Johnson Subgraph Problem Proposal

Tao Gaede

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Abstract

Summary

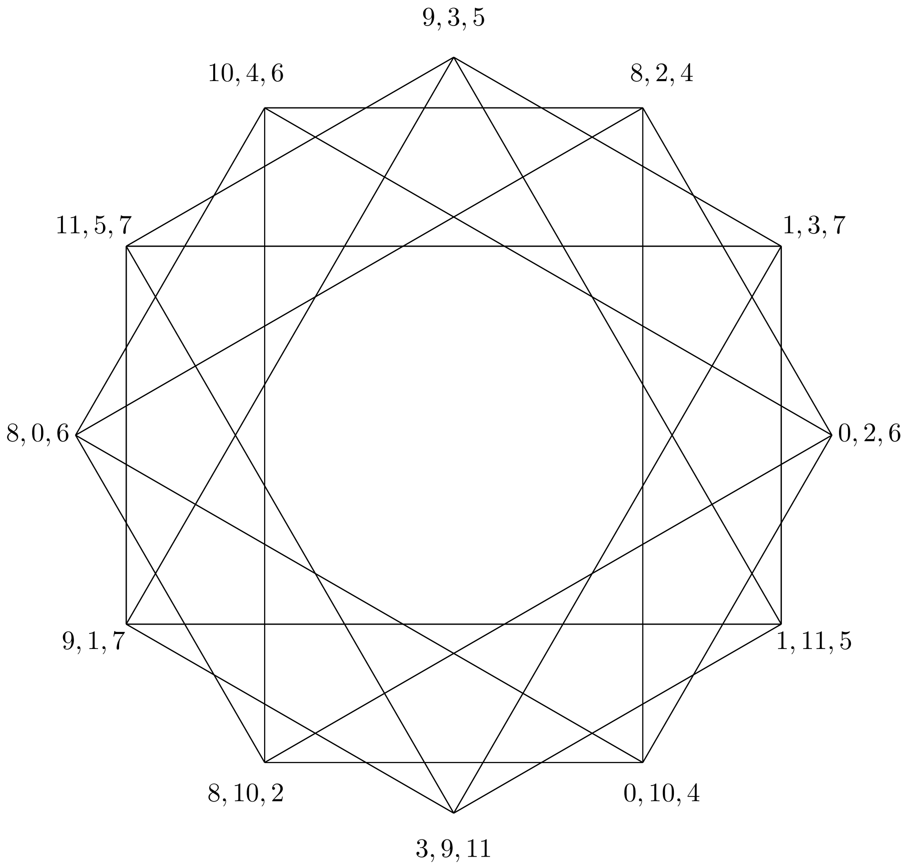
1 Preliminaries

Let $J = J(n,k,i)$ be the graph with vertex set $\binom{\mathbb{Z}_n}{k}$ and for every $u, v \in V(J)$, $uv \in E(J)$ if and only if $|u \cap v| = k - i$ or equivalently $|u \Delta v| = 2i$. We call $J(n,k,i)$ a *Johnson graph*.

We are interested in a particular class of induced subgraphs of $J(n,k,i)$, but we briefly introduce general Johnson induced subgraphs (JISs). General induced subgraphs of $J(n,k,i)$ have apparently only been studied recently in the last 15 years or so. [Brief survey of literature on JISs]

The class of JISs we aim to study are those whose vertices are characterized by a given pattern of consecutive differences. By "pattern of consecutive differences", we mean the following: Let $X \subseteq \mathbb{Z}_n$ and suppose $X = \{x_0, \ldots, x_{k-1}\}$ where $x_0 \leq \cdots \leq x_{k-1}$. Then the pattern of consecutive differences for X, denoted N_X, is $N_X = (x_1 - x_0, x_2 - x_1, \ldots, x_{k-1} - x_{k-2}, x_0 - x_{k-1})$; additionally, we consider N_X to be rotationally invariant, so all we care about is the cyclic ordering of the consecutive differences. Note that N_X is an *integer necklace with k beads of at most n colours*. Let N be an integer necklace with k beads of at most n colours. Then define $\mathcal{F}_N \subset \binom{\mathbb{Z}_n}{k}$ where for every $X \in \mathcal{F}_N$, $N_X = N$. Observe that $\mathcal{F}_N$ is closed under taking translates, so for every $X, Y \in \mathcal{F}_N$, $Y = X + t$ for some $t \in \mathbb{Z}_n$ and $|\mathcal{F}_N| = n$. The primary structure of interest is the subgraph of $J(n,k,i)$ induced by $\mathcal{F}_N$ for a given n, k, i, and N; we denote this subgraph by $JS(\mathcal{F}_N, i)$.

[Examples of JISs for]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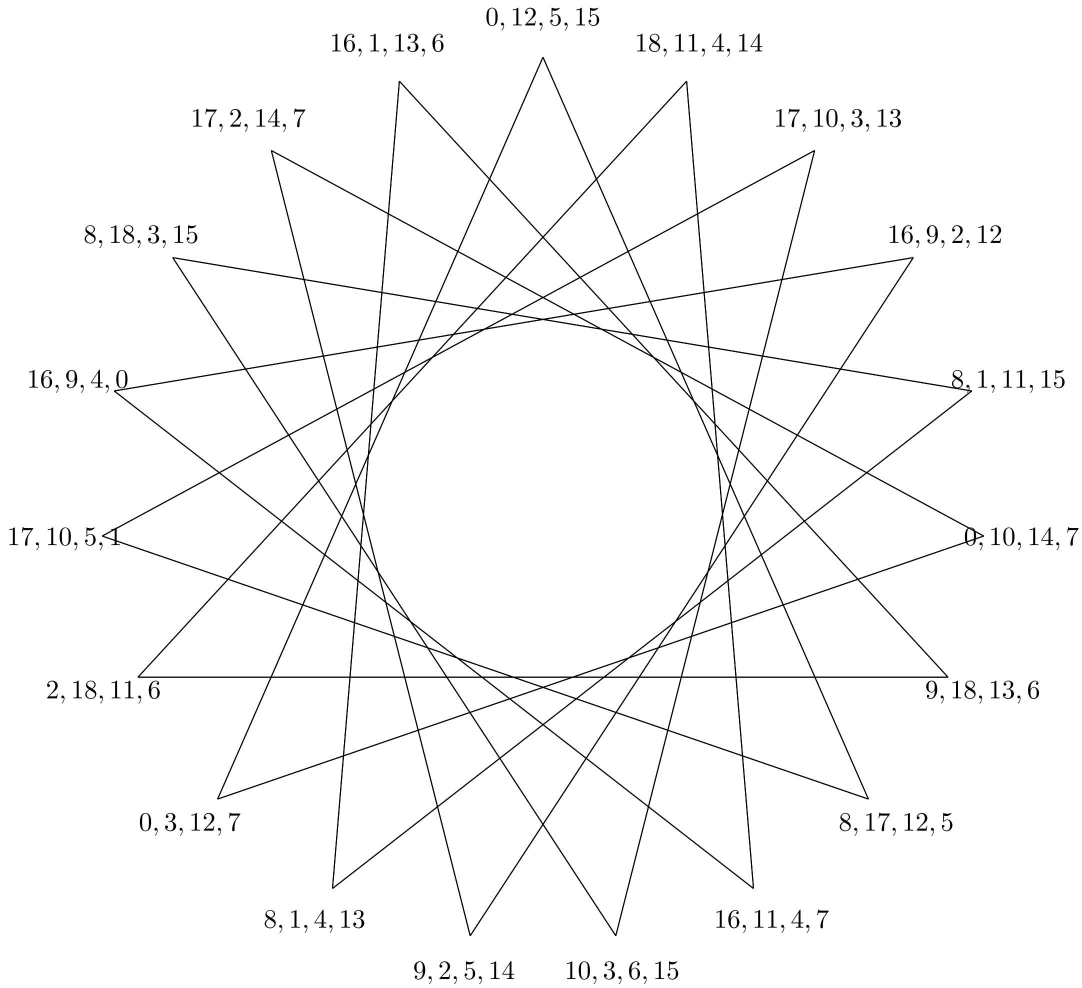
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8, 1, 11, 15
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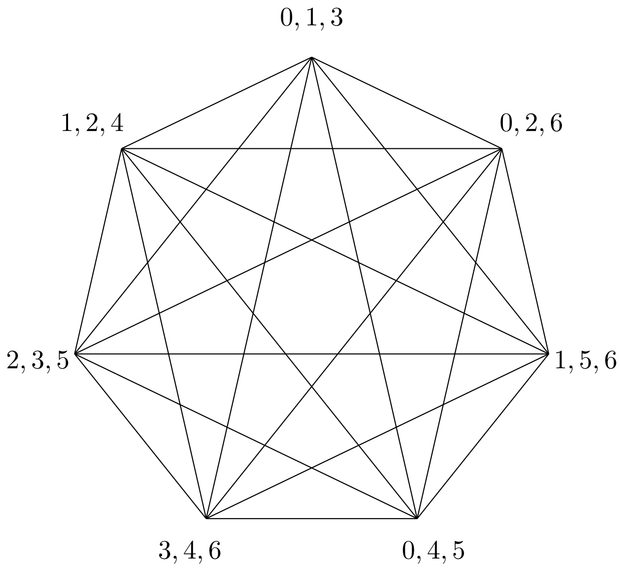
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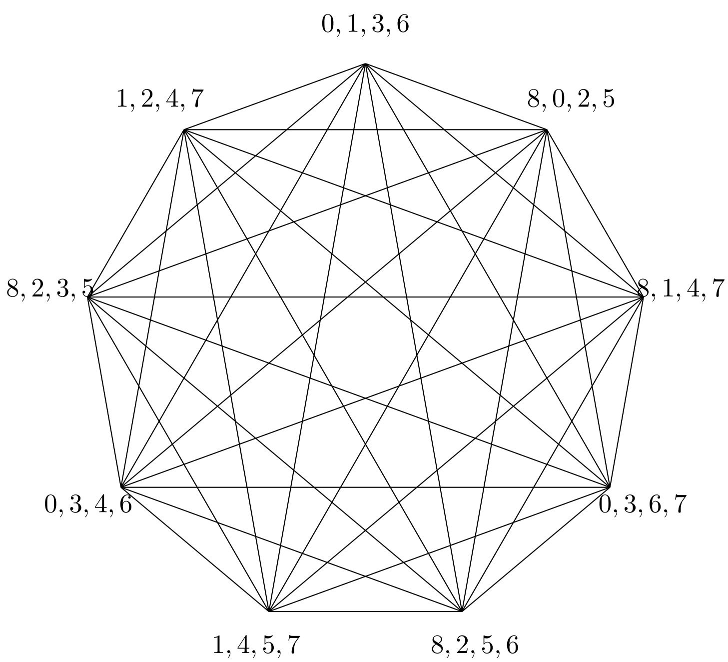
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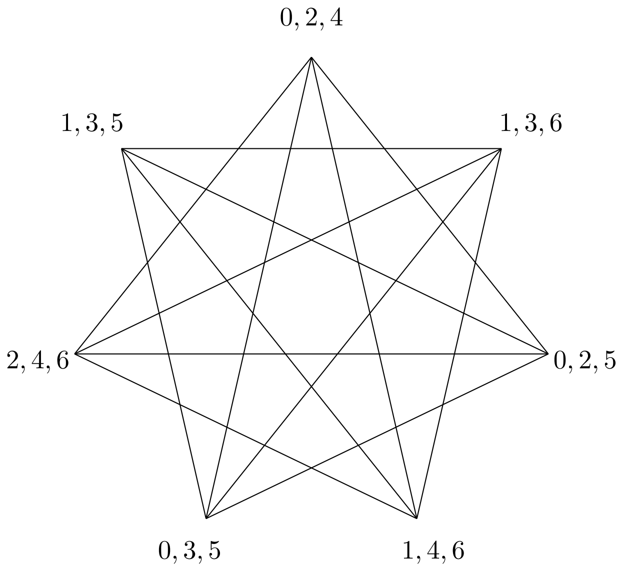
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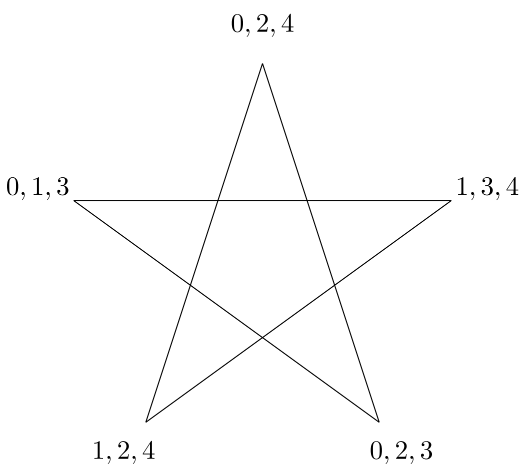
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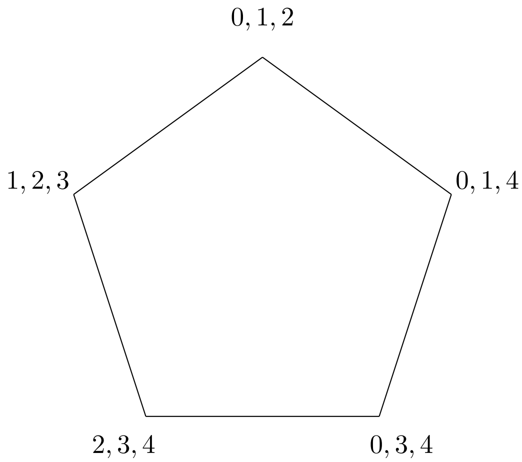
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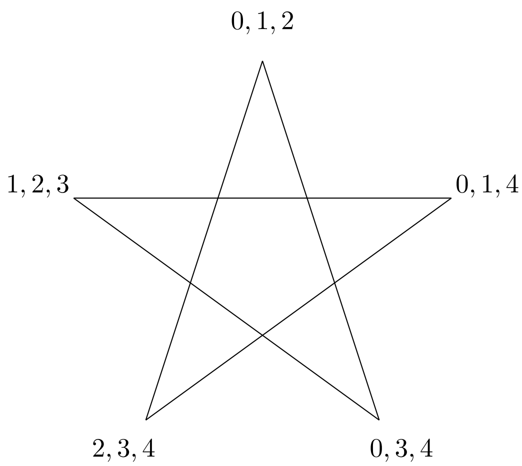
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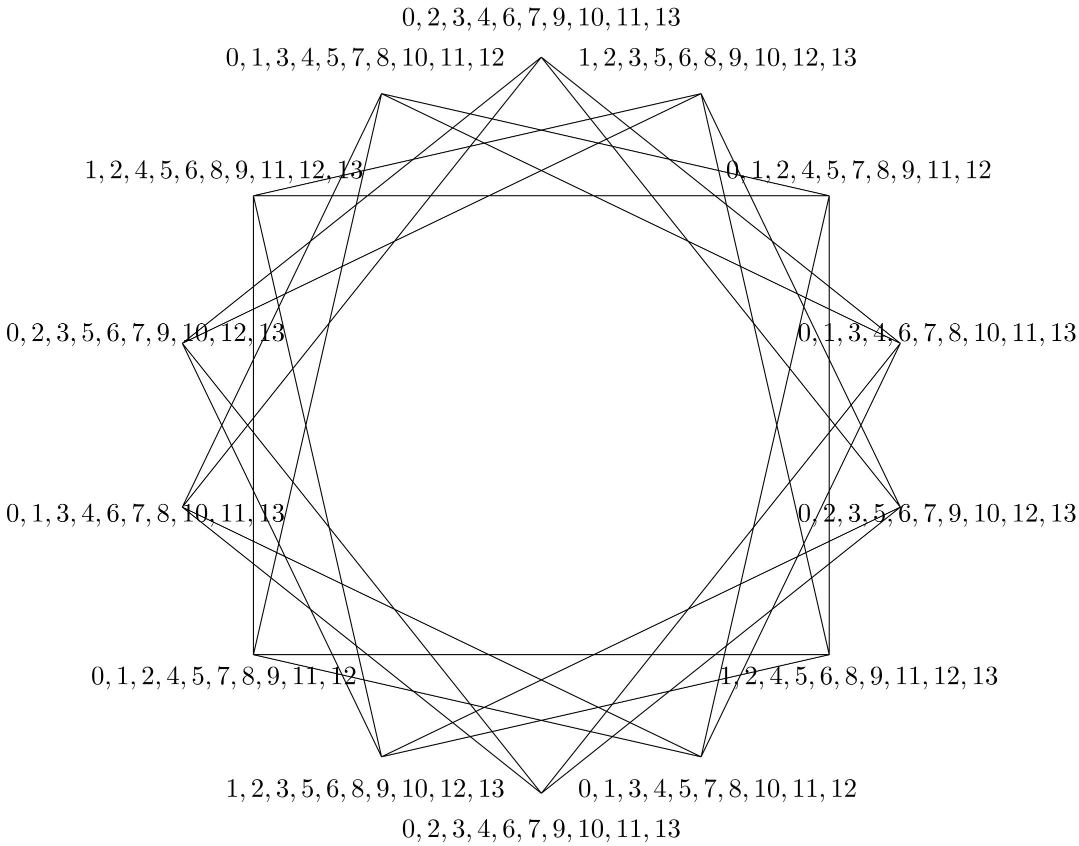

Observation 1. Let N be an integer necklace with k beads of at most n colours. Then

$$\bigcup_{i=1}^{k-1} JS(\mathcal{F}_N, i) \simeq K_k.$$

Conjecture 1.1. *The degree of $JS(\mathcal{F}_n, i)$, denoted $d(JS(\mathcal{F}_n, i)))$, is*

$$|\{\sum\}$$

2 Initial Questions

Question 1. *Let N_1 and N_2 be distinct integer necklaces with k beads of at most n colours. Let G be the subgraph of $J(n, k, i)$ induced by $\mathcal{F}_{N_1} \cup \mathcal{F}_{N_2}$. Does there exist an integer necklace N' with k' beads of at most n' colours such that $G \simeq H$ where H is the subgraph of $J(n', k', i')$ induced by $\mathcal{F}_{N'}$?*

Conjecture (Probably Easy)**.** *If every difference in $\{1, \ldots, \lfloor n/2 \rfloor\}$ is in N, then $JS(\mathcal{F}_N, k-1) \simeq K_k$.*

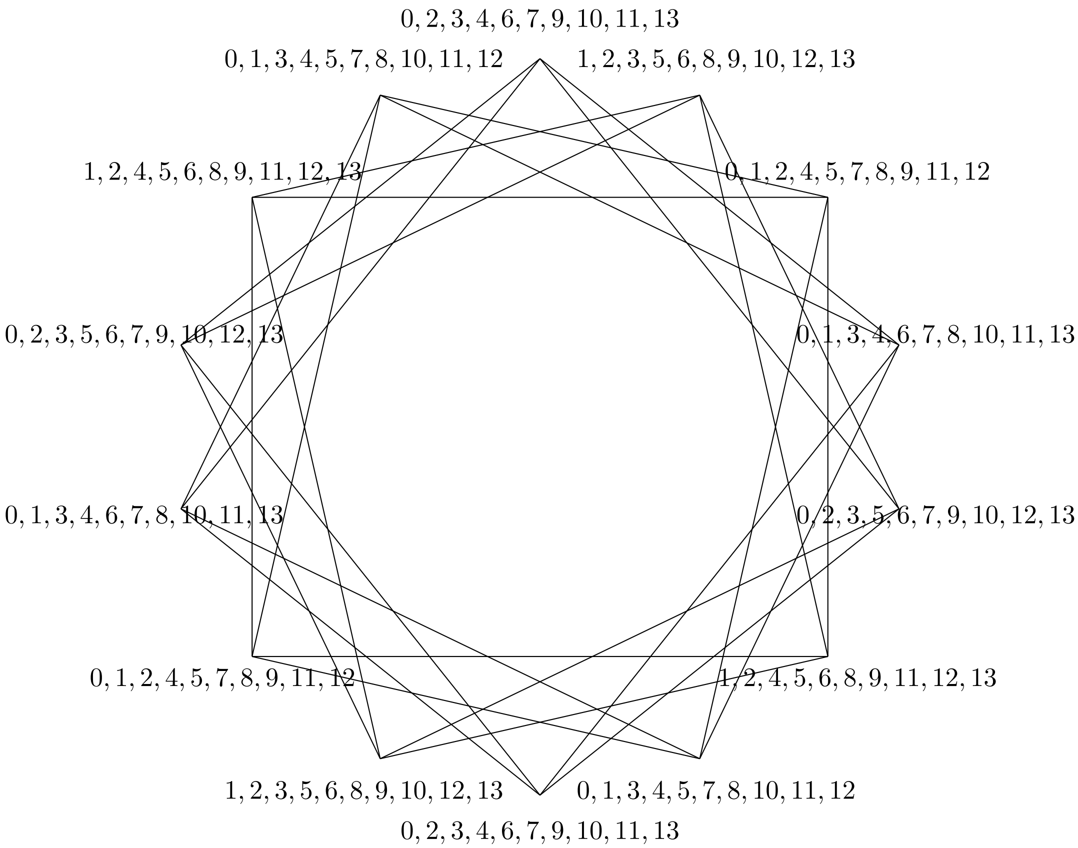

Figure 1: $N = [2, 1, 1, 2, 1, 2, 1, 1, 2, 1]; (n, k, i) = (14, 10, 2)$

References