Johnson Subgraphs and Consecutive Differences

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Let $J = J(n,k,i)$ be the graph with vertex set $\binom{\mathbb{Z}_n}{k}$, and for every $u, v \in V(J)$, $uv \in E(J)$ if and only if $|u \cap v| = k - i$ or equivalently $|u \Delta v| = 2i$. We call $J(n,k,i)$ a *Johnson graph*, and we let $i \in \{1, \dots, k\}$.

We are interested in a particular class of induced subgraphs of $J(n,k,i)$ characterized by a pattern of consecutive differences. General induced subgraphs of $J(n,k,i)$ have apparently only been studied recently in the last 15 years or so.

The class of Johnson induced subgraphs (JISs) we aim to study are those whose vertices are characterized by a given pattern of consecutive differences. By "pattern of consecutive differences", we mean the following: Let $X \subseteq \mathbb{Z}_n$ and suppose $X = \{x_0, \dots, x_{k-1}\}$ where $x_0 \leq \dots \leq x_{k-1}$. Then the pattern of consecutive differences for X, denoted N_X, is $N_X = (x_1 - x_0, x_2 - x_1, \dots, x_{k-1} - x_{k-2}, x_0 - x_{k-1})$; additionally, we consider N_X to be rotationally invariant, so all we care about is the cyclic ordering of the consecutive differences. Note that N_X is an *integer necklace with k beads of at most n colours*. Let N be an integer necklace with k beads of at most n colours. Then define $\mathcal{F}_N \subset \binom{\mathbb{Z}_n}{k}$ where for every $X \in \mathcal{F}_N$, $N_X = N$. Observe that $\mathcal{F}_N$ is closed under set translation, so for every $X, Y \in \mathcal{F}_N$, $Y = X + t$ for some $t \in \mathbb{Z}_n$ and $|\mathcal{F}_N| = n$. The primary structure of interest is the subgraph of $J(n,k,i)$ induced by $\mathcal{F}_N$ for a given n, k, i, and N; we denote this subgraph by $JS(\mathcal{F}_N, i)$.

Note we have the following fact:

Observation 1. Let N be an integer necklace with k beads of at most n colours. Then

$$\bigcup_{i=1}^{k} JS(\mathcal{F}_N, i) \simeq K_n.$$

Question 1 (Characterizing Unions)**.** *Let N_1 and N_2 be distinct integer necklaces with k beads of at most n_1 and n_2 colours, respectively. Does there exist an integer necklace N with k' beads of at most $|\mathcal{F}_{N_1} \cup \mathcal{F}_{N_2}|$ colours such that $JS(\mathcal{F}_{N_1} \cup \mathcal{F}_{N_2}, i) \simeq JS(\mathcal{F}_N, i)$? If so, what is N? How does the structure of N relate to that of N_1 and N_2? Can we define a binary operation $*$ on integer necklaces where $N_1 * N_2 = N$?*

Perhaps the place to start for Question 1 is to restrict N_2 to be the "flip" of N_1.

Question 2 (Structure of Degree Sequence)**.** *Let N be an integer necklace with k beads of at most n colours. For $i \in \{1, \dots, k-1\}$, let d_i be the degree of $JS(\mathcal{F}_N, i)$.*

- *What are the properties of the sequence $D = (d_1, \dots, d_{k-1})$?*
- *How are N and D related? Is it possible to determine D without having to generate all of the graphs $JS(\mathcal{F}_N, i)$ for all i?*
- *Does every necklace N have a unique degree sequence D? Or can there be necklaces N and N' such that each correspond to D?*

Conjecture (Probably Not Hard)**.** *If every difference in $\{1, \dots, \lfloor n/2 \rfloor\}$ is in N, then $JS(\mathcal{F}_N, k-1) \simeq K_k$.*

Below are examples of $JS(\mathcal{F}_N, i)$ for various values of n, k, i, and N.

$0, 4, 7$

$8, 1, 5$ $3, 11, 6$

$9, 2, 6$ $10, 2, 5$

$10, 3, 7$ $9, 4, 1$

$8, 11, 4$ $8, 0, 3$

$0, 9, 5$ $2, 11, 7$

$1, 10, 6$

Figure 1: $N = [4, 3, 5]; (n, k, i) = (12, 3, 1)$

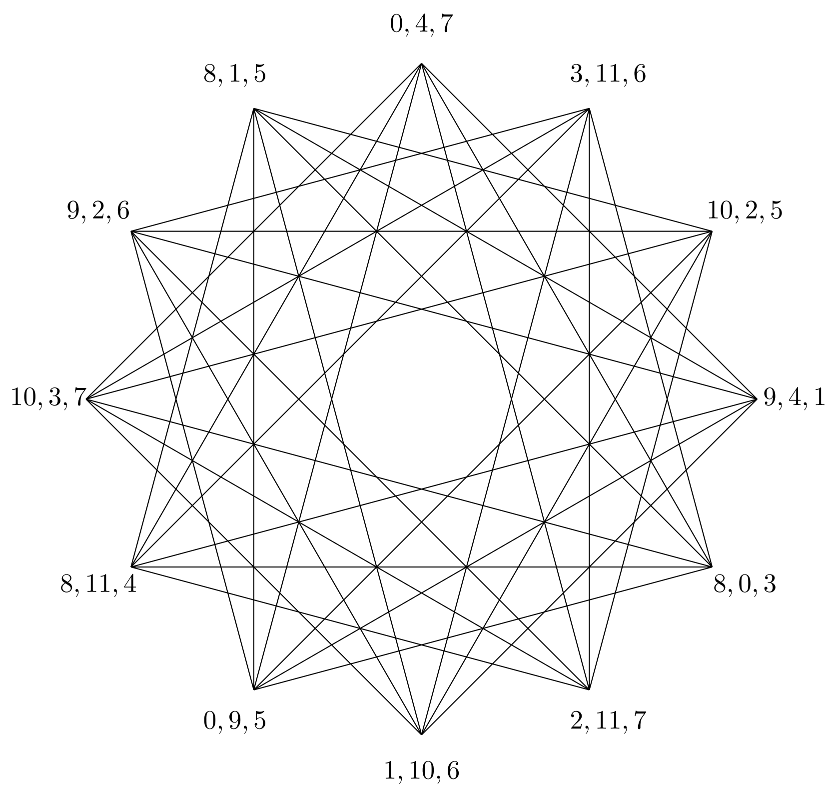

Figure 2: $N = [4, 3, 5]; (n, k, i) = (12, 3, 2)$

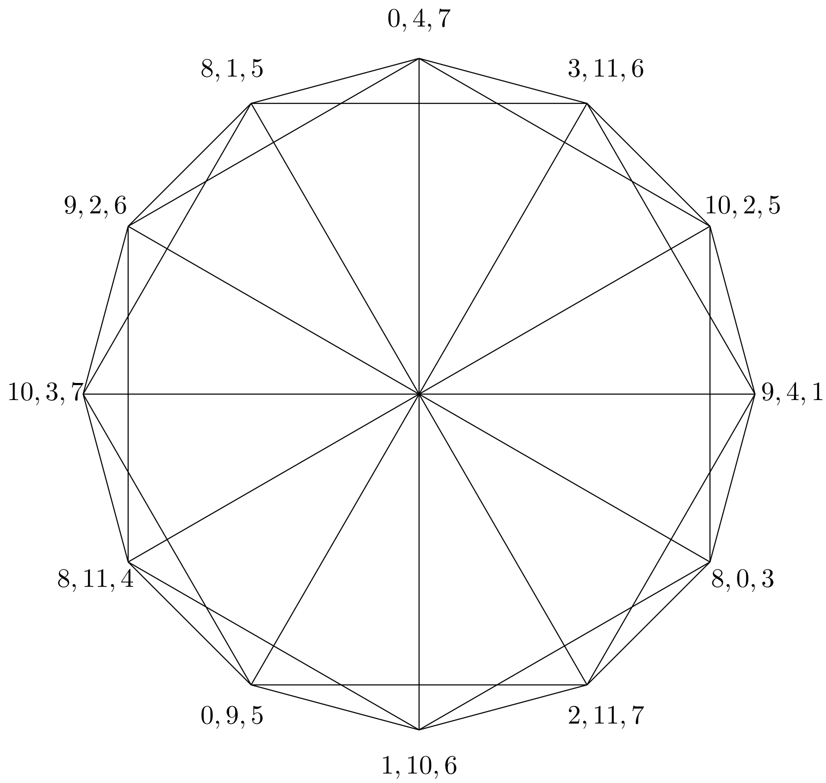

Figure 3: $N = [4, 3, 5]; (n, k, i) = (12, 3, 3)$

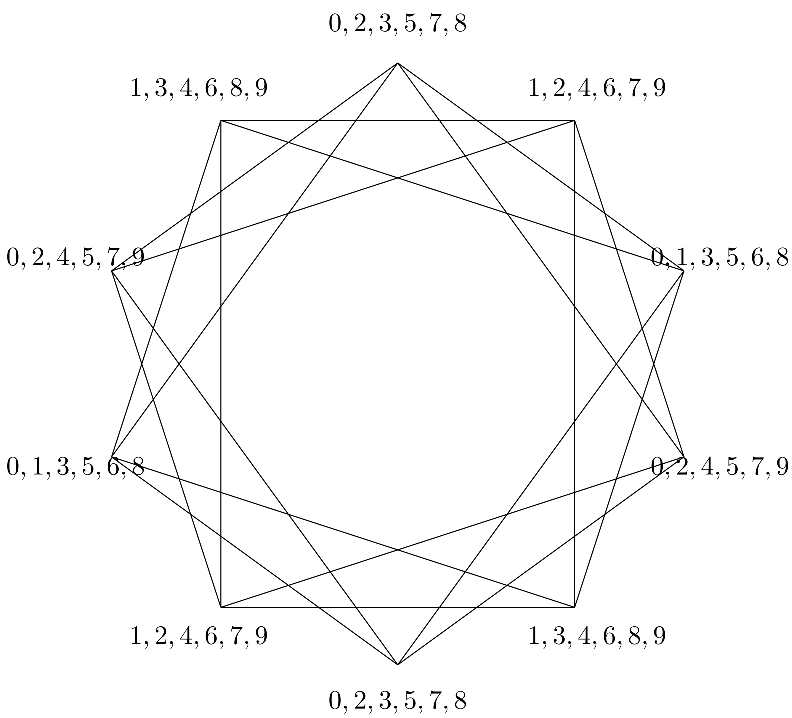

Figure 4: $N = [2, 1, 2, 2, 1, 2]; (n, k, i) = (10, 6, 2)$

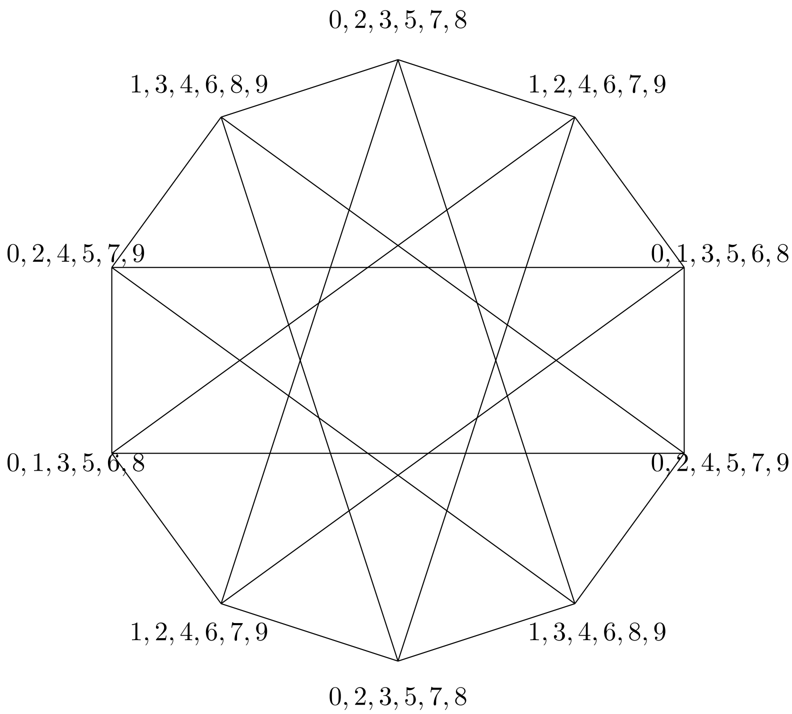

Figure 5: $N = [2, 1, 2, 2, 1, 2]; (n, k, i) = (10, 6, 4)$