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Note

# The Hosoya polynomial of distance-regular graphs

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ABSTRACT

In this note we obtain an explicit formula for the Hosoya polynomial of any distance-regular graph in terms of its intersection array. As a consequence, we obtain a very simple formula for the Hosoya polynomial of any strongly regular graph.



## 1. Introduction

Throughout this paper $G=(V,E)$ denotes a connected, simple and finite graph with vertex set $V=V(G)$ and edge set $E=E(G)$.

The *distance* $d(u,v)$ between two vertices $u$ and $v$ is the minimum of the lengths of paths between $u$ and $v$. The *diameter* $D$ of a graph $G$ is defined as

$$D := \max_{u,v\in V(G)}\{d(u,v)\}.$$

The *Wiener index* $W(G)$ of a graph $G$ with vertex set $\{v_1, v_2, \ldots, v_n\}$, defined as the sum of distances between all pairs of vertices of $G$,

$$W(G) := \frac{1}{2}\sum_{i=1,j=1}^{n} d(v_i, v_j),$$

is the first mathematical invariant reflecting the topological structure of a molecular graph.

This topological index has been extensively studied; for instance, a comprehensive survey on the direct calculation, applications, and the relation of the Wiener index of trees with other parameters of graphs can be found in [5]. Moreover, a list of 120 references of the main works on the Wiener index of graphs can be found in the referred survey.

The Hosoya polynomial of a graph was introduced in Hosoya's seminal paper [8] in 1988, and was there named "Wiener polynomial". It received a lot of attention afterwards. Some authors, e.g. Sagan, Yeh, and Zhang [13], continue to call it by its original name, but the later proposed name [10] "Hosoya polynomial" is nowadays used by the vast majority of researchers. The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance based

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graph invariants. For instance, knowing the Hosoya polynomial of a graph, it is straightforward to determine the Wiener index of a graph as the first derivative of the polynomial at the point $t = 1$. Cash [3] noticed that the hyper-Wiener index can be obtained from the Hosoya polynomial in a similar simple manner. A comprehensive survey on the main properties of the Hosoya polynomial, which includes a wide list of bibliographic references, can be found in Gutman et al. [7].

Let $G$ be a connected graph of diameter $D$ and let $d(G, k),\ k \geq 0$, be the number of vertex pairs at distance $k$. The *Hosoya polynomial* of $G$ is defined as

$$H(G, t) := \sum_{k=1}^{D} d(G, k) \cdot t^k.$$

As we pointed out above, the Wiener index of a graph $G$ is determined as the first derivative of the polynomial $H(G, t)$ at $t = 1$, *i.e.*,

$$W(G) = \sum_{k=1}^{D} k \cdot d(G, k).$$

The Hosoya polynomial has been obtained for trees, composite graphs, benzenoid graphs, tori, zig-zag open-ended nanotubes, certain graph decorations, armchair open-ended nanotubes, zigzag polyhex nanotorus, $TUC_4C_8(S)$ nanotubes, pentachains, polyphenyl chains, the circumcoronene series, Fibonacci and Lucas cubes, Hanoi graphs, etc., see the recent papers [4,6,9,11,14].

In this note we obtain an explicit formula for the Hosoya polynomial of any distance-regular graph. As a consequence, we obtain a very simple formula for the Hosoya polynomial of any strongly regular graph.

## 2. The Hosoya polynomial of distance-regular graphs

A *distance-regular* graph is a regular connected graph with diameter $D$, for which the following holds. There are natural numbers $b_0, b_1, \ldots, b_{D-1}, c_1 = 1, c_2, \ldots, c_D$ such that for each pair $(u, v)$ of vertices satisfying $d(u, v) = j$ we have

(1) the number of vertices in $G_{j-1}(v)$ adjacent to $u$ is $c_j (1 \leq j \leq D)$,
(2) the number of vertices in $G_{j+1}(v)$ adjacent to $u$ is $b_j (0 \leq j \leq D - 1)$,

where $G_i(v) = \{u \in V(G) : d(u, v) = i\}$. The array $\{b_0, b_1, \ldots, b_{D-1}; c_1 = 1, c_2, \ldots, c_D\}$ is the *intersection array* of $G$.

Classes of distance-regular graphs include complete graphs, cycle graphs, Hadamard graphs, hypercube graphs, Kneser graphs $K(n, 2)$, odd graphs and Platonic graphs [1,2].

**Theorem 1.** *Let $G$ be a distance-regular graph whose intersection array is*

$$\{b_0, b_1, \ldots, b_{D-1}; c_1 = 1, c_2, \ldots, c_D\}.$$

*Then we have*

$$H(G, t) = \frac{nb_0}{2} \left( t + \sum_{i=2}^{D} \frac{\prod_{j=1}^{i-1} b_j}{\prod_{j=2}^{i} c_j} \cdot t^i \right).$$

**Proof.** For any vertex $v \in V(G)$, each vertex of $G_{i-1}(v)$ is joined to $b_{i-1}$ vertices in $G_i(v)$ and each vertex of $G_i(v)$ is joined to $c_i$ vertices in $G_{i-1}(v)$. Thus

$$|G_{i-1}(v)| b_{i-1} = |G_i(v)| c_i. \tag{1}$$

Hence, it follows from (1) that the number of vertices at distance $i$ of a vertex $v$, namely $|G_i(v)|$, is obtained directly from the intersection array

$$|G_i(v)| = \frac{\prod_{j=0}^{i-1} b_j}{\prod_{j=2}^{i} c_j} (2 \leq i \leq D) \quad \text{and} \quad |G_1(v)| = b_0. \tag{2}$$

Now, since, $d(G,i) = \frac{1}{2}\sum_{v\in V(G)} |G_1(v)|$ and the value $|G_1(v)|$ does not depend on $v$, we obtain the following:

$$d(G,i) = \frac{n\prod_{j=0}^{i-1} b_j}{2\prod_{j=2}^{i} c_j} (2 \le i \le D) \quad \text{and} \quad |G_1(v)| = \frac{nb_0}{2}. \tag{3}$$

Therefore, the result is a direct consequence of the definition of the Hosoya polynomial. □

As an example, the hypercubes $Q_k$, $k \ge 2$, are distance-regular graphs whose intersection array is $\{k, k-1, \dots, 1; 1, 2, \dots, k\}$, [1]. Thus, from Theorem 1 we obtain that the Hosoya polynomial of the hypercube $Q_k$ is

$$H(Q_k, t) = 2^{k-1}\sum_{i=1}^{k}\binom{k}{i} t^i = 2^{k-1}\left((t+1)^k - 1\right).$$

As a direct consequence of Theorem 1 we deduce the formula on the Wiener index of a distance-regular graph, which was previously obtained in [12] for the general case of hypergraphs.

**Corollary 2** (*[12]*). *Let G be a distance-regular graph whose intersection array is*

$$\{b_0, b_1, \dots, b_{D-1}; c_1 = 1, c_2, \dots, c_D\}.$$

*Then we have*

$$W(G) = \frac{nb_0}{2}\left(1 + \sum_{i=2}^{D} i \frac{\prod_{j=1}^{i-1} b_j}{\prod_{j=2}^{i} c_j}\right).$$

A graph is said to be *k-regular* if all vertices have the same degree $k$. A $k$-regular graph $G$ of order $n$ is said to be *strongly regular*, with parameters $(n, k, \lambda, \mu)$, if the following conditions hold. Each pair of adjacent vertices has the same number $\lambda \ge 0$ of common neighbours, and each pair of non-adjacent vertices has the same number $\mu \ge 1$ of common neighbours (see, for instance, [1]). A distance-regular graph of diameter $D = 2$ is simply a strongly regular graph. In terms of the intersection array $\{b_0, b_1; 1, c_2\}$ we have that $\lambda = k - 1 - b_1$ and $\mu = c_2$, *i.e.*, the intersection array of any strongly regular graph with parameters $(n, k, \lambda, \mu)$ is $\{k, k-\lambda-1; 1, \mu\}$. Thus, as a consequence of Theorem 1 we deduce the following result.

**Corollary 3.** *Let G be a strongly regular graph with parameters* $(n, k, \lambda, \mu)$. *Then we have*

$$H(G,t) = \frac{nk}{2}\left(t + \frac{k-\lambda-1}{\mu}\cdot t^2\right).$$

It is well-known that the parameters $(n, k, \lambda, \mu)$ of any strongly regular graph are not independent and must obey the following relation:

$$(v-k-1)\mu = k(k-\lambda-1).$$

As a result, we can express the Hosoya polynomial of any strongly regular graph in the following manner

$$H(G,t) = \frac{n}{2}\left(kt + (n-k-1)t^2\right);$$

this is not surprising because for every vertex $x$ there are $k$ vertices at distance 1 from $x$ and $n-k-1$ at distance 2 (since a strongly regular graph has diameter $D = 2$).

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