University of Victoria
Department of Mathematics and Statistics

Comprehensive Exam in Graph Theory
January 8, 2007
Examiners: Peter Dukes, Denis Hanson and Gary MacGillivray

TIME: 3 HOURS.
DO ANY 4 OF QUESTIONS 1-7 AND ANY 3 OF QUESTIONS 8-12.

1 (a) Let r and n be integers such that $0 \leq r < n$. Prove that there exists an r-regular (simple) graph on n vertices if and only if rn is even.

(b) Find an example of integers r, s, a, b such that $ar + bs$ is even, $0 \leq r, s < a + b$, and yet there is no (simple) graph with exactly a vertices of degree r and b vertices of degree s.

2. Determine the number of labelled spanning trees of $K_{2,n}$, and also the number of isomorphism classes of such trees.

3 (a) State Menger's Theorem.

(b) Prove that families $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ have a common system of distinct representatives if and only if

$$\left| \left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) \right| \geq |I| + |J| - n$$

for all $I, J \subseteq \{1, \ldots, n\}$.

4. Describe in detail the structure of graphs G that are maximal in the following sense: G has the property that any added edge will decrease its diameter. If G has n vertices and diameter d then how many edges could G have?

5. A Latin rectangle is an $m \times n$ matrix $L = (\ell_{ij})$ whose entries are integers satisfying: $(i) 1 \leq \ell_{ij} \leq n$, and (ii) no two entries in the same row or column are equal. If $m = n$ we have a Latin square. Show that a Latin rectangle L with $m < n$ can be extended to a Latin square by the addition of $m - n$ new rows.

6. The graph G^3 has the same vertex set as G but two vertices are adjacent iff their distance is at most three in G. A graph is hamiltonian connected if there exists a hamilton path between x and y for any pair of vertices x and y. Prove that if G is connected then G^3 is hamiltonian connected.

7. A graph G with domination number $\gamma(G)$ is *γ-vertex-critical* if $\gamma(G - x) < \gamma(G)$ for all vertices x of G, and *γ-edge-critical* if $\gamma(G + uv) < \gamma(G)$ whenever u and v are non-adjacent vertices of G.

(a) Prove that 2-vertex-critical graphs are the complement of a perfect matching.

(b) Prove that the 2-edge-critical graphs are the complement of a disjoint union of stars.

8. Take $2n$ points in general position in the plane (no 3 collinear). How many line segments joining pairs of points can you draw without forming a triangle? Explain.

9. A graph G is *k-critical* if its chromatic number, $\chi(G)$, satisfies $\chi(H) < \chi(G) = k$ for every proper subgraph H of G. The *Mycielski construction* is as follows: Let G_3 be a 5-cycle (which is 3-critical and triangle-free); For $r \geq 3$, given G_r construct G_{r+1} by taking a copy of G_r, duplicating each vertex x_i of G_r by creating a new vertex y_i and joining it precisely to the neighbours of x_i in G_r, and finally adding a vertex u joined only to the vertices y_i. Show that this procedure constructs k-critical triangle-free graphs (*i.e.* G_k is k-critical and triangle free for every $k \geq 3$).

10 (a) Show that a plane graph G is 2-face-colourable if and only if G is eulerian.

(b) Show that a plane triangulation G is 3-vertex-colourable if and only if G is eulerian.

11. A tournament is *irreducible* (or *strong*) if the vertices cannot be partitioned into two sets A and B in such a way that the vertices in one of these sets are each adjacent *to* all of the vertices in the other set.

The following is due to Foulkes (1960):

Theorem. If an irreducible tournament contains at least three nodes, then it contains a spanning (i.e. directed Hamilton) cycle.

Proof. An inductive proof will be given. Assume that the theorem is true for n nodes and arrange these around a circle with the . . . [cycle] proceeding in a clockwise direction. Place the $(n+1)$st node in the center. . . . [Since the tournament is irreducible] there must be at least one . . . arc proceeding from a node on the circle to the center. Call this node 1 and number the nodes 1, 2, 3, etc., proceeding in a clockwise direction. Again there must be at least one . . . arc going from the center to a node on the circle. Let the first . . . arc of this type encountered in going around the circle . . . go to j. Then the . . . spanning cycle $1, 2, 3, ...(j-1), (n+1), j, j+1, ..., (n-1), n, 1$ exists. Hence the theorem is true for $(n+1)$ nodes. It is true for three nodes, and thus it is true for any n. □

Comment briefly on the validity of this argument explaining yourself clearly.

12. Describe Kruskal's algorithm and prove that it constructs a minimum weight spanning tree of the edge-weighted graph (G, w).