Comprehensive Examination in Graph Theory

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Duration: 3 hours

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Instructions: There are two pages, and ten questions of equal value. Do any *seven* questions. Write your answers in the booklets provided.

1. (a) State and prove Euler's formula that relates the number of vertices, edges and regions of a connected plane graph.

   (b) Prove that a planar graph of order $n \geq 3$ has at most $3n - 6$ edges.

   (c) Deduce that every planar graph has a vertex of degree at most five.

2. (a) Prove that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ for every graph G, where κ and κ' are the connectivity and edge-connectivity, respectively.

   (b) Show that $\kappa(G) = \delta(G)$ if G is a cubic graph.

3. (a) Prove Hall's Theorem: If G is a bipartite graph with partite sets U and W, then U can be matched to a subset of W if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$.

   (b) Prove that every r-regular bipartite graph, $r \geq 1$, has a perfect matching.

4. State and prove Brooks's upper bound for the chromatic number of a connected graph.

5. (a) Prove the Gallai-Roy-Hasse-Vitaver theorem: If D is an orientation of a graph G, then D has a directed path of length $\chi(G) - 1$.

   (b) Prove that every tournament has a directed Hamiltonian path.

   (c) Let G be an undirected graph. Describe how to obtain an orientation D of G so that the length of a longest directed path in D is minimized.

6. Let (G, w) be a connected, weighted graph in which all weights are positive. Let $x \in V(G)$. Describe Dijkstra's algorithm and prove that it finds the length a shortest path from x to v for all $v \in V(G) \setminus \{x\}$.

7. Let G be a graph. State three different conditions which are equivalent to the statement "*G is a tree*", and prove that the *four* statements are equivalent.

8. (a) State the theorem of Berge that characterizes maximum matchings.

   (b) Let G be a graph with two different perfect matchings. Prove that G contains a cycle.

   (c) Give an example to show that the statement in (b) is not true if "perfect matchings" is replaced by "maximum matchings"

9. (a) Define a *network*, a *flow* in a network, a *cut* in a network, and an *augmenting (or incrementing) path for a flow f.*

   (b) State and prove the Max-flow Min-cut Theorem.

   (c) Prove that if the capacities are integers, then there is a maximum flow, f, which is integer valued (i.e. $f(e) \in \mathbb{Z}$ for every arc e).

10. Prove that the following statements are equivalent for a connected graph G.

   (a) G is Eulerian (i.e. G has a closed Euler trail).

   (b) The edge set of G can be decomposed into edge-disjoint cycles.

   (c) Every minimal edge-cut of G contains an even number of edges.

   (d) Every vertex of G has even degree.