot 11 \cdot 13 \cdot 17$.

66. Let Z_n denote the n-element "zigzag poset" or *fence*, with elements $t_1, \ldots, t_n$ and cover relations $t_{2i-1} < t_{2i}$ and $t_{2i} > t_{2i+1}$.

a. [2] How many order ideals does Z_n have?

b. [2+] Let $W_n(q)$ denote the rank-generating function of $J(Z_n)$, so $W_0(q) = 1$, $W_1(q) = 1+q$, $W_2(q) = 1+q+q^2$, $W_3(q) = 1+2q+q^2+q^3$, etc. Find a simple explicit formula for the generating function

$$F(x) := \sum_{n \ge 0} W_n(q) x^n.$$

c. [2] Find the number $e(Z_n)$ of linear extensions of Z_n.

d. [3–] Let $\Omega_{Z_n}(m)$ be the order polynomial of Z_n. Set

$$G_m(x) = 1 + \sum_{n \ge 0} \Omega_{Z_n}(m) x^{n+1}, \;\; m \ge 1.$$

Find a recurrence relation expressing $G_m(x)$ in terms of $G_{m-2}(x)$, and give the intitial conditions $G_1(x)$ and $G_2(x)$.

67. [3] For $p \le q$ define a poset P_{pq} to consist of three chains $s_1 > \cdots > s_p$, $t_1 > \cdots > t_q$, and $u_1 > \cdots > u_q$, with $s_i < u_i$ and $t_i < u_i$. Show that the number of linear extensions of P_{pq} is given by

$$e(P_{pq}) = \frac{2^{2p}(p+2q)!\,(2q-2p+2)!}{p!\,(2q+2)!\,(q-p)!\,(q-p+1)!}.$$

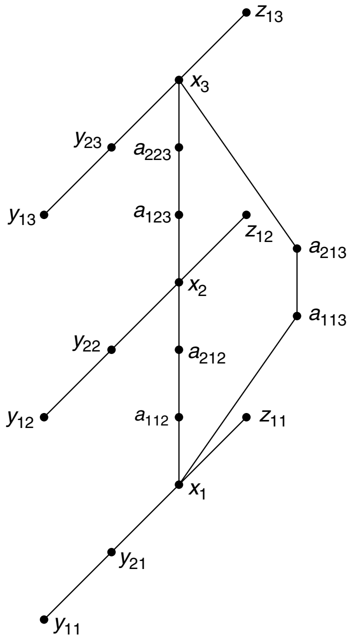

Figure 3.47 The “Selberg poset” $P(2,1,2,3)$.

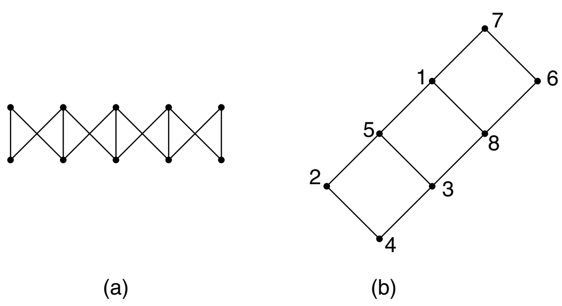

Figure 3.48 A garland and an alternating labeling of $\mathbf{2}\times\mathbf{4}$.

68. The *garland* or *double fence* G_n is the poset with vertices $s_1,\dots,s_n,t_1,\dots,t_n$ and cover relations $s_i < t_i$ $(1\le i\le n)$, $s_i < t_{i-1}$ $(2\le i\le n)$, and $s_i < t_{i+1}$ $(1\le i\le n-1)$. Figure 3.48(a) shows the garland G_5. An *alternating labeling* of an m-element poset P is a bijection $f\colon P\to[m]$ such that every maximal chain $t_1<t_2<\cdots<t_k$ has alternating labels, (i.e., $f(t_1)>f(t_2)<f(t_3)>f(t_4)<\cdots$). Figure 3.48(b) shows an alternating labeling of the poset $\mathbf{2}\times\mathbf{4}$.

a. [2+] Let $j(n,k)$ denote the number of k-element order ideals of G_n. Show that

$$\sum_{n\ge 0}\sum_{k\ge 0} j(n,k)x^k y^n = \frac{1-x^2y^2}{1-(1+x+x^2)y+x^2y^2+x^3y^3}.$$

b. [2]* Show that $e(G_n)$ is the number of alternating labelings of $\mathbf{2}\times\boldsymbol{n}$.

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c. [5–] Find a nice formula or generating function for $e(G_n)$. The values $e(G_n)$ for $1 \le n \le 6$ are

$$1,\ 4,\ 44,\ 896,\ 29392,\ 1413792.$$

69. **a.** [3] Fix an element t of a p-element poset P, and let $\mathcal{L}(P)$ denote the set of all linear extensions $f\colon P \to [p]$. Show that the polynomial

$$P_t(x) = \sum_{f \in \mathcal{L}(P)} x^{f(t)}$$

is log-concave, as defined in Exercise 1.50.

b. [3+] Suppose that the finite poset P is not a chain. Show that there exist elements $s,t \in P$ such that $f(s) < f(t)$ in more than a fraction $\frac{5-\sqrt{5}}{10} = 0.276\cdots$ and less than a fraction $\frac{5+\sqrt{5}}{10} = 0.723\cdots$ of the linear extensions f of P.

70. **a.** [2–]* Let E_n denote the poset of all subsets of $[n]$ whose elements have even sum, ordered by inclusion. Find $\#E_n$.

b. [2+]* Compute $\mu(S,T)$ for all $S \le T$ in E_n.

c. [3–] Generalize (b) as follows. Let $k \ge 3$, and let P_k denote the poset of all subsets of $\mathbb{P}$ whose elements have sum divisible by k. Given $T \le S$ in P_k, let

$$i_j = \#\{n \in T - S \,:\, n \equiv j \,(\mathrm{mod}\, k)\}.$$

Clearly $\mu(S,T)$ depends only on the k-tuple $(i_0, i_1, \ldots, i_{k-1})$, so write $\mu(i_0,\ldots,i_{k-1})$ for $\mu(S,T)$. Show that

$$\sum_{i_0,\ldots,i_{k-1} \ge 0} \mu(i_0,\ldots,i_{k-1}) \frac{x_0^{i_0} \cdots x_{k-1}^{i_{k-1}}}{i_0! \cdots i_{k-1}!}$$

$$= k \left[\sum_{j=0}^{k-1} \exp\left(x_0 + \zeta^j x_1 + \zeta^{2j} x_2 + \cdots + \zeta^{(k-1)j} x_{k-1}\right) \right]^{-1},$$

where ζ is a primitive kth root of unity.

71. [3–] Let P be a finite poset. The *free distributive lattice* $\mathrm{FD}(P)$ generated by P is, intuitively, the largest distributive lattice containing P as a subposet and generated (as a lattice) by P. More precisely, if L is any distributive lattice containing P and generated by P, then there is a (surjective) lattice homomorphism $f\colon \mathrm{FD}(P) \to L$ that is the identity on P. Show that $\mathrm{FD}(P) \cong J(J(P)) - \{\hat{0}, \hat{1}\}$. In particular, $\mathrm{FD}(P)$ is finite. When $P = n\mathbf{1}$ (an n-element antichain), we write $\mathrm{FD}(P) = \mathrm{FD}(n)$, the free distributive lattice with n generators, so that $\mathrm{FD}(n) \cong J(B_n) - \{\hat{0}, \hat{1}\}$.
NOTE. Sometimes one defines $\mathrm{FD}(P)$ to be the free *bounded* distributive lattice generated by P. In this case, we need to add an extra $\hat{0}$ and $\hat{1}$ to $\mathrm{FD}(P)$, so one sometimes sees the statement $\mathrm{FD}(P) \cong J(J(P))$ and $\mathrm{FD}(n) \cong J(B_n)$.

72. **a.** [2] Let P be a finite poset with largest antichain of cardinality k. Every antichain A of P corresponds to an order ideal

$$\langle A \rangle = \{s \,:\, s \le t \text{ for some } t \in A\} \in J(P).$$

Show that the set of all order ideals $\langle A \rangle$ of P with $\#A = k$ forms a sublattice $M(P)$ of $J(P)$.

b. [3–] Show that every finite distributive lattice is isomorphic to $M(P)$ for some P.

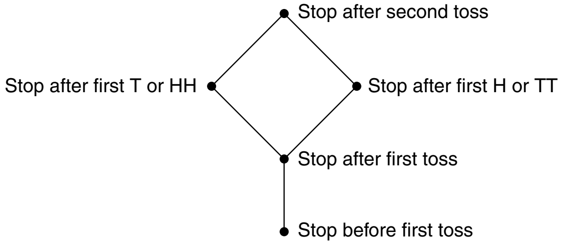

Figure 3.49 The binary stopping rule poset L_2.

73. **a.** [2+] Let P be a finite poset, and define $G_P(q,x) = \sum_I q^{\#I} x^{m(I)}$, where I ranges over all order ideals of P and where $m(I)$ denotes the number of maximal elements of I. (Thus, $G_P(q,1)$ is the rank-generating function of $J(P)$.) Let Q be an n-element poset. Show that

$$G_{P\otimes Q}(q,x) = G_P(q^n, q^{-n}(G_Q(q,x)-1)),$$

where $P \otimes Q$ denotes ordinal product.

b. [2+] Show that if $\#P = p$, then

$$G_P\left(q, \frac{q-1}{q}\right) = q^p.$$

74. [2+] A *binary stopping rule* of length n is (informally) a rule for telling a person when to stop tossing a coin, so that he is guaranteed to stop within n tosses. Two rules are considered the same if they result in the same outcome. For instance, “toss until you get three consecutive heads or four consecutive tails, or else after n tosses” is a stopping rule of length n. Partially order the stopping rules of length n by $A \le B$ if the tosser would never stop later using rule A rather than rule B. Let L_n be the resulting poset. For example, L_2 is shown in Figure 3.49. Show that L_n is a distributive lattice, and compute its poset of join-irreducibles. Find a simple recurrence for the rank-generating function $F(L_n,q)$ in terms of $F(L_{n-1},q)$.

75. Let G be a finite connected graph, allowing multiple edges but not loops. Fix a vertex v of G, and let $\mathrm{ao}(G,v)$ be the set of acyclic orientations of G such that v is a sink. If $\mathfrak{o}$, $\mathfrak{o}' \in \mathrm{ao}(G,v)$, then define $\mathfrak{o} \le \mathfrak{o}'$ if we can obtain $\mathfrak{o}'$ from $\mathfrak{o}$ by a sequence of operations that consist of choosing a *source* vertex w not adjacent to v and orienting all edges of G incident to w toward w, keeping the rest of $\mathfrak{o}$ unchanged.

a. [2+] Show that $(\mathrm{ao}(G,v),\le)$ is a poset.

b. [2–] Let G be a 6-cycle. By symmetry the choice of v is irrelevant. Show that

$$\mathrm{ao}(G,v) \cong J(\mathbf{2}\times\mathbf{2}) + \mathbf{4} + \mathbf{4} + \mathbf{1} + \mathbf{1}.$$

c. [3–] Show that every connected component of $(\mathrm{ao}(G,v),\le)$ is a distributive lattice.

76. In this exercise, P and Q denote locally finite posets and $I(P)$, $I(Q)$ their incidence algebras over a field K.

a. [2] Show that the (Jacobson) radical of $I(P)$ is $\{f : f(t,t) = 0 \text{ for all } t \in P\}$. The Jabcobson radical can be defined as the intersection of all maximal right ideals of $I(P)$.

b. [2+] Show that the lattice of two-sided ideals of $I(P)$ is isomorphic to the set of all order ideals A of $\mathrm{Int}(P)$ (the poset of intervals of P), ordered by reverse inclusion.

c. [3–] Show that if $I(P)$ and $I(Q)$ are isomorphic as K-algebras, then P and Q are isomorphic.

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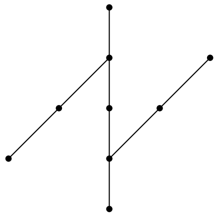

Figure 3.50 A poset for Exercise 3.77.

d. [3] Describe the group of K-automorphisms and the space of K-derivations of $I(P)$.

77. **a.** [3] Let P be a p-element poset, and define nonnegative integers λ_i by setting $\lambda_1 + \cdots + \lambda_i$ equal to the maximum size of a union of i chains in P. For instance, the poset P of Figure 3.50 satisfies $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 1$, and $\lambda_i = 0$ for $i \geq 4$. Note that the largest chain has five elements, but that the largest union of two chains does not contain a five-element chain. Show that $\lambda_1 \geq \lambda_2 \geq \cdots$ (i.e., if we set $\lambda = (\lambda_1, \lambda_2, \ldots)$ then $\lambda \vdash p$).

b. [3] Define μ_i's analogously by letting $\mu_1 + \cdots + \mu_i$ be the maximum size of a union of i antichains. For the poset of Figure 3.50 we have $\mu = (\mu_1, \mu_2, \ldots) = (3, 2, 2, 1, 1, 0, 0, \ldots)$. Show that $\mu \vdash p$.

c. [3] Show that $\mu = \lambda'$, the conjugate partition to λ.

d. [2–] Deduce from (c) *Dilworth's theorem*: The minimum k for which P is a union of k chains is equal to the size of the largest antichain of P.

e. [2] Prove directly that $\lambda_1 = \mu'_1$ (i.e., the size of the longest chain of P is equal to the minimum k for which P is a union of k antichains).

f. [3] Let A be the matrix whose rows and columns are indexed by P, with

$$A_{st} = \begin{cases} x_{st}, & \text{if } s < t, \\ 0, & \text{otherwise,} \end{cases}$$

where the x_{st}'s are independent indeterminates. It is clear that A is nilpotent (i.e., every eigenvalue is 0). Show that the Jordan block sizes of A are the numbers $\lambda_i > 0$ of (a).

78. **a.** [3–] Find the partitions λ and μ of Exercise 1.77 for the boolean algebra B_n.

b. [5] Do the same for the partition lattice Π_n.

c. [5] Do the same for the lattice $\mathrm{Par}(n)$ of partitions of n ordered by dominance (defined in Exercise 3.136).

79. [3–] Let P be a finite poset on the set $[p]$, such that if $s < t$ in P then $s < t$ in $\mathbb{Z}$. A linear extension of P can therefore be regarded as a permutation $w = a_1 a_2 \cdots a_p \in \mathfrak{S}_p$ such that if $a_i < a_j$ in P, then $i < j$ in $\mathbb{Z}$. Define the *comajor index* $\mathrm{comaj}(w) = \sum_{i \in D(w)} (p - i)$, where $D(w)$ denotes the descent set of w. A *P-domino tableau* is a chain $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_r = P$ of order ideals of P such that $I_i - I_{i-1}$ is a two-element chain for $2 \leq i \leq r$, while I_1 is either a two-element or one-element chain (depending on whether p is even or odd). In particular, $r = \lceil p/2 \rceil$. Show that the following three quantities are equal.

i. The sum $w(P) = \sum_{w \in \mathcal{L}(P)} (-1)^{\mathrm{comaj}(w)}$. Note. If p is even, then $\mathrm{comaj}(w) \equiv \mathrm{maj}(w) \pmod 2$. In this case $w(P) = W_P(-1)$ in the notation of Section 3.15.

ii. The number of P-domino tableaux.

iii. The number of self-evacuating linear extensions of P (i.e., linear extensions f satisfying $f\epsilon = f$, where ϵ denotes evacuation).

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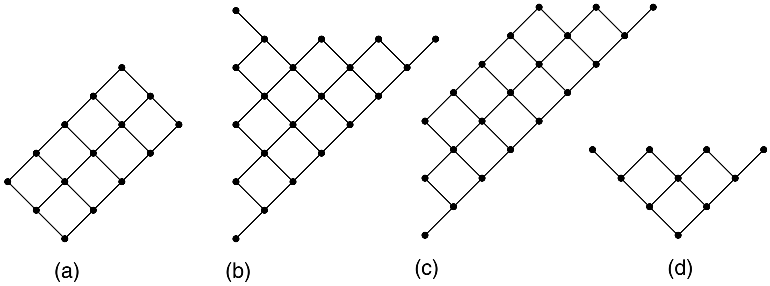

Figure 3.51 Four posets with nice promotion properties.

80. **a.** [3] Show that for the following p-element posets P we have $f\partial^p = f$, where ∂ is the promotion operator and f is a linear extension of P. (We give an example of each type of poset, from which the general definition should be clear.)
i. Rectangles: Figure 3.51(a).
ii. Shifted double staircases: Figure 3.51(b).
iii. Shifted trapezoids: Figure 3.51(c).
b. [3] Show that if P is a staircase (illustrated in Figure 3.51(d)), then $f\partial^p$ is obtained by reflecting P (labeled by f) about a vertical line. Thus, $f\partial^{2p} = f$.

81. An n-element poset P is *sign-balanced* if the set $\mathcal{E}_P$ of linear extensions of P (regarded as a permutation of the elements of P with respect to some fixed ordering of the elements) contains the same number of even permutations as odd permutations. (This definition does not depend on the fixed ordering of the elements of P, since changing the ordering simply multiplies the elements of $\mathcal{E}_P$ by a fixed permutation in $\mathfrak{S}_n$).
a. [2–] Suppose that $n \geq 2$. Show that if every nonminimal element of P is greater than at least two minimal elements, then P is sign-balanced. For instance, atomic lattices with at least three elements are sign-balanced (since we can clearly remove $\hat{0}$ without affecting the property of being sign-balanced).
b. [2+] Suppose that the length $\ell(C)$ of every maximal chain C of P satisfies $\ell(C) \equiv n \pmod 2$. Show that P is sign-balanced.

82. [3] Show that a product $\boldsymbol{p} \times \boldsymbol{q}$ of two chains is sign-balanced if and only if $p, q > 1$ and $p \equiv q \pmod 2$.

83. [2] Show that P is sign-balanced if $\#P$ is even and there does not exist a P-domino tableau, as defined in Exercise 3.79.

84. [2+] A mapping $t \mapsto \bar{t}$ on a poset P is called a *closure operator* (or *closure*) if for all $s, t \in P$,

$$\begin{aligned} &t \leq \bar{t}, \\ &s \leq t \Rightarrow \bar{s} \leq \bar{t}, \\ &\bar{\bar{t}} = \bar{t}. \end{aligned}$$

An element t of P is *closed* if $t = \bar{t}$. The set of closed elements of P is denoted $\overline{P}$, called the *quotient* of P relative to the closure $\bar{\ }$. If $s \leq t$ in P, then define $\bar{s} \leq \bar{t}$ in $\overline{P}$. It is easy to see that $\bar{P}$ is a poset.

Let P be a locally finite poset with closure $t \mapsto \bar{t}$ and quotient $\overline{P}$. Show that for all $s, t \in P$,

$$\sum_{\substack{u \in P \\ \bar{u} = \bar{t}}} \mu(s, u) = \begin{cases} \mu_{\overline{P}}(\bar{s}, \bar{t}), & \text{if } s = \bar{s} \\ 0, & \text{if } s < \bar{s}. \end{cases}$$

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85. [2+]* Let P be a finite poset. Show that the following two conditions are equivalent:
i. For all $s < t$, the interval $[s,t]$ has an odd number of atoms.
ii. For all $s < t$, the interval $[s,t]$ has an odd number of coatoms.
Hint. Consider $\mu(s,t)$ modulo 2.

86. [2+] Let f and g be functions on a finite lattice L, with values in a field of characteristic 0, satisfying

$$f(s) = \sum_{\substack{t \\ s\wedge t=\hat{0}}} g(t). \tag{3.117}$$

Show that if $\mu(\hat{0},u) \neq 0$ for all $u \in L$, then equation (3.117) can be inverted to yield

$$g(s) = \sum_t \alpha(s,t) f(t),$$

where

$$\alpha(s,t) = \sum_u \frac{\mu(s,u)\mu(t,u)}{\mu(\hat{0},u)}.$$

87. **a.** [2+] Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let μ be its Möbius function. Let $f\colon P \to \mathbb{C}$. Show that

$$\sum (f(t_1)-1)(f(t_2)-1)\cdots(f(t_k)-1)$$
$$= \sum (-1)^{k+1}\mu(\hat{0},t_1)\mu(t_1,t_2)\cdots\mu(t_k,\hat{1})f(t_1)f(t_2)\cdots f(t_k),$$

where both sums range over all chains $\hat{0} < t_1 < \cdots < t_k < \hat{1}$ of P.
b. [1+] Deduce that

$$\sum_{\hat{0}=t_0<t_1<\cdots<t_k=\hat{1}} (-1)^k \mu(t_0,t_1)\mu(t_1,t_2)\cdots\mu(t_{k-1},t_k) = 1.$$

c. [2] Give a proof of (b) using incidence algebras.
d. [2–] Deduce equation (3.113) from (a) when L is a meet-distributive lattice.

88. [2] Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and with Möbius function μ. Show that

$$\sum_{s\le t} \mu(s,t) = 1.$$

89. [2]* For a finite lattice L, let $f_L(m)$ be the number of m-tuples $(t_1,\ldots,t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \cdots \wedge t_m = \hat{0}$. Give two proofs that

$$f_L(m) = \sum_{t\in L} \mu(\hat{0},t)(\#V_t)^m.$$

The first proof should be by direct Möbius inversion, and the second by considering $\left(\sum_{t\in L} t\right)^m$ in the Möbius algebra $A(L,\mathbb{R})$.

90. [2]* Let P be a finite graded poset, and let $m(s,t)$ denote the number of maximal chains of the interval $[s,t]$. Define $f \in I(P,\mathbb{C})$ by

$$f(s,t) = \frac{m(s,t)}{\ell(s,t)!}.$$

Show that

$$f^{-1}(s,t) = (-1)^{\ell(s,t)} f(s,t).$$

91. **a.** [3–]* Let L be a finite lattice with n atoms. Show that

$$|\mu(\hat{0},\hat{1})| \leq \binom{n-1}{\lfloor (n-1)/2 \rfloor},$$

and that this result is best possible.

b. [3+]* Assume also that the longest chain of L has length at most ℓ. show that

$$|\mu(\hat{0},\hat{1})| \leq \binom{n-1}{k},$$

where $k = \min(\ell - 1, \lfloor (n-1)/2 \rfloor)$, and that this result is best possible.

92. [3–] Assume that L is a finite lattice and fix $t \in L$. Show that

$$\mu(\hat{0},\hat{1}) = \sum_{u,v} \mu(\hat{0},u)\zeta(u,v)\mu(v,\hat{1}),$$

where u,v range over all pairs of complements of t. Deduce that if $\mu(\hat{0},\hat{1}) \neq 0$, then L is complemented.

93. **a.** [2] Let L be a finite lattice such that for every $t > \hat{0}$, the interval $[\hat{0},t]$ has even cardinality. Use Exercise 3.92 to show that L is complemented.

b. [3–] Find a simple proof that avoids Möbius functions.

94. [2+] Let $L = J(P)$ be a finite distributive lattice. A function $v\colon L \to \mathbb{C}$ is called a *valuation* (over $\mathbb{C}$) if $v(\hat{0}) = 0$ and $v(s) + v(t) = v(s \wedge t) + v(s \vee t)$ for all $s,t \in L$. Prove that v is uniquely determined by its values on the join-irreducibles of L (which we may identify with P). More precisely, show that if I is an order ideal of P, then

$$v(I) = -\sum_{t \in I} v(t)\mu(t,\hat{1}),$$

where μ denotes the Möbius function of I (considered as a subposet of P) with a $\hat{1}$ adjoined.

95. [3–] Let L be a finite lattice and fix $z \in L$. Show that the following identity holds in the Möbius algebra of L (over some field):

$$\sum_{t \in L} \mu(\hat{0},t)t = \left(\sum_{u \leq z} \mu(\hat{0},u)u \right) \cdot \left(\sum_{v \wedge z = \hat{0}} \mu(\hat{0},v)v \right).$$

96. **a.** [3–] Let L be a finite lattice (or meet-semilattice), and let $f(s,v)$ be a function (say with values in a commutative ring) defined for all $s,v \in L$. Set $F(s,v) = \sum_{u \leq s} f(u,v)$. Show that

$$\det [F(s \wedge t, s)]_{s,t \in L} = \prod_{s \in L} f(s,s).$$

b. [2] Deduce that

$$\det [\gcd(i,j)]_{i,j=1}^{n} = \prod_{k=1}^{n} \phi(k),$$

where ϕ is the Euler totient (or ϕ) function.

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c. [2] Choose $f(s,v) = \mu(\hat{0},s)$ to deduce that if L is a finite meet-semilattice such that $\mu(\hat{0},s) \neq 0$ for all $s \in L$, then there exists a permutation $w\colon L \to L$ satisfying $s \wedge w(s) = \hat{0}$ for all $s \in L$.

d. [2] Let L be a finite geometric lattice of rank n with W_i elements of rank i. Deduce from (c) (more precisely, the dualized form of (c)) that for $k \leq n/2$,

$$W_1 + \cdots + W_k \leq W_{n-k} + \cdots + W_{n-1}. \tag{3.118}$$

In particular, $W_1 \leq W_{n-1}$.

e. [3–] If equality holds in equation (3.118) for any one value of k, then show that L is modular.

f. [5] With L as in (d), show that $W_k \leq W_{n-k}$ for all $k \leq n/2$.

97. [3–] Let L be a finite lattice such that $\mu(t,\hat{1}) \neq 0$ and $\mu(\hat{0},t) \neq 0$ for all $t \in L$. Prove that there is a permutation $w\colon L \to L$ such that for all $t \in L$, t and $w(t)$ are complements. Show that this conclusion is false if one merely assumes that $\mu(\hat{0},t) \neq 0$ for all $t \in L$.

98. [2+]* Let L be a finite geometric lattice, and let t be a coatom of L. Let $\eta(t)$ be the number of atoms $s \in L$ satisfying $s \not\leq t$. Show that

$$|\mu(\hat{0},\hat{1})| \leq |\mu(\hat{0},t)| \cdot \eta(t).$$

99. [2+]* Let L be a finite geometric lattice of rank n. Let $L' = L - \{\hat{0}\}$, and let $f : L' \to A$ be a function from L' to the set A of atoms of L satisfying $f(t) \leq t$ for all $t \in L'$. Let $\alpha(L,f)$ be the number of maximal chains $\hat{0} = t_0 < t_1 < \cdots < t_n = \hat{1}$ of L such that

$$f(t_1) \vee \cdots \vee f(t_n) = \hat{1}.$$

Show that $\alpha(L,f) = (-1)^n \mu(\hat{0},\hat{1})$.

100. Let L be a finite geometric lattice.

a. [2] Show that every element of L is a meet of coatoms (where we regard $\hat{1}$ as being the meet of the empty set of coatoms).

b. [2] Show that Proposition 3.10.1 has the following improvement for geometric lattices: the Möbius function of L *strictly* alternates in sign. In other words, if $s \leq t$ in L then $(-1)^{\rho(t)-\rho(s)}\mu(s,t) > 0$.

c. [2+] Show that if $\mu(s,t) = \pm 1$, then the interval $[s,t]$ is a boolean algebra.

d. [3–]* Let $n \in \mathbb{P}$. Show that there exist finitely many geometric lattices $L_1,\dots,L_k$ such that if L is any finite geometric lattice satisfying $|\mu(\hat{0},\hat{1})| = n$, then $L \cong L_i \times B_d$ for some i and d.

101. a. [3] Let L be a finite lattice and A,B subsets of L. Suppose that for all $t \notin A$ there exists $t^* > t$ such that $\mu(t,t^*) \neq 0$ and $t^* \neq t \vee u$ whenever $u \in B$. (Thus, $\hat{1} \in A$.) Show that there exists an injective map $\phi\colon B \to A$ satisfying $\phi(s) \geq s$ for all $s \in B$.

b. [2+] Let K be a finite modular lattice. Show the following: (i) If $\hat{1}$ is a join of atoms of K, then K is a geometric lattice and hence $\mu(\hat{0},\hat{1}) \neq 0$. (ii) With K as in (i), K has the same number of atoms as coatoms. (iii) For any $a,b \in K$, the map $\psi_b\colon [a \wedge b, a] \to [b, a \vee b]$ defined by $\psi_b(t) = t \vee b$ is a lattice (or poset) isomorphism.

c. [2+] Let L be a finite modular lattice, and let J_k (respectively, M_k) be the set of elements of L that cover (respectively, are covered by) at most k elements. (Thus, $J_0 = \{\hat{0}\}$ and $M_0 = \{\hat{1}\}$.) Deduce from (a) and (b) the existence of an injective map $\phi\colon J_k \to M_k$ satisfying $\phi(s) \geq s$ for all $s \in J_k$.

d. [2–] Deduce from (c) that the number of elements in L covering exactly k elements equals the number of elements covered by exactly k elements.

e. [2] Let P_k be the subposet of elements of L that cover k elements, and let R_k be the subposet of elements that are covered by k elements. Show by example that we need not have $P_k \cong R_k$, unlike the situation for distributive lattices (Exercise 3.38).

f. Deduce Exercise 3.96(d) from (a).

102. a. [5] Let L be a finite lattice with n elements. Does there exist a join-irreducible t of L such that the principal dual order ideal $V_t := \{s \in L : s \geq t\}$ has at most $n/2$ elements?

b. [2+] Let L be any finite lattice with n elements. Suppose that there is a $t \neq \hat{0}$ in L such that $\#V_t > n/2$. Show that $\mu(\hat{0}, s) = 0$ for some $s \in L$.

103. [3] Let L be a finite lattice, and suppose that L contains a subset S of cardinality n such that (i) any two elements of S are incomparable (i.e., S is an antichain), and (ii) every maximal chain of L meets S. Find, as a function of n, the smallest and largest possible values of $\mu(\hat{0}, \hat{1})$. For instance, if $n = 2$, then $0 \leq \mu(\hat{0}, \hat{1}) \leq 1$, while if $n = 3$ then $-1 \leq \mu(\hat{0}, \hat{1}) \leq 2$.

104. a. [3–] Let P be an $(n+2)$-element poset with $\hat{0}$ and $\hat{1}$. What is the largest possible value of $|\mu(\hat{0}, \hat{1})|$?

b. [5] Same as (a) for n-element lattices L.

105. [5–] Let $k, \ell \in \mathbb{P}$. Find $\max_P |\mu(\hat{0}, \hat{1})|$, where P ranges over all finite posets with $\hat{0}$ and $\hat{1}$ and longest chain of length ℓ, such that every element of P is covered by at most k elements.

106. [2+] Let L be a finite lattice for which $|\mu_L(\hat{0}, \hat{1})| \geq 2$. Does it follow that L contains a sublattice isomorphic to the 5-element lattice $\mathbf{1} \oplus (\mathbf{1}+\mathbf{1}+\mathbf{1}) \oplus \mathbf{1}$?

107. [3–] Let $k \geq 0$, and let I be an order ideal of the boolean algebra B_n. Suppose that for any $t \in I$ of rank at most k, we have $\displaystyle\sum_{\substack{u \in I \\ u \geq t}} \mu(t, u) = 0$. Show that $\#I$ is divisible by 2^{k+1}.

108. Let G be a (simple) graph with finite vertex set V and edge set $E \subseteq \binom{V}{2}$. Write $p = \#V$. An *n-coloring* of G (sometimes called a *proper* n-coloring) is a function $f : V \to [n]$ such that $f(a) \neq f(b)$ if $\{a, b\} \in E$. Let $\chi_G(n)$ be the number of n-colorings of G. The function $\chi_G : \mathbb{N} \to \mathbb{N}$ is called the *chromatic polynomial* of G.

a. [2–]* A *stable partition* of V is a partition π of V such that every block B of π is stable (or independent), that is, no two vertices of B are adjacent. Let $S_G(j)$ be the number of stable partitions of V with k blocks. Show that

$$\chi_G(n) = \sum_j S_G(j)(n)_j.$$

Deduce that $\chi_G(n)$ is a monic polynomial in n of degree p with integer coefficients. Moreover, the coefficient of n^{p-1} is $-(\#E)$.

b. [2+] A set $A \subseteq V$ is *connected* if the induced subgraph on A is connected (i.e., for any two vertices $v, v' \in A$ there is a path from v to v' using only vertices in A). Let L_G be the poset (actually a geometric lattice) of all partitions π of V ordered by refinement, such that every block of V is connected. Show that

$$\chi_G(n) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) n^{\#\pi},$$

where $\#\pi$ is the number of blocks of π and μ is the Möbius function of L_G. It follows that the chromatic polynomial $\chi_G(n)$ and characteristic polynomial $\chi_{L_G}(n)$ are related by $\chi_G(n) = n^c \chi_{L_G}(n)$, where c is the number of connected components of G. Note that when G is the complete graph K_p (i.e., $E = \binom{V}{2}$), then we obtain equation (3.38).

c. [2+] Let $\mathcal{B}_G$ be the hyperplane arrangement in $\mathbb{R}^p$ with hyperplanes $x_i = x_j$ whenever $\{i, j\} \in E$. We call $\mathcal{B}_G$ a *graphical arrangement*. Show that $L_G \cong L(\mathcal{B}_G)$ (the intersection poset of $\mathcal{B}$). Deduce that $\chi_G = \chi_{\mathcal{B}_G}$.

d. [2+] Let e be an edge of G. Let $G - e$ (also denoted $G \backslash e$) denote G with e deleted, and let G/e denote G with e contracted to a point, and all resulting multiple edges replaced by a single edge (so that G/e is simple). Deduce from (c) and Proposition 3.11.5 that

$$\chi_G(n) = \chi_{G-e}(n) - \chi_{G/e}(n). \tag{3.119}$$

Give also a direct combinatorial proof.

e. [2+] Let $\varphi\colon \mathbb{Q}[n] \to \mathbb{Q}[x]$ be the $\mathbb{Q}$-linear function defined by $\varphi(n^k) = \sum_j S(k, j) x^j$, where $S(k, j)$ denotes a Stirling number of the second kind. Show that

$$\varphi(\chi_G(n)) = \sum_j S_G(j) x^j. \tag{3.120}$$

In particular, if B_G denotes the total number of stable partitions of G (a G-analogue of the Bell number $B(n)$), then we have the “umbral” formula $\chi_G(B) = B_G$. That is, expand $\chi_G(B)$ as a polynomial in B (regarding B as an indeterminate), and then replace B^k by $B(k)$.

109. Preserve the notation of the previous exercise. Let $\mathrm{ao}(G)$ denote the number of acyclic orientations of G, as defined in Exercise 3.60.

a. [2+] Use equation (3.119) to prove that

$$\mathrm{ao}(G) = (-1)^p \chi_G(-1). \tag{3.121}$$

b. [2+] Give another proof of equation (3.121) using Theorem 3.11.7.

110. [3] Let $w \in \mathfrak{S}_n$, and let $\mathcal{A}_w$ be the arrangement in $\mathbb{R}^n$ determined by the equations $x_i = x_j$ for all inversions (i, j) of w.

a. Show that $r(\mathcal{A}_w) \geq \#\Lambda_w$, where Λ_w is the principal order ideal generated by w in the Bruhat order on $\mathfrak{S}_n$ (as defined in Exercise 3.183).

b. Show that equality holds in (a) if and only if w avoids all the patterns 4231, 35142, 42513, and 351624.

111. [3–] Give a bijective proof that the number of regions of the Shi arrangement $\mathcal{S}_n$ is $(n+1)^{n-1}$ (Corollary 3.11.14).

112. A sequence $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ of arrangements is called an *exponential sequence of arrangements* (ESA) if it satisfies the following three conditions.

- $\mathcal{A}_n$ is in K^n for some field K (independent of n).
- Every $H \in \mathcal{A}_n$ is parallel to some hyperplane H' in the braid arrangement $\mathcal{B}_n$ (over K).
- Let S be a k-element subset of $[n]$, and define

$$\mathcal{A}_n^S = \{H \in \mathcal{A}_n \,:\, H \text{ is parallel to } x_i - x_j = 0 \text{ for some } i, j \in S\}.$$

Then $L(\mathcal{A}_n^S) \cong L(\mathcal{A}_k)$.

a. [1+]* Show that the braid arrangements $(\mathcal{B}_1, \mathcal{B}_2, \ldots)$ and Shi arrangements $(\mathcal{S}_1, \mathcal{S}_2, \ldots)$ form ESAs.

b. [3–] Let $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ be an ESA. Show that

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(x)\frac{z^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{z^n}{n!}\right)^{-x}.$$

c. [3–] Generalize (b) as follows. For $n \geq 1$ let $\mathcal{A}_n$ be an arrangement in $\mathbb{R}^n$ such that every $H \in \mathcal{A}_n$ is parallel to a hyperplane of the form $x_i = cx_j$, where $c \in \mathbb{R}$. Just as in (b), define for every subset S of $[n]$ the arrangement

$$\mathcal{A}_n^S = \{H \in \mathcal{A}_n : H \text{ is parallel to some } x_i = cx_j, \text{ where } i, j \in S\}.$$

Suppose that for every such S we have $L_{\mathcal{A}_n^S} \cong L_{\mathcal{A}_k}$, where $k = \#S$. Let

$$F(z) = \sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{z^n}{n!}$$

$$G(z) = \sum_{n\geq 0}(-1)^{\operatorname{rank}(\mathcal{A}_n)} b(\mathcal{A}_n)\frac{z^n}{n!}.$$

Show that

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(x)\frac{z^n}{n!} = \frac{G(z)^{(x+1)/2}}{F(z)^{(x-1)/2}}.$$

113. [2] Use the finite field method (Theorem 3.11.10) to give a proof of the Deletion-Restriction recurrence (Proposition 3.11.5) for arrangements defined over $\mathbb{Q}$.

114. For the arrangements $\mathcal{A}$ below (all in $\mathbb{R}^n$), show that the characteristic polynomials are as indicated.

a. [2–]* $x_i = x_j$ for $1 \leq i < j \leq n$ and $x_i = 0$ for $1 \leq i \leq n$. Then

$$\chi_{\mathcal{A}}(x) = (x-1)^2(x-2)(x-3)\cdots(x-n+1).$$

b. [2+]* $x_i = x_j$ for $1 \leq i < j \leq n$ and $x_1 + x_2 + \cdots + x_n = 0$. Then

$$\chi_{\mathcal{A}}(x) = (x-1)^2(x-2)(x-3)\cdots(x-n+1).$$

c. [3–]* $x_i = 2x_j$ and $x_i = x_j$ for $1 \leq i < j \leq n$, and $x_i = 0$ for $1 \leq i \leq n$. Then

$$\chi_{\mathcal{A}}(x) = (x-1)(x-n-1)^{n-1}.$$

115. For the arrangements $\mathcal{A}$ below (all in $\mathbb{R}^n$), show that the characteristic polynomials are as indicated.

a. [2+] The *Catalan arrangement* $\mathcal{C}_n$: $x_i - x_j = -1, 0, 1,\ \ 1 \leq i < j \leq n$. Then

$$\chi_{\mathcal{C}_n}(x) = x(x-n-1)(x-n-2)(x-n-3)\cdots(x-2n+1).$$

b. [3] The *Linial arrangement* $\mathcal{L}_n$: $x_i - x_j = 1,\ \ 1 \leq i < j \leq n$. Then

$$\chi_{\mathcal{L}_n}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(x-k)^{n-1}. \tag{3.122}$$

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c. [3–] The *threshold arrangement* $\mathcal{T}_n$: $x_i + x_j = 0, 1,\ \ 1 \le i < j \le n$. Then

$$\sum_{n \ge 0} \chi_{\mathcal{T}_n}(x) \frac{z^n}{n!} = (1+z)(2e^z - 1)^{(x-1)/2}.$$

d. [2] The *type B braid arrangement* $\mathcal{B}_n^B$: $x_i - x_j = 0,\ x_i + x_j = 0,\ \ 1 \le i < j \le n$, and $x_i = 0,\ \ 1 \le i \le n$. Then

$$\chi_{\mathcal{B}_n^B} = (x-1)(x-3)(x-5)\cdots(x-2n+1).$$

116. [3–] Let $v_1, \dots, v_k$ be "generic" points in $\mathbb{R}^n$. Let $\mathcal{C} = \mathcal{C}(v_1, \dots, v_k)$ be the arrangement consisting of the perpendicular bisectors of all pairs of the points. Thus, $\#\mathcal{C} = \binom{k}{2}$. Find the characteristic polynomial $\chi_{\mathcal{C}}(x)$ and number of regions $r(\mathcal{C})$.

117. [3–] Let $(t_1, \boldsymbol{x}_1), \dots, (t_k, \boldsymbol{x}_k)$ be "generic" events (points) in $(n+1)$-dimensional Minkowski space $\mathbb{R} \times \mathbb{R}^n$ with respect to some reference frame. Assume that the events are spacelike with respect to each other (i.e., there can be no causal connection among them). Suppose that $t_1 < \cdots < t_k$ (i.e., the events occur in the order $1, 2, \dots, k$). In another reference frame moving at a constant velocity $\boldsymbol{v}$ with respect to the first, the events may occur in a different order $a_1 a_2 \cdots a_k \in \mathfrak{S}_k$. What is the number of different orders in which observers can see the events? Express your answer in terms of the signless Stirling numbers $c(n,i)$ of the first kind.
NOTE. Write $\boldsymbol{v} = \tanh(\rho)\boldsymbol{u}$, where $\boldsymbol{u}$ is a unit vector in $\mathbb{R}^n$ and $\tanh(\rho)$ is the speed (with the speed of light $c = 1$). Part of the Lorentz transformation states that the coordinates $(t, \boldsymbol{x})$ and $(t', \boldsymbol{x}')$ of the two frames are related by

$$t' = \cosh(\rho)t - \sinh(\rho)\boldsymbol{x} \cdot \boldsymbol{u}. \tag{3.123}$$

118. a. [3+] Let $\mathcal{A} = \{H_1, \dots, H_\nu\}$ be a linear arrangement of hyperplanes in $\mathbb{R}^d$ with intersection lattice $L(\mathcal{A})$. Let $r = d - \dim(H_1 \cap \cdots \cap H_\nu) = \operatorname{rank} L(\mathcal{A})$. Define

$$\begin{aligned}\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathcal{A}) = \{\boldsymbol{p} = (p_1, \dots, p_d) \,:\, p_i \in \mathbb{R}[x_1, \dots, x_d],\ &\text{and for all } i \in [\nu] \\ &\text{and } \boldsymbol{\alpha} \in H_i \text{ we have } \boldsymbol{p}(\boldsymbol{\alpha}) \in H_i\}.\end{aligned}$$

Clearly, $\boldsymbol{\Omega}$ is a module over the ring $R = \mathbb{R}[x_1, \dots, x_d]$, that is, if $\boldsymbol{p} \in \boldsymbol{\Omega}$ and $q \in R$, then $q\boldsymbol{p} \in \boldsymbol{\Omega}$. One easily shows that $\boldsymbol{\Omega}$ has rank r, that is, $\boldsymbol{\Omega}$ contains r (and no more) elements linearly independent over R. Suppose that $\boldsymbol{\Omega}$ is a *free* R-module—that is, we can find $\boldsymbol{p}_1, \dots, \boldsymbol{p}_r \in \boldsymbol{\Omega}$ such that $\boldsymbol{\Omega} = \boldsymbol{p}_1 R \oplus \cdots \oplus \boldsymbol{p}_r R$. (The additional condition that $\boldsymbol{p}_i R \cong R$ as R-modules is automatic here.) We then call $\mathcal{A}$ a *free* arrangement. It is easy to see that we can choose each $\boldsymbol{p}_i$ so that all its components are homogeneous of the same degree e_i. Show that the characteristic polynomial of $L(\mathcal{A})$ is given by

$$\chi_{L(\mathcal{A})}(x) = \prod_{i=1}^{r} (x - e_i).$$

b. [3] Show that $\boldsymbol{\Omega}$ is free if L is supersolvable, and find a free $\boldsymbol{\Omega}$ for which L is not supersolvable.

c. [3] For $n \ge 3$ let $H_1, \dots, H_\nu$ $(\nu = \binom{n}{2} + \binom{n}{3})$ be defined by the equations

$$x_i = x_j,\ \ 1 \le i < j \le n,$$
$$x_i + x_j + x_k = 0,\ \ 1 \le i < j < k \le n.$$

Is $\boldsymbol{\Omega}$ free?

d. [5] Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are two linear hyperplane arrangements in $\mathbb{R}^d$ with corresponding modules $\mathbf{\Omega}$ and $\mathbf{\Omega}'$. If $L(\mathcal{A}) \cong L(\mathcal{A}')$ and $\mathbf{\Omega}$ is free, does it follow that $\mathbf{\Omega}'$ is free? In other words, is freeness a property of $L(\mathcal{A})$ alone, or does it depend on the actual position of the hyperplanes?

e. [3] Let $\mathcal{A} = \{H_1, \dots, H_\nu\}$ as in (a), and let $t \in L(\mathcal{A})$. With $\mathcal{A}_t$ as in equation (3.43), show that if $\mathbf{\Omega}(\mathcal{A})$ is free, then $\mathbf{\Omega}(\mathcal{A}_t)$ is free.

f. [3] Continuing (e), let $\mathcal{A}^t$ be as in equation (3.44). Give an example where $\mathbf{\Omega}(\mathcal{A})$ is free but $\mathbf{\Omega}(\mathcal{A}^t)$ is not free.

119. [2] Let V be an n-dimensional vector space over $\mathbb{F}_q$, and let L be the lattice of subspaces of V. Let X be a vector space over $\mathbb{F}_q$ with x vectors. By counting the number of injective linear transformations $V \to X$ in two ways (first way – direct, second way – Möbius inversion on L) show that

$$\prod_{k=0}^{n-1}(x - q^k) = \sum_{k=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-1)^k q^{\binom{k}{2}} x^{n-k}.$$

This is an identity valid for infinitely many x and hence valid as a polynomial identity (with x an indeterminate). Note that if we substitute $-1/x$ for x then we obtain equation (1.87) (the q-binomial theorem).

120. a. [3–] Let P be a finite graded poset of rank n, and let $q \geq 2$. Show that the following two conditions are equivalent:

- For every interval $[s,t]$ of length k we have $\mu(s,t) = (-1)^k q^{\binom{k}{2}}$.
- For every interval $[s,t]$ of length k and all $0 \leq i \leq k$, the number of elements of $[s,t]$ of rank i (where the rank is computed in $[s,t]$, not in P) is equal to the q-binomial coefficient $\binom{\boldsymbol{k}}{\boldsymbol{i}}$ (evaluated at the positive integer q).

b. [5–] Is it true that for n sufficiently large, such posets P must be isomorphic to $B_n(q)$ (the lattice of subspaces of $\mathbb{F}_q^n$)?

121. [3–] Fix $k \geq 2$. Let L_n' be the poset of all subsets S of $[n]$, ordered by inclusion, such that S contains no k consecutive integers. Let L_n be L_n with a $\hat{1}$ adjoined. Let μ_n denote the Möbius function of L_n. Find $\mu_n(\emptyset, \hat{1})$. Your answer should depend only on the congruence class of n modulo $2k+2$.

122. [2] A positive integer d is a *unitary divisor* of n if $d|n$ and $(d, n/d) = 1$. Let L be the poset of all positive integers with $a \leq b$ if a is a unitary divisor of b. Describe the Möbius function of L. State a unitary analogue of the classical Möbius inversion formula of number theory.

123. a. [2+] Let M be a monoid (semigroup with identity ε) with generators $g_1, \dots, g_n$ subject only to relations of the form $g_i g_j = g_j g_i$ for certain pairs $i \neq j$. Order the elements of M by $s \leq t$ if there is a u such that $su = t$. For instance, suppose that M has generators $1,2,3,4$ (short for $g_1, \dots, g_4$) with relations

$$13 = 31,\ 14 = 41,\ 24 = 42.$$

Then the interval $[\varepsilon, 11324]$ is shown in Figure 3.52. Show that any interval $[\varepsilon, w]$ in M is a distributive lattice L_w, and describe the poset P_w for which $L_w = J(P_w)$.

b. [1+] Deduce from (a) that the number of factorizations $w = g_{i_1} \cdots g_{i_\ell}$ is equal to the number $e(P_w)$ of linear extensions of P_w.

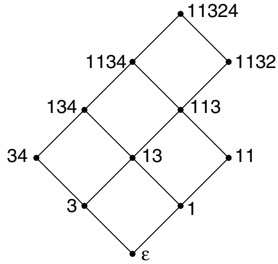

Figure 3.52 The distributive lattice L_{11324} when $13=31$, $14=41$, $24=42$.

c. [2–] Deduce from (a) that the Möbius function of M is given by

$$\mu(s,su) = \begin{cases} (-1)^r, & \text{if } u \text{ is a product of } r \text{ distinct} \\ & \text{pairwise commuting } g_i, \\ 0, & \text{otherwise.} \end{cases}$$

d. [2] Let $N(a_1,a_2,\dots,a_n)$ denote the number of *distinct* elements of M of degree a_i in g_i. (E.g., $g_1^2g_2g_1g_4^2$ has $a_1=3$, $a_2=1$, $a_3=0$, $a_4=2$.) Let $x_1,\dots,x_n$ be independent (commuting) indeterminates. Deduce from (c) that

$$\sum_{a_1\geq 0}\cdots\sum_{a_n\geq 0} N(a_1,\dots,a_n)x_1^{a_1}\cdots x_n^{a_n} = \left(\sum(-1)^r x_{i_1}x_{i_2}\cdots x_{i_r}\right)^{-1},$$

where the last sum is over all $(i_1,i_2,\dots,i_r)$ such that $1\leq i_1<i_2<\cdots<i_r\leq n$ and $g_{i_1},g_{i_2},\dots,g_{i_r}$ pairwise commute.

e. [2–] What identities result in (d) when no g_i and g_j commute ($i\neq j$), or when all g_i and g_j commute?

124. Let L be a finite supersolvable semimodular lattice, with M-chain $C:\hat{0}=t_0<t_1<\cdots<t_n=\hat{1}$.

a. [3–] Let a_i be the number of atoms s of L such that $s\leq t_i$ but $s\not\leq t_{i-1}$. Show that

$$\chi_L(q)=(q-a_1)(q-a_2)\cdots(q-a_n).$$

b. [3–] If $t\in L$ then define

$$\Lambda(t)=\{i\,:\,t\vee t_{i-1}=t\vee t_i\}\subseteq[n].$$

One easily sees that $\#\Lambda(t)=\rho(t)$ and that if u covers t then (in the notation of equation (3.56)) $\Lambda(u)-\Lambda(t)=\{\lambda(t,u)\}$. Now let P be any natural partial ordering of $[n]$ (i.e., if $i<j$ in P, then $i<j$ in $\mathbb{Z}$), and define

$$L_P=\{t\in L\,:\,\Lambda(t)\in J(P)\}.$$

Show that L_P is an R-labelable poset satisfying

$$\beta_{L_P}(S)=\sum_{\substack{w\in\mathcal{L}(P)\\ D(w)=S}}\beta_L(S),$$

where $\mathcal{L}(P)$ denotes the Jordan-Hölder set of P (defined in Section 3.13).

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In particular, taking $L = B_n(q)$ yields from Theorem 3.13.3 a q-analogue of the distributive lattice $J(P)$, satisfying

$$\beta_{L_P}(S) = \sum_{\substack{w \in \mathcal{L}(P) \\ D(w)=S}} q^{\mathrm{inv}(w)}.$$

Note that L_P depends not only on P as an abstract poset, but also on the choice of linear extension P (or maximal chain of $J(P)$) that defines the elements of P as elements of $[n]$.

125. [3–] Let L be a finite graded lattice of rank n. Show that the following two conditions are equivalent:

- L is supersolvable.
- L has an R-labeling for which the label of every maximal chain is a permutation of $1, 2, \ldots, n$.

126. Fix a prime p and integer $k \geq 1$, and define posets $L_k^{(1)}(p)$, $L_k^{(2)}(p)$, and $L_k^{(3)}(p)$ as follows:

- $L_k^{(1)}(p)$ consists of all subgroups of the free abelian group $\mathbb{Z}^k$ that have finite index p^m for some $m \geq 0$, ordered by reverse inclusion.
- $L_k^{(2)}(p)$ consists of all finite subgroups of $(\mathbb{Z}/p^\infty\mathbb{Z})^k$ ordered by inclusion, where

$$\mathbb{Z}/p^\infty\mathbb{Z} = \mathbb{Z}[1/p]/\mathbb{Z},$$
$$\mathbb{Z}[1/p] = \{\alpha \in \mathbb{Q} : p^m\alpha \in \mathbb{Z} \text{ for some } m \geq 0\}.$$

- $L_k^{(3)}(p) = \bigcup_n L_{n,k}(p)$, where $L_{n.k}(p)$ denotes the lattice of subgroups of the abelian group $(\mathbb{Z}/p^n/\mathbb{Z})^k$, and where we regard

$$L_{n,k}(p) \subset L_{n+1,k}(p)$$

via the embedding

$$\left(\mathbb{Z}/p^n\mathbb{Z}\right)^k \hookrightarrow \left(\mathbb{Z}/p^{n+1}/\mathbb{Z}\right)^k$$

defined by

$$(a_1, \ldots, a_k) \mapsto (pa_1, \ldots, pa_k).$$

a. [2+] Show that $L_k^{(1)}(p) \cong L_k^{(2)}(p) \cong L_k^{(3)}(p)$. Calling this poset $L_k(p)$, show that $L_k(p)$ is a locally finite modular lattice with $\hat{0}$ such that each element is covered by finitely many elements (and hence $L_k(p)$ has a rank function $\rho\colon L_k(p) \to \mathbb{N}$).

b. [2–] Show that for any $t \in L_k(p)$, the principal dual order ideal V_t is isomorphic to $L_k(p)$.

c. [3–] Show that $L_k(p)$ has $\binom{n+k-1}{k-1}$ elements of rank n, and hence has rank-generating function

$$F(L_k(p), x) = \frac{1}{(1-x)(1-px)\cdots(1-p^{k-1}x)}.$$

All q-binomial coefficients in this exercise are in the variable p.

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d. [1+] Deduce from (b) and (c) that if $S = \{s_1, s_2, \ldots, s_j\}_< \subset \mathbb{P}$, then

$$\alpha_{L_k(p)}(S) = \binom{s_1 + k - 1}{k - 1}\binom{s_2 - s_1 + k - 1}{k - 1} \cdots \binom{s_j - s_{j-1} + k - 1}{k - 1}.$$

e. [2+] Let N_k denote the set of all infinite words $w = e_1 e_2 \cdots$ such that $e_i \in [0, k-1]$ and $e_i = 0$ for i sufficiently large. Define $\sigma(w) = e_1 + e_2 + \cdots$, and as usual define the descent set

$$D(w) = \{i : e_i > e_{i+1}\}.$$

Use (d) to show that for any finite $S \subset \mathbb{P}$,

$$\alpha_{L_k(p)}(S) = \sum_{\substack{w \in N_k \\ D(w) \subseteq S}} p^{\sigma(w)}$$

$$\beta_{L_k(p)}(S) = \sum_{\substack{w \in N_k \\ D(w) = S}} p^{\sigma(w)}.$$

127. a. [2–]* How many maximal chains does Π_n have?

b. [2+]* The symmetric group $\mathfrak{S}_n$ acts on the partition lattice Π_n in an obvious way. This action induces an action on the set $\mathcal{M}$ of maximal chains of Π_n. Show that the number $\#\mathcal{M}/\mathfrak{S}_n$ of $\mathfrak{S}_n$-orbits on $\mathcal{M}$ is equal to the Euler number E_{n-1}. For instance, when $n = 5$, a set of orbit representatives is given by (omitting $\hat{0}$ and $\hat{1}$ from each chain, and writing e.g. 12-34 for the partition whose non-singleton blocks are $\{1,2\}$ and $\{3,4\}$): 12 < 123 < 1234, 12 < 123 < 123-45, 12 < 12-34 < 125-34, 12 < 12-34 < 12-345, 12 < 12-34 < 1234. Hint. Use Proposition 1.6.2.

c. [2]* Let Λ_n denote the subposet of Π_n consisting of all partitions of $[n]$ satisfying (i) if i is the least element of a nonsingleton block B, then $i + 1 \in B$, and (ii) if $i < n$ and $\{i\}$ is a singleton block, then $\{i + 1\}$ is also a singleton block. Figure 3.53 shows Λ_6, where we have omitted singleton blocks from the labels. Show that the number of maximal chains of Λ_n is E_{n-1}.

d. [2+]* Show that Λ_n is a supersolvable lattice of rank $n-1$. Hence for all $S \subseteq [n-2]$ we have by Example 3.14.4 that $\beta_{\Lambda_n}(S) \geq 0$.

e. [5–] Find an *elegant* combinatorial interpretation of $\beta_{\Lambda_n}(S)$ as the number of alternating permutations in $\mathfrak{S}_{n-1}$ with some property depending on S.

f. [2]* Show that the number of elements of Λ_n whose nonsingleton block sizes are $\lambda_1 + 1, \ldots, \lambda_\ell + 1$ is the number of partitions of a set of cardinality $m = \sum \lambda_i$ whose block sizes are $\lambda_1, \ldots, \lambda_\ell$ (given explicitly by equation (3.36)) provided that $m + \ell \leq n$, and is 0 if $m + \ell > n$. As a corollary, the number of elements of Λ_n of rank k is $\sum_{j=0}^{\min\{k,n-k\}} S(k,j)$, while the total number of elements of Λ_n is

$$\#\Lambda_n = \sum_{\substack{j+k \leq n \\ j \leq k}} S(k,j),$$

including the term $S(0,0) = 1$.

g. [2]* We can identify Λ_n with a subposet of Λ_{n+1} by adjoining a single block $\{n+1\}$ to each $\pi \in \Lambda_n$. Hence we can define $\Lambda = \lim_{n \to \infty} \Lambda_n$. Show that Λ has $B(n)$ elements of rank n, where $B(n)$ denotes a Bell number.

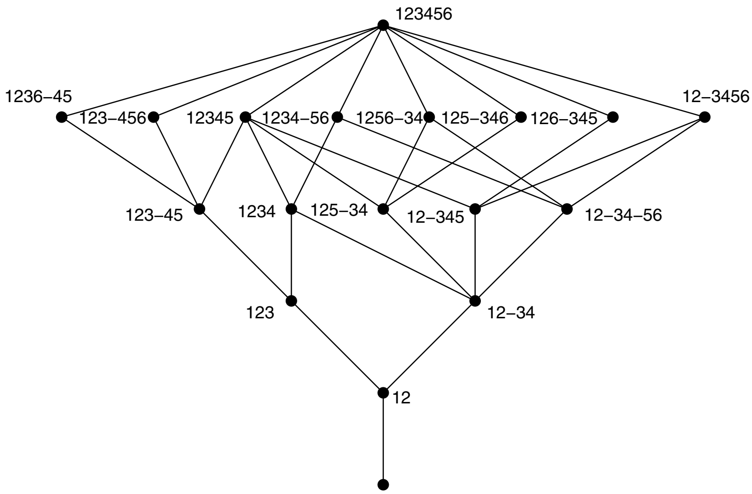

Figure 3.53 The poset Λ_6.

h. [2+]* Write

$$\exp \sum_{i \geq 1} E_i t_i \frac{x^i}{i!} = \sum_{k \geq 0} P_k(t_1, t_2, \ldots, t_k) \frac{x^k}{k!}.$$

Show the the coefficient of $t_1^{\alpha_1} \cdots t_k^{\alpha_k}$ in $P_k(t_1, \ldots, t_k)$ is the number of saturated chains in Λ from $\hat{0}$ to a partition with $\alpha_i + 1$ nonsingleton blocks of cardinality i. Thus, if $M(k, j)$ denotes the number of saturated chains in Λ from $\hat{0}$ to some element of rank k with j nonsingleton blocks, then

$$\sum_{j,k \geq 0} M(k, j) t^j \frac{x^k}{k!} = e^{t(\tan x + \sec x - 1)}.$$

128. [3–]* Let $\pi \in \Pi_n$ have type $(a_1, a_2, \ldots)$, with $\#\pi = \sum a_i = m$. Let $f(\pi)$ be the number of $\sigma \in \Pi_n$ satisfying $\pi \vee \sigma = \hat{1}$, $\pi \wedge \sigma = \hat{0}$, and $\#\sigma = n + 1 - \#\pi$. Show that

$$f(\pi) = 1^{a_1} 2^{a_2} \cdots n^{a_n} (n - m + 1)^{m-2}.$$

129. [2+]* Let P be a finite poset, and let μ be the Möbius function of $\widehat{P} = P \cup \{\hat{0}, \hat{1}\}$. Suppose that P has a fixed-point free automorphism $\sigma : P \to P$ of prime order p (i.e., $\sigma(t) \neq t$ and $\sigma^p(t) = t$ for all $t \in P$). Show that

$$\mu(\hat{0}, \hat{1}) \equiv -1 \,(\mathrm{mod}\, p).$$

What does this say in the case $\widehat{P} = \Pi_p$?

130. Let P be a finite poset satisfying: (i) P is graded of rank n and has a $\hat{0}$ and $\hat{1}$, and (ii) for $0 \leq i \leq n$, there is a poset Q_i such that $[t, \hat{1}] \cong Q_i$ whenever $n - \rho(t) = i$. In particular, $P \cong Q_n$. We call the poset P *uniform*.

a. [2+] Let $V(i, j)$ be the number of elements of Q_i that have rank $i - j$, and let

$$v(i, j) = \sum_t \mu(\hat{0}, t),$$

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where t ranges over all $t \in Q_i$ of rank $i-j$. (Thus, $V(i,j) = W_{i-j}$ and $v(i,j) = w_{i-j}$, where w and W denote the Whitney numbers of Q_i of the first and second kinds, as defined in Section 3.10.) Show that the matrices $[V(i,j)]_{0\le i,j\le n}$ and $[v(i,j)]_{0\le i,j\le n}$ are inverses of one another. (Note that Proposition 1.9.1 corresponds to the case $Q_i = \Pi_{i+1}$.)

b. [5] Find interesting uniform posets. Can all uniform geometric lattices be classified? (See Exercise 3.131(d).)

131. Let X be an n-element set and G a finite group of order m. A *partial partition* of X is a collection $\{A_1,\dots,A_r\}$ of nonempty, pairwise-disjoint subsets of X. A *partial G-partition* of X is a family $\alpha = \{a_1,\dots,a_r\}$ of functions $a_j\colon A_j \to G$, where $\{A_1,\dots,A_r\}$ is a partial partition of X. Define two partial G-partitions $\alpha = \{a_1,\dots,a_r\}$ and $\beta = \{b_1,\dots,b_s\}$ to be *equivalent* if their underlying partial partitions are the same (so $r=s$), say $\{A_1,\dots,A_r\}$, and if for each $1 \le j \le r$, there is some $w \in G$ (depending on j) such that $a_j(t) = w \cdot b_j(t)$ for all $t \in A_j$. Define a poset $Q_n(G)$ as follows. The elements of $Q_n(G)$ are equivalence classes of partial G-partitions. Representing a class by one of its elements, define $\alpha = \{a_1,\dots,a_r\} \le \beta = \{b_1,\dots,b_s\}$ in $Q_n(G)$ if every block A_i of the underlying partial partition $\{A_1,\dots,A_r\}$ of α is either (1) contained in a block B_j of the underlying partial partition σ of β, in which case there is a $w \in G$ for which $a_i(t) = w\cdot b_j(t)$ for all $t \in A_i$, or else (2) every block of σ is disjoint from A_i. (Thus, $Q_n(G)$ has a top element consisting of the empty set.)

a. [2–] Show that if $m=1$, then $Q_n(G) \cong \Pi_{n+1}$.

b. [3–] Show that $Q_n(G)$ is a supersolvable geometric lattice of rank n.

c. [2] Use (b) and Exercise 3.124 to show that the characteristic polynomial of $Q_n(G)$ is given by

$$\chi_{Q_n(G)}(t) = \prod_{i=1}^{n-1}(t-1-mi).$$

d. [2] Show that $Q_n(G)$ is uniform in the sense of Exercise 3.130.

132. [2+] Let P_n be the set of all sets $\{i_1,\dots,i_{2k}\} \subset \mathbb{P}$ where

$$0 < i_1 < i_2 < \cdots < i_{2k} < 2n+1,$$

and $i_1, i_2-i_1, \dots, i_{2k}-i_{2k-1}, 2n+1-i_{2k}$ are all odd. Order the elements of P_n by inclusion. Then P_n is graded of rank n, with $\hat{0}$ and $\hat{1}$. Compute the number of elements of P_n of rank k, the total number of elements of P_n, the Möbius function $\mu(\hat{0},\hat{1})$, and the number of maximal chains of P_n. Show that if $\rho(t)=k$ then $[\hat{0},t] \cong P_k$ while $[t,\hat{1}]$ is isomorphic to a product of P_i's. (Thus P_n^* is uniform in the sense of Exercise 3.130.)

133. Let L_n denote the lattice of all subgroups of the symmetric group $\mathfrak{S}_n$, ordered by inclusion. Let μ_n denote the Möbius function of L_n.

a. [2+] Show that

$$\sum \mu_n(\hat{0}, G) = (-1)^{n-1}(n-1)!,$$

where G ranges over all transitive subgroups of $\mathfrak{S}_n$.

b. [3] Show that $\mu_n(\hat{0},\hat{1})$ is divisible by $n!/2$.

c. [3] Let $\mathcal{C}_n$ denote the collection of transitive proper subgroups of $\mathfrak{S}_n$ that contain an odd involution (i.e., an involution with an odd number of 2-cycles). Show that

$$\mu_n(\hat{0},\hat{1}) = (-1)^{n-1}\frac{n!}{2} - \sum_{H\in\mathcal{C}_n}\mu_n(\hat{0},H).$$

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d. [3–] Let p be prime. Deduce from (c) that

$$\mu_p(\hat{0},\hat{1}) = (-1)^{p-1}\frac{p!}{2}.$$

e. [3–] Let $n = 2^a$ for some positive integer a. Deduce from (c) that

$$\mu_n(\hat{0},\hat{1}) = -\frac{n!}{2}.$$

f. [3] Let p be an odd prime and $n = 2p$. Deduce from (c) that

$$\mu_n(\hat{0},\hat{1}) = \begin{cases} -n!, & \text{if } n-1 \text{ is prime and } p \equiv 3\,(\mathrm{mod}\,4), \\ n!/2, & \text{if } n = 22, \\ -n!/2, & \text{otherwise.} \end{cases}$$

134. a. [3–]* Let A be a finite alphabet and A^* the free monoid generated by A. If $w = a_1a_2\cdots a_n$ is a word in the free monoid A^* with each $a_i \in A$, then a *subword* of w is a word $v = a_{i_1}a_{i_2}\cdots a_{i_k}$ where $1 \le i_1 < i_2 < \cdots < i_k \le n$. Partially order A^* by $u \le v$ if u is a subword of v. We call this partial ordering the *subword order* on A^*. Let μ be the Möbius function of A^*. Given $v = a_1a_2\cdots a_n$ where $a_i \in A$, call the letter a_i *special* if $a_i = a_{i-1}$. Show that

$$\mu(u,v) = (-1)^{\ell(v)-\ell(u)}s(u,v),$$

where $s(u,v)$ is the number of subwords of v isomorphic to u which use every special letter of v. For instance,

$$\mu(aba, aba\underline{a}ba) = -2$$

(where we have underlined the only special letter.) There is a simple proof using Philip Hall's theorem (Proposition 3.8.5), based on a sign-reversing involution acting on a subset of the set of chains of the open interval (u,v).

b. [3–]* Now define $u \le v$ if u is a factor of v, as defined in Example 4.7.7. We call this partial ordering the *factor order* on A^*. Given a word $w = a_1a_2\cdots a_n \in A^*$, $n \ge 2$, define $\iota w = a_2a_3\cdots a_{n-1}$ and φw to be the longest $v \ne w$ (possibly the empty word 1) which is both a left factor and right factor of w (so one can write $w = vw' = w''v$). The word w is *trivial* if $a_1 = a_2 = \cdots = a_n$. Show that the Möbius function of the factor order is determined recursively by

$$\mu(u,v) = \begin{cases} \mu(u,\varphi v), & \text{if } \ell(u,v) > 2 \text{ and } u \le \varphi v \not\le \iota v, \\ 1, & \text{if } \ell(u,v) = 2,\ v \text{ is nontrivial and } u = \iota v \text{ or } u = \varphi v, \\ (-1)^{\ell(u,v)}, & \text{if } \ell(u,v) < 2, \\ 0, & \text{in all other cases.} \end{cases}$$

In particular $\mu(u,v) \in \{0,+1,-1\}$.

135. Let Λ_n denote the set of all $p(n)$ partitions of the integer $n \ge 0$. Order Λ_n by refinement. This means that $\lambda \le \rho$ if the parts of λ can be partitioned into blocks so that the parts of ρ are precisely the sum of the elements in each block of λ. For instance, $(4,4,3,2,2,2,1,1) \le (9,4,4,2)$, corresponding to $9 = 4+2+2+1$, $4 = 4$, $4 = 3+1$, $2 = 2$.

a. [2–]* Show that Λ_n is graded of rank $n-1$.

b. [5] Determine the Möbius function $\mu(\lambda,\rho)$ of Λ_n. (This is trivial when $\lambda = \langle 1^n\rangle$ and easy when $\lambda = \langle 1^{n-2}2^1\rangle$.)

c. [3] Does the Möbius function μ of Λ_n alternate in sign; that is, $(-1)^{\ell}\mu(\lambda,\rho)\geq 0$ if $[\lambda,\rho]$ is an interval of length ℓ? Is Λ_n a Cohen-Macaulay poset?

136. [3] Let Λ_n be as in Exercise 3.135, but now order Λ_n by *dominance*. This means that $(\lambda_1,\lambda_2,\dots)\leq(\rho_1,\rho_2,\dots)$ if $\lambda_1+\lambda_2+\cdots+\lambda_i\leq\rho_1+\rho_2+\cdots+\rho_i$ for all $i\geq 1$. Find μ for this ordering.

137. [2] Let P and Q be finite posets. Express the zeta polynomial values $Z(P+Q,m)$, $Z(P\oplus Q,m)$, and $Z(P\times Q,m)$ in terms of $Z(P,j)$ and $Z(Q,j)$ for suitable values of j.

138. a. [2] Let P be a finite poset and $\mathrm{Int}(P)$ the poset of (nonempty) intervals of P, ordered by inclusion. How are the zeta polynomials $Z(P,n)$ and $Z(\mathrm{Int}(P),n)$ related?

b. [2] Suppose that P has a $\hat{0}$ and $\hat{1}$. Let Q denote $\mathrm{Int}(P)$ with a $\hat{0}$ adjoined. How are $\mu_P(\hat{0},\hat{1})$ and $\mu_Q(\hat{0},\hat{1})$ related?

139. [2+]* Let U_k denote the ordinal sum of k 2-element antichains, so $\#U_k=2k$. Show that

$$\sum_{k\geq 0} Z(U_k,n)x^k=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^n-\frac{1}{2}.$$

140. [2+] Let $\varphi\colon\mathbb{Q}[n]\to\mathbb{Q}[x]$ be the $\mathbb{Q}$-linear function on polynomials with rational coefficients that takes n^k to $\sum_j c_j(k)x^j$, where $c_j(k)=j!S(k,j)$, the number of ordered partitions of $[k]$ into j blocks, or equivalently, the number of surjective functions $[k]\to[j]$ (Example 3.18.9). (Set $\varphi(1)=1$.) Let $Z(P,n)$ denote the zeta polynomial of the poset P. Show that

$$\varphi Z(P,n+2)=\sum_{j\geq 1}c_j(P)x^{j-1},$$

where $c_j(P)$ is the number of j-element chains of P.

141. a. [2] Let P be a finite poset, and let $Q=\mathrm{ch}(P)$ denote the poset of nonempty chains of P, ordered by inclusion. Let Q_0 denote Q with a $\hat{0}$ (the empty chain of P) adjoined. Show that if $Z(P,m+1)=\sum_{i\geq 1}a_i\binom{m-1}{i}$, then $Z(Q_0,m+1)=1+\sum_{i\geq 1}a_i m^i$.

b. [2] Let $\widehat{P}$ and $\widehat{Q}$ denote P and Q, respectively, with a $\hat{0}$ and $\hat{1}$ adjoined. Express $\mu_{\widehat{Q}}(\hat{0},\hat{1})$ in terms of $\mu_{\widehat{P}}(\hat{0},\hat{1})$.

c. [2–] Let P be an Eulerian poset with $\hat{0}$ and $\hat{1}$ removed. Show that $\widehat{Q}$ is Eulerian.

d. [2+] Define $F_n(x)=\sum_{k=1}^{n}k!\,S(n,k)x^{k-1}$, where $S(n,k)$ denotes a Stirling number of the second kind. By letting $E=B_n$ in (c), deduce that

$$F_n(x)=(-1)^{n-1}F_n(-x-1). \tag{3.124}$$

142. a. [2] We say that a finite graded poset P of rank n is *chain-partitionable*, or just *partitionable*, if for every maximal chain K of P there is a chain $r(K)\subseteq K$ (the *restriction* of K) such that every chain (including $\emptyset$) of P lies in exactly one of the intervals $[r(K),K]$ of Q_0. Given a chain C of P, define its *rank set* $\rho(C)=\{\rho(t)\,:\,t\in C\}\subseteq[0,n]$. Show that if P is partitionable, then $\beta(P,S)$ is equal to the number of maximal chains K of P for which $\rho(r(K))=S$. Thus, a necessary condition that P is partitionable is that $\beta(P,S)\geq 0$ for all $S\subseteq[0,n]$.

b. [2+] Show that if P is a poset for which $\widehat{P}:=P\cup\{\hat{0},\hat{1}\}$ is R-labelable, then P is partitionable.

c. [5] Is every Cohen-Macaulay poset partitionable?

143. a. [3–] If P is a poset, then the *comparability graph* $\mathrm{Com}(P)$ is the graph whose vertices are the elements of P, and two vertices s and t are connected by an (undirected) edge if $s < t$ or $t < s$. Show that the order polynomial $\Omega_P(m)$ of a finite poset P depends only on $\mathrm{Com}(P)$.

b. [2] Give an example of two finite posets P, Q for which $\mathrm{Com}(P) \not\cong \mathrm{Com}(Q)$ but $\Omega_P(m) = \Omega_Q(m)$.

144. [2]* Let B_k denote a boolean algebra of rank k, and $\Omega_{B_k}(m)$ its order polynomial. Show that $\Omega_{B_{n+1}}(2) = \Omega_{B_n}(3)$.

145. [2+] Let $\Omega_P(n)$ denote the order polynomial of the finite poset P, so from Section 3.12 we have $\Omega_P(n) = Z(J(P), n)$. Let $p = \#P$. Use Example 3.9.6 to give another proof of the reciprocity theorem for order polynomials (Theorem 3.15.10) in the case of natural labelings, that is, for $n \in \mathbb{P}$, $(-1)^p \Omega_P(-n)$ is equal to the number of strict order-preserving maps $\tau : P \to \boldsymbol{n}$.

146. [1+] Compute $\Omega_P(n)$ and $(-1)^p \Omega_P(-n)$ explicitly when (i) P is a p-element chain, and (ii) P is a p-element antichain.

147. [1+] Compute $Z(L, n)$ when L is the lattice of faces of each of the five Platonic solids.

148. [2]* Let P be a p-element poset. Find a simple expression for $\sum_\omega \Omega_{P,\omega}(n)$, where ω ranges over all $p!$ labelings of P.

149. [3] Let Y be Young's lattice (defined in Section 3.4). Fix $\mu \le \lambda$ in Y, and let $Z(n) = \zeta^n(\mu, \lambda)$ be the zeta polynomial of the interval $[\mu, \lambda]$. Choose r so that $\lambda_{r+1} = 0$, and set $\binom{a}{b} = 0$ if $a < 0$ (in contravention to the usual definition). Show that

$$Z(n+1) = \det\left[\binom{\lambda_i - \mu_j + n}{i - j + n}\right]_{1 \le i,k \le r}.$$

150. a. [3] Let $S = \{a_1, \dots, a_j\}_< \subset \mathbb{P}$. Define $f_S(n)$ to be the number of chains $\lambda^0 < \lambda^1 < \cdots < \lambda^j$ of partitions λ^i in Young's lattice Y such that $\lambda^0 \vdash n$ and $\lambda^i \vdash n + a_i$ for $i \in [j]$. Thus in the notation of Section 3.13, we have $f_S(n) = \alpha_Y(T)$, where $T = \{n, n + a_1, \dots, n + a_j\}$. Set

$$\sum_{n \ge 0} f_S(n) q^n = P(q) A_S(q),$$

where $P(q) = \prod_{i \ge 1}(1 - q^i)^{-1}$. For instance, $A_\emptyset(q) = 1$. Show that $A_S(q)$ is a rational function whose denominator can be taken as

$$\phi_{a_j}(q) = (1-q)(1-q^2)\cdots(1-q^{a_j}).$$

b. [2+] Compute $A_S(q)$ for $S \subseteq [3]$.

c. [3–] Show that for $k \in \mathbb{P}$,

$$\sum_{S \subseteq [k]} (-1)^{k - \#S} A_S(q) = q^{\binom{k+1}{2}} \phi_k(q)^{-1}. \tag{3.125}$$

d. [2+] Deduce from (c) that if $\beta_Y(S)$ is defined as in Section 3.13, then for $k \in \mathbb{N}$ we have

$$\sum_{n \ge 0} \beta_Y([n, n+k]) q^{n+k} = P(q) \sum_{i=0}^{k} \left(\frac{q^{i(i+3)/2}(-1)^{k-i}}{\phi_i(q)} \right) - \frac{(-1)^k}{1-q}.$$

e. [2] Give a simple combinatorial proof that $A_{\{1\}}(q) = (1-q)^{-1}$.

151. a. [2+] Let P be a p-element poset, and let $S \subseteq [p-1]$ such that $\beta_{J(P)}(S) \neq 0$. Show that if $T \subseteq S$, then $\beta_{J(P)}(T) \neq 0$.

b. [5–] Find a "nice" characterization of the collections Δ of subsets of $[p-1]$ for which there exists a p-element poset P satisfying

$$\beta_{J(P)}(S) \neq 0 \Leftrightarrow S \in \Delta.$$

c. [2+] Show that (a) continues to hold if we replace $J(P)$ with any finite supersolvable lattice L of rank p.

152. a. [2+] Let P be a finite naturally labeled poset. Construct explicitly a simplicial complex Δ_P whose faces are the linear extensions of P, such that the dimension of the face w is $\mathrm{des}(w) - 1$. (In particular, the empty face $\emptyset \in \Delta_P$ is the linear extension $12 \cdots p$, where $p = \#P$.)

b. [2] Draw a picture (geometric realization) of Δ_P when P is a four-element antichain.

c. [3] Show that when $P = p\mathbf{1}$ (a p-element antichain), we have $\widetilde{H}_i(\Delta_P; \mathbb{Z}) \neq 0$ if and only if

$$\frac{p-4}{3} \leq i \leq \frac{2p-5}{3},$$

where $\widetilde{H}$ denotes reduced homology.

153. [2+] Let $p \in \mathbb{P}$ and $S \subseteq [p-1]$. What is the least number of linear extensions a p-element poset P can have if $\beta_{J(P)}(S) > 0$?

154. [3–] If L and L' are distributive lattices of rank n such that $\beta_L(S) = \beta_{L'}(S)$ for all $S \subseteq [n-1]$ (or equivalently $\alpha_L(S) = \alpha_{L'}(S)$ for all $S \subseteq [n-1]$), then are L and L' isomorphic?

155. a. [2+] Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$, and suppose that every interval of P is self-dual. Let $S = \{n_1, n_2, \ldots, n_s\}_< \subseteq [n-1]$. Show that $\alpha_P(S)$ depends only on the multiset of numbers $n_1, n_2 - n_1, n_3 - n_2, \ldots, n_s - n_{s-1}, n - n_s$ (not on their order).

b. [2]* Let L be a finite distributive lattice for which every interval is self-dual. Show that L is a product of chains. (For a stronger result, see Exercise 3.166.)

c. [5] Find all finite modular lattices for which every interval is self-dual.

156. [2+] Let $P = \mathbb{N} \times \mathbb{N}$. For any finite $S \subset \mathbb{P}$ we can define $\alpha_P(S)$ and $\beta_P(S)$ exactly as in Section 3.13 (even though P is infinite). Show that if $S = \{m_1, m_2, \ldots, m_s\}_< \subset \mathbb{N}$, then

$$\beta_{\mathbb{N}\times\mathbb{N}}(S) = m_1(m_2 - m_1 - 1) \cdots (m_s - m_{s-1} - 1).$$

157. Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$.

a. [2] Show that

$$\Delta^{k+1} Z(P, 0) = \sum_{\substack{S \subseteq [n-1] \\ \#S = k}} \alpha_P(S).$$

b. [2+] Show that

$$(1-x)^{n+1} \sum_{m \geq 0} Z(P, m) x^m = \sum_{k \geq 0} \beta_k x^{k+1},$$

where

$$\beta_k = \sum_{\substack{S\subseteq[n-1]\\ \#S=k}} \beta_P(S).$$

c. [2] Show that the characteristic polynomial of P is given by $\chi_P(q) = \sum_{k\geq 0} w_k q^{n-k}$, where

$$(-1)^k w_k = \beta_P([k-1]) + \beta_P([k]).$$

(Set $\beta_P([n]) = \beta_P([-1]) = 0$.)

158. **a.** [3–] Let $k,t \in \mathbb{P}$. Let $P_{k,t}$ denote the poset of all partitions π of the set $[kt] = \{1,2,\dots,kt\}$, ordered by refinement (i.e., $P_{k,t}$ is a subposet of Π_{kt}), satisfying the two conditions:

a. Every block of π has cardinality divisible by k.

b. If $a < b < c < d$ and if B and B' are blocks of π such that $a,c \in B$ and $b,d \in B'$, then $B = B'$.

Show combinatorially that the zeta polynomial of $P_{k,t}$ is given by

$$Z(P_{k,t}, n+1) = \frac{((kn+1)t)_{t-1}}{t!}.$$

b. [1+] Note that $P_{k,t}$ always has a $\hat{1}$, and that $P_{1,t}$ has a $\hat{0}$. Use (a) to show that $P_{1,t}$ has C_t elements and that $\mu_{P_{1,t}}(\hat{0},\hat{1}) = (-1)^{t-1}C_{t-1}$, where C_r denotes a Catalan number.

c. [3–] Show that $P_{2,t} \cong \mathrm{Int}(P_{1,t})$, the poset of intervals of $P_{1,t}$..

d. [3] Note that $P_{k,t}$ is graded of rank $t-1$. If $S = \{m_1,\dots,m_s\}_< \subseteq [0,t-2]$, then show thats

$$\alpha_{P_{k,t}}(S) = \frac{1}{t}\binom{t}{m_1}\binom{kt}{m_2 - m_1}\cdots\binom{kt}{m_s - m_{s-1}}\binom{kt}{t-1-m_s}.$$

e. [1] Deduce that $P_{k,t}$ has $\frac{1}{t}\binom{t}{m}\binom{kt}{m-1}$ elements of rank $t-m$ and has $k(kt)!^{t-2}$ maximal chains.

f. [3–] Let $\lambda \vdash t$. Show that the number N_λ of $\pi \in P_{1,t}$ of type λ (i.e., with blocks sizes λ_1, $\lambda_2,\dots$) is given by

$$N_\lambda = \frac{(n)_{\ell(\lambda)-1}}{m_1(\lambda)!\cdots m_n(\lambda)!},$$

where λ has $m_i(\lambda)$ parts equal to i.

g. [2+] Use Exercise 3.125 to show that $P_{1,t}$ is a supersolvable lattice (though not semimodular for $t \geq 4$).

159. [2+] Define a partial order $A(\mathfrak{S}_n)$ on the symmetric group $\mathfrak{S}_n$, called the *absolute order*, as follows. We say that $u < v$ in $A(\mathfrak{S}_n)$ if $v = (i,j)u$ for some transposition (i,j), and if v has fewer cycles (necessarily exactly one less) than u. See Figure 3.54 for the case $n = 4$. Clearly, the maximal elements of $A(\mathfrak{S}_n)$ are the n-cycles, while there is a unique minimal element $\hat{0}$ (the identity permutation). Show that if w is an n-cycle then $[\hat{0}, w] \cong P_{1,n}$, where $P_{1,n}$ is defined in Exercise 3.158.

160. [2] Let P be a p-element poset. Define two labelings $\omega,\omega'\colon P \to [p]$ to be *equivalent* if $\mathcal{A}(P,\omega) = \mathcal{A}(P,\omega')$. Clearly, this definition of equivalence is an equivalence relation. For instance, one equivalence class consists of the natural labelings. Show that the number of equivalence classes is equal to the number of acyclic orientations

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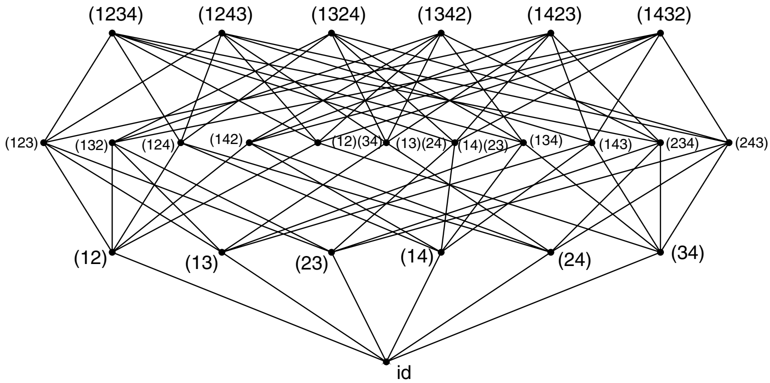

Figure 3.54 The absolute order on $\mathfrak{S}_4$.

of the Hasse diagram $\mathcal{H}$ of P, considered as an undirected graph. (See Exercise 3.109 for further information on the number of acyclic orientations of a graph.)

161. [2] Fix $j,k \geq 1$. Given two permutations $u = u_1 \cdots u_j$ and $v = v_1 \cdots v_k$ of disjoint finite sets U and V of integers, a *shuffle* of u and v is a permutation $w = w_1 \cdots w_{j+k}$ of $U \cup V$ such that u and v are subsequences of w. Let $\mathrm{Sh}(u,v)$ denote the set of shuffles of u and v. For instance, $\mathrm{Sh}(14,26) = \{1426, 1246, 1264, 2146, 2164, 2614\}$. In general, $\#\mathrm{Sh}(u,v) = \binom{j+k}{j}$. Let $S \subseteq [j+k-1]$. Show that the number of permutations in $\mathrm{Sh}(u,v)$ with descent set S depends only on $D(u)$ and $D(v)$ (the descent sets of u and v). (Use the theory of (P,ω)-partitions.)

162. a. [2+] Let P_1, P_2 be disjoint posets with $p_i = \#P_i$. Let ω be a labeling of $P_1 + P_2$ (disjoint union). Let ω_i be the labeling of P_i whose labels are in the same relative order as they are in the restriction of ω to P_i. Show that

$$W_{P_1+P_2,\omega}(x) = \binom{\boldsymbol{p_1+p_2}}{\boldsymbol{p_1}}_x W_{P_1,\omega_1}(x) W_{P_2,\omega_2}(x),$$

where $\binom{\boldsymbol{p_1+p_2}}{\boldsymbol{p_1}}_x$ indicates that the q-binomial coefficient should be taken in the variable x.

b. [2] Let $\{B_1, \ldots, B_k\} \in \Pi_n$, and let w^i be a permutation of B_i. Extending to k permutations the definition of shuffle in Exercise 3.161, define a *shuffle* of $w^1, \ldots, w^k$ to be a permutation $w = a_1 \cdots a_n$ of $[n]$ such that the subword of w consisting of letter from B_i is w^i. For instance 469381752 is a shuffle of 4812, 67, and 935. Let $\mathrm{sh}(w^1, \ldots, w^k)$ denote the set of all shuffles of $w^1, \ldots, w^k$, so if $\#B_i = b_i$ then

$$\#\mathrm{sh}(w^1, \ldots, w^k) = \binom{n}{b_1, \ldots, b_k}.$$

Show that

$$\sum_{w \in \mathrm{sh}(w^1, \ldots, w^k)} x^{\mathrm{maj}(w)} = x^{\alpha} \binom{\boldsymbol{n}}{\boldsymbol{b_1, \ldots, b_k}}_x, \tag{3.126}$$

where $\alpha = \sum_i \mathrm{maj}(w^i)$.

c. [2+]* Deduce from (b) that the minimum value of $\mathrm{maj}(w)$ for $w \in \mathrm{sh}(w^1, \ldots, w^k)$ is equal to $\sum \mathrm{maj}(w^i)$, and that this value is achieved for a unique w. Find an explicit description of this extremal permutation w.

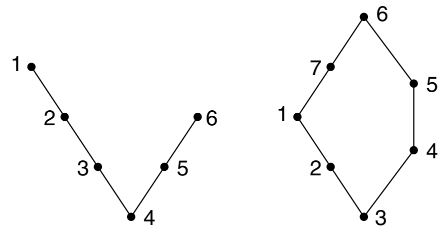

Figure 3.55 Two posets with simple order polynomials.

163. **a.** [2+] Let P be a p-element poset, with order polynomial $\Omega_P(m)$. Show that as $m \to \infty$ (with $m \in \mathbb{P}$), the function $\Omega_P(m)m^{-p}$ is eventually decreasing, and eventually *strictly* decreasing if P is not an antichain.
b. [5] Is the function $\Omega_P(m)m^{-p}$ decreasing for *all* $m \in \mathbb{P}$?

164. [2+] Let P be a finite poset. Does the order polynomial $\Omega_P(m)$ always have nonnegative coefficients?

165. [3+] Let (P,ω) be a finite labeled poset. Does the (P,ω)-Eulerian polynomial $A_{P,\omega}(x)$ have only real zeros? What if ω is a natural labeling?

166. Let $(P,\omega) = \{t_1,\dots,t_p\}$ be a labeled p-element poset. Define the formal power series $G_{P,\omega}(\boldsymbol{x})$ in the variables $\boldsymbol{x} = (x_0, x_1, \dots)$ by

$$G_{P,\omega}(x) = \sum_\sigma x_{\sigma(t_1)} \cdots x_{\sigma(t_p)} = \sum_\sigma x_0^{\#\sigma^{-1}(0)} x_1^{\#\sigma^{-1}(1)} \cdots,$$

where the sums range over all (P,ω)-partitions $\sigma : P \to \mathbb{N}$.
a. [3] Suppose that ω is natural, and write $G_P(\boldsymbol{x})$ for $G_{P,\omega}(\boldsymbol{x})$. Show that $G_P(\boldsymbol{x})$ is a symmetric function (i.e., $G_P(\boldsymbol{x}) = G_P(w\boldsymbol{x})$ for any permutation w of $\mathbb{N}$, where $w\boldsymbol{x} = (x_{w(0)}, x_{w(1)}, \dots)$) if and only if P is a disjoint union of chains.

NOTE. It is easily seen that $G_P(\boldsymbol{x})$ is symmetric if and only if for

$$S = \{n_1, n_2, \dots, n_s\}_< \subseteq [p-1],$$

the number $\alpha_{J(P)}(S)$ depends only on the multiset of numbers n_1, $n_2 - n_1, \cdots$, $n_s - n_{s-1}$, $p - n_s$ (not on their order). See Exercise 3.155.
b. [5] Show that $G_{P,\omega}(\boldsymbol{x})$ is symmetric if and only if P is isomorphic to a (finite) convex subset of $\mathbb{N} \times \mathbb{N}$, labeled so that $\omega(i,j) > \omega(i+1,j)$ and $\omega(i,j) < \omega(i,j+1)$.

167. **a.** [2+]* Let P be a finite poset that is a disjoint union of two chains, with a $\hat{0}$ or $\hat{1}$ or both added. Label P so that the labels i and $i+1$ always occur on two elements that form an edge of the Hasse diagram. This gives a labeled poset (P,ω). Two examples are shown in Figure 3.55.
Show that all $w \in \mathcal{L}(P,\omega)$ have the same number of descents, and as a consequence give an explicit formula for the (P,ω)-order polynomial $\Omega_{P,\omega}(m)$.
b. [3–] Show that if (P,ω) is a labeled poset such that all $w \in \mathcal{L}(P,\omega)$ have the same number of descents, then P is an ordinal sum of the posets of (a), and describe the possible labelings ω.

168. [3]* Let P be a finite naturally labeled poset. Suppose that every connected order ideal I of P is either principal, or else there is a *unique* way to write $I = I_1 \cup I_2$ (up to order) where I_1 and I_2 are connected order ideals properly contained in I. The poset P of Figure 3.56 is an example. Write as usual $(\boldsymbol{i}) = 1 + q + \cdots + q^{i-1}$ and $(\boldsymbol{n})! = (\boldsymbol{1})(\boldsymbol{2})\cdots(\boldsymbol{n})$. Show that

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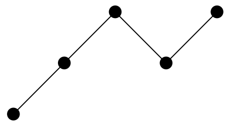

Figure 3.56 A poset P for which $W_P(x)$ has a nice product formula.

$$W_P(x) = (\boldsymbol{n})!\frac{\prod_{\{I_1,I_2\}}(\#\boldsymbol{I_1}+\#\boldsymbol{I_2})}{\prod_I(\#\boldsymbol{I})},$$

where I runs over all *connected* order ideals of P, and $\{I_1,I_2\}$ runs over all pairs of incomparable (in $J(P)$) connected order ideals such that $I_1\cap I_2\neq\emptyset$. For the poset P of Figure 3.56 we get

$$W_P(x) = (\boldsymbol{5})!\frac{(\boldsymbol{6})}{(\boldsymbol{1})(\boldsymbol{1})(\boldsymbol{2})(\boldsymbol{2})(\boldsymbol{4})(\boldsymbol{5})} = 1+x+2x^2+x^3+2x^4+x^5+x^6.$$

169. a. [2] Let $M=\{1^{r_1},2^{r_2},\dots,m^{r_m}\}$ be a finite multiset on $[m]$, and let $\mathfrak{S}_M$ be the set of all $\binom{r_1+\cdots+r_m}{r_1,\dots,r_m}$ permutations $w=(a_1,a_2,\dots a_r)$ of M, where $r=r_1+\cdots+r_m=\#M$. Let $\operatorname{des}(w)$ be the number of descents of w, and set

$$A_M(x) = \sum_{w\in\mathfrak{S}_M} x^{1+\operatorname{des}(w)},$$

$$\overline{A}_M(x) = \sum_{w\in\mathfrak{S}_M} x^{r-\operatorname{des}(w)}.$$

Show that

$$\sum_{n\geq 0}\left(\!\!\binom{n}{r_1}\!\!\right)\left(\!\!\binom{n}{r_2}\!\!\right)\cdots\left(\!\!\binom{n}{r_m}\!\!\right)x^n = \frac{A_M(x)}{(1-x)^{r+1}},$$

$$\sum_{n\geq 0}\binom{n}{r_1}\binom{n}{r_2}\cdots\binom{n}{r_m}x^n = \frac{\overline{A}_M(x)}{(1-x)^{r+1}}.$$

b. [2+] Find the coefficients of $A_M(x)$ explicitly in the case $m=2$.

170. Let us call a finite graded poset P (with rank function ρ) *pleasant* if the rank-generating function $F(L,q)$ of $L=J(P)$ is given by

$$F(L,q) = \prod_{t\in P}\frac{1-q^{\rho(t)+2}}{1-q^{\rho(t)+1}}.$$

In (a)–(g) show that the given posets P are pleasant. (Note that (a) is a special case of (b), and (c) is a special case of (d).)

a. [2] $P=\boldsymbol{m}\times\boldsymbol{n}$, where $m,n\in\mathbb{P}$.
b. [3] $P=\boldsymbol{l}\times\boldsymbol{m}\times\boldsymbol{n}$, where $l,m,n\in\mathbb{P}$.
c. [2] $P=J(\boldsymbol{2}\times\boldsymbol{n})$, where $n\in\mathbb{P}$.
d. [3+] $P=\boldsymbol{m}\times J(\boldsymbol{2}\times\boldsymbol{n})$, where $m,n\in\mathbb{P}$.
e. [3+] $P=J(\boldsymbol{3}\times\boldsymbol{n})$, where $n\in\mathbb{P}$.
f. [2+] $P=\boldsymbol{m}\times(\boldsymbol{n}\oplus(\boldsymbol{1}+\boldsymbol{1})\oplus\boldsymbol{n})$, where $m,n\in\mathbb{P}$.
g. [3–] $P=\boldsymbol{m}\times J(J(\boldsymbol{2}\times\boldsymbol{3}))$ and $P=\boldsymbol{m}\times J(J(J(\boldsymbol{2}\times\boldsymbol{3})))$, where $m\in\mathbb{P}$.
h. [5] Find a reasonable expression for $F(J(P))$, where $P=\boldsymbol{n_1}\times\boldsymbol{n_2}\times\boldsymbol{n_3}\times\boldsymbol{n_4}$ or $P=J(\boldsymbol{4}\times\boldsymbol{n})$. (In general, these posets P are not pleasant.)

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i. [5] Are there any other "nice" classes of connected pleasant posets? Can all pleasant posets be classified?

171. a. [2–] Let (P,ω) be a finite labeled poset and $m \in \mathbb{N}$. Define a polynomial

$$U_{P,\omega,m}(q) = \sum_{\sigma} q^{|\sigma|},$$

where σ ranges over all (P,ω)-partitions $\sigma : P \to [0,m]$. In particular, $U_{P,0}(q) = 1$ (as usual, the suppression of ω from the notation indicates that ω is natural) and $U_{P,\omega,m}(1) = \Omega_{P,\omega}(m+1)$. Show that $U_{P,m}(q) = F(J(\boldsymbol{m} \times P), q)$, the rank-generating function of $J(\boldsymbol{m} \times P)$.

b. [2+] If $\#P = p$ and $0 \le i \le p-1$, then define

$$W_{P,\omega,i}(q) = \sum_{w} q^{\mathrm{maj}(w)}, \tag{3.127}$$

where w ranges over all permutations in $\mathcal{L}(P,\omega)$ with exactly i descents. Note that $W_{P,\omega}(q) = \sum_i W_{P,\omega,i}(q)$. Show that for all $m \in \mathbb{N}$,

$$U_{P,\omega,m}(q) = \sum_{i=0}^{p-1} \binom{\boldsymbol{p+m-i}}{\boldsymbol{p}} W_{P,\omega,i}(q). \tag{3.128}$$

c. [2] Let ω^* be the labeling of the dual P^* defined by $\omega^*(t) = p+1-\omega(t)$. (Note that ω^* and $\overline{\omega}$ have the same values. However, ω^* is a labeling of P^*, while $\overline{\omega}$ is a labeling of P.) Show that

$$W_{P^*,\omega^*,i}(q) = q^{pi} W_{P,\omega,i}(1/q), \tag{3.129}$$

$$U_{P^*,\omega^*,m}(q) = q^{pm} U_{P,\omega,m}(1/q). \tag{3.130}$$

d. [1+]* The formula

$$\binom{\boldsymbol{a}}{\boldsymbol{b}} = \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)(1-q^{b-1})\cdots(1-q)}$$

allows us to define $\binom{\boldsymbol{a}}{\boldsymbol{b}}$ for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Show that

$$\begin{aligned}\binom{\boldsymbol{-a}}{\boldsymbol{b}} &= (-1)^b q^{-b(2a+b-1)/2} \binom{\boldsymbol{a+b-1}}{\boldsymbol{b}} \\ &= (-1)^b q^{\binom{b+1}{2}} \binom{\boldsymbol{a+b-1}}{\boldsymbol{b}}_{1/q}.\end{aligned}$$

e. [2+] Equation (3.96) and part (d) above allow us to define $U_{P,\omega,m}(q)$ for any $m \in \mathbb{Z}$. Show that for $m \in \mathbb{P}$,

$$U_{P,\omega,-m}(q) = (-1)^p \sum_{\tau} q^{-|\tau|},$$

where τ ranges over all $(P,\overline{\omega})$-partitions $\tau : P \to [m-1]$.

f. [3–] If $t \in P$, then let $\delta(t)$ and δ_i, $0 \le i \le \ell = \ell(P)$, be as in Section 3.15.4. Define

$$\Delta_r = \delta_r + \delta_{r+1} + \cdots + \delta_\ell, \quad 1 \le r \le \ell,$$

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and set
$$M(P) = [p-1] - \{\Delta_1, \Delta_2, \dots, \Delta_\ell\}.$$

Show that the degree of $W_{P,i}(q)$ is equal to the sum of the largest i elements of $M(P)$. Note also that if P is graded of rank ℓ, then

$$\Delta_r = \#\{t \in P \,:\, \rho(t) \le \ell - r\}.$$

172. Let $P = \{t_1, \dots, t_p\}$ be a finite poset. We say that P is *Gaussian* if there exist integers $h_1, \dots, h_p > 0$ such that for all $m \in \mathbb{N}$,

$$U_{P,m}(q) = \prod_{i=1}^{p} \frac{1 - q^{m+h_i}}{1 - q^{h_i}}, \tag{3.131}$$

where $U_{P,m}(q)$ is given by Exercise 3.171.

a. [3–] Show that P is Gaussian if and only if every connected component of P is Gaussian.

b. [3–] If P is connected and Gaussian, then show that every maximal chain of P has the same length ℓ. (Thus, P is graded of rank ℓ.)

c. [3] Let P be connected and Gaussian, with rank function ρ (which exists by (b)). Show that the multisets $\{h_1, \dots, h_p\}$ and $\{1 + \rho(t) \,:\, t \in P\}$ coincide.

NOTE. It follows easily from (c) that a finite connected poset P is Gaussian if and only if $P \times \boldsymbol{m}$ is pleasant (as defined in Exercise 3.170) for all $m \in \mathbb{P}$.

d. [2+] Suppose that P is connected and Gaussian, with $h_1, \dots, h_p$ labeled so that $h_1 \le h_2 \le \cdots \le h_p$. Show that $h_i + h_{p+1-i} = \ell(P) + 2$ for $1 \le i \le p$.

e. [2+] Let P be connected and Gaussian. Show that every element of P of rank one covers exactly one minimal element of P.

f. [3+] Show that the following posets are Gaussian:

i. $\boldsymbol{r} \times \boldsymbol{s}$, for all $r, s \in \mathbb{P}$,

ii. $J(2 \times \boldsymbol{r})$, for all $r \in \mathbb{P}$,

iii. the ordinal sum $\boldsymbol{r} \oplus (\mathbf{1} + \mathbf{1}) \oplus \boldsymbol{r}$, for all $r \in \mathbb{P}$,

iv. $J(J(\mathbf{2} \times \mathbf{3}))$,

v. $J(J(J(\mathbf{2} \times \mathbf{3})))$.

g. [5] Are there any other connected Gaussian posets? In particular, must a connected Gaussian poset be a distributive lattice?

173. Let (P, ω) be a labeled poset, and set

$$\mathcal{E}(P) = \{(s,t) \,:\, s \lessdot t\}.$$

Define $\epsilon = \epsilon_\omega \colon \mathcal{E}(P) \to \{-1, 1\}$ by

$$\epsilon(s,t) = \begin{cases} 1, & \omega(s) < \omega(t), \\ -1, & \omega(s) > \omega(t). \end{cases}$$

We say that ϵ is a *sign-grading* if for all maximal chains $t_0 \lessdot t_1 \lessdot \cdots \lessdot t_\ell$ in P the quantity $\sum_{i=1}^{\ell} \epsilon(t_{i-1}, t_i)$ is the same, denoted $r(\epsilon)$ and called the *rank* of ϵ. A labeled poset (P, ω) with a sign-grading ϵ is called a *sign-graded* poset. In that case, we have a *rank function* $\rho = \rho_\epsilon$ given by

$$\rho(t) = \sum_{i=1}^{m} \epsilon(t_{i-1}, t_i),$$

where $t_0 \lessdot t_1 \lessdot \cdots \lessdot t_m = t$ is a saturated chain from a minimal element t_0 to t. (The definition of sign-grading insures that $\rho(t)$ is well-defined.)

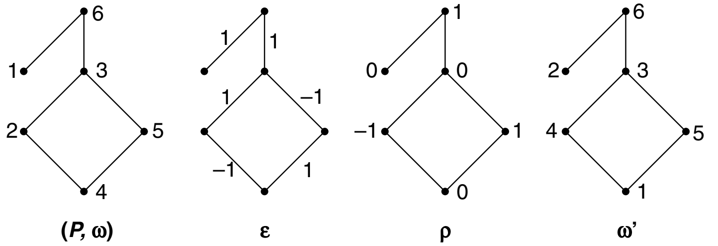

Figure 3.57 A labeling, sign-grading, rank function, and canonical labeling.

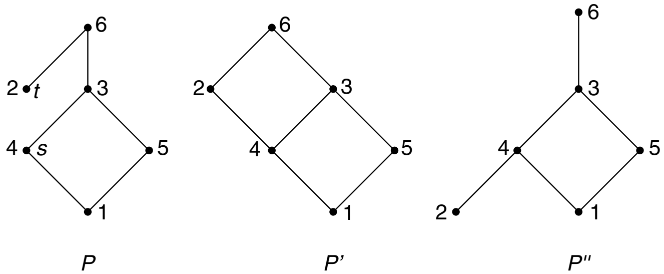

Figure 3.58 The procedure $P \to (P', P'')$.

a. [1+]* Suppose that ω is natural. Show that ϵ is a sign-grading if and only if P is graded.

b. [2]* Show that a finite poset P has a labeling ω for which (P,ω) is a sign-graded poset if and only if the lengths of all maximal chains of P have the same parity.

c. [2+]* Suppose that ω and ω' are labelings of P which both give rise to sign-gradings ϵ and ϵ'. Show that the (P,ω) and (P,ω')-Eulerian polynomials are related by

$$x^{r(\epsilon)/2} A_{P,\omega}(x) = x^{r(\epsilon')/2} A_{P,\omega'}(x).$$

d. [2]* Suppose that (P,ω) is a sign-graded poset whose corresponding rank function ρ takes on only the values 0 and 1, and $\rho(s) < \rho(t)$ implies $\omega(s) < \omega(t)$. We then call ω a *canonical* labeling of P. Show that every sign-graded poset (P,ω) has a canonical labeling. Figure 3.57 shows a poset (P,ω), the sign-grading ϵ_ω, the rank function ρ, and a canonical labeling ω'.

e. [2+]* Let (P,ω) be sign-graded, where ω is a canonical labeling. Let $s \parallel t$ in P, with $\rho(t) = \rho(s) + 1$. Define

$$P' = P \text{ with } s < t \text{ adjoined},$$
$$P'' = P \text{ with } s > t \text{ adjoined}.$$

Thus ω continues to be a labeling of P' and P''. Show that P' and P'' are sign-graded, and that ω is a canonical labeling for both. Figure 3.58 shows an example of this decomposition for the poset of Figure 3.57. We take s to be the element labeled 4 (of rank 1) and t to be the element labeled 2 (of rank 1).

f. [2–]* Show that $\mathcal{L}(P,\omega) = \mathcal{L}(P',\omega) \mathbin{\dot\cup} \mathcal{L}(P'',\omega)$.

g. [2+] Write $A'_j(x) = A_j(x)/x$, where $A_j(x)$ is an Eulerian polynomial. Iterate the procedure $P \to (P', P'')$ as long as possible. Deduce that if (P,ω) is sign-graded,

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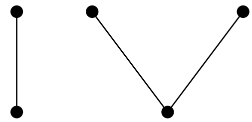


Figure 3.59 A poset for Exercise 3.173(i).

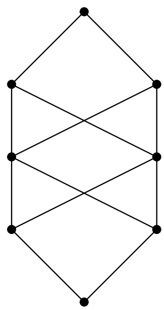


Figure 3.60 The poset $\mathbf{1} \oplus \mathbf{21} \oplus \mathbf{21} \oplus \mathbf{21} \oplus \mathbf{1}$.

then we can write the (P,ω)-Eulerian polynomial $A_{P,\omega}(x)$ as a sum of terms of the form $x^b A'_{a_1}(x) \cdots A'_{a_k}(x)$, where $b \in \mathbb{N}$. Moreover, all these terms are symmetric (in the sense of Exercise 3.50) with the same center of symmetry.

h. [2] Deduce from (g) that the coefficients of $A_{P,\omega}(x)$ are symmetric and unimodal.

i. [2–] Carry out the procedure of (g) for the poset of Figure 3.59, naturally labeled.

174. a. [2–] Show that a finite graded poset with $\hat{0}$ and $\hat{1}$ is semi-Eulerian if and only if for all $s < t$ in P except possibly $(s,t) = (\hat{0},\hat{1})$, the interval $[s,t]$ has as many elements of odd rank as of even rank. Show that P is Eulerian if in addition P has as many elements of odd rank as of even rank.

b. [2] Show that if P is semi-Eulerian of rank n, then

$$(-1)^n Z(P,-m) = Z(P,m) + m((-1)^n \mu_P(\hat{0},\hat{1}) - 1).$$

c. [2] Show that a semi-Eulerian poset of odd rank n is Eulerian.

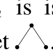
175. [2+] Suppose that P and Q are Eulerian, and let $P' = P - \{\hat{0}\}$, $Q' = Q - \{\hat{0}\}$, $R = (P' \times Q') \cup \{\hat{0}\}$. Show that R is Eulerian.

176. a. [2] Let P_n denote the ordinal sum $\mathbf{1} \oplus \mathbf{21} \oplus \mathbf{21} \oplus \cdots \oplus \mathbf{21} \oplus \mathbf{1}$ (n copies of $\mathbf{21}$). For example, P_3 is shown in Figure 3.60. Compute $\beta_{P_n}(S)$ for all $S \subseteq [n]$.

b. [1+] Use (a) and Exercise 3.157(b) to compute $\sum_{m \geq 0} Z(P_n,m)x^m$.

c. [2+] It is easily seen that P_n is Eulerian. Compute the polynomials $f(P_n,x)$ and $g(P_n,x)$ of Section 3.16.

177. a. [2] Let L_n denote the lattice of faces of an n-dimensional cube, ordered by inclusion. Show that L_n is isomorphic to the poset $\mathrm{Int}(B_n)$ with a $\hat{0}$ adjoined, where B_n denotes a boolean algebra of rank n.

b. [2] Show that L_n is isomorphic to Λ^n with a $\hat{0}$ adjoined, where Λ is the three-element poset .

c. [2] Let P_n be the poset of Exercise 3.176. Show that L_n is isomorphic to the poset of chains of P_n that don't contain $\hat{0}$ and $\hat{1}$ (including the empty chain), ordered by reverse inclusion, with a $\hat{0}$ adjoined.

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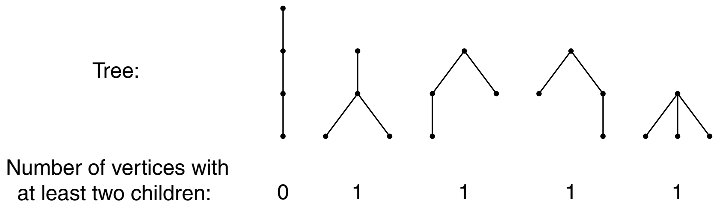

Figure 3.61 Plane trees with four vertices.

d. [3–] Let $S \subseteq [n]$. Show that

$$\beta_{L_n}(S) = \sum_{i=0}^{n} \binom{n}{i} D_{n+1}(\overline{S}, i+1),$$

where $D_m(T, j)$ denotes the number of permutations of $[m]$ with descent set T and last element j, and where $\overline{S} = [n] - S$.

e. [2+] Compute $Z(L_n, m)$.

f. [3] Since L_n is the lattice of faces of a convex polytope, it is Eulerian by Proposition 3.8.9. Compute the polynomial $g(L_n, x)$ of Section 3.16. Show in particular that

$$g(L_n, 1) = \frac{1}{n+1}\binom{2n}{n} \quad \text{and} \quad f(L_n, 1) = 2\binom{2(n-1)}{n-1}.$$

g. [3–] Use (f) to show that $g(L_n, x) = \sum a_i x^i$, where a_i is the number of plane trees with $n+1$ vertices such that exactly i vertices have at least two children. For example, see Figure 3.61 for $n = 3$, which shows that $g(L_3, x) = 1 + 4x$.

178. [2+]* Let $f(n)$ be the total number of chains containing $\hat{0}$ and $\hat{1}$ in the lattice L_n of the previous exercise. Show that

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \frac{e^x}{2 - e^{2x}}.$$

179. [4] Let L be the face lattice of a convex polytope $\mathcal{P}$. Show that the coefficients of $g(L, x)$ are nonnegative. Equivalently (since $f(L, 0) = g(L, 0) = 1$), the coefficients of $f(L, x)$ are nonnegative and *unimodal* (i.e., weakly increase to a maximum and then decrease).

180. **a.** [2+] Let $n, d \in \mathbb{P}$ with $n \geq d+1$. Define L'_{nd} to be the poset of all subsets S of $[n]$, ordered by inclusion, satisfying the following condition: S is contained in a d-subset T of $[n]$ such that whenever $1 \leq i \notin T$, $[i+1, i+k] \subseteq T$, and $n \geq i+k+1 \notin T$, then k is even. Let L_{nd} be L'_{nd} with a $\hat{1}$ adjoined. Show that L_{nd} is an Eulerian lattice of rank $d+1$. The lattice L_{42} is shown in Figure 3.62.

b. [2]* Show that L_{nd} has $\binom{n}{k}$ elements of rank k for $0 \leq k \leq \lfloor d/2 \rfloor$.

181. **a.** [3–] Let $L = L_0 \cup L_1 \cup \cdots \cup L_{d+1}$ be an Eulerian lattice of rank $d+1$. Suppose that the truncation $L_0 \cup L_1 \cup \cdots \cup L_{\lceil (d+1)/2 \rceil}$ is isomorphic to the truncation $M = M_0 \cup M_1 \cup \cdots \cup M_{\lceil (d+1)/2 \rceil}$, where M is a boolean algebra B_n of rank $n = \#L_1$. Does it follow that $n = d+1$ and that $L \cong M$? Note that by Exercise 3.180 this result is best possible (i.e., $\lceil (d+1)/2 \rceil$ cannot be replaced with $\lceil (d-1)/2 \rceil$).

b. [2] What if we only assume that L is an Eulerian poset?

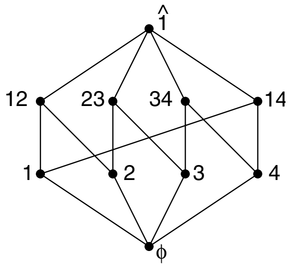

Figure 3.62 The Eulerian lattice L_{42}.

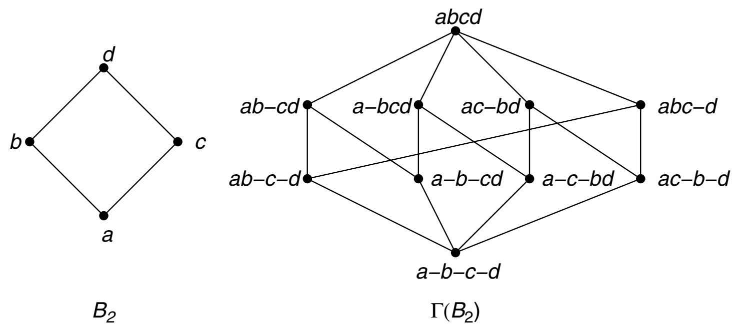

Figure 3.63 The Eulerian lattice Γ_{B_2}.

182. [3–] Let P be a finite poset, and let π be a partition of the elements of P such that every block of π is connected (as a subposet of P). Define a relation $\leq$ on the blocks of π as follows: $B \leq B'$ if for some $t \in B$ and $t' \in B'$ we have $t \leq t'$ in P. If this relation is a partial order, then we say that π is *P-compatible*. Let Γ_P be the set of all P-compatible partitions of P, ordered by refinement (so $\Gamma(P)$ is a subposet of Π_P). See Figure 3.63 for an example. Show that Γ_P is an Eulerian lattice.

183. a. [2–]* Define a partial order, called the *(strong) Bruhat order* on the symmetric group $\mathfrak{S}_n$, by defining its cover relations as follows. We say that w covers v if $w = (i, j)v$ for some transposition (i, j) and if $\mathrm{inv}(w) = 1 + \mathrm{inv}(v)$. For instance, 75618324 covers 73618524; here $(i, j) = (2,6)$. We always let the "default" partial ordering of $\mathfrak{S}_n$ be the Bruhat order, so any statement about the poset structure of $\mathfrak{S}_n$ refers to the Bruhat order. The poset $\mathfrak{S}_3$ is shown in Figure 3.64(a), while the solid and broken lines of Figure 3.65 show $\mathfrak{S}_4$. Show that $\mathfrak{S}_n$ is a graded poset with $\rho(w) = \mathrm{inv}(w)$, so that the rank-generating function is given by $F(\mathfrak{S}_n, q) = (\boldsymbol{n})!$.

b. [3–] Given $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$, define a left-justified triangular array T_w whose ith row consists of $a_1, \ldots, a_i$ written in increasing order. For instance, if $w = 31524$, then

$$T_w = \begin{array}{l} 3 \\ 1\,3 \\ 1\,3\,5 \\ 1\,2\,3\,5 \\ 1\,2\,3\,4\,5. \end{array}$$

Show that $v \leq w$ if and only if $T_v \leq T_w$ (component-wise ordering).

c. [3] Show that $\mathfrak{S}_n$ is Eulerian.

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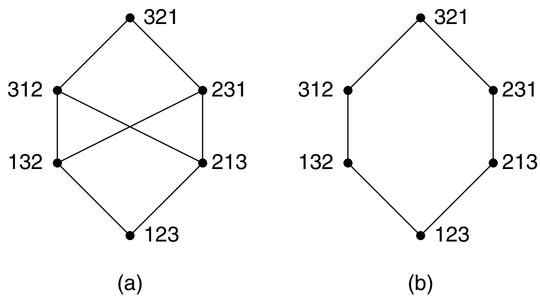

Figure 3.64 The Bruhat order $\mathfrak{S}_3$ and weak order $W(\mathfrak{S}_3)$.

d. [2+] Show that the number of cover relations in $\mathfrak{S}_n$ is $(n+1)!(H_{n+1}-2)+n!$, where $H_{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}$.

e. [2+]* Find the number of elements $w \in \mathfrak{S}_n$ for which the interval $[\hat{0}, w]$ is a boolean algebra. Your answer shouldn't involve any sums or products.

f. [5–] Find the total number of intervals of $\mathfrak{S}_n$ that are boolean algebras.

g. [3+] Let $v \lessdot w$ in $\mathfrak{S}_n$, so $w = (i,j)v$ for some $i<j$. Define the weight $\omega(v,w) = j-i$. Set $r = \binom{n}{2}$. If $C\colon \hat{0} = v_0 \lessdot v_1 \lessdot \cdots \lessdot v_r = \hat{1}$ is a maximal chain of $\mathfrak{S}_n$, then define

$$\omega(C) = \omega(v_0,v_1)\omega(v_1,v_2)\cdots\omega(v_{r-1},v_r).$$

Show that $\sum_C \omega(C) = r!$, where C ranges over all maximal chains of $\mathfrak{S}_n$.

184. [3] Let $\mathfrak{I}_n$ denote the subposet of $\mathfrak{S}_n$ (under Bruhat order) consisting of the involutions in $\mathfrak{S}_n$. Show that $\mathfrak{I}_n$ is Eulerian.

185. a. [2–]* Define a partial order $W(\mathfrak{S}_n)$, called the *weak (Bruhat) order* on $\mathfrak{S}_n$, by defining its cover relations as follows. We say that w covers v if $w = (i,i+1)v$ for some *adjacent* transposition $(i,i+1)$ and if $\mathrm{inv}(w) = 1+\mathrm{inv}(v)$. For instance, 75618325 covers 75613825. The poset $W(\mathfrak{S}_3)$ is shown in Figure 3.64(b), while the solid lines of Figure 3.65 show $W(\mathfrak{S}_4)$. Show that $W(\mathfrak{S}_n)$ is a graded poset with $\rho(w) = \mathrm{inv}(w)$, so that the rank-generating function is given by $F(W(\mathfrak{S}_n),q) = (\boldsymbol{n})!$.

b. [2+] Show that $W(\mathfrak{S}_n)$ is a lattice.

c. [2] Show that the number of cover relations in $W(\mathfrak{S}_n)$ is $(n-1)n!/2$.

d. [3–] Let μ denote the Möbius function of $W(\mathfrak{S}_n)$. Show that

$$\mu(v,w) = \begin{cases} (-1)^k, & \text{if } w \text{ can be obtained from } v \text{ by reversing the elements} \\ & \text{ in each of } k+1 \text{ disjoint increasing factors of } v, \\ 0, & \text{otherwise.} \end{cases}$$

e. [3] Show that the zeta polynomial of $W(\mathfrak{S}_n)$ satisfies

$$Z(W(\mathfrak{S}_n),-j) = (-1)^{n-1}j, \ \ 1 \le j \le n-1.$$

f. [2]* Characterize permutations $w \in W(\mathfrak{S}_n)$ for which the interval $[\hat{0},w]$ is a boolean algebra in terms of pattern avoidance.

g. [2+]* Find the number of elements $w \in W(\mathfrak{S}_n)$ for which the interval $[\hat{0},w]$ is a boolean algebra. Your answer shouldn't involve any sums or products. More generally, find a simple formula for the generating function $\sum_{n\ge 0}\sum_w q^{\mathrm{rank}(w)}x^n$, where w ranges over all elements of $W(\mathfrak{S}_n)$ for which the interval $[\hat{0},w]$ is a boolean algebra.

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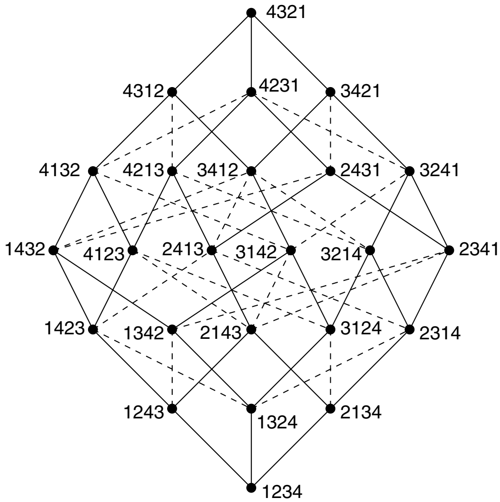


Figure 3.65 The Bruhat order $\mathfrak{S}_4$ and weak order $W(\mathfrak{S}_4)$.

h. [3–]* Let $f(n,i)$ denote the total number of intervals of $W(\mathfrak{S}_n)$ that are isomorphic to the boolean algebra B_i. Show that

$$\begin{aligned}\sum_{n\geq 0}\sum_{i\geq 0} f(n,i)q^i\frac{x^n}{n!} &= \frac{1}{1-x-\frac{qx^2}{2}}\\ &= 1+x+(q+2)\frac{x^2}{2!}+(6q+6)\frac{x^3}{3!}\\ &\quad +(6q^2+36q+24)\frac{x^4}{4!}+(90q^2+240q+120)\frac{x^5}{5!}\\ &\quad (90q^3+1080q^2+1800q+720)\frac{x^6}{6!}+\cdots.\end{aligned}$$

i. [3–]* Find the number of elements $w \in W(\mathfrak{S}_n)$ for which the interval $[\hat{0}, w]$ is a distributive lattice. Your answer shouldn't involve any sums or products.

j. [5–] Find the total number of intervals of $W(\mathfrak{S}_n)$ that are distributive lattices. The values for $1 \leq n \leq 8$ are 1, 2, 16, 124, 1262, 15898, 238572, 4152172.

k. [3+] Show that the number M_n of maximal chains of $W(\mathfrak{S}_n)$ is given by

$$M_n = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}.$$

l. [3+] Let $v \lessdot w$ in $W(\mathfrak{S}_n)$, so $w = (i, i+1)v$ for some i. Define the weight $\sigma(v,w) = i$. Set $r = \binom{n}{2}$. If $C : \hat{0} = v_0 \lessdot v_1 \lessdot \cdots \lessdot v_r = \hat{1}$ is a maximal chain of $W(\mathfrak{S}_n)$, then define

$$\sigma(C) = \sigma(v_0,v_1)\sigma(v_1,v_2)\cdots\sigma(v_{r-1},v_r).$$

Show that $\sum_C \sigma(C) = r!$, where C ranges over all maximal chains of $\mathfrak{S}_n$.

m. [5–] Is the similarity between (l) and Exercise 3.183(g) just a coincidence?

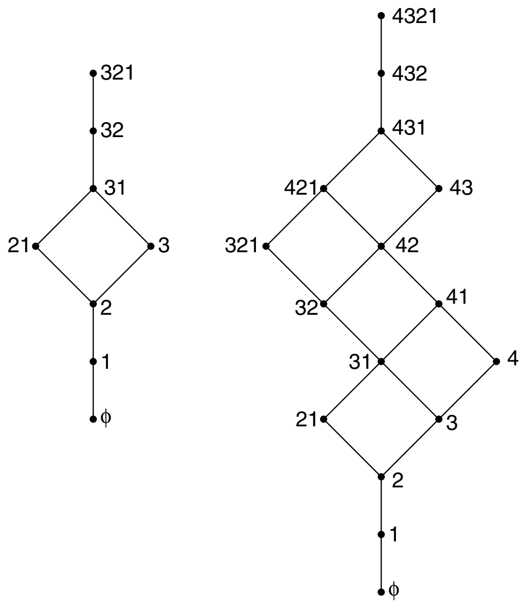


Figure 3.66 The distributive lattices $M(3)$ and $M(4)$.

186. a. [3–] Let $w \in \mathfrak{S}_n$ be separable, as defined in Exercise 3.14(b). Show that the rank-generating functions of the intervals $\Lambda_w = [\hat{0}, w]$ and $V_w = [w, \hat{1}]$ in $W(\mathfrak{S}_n)$ (where $\operatorname{rank}(w) = 0$ in V_w) satisfy

$$F(\Lambda_w, q)F(V_w, q) = (\boldsymbol{n})!.$$

b. [3–] Show that the polynomials $F(\Lambda_w, q)$ and $F(V_w, q)$ are symmetric and unimodal (as defined in Exercise 1.50).

c. [3–] Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ be 231-avoiding. Set $a_{n+1} = n+1$. Show that

$$F(\Lambda_w, q) = \prod_{i=1}^{n} (\boldsymbol{c_i}),$$

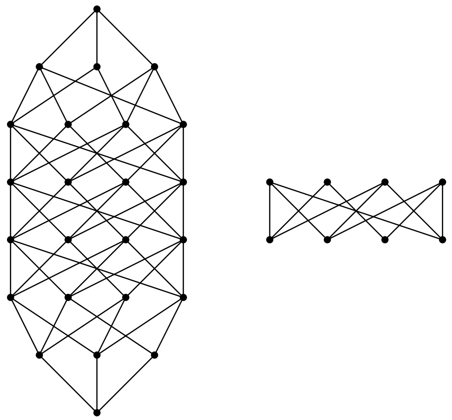
where c_i is the least positive integer for which $a_{i+c_i} > a_i$.

d. [5–] What can be said about other permutations $w \in \mathfrak{S}_n$ for which $F(\Lambda_w, q)$ is symmetric or, more strongly, is a divisor of $(\boldsymbol{n})!$?

187. a. [2]* For $n \geq 0$, define a partial order $M(n)$ on the set $2^{[n]}$ of all subsets of $[n]$ as follows. If $S = \{a_1, \ldots, a_s\}_> \in M(n)$ and $T = \{b_1, \ldots, b_t\}_> \in M(n)$, then $S \geq T$ if $s \geq t$ and $a_i \geq b_i$ for $1 \leq i \leq t$. The posets $M(3)$ and $M(4)$ are shown in Figure 3.66. Show that $M(n) \cong J(J(\boldsymbol{2} \times \boldsymbol{n}))$ and that the rank-generating function of $M(n)$ is given by

$$F(M(n), q) = (1+q)(1+q^2)\cdots(1+q^n).$$

b. [2+] Consider the following variation $\mathcal{G}_n$ of the weak order on $\mathfrak{S}_n$, which we call the *greedy weak order*. It is a partial order on a certain subset (also denoted $\mathcal{G}_n$) of $\mathfrak{S}_n$. First, we let $12\cdots n \in \mathcal{G}_n$. Suppose now that $w = a_1 a_2 \cdots a_n \in \mathcal{G}_n$ and that the permutations that cover w in the weak order $W(\mathfrak{S}_n)$ are obtained from w by transposing adjacent elements $(a_{i_1}, a_{i_1+1}), \ldots, (a_{i_k}, a_{i_k+1})$. In other words, the ascent



(a) (b)

Figure 3.67 An Eulerian poset with a simple cd-index.

set of w is $\{i_1,\ldots,i_k\}$. Then the permutations that cover w in $\mathcal{G}_n$ are obtained by transposing one of the pairs (a_{i_j},a_{i_j+1}) for which (a_{i_j},a_{i_j+1}) is *minimal* in the poset $\mathbb{P}\times\mathbb{P}$ among all the pairs $(a_{i_1},a_{i_1+1}),\ldots,(a_{i_k},a_{i_k+1})$. For instance, the elements that cover 342561 in weak order are obtained by transposing the pairs $(3,4)$, $(2,5)$, and $(5,6)$. The minimal pairs are $(3,4)$ and $(2,5)$. Hence in $\mathcal{G}_6$, 342561 is covered by 432561 and 345261. Show that $\mathcal{G}_n \cong M(n-1)$.

c. [2+] Describe the elements of the set $\mathcal{G}_n$.

188. [3–] Let $\boldsymbol{a}=(a_1,a_2,\ldots,a_n)$ be a finite sequence of integers with no two consecutive elements equal. Let $P=P(\boldsymbol{a})$ be the set of all subsequences $\boldsymbol{a}'=(a_{i_1},a_{i_2},\ldots,a_{i_m})$ (so $1\le i_1<i_2<\cdots<i_m\le n$) of $\boldsymbol{a}$ such that no two consecutive elements of $\boldsymbol{a}'$ are equal. Order P by the rule $\boldsymbol{b}\le\boldsymbol{c}$ if $\boldsymbol{b}$ is a subsequence of $\boldsymbol{c}$. Show that P is Eulerian.

189. [2+]* Let P be an Eulerian poset of rank $d+1$ with d atoms, such that $P-\{\hat{1}\}$ is a simplicial poset. Show that if d is even, then P has an even number of coatoms.

190. a. [3–] Let P_n be the poset of rank $n+1$ illustrated in Figure 3.67(a) for $n=6$. The restriction of P_n to ranks i and $i+1$, $2\le i\le n-2$, is the poset of Figure 3.67(b). Show that P_n is Eulerian with cd-index equal to the sum of all cd monomials of degree n (where $\deg c=1$, $\deg d=2$).

b. [2]* Let $M(n)$ denote the number of maximal chains of P_n. Show that

$$\sum_{n\ge 0} M(n)x^n = \frac{1}{1-2x-2x^2}.$$

191. a. [2]* Let P and Q be Eulerian posets. Show that $P*Q$ is Eulerian, where $P*Q$ is the join of P and Q as defined by equation (3.86).

b. [2]* Show that $\Phi_{P*Q}(c,d)=\Phi_P(c,d)\Phi_Q(c,d)$, where Φ denotes the cd-index.

192. [4–] Let P be a Cohen–Macaulay and Eulerian poset. Such posets are also called *Gorenstein* posets*, as in the topological digression of Section 3.16. Show that the cd-index $\Phi_P(c,d)$ has nonnegative coefficients.

193. a. [3–] Give an example of an Eulerian poset whose cd-index has a negative coefficient.

b. [3] Strengthening (a), give an example of an Eulerian poset P whose flag h-vector β_P has a negative value.

194. [5–] Let P be a finite graded poset of rank n, with p_i elements of rank i. We say that P is *rank-symmetric* if $p_i = p_{n-i}$ for $0 \le i \le n$. Find the dimension $d(n)$ of the space spanned, say over $\mathbb{Q}$, by the flag f-vectors (or equivalently flag h-vectors) of all Eulerian posets P of rank n with the additional condition that every interval of P is rank-symmetric.

195. [3] Fix $k \ge 1$, and let $\mathcal{F}_n(k)$ be the space spanned over $\mathbb{Q}$ by the flag f-vectors of all graded posets of rank n with $\hat{0}$ and $\hat{1}$ such that every interval of rank k is Eulerian. Such posets are called *k-Eulerian*. Since $\mathcal{F}_n(k) = \mathcal{F}_n(k+1)$ for k even by Exercise 3.174(c), we may assume that k is odd, say $k = 2j+1$. Let $d_n(k) = \dim \mathcal{F}_n(k)$. Show that

$$\sum_{n\ge 0} d_n(2j+1)x^n = 1 + \frac{x(1+x)(1+x^{2j})}{1-x-x^2-x^{2j+2}}.$$

196. a. [2] Show that if $B(n)$ is the factorial function of a binomial poset, then $B(n)^2 \le B(n-1)B(n+1)$.

b. [5] What functions $B(n)$ are factorial functions of binomial posets? In particular, can one have $B(n) = F_1F_2\cdots F_n$, where F_i is the ith Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$)?

197. [3–] Show that there exist an uncountable number of pairwise nonisomorphic binomial posets P_α such that (a) they all have the same factorial function $B(n)$, and (b) each P_α has a maximal chain $\hat{0} = t_0 \lessdot t_1 \lessdot t_2 < \cdots$ such that $P_\alpha = \bigcup_{n\ge 0}[\hat{0}, t_n]$.

198. [3–] Let P be an Eulerian binomial poset (i.e., a binomial poset for which every interval is Eulerian). Show that either every n-interval of P is a boolean algebra B_n, or else every n-interval is a "butterfly poset" or ladder $\mathbf{1} \oplus A_2 \oplus A_2 \oplus \cdots \oplus A_2 \oplus \mathbf{1}$, where $A_2 = \mathbf{1} + \mathbf{1}$, a two-element antichain.

199. [2–] Find all finite distributive lattices L that are binomial posets, except for the axiom of containing an infinite chain.

200. [2+] Let P_n be an n-interval of the $q = 2$ case of the binomial poset of Example 3.18.3(e), so $B(n) = 2^{\binom{n}{2}} n!$. Show that the zeta polynomial of P_n is given by

$$Z(P_n, m) = \sum_G \chi_G(m), \tag{3.132}$$

where G ranges over all simple graphs on the vertex set $[n]$, and where χ_G is the chromatic polynomial of G. (Note that Example 3.18.9 gives a generating function for $Z(P_n, m)$.)

201. a. [2] Let P be a locally finite poset with $\hat{0}$ for which every maximal chain is infinite and every interval $[s,t]$ is graded. Thus, P has a rank function ρ. Call P a *triangular poset* if there exists a function $B\colon \{(i,j) \in \mathbb{N}\times\mathbb{N} : i \le j\} \to \mathbb{P}$ such that any interval $[s,t]$ of P with $\rho(s) = m$ and $\rho(t) = n$ has $B(m,n)$ maximal chains. Define a subset T of the incidence algebra $I(P) = I(P,K)$, where $\mathrm{char}(K) = 0$, by

$$T(P) = \{f \in I(P) : f(s,t) = f(s',t') \text{ if } \rho(s) = \rho(s') \text{ and } \rho(t) = \rho(t')\}.$$

If $f \in T(P)$, then write $f(m,n)$ for $f(s,t)$ when $\rho(s) = m$ and $\rho(t) = n$. Show that $T(P)$ is isomorphic to the algebra of all infinite upper-triangular matrices $[a_{ij}]_{i,j\ge 0}$

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over K, the isomorphism being given by

$$f \mapsto \begin{bmatrix} \frac{f(0,0)}{B(0,0)} & \frac{f(0,1)}{B(0,1)} & \frac{f(0,2)}{B(0,2)} & \cdots \\ 0 & \frac{f(1,1)}{B(1,1)} & \frac{f(1,2)}{B(1,2)} & \cdots \\ 0 & 0 & \frac{f(2,2)}{B(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix},$$

where $f \in T(P)$.

b. [3–] Let L be a triangular lattice. Set $D(n) = B(n,n+1) - 1$. Show that L is (upper) semimodular if and only if for all $n \geq m+2$,

$$\frac{B(m,n)}{B(m+1,n)} = 1 + \sum_{i=0}^{n-m-2} D(m)D(m+1)\cdots D(m+i).$$

c. [2] Let L be a triangular lattice. If $D(n) \neq 0$ for all $n \geq 0$ then show that L is atomic. Use (b) to show that the converse is true if L is semimodular.

202. [3] The *shuffle poset* W_{mn} with respect to alphabets A and B is defined in Exercise 7.48(g). Let $[u,v]$ be an interval of W_{mn}, where $u = u_1 \cdots u_r$ and $v = v_1 \cdots v_s$. Let $u_{i_1} \cdots u_{i_t}$ and $v_{j_1} \cdots v_{j_t}$ be the subwords of u and v, respectively, formed by the letters in common to both words. Because $u \leq v$, the shuffle property implies $u_{i_p} = v_{i_p}$ for each $p = 1,\ldots,t$. Moreover, the remaining letters of u belong to A, and the remaining letters of v belong to B. Therefore the interval $[u,v]$ is isomorphic to the product of shuffle posets $W_{i_p - i_{p-1}-1, j_p - j_{p-1}-1}$ for $p = 1,2,\ldots,t+1$, where we set $i_0 = j_0 = 0$, $i_{t+1} = r+1$ and $j_{t+1} = s+1$. We write

$$[u,v] \simeq_c \prod_p W_{i_p - i_{p-1}-1, j_p - j_{p-1}-1}, \tag{3.133}$$

the *canonical isomorphism type* of the interval $[u,v]$. (The reason for this terminology is that some of the factors in equation (3.133) can be one-element posets, any of which could be omitted without affecting the isomorphism type.) Consider now the poset $W_{\infty\infty}$ whose elements are shuffles of finite words using the lower alphabet $A = \{a_i : i \in \mathbb{P}\}$ and the upper alphabet $B = \{b_i : i \in \mathbb{P}\}$, with the same definition of $\leq$ as for finite alphabets. A *multiplicative function* on $W_{\infty\infty}$ is a function f in the incidence algebra $I(W_{\infty\infty}, \mathbb{C})$ for which $f_{00} = 1$ and which has the following two properties:

- If $[u,v]$ and $[u',v']$ are two intervals both canonically isomorphic to W_{ij}, then $f(u,v) = f(u',v')$. We denote this value by f_{ij}.
- If $[u,v] \simeq_c \prod_{i,j} W_{ij}^{c_{ij}}$, then $f(u,v) = \prod_{ij} f_{ij}^{c_{ij}}$.

Let f and g be two multiplicative functions on $W_{\infty\infty}$, and let

$$F = F(x,y) = \sum_{i,j \geq 0} f_{ij} x^i y^j,$$

$$G = G(x,y) = \sum_{i,j \geq 0} g_{ij} x^i y^j,$$

$$F * G = (F * G)(x,y) = \sum_{i,j \geq 0} (f * g)_{ij} x^i y^j,$$

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where $*$ denotes convolution in the incidence algebra $I(W_{\infty\infty},\mathbb{C})$. Let $F_0 = F(x,0)$, $G_0 = G(0,y)$, and

$$\widetilde{F}(x,y) = F(x, G_0 y),$$
$$\widetilde{G}(x,y) = G(F_0 x, y).$$

Show that
$$\frac{1}{F*G} = \frac{1}{\widetilde{F}G_0} + \frac{1}{F_0\widetilde{G}} - \frac{1}{F_0G_0}.$$

203. [3–] Fix an integer sequence $0 \le a_1 < a_2 < \cdots < a_r < m$. For $k \in [r]$, let $f_k(n)$ denote the number of permutations $b_1b_2\cdots b_{mn+a_k}$ of $[mn+a_k]$ such that $b_j > b_{j+1}$ if and only if $j \equiv a_1,\dots,a_r \pmod m$. Let

$$F_k \;=\; F_k(x) = \sum_{n\ge 0}(-1)^{nr+k} f_k(n)\frac{x^{mn+a_k}}{(mn+a_k)!},$$
$$\Phi_j(x) = \sum_{n\ge 0}\frac{x^{mn+j}}{(mn+j)!}.$$

Let $\bar{a}$ denote the least nonnegative residue of $a \pmod m$, and set $\psi_{ij} = \Phi_{\overline{a_i - a_j}}(x)$. Show that

$$\begin{array}{ccccccccc}
F_1\psi_{11} & + & F_2\psi_{12} & + & \cdots & + & F_r\psi_{1r} & = & 1\\
F_1\psi_{21} & + & F_2\psi_{22} & + & \cdots & + & F_r\psi_{2r} & = & 0\\
 & & & & & & & \vdots & \\
F_1\psi_{r1} & + & F_2\psi_{r2} & + & \cdots & + & F_r\psi_{rr} & = & 0.
\end{array}$$

Solve these equations to obtain an explicit expression for $F_k(x)$ as a quotient of two determinants.

204. a. [2+] Let P be a locally finite poset for which every interval is graded. For any $S \subseteq \mathbb{P}$ and $s \le t$ in P, define $[s,t]_S$ as in equation (3.93), and let $\mu_S(s,t)$ denote the Möbius function of the poset $[s,t]_S$ evaluated at the interval $[s,t]$. Let z be an indeterminate, and define $g,h \in I(P)$ by

$$g(s,t) = \begin{cases} 1, & \text{if } s = t,\\ (1+z)^{n-1}, & \text{if } \ell(s,t) = n \ge 1,\end{cases}$$

$$h(s,t) = \begin{cases} 1, & \text{if } s = t,\\ \displaystyle\sum_S \mu_S(s,t) z^{n-1-\#S}, & \text{if } s < t, \text{ where } \ell(s,t) = n \ge 1 \text{ and}\\ & \;S \text{ ranges over all subsets of } [n-1].\end{cases}$$

Show that $h = g^{-1}$ in $I(P)$.

b. [1+] For a binomial poset P write $h(n)$ for $h(s,t)$ when $\ell(s,t) = n$, where h is defined in (a). Show that

$$1 + \sum_{n\ge 1} h(n)\frac{x^n}{B(n)} = \left[1 + \sum_{n\ge 1}(1+z)^{n-1}\frac{x^n}{B(n)}\right]^{-1}.$$

c. [2] Define
$$G_n(q,z) = \sum_{w\in\mathfrak{S}_n} z^{\mathrm{des}(w)} q^{\mathrm{inv}(w)},$$

where $\mathrm{des}(w)$ and $\mathrm{inv}(w)$ denote the number of descents and inversions of w, respectively. Show that

$$1+z\sum_{n\geq 1} G_n(q,z)\frac{x^n}{(\boldsymbol{n})!} = \left[1-z\sum_{n\geq 1}(z-1)^{n-1}\frac{x^n}{(\boldsymbol{n})!}\right]^{-1}.$$

In particular, setting $q=1$ we obtain Proposition 1.4.5:

$$1+\sum_{n\geq 1} z^{-1}A_n(z)\frac{x^n}{n!} = \left[1-\sum_{n\geq 1}(x-1)^{n-1}\frac{x^n}{n!}\right]^{-1}$$

$$= \frac{1-z}{e^{x(z-1)}-z},$$

where $A_n(z)$ denotes an Eulerian polynomial.

205. a. [2+] Give an example of a 1-differential poset that is not isomorphic to Young's lattice Y nor to $\Omega_1^\infty Y[n]$ for any n, where $Y[n]$ denotes the rank n truncation of Y (i.e., the subposet of Y consisting of all elements of rank at most n).

b. [3] Show that there are two nonisomorphic 1-differential posets up to rank 5, five up to rank 6, 35 up to rank 7, 643 up to rank 8, and 44605 up to rank 9.

c. [3–] Give an example of a 1-differential poset that is not isomorphic to Y nor to a poset $\Omega_1^\infty P$, where P is 1-differential up to some rank n.

206. [3] Show that the only 1-differential lattices are Y and Z_1.

207. [2+] Let P be an r-differential poset, and let $A_k(q)$ be as in equation (3.110). Write $\alpha(n-2\to n\to n-1\to n)$ for the number of Hasse walks $t_0 \lessdot t_1 \lessdot t_2 \gtrdot t_3 \lessdot t_4$ in P, where $\rho(t_0)=n-2$. Show that

$$\sum_{n\geq 0}\alpha(n-2\to n\to n-1\to n)q^n = F(P,q)(2rq^2A_2(q)+rq^3A_3(q)+q^4A_4(q)).$$

208. a. [2+] Let P be an r-differential poset, and let $t\in P$. Define a word (noncommutative monomial) $w=w(U,D)$ in the letters U and D to be a *valid t-word* if $\langle w(U,D)\hat{0},t\rangle \neq 0$. Note that if $s\in P$, then a valid t-word is also a valid s-word if and only if $\rho(s)=\rho(t)$. Let $w=w_1\cdots w_l$ be a valid t-word. Let $S=\{i : w_i=D\}$. For each $i\in S$, let a_i be the number of D's in w to the right of w_i, and let b_i be the number of U's in w to the right of w_i. Show that

$$\langle w\hat{0},t\rangle = e(t)r^{\#S}\prod_{i\in S}(b_i-a_i),$$

where $e(t)$ is defined in Example 3.21.5.

b. [2–]* Deduce from (a) that if $n=\rho(t)$ then

$$\langle w\hat{0},\boldsymbol{P}\rangle = \alpha(0\to n)r^{\#S}\prod_{i\in S}(b_i-a_i).$$

c. [2–]* Deduce the special case $\langle UDUU\hat{0},\boldsymbol{P}\rangle = 2r^2(r+1)$. Also deduce this result from Exercise 3.207.

209. [2] Let U and D be operators (or indeterminates) satisfying $DU-UD=1$. Show that

$$(UD)^n = \sum_{k=0}^{n} S(n,k)U^kD^k, \tag{3.134}$$

where $S(n,k)$ denotes a Stirling number of the second kind.

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210. [2+] A word w in U and D is *balanced* if it contains the same number of U's as D's. Show that if $DU - UD = 1$, then any two balanced words in U and D commute.

211. [3–]* Let P be an r-differential poset. Let $c(t)$ denote the number of elements covering $t \in P$, and set $f(n) = \sum_{t \in P_n} c(t)^2$. Show that

$$\sum_{n \geq 0} f(n) q^n = \frac{r^2 + (r+1)q - q^2}{(1-q)(1-q^2)} F(P,q).$$

212. Let P be an r-differential poset, and fix $k \in \mathbb{N}$. Let $\kappa(n \to n+k \to n)$ denote the number of *closed* Hasse walks in P of the form $t_0 \lessdot t_1 \lessdot \cdots \lessdot t_k \gtrdot t_{k+1} \gtrdot \cdots \gtrdot t_{2k}$ (so $t_0 = t_{2k}$) such that $\rho(t_0) = n$.

a. [2–]* Show that

$$\begin{aligned} \kappa(n \to n+k \to n) &= \sum_{t \in P_n} \langle D^k U^k t, t \rangle \\ &= \sum_{s \in P_n} \sum_{t \in P_{n+k}} e(s,t)^2, \end{aligned}$$

where $e(s,t)$ denotes the number of saturated chains $s = s_0 \lessdot s_1 \lessdot \cdots \lessdot s_k = t$.

b. [2+]* Show that

$$\sum_{n \geq 0} \kappa(n \to n+k \to n) q^n = r^k k! (1-q)^{-k} F(P,q).$$

213. [2+] Let P be an r-differential poset, and let $\kappa_{2k}(n)$ denote the total number of closed Hasse walks of length $2k$ starting at some element of P_n. Show that for fixed k,

$$\sum_{n \geq 0} \kappa_{2k}(n) q^n = \frac{(2k)! r^k}{2^k k!} \left(\frac{1+q}{1-q} \right)^k F(P,q).$$

214. **a.** [3–] Show that the "fattest" r-differential poset is Z_r (i.e., has at least as many elements of any rank i as any r-differential poset).

b. [5] Show that the "thinnest" r-differential poset is Y^r.

215. **a.** [2] Let P be an r-differential poset, and let $p_i = \#P_i$. Show that $p_0 \leq p_1 \leq \cdots$. Hint. Use linear algebra.

b. [2+]* Show that $\lim_{i \to \infty} p_i = \infty$.

c. [5] Show that $p_i < p_{i+1}$ except for the case $i = 0$ and $r = 1$.

Solutions to Exercises

1. Itinerant salespersons who take revenge on customers who don't pay their bills are retaliatory peddlers, and "retaliatory peddlers" is an anagram of "partially ordered set" (i.e., they have the same multiset of letters).

2. Routine. See [3.13], Lemma 1 on page 21.

3. The correspondence between finite posets and finite topologies (or more generally arbitrary posets and topologies for which any intersection of open sets is open) seems first to have been considered by P. S. Alexandroff, *Mat. Sb. (N.S.)* **2** (1937), 510–518, and has been rediscovered many times.

4. Let $\mathcal{S} = \{\Lambda_t : t \in P\}$, where $\Lambda_t = \{s \in P : s \leq t\}$. This exercise is the poset analogue of Cayley's theorem that every group is isomorphic to a group of permutations of a set.

5. **a.** The enumeration of n-element posets for $1 \leq n \leq 7$ appears in John A. Wright, thesis, Univ. of Rochester, 1972. Naturally computers have allowed the values of n to be considerably extended. At the time of this writing, the most recent paper on this topic is G. Brinkmann and B. D. McKay, *Order* **19** (2002), 147–179.

c. The purpose of this seemingly frivolous exercise is to point out that some simply stated facts about posets may be forever unknowable.

d. D. J. Kleitman and B. L. Rothschild, *Proc. Amer. Math. Soc.* **25** (1970), 276–282. The lower bound for this estimate is obtained by considering posets of rank one with $\lfloor n/2 \rfloor$ elements of rank 0 and $\lceil n/2 \rceil$ elements of rank 1.

e. D. J. Kleitman and B. L. Rothschild, *Trans. Amer. Math. Soc.* **205** (1975), 205–220. The asymptotic formula given there is more complicated but can be simplified to that given here. It follows from the proof that almost all posets have longest chain of length two.

6. **a.** The function f is a permutation of a finite set, so $f^n = 1$ for some $n \in \mathbb{P}$. But then $f^{-1} = f^{n-1}$, which is order-preserving.

b. Let $P = \mathbb{Z} \cup \{t\}$, with $t < 0$ and t incomparable with all $n < 0$. Let $f(t) = t$ and $f(n) = n + 1$ for $n \in \mathbb{Z}$.

7. **a.** An example is shown in Figure 3.68. There are four other 6-element examples, and none smaller. For the significance of this exercise, see Corollary 3.15.18(a).

b. Use induction on ℓ, removing all minimal elements from P. This proof is due to D. West. The result (with a more complicated proof) first appeared in [2.19, pp. 19–20].

8. The poset Q of Figure 3.69 was found by G. Ziegler, and with a $\hat{0}$ and $\hat{1}$ adjoined is a lattice. Ziegler also has an example of length one with 24 elements, and an example which is a graded lattice of length three with 26 elements. Another example is presumably the one referred to by Birkhoff in [3.13, Exer. 10, p. 54].

9. False. Non-self-dual posets come in pairs P, P^*, so the number of each order is even. The actual number is 16506. The number of self-dual 8-element posets is 493.

10. b. Suppose that $f : \mathrm{Int}(P) \to \mathrm{Int}(Q)$ is an isomorphism. Let $f([\hat{0},\hat{0}]) = [s,s]$, where $\hat{0} \in P$ and $s \in Q$. Define A to be the subposet of Q of all elements $t \geq s$, and define B to be all elements $t \leq s$. Check that $P \cong A \times B^*$, $Q \cong A \times B$.

This result is due independently to A. Gleason (unpublished) and M. Aigner and G. Prins, *Trans. Amer. Math. Soc.* **166** (1972), 351–360.

c. (A. Gleason, unpublished) See Figure 3.70. The poset P may be regarded as a "twisted" direct product (not defined here) of the posets A and B of Figure 3.71, and Q a twisted direct product of A and C. These twisted direct products exist since the poset A is, in a suitable sense, not simply-connected but has the covering poset $\widetilde{A}$ of Figure 3.71. A general theory was presented by A. Gleason at an M.I.T. seminar in December 1969.

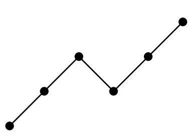

Figure 3.68 A solution to Exercise 3.7.

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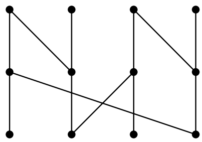

Figure 3.69 A self-dual poset with no involutive antiautomorphism.

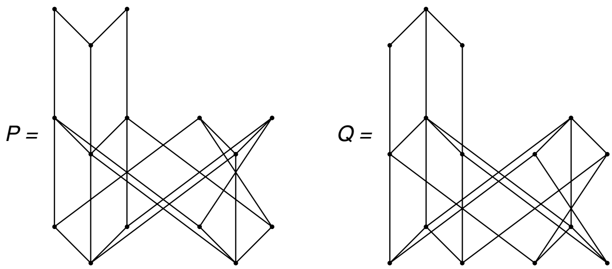

Figure 3.70 A solution to Exercise 3.10(c).

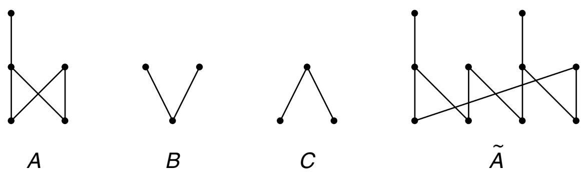

Figure 3.71 Four posets related to Exercise 3.10(c).

For the determination of which posets have isomorphic posets of *convex* subposets, see G. Birkhoff and M. K. Bennett, *Order* **2** (1985), 223–242 (Theorem 13).

11. **a.** See Birkhoff [3.13, Thm. 2, p. 57].

b. See [3.13, Thm. 2, pp. 68–69].

c. See [3.13, p. 69]. If P is any connected poset with more than one element, then we can take $P_1 = \mathbf{1} + P^3$, $P_2 = \mathbf{1} + P + P^2$, $P_3 = \mathbf{1} + P^2 + P^4$, $P_4 = \mathbf{1} + P$, where $\mathbf{1}$ denotes the one-element poset. There is no contradiction, because although $\mathbb{Z}[x_1, x_2, \dots]$ is a UFD, this does not mean that $\mathbb{N}[x_1, x_2, \dots]$ is a unique factorization semiring. In the ring B, we have (writing Q for $[Q]$)

$$P_1 P_2 = P_3 P_4 = (\mathbf{1} + P)(\mathbf{1} - P + P^2)(\mathbf{1} + P + P^2).$$

12. True. If the number of maximal chains of P is finite, then P is clearly finite, so assume that P has infinitely many maximal chains. These chains are all finite, so in particular every maximal chain containing a nonmaximal element t of P must contain an element covering t. The set $C^+(t)$ of elements covering $t \in P$ is an antichain, so $C^+(t)$ is finite. Since P has only finitely many minimal elements (since they form an antichain), infinitely many maximal chains C contain the same minimal element t_0. Since $C^+(t_0)$ is finite, infinitely many of the chains C contain the same element $t_1 \in C^+(t_0)$. Continuing in this way, we obtain an infinite chain $t_0 < t_1 < \cdots$, a contradiction.

13. **a.** This result is a consequence of G. Higman, *Proc. London Math. Soc. (3)* **2** (1952), 326–336.

b. Let $P = P_1 + P_2$, where each P_i is isomorphic to the rational numbers $\mathbb{Q}$ with their usual linear order. Every antichain of P has at most two elements. For any real

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$\alpha > 0$, let

$$I_\alpha = \{a \in P_1 : a < -\alpha\} \cup \{b \in P_2 : b < \alpha\}.$$

Then the I_α's form an infinite (in fact, uncountable) antichain in $J(P)$.

15. a. Straightforward proof by induction on $\#P$. This result is implicit in work of A. Ghouila-Houri *C. R. Acad. Sci. Paris* **254** (1962), 1370–1371, and P. C. Gilmore and A. J. Hoffman *Canad. J. Math.* **16** (1964), 539–548. The first explicit statement was given by P. C. Fishburn, *J. Math. Psych.* **7** (1970), 144–149. Two references with much further information on interval orders are P. C. Fishburn, *Interval Orders and Interval Graphs*, Wiley-Interscience, New York, 1985, and W. T. Trotter, *Combinatorics and Partially Ordered Sets*, The Johns Hopkins University Press, Baltimore, 1992. In particular, Fishburn (pp. 19–22) discusses the history of interval orders and their applications to such areas as psychology.

b. This result is due to D. Scott and P. Suppes, *J. Symbolic Logic* **23** (1958), 113–128. Much more information on semiorders may be found in the books of Fishburn and Trotter cited in (a).

c. See M. Bousquet-Mélou, A. Claesson, M. Dukes, and S. Kitaev, *J. Combinatorial Theory Ser. A* **117** (2010), 884–909 (Theorem 13). This paper mentions several other objects (some of which were already known) counted by $t(n)$, in particular, the *regular linearized chord diagrams* (RLCD) (not defined here) of D. Zagier, *Topology* **40** (2001), 945–960.

The paper of Zagier also contains the remarkable result that if we set $F(x) = \sum_{n\geq 0} t(n)x^n$, then

$$F(1-e^{-24x}) = e^x \sum_{n\geq 0} u(n)\frac{x^n}{n!},$$

where

$$\sum_{n\geq 0} u(n)\frac{x^{2n+1}}{(2n+1)!} = \frac{\sin 2x}{2\cos 3x}.$$

d. This result is a consequence of equation (4) of Zagier, ibid., and the bijection between interval orders and RLCD's given by Bousquet-Mélou et al., op. cit. Note that if we set $F(x) = \sum_{n\geq 0} t(n)x^n$ and $G(x) = \sum_{n\geq 0} u(n)\frac{x^n}{n!}$, then $G(x) = F(1-e^{-x})$. This phenomenon also occurs for semiorders (see equation (6.57)) and other objects (see Exercise 3.17 and the solution to Exercise 6.30).

e. A series-parallel interval order can clearly be represented by intervals such that for any two of these intervals, they are either disjoint or one is contained in the other. Conversely any such finite set of intervals represents a series-parallel interval order. Now use Exercise 6.19(o). For a refinement of this result see J. Berman and P. Dwinger, *J. Combin. Math. Combin. Comput.* **16** (1994), 75–85. An interesting characterization of series-parallel interval orders was given by M. S. Rhee and J. G. Lee, *J. Korean Math. Soc.* **32** (1995), 1–5.

f. (sketch) Let $\mathcal{G}_n$ denote the arrangement in part (e). Putting $x = -1$ in Proposition 3.11.3 gives

$$r(\mathcal{G}_n) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{G}_n \\ \mathcal{B}\text{ central}}} (-1)^{\#\mathcal{B}-\mathrm{rank}(\mathcal{B})}. \tag{3.135}$$

Given a central subarrangement $\mathcal{B} \subseteq \mathcal{G}_n$, define a digraph $G_\mathcal{B}$ on $[n]$ by letting $i \to j$ be a (directed) edge if the hyperplane $x_i - x_j = \ell_i$ belongs to $\mathcal{B}$. One then shows that as an undirected graph $G_\mathcal{B}$ is bipartite. Moreover, if B is a block of $G_\mathcal{B}$ (as defined in Exercise 5.20), say with vertex bipartition (U_B, V_B), then either all edges of B are directed from U_B to V_B, or all edges are directed from V_B to U_B.

It can also be seen that all such directed bipartite graphs can arise in this way. It follows that equation (3.135) can be rewritten

$$r(\mathcal{G}_n) = (-1)^n \sum_G (-1)^{e(G)+c(G)} 2^{b(G)}, \tag{3.136}$$

where G ranges over all (undirected) bipartite graphs on $[n]$, $e(G)$ denotes the number of edges of G, and $b(G)$ denotes the number of blocks of G.
Equation (3.136) reduces the problem of determining $r(\mathcal{G}_n)$ to a (rather difficult) problem in enumeration, whose solution may be found in A. Postnikov and R. Stanley, *J. Combinatorial Theory Ser. A* **91** (2000), 544–597 (§6).

16. This result is due to J. Lewis and Y. Zhang, Enumeration of graded (3+1)-avoiding posets, `arXiv:1106.5480`.

17. **a.** Hint. First show that the formula $G(x) = F(1-e^{-x})$ is equivalent to

$$n!f(n) = \sum_{k=1}^{n} c(n,k)g(k),$$

where $c(n,k)$ denotes a signless Stirling number of the first kind.
The special case where $\mathcal{T}$ consists of the nonisomorphic finite semiorders is due to J. L. Chandon, J. Lemaire, and J. Pouget, *Math. et Sciences Humaines* **62** (1978), 61–80, 83. (See Vol. II, Exercise 6.30.) The generalization to the present exercise (and beyond) is due to Y. Zhang, in preparation (2011).

b. $F(x) = (1-x)/(1-2x)$, the ordinary generating function for the number of compositions of n, and $G(x) = 1/(2-e^x)$, the exponential generating function for the number of ordered partitions of $[n]$. See Example 3.18.10.

c. These results follow from two properties of interval orders and semiorders P: (i) any automorphism of P is obtained by permuting elements in the same autonomous subset (as defined in the solution to Exercise 3.143), and (ii) replacing elements in an interval order (respectively, semiorder) by antichains preserves the property of being an interval order (respectively, semiorder).

18. Originally this result was proved using symmetric functions (R. Stanley, *Discrete Math.* **193** (1998), 267–286). Later M. Skandera, *J. Combinatorial Theory (A)* **93** (2001), 231–241, showed that for a certain ordering of the elements of P, the *square* of the anti-incidence matrix of Exercise 3.22, with each $t_i = 1$, is totally nonnegative (i.e., every minor is nonnegative). The result then follows easily from Exercise 3.22 and the standard fact that totally nonnegative square matrices have real eigenvalues. Note that if $P = \mathbf{3}+\mathbf{1}$ then $C_P(x) = x^3+3x^2+4x+1$, which has the approximate nonreal zeros $-1.34116 \pm 1.16154i$.

19. **a,c.** These results (in the context of finite topological spaces) are due to R. E. Stong, *Trans. Amer. Math. Soc.* **123** (1966), 325–340 (see page 330). For (a), see also D. Duffus and I. Rival, in *Colloq. Math. Soc. János Bolyai* (A. Hajnal and V. T. Sós, eds.), vol. 1, North-Holland, New York, pp. 271–292 (page 272), and J. D. Farley, *Order* **10** (1993), 129–131. For (c), see also D. Duffus and I. Rival, *Discrete Math.* **35** (1981), 53–118 (Theorem 6.13). Part (c) is generalized to infinite posets by K. Baclawski and A. Björner, *Advances in Math.* **31** (1979), 263–287 (Thm. 4.5). For a general approach to results such as (a) where any way of carrying out a procedure leads to the same outcome, see K. Eriksson, *Discrete Math.* **153** (1996), 105–122; *Europ. J. Combinatorics* **17** (1996), 379–390; and *Discrete Math.* **139** (1995), 155–166.

20. **a,b.** The least d for which (i) or (ii) holds is called the *dimension* of P. For a survey of this topic, see D. Kelly and W. T. Trotter, in [3.57, pp. 171–211]. In particular, the equivalence of (i) and (ii) is due to Ore, while (iii) is an observation of Dushnik and Miller. Note that a 2-dimensional poset P on $[n]$ which is compatible with the usual ordering of $[n]$ (i.e., if $s < t$ in P, then $s < t$ in $\mathbb{Z}$) is determined by the permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$ for which P is the intersection of the linear orders $1 < 2 < \cdots < n$ and $a_1 < a_2 < \cdots < a_n$. We call $P = P_w$ the *inversion poset* of the permutation w. In terms of w, we have that $a_i < a_j$ in P if and only if $i < j$ and $a_i < a_j$ in $\mathbb{Z}$. For further results on posets of dimension 2, see K. A. Baker, P. C. Fishburn, and F. S. Roberts, *Networks* **2** (1972), 11–28. Much additional information appears in P. C. Fishburn, *Interval Orders and Interval Graphs*, John Wiley, New York, 1985, and in W. T. Trotter, *Combinatorics and Partially Ordered Sets: Dimension Theory*, The John Hopkins University Press, Baltimore, 1992.

21. The statement is false. It was shown by S. Felsner, W. T. Trotter, and P. C. Fishburn, *Discrete Math.* **201** (1999), 101–132, that the poset $\boldsymbol{n}^3$, for n sufficiently large, is not a sphere order.

22. This result is an implicit special case of a theorem of D. M. Jackson and I. P. Goulden, *Studies Appl Math.* **61** (1979), 141–178 (Lemma 3.12). It was first stated explictly by R. Stanley, *J. Combinatorial Theory Ser. A* **74** (1996), 169–172 (in the more general context of acyclic digraphs). To prove it directly, use the fact that the coefficient of x^j in $\det(I + xDA)$ is the sum of the principal $j \times j$ minors of DA. Let $DA[W]$ denote the principal submatrix of DA whose rows and columns are indexed by $W \subseteq [p]$. It is not difficult to show that

$$\det DA[W] = \begin{cases} \prod_{i \in W} t_i, & \text{if } W \text{ is the set of vertices of a chain,} \\ 0, & \text{otherwise,} \end{cases}$$

and the proof follows.

23. Of course $\emptyset < [1] < [2] < \cdots$ is a countable maximal chain. Now clearly $B_{\mathbb{P}} \cong B_{\mathbb{Q}}$ since $\mathbb{P}$ and $\mathbb{Q}$ are both countable infinite sets. For each $\alpha \in \mathbb{R}$, define $t_\alpha \in B_{\mathbb{Q}}$ by $t_\alpha = \{s \in \mathbb{Q} : s < \alpha\}$. Then the elements t_α, together with $\hat{0}$ and $\hat{1}$, form an uncountable maximal chain.

24. For an extension to all n-element posets having at least $(7/16)2^n$ order ideals, see R. Stanley, *J. Combinatorial Theory Ser. A* **10** (1971), 74–79. For further work on the number of n-element posets with k order ideals, see M. Benoumhani, *J. Integer Seq.* **9** (2006), 06.2.6 (electronic), and K. Ragnarsson and B. Tenner, *J. Combinatorial Theory Ser. A* **117** (2010), 138–151.

25. None.

26. Perhaps the simplest example is $\mathbf{1} \oplus (\mathbf{1} + \mathbf{1}) \oplus \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the dual of $\mathbb{N}$ with the usual linear order. We could replace $\mathbb{N}^*$ with $\mathbb{Z}$.

27. Let B be the boolean algebra of all subsets of $\mathrm{Irr}(L)$, and let L' be the meet-semilattice of B generated by the principal order ideals of $\mathrm{Irr}(L)$. One can show that L is isomorphic to L' with a $\hat{1}$ adjoined.
In fact, L is the *MacNeille completion* (e.g., [3.13, Ch. V.9]) of $\mathrm{Irr}(L)$, and this exercise is a result of B. Banaschewski, *Z. Math. Logik* **2** (1956), 117–130. An example is shown in Figure 3.72.

28. Let L be the sub-meet-semilattice of the boolean algebra B_6 generated by the subsets 1234, 1236, 1345, 2346, 1245, 1256, 1356, 2456, with a $\hat{1}$ adjoined. By definition L is coatomic. One checks that each singleton subset $\{i\}$ belongs to L, $1 \leq i \leq 6$, so L is atomic. However, the subset $\{1,2\}$ has no complement.
This example was given by I. Rival (personal communication) in February 1978. See *Discrete Math.* **29** (1980), 245–250 (Fig. 5).

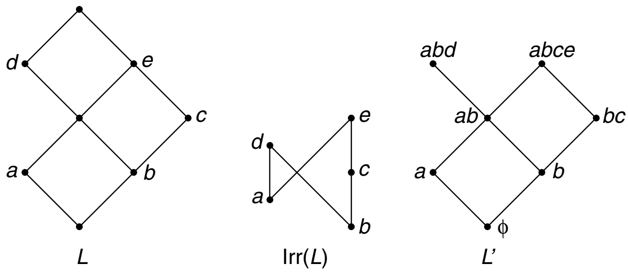

Figure 3.72 The MacNeille completion of Irr(L).

29. D. Kleitman has shown (unpublished) that

$$\binom{n}{\lfloor n/2 \rfloor}\left(1+\frac{1}{n}\right) < f(n) < \binom{n}{\lfloor n/2 \rfloor}\left(1+\frac{1}{\sqrt{n}}\right),$$

and conjectures that the lower bound is closer to the truth.

30. **a,b.** Since sublattices of distributive (respectively, modular) lattices are distributive (respectively, modular), the "only if" part is immediate from the nonmodularity of the first lattice in Figure 3.43 and the nondistributivity of both lattices. For the "if" part, it is not hard to check that the failure of the distributive law (3.8) for a triple (s,t,u) forces the sublattice generated by s,t,u to contain (as a sublattice) one of the two lattices of Figure 3.43. Similarly the failure of the modular law (3.7) forces the first lattice of Figure 3.43. This result goes back to R. Dedekind, *Festschrift Techn. Hoch. Braunschweig* (1897), 1–40; reprinted in *Ges. Werke*, vol. 2, 103–148, and *Math. Ann.* **53** (1900), 371–403; reprinted in *Ges. Werke*, vol. 2, 236–271.

31. **b.** See J. D. Farley and S. E. Schmidt, *J. Combinatorial Theory Ser. A* **92** (2000), 119–137.

c. This result was originally conjectured by R. Stanley (unpublished) and proved by D. J. Grabiner, *Discrete Math.* **199** (1999), 77–84.

d. This result is originally due to R. Stanley (unpublished).

32. **a.** *Answer.* $f(B_n) = \lceil n/2 \rceil$. See C. Biró, D. M. Howard, M. T. Keller, W. T. Trotter, and S. J. Young, *J. Combinatorial Theory Ser. A* **117** (2010), 475–482. See Exercise 3.142 for a more general context to this topic.

b–e. See Y. H. Wang, The new Stanley depth of some power sets of multisets, `arXiv:0908.3699`.

33. *Answer* (in collaboration with J. Shareshian). $n \neq 1,3,5,7$.

35. **a.** By Theorem 3.4.1, $f_2(n)$ is equal to the number of distributive lattices L of rank n with exactly two elements of every rank $1, 2, \dots, n-1$. We build L from the bottom up. Ranks 0, 1, 2 must look (up to isomorphism) like the diagram in Figure 3.73(a), where we have also included $u = s \vee t$ of rank 3. We have two choices for the remaining element v of rank 3—place it above s or above t, as shown in Figure 3.73(b). Again we have two choices for the remaining element of rank 4—place it above u or above v. Continuing this line of reasoning, we have two independent choices a total of $n-3$ times, yielding the result. When, for example, $n = 5$, the four posets are shown in Figure 3.74.

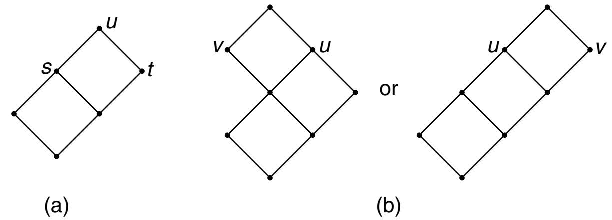

Figure 3.73 A construction used in the solution to Exercise 3.35.

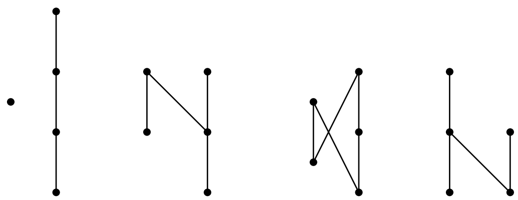

Figure 3.74 The four posets enumerated by $f_2(5)$.

b. Similar to (a).

c. See J. D. Farley and R. Klippenstine, *J. Combinatorial Theory Ser. A* **116** (2009), 1097–1119.

d. (suggested by P. Edelman) $f_k(n) = 0$ for $k > 3$ since $\binom{k}{2} > k$.

36. a. Clearly, L' is a join-semilattice of L with $\hat{0}$; hence by Proposition 3.3.1, L' is a lattice. By definition L' is atomic. Suppose t covers s in L'. Then $t = s \vee a$ for some atom a of L. The semimodularity property of Proposition 3.3.2(ii) is inherited from L by L'. Thus L' is geometric.

b. No. Let K be the boolean algebra B_5 of all subsets of $[5]$, with all four-element subsets removed. Let L consist of K with an additional element t adjoined such that t covers $\{1\}$ and is covered by $\{1,2,3\}$ and $\{1,4,5\}$. Then $t \notin L'$ but t belongs to the sublattice of L generated by L'.

37. a. This result is an immediate consequence of a much more general result of W. T. Tutte, *J. Res. Natl. Bur. Stand., Sect. B* **69** (1965), 1–47. For readers who know some matroid theory we provide some more details. Tutte shows (working in the broader context of "chain groups") that the set of minimal nonempty supports are the set of circuits of a matroid. Since $\mathrm{char}(K) = 0$ the support sets coincide with the unions of minimal nonempty supports. This means that the supports coincide with the sets of unions of circuits. The complements of circuits are hyperplanes of the dual matroid. The proper flats of a matroid coincide with the intersections of hyperplanes so their complements are unions of circuits of the dual, and the present exercise follows.

b. Let K^E denote the vector space of all functions $E \to K$, and let V denote the vertex set of G. Choose an orientation $\mathfrak{o}$ of the edges of G. For each vertex v, let v^+ denote the set of edges pointing out of v, and v^- the set of edges pointing in (with respect to $\mathfrak{o}$). Let

$$W = \left\{ f \in K^E : \forall v \in V \sum_{e \in v^+} f(e) = \sum_{e \in v^-} f(e) \right\}.$$

Elements of W are called *flows*. It is not hard to check that a spanning subgraph of G is the support of a flow if and only if it has no isthmus, and the proof follows from (a).

38. If $t \in P_k$, then define

$$\begin{aligned}\phi(t) = \sup\{u : u \not\geq \text{ any join-irreducible } t_i \text{ such that} \\ t = t_1 \vee \cdots \vee t_n \text{ is the (unique) irredundant} \\ \text{expression of } t \text{ as a join of join-irreducibles}\}.\end{aligned} \tag{3.137}$$

In particular, if $t \in P_1$ then $\phi(t) = \sup\{u : u \not\geq t\}$.
It is fairly easy to see that ϕ has the desired properties by dealing with the poset P for which $L = J(P)$, rather than with L itself.

40. **a.** *Answer.* $\dfrac{2^{k-1}(2^{r(k-1)}-1)}{2^{k-1}-1} + 2^{r(k-1)}$. This result is a special case of a more general result, where the number of elements of every rank is specified, of R. Stanley, *J. Combinatorial Theory* **14** (1973), 209–214 (Corollary 1).

b. See R. Stanley, ibid. (special case of Theorem 2). For further information on the extremal lattice $J(P+\mathbf{1})$, see [3.73].

c. This inequality, if true, is best possible, as seen by taking $L = J_f(P+\mathbf{1})$ as in (b). Note that $J_f(P+\mathbf{1})$ is *maximal* with respect to having two join-irreducibles at each positive rank, and is conjectured to be *minimal* with respect to having an antichain passing through each positive rank.

41. **a.** Let $t_1, \ldots, t_p$ be a linear extension of L, regarded as a permutation of the elements of L. Let $\sigma_i = (\Gamma^i(t_1), \ldots, \Gamma^i(t_p))$. All the sequences σ_i have the same sum of their terms. Moreover, if $\sigma_i \neq \sigma_{i+1}$ then $\sigma_i < \sigma_{i+1}$ in dominance order. It follows that eventually we must have $\sigma_n = \sigma_{n+1}$.

b. The "if" part of the statement is equivalent to Problem A3 on the 69th William Lowell Putnam Mathematical Competition (2008). The "only if" part follows easily from Exercise 3.30(a). The "only if" part was observed by T. Belulovich and is discussed at the Putnam Archive,

⟨`www.unl.edu/amc/a-activities/a7-problems/`

`putnamindex.shtml`⟩.

c. This result was shown by F. Liu and R. Stanley, October 2009.

d. This observation is due to R. Ehrenborg, October 2009.

e. Use (d) and the fact that $\#\{w \in \mathfrak{S}_{n+1} : w(i) > i\} = (n+1-i)n!$. This result is due to R. Ehrenborg, October 2009.

43. First show that $J_f(P)$ can be identified with the subposet of $\mathbb{N} \times \mathbb{N}$ consisting of all (i,j) for which $0 \leq j \leq \lfloor i/2 \rfloor$. Then show that $J_f(J_f(P))$ can be identified with the subposet (actually a sublattice) of Young's lattice consisting of all partitions whose parts differ by at least 2. It follows from Exercise 1.88 that

$$F_{J_f(J_f(P))}(q) = \frac{1}{\prod_{k\geq 0}(1-q^{5k+1})(1-q^{5k+4})}.$$

47. b. Induction on $\#L$. Trivial for $\#L = 1$. Now let $\#L \geq 2$, and let t be a maximal element of L. Suppose that t covers j elements of L, and set $L' = L - \{t\}$. The meet-distributivity hypothesis implies that the number of $s \leq t$ for which $[s,t] \cong B_k$

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is equal to $\binom{j}{k}$. Hence,

$$\sum_{k\geq 0} g_k(L)x^k = x^j + \sum_{k\geq 0} g_k(L')(1+x)^k, \text{ and}$$

$$\sum_{k\geq 0} f_k(L)x^k = \sum_{k=0}^{j}\binom{j}{k}x^k + \sum_{k\geq 0} f_k(L')x^k$$
$$= (1+x)^j + \sum_{k\geq 0} f_k(L')x^k,$$

and the proof follows by induction since L' is meet-distributive.

Note that in the special case $L = J(P)$, $g_k(L)$ is equal to the number of k-element antichains of P.

c. Let $x = -1$ in (b). This result was first proved (in a different way) for distributive lattices by S. K. Das, *J. Combinatorial Theory Ser. B* **26** (1979), 295–299. It can also be proved using the identity $\zeta\mu\zeta = \zeta$ in the incidence algebra of the lattice $L \cup \{\hat{1}\}$.

Topological Remark. This exercise has an interesting topological generalization (done in collaboration with G. Kalai). Given L, define an abstract cubical complex $\Omega = \Omega(L)$ as follows: the vertices of Ω are the elements of L, and the faces of Ω consist of intervals $[s,t]$ of L isomorphic to boolean algebras. (It follows from Exercise 3.177(a) that Ω is indeed a cubical complex.)

Proposition. *The geometric realization* $|\Omega|$ *is contractible.* In fact, Ω is collapsible.

Sketch of Proof. Let t be a maximal element of L, let $L' = L - \{t\}$, and let s be the meet of elements that t covers, so $[s,t] \cong B_k$ for some $k \in \mathbb{P}$. Then $|\Omega(L')|$ is obtained from $|\Omega(L)|$ by collapsing the cube $|[s,t]|$ onto its boundary faces that don't contain t. Thus by induction, $\Omega(L)$ is collapsible, so $|\Omega(L)|$ is contractible. $\square$

The formula $\sum(-1)^k f_k = 1$ asserts merely that the Euler characteristic of $\Omega(L)$ or $|\Omega(L)|$ is equal to 1; the statement that $|\Omega(L)|$ is contractible is much stronger. For some further results along these lines involving homotopy type, see P. H. Edelman, V. Reiner, and V. Welker, *Discrete & Computational Geometry* **27**(1) (2002), 99–116.

d. A k-element antichain A of $\boldsymbol{m} \times \boldsymbol{n}$ has the form

$$A = \{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\},$$

where $1 \leq a_1 < a_2 < \cdots < a_k \leq m$ and $n \geq b_1 > b_2 > \cdots > b_k \geq 1$. Hence, $g_k = \binom{m}{k}\binom{n}{k}$.

It is easy to compute, either by a direct combinatorial argument or by (b) and Vandermonde's convolution (Example 1.1.17), that $f_k = \binom{m}{k}\binom{m+n-k}{m}$.

e. This result was proved independently by J. R. Stembridge (unpublished) and R. A. Proctor, *Proc. Amer. Math. Soc.* **89** (1983), 553–559 (Theorem 2). Later Stembridge gave another proof in *Europ. J. Combinatorics* **7** (1986), 377–387 (Corollary 2.2).

f. R. A. Proctor, op. cit., Theorem 1.

g. This result was conjectured by P. H. Edelman for $m = n$, and first proved in general by R. Stanley and J. Stembridge using the theory of "jeu de taquin" (see Chapter 7, Vol. II, Appendix A1.2). An elementary proof was given by M. Haiman (unpublished). See J. R. Stembridge, *Europ. J. Combinatorics* **7** (1986), 377–387, for details and additional results (see in particular Corollary 2.4).

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48. Induction on $\rho(t)$. Clearly true for $\rho(t) \le 1$. Assume true for $\rho(t) < k$, and let $\rho(t) = k$. If t is join-irreducible, then the conclusion is clear. Otherwise t covers $r > 1$ elements. By the Principle of Inclusion-Exclusion and the induction hypothesis, the number of join-irreducibles $s \le t$ is

$$r(k-1) - \binom{r}{2}(k-2) + \binom{r}{3}(k-3) - \cdots \pm \binom{r}{k-1} = k.$$

For further information on this result and on meet-distributive lattices in general, see B. Monjardet, *Order* **1** (1985), 415–417, and P. H. Edelman, *Contemporary Math.* **57** (1986), 127–150. Other references include C. Greene and D. J. Kleitman, *J. Combinatorial Theory Ser. A* **20** (1976), 41–68 (Thm. 2.31); P. H. Edelman, *Alg. Universalis* **10** (1980), 290–299; and P. H. Edelman and R. F. Jamison, *Geometriae Ded.* **19** (1985), 247–270.

49. Routine. For more information on the posets L_p, see R. A. Dean and G. Keller, *Canad. J. Math.* **20** (1968), 535–554.

50. The left-hand side of equation (3.114) counts the number of pairs (s, S) where s is an element of L of rank i and S is a set of j elements that s covers. Similarly the right-hand side is equal to the number of pairs (t, T) where $\rho(t) = i - j$ and T is a set of j elements that cover t. We set up a bijection between the pairs (s, S) and (t, T) as follows. Given (s, S), let $t = \bigwedge_{w \in S} w$, and define T to be set of all elements in the interval $[t, s]$ that cover t.

51. **a.** Let L be a finitary distributive lattice with cover function f. Let L_k denote the sublattice of L generated by all join-irreducibles of rank at most k. We prove by induction on k that L_k is unique (if it exists). Since $L = \bigcup L_k$, the proof will follow.

True for $k = 0$, since L_0 is a point. Assume for k. Now L_k contains all elements of L of rank at most k. Suppose that t is an element of L_k of rank k covering n elements, and suppose that t is covered by c_t elements in L_k. Let $d_t = f(n) - c_t$. If $d_t < 0$ then L does not exist, so assume $d_t \ge 0$. Then the d_t elements of $L - L_k$ that cover t in L must be join-irreducibles of L. Thus for each $t \in L_k$ of rank k, attach d_t join-irreducibles covering t, yielding a meet-semilattice L'_k. Let P_{k+1} denote the poset of join-irreducibles of L'_k. Then P'_{k+1} must coincide with the poset of join-irreducibles of L_{k+1}. Hence, $L_{k+1} = J(P_{k+1})$, so L_{k+1} is uniquely determined.

b. See Proposition 2 on page 226 of [3.73].

c. If $f(n) = k$ then $L = \mathbb{N}^k$. If $f(n) = n + k$, then $L = J_f(\mathbb{N}^2)^k$.

d. Use Exercise 3.50 to show that

$$u(5,1) = -(k/3)(2a^3 - 2a^2 - 3).$$

Hence, $u(5,1) < 0$ if $a \ge 2$ and $k \ge 1$, so L does not exist.

e. See J. D. Farley, *Graphs and Combinatorics* **19** (2003), 475–491 (Theorem 11.1).

55. **a.** See K. Saito, *Advances in Math.* **212** (2007), 645–688 (Theorem 3.2).

b. Essentially this question was raised by Saito, ibid. (Remark 4).

56. Let E be the set of all (undirected) edges of the Hasse diagram of P. Define $e, f \in E$ to be *equivalent* if e has vertices s, u and f has vertices t, u, such that either both $s < u$ and $t < u$, or both $s > u$ and $t > u$. Extend this equivalence to an equivalence relation using reflexivity and transitivity. The condition on P implies that the equivalence classes are paths and cycles. We obtain a partition of P into disjoint saturated chains by choosing a set of edges, no two consecutive, from each equivalence class. If an element t of P does not lie on one of the chosen edges, then it forms a one-element saturated chain. The number of ways to choose a set of edges, no two consecutive, from a path of length ℓ is the Fibonacci number $F_{\ell+2}$. The number of ways to choose a

set of edges, no two consecutive, from a cycle of length ℓ is the Lucas number L_ℓ (see Exercise 1.40), and the proof follows. This result is due to R. Stanley, *Amer. Math. Monthly* **99** (1992); published solution by W. Y. C. Chen, **101** (1994), 278–279.
For $P = \boldsymbol{m} \times \boldsymbol{n}$ the equivalence classes consist of all cover relations between two consecutive ranks. Assuming $m \leq n$, we obtain

$$f(\boldsymbol{m} \times \boldsymbol{n}) = F_{2m+3}^{n-m} \prod_{i=1}^{m} F_{2i+2}^2.$$

59. It is straightforward to prove by induction on n that a_i is the number of strict surjective maps $\tau : P \to \boldsymbol{i}$, i.e., τ is surjective, and if $s < t$ in P then $\tau(s) < \tau(t)$. See R. Stanley, *Discrete Math.* **4** (1973), 77–82.

60. **b.** This result is implicit in J. R. Goldman, J. T. Joichi, and D. E. White, *J. Combinatorial Theory Ser. B* **25** (1978), 135–142 (put $x = -1$ in Theorem 2) and J. P. Buhler and R. L. Graham, *J. Combinatorial Theory Ser. A* **66** (1994), 321–326 (put $\lambda = -1$ and use our equation (3.121) in the theorem on page 322), and explicit in E. Steingrímsson, Ph.D. thesis, M.I.T., 1991 (Theorem 4.12). For an application see R. Stanley, *J. Combinatorial Theory Ser. A* **100** (2002), 349–375 (Theorem 4.8).
Sketch of Proof. Given the dropless labeling $f : P \to [p]$, define an acyclic orientation $\mathfrak{o} = \mathfrak{o}(f)$ as follows. If st is an edge of $\mathrm{inc}(P)$, then let $s \to t$ in $\mathfrak{o}$ if $f(s) < f(t)$. Clearly $\mathfrak{o}$ is an acyclic orientation of $\mathrm{inc}(P)$. Conversely, let $\mathfrak{o}$ be an acyclic orientation of $\mathrm{inc}(P)$. The set of sources (i.e., vertices with no arrows into them) form a chain in P since otherwise two are incomparable, and there is an arrow between them that must point into one of them. Let s be the minimal element of this chain (i.e., the unique minimal source). If f is a dropless labeling of P with $\mathfrak{o} = \mathfrak{o}(f)$, then we claim $f(s) = 1$. Suppose to the contrary that $f(s) = i > 1$. Let j be the largest integer satisfying $j < i$ and $t := f^{-1}(j) \not< s$. Note that j exists since $f^{-1}(1) > s$. We must have $t > s$ since s is a source. But then $f^{-1}(j+1) \leq s < t = f^{-1}(j)$, contradicting the fact that f is dropless. Thus, we can set $f(s) = 1$, remove s from $\mathrm{inc}(P)$, and proceed inductively to construct a unique f satisfying $\mathfrak{o} = \mathfrak{o}(f)$.

61. Write $\mathrm{Comp}(n)$ for the set of compositions of n. Regarding n as given, and given a set $S = \{i_1, i_2, \ldots, i_j\}_< \subseteq [n-1]$, define the composition

$$\sigma_S = (i_1, i_2 - i_1, \ldots, i_j - i_{j-1}, n - i_j) \in \mathrm{Comp}(n).$$

Given a sequence $u = b_1 \cdots b_k$ of distinct integers, let $D(u) = \{i_1, i_2, \ldots, i_j\}_< \subseteq [k-1]$ be its descent set. Now given a permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$, let $w[k] = a_1 \cdots a_k$. It can be checked that

$$\sigma_{D(w[1])} \lessdot \sigma_{D(w[2])} \lessdot \cdots \lessdot \sigma_{D(w[n])}$$

is a saturated chain $\mathfrak{m}$ in $\mathcal{C}$ from 1 to $\sigma = \sigma(w[n])$, and that the map $w \mapsto \mathfrak{m}$ is a bijection from $\mathfrak{S}_n$ to saturated chains in $\mathcal{C}$ from 1 to a composition of n. Hence, the number of saturated chains from 1 to $\sigma \in \mathrm{Comp}(n)$ is $\beta_n(S)$, the number of $w \in \mathfrak{S}_n$ with descent set S, where $\sigma = \sigma_S$. In particular, the total number of saturated chains from 1 to some composition of n is $\sum_S \beta_n(S) = n!$. This latter fact also follows from the fact that every $\alpha \in \mathrm{Comp}(n)$ is covered in $\mathcal{C}$ by exactly $n+1$ elements.
The poset $\mathcal{C}$ was first defined explicitly in terms of compositions by Björner and Stanley (unpublished). It was pointed out by S. Fomin that $\mathcal{C}$ is isomorphic to the subword order on all words in a two-letter alphabet (see Exercise 3.134). A generalization was given by B. Drake and T. K. Petersen, *Electronic J. Combinatorics* **14**(1) (2007), #R23.

62. **f.** Let k_i be the number of λ_j's that are equal to i in a protruded partition (λ,μ). If some $a_j=i$, then μ_j can be any of $0,1,\dots,i$, so a_j+b_j is one of $i,i+1,\dots,2i$. Hence,

$$\begin{aligned}U_{P_n}(x) &= \prod_{i=1}^{n}\left(\sum_{k\geq 0}(x^i+x^{i+1}+\cdots+x^{2i})^k\right)\\ &= \prod_{i=1}^{n}(1-x^i-x^{i+1}-\cdots-x^{2i})^{-1}.\end{aligned}$$

g. Write

$$\sum_{n\geq 0}U_{P_n}(x)q^n = P(q,x)\sum_{j\geq 0}W_j(x)q^j.$$

The poset P satisfies $P\cong \mathbf{1}\oplus(\mathbf{1}+P)$. This leads to the recurrence

$$W_j(x)=\frac{x^{2j}}{1-x}W_{j-1}(x)+x^j\,\frac{1-x^{j+1}}{1-x}W_j(x),\quad W_0(x)=1.$$

Hence $W_j(x)=x^{2j}W_{j-1}(x)/(1-x^j)(1-x-x^{j+1})$, from which the proof follows. Protruded partitions are due to Stanley [2.19, Ch. 5.4][2.20, §24], where more details of the above argument can be found. For a less combinatorial approach, see Andrews [1.2, Exam. 18, p. 51].

65. The fact that Exercise 1.11 can be interpreted in terms of linear extensions is an observation of I. M. Pak (private communication). NOTE. Equation (3.116) continues to hold if $2t$ is an odd integer, provided we replace any factorial $m!$ with the corresponding Gamma function value $\Gamma(m+1)$.

66. **a.** The Fibonacci number F_{n+2}—a direct consequence of Exercise 1.35(e).

b. Simple combinatorial proofs can be given of the recurrences

$$\begin{aligned}W_{2n} &= W_{2n-1}+q^2W_{2n-2},\ n\geq 1,\\ W_{2n+1} &= qW_{2n}+W_{2n-1},\ n\geq 1.\end{aligned}$$

It follows easily from multiplying these recurrences by x^{2n} and x^{2n+1}, respectively, and summing on n, that

$$F(x)=\frac{1+(1+q)x-q^2x^3}{1-(1+q+q^2)x^2+q^2x^4}.$$

c. A bijection $\sigma: Z_n\to[n]$ is a linear extension if and only if the sequence $n+1-\sigma(t_1),\dots,n+1-\sigma(t_n)$ is an alternating permutation of $[n]$ (as defined in Section 1.4). Hence $e(Z_n)$ is the Euler number E_n, and by Proposition 1.6.1 we have

$$\sum_{n\geq 0}e(Z_n)\frac{x^n}{n!}=\tan x+\sec x.$$

d. Adjoin an extra element t_{n+1} to Z_n to create Z_{n+1}. We can obtain an order-preserving map $f: Z_n\to \boldsymbol{m+2}$ as follows. Choose a composition $a_1+\cdots+a_k=n+1$, and associate with it the partition $\{t_1,\dots,t_{a_1}\},\{t_{a_1+1},\dots,t_{a_1+a_2}\},\dots$ of Z_{n+1}. For example, choosing $n=17$ and $3+1+2+4+1+2+2+3=18$ gives the partition shown in Figure 3.75. Label the last element t of each block by 1 or $m+2$, depending on whether t is a minimal or maximal element of Z_{n+1}, as shown in

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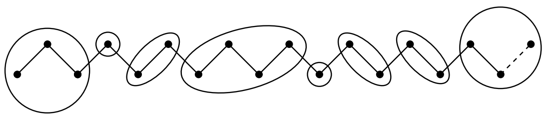

Figure 3.75 Illustration of the solution to Exercise 3.66(d).

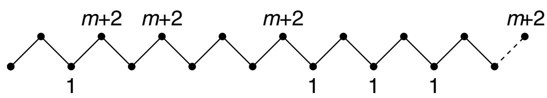

Figure 3.76 Continuing the solution to Exercise 3.66(d).

Figure 3.76. Removing these labeled elements from Z_{n+1} yields a disjoint union $Y_1 + \cdots + Y_k$, where Y_i is isomorphic to Z_{a_i-1} or $Z^*_{a_i-1}$ (where * denotes dual). For each i choose an order-preserving map $Y_i \to [2, m+1]$ in $\Omega_{Z_{a_i-1}}(m)$ ways. There is one additional possibility. If some $a_i = 2$, then we can also assign the unique element t of Y_i the same label (1 or $m+2$) as the remaining element s in the block containing t (so t is labeled 1 if it is a maximal element of Z_{n+1} and $m+2$ if it is minimal). This procedure yields each order-preserving map $f\colon Z_n \to \boldsymbol{m+2}$ exactly once. Hence,

$$\begin{aligned}\Omega_{Z_n}(m+2) &= \sum_{a_1+\cdots+a_k=n+1} \prod_{i=1}^{k} (\Omega_{Z_{a_i-1}}(m) + \delta_{2,a_i})\\ \Rightarrow G_{m+2}(x) &= \sum_{k\geq 0} (G_m(x) - 1 + x^2)^k\\ &= (2 - x^2 - G_m(x))^{-1}.\end{aligned}$$

The initial conditions are $G_1(x) = 1/(1-x)$ and $G_2(x) = 1/(1-x-x^2)$. An equivalent result was stated without proof (with an error in notation) in Example 3.2 of R. Stanley, *Annals of Discrete Math.* **6** (1980), 333–342. Moreover, G. Ziegler has shown (unpublished) that

$$G_{m+1}(x) = \frac{1 + G_m(x)}{3 - x^2 - G_m(x)}.$$

67. A complicated proof was first given by G. Kreweras, *Cahiers Bur. Univ. Rech. Operationnelle*, no. 6, 1965 (eqn. (85). Subsequent proofs were given by H. Niederhausen, Proc. West Coast Conf. on Combinatorics, Graph Theory, and Computing (Arcata, Calif., 1979), *Utilitas Math.*, Winnipeg, Man., 1980, pp. 281–294, and Kreweras and Niederhausen, *Europ. J. Combinatorics* **4** (1983), 161–167.

68. **a.** See E. Munarini, *Ars Combin.* **76** (2005), 185–192. For further properties of order ideals and antichains of garlands, see E. Munarini, *Integers* **9** (2009), 353–374.

69. **a.** This result is due to R. Stanley, *J. Combinatorial Theory Ser. A* **31** (1981), 56–65 (see Theorem 3.1). The proof uses the Aleksandrov-Fenchel inequalities from the theory of mixed volumes.

b. This result was proved by J. N. Kahn and M. Saks, *Order* **1** (1984), 113–126, with $\frac{5\pm\sqrt{5}}{10}$ replaced with $\frac{3}{11}$ and $\frac{8}{11}$. The improvement to $\frac{5\pm\sqrt{5}}{10}$ is due to G. R. Brightwell, S. Felsner, and W. T. Trotter, *Order* **12** (1995), 327–349. Both proofs use (a). It is conjectured that there exist s,t in P such that $f(s) < f(t)$ in no fewer than $\frac{1}{3}$ and no more than $\frac{2}{3}$ of the linear extensions of P. The poset $\mathbf{2}+\mathbf{1}$ shows that this

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result, if true, would be best possible. On the other hand, Brightwell, Felsner, and Trotter show that their result is best possible for a certain class of countably infinite posets, called *thin* posets.

70. **c.** Due to Ethan Fenn, private communication, November 2002.

71. The result for $\mathrm{FD}(n)$ is due to Dedekind. See [3.13, Ch. III, §4]. The result for $\mathrm{FD}(P)$ is proved the same way. See, for example, Corollary 6.3 of B. Jónsson, in [3.57, pp. 3–41]. For some related results, see J. V. Semegni and M. Wild, Lattices freely generated by posets within a variety. Part I: Four easy varieties, `arXiv:1004.4082`; Part II: Finitely generated varieties, `arXiv:1007.1643`.

72. **a.** The proof easily reduces to the following statement: if A and B are k-element antichains of P, then $A \cup B$ has k maximal elements. Let C and D be the set of maximal and minimal elements, respectively, of $A \cup B$. Since $t \in A \cap B$ if and only if $t \in C \cap D$, it follows that $\#C + \#D = 2k$. If $\#C < k$, then D would be an antichain of P with more than k elements, a contradiction.

This result is due to R. P. Dilworth, in *Proc. Symp. Appl. Math.* (R. Bellman and M. Hall, Jr., eds.), American Mathematical Society, Providence, R.I., 1960, pp. 85–90. An interesting application appears in §2 of C. Greene and D. J. Kleitman, in *Studies in Combinatorics* (G.-C. Rota, ed.), Mathematical Association of America, 1978, pp. 22–79.

b. R. M. Koh, *Alg. Univ.* **17** (1983), 73–86, and **20** (1985), 217–218.

73. **a.** Let $p\colon P \otimes Q \to P$ be the projection map onto P (i.e., $p(s,t) = s$), and let I be an order ideal of $P \otimes Q$. Then $p(I)$ is an order ideal of P, say with m maximal elements $t_1, \dots, t_m$ and k nonmaximal elements $s_1, \dots, s_k$. Then I is obtained by taking $p^{-1}(s_1) \cup \cdots \cup p^{-1}(s_k)$ together with a *nonempty* order ideal I_i of each $p^{-1}(t_i) \cong Q$. We then have $\#I = kn + \sum \#I_i$ and $m(I) = \sum m(I_i)$. Hence,

$$\begin{aligned}\sum_{I \in J(P \otimes Q)} q^{\#I} x^{m(I)} &= \sum_{T \in J(P)} q^{n(\#T - m(T))} (G_Q(q,x) - 1)^{m(T)} \\ &= G_P(q^n, q^{-n}(G_Q(q,x) - 1)).\end{aligned}$$

b. Let t be a maximal element of P, and let $\Lambda_t = \{s \in P : s \leq t\}$. Set $P_1 = P - t$ and $P_2 = P - \Lambda_t$. write $G(P) = G_P(q, (q-1)/q)$. One sees easily that

$$G(P) = G(P_1) + (q-1) q^{\#\Lambda_t - 1} G(P_2),$$

by considering for each $I \in J(P)$ whether $t \in I$ or $t \notin I$. By induction we have $G(P_1) = q^{p-1}$ and $G(P_2) = q^{\#(P - \Lambda_t)}$, so the proof follows.
This exercise is due to M. D. Haiman.

74. If $L_n = J(P_n)$, then P_n is the complete dual binary tree of height n, as illustrated in Figure 3.77. An order ideal I of P_n defines a stopping rule as follows: start at $\hat{0}$, and move up one step left (respectively, right) after tossing a tail (respectively, head). Stop as soon as you leave I or have reached a maximal element of P_n.
Since $P_n = \mathbf{1} \oplus (P_{n-1} + P_{n-1})$, it follows easily that

$$F(L_n, q) = 1 + q F(L_{n-1}, q)^2.$$

75. This result is due to J. Propp, Lattice structure for orientations of graphs, preprint, 1993, and was given another proof using hyperplane arrangements by R. Ehrenborg and M. Slone, *Order* **26** (2009), 283–288.

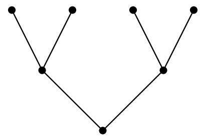

Figure 3.77 The complete dual binary tree P_3.

76. See R. Stanley, *Bull. Amer. Math. Soc.* **76** (1972), 1236–1239; [3.25, §3]; K. Baclawski, *Proc. Amer. Math. Soc.* **36** (1972), 351–356; R. B. Feinberg, *Pacific J. Math.* **65** (1976), 35–45; R. B. Feinberg, *Discrete Math.* **17** (1977), 47–70; M. Wild, *Linear Algebra Appl.* **430** (2009), 1007–1016; Y. Drozd and P. Kolesnik, *Comm. Alg.* **35** (2007), 3851–3854. A wealth of additional material can be found in Spiegel and O'Donnell [3.66].

77. **a–c.** These results are part of a beautiful theory of chains and antichains developed originally by C. Greene and D. J. Kleitman, *J. Combinatorial Theory Ser. A* **20** (1976), 41–68, and C. Greene, *J. Combinatorial Theory Ser. A* **20** (1976), 69–70. They were rediscovered by S. Fomin, *Soviet Math. Dokl.* **19** (1978), 1510–1514. Subsequently, two other elegant approaches were discovered, the first based on linear algebra by E. R. Gansner, *SIAM J. Algebraic Discrete Methods* **2** (1981), 429–440, and the second based on network flows by A. Frank, *J. Combinatorial Theory, Ser. B* **29** (1980), 176–184. A survey of this latter method (with much additional information) appears in T. Britz and S. Fomin, *Advances in Math.* **158** (2001), 86–127.

d. Clearly, $k = \ell(\lambda)$, and by (c) we have $\ell(\lambda) = \mu_1$. This famous result, which can be regarded as a special case of the duality theorems of network flows and linear programming, is due to R. P. Dilworth, *Ann. Math.* **51** (1950), 161–166.

e. Clearly $\mu_1' \geq \lambda_1$, since an antichain intersects a chain in at most one element. On the other hand, we have $P = P_1 \cup \cdots \cup P_{\lambda_1}$, where P_1 is the set of minimal elements of P, P_2 is the set of minimal elements of $P - P_1$, and so on. Each P_i is an antichain, so $\mu_1' \leq \lambda_1$. Note that this "dual" version of Dilworth's theorem is much easier to prove than Dilworth's theorem itself.

f. See M. Saks, *SIAM J. Algebraic Discrete Methods* **1** (1980), 211–215, and *Discrete Math.* **59** (1986), 135–166, and E. R. Gansner, op. cit. An erroneous determination of the Jordan block sizes of A was earlier given by A. C. Aitken, *Proc. London Math. Soc. (2)* **38** (1934), 354–376, and D. E. Littlewood *Proc. London Math. Soc. (2)* **40** (1936), 370–381, and Vol. II, [7.88, §10.2].

78. **a.** The lattice B_n has the property, known as the *strong Sperner property*, that a maximum size union of k antichains consists of the union of the k largest ranks. Hence μ_i is just the ith largest binomial coefficient $\binom{n}{j}$. Some other posets with the strong Sperner property are any finite product of chains, $B_n(q)$, $J(\boldsymbol{m} \times \boldsymbol{n} \times \boldsymbol{r})$ for any $m,n,r \geq 1$, and $J(J(\boldsymbol{2} \times \boldsymbol{n}))$. On the other hand, it is unknown whether $J(\boldsymbol{m} \times \boldsymbol{n} \times \boldsymbol{r} \times \boldsymbol{s})$ has the strong Sperner property. For further information see K. Engel, *Sperner Theory*, Cambridge University Press, Cambridge, 1997.

b. G.-C. Rota, *J. Combinatorial Theory* **2** (1967), 104, conjectured that the size of the largest antichain in Π_n was the maximum Stirling number $S(n,k)$ (i.e., the largest rank in Π_n was a maximum size antichain). This conjecture was disproved by E. R. Canfield, *Bull. Amer. Math. Soc.* **84** (1978), 164. For additional information, see E. R. Canfield, *J. Combinatorial Theory Ser. A* **83** (1998), 188–201.

c. By Vol. II, Exercise 7.2(f) we have $\lambda_1 = \frac{1}{3}m(m^2 + 3r - 1)$, where $n = \binom{m+1}{2} + r$, $0 \leq r \leq m$. E. Early, Ph.D. thesis, M.I.T., 2004 (§2), showed that $\lambda_2 = \lambda_1 - 6$

for $n > 16$, and $\lambda_3 = \lambda_2 - 6$ for $n > 135$. Early conjectures that for large n, $\lambda_i - \lambda_{i+1}$ depends only on i. It is an interesting open problem to determine μ_1. Some observations on this problem are given by Early, ibid.

79. **i=ii.** Let $w = a_1 \cdots a_p \in \mathcal{L}(P)$. Let i be the least nonnegative integer (if it exists) for which

$$w' := a_1 \cdots a_{p-2i-2} a_{p-2i} a_{p-2i-1} a_{p-2i+1} \cdots a_p \in \mathcal{L}(P).$$

Note that $w'' = w$. Now exactly one of w and w' has the descent $p-2i-1$. The only other differences in the descent sets of w and w' occur (possibly) for the numbers $p-2i-2$ and $p-2i$. Hence, $(-1)^{\mathrm{comaj}(w)} + (-1)^{\mathrm{comaj}(w')} = 0$. The surviving permutations $w = b_1 \cdots b_p$ in $\mathcal{L}(P)$ (those for which i does not exist) are exactly those for which the chain of order ideals

$$\emptyset \subset \cdots \subset \{b_1, b_2, \dots, b_{p-4}\} \subset \{b_1, b_2, \dots, b_{p-2}\} \subset \{b_1, b_2, \dots, b_p\} = P$$

is a P-domino tableau. We call w a *domino linear extension*; they are in bijection with domino tableaux. Such permutations w can only have descents in positions $p-j$ where j is even, so $(-1)^{\mathrm{comaj}(w)} = 1$. Hence, (i) and (ii) are equal. This result, stated in a dual form, appears in R. Stanley, *Advances Appl. Math.* **34** (2005), 880–902 (Theorem 5.1(a)).

ii=iii. Let τ_i be the operator on $\mathcal{L}(P)$ defined by equation (3.102). Thus, w is self-evacuating if and only if

$$w = w\tau_1\tau_2 \cdots \tau_{p-1} \cdot \tau_1 \cdots \tau_{p-2} \cdots \tau_1\tau_2\tau_3 \cdot \tau_1\tau_2 \cdot \tau_1.$$

On the other hand, note that w is a domino linear extension if and only if

$$w\tau_{p-1}\tau_{p-3}\tau_{p-5} \cdots \tau_h = w,$$

where $h = 1$ if p is even, and $h = 2$ if p is odd. We claim that w is a domino linear extension if and only if

$$\widetilde{w} := w\tau_1 \cdot \tau_3\tau_2\tau_1 \cdot \tau_5\tau_4\tau_3\tau_2\tau_1 \cdots \tau_m\tau_{m-1} \cdots \tau_1$$

is self-evacuating, where $m = p-1$ if p is even, and $m = p-2$ if p is odd. The proof follows from this claim since the map $w \mapsto \widetilde{w}$ is then a bijection between domino linear extensions and self-evacuating linear extensions of P.

The claim is proved by an elementary argument analogous to the proof of Theorem 3.20.1. The cases p even and p odd need to be treated separately. We won't give the details here but will prove the case $p = 6$ as an example. For notational simplicity we write simply i for τ_i. We need to show that the two conditions

$$w = w135 \tag{3.138}$$

$$w132154321 = w132154321 \cdot 123451234123121 \tag{3.139}$$

are equivalent. (The first condition says that w is a domino linear extension, and the second that $w132154321$ is self-evacuating.) The internal factor $32154321 \cdot 12345123$ cancels out of the right-hand side of equation (3.139). We can also cancel the rightmost 21 on both sides of (3.139). Thus, (3.139) is equivalent to $w1321543 = w141231$. Now $w1321543 = w1352143$ and $w141231 = w112143 = w2143$. Cancelling 2143 from the right of both sides yields $w135 = w$. Since all steps are reversible, the claim is proved for $p = 6$.

The equality of (ii) and (iii) was first proved by J. R. Stembridge, *Duke Math. J.* **82** (1996), 585–606, for the special case of standard Young tableaux (i.e., when P is a finite order ideal of $\mathbb{N} \times \mathbb{N}$). Stembridge's proof was based on representation theory.

He actually proved a more general result involving *semistandard* tableaux that does not seem to extend to other posets. The bijective argument given here, again for the case of semistandard tableaux, is due to A. Berenstein and A. N. Kirillov, *Discrete Math.* **225** (2000), 15–24.

The equivalence of (i) and (iii) is an instance of Stembridge's "$q = -1$ phenomenon." Namely, suppose that an involution ι acts on a finite set S. Let $f : S \to \mathbb{Z}$. (Usually f will be a "natural" combinatorial or algebraic statistic on S.) Then we say that the triple (S, ι, f) exhibits the $q = -1$ phenomenon if the number of fixed points of ι is given by $\sum_{t\in S}(-1)^{f(t)}$. See J. R. Stembridge, *J. Combinatorial Theory Ser. A* **68** (1994), 373–409; *Duke Math. J.* **73** (1994), 469–490; and *Duke Math. J.* **82** (1996), 585–606. The $q = -1$ phenomenon has been generalized to the action of cyclic groups by V. Reiner, D. Stanton, and D. E. White, *J. Combinatorial Theory Ser. A* **108** (2004), 17–50, where it is called the "cyclic sieving phenomenon." For further examples of the cyclic sieving phenomenon, see C. Bessis and V. Reiner, *Ann. Combinatorics*, submitted, `arXiv:math/0701792`; H. Barcelo, D. Stanton, and V. Reiner, *J. London Math. Soc. (2)* **77** (2009), 627–646; and B. Rhoades, Cyclic sieving and promotion, preprint.

80. Part (a)(i) follows easily from work of Schützenberger, whereas (a)(ii)–(a)(iii) are due to Haiman and (b) to Edelman and Greene. For further details, see R. Stanley, *Electronic J. Combinatorics* **15**(2) (2008–2009), #R9 (§4).

81. **a.** If $a_1a_2a_3\cdots a_n \in \mathcal{L}(P)$, then $a_2a_1a_3\cdots a_n \in \mathcal{L}(P)$.
b. Hint. Show that the promotion operator $\partial\colon \mathcal{L}(P) \to \mathcal{L}(P)$ always reverses the parity of the linear extension f to which it is applied. See R. Stanley, *Advances in Appl. Math.* **34** (2005), 880–902 (Corollary 2.2). Corollary 2.4 of this reference gives another result of a similar nature.

82. This result is due to D. White, *J. Combinatorial Theory Ser. A* **95** (2001), 1–38 (Corollary 20 and §8). White also computes the "sign imbalance" $\left|\sum_{w\in\mathcal{E}_{\boldsymbol{p}\times\boldsymbol{q}}} \operatorname{sgn}(w)\right|$ when $\boldsymbol{p}\times\boldsymbol{q}$ in not sign-balanced. A conjectured generalization for any finite-order ideal of $\mathbb{N}\times\mathbb{N}$ appears in R. Stanley, *Advances in Appl. Math.* **34** (2005), 880–902 (Conjecture 3.6).

83. Let $\#P = 2m$, and suppose that there does not exist a P-domino tableau. Let $w = a_1a_2\cdots a_{2m} \in \mathcal{E}_P$. Since there does not exist a P-domino tableau, there is a least i for which a_{2i-1} and a_{2i} are incomparable. Let w' be the permutation obtained from w by transposing a_{2i-1} and a_{2i}. Then the map $w \mapsto w'$ is an involution on $\mathcal{E}_P$ that reverses parity, and the proof follows. This result appears in R. Stanley, ibid. (Corollary 4.2), with an analogous result for $\#P$ odd.

84. We have

$$\begin{aligned}
\sum_{\substack{u\in P\\ \bar{u}=\bar{t}}} \mu(s,u) &= \sum_{u} \mu(s,u)\delta_{\overline{P}}(\bar{u},\bar{t})\\
&= \sum_{u,\bar{v}} \mu(s,u)\zeta_{\overline{P}}(\bar{u},\bar{v})\mu_{\overline{P}}(\bar{v},\bar{t})\\
&= \sum_{u,\bar{v}} \mu(s,u)\zeta(u,\bar{v})\mu_{\overline{P}}(\bar{v},\bar{t}) \quad (\text{since } u\le\bar{v} \Leftrightarrow \bar{u}\le\bar{v})\\
&= \sum_{\bar{v}\in\overline{P}} \delta(s,\bar{v})\mu_{\overline{P}}(\bar{v},\bar{t}).
\end{aligned}$$

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This fundamental result was first given by H. Crapo, *Archiv der Math.* **19** (1968), 595–607 (Thm. 1), simplifying some earlier work of G.-C. Rota in [3.58]. For an exposition of the theory of Möbius functions based on closure operators, see Ch. IV.3 of M. Aigner, *Combinatorial Theory*, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

86. Let $G(s) = \sum_{t \geq s} g(t)$. It is easy to show that

$$\sum_{\hat{0} \leq u \leq s} \mu(\hat{0}, u) G(u) = \sum_{\substack{t \\ s \wedge t = \hat{0}}} g(t) = f(s).$$

Now use Möbius inversion to obtain

$$\mu(\hat{0}, t) G(t) = \sum_{u \leq t} \mu(u, t) f(u). \tag{3.140}$$

On the other hand, Möbius inversion also yields

$$g(s) = \sum_{t \geq s} \mu(s, t) G(t). \tag{3.141}$$

Substituting the value of $G(t)$ from equation (3.140) into (3.141) yields the desired result.
This formula is a result of P. Doubilet, *Studies in Appl. Math.* **51** (1972), 377–395 (lemma on page 380).

87. **a.** Given $C\colon \hat{0} < t_1 < \cdots < t_k < \hat{1}$, the coefficient of $f(t_1) \cdots f(t_k)$ on the left-hand side is

$$\sum_{C' \supseteq C} (-1)^{\#(C'-C)} = (-1)^{k+1} \mu(\hat{0}, t_1) \mu(t_1, t_2) \cdots \mu(t_k, \hat{1}),$$

by Proposition 3.8.5. Here C' ranges over all chains of $P - \{\hat{0}, \hat{1}\}$ containing C. Essentially the same result appears in Ch. II, Lemma 3.2, of [3.67].

b. Put each $f(t) = 1$. All terms on the left-hand side are 0 except for the term indexed by the chain $\hat{0} < \hat{1}$ (an empty product is equal to 1).

c. We have

$$\begin{aligned} \sum_{\hat{0} = t_0 < t_1 < \cdots < t_k = \hat{1}} & (-1)^k \mu(t_0, t_1) \mu(t_1, t_2) \cdots \mu(t_{k-1}, t_k) \\ &= (1 - (\mu - 1) + (\mu - 1)^2 - (\mu - 1)^3 + \cdots)(\hat{0}, \hat{1}) \\ &= (1 + (\mu - 1))^{-1}(\hat{0}, \hat{1}) \\ &= \zeta(\hat{0}, \hat{1}) \\ &= 1. \end{aligned}$$

d. By Example 3.9.6 we have

$$\mu(t_0, t_1) \mu(t_1, t_2) \cdots \mu(t_{k-1}, t_k) = \begin{cases} (-1)^{\ell}, & \text{if the chain } t_0 < t_1 < \cdots < t_k \text{ is boolean}, \\ 0, & \text{otherwise}, \end{cases}$$

and the proof follows easily from (a).

88. Consider $\zeta \mu \zeta(\hat{0}, \hat{1})$ in the incidence algebra $I(P, \mathbb{C})$. For a similar trick, see the solution to Exercise 3.47(b). A solution can also be given based on Philip Hall's theorem (Proposition 3.8.5).

92. This result is known as the "Crapo complementation theorem." See H. H. Crapo, *J. Combinatorial Theory* **1** (1966), 126–131 (Thm. 3). For topological aspects of this result, see A. Björner, *J. Combinatorial Theory Ser. A* **30** (1981), 90–100.

93. **a.** By the inductive definition (3.15) of the Möbius function, it follows that $\mu_L(\hat{0},t)$ is *odd* (and therefore nonzero) for all $t \in L$. Now use Exercise 3.92.

b. See R. Freese and Univ. of Wyoming Problem Group, *Amer. Math. Monthly* **86** (1979), 310–311.

94. This result (stated slightly differently) is due to G.-C. Rota, in *Studies in Pure Mathematics* (L. Mirsky, ed.), Academic Press, London, 1971, pp. 221–233 (Thm. 2). Related papers include G.-C. Rota, in *Proc. Univ. Houston Lattice Theory Conf.*, 1973, pp. 575–628; L. D. Geissinger, *Arch. Math. (Basel)* **24** (1973), 230–239, 337–345, and in *Proc. Third Caribbean Conference on Combinatorics and Computing*, University of the West Indies, Cave Hill, Barbados, pp. 125–133; R. L. Davis, *Bull. Amer. Math. Soc.* **76** (1970), 83–87; H. Dobbertin, *Order* **2** (1985), 193–198. See also Exercise 4.58.

95. See Greene [3.34, Thm. 5].

96. Our exposition for this entire exercise is based on Greene [3.35].

a. Define a matrix $M = [M(s,t)]$ by setting $M(s,t) = \zeta(s,t)f(s,t)$. Clearly if we order the rows and columns of M by some linear extension of L, then M is triangular and $\det M = \prod_s f(s,s)$. On the other hand (writing ζ for the matrix of the ζ-function of L with respect to the basis L, that is, ζ is the incidence matrix of the relation L),

$$\begin{aligned} M^t\zeta &= \left[\sum_u f(u,s)\zeta(u,s)\zeta(u,t)\right]_{s,t\in L} \\ &= \left[\sum_{u\le s\wedge t} f(u,s)\right]_{s,t\in L} = [F(s\wedge t),s]. \end{aligned}$$

Thus, $\det[F(s\wedge t,s)] = \det M^t\zeta = \det M$.

This formula is a result of B. Lindström, *Proc. Amer. Math. Soc.* **20** (1969), 207–208, and (in the case where $F(s,v)$ depends only on s) H. S. Wilf, *Bull. Amer. Math. Soc.* **74** (1968), 960–964.

b. Take L to be the set $[n]$ ordered by divisibility, and let $f(s,v) = \phi(s)$ (so $F(s,v) = s$). For a proof from scratch, see G. Pólya and G. Szegö, *Problems and Theorems in Analysis II*, Springer-Verlag, Berlin/Heidelberg/New York, 1976 (Part VIII, Ch. 1, no. 33).

c. When $f(s,v) = \mu(\hat{0},s)$, we have (suppressing v)

$$F(s\wedge t) = \sum_{u\le s\wedge t} \mu(\hat{0},u) = \delta(\hat{0},s\wedge t).$$

Hence the matrix $R = [F(s\wedge t)]$ is just the incidence matrix of the relation $s\wedge t = \hat{0}$. By (a), $\det R \neq 0$. Hence some term in the expansion of $\det R$ must be nonzero, and this term yields the desired permutation w.

This result is due to T. A. Dowling and R. M. Wilson, *Proc. Amer. Math. Soc.* **47** (1975), 504–512 (Thm. 2*).

d. By Exercise 3.100(b) we have that $\mu(s,t) \neq 0$ for all $s \le t$ in a geometric lattice L. Apply (c) to the dual L^*. We get a permutation $w\colon L \to L$ such that $s \vee w(s) = \hat{1}$ for all $s \in L$. Semimodularity implies $\rho(s) + \rho(w(s)) \ge n$, so w maps elements of rank at most k injectively into elements of rank at least $n-k$.

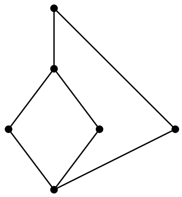

Figure 3.78 A lattice with no complementing permutation.

This result is also due to T. A. Dowling and R. M. Wilson, ibid. (Thm. 1). The case $k = 1$ was first proved by C. Greene, *J. Combinatorial Theory* **2** (1970), 357–364.

e. T. A. Dowling and R. M. Wilson, op. cit. (Thm. 1).

97. See T. A. Dowling, *J. Combinatorial Theory Ser. B* **23** (1977), 223–226. The following elegant proof is due to R. M. Wilson (unpublished). Let ζ be the matrix in the solution to Exercise 3.96(a), and let

$$\Delta_0 = \operatorname{diag}(\mu(\hat{0},t) : t \in L),$$
$$\Delta_1 = \operatorname{diag}(\mu(t,\hat{1}) : t \in L).$$

By the solution to Exercise 3.96(c) (and its dual), we have that

$$\left[\zeta^t \Delta_0 \zeta\right]_{uv} = \delta(\hat{0}, u \wedge v),$$
$$\left[\zeta \Delta_1 \zeta^t\right]_{uv} = \delta(u \vee v, \hat{1}).$$

Let $C = \zeta\Delta_1\zeta^t\Delta_0\zeta$. Since $C = (\zeta\Delta_1\zeta^t)\Delta_0\zeta = \zeta\Delta_1(\zeta^t\Delta_0\zeta)$, it follows that $C_{uv} = 0$ unless u and v are complements. But the hypothesis on L implies that $\det C \neq 0$, and so a nonzero term in the expansion of $\det C$ gives the desired permutation w.

For an example where $\mu(\hat{0},t) \neq 0$ for all $t \in L$, yet a "complementing permutation" does not exist, see Figure 3.78.

100. **a.** Suppose that s is an element of L of maximal rank that is not a meet of coatoms. Thus, s is covered by a unique element t [why?]. Clearly, $t \neq \hat{1}$; else s would be a coatom. Since L is atomic, there is an atom a of L such that $a \not\leq t$; else $\hat{1}$ would not be a join of atoms. Since L is semimodular, $s \vee a$ covers a. Since $a \not\leq t$ we have $s \vee a \neq t$, a contradiction.

b. Consider equation (3.33). By (a) the sum is not empty, so the proof follows by induction on the rank of L (the case $\operatorname{rank}(L) = 1$ being trivial).

c. Induction on $n = \operatorname{rank}(L)$, the case $n = 1$ being trivial. Now assume for $\operatorname{rank}(L) = n-1$, and let $\operatorname{rank}(L) = n$. Let a be an atom of L. By (b), the sum over t in equation (3.33) has exactly one term. Thus, there is exactly one coatom t of L not lying above a, and moreover by the induction hypothesis the interval $[\hat{0},t]$ is isomorphic to B_{n-1}. Since this result holds for all atoms a, it is easy to see that all subsets of the atoms have different joins. Hence, $L \cong B_n$.

101. **a.** Let $f : L \to \mathbb{Q}$, and define $\widehat{f} : L \to \mathbb{Q}$ by

$$\widehat{f}(t) = \sum_{s \leq t} f(s).$$

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For any $t \le t^*$ in L, we have

$$\begin{aligned}\sum_{t\le s\le t^*} \widehat{f}(s)\mu(s,t^*) &= \sum_{t\le s\le t^*} \mu(s,t^*)\sum_{u\le s} f(u)\\ &= \sum_u f(u) \sum_{\substack{t\le s\le t^*\\ u\le s}} \mu(s,t^*)\\ &= \sum_u f(u) \sum_{t\vee u\le s\le t^*} \mu(s,t^*)\\ &= \sum_{\substack{u\\ t\vee u=t^*}} f(u).\end{aligned}$$

Now suppose that $f(t) = 0$ unless $t \in B$. We claim that the restriction $\widehat{f_A}$ of $\widehat{f}$ to A determines $\widehat{f}$ (and hence f since $f(t) = \sum_{s\le t} \widehat{f}(s)\mu(s,t)$ by Möbius inversion). We prove the claim by induction on the length $\ell(t,\hat{1})$ of the interval $[t,\hat{1}]$. If $t = \hat{1}$ that $\hat{1} \in A$ by hypothesis, so $\widehat{f}(\hat{1}) = \widehat{f_A}(\hat{1})$. Now let $t < \hat{1}$. If $t \in A$, then there is nothing to prove, since $\widehat{f}(t) = \widehat{f_A}(t)$. Thus, assume $t \notin A$. Let t^* be as in the hypothesis. Then

$$\sum_{\substack{u\\ t\vee u=t^*}} f(u) = 0 \ \text{ (empty sum)},$$

so

$$\sum_{t\le s\le t^*} \widehat{f}(s)\mu(s,t^*) = 0.$$

By induction, we know $\widehat{f}(s)$ for $t < s$. Since $\mu(t,t^*) \ne 0$, we can then solve for $\widehat{f}(t)$. Hence, the claim is proved.

It follows that the matrix $[\zeta(s,t)]_{\substack{s\in B\\ t\in A}}$ has rank $b = \#B$. Thus, some $b\times b$ submatrix has nonzero determinant. A nonzero term in the expansion of this determinant defines an injective function $\phi\colon B \to A$ with $\phi(s) \ge s$, and the proof follows.

This result and the following applications are due to J. P. S. Kung, *Order* **2** (1985), 105–112; *Math. Proc. Cambridge Phil. Soc.* **101** (1987), 221–231. The solution given here was suggested by C. Greene.

b. These are standard results in lattice theory; for example [3.13], Theorem 13 on page 13 and §IV.6–IV.7.

c. Choose $A = M_k$ and $B = J_k$ in (a). Given $t \in L$, let t^* be the join of elements covering t. By (i) from (b) we have $\mu(t,t^*) \ne 0$. Moreover by (ii) if t is covered by j elements of L then t^* covers j elements of $[t,t^*]$. Thus, if $t \notin A = M_k$ then T^* covers more than k elements of $[t,t^*]$. Let $u \in B = J_k$. Then by (iii), $[t\wedge u,u] \cong [t,t\vee u]$. Hence, $t\vee u \ne t^*$, so the hypotheses of (a) are satisfied, and the result follows.

d. By (c), $\#J_k \le \#M_k$. Since the dual of a modular lattice is modular, we also have $\#M_k \le \#J_k$, and the result follows. This result was first proved (in a more complicated way) by R. P. Dilworth, *Ann. Math. (2)* **60** (1954), 359–364, and later by B. Ganter and I. Rival, *Alg. Universalis* **3** (1973), 348–350.

e. See Figure 3.79.

f. With L as in Exercise 3.96(d), choose

$$A = \{t \in L \,:\, \rho(t) \ge n-k\},$$
$$B = \{t \in L \,:\, \rho(t) \le k\}.$$

Define $t^* = \hat{1}$ for all $t \in L$. The hypotheses of (a) are easily checked, so in particular $\#B \le \#A$ as desired.

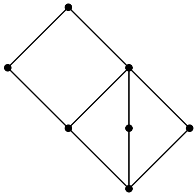

Figure 3.79 A modular lattice for which $P_1 \not\cong Q_1$.

102. a. This diabolical problem is equivalent to a conjecture of P. Frankl. See page 525 of *Graphs and Order* (I. Rival, ed.), Reidel, Dordrecht/Boston, 1985. For some further work on this conjecture, see R. Morris, *Europ. J. Combinatorics* **27** (2006), 269–282, and the references therein.

b. If not, then by Exercise 3.96(c) there is a permutation $w\colon L \to L$ for which $s \wedge w(s) = \hat{0}$ for all $s \in L$. But if $\#V_t > n/2$, then $V_t \cap w(V_t) \neq \emptyset$, and any $u \in V_t \cap w(V_t)$ satisfies $u \wedge w(u) \geq t$. One can also give a simple direct proof (avoiding Möbius inversion) of the following stronger result. Let L be a finite lattice with n elements, such that for all $\hat{0} < s \leq t$ in L, there exists $u \neq t$ for which $s \vee u = t$. Then every $t > \hat{0}$ satisfies $\#V_t \leq n/2$.

103. *Answer.* If $n \geq 3$ then

$$-\binom{n-1}{2\lfloor (n-1)/4\rfloor} \leq \mu(\hat{0},\hat{1}) \leq \binom{n-1}{2\lfloor (n-1)/4\rfloor + 1}.$$

See H. Scheid, *J. Combinatorial Theory* **13** (1972), 315–331 (Satz 5).

104. a. E. E. Maranich, *Mat. Zametki* **44** (1988), 469–487, 557; translation in *Math. Notes* **44** (1988), 736–447 (1989), and independently G. M. Ziegler, *J. Combinatorial Theory Ser. A* **56** (1991), 203–222, have shown by induction on the length of P that the answer is

$$\max\,(a_1-1)\cdots(a_k-1),$$

where the maximum is taken over all partitions $a_1 + \cdots + a_j = n$. (One can show that the maximum is obtained by taking at most four of the a_i's not equal to five.) This bound is achieved by taking P to be the ordinal sum $\mathbf{1} \oplus a_1\mathbf{1} \oplus \cdots \oplus a_k\mathbf{1} \oplus \mathbf{1}$. For some additional results, see D. N. Kozlov, *Combinatorica* **19** (1999), 533–548.

b. One can achieve $n^{2-\epsilon}$ (for any $\epsilon > 0$ and sufficiently large n) by taking L to be the lattice of subspaces of a suitable finite-dimensional vector space over a finite field. It seems plausible that $n^{2-\epsilon}$ is best possible. This problem was suggested by L. Lovász. A subexponential upper bound is given by Ziegler, op. cit.

105. This problem was suggested by P. H. Edelman. It is plausible to conjecture that the maximum is obtained by taking P to be the ordinal sum $\mathbf{1} \oplus k\mathbf{1} \oplus k\mathbf{1} \oplus \cdots \oplus k\mathbf{1} \oplus \mathbf{1}$ ($\ell - 1$ copies of $k\mathbf{1}$ in all), yielding $|\mu(\hat{0},\hat{1})| = (k-1)^{\ell-1}$, but this conjecture is false. The first counterexample was given by Edelman; and G. M. Ziegler, op. cit., attained $|\mu(\hat{0},\hat{1})| = (k-1)(k^{\ell-1}-1)$, together with some related results.

106. No, an example being given in Figure 3.80. The first such example (somewhat more complicated) was given by C. Greene (private communication, 1972).

107. This result is due to R. Stanley (proposer), Problem 11453, *Amer. Math. Monthly* **116** (2009), 746. The following solution is due to R. Ehrenborg.

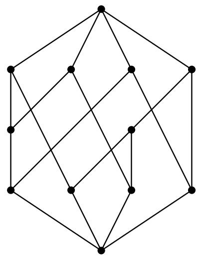

Figure 3.80 A lattice with no sublattice $\mathbf{1} \oplus (\mathbf{1}+\mathbf{1}+\mathbf{1}) \oplus \mathbf{1}$.

We can rewrite the identity (regarding elements of I as subsets of $[n]$) as

$$\sum_{\substack{u\in I\\ u\geq t}} (-1)^{\#u} = 0.$$

Sum the given identity over all sets v in I of cardinality j, where $0 \leq j \leq k$:

$$\begin{aligned} 0 &= \sum_{\substack{v\in I\\ \#v=j}} \sum_{\substack{u\in I\\ u\geq v}} (-1)^{\#u} \\ &= \sum_{u\in I} \sum_{\substack{v\geq u\\ \#v=j}} (-1)^{\#u} \\ &= \sum_{u\in I} \binom{\#u}{j} (-1)^{\#u}. \end{aligned}$$

Multiply this equation by $(-2)^j$ and take modulo 2^{k+1}, to obtain

$$0 \equiv \sum_{u\in I} \binom{\#u}{j} (-1)^{\#u-j} 2^j \bmod 2^{k+1}.$$

Observe that this congruence is also true for $j > k$, that is, it holds for all nonnegative integers j. Now summing over all j and using the binomial theorem, we have modulo 2^{k+1} that

$$\begin{aligned} 0 &\equiv \sum_{j\geq 0} \sum_{u\in I} \binom{\#u}{j} (-1)^{\#u-j} 2^j \\ &\equiv \sum_{u\in I} \sum_{j\geq 0} \binom{\#u}{j} (-1)^{\#u-j} 2^j \\ &\equiv \sum_{u\in I} (-1+2)^{\#u} \\ &\equiv \sum_{u\in I} 1 \\ &\equiv \#I. \end{aligned}$$

This result is the combinatorial analogue of a much deeper topological result of G. Kalai, in *Computational Commutative Algebra and Combinatorics*, Advanced Studies in Pure Mathematics **23** (2002), 121–163 (Theorem 4.2), a special case of which can be stated as follows. Let Δ be a finite simplicial complex, or equivalently, an order

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ideal I of B_n. Suppose that for any face F of dimension at most $k-1$ (including the empty face of dimension -1), the link (defined in equation (3.26)) of F is acyclic (i.e., has vanishing reduced homology). Let f_i denote the number of i-dimensional faces of Δ. Then there exists a simplicial complex Γ with g_i i-dimensional faces such that

$$\sum_{i\geq -1} f_i x^i = (1+x)^{k+1} \sum_{i\geq -1} g_i x^i. \tag{3.142}$$

(Note that equation (3.142) does not imply the present exercise because the hypothesis on Δ is stronger for (3.142).) An even stronger result was conjectured by Stanley, *Discrete Math.* **120** (1993), 175–182 (Conjecture 2.4), as follows. Let L be the poset (or meet-semilattice) of faces of Δ. Then there exists a partitioning of L into intervals $[s,t]$ of rank $k+1$ such that the bottom elements s of the intervals form an order ideal of L. The case $k=0$ was proved by Stanley, *Discrete Math.* **120** (1993), 175–182, and some generalizations by A. M. Duval, *Israel J. Math.* **87** (1994), 77–87, and A. M. Duval and P. Zhang, *Israel J. Math.* **121** (2001), 313–331. A stronger conjecture than the one just stated is due to Kalai, op. cit. (Conjecture 22).

108. b. If σ is a partition of V, then let $\chi_\sigma(n)$ be the number of maps $f: V \to [n]$ such that (i) if a and b are in the same block of σ then $f(a) = f(b)$, and (ii) if a and b are in different blocks and $\{a,b\} \in E$, then $f(a) \neq f(b)$. Given *any* $f: V \to [n]$, there is a unique $\sigma \in L_G$ such that f is one of the maps enumerated by $\chi_\sigma(n)$. It follows that for any $\pi \in L_G$, we have $n^{\#\pi} = \sum_{\sigma\geq\pi} \chi_\sigma(n)$. By Möbius inversion $\chi_\pi(n) = \sum_{\sigma\geq\pi} n^{\#\sigma}\mu(\pi,\sigma)$. But $\chi_{\hat{0}}(n) = \chi_G(n)$, so the proof follows.
This interpretation of $\chi_G(n)$ in terms of Möbius functions is due to G.-C. Rota [3.58, §9].

c. Denote the hyperplane with defining equation $x_i - x_j$ by H_e, where e is the edge with vertices i and j. Let i_T be an intersection of some set T of hyperplanes of the arrangement $\mathcal{B}_G$. Let G_T be the spanning subgraph of G with edge set $\{e : H_e \in T\}$. If e' is an edge of G such that its vertices belong to the same connected component of G_T, then it is easy to see that $i_T = i_{T\cup\{e'\}}$. From this observation, it follows that $L_{\mathcal{B}_G}$ is isomorphic to the set of connected partitions of G ordered by refinement, as desired. It follows from (b) that χ_G and $\chi_{\mathcal{B}_G}$ differ at most by a power of q. Equality then follows, for example, from the fact that both have degree equal to $\#V$.

d. It is routine to verify equation (3.119) from (c) and Proposition 3.11.5. To give a direct combinatorial proof, let $e = \{u,v\}$. Show that $\chi_G(n)$ is the number of proper colorings f of $G-e$ such that $f(u) \neq f(v)$, while $\chi_{G/e}(n)$ is the number of proper colorings f of $G-e$ such that $f(u) = f(v)$.

e. It follows from equation (1.96) and Proposition 1.9.1(a) that $\varphi((n)_k) = x^k$. Now use (a). Chung Chan has pointed out that this result can also be proved from (d) by first showing that if we set $g_G = \sum_j S_G(j)x^j$, then $g_G = g_{G-e} - g_{G/e}$ for any edge e of G. Equation (3.120) is equivalent to an unpublished result of Rhodes Peele.

109. a. We need to prove that

$$\mathrm{ao}(G) = \mathrm{ao}(G-e) + \mathrm{ao}(G/e),$$

together with the initial condition $\mathrm{ao}(G) = 1$ if G has no edges. Let $\mathfrak{o}$ be an acyclic orientation of $G-e$, where $e = \{u,v\}$. Let $\mathfrak{o}_1$ be $\mathfrak{o}$ with $u \to v$ adjoined, and $\mathfrak{o}_2$ be $\mathfrak{o}$ with $v \to u$ adjoined, so $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are orientations of G. The key step is to show the following: Exactly one of $\mathfrak{o}_1$ and $\mathfrak{o}_2$ is acyclic, except for $\mathrm{ao}(G/e)$ cases for which both $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are acyclic. See R. Stanley, *Discrete Math.* **5** (1973), 171–178.

b. A region of the graphical arrangement $\mathcal{B}_G$ is obtained by specifying for each edge $\{i,j\}$ of G whether $x_i < x_j$ or $x_i > x_j$. Such a specification is consistent if and only if the following condition is satisfied: Let $\mathfrak{o}$ be the orientation obtained by letting $i \to j$ whenever we choose $x_i < x_j$. Then $\mathfrak{o}$ is acyclic. Hence, the number of regions of $\mathcal{B}_G$ is $\mathrm{ao}(G)$. Now use Exercise 3.108(c) and Theorem 3.11.7. This proof is due to G. Greene and T. Zaslavsky, *Trans. Amer. Math. Soc.* **280** (1983), 97–126.

110. Part (a) was proved and (b) was conjectured by A. E. Postnikov, Total positivity, Grassmannians, and networks, `arXiv:math/0609764` (Conjecture 24.4(1)). Postnikov's conjecture was proved by A. Hultman, S. Linusson, J. Shareshian, and J. Sjöstrand, *J. Combinatorial Theory Ser. A* **116** (2009), 564–580.

111. See C. A. Athanasiadis and S. Linusson, *Discrete Math.* **204** (1999), 27–39; and R. Stanley, in *Mathematical Essays in Honor of Gian-Carlo Rota* (B. Sagan and R. Stanley, eds.), Birkhäuser, Boston/Basel/Berlin, 1998, pp. 359–375.

112. b. By Whitney's theorem (Proposition 3.11.3) we have for any arrangement $\mathcal{A}$ in K^n that

$$\chi_{\mathcal{A}}(x) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \mathcal{B}\,\text{central}}} (-1)^{\#\mathcal{B}} x^{n-\mathrm{rank}(\mathcal{B})}.$$

Let $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$, and let $\mathcal{B} \subseteq \mathcal{A}_n$ for some n. Define $\pi(\mathcal{B}) \in \Pi_n$ to have blocks that are the vertex sets of the connected components of the graph G on $[n]$ with edges

$$E(G) = \{ij : \exists x_i - x_j = c \text{ in } \mathcal{B}\}. \tag{3.143}$$

Define

$$\widetilde{\chi}_{\mathcal{A}_n}(x) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \mathcal{B}\,\text{central} \\ \pi(\mathcal{B})=\{[n]\}}} (-1)^{\#\mathcal{B}} x^{n-\mathrm{rank}(\mathcal{B})}.$$

Then

$$\begin{aligned}
\chi_{\mathcal{A}_n}(x) &= \sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_n} \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \mathcal{B}\,\text{central} \\ \pi(\mathcal{B})=\pi}} (-1)^{\#\mathcal{B}} x^{n-\mathrm{rank}(\mathcal{B})} \\
&= \sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_n} \widetilde{\chi}_{\mathcal{A}_{\#B_1}}(x)\widetilde{\chi}_{\mathcal{A}_{\#B_2}}(x)\cdots\widetilde{\chi}_{\mathcal{A}_{\#B_k}}(x).
\end{aligned}$$

Thus by the exponential formula (Corollary 5.1.6),

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(x)\frac{z^n}{n!} = \exp \sum_{n\geq 1} \widetilde{\chi}_{\mathcal{A}_n}(x)\frac{z^n}{n!}.$$

But $\pi(\mathcal{B}) = \{[n]\}$ if and only if $\mathrm{rank}(\mathcal{B}) = n-1$, so $\widetilde{\chi}_{\mathcal{A}_n}(x) = c_n x$ for some $c_n \in \mathbb{Z}$. We therefore get

$$\begin{aligned}
\sum_{n\geq 0} \chi_{\mathcal{A}_n}(x)\frac{z^n}{n!} &= \exp x \sum_{n\geq 1} c_n \frac{z^n}{n!} \qquad (3.144)\\
&= \left(\sum_{n\geq 0} b_n \frac{z^n}{n!}\right)^x,
\end{aligned}$$

where $\exp \sum_{n\geq 1} c_n \frac{z^n}{n!} = \sum_{n\geq 0} b_n \frac{z^n}{n!}$. Put $x = -1$ to get

$$\sum_{n\geq 0} (-1)^n r(\mathcal{A}_n)\frac{z^n}{n!} = \left(\sum_{n\geq 0} b_n \frac{z^n}{n!}\right)^{-1},$$

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from which it follows that

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(x)\frac{z^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{z^n}{n!}\right)^{-x}.$$

This result was stated without proof by R. Stanley, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625 (Theorem 1.2), and proved in [3.83, Thm. 5.17].

c. Similarly to equation (3.144), we get

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(x)\frac{z^n}{n!} = \exp \sum_{n\geq 1}(c_n x + d_n)\frac{z^n}{n!}$$
$$= A(z)^x B(z),$$

say, where $A(z)$ and $B(z)$ are independent of x. Put $x=-1$ and $x=1$, and solve for $A(z)$ and $B(z)$ to complete the proof. This result appears without proof in [3.83, Exer. 5.10].

113. Let $\mathcal{A}_q$ lie in $\mathbb{F}_q^n$. Suppose that $\mathcal{A}'_q = \mathcal{A}_q - \{H_0\}$. The points of $\mathbb{F}_q^n$ that do not lie in any $H \in \mathcal{A}'$ are a disjoint union of those points that do not lie on any $H \in \mathcal{A}_q$, together with the points $\alpha \in H_0$ that do not lie on any $H \in \mathcal{A}_q$. These points α are just those points in H_0 that do not lie on $\mathcal{A}''_q$, so the proof follows. This proof was suggested by A. Postnikov, private communication, February 2010.

115. a. Let p be a large prime. By Theorem 3.11.10 we want the number of ways to choose an n-tuple $(a_1,\dots,a_n) \in \mathbb{F}_p^n$ such that no $a_i - a_j = 0, \pm 1$ $(i \neq j)$. Once we choose a_1 in p ways, we need to choose $n-1$ points (in order) from $[p-3]$ so that no two are consecutive. Now use Exercise 1.34 for $j=2$. This arrangement is called the "Catalan arrangement" because the number of regions is $n!C_n$. Perhaps the first explicit appearance of this arrangement and determination of the number of regions is R. Stanley, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625 (special case of Theorem 2.2). The evaluation of $\chi_{\mathcal{C}_n}(x)$ appears in C. A. Athanasiadis, *Advances in Math.* **122** (1996), 193–233 (special case of Theorem 5.1).

b. The case $x=-1$ (i.e., the number of regions of $\mathcal{L}_n$) was raised by N. Linial. Equation (3.122) was first proved by C. Athanasiadis, ibid. (Theorem 5.2), generalized further in *J. Alg. Comb.* **10** (1999), 207–225 (§3), using the finite field method (Theorem 3.11.10). A proof based on Whitney's theorem (Proposition 3.11.3) was given by A. E. Postnikov, Ph.D. thesis, M.I.T., 1997. Numerous generalizations appear in A. E. Postnikov and R. Stanley, *J. Combinatorial Theory, Ser. A* **91** (2000), 544–597. See also Exercise 5.41 for some combinatorial interpretations of $r(\mathcal{L}_n)$.

c. This result is a special case of Exercise 3.112(c). It first appeared (without proof) as Exercise 5.25 of [3.83]. The arrangement $\mathcal{T}_n$ is called the "threshold arrangement" because the number of regions is equal to the number of threshold graphs with vertex set $[n]$ (see Exercise 5.4).

d. Let p be a large prime ($p>2$ will do). Choose $a_1 \neq 0$ in $p-1$ ways. Since p is odd, we can choose $a_2 \neq 0, \pm a_1$ in $p-3$ ways. We can then choose $a_3 \neq 0, \pm a_1, \pm a_2$ in $p-5$ ways, and so on, giving

$$\chi_{\mathcal{B}_n^B}(x) = (x-1)(x-3)(x-5)\cdots(x-2n+1).$$

A nice introduction to the combinatorics of hyperplane arrangements related to root systems is T. Zaslavsky, *Amer. Math. Monthly* **88** (1981), 88–105.

116. It is not so difficult to show that the intersection poset $L(\mathcal{C})$ is isomorphic to the rank k truncation of the partition lattice Π_n (i.e., the order ideal of Π_n consisting of all partitions with at least $n-k$ blocks). It follows from Proposition 1.3.7 and equations (3.38) and (3.46) that

$$\chi_{\mathcal{C}}(x) = \sum_{i=0}^{k}(-1)^i c(n,n-i)x^{n-i},$$

$$r(\mathcal{C}) = c(n,n) + c(n,n-1) + \cdots + c(n,n-k).$$

This problem was first considered by I. J. Good and T. N. Tideman, *J. Combinatorial Theory Ser. A* **23** (1977), 34–45, in connection with voting theory. They obtained the formula for $r(\mathcal{C})$ by a rather complicated induction argument. Later Zaslavsky, *Discrete Comput. Geom.* **27** (2002), 303–351, corrected an oversight in the proof of Good and Tideman and reproved their result by using standard techniques from the theory of arrangements (working in a more general context than here). H. Kamiya, P. Orlik, A. Takemura, and H. Terao, *Ann. Combinatorics* **10** (2006), 219–235, considered additional aspects of this topic in an analysis of ranking patterns.

117. It follows from equation (3.123) that in a reference frame at velocity $\boldsymbol{v}$, the events $\boldsymbol{p_i} = (t_i, \boldsymbol{x_i})$ and $\boldsymbol{p_j} = (t_j, \boldsymbol{x_j})$ occur at the same time if and only if

$$t_1 - t_2 = (\boldsymbol{x_1} - \boldsymbol{x_2}) \cdot \boldsymbol{v}.$$

The set of all such $\boldsymbol{v} \in \mathbb{R}^n$ forms a hyperplane. The set of all such $\binom{k}{2}$ hyperplanes forms an arrangement $\mathcal{E} = \mathcal{E}(\boldsymbol{p_1}, \dots, \boldsymbol{p_k})$, which we call the *Einstein arrangement*. The number of different orders in which the events can be observed is therefore $r(\mathcal{E})$. As in the previous exercise, the intersection poset $L(\mathcal{E})$ is isomorphic to the rank k truncation of Π_n, so we obtain as earlier that

$$r(\mathcal{E}) = c(n,n) + c(n,n-1) + \cdots + c(n,n-k).$$

For instance, when $n = 3$, we get

$$r(\mathcal{E}) = \frac{1}{48}\left(k^6 - 7k^5 + 23k^4 - 37k^3 + 48k^2 - 28k + 48\right).$$

For further details, see R. Stanley, *Advances in Appl. Math.* **37** (2006), 514–525. Some additional results are due to M. I. Heiligman, Sequentiality restrictions in special relativity, preprint dated February 4, 2010.

118. **a.** This remarkable result is equivalent to the main theorem of H. Terao, *Invent. Math.* **63** (1981), 159–179. For an exposition, see Orlik and Terao [3.53, Thm. 4.6.21].

b. The result that $\boldsymbol{\Omega}$ is free when L is supersolvable (due independently to R. Stanley and to M. Jambu and H. Terao, *Advances in Math.* **52** (1984), 248–258) can be proved by induction on ν using the Removal Theorem of H. Terao, *J. Fac. Sci. Tokyo (IA)* **27** (1980), 293–312, and the fact that if $L = L(H_1, \dots, H_\nu)$ is supersolvable, then for some $i \in [\nu]$ we have that $L(H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_\nu)$ is also supersolvable. Examples of free $\boldsymbol{\Omega}$ when L is not supersolvable appear in the previous reference and in H. Terao, *Proc. Japan Acad. (A)* **56** (1980), 389–392.

c. This question was raised by Orlik–Solomon–Terao, who verified it for $n \le 7$. The numbers $(e_1, \dots, e_n)$ for $3 \le n \le 7$ are given by $(1,1,2)$, $(1,2,3,4)$, $(1,3,4,5,7)$, $(1,4,5,7,8,10)$, and $(1,5,7,9,10,11,13)$. However, G. M. Ziegler showed in *Advances in Math.* **101** (1993), 50–58, that the arrangement is not free for $n \ge 9$. The case $n = 8$ remains open.

d. This question is alluded to on page 293 of H. Terao, *F. Fac. Sci. Tokyo (IA)* **27** (1980), 293–312. It is a central open problem in the theory of free arrangements, though most likely the answer is negative.

e. See H. Terao, *Invent. Math.* **63** (1981), 159–179 (Prop. 5.5), and Orlik and Terao [3.53, Thm. 4.2.23]. Is there a more elementary proof?

f. The question of the freeness of $\mathcal{A}^t$ was raised by P. Orlik. A counterexample was discovered by P. H. Edelman and V. Reiner, *Proc. Amer. Math. Soc.* **118** (1993), 927–929.

119. Let $N(V,X)$ be the number of injective linear transformations $V \to X$. It is easy to see that $N(V,X) = \prod_{k=0}^{n-1}(x-q^k)$. On the other hand, let W be a subspace of V and let $F_=(W)$ be the number of linear $\theta\colon V \to X$ with kernel (null space) W. Let $F_\geq(W)$ be the number with kernel containing W. Thus $F_\geq(W) = \sum_{W' \geq W} F_=(W)$, so by Möbius inversion we get

$$N(V,X) = F_=(\{0\}) = \sum_{W'} F_\geq(W')\mu(\hat{0},W').$$

Clearly, $F_\geq(W') = x^{n-\dim W'}$, whereas by equation (3.34) $\mu(\hat{0},W') = (-1)^k q^{\binom{k}{2}}$, where $k = \dim W'$. Since there are $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ subspaces W' of dimension k, we get

$$N(V,X) = \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\binom{\boldsymbol{n}}{\boldsymbol{k}} x^{n-k}.$$

120. See R. Stanley, *J. Amer. Math. Soc.* **5** (1992), 805–851 (Proposition 9.1). This exercise suggests that there is no good q-analogue of an Eulerian poset.

121. First Solution. Let $f(i,n)$ be the number of i-subsets of $[n]$ with no k consecutive integers. Since the interval $[\emptyset, S]$ is a boolean algebra for $S \in L_n'$, it follows that $\mu(\emptyset,S) = (-1)^{\#S}$. Hence, setting $a_n = \mu_n(\emptyset,\hat{1})$,

$$-a_n = \sum_{i=0}^{n}(-1)^i f(i,n).$$

Define $F(x,y) = \sum_{i\geq 0}\sum_{n\geq 0} f(i,n)x^i y^n$. The recurrence

$$f(i,n) = f(i,n-1) + f(i-1,n-2) + \cdots + f(i-k+1,n-k)$$

(obtained by considering the largest element of $[n]$ omitted from $S \in L_n'$) yields

$$F(x,y) = \frac{1+xy+x^2y^2+\cdots+x^{k-1}y^{k-1}}{1-y(1+xy+\cdots+x^{k-1}y^{k-1})}.$$

Since $-F(-1,y) = \sum_{n\geq 0} a_n y^n$, we get

$$\begin{aligned}\sum_{n\geq 0} a_n y^n &= \frac{-(1-y+y^2-\cdots\pm y^{k-1})}{1-y(1-y+y^2-\cdots\pm y^{k-1})}\\ &= \frac{1+(-1)^{k-1}y^k}{1+(-1)^k y^{k+1}}\\ &= -(1+(-1)^{k-1}y^k)\sum_{i\geq 0}(-1)^i(-1)^{ki}y^{i(k+1)}\end{aligned}$$

$$\Rightarrow a_n = \begin{cases} -1, & \text{if } n \equiv 0,-1 \pmod{2k+2},\\ (-1)^k, & \text{if } n \equiv k,k+1 \pmod{2k+2},\\ 0, & \text{otherwise.}\end{cases}$$

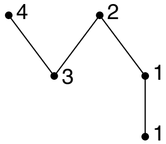

Figure 3.81 The poset P_{11324} when $13 = 31$, $14 = 41$, $24 = 42$.

Second Solution (E. Grimson and J. B. Shearer, independently). Let $\emptyset \neq a \in L'_n$. The dual form of Corollary 3.9.3 asserts that

$$\sum_{t \vee a = \hat{1}} \mu(\emptyset, t) = 0.$$

Now $t \vee a = \hat{1} \Rightarrow t = \hat{1}$ or $t = \{2, 3, \ldots, k\} \cup A$ where $A \subseteq \{k+2, \ldots, n\}$. It follows easily that

$$a_n - (-1)^{k-1} a_{n-k-1} = 0.$$

This recurrence, together with the initial conditions $a_0 = -1$, $a_i = 0$ if $i \in [k-1]$, and $a_k = (-1)^k$ determine a_n uniquely.

122. An interval $[d,n]$ of L is isomorphic to the boolean algebra $B_{\nu(n/d)}$, where $\nu(m)$ denotes the number of distinct prime divisors of m. Hence $\mu(d,n) = (-1)^{\nu(n/d)}$. Write $d \parallel n$ if $d \leq n$ in L. Given $f, g \colon \mathbb{P} \to \mathbb{C}$, we have

$$g(n) = \sum_{d \parallel n} f(d), \quad \text{for all } n \in \mathbb{P},$$

if and only if

$$f(n) = \sum_{d \parallel n} (-1)^{\nu(n/d)} g(d), \quad \text{for all } n \in \mathbb{P}.$$

123. a,b. Choose a factorization $w = g_{i_1} \cdots g_{i_\ell}$. Define P_w to be the multiset $\{i_1, \ldots, i_\ell\}$ partially ordered by letting $i_r < i_s$ if $r < s$ and $g_{i_r} g_{i_s} \neq g_{i_s} g_{i_r}$, or if $r < s$ and $i_r = i_s$. For instance, with $w = 11324$ as in Figure 3.52, we have P_w as in Figure 3.81. One can show that I is an order ideal of P_w if and only if for some (or any) linear extension $g_{i_1}, \ldots, g_{i_k}$ of I, we have $w = g_{i_1} \cdots g_{i_k} z$ for some $z \in M$. It follows readily that $L_w = J(P_w)$, and (b) is then immediate.

The monoid M was introduced and extensively studied by P. Cartier and D. Foata, *Lecture Notes in Math.*, no. 85, Springer-Verlag, Berlin/Heidelberg/New York, 1969. It is known as a *free partially commutative monoid* or *trace monoid*. The first explicit statement that $L_w = J(P_w)$ seems to have been made by I. M. Gessel in a letter dated February 8, 1978. This result is implicit, however, in Exercise 5.1.2.11 of D. E. Knuth [1.48]. This exercise of Knuth is essentially the same as our (b), though Knuth deals with a certain representation of elements of M as multiset permutations. An equivalent approach to this subject is the theory of *heaps*, developed by X. G. Viennot [4.60] after a suggestion of A. M. Garsia. For the connection between factorization and heaps, see C. Krattenthaler, appendix to electronic edition of Cartier-Foata, ⟨`www.mat.univie.ac.at/~slc/books/cartfoa.pdf`⟩.

c. The intervals $[v, vw]$ and $[\varepsilon, w]$ are clearly isomorphic (*via* the map $x \mapsto vx$), and it follows from (a) that P_w is an antichain (and hence $[\varepsilon, w]$ is a boolean algebra) if and only if w is a product of r distinct pairwise commuting g_i. The proof follows from Example 3.9.6.

A different proof appears in P. Cartier and D. Foata, op. cit, Ch. II.3.

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d. If $w \in M$, then let x^w denote the (commutative) monomial obtained by replacing in w each g_i by x_i. By (c) we want to show that

$$\left(\sum_{w\in M} x^w\right)\left(\sum_{v\in M}\mu(\varepsilon,v)x^v\right) = 1. \tag{3.145}$$

Expand the left-hand side of equation (3.145), take the coefficient of a monomial x^u, and use the defining recurrence (3.15) for μ to complete the proof.

(e) We have

$$\sum_{a_1\geq 0}\cdots\sum_{a_n\geq 0}\binom{a_1+\cdots+a_n}{a_1,\ldots,a_n}x_1^{a_1}\cdots x_n^{a_n} = \frac{1}{1-(x_1+\cdots+x_n)}$$

and

$$\sum_{a_1\geq 0}\cdots\sum_{a_n\geq 0}x_1^{a_1}\cdots x_n^{a_n} = \frac{1}{(1-x_1)\cdots(1-x_n)},$$

respectively.

124. a. See [3.68, Thm. 4.1].

b. This exercise is jointly due to A. Björner and R. Stanley. Given $t \in L$, let $D_t = J(Q_t)$ be the distributive sublattice of L generated by C and t. The M-chain C defines a linear extension of Q_t and hence defines Q_t as a natural partial ordering of $[n]$. One sees easily that $L_P \cap D_t = J(P\cap Q_t)$. From this, all statements follow readily. Let us mention that it is not always the case that L_P is a lattice.

125. See P. McNamara, *J. Combinatorial Theory, Ser. A* **101** (2003), 69–89. McNamara shows that there is a third equivalent condition: L admits a good local action of the 0-Hecke algebra $\mathcal{H}_n(0)$. This condition is too technical to be explained here.

126. a. The isomorphism $L_k^{(2)}(p)\cong L_k^{(3)}(p)$ is straightfoward, whereas $L_k^{(1)}(p)\cong L_k^{(2)}(p)$ follows from standard duality results in the theory of abelian groups (or more generally abelian categories). A good elementary reference is Chapter 2 of P. J. Hilton and Y.-C. Wu, *A Course in Modern Algebra*, Wiley, New York, 1974. In particular, the functor taking G to $\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Z}/p^\infty\mathbb{Z})$ is an order-reversing bijection between subgroups G of index p^m (for some $m\geq 0$) in $\mathbb{Z}^k$ and subgroups of order p^m in $(\mathbb{Z}/p^\infty\mathbb{Z})^k \cong \mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Z}/p^\infty\mathbb{Z})$.
The remainder of (a) is routine.

b. Follows, for example, from the fact that every subgroup of $\mathbb{Z}^k$ of finite index is isomorphic to $\mathbb{Z}^k$.

c. This result goes back to Eisenstein, Königl. Preuss. Akad. Wiss. Berlin (1852), 350–359, and Hermite, *J. Reine u. angewandte Mathematik* **41** (1851), 191–216. The proof follows directly from the theory of Hermite normal form (see, e.g., §6 of M. Newman, *Integral Matrices*, Academic Press, New York, 1972), which implies that every subgroup G of $\mathbb{Z}^k$ of index p^n has a unique $\mathbb{Z}$-basis $y_1,\ldots,y_k$ of the form

$$y_i = (a_{i1},a_{i2},\ldots,a_{ii},0,\ldots,0),$$

where $a_{ii}>0$, $0\leq a_{ij}<a_{ii}$ if $j<i$, and $a_{11}a_{22}\cdots a_{kk}=p^n$. Hence, the number of such subgroups is

$$\sum_{b_1+\cdots+b_k=n} p^{b_2+2b_3+\cdots+(k-1)b_k} = \binom{\boldsymbol{n+k-1}}{\boldsymbol{k-1}}.$$

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For some generalizations, see L. Solomon, *Advances in Math.* **26** (1977), 306–326, and L. Solomon, in *Relations between Combinatorics and Other Parts of Mathematics* (D.-K. Ray-Chaudhuri, ed.), Proc. Symp. Pure Math., vol 34, American Mathematical Society, Providence, R.I., 1979, pp. 309–329.

d. If $t_1 < \cdots < t_j$ in $L_k(p)$ with $\rho(t_i) = s_i$, then t_1 can be chosen in $\binom{s_1+k-1}{k-1}$ ways, next t_2 in $\binom{s_2-s_1+k-1}{k-1}$ ways, and so on.

e. A word $w = e_1 e_2 \cdots \in N_k$ satisfies $D(w) \subseteq S = \{s_1, \ldots, s_j\}_<$ if and only if $e_1 \leq e_2 \leq \cdots \leq e_{s_1}, e_{s_1+1} \leq \cdots \leq e_{s_2}, \ldots, e_{s_{j-1}+1} \leq \cdots \leq e_{s_j}, e_{s_j+1} = e_{s_j+2} = \cdots = 0$. Now for fixed i and k,

$$\sum_{0 \leq d_1 \leq \cdots \leq d_i \leq k-1} p^{d_1+\cdots+d_i} = \binom{i+k-1}{k-1},$$

and the proof follows easily.

The problem of computing $\alpha_{L_\lambda}(S)$ and $\beta_{L_\lambda}(S)$, where L_λ is the lattice of subgroups of a *finite* abelian group of type $\lambda = (\lambda_1, \ldots, \lambda_k)$ (or more generally, a q*-primary lattice* as defined in R. Stanley, *Electronic J. Combinatorics* **3**(2) (1996), #R6 (page 9)) is more difficult. (The present exercise deals with the "stable" case $\lambda_i \to \infty$, $1 \leq i \leq k$.) One can show fairly easily that $\beta_{L_\lambda}(S)$ is a polynomial in p, and the theory of symmetric functions can be used to give a combinatorial interpretation of its coefficients that shows they are nonnegative. An independent proof of this fact is due to L. M. Butler, Ph.D. thesis, M.I.T., 1986, and *Memoirs Amer. Math. Soc.* **112**, no. 539 (1994) (Theorem 1.5.5).

130. a. For any fixed $t \in Q_i$ we have

$$0 = \sum_{s \leq t} \mu(\hat{0}, s) = \sum_j \left(\sum_{\substack{s \leq t \\ \rho(t) = i-j}} \mu(\hat{0}, s) \right).$$

Sum on all $t \in Q_i$ of fixed rank $i - k > 0$ to get (since $[x, \hat{1}] \cong Q_j$)

$$\begin{aligned} 0 &= \sum_j \left(\sum_{\substack{s \in Q_i \\ \rho(s) = i-j}} \mu(\hat{0}, s) \right) \left(\sum_{\substack{t \in Q_j \\ \rho(t) = j-k}} 1 \right) \\ &= \sum_j v(i,j) V(j,k). \end{aligned}$$

On the other hand, it is clear that $\sum_j v(i,j) V(j,i) = 1$, and the proof follows. This result (for geometric lattices) is due to T. A. Dowling, *J. Combinatorial Theory Ser. B* **14** (1973), 61–86 (Thm. 6).

b. See M. Aigner, *Math. Ann.* **207** (1974), 1–22; M. Aigner, *Aeq. Math.* **16** (1977), 37–50; and J. R. Stonesifer, *Discrete Math.* **32** (1980), 85–88. For some related results, see J. N. Kahn and J. P. S. Kung, *Trans. Amer. Math. Soc.* **271** (1982), 485–489, and J. P. S. Kung, *Geom. Dedicata* **21** (1986), 85–105.

131. See T. A. Dowling, *J. Combinatorial Theory Ser. B* **14** (1973), 61–86. Erratum, same journal **15** (1973), 211.

A far-reaching extension of these remarkable "Dowling lattices" appears in the work of Zaslavsky on signed graphs (corresponding to the case $\#G = 2$) and gain graphs (arbitrary G). Zaslavsky's work on the calculation of characteristic polynomials and

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related invariants appears in *Quart. J. Math. Oxford (2)* **33** (1982), 493–511. A general reference for enumerative results on gain graphs is T. Zaslavsky, *J. Combinatorial Theory Ser. B* **64** (1995), 17–88.

132. Number of elements of rank k is $\binom{n+k}{2k}$:
$\#P_n = F_{2n+1}$ (Fibonacci number),
$(-1)^n\mu(\hat{0},\hat{1}) = \frac{1}{n+1}\dbinom{2n}{n}$ (Catalan number),
number of maximal chains is $1\cdot 3\cdot 5\cdots(2n-1)$.
This exercise is due to K. Baclawski and P. H. Edelman.

133. **a.** Define a closure operator (as defined in Exercise 3.84) on L_n by setting $\overline{G} = \mathfrak{S}(\mathcal{O}_1)\times\cdots\times\mathfrak{S}(\mathcal{O}_k)$, where $\mathcal{O}_1,\dots,\mathcal{O}_k$ are the orbits of G and $\mathfrak{S}(\mathcal{O}_i)$ denotes the symmetric group on $\mathcal{O}_i$. Then $\overline{L}_n \cong \Pi_n$. In Exercise 3.84 choose $s=\hat{0}$ and $t=\hat{1}$, and the result follows from equation (3.37).

b. A generalization valid for any finite group G is given in Theorem 3.1 of C. Kratzer and J. Thévenaz, *Comment. Math. Helvetici* **59** (1984), 425–438.

c–f. See J. Shareshian, *J. Combinatorial Theory Ser. A* **78** (1997), 236–267. For a topological refinement, see J. Shareshian, *J. Combinatorial Theory Ser. A* **104** (2003), 137–155.

135. **b.** The poset Λ_n is defined in Birkhoff [3.12, Ch. I.8, Ex. 10]. The problem of computing the Möbius function is raised in Exercise 13 on p. 104 of the same reference. (In this exercise, 0 should be replaced with the partition $\langle 1^{n-2}2\rangle$).

c. It was shown by G. M. Ziegler, *J. Combinatorial Theory Ser. A* **42** (1986), 215–222, that Λ_n is not Cohen–Macaulay for $n\geq 19$, and that the Möbius function does not alternate in sign for $n\geq 111$. (These bounds are not necessarily tight.) For some further information on Λ_n, see F. Bédard and A. Goupil, *Canad. Math. Bull.* **35** (1992), 152–160.

136. See T. H. Brylawski, *Discrete Math.* **6** (1973), 201–219 (Prop. 3.10), and C. Greene, *Europ. J. Combinatorics* **9** (1988), 225–240. For further information on this poset, see Exercises 3.78(c) and 7.2, as well as A. Björner and M. L. Wachs, *Trans. Amer. Math. Soc.* **349** (1997), 3945–3975 (§8); J. N. Kahn, *Discrete and Comput. Geometry* **2** (1987), 1–8; and S. Linusson, *Europ. J. Combinatorics* **20** (1999), 239–257.

137. *Answer.* $Z(P+Q,m) = Z(P,m)+Z(Q,m)$,

$$\begin{aligned}
Z(P\oplus Q,m) &= \sum_{j=2}^{m-1} Z(P,j)Z(Q,m+1-j) + Z(P,m)+Z(Q,m),\\
&\quad m\geq 2,\\
Z(P\times Q,m) &= Z(P,m)Z(Q,m).
\end{aligned}$$

138. **a.** By definition, $Z(\mathrm{Int}(P),n)$ is equal to the number of multichains

$$[s_1,t_1]\leq[s_2,t_2]\leq\cdots\leq[s_{n-1},t_{n-1}]$$

of intervals of P. Equivalently,

$$s_{n-1}\leq s_{n-2}\leq\cdots\leq s_1\leq t_1\leq t_2\leq t_{n-1}.$$

Hence, $Z(\mathrm{Int}(P),n) = Z(P,2n-1)$.

b. It is easily seen that

$$Z(Q,n)-Z(Q,n-1) = Z(\mathrm{Int}(P),n).$$

Put $n = 0$ and use Proposition 3.12.1(c) together with (a) to obtain $\mu_Q(\hat{0},\hat{1}) = -Z(P,-1) = -\mu_P(\hat{0},\hat{1})$. When P is the face lattice of a convex polytope, much more can be said about Q. This is unpublished work of A. Björner, though an abstract appears in the Oberwolfach Tagungsbericht 41/1997, pp. 7–8, and a shorter version in Abstract 918-05-688, *Abstracts Amer. Math. Soc.* **18:1** (1997), 19.

140. Since $n^k = \sum_j j!S(k,j)\binom{n}{j}$ by equation (1.94d), and since the polynomials $\binom{n}{j}$ are linearly independent over $\mathbb{Q}$, it follows that $\varphi\left(\binom{n}{j}\right) = x^j$. But by equation (3.51) we have

$$Z(P,n+2) = \sum_{j\geq 1} c_j(P)\binom{n}{j-1}.$$

Applying φ to both sides completes the proof. Note the similarity to Exercise 3.108(e).

141. **a.** For any chain C of P, let $Z_C(Q_0,m+1)$ be the number of multichains $C_1 \leq C_2 \leq \cdots \leq C_m = C$ in Q_0. Since the interval $[\emptyset, C]$ in Q_0 is a boolean algebra, we have by Example 3.12.2 that $Z_C(Q_0,m+1) = m^{\#C}$. Hence, $Z(Q_0,m+1) = \sum_{C\in Q_0} m^{\#C} = \sum a_i m^i$, where P has a_i i-chains, and the proof follows from Proposition 3.12.1(a).

b. *Answer.* $\mu_{\widehat{P}}(\hat{0},\hat{1}) = \mu_{\widehat{Q}}(\hat{0},\hat{1})$. Topologically, this identity reflects the fact that a finite simplicial complex and its first barycentric subdivision have homeomorphic geometric realizations and therefore equal Euler characteristics.

c. Follows easily from (b).

d. It is easy to see that the number of elements of $\widehat{Q}$ of rank $k-1$ is $k!\,S(n,k)$, $1 \leq k \leq n$. It is not hard to see that equation (3.124) is then a consequence of Theorem 3.16.9. The formula (3.124) was first observed empirically by M. Bóna (private communication, dated 27 October 2009). Note. The dual poset $\widehat{Q}^*$ is the face lattice of the *permutohedron*, the polytope of Exercise 4.64(a).

142. **a.** Let $\gamma_P(S)$ denote the number of intervals $[r(K),K)]$ for which $\rho(r(K)) = S$. If C is any chain of P with $\rho(C) = S$, then C is contained in a unique interval $[r(K),K]$ such that $\rho(r(K)) \subseteq S$; and conversely an interval $[r(K),K]$ such that $\rho(r(K)) \subseteq S$ contains a unique chain C of P such that $\rho(C) = S$. Hence,

$$\sum_{T\subseteq S} \gamma_P(T) = \alpha_P(S),$$

and the proof follows from equation (3.53).

The concept of chain-partitionable posets is due independently to J. S. Provan, thesis, Cornell Univ., 1977 (Appendix 4); R. Stanley [3.76, p. 149]; and A. M. Garsia, *Advances in Math.* **38** (1980), 229–266 (§4). The first two of these references work in the more general context of simplicial complexes, whereas the third uses the term “ER-poset” for our (chain-)partitionable poset.

b. Let $\lambda\colon \mathcal{H}(\widehat{P}) \to \mathbb{Z}$ be an R-labeling and $K: t_1 < \cdots < t_{n-1}$ a maximal chain of P, so $\hat{0} = t_0 < t_1 < \cdots < t_{n-1} < t_n = \hat{1}$ is a maximal chain of $\widehat{P}$. Define

$$r(K) = \{t_i : \lambda(t_{i-1},t_i) > \lambda(t_i,t_{i+1})\}.$$

Given any chain $C: s_1 < \cdots < s_k$ of P, define K to be the (unique) maximal chain of P that consists of increasing chains of the intervals $[\hat{0},s_1],[s_1,s_2],\ldots,[s_k,\hat{1}]$, with $\hat{0}$ and $\hat{1}$ removed. It is easily seen that $C \in [r(K),K]$, and that K is the only maximal chain of P for which $C \in [r(K),K]$. Hence, P is partitionable.

c. The posets in a special class of Cohen–Macaulay posets called “shellable” are proved to be partitionable in the three references given in (a). It is not known whether all Cohen–Macaulay shellable posets (or in

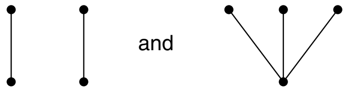

Figure 3.82 Two posets with the same order polynomial.

fact all Cohen–Macaulay posets) are R-labelable. On the other hand, it seems quite likely that there exist Cohen–Macaulay R-labelable posets that are not shellable, though this fact is also unproved. (Two candidates are Figures 18 and 19 of Björner–Garsia–Stanley [3.16].) A very general ring-theoretic conjecture that would imply that Cohen–Macaulay posets are partitionable appears in R. Stanley, *Invent. Math.* **68** (1982), 175–193 (Conjecture 5.1). For some progress on this conjecture, see for instance I. Anwar and D. Popescu, *J. Algebra* **318** (2007), 1027–1031; Y. H. Shen, *J. Algebra* **321** (2009), 1285–1292; D. Popescu, *J. Algebra* **321** (2009), 2782-2797; M. Cimpoeaş, *Matematiche (Catania)* **63** (2008), 165–171; J. Herzog, M. Vladoiu, and X. Zheng, *J. Algebra 322* (2009), 3151-3169; D. Popescu and M. I. Qureshi, *J. Algebra* **323** (2010), 2943–2959; and the two surveys D. Popescu, Stanley depth, ⟨www.univ-ovidius.ro/math/sna/17/PDF/17_Lectures.pdf⟩ and S. A. Seyed, M. Tousi, and S. Yassemi, *Notices Amer. Math. Soc.* **56** (2009), 1106–1108.

143. a. *First Proof.* It is implicit in the work of several persons (e.g., Faigle-Schrader, Gallai, Golumbic, Habib, Kelly, Wille) that two finite posets P and Q have the same comparability graph if and only if there is a sequence $P = P_0, P_1, \dots, P_k = Q$ such that P_{i+1} is obtained from P_i by "turning upside-down" (dualizing) a subset $T \subseteq P_i$ such that every element $t \in P_i - T$ satisfies either (a) $t < s$ for all $s \in T$, or (b) $t > s$ for all $s \in T$, or (c) $s \| t$ for all $s \in T$. (Such subsets T are called *autonomous subsets*.) The first explicit statement and proof seem to be in B. Dreesen, W. Poguntke, and P. M. Winkler, *Order* **2** (1985), 269–274 (Thm. 1). A further proof appears in D. A. Kelly, *Order* **3** (1986), 155–158. It is easy to see that P_i and P_{i+1} have the same order polynomial, so the proof of the present exercise follows.

Second Proof. Let $\Gamma_P(m)$ be the number of maps $g\colon P \to [0, m-1]$ satisfying $g(t_1) + \cdots + g(t_k) \le m - 1$ for every chain $t_1 < \cdots < t_k$ of P. We claim that $\Omega_P(m) = \Gamma_P(m)$. To prove this claim, given g as above define for $t \in P$

$$f(t) = 1 + \max\{g(t_1) + \cdots + g(t_k) : t_1 < \cdots < t_k = t\}.$$

Then $f\colon P \to [m]$ is order-preserving. Conversely, given f then

$$g(t) = \min\{f(t) - f(s) : t \text{ covers } s\}.$$

Thus, $\Omega_P(m) = \Gamma_P(m)$. But by definition $\Gamma_P(m)$ depends only on $\mathrm{Com}(P)$. This proof appears in R. Stanley, *Discrete Comput. Geom.* **1** (1986), 9–23 (Cor. 4.4).

b. See Figure 3.82.

For a general survey of comparability graphs of posets, see D. A. Kelly, in *Graphs and Order* (I. Rival, ed.), Reidel, Dordrecht/Boston, 1985, pp. 3–40.

145. We have $\Omega_P(-n) = Z(J(P), -n) = \mu^n_{J(P)}(\hat{0}, \hat{1})$. By Example 3.9.6.

$$\mu^n(\hat{0}, \hat{1}) = \sum (-1)^{\#(I_1 - I_0) + \cdots + \#(I_n - I_{n-1})},$$

summed over all multichains $\emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = P$ of order ideals of P such that each $I_i - I_{i-1}$ is an antichain of P. Since $\#(I_1 - I_0) + \cdots + \#(I_n - I_{n-1}) = p$, we have that $(-1)^p \mu^n(\hat{0}, \hat{1})$ is equal to the number of such multichains. But such a multichain

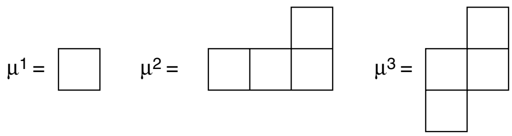

Figure 3.83 A sequence of skew shapes.

corresponds to the strict order-preserving map $\tau : P \to \boldsymbol{n}$ defined by $\tau(t) = i$ if $t \in I_i - I_{i-1}$, and the proof follows. This proof appeared in Stanley [3.67, Thm. 4.2].

146.

$$\Omega_{p\mathbf{1}}(n) \;=\; (-1)^p \Omega_{p\mathbf{1}}(-n) = n^p,$$

$$\Omega_p(n) = \left(\!\!\binom{n}{p}\!\!\right) \;=\; \binom{n+p-1}{p},$$

$$(-1)^p \Omega_p(-n) = \binom{n}{p}.$$

147. Tetrahedron: $Z(L,n) = n^4$.
cube or octahedron: $Z(L,n) = 2n^4 - n^2$. icosahedron or dodecahedron: $Z(L,n) = 5n^4 - 4n^2$.
Note that in all cases $Z(L,n) = Z(L,-n)$, a consequence of Proposition 3.16.1.

149. The case $\mu = \emptyset$ is equivalent to a result of P. A. MacMahon [1.55] (put $x = 1$ in the implied formula for $GF(p_1, p_2, \ldots, p_m; n)$ on page 243) and has been frequently rediscovered in various guises. The general case is due to G. Kreweras, *Cahiers du BURO*, no. 6, Institut de Statistique de L'Univ. Paris, 1965 (Section 2.3.7) and is also a special case (after a simple preliminary bijection) of Theorem 2.7.1. When $\mu = \emptyset$ and λ has the form $(M-d, M-2d, \ldots, M-\ell d)$ the determinant can be explicitly evaluated; see Vol. II, Exercise 7.101(b). A different approach to these results was given by I. M. Gessel, *J. Stat. Planning and Inference* **14** (1986), 49–58, and by R. A. Pemantle and H. S. Wilf, *Electronic J. Combinatorics* **16** (2009), #R60. For an extensive survey of the evaluation of combinatorial determinants, see C. Krattenthaler, *Sém. Lotharingien Combin.* 42 (1999), article B42q, and *Linear Algebra Appl.* **411** (2005), 68-166.

150. a. When the Young diagram λ^0 is removed from λ^j, there results an ordered disjoint union (the order being from lower left to upper right) of rookwise connected skew diagrams (or skew shapes, as defined in Section 7.10) $\mu^1, \ldots, \mu^r$. For example, if $\lambda^0 = (5,4,4,4,3,1)$ and $\lambda^j = (6,6,5,4,4,4,1)$, then we obtain the sequence of skew diagrams shown in Figure 3.83. Since $|\mu^1| + \cdots + |\mu^r| = a_j$, there are only finitely many possible sequences $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$ for fixed S. Thus, if we let $f_S(\boldsymbol{\mu}, n)$ be the number of chains $\lambda^0 < \lambda^1 < \cdots < \lambda^j$ under consideration yielding the sequence $\boldsymbol{\mu}$, then it suffices to show that the power series $A_S(\boldsymbol{\mu}, q)$ defined by

$$\sum_{n \geq 0} f_S(\boldsymbol{\mu}, n) q^n = P(q) A_S(\boldsymbol{\mu}, q) \qquad (3.146)$$

is rational with numerator $\phi_{a_j}(q)$.
We illustrate the computation of $A_S(\boldsymbol{\mu}, q)$ for $\boldsymbol{\mu}$ given by Figure 3.83 and leave the reader the task of seeing that the argument works for arbitrary $\boldsymbol{\mu}$. First, it is easy to see that there is a constant $c_S(\boldsymbol{\mu}) \in \mathbb{P}$ for which $A_S(\boldsymbol{\mu}, q) = c_S(\boldsymbol{\mu}) A_{\{a_j\}}(\boldsymbol{\mu}, q)$, so we may assume that $S = \{a_j\} = \{9\}$. Consider a typical λ^j, as shown in Figure 3.84. Here a, b, c mark the lengths of the indicated rows, so $c \geq b+2 \geq a+5$. When the

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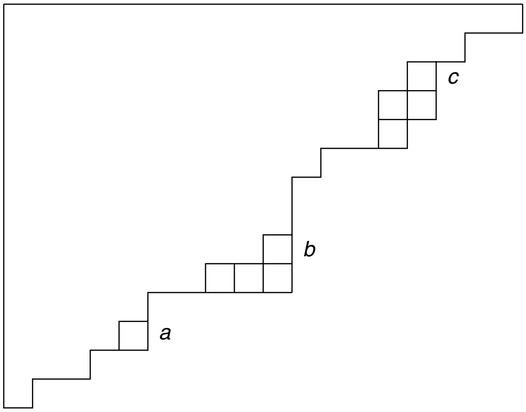

Figure 3.84 An example of the computation of $A_S(\boldsymbol{\mu},q)$.

rows intersecting some μ^i are removed from λ^j, there results a partition ν with no parts equal to $b-1$, $b-2$, or $c-1$, and every such ν occurs exactly once. Hence,

$$\sum_{n\geq 0} f_{\{9\}}(\boldsymbol{\mu},n)q^{n+9}$$

$$= P(q) \sum_{c\geq b+2\geq a+5\geq 6} q^{a+2b+(3c-1)}(1-q^{b-1})(1-q^{b-2})(1-q^{c-1}).$$

To evaluate this sum, expand the summand into eight terms, and sum on c,b,a in that order. Each sum will be a geometric series, introducing a factor $1-q^i$ in the denominator and a monomial in the numerator. Since among the eight terms the maximum sum of coefficients of a,b,c in the exponent of q is $a_j = 9$ (coming from $q^{a+4b+4c-5}$), it follows that the eight denominators will consist of distinct factors $1-q^i$, $1\leq i\leq 9$. Hence, they have a common denominator $\phi_9(q)$, as desired. Is there a simpler proof?

b. Let $A_S(q) = B_S(q)/\phi_{a_j}(q)$. Then

$$B_\emptyset = 1,\ \ B_1 = 1,\ \ B_2 = 2-q,\ \ B_3 = 3-q-q^2,\ \ B_{1,2} = 2,$$

$$B_{1,3} = 3+2q-q^2-q^3,\ \ B_{2,3} = 4-q+2q^2-2q^3, \text{ and}$$

$$B_{1,2,3}(q) = 2(2-q)(1+q+q^2).$$

Is there a simple formula for $B_{[n]}(q)$?

c. (with assistance from L. M. Butler) First check that the coefficient $g(n)$ of q^n, in the product of the left-hand side of equation (3.125) with $P(q)$, is equal to $\beta_Y([n,n+k]) + \beta_Y([n+1,n+k])$. We now want to apply Theorem 3.13.1. Regard $\mathbb{N}^2$ with the usual product order as a coarsening of the total (lexicographic) order

$$(i,j) \leq (i',j') \text{ if } i<i' \text{ or if } i=i',\ j\leq j'.$$

By Theorem 3.13.1, $g(n)$ is equal to the number of chains $\boldsymbol{\nu}\colon \nu^0 < \nu^1 < \cdots < \nu^k$ of partitions ν^i such that (1) $\nu_i \vdash n+i$; (2) ν^{i+1} is obtained from ν^i by adding a square (in the Young diagram) strictly above the square that was added in obtaining ν^i from ν^{i-1}; and (3) the square added in ν^k from ν^{k-1} is not in the top row. (This last condition guarantees a descent at $n+k$.) Here ν^0 can be arbitrary and ν^1 can be obtained by adding any square to ν^0. (If the square added to ν^0 starts a new row

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or is in the bottom row of ν^0, then the chain $\boldsymbol{\nu}$ contributes to $\beta_Y([n+1,n+k])$; otherwise, it contributes to $\beta_Y([n,n+k])$. We can now argue as in the solution to (a); namely, the added k squares belong to columns of length $2 \le i_1 < i_2 < \cdots < i_k$, and when these rows are removed, any partition can be left. Hence,

$$\sum_{n\ge 0} g(n)q^{n+k} = P(q) \sum_{2\le i_1<i_2\le\cdots\le i_k} q^{i_1+\cdots+i_k}$$
$$= q^{k+\binom{k+1}{2}} P(q)\phi_k(q),$$

and the proof follows. Is there a simple proof avoiding Theorem 3.13.1?

d. Follows readily from the first sentence of the solution to (c), upon noting that

$$\beta_Y([n,n+k]) = \sum_{i=0}^{k} (-1)^i \left(\beta_Y([n+i,n+k]) + \beta_Y([n+i+1,n+k))\right) - (-1)^k.$$

(The term $-(-1)^k$ is needed to cancel the term $(-1)^k\beta_Y[n+k+1,n+k]) = (-1)^k\beta_Y(\emptyset) = (-1)^k$ arising in the summand with $i = k$.)

e. We want to show that the number $f(n)$ of chains $\lambda < \mu$ with $\lambda \vdash n$ and $\mu \vdash n+1$ is equal to $p(0)+p(1)+\cdots+p(n)$, where $p(j)$ is the number of partitions of j. Now see Exercise 1.71. (This bijection is implicit in the proof of (a) or (c).)

151. a. By Theorem 3.13.1 there exists a permutation $w = a_1a_2\cdots a_n \in \mathcal{L}(P)$ with descent set $S = \{i_1,\ldots,i_k\}_<$. Set $i_0 = 0$, $i_{k+1} = p$, and choose $1 \le r \le k$. Rearrange all the elements $a_{i_{r-1}+1}, a_{i_{r-1}+1}, \ldots, a_{i_{r+1}}$ in increasing order, obtaining a permutation w'. Since P is naturally labeled we have $w' \in \mathcal{L}(P)$. Moreover, $D(w') = D(w) - \{i_r\}$, from which the proof is immediate. This result is due to Stanley [3.67, Ch. III, Cor. 1.2][3.68, Cor. 15.2].

b. Some necessary conditions on Δ are given in [3.68, §16].

c. We use the characterization of supersolvable lattices given by Exercise 3.125. The proof then parallels that of (a) with linear extensions replaced with labels of maximal chains. Specifically, let $\mathfrak{m}\colon \hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_p = \hat{1}$ be a maximal chain with label $\lambda(\mathfrak{m}) = (\lambda_1,\ldots,\lambda_p) \in \mathfrak{S}_p$ such that $D(\lambda(\mathfrak{m})) = S$. Using the notation of (a), replace $t_{i_{r-1}+1}, t_{i_{r-1}+2}, \ldots, t_{i_{r+1}}$ with the unique increasing chain between $t_{i_{r-1}+1}$ and $t_{i_{r+1}}$, obtaining a new maximal chain $\mathfrak{m}'$. Because the labels of maximal chains are permutations of $1,2,\ldots,p$ it follows that $D(\lambda(\mathfrak{m}')) = D(\lambda(\mathfrak{m})) - \{i_r\}$, and the proof is immediate as in (a).

Note. The result we have just proved is true under the even more general hypothesis that L is a finite Cohen–Macaulay poset, but the proof now involves algebraic techniques. See R. Stanley [3.76, Cor. 4.5] and Björner-Garsia-Stanley [3.16, p. 24].

152. a. Let $w = a_1\cdots a_n$ be a linear extension of P. Define the vertices of w to be the linear extensions obtained by choosing $i \in D(w)$ and writing the elements $a_1,\ldots,a_i$ in increasing order, followed by writing the elements $a_{i+1},\ldots,a_n$ in increasing order. It is easy to check that we obtain a simplicial complex Δ_P with the desired properties.

Example. If $w = 3642175$, then the vertices of w are 3612457, 3461257, 2346157, and 1234675. The simplicial complex Δ_P was investigated (in a more general context) by P. H. Edelman and V. Reiner, *Advances in Math.* **106** (1994), 36–62.

b. See Figure 3.85.

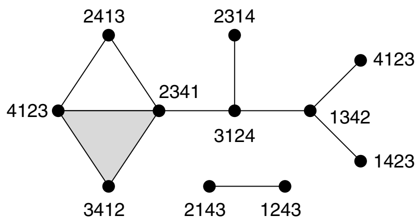

Figure 3.85 A simplicial complex whose faces are the permutations $w \in \mathfrak{S}_4$.

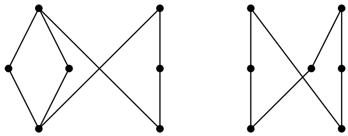

Figure 3.86 Posets with equal values of $\beta_{J(P)}(S)$.

c. See P. L. Hersh, *J. Combinatorial Theory, Ser. A* **105** (2004), 111–126.

153. *Answer.* Let $[p-1]-S = \{i_1,\dots,i_k\}_<$. Then the minimum value of $e(P)$ is

$$\min e(P) = i_1!(i_2-i_1)!(i_3-i_2)!\cdots(p-i_k)!,$$

achieved uniquely by $P = \boldsymbol{i_1} \oplus (\boldsymbol{i_2 - i_1}) \oplus \cdots \oplus (\boldsymbol{p - i_k})$.

154. The answer is affirmative for $n \le 6$. Of the 2045 nonisomorphic seven-element posets, it was checked by J. R. Stembridge that there is a unique pair P, Q that satisfy $\beta_{J(P)}(S) = \beta_{J(Q)}(S)$ for all $S \subseteq [6]$. The Hasse diagrams of P and Q are given in Figure 3.86.

155. a. Let $1 \le k \le n$, and define in the incidence algebra $I(P)$ (over $\mathbb{R}$, say) a function η_k by

$$\eta_k(s,t) = \begin{cases} 1, & \text{if } \rho(t)-\rho(s) = k, \\ 0, & \text{otherwise.} \end{cases}$$

The self-duality of $[s,t]$ implies that $\eta_j\eta_k(s,t) = \eta_k\eta_j(s,t)$ for all j and k, so η_j and η_k commute. But

$$\alpha_P(S) = \eta_{n_1}\eta_{n_2-n_1}\cdots\eta_{n-n_k}(\hat{0},\hat{1}),$$

and the proof follows since the various η_j's can be permuted arbitrarily.

c. It follows from a result of F. Regonati, *J. Combinatorial Theory, Ser. A* **60** (1992), 34–49 (theorem on page 45) that such lattices are products of certain modular lattices known as *q-primary* (though not conversely). See also Theorem 3.4 of R. Stanley, *Electronic J. Combinatorics* **3**, #R6 (1996); reprinted in *The Foata Festschrift* (J. Désarménien, A. Kerber, and V. Strehl, eds.), Imprimerie Louis-Jean, Gap, 1996, pp. 165–186. There is an almost complete classification of primary modular lattices (which includes the q-primary modular lattices) by Baer, Inaba, and Jónsson-Monk; see B. Jónsson and G. S. Monk, *Pacific J. Math.* **30** (1969), 95–139. A complete classification of finite modular lattices for which every interval is self-dual (or the more general products of q-primary lattices) seems hopeless since it involves such problems as the classification of finite projective planes. For some further work related to primary modular lattices, see F. Regonati and S. D. Sarti, *Ann. Combinatorics* **4** (2000), 109–124.

156. We have $\mathbb{N}\times\mathbb{N} = J_f(Q)$, where the elements of Q are $s_1 < s_2 < \cdots$ and $t_1 < t_2 < \cdots$. Regard Q as being contained in the total order where $s_i < t_j$ for all i, j. By Theorem 3.13.1 (extended in an obvious way to finitary distributive lattices), we have that

$\beta_{\mathbb{N}\times\mathbb{N}}(S)$ is equal to the number of linear orderings $u_1,u_2,\ldots$ of Q such that the s_i's appear in increasing order, the t_i's appear in increasing order, and a t_i is immediately followed by an s_j if and only if $t_i = u_k$ where $k \in S$. Thus, $u_1,\ldots,u_{m_1}$ can be chosen as $s_1,\ldots,s_i$, $t_1,\ldots,t_{m_1-i}$ $(0 \le i \le m_1 - 1)$ in m_1 ways. Then $u_{m_1+1} = s_{i+1}$, whereas $u_{m_1+2},\ldots,u_{m_2}$ can be chosen in $m_2 - m_1 - 1$ ways, and so on, giving the desired result. A less combinatorial proof appears in [3.68, Prop. 23.7].

157. a. Let $\alpha_k = \sum_{\#S=k}\alpha_P(S)$. Now $Z(P,m)$ is equal to the number of multichains $\hat{0} = t_0 \le t_1 \le \cdots \le t_m = \hat{1}$. Such a multichain K is obtained by first choosing a chain $C\colon \hat{0} < u_1 < \cdots < u_k < \hat{1}$ in α_k ways, and then choosing K whose support (underlying set) is C in $\left(\!\binom{k+2}{m-1-k}\!\right) = \binom{m}{k+1}$ ways. Hence, $Z(P,m) = \sum_k \binom{m}{k+1}\alpha_k$; that is, $\Delta^{k+1}Z(P,0) = \alpha_k$.

b. Divide both sides of the desired equality by $(1-x)^{n+1}$ and take the coefficient of x^m. Then we need to show that

$$\begin{aligned} Z(P,m) &= \sum_j \beta_j(-1)^{m-j-1}\binom{-n-1}{m-j-1} \\ &= \sum_j \beta_j\binom{n+m-j-1}{n}. \end{aligned}$$

Now

$$\begin{aligned} \alpha_k &= \sum_{\#S=k}\sum_{T\subseteq S}\beta_P(T) \\ &= \sum_j\sum_{\#T=j}\binom{n-1-j}{n-1-k}\beta_P(T) \\ &= \sum_j\binom{n-1-j}{n-1-k}\beta_j. \end{aligned}$$

Hence from (a),

$$\begin{aligned} Z(P,m) &= \sum_k\binom{m}{k+1}\alpha_k \\ &= \sum_{j,k}\binom{m}{k+1}\binom{n-1-j}{n-1-k}\beta_j. \end{aligned}$$

But

$$\sum_k\binom{m}{k+1}\binom{n-1-j}{n-1-k} = \binom{n+m-j-1}{n}$$

(e.g., by Example 1.1.17), and the proof follows.

A more elegant proof can be given along the following lines. Introduce variables $x_1,\ldots,x_{n-1}$, and for $S \subseteq [n-1]$ write $x_S = \prod_{i\in S}x_i$. Moreover, for a multichain $K\colon t_1 \le \cdots \le t_m$ of $P - \{\hat{0},\hat{1}\}$, write $x_K = \prod_{i=1}^m x_{\rho(t_i)}$. One easily sees that

$$\begin{aligned} \sum_K x_K &= \sum_S \alpha_P(S)\left(\prod_{i\in S}\frac{x_i}{1-x_i}\right) \\ &= \frac{\sum_S\beta_P(S)x_S}{(1-x_1)(1-x_2)\cdots(1-x_{n-1})}. \end{aligned}$$

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Set each $x_i = x$ and multiply by $(1-x)^{-2}$ (corresponding to adjoining $\hat{0}$ and $\hat{1}$) to obtain (a) and (b).

NOTE. If $f(m)$ is any polynomial of degree n, then Section 4.3 discusses the generating function $\sum_{m\geq 0} f(m)x^m$, in particular, its representation in the form $P(x)(1-x)^{-n-1}$. Hence, the present exercise may be regarded as "determining" $P(x)$ when $f(m) = Z(P,m)$.

c. By definition of $\chi_P(q)$, we have

$$\begin{aligned} w_k &= \sum_{\substack{t\in P\\ \rho(t)=k}} \mu(\hat{0},t) \\ &= \sum_{\rho(t)\leq k} \mu(\hat{0},t) - \sum_{\rho(t)\leq k-1} \mu(\hat{0},t). \end{aligned}$$

Letting μ_S denote the Möbius function of the S-rank-selected subposet P_S of P as in Section 3.13, then by the defining recurrence (3.15) for μ we get

$$w_k = -\mu_{[k]}(\hat{0},\hat{1}) + \mu_{[k-1]}(\hat{0},\hat{1}).$$

The proof follows from equation (3.54).

158. In the case $k = 1$, a noncombinatorial proof of (a) was first given by G. Kreweras, *Discrete Math.* **1** (1972), 333–350, followed by a combinatorial proof by Y. Poupard, *Discrete Math.* **2** (1972), 279–288. The case of general k, as well as (c) and (d), is due to P. H. Edelman, *Discrete Math.* **31** (1980), 171–180. See also P. H. Edelman, *Discrete Math.* **40** (1982), 171–179. Of course (b) follows from (a) by taking $n = 1$ and $n = -2$, whereas (e) follows from (d) by taking $S = \{t-m\}$ and $S = [0, t-2]$. Part (f) is due to Kreweras, op. cit. (Thm. 4), while (g) first appeared in P. L. Hersh, Ph.D. thesis, M.I.T., 1999 (Theorem 4.3.2), and *J. Combinatorial Theory Ser. A* **103** (2003), 27–52 (Theorem 6.3). To solve (g) using Exercise 3.125, define a labeling $\lambda\colon \mathcal{H}(P_{1,t}) \to \mathbb{Z}$ as follows. If $\pi \lessdot \sigma$ in $P_{1,t}$, then σ is obtained from π by merging two blocks B, B'. Define

$$\lambda(\pi,\sigma) = \max(\min B, \min B') - 1.$$

It is routine to check that λ has the necessary properties. This labeling is due to P. H. Edelman and A. Björner, and appears in A. Björner, *Trans. Amer. Math. Soc.* **260** (1980), 159–183 (page 165). A different edge labeling related to parking functions appears in R. Stanley, *Electronic J. Combinatorics* **4** (1997), #R20. Note that $P_{1,t}$ is a lattice by Proposition 3.3.1 because it is a meet-semilattice of Π_t with $\hat{1}$.
Partitions π satisfying (ii) are called *noncrossing partitions* and have received much attention. For some additional information and references, see Vol. II, Exercises 5.35, 6.19(pp), and 7.48(f).

159. By symmetry it suffices to take $w = (1,2,\ldots,n)$. In this case, it can be checked that an isomorphism $\varphi\colon [\hat{0},w] \to P_{1,n}$ is obtained by taking the set of elements of each cycle of $u \in [\hat{0},w]$ to be the blocks of $\varphi(w)$. This result is due to P. Biane, *Discrete Math.* **175** (1997), 41–53 (Theorem 1). For further information on the absolute order, see C. A. Athanasiadis and M. Kallipoliti, *J. Combinatorial Theory Ser. A* **115** (2008), 1286–1295.

160. Given a labeling ω, define an orientation $\mathfrak{o}_\omega$ of $\mathcal{H}$ by directing an edge ij from i to j if $i < j$. Clearly, $\mathfrak{o}_\omega$ is acyclic, and it is easy to check that ω and ω' are equivalent if and only if $\mathfrak{o}_\omega = \mathfrak{o}_{\omega'}$. The problem of counting the number of equivalence classes was raised by Stanley [3.67, p. 25], with the answer stated without proof in [3.68, p. 7].

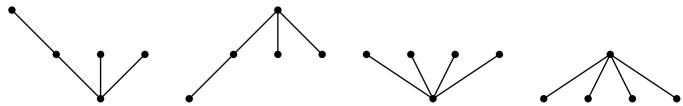

Figure 3.87 Posets for which $\Omega_P(m)$ has a negative coefficient.

161. Without loss of generality we may assume $U \cup V = [j+k]$. Let P be the j-element chain with elements labeled $u_1,\dots,u_j$ from bottom to top, and similarly Q with elements labeled $v_1,\dots,v_k$. Denote this labeling of $P+Q$ by ω. Note that $\mathcal{A}(P+Q,\omega)$ depends only on $D(u)$ and $D(v)$, and that $\mathcal{L}(P,\omega) = \mathrm{sh}(u,v)$. The proof follows easily from Lemmas 3.15.3 and 3.15.4. Exercises 7.93 and 7.95 are related.

162. **a.** Although this problem can be done using the formula (3.13) for $W_{P,\omega}(x)$, it is easier to observe that

$$G_{P_1+P_2,\omega}(x) = G_{P_1,\omega_1}(x)G_{P_2,\omega_2}(x)$$

and then use equation (3.63).

b. Let $P = P_1 + \cdots + P_k$, where P_k is a chain labeled by the word w^i from bottom to top. Then $\mathcal{L}(P,\omega) = \mathrm{sh}(w^1,\dots,w^k)$. Moreover, $W_{P_i,\omega_i} = q^{\mathrm{maj}(w^i)}$, so the proof follows from iterating (a).

163. **a.** If P is an antichain then $\Omega_P(m) = m^p$, and the conclusion is clear. It thus suffices to show that when P is not an antichain, the coefficient of m^{p-1} in $\Omega_P(m)$ is positive. The coefficient is equal to $2e_P(p-1) - (p-1)e(p)$ (as defined preceding Theorem 3.15.8). Let A be the set of all ordered pairs (σ,i), where $\sigma\colon P \to \boldsymbol{p}$ is a linear extension and $i \in [p-1]$. Let B be the set of all ordered pairs (τ,j), where $\tau\colon P \to \boldsymbol{p-1}$ is a surjective order-preserving map and $j = 1$ or 2. Since $\#A = (p-1)e(p)$ and $\#B = 2e(p-1)$, it suffices to find an injection $\phi\colon A \to B$ that is not surjective. Choose an indexing $\{t_1,\dots,t_p\}$ of the elements of P. Given $(\sigma,i) \in A$, define $\phi(\sigma,i) = (\tau,j)$, where

$$\tau(t) = \begin{cases} \sigma(t), & \text{if } \sigma(t) \le i, \\ \sigma(t)-1, & \text{if } \sigma(t) > i, \end{cases}$$

$$j = \begin{cases} 1, & \text{if } \sigma(t_r) = i,\ \sigma(t_s) = i+1, \text{ and } r < s, \\ 2, & \text{if } \sigma(t_r) = i,\ \sigma(t_s) = i+1, \text{ and } r > s. \end{cases}$$

It is easily seen that ϕ is injective. If t covers s in P and $\tau\colon P \to \boldsymbol{p-1}$ is an order-preserving surjection for which $\tau(s) = \tau(t)$ (such a τ always exists), then one of $(\tau,1)$ and $(\tau,2)$ cannot be in the image of ϕ. Hence, ϕ is not surjective.

b. This problem was raised by J. N. Kahn and M. Saks, who found the foregoing proof of (a) independently from this writer.

164. No. There are four 5-element posets for which $\Omega_P(m)$ has a negative coefficient, and none smaller. These four posets are shown in Figure 3.87.

165. J. Neggers, *J. Combin. Inform. System Sci.* **3** (1978), 113–133, made a conjecture equivalent to $A_P(x)$ having only real zeros (the naturally labeled case). In 1986 Stanley (unpublished) suggested that this conjecture could be extended to arbitrary labelings. The first published reference seems to be F. Brenti, *Mem. Amer. Math. Soc.*, no. 413 (1989). These conjectures became known as the *poset conjecture* or the *Neggers–Stanley conjecture*. Counterexamples to the conjecture of Stanley were obtained by P. Brändén, *Electron. Res. Announc. Amer. Math. Soc.* **10** (2004), 155–158. Finally, J.

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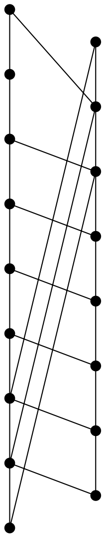

Figure 3.88 A counterexample to the poset conjecture.

R. Stembridge, *Trans. Amer. Math. Soc.* **359** (2007), 1115–1128, produced counterexamples to the original conjecture of Neggers. Stembridge's smallest counterexample has 17 elements. One such poset P is given by Figure 3.88, for which

$$A_P(x) = x + 32x^2 + 336x^3 + 1420x^4 + 2534x^5 + 1946x^6 + 658x^7 + 86x^8 + 3x^9,$$

which has zeros near $-1.858844 \pm 0.149768i$. It is still open whether every *graded* natural poset satisfies the poset conjecture.

166. a. The "if" part is easy; we sketch a proof of the "only if" part. Let P be the smallest poset for which $G_P(\boldsymbol{x})$ is symmetric and P is not a disjoint union of chains. Define $\overline{G}_P(\boldsymbol{x}) = \sum_\tau x_{\tau(t_1)} \cdots x_{\tau(t_p)}$, where τ ranges over all *strict* P-partitions $\tau\colon P \to \mathbb{N}$. The technique used to prove Theorem 3.15.10 shows that $G_P(\boldsymbol{x})$ is symmetric if and only if $\overline{G}_P(\boldsymbol{x})$ is symmetric. Let M be the set of minimal elements of P. Set $m = \#M$ and $P_1 = P - M$. The coefficient of x_0^m in $\overline{G}_P(\boldsymbol{x})$ is $\overline{G}_{P_1}(\boldsymbol{x'})$, where $\boldsymbol{x'} = (x_1, x_2, \ldots)$. Hence, $\overline{G}_{P_1}(\boldsymbol{x})$ is symmetric, so P_1 is a disjoint union of chains. Similarly, if M' denotes the set of maximal elements of P, then $P - M'$ is a disjoint union of chains.

Now note that m is the largest power of x_0 that can appear in a monomial in $\overline{G}_P(\boldsymbol{x})$. Hence, m is the largest power of *any* x_i that can appear in a monomial in $\overline{G}_P(\boldsymbol{x})$. Let A be an antichain of P. We can easily find a strict P-partition that is constant on A, so $\#A \leq m$. Hence, the largest antichain of P has size m. By Dilworth's theorem (Exercise 3.77(d)), P is a union of m chains. Each such chain intersects M and M'. It is easy to conclude that P is a disjoint union of chains $C_1, \ldots, C_k$, together with relations $s < t$, where s is a minimal element of some C_i and t a maximal element of some C_j, $i \neq j$.

Next note that the coefficient of $x_0^m x_1 x_2 \cdots x_{p-m}$ in $\overline{G}_P(\boldsymbol{x})$ is equal to $e(P_1)$, the number of linear extensions of P_1, so the coefficient of $x_0 x_1 \cdots x_{i-1} x_i^m x_{i+1} \cdots x_{p-m}$ is also $e(P_1)$ for any $0 \leq i \leq p - m$. Let $Q = C_1 + \cdots + C_k$. Then the coefficient of $x_0^m x_1 x_2 \cdots x_{p-m}$ in $\overline{G}_Q(\boldsymbol{x})$ is again equal to $e(P_1)$, since $P_1 \cong Q - \{\text{minimal elements of } Q\}$. Thus, the coefficient of $x_0 x_1 \cdots x_{i-1} x_i^m x_{i+1} \cdots x_{p-m}$ in $\overline{G}_Q(\boldsymbol{x})$ is $e(P_1)$. Since P is a refinement of Q it follows that if $\tau\colon P \to [0, p-m]$

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is a strict Q-partition such that $\tau^{-1}(j)$ has one element for all $j \in [0, p-m]$ with a single exception $\#\tau^{-1}(i) = m$, then (regarding P as a refinement of Q) $\tau\colon P \to [0, p-m]$ is a strict P-partition. Now let $s < t$ in P but $s \parallel t$ in Q. One can easily find a strict Q-partition $\tau\colon Q \to [0, p-m]$ with $\tau(s) = \tau(t) = i$, say, and with $\#\tau^{-1}(i) = m$, $\#\tau^{-1}(j) = 1$ if $j \neq i$. Then $\tau\colon P \to [0, p-m]$ is not a strict P-partition, a contradiction.

b. This conjecture is due to R. Stanley, [3.68, p. 81]. For a proof of the "if" part, see Vol. II, Theorem 7.10.2. An interesting special case (different from (a)) is due to C. Malvenuto, *Graphs and Combinatorics* **9** (1993), 63–73.

167. b. The idea is to rule out subposets of P until the only P that remain have the desired form. For instance, P cannot have a three-element antichain A. For let i, j, k be the labels of the elements of A. Then there are linear extensions of P of the form $\sigma i' j' k' \tau$ for fixed σ and τ, where $i' j' k'$ is any permutation of ijk. One can check that these six linear extensions cannot all have the same number of descents.

169. a. Apply Theorem 3.15.8 to the case $P = \boldsymbol{r}_1 + \cdots + \boldsymbol{r}_{\boldsymbol{m}}$, naturally labeled.

b. Suppose that $w \in \mathfrak{S}_M$ with $\operatorname{des}(w) = k-1$. Then w consists of x_1 1's, then y_1 2's, and so on, where $x_1 + \cdots + x_k = r_1$, $y_1 + \cdots + y_k = r_2$, and $x_1 \in \mathbb{N}$, $x_i \in \mathbb{P}$ for $2 \le i \le k$, $y_i \in \mathbb{P}$ for $1 \le i \le k-1$, $y_k \in \mathbb{N}$. Conversely, any such x_i's and y_i's yield a $w \in \mathfrak{S}_M$ with $\operatorname{des}(w) = k-1$. There are $\binom{r_1}{k-1}$ ways of choosing the x_i's and $\binom{r_2}{k-1}$ ways of choosing the y_i's. Hence,

$$A_M(x) = \sum_{k=0}^{r_1+r_2} \binom{r_1}{k}\binom{r_2}{k} x^{k+1}.$$

A q-analogue of this result appears in [3.68, Cor. 12.8]. Exercise 3.156 is related.

170. a. An order ideal of $J(\boldsymbol{m} \times \boldsymbol{n})$ of rank r can easily be identified with a partition of r into at most m parts, with largest part at most n. Now use Proposition 1.7.3 to show that $F(L,q) = \binom{\boldsymbol{m}+\boldsymbol{n}}{\boldsymbol{m}}$, which is equivalent to pleasantness.

b. Equivalent to a famous result of MacMahon. See Vol. II, Theorem 7.21.7 and the discussion following it. A further reference is R. Stanley, *Studies in Applied Math.* **50** (1971), 167–188, 259–279.

c. An order ideal of $J(\boldsymbol{2} \times \boldsymbol{n})$ of rank r can easily be identified with a partition of r into *distinct parts*, with largest part at most n, whence $F(L,q) = (1+q)(1+q^2)\cdots(1+q^n)$.

d. This result is equivalent to a conjecture of Bender and Knuth, shown by G. E. Andrews, *Pacific J. Math.* **72** (1977), 283–291, to follow from a much earlier conjecture of MacMahon. MacMahon's conjecture was proved independently by G. E. Andrews, *Adv. Math. Suppl. Studies* **1** (1978), 131–150; B. Gordon, *Pacific J. Math.* **108** (1983), 99–113; and I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, Oxford, 1979 (Ex. 19 on p. 53), 2nd ed., 1995 (Ex. 19 on p. 86). MacMahon's conjecture and similar results can be unified by the theory of minuscule representations of finite-dimensional complex semisimple Lie algebras; see R. A. Proctor, *Europ. J. Combinatorics* **5** (1984), 331–350.

e. This result is equivalent to the conjectured "q-enumeration of totally symmetric plane partitions," alluded to by G. E. Andrews, *Abstracts Amer. Math. Soc.* **1** (1980), 415, and D. P. Robbins (unpublished), and stated more explicitly in R. Stanley, *J. Combinatorial Theory, Ser. A* **43** (1986), 103–113 (equation (2)). The $q = 1$ case was first proved by J. R. Stembridge, *Advances in Math.* **111** (1995), 227–243, and later by G. E. Andrews, P. Paule, and C. Schneider, *Advances in Appl. Math.* **34** (2005), 709–739. A proof of the general case was finally given by C. Koutschan, M. Kauers, and D. Zeilberger, `arXiv:1002.4384`, 23 February 2010. Several persons have shown that $F(L,q)$ is also equal to $\sum_A(\det A)$, where A ranges over all square submatrices (including the empty matrix $\emptyset$, with $\det\emptyset = 1$) of the $(n+1)\times(n+1)$ matrix

$$\left[q^{i+1+\binom{j+1}{2}} \binom{i}{j} \right]_{i,j=0}^{n}.$$

f, g. Follows from either Theorem 6 or the proof of Theorem 8 of R. A. Proctor, *Europ. J. Combinatorics* **5** (1984), 331–350. (It is not difficult to give a direct proof of (f).) The proof of Proctor's Theorem 8 involves the application of the techniques of our Section 3.15 to these posets.

171. a. Follows from the bijection given in the proof of Proposition 3.5.1.

b. This result appears in [3.68, Prop. 8.2] and is proved in the same way as Theorems 3.15.7 or 3.15.16.

c. Equation (3.129) follows directly from the definition (3.127); see [3.68, Prop. 12.1]. Equation (3.130) is then a consequence of (3.96) and (3.97). Alternatively, (3.130) follows directly from (a).

(e) Analogous to the proof of Theorem 3.15.10.

(f) See [3.68, Prop. 17.3(ii)].

172. a. First note that

$$\binom{p+m-i}{p} = \frac{(1-q^{p-i}y)(1-q^{p-i-1}y)\cdots(1-q^{-i+1}y)}{(1-q^{p})(1-q^{p-1})\cdots(1-q)},$$

where $y = q^m$. It follows from Exercise 3.171(b) that there is a polynomial $V_P(y)$ of degree p in y, whose coefficients are rational functions of q, such that

$$U_{P,m}(q) = V_P(q^m).$$

The polynomial $V_P(y)$ is unique since it is determined by its values on the infinite set $\{1, q, q^2, \dots\}$.

Since $U_{P_1+P_2,m}(q) = U_{P_1,m}(q)U_{P_2,m}(q)$, it follows that if each component of P is Gaussian, then so is P. Conversely, suppose that $P_1 + P_2$ is Gaussian. Thus,

$$V_{P_1+P_2}(y) = R(q)\prod_{i=1}^{p}\left(1 - yq^{h_i}\right),$$

where $R(q)$ depends only on q (not on y). But clearly $V_{P_1+V_2}(y) = V_{P_1}(y)V_{P_2}(y)$. Since each factor $1 - yq^{h_i}$ is irreducible (as a polynomial in y) and since $\deg V_{P_i}(y) = \#P_i$, we must have

$$V_{P_i}(y) = R_i(q)\prod_{j\in S_i}\left(1 - yq^{h_i}\right),$$

where j ranges over some subset S_i of $[p]$. Since $U_{P_i,0}(q) = V_{P_i}(1) = 1$, it follows that $R_i(q) = \prod_{j\in S_i}\left(1 - q^{h_i}\right)^{-1}$, so P_i is Gaussian.

b. Clearly for any finite poset P, we have

$$\lim_{m\to\infty} U_{P,m}(q) = G_p(q),$$

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as defined by equation (3.62). Hence, if P is Gaussian, we get

$$G_P(q) = \frac{W_P(q)}{(1-q)\cdots(1-q^p)} = \prod_{i=1}^{p}\left(1-q^{h_i}\right)^{-1}. \tag{3.147}$$

Hence, $W_P(q) = q^{d(P)}W_P(1/q)$, where $d(P) = \deg W_P(q)$, so by Theorem 3.15.16 P satisfies the δ-chain condition.
Now by equation (3.130), we have

$$U_{m,P}(q) = q^{pm}U_{P,m}(1/q) = U_{P,m}(q).$$

It follows that P^* is also Gaussian, and hence P^* satisfies the δ-chain condition. But if P is connected, then both P and P^* satisfy the δ-chain condition if and only if P is graded, and the proof follows.

c. Suppose that a_i of the h_j's are equal to i. Then by equation (3.128) we have

$$\frac{(\boldsymbol{p})!}{(\mathbf{1})^{a_1}\cdots(\boldsymbol{p})^{a_p}}(1-qy)^{a_1}(1-q^2y)^{a_2}\cdots(1-q^py)^{a_p}$$

$$= \sum_{i=0}^{p-1}(1-q^{p-i}y)(1-q^{p-i-1}y)\cdots(1-q^{-i+1}y)W_{P,i}(q). \tag{3.148}$$

Pick $1 \le j \le p+1$, and let $b_i = a_i$ if $i \ne j$, and $b_j = a_j + 1$ (where we set $a_{p+1} = 0$). Set

$$\frac{(\boldsymbol{p+1})!}{(\mathbf{1})^{b_1}\cdots(\boldsymbol{p+1})^{b_{p+1}}}(1-qy)^{b_1}\cdots(1-q^{p+1}y)^{b_{p+1}}$$

$$= \sum_{i=0}^{p}(1-q^{p+1-i}y)\cdots(1-q^{-i+1}y)X_i(P,q).$$

This equation uniquely determines each $X_i(P,q)$.
Now we note the identity

$$\begin{aligned}(1-q^{p+1}y)(1-q^jy) = {} & (1-q^{i+j})(1-q^{p+1-i}y)\\ & + (q^{i+j}-q^{p+1})(1-q^{-i}y).\end{aligned} \tag{3.149}$$

Multiply equation (3.148) by (3.149) to obtain

$$\begin{aligned}&(1-q^j)\sum_{i=0}^{p}(1-q^{p+1-i}y)\cdots(1-q^{-i+1}y)X_i(P,q)\\ &\quad = \sum_{i=0}^{p-1}\big[(1-q^{i+j})(1-q^{p+1-i}y)\cdots(1-q^{-i+1}y)\\ &\qquad + (q^{i+j}-q^{p+1})(1-q^{p-i}y)\cdots(1-q^{-i}y)\big]W_{P,i}(q).\end{aligned}$$

It follows that

$$(1-q^j)X_i = (1-q^{i+j})W_i + (q^{i+j-1}-q^{p+1})W_{i-1}. \tag{3.150}$$

Next define

$$[p-1] - \{a_1 + a_2 + \cdots + a_i : i \ge 1\} = \{c_1, \ldots, c_k\}_>.$$

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If we assume by induction that if we know $\deg W_{i-1}$ and $\deg W_i$ in equation (3.150), we can compute $\deg X_i$. It then follows by induction that

$$\deg W_i = c_1 + \cdots + c_i,\ \ 0 \leq i \leq k.$$

Comparing with Exercise 3.171(f) completes the proof.

d. If $U_{P,m}(q)$ is given by equation (3.131), then

$$q^{pm} U_{P,m}(1/q) = U_{P,m}(q).$$

Comparing with equation (3.130) shows that $U_{P,m}(q) = U_{P^*,m}(q)$. Let ρ^* denote the rank function of P^*. It follows from (c) that

$$\begin{aligned}\{1+\rho(t) : t \in P\} &= \{1+\rho^*(t) : t \in P\}\\ &= \{\ell(P)+1-\rho(t) : t \in P\}\end{aligned}$$

(as multisets). Hence by (c), the multisets $\{h_1,\ldots,h_p\}$ and $\{\ell(P)+2-h_1,\ldots,\ell(P)+2-h_p\}$ coincide, and the proof follows. (This result was independently obtained by P. J. Hanlon.)

e. Let P have W_i elements of rank i. Using equation (3.131) and (c), one computes that the coefficient of q^2 in $U_{P,1}(q)$ is $\binom{W_0}{2} + W_1$. By Exercise 3.171(a), this number is equal to the number of two-element order ideals of P. Any of the $\binom{W_0}{2}$ two-element subsets of minimal elements forms such an order ideal. The remaining W_1 two-element order ideals must consist of an element of rank one and the unique element that it covers, completing the proof.

f. A uniform proof of (i)-(v), using the representation theory of semisimple Lie algebras, is due to R. A. Proctor, *Europ. J. Combinatorics* **5** (1984), 313–321. For ad hoc proofs (using the fact that a connected poset P is Gaussian if and only if $P \times \boldsymbol{m}$ is pleasant for all $m \in \mathbb{P}$). see the solution to Exercise 3.170(b,d,f,g).

NOTE. Posets P satisfying equation (3.147) are called *hook length posets*. R. A. Proctor and D. Peterson found many interesting classes of such posets. See Proctor, *J. Algebra* **213** (1999), 272–303 (§1). Proctor discusses a uniform proof based on representation theory and calls these posets *d-complete*. For a classification of d-complete posets, see Proctor, *J. Algebraic Combinatorics* **9** (1999), 61–94. For a further important property of d-complete posets, see Proctor, preprint, `arXiv:0905.3716`.

173. This beautiful theory is due to P. Brändén, *Electronic J. Combinatorics* **11(2)** (2004), #R9. Note that as a special case of (h), $A_{P,\omega}(x)$ has symmetric unimodal coefficients if P is graded and ω is natural. (Symmetry of the coefficients also follows from Corollary 3.15.18 and Corollary 4.2.4 (iii).) In this special case unimodality was shown by V. Reiner and V. Welker, *J. Combinatorial Theory Ser. A* **109** (2005), 247–280 (Corollary 3.8 and Theorem 3.14), and later as part of more general results by C. A. Athanasiadis, *J. reine angew. Math.* **583** (2005), 163–174 (Lemma 3.8), and *Electronic J. Combinatorics* **11** (2004), #R6 (special case of Theorem 4.1), by using deep results on toric varieties. A combinatorial proof using a complicated recursion argument was given by J. D. Farley, *Advances in Applied Math.* **34** (2005), 295–312.

g. For a canonical labeling ω the procedure will end when each poset is an ordinal sum $Q_1 \oplus \cdots \oplus Q_k$ of antichains, labeled so that every label of elements of Q_i is either less than or greater than every label of elements in Q_{i+1}, depending on whether i is odd or even. From this observation the proof follows easily (using (c) to extend the result to any labeling ω for which (P,ω) is sign-graded).

h. Use Exercise 1.50(c,e).

i. We obtain

$$A_P(x) = (1+x)(1+4x+x^2)+4(x+x^2) = 1+9x+9x^2+x^3.$$

174. a. The statement that the interval $[s,t]$ has as many elements of odd rank as of even rank is equivalent to $\sum_{u\in[s,t]}(-1)^{\rho(u)-\rho(s)} = 0$. The proof now follows easily from the defining recurrence (3.15) for μ.

b. Analogous to Proposition 3.16.1.

c. If n is odd, then by (b),

$$Z(P,m)+Z(P,-m) = -m((-1)^n\mu_P(\hat{0},\hat{1})-1).$$

The left-hand side is an even function of m, whereas the right-hand side is even if and only if $\mu_P(\hat{0},\hat{1}) = (-1)^n$. (There are many other proofs.)

175. By Proposition 3.8.2, $P\times Q$ is Eulerian. Hence, every interval $[z',z]$ of R with $z'\neq\hat{0}_R$ is Eulerian. Thus by Exercise 3.174(a), it suffices to show that for every $z=(s,t)>\hat{0}_R$ in R, we have

$$\sum_{z'\leq z}(-1)^{\rho_R(z')} = 0,$$

where ρ_R denotes the rank function in R. Since for any $v\neq\hat{0}_R$, we have $\rho_R(v) = \rho_{P\times Q}(v)-1$, there follows

$$\begin{aligned}\sum_{\substack{z\leq z'\\ \text{in } R}}(-1)^{\rho_R(z')} &= \sum_{\substack{u\leq z\\ \text{in } P\times Q}}(-1)^{\rho_{P\times Q}(u)-1}\\ &\quad -\sum_{\substack{\hat{0}_P\neq s'\leq s\\ \text{in } P}}(-1)^{\rho_P(s')-1} - \sum_{\substack{\hat{0}_Q\neq t'\leq t\\ \text{in } Q}}(-1)^{\rho_Q(t')-1}\\ &\quad +(-1)^{\rho_{P\times Q}(\hat{0}_{P\times Q})-1}+(-1)^{\rho_R(\hat{0}_R)}\\ &= 0-1-1+1+1 \ = \ 0.\end{aligned}$$

For further information related to the poset R, see M. K. Bennett, *Discrete Math.* **79** (1990), 235–249.

176. a. *Answer.* $\beta_{P_n}(S) = 1$ for all $S\subseteq[n]$.

b. By Exercise 3.157(b),

$$\sum_{m\geq 0} Z(P_n,m)x^m = \frac{x(1+x)^n}{(1-x)^{n+2}}.$$

(One could also appeal to Exercise 3.137.)

c. Write $f_n = f(P_n,x)$, $g_n = g(P_n,x)$. The recurrence (3.76) yields

$$f_n = (x-1)^n + 2\sum_{i=0}^{n-1} g_i(x-1)^{n-1-i}. \tag{3.151}$$

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Equations (3.75) and (3.151), together with the initial conditions $f_0 = g_0 = 1$, completely determine f_n and g_n. Calculating some small cases leads to the guess

$$g_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-1}{k} - \binom{n-1}{k-2} \right] x^k, \tag{3.152}$$

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-1}{k} - \binom{n-1}{k-1} \right] (x^k + x^{n-k}).$$

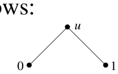
It is not difficult to check that these polynomials satisfy the necessary recurrences. Note also that $g_{2m} = (1-x)g_{2m-1}$ and $f_{2m+1} = (1-x)^{2m}(1+x)$.

177. a. Let $C_n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : 0 \le x_i \le 1\}$, an n-dimensional cube. A nonempty face F of C_n is obtained by choosing a subset $T \subseteq [n]$ and a function $\phi\colon T \to \{0,1\}$, and setting

$$F = \{(x_1,\dots,x_n) \in C_n : x_i = \phi(i) \text{ if } i \in T\}.$$

Let F correspond to the interval $[\phi^{-1}(1), \phi^{-1}(1) \cup ([n]-T)]$ of B_n. This yields the desired (order-preserving) bijection.

b. Denote the elements of Λ as follows:



Let F be as in (a), and correspond to F the n-tuple $(y_1,\dots,y_n) \in \Lambda^n$ where $y_i = \phi(i)$ if $i \in T$ and $y_i = u$ if $i \notin T$. This yields the desired (order-preserving) bijection.

c. Denote the two elements of P_n of rank i by a_i and b_i, $1 \le i \le n$. Associate with the chain $z_1 < z_2 < \cdots < z_k$ of $P_n - \{\hat{0},\hat{1}\}$ the n-tuple $(y_1,\dots,y_n) \in \Lambda^n$ as follows:

$$y_i = \begin{cases} 0, & \text{if some } z_j = a_i, \\ 1, & \text{if some } z_j = b_i, \\ u, & \text{otherwise.} \end{cases}$$

This yields the desired bijection.

d. Follows from (c), Exercise 3.176(a), and [3.77, Thm. 8.3].

e. With Λ as in (b), we have $Z(\Lambda,m) = 2m-1$, so $Z(\Lambda^n,m) = (2m-1)^n$ by Exercise 3.137. It follows easily that

$$Z(L_n,m) = 1^n + 3^n + 5^n + \cdots + (2m-1)^n.$$

f. *Answer.* $g(L_n,x) = \sum_{k \ge 0} \frac{1}{n-k+1}\binom{n}{k}\binom{2n-2k}{n}(x-1)^k$ (obtained in collaboration with I. M. Gessel). A generating function for $g(L_n,x)$ was given in R. Stanley, *J. Amer. Math. Soc.* **5** (1992), 805–851 (Proposition 8.6), namely,

$$\sum_{n \ge 0} g(L_n,x)\frac{y^n}{n!} = e^{2y} \sum_{n \ge 0} (-1)^n g_n \frac{y^n}{n!},$$

where g_n is given by equation (3.152).

g. This result was deduced from (f) by L. W. Shapiro (private communication). For further work in this area, see G. Hetyei, A second look at the toric h-polynomial of a cubical complex, `arXiv:1002.3601`.

179. For *rational* polytopes (i.e., those whose vertices have rational coordinates), this result follows from the hard Lefschetz theorem for the intersection homology of projective toric varieties; see Stanley [3.79]. For arbitrary convex polytopes the notion of intersection homology needs to be defined despite the absence of a corresponding variety, and the hard Lefschetz theorem must be proved in this context. The theory of "combinatorial intersection homology" was developed by G. Barthel, J.-P. Brasselet, K.-H. Fiesler, and L. Kaup, *Tohoku Math. J.* **54** (2002), 1–41, and independently by P. Bressler and V. A. Lunts, *Compositio Math.* **135:3** (2003), 245–278. K. Karu, *Invent. math.* **157** (2004), 419–447, showed that the hard Lefschetz theorem held for this theory, thereby proving the nonnegativity of the coefficients of $g(L,x)$. An improvement to Karu's result was given by Bressler and Lunts, *Indiana Univ. Math. J.* **54** (2005), 263–307. A more direct approach to the work of Bressler and Lunts was given by Barthel, Brasselet, Fiesler, and Kaup, *Tohoku Math. J.* **57** (2005), 273–292. It remains open to prove the nonnegativity of the coefficients of $g(P,x)$ (or even $f(P,x)$) when P is both Cohen–Macaulay and Eulerian.

180. L_{nd} is in fact the lattice of faces of a certain d-dimensional convex polytope $C(n,d)$ called a *cyclic polytope*. Hence by Proposition 3.8.9, L_{nd} is an Eulerian lattice of rank $d+1$. The combinatorial description of L_{nd} given in the problem is called "Gale's evenness condition." See, for example, page 85 of P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge Univ. Press, London, 1971, or [3.37, p. 62], or G. M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York, 1995 (Theorem 0.7).

181. a. If L is the face lattice of a convex d-polytope $\mathcal{P}$, then the result goes back to Carathéodory. For a direct proof, see B. Grünbaum, *Convex Polytopes*, 2nd ed., Springer-Verlag, New York, 2003 (item 4 on page 123). The extension to Eulerian lattices is due to H. Bidkhori, Ph.D. thesis, M.I.T., 2010 (Section 3.5). The proof first shows by induction on d that $L-\{\hat{1}\}$ is simplicial (as defined in Section 3.16). It then follows from equation (3.73) that L has $\binom{d+1}{k}$ elements of rank k for all k. Since L is atomic (e.g., by Corollary 3.9.5) it must be a boolean algebra.

b. In Exercise 3.191 let $P=B_d$ and $Q=B_2$. Then $P*Q$ (defined by equation (3.86)) is Eulerian of rank $d+1$ whose truncation $(P*Q)_0\cup(P*Q)_1\cup\cdots\cup(P*Q)_{d-1}$ is a truncated boolean algebra, yet $P*Q$ itself is not a boolean algebra.

182. Let $P=\{t_1,\dots,t_n\}$, and define

$$\mathcal{P}=\{(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n : 0\le\alpha_i\le 1, \text{ and } t_i\le t_j\Rightarrow\alpha_i\le\alpha_j\}.$$

Then $\mathcal{P}$ is a convex polytope, and it is not difficult to show (as first noted by L. D. Geissinger, in *Proc. Third Carribean Conf. on Combinatorics*, 1981, pp. 125–133) that Γ_P is isomorphic to the dual of the lattice of faces of $\mathcal{P}$ and hence is an Eulerian lattice. For further information on the polytope $\mathcal{P}$, see R. Stanley, *J. Disc. and Comp. Geom.* **1** (1986), 9–23.

183. b. This description of the Bruhat order goes back to C. Ehresmann, *Ann. Math.* **35** (1934), 396–443, who was the first person to define the order. For an exposition, see A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer, New York, 2005 (Chapter 2). This book is the standard reference on the combinatorics of Coxeter groups, which we will refer to as B-B for the remainder of this exercise and in Exercise 3.185.

c. The Bruhat order can be generalized to arbitrary Coxeter groups. In this context, $\mathfrak{S}_n$ was shown to be Eulerian by D.-N. Verma, *Ann. Sci. Éc. Norm. Sup.* **4** (1971), 393–398, and V. V. Deodhar, *Invent. Math.* **39** (1977), 187–198. See B-B, Corollary 2.7.10. More recent proofs were given by J. R. Stembridge, *J. Algebraic Combinatorics* **25** (2007), 141–148, B. C. Jones, *Order* **26** (2009), 319–330, and M. Marietti, *J. Algebraic Combinatorics* **26** (2007), 363–382. This last paper introduces a new class of Eulerian posets called *zircons*, which give a combinatorial generalization of Bruhat order. A far-reaching topological generalization of the present exercise is due to A. Björner and M. L. Wachs [3.17]. A survey of Bruhat orders is given by A. Björner, *Contemp. Math.* **34** (1984), pp. 175–195.

d. First show that for fixed $i < j$, the number of permutations v for which $v < (i,j)v$ is $n!/(j-i+1)$. Then sum on $1 \le i < j \le n$. This argument is due to D. Callan, as reported in *The On-Line Encyclopedia of Integer Sequences*, A002538.

g. This result goes back to Chevalley in 1958 (for arbitrary finite Coxeter groups), but the first explicit statement seems to be due to J. R. Stembridge, *J. Algebraic Combinatorics* **15** (2002), 291–301. For additional information see A. Postnikov and R. Stanley, *J. Algebraic Combinatorics* **29** (2009), 133–174.

184. See F. Incitti, *J. Algebraic Combinatorics* **20** (2004), 243–261. For further work on this poset, see A. Hultman and K. Vorwerk, *J. Algebraic Combinatorics* **30** (2009), 87–102.

185. b. Given $w = a_1a_2\cdots a_n \in \mathfrak{S}_n$, let $I_w = \{(a_i,a_j) : i < j,\ a_i > a_j\}$, the *inversion set* of w. It is easy to see that $v \le w$ in $W(\mathfrak{S}_n)$ if and only if $I_v \subseteq I_w$. From this observation it follows readily that $v \vee w$ is defined by $I_{v\vee w} = \overline{I_v \cup I_w}$, where the overline denotes transitive closure. Hence, $W(\mathfrak{S}_n)$ is a join-semilattice. Since it has a $\hat{0}$ (or, in fact, since it is self-dual *via* the anti-automorphism $a_1a_2\cdots a_n \mapsto a_n\cdots a_2a_1$), it follows that $W(\mathfrak{S}_n)$ is a lattice. This argument appears in C. Berge, *Principles of Combinatorics*, Academic Press, New York, 1971 (§4.4, Prop. 3). For further information see A. Hammett and B. G. Pittel, Meet and join in the weak order lattice, preprint, 2006. An exposition of the weak order for arbitrary Coxeter groups appears in B-B, Chapter 3.

c. Every vertex in the Hasse diagram of $W(\mathfrak{S}_n)$ has degree $n-1$, from which the result is immediate.

d. Follows from Corollary 3 on page 185 of A. Björner, *Contemp. Math.* **34** (1984), pp. 175–195. A topological generalization appears in B-B, Corollary 3.2.8.

e. This result was shown by P. H. Edelman, Geometry and the Möbius function of the weak Bruhat order of the symmetric group, unpublished.

f. This result was first proved by R. Stanley, *Europ. J. Combinatorics* **5** (1984), 359–372 (Corollary 4.3). Subsequent proofs were announced in P. H. Edelman and C. Greene, *Contemporary Math.* **34** (1984), 155–162, and A. Lascoux and M.-P. Schützenberger, *C. R. Acad. Sc. Paris* **295**, Série I (1982), 629–633. The proof of Edelman and Greene appears in *Advances in Math.* **63** (1987), 42–99. An interesting exposition was given by A. M. Garsia, Publications du LaCIM, Université du Québec á Montréal, Montréal, vol. 29, 2002. The number M_n is just the number of standard Young tableaux of the staircase shape $(n-1, n-2, \dots, 1)$; see Exercise 7.22.

g. This result was first proved by I. G. Macdonald, *Notes on Schubert polynomials*, Publications du LaCIM, Université du Québec à Montréal, Montréal, vol. 6, 1991 (equation (6.11)). A simpler proof, as well as a proof of a q-analogue conjectured by Macdonald, was given by S. Fomin and R. Stanley, *Advances in Math.* **103** (1994), 196–207 (§2).

186. a–c. Fan Wei, The weak Bruhat order and separable permutations, `arXiv:1009:5740`.

d. It has been checked for $n \le 8$ that if $w \in \mathfrak{S}_n$ and $F(\Lambda_w,q)$ is symmetric, then every zero of $F(\Lambda_w,q)$ is a root of unity.

187. b. For every set $S \subseteq [n-1]$, there exists a unique permutation $w \in \mathcal{G}_n$ with descent set $D(w) = S$. The map $w \mapsto D(w)$ is an isomorphism from $\mathcal{G}_n$ to $M(n)$.

c. These permutations $w = a_1\cdots a_n$ are just those of Exercise 1.114(b) (i.e., for all $1 \le i \le n$, the set $\{a_1,a_2,\dots,a_i\}$ consists of consecutive integers (in some order)). Another characterization of such permutations w is the following. For $1 \le i \le n$,

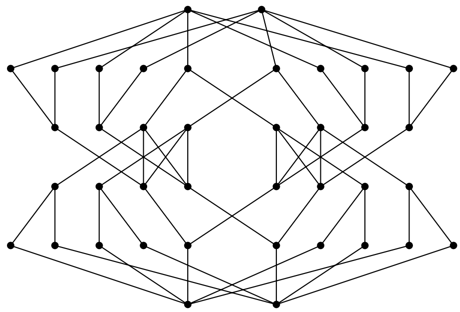


Figure 3.89 A poset Q for which $Q \otimes (\mathbf{1}+\mathbf{1}) \cup \{\hat{0}, \hat{1}\}$ has a nonpositive flag h-vector.

let μ_i be the number of terms of w that lie to the left of i and that are greater than i (a variation of the inversion table of w). Then $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is a partition into distinct parts (i.e., for some k we have $\mu_1 > \mu_2 > \cdots > \mu_k = \mu_{k+1} = \cdots = \mu_n = 0$). Note that we also have $D(w) = \{\mu_1, \cdots, \mu_{k-1}\}$ and $\mathrm{maj}(w) = \mathrm{inv}(w)$. These permutations are also the possible *ranking patterns* as defined by H. Kamiya, P. Orlik, A. Takemura, and H. Terao, *Ann. Combinatorics* **10** (2006), 219–235.

188. The poset P is an interval of the poset of *normal words* introduced by F. D. Farmer, *Math. Japonica* **23** (1979), 607–613. It was observed by A. Björner and M. L. Wachs [3.19, §6] that the poset of all normal words on a finite alphabet $S = \{s_1, \dots, s_n\}$ is just the Bruhat order of the Coxeter group $W = \langle S : s_i^2 = 1 \rangle$. Hence, P is Eulerian by the Verma-Deodhar result mentioned in the solution to Exercise 3.183. A direct proof can also be given.

190. a. This result is due to R. Ehrenborg, G. Hetyei, and M. A. Readdy, Level Eulerian posets, preprint dated June 12, 2010 (Corollary 8.3), as a special case of a much more general situation.

192. This deep result was proved by K. Karu, *Compositio Math.* **142** (2006), 701–708. Karu gives another proof for a special case (complete fans) in Lefschetz decomposition and the cd-index of fans, preprint, `math.AG/0509220`. Ehrenborg and Karu, *J. Algebraic Combin.* **26** (2007), 225–251, continue this work, proving in particular a conjecture of Stanley that the cd-index of a Gorenstein* lattice is minimized on boolean algebras.

193. a. The simplest example is obtained by taking two butterfly posets (as defined in Exercise 3.198) of rank 5 and identifying their top and bottom elements. For this poset P, we have

$$\Phi_P(c,d) = c^4 + 2c^2d + 2cd^2 - 4d^2.$$

For further information on negative coefficients of the cd-index, see M. M. Bayer, *Proc. Amer. Math. Soc.* **129** (2001), 2219–2225.

b. It follows from the work of M. M. Bayer and G. Hetyei, *Europ. J. Combinatorics* **22** (2001), 5–26, that such a poset must have rank at least seven. For an example of rank 7, let Q be the poset of Figure 3.89, and let $P = Q \otimes (\mathbf{1}+\mathbf{1})$, with a $\hat{0}$ and $\hat{1}$ adjoined (where $\otimes$ denotes ordinal product). Then it can be checked that P is Eulerian, with $\beta_P(4,5,6) = -2$. The poset Q appears as Figure 2 in Bayer and Hetyei, ibid.

194. If $S = \{a_1, a_2, \dots, a_k\}_{\le} \subseteq [n-1]$, then let $\rho = (a_1, a_2 - a_1, a_3 - a_2, \dots, n - a_k)$, a composition of n. We write $\alpha_P(\rho)$ for $\alpha_P(S)$. By Exercise 3.155(a), we have $\alpha_P(\rho) = \alpha_P(\sigma)$ if ρ and σ have the same multiset of parts. By a result of Bayer and Billera [3.5, Prop. 2.2], α_P is determined by its values on those ρ with no part

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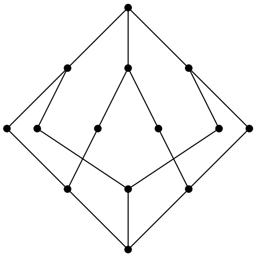

Figure 3.90 An interval in a putative "Fibonacci binomial poset".

equal to 1. From this we get $d(n) \leq p(n) - p(n-1)$, where $p(n)$ denotes the number of partitions of n. On the other hand, the work of T. Bisztriczky, *Mathematika* **43** (1996), 274–285, gives a lower bound for $d(n)$, though it is far from the upper bound $p(n) - p(n-1)$. For the lower bound see also M. M. Bayer, A. Bruening, and J. Stewart, *Discrete Comput. Geom.* **27** (2002), 49–63.

195. See R. Ehrenborg, *Order* **18** (2001), 227–236 (Prop. 4.6).

196. **a.** Let $[s,t]$ be an $(n+1)$-interval of P, and let u be a coatom (element covered by t) of $[s,t]$ Then $[s,t]$ has $A(n+1) = B(n+1)/B(n)$ atoms, while $[s,u]$ has $A(n) = B(n)/B(n-1)$ atoms. Since every atom of $[s,u]$ is an atom of $[s,t]$ we have $A(n+1) \geq A(n)$, and the proof follows.

b. The poset of Figure 3.90 could be a 4-interval in a binomial poset where $B(n) = F_1 F_2 \cdots F_n$. It is known that the *Fibonomial coefficient*

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1}$$

is an integer, a necessary condition for the existence of a binomial poset with $B(n) = F_1 F_2 \cdots F_n$. For a combinatorial interpretation of $\binom{n}{k}_F$, see A. T. Benjamin and S. S. Plott, *Fib. Quart.* **46/47** (2008/2009), 7–9.

197. See J. Backelin, Binomial posets with non-isomorphic intervals, `arXiv:math/0508397`. Backelin's posets have factorial function $B(1) = 1$ and $B(n) = 2^{n-2}$ for $n \geq 2$.

198. See R. Ehrenborg and M. A. Readdy, *J. Combinatorial Theory Ser. A* **114** (2007), 339–359. For further work in this area, see H. Bidkhori, Ph.D. thesis, M.I.T., 2010, and Finite Eulerian posets which are binomial, Sheffer or triangular, `arXiv:1001.3175`.

199. *Answer:* L is a chain or a boolean algebra.

200. Equation (3.132) is equivalent to a result of R. C. Read, *Canad. J. Math.* **12** (1960), 410–414 (also obtained by E. A. Bender and J. R. Goldman [3.7]). The connection with binomial posets was pointed out by Stanley, *Discrete Math.* **5** (1973), 171–178 (§3). Note that equation (3.88) (the chromatic generating function for the number of acyclic digraphs on $[n]$) follows immediately from equations (3.132) and (3.121).

201. **a.** See [3.25, Prop. 9.1]. This result is proved in exact analogy with Theorem 3.18.4.
b,c. See [3.25, Prop. 9.3].

202. See Theorem 5.2 of R. Simion and R. Stanley, *Discrete Math.* **204** (1999), 369–396.

203. As the notation becomes rather messy, let us illustrate the proof with the example $a_1 = 0$, $a_2 = 3$, $a_3 = 4$, $m = 6$. Let

$$\begin{aligned} S_n &= \{6i, 6i+3, 6i+4 : 0 \le i \le n\}, \\ S_n' &= S_n \cup \{6n\}, \\ S_n'' &= S_n \cup \{6n, 6n+3\}. \end{aligned}$$

Let P be the binomial poset $\mathbb{B}$ of all finite subsets of $\mathbb{N}$, ordered by inclusion, and let $\mu_S(n)$ be as in Section 3.19. Then by Theorem 3.13.1, we have

$$\begin{aligned} (-1)^n f_1(n) &= \quad \mu_{S_n}(6n) := g_1(n), \\ (-1)^{n+1} f_2(n) &= \mu_{S_n'}(6n+3) := g_2(n), \\ (-1)^{n+2} f_3(n) &= \mu_{S_n''}(6n+4) := g_3(n). \end{aligned}$$

By the defining recurrence (3.15) we have

$$g_1(n) = -\sum_{i=0}^{n-1}\left[\binom{6n}{6i} g_1(i) + \binom{6n}{6i+3} g_2(i) + \binom{6n}{6i+4} g_3(i)\right], \quad n > 0,$$

$$g_2(n) = -\sum_{i=0}^{n}\binom{6n+3}{6i} g_1(i) - \sum_{i=0}^{n-1}\binom{6n+3}{6i+3} g_2(i) - \sum_{i=0}^{n-1}\binom{6n+3}{6i+4} g_3(i),$$

$$g_3(n) = -\sum_{i=0}^{n}\binom{6n+4}{6i} g_1(i) - \sum_{i=0}^{n}\binom{6n+4}{6i+3} g_2(i) - \sum_{i=0}^{n-1}\binom{6n+4}{6i+4} g_3(i).$$

These formulas may be rewritten (incorporating also $g_1(0) = 1$)

$$\delta_{0n} = \sum_{i=0}^{n}\left[\binom{6n}{6i} g_1(i) + \binom{6n}{6i+3} g_2(i) + \binom{6n}{6i+4} g_3(i)\right],$$

$$0 = \sum_{i=0}^{n}\left[\binom{6n+3}{6i} g_1(i) + \binom{6n+3}{6i+3} g_2(i) + \binom{6n+3}{6i+4} g_3(i)\right],$$

$$0 = \sum_{i=0}^{n}\left[\binom{6n+4}{6i} g_1(i) + \binom{6n+4}{6i+3} g_2(i) + \binom{6n+4}{6i+4} g_3(i)\right].$$

Multiplying the three equations by $x^{6n}/(6n)!$, $x^{6n+3}/(6n+3)!$, and $x^{6n+4}/(6n+4)!$, respectively, and summing on $n \ge 0$ yields

$$\begin{aligned} F_1\Phi_0 + F_2\Phi_3 + F_3\Phi_2 &= 1, \\ F_1\Phi_3 + F_2\Phi_0 + F_3\Phi_5 &= 0, \\ F_1\Phi_4 + F_2\Phi_1 + F_3\Phi_0 &= 0, \end{aligned}$$

as desired. We leave to the reader to see that the general case works out in the same way. Note that we can replace $f_k(n)$ by the more refined $\sum_w q^{\mathrm{inv}(w)}$, where w ranges over all permutations enumerated by $f_k(n)$, simply by replacing $\mathbb{B}$ by $\mathbb{B}_q$ and thus $a!$ by $(\boldsymbol{a})!$ and $\binom{a}{b}$ by $\binom{\boldsymbol{a}}{\boldsymbol{b}}$ throughout.

An alternative approach to this problem is given by D. M. Jackson and I. P. Goulden, *Advances in Math.* **42** (1981), 113–135.

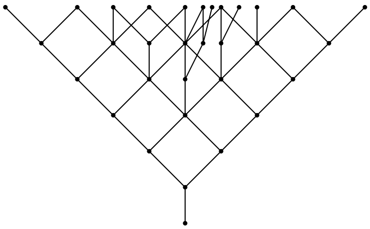

Figure 3.91 A 1-differential poset up to rank 6.

204. a. See [3.74, Lemma 2.5].

b. Apply Theorem 3.18.4 to (a). See [3.74, Cor. 2.6].

c. Specialize (b) to $P=\mathbb{B}(q)$, and note that by Theorem 3.13.3 we have $G_n(q,z)=(-1)^n h(n)|_{z\to -z}$. A more general result is given in [3.74, Cor. 3.6].

205. a. See Figure 3.91 for a 1-differential poset P up to rank 6 that is not isomorphic to $\Omega^i Y[6-i]$ for any $0\le i\le 6$. Then $\Omega^\infty P$ is the desired example.

b. These results were computed by Patrick Byrnes, private communication, dated 7 March 2008.

c. Examples of this nature appear in J. B. Lewis, On differential posets, undergraduate thesis, Harvard University, 2007;

⟨`http://math.mit.edu/~jblewis/`

`JBLHarvardSeniorThesis.pdf`⟩.

206. This result was proved by P. Byrnes, preprint, 2011, based on work of Y. Qing, Master's thesis, M.I.T., 2008. It is reasonable to conjecture that the only r-differential lattices are direct products of a suitable number of copies of Y and Z_k's, $k\ge 1$.

207. With γ as in the proof of Theorem 3.21.11, we have

$$\gamma(UDUU\boldsymbol{P})=\sum_{n\ge 0}\alpha(n-2\to n\to n-1\to n)q^n.$$

Repeated applications of $DU=UD+rI$ gives $UDUU=2rU^2+U^3D$. Then use $D\boldsymbol{P}=(U+r)\boldsymbol{P}$ (Proposition 3.21.3) to get

$$UUDU\boldsymbol{P}=(2rU^2+rU^3+U^4)\boldsymbol{P}.$$

The proof follows easily from Theorem 3.21.11. This result appeared in [3.80, Exam. 3.5] as an illustration of a more general result, where $UUDU$ is replaced by any word in U and D.

208. a. Use the relation $DU=UD+rI$ to put w in the form

$$w=\sum_{i,j}c_{ij}(w)U^iD^j, \tag{3.153}$$

where $c_{ij}(w)$ is a polynomial in r, and where if $c_{ij}(w)\ne 0$ then $i-j=\rho(t)$. It is easily seen that this representation of w is unique. Apply U on the left to equation (3.153). By uniqueness of the c_{ij}'s there follows [why?]

$$c_{ij}(Uw)=c_{i-1,j}(w). \tag{3.154}$$

Now apply D on the left to equation (3.153). Using $DU^i=U^iD+riU^{i-1}$ we get [why?]

$$c_{ij}(Dw)=c_{i,j-1}(w)+r(i+1)c_{i+1,j}(w). \tag{3.155}$$

Setting $j = 0$ in equations (3.154) and (3.155) yields

$$c_{i0}(Uw) = c_{i-1,0}(w), \tag{3.156}$$

$$c_{i0}(Dw) = r(i+1)c_{i+1,0}. \tag{3.157}$$

Now let (3.153) operate on $\hat{0}$. We get $w(\hat{0}) = c_{n0}(w)U^n(\hat{0})$. Thus, the coefficient of t in $w(\hat{0})$ is given by

$$\langle w(\hat{0}), t\rangle = c_{n0}(w)e(t).$$

It is easy to see from equations (3.156) and (3.157) that

$$c_{n0}(w) = r^{\#S}\prod_{i\in S}(b_i - a_i),$$

and the proof follows.

209. Easily proved by induction on n. In particular, assume for n, multiply by UD on the left, and use the identity $DU^k = kU^{k-1} + U^kD$. See [3.80, Prop. 4.9].

210. Using $DU = UD + 1$, we can write a balanced word $w = w(U,D)$ as a linear combination of words U^kD^k. By Proposition 1.9.1, we can invert equation (3.134) to get

$$U^nD^n = \sum_{k=0}^{n} s(n,k)(UD)^k = UD(UD-1)\cdots(UD-n+1).$$

It follows that every balanced word is a polynomial in UD. Since any two polynomials in UD commute, the proof follows. This result appeared in [3.80, Cor. 4.11(a)].

213. We have (using equation (3.105))

$$\begin{aligned}\sum_{n\geq 0}\sum_{k\geq 0}\kappa_{2k}(n)\frac{q^nx^{2k}}{(2k)!} &= \sum_{t\in P}\left\langle e^{(D+U)x}t, t\right\rangle q^{\rho(t)}\\ &= e^{rx^2/2}\sum_{t\in P}\left\langle e^{Ux}e^{Dx}t, t\right\rangle q^{\rho(t)}.\end{aligned}$$

From Exercise 3.212(b), it is easy to obtain

$$\sum_{n\geq 0}\sum_{k\geq 0}\kappa_{2k}(n)\frac{q^nx^{2k}}{(2k)!} = F(P,q)\exp\left(\frac{1}{2}rx^2 + \frac{rqx^2}{1-q}\right).$$

Extracting the coefficient of $x^{2k}/(2k)!$ on both sides completes the proof. This result first appeared in [3.80, Cor. 3.14].

214. This is a result of P. Byrnes, preprint, 2011.

215. a. We showed in the proof of Theorem 3.21.12 that the linear transformation $U_i : \mathbb{C}P_i \to \mathbb{C}P_{i+1}$ is injective. Hence,

$$p_i = \dim \mathbb{C}P_i \leq \dim \mathbb{C}P_{i+1} = p_{i+1}.$$