Comprehensive Examination in Graph Theory

Time allowed: 3 hours

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This examination paper has **two** pages and **ten** questions, all of equal value.

Do any **seven** questions. Write your answers in the booklets provided.

All undirected graphs considered in this examination should be understood to be finite and simple.

1. A graph is *minimally k-connected* if G is k-connected and, for every edge e of G, $\kappa(G-e) < \kappa(G)$. Prove that if G is minimally 2-connected, then G must have a vertex of degree 2.

2. Show that every 3-regular graph without cut-edges can be decomposed into copies of P_3 (the path of *length* 3).

3. Prove that every chordal graph contains two nonadjacent simplicial vertices, unless it is a complete graph.

4. (a) Prove the following theorem of Brooks:

> *If G is a connected graph that is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

(b) Discuss the sharpness of Brooks' Theorem.

(c) Use Brooks Theorem to show that if $\Delta = 3$ then the edge-chromatic number of G is at most 4.

5. Prove that every vertex of a strong (ı.e. diconnected) tournament T with $n \geq 3$ vertices is contained in in a directed k-cycle, $3 \leq k \leq n$.

6. (a) State Euler's Theorem relating the number of vertices, edges and faces of a connected planar graph.

(b) Prove that every connected planar graph G with $n \geq 3$ vertices has at most $3n-6$ edges.

(c) Without appealing to Kuratowski's Theorem, prove that $K_{3,3}$ is non-planar.

7. (a) State Tutte's Theorem on the existence of 1-factors.

(b) Prove that every cubic graph without cut edges has a 1-factor.

(c) Use Tutte's Theorem to establish whether or not the following graph has a 1-factor. (Or, you could if I had X-fig on this computer ...)

8. (a) State the Max-flow Min-cut Theorem. Define all terms necessary to make your statement meaningful.

(b) Show that in any network N with integer capacities there is a maximum flow f such that $f(a)$ is an integer for every arc a.

(c) Show how to use the Max-flow Min-cut Theorem to find a maximum matching in a bipartite graph G.

9. Prove that every graph G with n vertices and minimum degree $\delta > 1$ has a dominating set of size at most $\frac{n(1+\ln(\delta+1))}{\delta+1}$.

10. (a) Prove Cayley's Theorem that the number of spanning trees of the complete graph with n vertices is n^{n-2}.

(b) Determine the number of non-isomorphic spanning trees of K_4.