Qualitative Reasoning

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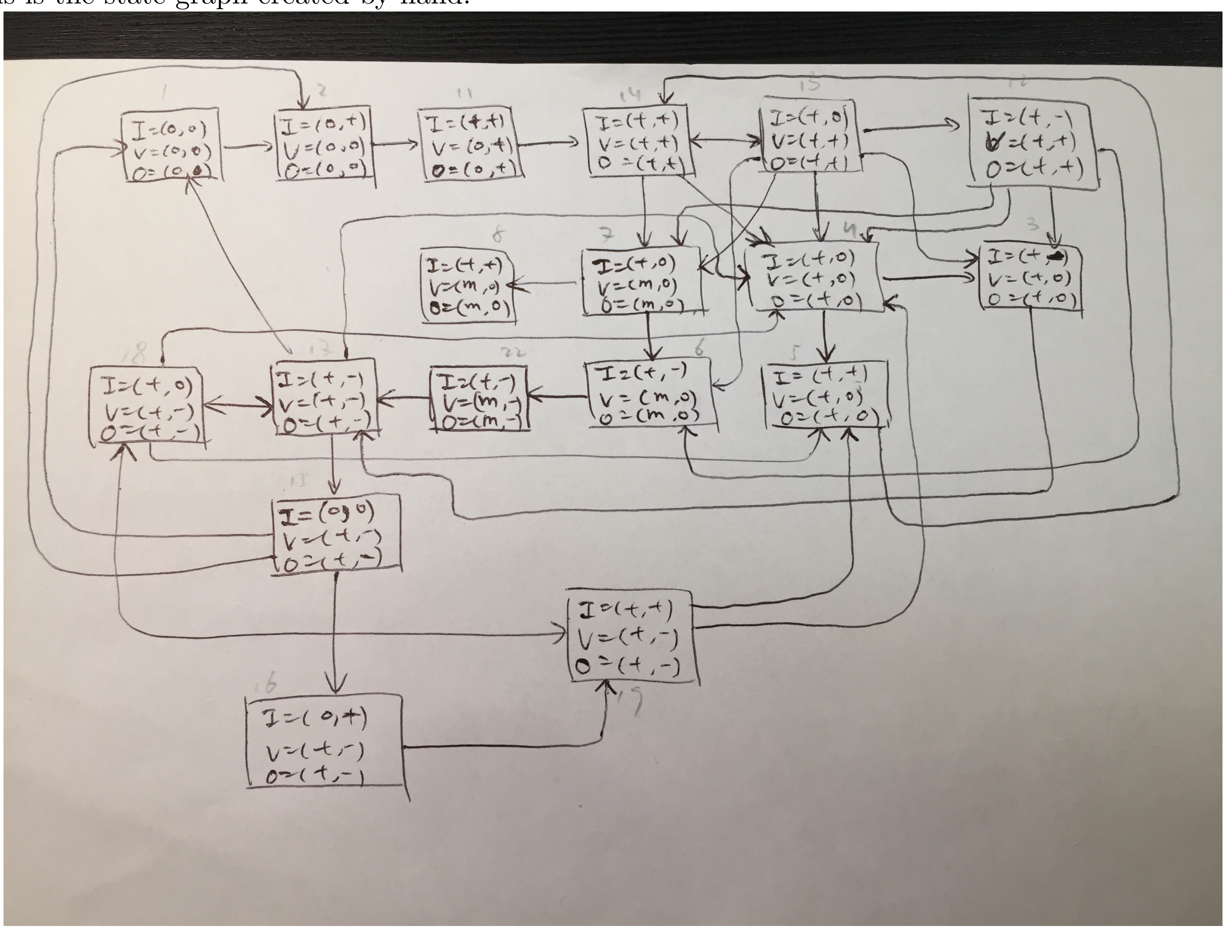
1 Introduction

In this report, we have analyzed a simple input-output system using qualitative reasoning. Our main algorithm is model product. Then we restrict our model by certain assumptions. Based on that, an insightful trace is created. Finally we compare it to our original state diagram drawing.

2 Hand-drawing State Diagram

This is the state-graph created by hand:



Still, for a better view, we refer the reader to the file hand_drawing.jpeg in our zip.

3 Causal Model

We mainly focus on the system $\mathcal{S}$ consisting of: an inflow tap, a container and an outflow pipe. For simplicity, we make some assumption on the system.

1. The inflow tap has non-negative inflow, whose inflow sped can be modified. Besides, it has no upper bound;
2. The container is a cylinder with a limited volume;
3. The outflow pipe is at the bottom of the container, which means that the outflow speed is proportional to the amount of water in the container.

Importantly, all the changes are continuous. The detailed exposition of the continuity will be given below.

3.1 Quantity

We consider the following quantities of the system:

- The inflow $I = (I_0, I_1)$, where I_0 is the inflow speed, and I_1 the inflow acceleration (i.e. derivative [1] of speed);
- The outflow $O = (O_0, O_1)$, where O_0 is the outflow speed, and O_1 the outflow acceleration (i.e. derivative of speed);
- The container volume $V = (V_0, V_1)$, where V_0 is the volume, and V_1 the speed of the volume change (i.e. the derivative of volume).
- The height $H = (H_0, H_1)$, where H_0 is the height of the water in the container, and H_1 the derivative;
- The pressure $P = (P_0, P_1)$, where P_0 is the pressure at the bottom of the container from the water, and P_1 its derivative.

with the respective quantity space:

$$\begin{array}{rr}
I_0 \in \{0, +\} & I_1 \in \{-, 0, +\} \\
O_0 \in \{0, +, \max\} & O_1 \in \{-, 0, +\} \\
V_0 \in \{0, +, \max\} & V_1 \in \{-, 0, +\} \\
H_0 \in \{0, +, \max\} & H_1 \in \{-, 0, +\} \\
P_0 \in \{0, +, \max\} & P_1 \in \{-, 0, +\}
\end{array}$$

So any state S of the system $\mathcal{S}$ consists of tuple

$$S = (I, V, O, H, P)$$

For the derivative, we understand it as the derivative at the current state. This, together with the continuity of changes, mean that at a certain state, the derivative should affect the current quantity magnitude.

[1] Unless mentioned specifically, all the derivatives here are over time t

Having made the basic model setup, we go on to define the dependencies, consisting of two parts: inner-state and inter-state. As the name suggests, the inner-state dependency considers the relation of these quantities in one single state, and the inter-state dependency takes into account the quantity relations in state transition. From a model point of view, we can understand the former as describing which states are legal, while the latter tells which transitions are acceptable.

3.2 Inner-state Dependency

Now let's fix one state and consider the dependencies of different quantities at this single state.

Some remarks before heading on. As we also have $+$ and $-$ in the quantity space which represent positive and negative, we write the algebra relations "plus" and "minus" as $\dot{+}$ and $\dot{-}$ instead, respectively. Besides, note that here the algebra relations (e.g. $+$, $=$) are modified to fit the qualitative setting. For example, $0 \dot{+} + = +$, $+ \dot{-} - = +$, etc.

First we give the (in)equalities, and the argument for their soundness follows:

1. $V_1 = I_0 - O_0$;
2. $V_0 = P_0 = H_0$;
3. $V_1 = P_1 = H_1$.

(1) means that the change of the volume depends on the inflow and outflow speed, namely if inflow speed exceeds outflow speed then the volume has a trend to increase, and vice versa. (2) says that the volume of water, the pressure of water to the container and the height of water in the container are the same (module quality). This is immediate from the given assumption that the container is a cylinder, the volume formula for cylinders $V = S \times H$ (where S stands for the bottom size and H stands for height), and the formula for liquid pressure $P = \rho g h$ (where ρ stands for the liquid density, g the gravity and h the height). From (2) we can derive that their derivatives also coincide, namely we have (3). Therefore we will neglect P and H in our discussion below, since they coincide with V.

Next we have the co-occurence:

1. $V_0 = 0 \iff O_0 = 0$;
2. $V_1 = 0 \iff O_1 = 0$;
3. $V_0 = \max \iff O_0 = \max$;
4. $V_0 = + \iff O_0 = +$

These conditions should also be straightforward. (1) catches the intuition that, if the container is empty, then the outflow is 0, and since the outflow totally caused by the pressure of the water, that the outflow is 0 implies that the container is empty. (2) says further that, if the volume of the water in the container has no trend of changing, then the outflow should also be stable, and vice versa. Again this is guaranteed by our simplified system. (3) also follows directly from our basic setting, namely the volume and the outflow reach their maximum at the same time, since outflow is totally determined by the volume. (4) can be understood as entailed from (2) and (3), since the states for V_0 and O_0 are both exhausted by $0, +$ and max.

Then we have the rules for the special values, namely 0 and max. Intuitively, the derivative cannot be positive (negative) when the maximum (zero) point is reached. Formally, it is that for any $X \in \{I, V, O.H, P\}$ we have :

1. $X_0 = 0 \Longrightarrow X_1 \neq -$;
2. $X_0 = \max \Longrightarrow X_1 \neq +$.

Finally we have the influence, where $\mathfrak{I}$ stands for influence and $\mathfrak{P}$ stand for proportionality, and the superscripts naturally means positive/negative influence:

1. $I \xrightarrow{\mathfrak{I}^+} V$;
2. $O \xrightarrow{\mathfrak{I}^-} V$;
3. $V \xrightarrow{\mathfrak{P}^+} O$;

For the influence and proportionality, we shall be a bit more careful. (1) says that I_0 being positive entails V_1 is positive, ceteris paribus. The intuition behind is that, if we consider only I and V (without O), then a positive inflow should lead to a positive trend of change of the volume, namely $I_0 > 0 \Rightarrow V_1 > 0$. A similar argument for the relation between outflow and water volume in the container should lead to (2). Condition (3) says that, as V_0 increases (decreases), O_0 should also increase (decrease), which again is by the basic setting of our system that the outflow is determined.

In fact, from the above dependencies, we can further derive an equivalence which is not obvious from first sight, but which greatly simplifies the model restriction:

$$V = O$$

This equation boils down to $V_0 = O_0$ and $V_1 = O_1$, at arbitrary state, for which we offers a simple proof:

Proof:($V = O$)

1. For $V_0 = O_0$, note that we have $V_0 = 0 \iff O_0 = 0$, $V_0 = + \iff O_0+$ and $V_0 = \max \iff O_0 = \max$ as the co-occurence, which exhaust the domain of V_0 and O_0.
2. For $V_1 = O_1$, first note that $V_1 = 0 \iff O_1 = 0$ is stated as the co-occurence condition. If $V_1 = +$, then $V \xrightarrow{\mathfrak{P}+} O$ entails that $O_1 = +$. If $V_1 = -$, then $V \xrightarrow{\mathfrak{P}+} O$ entails that $O_1 = -$ as well. But then both directions are down: for example, if $O_1 = +$, then $V_1 \neq -$ and $V_1 \neq 0$, and the only alternative is that $V_1 = +$. Therefore we can conclude that $V_1 = O_1$.

■

3.3 Inter-state Dependency

Now let's go on to the conditions for state transitions. As said in the introduction, our algorithm does model product, so it is better if we make restrictions on what transitions are *not* allowed. For convenience, throughout this section we will consider two states, say $S = (I, V, O, H, P)$ and $S' = (I', V', O', H', P')$, and offer the conditions for the transition $S \to S'$. There are three classes of inter-state rules in our causal model. Such classification is not forced; but we think it helps to clarify the intuitions.

First we have the continuity rules:

1. $X_0 = \max \Longrightarrow X'_0 \neq 0$;
2. $X_1 = + \Longrightarrow X'_1 \neq -$;

3. $X_1 = - \Longrightarrow X_1' \neq +$.

where $X \in \{I, V, O, H, P\}$. The basic idea for continuity rules, as the name suggests, is that for arbitrary quantity, it cannot change discontinuously in two successive states. For example, the outflow cannot change from max to 0 without first becoming $+$.

Second we have the class of rules for single quantity, namely for any $X \in \{I, V, O\}$:

1. It cannot be the case that $X_0 \neq X_0'$ and $X_1 \neq X_1'$, except for the two limit cases:
   (i) $X = (+, -)$ and $X' = (0, 0)$;
   (ii) $X = (+, +)$ and $X' = (\max, 0)$.
2. If $X_1 = 0$, then $X_0' = X_0$;
3. If $X_0 \geqslant +$ and $X_1 = +$, then $X_0' \geqslant X_0$;
4. If $X_0 \geqslant +$ and $X_1 = -$, then $X_0' \leqslant X_0$;
5. If $X = (0, +)$, then $X_0' = +$;
6. If $X = (\max, -)$, then $X_0' = +$.

Let's explain the above restrictions. (1) is a corollary of the continuity assumption. Note that except for the mentioned two limit cases, in other scenario the derivative should have affected the magnitude before it changes. For example, starting from $(0, +)$, by our assumption the positive derivative is the trend of changing at the current state, so the magnitude must be affected (namely increase) before the derivative can be changed. (2) means that if the derivative is 0, then the quantity should not change by the definition of derivative. (3) and (4) can be seen as a group of ideas: if X_0 is positive and the derivative says its increasing (decreasing), then at the next stage X_0' must be at least (at most) X_0. (5) and (6) is each other's dual, both depicting a limit situation w.r.t. continuity. For example, (5) says that if $X_0 = 0$ and the derivative shows a trend of increasing, then at the next stage X_0' should be $+$.

Thirdly we have the restriction where interrelation between different quantities are involved. And we can start from the easiest situation, namely at the first state S, the volume is stable (i.e. $V_1 = 0$):

1. If $V_1 = 0$ and $I = I'$, then $V_1' = I_1$ and $V_0' = V_0$;
2. If $V_1 = 0$, $I_1 \neq I_1'$ and $I_1 = 0$, then $V_1' = 0$;
3. If $V_1 = 0$, $I_1 \neq I_1'$ and $I_1 \neq 0$, then $V_1' = I_1$.

All of the rules consider the initial state with $V_1 = 0$; in words, it means that the volume has no trend of changing, or that the inflow and the outflow are exactly the same (quantitatively, not only qualitatively). Then we consider how the change of inflow can have impact on the derivative of volume [2]. (1) considers the simplest situation, namely inflow is qualitatively stable. Then the derivative of volume in the second state should correspond to the derivative of inflow. Besides, the volume magnitude has not changed yet. (2) and (3) consider the situations when the inflow changes,

[2] One may wonder why don't we consider the impact of outflow changes. Our answer is that it should be the other way around: the outflow cannot be changed without the volume change.

where there are two subcases. (2) aims at the situation when the derivative of inflow changes from 0 (namely $I_1 = 0$ while $I_1' \neq 0$). In this case V_1' must be 0 since the inflow is quantitatively stable (still equals the outflow). Similarly, (3) looks at the situation where the derivative becomes zero. By that $I_1 \neq I_1'$ and the continuity, the inflow must have quantitatively changed (corresponding to I_1). Then the volume should also start to change accordingly, as it is stable in the first state.

Then we can go on to some more complex cases.

1. If $I = (+, -)$ and $V = (\max, 0)$, then $I' = (+, -)$ and $V' = (\max, -)$;
2. If $I = (0, +)$ and $V = (0, 0)$, then $I' = (+, +)$ and $V' = (0, +)$;

These two are descriptions of two limit scenarios. They say that there is only one possible successive qualitatively different state from certain limit initial state, respectively.

One last remark. Even though we have listed a bunch of rules for restrictions and dependencies, the resulting state diagram still have some extra transitions compared with our original hand-made picture. After careful comparison, we find these extra transitions redundant. However, we did not manage to find some general rules which exactly eliminate them, and have to delete them case by case. This is a potential work to be done.

4 Algorithm

As said above, our main algorithm is getting total envisionment by so-called model products. The basic idea is as following. To construct a state diagram, namely a collection of states (each with many quantities) and the accessible relations between those states, we may first focus on each quantity and construct the accessible relation, then get the whole diagram by the product of those quantities and their accessible relations, plus some interactive restriction rules. To illustrate the idea, note that in modal logic, this corresponds to:

Definition 1. Given two relation frame $\mathcal{S}_0 = (S_0, R_0)$ and $\mathcal{S}_1 = (S_1, R_1)$, where S_i is a set of states and $R_i \subseteq S_i \times S_i$ is the set of relations, then the product frame $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1$ is $\mathcal{S} = (S, R)$, where:

- $S = S_0 \times S_1$;
- For any $(s, t), (s', t') \in S$, $(s, t)R(s', t')$ *iff* sR_0s' and tR_1t'.

And here we require some more restrictions of the relations, as listed in the previous sections (namely in the **Inter-state Dependency** part).

In our case, we first construct the pairs of (*magnitude,derivative*) for each quantity. Then we build up admissible accessible relations of these pairs, for each quantity, based on the inter-state rules for fixed quantity. Finally we construct state of the whole procedure by taking the product of the admissible quantity states and of the admissible quantity transitions, also by adding some extra inter-state rules we get the transitions from the admissible quantity state transitions. Finally, since we want to restrict attention to only the admissible states which are possible to be reached from $S = ((0, 0), (0, 0), (0, 0))$, we prune those unreachable from any other state, and this turns out to be sufficient.

We say that this algorithm is a *total envisionment*, in that it generates the state diagram *not* starting from particular state but as a whole. This can be seen from the diagram that no state appears twice, and they admit different income arrow.

Algorithm 1 Admissible States

```
admissibleState = [] // The list of all admissible states;
for state in possibleStates do
    judge = True
    for i = 0, 1, 2 do // For inflow,volume and outflow;
        if state[i] violates rule i then // Rule i is the rule for quantitive i;
            judge = False
    if judge = True then
        admissibleState += state // If all quantities of the current state pass, then add it to admissbleState;
    return admissibleState // Return all the admissible states;
```

Algorithm 2 Admissible Transitions

```
admissibleTrans = [] // The list of all admissible one-step state transitions;
for state0, state1 in admissibleState do
    judge = True
    for i=0,1,2 do // For inflow, volume, outflow;
        if (state0[i],state1[i]) violates inter-state rule-i then
            judge = false
    if judge == True then
        admissibleTrans += (state0,state1)
return admissibleTrans
for state in admissibleState do // Now we prune unreachable states;
    judge = 0
    for (state0,state1) in admissibleTrans do
        if state == state1 then
            judge = 1
    if judge == 0 then
        admissibleState -= state // We delete those states in admissibleState which is not reach by any state in admissibleState;
for (state0,state1) in admissibleTrans do // Finally we confine our transitions only to those admissible states;
    if state0 not in admissibleState then
        admissibleTrans -= (state0,state1) // If state0 is not admissible, then we delete the whole transition from admissibleState
return admissibleTrans
```

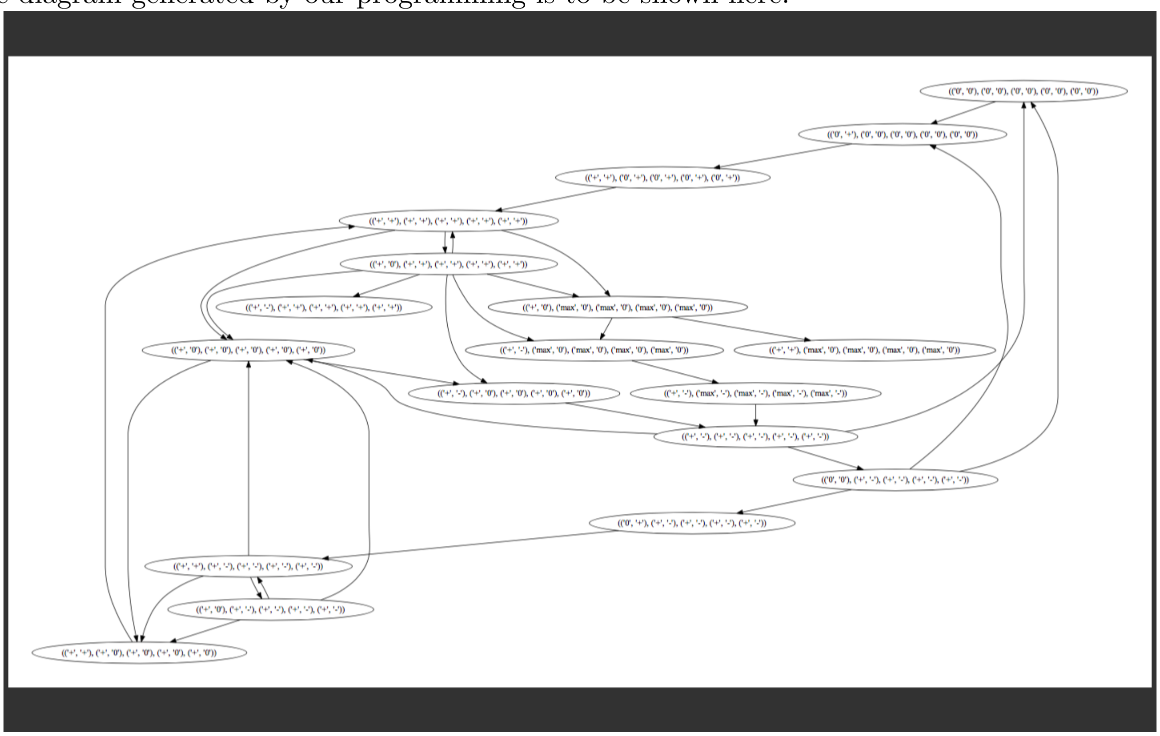
One remark of the differences between those rules in the previous section and those in our programming (see the code). You may notice that there seem to be far more rules than that we have here, but in fact they are the same. Some of the rules becomes quite complex when spelled out in python language; some are modified to have simpler coding.

Now having explained what our algorithm *is*, we go on to justify it. First of all, that our algorithm works is essentially based on the assumption that at any state, the admissible next-step-state is totally determined by the current state. This means, for example, that the history of how this state is reached should *not* matter the states that one can reach from this state.

Then, we claim that our restriction on the transition relations are sound and complete. Since our rules basically prunes those unreasonable transitions, soundness means all such pruning do remove the wrong transitions; while completeness means that all unreasonable transitions are removed. The soundness of each rule is, more or less, argued after the exposition. And for the completeness, we basically look at all the transitions and check if they are reasonable w.r.t. our assumptions. This manual method is troublesome and cannot be free of doubt. So a future improvement is to find a way to prove or guarantee the completeness formally or systematically.

5 State Diagram

The diagram generated by our programming is to be shown here:



Due to the complexity of the graph and the limited space here, we suggest the reader to Stategraph.pdf in our zip file for more details.

6 Insightful Trace

We generated this by our program (see the last part). The idea is that, instead of writing the description of each transition one by one, we uniformly describe the transition of the components of the state, namely the transition between inflow, volume, outflow, etc. See Insightful_trace.txt for the result.