Ph.D. Working Notes

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1 Crescent Labelled Trees

Let T be a tree of order n. A crescent labelling of T is a map $L : E(T) \mapsto \{1, 2, \ldots, t\}$, such that the distance multiset of $L(T)$ is of the form $\{d_1^1, d_2^2, \ldots, d_{n-1}^{n-1}\}$. The diameter of T, denoted $\mathrm{Diam}(T)$, is the length of the (u, v)-path in T. The max degree of T is denoted $\Delta(T)$.

Lemma 1 (Basic Diameter Lower Bound)**.** *Let t be a positive integer. If $L(T)$ is a crescent labelling of the tree T with weights $\{1, 2, \ldots, t\}$, then* $\mathrm{Diam}(T) \geq \frac{n-1}{t}$.

Proof. Since there are at least $n-1$ distinct distances, there is a distance d with value at least $n-1$. Let $u, v \in V(T)$ such that $d(u, v) = d$, then since t is the max edge weight, this means that the number of edges on a (u, v)-path is at least $\frac{d}{t} \geq \frac{n-1}{t}$. □

For a pair of vertices $u, v \in V(T)$, we denote the (u, v)-path in T as $P(u, v)$. Lemma 2 below generalizes the observation underlying the maximum degree upper bound of $\sim \sqrt{2n}$.

Lemma 2. *Let T be a tree of order n. For every $i \in [1, n-1]$, $M \in V(T)$, and $j \in \mathcal{N}(M)$, define*

$$D_j := \{u \in V(T) \setminus \{M\} : d(u, M) = d_i, j \in P(u, M)\}.$$

Then distance $2d_i$ occurs with multiplicity at least $\sum_{j<k}^{\deg(M)} |D_j| \cdot |D_k|$.

Proof. Let $M \in V(T)$ and $i \in [1, n-1]$. Since T is a tree, there is always a unique (u, v)-path for all $u, v \in V(T)$. So, for each $u \in D_j$ and $v \in D_k$, the (u, v)-path must go through M, which means $d(u, v) = d(u, M) + d(M, v) = 2d_i$. There are $|D_j| \cdot |D_k|$ such u and v pairs, so indeed $2d_i$ has multiplicity at least $\sum_{j<k}^{\deg(M)} |D_j| \cdot |D_k|$. □

Now we apply the lemma to get a condition on crescent labelled trees.

Proposition 1.1 (Max Multiplicity Condition)**.** *Let $L(T)$ be a crescent labelling of a tree T. Then for every $i \in [1, n-1]$, $M \in V(T)$, and $j \in \mathcal{N}(M)$,*

$$\sum_{j<k}^{\deg(M)} |D_j| \cdot |D_k| < n.$$

Proof. Since $L(T)$ is a crescent labelling of T, no distance can have multiplicity greater than $n-1$and T is a tree. Since T is a tree, it follows by Lemma 2 that for each vertex $M \in V(T)$, $i \in [1, n-1]$, $\sum_{j<k}^{\deg(M)} |D_j| \cdot |D_k| < n$. □

Next is a general lemma on $\{1, 2, \ldots, t\}$-words containing subwords with $t-1$ consecutive $1s$.

Lemma 3 (Arithmetic Condition)**.** *Let $k \geq 2t$. Let $\mathbf{w}$ be a $\{1, 2, \ldots, t\}$-word with length k. If $w_{a-t+2} = w_{a-t+3} = \cdots = w_a = 1$ for some $a \in \{1, 2, \ldots, k\}$, then each value*

$$1, 2, \ldots, \max\left\{\sum_{i=1}^{a} w_i, \sum_{i=a-t+2}^{k} w_i\right\}$$

occurs as a partial sum in $\mathbf{w}$.

Proof. Suppose without loss of generality that $\sum_{i=1}^{a} w_i \leq \sum_{i=a-t+2}^{k} w_i$. Then it is sufficient to show that every value $1, 2, \ldots, \sum_{i=a-t+2}^{k} w_i$ occurs as a partial sum in $\mathbf{w}$. Call $w_{a-t+2}, w_{a-t+3}, \ldots, w_a$ the *unit segment* of $\mathbf{w}$ and $w_{a+1}, w_{a+2}, \ldots, w_k$ the *non-unit segment* of $\mathbf{w}$. We proceed by induction on the number of terms r in the non-unit segment of $\mathbf{w}$. When $r = 1$, $w_{a+r} \in \{1, \ldots, t\}$, and since the unit segment has $t-1$ 1s, for each $j \in \{1, 2, \ldots, t-1\}$, we have the partial sums $j = \sum_{i=0}^{j-1} w_{a-i}$. Then the values between w_{a+r} and $\sum_{i=a-t+2}^{a+r} w_i$ are of the form $w_{a+r} + \sum_{i=0}^{j-1} w_{a-i}$. For the inductive step, the values $1, 2, \ldots, \sum_{i=a-t+2}^{a+r-1} w_i$ occur at least once by inductive hypothesis. We have that $w_{a+r} \in \{1, 2, \ldots, t\}$ and the values between $\sum_{i=a+1}^{a+r-1} w_i$ and $\sum_{i=a+1}^{a+r} w_i$ can be obtained from $\sum_{i=a+1-j}^{a+r-1} w_i$ for each $j \in \{1, 2, \ldots, t-1\}$. Then similarly the values between $\sum_{i=a+1}^{a+r} w_i$ and $\sum_{i=a-t+2}^{a+r} w_i$ are $\sum_{i=a+1-j}^{a+r} w_i$ for $j \in \{1, 2, \ldots, t-1\}$. □

We now apply this arithmetic lemma to crescent labelled trees to show that when there are many consecutive 1s on a path, the path cannot be too long with many large weight edges.

Proposition 1.2. *Let $L(T)$ be a crescent labelling of a tree T with edge weights in $\{1, 2, \ldots, t\}$. Then for every path $P = (v_1v_2, v_2v_3, \ldots, v_{t-1}v_t)$ in T such that $w(v_iv_{i+1}) = 1$ for $i \in \{1, 2, \ldots, t-1\}$, it follows that $\max\{d(v_1, u) : u \in V(T)\} < n$ and $\max\{d(v_t, u) : u \in V(T)\} < n$.*

Proof. Let T be a tree with a path P specified in the proposition statement and $L(T)$ a crescent labelling. It is sufficient to show that $\max\{d(v_1, u) : u \in V(T)\} < n$ since the case for v_t is similar. Let $u' \in V(T)$ such that $d(v_1, u') = \max\{d(v_1, u) : u \in V(T)\}$. By Lemma 3, every distance $1, 2, \ldots, d(v_1, u')$ occurs at least once. Since $L(T)$ is a crescent labelling, there can be at most $n-1$ distinct distances, so $d(v_1, u') < n$ as desired. □

The implication for when $t = 2$ is quite strong since this imposes a max distance condition on vertices incident to edges with weight 1.

Corollary 1.3. *Let $L(T)$ be a crescent labelling of a tree T. If $t = 2$, then every vertex incident to an edge with weight 1 has max distance at most $n-1$.*

What follows is a basic lemma about trees that may turn out to be useful in case parameterizing by number of leaves becomes sensible.

Lemma 4 (From Chartrand and Lesniak's text "Graphs and Digraphs" 4th edition)**.** *Let T be a tree with n_i vertices with degree i, where $i \in \{1, 2, \ldots, \Delta(T)\}$. Then $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta(T) - 2)n_{\Delta(T)} + 2$.*

Proof. Note that $n = \sum_{i=1}^{\Delta(T)} n_i$. Since T is a tree,

$$\sum_{i=1}^{\Delta(T)} in_i = \sum_{v \in V(T)} \deg(v) = 2(n-1) = 2\left(\sum_{i=1}^{\Delta(T)} n_i\right) - 2.$$

Rearranging gives $2 + \sum_{i=1}^{\Delta(T)} (i-2)n_i = 0$. □

Corollary 1.4. *If T is a tree, then $\sum_{i=3}^{\Delta(T)} (i-2)n_i < n_1$*