General Necessary Conditions on Relation Partitions Using the Combinatorial Nullstellensatz

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1 Summary

The basic structure of interest here is a finite set X paired with a partition of its unordered pairs into symmetric relations $R_1, \dots, R_r$. We denote this partition with P, and we call the pair (X, P) a *relation partition*. First, we present Lemma 2, which is a general cardinality condition on a subset of $\{R_1, \dots, R_r\}$ in term of positive multiples of large primes and laplacian eigenvalue partial sums of graphs corresponding to the R_ks. We prove this lemma using the combinatorial nullstellensatz and eigenvalue interlacing. Using Lemma 2, we briefly discuss implications on the structure of crescent configurations in $\mathbb{R}^2$ of order 9. Then we move on to a multiple set variation, wherein we have a family of relation partitions $(X_1, P_1), \dots, (X_s, P_s)$ satisfying $P_i \subseteq \{R_1, \dots, R_r\}$. We obtain Theorem 3.1 using Lemma 2, which provides general necessary conditions on both the structure and cardinalities of $R_1, \dots, R_r$. We discuss a technique for how Theorem 3.1 could be used to derive stronger conditions in more particular contexts. We include a brief discussion on entropy as a natural way to describe the complexity of a relation partition.

2 Relation Partition and Their Cardinalities

Let X be a finite set of n elements, called points. Let $R_1, \dots, R_r$ be symmetric relations that partition the unordered pairs of points in X. We call such a partition a *relation partition of* X. Then for each $k \in [r]$, we define G_k to be the graph corresponding to R_k; that is, G_k has vertex set X where $u \sim v$ if and only if $\{u, v\} \in R_k$.

We are interested in the set $\{|R_k| : k \in [r]\}$. Observe that $|R_k| = \frac{1}{2}Tr(L(G_k))$, where L denotes the Laplacian matrix. We will use the combinatorial nullstellensatz (Lemma 1) and eigenvalue interlacing.

Lemma 1 (Combinatorial Nullstellensatz)**.** *Let $\mathbb{F}$ be a field and let $f \in \mathbb{F}[x_1, x_2, \dots, x_n]$ be a polynomial such that $\deg(f) = \sum_{i=1}^n t_i$ and the coefficient of $\prod_{i=1}^n x_i^{t_i}$ is non-zero. Let $S_1, S_2, \dots, S_n$ be subsets of $\mathbb{F}$ such that $|S_i| > t_i$ for all $i \in [n]$. Then there exists $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ such that $f(s_1, s_2, \dots, s_n) \neq 0$.*

We say that a relation partition $R_1, \dots, R_r$ of X is *dependent* if there exists a subset $S \subseteq X$ such that for some $v \in S$ there is an $x \in X$ satisfying $R_k(v, x)$, and $u, w \in X \setminus S$ satisfying $R_{k'}(u, w)$ such that $R_k(v, x) \Leftrightarrow R_{k'}(u, w)$. That is, deleting the vertices of S causes point pairs outside of S to no longer be related. For example, if $(X; R_1, \dots, R_r)$ is a metric space for a graph in which R_k corresponds to the distinct distance k, any vertex cut destroys geodetic paths between vertex pairs in the different resulting components. Note that dependence occurs here because the geodetic paths are in X itself rather than an underlying space. If on the other hand, X is a subset of an underlying metric space $\mathcal{X}$ whereby the distances are calculated with respect to geodetic paths in $\mathcal{X}$, then no $(X \subseteq \mathcal{X}; R_1, \dots, R_r)$ can be dependent. We say that $(X : R_1, \dots, R_r)$ is *independent* if it is not dependent.

Lemma 2. *Let X be a finite set of n points with an independent relation partition $R_1, \dots, R_r$. Then for any prime p satisfying $p - 1 < \frac{n(n-1)}{n+r}$, there exist nonempty subsets $D \subseteq [r]$ and $P \subseteq X$, such that for all $k \in D$, there is a positive integer ℓ_k such that $\ell_k p \leq 2|R_k|$, and if $X \setminus P$ is independent, $\ell_k p \leq \sum_{j=1}^{|P|} \lambda_j^{(k)}$, where $\lambda_j^{(k)}$ is the j-th largest eigenvalue of $L(G_k)$.*

Proof of Lemma 2. Let $\mathcal{M}$ be the bipartite multigraph with point vertices X in one part and relation vertices for each $R_k, k \in [r]$ for the other part. For each $k \in [r]$, and for each $u, v \in X$ satisfying $R_k(u, v)$ we include the edges $u \sim R_k$ and $v \sim R_k$.

Claim (Variation of Theorem 6.1 in [1])**.** *The multigraph $\mathcal{M}(X)$ contains a nonempty subgraph $\mathcal{U}$ such that for every $u \in V(\mathcal{U})$, $\deg(u) \in \{kp : k \in \mathbb{Z}^+\}$.*

Proof of Claim. We define a polynomial f with degree $|E(\mathcal{M})|$ over $\mathbb{F}_2$, and using the fact that $a^{p-1} \pmod p \equiv 1$ for all $a \not\equiv 0 \pmod p$, we show the existence of the desired subgraph using the nullstellensatz directly. Define the polynomial

$$f(x_e : e \in E(\mathcal{M})) := \prod_{v \in V(\mathcal{M})} \left[1 - \Big(\sum_{\substack{e \in E(\mathcal{M}) \\ v \in e}} x_e \Big)^{p-1} \right] - \prod_{e \in E(\mathcal{M})} (1 - x_e).$$

The degree of f is $|E(\mathcal{M})|$ because every other term has degree at most

$$|V(\mathcal{M})|(p-1) = (n+r)(p-1) < n(n-1) = |E(\mathcal{M})|.$$

Note that the max degree term of f, $(-1)^{|E(\mathcal{M})|+1} \prod_{e \in E(\mathcal{M})} x_e$, has a non-zero coefficient. To apply the nullstellensatz, we consider solutions to f of the form $(s_1, s_2, \dots, s_{|E(\mathcal{M})|}) \in \{0, 1\}^{|E(\mathcal{M})|}$ (where $t_i = 1$ for all $i \in [|E(\mathcal{M})|]$). Thus by Lemma 1, there exists a edge vector $\mathbf{u} = (u_e : e \in E(\mathcal{M}))$ such that $f(\mathbf{u}) \neq 0$. By the definition of

f, $\mathbf{u} \neq \mathbf{0}$ because $f(\mathbf{0}) = 0$, so some of its entries are 1. This means that the latter product in f vanishes when evaluated at $\mathbf{u}$. The former product in f can be non-zero only when

$$\Big(\sum_{\substack{e \in E(\mathcal{M}) \\ v \in e}} u_e\Big)^{p-1} \equiv 0 \pmod{p}.$$

It follows that $\mathbf{u}$ corresponds to a subgraph $\mathcal{U}$ of $\mathcal{M}(X)$ whose vertex degrees are congruent to $0 \pmod{p}$. Since $\mathbf{u} \neq \mathbf{0}$, there exists a vertex $v \in \mathcal{U}$ such that $\deg_{\mathcal{U}}(v) \in \{kp : k \in \mathbb{Z}^+\}$. Note that since $\mathcal{U}$ is a subgraph of $\mathcal{M}$, which is bipartite, the degree sums in each part need to be equal; therefore, the vertices of $\mathcal{U}$ all have degrees being a positive multiple of p, and these positive degree vertices are in both parts. $\square$

Notice that the claim implies that for each $k \in D$, there exists a positive integer ℓ_k such that $\deg_{\mathcal{U}}(R_k) = \ell_k p$, which means in general $2|R_k| \geq \ell_k p$.

Suppose $X \setminus P$ is independent. Our goal is to lower bound the degrees of a subset of the relation vertices of $\mathcal{M}$. Using what we know about the degrees of $\{R_k : k \in D\}$ from the subgraph $\mathcal{U}$ we obtained in the claim above, we now apply eigenvalue interlacing on the laplacians of the G_k graphs for $k \in D$ on the complement of the point vertices of $\mathcal{U}$.

Let Y be the point vertices of $\mathcal{U}$, and set $m := |Y|$. For each $k \in D$, let $\mu_1^{(k)} \geq \mu_2^{(k)} \geq \cdots \geq \mu_{n-m}^{(k)}$ be the laplacian eigenvalues of $G_k \setminus Y$. Then by eigenvalue interlacing, we have that for each $j \in [n-m]$, $\lambda_{m+j}^{(k)} \leq \mu_j^{(k)}$. For each $v \in Y$, let $N_k(v)$ be the k-neighbourhood of v in X; that is, $N_k(v)$ is the set of other points $u \in X$ satisfying $R_k(v,u)$. Note that $\deg_{G_k}(v) = |N_k(v)|$. Since $\sum_{j=1}^{n-m} \mu_j^{(k)}$ counts the degree of R_k in $\mathcal{M}$ involving edges incident to point vertices only in $X \setminus Y$ (we don't lose count of the crossing edges here because the diagonal entries of the principal submatrix of the Laplacian are unchanged), it follows that

$$\sum_{j=1}^{n} \lambda_j^{(k)} = \sum_{j=1}^{n-m} \mu_j^{(k)} + \sum_{v \in Y} \deg_{G_k}(v).$$

Since $\ell_k p = \deg_{\mathcal{U}}(R_k) \leq \sum_{v \in Y} \deg_{G_k}(v)$, it follows that for each $k \in D$,

$$\sum_{j=m+1}^{n} \lambda_j^{(k)} \leq \sum_{j=1}^{n} \lambda_j^{(k)} - \ell_k p \Leftrightarrow \ell_k p \leq \sum_{j=1}^{m} \lambda_j^{(k)},$$

as desired. $\square$

The value that the nullstellensatz approach in Lemma 2 is providing is that we get a set of relations of larger cardinality, rather than only one relation. Note that PHP implies that there is some relation R_k with cardinality at least $\lceil \frac{n(n-1)}{4r} \rceil$. Lemma 2 gives us a **set** D of relations with "large" cardinality, and we can control the size of D to ensure that we have a larger set of relations with this cardinality. What follows are conditions that allow us to control $|Y|$ and $|D|$. This will help us prove necessary structure on relation partitions in particular contexts (see Example 1 for a brief discussion on implications for crescents in $\mathbb{R}^2$ of order 9).

There is a special case when $n/2$ or $(n+1)/2$ is prime since in this case the point vertices of the subgraph of $\mathcal{M}$ given by Lemma 2 must have degree p.

Corollary 2.1. *If $\lceil n/2 \rceil$ is prime, then $|Y| \geq |D|$ and*

$$|Y|p \leq \sum_{k \in D} \sum_{j=1}^{|Y|} \lambda_j^{(k)}.$$

Proof. Since no point can be related with more than $n-1$ other points, the point vertices in the multi-subgraph $\mathcal{U}$ from the proof of Lemma 2 have degree exactly $p = \lceil n/2 \rceil$. Additionally, since $\mathcal{U}$ is bipartite, the degree sums in both the point and relation parts need to be equal, so $|Y| = \sum_{k \in D} \ell_k$. □

The next corollary introduces a natural condition that seems to show up in a variety of relation partition contexts, including in the context of geometric finite metric spaces where the condition of general position with respect to spheres is considered. For example, suppose X is a finite metric space in $\mathbb{R}^{d-1}$ such that we forbid any $d+1$ points being on a $(d-2)$-sphere $\mathcal{S}^{d-2}$; then, this implies that no point in X can be at equal distance with more than d other points in X.

Corollary 2.2. *If for each $k \in [r]$, it holds that $\Delta(G_k) \leq d$ for some positive integer d, then $|D| \geq \lceil p/d \rceil$.*

Proof. Each point vertex v in Y has degree at least p, and there are at most d edges between v and any relation vertex in D. □

The purpose of the following example is to show how Lemma 2 can imply necessary substructure in X, if X and its relation partition are instantiated with particular properties (that is, as a metric space, or proper edge colouring, an equivalence relation, or whatever).

Example 1 (Crescent Configurations of Order 9 in the plane)**.** Let X be a crescent configuration in general position with respect to the circle in $\mathbb{R}^2$ with 9 points. Let $d_1, \ldots, d_8$ be the distinct distances of X. Note that we are in the case $p = 5$ and $d = 3$. Since distances are calculated in the underlying space of $\mathbb{R}^2$, all subsets of X are independent. Thus by Lemma 2 there exist non-empty subsets $D \subseteq [8]$ and $Y \subseteq X$ with $|Y| \geq |D|$ such that for all $k \in D$, there exists a positive integer ℓ_k satisfying $5\ell_k \leq \sum_{j=1}^{|Y|} \lambda_j^{(k)}$, where $\lambda_j^{(k)}$ is the j-th largest laplacian eigenvalue of the distance graph G_k corresponding to distance d_k. Since the maximum multiplicity of each distance is $9 - 1 = 8$, the distance graphs are either trees or disconnected.

Notice that $p = 5$ and the max multiplicity of 8 condition implies $1 \leq |D| \leq 6$ (there is a distance in D with multiplicity at least $\lceil p/2 \rceil = 3$). Indeed $|D| \geq 2$, since each point vertex in Y has degree exactly $p = 5$ and $d = 3$. In the case $|Y| = |D| = 2$, we have a particular "distance saturated" configuration occurring in X whereby two points $x_1, x_2 \in Y$ and two distances $d_1, d_2 \in D$ satisfy the properties that x_1 is at distance d_1 and d_2 with three and two other points, respectively, and similarly x_2 is at distance d_2 and d_2 with three and two other points, respectively. Note that if $d(x_1, x_2) \notin \{d_1, d_2\}$, then the multiplicity of both d_1 and d_2 is at least 5. Also, if $|Y| > |D|$, then there exists a distance d_k with $\ell_k \geq 2$, giving it multiplicity at least 5; we also have that $\ell_k \leq 3$ and at most one distance can satisfy $\ell_k = 3$ since $\lceil 3p/2 \rceil = 8$, which is the max multiplicity. There are probably other implications of Lemma 2.

It could be fruitful to examine the possible distance graphs of order 9 crescent configurations and see if one could narrow down the possible candidates using the structure of Y and D with the laplacian eigenvalue necessary condition. There seems to be a fair amount of structure to work with here.

Remark 2.3 (Laplacian partial sums)**.** There are known general upper bounds on partial sums of Laplacian eigenvalues, which could be used if needed. There's a conjectured general upper bound of $\sum_{j=1}^{m} \lambda_j \leq (\#\text{ of edges}) + \binom{m+1}{2}$, which is known to hold for trees and all graphs of order at most 10.

3 Multiple Set Version

Let $X_1, \ldots, X_s$ be finite sets with symmetric relation partitions $P_1, \ldots, P_s \subseteq \{R_1, \ldots, R_r\}$. Observe that $R_1, \ldots, R_r$ are symmetric relations that partition $\bigcup_{i \in [s]} \binom{X_i}{2}$. Our goal is to understand the cardinalities of $R_1, \ldots, R_r$. In Theorem 3.1 below, we apply Lemma 2 to each constituent (point set, partition) pair (X_i, P_i) to obtain "dense" relation sets D_i for each $i \in [s]$. Then using these sets with the max degree d condition on each set, we determine that there must exist some positive integer t satisfying four necessary conditions on the structure of $R_1, \dots, R_r$. The idea for how to use Theorem 3.1 in a particular context is to determine the permissible values of t in this context, which then in turn provides a set of necessary conditions on the relation cardinalities (ideally these are quite restricted conditions given the particular context).

Recall we say that a set X_i has *max degree* d in P_i if for each $k \in [r]$ and $v \in X$, there are at most d points $u \in X \setminus \{v\}$ such that $\{u, v\} \in R_k$. Again, this condition is equivalent to the max degree of the graph G_k being at most d.

Theorem 3.1. *Let s, r, and d be positive integers and $X_1, \dots, X_s$ be finite sets with max degree d and symmetric relation partitions $P_1, \dots, P_s$, respectively, where each $P_i \subseteq \{R_1, \dots, R_r\}$. Let p be the largest prime satisfying $p \leq \min_{i \in [s]} \left(\frac{|X_i|(|X_i|-1)}{|X_i|+|P_i|}\right)$. Then there exists a positive integer t and an integer $\ell \in [0, r - \frac{sp^2}{2td}]$ such that the following conditions hold:*

1. *the inequality $r \geq \frac{sp^2}{2td}$ holds;*
2. *there is some $k \in [r]$ such that $|R_k| > t$;*
3. *there are $\frac{sp^2}{2td} + \ell$ relations with cardinality at least $p/2$; and*
4. *there is a set of relations $\{R_{j_q} : q \in [\frac{sp^2}{2td}]\}$ such that*

$$|R_{j_q}| \geq \left(\frac{\frac{sp}{d} - (q-1)2t/p}{\frac{sp^2}{2td} + \ell - (q-1)} \right) \frac{p}{2}.$$

Proof. Let $\{(Y_i, D_i) : i \in [s]\}$ be the bipartite multigraphs obtained from Lemma 2 applied to each (X_i, P_i) with the prime p. We have by max degree that each relation vertex in (Y_i, D_i) has at most d edges with each point vertex, so $|D_i| \geq \lceil p/d \rceil$. Suppose t is maximal satisfying the property that there is a subcollection $\mathcal{D}$ of $\{D_1, \dots, D_s\}$ of size at least $2t/p$ such that $\bigcap_{D \in \mathcal{D}} D \neq \emptyset$. Then there exists a $k \in [r]$ satisfying $|R_k| > t$. By maximality of t, each relation $R_1, \dots, R_r$ is contained in fewer than $2t/p$ of the sets $D_1, \dots, D_s$. Consider the $(0, 1)$ incidence matrix A with rows corresponding to the relations in $\bigcup_{i \in [s]} D_i$ and columns the sets $D_1, \dots, D_s$. The row sums of A are at most $2t/p$ and the column sums are at least p/d, and since there are s columns, we need enough rows r' to satisfy $r'(2t/p) \geq sp/d$. Thus $r \geq |\bigcup_{i \in [s]} D_i| \geq \frac{sp^2}{2td}$, and recall that all relations in $\bigcup_{i \in [s]} D_i$ have cardinality at least $p/2$. Suppose $r' = \frac{sp^2}{2td} + \ell$, and note that all of these rows are non-zero. Rearrange the rows of A by row sum from highest to lowest. The average number of 1s in these r' rows is $\frac{sp}{dr'}$, so there exists a row r_1 with this many 1s; suppose r_1 has max row sum. Delete r_1 from A and what remains is a matrix with at most $sp/d - 2t/p$ 1s and $r' - 1$ rows. By induction on q, for all $q \in [r' + \ell]$, there is a row r_q with at least $(\frac{sp}{d} - (q-1)2t/p)\frac{1}{(r'-(q-1))}$ 1s. Therefore, there exists a set of distinct relations $\{R_{j_q} : q \in [r']\}$ such that $|R_{j_q}| \geq (\frac{sp}{d} - (q-1)2t/p)\frac{p}{2(r'-(q-1))}$. □

Very little is assumed about $R_1, \ldots, R_r$ or $P_1, \ldots, P_s$, other than that the relation graphs, when restricted to (X_i, P_i) have max degree d. That is, we haven't specified any cardinality lists for the relations in each P_i. The format of the theorem suggests a technique: Given a relation partition of a family of point sets $X_1, \ldots, X_s$, find the values of t that satisfy the conditions of Theorem 3.1; then, these permissible t values constrain the cardinalities of the relations. Since Theorem 3.1 is so general, the idea would be to constrain to relation partitions of set families with specific structure and then find permissible t values, which imply necessary conditions on the relation cardinalities by Theorem 3.1. Below are some examples of what I mean by "specific structure". What are the permissible values of t if

1. each point set is an AP in $\mathbb{Z}$ or $\mathbb{Z}_n$ (each relation graph is a forest – and I think they tend to be paths? So, $d \leq 2$);
2. each (X_i, P_i) is an optimal proper edge colouring of $K_{|X_i|}$ using a subset P_i of r colours (note proper implies $d = 1$ here);
3. each (X_i, P_i) is a metric space of points in $\mathbb{R}^{d-1}$ that are in general position with respect to the sphere $\mathcal{S}^{d-2}$ (so for $\mathbb{R}^2$, $d = 3$); or
4. what if each (X_i, P_i) is a coherent configuration? (I don't know anything about this, but it seems quite related, and I'd be interested to learn more!)

The max degree parameter d seems to come up in a variety of contexts, and it's often just a small constant. This suggests to me that the max degree condition is a natural one. Also, it's neat how there's this general interplay between the structure of the relation graphs G_k (max degree and the laplacian), the degrees of the bipartite multigraph $\mathcal{M}$ (nullstellensatz) and the actual combinatorics of the underlying relation partitions (metric spaces, edge colourings, and so on).

For all this to work, we needed a "dense subgraph existence" argument like Lemma 2 to get us a **subset** of large cardinality relations for each pair (X_i, P_i). I think Lemma 2 is a proof of concept of this idea, and it would be interesting to explore other techniques or conditions that enable us to find dense subgraphs of $\mathcal{M}$ to make a similar but stronger version of Theorem 3.1 work.

4 Entropy Connection to Relation Cardinalities

The normalized relation cardinalities can be interpreted as a probability vector (each value is essentially the counting measure of one of the relations). Let $p_1, \ldots, p_r$ be these probabilities for relations $R_1, \ldots, R_r$. So, for each $k \in [r]$, the probability that a randomly chosen unordered pair is in R_k is p_k. Then the entropy of this probability vector is $-\sum_{k=1}^{r} p_k \log(p_k)$, which by Jensen's inequality is at most $\log(r)$ with equality in the case when all relation cardinalities are equal. The idea is that the entropy of the relation partition can be interpreted as a statistic describing the complexity of the partition, and this complexity depends on the relation cardinalities. Suppose one has found necessary relation cardinality bounds on the relation partitions that satisfy specific structure, like being particular class of metric space, or a coherent configuration, or anything. Then these cardinality bounds imply entropy bounds, which means one can in a sense "bound the complexity" of a class of structures that are defined in terms of relation partitions.

For example, this perspective actually provides a neat motivation for characterizing cases when uniform relation cardinalities occur – a while back, I noticed a little proof that trees can't have uniform distance multiplicities, except for K_2 and $K_{1,3}$, which means that trees as metric spaces have entropy strictly less than log(diam). Odd cycles have uniform distance multiplicities, so they have entropy log(diam). Crescents (cardinalities being exactly $1, \ldots, r$) have entropy $-\sum_{k=1}^{r} \frac{k}{\binom{r+1}{2}} \log(\frac{2k}{r(r+1)})$, which I think is close, but not equal to $\log(r)$.

Inversely, suppose we assume that the entropy of some class of relation partitions is g. What does this tell us about the relation cardinalities? It might be possible to use the statistic of entropy as a way to compare relation partitions.

5 Conclusion

I think Lemma 2 and Theorem 3.1 provide a decent proof of concept for a set of tools, derived from techniques (like the nullstellensatz) for finding dense subgraphs of the bipartite multigraph $\mathcal{M}$. Such tools enable one to find necessary conditions on the structure and cardinalities of relation partitions, both for single sets and set families. These results could probably be extended to non-symmetric relations and also to b-ary relation partitions wherein we partition the b-subsets (or b-tuples). There appears to be a rather deep connection between the graph structure in the relation graphs of a relation partition, the combinatorics of particular classes of relation partitions, and the structure of the bipartite multigraph $\mathcal{M}$.

References

[1] Alon, N. (1999). Combinatorial Nullstellensatz. Combinatorics, Probability and Computing, 8(1-2), 7-29. doi:10.1017/S0963548398003411