1. Prove that a graph G with n vertices and m edges is a tree if and only if it satisfies any two of the following properties:

$$(1)\ G \text{ is connected}, \qquad (2)\ G \text{ is acyclic}, \qquad (3)\ m = n - 1.$$

2. (a) Prove that any graph G with at least three vertices, connectivity $\kappa(G)$ and independence number $\alpha(G)$, where $\kappa(G) \geq \alpha(G)$, is Hamiltonian.

   (b) Show that if G is Hamiltonian, then for every proper, nonempty vertex subset S of G, $G - S$ has at most $|S|$ components.

3. (a) Prove that every planar graph is 5-colourable.

   (b) Give an example of a planar graph $G \not\cong K_4$ that is not 3-colourable. Explain why G is not 3-colourable.

4. A graph G is called *colour-critical* if $\chi(G - v) < \chi(G)$ for each vertex v of G.

   By using a construction known as *Mycielski's construction*, it can be shown that for every integer $k \geq 1$ there exists a k-chromatic triangle-free graph.

   Describe Mycielski's construction and prove that if G is colour-critical, then the graph G' obtained from G by Mycielski's construction is also colour-critical.

5. (a) Prove that every bridgeless cubic graph has a perfect matching.

   (b) Draw a cubic graph with bridges that has a perfect matching.

   (c) Prove that every tree has at most one perfect matching.

6. (a) Show that every simple graph G has a bipartite spanning subgraph H such that $\deg_H(v) \geq \deg_G(v)$ for all $v \in V$.

   (b) Let T be a tree with k vertices. Prove that if G is a simple graph with $\delta(G) \geq k - 1$, then G contains a subgraph isomorphic to T.

7. Describe Dijkstra's Algorithm and prove that it finds a shortest path between a given vertex v and all other vertices of a weighted graph (G, w) in which all edge weights are positive. The graph G can be assumed to be simple.

8. (a) State and prove the Max-Flow - Min-Cut Theorem. Include in your statement the definition of all terms necessary to make it meaningful.

   (b) Prove that if all capacities are integers, then there exists a maximum flow f such that $f(e)$ is an integer for every arc e.

9. (a) Prove that a directed graph G has a directed path of length $\chi - 1$, where χ is the chromatic number of the underlying simple graph.

   (b) Deduce the Corollary that every tournament has a directed Hamilton path.

10. Let G be a bipartite graph with bipartition (X, Y).

    (a) Prove that the edge chromatic number $\chi'(G) = \Delta(G)$.

(b) Prove that, if G is regular, then G has a perfect matching.

11. (a) Prove that a simple graph has a perfect elimination ordering (also known as a simplicial elimination ordering) if and only if it is chordal.

    (b) Prove that every chordal graph is perfect.

12. (a) Prove that every connected graph has a dominating set of size at most $n/2$, where $n = |V|$.

    (b) The *corona* of a graph G with n vertices is the graph G' on $2n$ vertices obtained from $G \cup \overline{K}_n$ by joining the i-th vertex of G to the i-th vertex of $\overline{K}_n$. Prove that, for any graph G with n vertices, the domination number $\gamma(G') = n$.

    (c) Prove that if H is a connected graph with n vertices and $\gamma(H) = n/2$, then either $H \cong C_4$, or H is the corona of some connected graph G.