Relation Partitions

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1 Summary

2 Preliminaries

Bipartite Distance Multigraph, Distance Graphs, Distance Multiplicities as adj mat (or Laplacian) traces,

Combinatorial Nullstellensatz, Eigenvalue interlacing theorem

3 Nullstellensatz Shows Existence of Dense Subgraph

4 General Position (Max Distance Degree) Permits Large Order

Theorem 4.1. *Let X be a finite metric space with n points and r distinct distances $d_1, \ldots, d_r$ whereby either $n/2$ or $(n+1)/2$ is a prime p. Let $d \in \mathbb{Z}^+$. Suppose $\Delta(D_{d_k}) \leq d$ for all $k \in [1, r]$. Then there is a set of P points and D distances satisfying $|P| \geq |D| \geq p/d$ such that*

1. *for each $v \in P$, there are p points $u \in P \setminus \{v\}$ such that $d(u, v) \in D$; and*
2. *for each $d_k \in D$, there is an $\ell_k \in \mathbb{Z}^+$ such that there are $\ell_k p$ points P at distance d_k with some other point in X.*

Corollary 4.2. *The average multiplicity of distances in D is at least $\frac{|P|p}{2|D|}$.*

5 Eigenvalue Interlacing and Distance Multiplicity Stability

Let A_k be the adjacency matrix for distance d_k in X. Then $2m(d_k) = Tr(A_k^2) = \deg_{\mathcal{M}}(d_k)$. We say that a point $v \in X$ is *distance independent* if for all $u, w \in X \setminus \{v\}$, $d_{X\setminus\{v\}}(u, w) = d_X(u, w)$.

Let v be a distance independent vertex of X. Let $\alpha_1^{(k)} \geq \alpha_2^{(k)} \geq \cdots \geq \alpha_n^{(k)}$, and $\beta_1^{(k)} \geq \beta_2^{(k)} \geq \cdots \geq \beta_{n-1}^{(k)}$ be the eigenvalues of $A_k(X)$ and $A_k(X \setminus \{v\})$, respectively. Similarly, let $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_n^{(k)}$, and $\mu_1^{(k)} \geq \mu_2^{(k)} \geq \cdots \geq \mu_{n-1}^{(k)}$ be the eigenvalues of $\mathcal{L}(A_k(X))$ and $\mathcal{L}(A_k(X \setminus \{v\}))$, respectively.

Theorem 5.1. *Let X and Y finite metric spaces that differ by a single distance independent point. Then for every $k \in [1, r]$,*

$$|m_X(k) - m_Y(k)| < \tfrac{1}{2}(\alpha_1^2 + \alpha_n^2),$$

and

$$|m_X(k) - m_Y(k)| \leq \tfrac{\lambda_1}{2}.$$

Proof. Without loss of generality, suppose $Y = X \setminus \{v\}$, where v is a distance independent point. For ease of notation, we omit the (k) superscripts in the eigenvalues for this proof.

For the Laplacians $\mathcal{L}(A_k(X))$ and $\mathcal{L}(A_k(Y))$, we have by eigenvalue interlacing that $\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$, which immediately implies

$$Tr\mathcal{L}(A_k(X)) - \lambda_1 = \sum_{j=2}^{n} \lambda_j \leq \sum_{j=1}^{n-1} \mu_j \leq \sum_{j=1}^{n-1} \lambda_j = Tr\mathcal{L}(A_k(X)) - \lambda_n.$$

Since $Tr\mathcal{L}(A_k(Y)) = \sum_{j=1}^{n-1} \mu_j$, we have that $|m_X(k) - m_Y(k)| \leq \frac{1}{2}(\lambda_1 - \lambda_n)$. But the laplacian is singular with all eigenvalues non-negative, so $\lambda_n = 0$.

For the adjacency squares, since v is independent, by eigenvalue interlacing, we again have

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n.$$

Let i be the smallest integer satisfying $\alpha_i < 0$. Then for all $j \in [i, n-1]$, $\alpha_j^2 \leq \beta_j^2 \leq \alpha_{j+1}^2$, and similarly, for all $j \in [1, i-2]$, $\alpha_j^2 \geq \beta_j^2 \geq \alpha_{j+1}^2$. Note that if $i = 2$, then β_1 and α_1 are the only positive eigenvalue of $A_k(Y)$ and $A_k(X)$, respectively, which means that $\beta_1 = \sum_{j=2}^{n-1} |\beta_j| \geq \sum_{j=2}^{n-1} |\alpha_j| \geq |\alpha_2|$. Thus we have that

$$\sum_{j=1}^{n} \alpha_j^2 - \alpha_1^2 - \alpha_n^2 \leq \sum_{j=1}^{n-1} \beta_j^2 = \sum_{j=1}^{i-1} \beta_j^2 + \sum_{j=i}^{n-1} \beta_j^2 \leq \sum_{j=1}^{n} \alpha_j^2 - \alpha_i^2.$$

Thus the multiplicity gap for distance d_k between X and Y is

$$|m_X(k) - m_Y(k)| \leq \frac{1}{2}(\alpha_1^2 + \alpha_n^2 - \alpha_i^2) < \frac{1}{2}(\alpha_1^2 + \alpha_n^2).$$

Note that α_i is one of the eigenvalues of $A_k(X)$ with smallest magnitude. $\square$

Theorem 5.2. *Let X be a finite metric space with n points and r distinct distances $d_1, d_2, \ldots, d_r$. Let p be a prime satisfying $p - 1 < \frac{n(n-1)}{n+r}$. Let $\mathcal{U} = (P, D)$ be a subgraph of $\mathcal{M}(X)$ such that for all $v \in P$, $\deg_{\mathcal{U}}(v) = h_v p$ with $h_v \in \mathbb{Z}^+$ and for all $d_k \in D$, $\deg_{\mathcal{U}}(d_k) = \ell_k p$ with $\ell_k \in \mathbb{Z}^+$. Then for all $d_k \in D$, it holds that*

$$\sum_{j=n-|P|+1}^{n} \lambda_j^{(k)} \le \ell_k p.$$

6 Relation Partition and Their Cardinalities

Let X be a finite set of n elements, called points. Let $R_1, \ldots, R_r$ be symmetric relations that partition the unordered pairs of points in X. We call such a partition a *relation partition of* X. Then for each $k \in [r]$, we define G_k to be the graph corresponding to R_k; that is, G_k has vertex set X where $u \sim v$ if and only if $\{u, v\} \in R_k$.

We are interested in the set $\{|R_k| : k \in [r]\}$. Observe that $|R_k| = \frac{1}{2} Tr(L(G_k))$, where L denotes the Laplacian matrix. We will use the following well known tools of Lemma 1 and Lemma 2.

Lemma 1 (Combinatorial Nullstellensatz). *Let $\mathbb{F}$ be a field and let $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a polynomial such that $\deg(f) = \sum_{i=1}^{n} t_i$ and the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ is non-zero. Let $S_1, S_2, \ldots, S_n$ be subsets of $\mathbb{F}$ such that $|S_i| > t_i$ for all $i \in [n]$. Then there exists $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ such that $f(s_1, s_2, \ldots, s_n) \neq 0$.*

Lemma 2 (Eigenvalue Interlacing). *...*

We say that a relation partition $R_1, \ldots, R_r$ of X is *dependent* if there exists a subset $S \subseteq X$ such that for some $v \in S$ there is an $x \in X$ satisfying $R_k(v, x)$, and $u, w \in X \setminus S$ satisfying $R_{k'}(u, w)$ such that $R_k(v, x) \Leftrightarrow R_{k'}(u, w)$. That is, deleting the vertices of S causes point pairs outside of S to no longer be related. For example, if $(X; R_1, \ldots, R_r)$ is a metric space for a graph in which R_k corresponds to the distinct distance k, any vertex cut destroys geodetic paths between vertex pairs in the different resulting components. Note that dependence occurs here because the geodetic paths are in X itself rather than an underlying space. If on the other hand, X is a subset of an underlying metric space $\mathcal{X}$ whereby the distances are calculated with respect to geodetic paths in $\mathcal{X}$, then no $(X \subseteq \mathcal{X}; R_1, \ldots, R_r)$ can be dependent. We say that $(X : R_1, \ldots, R_r)$ is *independent* if it is not dependent.

Lemma 3. *Let X be a finite set of n points with an independent relation partition $R_1, \ldots, R_r$. Then for any prime p satisfying $p - 1 < \frac{n(n-1)}{n+r}$, there exist nonempty subsets $D \subseteq [r]$ and $P \subseteq X$, such that for all $k \in D$, there is a positive integer ℓ_k such that*

$$\ell_k p \leq \sum_{j=1}^{|P|} \lambda_j^{(k)},$$

where $\lambda_j^{(k)}$ is the j-th largest eigenvalue of $L(G_k)$.

Proof of Lemma 3. Let $\mathcal{M}$ be the bipartite multigraph with point vertices X in one part and relation vertices for each $R_k, k \in [r]$ for the other part. For each $k \in [r]$, and for each $u, v \in X$ satisfying $R_k(u, v)$ we include the edges $u \sim R_k$ and $v \sim R_k$.

Claim (Variation of Theorem 6.1 in [**?**])**.** *The multigraph $\mathcal{M}(X)$ contains a non-empty subgraph $\mathcal{U}$ such that for every $u \in V(\mathcal{U})$, $\deg(u) \in \{kp : k \in \mathbb{Z}^+\}$.*

Proof of Claim. We define a polynomial f with degree $|E(\mathcal{M})|$ over $\mathbb{F}_2$, and using the fact that $a^{p-1} \pmod p \equiv 1$ for all $a \not\equiv 0 \pmod p$, we show the existence of the desired subgraph using the nullstellensatz directly. Define the polynomial

$$f(x_e : e \in E(\mathcal{M})) := \prod_{v \in V(\mathcal{M})} \left[1 - \Big(\sum_{\substack{e \in E(\mathcal{M}) \\ v \in e}} x_e\Big)^{p-1}\right] - \prod_{e \in E(\mathcal{M})} (1 - x_e).$$

The degree of f is $|E(\mathcal{M})|$ because every other term has degree at most

$$|V(\mathcal{M})|(p-1) = (n+r)(p-1) < n(n-1) = |E(\mathcal{M})|.$$

Note that the max degree term of f, $(-1)^{|E(\mathcal{M})|+1} \prod_{e \in E(\mathcal{M})} x_e$, has a non-zero coefficient. To apply the nullstellensatz, we consider solutions to f of the form $(s_1, s_2, \ldots, s_{|E(\mathcal{M})|}) \in \{0,1\}^{|E(\mathcal{M})|}$ (where $t_i = 1$ for all $i \in [|E(\mathcal{M})|]$). Thus by Lemma **??**, there exists a edge vector $\mathbf{u} = (u_e : e \in E(\mathcal{M}))$ such that $f(\mathbf{u}) \neq 0$. By the definition of f, $\mathbf{u} \neq \mathbf{0}$ because $f(\mathbf{0}) = 0$, so some of its entries are 1. This means that the latter product in f vanishes when evaluated at $\mathbf{u}$. The former product in f can be non-zero only when

$$\Big(\sum_{\substack{e \in E(\mathcal{M}) \\ v \in e}} u_e\Big)^{p-1} \equiv 0 \pmod p.$$

It follows that $\mathbf{u}$ corresponds to a subgraph $\mathcal{U}$ of $\mathcal{M}(X)$ whose vertex degrees are congruent to $0 \pmod p$. Since $\mathbf{u} \neq \mathbf{0}$, there exists a vertex $v \in \mathcal{U}$ such that $\deg_{\mathcal{U}}(v) \in \{kp : k \in \mathbb{Z}^+\}$. Note that since $\mathcal{U}$ is a subgraph of $\mathcal{M}$, which is bipartite, the degree sums in each part need to be equal; therefore, the vertices of $\mathcal{U}$ all have degrees being a positive multiple of p, and these positive degree vertices are in both parts. $\square$

Notice that the claim implies that for each $k \in D$, there exists a positive integer ℓ_k such that $\deg_{\mathcal{U}}(R_k) = \ell_k p$. Our goal is to lower bound the degrees of a subset of the relation vertices of $\mathcal{M}$. Using what we know about the degrees of $\{R_k : k \in D\}$ from the subgraph $\mathcal{U}$ we obtained in the claim above, we now apply eigenvalue interlacing on the laplacians of the G_k graphs for $k \in D$ on the complement of the point vertices of $\mathcal{U}$.

Let P be the point vertices of $\mathcal{U}$, and set $m := |P|$. For each $k \in D$, let $\mu_1^{(k)} \geq \mu_2^{(k)} \geq \cdots \geq \mu_{n-m}^{(k)}$ be the laplacian eigenvalues of $G_k \setminus P$. Then by Lemma 2, we have that for each $j \in [n-m]$, $\lambda_{m+j}^{(k)} \leq \mu_j^{(k)}$. For each $v \in P$, let $N_k(v)$ be the

k-neighbourhood of v in X; that is, $N_k(v)$ is the set of other points $u \in X$ satisfying $R_k(v,u)$. Note that $\deg_{G_k}(v) = |N_k(v)|$. Since $\sum_{j=1}^{n-m} \mu_j^{(k)}$ counts the degree of R_k in $\mathcal{M}$ involving edges incident to point vertices only in $X \setminus P$, it follows that

$$\sum_{j=1}^{n} \lambda_j^{(k)} = \sum_{j=1}^{n-m} \mu_j^{(k)} - \sum_{v \in P} \deg_{G_k}(v).$$

Since $\ell_k p = \deg_{\mathcal{U}}(R_k) \leq \sum_{v \in P} \deg_{G_k}(v)$, it follows that for each $k \in D$,

$$\sum_{j=m+1}^{n} \lambda_j^{(k)} \leq \sum_{j=1}^{n} \lambda_j^{(k)} - \ell_k p \Leftrightarrow \ell_k p \leq \sum_{j=1}^{m} \lambda_j^{(k)},$$

as desired. $\square$

Lemma 3 is substantially more powerful than a pigeonhole argument, because it provides us with a set of relations of larger cardinality, rather than only one. Note that PHP implies that there is some relation R_k with cardinality at least $\lceil \frac{n(n-1)}{4r} \rceil$. Lemma 3 gives us a **set** D of relations with large cardinality, and we can control the cardinality of D to ensure that we have larger set of relations with this cardinality. What follows are conditions that allow us to control $|Y|$ and $|D|$. What follows are conditions that enable us to control these quantities, which will enable us to prove necessary lower bounds on the relation cardinalities. In particular, we prove a necessary lower bound on the maximum relation cardinality in various contexts. Note that in metric spaces, this is the same thing as lower bounding the maximum multiplicity of a distinct distance.

There is a special case when $n/2$ or $(n+1)/2$ is prime since in this case the point vertices of the subgraph of $\mathcal{M}$ given by the nullstellensatz must have degree p.

Corollary 6.1. *If $\lceil n/2 \rceil$ is prime, then $|Y| \geq |D|$ and*

$$|Y|p \leq \sum_{k \in D} \sum_{j=1}^{|Y|} \lambda_j^{(k)}.$$

Proof. Since no point can be related with more than $n-1$ other points, the point vertices in the multi-subgraph $\mathcal{U}$ from the proof of Lemma 3 have degree exactly $p = \lceil n/2 \rceil$. Additionally, since $\mathcal{U}$ is bipartite, the degree sums in both the point and relation parts need to be equal, so $m = \sum_{k \in D} \ell_k$. $\square$

The next corollary introduces a natural condition in the context of geometric finite metric spaces where the condition of general position with respect to spheres is considered. For example, suppose X is a finite metric space in $\mathbb{R}^{d-1}$ such that we forbid any $d+1$ points being on a $(d-2)$-sphere $\mathcal{S}^{d-2}$; then, this implies that no point in X can be at equal distance with more than d other points in X.

Corollary 6.2. *If for each $k \in [r]$, it holds that $\Delta(G_k) \leq d$ for some positive integer d, then $m \geq \lceil |D|/d \rceil$.*

7 Multiple Set Version

Let $X_1, \dots, X_s$ be finite sets with cardinalities in $[N/2, N]$. Let $R_1, \dots, R_r$ be symmetric relations that partition $\bigcup_{i\in[s]} \binom{X_i}{2}$. Our goal is to understand the cardinalities of $R_1, \dots, R_r$.

We say that a set X has *max degree* d in $\{R_1, \dots, R_r\}$ if for each $k \in [r]$ and $v \in X$, there are at most d points $u \in X \setminus \{v\}$ such that $\{u, v\} \in R_k$. Note that this condition is equivalent to the max degree of the graph G_k being at most d.

Theorem 7.1. *Let s, r, and d be positive integers and $X_1, \dots, X_s$ be finite sets with max degree d and symmetric relation partitions $P_1, \dots, P_s$, respectively, where each $P_i \subseteq \{R_1, \dots, R_r\}$. Let p be the largest prime satisfying $p \leq \min_{i\in[s]} \left(\frac{|X_i|(|X_i|-1)}{|X_i|+|P_i|}\right)$. Then either there exists a non-negative integer ℓ and a set of distinct relations $\{R_{j_q} : q \in [\frac{sp^2}{2td} + \ell]\}$ such that*

$$|R_{j_q}| \geq \left(\frac{\frac{sp}{d} - (q-1)2t/p}{\frac{sp^2}{2td} + \ell - (q-1)}\right)\frac{p}{2},$$

or there is some $k \in [r]$ such that $|R_k| > t$.

Proof. Let $\{(Y_i, D_i) : i \in [s]\}$ be the bipartite multigraphs obtained from Lemma 3 applied to each (X_i, P_i) with the prime p. We have by max degree that each relation vertex in (Y_i, D_i) has at most d edges with each point vertex, so $|D_i| \geq \lceil p/d \rceil$. Suppose there is a subcollection $\mathcal{D}$ of $\{D_1, \dots, D_s\}$ of size at least $2t/p$ such that $\bigcap_{D\in\mathcal{D}} D \neq \emptyset$. Then there exists a $k \in [r]$ satisfying $|R_k| > t$. Otherwise, each relation $R_1, \dots, R_r$ is contained in fewer than $2t/p$ of the sets $D_1, \dots, D_s$. Consider the $(0,1)$ incidence matrix A with rows corresponding to the relations in $\bigcup_{i\in[s]} D_i$ and columns the sets $D_1, \dots, D_s$. The row sums of A are at most $2t/p$ and the column sums are at least p/d, and since there are s columns, we need enough rows r' to satisfy $r'(2t/p) \geq sp/d$. Thus $|\bigcup_{i\in[s]} D_i| \geq \frac{sp^2}{2td}$. Suppose $r' = \frac{sp^2}{2td} + \ell$, and note that all of these rows are non-zero. Rearrange the rows of A by row sum from highest to lowest. The average number of 1s in these r' rows is $\frac{sp}{dr'}$, so there exists a row r_1 with this many 1s; suppose r_1 has max row sum. Delete r_1 from A and what remains is a matrix with at most $sp/d - 2t/p$ 1s and $r' - 1$ rows. By induction on q, for all $q \in [r' + \ell]$, there is a row r_q with at least $(\frac{sp}{d} - (q-1)2t/p)\frac{1}{(r'-(q-1))}$ 1s. Therefore, there exists a non-negative integer ℓ and a set of distinct relations $\{R_{j_q} : q \in [r']\}$ such that $|R_{j_q}| \geq (\frac{sp}{d} - (q-1)2t/p)\frac{p}{2(r'-(q-1))}$. □

A nice thing about this theorem is that very little is assumed about $R_1, \dots, R_r$, other than that their graphs have max degree d. That is, we haven't specified any cardinality lists for the relations in each P_i. But nonetheless, we can still lower bound the max cardinality of a relation.

As mentioned above, we need a "dense subgraph existence" argument like Lemma 3 to get us a **subset** of large cardinality relations for each pair (X_i, P_i). PHP with the max degree condition alone cannot give us such relation sets to then find a non-empty intersection. I think Lemma 3 is a proof of concept of this idea, and it would be interesting to explore other techniques or conditions that enable us to find dense subgraphs of $\mathcal{M}$ to make the PHP argument in Theorem 7.1 go through.

Notice that the condition in Theorem 7.1 cannot be met if $\frac{2trd}{p^2} \geq s$. While I currently don't see why one would have d grow as a function of s, there are cases when both t and r grow with s. For example in crescent families of sets X_i with max degree d (note the constituent sets are not assumed to be crescents here), if we wanted to contradict this structure by showing that there is some $t > r$, we would run into trouble because the binomial condition requires $r(r-1) = \sum_{i \in [s]} |X_i|(|X_i| - 1)$. If we assume the balanced case (to maximize p, say) where we assume $|X_i| \sim N$ and so at best $p \sim N/2$, we would have roughly that $r \sim \sqrt{s}N$, and since we want $t \geq r$, we have that $s \leq \frac{2trd}{p^2} \leq 8sd$, which means the condition in Theorem 7.1 cannot be met. Although, I think the condition is close to being met here; so I suspect there's a way to somehow improve Theorem 7.1 (and possibly add a condition) to get some kind of upper bound on the number of sets in a crescent family.

The following is a necessary lower bound on the maximum relation cardinality for set families with constituent sets having max degree d.

Corollary 7.2. *If $X_1, \ldots, X_s$ are finite sets with symmetric relation partitions $P_1, \ldots, P_s \subseteq \{R_1, \ldots, R_r\}$ with max degree d. Let p be the largest prime satisfying $p \leq \min_{i \in [s]} \left(\frac{|X_i|(|X_i|-1)}{|X_i|+|P_i|} \right)$. Then $\max_{k \in [r]}(|R_k|) > \frac{sp^2}{2rd}$.*

Proof. We contradict Theorem 7.1 otherwise. $\square$