Comprehensive Examination in Graph Theory

August 31, 2000

Time allowed: 3 hours

Examiners: E.J. Cockayne, J. Huang, and G. MacGillivray

This examination paper has **two** pages and **ten** questions, all of equal value.

Do any **seven** questions. Write your answers in the booklets provided.

All undirected graphs considered in this examination should be understood to be finite and simple. The number of vertices is denoted by ν, and the number of edges by ϵ.

1. State and prove Turan's Theorem.

2 (a) Let G be a graph in which any two k-critical subgraphs intersect. Prove that

$$\chi(G) \leq 2k - 1.$$

(b) For each integer $k \geq 1$, give an example of a graph G that achieves the bound in (a).

3. Prove that if G has k components, then

$$\epsilon \leq \binom{\nu - k + 1}{2}$$

4 (a) State Ore's Theorem and Dirac's Theorem concerning Hamilton cycles in graphs.

(b) Let G be a graph, and let x and y be non-adjacent vertices of G such that

$$d(x) + d(y) \geq \nu.$$

Prove that G is hamiltonian if and only if $G + xy$ is hamiltonian.

(c) Use the result in (b) to prove the two theorems in (a).

5. Recall that in the Graph Reconstruction Problem you are given a deck of ν cards, each of which contains one of the ν vertex-deleted subgraphs of an unknown graph G. The vertices of these subgraphs are not labelled. Show how the deck of cards can be used to obtain the following information:

(a) The number of vertices of G and the number of edges of G.

(b) The degree sequence of G.

(c) The number of components of G.

(d) Whether G is a tree.

6 (a) Prove that a matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

(b) State Hall's Theorem, König's Theorem, and Tutte's Theorem (all regarding matchings).

(c) Prove that a tree has at most one perfect matching.

7 (a) Define an *outerplanar graph*.

(b) Prove that any outerplanar graph with $\nu \geq 4$ contains two non-adjacent vertices of degree at most two. State any theorems used.

(c) Use the result in (b) to determine a sharp upper bound for the number of edges in an outerplanar graph with ν vertices.

8. Prove that a tournament T is strongly connected if and only if it is Hamiltonian.

9. Prove that any graph G has a spanning bipartite subgraph H such that, for every vertex x, $d_H(x) \geq \frac{1}{2}d_G(x)$.

10 (a) Let G be a connected graph with at least one cut-vertex. Prove that G has at least two blocks that each contain exactly one cut-vertex.

(b) Prove that if G is 3-regular, then $\kappa = \kappa'$.

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