Comprehensive Examination in Graph Theory January 28, 2022

Duration: 3 hours Examiners: Kieka Mynhardt, Jing Huang and Gary MacGillivray

Instructions: There is one page, and ten questions of equal value. Do any *seven* questions. Write your answers in the booklets provided.

1. A set X of vertices of a graph G is *irredundant* if, for each $x \in X$, the closed neighbourhood of x contains a vertex that does not belong to the closed neighbourhood of any vertex $c' \in X \setminus \{x\}$. Prove that an independent set $X \subseteq V(G)$ is maximal independent if and only if it is dominating, and a dominating set X is minimal dominating if and only if it is irredundant. Give, with reasons, an example of a graph that has a maximal irredundant set that is not dominating.

2. Let G be a graph (not necessarily simple) with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let A be the matrix whose (i, j)-entry is the number of edges joining v_i and v_j.

   (a) Prove that for $k > 0$ the (i, j)-entry of A^k is the number of walks in G from v_i to v_j.

   (b) The *odd girth* of G is the length of a shortest odd cycle in G, or ∞ if G has no odd cycle. Describe a method based on (a) for determining the odd girth of a graph G, and justify that it works.

   (c) The *even girth* of G is defined similarly as in part (b). Is there a similar method for determining the even girth of G? Why or why not?

3. Prove that each vertex of a strong tournament of order $n \geq 3$ lies on a directed k-cycle for each k such that $3 \leq k \leq n$.

4. Prove that if every vertex of G has even degree, then the edges of G can be oriented so that, at each vertex, the indegree equals the outdegree.

5. Prove that any graph G with order at least three, connectivity $\kappa(G)$ and independence number $\alpha(G)$, where $\kappa(G) \geq \alpha(G)$, is Hamiltonian.

6. (a) Prove that every bridgeless cubic graph (i.e. a cubic graph with no cut edge) has a perfect matching.

   (b) Draw a cubic graph with bridges that has a perfect matching.

7. Prove that a graph G is 2-factorable if and only if G is $2k$-regular for some integer $k \geq 1$ (Petersen's Theorem).

8. (a) State Menger's Theorem and the Max-Flow Min-Cut Theorem, define any terms necessary to make your statements meaningful except for path, cut, source and sink.

   (b) Use the Max-Flow Min-Cut Theorem to prove Menger's Theorem. State any other results used.

9. Let G be a graph with $\chi(G) > k$, and let X, Y be a partition of $V(G)$. Suppose that $G[X]$ and $G[Y]$ are k-colourable. Prove that the edge cut $[X, Y]$ has at least k edges.

10. Let G be a simple 2-edge-connected 3-regular plane graph and let G^* be the dual graph of G. Prove that G is 3-edge-colourable if and only if G^* is 4-colourable.