University of Victoria

Department of Mathematics and Statistics

Comprehensive Exam in Graph Theory

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TIME: 3 HOURS.

This question paper has 10 questions on two pages. Each question is worth 10 marks.
Do any seven questions. A minimum of 40 marks is required to pass.

1. (a) Describe three methods for constructing a minimum weight spanning tree of an edge-weighted graph (G, w).

   (b) Prove correctness of any one of the methods you stated in (a).

2. (a) State and prove the max-flow min-cut theorem.

   (b) Prove that if the capacity function of the network N is integer valued, then there exists a maximum flow which is integer valued.

3. (a) State each of the following theorems: Berge's Theorem that characterizes maximum matchings, König's Theorem about vertex coverings of bipartite graphs, and Tutte's Theorem that characterizes graphs having a perfect matching.

   (b) Prove any one of the above theorems.

4. Prove Moon's Theorem that every vertex of a strong tournament on n vertices belongs to a directed cycle of each length k, $3 \leq k \leq n$.

5. (a) Let G be a graph with at least three vertices. Prove that if $\kappa(G) \geq \beta(G)$, where $\beta(G)$ denotes the independence number of G, then G is hamiltonian. State any theorem(s) you use.

   (b) Construct an infinite class of nonhamiltonian graphs G such that $\kappa(G) = \beta(G) - 1$.

6. Prove that, for any graph G, $\kappa \leq \kappa' \leq \delta$.

7. (a) A university has a collection of exams to schedule into a collection of rooms and time slots subject to the following constraints:
    - No student or professor has two exams in the same time slot.
    - Each exam is scheduled into a room with at least twice as many seats as students.
    - There is at most one exam scheduled into any room in any time slot.

    Assuming that the largest room available has at least twice as many seats as students registered in the largest class holding an exam, explain how to model this problem using list colourings of graphs. It can be assumed that every room is available in every time slot.

    (b) Prove that if each vertex x of a tree is assigned a list $\ell(x)$ of two colours, where the lists need not be identical, then there exists a proper colouring of the tree in which every vertex is assigned a colour from its list.

    (c) Show that the result in (b) does not hold for all bipartite graphs.

8. (a) Prove that, for $n \geq 1$, the edges of K_{2n} can be partitioned into Hamilton paths.

    (b) Prove that, for $n \geq 1$, the edges of K_{2n+1} can be partitioned into Hamilton cycles.

    (c) Prove that every cubic graph with no cut edges has a 2-factor.

9. (a) Prove that a nonempty graph G is chordal if and only if G is complete or G can be obtained from two chordal graphs G_1 and G_2 of orders less than the order of G by identifying two cliques of the same order in G_1 and G_2.

    (b) Prove that if G is chordal, then G has equal clique and chromatic numbers.

10. Prove that if every vertex of G has even degree, then the edges of G can be oriented so that, at each vertex, the indegree equals the outdegree.