Comprehensive Examination in Graph Theory

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Time allowed: 3 hours

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This examination paper has **two** pages and **ten** questions, worth ten marks each.

Do any **seven** questions. Write your answers in the booklets provided.

1. (a) Prove that $\tau(K_n) = n^{n-2}$.

(b) State the Matrix-tree Theorem. Include any definitions necessary to make the statement meaningful.

2. Denote the edge-chromatic number of G by $\chi'(G)$.

(a) Prove that for any nonempty graph G and any $d \geq \Delta(G)$, if the set of vertices of G of degree d is empty or independent, then $\chi'(G) \leq d$.

(b) State the result due to Vizing that can be obtained from the result in (a).

(c) Prove that if G has m edges and edge independence number $\beta_1(G)$, and

$$m > \Delta(G) \cdot \beta_1(G),$$

then G is of class two (i.e. $\chi'(G) = \Delta(G) + 1$).

3. Let G be a graph with n vertices and m edges. Prove that if $n \geq 6\delta$ and $m > \binom{n-\delta}{2} + \delta^2$, then G is hamiltonian.

4. Prove Havel's Theorem: The sequence $\mathbf{d} = d_1, d_2, \ldots d_n$ of non-negative integers with

$$d_1 \geq d_2 \geq \cdots \geq d_n \quad (n \geq 2, d_1 \geq 1$$

is graphic (that is, there is a simple graph with degree sequence $\mathbf{d}$) if and only if the sequence

$$d_1 - 1, d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphic.

5. Prove Brooks' Theorem: If G is a connected simple graph and neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

6. (a) Prove that if G is a connected plane graph with n vertices, m edges and f faces, then $n - m + f = 2$.

(b) Deduce that a maximal planar graph with $n \geq 3$ vertices and m edges, than $m = 3n-6$.

(c) Prove that K_5 is non-planar and sketch an embedding of K_5 on the torus.

7. (a) Prove that a dominating set D is minimal if and only if D is irredundant.

(b) Prove that an independent set I is maximal if and only if it is dominating.

8. (a) State Hall's Theorem, König's Theorem, and Tutte's Theorem (all regarding matchings). Include any definitions necessary to make the statements meaningful.

(b) Prove any one of the theorems you stated in (a).

9. (a) State the Max flow - min cut Theorem. Include any definitions necessary to make the statement meaningful.

(b) Prove that a flow f is a maximum flow if and only if there is no f-incrementing path.

10. (a) Two vertices u and v of a graph G are said to be *similar* if there is an automorphism ϕ of G such that $\phi(u) = v$. It is easy to see that if u and v are similar, then $G - u \cong G - v$. Give an example to show that the converse need not be true.

A graph G with $V(G) = \{v_1, ..., v_n\}$ is *reconstructible* if for every graph H with $V(H) = \{u_1, ..., u_n\}$, $G - v_i = H - u_i$ for each i implies that $G = H$. (Thus G is reconstructible if the subgraphs $G - v_i$ determine G uniquely.)

(b) State the Reconstruction Conjecture.

(c) Prove that the number of vertices, n, and number of edges, m, can be determined from the n subgraphs $G - v$, $v \in V(G)$. Prove futher that the degrees of the vertices of G can also be determined.

(e) Prove that every regular graph with at least three vertices is reconstructible.

(f) Prove that G with $n \geq 3$ vertices is connected if and only if at least of the two subgraphs $G - v$, $v \in V(G)$, are connected.

(g) Prove that disconnected graphs with at least three vertices are reconstructible.