Combinatorics Comprehensive Exam Preparation

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Contents

1 Topic Notes

1.1 Important Proofs

1.1.1 Pigeonhole Principle

Theorem 1.1. *If n objects are placed into k containers, then at least one container contains at least $\lceil n/k \rceil$ objects and at least one container contains at most $\lfloor n/k \rfloor$ objects.*

Proof. We distribute the n objects uniformly in the k containers. If $k \mid n$, then the statement conclusion follows. If $k \nmid n$, then there is a container with more than n/k objects and at least one container with fewer than n/k objects. $\square$

Theorem 1.2. *If $k^2 + 1$ points are placed in an equilateral triangle with side lengths k, then there are at least two points at distance less than 1.*

Theorem 1.3. *If $k^d + 1$ points are placed in a d-dimensional hypercube with side lengths k, then there are at least two points at distance less than $\sqrt{d}$.*

Theorem 1.4 (Erdős-Szekeres). *Let v, m, n be positive integers such that $v > mn$. Let $a_1, a_2, \ldots, a_v$ be a sequence of distinct real numbers such that the number of terms of every decreasing subsequence is at most m. Then there exists an increasing subsequence of more than n terms.*

Proof (A. Seidenberg 1959). To each a_i assign a pair (m_i, n_i), where m_i is the largest length of a decreasing subsequence beginning at a_i and n_i the same but for increasing subsequences. For each $i < j$, (m_i, n_i) and (m_j, n_j) are distinct. For otherwise if $a_i < a_j$ then $n_i > n_j$, and if $a_j > a_i$ then $m_i > m_j$. Thus there are $v > mn$ such distinct pairs. But by pigeonhole principle, if each $n_i \leq n$, then since each $m_i \leq m$, there would be at most mn distinct pairs, a contradiction. □

1.1.2 Principle of Inclusion and Exclusion

Theorem 1.5. *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left|\bigcap_{i \in I} A_i\right|.$$

Proof. Since every element in $\bigcup_{i=1}^{n} A_i$ is counted at least once in the RHS, we show that in fact each element is counted at most once. Suppose $x \in \bigcup_{i=1}^{n} A_i$ belongs to t of the sets $A_1, A_2, \ldots, A_n$ where $1 \leq t \leq n$. Then on the RHS, x is counted t times when $k = 1$, $\binom{t}{2}$ times when $k = 2$, and in general $\binom{t}{k}$ times for all $1 \leq k \leq t$. By the binomial theorem,

$$-\sum_{k=1}^{t} (-1)^k \binom{t}{k} = -\left[(1 + (-1))^t - \binom{t}{0}(-1)^0\right] = 1.$$

So the RHS also counts x exactly once. □

Theorem 1.6. *The number of onto functions from $[n]$ to $[m]$ is*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

Proof. The term in the sum at $k = 0$ is m^n, which is the number of all functions from $[n]$ to $[m]$, so the remaining terms count the number of non-onto functions. A non-onto function is one that doesn't map to an element in the codomain $[m]$, so $\binom{m}{k}$ counts the number of k-subsets of elements in the codomain for $(m-k)^n$ functions to ignore. So for each $1 \leq i \leq m$, the set A_i is the number of functions that ignore element $i \in [m]$, and their union is the number of non-onto functions. By PIE,

$$\left|\bigcup_{i=1}^{m} A_i\right| = -\sum_{k=1}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

So the number of onto functions is m^n less this quantity. □

Theorem 1.7. *The number of derangements in $\mathcal{S}_n$ is*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

Proof (See proof about onto functions). The proof is the same as the onto function proof above. When $k = 0$ we count the total number of objects with and without the property of interest. Then for $k \geq 1$, we only count the objects without the POI, apply PIE and subtract this from the total. At $k = 0$ we have all $n!$ permutations, then for each of the $\binom{n}{k}$ ways to fix k points, there are $(n-k)!$ permutations with these fixed points. These $(n-k)!$ permutations include permutations with more than k fixed points, which is why we are using PIE. $\square$

1.1.3 Möbius Inversion

1.1.4 Mirsky's Theorem

1.1.5 Dilworth's Theorem

1.1.6 Hall's Theorem

Theorem 1.8. *Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be subsets of $[n]$. Then $\mathcal{F}$ has an SDR if and only iffor every $I \in \binom{[m]}{k}$, $|\bigcup_{i \in I} A_i| \geq k$.*

1.1.7 König's Theorem

1.1.8 Orbit-Stabilizer Theorem

1.1.9 Burnside's Theorem

1.1.10 Ramsey Theorems

Theorem 1.9 (General Ramsey Theorem)**.** *For $r, k \in \mathbb{Z}^+$, $a_1, a_2, \ldots, a_r \geq k$, $r \geq 2$, there is a least integer $R := R_k(a_1, a_2, \ldots, a_r)$ such that for each $n \geq R$, if the $\binom{n}{k}$ k-subsets of an n-set are partitioned (coloured) into r classes $C_1, C_2, \ldots, C_r$, then some a_i-set has all of its k-subsets in class C_i for some i.*

Theorem 1.10 (Upper Bound 1)**.**

$$R(a, b) \leq R(a-1, b) + R(a, b-1)$$

with strict inequality when both terms on the RHS are even.

Theorem 1.11 (Upper Bound 2)**.**

$$R(a, b) \leq \binom{a+b-2}{a-1}.$$

Theorem 1.12 (Symmetric Ramsey Lower Bound)**.** *Let $k \geq 2$. Then*

$$R(k, k) \geq 2^{k/2}$$

Theorem 1.13. *For every $k, a, b \geq 1$,*

$$R_k(a, b) \leq R_{k-1}(R_k(a-1, b), R_k(a, b-1)) + 1.$$

1.1.11 Designs

Theorem 1.14 (Symmetric Designs). *Let A be the incidence matrix for a symmetric (v,k,λ) design. Then*

$$A^T A = AA^T = (k-\lambda)I + \lambda J.$$

Theorem 1.15 (Bruck-Ryser-Chowla Theorem (Even Case)). *If there exists a symmetric (v,k,λ) design and v is even, then $k-\lambda$ is a square.*

Theorem 1.16 (Bruck-Ryser-Chowla Theorem (Odd Case)). *If there is a symmetric (v,k,λ) design and v is odd, then*

$$z^2 = (k-\lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$$

has a non-trivial solution.

Theorem 1.17. *If $v > k$ and $(V,\mathcal{B})$ is a resolvable (v,k,λ) design, then $b \geq v + r - 1$.*

1.2 Enumeration

1.2.1 Basics

Counting Fact 1 (Number of k-subsets). *The number of k-subsets of a ground set with n elements is $\binom{n}{k}$*

Proof. □

Definition 1.18. The rising factorial function is defined by $[x]^0 = 1$ and $[x]^n = \prod_{i=0}^{n-1}(x+i)$

Definition 1.19. The Stirling numbers of the first kind, denoted $\left[{n \atop k}\right]$ can be defined as the coefficient of x^k in $[x]^n$.

Counting Fact 2 (Stirling Numbers of the First Kind). *The number of permutations in $\mathcal{S}_n$ with k cycles is $\left[{n \atop k}\right]$.*

Counting Fact 3 (Stirling Numbers of the Second Kind). *The number of partitions of an n-set into exactly r nonempty sets is $\left\{{n \atop k}\right\}$.*

Counting Fact 4. *The number of onto functions from an n-set to an k-set is*

$$k!\left\{{n \atop k}\right\}.$$

Proof. □

Counting Fact 5. *The number of functions from an n-set to a k-set is given by*

$$k^n = \sum_{i=1}^{k} \left\{{n \atop i}\right\}\binom{k}{i} i!$$

Proof. □

Definition 1.20 (Bell Number)**.** The n-th Bell number, denoted B_n is the number of unordered partitions of an n-set.

Counting Fact 6.

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

Definition 1.21. The Catalan numbers, denoted by c_n, is given by $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Counting Fact 7 (Catalan Numbers)**.**

a) The number of rooted and ordered binary trees on n vertices is c_n.

b) The number of distinct triangulations of an n-gon is c_{n+2}.

c) The number of lattice paths from $(1,0)$ to $(n+1,n)$ that lie below the line $y = x$ is c_n

Proof. □

Counting Fact 8. *The number of ways that m distinct numbers from the set $\{1, 2, \ldots, n\}$ can be arranged in a circle is*

$$\frac{n!}{m(n-m)!}$$

where arrangements which differ only by rotation are considered the same.

Proof. □

Counting Fact 9. *Stirling's approximation is as follows*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

1.2.2 List of Combinatorial Identities

In this section, I organize a bunch identities that I've come across in upper-level undergraduate enumeration courses and texts. I mainly include combinatorial proofs of each; and if I am aware of others, I include those as well.

Identity 1 (Binomial Theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof (Combinatorial). We prove a weaker version in which we assume x and y are nonnegative integers. For the RHS, we split a domain into two sets – one of size k and the other of size $n-k$, and then map these domains to ranges X and Y of sizes x and y, respectively. We show that ordered pairs of these functions correspond to functions with domain of size n and range of size $x+y$. For each k-subset A, we pair each function f with A as domain and a range X with each function g having $[n] \setminus A$ as domain and range Y. For each pair (f, g), define the function h with domain $\mathcal{D}(f) \cup \mathcal{D}(g) = [n]$ and range $X \cup Y$. There are $(x+y)^n$ such functions of the form of h. □

Identity 2 (Multinomial Theorem)**.**

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$

Proof (Combinatorial). □

Identity 3 (Multinomial Triangle Identity)**.**

$$\binom{n}{n_1, n_2, \ldots, n_r} = \sum_{i=1}^{r} \binom{n-1}{n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r}.$$

Proof (Combinatorial). □

Identity 4.

$$\binom{2n}{n} = (-1)^n 4^n \binom{-1/2}{n}.$$

Proof.

$$\begin{aligned}
(-1)^n \binom{-1/2}{n} &= (-1)^n \frac{(-1/2)(-1/2-1)(-1/2-2)\cdots(-1/2-(n-1))}{n!} \\
&= \frac{(1/2)(1/2+1)(1/2+2)\cdots(1/2+n-1)}{n!} \\
&= \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})\cdots(\frac{2n-1}{2})}{n!} \\
&= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} \\
&= \frac{2\cdot 4\cdot 6\cdots 2n}{2\cdot 4\cdot 6\cdots 2n} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} \\
&= \left(\frac{2\cdot 4\cdot 6\cdots 2n}{2^n n!}\right)\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\right) \\
&= \left(\frac{1}{4^n}\right)\left(\frac{(2n)!}{n!n!}\right) \\
&= \frac{\binom{2n}{n}}{4^n}.
\end{aligned}$$

□

Identity 5 (Pascal)**.**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof (Combinatorial). $\binom{n-1}{k-1}$ counts the number of k-subsets of $[n]$ containing n, and $\binom{n-1}{k}$ counts the number of k-subsets of $[n]$ not containing n. □

Identity 6.

$$\binom{n}{r}\binom{n-r}{k} = \binom{n}{k}\binom{n-k}{r} = \binom{n}{r+k}\binom{r+k}{r}.$$

Proof (Combinatorial). Count ordered triples of subsets of $[n]$ that partition $[n]$. Note $\binom{n}{r+k} = \binom{n}{n-r-k}$, so if A, B, and C are subsets of $[n]$ with r, k, and $n-r-k$ elements, respectively. Then the first expression counts triples of the form (A, B, C) and the latter two expression count triples of the forms (B, A, C) and (C, A, B). We could be annoying here and include the three other equivalent expressions, but nope :). $\square$

Identity 7.

$$\binom{n-1}{m-1} = \sum_{i=0}^{n-m} (-1)^i \binom{n}{m+i}.$$

Proof (Combinatorial). $\square$

Identity 8 (Pascal Triangle Shallow Diagonals Identity)**.** *Let f_n be the n-th Fibonacci number where $f_0 = 0$ and $f_1 = 1$. Then*

$$f_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Proof (Combinatorial). $\square$

Identity 9.

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Proof (Combinatorial). $k\binom{n}{k}$ counts the number of strings of length n over alphabet $\{0, 1, \ldots, n-1\}$ with $n-k$ 0s, first non-zero entry in $[k]$, and the rest of the $k-1$ entries with value 1. Summing over all $1 \leq k \leq n$ counts all strings of length n whose first non-zero entry has a value in $[n]$ and the rest of the $n-1$ entries have value either 0 or 1. $\square$

Identity 10.

$$\sum_{k=1}^{n} k^2\binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}.$$

Proof (Combinatorial). $k^2\binom{n}{k}$ counts the number of length n strings over alphabet $\{0, 1, \ldots, n-1\}$ such that there are $n-k$ 0s, the first two elements are in $[k]$ and the rest have value 1. The number $n2^{n-1}$ counts all strings of length n over $\{0, 1, \ldots, n-1\}$ such that the first non-zero entry has value in $[n-1]$, the second non-zero entry has value 1, and the rest of the $n-2$ entries have value in $\{0, 1\}$. The number $n(n-1)2^{n-2}$, counts the case when the second non-zero entry has value in $[k] \setminus \{1\}$. $\square$

Identity 11.

$$\sum_{k=1}^{n} k^r\binom{n}{k} = \sum_{s=1}^{r} \frac{n!}{(n-s)!}2^{n-s}.$$

Combinatorial Proof. $k^r\binom{n}{k}$ counts the number of strings of length n over alphabet $\{0, 1, \ldots, n-1\}$ such that the first r non-zero entries are in $[k]$, there are $n-k$ entries with value 0, and the rest of the $n-k-r$ entries have value 1. The number $\frac{n!}{(n-s)!}2^{n-s}$ counts the number of length n strings over

$\{0, 1, \ldots, n-1\}$ such that for each $1 \leq j \leq s$, the j-th non-zero entry has value in $[n-1] \setminus [j-1]$, and the rest have value in $\{0, 1\}$. Summing over $1 \leq s \leq r$ counts all strings of length n over $\{0, 1, \ldots, n-1\}$ such that the first r non-zero entries are in $[n-1]$ and the rest are in $\{0, 1\}$. $\square$

Identity 12.

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}.$$

Proof (Combinatorial). $\square$

Identity 13 (Vandermonde's Convolution)**.**

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}.$$

Proof (Combinatorial). $\binom{n}{i}\binom{m}{k-i}$ counts the number of ways of choosing i elements from set N and $k-i$ elements from set M. But summing over all $0 \leq i \leq k$ means we count all combinations of size k from $N \cup M$. Note that when $i = 0$, we get all k-subsets from M (similarly for N when $i = k$), and otherwise we count all k-set combinations between the sets. The distinction we have made between sets N and M has no effect on counting combinations of elements from the two sets. $\square$

Identity 14 (Lattice Path Identity)**.**

$$\binom{n+1}{r+1} = \sum_{i=0}^{n-r} \binom{r+i}{r}.$$

Proof (Combinatorial). $\square$

Identity 15.

$$\sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i} = \frac{2^{n+1}-1}{n+1}.$$

Proof (Combinatorial). Count binary strings of length $n+1$, excluding the all 0s string, with rotational equivalence. $\square$

Identity 16 (Shows up in a Proof of Cayley's Tree Theorem (see proofs from the book))**.**

$$n^{n-k} = \sum_{i=0}^{n-k} \binom{n-k}{i}(n-1)^i.$$

Proof (Combinatorial). $\binom{n-k}{i}(n-1)^i$ counts the number of ways of choosing i of the $n-k$ objects to place into $n-1$ bins, with the remaining $n-k-i$ objects going into another bin. Summing $0 \leq i \leq n-k$ counts the number of ways of placing $n-k$ objects into n bins. $\square$

Identity 17.

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

Proof (Combinatorial). 2^{n-1} is the number of subsets of $[n]$ not containing n, 2^{n-2} is the number of subsets containing n but not $n-1$, 2^{n-3} is the number of subsets of $[n]$ containing n but not $n-1$ nor $n-2$. Thus $\sum_{i=0}^{n} 2^i$ is the number of subsets of $[n]$ containing n, except for the set $[n]$. Note the empty set is counted by 2^{n-1}. Thus all subsets of $[n]$, except $[n]$ itself, are accounted for by the LHS, so the result follows. $\square$

Identity 18 (Stirling Number of the First Kind Triangle Identity)**.**

$$\begin{bmatrix} n \\ r \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ r \end{bmatrix} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}.$$

Proof (Combinatorial). Recall that $\left[{n \atop r}\right]$ is the number of permutations in $\mathcal{S}_n$ with exactly r cycles, so $(n-1)\left[{n-1 \atop r}\right]$ is the number of permutations of $\mathcal{S}_{n-1}$ with exactly r cycles $\square$

Identity 19.

$$\binom{n}{k}k! = \frac{n!}{(n-k)!}.$$

Proof (Combinatorial). The LHS counts all permutations of each k-subset of elements in $[n]$. The RHS counts all permutations of $\mathcal{S}_n$ such that $n-k$ of the elements are treated as fixed points, which are the permutations of all k-subsets of $[n]$.

The two expressions also both count the number of injective functions with domain of size k and range of size n. For the LHS, choose a k-subset of the range to map the domain to, then take all permutations to get all functions to that particular k-subset. For the RHS, we count the number of injections by mapping one element in the domain at a time; there are n values for the first element, $n-1$ for the second, and so on, and $n-(k-1)$ values for the k-th element. Thus there are $n(n-1)\cdots(n-(k-1))$ ways of mapping the elements in the domain to distinct elements in the range. $\square$

Identity 20 (First Stirlings Sum)**.**

$$\sum_{r=0} \begin{bmatrix} n \\ r \end{bmatrix} = n!$$

Proof (Combinatorial). Every permutation in $\mathcal{S}_n$ has some number of cycles between 1 and n, and $\left[{n \atop 0}\right] = 0$ for all $n \geq 1$. So when $n \geq 1$, $\sum_{r=0} \left[{n \atop r}\right] = n!$. When $n = 0$, then the equality still holds because $\left[{0 \atop 0}\right] = 0! = 1$ by definition. $\square$

Identity 21 (Rising Factorial Binomial Theorem)**.**

$$[x+y]^n = \sum_{k=0}^{n} \binom{n}{k} [x]^k [y]^{n-k}.$$

Proof (Combinatorial). $\square$

Identity 22 (Stirling Numbers of the Second Kind Triangle Identity)**.**

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = r\begin{Bmatrix} n-1 \\ r \end{Bmatrix} + \begin{Bmatrix} n-1 \\ r-1 \end{Bmatrix}.$$

Proof (Combinatorial). Stirling numbers of the second kind, denoted $\left\{ {n \atop r} \right\}$, count the number of unordered partitions of an n-set involving r classes. As an aside, note that multinomial coefficients are similar except that they impose an order on the classes in the partition. So, $r\left\{ {n-1 \atop r} \right\}$ counts the number of ways of partitioning $[n]$ into r classes where n is placed in one of these classes (a class of size larger than 1), and $\left\{ {n-1 \atop r-1} \right\}$ counts the number of partitions of $[n]$ where n is a singleton. These are the only two cases for partitions of $[n]$ into r classes, so the result follows by the addition principle (if the set of objects being counted is the union of subsets of objects, which are non-intersecting, then the size of the superset is the sum of sizes of the non-intersecting subsets). □

Identity 23.

$$\left\{ {n+1 \atop r} \right\} = \sum_{i=0}^{n} \binom{n}{i} \left\{ {i \atop r-1} \right\}.$$

Proof (Combinatorial). □

Identity 24 (Stirling Numbers Matrix Identity)**.** *Let A and B be $n \times n$ matrices with entries $a_{i,j} = \left\{ {i \atop j} \right\}$ and $b_{i,j} = (-1)^{i-j} \left[{i \atop j} \right]$, respectively. Then*

$$AB = I_n.$$

Proof (Combinatorial). □

Identity 25 (Bell Numbers Triangle Identity)**.**

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Proof (Combinatorial). □

Identity 26 (Catalan Recursion Identity 1 (Triangulations))**.**

$$c_{n+3} = \sum_{i=1}^{n-1} c_{i+3} c_{n-i+3}.$$

Proof. □

Identity 27 (Catalan Recursion Identity 2 (Triangulations))**.**

$$(n-3)c_{n+2} = \frac{n}{2} \sum_{i=3}^{n-1} c_{i+2} c_{n+4-i}.$$

Proof. □

Identity 28.

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

Combinatorial. □

Identity 29.

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}.$$

Combinatorial Proof. $\square$

Identity 30. *Find the closed form expression for*

$$\sum_{k=0}^{n} \left(\frac{1}{k+1}\right)^2 \binom{n}{k}.$$

Combinatorial Proof. $\square$

Identity 31.

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

Combinatorial Proof. $\square$

Identity 32.

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

Combinatorial Proof. $\square$

Identity 33.

$$\sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-2k}{k} < \left(\frac{3}{2}\right)^n.$$

Combinatorial Proof. $\square$

Identity 34.

$$(r+1)^n = \sum_{k=0}^{n} \binom{n}{k} r^k.$$

1.2.3 Miscellaneous Interesting Counting Arguments

Theorem 1.22. *The number of different trees with vertex set $\{1, 2, \ldots, n\}$ and degree sequence $d_1, d_2, \ldots, d_n$ is the multinomial coefficient*

$$\binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1}.$$

Proof. $\square$

Theorem 1.23. *The number of odd terms in the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is a power of two.*

Proof. $\square$

Theorem 1.24. *Every element in the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is odd if and only if $n = 2^t - 1$ for some integer $t \geq 0$.*

Proof. $\square$

1.3 Generating Functions

Exercise 1. *Prove that the generating function of the sequence* $0, 1, 4, 9, 16, \ldots$ *is* $\frac{x(x+1)}{(1-x)^3}$.

Proof. The generating function for $0, 1, 2, \ldots$ is $\frac{x}{(1-x)^2}$ since

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + \cdots)$$

$$\ldots$$

□

2 Questions

Question 1. *In this question there are* b *balls and* c *containers. Determine the number of distributions of the balls among the containers in each off the scenarios given. The order in which the balls are placed into the containers is unimportant, except in the last part of the question. Your answers may be given in terms of other quantities like binomial coefficients or Stirling numbers of the second kind, etc. A brief explanation of your reasoning is required when indicated, as well as the definition of any symbol or quantity other than* $\binom{n}{k}$.

1. *The balls are indistinguishable and the containers are indistinguishable. Explain your reasoning.*
2. *The balls are indistinguishable and the containers are labelled* $1, 2, \ldots, c$.
3. *The balls are indistinguishable, the containers are labelled* $1, 2, \ldots, c$, *and each container holds at most one ball.*
4. *The balls are labelled* $1, 2, \ldots, b$ *and the containers are indistinguishable. Explain your reasoning.*
5. *The balls are labelled* $1, 2, \ldots, b$ *and the containers are labelled* $1, 2, \ldots, c$.
6. *The balls are labelled* $1, 2, \ldots, b$ *the containers are labelled* $1, 2, \ldots, c$, *and no container is empty.*
7. *The balls are labelled* $1, 2, \ldots, b$, *the containers are labelled* $1, 2, \ldots, c$, *and the order in which the balls are placed into the containers matters (i.e., the balls are ordered within each container). Explain your reasoning.*

Question 2. *Use any method to evaluate*

$$\sum_{k=0}^{n} k^2 \binom{n}{k}.$$

Then, give a proof of the resulting identity using a combinatorial argument. (It is possible to do both parts simultaneously.)

Question 3. *State both the Pigeonhole Principle and Ramsey's Theorem (finite version, in full generality), and show how Ramsey's Theorem is a generalization of the Pigeonhole Principle. Define all terms necessary to make your statements meaningful.*

Question 4. *A pair of two distinct dice is rolled six times. Suppose none of the ordered pairs of values $(1,5),(2,6),(3,4),(5,5),(5,3),(6,1)$ occur. What is the probability that all six values on the first die and all six values on the second die occur once in the six rolls of the two dice.*

Question 5. *Let $a(n,k)$ be the number of k-permutations of n distinct objects (that is, the number of linear arrangements of k of the n objects), and define $a(n,k)$ to be zero if $k > n$.*

a) *Find a recurrence relation and initial conditions for $a(n,k)$.*

b) *Let*

$$G_n(x) = \sum_{k=0}^{\infty} a(n,k) \frac{x^k}{k!}.$$

Find a closed form for $G_n(x)$ and use it to determine $a(n,k)$.

Question 6. *Let A be a $(0,1)$-matrix. Prove that the minimum number of lines containing all 1s of A equals the maximum number of 1s, no two on a line. State any theorems used.*

Question 7. *A scientist is studying the effect of soil, temperature, and fertilizer on development of five different varieties of strawberries. She wants to compare the effects of five types of soil, five types of fertilizer, and five different temperatures on the growth of the strawberries. A comprehensive study would test each possible combination of variety, soil, temperature, and fertilizer, and would require $5^4 = 625$ different plots. Due to budget and space constraints she has only five small greenhouse units, each with five boxes in which plants can be grown, available for her study. Each greenhouse has its own heat control and can be kept at a different temperature. She decides that it is most important that each pairing of strawberry variety and fertilizer is tested (at some temperature) and, further, each type of soil is tested at least once at each temperature. Show how Latin squares can be used to obtain the desired experimental design.*

Question 8.

a) *State necessary and sufficient conditions for the existence of a Steiner Triple System. Prove that the conditions you state are necessary.*

b) *Show how to use a Steiner Triple System on v points to construct a Steiner Triple System on $3v$ points.*

Question 9.

a) *State the Hamming bound on the number of words in a code with length n and minimum distance $2d+1$ or $2d+2$.*

b) *Show that, for $r \geq 3$, the Hamming code of length $2^r - 1$ is a perfect single error correcting code.*

Question 10.

a) *State Polya's Enumeration Theorem and define any terms necessary to make your statement meaningful.*

b) *You are given a large supply of beads of* 5 *different colours. How many difference necklaces of* 9 *beads can be made? Two necklaces are the same if one can be rotated and/or flipped to obtain the other.*

c) *How many necklaces in* (*b*) *use* 4 *red beads,* 3 *white beads, and* 2 *black beads?*

Question 11. *Define the Ramsey number $N(q_1, q_2, \ldots, q_t; r)$ and prove that $N(4, 3; 2) \leq 9$.*

Proof. **Definition:** $N(q_1, q_2, \ldots, q_t; r)$ is the smallest number of vertices required in a hypergraph such that any t-colouring of the hyper-edges of size r produces a monochromatic r-hyper clique of size q_i for some $i \in [1, t]$.

To prove $N(4, 3; 2) \leq 9$, we will use the fact that $N(a, b) \leq N(a, b-1) + N(a-1, b)$ with strict inequality when the RHS terms are both even. Then since $N(3, 3) = 6$ and $N(n, 2) = n$, we will be done.

Lemma 1. *$N(a, b) \leq N(a, b-1) + N(a-1, b)$, where strict inequality holds when the RHS terms are both even.*

Proof. Let $n = N(a, b)$, $n_1 = N(a, b-1)$, $n_2 = N(a-1, b)$, and consider a graph G of order n. Then $n = n_1 + n_2$. Let $v \in V(G)$ and define $S = G \setminus [N(v)]$. If $\deg(v) \geq n_1$, then $[N(v)]$ contains a monochromatic K_a or K_{b-1}^C. If $\deg(v) < n_1$, then $|S| \geq n_2$ and so S contains a monochromatic K_{a-1} or K_b^C. Since we only want an upper bound, we may assume these cases occur independently (where $[N(v)]$ and S are treated as disjoint, say) to get that $N(a, b) \leq N(a, b-1) + N(a-1, b)$.

For the strict inequality: assume n_1 and n_2 are both even, then $n := n_1 + n_2 - 1$ is odd. By handshaking G cannot have an odd number of odd vertices, so there exists an even vertex $v \in V(G)$. If $\deg(v) \geq n_1$, then same as above case. If $\deg(v) < n_1$, then indeed $\deg(v) \leq n_1 - 2$, which means that $|S| \geq n_1 + n_2 - 1 - (n_1 - 2) = n_2 + 1$, thus $N(a, b) \leq n_1 + n_2 - 1$. □

$N(3, 3) \leq 6$ because pick a vertex, PHP gives it a monochromatic neighbourhood of size 3, try not to make a monochromatic triangle, must fail. Lower bound: C_5. Obviously, $N(n, 2) = n$.

Thus $N(4, 3; 2) < N(3, 3; 2) + N(4, 2; 2) = 6 + 4$, as desired. □

Question 12. *Prove that for all nonnegative integers m, n,*

a) $\sum_{k=0}^{n} \binom{m}{k}\binom{m-k}{n-k} = 2^n \binom{m}{n}$.

b) $\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{m-k}{n-k} = \binom{m-n}{n}$.

Proof of Part a). This is just a calculation. Can tinker with factors on the LHS to get $\binom{m}{n}\sum_{k=0}^{n}\binom{n}{k} = \binom{m}{n}2^n$. □

Proof of Part b). Let M be an m-set and $N \in \binom{M}{n}$. Then RHS counts the number of n-subsets of $M \setminus N$. Use PIE to show the same for the LHS; given a fixed k-subset K of N, $\binom{m-k}{n-k}$ counts the number of n-subsets of M that ignore K... □

Question 13.

a) What is the generating function for $p(n)$, the number of integer partitions of n? (Your answer should not involve $p(n)$)

b) Prove that the number of partitions of n into even summands is equal to the number of partitions of n with each summand appearing an even number of times.

Proof of Part a.

$$\begin{aligned}
\sum_{n\geq 0} p(n)x^n &= \sum_{n\geq 0} \sum_{\substack{m_1+m_2+\cdots=n \\ m_i\geq 0}} x^{m_1+m_2+\cdots} \\
&= \prod_{i\geq 1}\Big(\sum_{m_i\geq 0} x^{m_i}\Big) \\
&= \prod_{i\geq 1}\frac{1}{1-x^i}
\end{aligned}$$

□

Proof of Part b. Let P be a partition of n with only even parts. The Ferrer diagram of P is essentially a $0-1$ matrix with rows corresponding to part sizes in weakly decreasing order from top to bottom. If P has k parts, then each row has even number of consecutive 1s. A conjugate partition is characterized by the transpose of a Ferrer diagram. The conjugate partition of P has an even number of rows in its Ferrer diagram, and so it has an even number of parts. Each part size occurs an even number of times, because the i-th and $(i-1)$-th rows have the same number of 1s for all even i. Transpose of Ferrer diagrams is invertible. □

Question 14. *A Hadamard matrix is an $n\times n$ matrix H such that every entry is ± 1 and $HH^T = nI$. That is, the dot product of distinct rows are zero. Suppose there is an $n\times n$ Hadamard matrix. Prove that if $n\geq 4$, then $n\equiv 0 \pmod 4$.*

Proof. Look at any 3 rows and take their inner products. Let $H = (a_{i,j})$, then $\sum_{j=1}^n a_{1,j}a_{2,j} = 0$ iff half the terms are $(1)(-1)$ products and the other half are $(1)(1)$ or $(-1)(-1)$ products $\Rightarrow n\equiv 0 \pmod 2$. Since $\sum_{j=1}^n a_{1,j}a_{3,j} = 0$ it follows that

$$\begin{aligned}
\sum_{j=1}^n a_{1,j}a_{2,j} &= \sum_{j=1}^n a_{1,j}a_{3,j} \\
\Rightarrow \sum_{j=1}^n a_{1,j}(a_{2,j} - a_{3,j}) &= 0,
\end{aligned}$$

which means that there are $n/2$ terms where $a_{2,j} - a_{3,j} = 0$ and $n/4$ where $a_{2,j} - a_{3,j} = -2a_{1,j}$, and $n/4$ where $a_{2,j} - a_{3,j} = 2a_{1,j}$. There are $n/4$ differences of the form $(-1)-(-1) = 0$, $n/4$ of the form $(1)-(1) = 0$, $n/4$ of the form $(1)-(-1) = 2$ and similarly for $(-1)-(1) = -2$. Thus $n\equiv 0 \pmod 4$. □

Question 15.

a) Give the definition of a $2-(v,k,\lambda)$ block design.

b) How many blocks are in such a design?

c) Give a construction for a $2-(7,4,2)$ design.

d) State Fisher's inequality.

Proof. **Definition:** Let X be a v-set. A $2-(v,k,\lambda)$ block design is a pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a collection of k-subsets of X, called blocks, such that each pair $p \in \binom{X}{2}$ occurs in exactly λ blocks.

Let b be the number of blocks in a (v,k,λ)-design. To find the number of block in a design, just count pairs in two ways: there are $\binom{v}{2}\lambda$ pairs in the design (Top-down perspective), and now look at the blocks in $\mathcal{B}$ (Bottom-up perspective): there are b blocks and $\binom{k}{2}$ pairs in each block. Thus we have $\binom{v}{2}\lambda = b\binom{k}{2} \Leftrightarrow b = \frac{v(v-1)\lambda}{k(k-1)}$.

To construct a $2-(7,4,2)$-design, just take the complement of the finite projective plane of order 2.

Fisher's inequality is $b \geq v$. It can be proven using a rank argument on the incidence matrix, I think. Note that this inequality is not at all obvious because suppose $v = Cb$ and , then

$$b \sim \tfrac{v^2}{k^2}\lambda \sim \tfrac{C^2b^2}{k^2}\lambda \Leftrightarrow b \sim \tfrac{k^2}{C^2\lambda}.$$

I think it's not so hard to imagine this working for C equal to something like 1.5. □

Question 16. *Let $I(n,k)$ be the number of permutations of $1,2,\ldots,n$ with exactly k inverses (an inverse of a permutation π is an ordered pair (i,j) such that $i<j$ and $\pi(i)>\pi(j)$). For fixed n, find a generating function for $I(n,k)$ and from this deduce a recurrence for these numbers.*

Proof. q-analogues! We count according to the possible gap sizes between i and $\pi(i)$. Note that there $\pi(i)-i$ inverses in a permutation π contributed to by the inverse $(i,\pi(i))$.

The generating function is $\sum_{k\geq 0} I_{n,k}x^k = (n!)_q = \prod_{j=1}^{n-1}\frac{1-q^j}{1-q}\cdots$ □

Question 17. *Show that an automorphism of a projective plane that fixes all points on some line must fix each line through some point.*

Proof. Let L be a line whose points are fixed under some automorphism Q. Then since we're in a projective plane, not all points are in L and there are other lines $L_1, L_2, \ldots, L_{n^2+n}$. Each pair of lines intersects at exactly one point and every pair of points are on a unique line. Thus for each L_i there is a point $x_i \in L \cap L_i$ (not all distinct of course since $|L| = n+1$), such that x_i is fixed in L_i. Thus there is at least one point on each line that is fixed under Q. □

Question 18.

a) *State a theorem that characterizes distributive lattices.*

b) *State a theorem that characterizes modular lattices.*

c) *Prove either of the theorems of (a) and (b).*

Question 19. *State and prove Dilworth's theorem.*

Proof. **Statement:** In any poset P, a largest antichain in P has the same size as a minimum sized chain decomposition of P.

Let P be a poset and C a longest chain of P. Let M_P be the size of a largest antichain in P and m_P the size of a smallest chain decomposition in P.

Note that $M_P \leq m_P$ because otherwise there would be two elements in an antichain contained in a chain of a minimum sized chain decomposition, a contradiction.

We proceed by induction on the order of P. The theorem holds for the order 1 poset. Suppose for all posets of order less than that of P that the theorem holds.

Consider the poset $P - C$. Note that $M_P - 1 \le M_{P-C} \le M_P$ since a maximum antichain in P can only have at most one element in C; so LH inequality follows in the case where the P max antichain intersects C while the RH inequality is the case when the P max antichain doesn't intersect C.

Suppose $M_P - 1 = M_{P-C}$. By the inductive hypothesis, $M_P = m_{P-C} + 1$, which can only hold if a largest antichain in P intersects C; this is because exactly one element in the antichain must be in each chain in a minimum size chain decomposition of $P - C$. But in this case, $m_{P-C} + 1 = m_P$.

Suppose $M_{P-C} = M_P$. Let $Y = \{y_1, y_2, \dots, y_{M_{P-C}}\}$ be a largest antichain in $P - C$. Note that Y is a largest antichain in P as well. Let A and B be the upset and downset of Y:

$$A := \{x \in P : x \ge y, y \in Y\}$$
$$B := \{x \in P : x \le y, y \in Y\}$$

Note that $Y = A \cap B$. Indeed $P = A \cup B$ because otherwise if $z \in P \setminus (A \cup B)$, then $Y \cup \{z\}$ is a larger antichain than Y, a contradiction. Recall that $Y \cap C = \emptyset$. Now let $g = \max(C)$ and $\ell = \min(C)$; since C is a longest chain in P, $g \notin B$ and $\ell \notin A$. Since $A, B \subset P$, by the inductive hypothesis, A and B both have a minimum sized chain decomposition of size $M_{P-C} = M_P$ whose chains are in bijection with Y, and since $A \cup B = P$, it follows that P has a minimum sized chain decomposition of size M_P, as desired. □

Question 20. *State Hall's theorem and show that it follows from Dilworth's theorem.*

Proof. **Statement:** Let $m \ge n$. The sets $A_1, A_2, \dots, A_n \subseteq [m]$ have an SDR iff for every $I \in 2^{[n]}$, $|\bigcup_{i \in I} A_i| \ge |I|$.

Forward: Suppose the family has an SDR $\{s_1, s_2, \dots, s_n\}$, where s_i represents A_i. Then

$$\left|\bigcup_{i \in I} A_i\right| \ge \left|\bigcup_{i \in I} \{s_i\}\right| \ge |I|.$$

Backward: Suppose Hall's condition holds. Let $\bigcup_{i \in I} A_i = \{x_1, x_2, \dots, x_m\}$. Then define a poset $P = \{A_1, A_2, \dots, A_n\} \cup \{\{x_1\}, \{x_2\}, \dots, \{x_m\}\}$, where for all $U, V \in P$, $U \le V$ iff $U \subseteq V$. Note that $\{x_1, x_2, \dots, x_m\}$ is an antichain. Suppose for some $\mathcal{X} \subseteq \{\{x_1\}, \{x_2\}, \dots, \{x_m\}\}$ and $\mathcal{A} \subseteq \{A_1, A_2, \dots, A_n\}$, $\mathcal{X} \cup \mathcal{A}$ is a largest antichain, then for all $X \in \mathcal{X}$, $X \cap \bigcup_{A \in \mathcal{A}} A = \emptyset$, but Hall's condition implies that $|\bigcup_{A \in \mathcal{A}} A| \ge |\mathcal{A}|$. Thus $|\mathcal{X}| \le m - |\mathcal{A}| \Leftrightarrow |\mathcal{X}| + |\mathcal{A}| \le m$, which means that $\{x_1, x_2, \dots, x_m\}$ is a largest antichain in P. So by Dilworth's theorem, a smallest chain decomposition has size m, and the only way for this to be true is if there are $m - n$ singletons and n chains of size 2, which means that the chains of size 2 are of the form $\{A_i, x_j\}$ where $i \in [n]$ and $j \in [m]$. An SDR is the set of minimum elements in each chain of length 2 in a smallest chain decomposition of P. □

Question 21. *Let $\{F_k\}$ be the Fibonacci sequence, with $F_0 = 0$, and let n be a positive integer.*

a) *Show that F_{n-1} counts the number of tilings of a $1 \times n$ grid with 1×1 and 1×2 tiles.*

b) *Use (a) to show that*

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_{n-1}.$$

c) *Prove that*

$$\binom{n}{0} + \binom{n-2}{1} + \binom{n-4}{2} + \cdots < \left(\frac{3}{2}\right)^n.$$

Question 22 (Multiplicity: 2)**.**

1. *State and prove Burnside's Lemma regarding the number of distinct orbits associated with a group of permutations. Define all terms necessary to make the statement meaningful.*
2. *In how many ways may the vertices of a regular pentagon be coloured with red and blue, if colourings that differ by rotation and/or reflection are considered the same.*

Question 23.

a) *Define the conjugate of an integer partition.*

b) *Show that the number of self-conjugate integer partitions of n is equal to the number of partitions of n into distinct odd parts.*

c) *Give a closed-form generating function for $p(n)$, the number of integer partitions of n. (Of course, do not use $p(n)$ itself in the answer.)*

Question 24. *Use generating functions to solve the following:*

a) *How many lines of length n containing Englishmen, Irishmen, Scotsmen, and Welshmen contain an even number of Englishmen and an odd number of Welshmen?*

b) *How many 5-subsets of $\{1, 2, \ldots, 12\}$ contain no three consecutive integers?*

Question 25. *Let A be an $n \times n$ $(0, 1)$-matrix. Prove that A has a collection of ones in distinct rows and columns if and only if A has no $r \times s$ all-zero submatrix for $r + s > n$. State any major theorems used in your proof.*

Question 26. *Use a 'rank argument' to prove that if a (v, b, r, k, λ)-design with $v > k$ is resolvable, then $b \geq v + r - 1$.*

Question 27.

1. *State the definition of a $t-(v, k, \lambda)$ design, and write the number of blocks, b, and replication number, r, in terms of the other parameters.*
2. *Let $\mathcal{D}$ be a $t-(v, k, \lambda)$ design and $0 \leq s \leq t$. Show that there exists λ_s such that $\mathcal{D}$ is also an $s-(v, k, \lambda_s)$ design.*

3. *A Steiner quadruple system $SQS(n)$ is a $3-(n,4,1)$ design. Prove that if an $SQS(n)$ exists, then so does an $SQS(2n)$.*

Question 28. *Let n be a positive integer. Consider the poset P of all positive integer divisors of $3\cdot 2^{n-1}$, equipped with the partial order of divisibility, namely $\mid \subseteq P\times P$ defined by $x \mid y$ if and only if $y = xt$ for some integer t. Prove that the number of total orders containing $\mid$ is equal to the nth Catalan number.*

Question 29. *Define a function $\lambda(n)$ on the positive integers by $\sum_{d|n}\lambda(d) = \log(n)$. Show that if $n = p^k$ is a prime power (p is prime), then $\lambda(n) = \log(p)$, and $\lambda(n) = 0$ otherwise.*

Question 30.

a) *Define the graph Ramsey number $R(G_1, G_2, \ldots, G_k)$.*

b) *Prove that, for any tree T_n on n vertices and star $K_{1,n}$,*

$$R(T_n, K_{1,n}) = 2n-1.$$

Question 31.

a) *State the enumeration theorem of inclusion and exclusion. Define all terms necessary to make your statement meaningful.*

b) *Give a formula for the number of functions from an n-set onto an m-set.*

c) *Express the count in (b) using the Stirling numbers of the second kind.*

Question 32.

1. *State the Vandermonde identity of binomial coefficients.*
2. *Give a combinatorial proof of it.*
3. *Use it to deduce a closed form for*

$$\sum_{k=0}^{n}\binom{n}{k}^2.$$

Question 33. *Let r be an arbitrary positive integer. Use the Pigeonhole principle to assert the existence of an integer n such that both:*

- *r divides n and*
- *the only (base ten) digits in n are 0 and 1.*

Question 34. *Let n be a fixed positive integer. The rank of a permutation $\pi = p_1p_2\ldots p_n$ is the position in which π appears in the dictionary order, starting from 0. For instance, if $n = 6$, the rank of 123456 is 0 and the rank of 654321 is $6! - 1 = 719$.*

a) *Outline a recursive algorithm that computes the rank of permutation.*

b) *For general n, describe the permutation whose rank is $n!/2$.*

Question 35. *A one-factorization of order $2n$ is a partition of the $\binom{2n}{n}$ unordered pairs of a $(2n)$-set into partitions of the set. In other words, it is a resolvable $(2n, 2, 1)$ design.*

1. *Prove that a one-factorization of order $2n$ exists for all positive integers n.*
2. *A one-factorization is perfect if the union of any two parallel classes, regarded as a set of edges, induces a (Hamiltonian) $2n$-cycle. Prove that a perfect one-factorization of order $p+1$ exists for all primes $p > 2$.*

Question 36. *Consider $(v = 6, k = 3, \lambda = 2)$ block designs.*

1. *How many blocks are in such a design?*
2. *Prove that such a design cannot have two disjoint blocks.*
3. *Construct such a design.*

Question 37.

1. *State Hall's theorem.*
2. *State Dilworth's theorem.*
3. *Show how Hall's theorem follows from Dilworth's theorem.*

Question 38. *Determine a closed form for the number of ways in which a $2 \times n$ chessboard can be tiled using a supply of 1×1, 2×1, or 1×2 tiles.*

Question 39.

1. *State precisely what is meant by the Ramsey number $N(3, 4, 5; 3)$.*
2. *Prove that $N(p, q; 2) \leq \binom{p+q-2}{p-1}$.*
3. *Find an upper bound on $N(4, 4, ; 3)$.*
4. *Prove that $N(4, 5; 3) \leq N(5, 19; 2) + 1$.*

Question 40 (Multiplicity: 2)**.**

a) *State and prove Burnside's Lemma regarding te number of distinct orbits associated with a group of permutations. Define all terms necessary to make the statement meaningful.*

b) *In how many ways can the vertices of a regular pentagon be coloured with Red and Blue, if colourings that differ by rotation and/or reflection are considered the same?*

Question 41.

a) *Solve the difference equation $u_{n+3} = 3u_{n+2} - 4u_n$, ($n \geq 0$), given that $u_0 = 1$, $u_1 = 1$, and $u_2 = 3$.*

b) *Derive a recurrence relation and initial conditions for t_n, the number of ways to triangulate the interior of a convex n-gon. Given a non-recursive formula for t_n (you need not prove it). State any results used.*

Question 42.

a) *Define $S(n,m)$, the Stirling number of the second kind, and prove a relationship between $S(n,m)$ and the number of functions from an n-set onto an m-set.*

b) *Using (a) or otherwise prove*

$$S(n,m) = \frac{1}{m!}\sum_{r=0}^{m-1}(-1)^r\binom{m}{r}(m-r)^n.$$

Question 43.

1. *State the enumeration theorem of inclusion and exclusion. Define any terms necessary to make our statement meaningful.*

2. *A man has six friends. He has met each of them at dinner 12 times, every two of them six times, every three of them four times, every four of them three times, every five twice, and all six only once. He has dined out eight times without meeting any of them. How many times has he dined out altogether?*

3. *$5n$ men, standing in a line, are from n countries (five men from each country). Find the number of ways in which the line can be formed so that every man is standing next to a compatriot.*

Question 44.

a) *Find a generating function for the number a_n of sequences of length n from the set $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ in which the letter α occurs an even number of times an the letter β occurs an odd number of times. Hence, find a_n.*

b) *Find the number of 7-subsets of $\{1, 2, \ldots, n\}$ which contain no three consecutive integers.*

Question 45.

1. *State precisely what is meant by the Ramsey number $N(3,4,5;3)$.*

2. *Prove that $N(k,k;2) \geq 2^{k/2}$.*

Question 46 (Multiplicity = 2)**.** *A Steiner quadruple system, $SQS(n)$, is a $3-(n,4,1)$ design. Prove that if an $SQS(n)$ exists, then so does an $SQS(2n)$.*

Question 47 (Multiplicity = 3)**.** *Define a function $\lambda(n)$ o the positive integers by $\sum_{d|n}\lambda(d) = \log(n)$. Show that if $n = p^k$ is a prime power (p is prime), then $\lambda(n) = \log(p)$, and $\lambda(n) = 0$ otherwise.*

Question 48.

a) *Prove that*

$$\binom{2n}{n} + 2\binom{2n-1}{n} + 2^2\binom{2n-2}{n} + \cdots + 2^n\binom{n}{n} = 2^{2n}.$$

b) Prove that

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k}\binom{n-k}{m-k} = 0.$$

Question 49 (Multiplicity = 2)**.**

1. *State Polya's Enumeration Theorem and define any terms necessary to make your statement meaningful.*

2. *A circle is divided into eight identical sectors. In how many ways can these be painted with three colours?*

3. *How many of these paintings have one white sector, three blue sectors, and four red sectors?*

Question 50. *State and prove Dilworth's theorem for finite posets. Define any terms necessary in the statement and proof.*

Question 51.

a) *Determine the number of arrangements of $a_1, a_1, a_2, a_2, \ldots, a_n, a_n$ where no two equal letters are adjacent.*

b) *State and prove the enumeration theorem known as the Principle of Inclusion and Exclusion.*

Question 52 (Multiplicity = 2)**.**

a) *State and prove Burnside's lemma regarding the number of distinct orbits associated with a group of permutations. Define all terms necessary to make the statement meaningful.*

b) *In how many ways may the vertices of a regular pentagon be coloured with red and blue, if colourings that differ by rotation and/or reflection are considered the same?*

Question 53.

a) *State Ramsey's theorem (the finite version, in its full generality). Define all terms necessary to make the statement meaningful.*

b) *Prove that $N(a, b; 2) \leq N(a, b-1; 2) + N(a-1, b; 2)$.*

c) *Prove the following statement concerning Ramsey numbers: If both $N(a, b-1; 2)$ and $N(a-1, b; 2)$ are even, then $N(a, b; 2) < N(a, b-1; 2) + N(a-1, b; 2)$.*

d) *Show that $N(3, 5; 2) \leq 14$.*

Question 54. *Use generating functions to solve:*

a) *How many lines of length n containing Englishmen, Irishmen, Scotsmen, and Welshmen contain an even number of Englishmen and an odd number of Welshmen?*

b) *How many 5-subsets of $\{1, 2, \ldots, 12\}$ contain no three consecutive integers?*

Question 55. *Prove that there exists a projective plane of order n if and only if there exists a collection of $n-1$ pairwise orthogonal latin squares of order n.*

Question 56. *Show that if every chain and every antichain of a poset P is finite, then P is finite.*

Question 57.

a) *State the definition of a $t-(v, b, r, k, \lambda)$ design (usually called a $t-(v, k, \lambda)$ design), and show that the number of blocks, b, and replication number, r, can be determined from the other parameters.*

b) *Let $\mathcal{D}$ be a $t-(v, k, \lambda)$ design and $s \leq t$. Show that there exists λ_s such that $\mathcal{D}$ is also an $s-(v, k, \lambda_s)$ design.*

c) *Prove that necessary conditions for the existence of a $2-(v, k, \lambda)$ design are:*

$$(v-1)\lambda \equiv 0 \pmod{k-1} \text{ and } v(v-1)\lambda \equiv 0 \pmod{k(k-1)}.$$

d) *State Wilson's theorem regarding designs.*

Question 58.

a) Prove that

$$\binom{2n}{n} + 2\binom{2n-1}{n} + 2^2\binom{2n-2}{n} + \cdots + 2^n\binom{n}{n} = 2^{2n}.$$

b) Prove that

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}.$$

Question 59 (Multiplicity = 3)**.** *Define a function $\lambda(n)$ o the positive integers by $\sum_{d|n} \lambda(d) = \log(n)$. Show that if $n = p^k$ is a prime power (p is prime), then $\lambda(n) = \log(p)$, and $\lambda(n) = 0$ otherwise.*

Question 60.

1. *Derive a recurrence for d_n, the number of derangements of n distinct objects.*
2. *Derive a recurrence for the number of ways of parenthasizing the product $x_1 x_2 \cdots x_n$.*
3. *Solve the recurrence $a_0 = 30$ and $a_1 = 33$, and for $n \geq 2$, $a_n = 4a_{n-1} - 3a_{n-2} - 200$.*