Note on Expressing Multiplicities as Matrix Vector Product in Subclass of Trees

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Let T be a tree with diameter d and r internal vertices. Let $m(x)$ be the multiplicity of distance x in T. Let I_T be the tree induced by the internal vertices of T. Suppose that T satisfies the property that I_T has max degree 3 and at most one vertex with degree 3, with the remaining vertices having degree at most 2. Note that T can be any caterpillar as well as a tree constructed from three caterpillars joined at the ends of their internal trees. Note that a tree T has the corollary property that the multiplicities of distances in I_T never exceed r. These properties turn out to be important for representing the multiplicity vector $\mathbf{m} = [m(3), m(4), \dots, m(d)]^T$ of T as a matrix vector product.

Observation Recall for any distance $x \in [3, d]$, we have

$$m(x) = \sum_{\substack{\{u,v\} \in \binom{V(T)}{2} \\ d(u,v) = x-2}} (\deg(u) - 1)(\deg(v) - 1).$$

Note that this is just a dot product of internal vertex degree vectors. Since the max multiplicity of distances in I_T is at most r, there are at most r terms (possibly with value 0) in $m(x)$.

Making the vector and matrix Suppose we order (in some generally standardized way) the internal vertices of T as $u_1, u_2, \dots, u_r$. Then set $a_i := \deg(u_i) - 1$. Define the vector $\mathbf{v} = [a_1, a_2, \dots, a_r]^T$. Then there exists a $(d-2) \times (r)$ matrix B with entries coming from the set $\{0, a_1, a_2, \dots, a_r\}$ such that $\mathbf{m} = B\mathbf{v}$.

The (i, j) entry of B corresponds to paths of length i in I_T involving vertex u_j. We have that $B_{i,j} = 0$ iff there does not exist a path of length i involving u_j in I_T. Otherwise, if $B_{i,j} = a_k$ for some $k \in [r]$, then there exists a path of length i between u_j and u_k in I_T. The special degree property of T ensures that the distance i graph of I_T has max degree 2, which means it's a union of paths and triangles (i.e. there are no cliques of order greater than 3). This means that every path of length i in I_T can be accounted for as a unique non-zero entry of B. So, it is always possible to place the a_ks in the appropriate entries of B to produce the desired dot products with $\mathbf{v}$. Thus all of the multiplicity for distance $i + 2$ is accounted for in $B\mathbf{v}$.

Idea Suppose we are given $\mathbf{v}$. Then search the possible ways to assign values from $\{0, a_1, a_2, \dots, a_r\}$ to entries of B to see how B transforms the degree vector $\mathbf{v}$ into a distance multiplicity vector.