Comprehensive Examination in Graph Theory

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Duration: 3 hours
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Instructions: There are ten questions of equal value. Do any seven questions. When you are done, please scan your solutions and email the PDF to gmacgill@uvic.ca.

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1. A sequence $s : d_1, d_2, ..., d_n$ of nonnegative integers is called *graphical* if there exists a simple graph with degree sequence s.

   Prove that a sequence $s : d_1, d_2, ..., d_n$ of nonnegative integers with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and $\Delta \geq 1$ is graphical if and only if the sequence $s_1 : d_2 - 1, d_3 - 1, ..., d_{\Delta+1} - 1, d_{\Delta+2}, ..., d_n$ is graphical.

2. Prove that a nontrivial graph is bipartite if and only if it contains no odd cycles.

3. State Prim's Algorithm for finding a minimum spanning tree of a connected weighted graph G. Prove that the algorithm produces a minimum spanning tree of G.

4. (a) State Menger's Theorem.
   (b) Let G be a graph with at least three vertices. Without using Menger's Theorem, prove that G is 2-connected if and only if every two vertices of G lie on a common cycle.

5. Let G be a graph in which the degree of every vertex is even. Show that the edge set of G can be partitioned into disjoint subsets each of which induces a cycle.

6. Let G be a bipartite graph with bipartition (X, Y). Let $d = \max_{S \subseteq X} \{|S| - |N(S)|\}$. Prove that the number of edges in a maximum matching of G is $|X| - d$.

7. Let G be a simple graph such that $\deg(x) + \deg(y) \geq |V| - 1$ for any two non-adjacent vertices x and y. Prove that G has a Hamilton path.

8. A *plane graph* is a planar graph that has been drawn in the plane so that edges meet only at their ends. A plane graph is called *self-dual* if it is isomorphic to its dual.

   (a) Show that if G is self-dual, then $|E| = 2|V| - 2$.
   (b) For each $n \geq 4$, find a self-dual plan graph on n vertices.

9. (a) Denote the clique number of a graph G by $\omega(G)$. Without using the Perfect Graph Theorem, the Strong Perfect Graph Theorem, or the fact that chordal graphs are perfect, prove that if G is a chordal graph, then $\chi(G) = \omega(G)$.
   (b) Deduce that chordal graphs are perfect.

10. Let $m, n \in \mathbb{Z}$ and $r = m + n$. Let G be an r-regular graph. Prove that G has an orientation so that every vertex has out-degree either m or n.