University of Victoria
Department of Mathematics and Statistics
Comprehensive Exam in Graph Theory

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TIME: 3 HOURS.

Each question is worth 10 marks. A minimum of 40 marks, and four questions substantially correct, is required to pass. Unless explicitly stated otherwise, any use of the word *graph* in what follows should be understood to mean *finite, simple, undirected graph*.

1. (a) Prove that the automorphism group of a graph G equals the automorphism group of $\overline{G}$.

   (b) For each positive integer $n \geq 3$, find, with proof, a graph G_n whose automorphism group is the cyclic group $\mathbb{Z}_n$.

   (c) Find, with proofs, an example of a graph which is vertex-transitive but not edge-transitive, and and example of a graph which is edge-transitive but not vertex-transitive.

2. The *tree graph* of a connected graph G with n vertices has as its vertices the spanning trees of G, with vertex T_i adjacent to vertex T_j if and only if T_i and T_j have $n-2$ edges in common. Prove that the tree graph of a connected graph is connected.

3. (a) Prove that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ for every graph G, where κ and κ' are the connectivity and edge-connectivity, respectively.

   (b) Prove that if a graph G has $n \geq 3$ vertices and $\kappa(G) \geq \alpha(G)$ (the independence number), then G is Hamiltonian.

   (c) For each $n \geq 6$, give an example of a graph G on n vertices for which the converse of the result in (c) does not hold.

4. (a) State and prove the theorem of Berge that characterizes maximum matchings.

   (b) Prove that if G is a bipartite graph with bipartition (X, Y), then the number of edges in a maximum matching of G is

   $$|X| - \max_{S \subseteq X} \{|S| - |N(S)|\}.$$

5. Let $T_{n,r}$ be the complete r-partite graph with n vertices that has b parts of size $\alpha + 1$ and $r - b$ parts of size α, where $\alpha = \lfloor n/r \rfloor$ and $b = n - r\alpha$.

   Show that $T_{n,r}$ is the unique graph of with n vertices, the maximum number of edges, and no $(r+1)$-clique.

6. (a) Let (X, Y) be a partition of $V(G)$ such that the induced subgraphs $G[X]$ and $G[Y]$ are both k-colourable. Show that if the edge cut $[X, Y]$ has at most $k-1$ edges, then G is also k-colourable.
   (b) Deduce that every k-critical graph is $(k-1)$-edge connected.

7. (a) Let G be a connected plane graph with n vertices, m edges and r regions. State and prove Euler's identity that relates n, m and r.
   (b) Deduce that if G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n-6$.
   (c) State and prove a necessary condition for a planar graph to be Hamiltonian.

8. (a) Prove the theorem of Gallai, Roy, Hasse and Vitaver that if D is a directed graph with underlying graph G, then D has a directed path of length $\chi(G)-1$.
   (b) Prove that every tournament has a directed Hamilton path.

9. (a) State the Max-flow Min-cut theorem.
   (b) Prove that if the network N has integer (arc) capacities, then there is a maximum flow f such that $f(a)$ is an integer for every arc a.

10. (a) Prove that any induced subgraph of a chordal graph is chordal.
    (b) Prove *directly* that if G is chordal, then $\chi(G) = \omega(G)$, where $\omega(G)$ is the clique number of G. Do not use the Perfect Graph Theorem, the Strong Perfect Graph Theorem, or the fact that chordal graphs are perfect.
    (c) Deduce that chordal graphs are perfect.