University of Victoria
Department of Mathematics and Statistics
Comprehensive Exam in Graph Theory

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TIME: 3 HOURS.

Each question is worth 10 marks. A minimum of 40 marks, and four questions substantally correct, is required to pass.

1. **Part 1.** Do any two questions. In each question, include all definitions other than "graph" necessary to make the statement meaningful.
   (a) State and prove Menger's Theorem.
   (b) State and prove Turan's Theorem.
   (c) State and prove Vizing's Theorem.

2. **Part 2.** Do any two questions.
   (a) Prove that a connected graph is Eulerian if and only if every minimal edge cut contains an even number of edges.
   (b) Describe Kruskal's Algorithm and prove that it constructs a minimum weight spanning tree of a connected, weighted graph (G, w).
   (c) Prove that every k-connected graph with independence number at most $k-1$ has a Hamilton path.

3. **Part 3.** Do any three questions.
   (a) i. Prove that every cubic graph without cut edges has a 2-factor. Give an example to show that the statement is not true for graphs with cut edges.
       ii. Let G be an r-regular bipartite graph. Prove that G has a k-factor for every k with $1 \leq k \leq r$.
   (b) i. Prove that every tree has at most one perfect matching.
       ii. Prove that a tree T has a perfect matching if and only if $T - v$ has exactly one odd component for every $v \in V$.
   (c) Without using the four colour theorem, prove that every planar graph is five colourable.
   (d) A *King* in a tournament is a vertex x so that, for every $y \in V - \{x\}$ there is a directed path of length at most two from x to y. Prove that every tournament has a King.