Relation Partitions

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1 Summary

2 Preliminaries

Bipartite Distance Multigraph, Distance Graphs, Distance Multiplicities as adj mat (or Laplacian) traces,

Combinatorial Nullstellensatz, Eigenvalue interlacing theorem

3 Nullstellensatz Shows Existence of Dense Subgraph

4 General Position (Max Distance Degree) Permits Large Order

Theorem 4.1. *Let X be a finite metric space with n points and r distinct distances $d_1, \ldots, d_r$ whereby either $n/2$ or $(n+1)/2$ is a prime p. Let $d \in \mathbb{Z}^+$. Suppose $\Delta(D_{d_k}) \leq d$ for all $k \in [1, r]$. Then there is a set of P points and D distances satisfying $|P| \geq |D| \geq p/d$ such that*

1. *for each $v \in P$, there are p points $u \in P \setminus \{v\}$ such that $d(u, v) \in D$; and*
2. *for each $d_k \in D$, there is an $\ell_k \in \mathbb{Z}^+$ such that there are $\ell_k p$ points P at distance d_k with some other point in X.*

Corollary 4.2. *The average multiplicity of distances in D is at least $\frac{|P|p}{2|D|}$.*

5 Eigenvalue Interlacing and Distance Multiplicity Stability

Let A_k be the adjacency matrix for distance d_k in X. Then $2m(d_k) = Tr(A_k^2) = \deg_{\mathcal{M}}(d_k)$. We say that a point $v \in X$ is *distance independent* if for all $u, w \in X \setminus \{v\}$, $d_{X \setminus \{v\}}(u, w) = d_X(u, w)$.

Let v be a distance independent vertex of X. Let $\alpha_1^{(k)} \geq \alpha_2^{(k)} \geq \cdots \geq \alpha_n^{(k)}$, and $\beta_1^{(k)} \geq \beta_2^{(k)} \geq \cdots \geq \beta_{n-1}^{(k)}$ be the eigenvalues of $A_k(X)$ and $A_k(X \setminus \{v\})$, respectively. Similarly, let $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_n^{(k)}$, and $\mu_1^{(k)} \geq \mu_2^{(k)} \geq \cdots \geq \mu_{n-1}^{(k)}$ be the eigenvalues of $\mathcal{L}(A_k(X))$ and $\mathcal{L}(A_k(X \setminus \{v\}))$, respectively.

Theorem 5.1. *Let X and Y finite metric spaces that differ by a single distance independent point. Then for every $k \in [1, r]$,*

$$|m_X(k) - m_Y(k)| < \tfrac{1}{2}(\alpha_1^2 + \alpha_n^2),$$

and

$$|m_X(k) - m_Y(k)| \leq \tfrac{\lambda_1}{2}.$$

Proof. Without loss of generality, suppose $Y = X \setminus \{v\}$, where v is a distance independent point. For ease of notation, we omit the (k) superscripts in the eigenvalues for this proof.

For the Laplacians $\mathcal{L}(A_k(X))$ and $\mathcal{L}(A_k(Y))$, we have by eigenvalue interlacing that $\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$, which immediately implies

$$Tr\mathcal{L}(A_k(X)) - \lambda_1 = \sum_{j=2}^{n} \lambda_j \leq \sum_{j=1}^{n-1} \mu_j \leq \sum_{j=1}^{n-1} \lambda_j = Tr\mathcal{L}(A_k(X)) - \lambda_n.$$

Since $Tr\mathcal{L}(A_k(Y)) = \sum_{j=1}^{n-1} \mu_j$, we have that $|m_X(k) - m_Y(k)| \leq \frac{1}{2}(\lambda_1 - \lambda_n)$. But the laplacian is singular with all eigenvalues non-negative, so $\lambda_n = 0$.

For the adjacency squares, since v is independent, by eigenvalue interlacing, we again have

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n.$$

Let i be the smallest integer satisfying $\alpha_i < 0$. Then for all $j \in [i, n-1]$, $\alpha_j^2 \leq \beta_j^2 \leq \alpha_{j+1}^2$, and similarly, for all $j \in [1, i-2]$, $\alpha_j^2 \geq \beta_j^2 \geq \alpha_{j+1}^2$. Note that if $i = 2$, then β_1 and α_1 are the only positive eigenvalue of $A_k(Y)$ and $A_k(X)$, respectively, which means that $\beta_1 = \sum_{j=2}^{n-1} |\beta_j| \geq \sum_{j=2}^{n-1} |\alpha_j| \geq |\alpha_2|$. Thus we have that

$$\sum_{j=1}^{n} \alpha_j^2 - \alpha_1^2 - \alpha_n^2 \leq \sum_{j=1}^{n-1} \beta_j^2 = \sum_{j=1}^{i-1} \beta_j^2 + \sum_{j=i}^{n-1} \beta_j^2 \leq \sum_{j=1}^{n} \alpha_j^2 - \alpha_i^2.$$

Thus the multiplicity gap for distance d_k between X and Y is

$$|m_X(k) - m_Y(k)| \leq \frac{1}{2}(\alpha_1^2 + \alpha_n^2 - \alpha_i^2) < \frac{1}{2}(\alpha_1^2 + \alpha_n^2).$$

Note that α_i is one of the eigenvalues of $A_k(X)$ with smallest magnitude. $\square$

Theorem 5.2. *Let X be a finite metric space with n points and r distinct distances $d_1, d_2, \ldots, d_r$. Let p be a prime satisfying $p - 1 < \frac{n(n-1)}{n+r}$. Let $\mathcal{U} = (P, D)$ be a subgraph of $\mathcal{M}(X)$ such that for all $v \in P$, $\deg_{\mathcal{U}}(v) = h_v p$ with $h_v \in \mathbb{Z}^+$ and for all $d_k \in D$, $\deg_{\mathcal{U}}(d_k) = \ell_k p$ with $\ell_k \in \mathbb{Z}^+$. Then for all $d_k \in D$, it holds that*

$$\sum_{j=n-|P|+1}^{n} \lambda_j^{(k)} \leq \ell_k p.$$

6 Relation Partition and Their Cardinalities

Let X be a finite set of n elements, called points. Let $R_1, \dots, R_r$ be symmetric relations that partition the unordered pairs of points in X. We call such a partition a *relation partition of* X. Then for each $k \in [r]$, we define G_k to be the graph corresponding to R_k; that is, G_k has vertex set X where $u \sim v$ if and only if $\{u, v\} \in R_k$.

We are interested in the set $\{|R_k| : k \in [r]\}$. Observe that $|R_k| = \frac{1}{2} Tr(L(G_k))$, where L denotes the Laplacian matrix.

Lemma 1. *For any finite set X and relation partition of X, for any prime p satisfying $p - 1 < \frac{n(n-1)}{n+r}$, there exist nonempty subsets $D \subseteq [r]$ and $P \subseteq X$, such that for all $k \in D$, there is a positive integer ℓ_k such that*

$$\ell_k p \le \sum_{j=1}^{|P|} \lambda_j^{(k)},$$

where $\lambda_j^{(k)}$ is the j-th largest eigenvalue of $L(G_k)$.

Proof. [nullstellensatz]

[eigenvalue interlacing] □

Lemma 1 on its own is not so powerful because we cannot in general control $|P|$ or $|D|$. What follows are conditions that enable us to control these quantities, which will enable us to prove necessary lower bounds on the relation cardinalities.

There is a special case when $n/2$ or $(n+1)/2$ is prime since in this case the point vertices of the subgraph of $\mathcal{M}$ given by the nullstellensatz must have degree p.

Corollary 6.1. *If $\lceil n/2 \rceil$ is prime, then $m \ge |D|$ and*

$$mp \le \sum_{k \in D} \sum_{j=1}^{m} \lambda_j^{(k)}.$$

Proof. We have $m = \sum_{k \in D} \ell_k$. □

Let $X_1, \dots, X_s$ be finite sets with cardinalities $n_1, \dots, n_s$, respectively. Let $R_1, \dots, R_r$ be a relation partition of the union of unordered pairs from each of $X_1, \dots, X_s$. Let p be the largest prime satisfying $p - 1 < \frac{n_i(n_i-1)}{n_i+r}$ for all $i \in [s]$. Then there exists a $D \subseteq [r]$ and $m \geq 1$ such that for all $k \in D$, there exists a nonnegative integer sequence $(\ell_k^{(1)}, \ldots, \ell_k^{(s)})$, not all 0 such that

$$\sum_{i=1}^{s} \ell_k^{(i)} p \leq \sum_{j=1}^{m} \lambda_j^{(k)}.$$

[proof of something like this]

Under what conditions can we ensure that all subgraphs given by the nullstellensatz share a relation vertex? Such conditions provide a natural lower bound on a maximum cardinality of a relation. The next theorem presents such a condition.

Theorem 6.2. *Suppose there exists an integer t satisfying $t(t-1) = \sum_{i=1}^{s} n_i(n_i-1)$. Suppose each G_k has max degree d, $r = t-1$, and $s \geq Cd^2 \ldots$ other conditions needed. Then there exists a $k \in [r]$ such that $|R_k| \geq t$.*

Remark 6.3. Notice that the above theorem provides a general necessary condition for the existence of a crescent family.