Comprehensive Examination in Graph Theory 2006

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Duration: 3 hours

Instructions: There are ten questions, all of equal value. Do any *seven* questions. Write your answers in the booklets provided.

1. (a) State and prove Euler's formula for connected plane graphs.

   (b) Use (a) to show that K_5 is not planar.

   (c) State Kuratowski's theorem and use it to show that the Petersen graph is not planar.

   (d) Show that a plane graph is 2-face colourable if every vertex has even degree. Is the converse true? Explain.

2. Let $A(G)$ be the adjacency matrix of graph G. Prove that if the eigenvalues of $A(G)$ are all distinct, then the automorphism group of G is abelian.

3. Prove the theorem of Gallai and Milgram which states that the vertices of a digraph D can be covered using at most $\alpha(D)$ disjoint paths, where $\alpha(D)$ is the maximum size of an independent set.

4. Prove that any graph G has a spanning bipartite subgraph H such that, for every vertex x, $d_H(x) \geq \frac{1}{2}d_G(x)$.

5. Prove the theorem of Roy and Gallai which states that a digraph D contains a directed path of length $\chi - 1$, where χ is the chromatic number of the underlying graph of D.

6. A graph G is *minimally* (resp. *critically*) k*-connected* if for every edge e (resp. vertex v), $G - e$ (resp. $G - v$) is not k-connected. Prove that if G has at least three vertices and is minimally (resp. critically) 2-connected, then G contains a vertex of degree two. (There are two questions to be answered.)

7. Let $G = (V, E)$ be a connected graph which is not a tree, i.e., which contains at least one cycle. Suppose that $f(v) : V \to \mathbf{N}$ is a function. Show that G must contain a spanning subgraph H such that every vertex v has degree $d_H(v)$ (in H) distinct from $f(v)$. What can you say if G is a tree?

8. (a) State Ore's Theorem and Dirac's Theorem concerning Hamilton cycles in graphs.

   (b) Let G be a graph with n vertices, and let x and y be non-adjacent vertices of G such that
   $$d(x) + d(y) \geq n.$$
   Prove that G is hamiltonian if and only if $G + xy$ is hamiltonian.

   (c) Use the result in (b) to prove the two theorems in (a).

9. (a) State Hall's Theorem, König's Theorem, and Tutte's Theorem (all regarding matchings).

   (b) Prove that if G is a graph with n vertices and $\delta(G) \geq 1$, then

$$\alpha' + \beta' = n,$$

   where α' is the size of a maximum matching and β' is the size of a minimum edge cover in G.

10. A *kernel* in a digraph D is an independent set S such that for every vertex $u \notin S$ there exists a vertex $v \in S$ such that uv is an arc of D. Show that if D contains no directed cycle of odd length then D contains a kernel.