5 Distance Multisets of Specific Graphs

5.1 Paths, Cycles, and Grid Graphs

Observation 1. Paths of order n have distance multiset $\{1^{n-1}, 2^{n-2}, \ldots, (n-1)^1\}$.

Observation 2. Odd cycles of order n have distance multiset $\{1^n, 2^n, \ldots, \lfloor \frac{n}{2} \rfloor^n\}$. Even cycles of order n have distance multiset $\{1^n, 2^n, \ldots, (\frac{n}{2}-1)^n, (\frac{n}{2})^{\frac{n}{2}}\}$.

Proposition 5.1. *Let a and b be positive integers such that $a \geq b$. Then the multiplicity of each distance x in the a by b grid graph is given by*

$$\mathrm{mult}(x) = \begin{cases} (2bx - x^2)a + 2\binom{x+1}{3} - bx^2, & \textit{if } x \in [1, b-1]; \\ b^2(a-x) + 2\binom{b+1}{3}, & \textit{if } x \in [b, a-1]; \\ 2\binom{b+1-(x-a)}{3}, & \textit{if } x \in [a, a-1+b-1]. \end{cases}$$

Proof. Let $G_{a,b}$ be the grid graph with dimensions a and b. We first find a recurrence relation by calculating the distances between two vertex disjoint induced subgraphs of $G_{a,b}$: $G_{a,b-1}$ and a path of length $a-1$, which we call P_a. We call such distances "external distances", and the distances between vertices within $G_{a,b-1}$ and within P_a are called "internal distances". Throughout this proof, for a distance $x \in [1, a-1+b-1]$, we often distinguish between the internal and external multiplicities of x. It is straightforward to see that the distances of $G_{a,b}$ are the union of external and internal distances between $G_{a,b-1}$ and P_a. Once we know how to find these external distances, we will use this relation in a proof by induction to obtain the closed form expression for the distance multiset of $G_{a,b}$.

Let $v_1, v_2, \ldots, v_a$ be the vertices of P_a, and let $P(j)$ denote the path subgraph of $G_{a,b}$ with length $b-1$ such that v_j is one of its end-vertices. Note that $\{P(j) : j \in [a]\}$ partitions $V(G_{a,b})$ and there are a^2 ways to pair these paths with the vertices of P_a.

For a distance $x \in [1, b-1]$, there are $2(a-d)$ ordered pairs $(i,j) \in [a]^2$ such that $|i-j| = d$, and each of these corresponds to an external distance x. The external multiplicity of x is given by $a + \sum_{d=1}^{x-1} 2(a-d)$, where the upper limit of $x-1$ ensures that we are ignoring internal distances within P_a. The a term comes from the a pairs (i,j) such that $|i-j| = 0$.

When $x \in [b, a-1]$, the external multiplicity is given by $\sum_{d=x-(b-1)}^{x-1} 2(a-d)$. In this case, we have to ignore smaller differences (*i.e.*, those less than $x-(b-1)$) in P_a since $x \geq b-1$. When $x \in [a, a-1+b-1]$, the external multiplicity of x is $\sum_{d=x-(b-1)}^{a-1} 2(a-d)$. [**Probably should include pictures to illustrate these three cases**] We can simplify these expressions as follows:

Case 1: When $x \in [1, b-1]$,

$$a + \sum_{d=1}^{x-1} 2(a-d) = a + 2a(x-1) - x(x-1) = a + (x-1)(2a-x).$$

Case 2: When $x \in [b, a-1]$,

$$\sum_{d=x-(b-1)}^{x-1} 2(a-d) = 2a(b-1) - 2(b-1)(x-(b-1)) - (b-1)(b-2) = (b-1)(2a-2x+b).$$

Case 3: When $x \in [a, a-1+b-1]$,

$$\sum_{d=x-(b-1)}^{a-1} 2(a-d) = 2a(a-x+b-1) - 2(a-x+b-1)(x-(b-1)) - (a-x+b-1)(a-x+b-2)$$
$$= (a-x+b-1)(2a-2(x-b+1)-a+x-b+2)$$
$$= (a+b-x-1)(a+b-x)$$

Altogether, the external distances between $G_{a,b-1}$ and P_a are:

$$\text{mult}(x)_{Ext} = \begin{cases} a+(x-1)(2a-x), & \text{if } x \in [1, b-1]; \\ (b-1)(2a-2x+b), & \text{if } x \in [b, a-1]; \\ 2\binom{a+b-x}{2}, & \text{if } x \in [a, a-1+b-1]. \end{cases}$$

Now we use this to find the distances in $G_{a,b}$. Below are the distance multisets of $G_{a,2}$, $G_{a,3}$, $G_{a,4}$, and $G_{a,5}$.

$G_{a,2} \Rightarrow \{1^{3a-2}\} \cup \{x^{4(a-x)+2} : x \in [2, a-1]\} \cup \{a^2\}$.
$G_{a,3} \Rightarrow \{1^{5a-3}, 2^{8a-10}\} \cup \{x^{9(a-x)+8} : x \in [3, a-1]\} \cup \{a^8, (a+1)^2\}$.
$G_{a,4} \Rightarrow \{1^{7a-4}, 2^{12a-14}, 3^{15a-28}\} \cup \{x^{16(a-x)+20} : x \in \{4, a-1\}\} \cup \{a^{20}, (a+1)^8, (a+2)^2\}$.
$G_{a,5} \Rightarrow \{1^{9a-5}, 2^{16a-18}, 3^{21a-37}, 4^{24a-60}\} \cup \{x^{25(a-x)+40} : x \in [5, a-1]\} \cup \{a^{40}, (a+1)^{20}, (a+2)^8, (a+3)^2\}$.

We claim that the distance multiset of $G_{a,b}$ is given by

$$\text{mult}(x) = \begin{cases} (2bx-x^2)a + 2\binom{x+1}{3} - bx^2, & \text{if } x \in [1, b-1]; \\ b^2(a-x) + 2\binom{b+1}{3}, & \text{if } x \in [b, a-1]; \\ 2\binom{b+1-(x-a)}{3}, & \text{if } x \in [a, a-1+b-1]. \end{cases}$$

This is our inductive hypothesis. Note that the base cases are satisfied by our claimed solution. We now begin the inductive step of the proof. By the inductive hypothesis, the cumulative internal distance multiplicities of $G_{a,b-1}$ and P_a for distance x are

$$\text{mult}(x)_{Int} = \begin{cases} (2(b-1)x-x^2)a + 2\binom{x+1}{3} - (b-1)x^2 + (a-x), & \text{if } x \in [1, b-2]; \\ (b-1)^2(a-x) + 2\binom{b}{3} + (a-x), & \text{if } x \in [b-1, a-1]; \\ 2\binom{b-(x-a)}{3}, & \text{if } x \in [a, a-1+b-2]. \end{cases}$$

Note that we simply add $(a-x)$ (the internal multiplicity of x within P_a) to the internal multiplicities of $G_{a,b-1}$ when $x \in [1, a-1]$. We showed earlier that the external distance multiplicities between $G_{a,b-1}$ and P_a are

$$\text{mult}(x)_{Ext} = \begin{cases} a+(x-1)(2a-x), & \text{if } x \in [1, b-1]; \\ (b-1)(2a-2x+b), & \text{if } x \in [b, a-1]; \\ 2\binom{a+b-x}{2}, & \text{if } x \in [a, a-1+b-1]. \end{cases}$$

Now all we do is add these multiplicities together. When $x \in [a, a-1+b-1]$, we have

$$\begin{aligned}
\mathrm{mult}(x) &= 2\binom{b-(x-a)}{3} + 2\binom{a+b-x}{2} \\
&= \frac{(b-x+a)(b-x+a-1)(b-x+a-2)}{3} + (b-x+a)(b-x+a-1) \\
&= (b-x+a)(b-x+a-1)(\frac{b-x+a-2}{3}+1) \\
&= (b-x+a)(b-x+a-1)(\frac{b-x+a+1}{3}) \\
&= 2\binom{b+1-(x-a)}{3},
\end{aligned}$$

as desired.

When $x \in [b, a-1]$, we have

$$\begin{aligned}
\mathrm{mult}(x) &= (b-1)^2(a-x) + 2\binom{b}{3} + (a-x) + (b-1)(2a-2x+b) \\
&= (b-1)(a-x)(b-1+2) + b(b-1) + (a-x) + 2\binom{b}{3} \\
&= (b-1)(a-x)(b+1) + b(b-1) + (a-x) + \frac{b(b-1)(b-2)}{3} \\
&= (a-x)(1+(b-1)(b+1))b(b-1) + \frac{b(b-1)(b-2)}{3} \\
&= b^2(a-x) + b(b-1)(1+\frac{b-2}{3}) \\
&= b^2(a-x) + \frac{(b+1)b(b-1)}{3} \\
&= b^2(a-x) + 2\binom{b+1}{3},
\end{aligned}$$

as desired.

When $x = b-1$, we have

$$\begin{aligned}
\mathrm{mult}(x) &= (b-1)^2(a-x) + 2\binom{b}{3} + (a-x) + a + (x-1)(2a-x) \\
&= (b-1)^2(a-x) + 2\binom{b}{3} + a - x + a + 2ax - x^2 - 2a + x \\
&= (b-1)^2(a-x) + 2\binom{b}{3} + 2ax - x^2 \\
&= x^2(a-x) + 2\binom{x+1}{3} + 2ax - x^2 \\
&= a(x^2+2x) + 2\binom{x+1}{3} - x^2(x+1) \\
&= (x^2+2x)a + 2\binom{x+1}{3} - bx^2 \\
&= (2(x+1)x - x^2) + 2\binom{x+1}{3} - bx^2 \\
&= (2bx - x^2)a + 2\binom{x+1}{3} - bx^2,
\end{aligned}$$

as desired.

When $x \in [1, b-2]$, we have

$$\begin{aligned}
\mathrm{mult}(x) &= (2(b-1)x - x^2)a + 2\binom{x+1}{3} - (b-1)x^2 + a + (x-1)(2a-x) + (a-x) \\
&= a(2(b-1)x - x^2) + 2\binom{x+1}{3} - bx^2 + x^2 + a + 2ax - x^2 - 2a + x + a - x \\
&= a(2(b-1)x - x^2) + 2\binom{x+1}{3} - bx^2 + 2ax \\
&= a(2(b-1)x - x^2 + 2x) + 2\binom{x+1}{3} - bx^2 \\
&= a(2bx - x^2) + 2\binom{x+1}{3} - bx^2,
\end{aligned}$$

as desired. Thus the proposition holds. □

I suspect the distance multisets of $C_a \times P_b$ and $C_a \times C_b$ are actually simpler to determine than $P_a \times P_b$ above. These would be fun next steps. It should be doable to find the distance multisets of a d dimensional grid, certainly for $d = 3$, but maybe even general d. The expressions would be pretty nasty, but I think it's doable by a more general recursion and induction approach.