Elementary Graph Theory

Tao Hou

Outline

- Graphs: definitions (Review+New)
- Representations (Review)
- Topological sort
- DFS (mostly *New*)

- A ***graph***

$$G = (V, E)$$

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- E is the set of ***edges***
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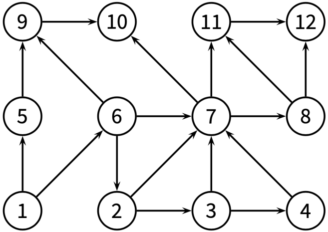
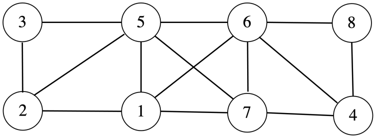
- In this course, unless otherwise noted, we assume graphs are ***simple graphs***, i.e., no *self loops* or *parallel edges*.

Examples





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- A ***cycle*** is a path starting and ending at the same vertex
  - A cycle is called ***simple*** if there are no duplicate vertices on the cycle other than the starting and ending vertices

For a *directed* graph $G = (V, E)$,

- The ***out-degree*** of a vertex $x \in V$ is the number of edges starting with x, i.e,

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The version of 'connected components' for ***directed*** graphs are called ***strongly connected components***, which we do not touch

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  - Most 'tree data structures' are indeed rooted trees, e.g., binary trees, heaps, B-tree
- More on rooted tree:
  - Each vertex has exactly one in-coming edge from its ***parent*** except the root, which has no in-coming edges.
  - If there is a path from u to v, then u is an ***ancestor*** of v and v is a ***descendant*** of u

Some facts about trees

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  - Since the tree has no cycle, only situation (1) can happen.
  - So after adding the $n - 1$ edges, there is only one connected component.
  - This means that when we add the n-th edge, it must create a cycle.

Fact

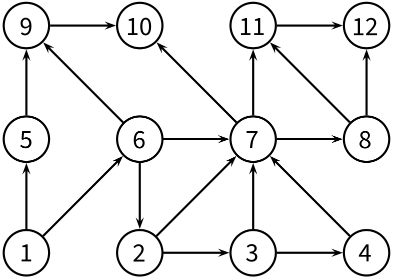
A connected, undirected graph with n vertices and $n - 1$ edges is a tree

- How do we represent a graph $G = (V, E)$ in a computer?

Adjacency-list representation:

- $V = \{1, 2, \ldots, |V|\}$
- G consists of an array Adj
- A vertex $u \in V$ is represented by an element in the array Adj
- $Adj[u]$ is the ***adjacency list*** of vertex u
    - the list of the vertices that are adjacent to u
    - i.e., the list of all v such that $(u, v) \in E$
    - Notice the difference between *directed* and *undirected* graphs

Example



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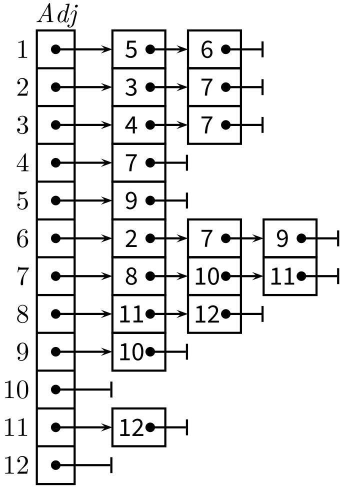

Adj
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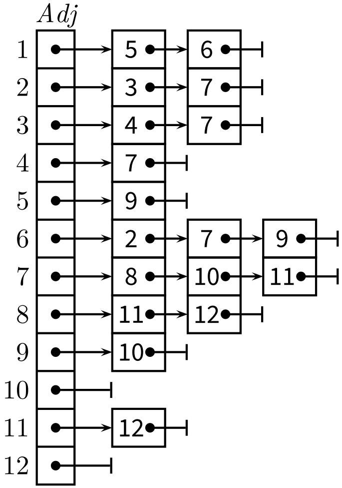
Using the Adjacency List (Review)

- Iteration through E? $O(|V| + |E|)$
  - okay (not optimal)
- Checking $(u, v) \in E$? $O(|V|)$
  - looks bad, but it depends

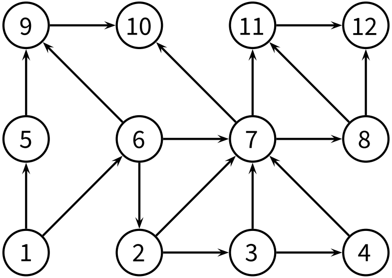


Adjacency-matrix representation:

- $V = \{1, 2, \ldots |V|\}$
- G consists of a $|V| \times |V|$ matrix A
- $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Example



	1	2	3	4	5	6	7	8	9	10	11	12
1					1	1						
2			1				1					
3				1			1					
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5									1			
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10												
11												1
12												

Using the Adjacency Matrix (Review)

- Iteration through E? $O(|V|^2)$
  - possibly very bad
- Checking $(u, v) \in E$? $O(1)$
  - optimal

	1	2	3	4	5	6	7	8	9	10	11	12
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Space Complexity (Review)

- Adjacency-list representation

$$\boxed{O(|V| + |E|)}$$

  optimal

- Adjacency-matrix representation

$$\boxed{O(|V|^2)}$$

  possibly very bad

Choosing a Graph Representation (Review)

- Adjacency-list representation
  - generally good, especially for its optimal space complexity
  - bad for ***dense*** graphs and algorithms that require random access to edges
  - preferable for ***sparse*** graphs or graphs with ***low degree***
- Adjacency-matrix representation
  - suffers from a bad space complexity
  - good for algorithms that require random access to edges
  - preferable for ***dense*** graphs
- Sparse vs. dense graph
  - ***Sparse*** graph: $|E| = O(|V|)$
  - ***Dense*** graph: $|E| = \Theta(|V|^2)$

- **Problem:** (topological sort)

  Given a *directed acyclic graph* (DAG)

  - find an ordering of vertices such that you only end up with *forward edges*
  - in another word, if there is an edge (u, v), then u appears before v in the ordering (that's also the reason why we can do this *only* on DAG instead of general graphs)

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- **Example:** dependencies in software packages

  - find an installation order for a set of software packages
  - such that every package is installed only after all the packages it depends on

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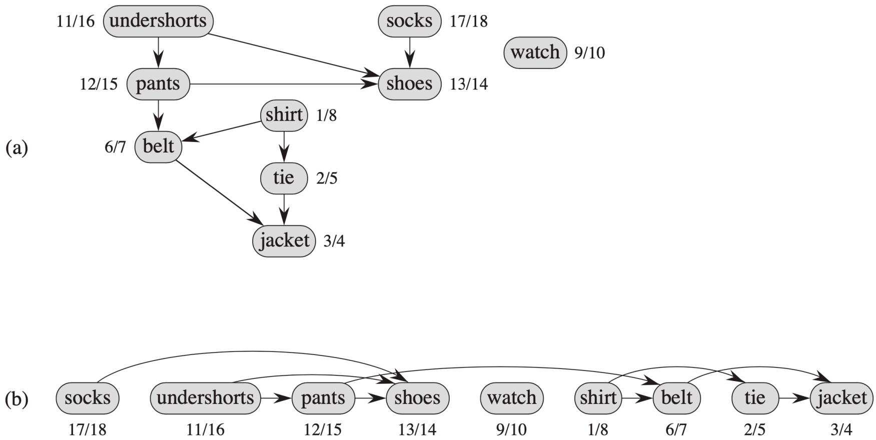

(Example from CLRS)

Topological Sort Algorithm

TOPOLOGICAL-SORT(G)

while $\exists v \in V$ s.t. $\mathit{in}\text{-}\mathit{deg}(v) = 0$
    output v
    remove v and all its out-going edges from G

```
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while ∃v ∈ V s.t. in-deg(v) = 0
    output v
    remove v and all its out-going edges from G
```

Argument of correctness:

- We remove an edge only when its starting vertex has been output in the order
- Thus, when a vertex *v* has in-degree 0, this means that all vertices pointing to *v* (if any) have been output, so that we can also safely output v

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Question:

- Why should there always be a vertex with 0 in-degree?

Topological Sort: Alternative Algorithm

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TOPOLOGICAL-SORT(G)
1  DFS(G)
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Topological Sort: Alternative Algorithm

Topological-Sort(G)

DFS(G)
output V sorted in reverse order of $f[\cdot]$

We will see why this algorithm works later on.
Some comments:

- The first algorithm is mainly of theoretical value (helps you to understand the whole procedure)
- In practice, you should utilize DFS to compute topological sorting for DAGs because it's much simpler (you don't need to bother to delete the edges)
- So topological sort can be done in $O(|V| + |E|)$ time

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- Explores the graph starting from s, touching all vertices that are reachable from s

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  - ***white***: not yet visited
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  - ***black***: finished visiting
- Associates ***two time-stamps*** to each vertex
  - $d[u]$ records when DFS starts visiting u (turns *grey*)
  - $f[u]$ records when DFS finishes visiting u and therefore backtracks from u (turns *black*)

DFS Algorithm

```
DFS(G)
for each vertex u ∈ V(G)
        color[u] = WHITE
        π[u] = NIL
time = 0 // "global" variable
for each vertex u ∈ V(G)
        if color[u] == WHITE
                DFS-VISIT(u)
```

```
DFS-VISIT(u)
color[u] = GREY
time = time + 1
d[u] = time
for each v ∈ Adj[u]
        if color[v] == WHITE
                π[v] = u
                DFS-VISIT(v)
color[u] = BLACK
time = time + 1
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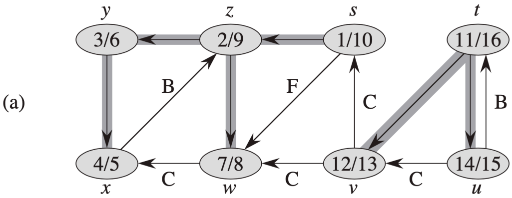
- A first very silly question: Can DFS ever end?

DFS: Example



(Example from CLRS)

Complexity of DFS

- The loop in **DFS-VISIT**(u) (lines 4–7) executes for $O(\text{out-deg}(u))$ times
- We call **DFS-VISIT**(u) *once* for each vertex u
  - either in **DFS**, or recursively in **DFS-VISIT**
  - because we call it only if $color[u] = $ WHITE, but then we immediately set $color[u] = $ GREY
- So, the overall complexity is $\Theta(|V| + |E|)$

Parenthesis Theorem

In a DFS on a (directed or undirected) graph G, for any two vertices u and v, exactly one of the following two holds:

1. The intervals $\left[d[u], f[u]\right]$ and $\left[d[v], f[v]\right]$ are entirely disjoint, and neither one is a descendant of the other in the DFS forest
2. One interval is entirely contained in the other interval, and the vertex is a descendant of another (e.g., $\left[d[v], f[v]\right] \subseteq \left[d[u], f[u]\right]$ and v is a descendant of u)

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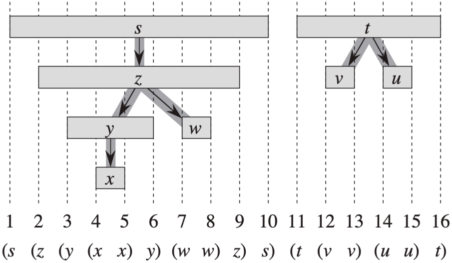
Example (from CLRS):

(b)



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- Observe: *the grey vertices in DFS always form a linear chain of descendants (in the DFS tree) corresponding to the stack of active DFS-VISIT invocations*
- This means that v is a descendant of u in the DFS forest
- Also, the visiting of u cannot finish before we finish visiting u (this is how recursive calls work), so $f[v] < f[u]$ (aka. $d[u] < d[v] < f[v] < f[u]$)

- Now consider $d[v] > f[u]$
- Obviously, $d[u] < f[u] < d[v] < f[v]$, so the two intervals are disjoint

White-Path Theorem

In a DFS forest of a (directed or undirected) graph G, a vertex v is a descendant of a vertex u if and only if at time $d[u]$, there is a path from u to v on G consisting of *only* white vertices

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proof:

- "$\Rightarrow$": let w be any descendant of u in the DFS tree
- By the previous Parenthesis Theorem, we have that $d[u] < d[w]$, so when u is discovered, w is still white
- Notice that on the path from u to v in the DFS tree, all vertices are descendants of v, so all of them are white at time $d[u]$

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proof (continue):

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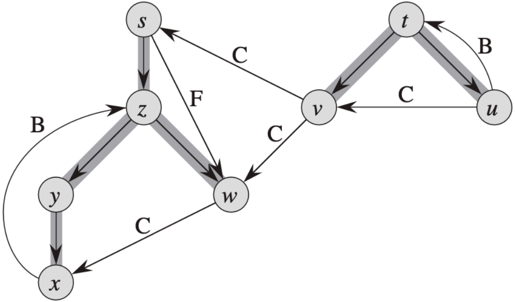
- But if this is true, then x must be a descendant of w and in turn a descendant of u (a contradiction)

Four Types of Edges in DFS on Directed Graphs

- ***Tree edge***: Edges on the DFS forest
- ***Back edge***: Connecting a vertex to its *ancestor* in the DFS forest
- ***Forward edge***: Non-tree edges connecting a vertex to its *descendant* in the DFS forest
- ***Cross edge***: all other edges



(Example from CLRS)

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- By the White-Path Theorem, u must be a descendant of v in the depth-first forest

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- When v is discovered, we have that all vertices on path from v to u (on c) are white (undiscovered)
- By the White-Path Theorem, u must be a descendant of v in the depth-first forest
- Therefore, (u, v) is a back edge

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- When we are visiting u and exploring the edge (u, v) in DFS, we have that v cannot be grey because otherwise (u, v) would be a back edge (notice that u must be at the stack top when we are exploring (u, v)), contradicting the previous Lemma saying that DFS on DAG yields no back edges

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- If v is black, we have already finished visiting v. But the visiting of u is not finished. So we obviously have $f[v] < f[u]$.

Observation

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Proof:

- Consider a path P connecting u, v in G
- Let x be the first vertex on P visited by DFS.
- When we discover x, we have that there is white path from x to u and from x to v.
- Therefore, u, v, x are all in the same DFS tree by the White Path Theorem.