Ph.D. Working Notes

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1 Trees

Let T be a tree of order n. A crescent labelling of T is a map $L : E(T) \mapsto \{1, 2, \ldots, t\}$, such that the distance multiset of $L(T)$ is of the form $\{d_1^1, d_2^2, \ldots, d_{n-1}^{n-1}\}$. The diameter of T, denoted $\mathrm{Diam}(T)$, is the length of the maximum shortest (u, v)-path in T. The max degree of T is denoted $\Delta(T)$.

1.1 Path Perspective: Crescent Labellings and r-Bounding Paths

This section establishes a simple arithmetic condition on long paths in a crescent labelled tree.

Lemma 1 (Arithmetic Condition). *Let $k \geq 2t$. Let $\mathbf{w}$ be a $\{1, 2, \ldots, t\}$-word with length k containing at least two disjoint subwords of $t-1$ 1s. Let $w_{a-t+2} = w_{a-t+3} = \cdots = w_a = 1 = w_b = w_{b+1} = \cdots = w_{b+t-2}$, where a and b are the smallest and largest satisfying $a < b$, respectively. Then all distances $1, 2, \ldots, \sum_{i=a}^{b-1} w_i$ occur with multiplicity at least 2.*

Proof. Let $m = \sum_{i=a}^{b-1} w_i$ and observe that because $w_a = w_b = 1$, $m = \sum_{i=a+1}^{b} w_i$ as well. It is sufficient to show that for any such $\mathbf{w}$, the following set inclusion holds:

$$\{1, 2, \ldots, m\} \subseteq \{d(w_{a+1-j}, w_i) : j \in [0, t-1], i \in [a-j+2, b-1]\}.$$

This is because, if true, the same property will hold for the distances between the all 1s subwords involving w_b, w_{b+1}, …, w_{b+t-2}, which means that each such distance occurs at least twice. We show this by induction on the length of the word $\mathbf{w}' = (w_{a-t+2}, w_{a-t+3}, \ldots, w_a, w_{a+1}, \ldots, w_{a+r-1})$, which we denote by $r+t-1$. The statement clearly holds for $r = 0$. If $r = 1$, then $\mathbf{w}' = (1^{t-1}, 1)$ or $\mathbf{w}' = (1^{t-1}, u)$ for some $u \in [2, t]$, where concatenation means repeated consecutive word elements. In the former case, distances $1, 2, \ldots, t$ clearly occur at least once, and in the latter, distances $1, 2, \ldots, t-1+u$ each occur as well since for any $v \in [0, u-1]$, $t-1+u-v$ is the sum of all but the first v elements of $\mathbf{w}'$. Suppose the inductive hypothesis holds for some $r \geq 1$ and set $m' = \sum_{i=a-t+2}^{a+r-1} w_i$. That is, we assume that the $\{1, 2, \ldots, t\}$-word $\mathbf{w}' = (w_{a-t+2}, w_{a-t+3}, \ldots, w_a, w_{a+1}, \ldots, w_{a+r-1})$ attains the distances $1, 2, \ldots, m'$. Then the length $(r+1)+t-1$ word $\mathbf{w}'$ with a 1 appended to the end clearly attains all distances $1, 2, \ldots, m'+1$. The other case is when u is appended to the end of $\mathbf{w}'$ where $u \in [2, t]$. In this case, the word attains the distance $\sum_{i=a-t+2+v}^{a+r} w_i = m' + v$ for each $v \in [2, u]$, which is what we wanted to show.

Thus in particular, the distances $1, 2, \ldots, m$ each occur at least once in

$$\{d(w_{a+1-j}, w_i) : j \in [0, t-1], i \in [a-j+2, b-1]\}$$

as well as at least once in

$$\{d(w_{b-1+j}, w_i) : j \in [0, t-1], i \in [a+1, b+j-2]\},$$

as desired. □

Lemma 2 (Basic Diameter Lower Bound). *Let t be a positive integer. If $L(T)$ is a crescent labelling of the tree T with weights $\{1, 2, \ldots, t\}$, then* $\mathrm{Diam}(T) \geq \frac{n-1}{t}$.

Proof. Since there are at least $n-1$ distinct distances, there is a distance d with value at least $n-1$. Let $u, v \in V(T)$ such that $d(u,v) = d$, then since t is the max edge weight, this means that the number of edges on a (u,v)-path is at least $\frac{d}{t} \geq \frac{n-1}{t}$. □

1.1.1 t-Bounded Paths

Let $L(T)$ be a labelling of T with edge weights from $\{1, 2, \ldots, t\}$. If a path $P \subseteq T$ is labelled by L in such a way that **(1)** all edge weights are in $\{1, 2, \ldots, r\}$ for some $1 \leq r \leq t$, and **(2)** it is appended on each side by $r-1$ consecutive edges with weight 1, then we call P an *r-bounded path* of T. Define $b_L(T, r)$ to be the maximum weight sum over all edges of a r-bounded path of T under L.

Proposition 1.1. *Let $L(T)$ be a crescent labelling of T with weights $\{1, 2, \ldots, t\}$. Suppose $L(T)$ contains a r-bounded path $P = (e_1, e_2, \ldots, e_k)$ of T. Then every distance $1, 2, \ldots, b_L(T, r)$ occurs with multiplicity at least 2.*

Proof. The statement follows immediately from Lemma 1. □

Proposition 1.2. *Let $L(T)$ be a crescent labelling of T with weights $\{1, 2, \ldots, t\}$. Then $b_L(T, r) \leq n-1$ for every $r \in \{1, 2, \ldots, t\}$.*

Proof. Suppose $b_L(T, r) \geq n-1$. Then by the proposition, every distance $1, 2, \ldots, n-1$ has multiplicity at least 2, which contradicts $L(T)$ inducing $n-1$ distinct distances such that exactly one is unique. □

Corollary 1.3. *Let $L(T)$ be a crescent labelling of T with weights $\{1, 2, \ldots, t\}$. Define a_i to be the number of weight i edges on an r-bounding path of T with length k and maximum weight sum under L. Then $\sum_{i=1}^{r} i a_i \leq n-1$ and $\sum_{i=1}^{r} a_i = k$.*

Proposition 1.4. *Let $L(T)$ be a crescent labelling of T with weights $\{1, 2, \ldots, t\}$. If $P = (e_1,\ e_2,\ \ldots,\ e_k)$ is an r-bounding path of T under L, then $k \leq n - 1 - \sum_{i=2}^{r} (i-1) a_i$.*

The next proposition suggests that, when $t = 2$, the larger the maximum distance becomes (over $n-1$), the more weight 2 edges there should be on the tails of the path inducing maximum distance.

Proposition 1.5. *Let $L(T)$ be a crescent labelling with weights $\{1, 2, \ldots, t\}$. For every path $F = (v_1v_2, v_2v_3, \ldots, v_{t-1}v_t)$ satisfying $w(v_iv_{i+1}) = 1$ for each $i \in [1, t-1]$, define X and Y to be the vertices of T connected to v_1 and v_t, respectively, but not through those in F. Then for every $x \in X$, $d(v_1, x) \leq n$ and for every $y \in Y$, $d(v_t, y) \leq n$.*

Proof. Let $P = (e_1, e_2, \ldots, e_k)$ be a path such that the longest distance in $L(T)$ is $\sum_{i=1}^{k} w(e_i)$. Let $P' = (e_a, e_{a+1}, \ldots, e_b)$ be any smallest path satisfying $\sum_{i=a}^{b} w(e_i) \geq n$ for some $1 \leq a < b \leq k$. If there is a $j \in \{1, 2, \ldots, a-1\} \cup \{b+1, b+2, \ldots, k\}$ such that $w(e_j) = w(e_{j+1}) = \cdots = w(e_{j+t-1}) = 1$, then every distance $1, 2, \ldots, n$ occurs at least once (by similar reasoning to the proof of Lemma 1, a contradiction. So, if $t = 2$, then all edges connected to $P \setminus P'$ have to have weight 2 since otherwise there would be a path beginning with a weight 1 edge containing P', which again means there would be more than $n-1$ distinct distances. In general, there can be no path of $t-1$ consecutive weight 1 edges in $P \setminus P'$. □

Proposition 1.6. *Let $L(T)$ be a crescent labelling with both weights $\{1, 2\}$. Let P be a path in T where $P = (e_1, e_2, \ldots, e_k)$ such that the following two properties hold:*

1. $\sum_{i=1}^{k} w(e_i) = d_{max}$.
2. *P contains a subpath $P' = (e_a, e_{a+1}, \ldots, e_{a+b-1})$ satisfying $\sum_{i=a}^{a+b-1} w(e_i) \in [n + 1, n + 2]$ where a and b are chosen so that the maximum number of weight 2 edges are to the "right" of b.*

Then $k \leq \frac{3n}{4}$.

Proof. Note by proposition, no edge e_h for any $h \in [1, a-1] \cup [a+b, k]$ satisfies $w(e_h) = 1$. Suppose for a contradiction that $k > 3n/4$. Then either P' contains an edge with weight 1 or it does not. If P' contains only weight 2 edges, then since $k \geq 3n/4$ and there exists an edge with weight 1, Suppose for some $j \in [a, a+b-1]$, $w(e_j) = 1$. Thus the sums $\sum_{i=1}^{j} w(e_i) \leq n$ and $\sum_{i=j}^{k} w(e_i) \leq n$ both hold. But since $\sum_{i=a}^{a+b-1} w(e_i) \in [n + 1, n + t]$, we have that, say, $n + 1 = a_1 + 2a_2$, where a_1 and a_2 are the number of 1s and 2s in P'. Since $b - a > n/2$ and $k > 3n/4$, either $w(e_1) = \cdots = w(e_{n/4}) = 2$ or $w(e_k) = w(e_{k-1}) = \cdots = w(e_{k-1-n/4}) = 2$. Since $w(e_j) = 1$ and $j \in [a, a + b - 1]$, each of these tail edges with weight 2 correspond to at least 2 distinct distances each, implying that there are at least $n/2$ distinct distances from these tails. Then for each $h \in [1, j]$ there is a distinct distance as well. That is, if $j = n/C$, and if J_1 and J_2 are the number of weights 1 and 2 edges with index larger than j, respectively, then there are $2J_2 + J_1$ distinct distances to the right of j and n/C to the left. Thus we require $n - 1 \geq 2J_2 + J_1 + n/C$. If there are $n/4$ twos and $n/4$ ones and $C > n/4$, then we get a contradiction. Note that we require $J_2 + J_1 + n/C = k$ and we know for sure that $J_2 > n/4$. □

Lemma 3. *Let $L(T)$ be a crescent labelling of T using weights $\{1, 2, \ldots, t\}$. Then for every $r \in \{2, 3, \ldots, t\}$, $L(T)$ cannot contain a subpath $P = (e_1, e_2, \ldots, e_{(n-1)/r})$ such that $w(e_i) \in \{1, 2, \ldots, r\}$ where $i \in \{1, 2, \ldots, (n - 1)/r\}$ and P is preceded by a path of $r - 1$ weight 1 edges.*