Qualifying Examination in Graph Theory

June 29, 1994 Time: 3 hours

Each question is worth 10 marks. Fifty marks are necessary and sufficient for a pass. In each question, if you appeal to a theorem within your proof, you must carefully state that theorem.

1. State 3 different equivalent definitions of a tree, and prove that they are equivalent.
2. Prove that K_5 and $K_{3,3}$ are not planar. Sketch an embedding of K_5 on the torus.
3. Prove that if every vertex of G has even degree, then the edges of G can be oriented so that, at each vertex, the indegree equals the outdegree.
4. Let A be the adjacency matrix of G, where $V(G) = \{1, 2, \ldots, n\}$. Prove that in the matrix A^k the entry in row i and column j equals the number of (i, j)-walks of length k in G.
5. Define the *edge connectivity* of a graph G, and prove that G is k-edge-connected if and only if any two vertices of G are joined by k edge-disjoint paths.
6. Prove that, if Γ is a finite group, then there exists a finite graph G such that *Aut(G)* is isomorphic to Γ. Describe a graph whose automorphism group is the cyclic group of order 3.
7. Let G be a graph in which any two odd cycles intersect. Prove that G is 5-colourable. Give an example to show that four colours do not suffice.
8. Show that there exist a real constant $c \in (0, 1)$ and an integer constant $d \geq 6$ such that any connected planar graph G with n vertices contains cn independent vertices with degree at most d.
9. Prove that a graph G has a spanning subgraph consisting of vertex disjoint edges and cycles if and only if it has such a spanning subgraph consisting of vertex disjoint edges and odd cycles. Show that a necessary an sufficient condition for such a spanning subgraph to exist is that no removal of k vertices of G results in more than k isolated vertices.
10. Prove that a tournament T is strong if and only if it has a directed Hamilton cycle.