December 2010 Comprehensive Exam in Graph Theory

Examiners: G. MacGillivray and K. Mynhardt
Time: 3 hours

This question paper has 10 questions on two pages. Each question is worth 10 marks. **Do any seven questions**. A minimum of 40 marks is required to pass.

1. State four equivalent characterizations of trees (one being the definition) and prove that they are equivalent.

2. (a) Let G be a graph with at least three vertices. Prove that if $\kappa(G) \geq \beta(G)$, where $\beta(G)$ denotes the independence number of G, then G is hamiltonian. State any theorem(s) you use.

   (b) Construct an infinite class of nonhamiltonian graphs G such that $\kappa(G) = \beta(G) - 1$.

   (c) Let G be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$ and $\deg v_i = d_i$. Construct H by replacing each vertex v_i by $H_i = K_{d_i}$ and each edge $v_i v_j$ by an edge joining one vertex of H_i to one vertex of H_j, in such a way that each vertex of H_i is joined to exactly one vertex of $H - H_i$. Suppose G is connected and eulerian. Is H hamiltonian? Explain.

3. (a) State each of the following: Berge's Theorem that characterizes maximum matchings, Hall's Theorem about matchings in bipartite graphs, König's Theorem about (vertex) coverings of bipartite graphs, and Tutte's Theorem that characterizes the graphs having a perfect matching.

   (b) Prove any one of the above theorems.

4. Write notes on edge colourings of graphs. Address the following points in your notes, including one substantial proof.

   (a) bounds for $\chi_1(G)$;

   (b) the Classification Problem, with examples of graphs of either class;

   (c) a link between the Four Colour Theorem and edge colourings.

5. Prove:

   (a) In every network, the maximum value of a feasible flow equals the minimum capacity of a source/sink cut.

   (b) If all capacities in a network are integers, then there is a maximum flow assigning integral flow to each edge, and some maximum flow can be partitioned into flows of unit value along paths from source to sink.

6. The *Turán graph* $T_{n,r}$ is the complete r-partite graph of order n that has b parts of size $a+1$ and $r-b$ parts of size a, where $a = \left\lfloor \frac{n}{r} \right\rfloor$ and $b = n - ra$.

   Prove that amongst the n-vertex (simple) K_{r+1}-free graphs, $T_{n,r}$ has the maximum number of edges.

7. (a) Define the (graph) Ramsey number $r(k, \ell)$.

   (b) Prove that $r(k, \ell) \leq r(k-1, \ell) + r(\ell - 1, k)$.

   (c) Show that $r(3, 4) = 9$. State any results that you use in addition to the theorem in (b).

8. Recall that a sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is called *graphic* if there is a simple graph with degree sequence $\mathbf{d}$. Prove the theorem of Havel and Hakimi that a *non-increasing* sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is graphic if and only if the sequence $\mathbf{d}' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.

9. Prove the theorem of Moon that every vertex of a strongly connected tournament with $n \geq 3$ vertices is contained in a directed cycle of each length k, $3 \leq k \leq n$.

10. (a) Describe the graph reconstruction problem.

    (b) Prove Kelly's Lemma: For any two graphs F and G such that F has fewer vertices than G, the parameter $\binom{G}{F}$, i.e. the number of copies of F in G, is reconstructible.

    (c) Prove that the degree sequence of a graph is reconstructible.

    (d) Is the number of components of a graph reconstructible? Why or why not?

- - - E N D - - -