

Persistent Homology: Interpretation and Stability

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- We also looked at several common ways for building filtration out of data that are practical in different applications
- Notice that our ultimate goal is to use persistent homology to infer the “shape” of data (i.e., homology inference)
- To do this, we need to fully understand the meanings of PD or barcode, from different aspects
- Moreover, we shall also see some properties that are critical to showing that persistent homology is a “reliable” way for inferring the shape

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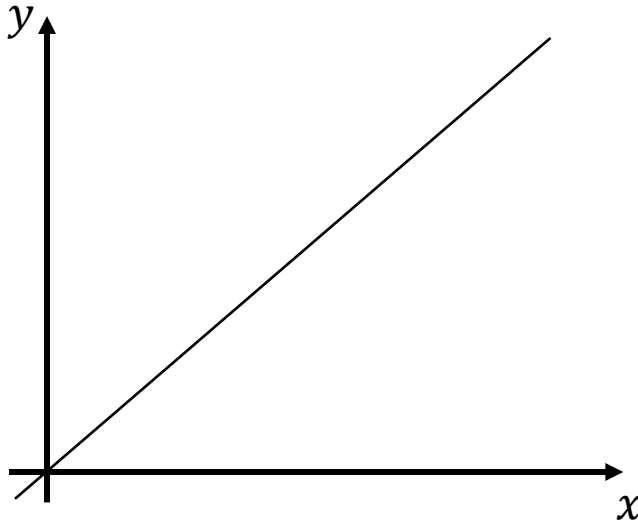
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- Why above the diagonal?

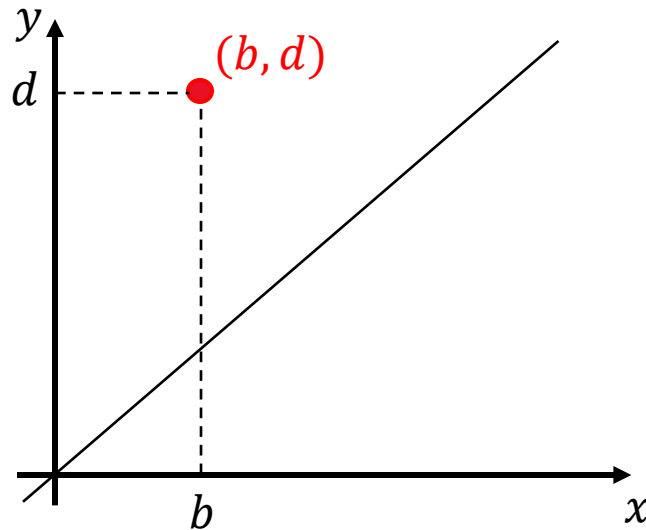
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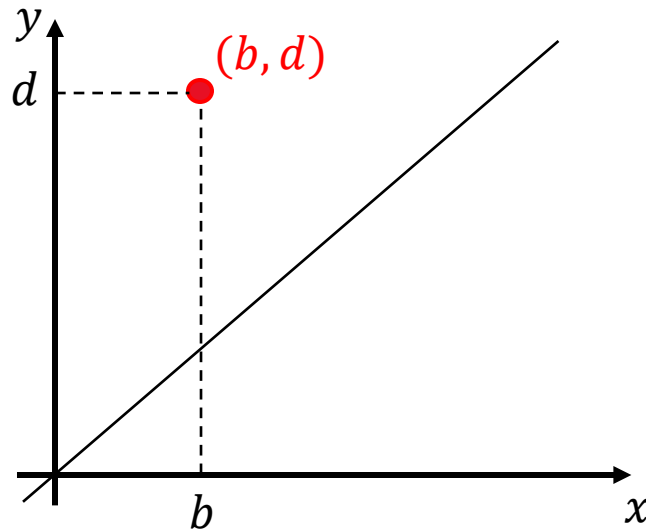
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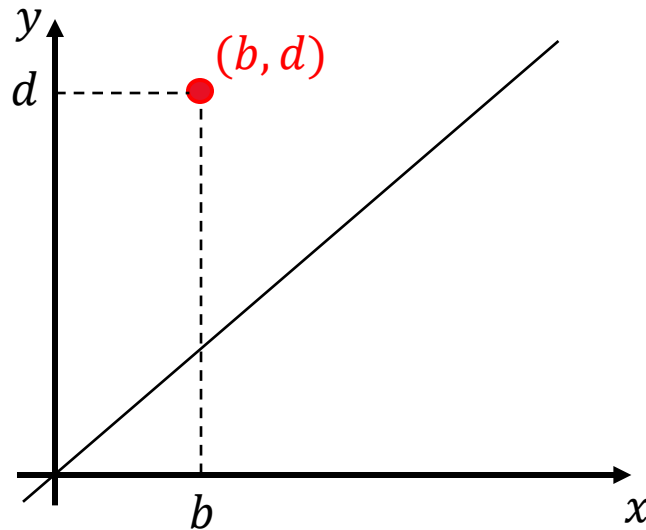
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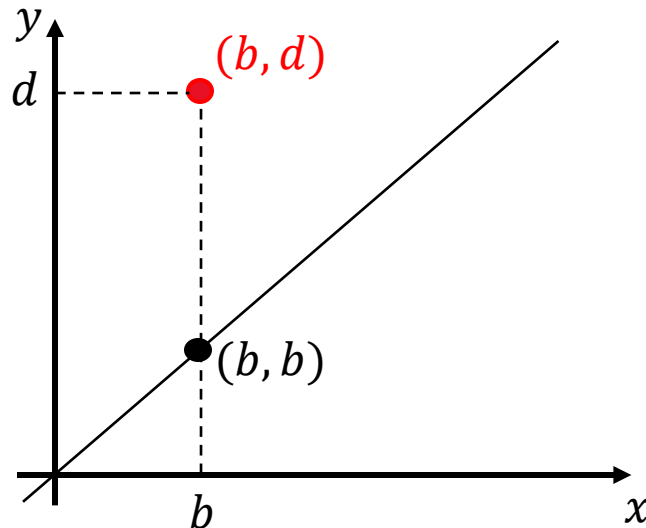
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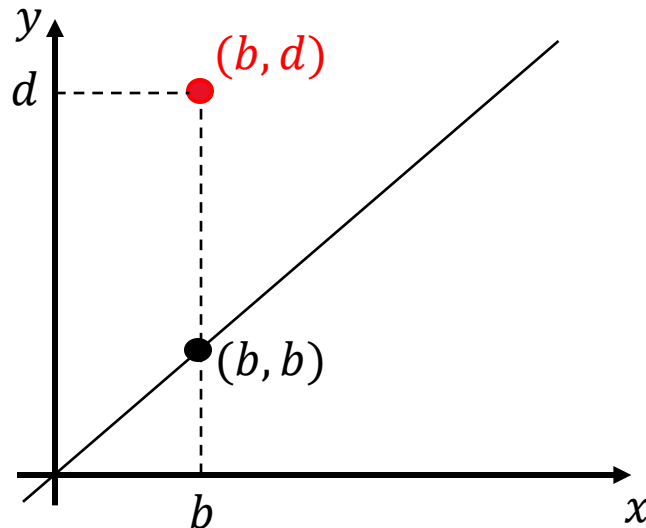
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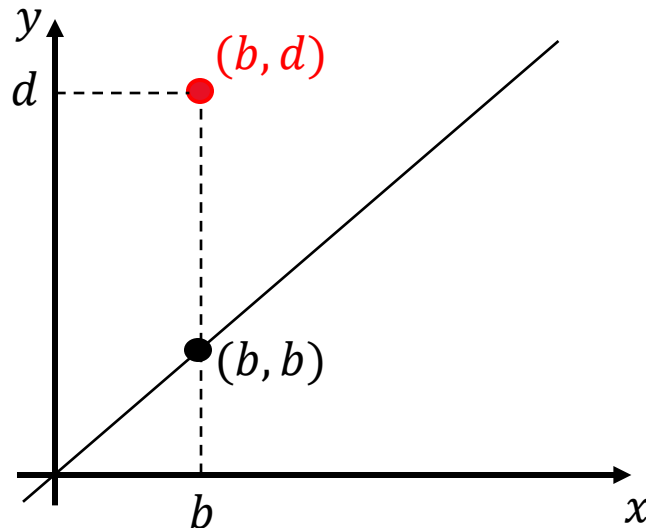
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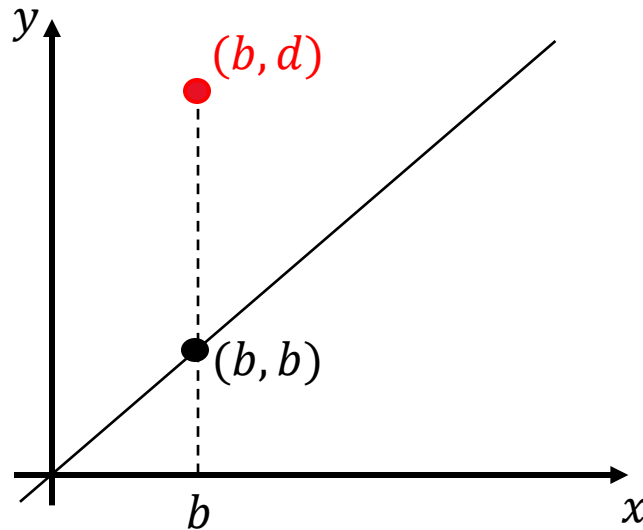
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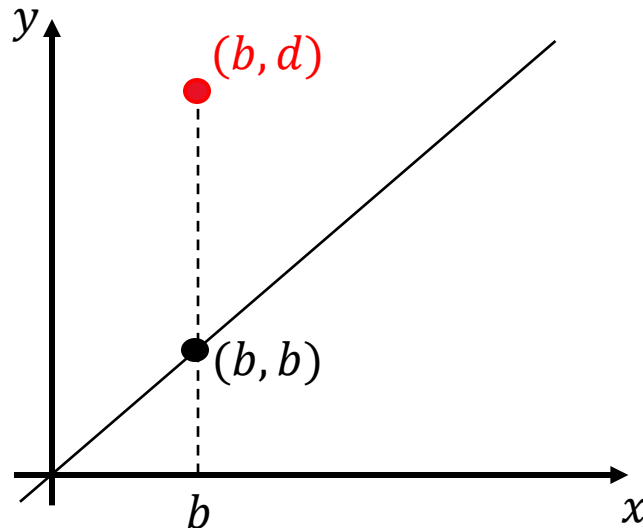
Length of an interval manifested on PD

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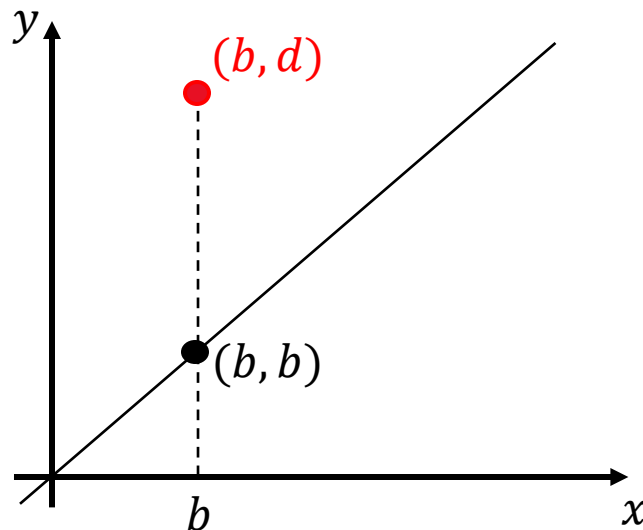
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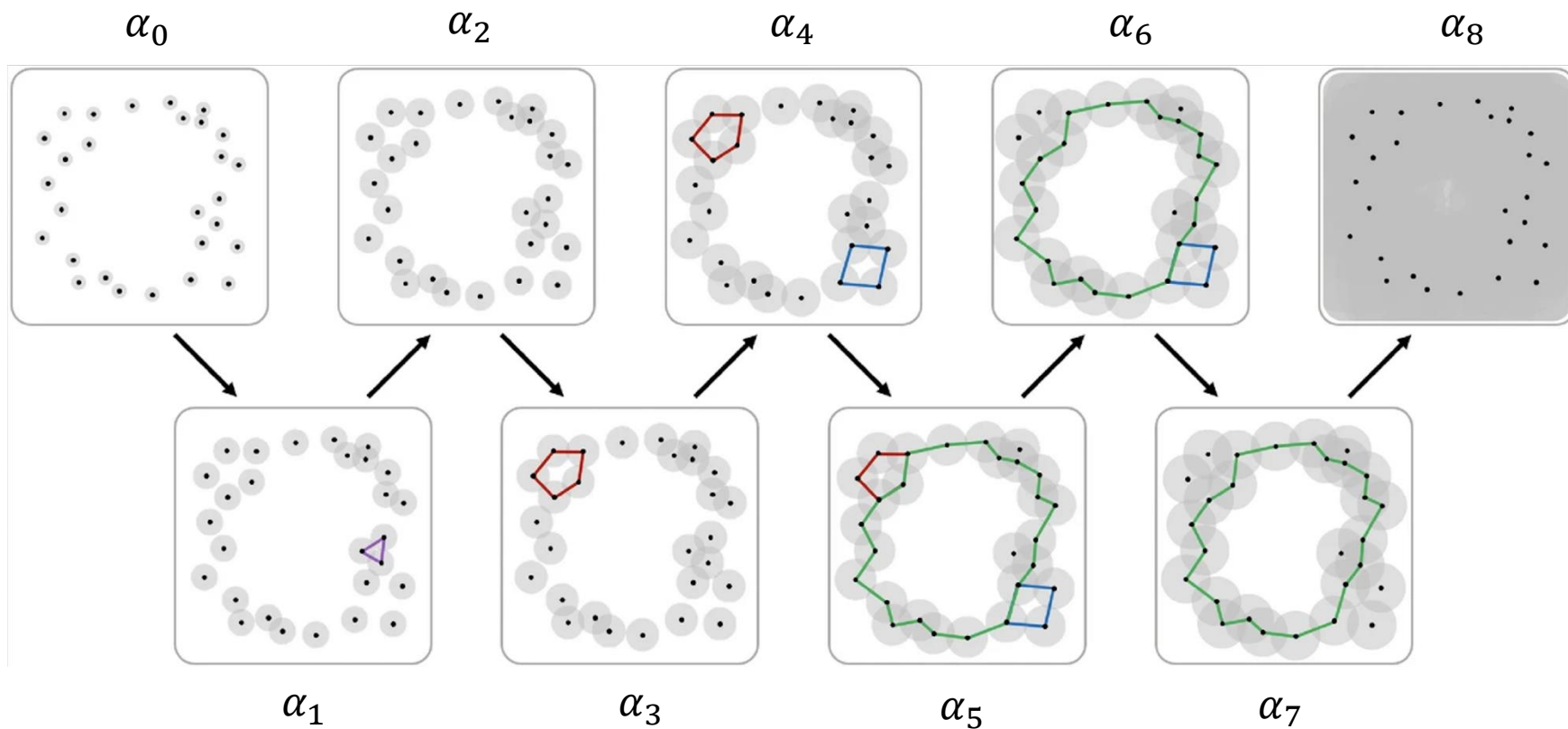


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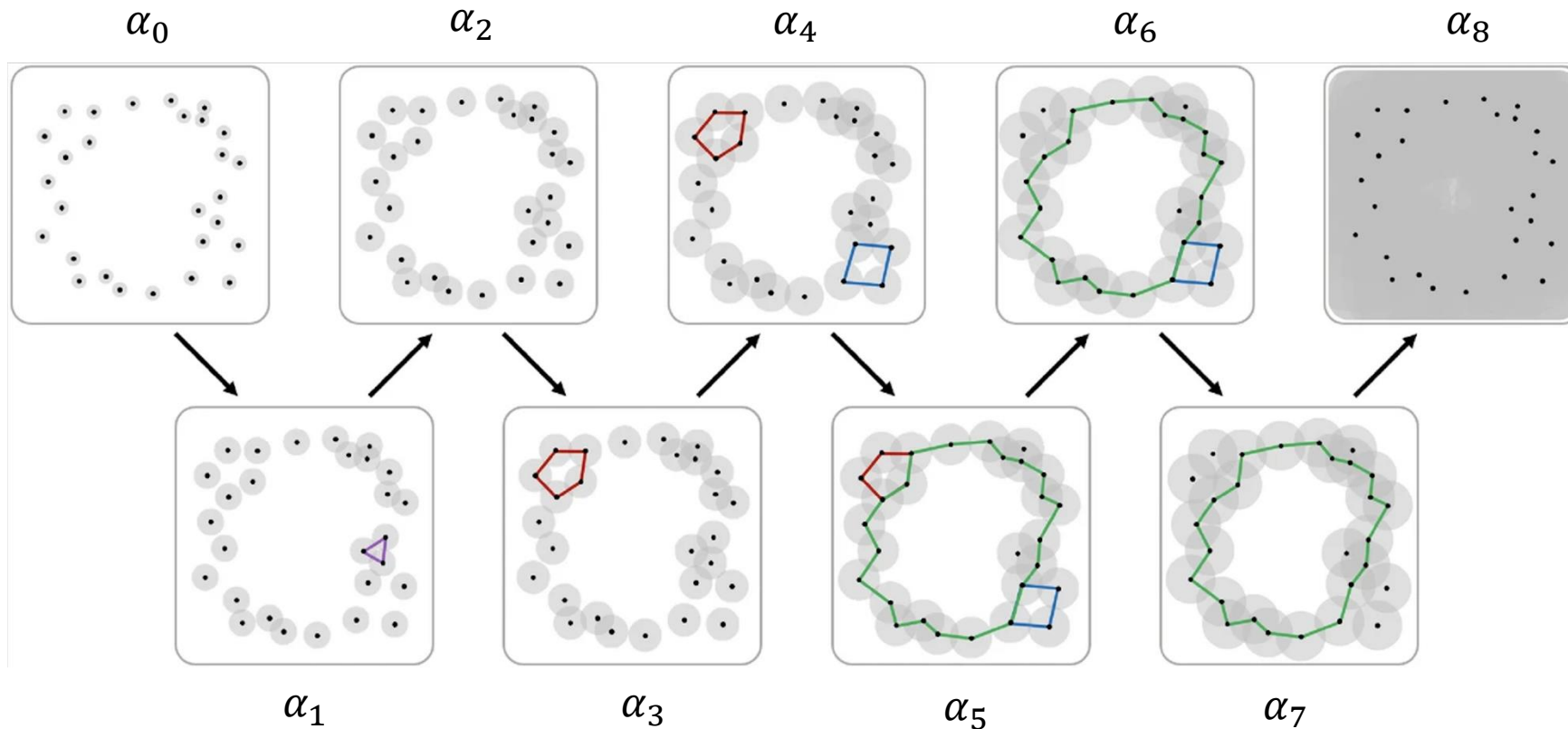
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- So the distance of a point in PD to the diagonal indicates the length of the corresponding interval
- A general fact is that a longer interval indicate a greater significance of the homological feature



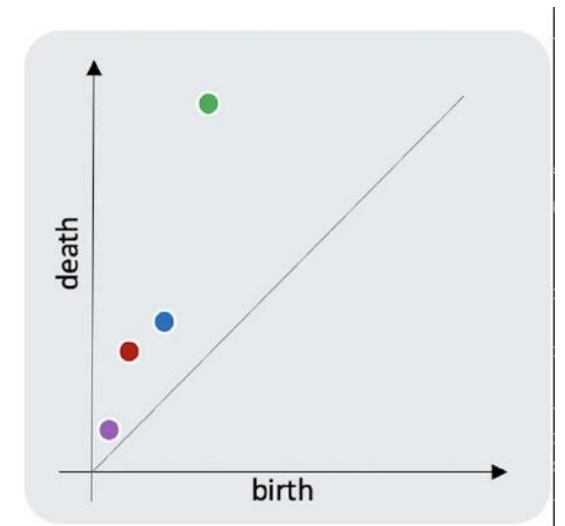
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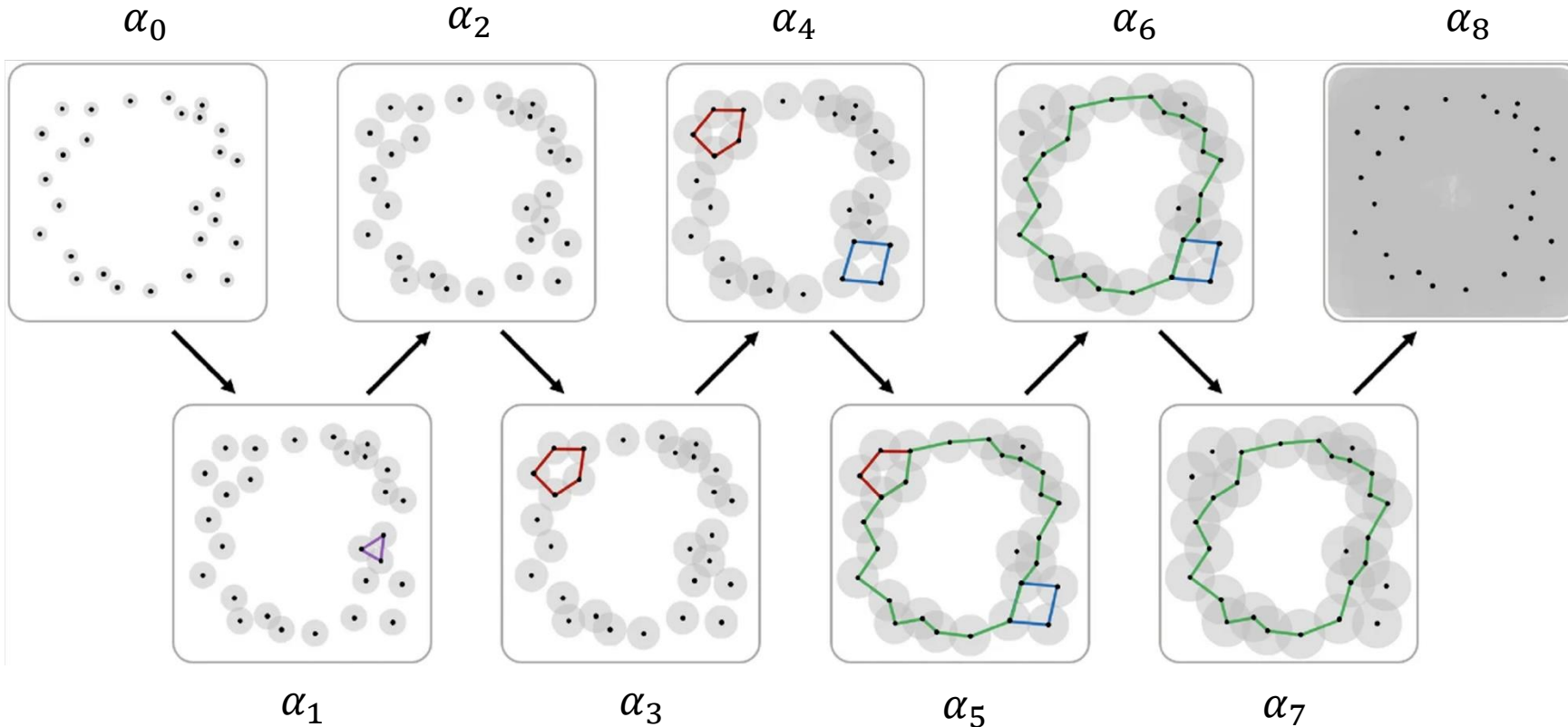


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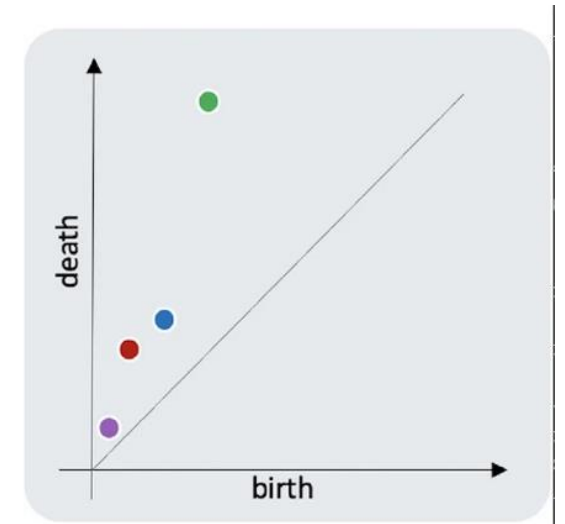


Meaning of “length” of a PD interval

- Observe the longer an interval is, the more significant its homology hole is



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- For this, we need a way to **measure**:
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- Mathematically, the tool to measure the difference of two objects is called a **distance** function (or **metric**)

Distance function (as on Wikipedia)

Definition [\[edit\]](#)

Formally, a **metric space** is an [ordered pair](#) (M, d) where M is a set and d is a **metric** on M , i.e., a [function](#)

$$d: M \times M \rightarrow \mathbb{R}$$

satisfying the following axioms for all points $x, y, z \in M$:[\[4\]](#)[\[5\]](#)

1. The distance from a point to itself is zero:

$$d(x, x) = 0$$

2. (Positivity) The distance between two distinct points is always positive:

$$\text{If } x \neq y, \text{ then } d(x, y) > 0$$

3. ([Symmetry](#)) The distance from x to y is always the same as the distance from y to x :

$$d(x, y) = d(y, x)$$

4. The [triangle inequality](#) holds:

$$d(x, z) \leq d(x, y) + d(y, z)$$

This is a natural property of both physical and metaphorical notions of distance: you can arrive at z from x by taking a detour through y , but this will not make your journey any shorter than the direct path.

If the metric d is unambiguous, one often refers by [abuse of notation](#) to "the metric space M ".

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- L_∞ -distance: $d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

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- What is the data we try to measure here?
- It turns out that “functions” are a quite universal type of data (reason will be made clear later)
- Specifically, for measuring the difference of two functions, we **assume the domain to be the same**, aka. we measure two functions of the following form:
 - $f: X \rightarrow \mathbb{R}$
 - $g: X \rightarrow \mathbb{R}$

Distance for functions

- The idea of our distance $d(f, g)$ for the two functions f, g is to measure the maximum of difference of the function values at each point in the domain X

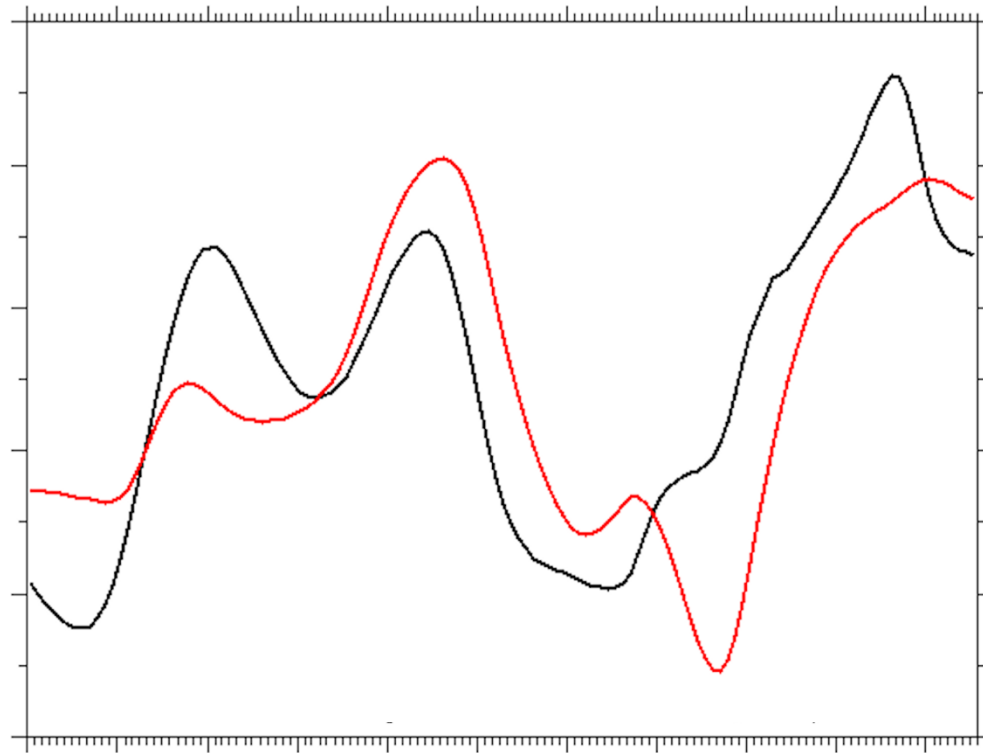
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- The distance is also denoted $\|f - g\|_\infty$:

$$\|f - g\|_\infty = \max_{x \in X} \{|f(x) - g(x)|\}$$

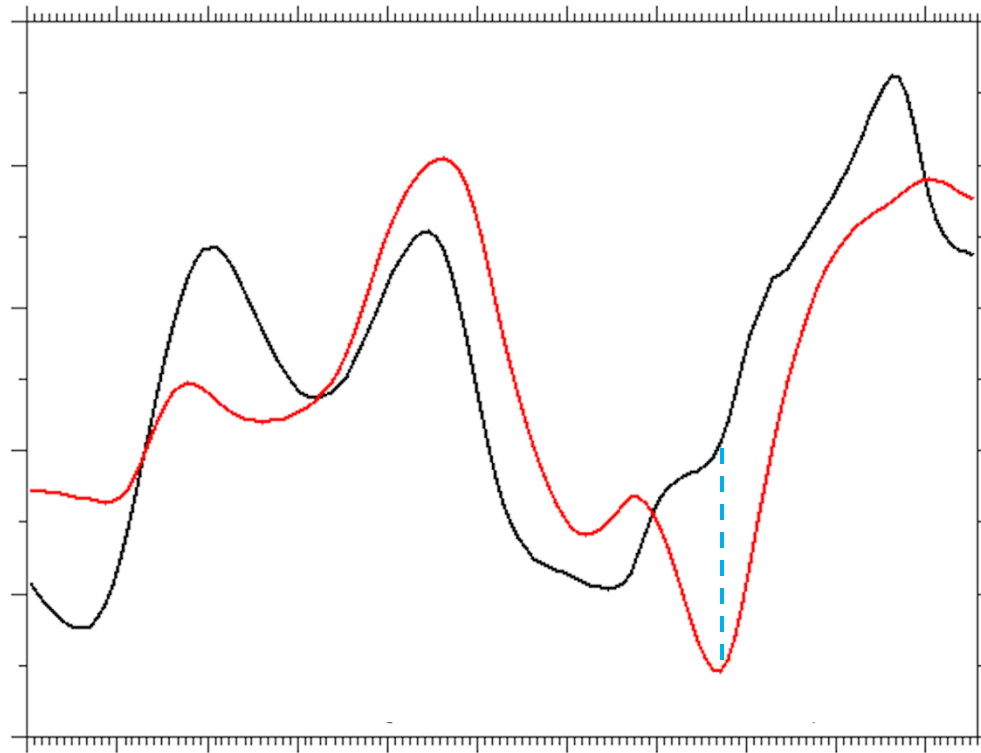
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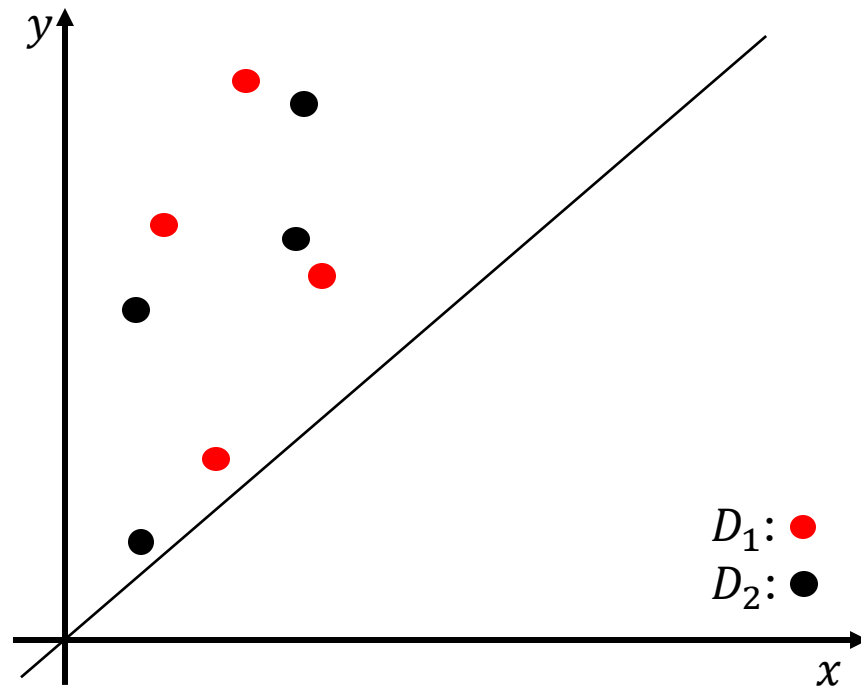


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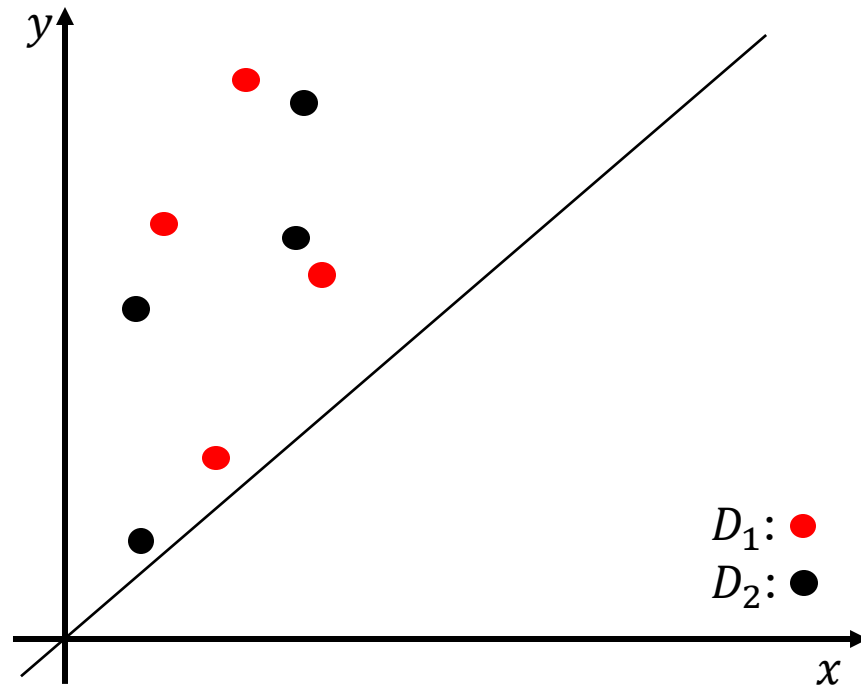
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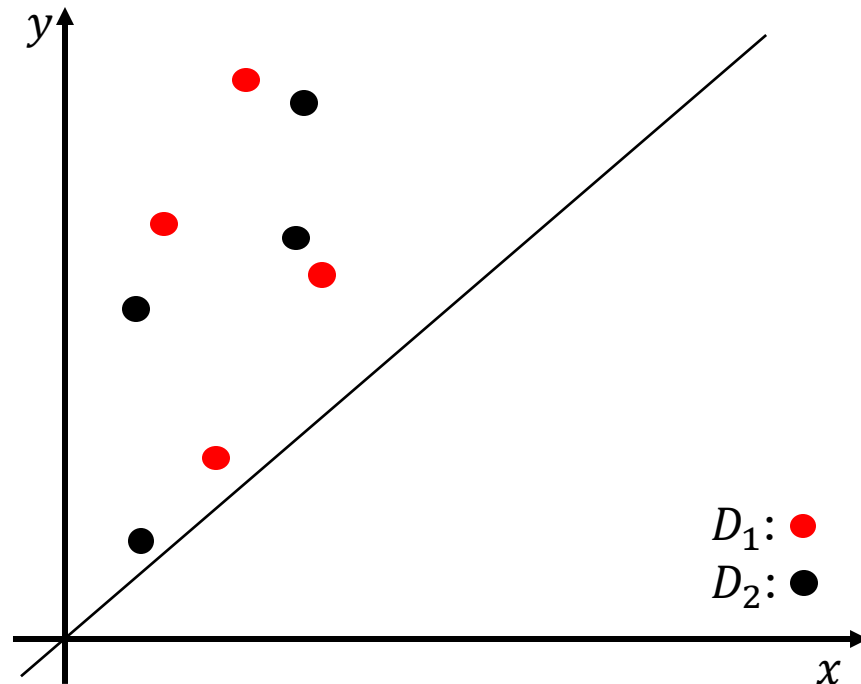
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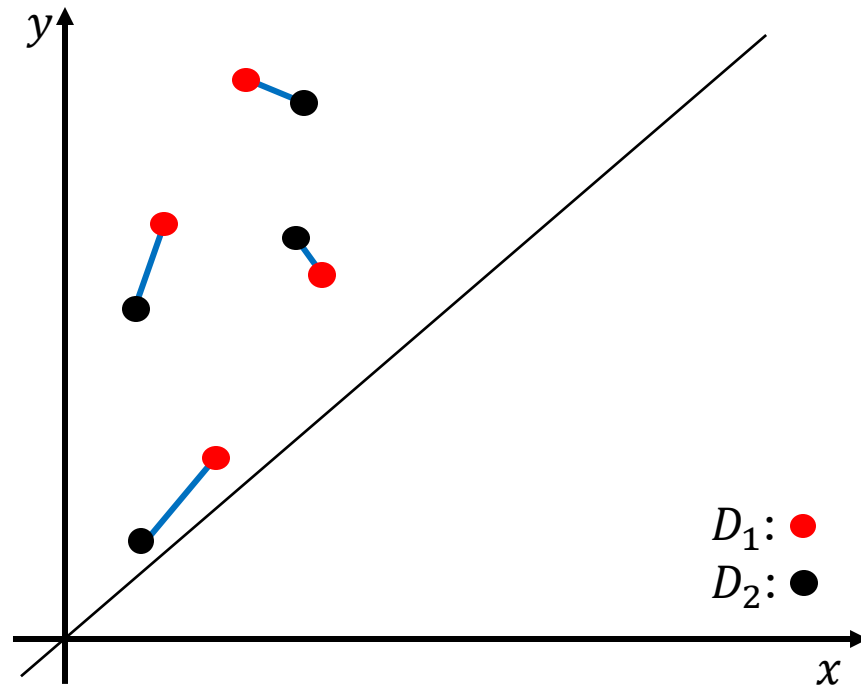
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- Furthermore, we want the matched points in the two PDs to be as close as possible, which indicates that overall the two PDs are “close” to each other



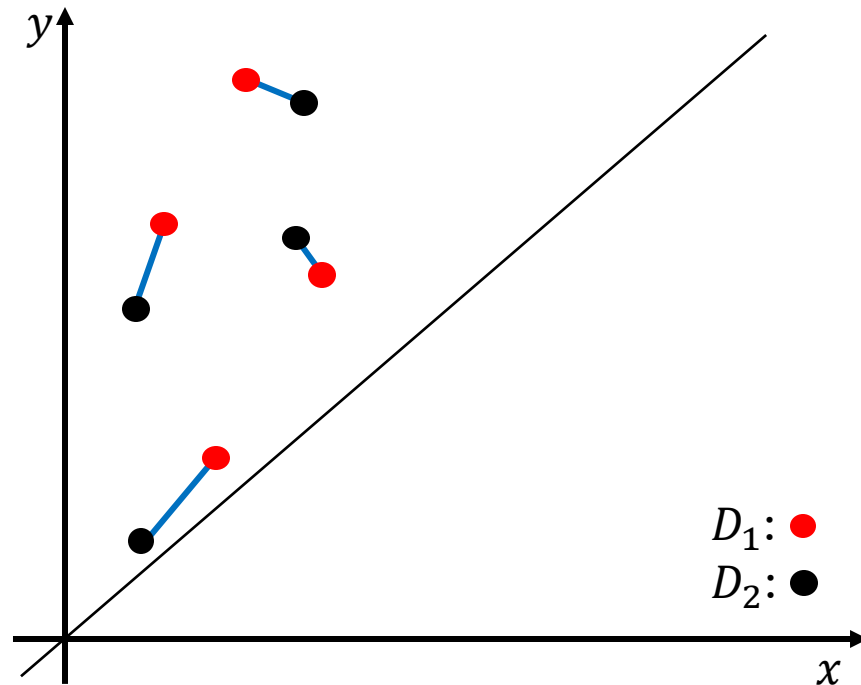
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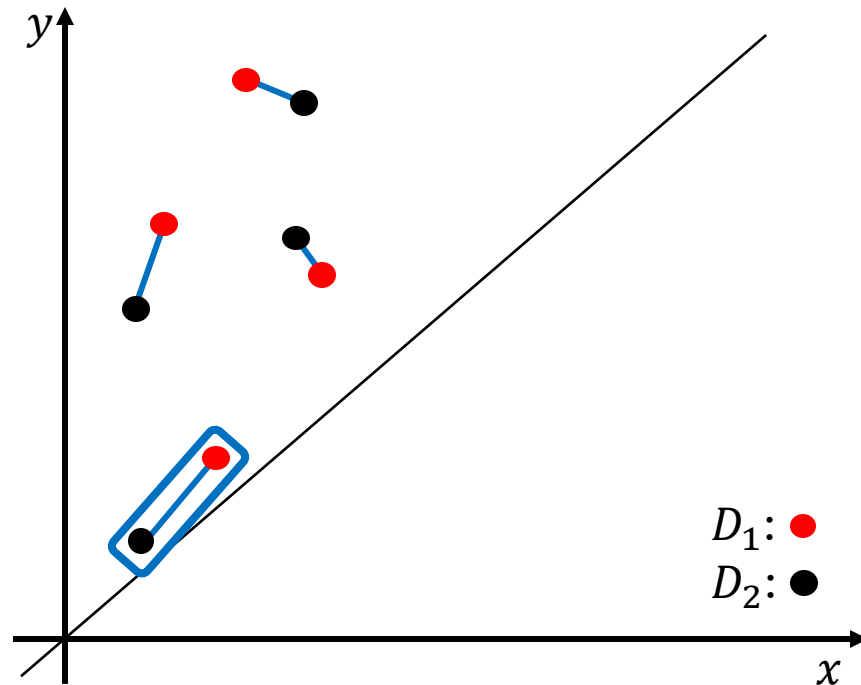
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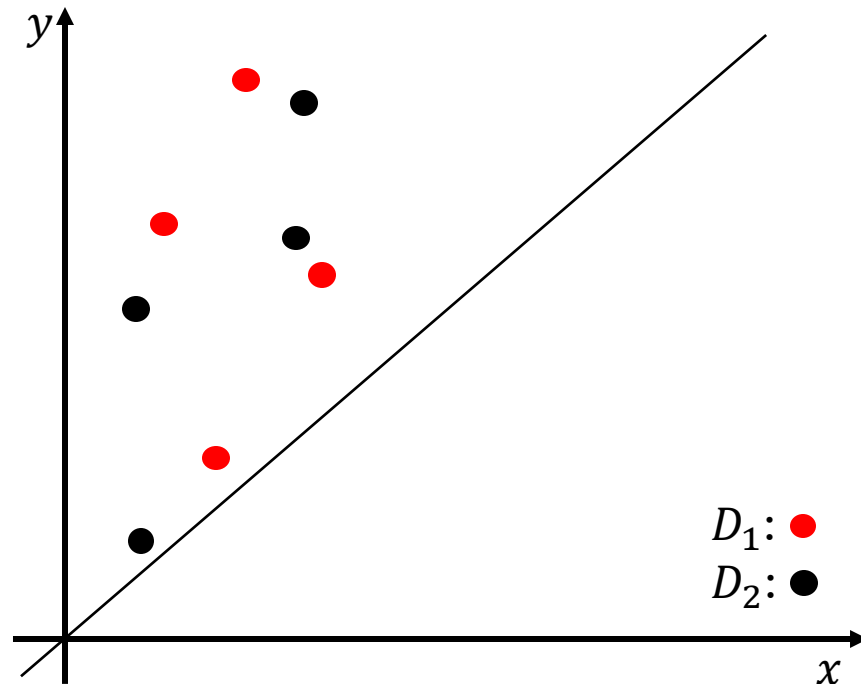
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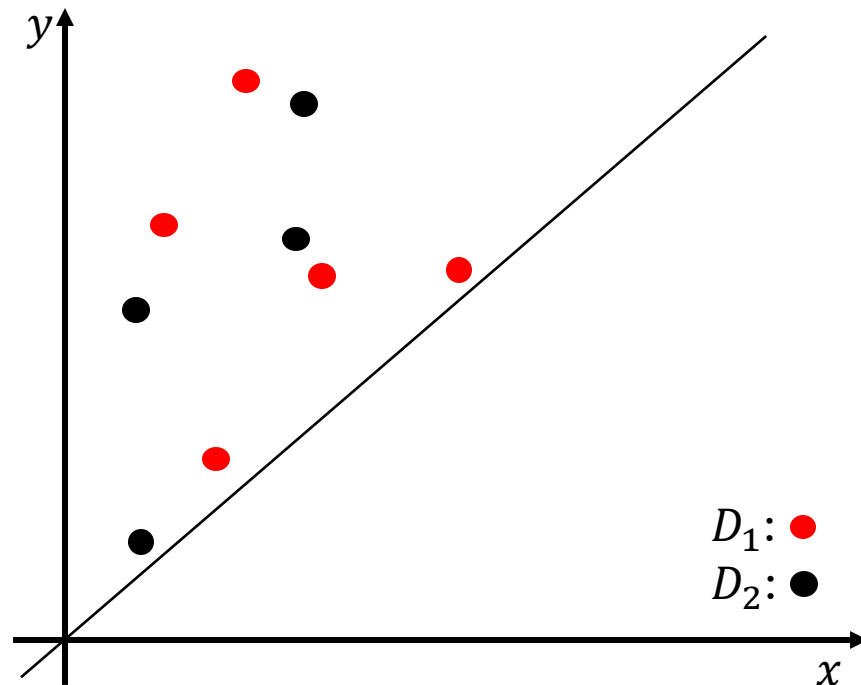
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 - We measure the “cost” of “ignoring” these unmatched points by taking their distance to the diagonal

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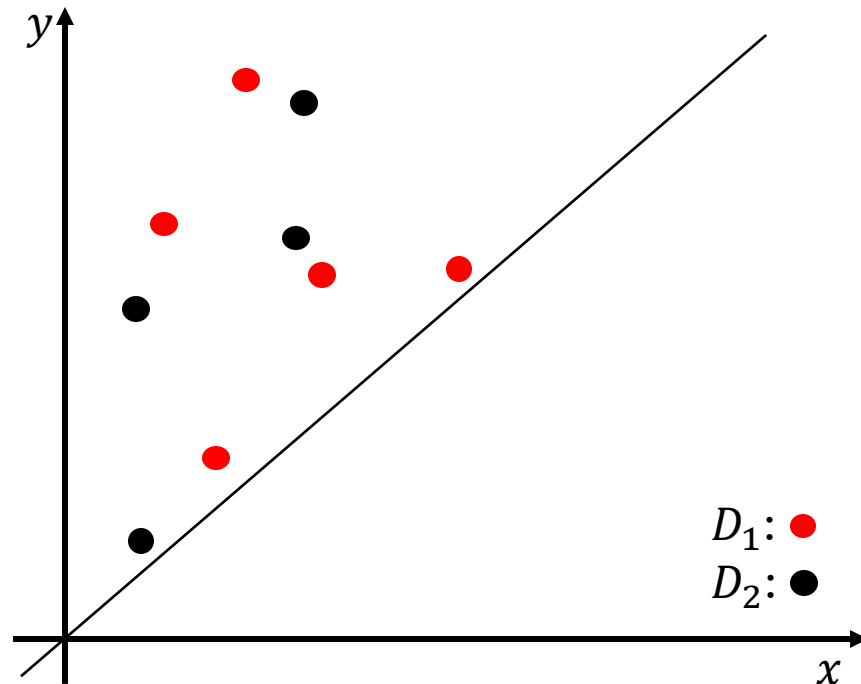
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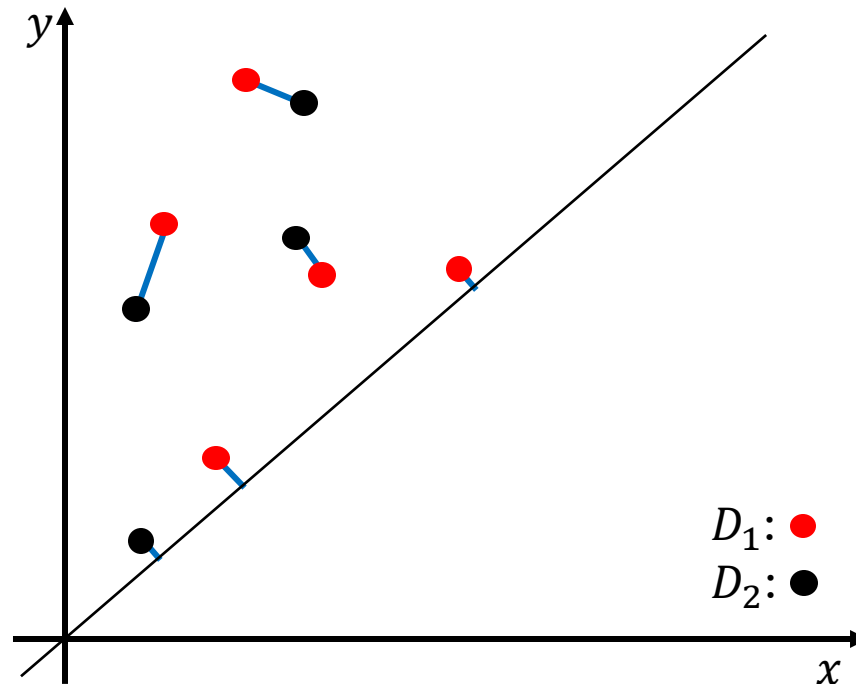
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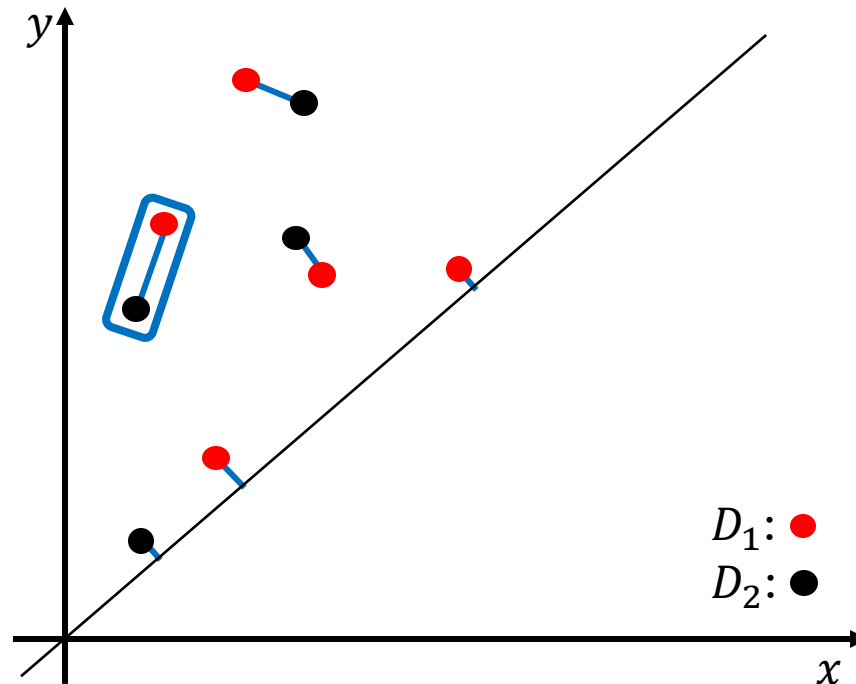
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The **bottleneck distance** is then defined as follows:

$$d_B(D_1, D_2) = \min_{\eta \text{ over all partial matchings}} \{\text{cost}(\eta)\}$$

aka. the minimum cost of all partial matchings

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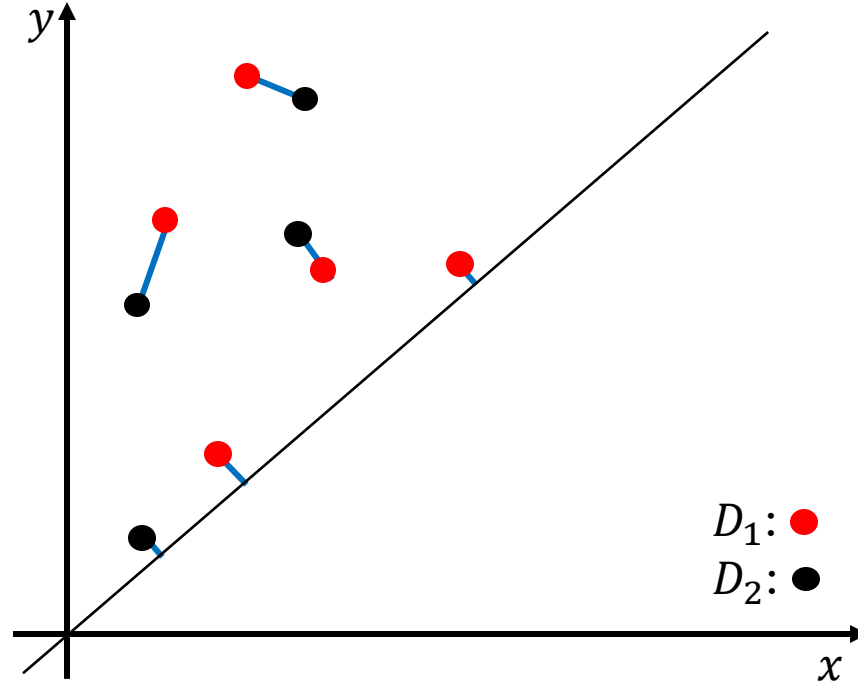
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- So overall the two PDs are close

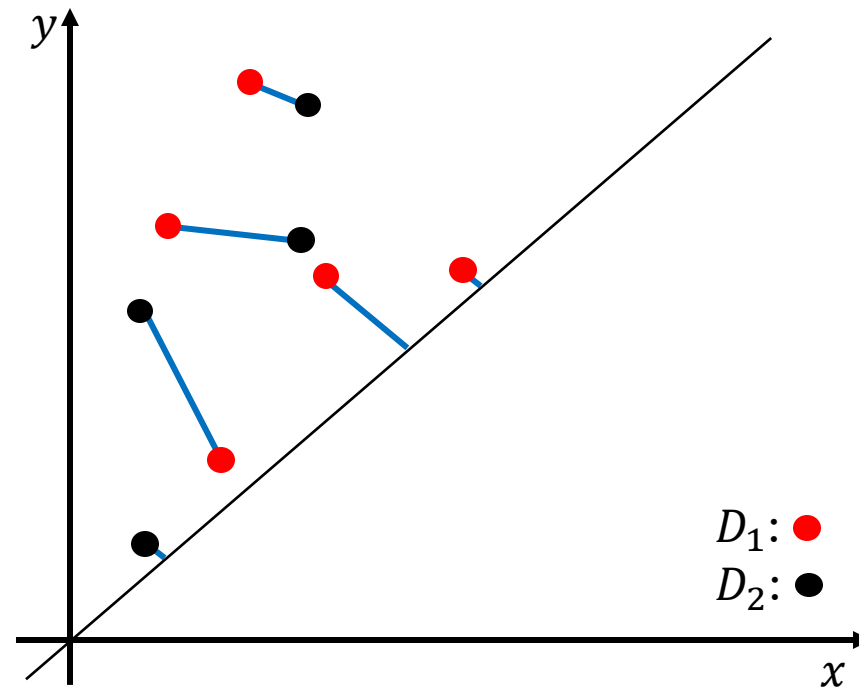
Distance for PDs

- The following is a “close” partial matching (with small cost) which achieves the bottleneck distance between D_1 , D_2 (aka. $d_B(D_1, D_2) = \text{cost}(\eta)$)



Distance for PDs

- The following is a partial matching where the max distance between matched points is high



Stability Theorem for Persistent Homology

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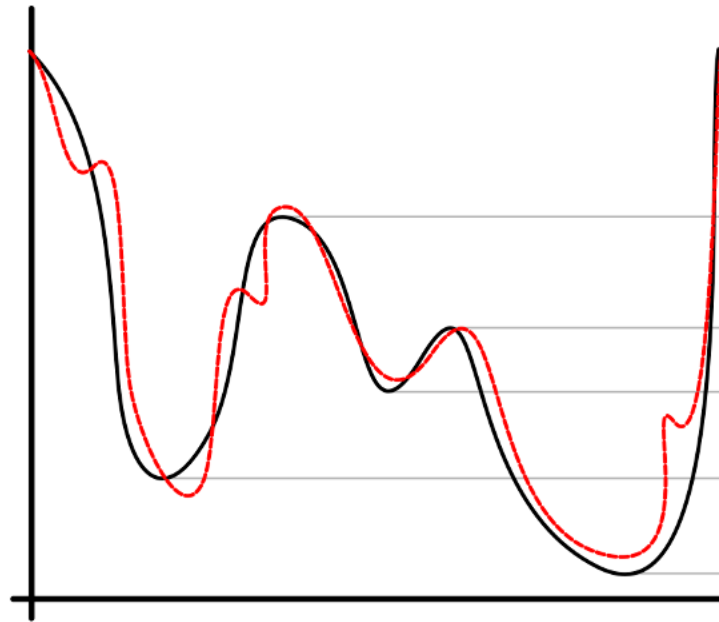
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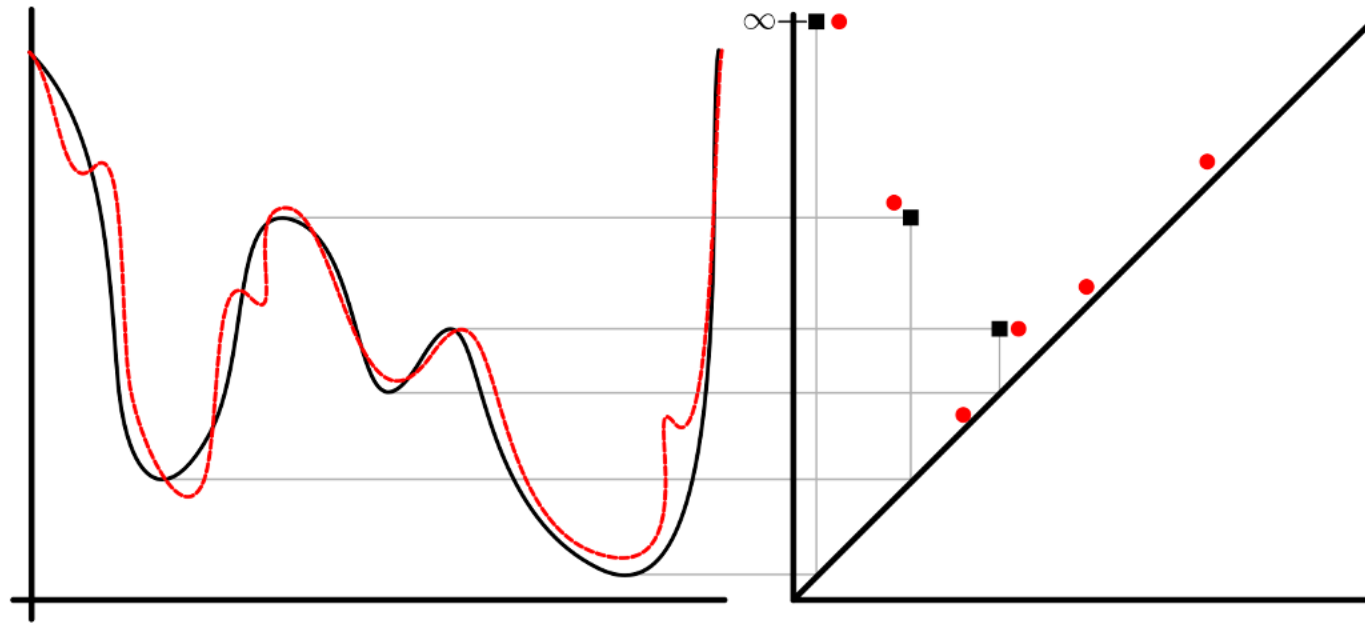


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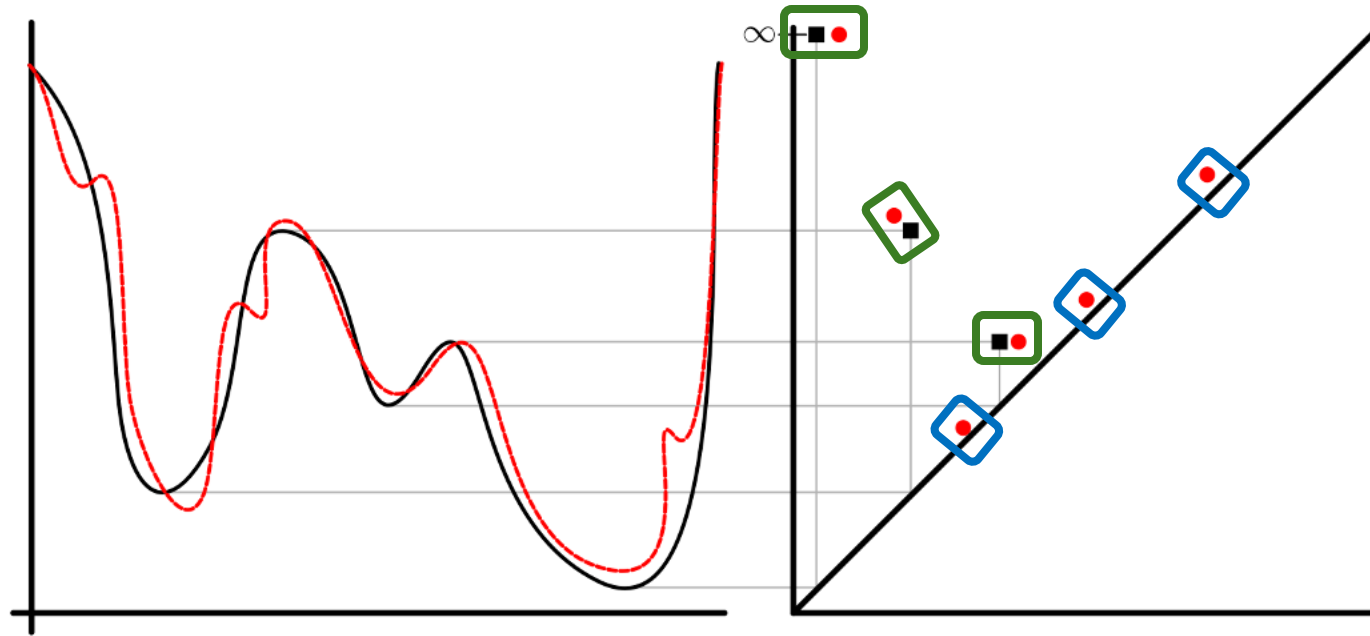


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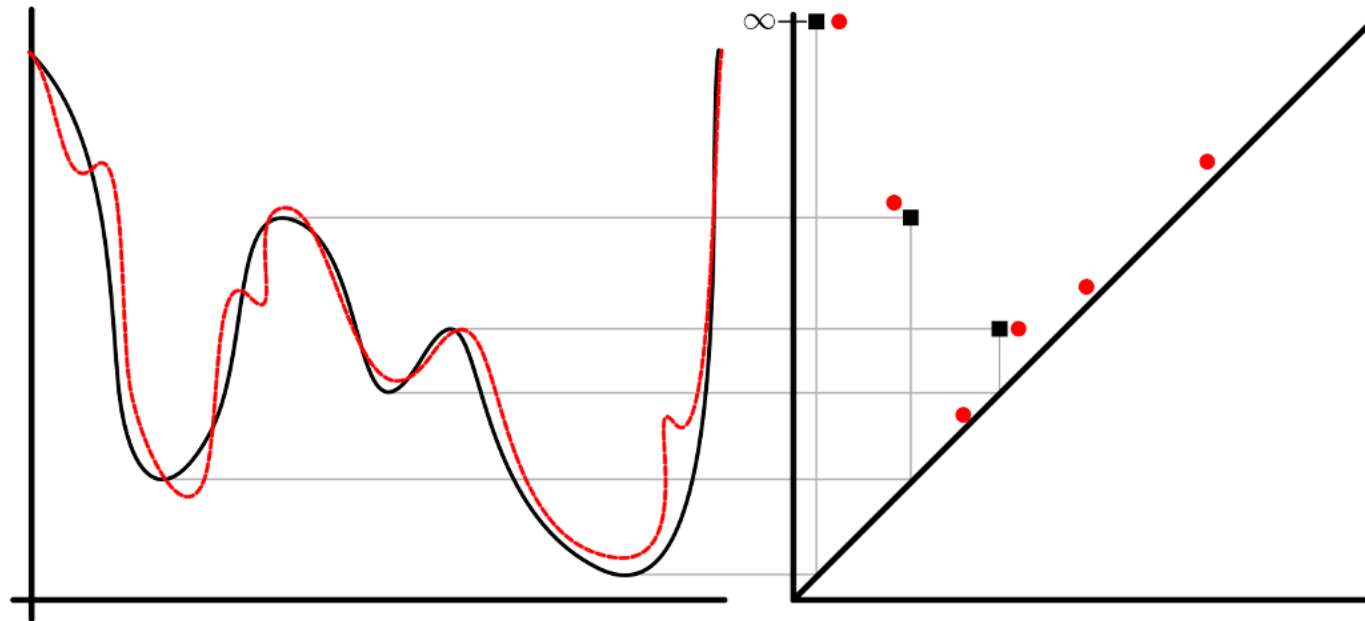
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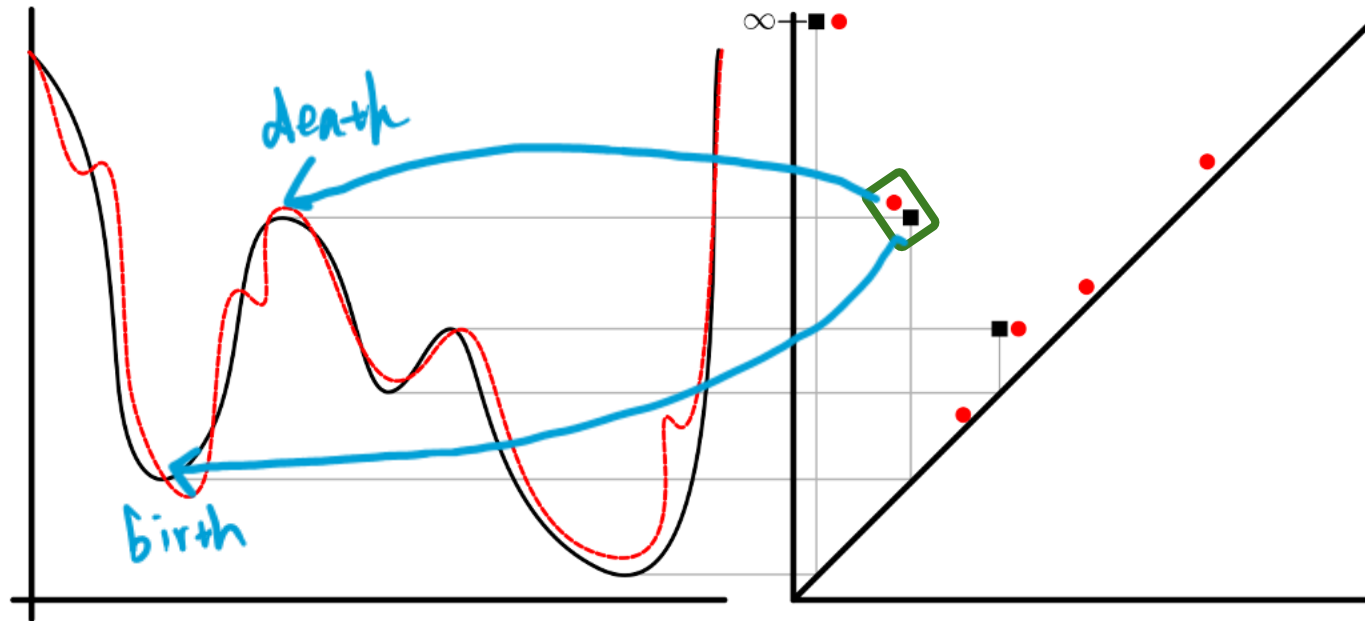
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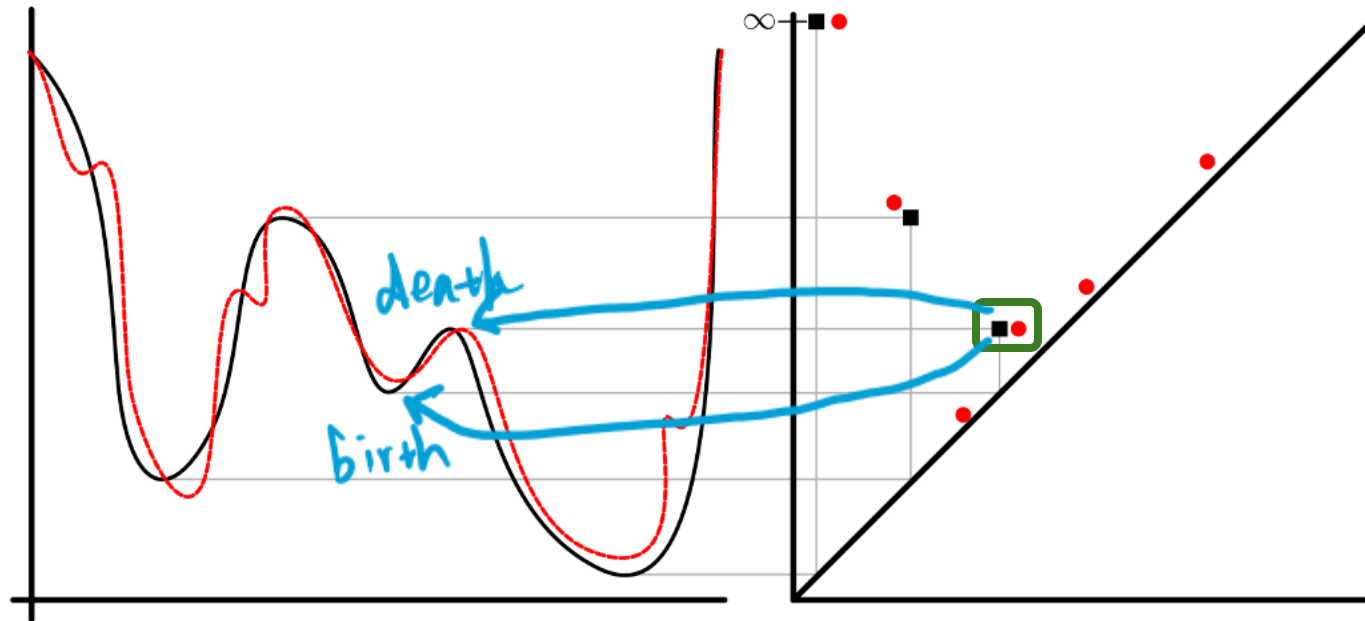
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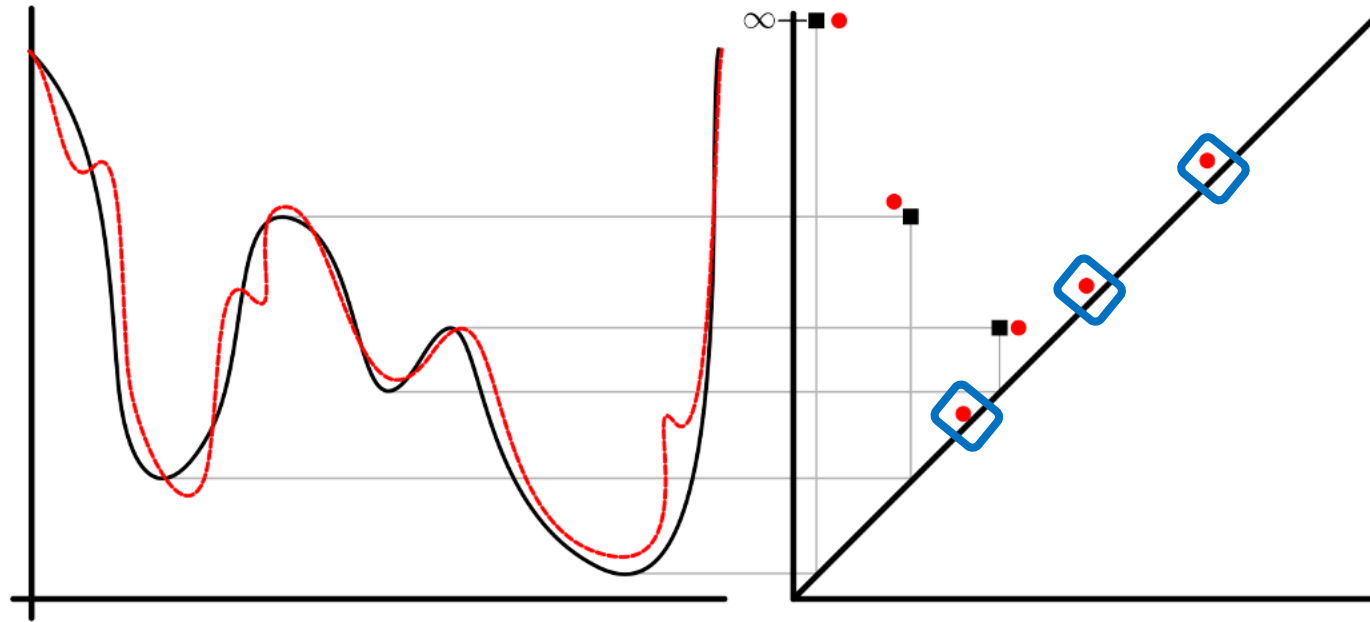
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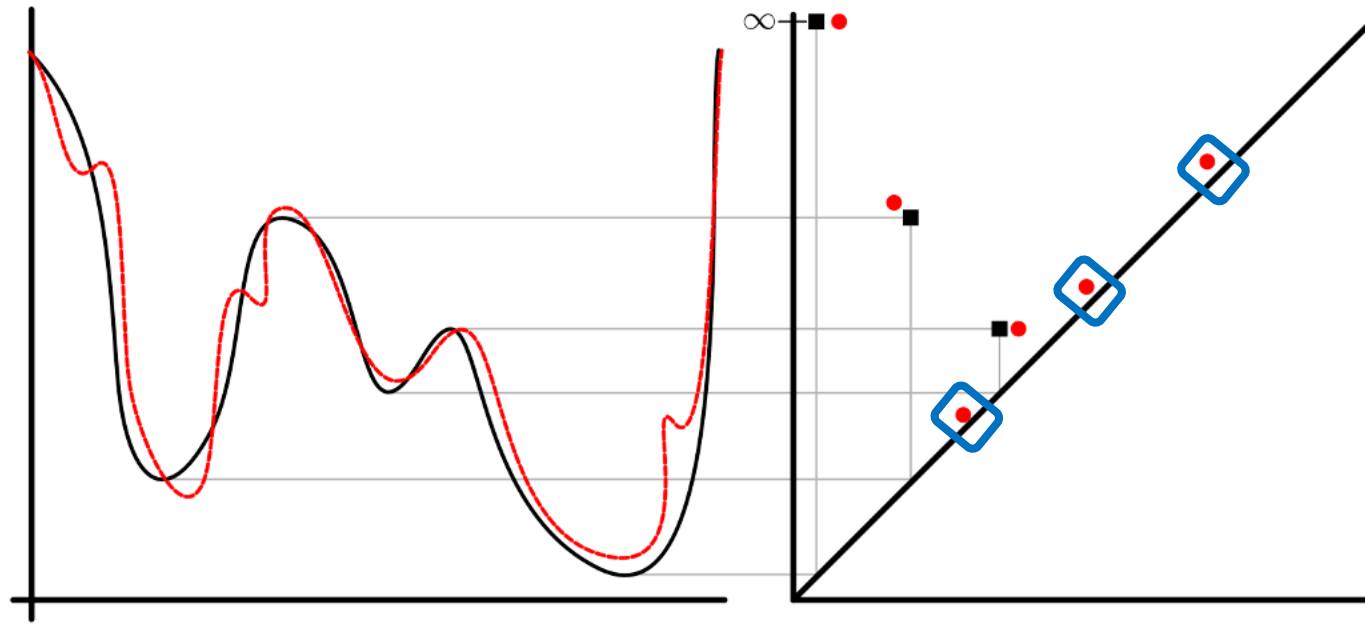
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- This also corresponds to the fact that the red curve is more noisy than the black one



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- But there is another important type of data which is **point cloud** ---- a stability result for it would also be helpful
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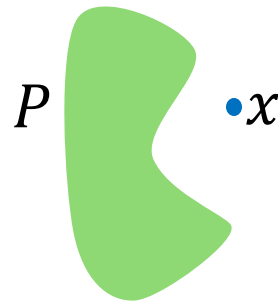
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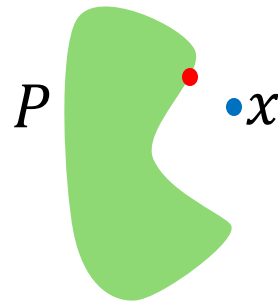
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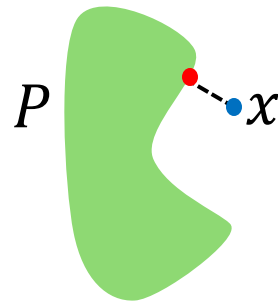
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- **Corollary:** The above observation also means that **$PD(f_P)$ equals the PD of the Čech filtration of P**

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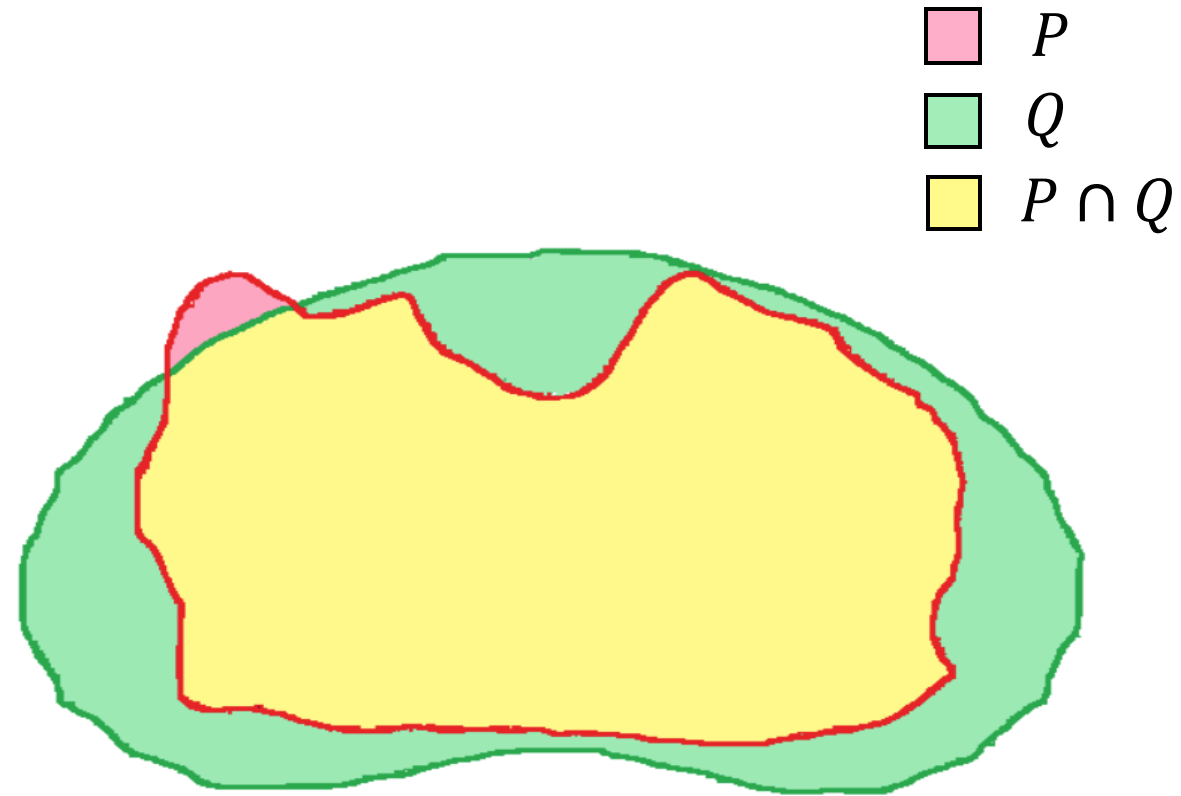
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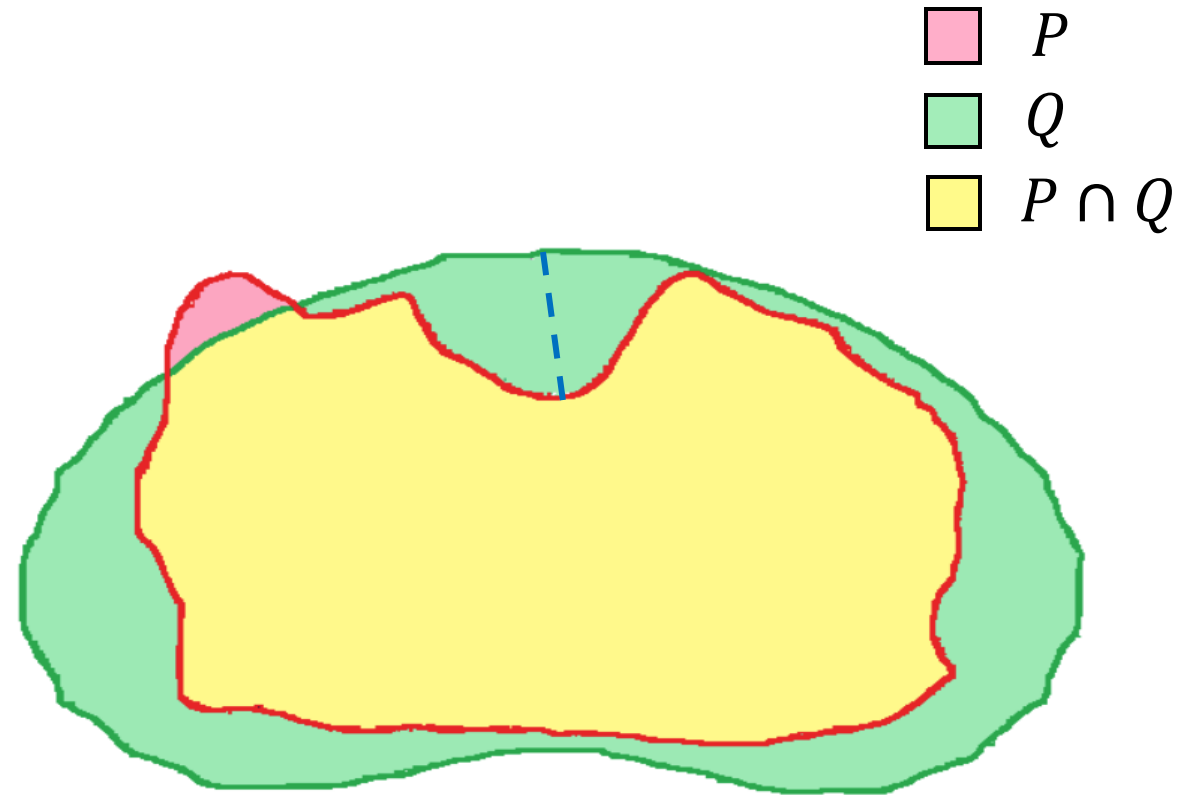
- The **Hausdorff** distance is then the maximum of the two:

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- Based on the definition of Hausdorff distance, we have

$$d(p, q) = d(p, Q) \leq d_H(P, Q)$$

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- We then have

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where the middle inequality is by previous stability theorem

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Stability for point clouds

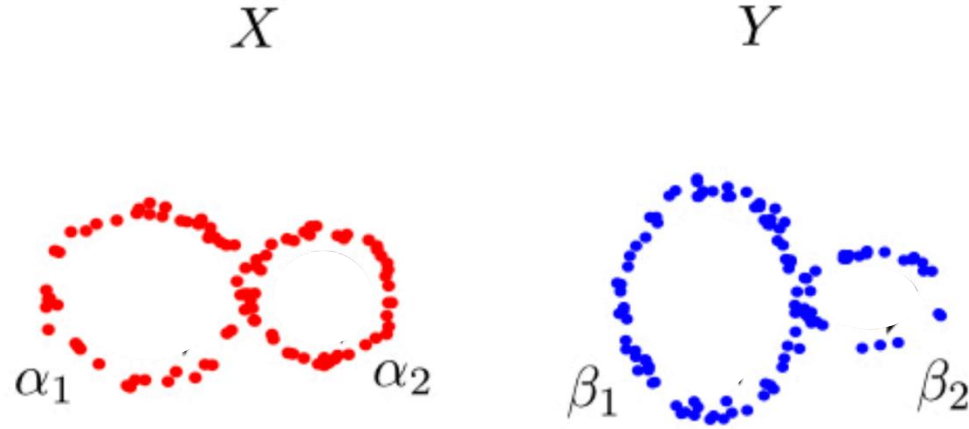
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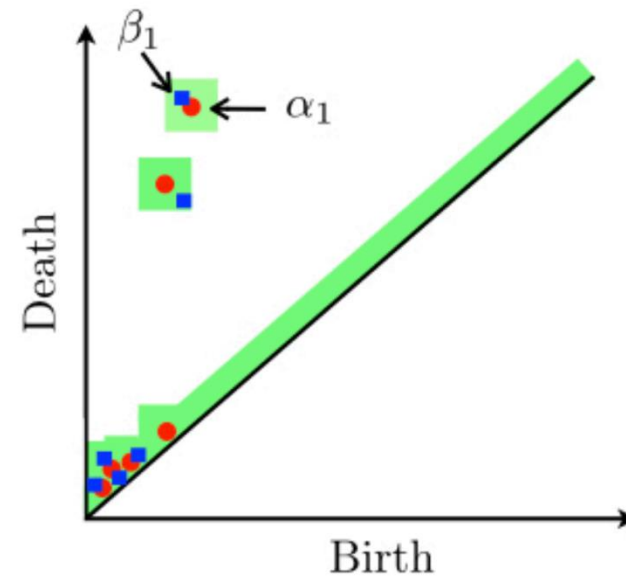
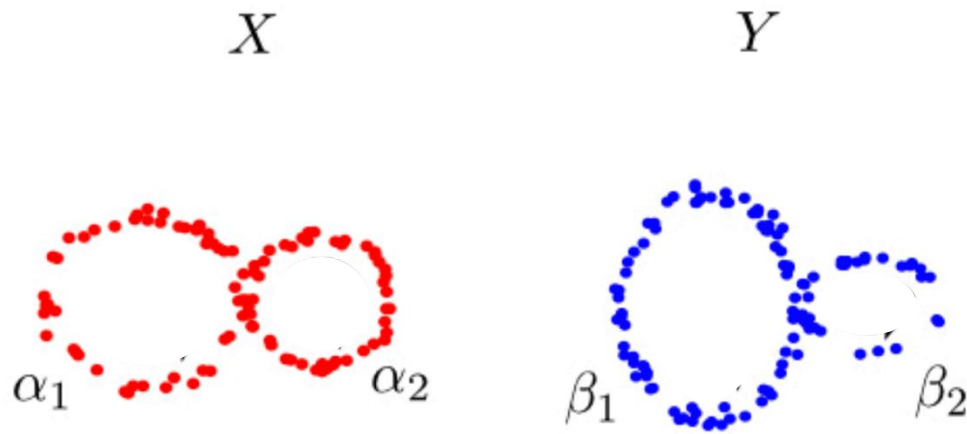
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- Proof of this needs the more advanced notion of “interleaving” stability and is beyond scope

Stability for point clouds



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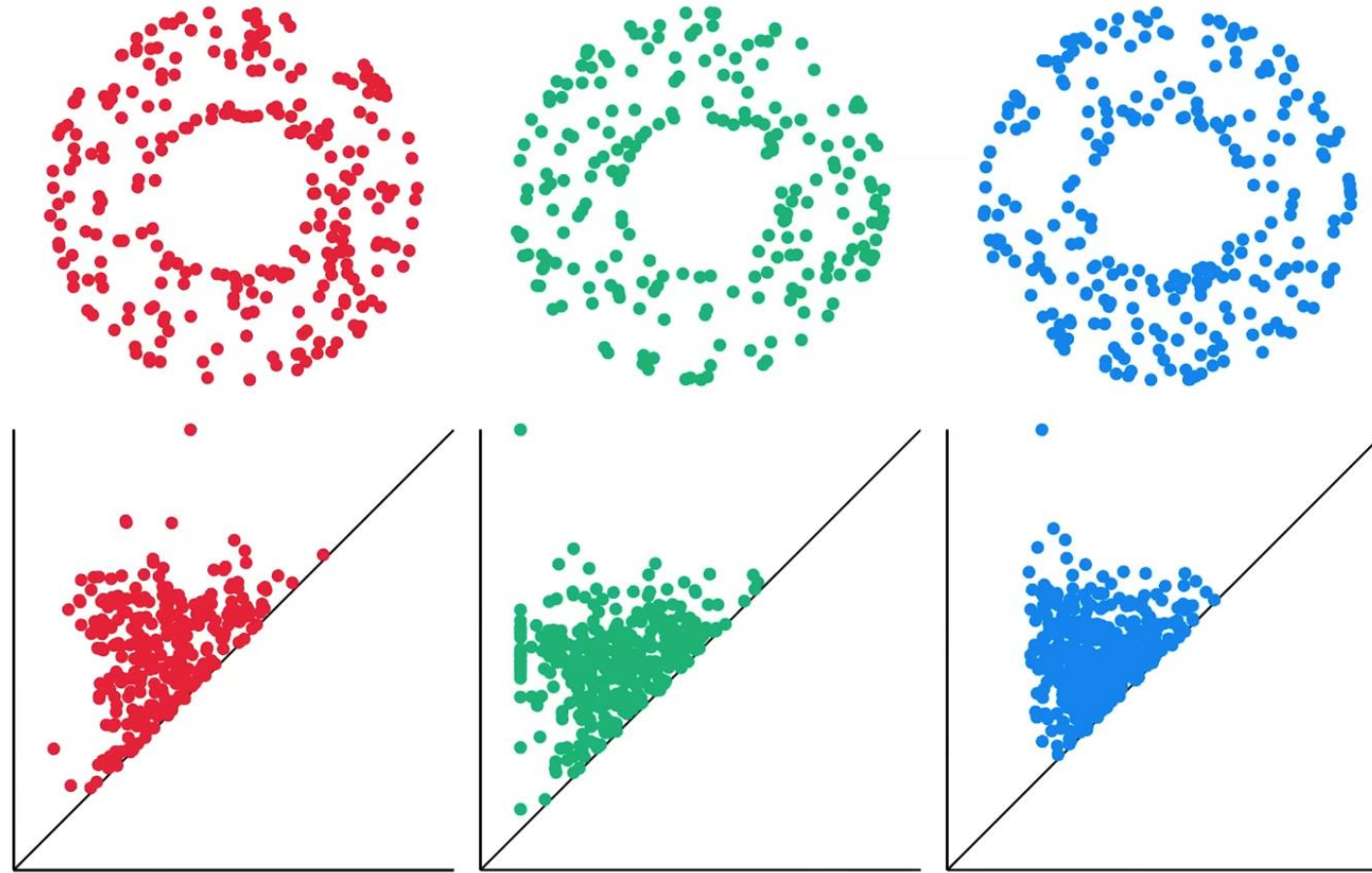


Figure from: Bastian Rieck: Topological Data Analysis for Machine Learning III: Topological Descriptors & How to Use Them (<https://www.youtube.com/watch?app=desktop&v=7i1kabhl5IU>)

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$$d_B(PD_{\check{c}ech}^{\log}(P), PD_{Rips}^{\log}(P)) \leq \log 2$$

- The theorem follows from the previous “interleaving” between Čech and Rips complexes (but details omitted)

$$\mathbb{C}^{\alpha}(P) \subseteq \mathbb{VR}^{\alpha}(P) \subseteq \mathbb{C}^{2\alpha}(P)$$

Computing bottleneck distance

- <https://github.com/nihell/tutorialathon/blob/master/BottleneckTutorial.ipynb>
- <https://www.youtube.com/watch?v=4WswT9snTjc>

Bottleneck distance

- Define a “**partial matching**” η :
 1. Select the same number of points from D_1 and D_2 : let these points “perfectly” match to each other
 2. For the remaining points, we let them be “unmatched”

Bottleneck distance

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- Define $cost(\eta)$ as the maximum of the two below:
 1. Maximum L_∞ -distance of each two matched point p and q in D_1, D_2 :
$$\max_{x,y \text{ matched in } \eta} \{d_\infty(x, y)\}$$
where $d_\infty(x, y) = \max\{|b_1 - d_1|, |b_2 - d_2|\}$ for $x = (b_1, d_1), y = (b_2, d_2)$
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- The **bottleneck distance** is then defined as follows:

$$d_B(D_1, D_2) = \min_{\eta \text{ over all partial matchings}} \{cost(\eta)\}$$

aka. the minimum cost of all partial matchings

Wasserstein distance

- Define a “**partial matching**” η :
 1. Select the same number of points from D_1 and D_2 : let these points “perfectly” match to each other
 2. For the remaining points, we let them be “unmatched”

- Define $cost_p(\eta)$ as:

$$\sqrt[p]{\sum_{x,y \text{ matched in } \eta} (d_p(x,y))^p + \sum_{(b,d) \text{ unmatched in } \eta} (d-b)^p}$$

where $(d_p(x,y))^p = (b_1 - d_1)^p + (b_2 - d_2)^p$ for $x = (b_1, d_1)$, $y = (b_2, d_2)$

- The p -th **Wasserstein distance** is then defined as follows:

$$d_p^W(D_1, D_2) = \min_{\eta \text{ over all partial matchings}} \{cost_p(\eta)\}$$

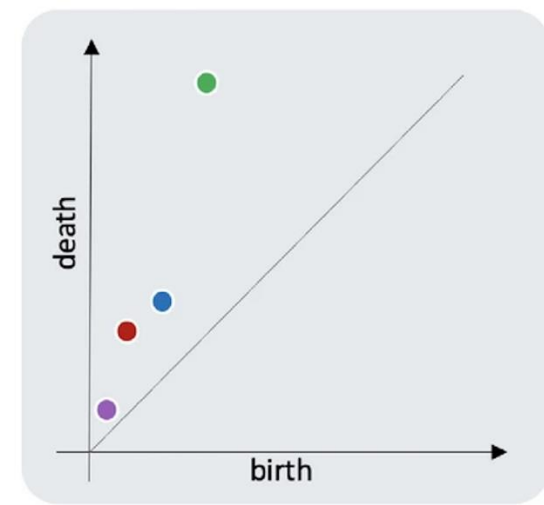
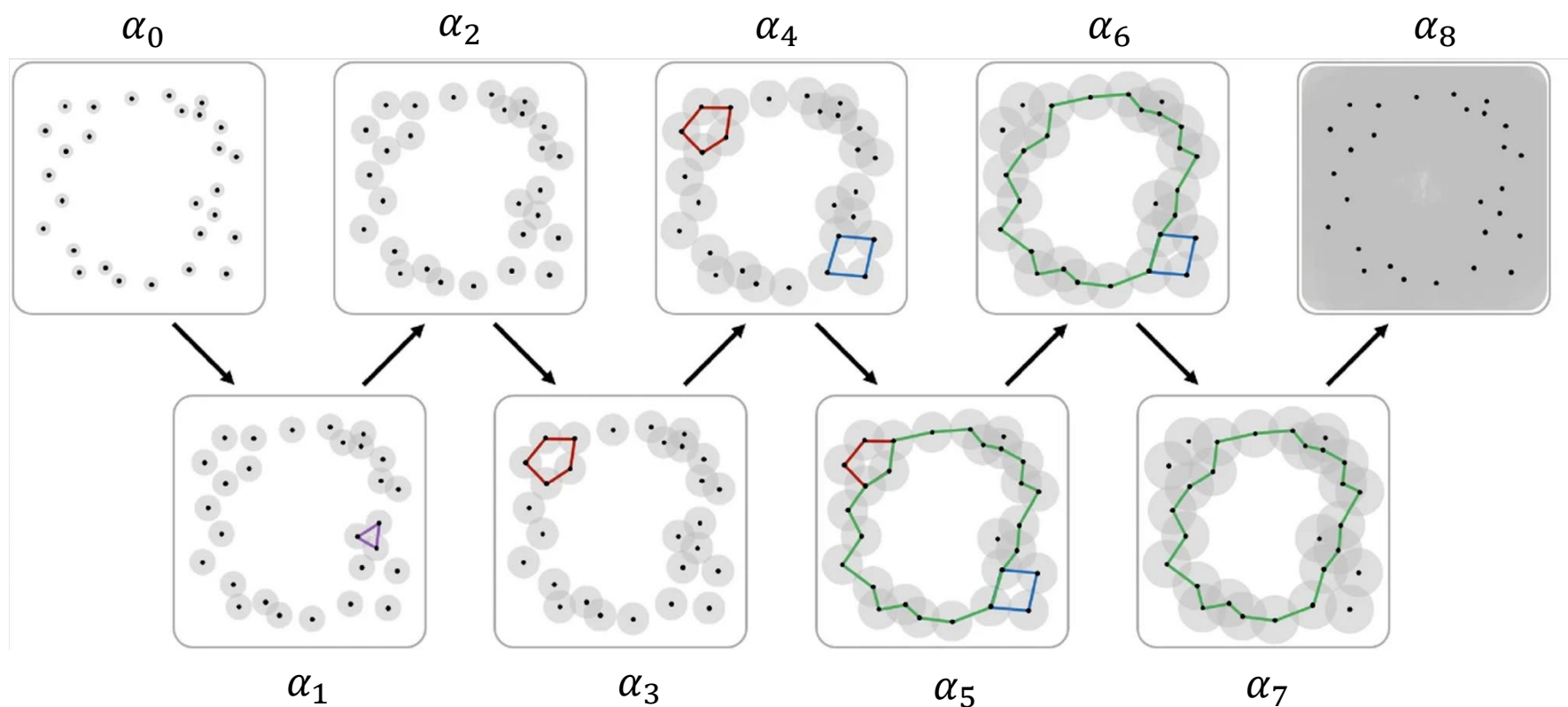
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A practical use of persistence barcode

- A practical use of drawing a PD as a barcode is that barcode provides a way to “visualize” the Betti number across the different range (value α)

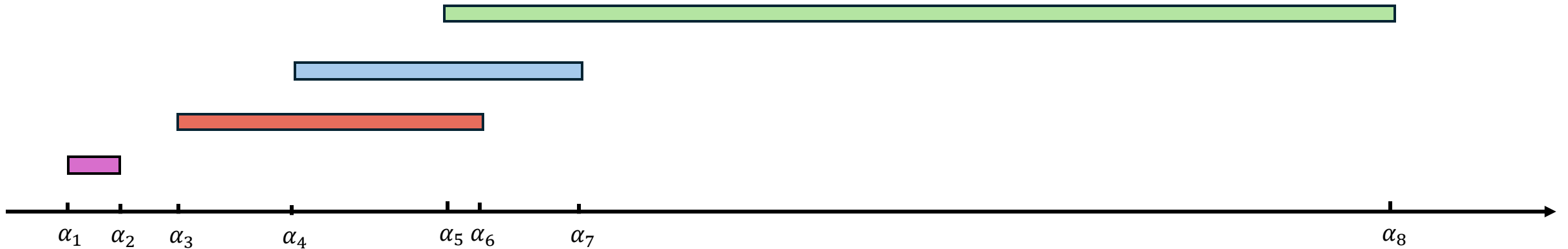
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- For the previous filtration on a point cloud and its 1d PD



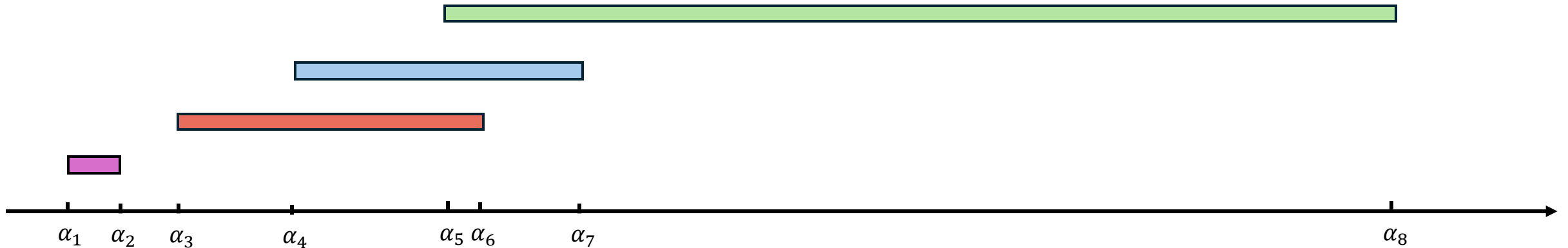
A practical use of persistence barcode

- The following is its 1d barcode



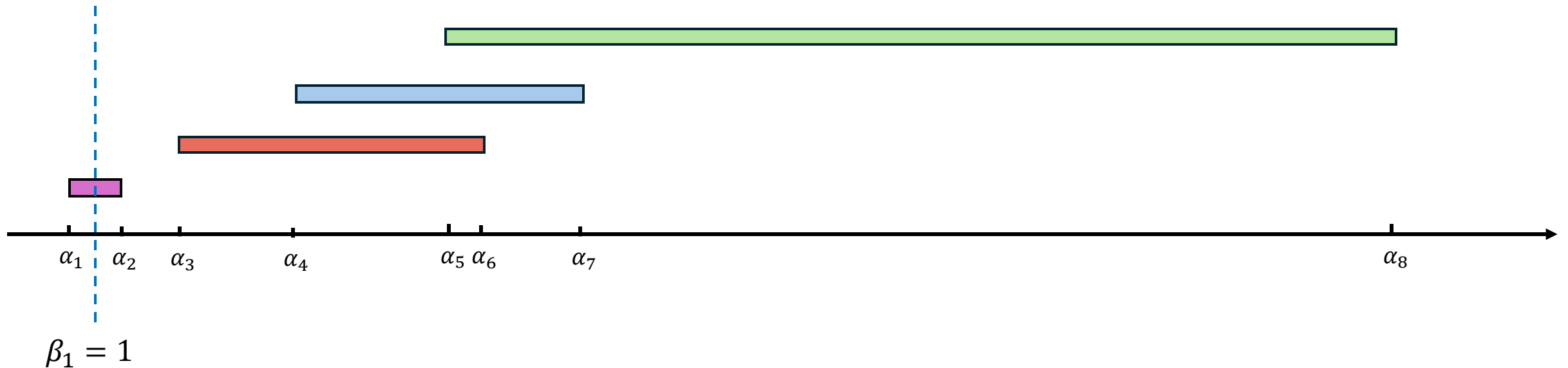
A practical use of persistence barcode

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- Observe, if you count the number of intervals (bars) containing a certain value, then it gives you [the 1st Betti for the complex corresponding to the range](#) (value α) in the filtration



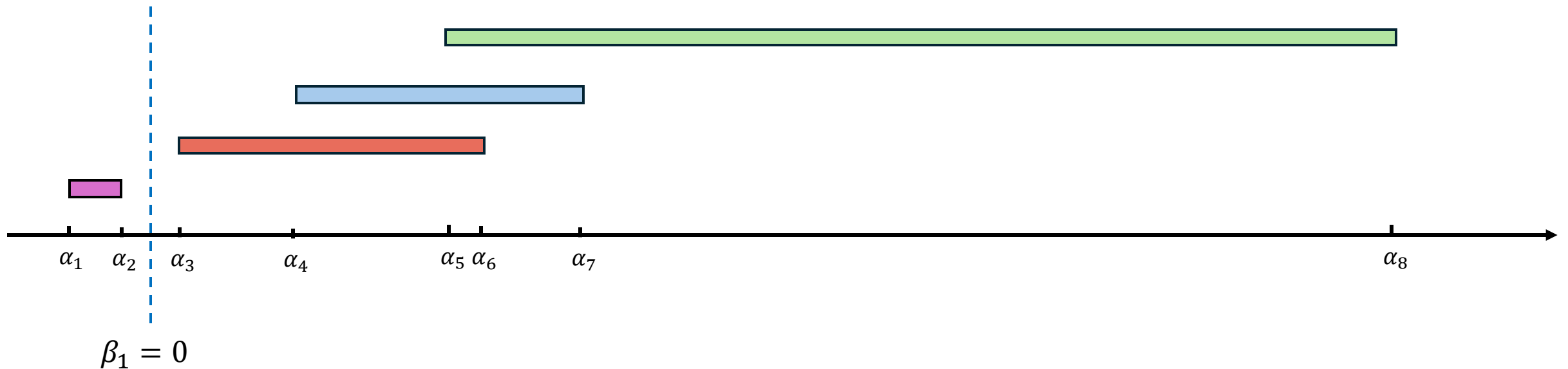
A practical use of persistence barcode

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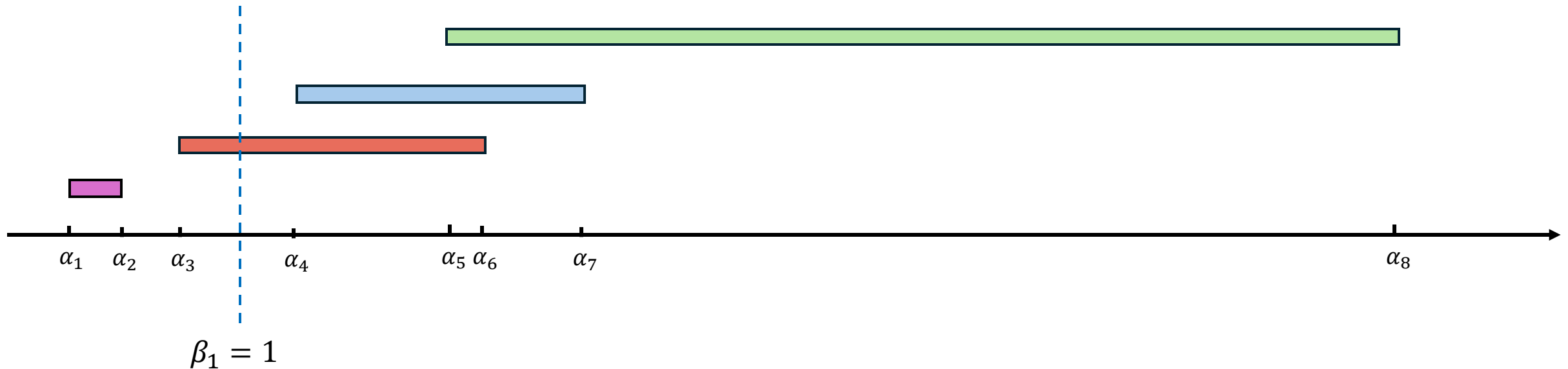
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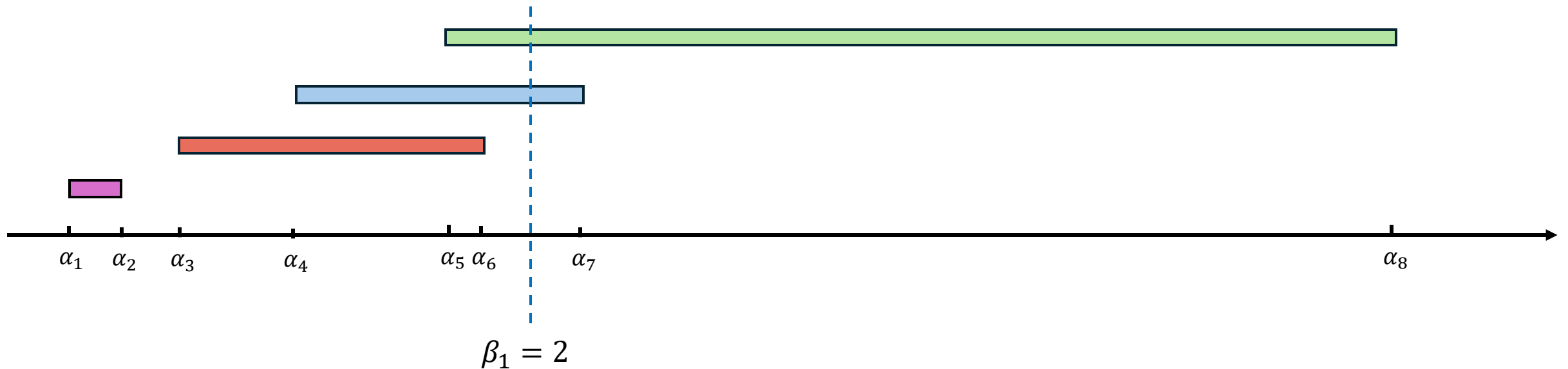
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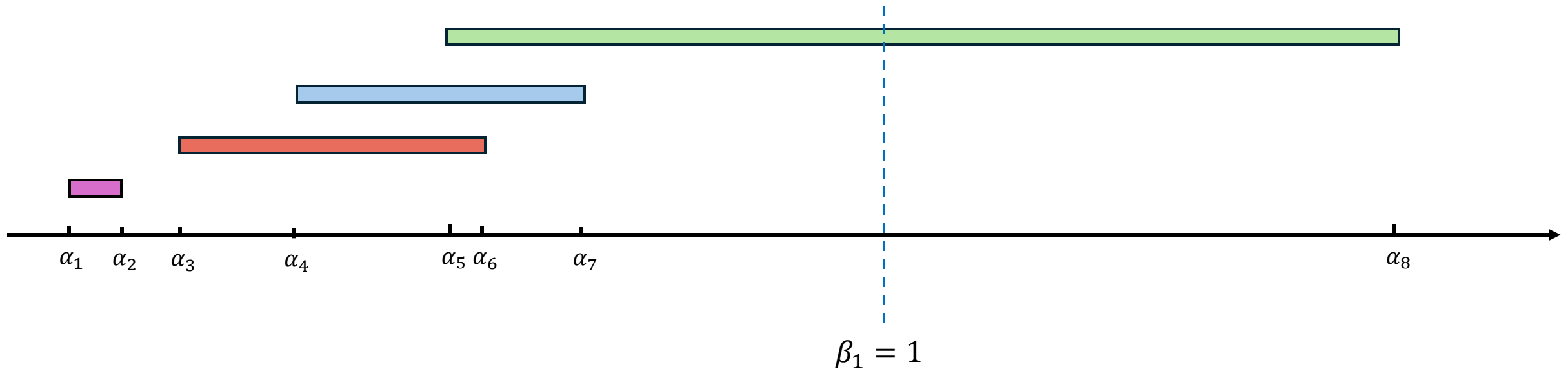
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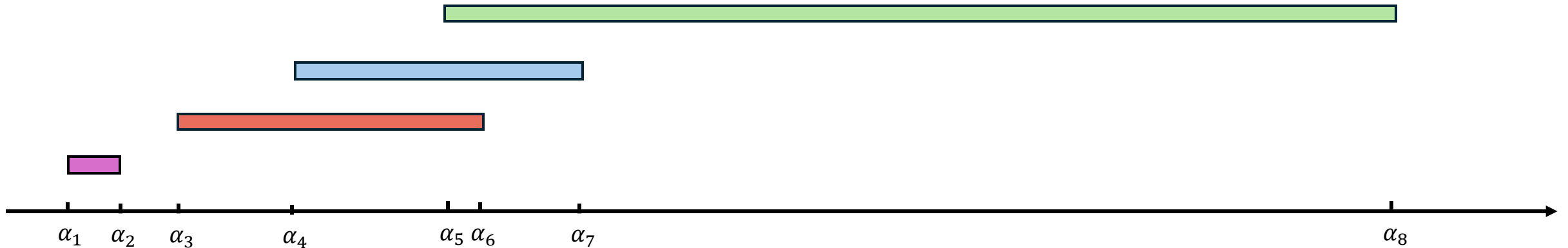
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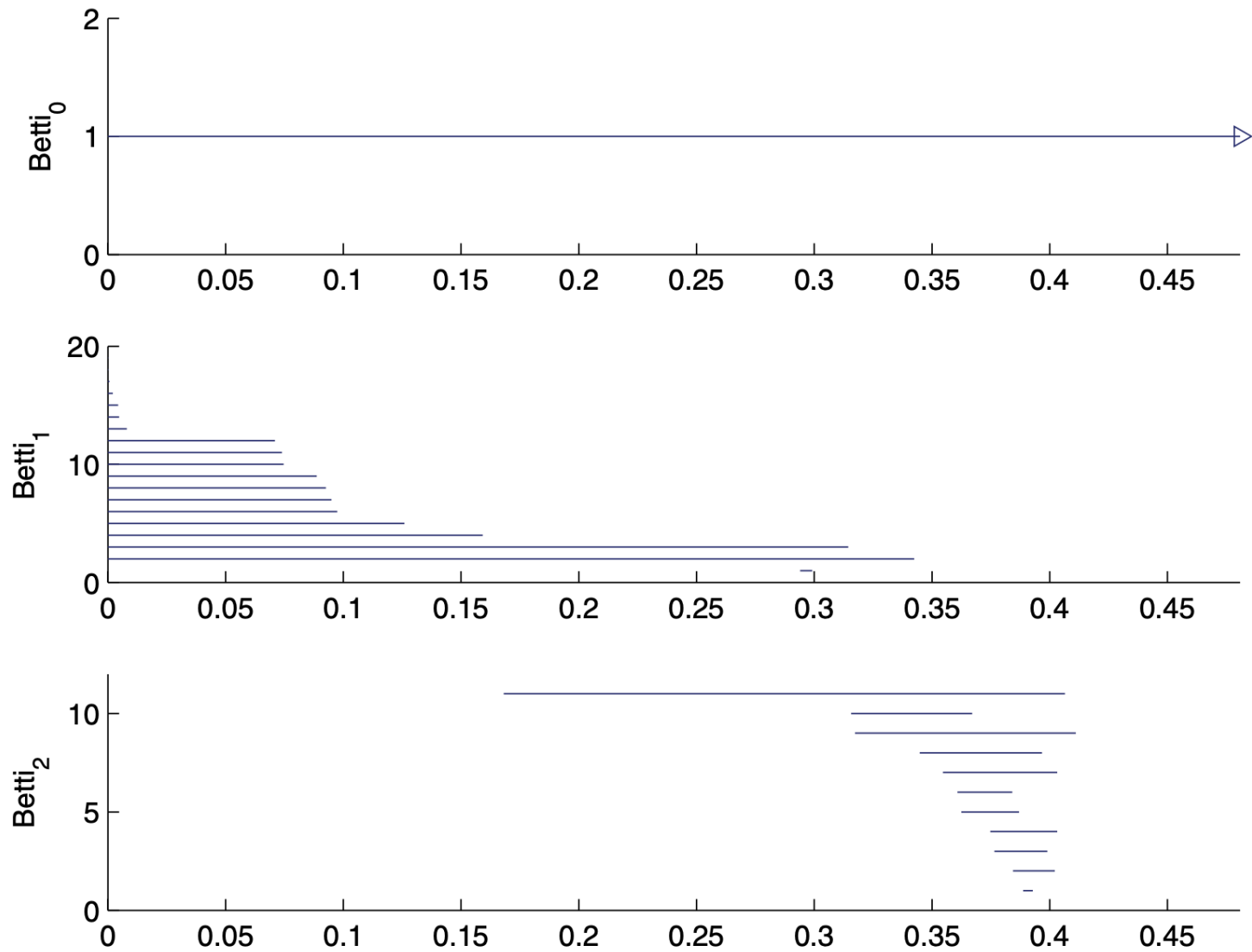


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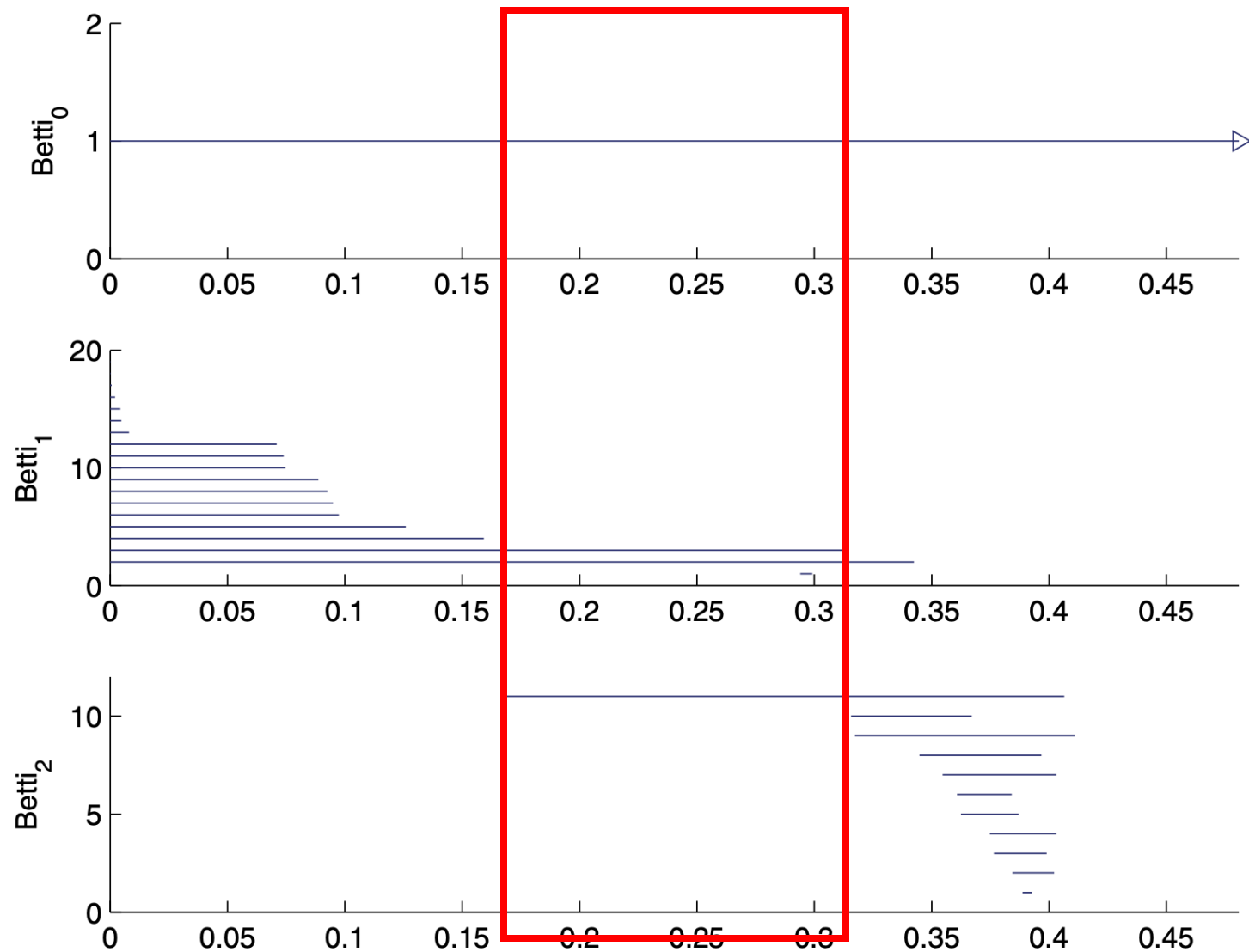
- More powerfully, if you look at the range of the values where the Betti number stays the same, and take **the longest range**, that would give you the most probably inference of the Betti number
- So most probably, for the previous point cloud, $\beta_1 = 1$ (complying out intuitions)



Another example



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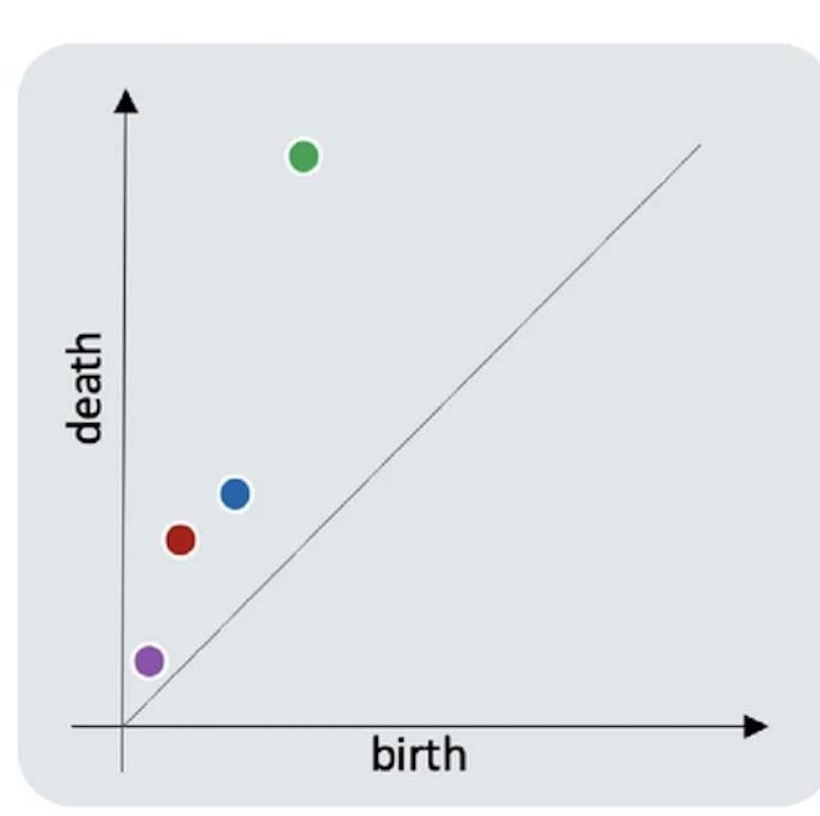


Counting Betti number using PD

- Now we know that taking the intersection of a vertical line with the intervals in the barcode gives you the Betti number at the value of the vertical line, what about PD? Can we do similar things in PD?

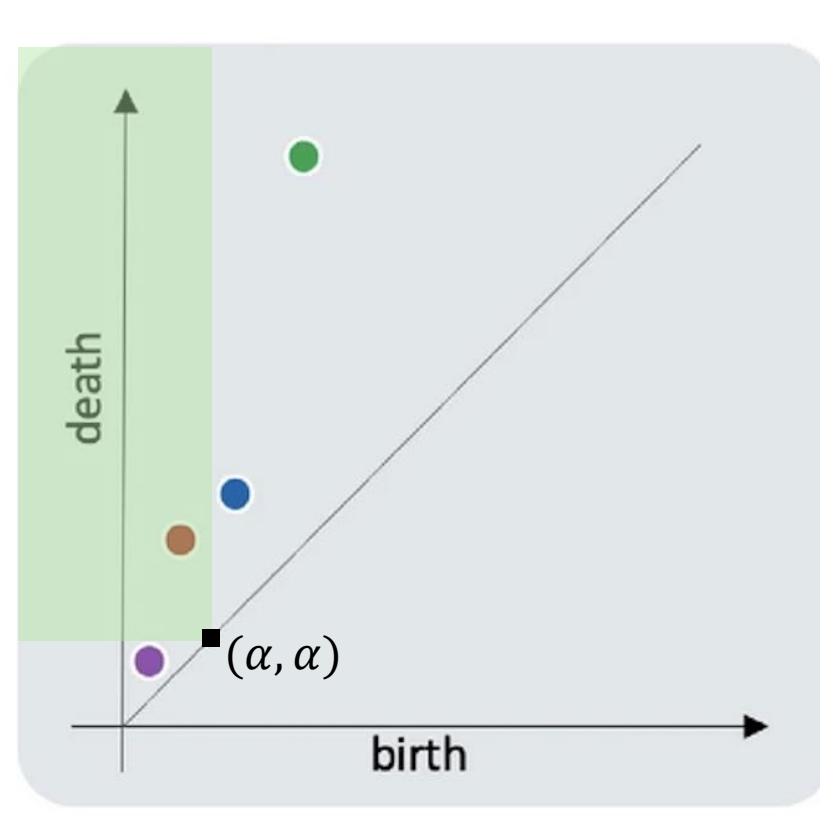
Counting Betti number using PD

- It turns out that, the number of intervals intersecting a value α as manifested on the PD is the **number of pointing in the upper-left quadrant of the point (α, α) on the diagonal**



Counting Betti number using PD

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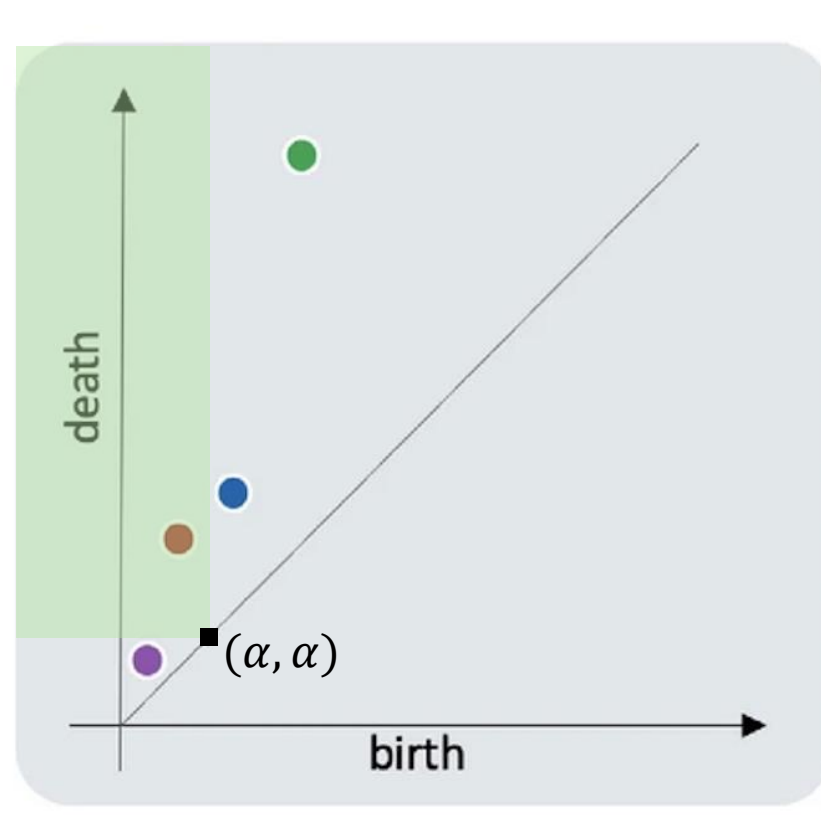


On barcode

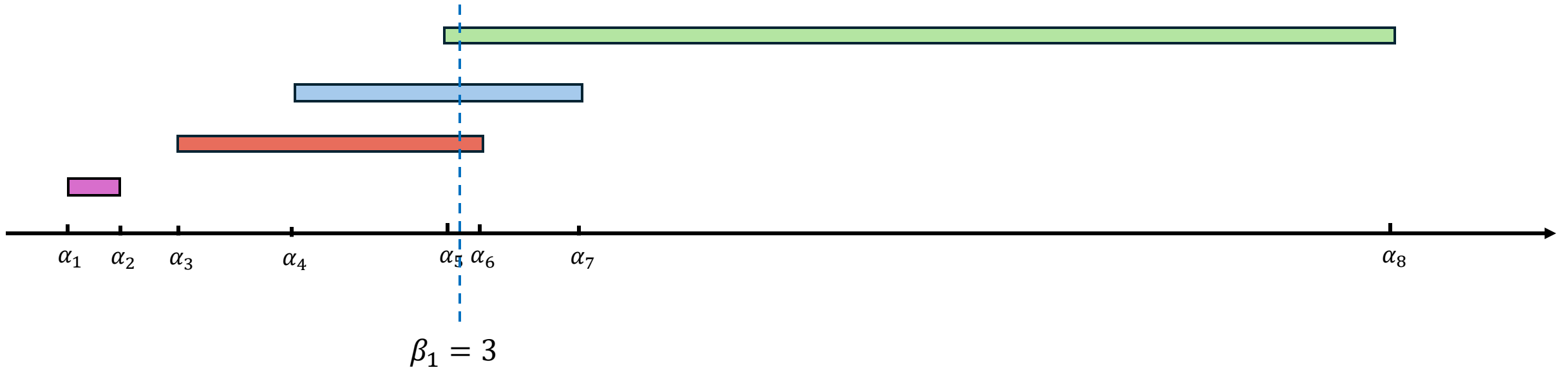


On PD

- $\alpha_3 < \alpha < \alpha_4$

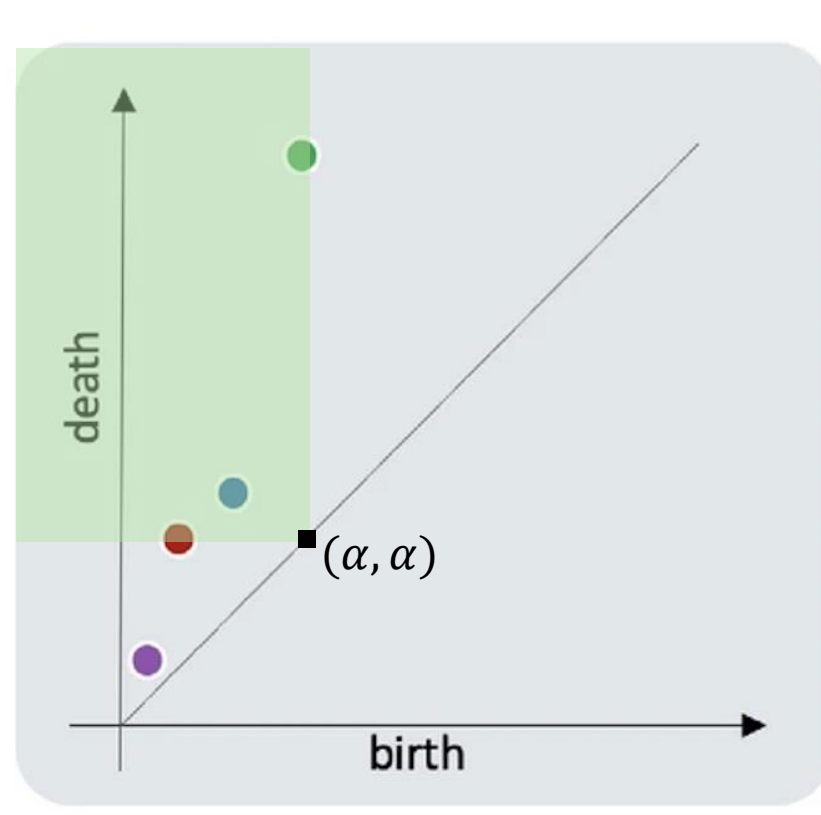


On Barcode



On PD

- $\alpha_5 < \alpha < \alpha_6$

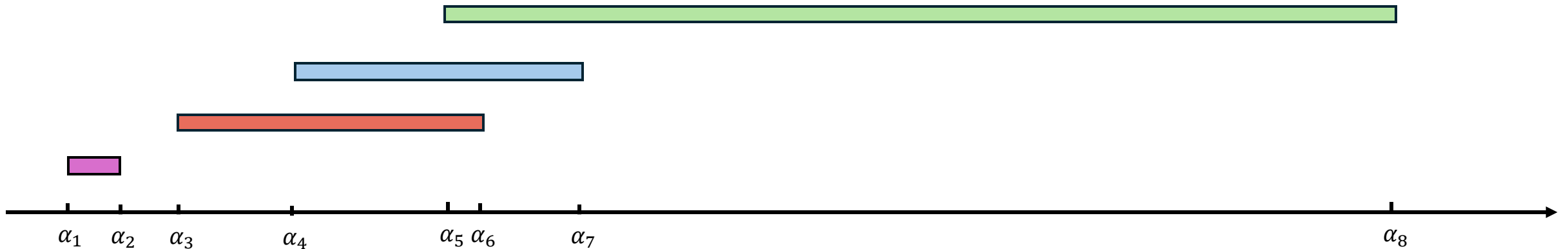


Length “cut-off” for intervals

- A typically belief in TDA is that people often think of features (intervals) with long lifespans as robust, important features, whereas a short lifespan may be an indication that the feature is less essential and may in fact be due to noise in data
- A commonly adopted approach in using PD/barcode for inference is to find a “cut-off” for the short intervals (noise) , and focus on the long intervals (actual features) only

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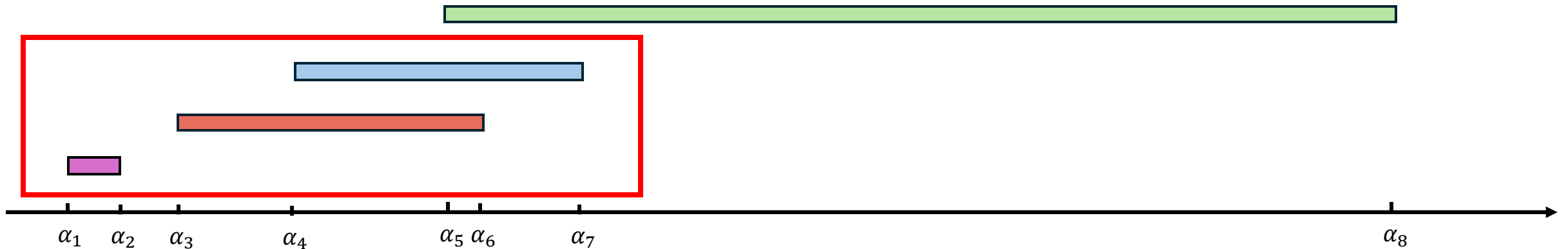
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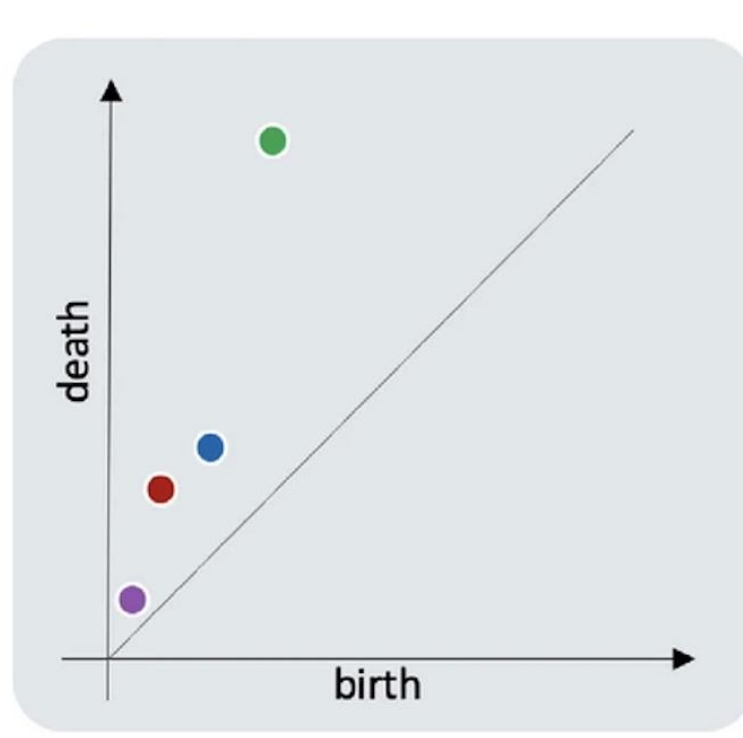
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Length “cut-off” for intervals

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- This is equivalent to take the lower-half space of a 45° degree line (parallel to diagonal)

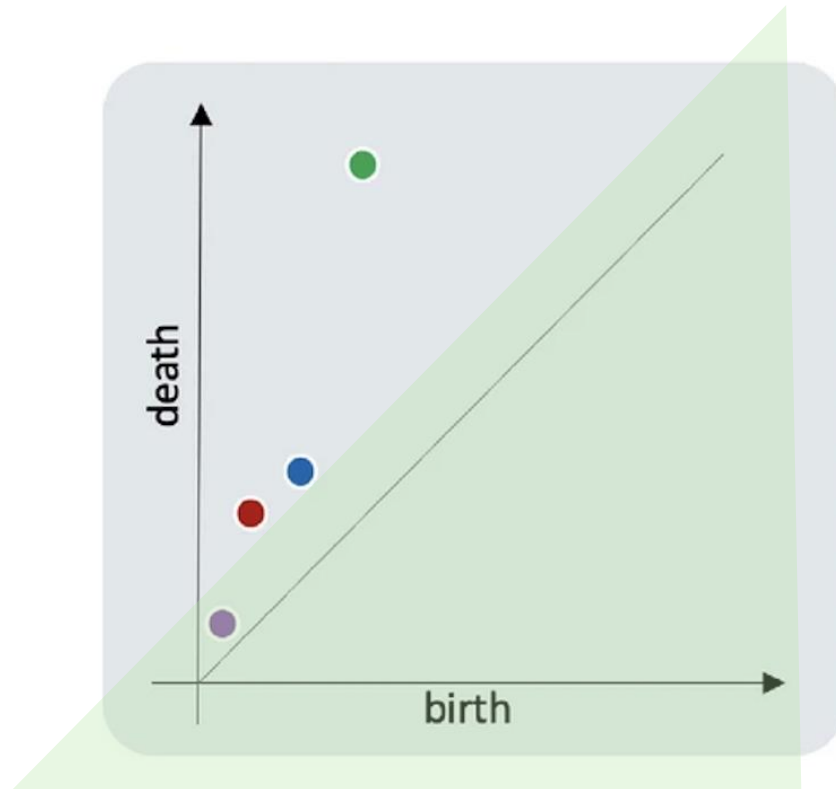
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