Persistent Homology: Intro

Tao Hou, University of Oregon

Outline for studying persistent homology

1. Intro to persistent homology
    - Build intuitions of persistent homology: what it does, what it produces
2. Formalizing persistent homology
    - Introduce its input (filtration) and study an algorithm for computation
3. Different ways for building filtrations
    - Vietoris-Rips filtration, sub-levelset filtration
    - Cubical complexes (for images)
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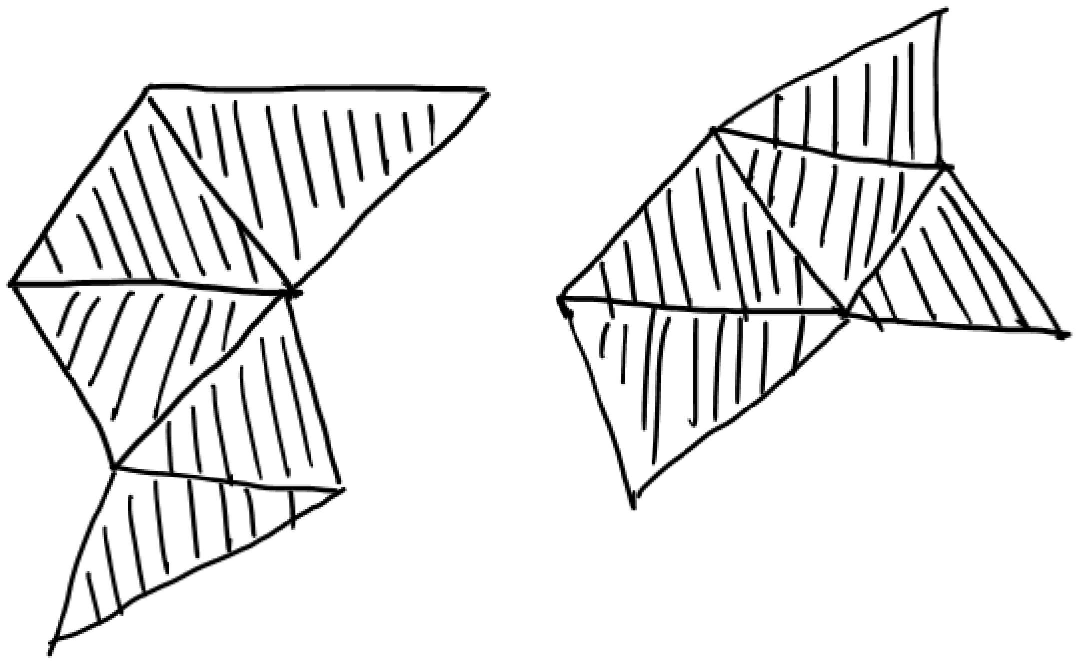
- **Ex**: There is a 0-dimensional hole of the following complex because of the gap between the two connected components



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- **Ex**: The homology basis for the 1-cycles in the below simplicial complex contains the single red 1-cycle.
  - So that we can use the red cycle to represent the 1-dimensional "homological features" of the space

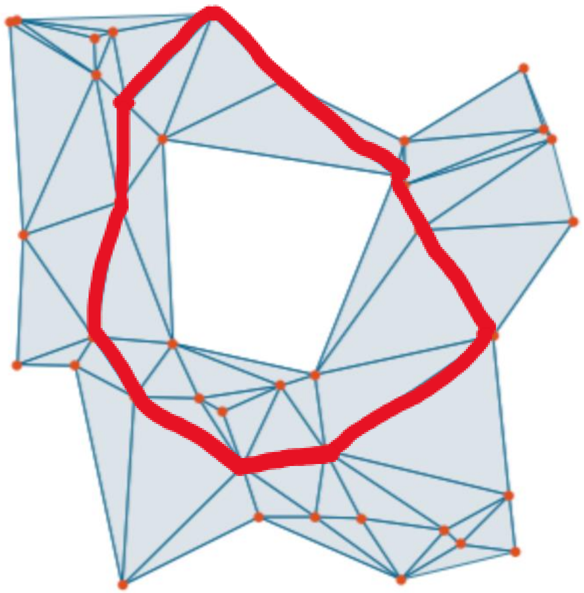

Image source: Yan et al. Persistence Landscape based Topological Data Analysis for Personalized Arrhythmia Classification

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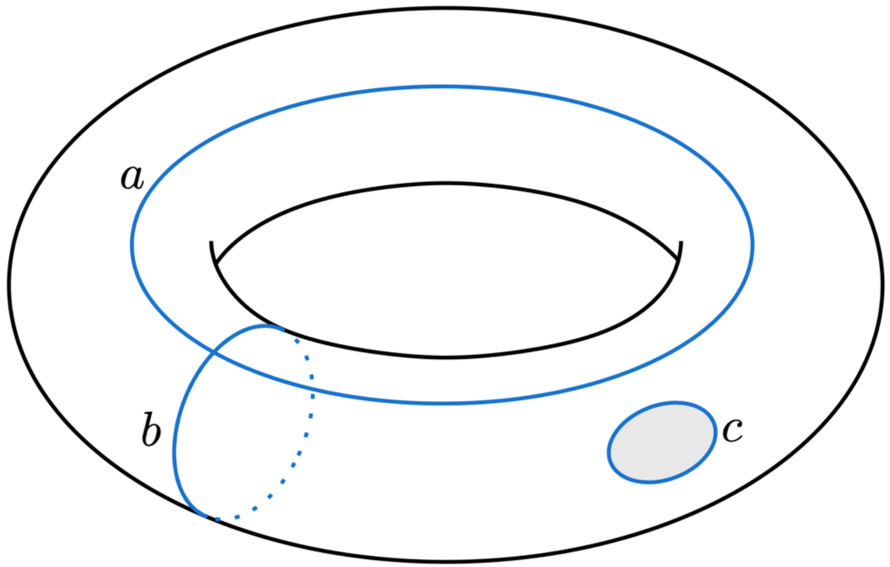
- **Ex**: The 1-dimensional homological features of a torus can be characterized by two cycles:
  - a (longitude) and b (meridian)



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- We shall look at at least two problems with it

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- Simplicial complex is "highly structured" data, while in practice we don't have the luxury of always having data rich structure
- Typically, data come in as "unstructured" (e.g., point clouds)
- For the right point cloud (which is unstructured), everyone could see that it consists of two rings (1-cycles)
- But we have to **construct a simplicial complex from the point cloud** first to infer this information

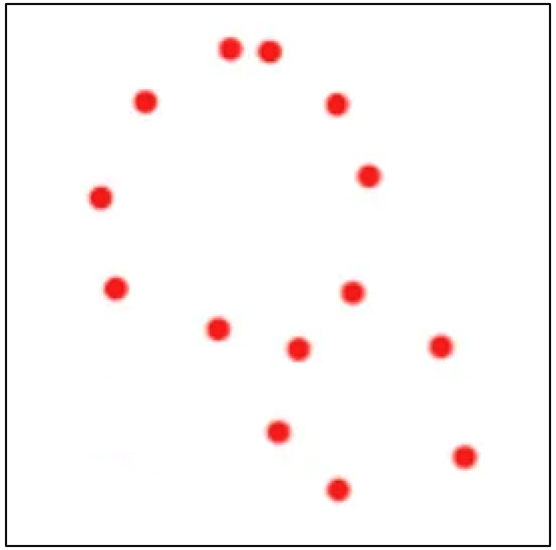

Image source: https://medium.com/@deltorobarba/quantum-topological-data-analysis-the-most-powerful-quantum-machine-learning-algorithm-part-1-c6d055f2a4de

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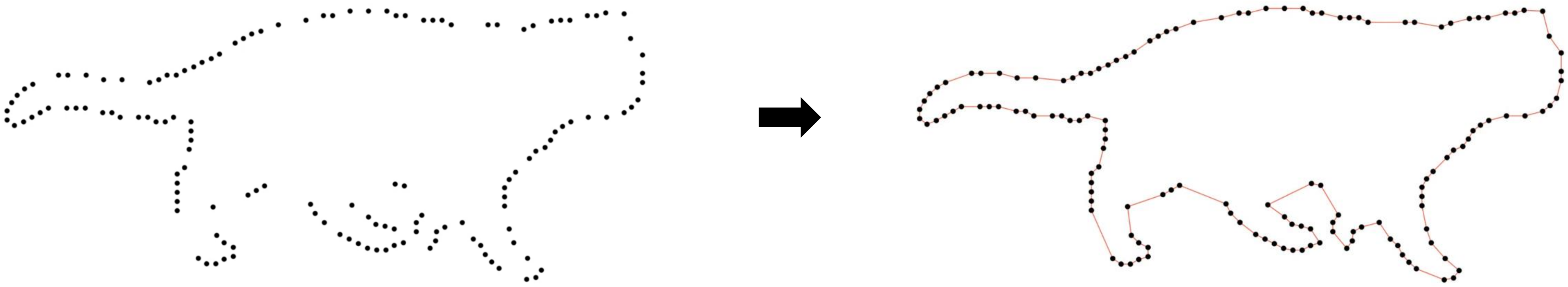

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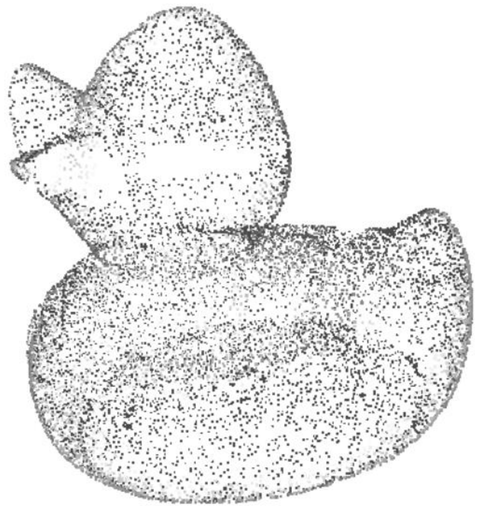

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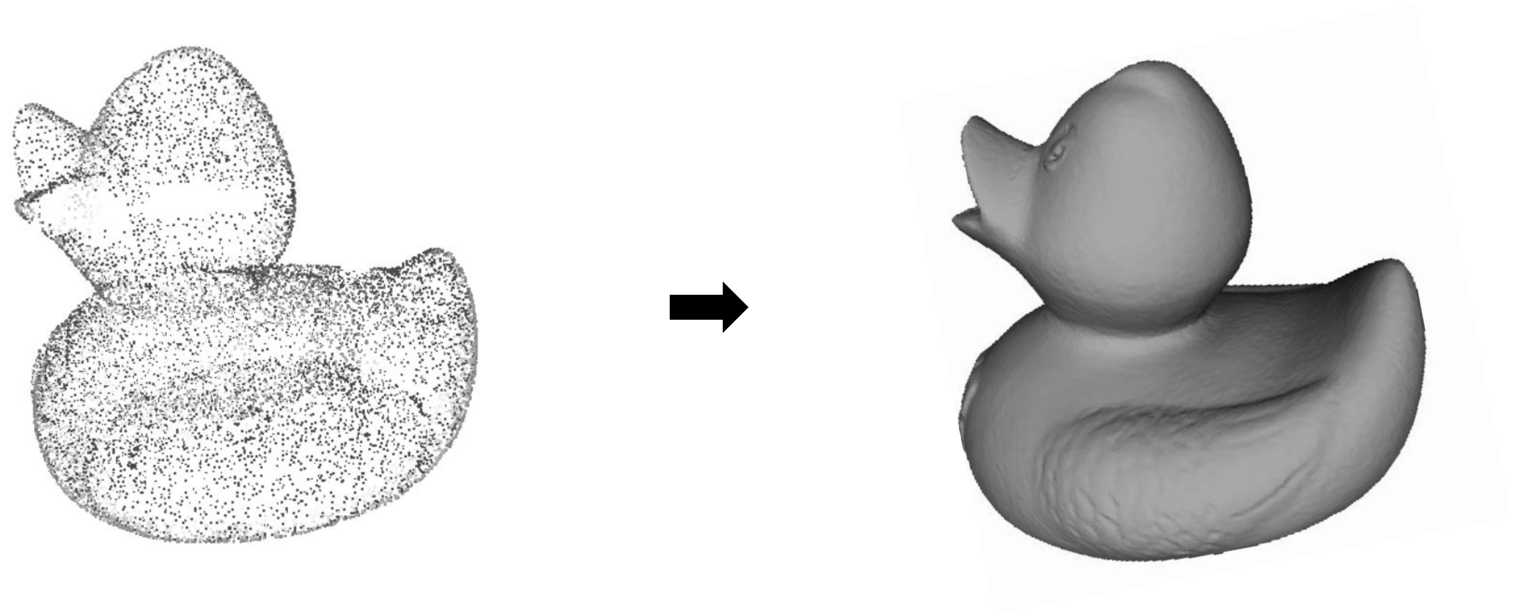

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- But there still are problems:
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  2. There are probably more information in the original unstructured data than is reconstructed
  3. Reconstruction from point clouds which are not nicely shaped is very hard if at all possible

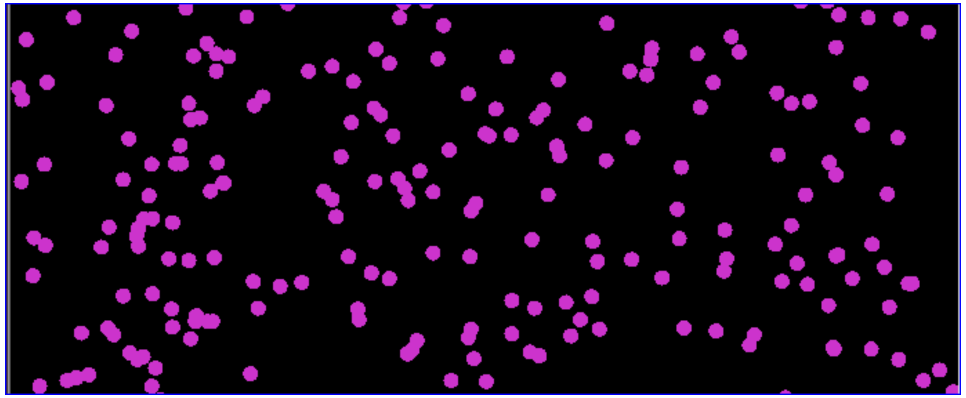

Image source: https://prototechsolutions.com/cad-notes/lasso-selection-tool/

Homology inference: Problem 2

- In the following space, there are seven 1-dimensional holes (i.e., *homology basis contains seven non-trivial 1-cycles*)

Image source: https://kids.frontiersin.org/articles/10.3389/frym.2021.551557

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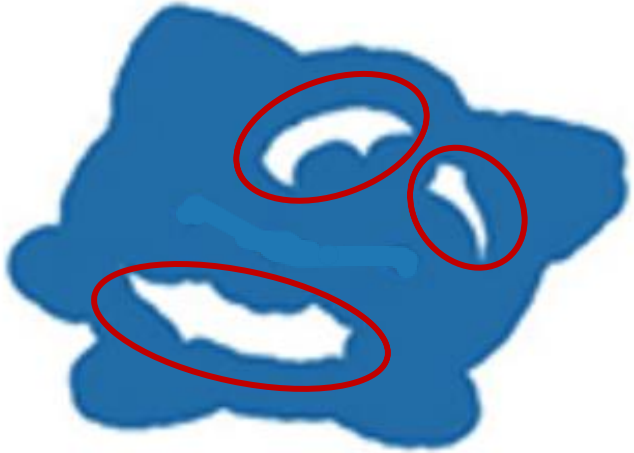

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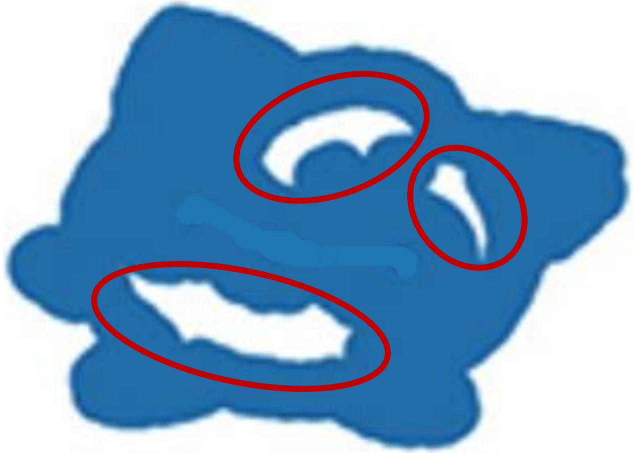
- i.e., with some "perturbation" on the data, the four small holes could simply disappear
- But using homology basis we could not differentiate the "more significant holes" from the "less significant ones"



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Homology inference: Problem 2

- Similarly, in the following space, there are three 1-dimensional holes, but there is clearly a “more significant” one and two “less significant” ones which also be some artifacts
- Again, using just homology basis we could not differentiate them

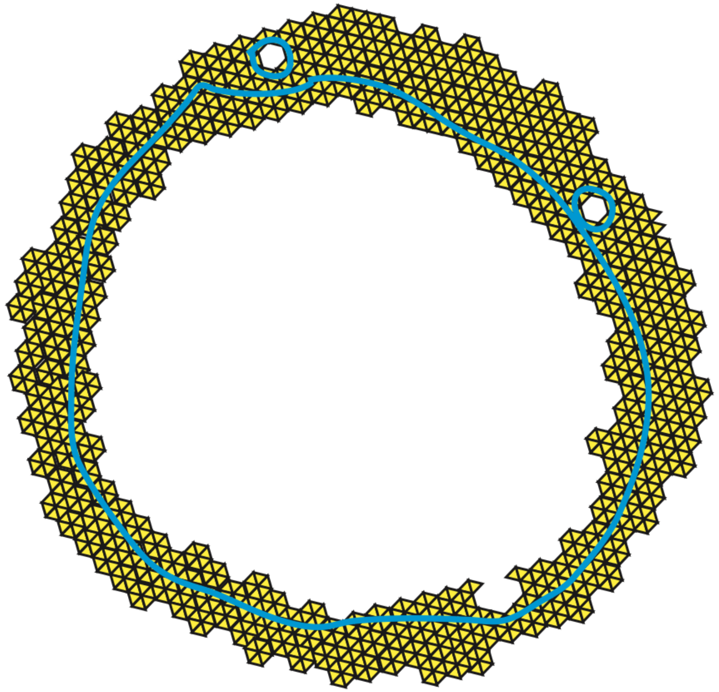

Image source: Gunnar Carlsson, Topology and Data

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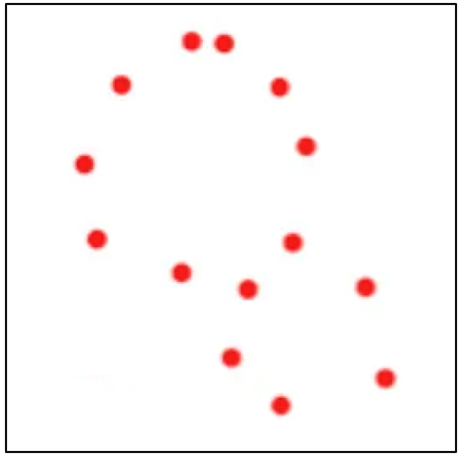

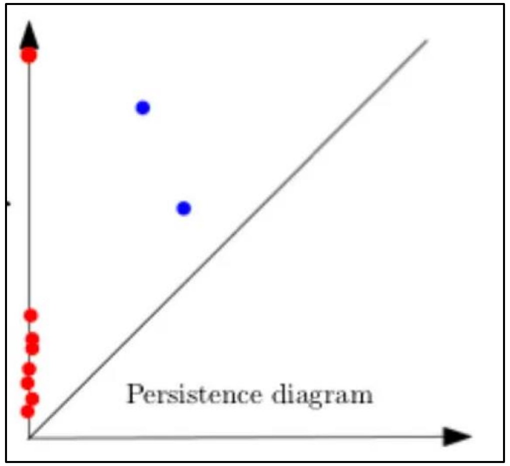

- For the point cloud, persistent homology produces a "topological signature" called **persistence diagram**
- In the diagram, the **blue dots** represents the two rings, thus correctly inferring the topological structure of the point cloud

Image source: https://medium.com/@deltorobarba/quantum-topological-data-analysis-the-most-powerful-quantum-machine-learning-algorithm-part-1-c6d055f2a4de

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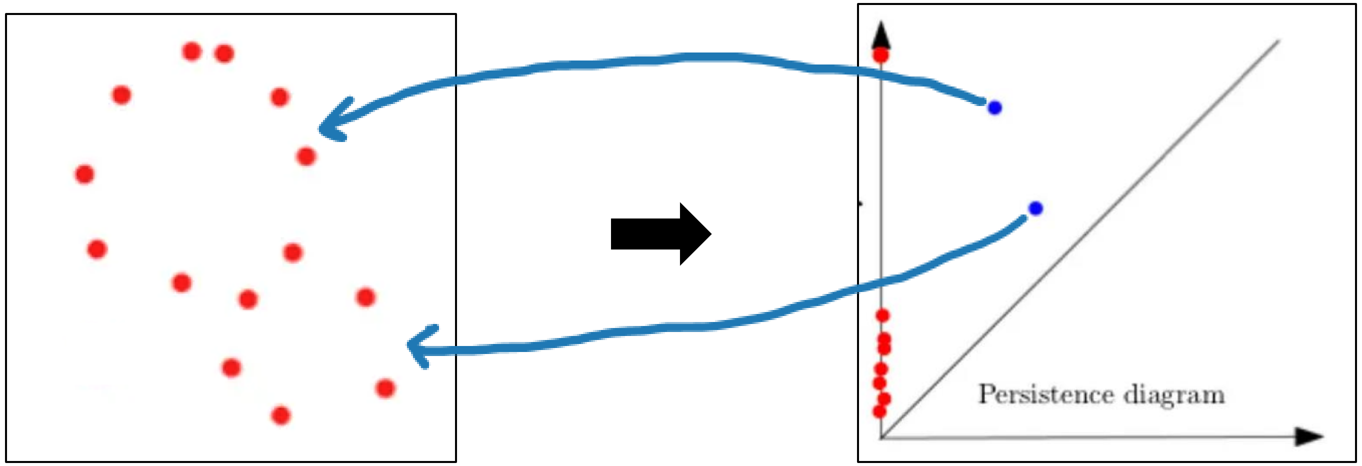

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- Solving problem 2:

- For the above shape, its **persistence diagram** provides a measure of the “size” (i.e., “significance”) of the 1-dimensional holes so that we can differentiate the three more significant ones from the remaining

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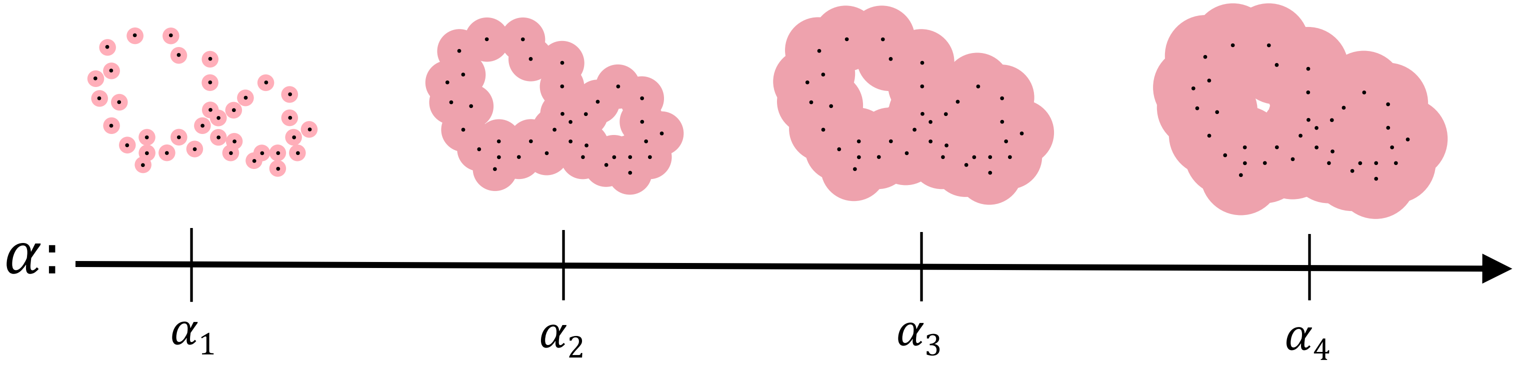
Persistent homology, more formally



- The input to persistent homology is a **growing topological space**

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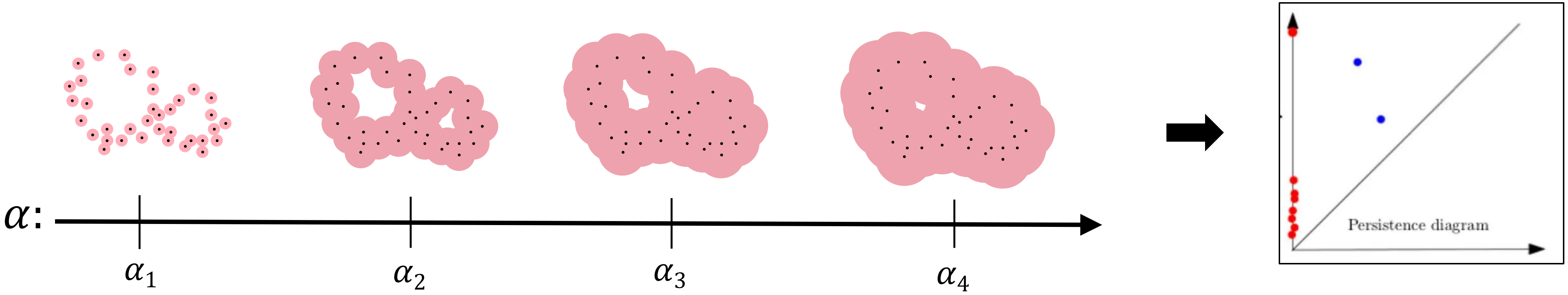

- The input to persistent homology is a **growing topological space**
- Given this, it produces a persistence diagram, which is a robust (i.e., stable) “topological signature” that captures the multi-scale topological features (aka. holes) of the data in arbitrary dimensions

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Idea:

- The growing space can be more formally defined as follows:

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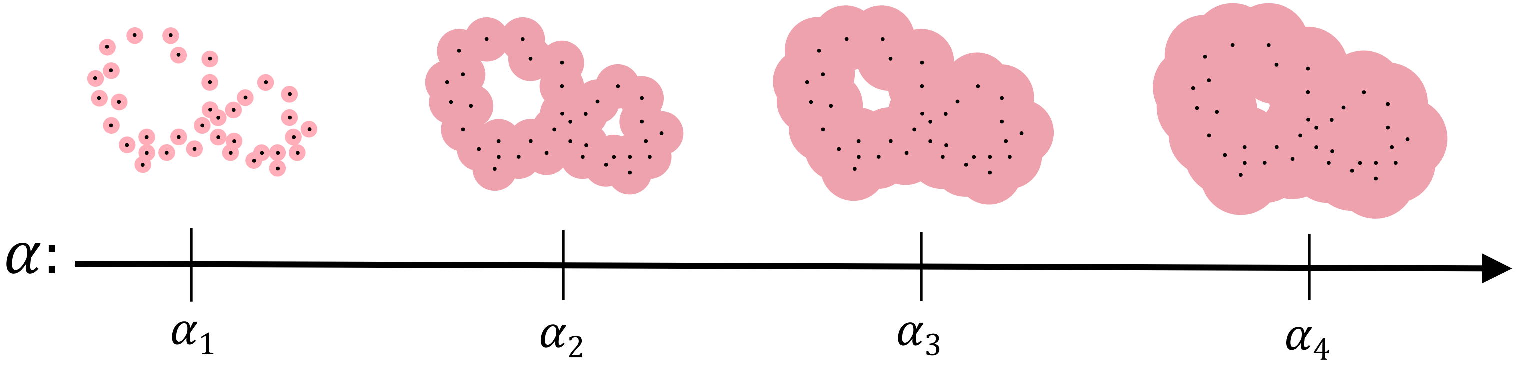

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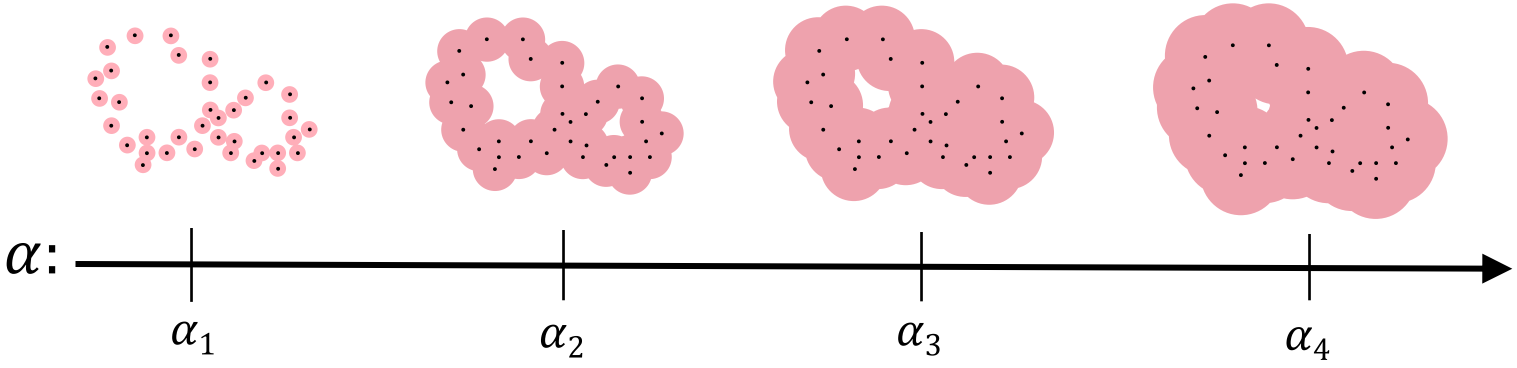

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- Then, as α increase, we track the changes of the homology features of the corresponding spaces

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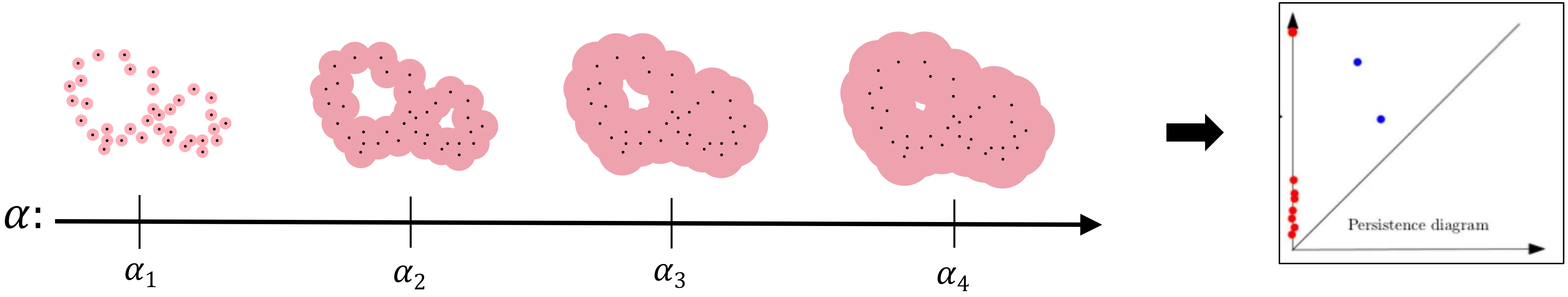

Examples:

- https://gjkoplik.github.io/pers-hom-examples/0d_pers_2d_data_widget.html
- https://gjkoplik.github.io/pers-hom-examples/1d_pers_2d_data_widget.html

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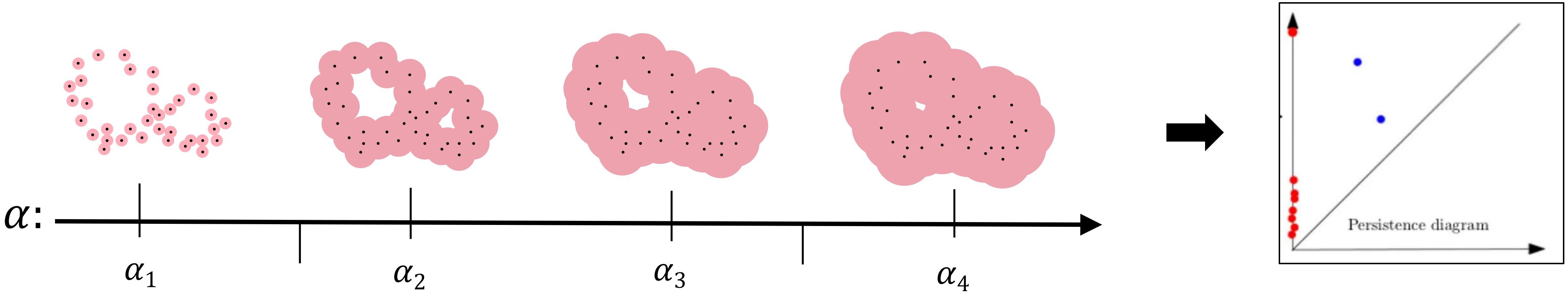

- **Definition:** A *persistence diagram (PD)* is a set of points on the 2D plane above the diagonal such that:

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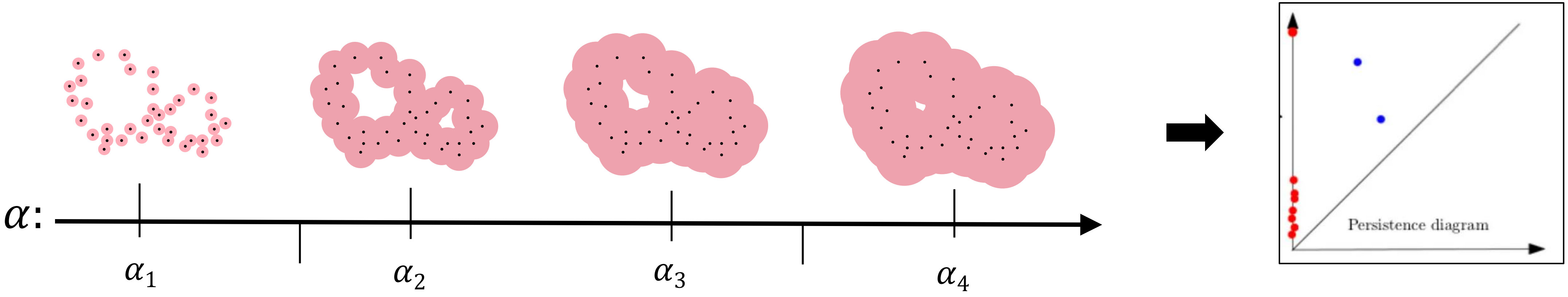

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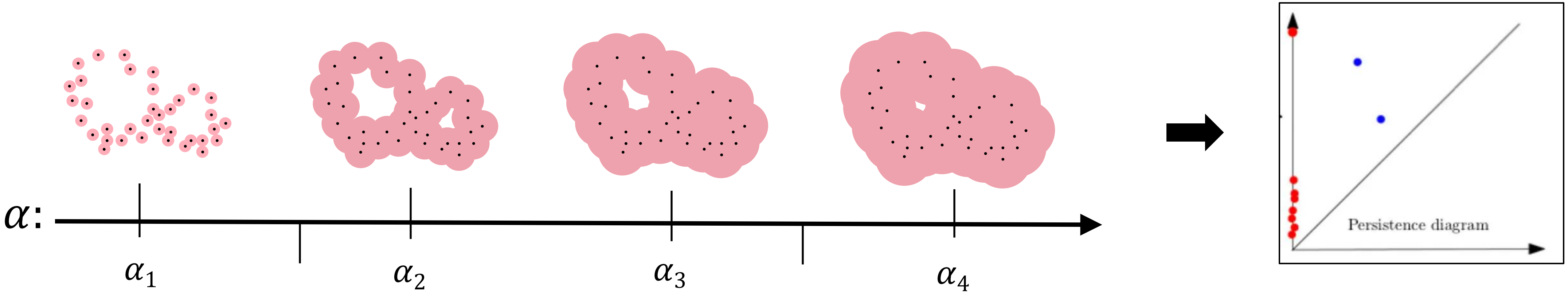

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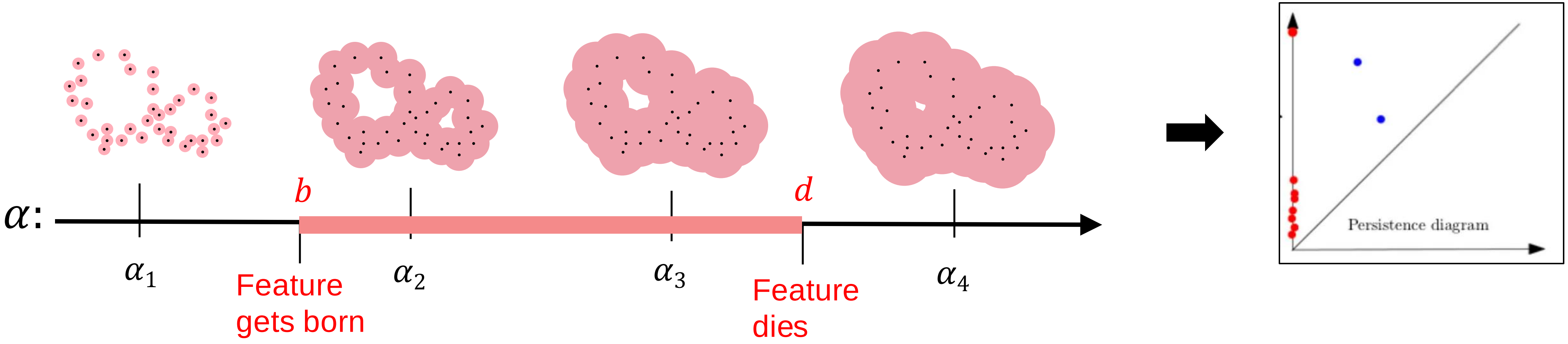

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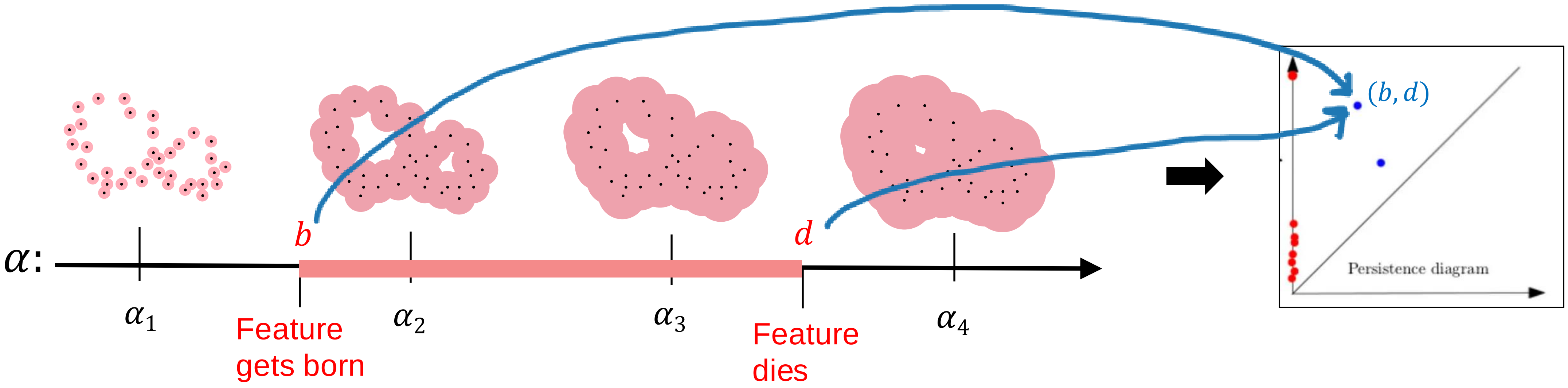

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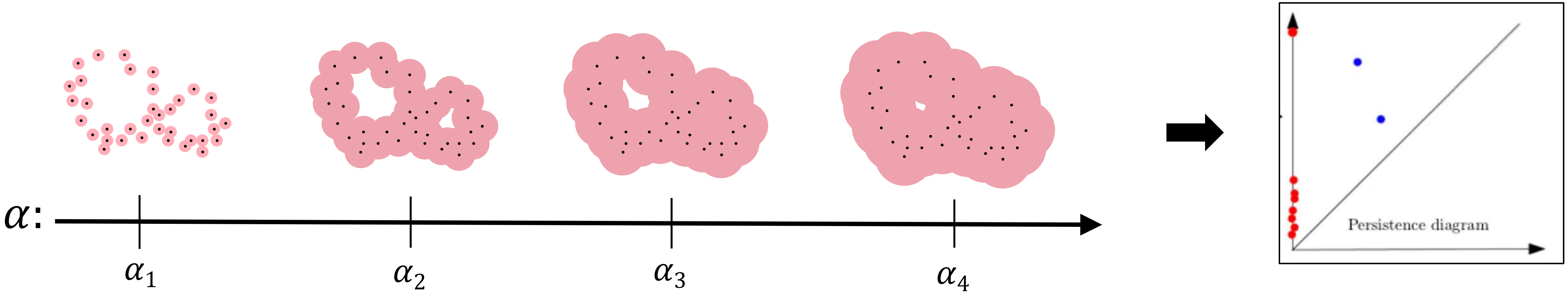

- Notice that homology features / holes are in different dimensions
- The PD where points corresponding to d-dimensional holes is also called the d-dimensional / d-th PD which is typically denoted as PD_d
- And of course, we could also have the PD in all dimensions (this is the PD by default)

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Persistent homology: History

- Persistent homology is proposed roughly around 2000 (or earlier) by several works
- The following is *by no means a comprehensive list of works*:
  - Edelsbrunner, Letscher and Zomorodian, 2002. Topological persistence and simplification.
  - Zomorodian, A. and Carlsson, G., 2004, June. Computing persistent homology.
  - Carlsson, G., 2009. Topology and data.
  - Ghrist, R., 2008. Barcodes: the persistent topology of data.
  - Singh, G., Mémoli, F. and Carlsson, G.E., 2007. Topological methods for the analysis of high dimensional data sets and 3d object recognition.

Motivation: Homology inference from points clous

- We try to infer the homology for the following point cloud data

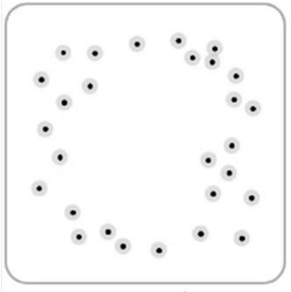

Image source: Bobrowski O, Skraba P. A universal null-distribution for topological data analysis.

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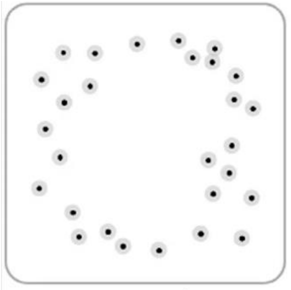

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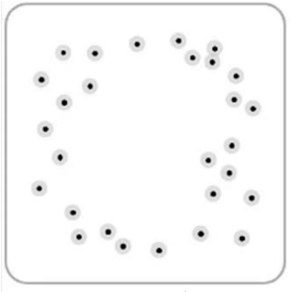

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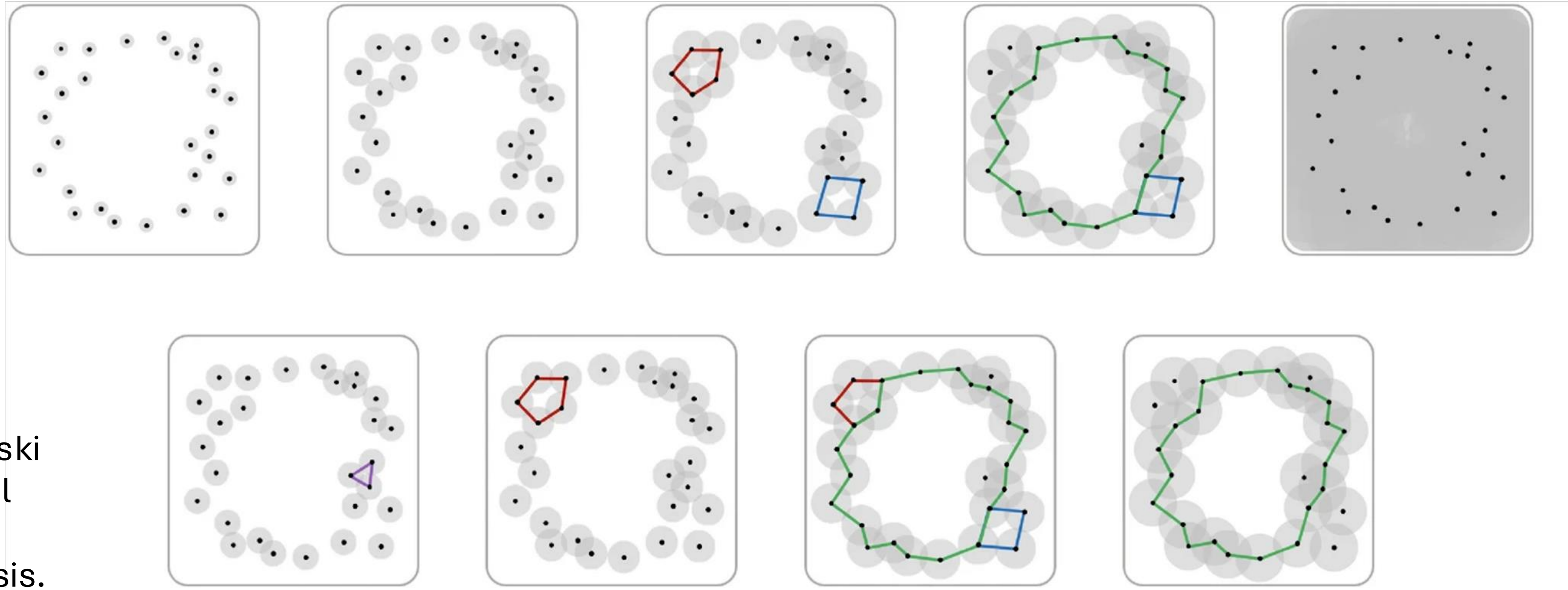
- Notice that there are different choices of the size



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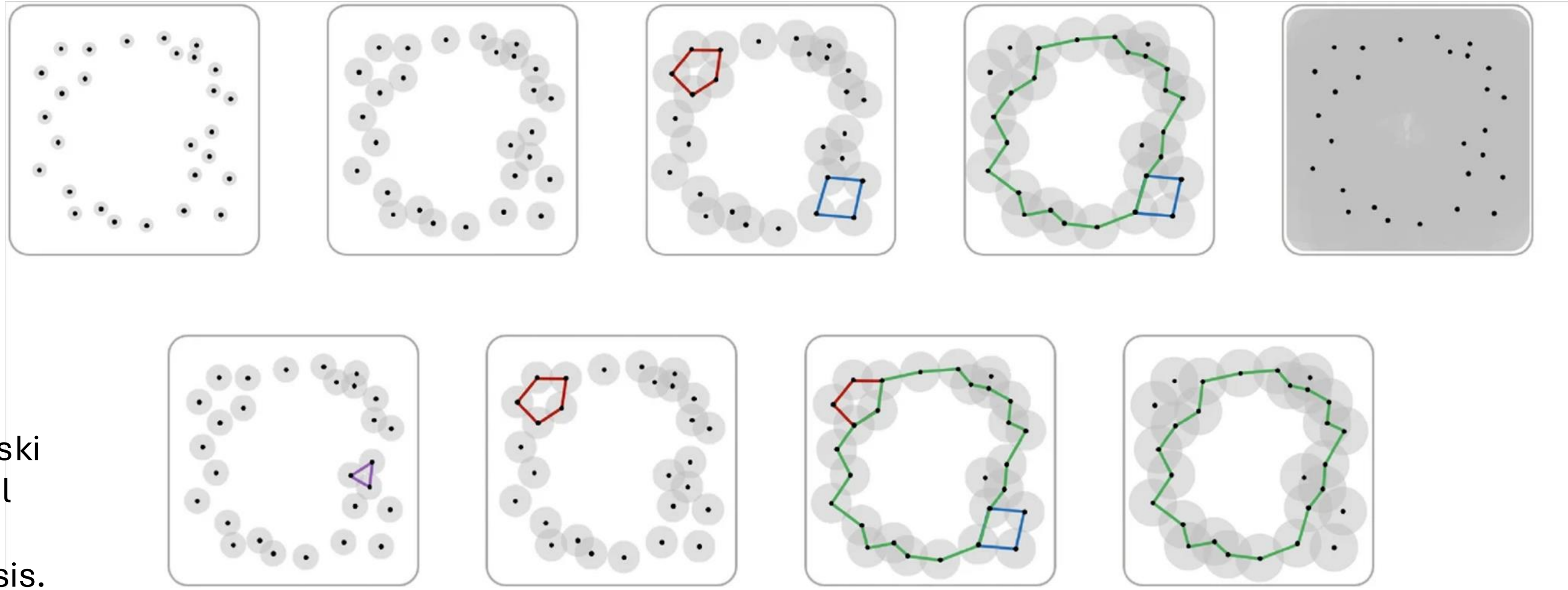

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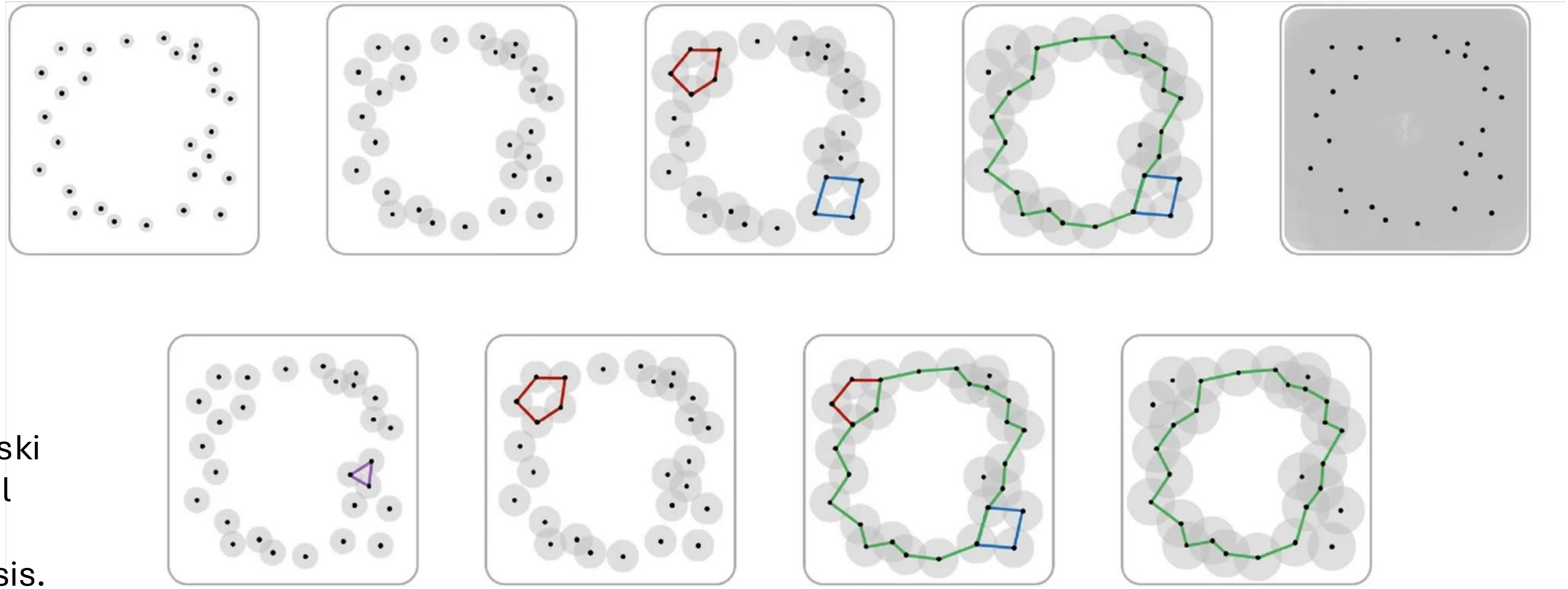
- For each different radius, the homology can be **vastly different**, with different cycles in the homology basis corresponding to the different radii
  - We focus on the 1-cycles (1-dimensional holes) in the example
  - For each radius, the colored cycles form the homology basis



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Motivation: Homology inference from points clous

- **Question**: What is a correct radius to infer the shape of the point cloud?

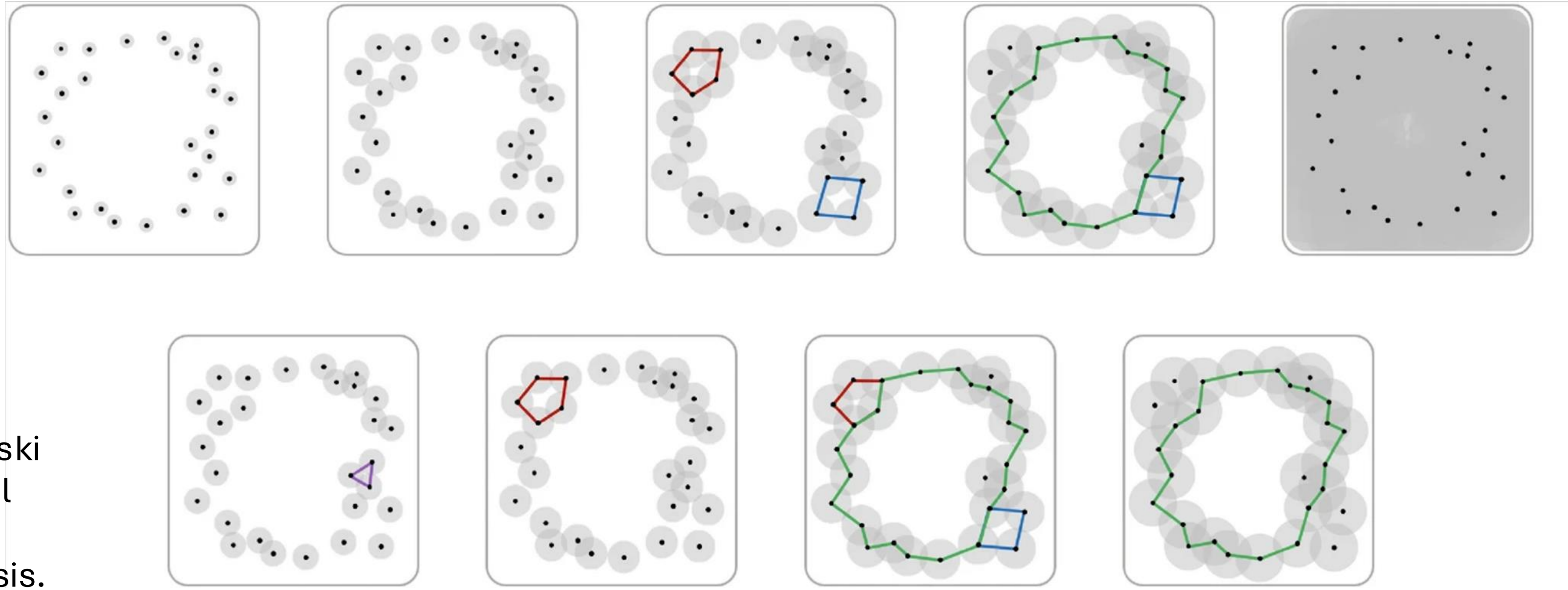
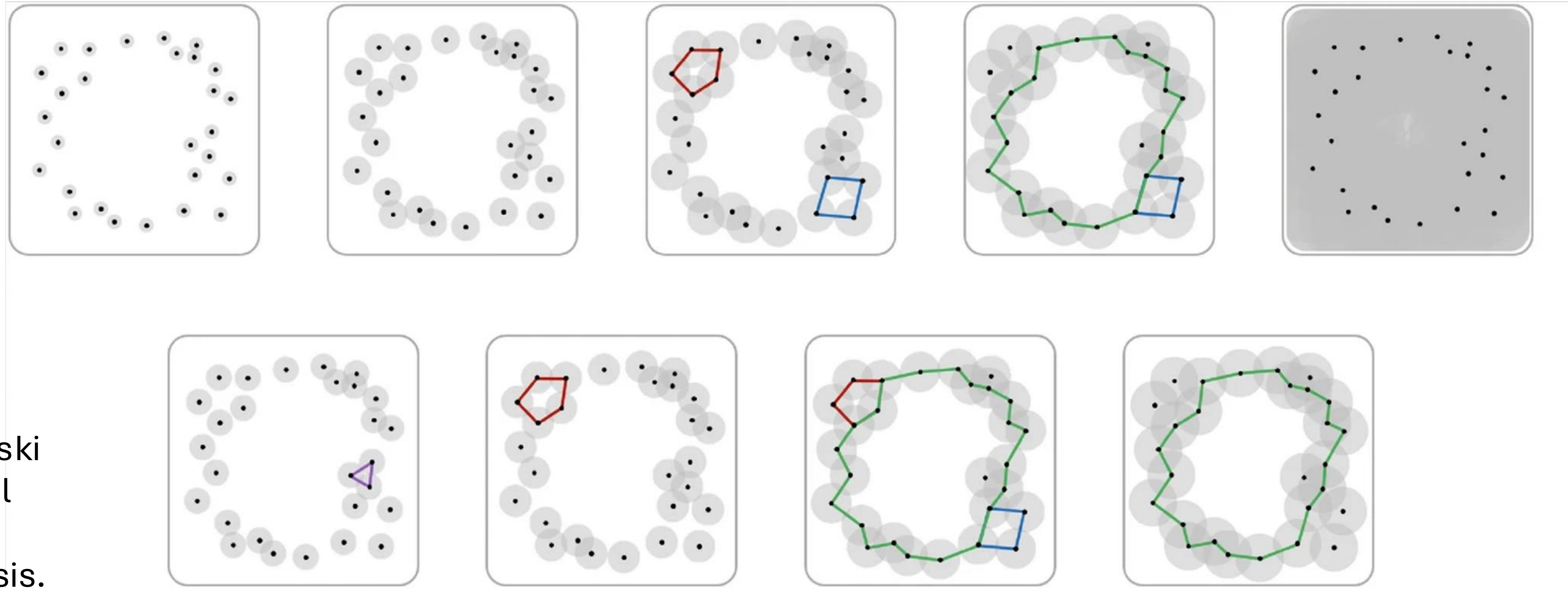


Image source: Bobrowski O, Skraba P. A universal null-distribution for topological data analysis.

Motivation: Homology inference from points clous

- **Question**: What is a correct radius to infer the shape of the point cloud?
- **Answer**: It's really hard to know, and there probably is no such "correct" radius



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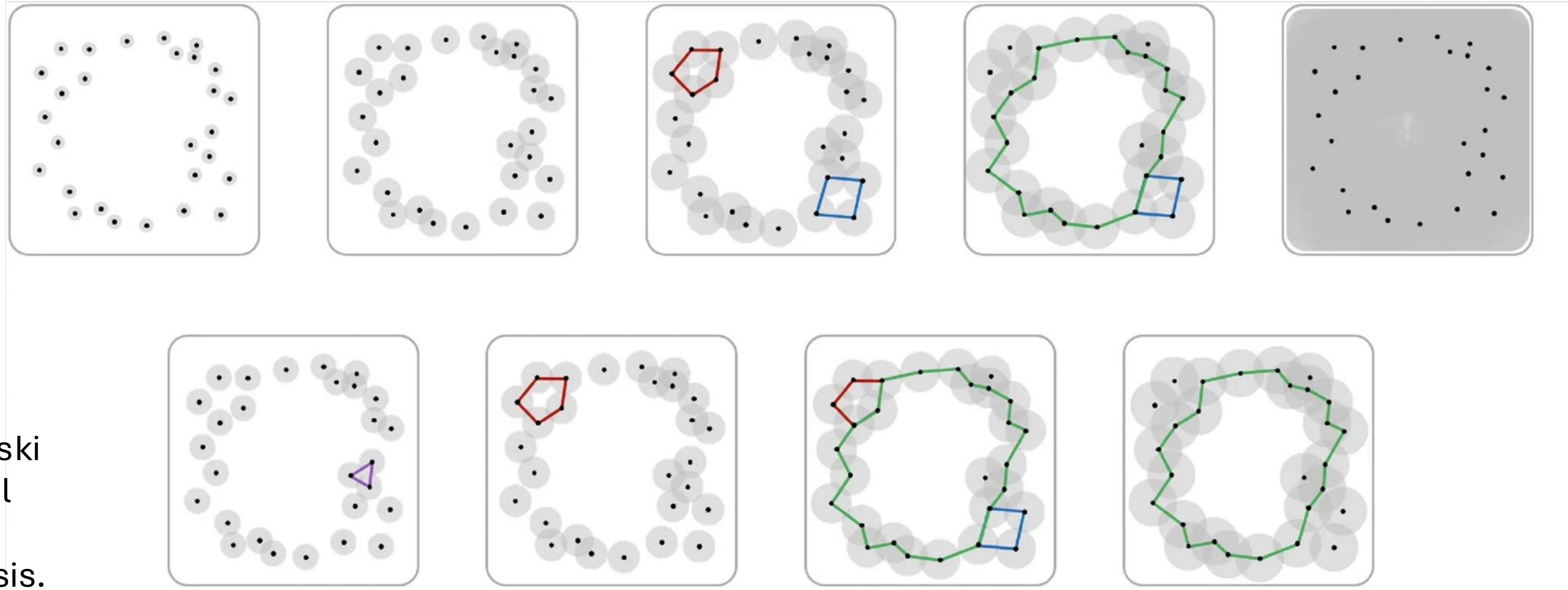
Motivation: Homology inference from points clous

- **Solution**: Consider all radius, and track the changes of the 1-cycles in the homology basis as we increase the radius



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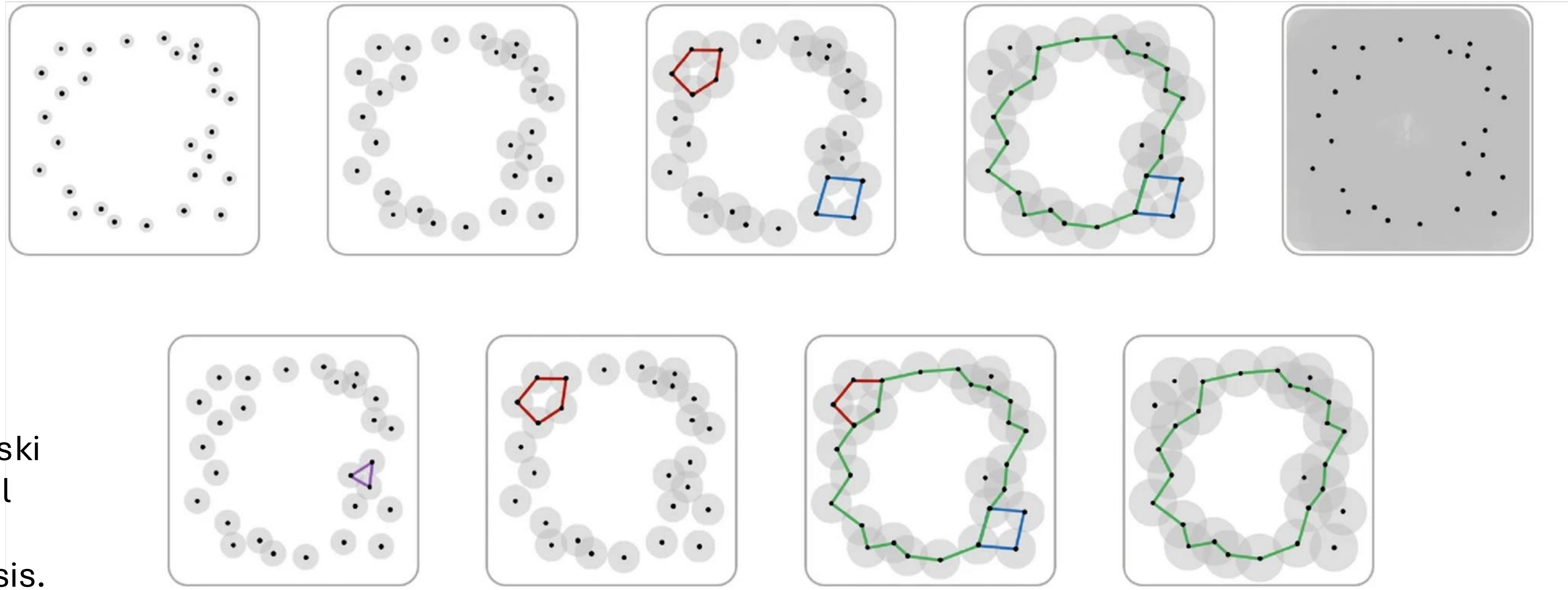
- As the radius increases, different cycles in the basis could appear (getting **born**) or becomes trivial (**dies**).



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Motivation: Homology inference from points clous

- **Solution**: Consider all radius, and track the changes of the 1-cycles in the homology basis as we increase the radius
- As the radius increases, different cycles in the basis could appear (getting **born**) or becomes trivial (**dies**).
- We pair the **births** and **deaths**, which are the points in the PD

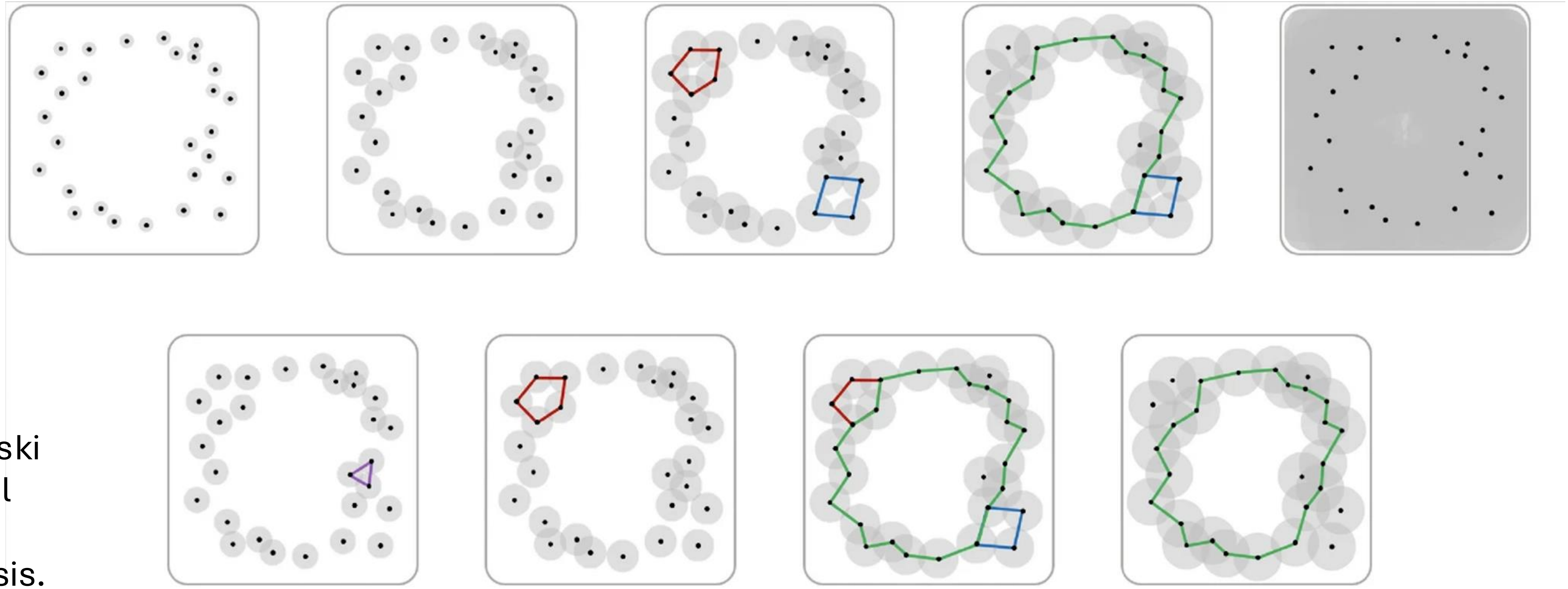
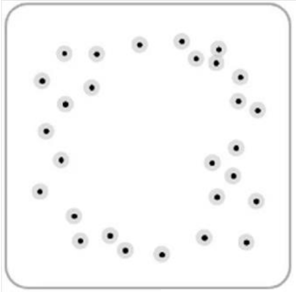


Image source: Bobrowski O, Skraba P. A universal null-distribution for topological data analysis.

- α_0: nothing happens.

α_0



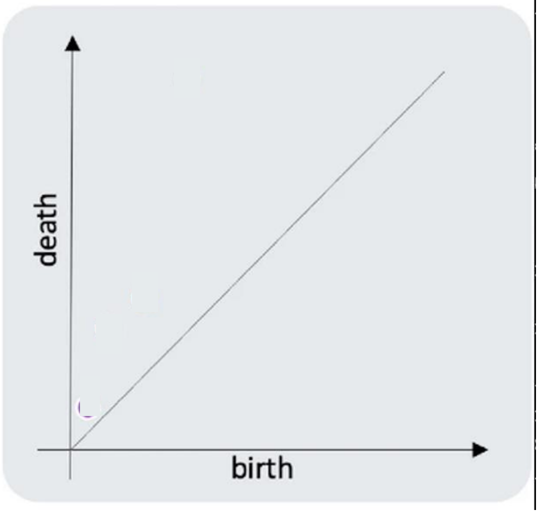
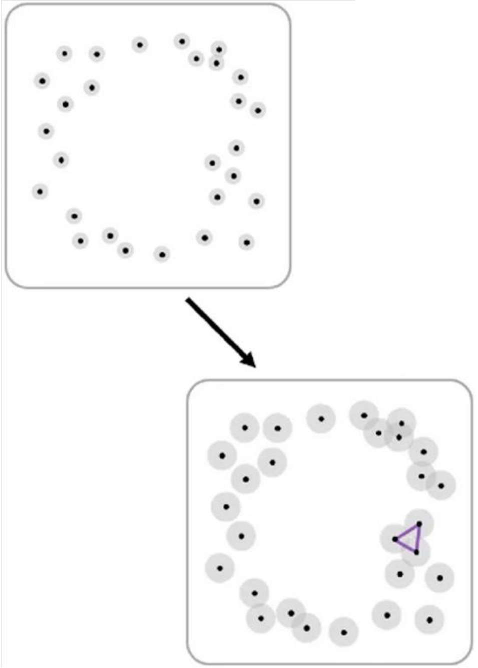


Image: Bobrowski, Skraba. A universal null-distribution for topological data analysis

- α_0: nothing happens.
- α_1: purple cycle born

α_0



α_1

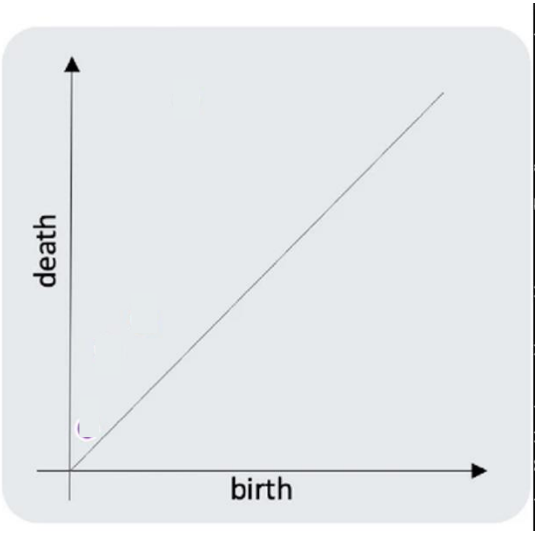
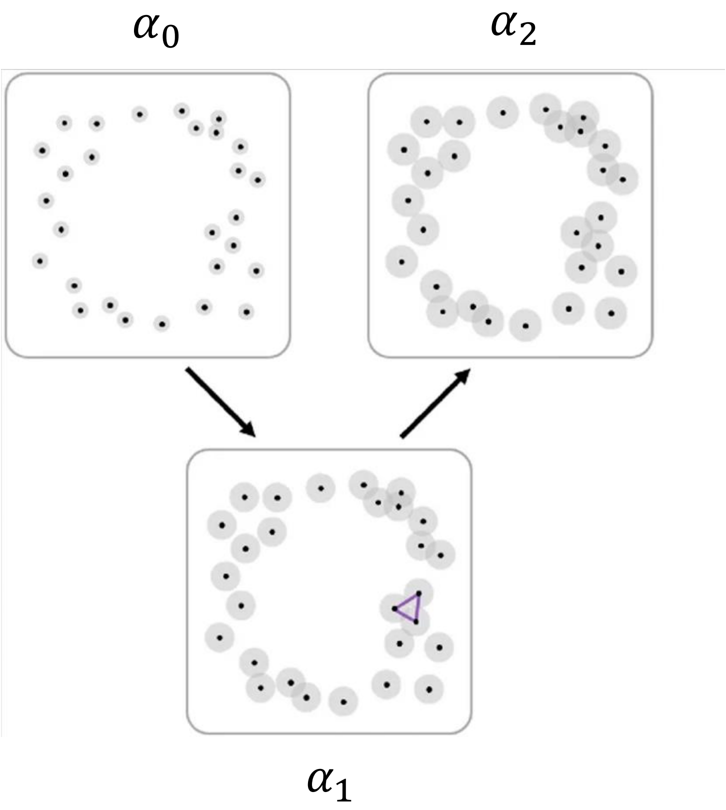
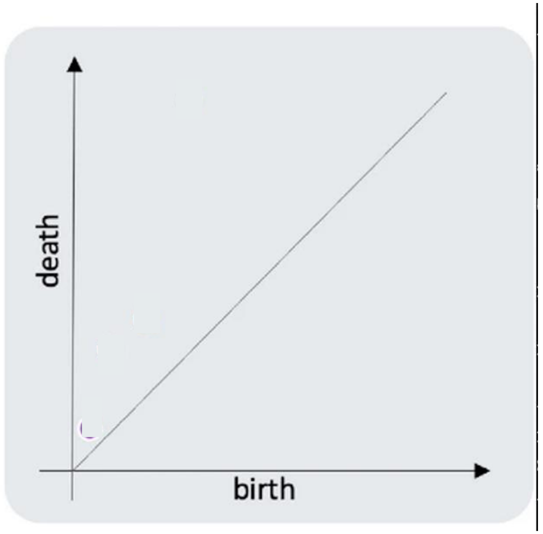


Image: Bobrowski, Skraba. A universal null-distribution for topological data analysis

- α_0: nothing happens.
- α_1: purple cycle born
- α_2: purple cycle dies





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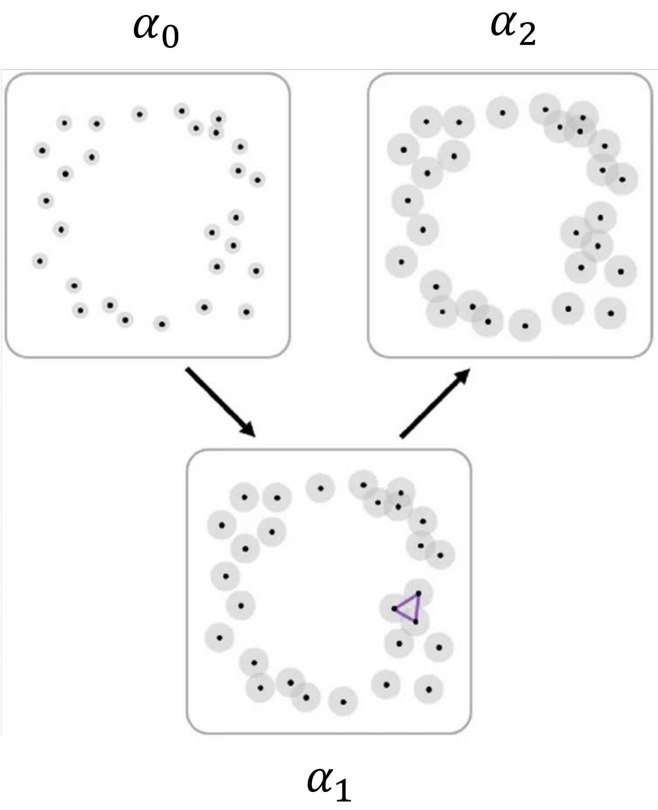
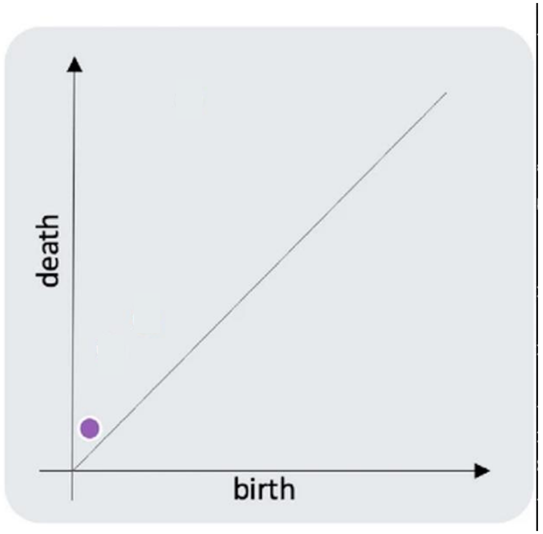
  $\Rightarrow (\alpha_1, \alpha_2)$





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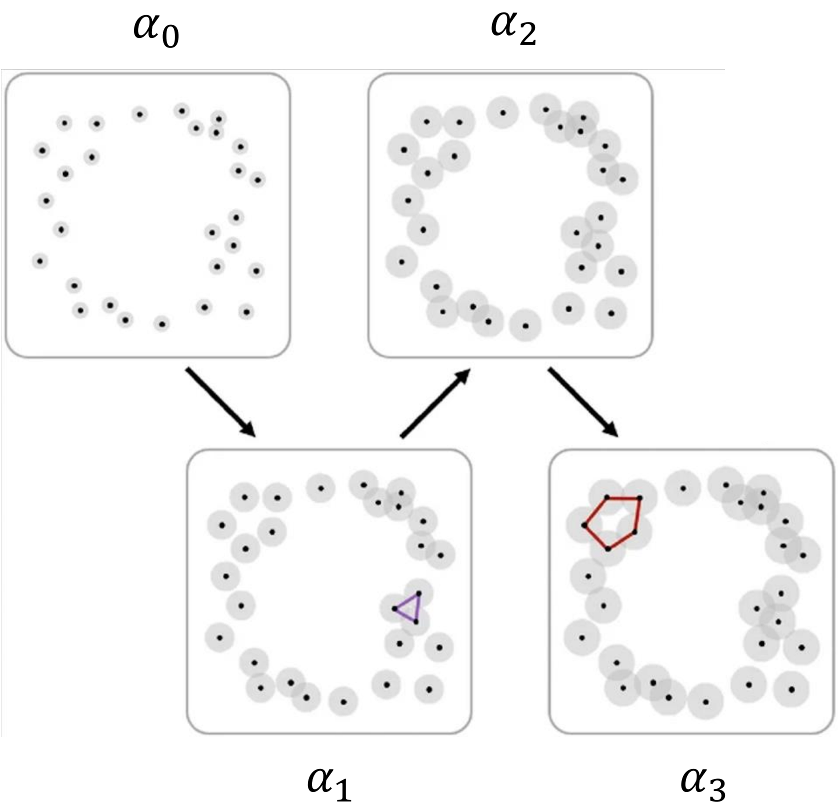
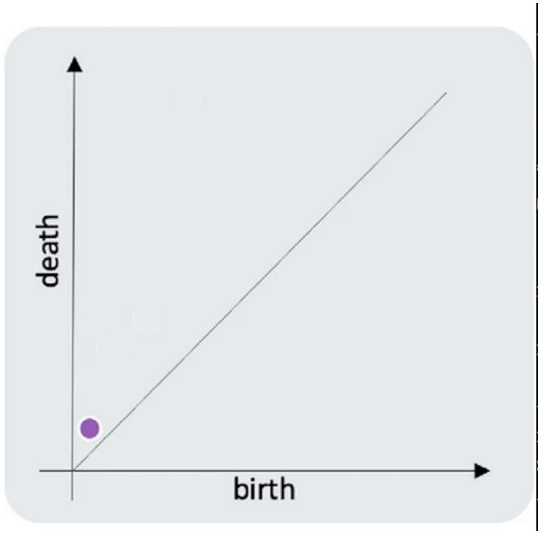
  $\Rightarrow (\alpha_1, \alpha_2)$

- α_3: red cycle born





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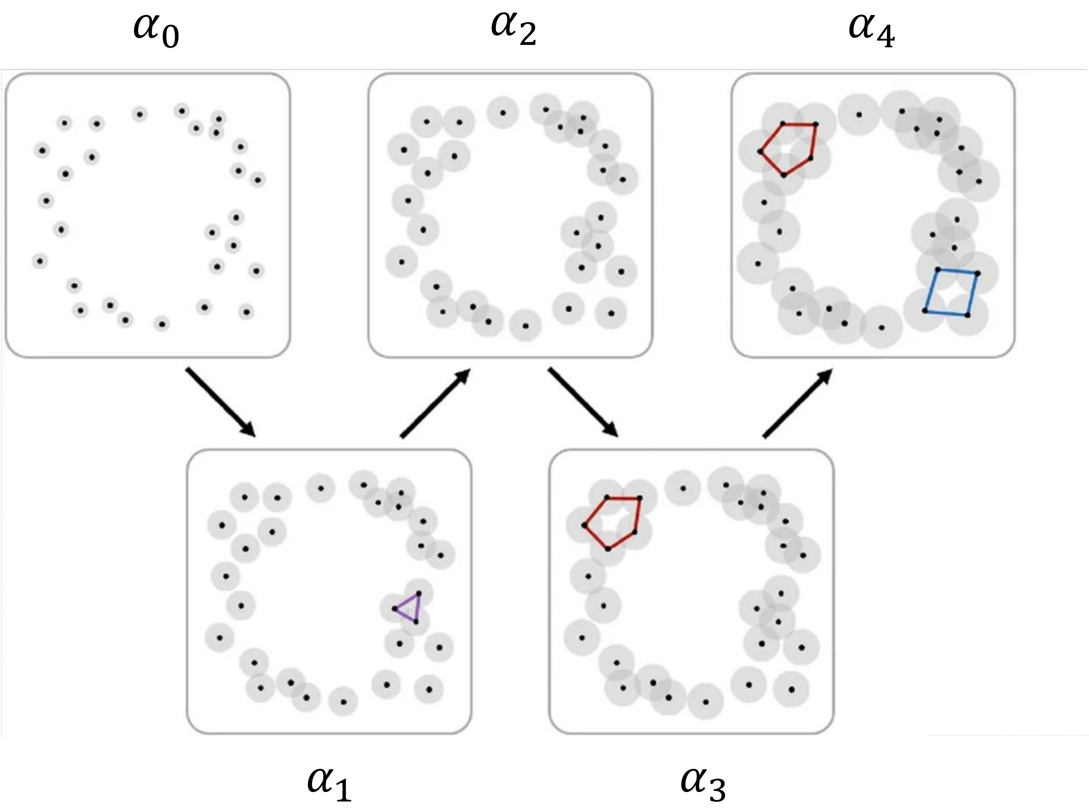
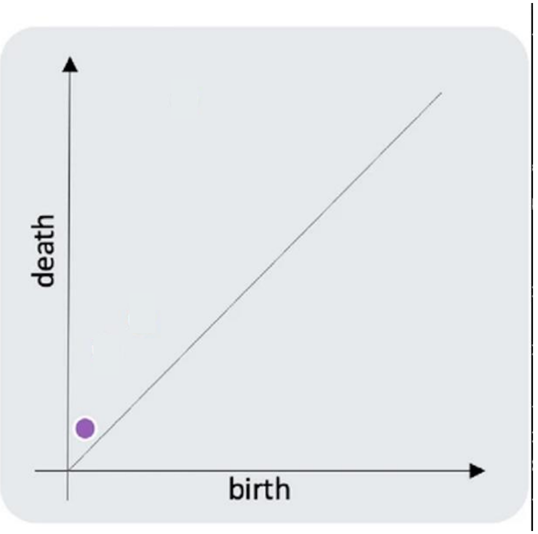
- α_4: blue cycle born





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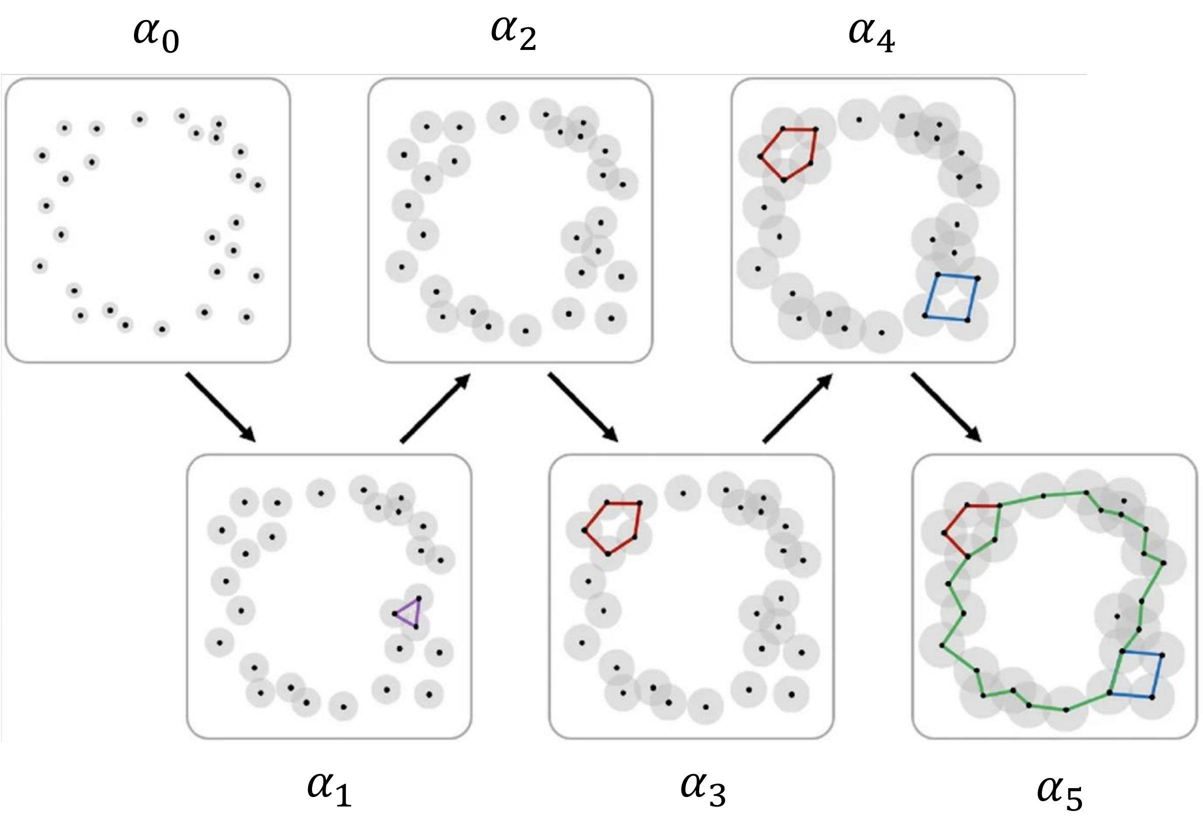
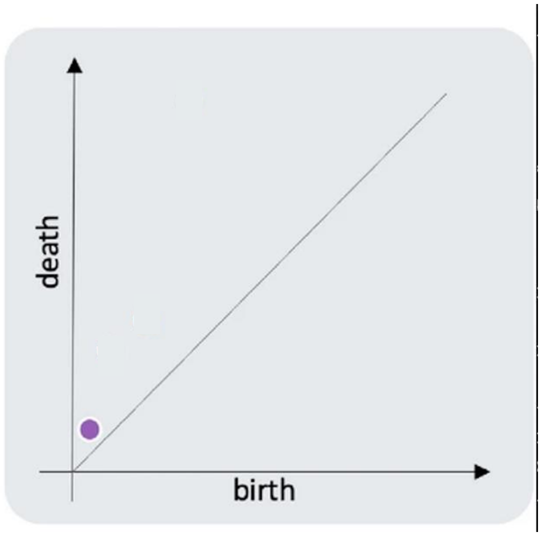
- α_4: blue cycle born
- α_5: green cycle born





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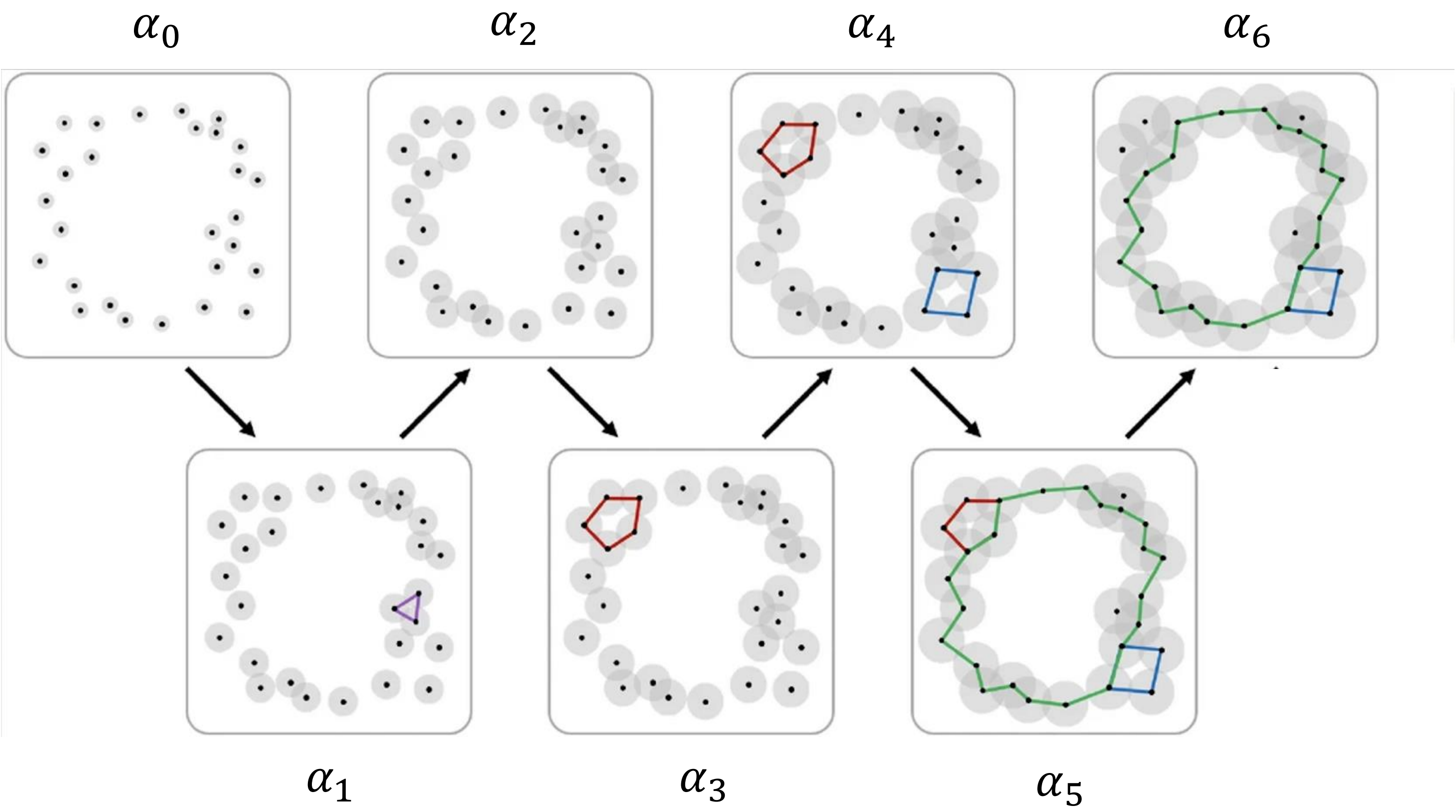
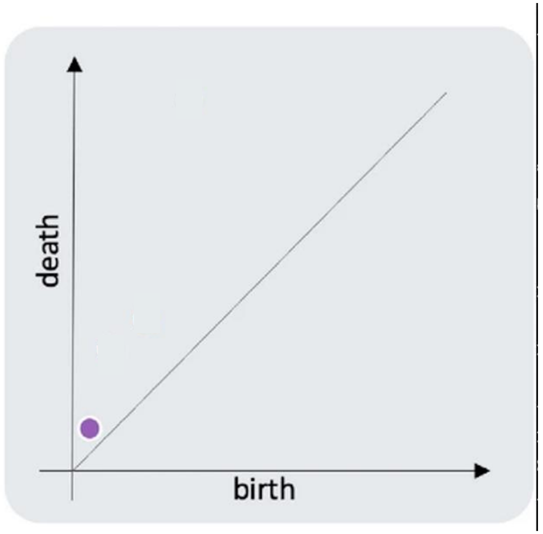
- α_3: red cycle born
- α_4: blue cycle born
- α_5: green cycle born
- α_6: red cycle dies





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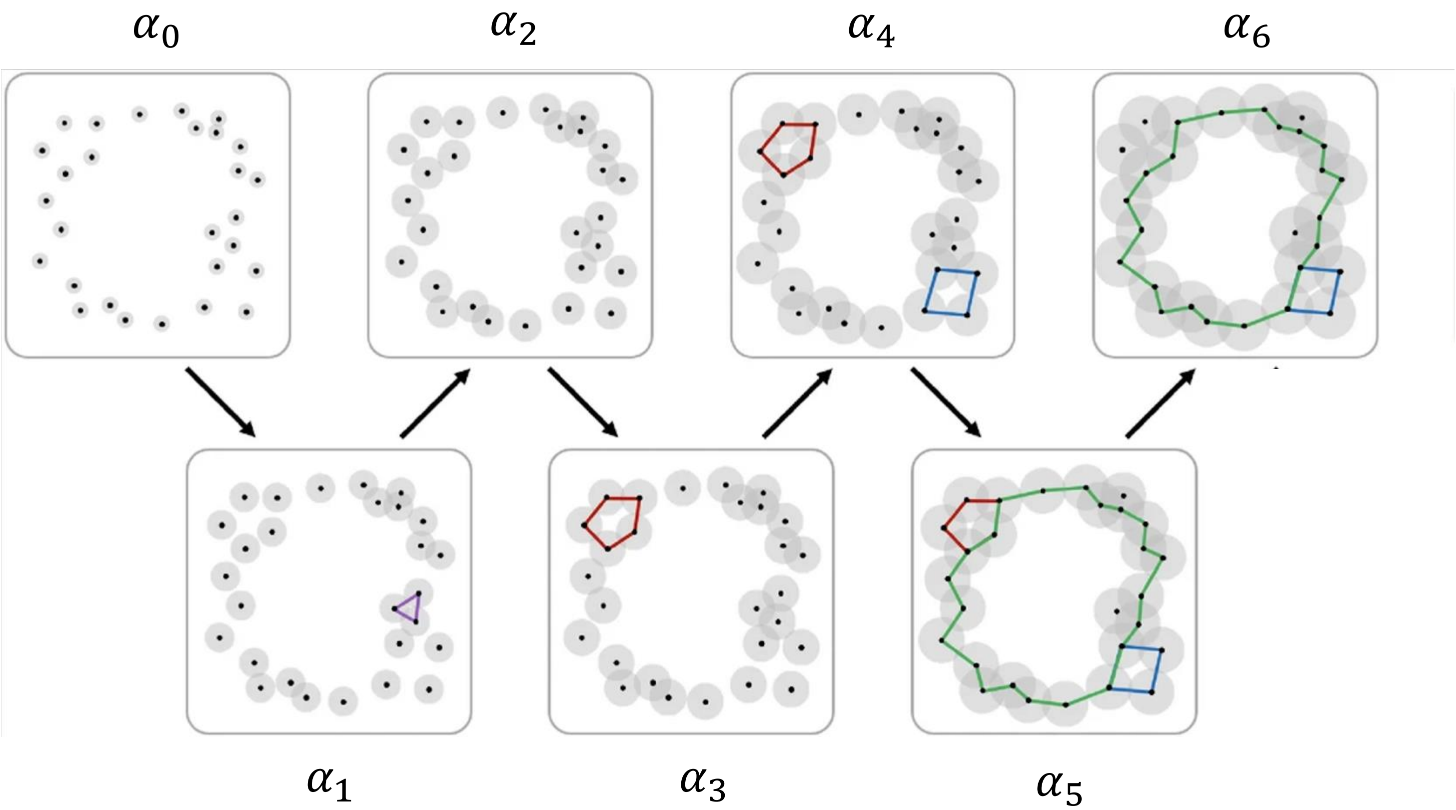

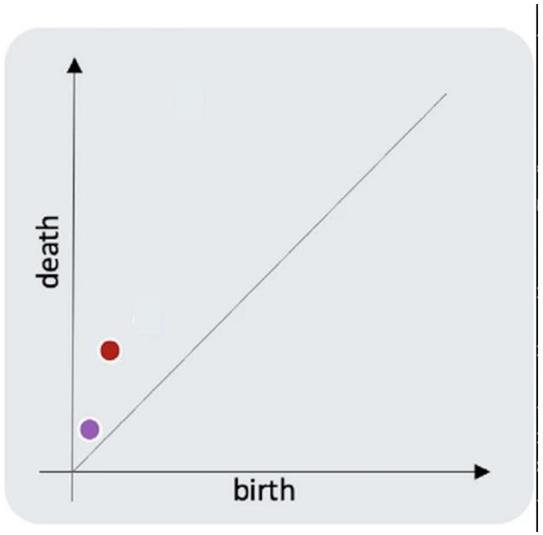

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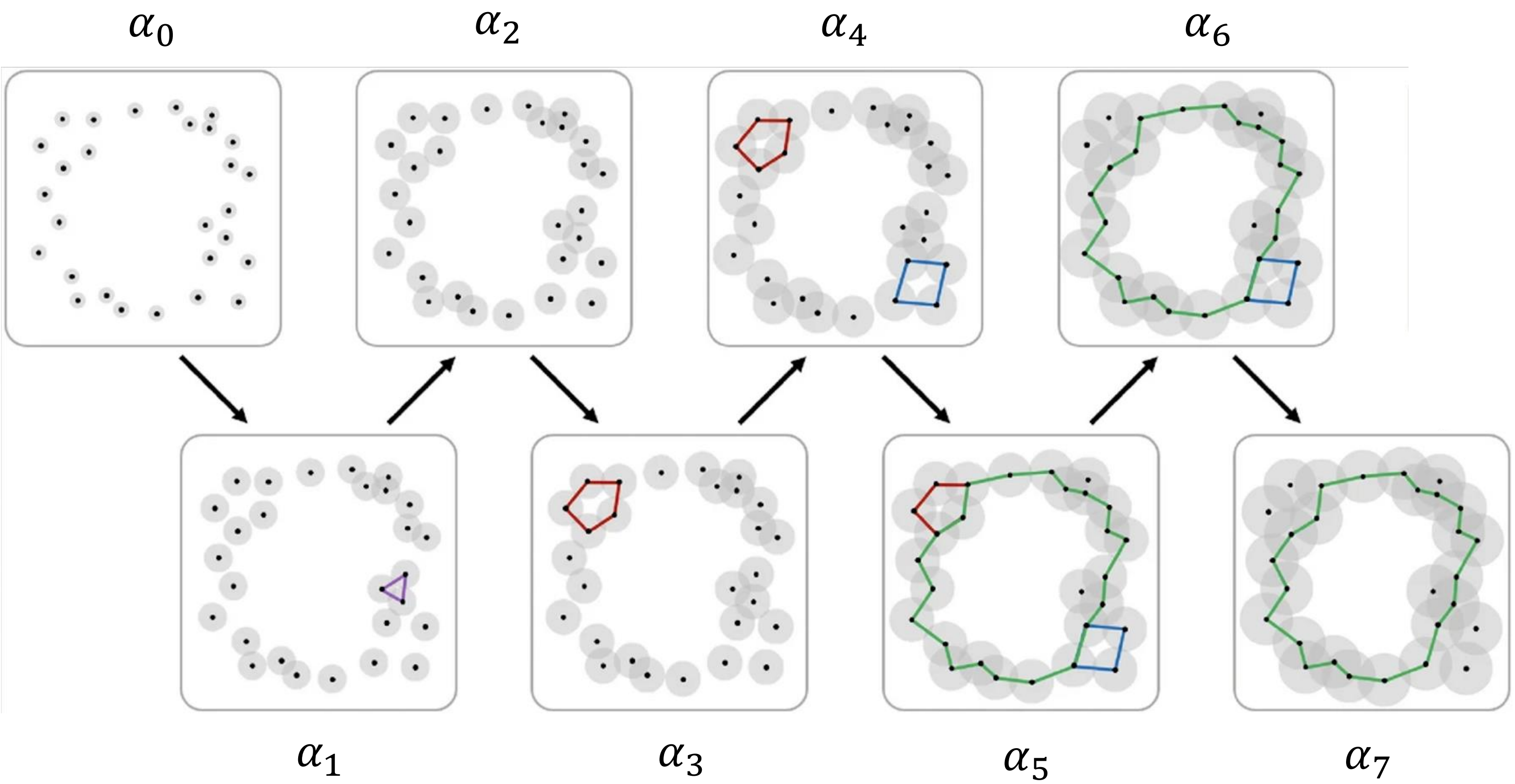
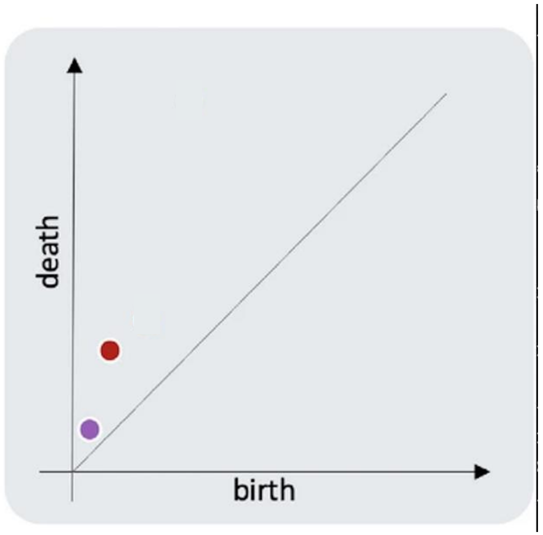
- α_6: red cycle dies $\Rightarrow (\alpha_3, \alpha_6)$
- α_7: blue cycle dies





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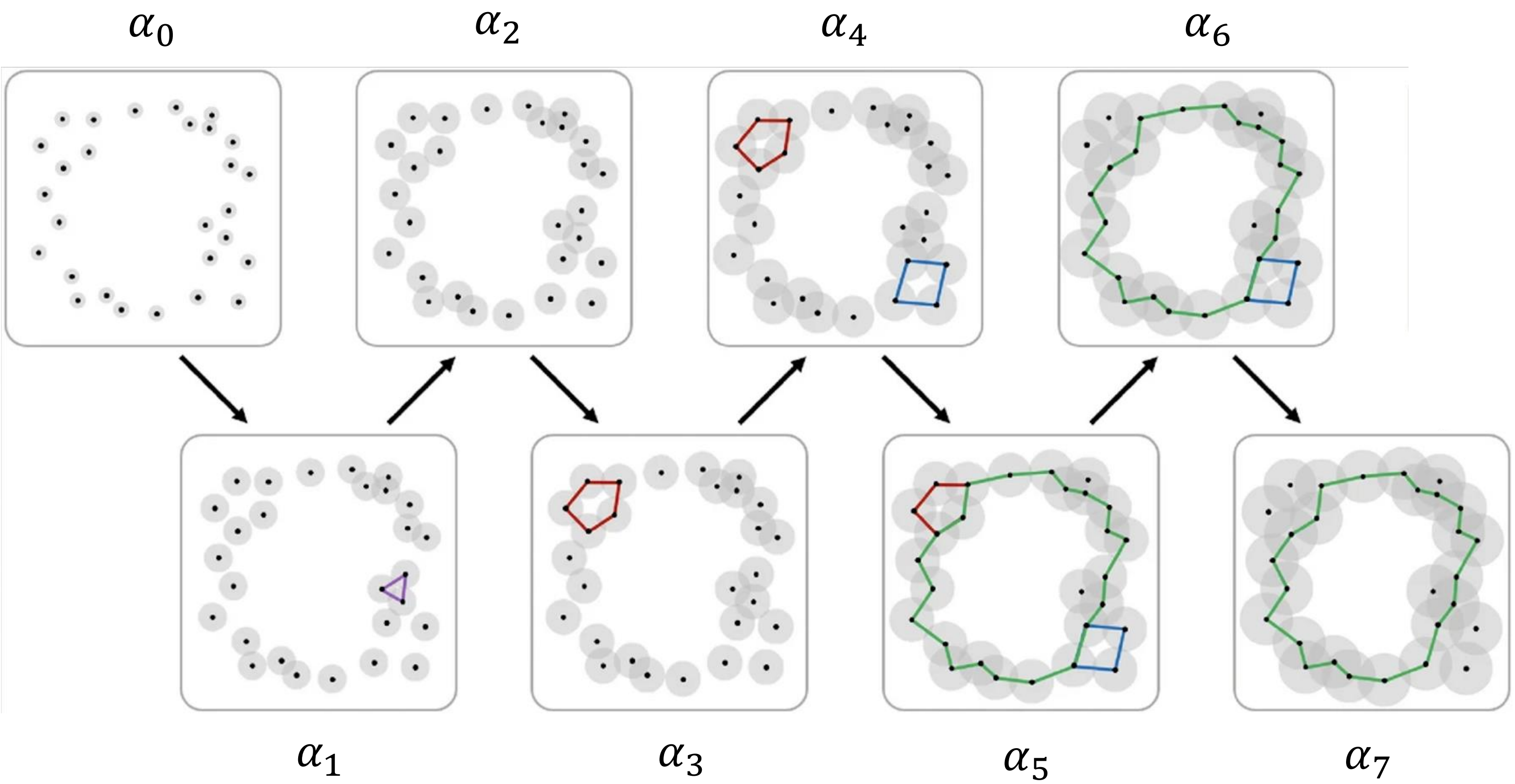
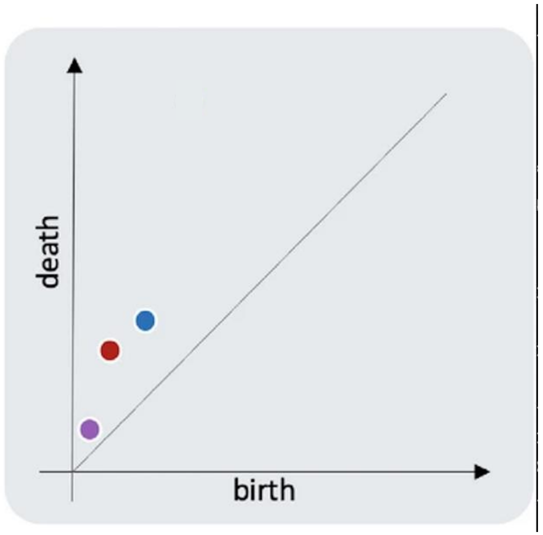
- α_7: blue cycle dies $\Rightarrow (\alpha_4, \alpha_7)$





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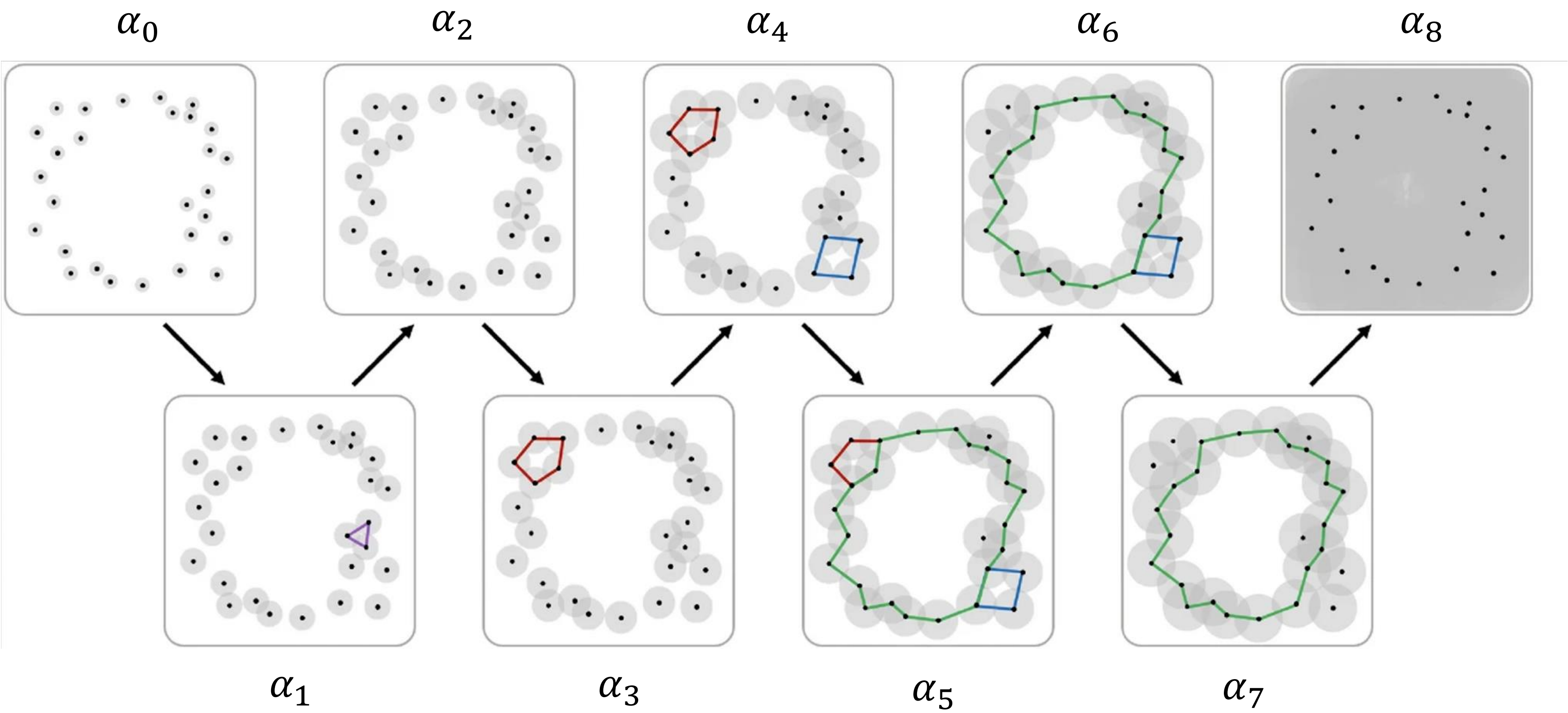
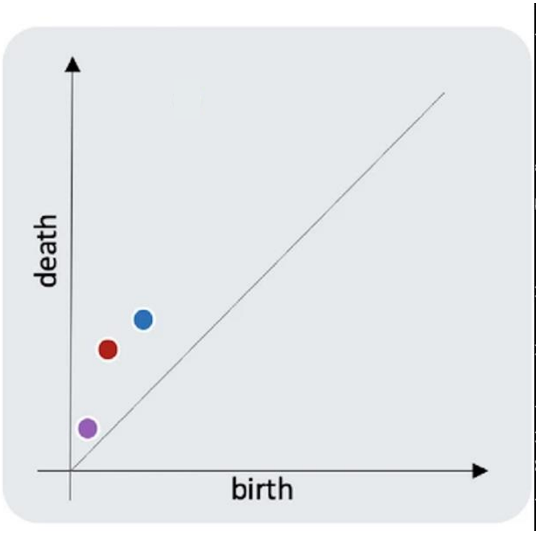
- α_7: blue cycle dies $\Rightarrow (\alpha_4, \alpha_7)$
- α_8: green cycle dies





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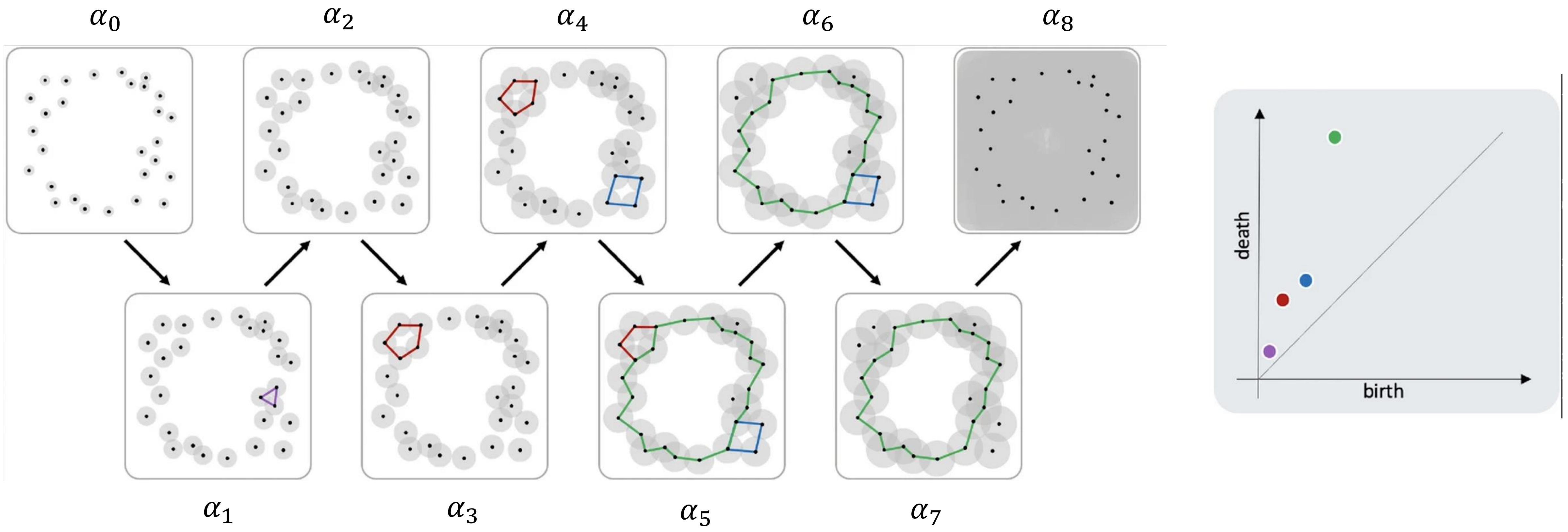

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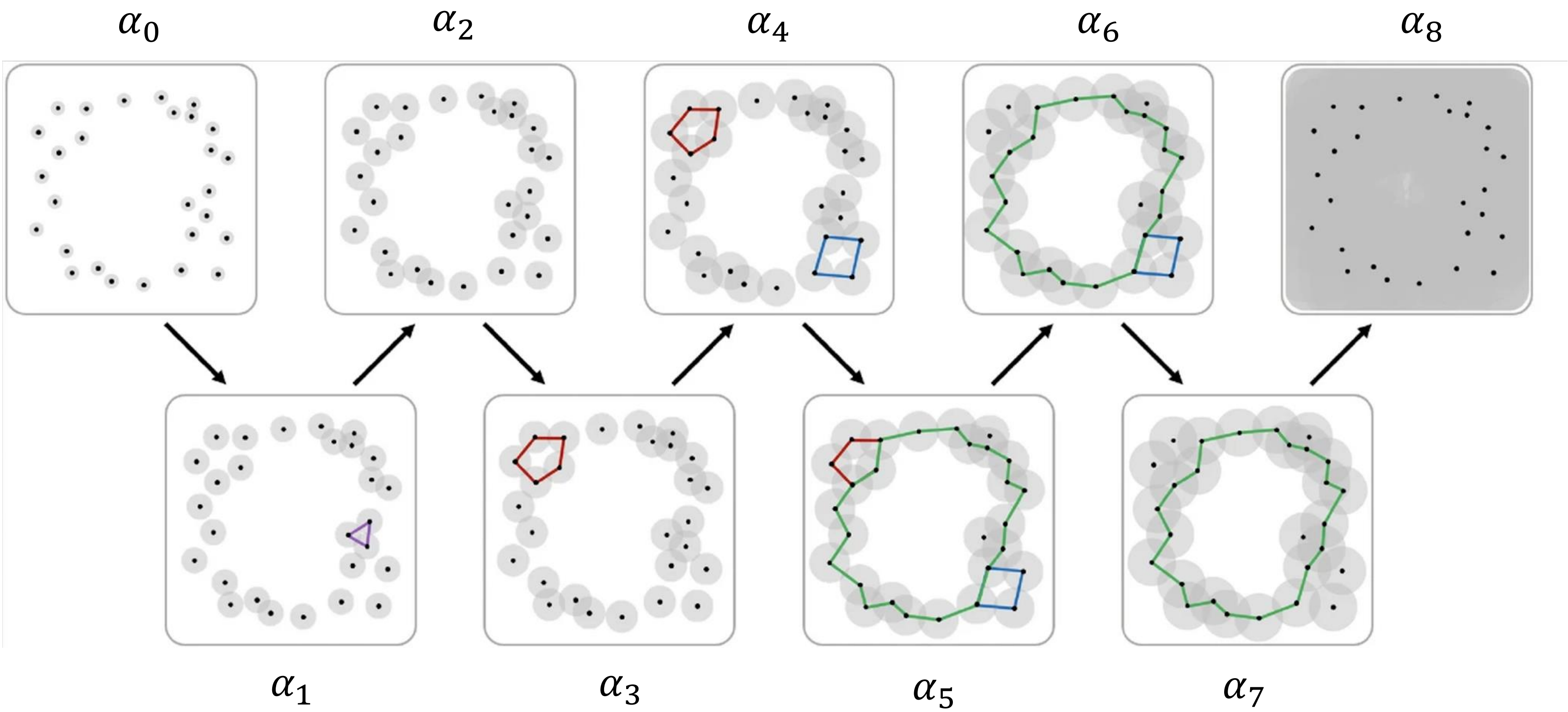
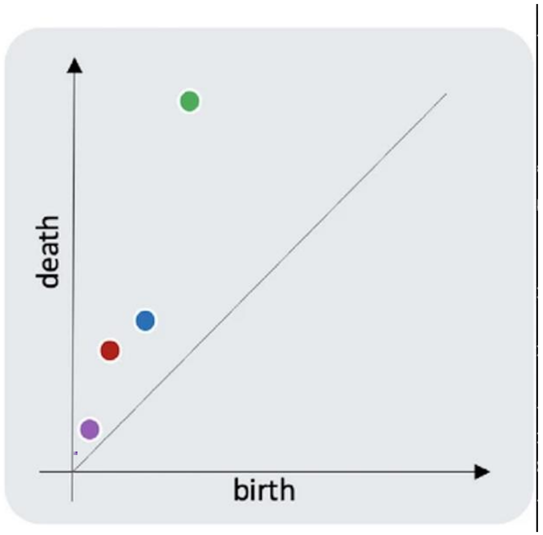
- So we have a 1-dimensional PD on the left with the four points corresponding to the different cycles born and died in the growing spaces with different α value, matching the colors





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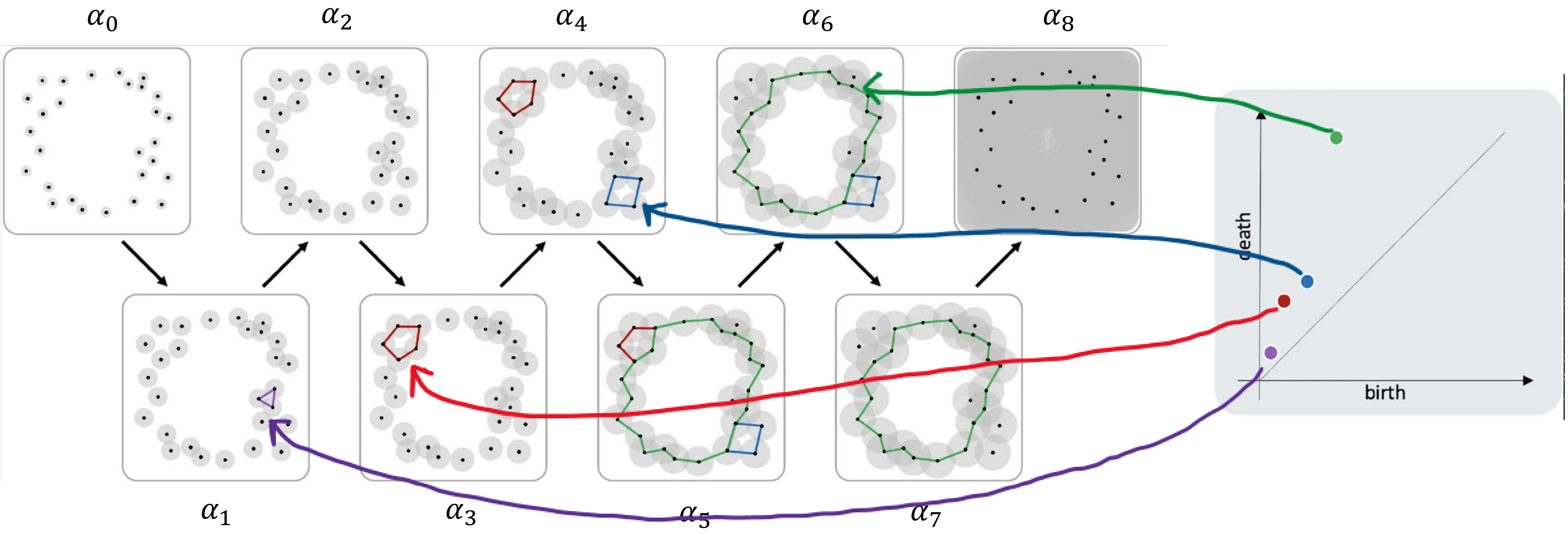

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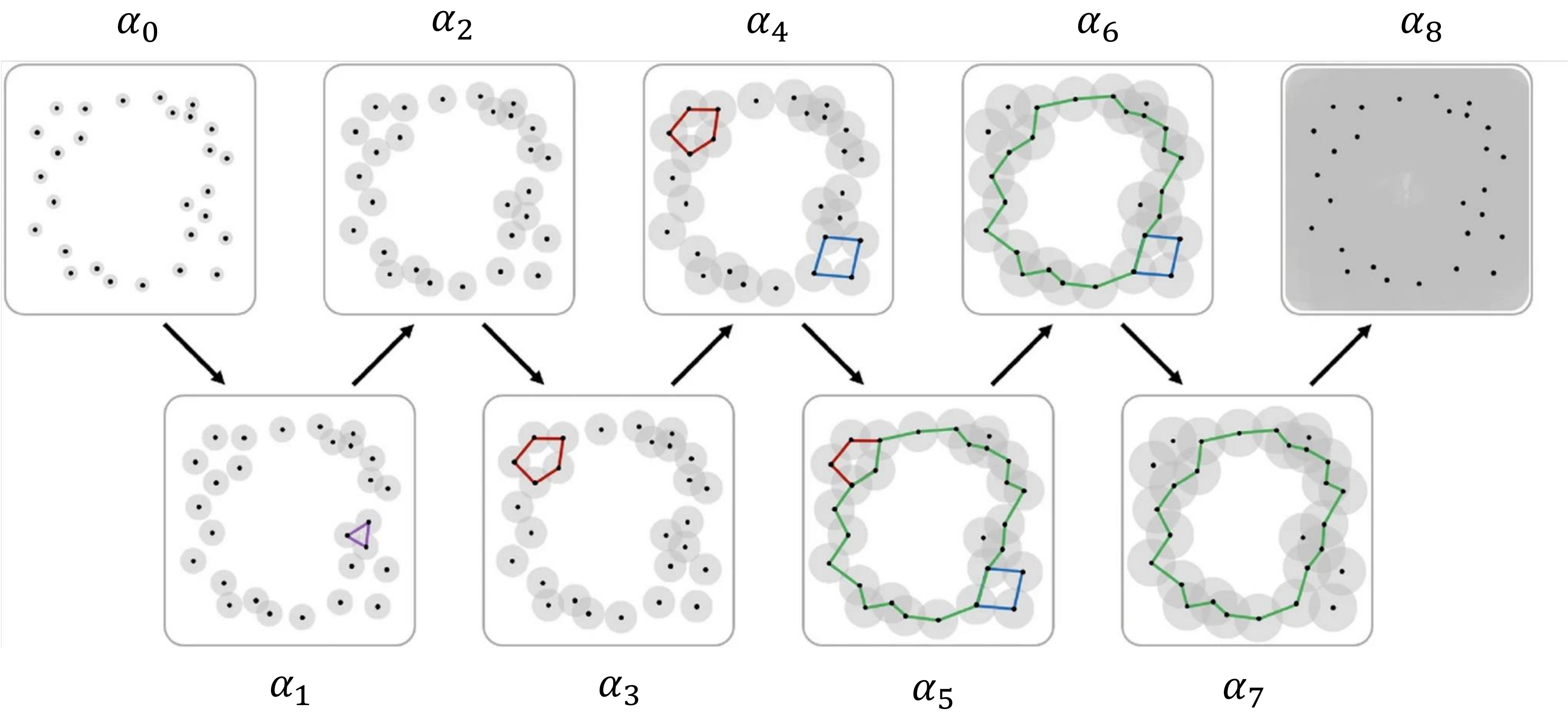
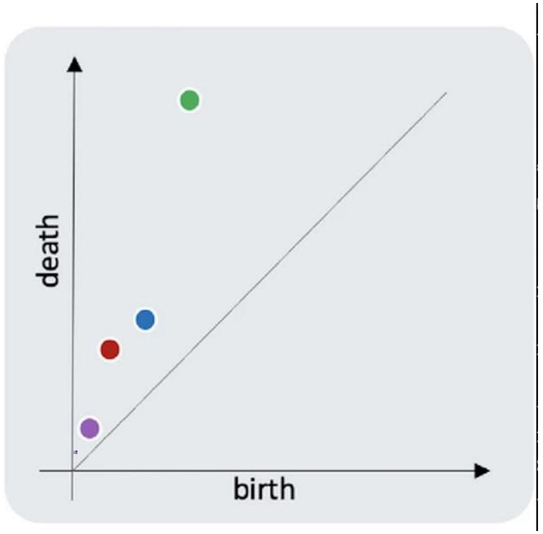
- Furthermore, we have that distances of the points to diagonal indicate the difference of birth and death (how long a cycle persist), which in turn indicate the significance of the feature





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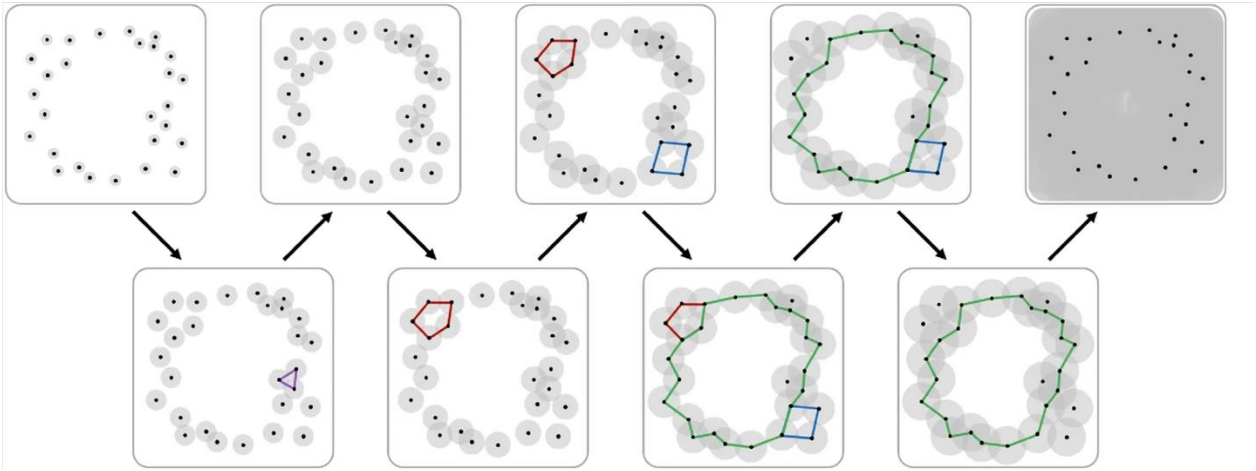
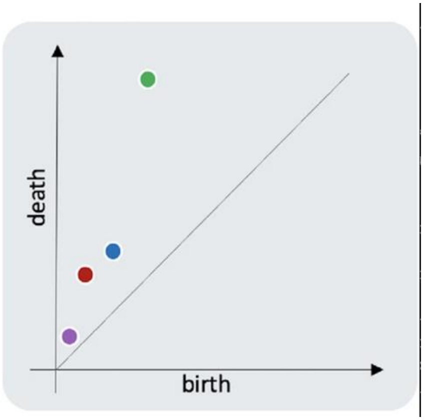
Persistent homology: Brief Summary



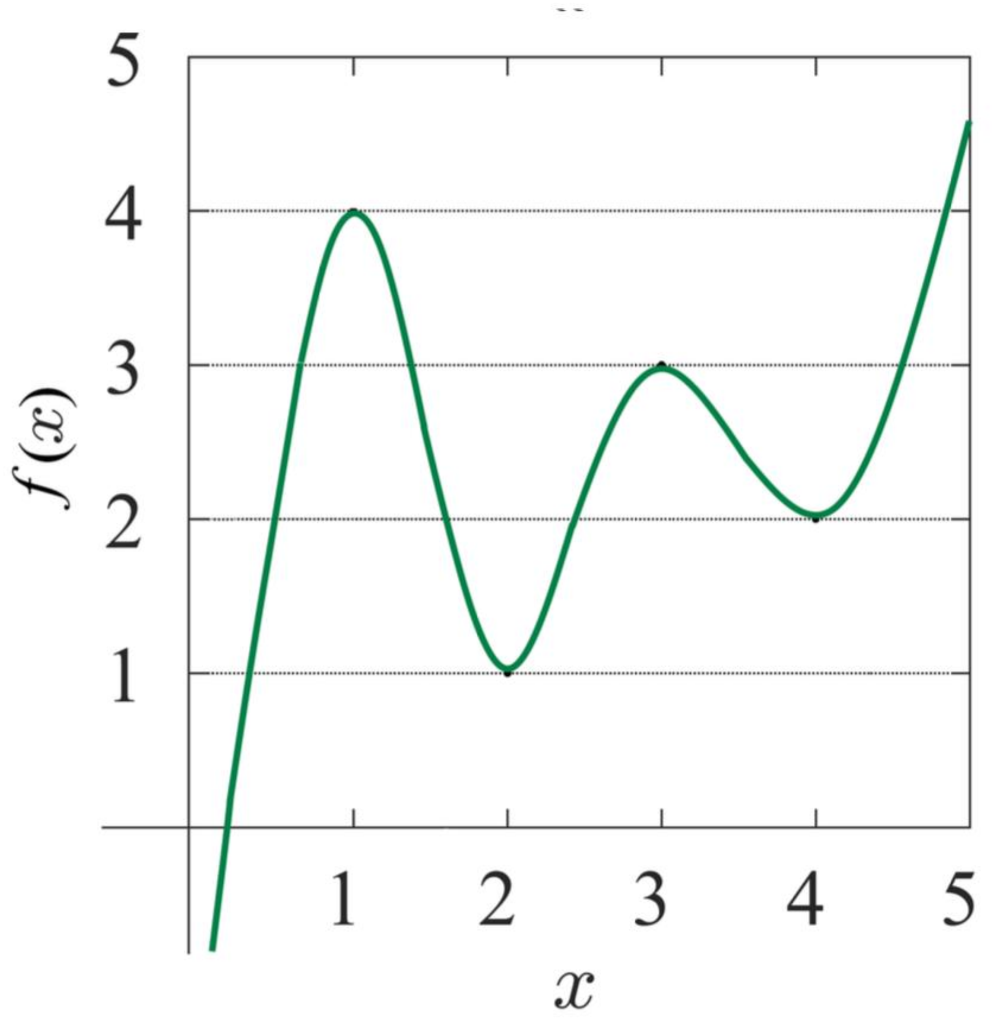


- Given a **growing topological space**, produce a **set of points on the 2D plane** (above the diagonal) called **persistence diagram (PD)** such that:
  - each point in the PD represents a homological feature (aka. cycle / hole) of the data in a certain dimension.

Image: Bobrowski, Skraba. A universal null-distribution for topological data analysis

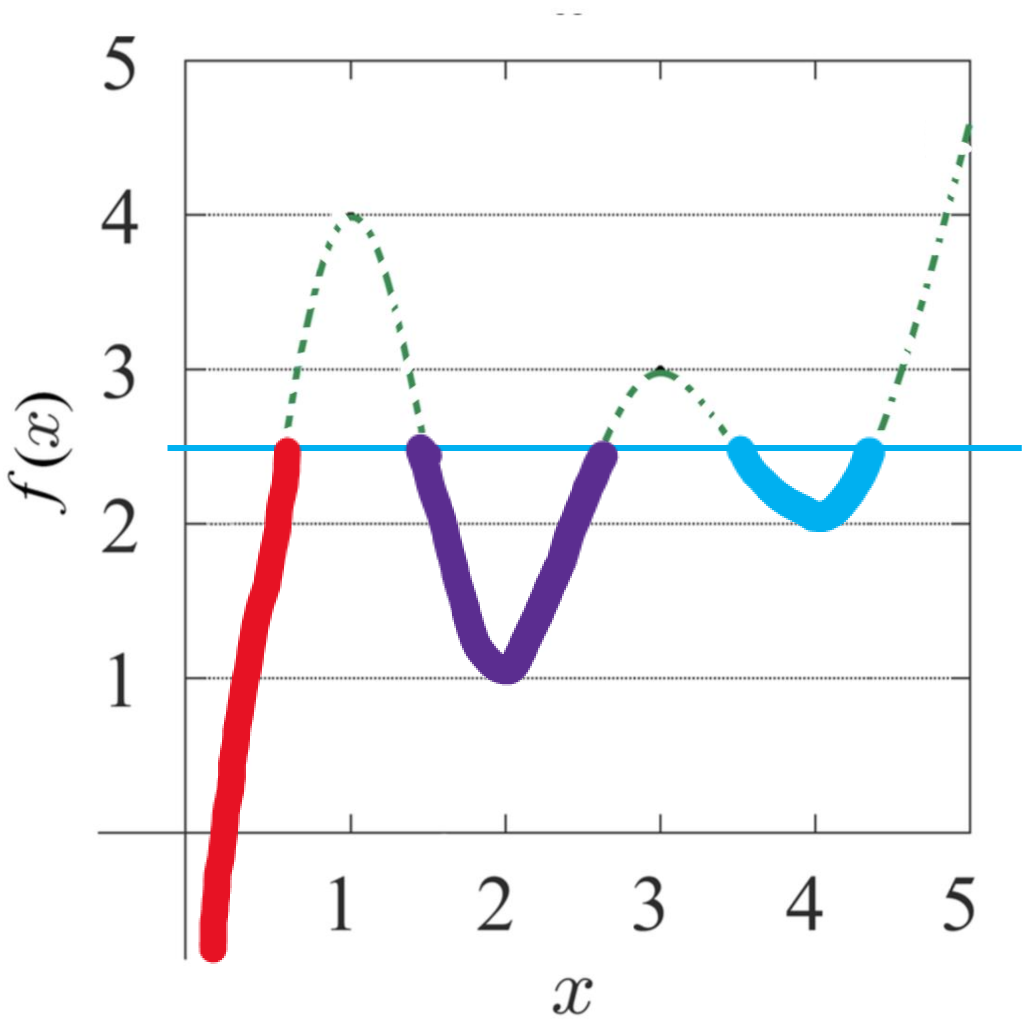
Online resources

- A webpage for visualizing 1–dim PD: https://gjkoplik.github.io/pers-hom-examples/1d_pers_2d_data_widget.html



- For another example of persistent homology, we look at the left curve $y = f(x)$
- Again, we consider a growing space
- Each space in the growing sequence is part of the curve below a certain horizontal line

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



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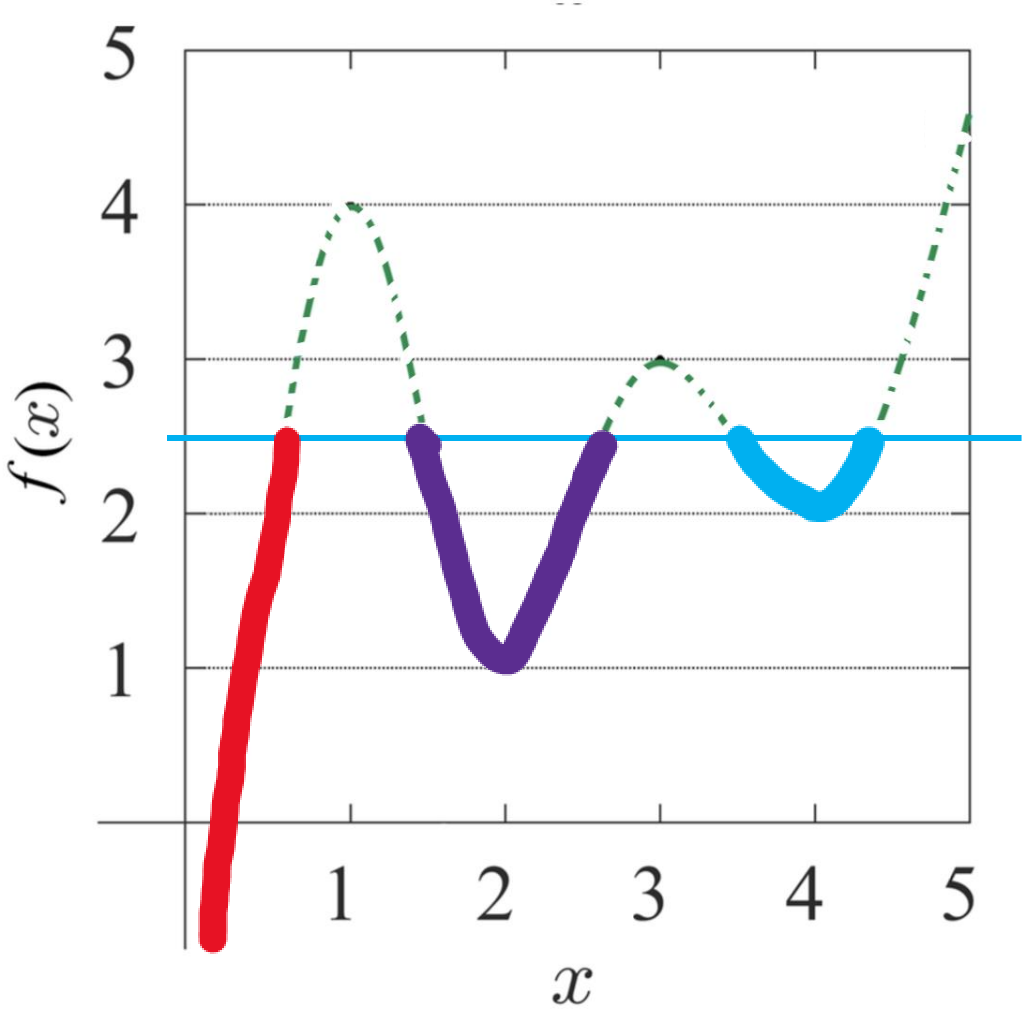
- Left is an example for horizontal line $y = 2.5$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



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- As the space grows, we track the changes of *0-dimensional homology*

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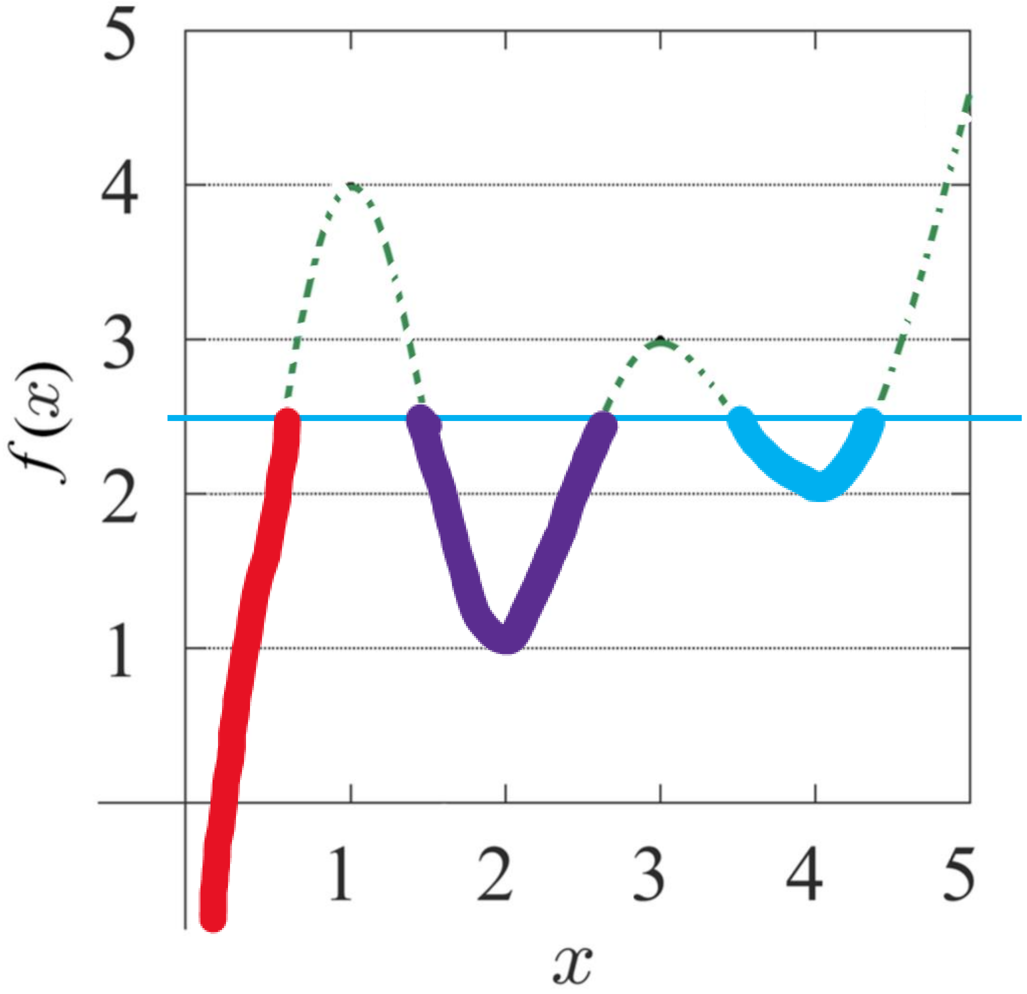

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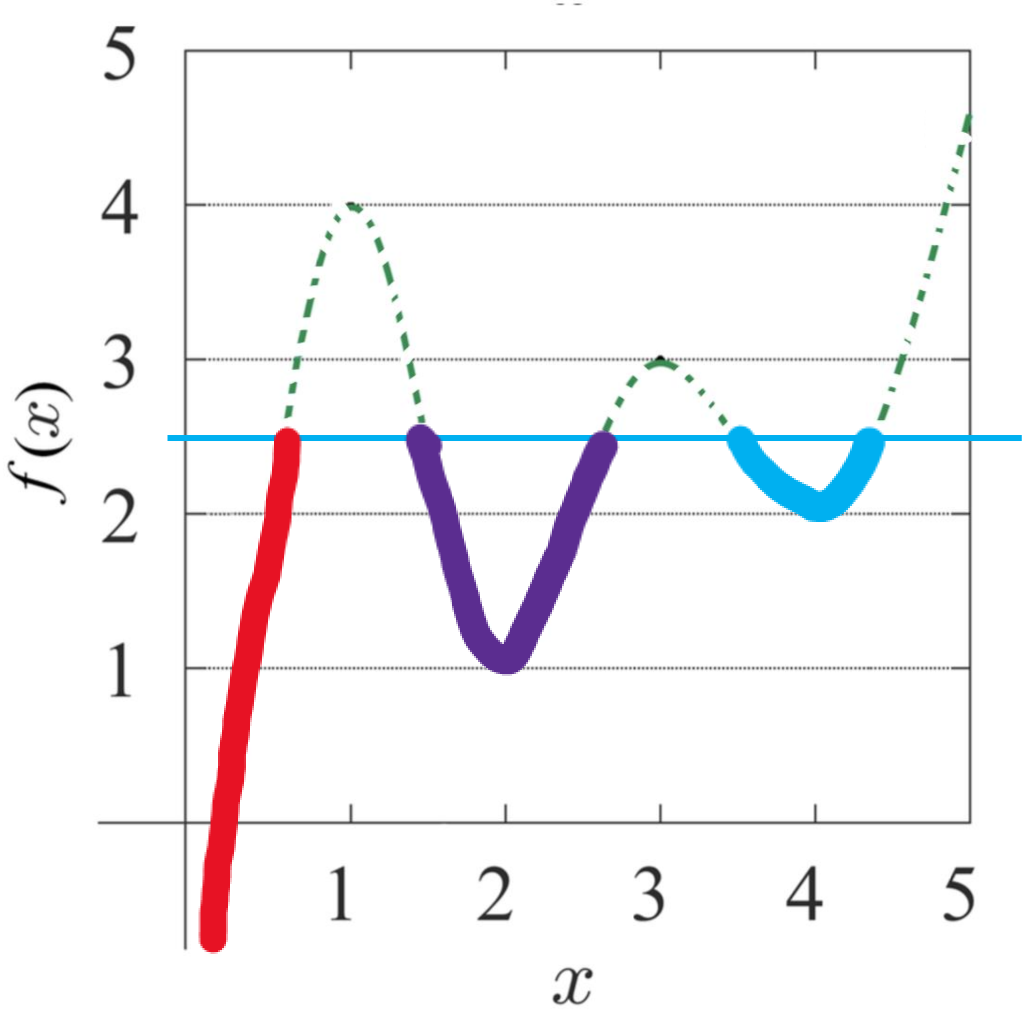
- Each space in the growing sequence is part of the curve below a certain horizontal line
- Left is an example for horizontal line $y = 2.5$
- As the space grows, we track the changes of *0-dimensional homology*
- i.e., we track the **changes of the connected components and the gaps in between**

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



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- Again, we consider a growing space
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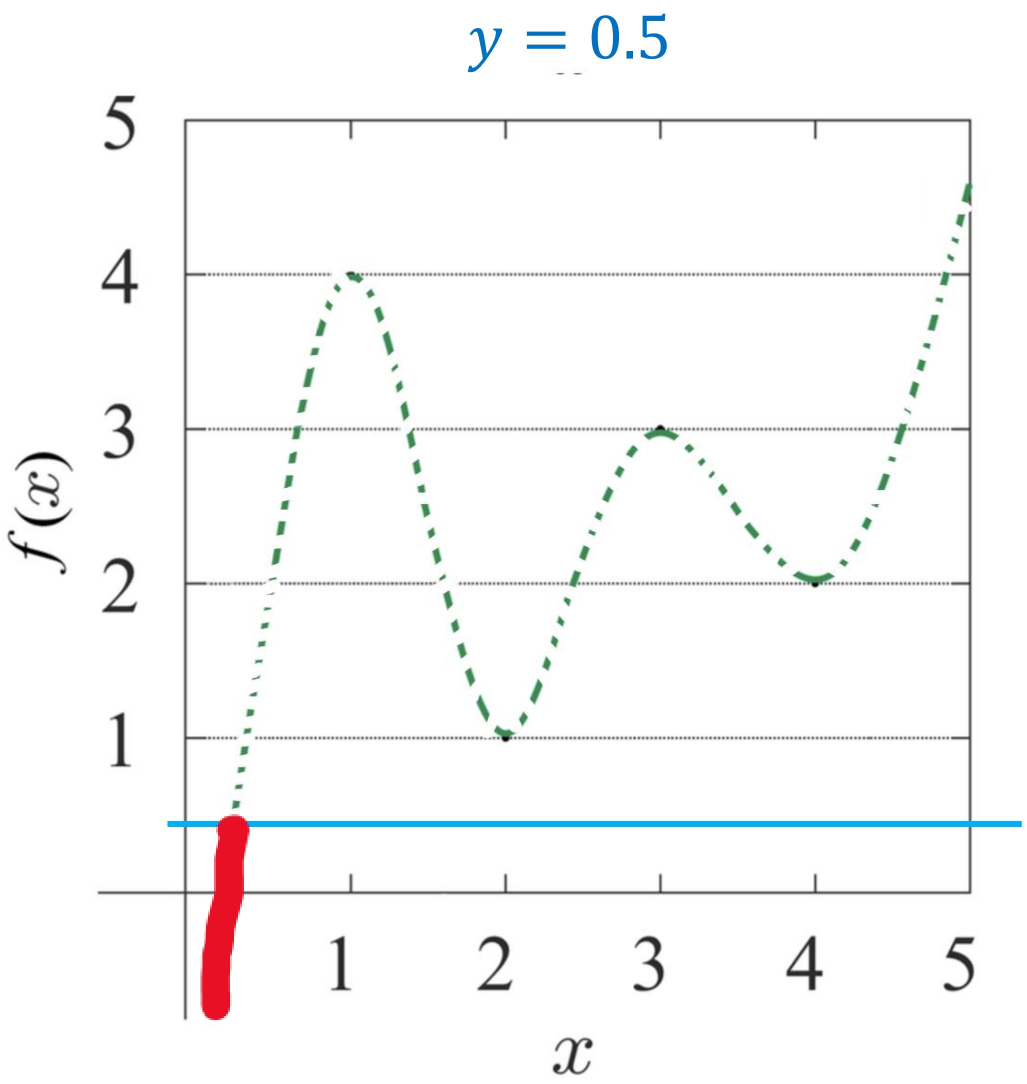
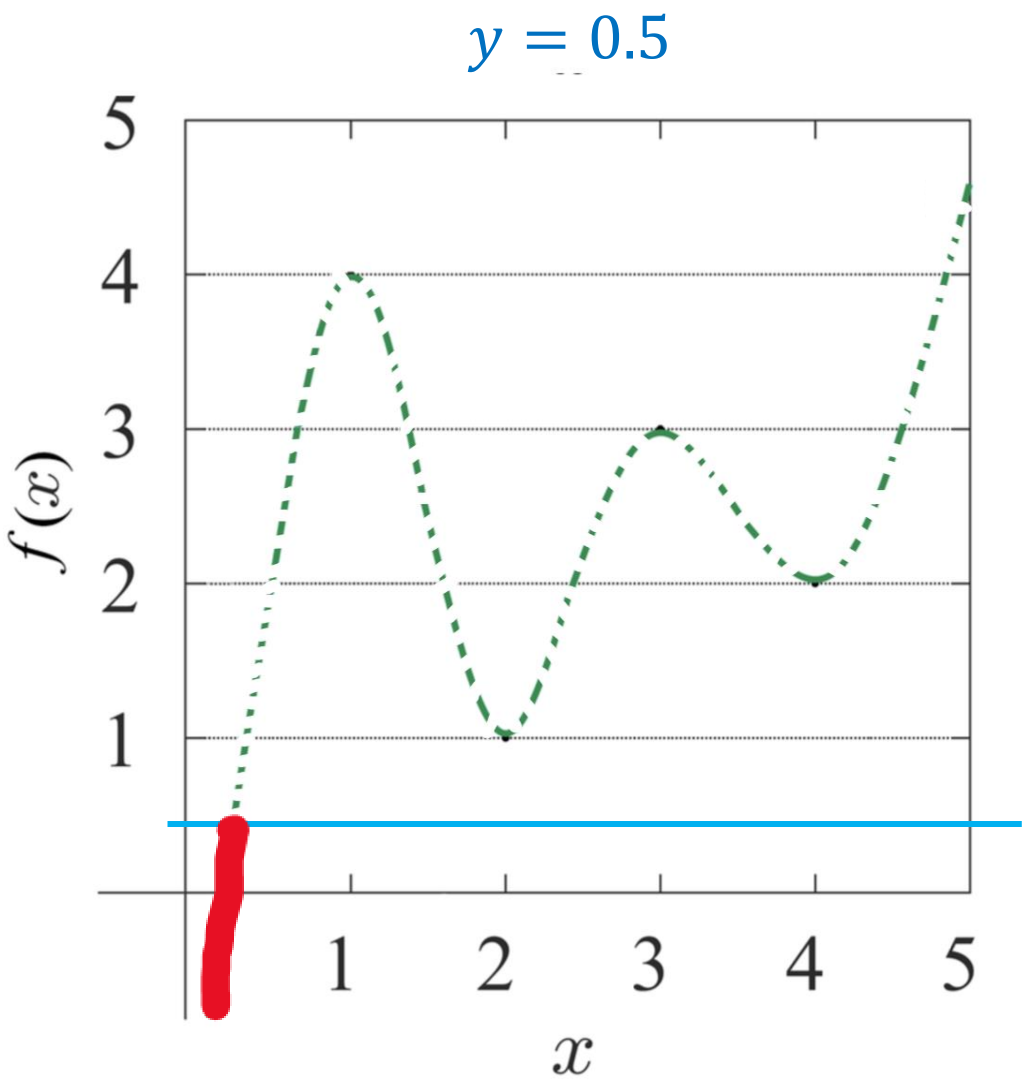
- As the space grows, we track the changes of *0-dimensional homology*
- i.e., we track the **changes of the connected components and the gaps in between**
- On the left, there are *three connected components with two gaps in between*

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



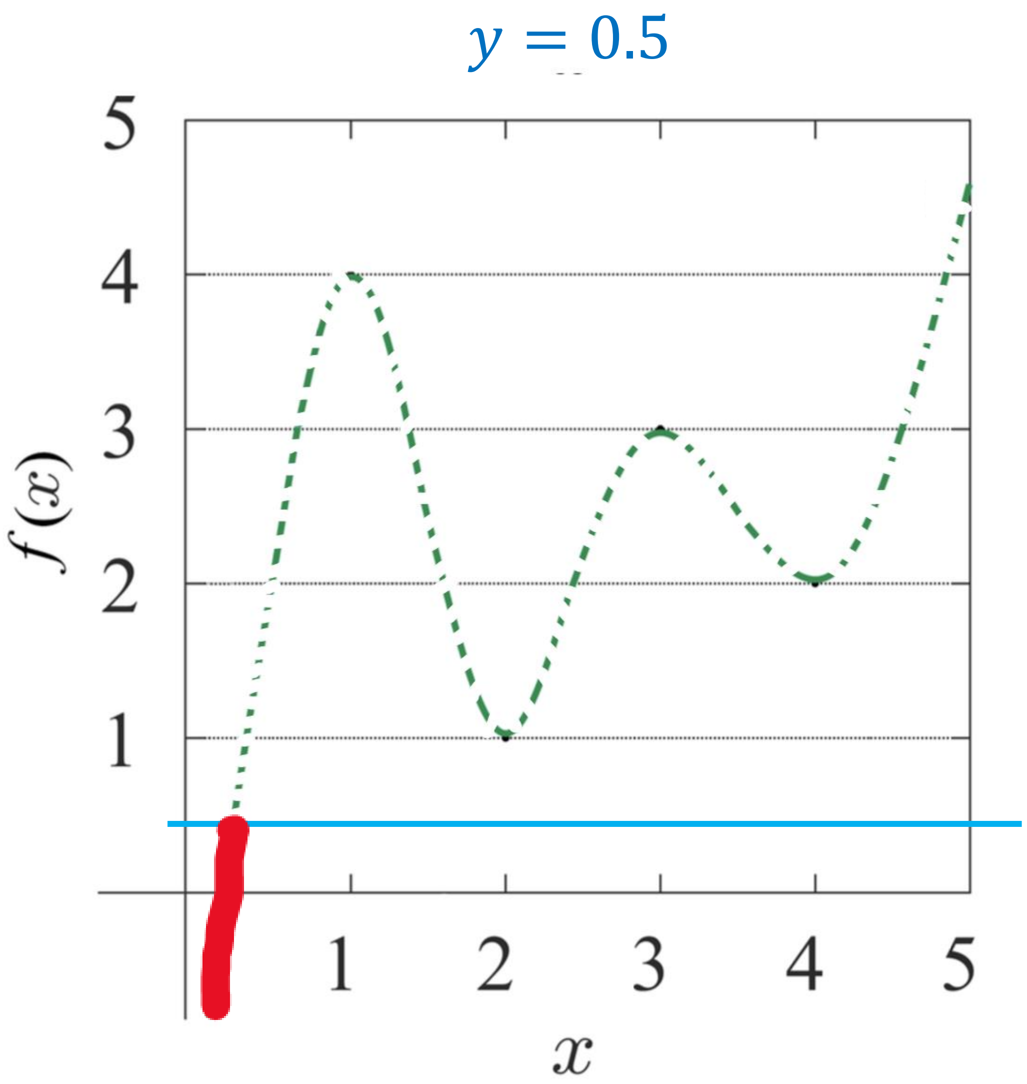
- We have that there is a single connected component (red) below the line $y = 0.5$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- We have that there is a single connected component (red) below the line $y = 0.5$
- In general, suppose that $f(x)$ approaches -∞ as x approaches 0, we have that there is a single connected component below the line $y = \alpha$ for any $\alpha \leq 0.5$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



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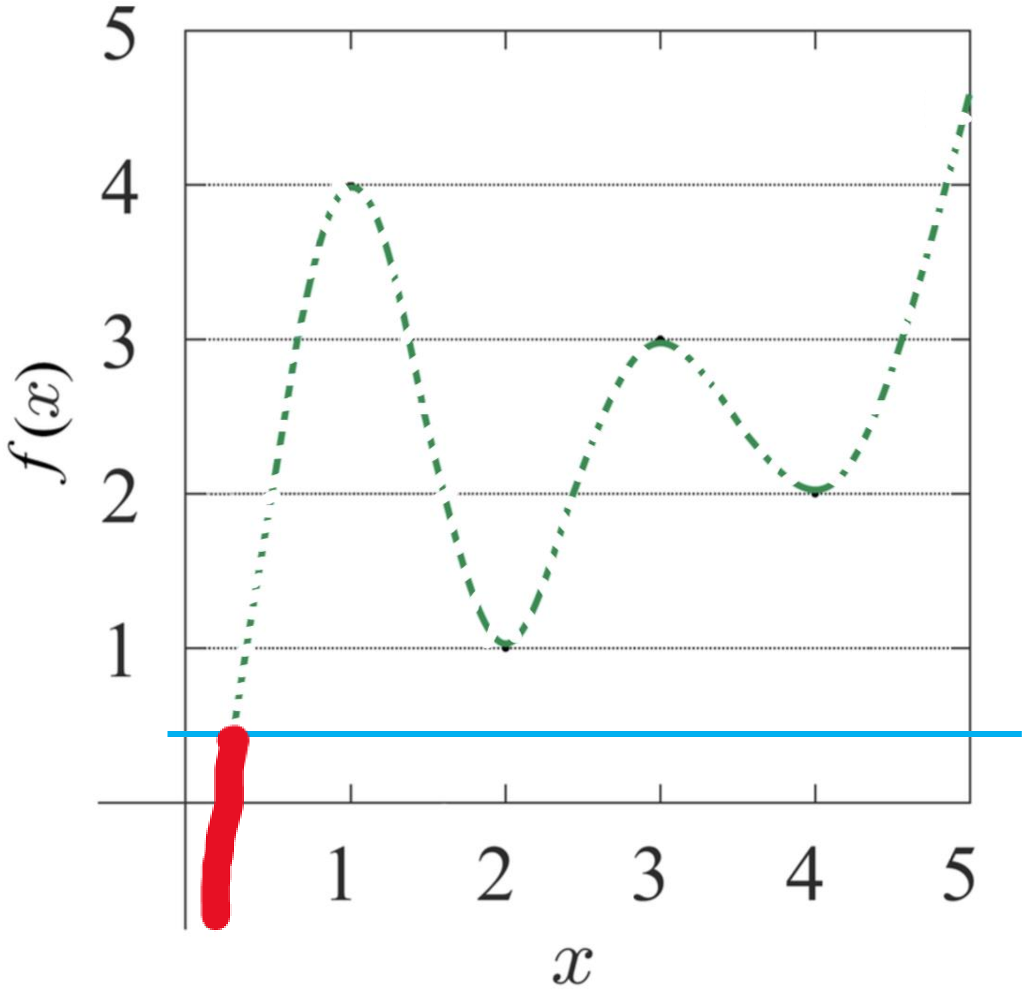
- So we can assume the red connected component is born at the value -∞

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

$y = 0.5$



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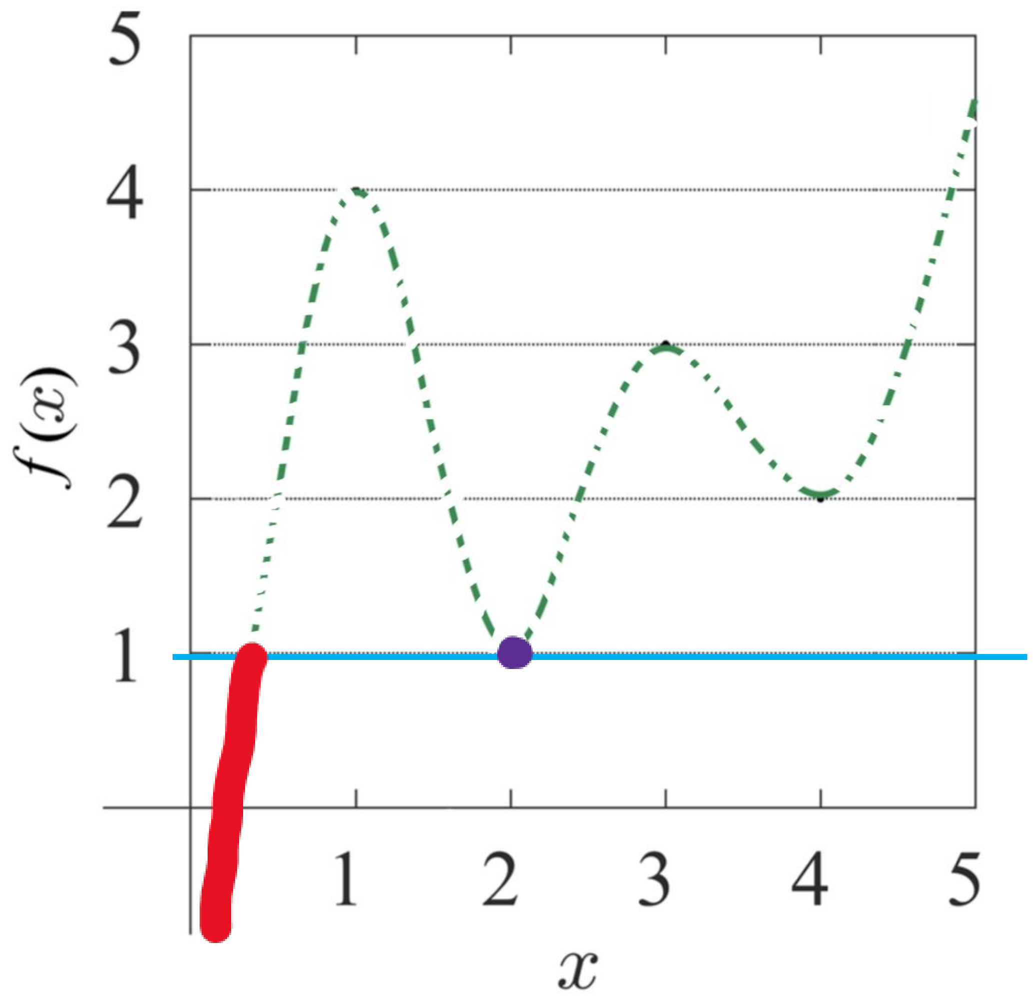
- So we can assume the red connected component is born at the value -∞

- Red: born at -∞

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

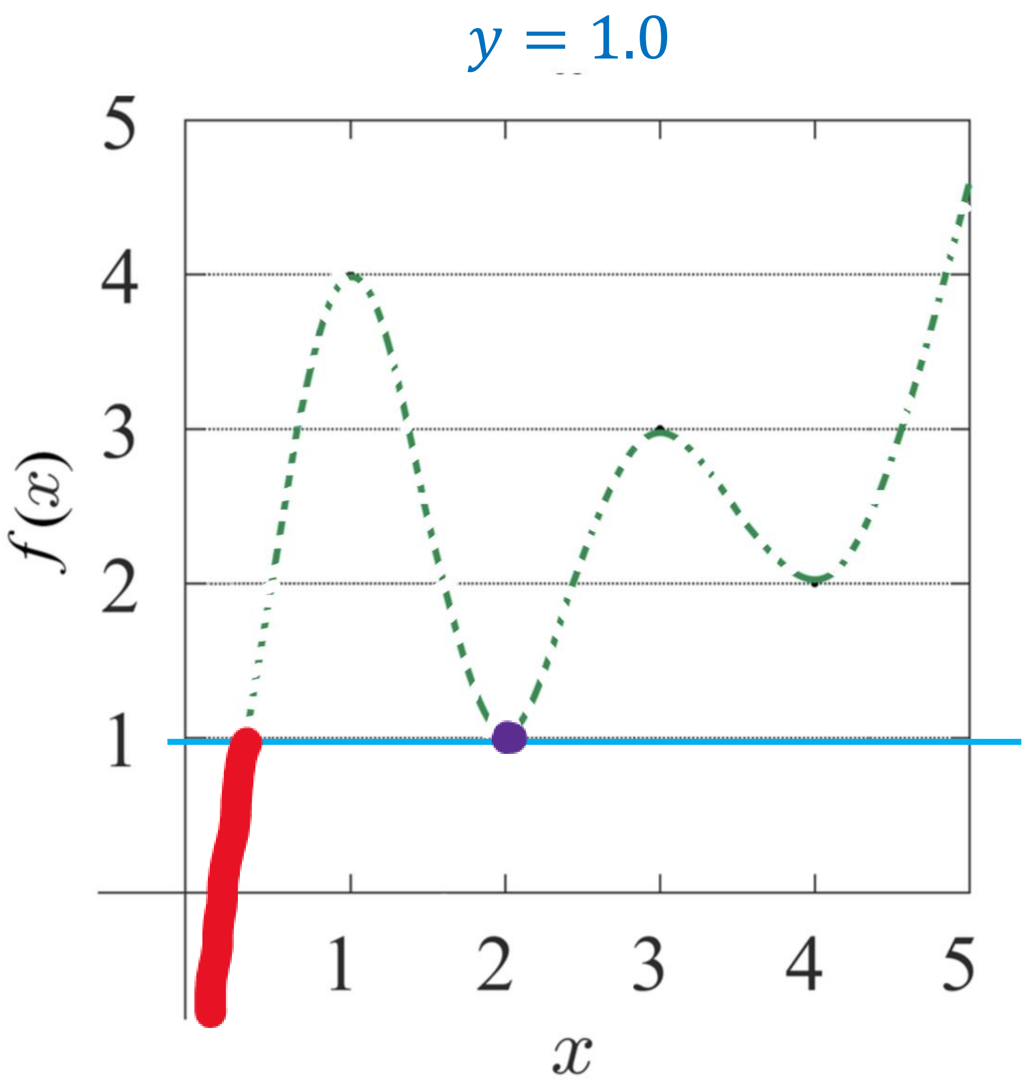
$y = 1.0$



- Red component continues
- A new purple component is born

- Red: born at -∞

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



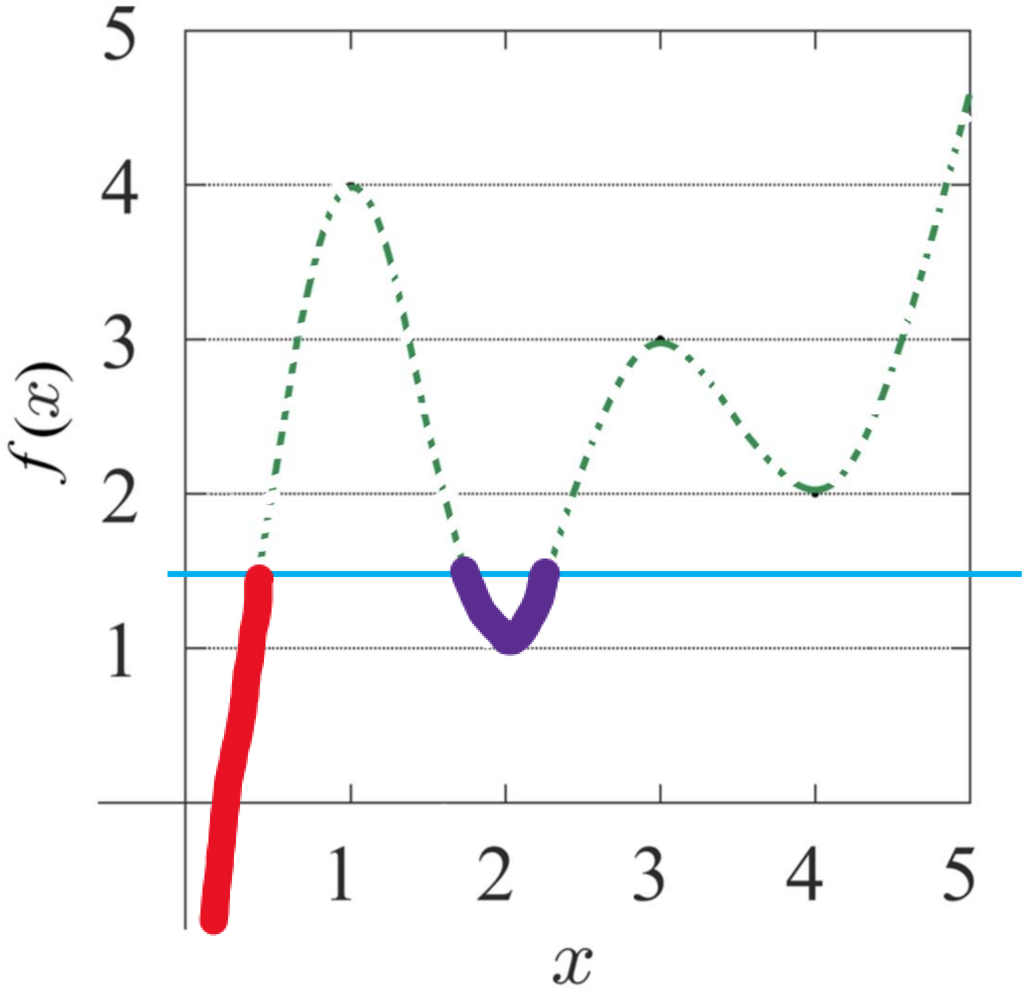
- Red component continues
- A new purple component is born

- Red: born at -∞

- Purple: born at 1.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

$y = 1.5$

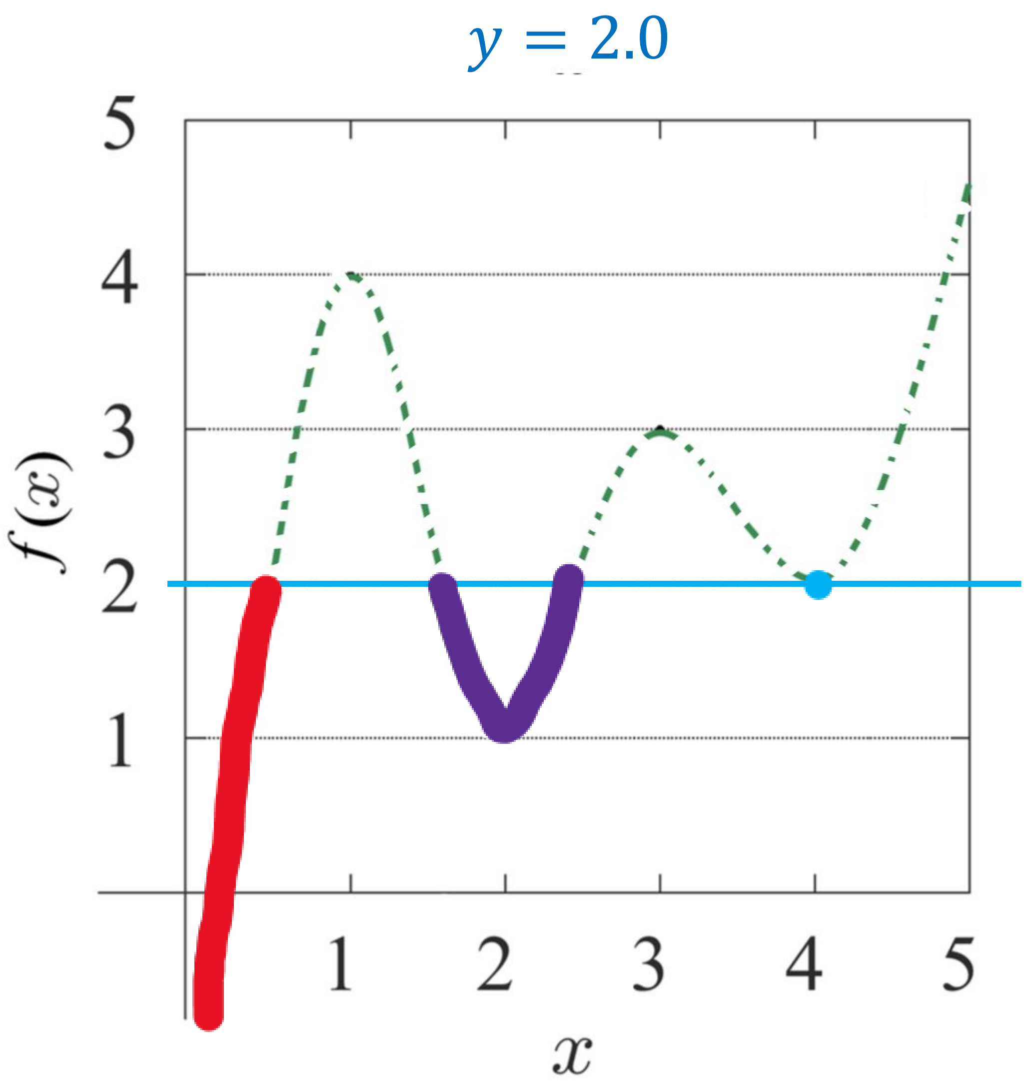


- Red and purple components continue

- Red: born at -∞

- Purple: born at 1.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

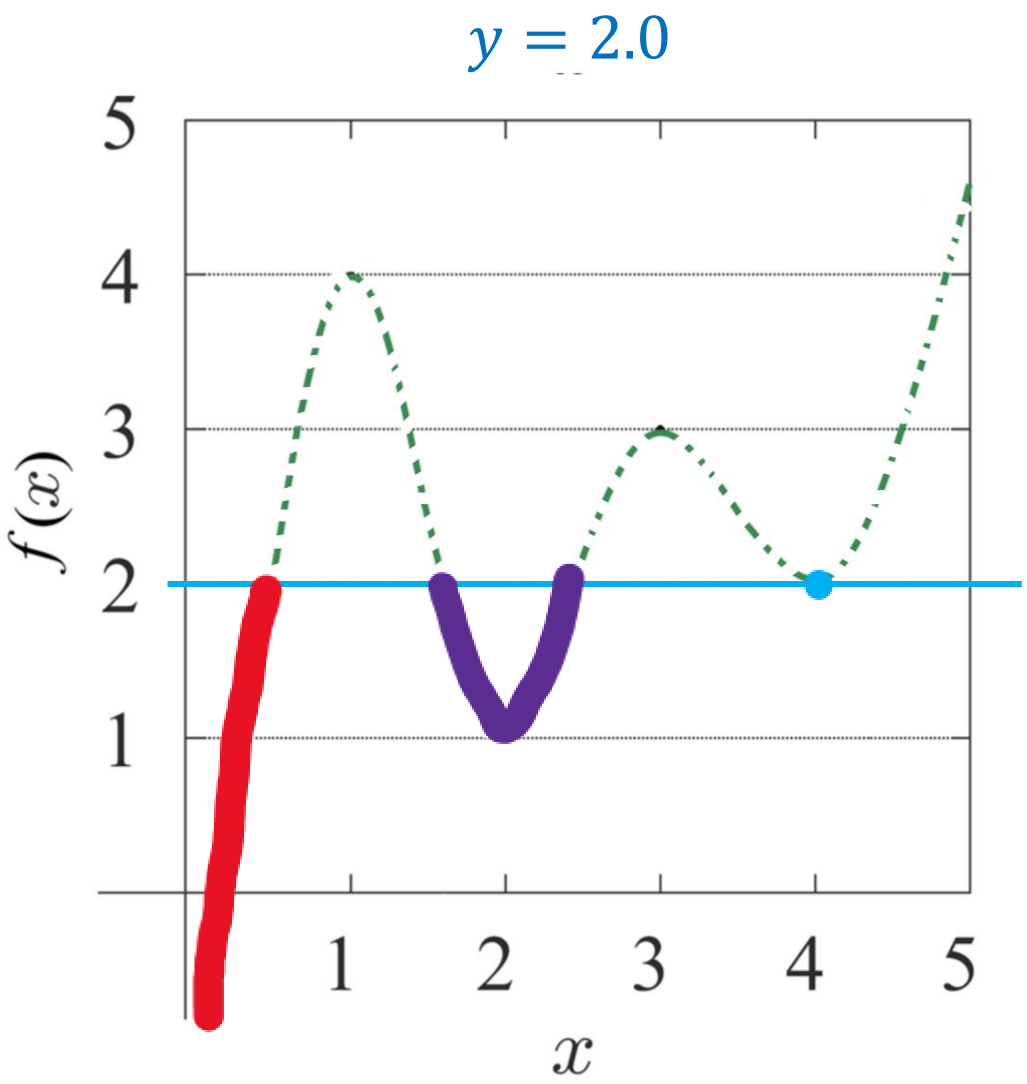


- Red and purple components continue
- A new blue component is born

- Red: born at -∞

- Purple: born at 1.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



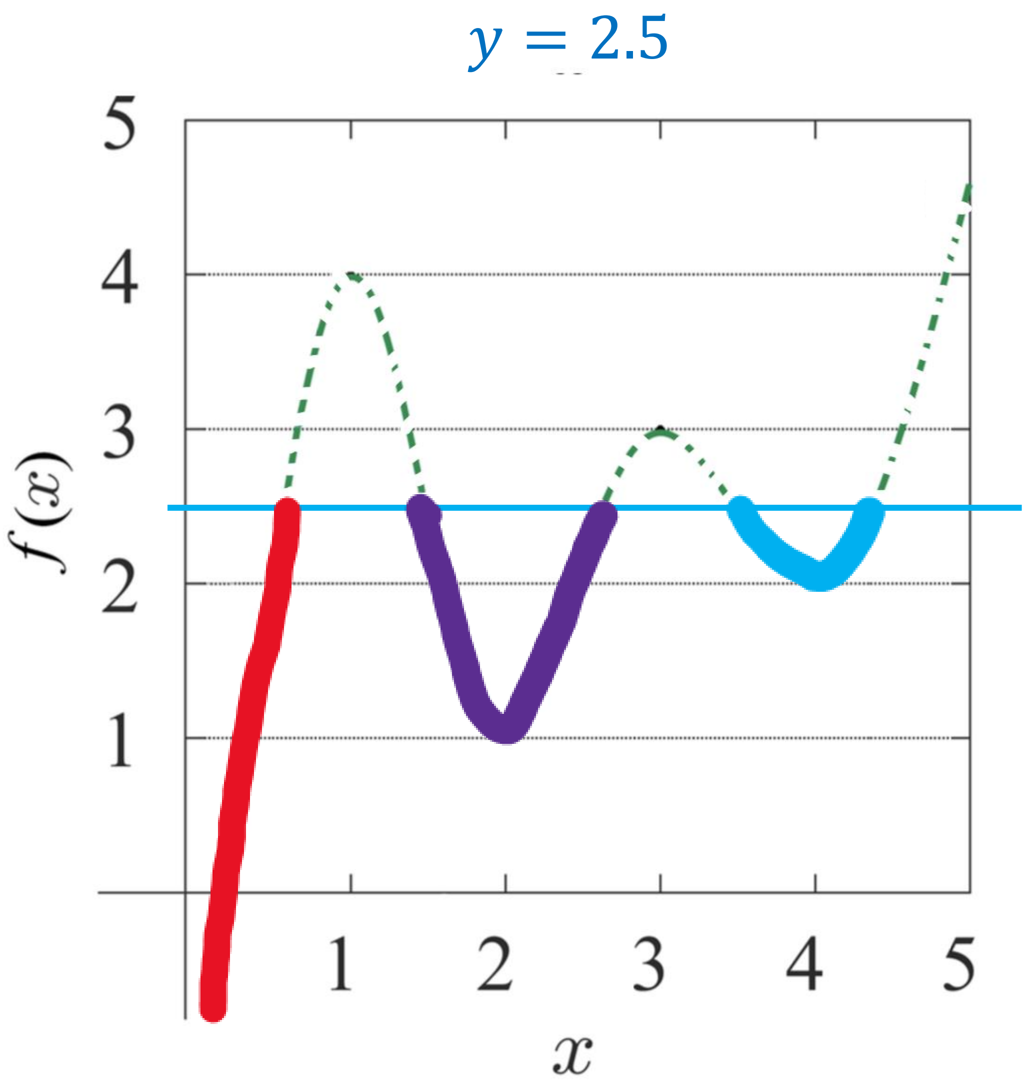
- Red and purple components continue
- A new blue component is born

- Red: born at -∞

- Purple: born at 1.0

- Blue: born at 2.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



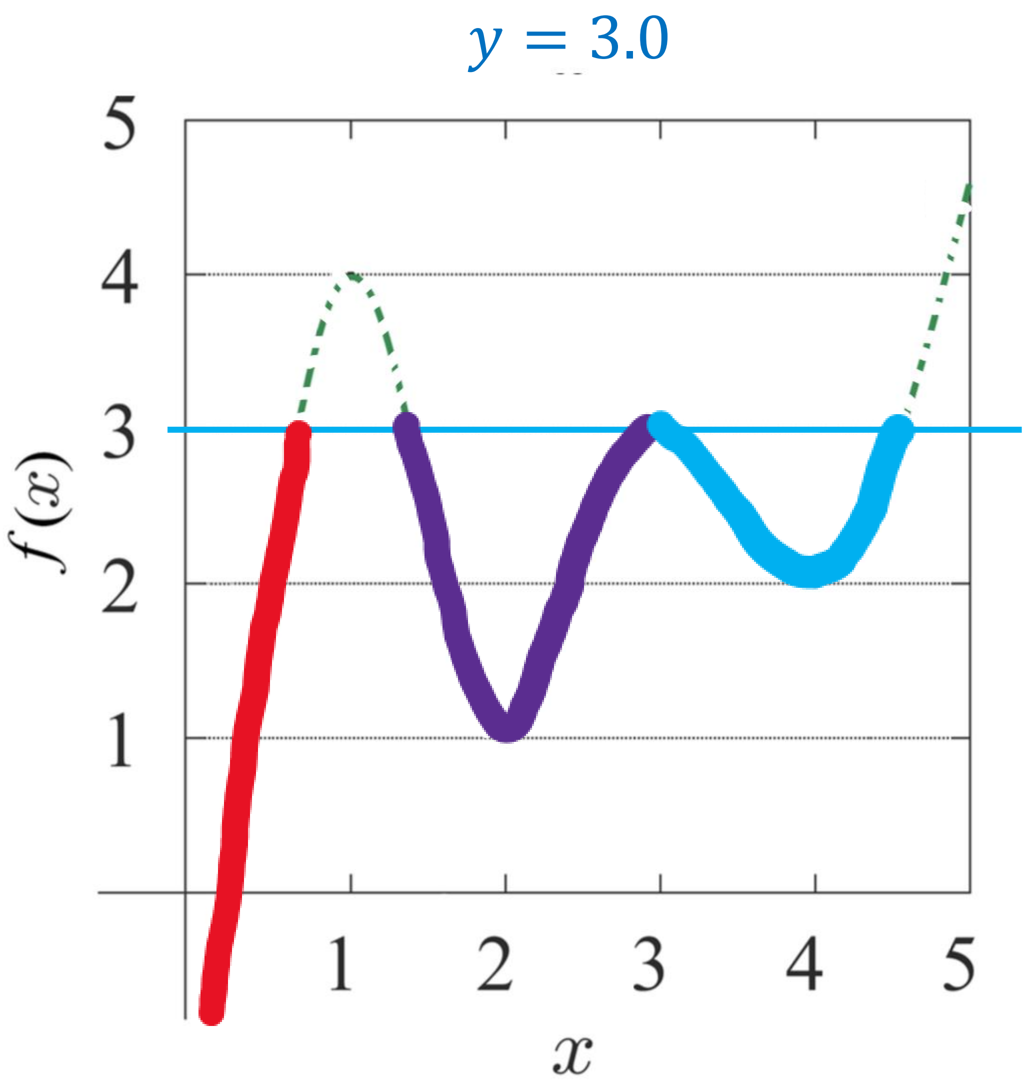
- Three components continue

- Red: born at -∞

- Purple: born at 1.0

- Blue: born at 2.0

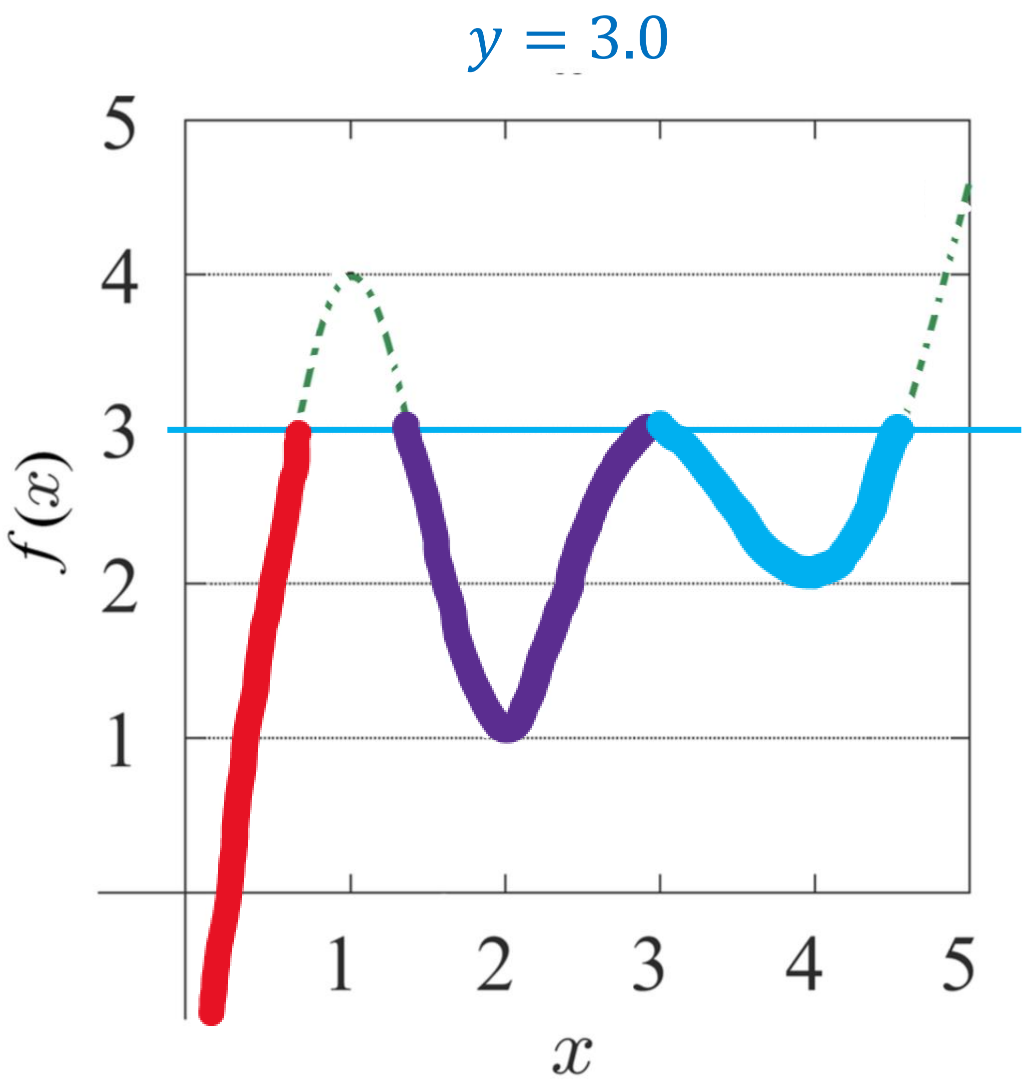
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- The purple and blue components merge into one (gaps between them disappear)

- Red: born at -∞
- Purple: born at 1.0
- Blue: born at 2.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- The purple and blue components merge into one (gaps between them disappear)
- The means that a 0-dimensional homology hole disappears (dies)

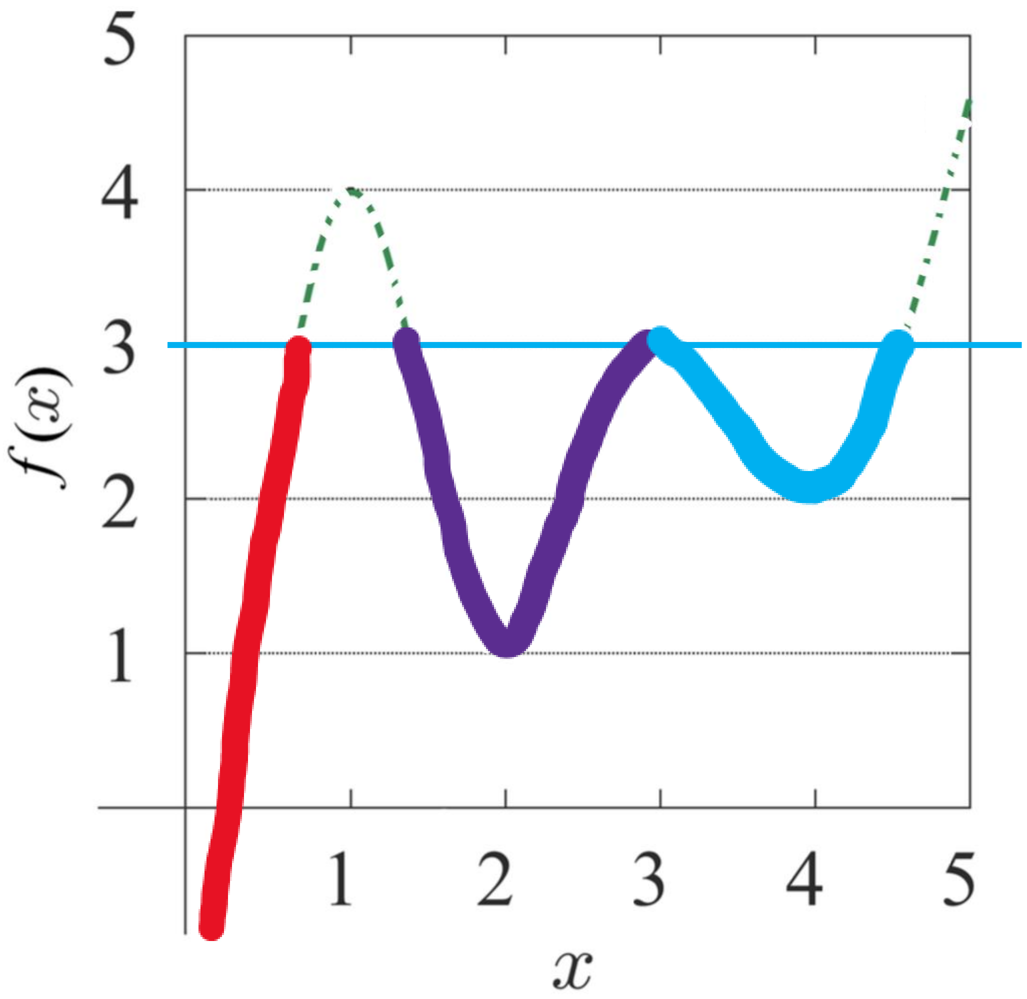
- Red: born at -∞

- Purple: born at 1.0

- Blue: born at 2.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

$y = 3.0$



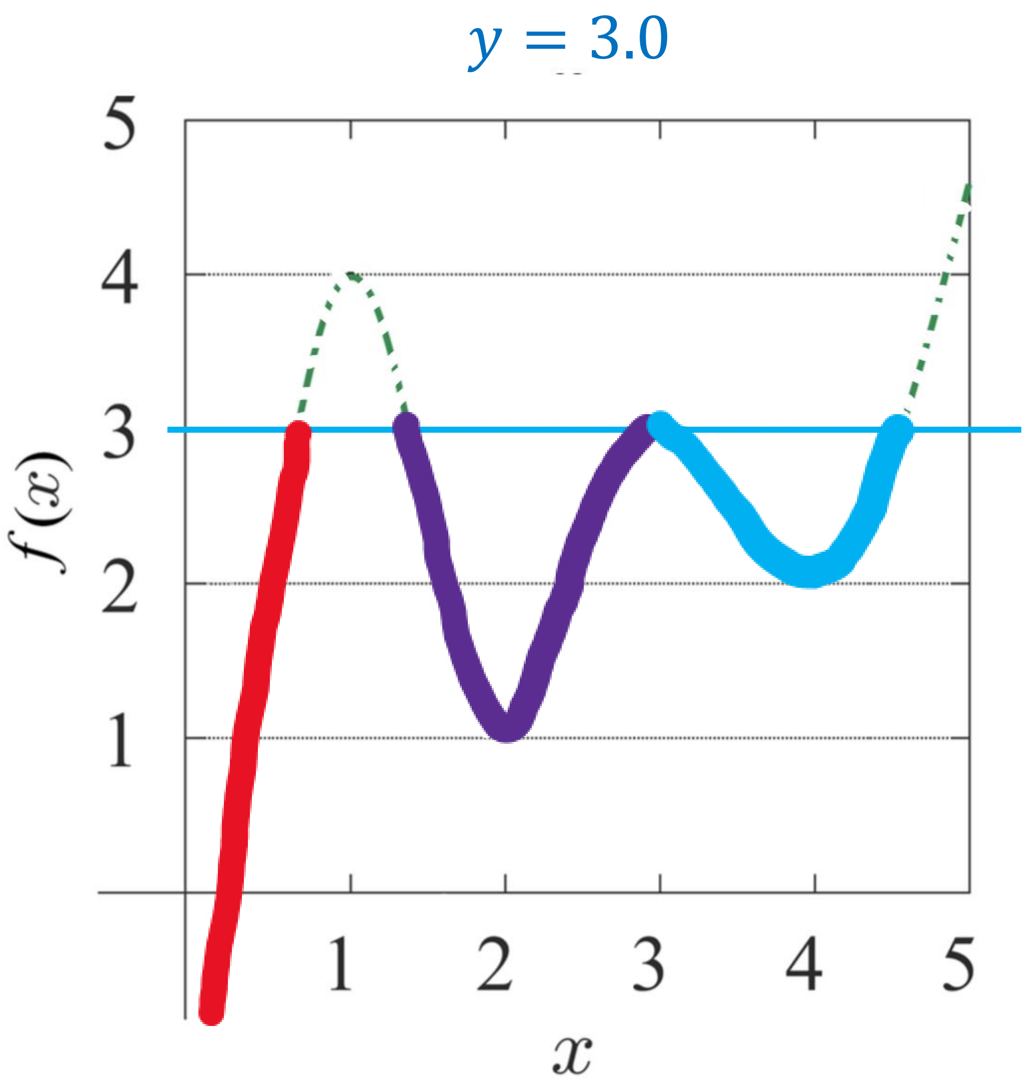
- The purple and blue components merge into one (gaps between them disappear)
- The means that a 0-dimensional homology hole disappears (dies)
- The gap between purple and blue components appears because of birth of the blue component

- Red: born at -∞

- Purple: born at 1.0

- Blue: born at 2.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



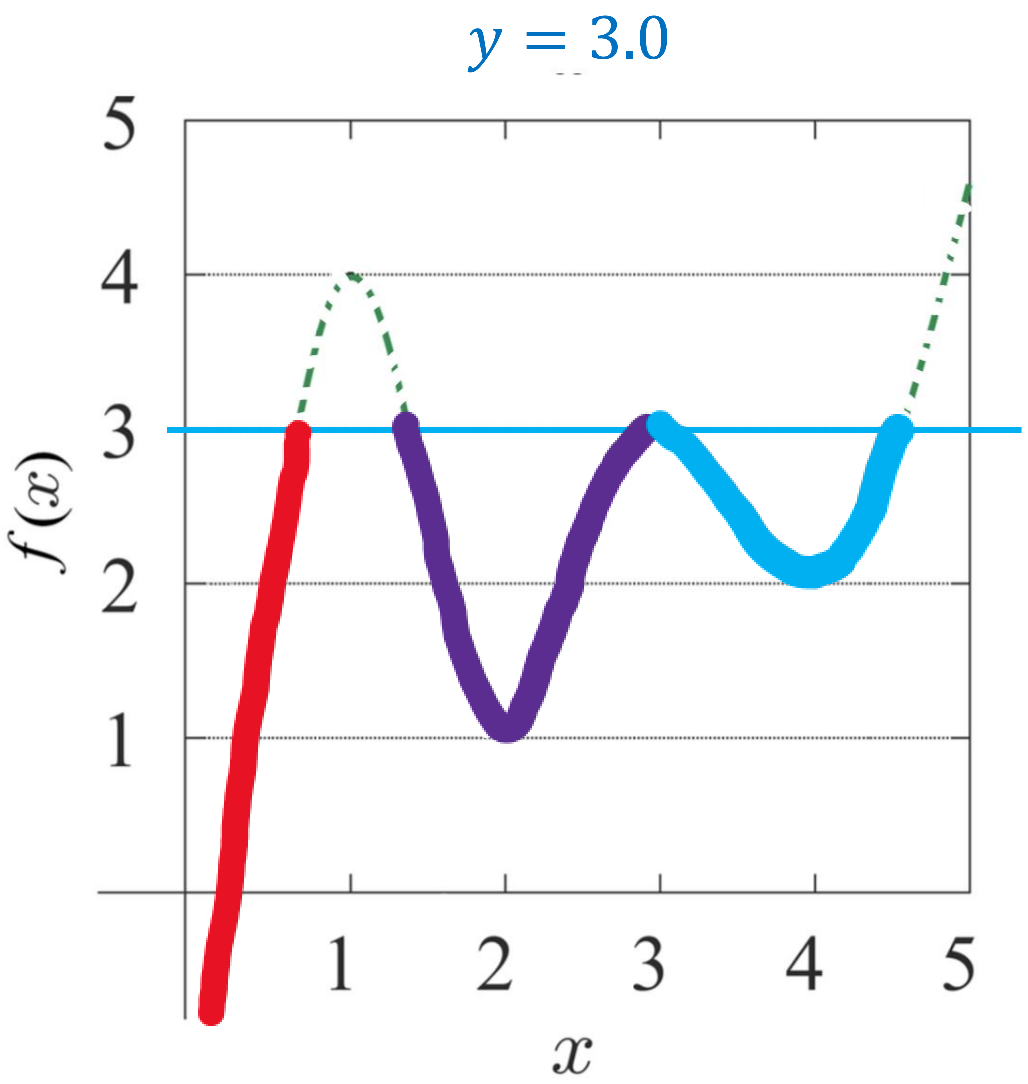
- The purple and blue components merge into one (gaps between them disappear)
- The means that a 0-dimensional homology hole disappears (dies)
- The gap between purple and blue components appears because of birth of the blue component
- So we consider the gap to be born when the blue component is born, i.e., at 2.0

- Red: born at -∞

- Purple: born at 1.0

- Blue: born at 2.0

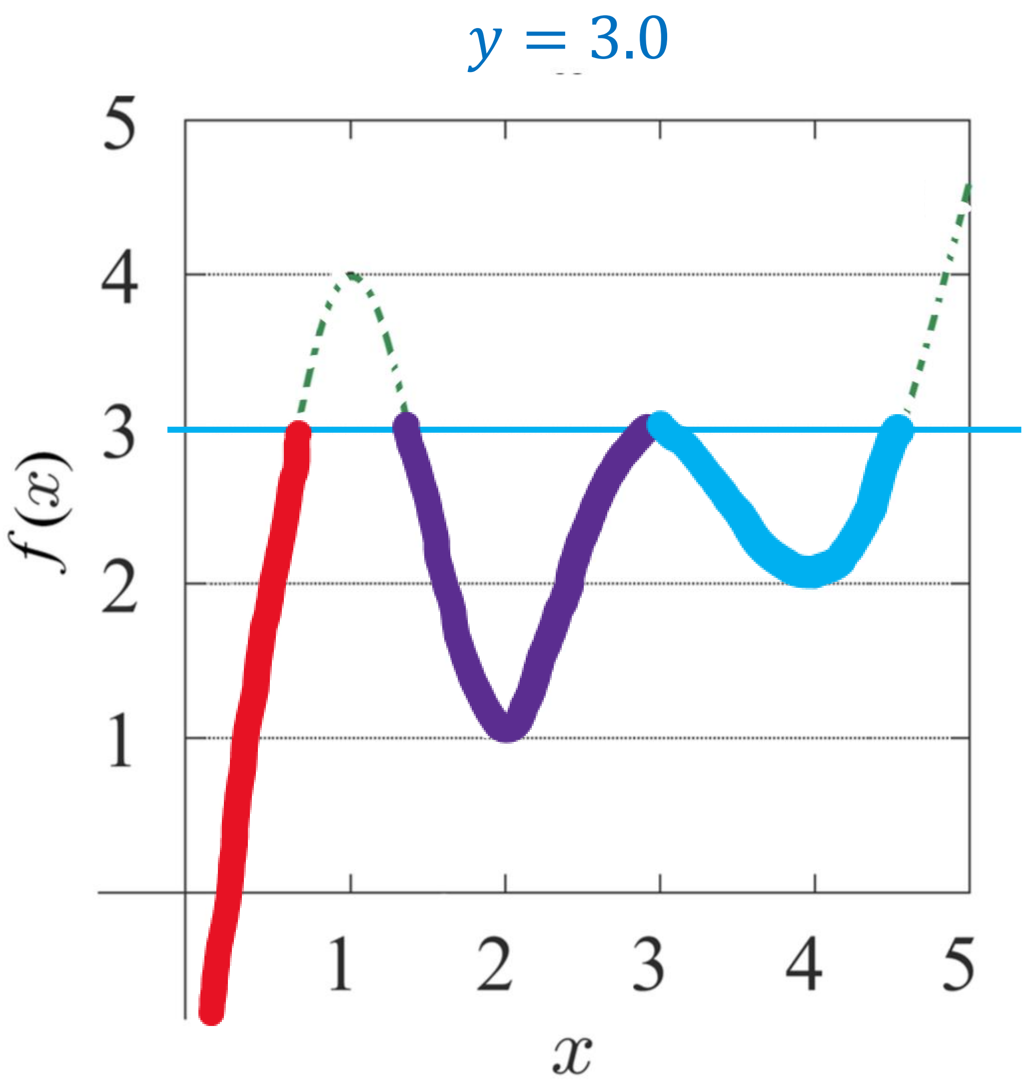
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- The purple and blue components merge into one (gaps between them disappear)
- The means that a 0-dimensional homology hole disappears (dies)
- The gap between purple and blue components appears because of birth of the blue component
- So we consider the gap to be born when the blue component is born, i.e., at 2.0
- So we have a 0-dimensional hole born at 2.0 and dies at 3.0

• Red: born at -∞	• Purple: born at 1.0	• Blue: born at 2.0

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



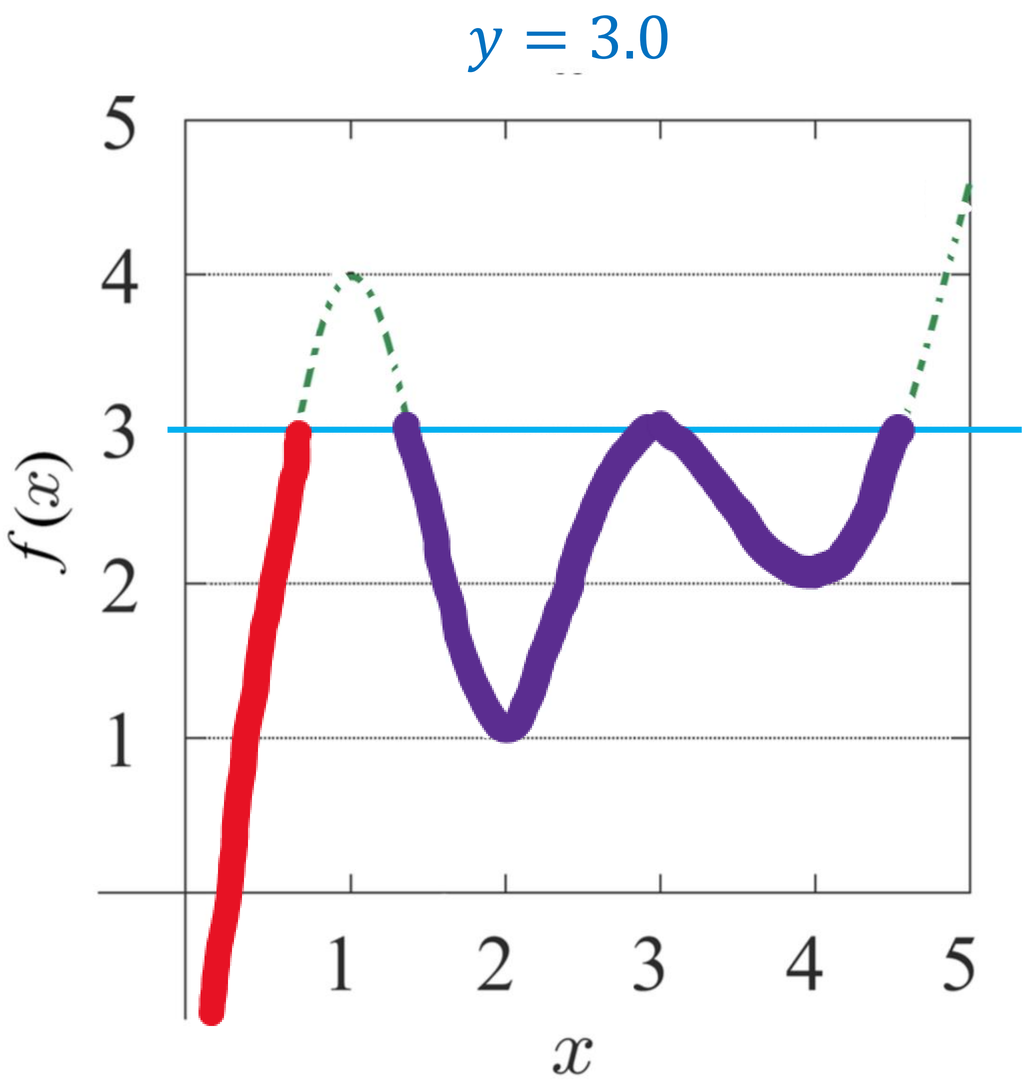
- The purple and blue components merge into one (gaps between them disappear)
- The means that a 0-dimensional homology hole disappears (dies)
- The gap between purple and blue components appears because of birth of the blue component
- So we consider the gap to be born when the blue component is born, i.e., at 2.0
- So we have a 0-dimensional hole born at 2.0 and dies at 3.0

- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- For the merged component, we keep the one born earlier (purple), and kill the one born later (blue)

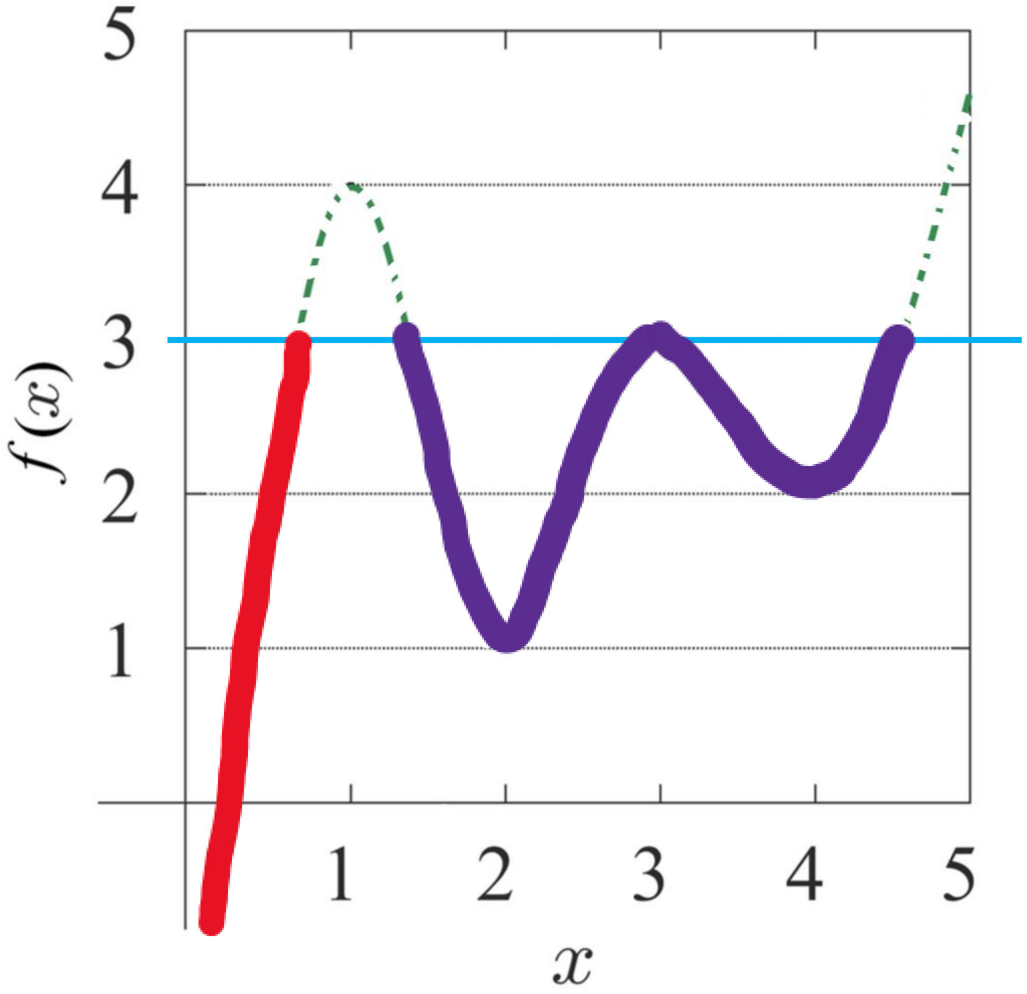
- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

$y = 3.0$



- For the merged component, we keep the one born earlier (purple), and kill the one born later (blue)
- So we have a larger purple component born at 1.0

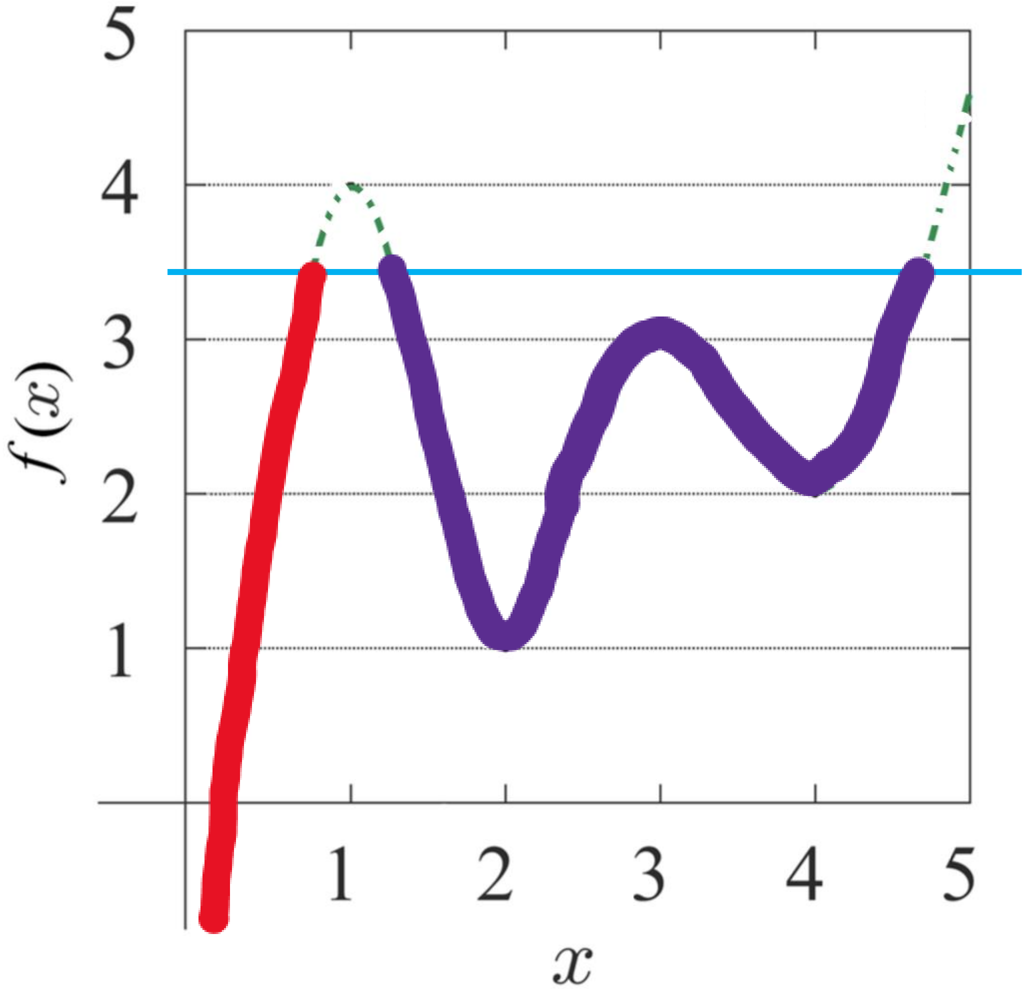
- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

$y = 3.5$



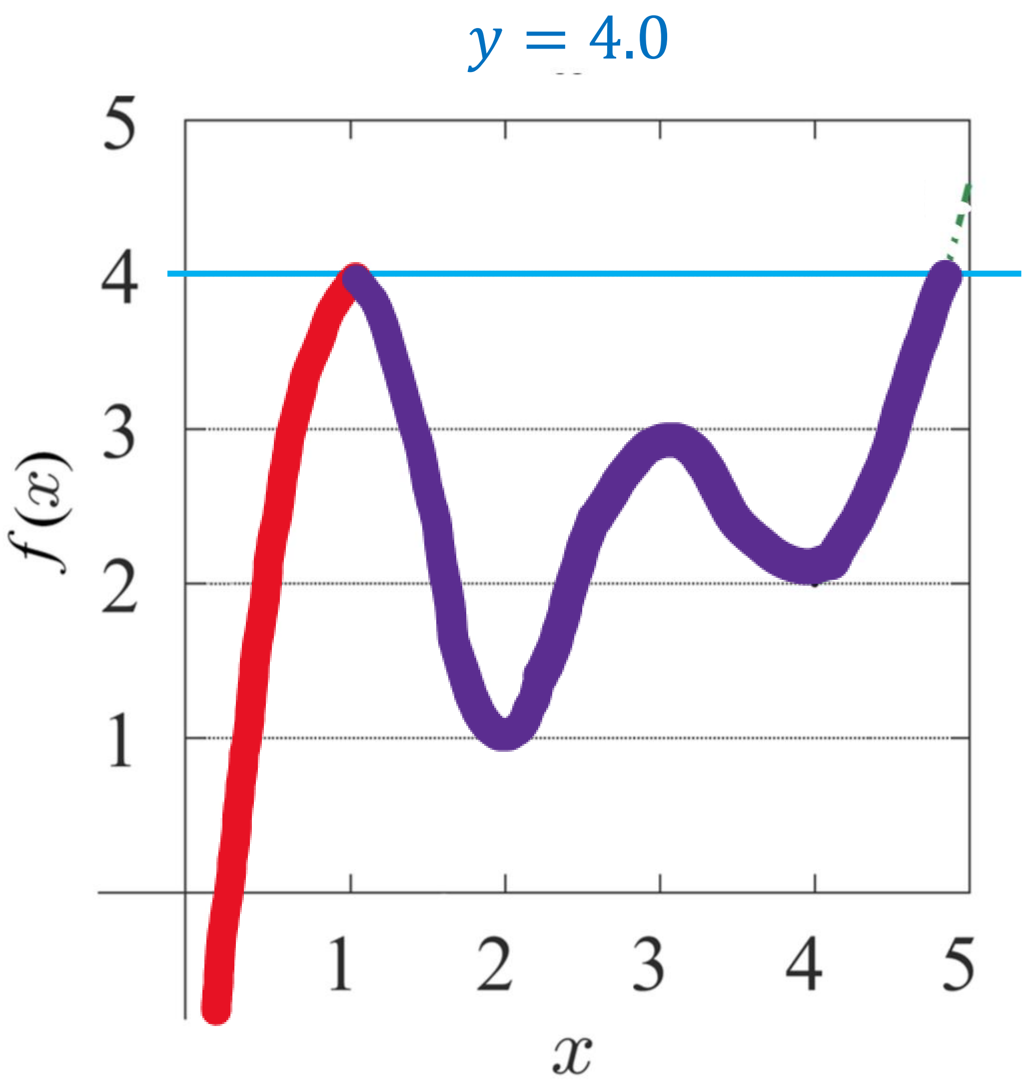
- Two components continue

- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



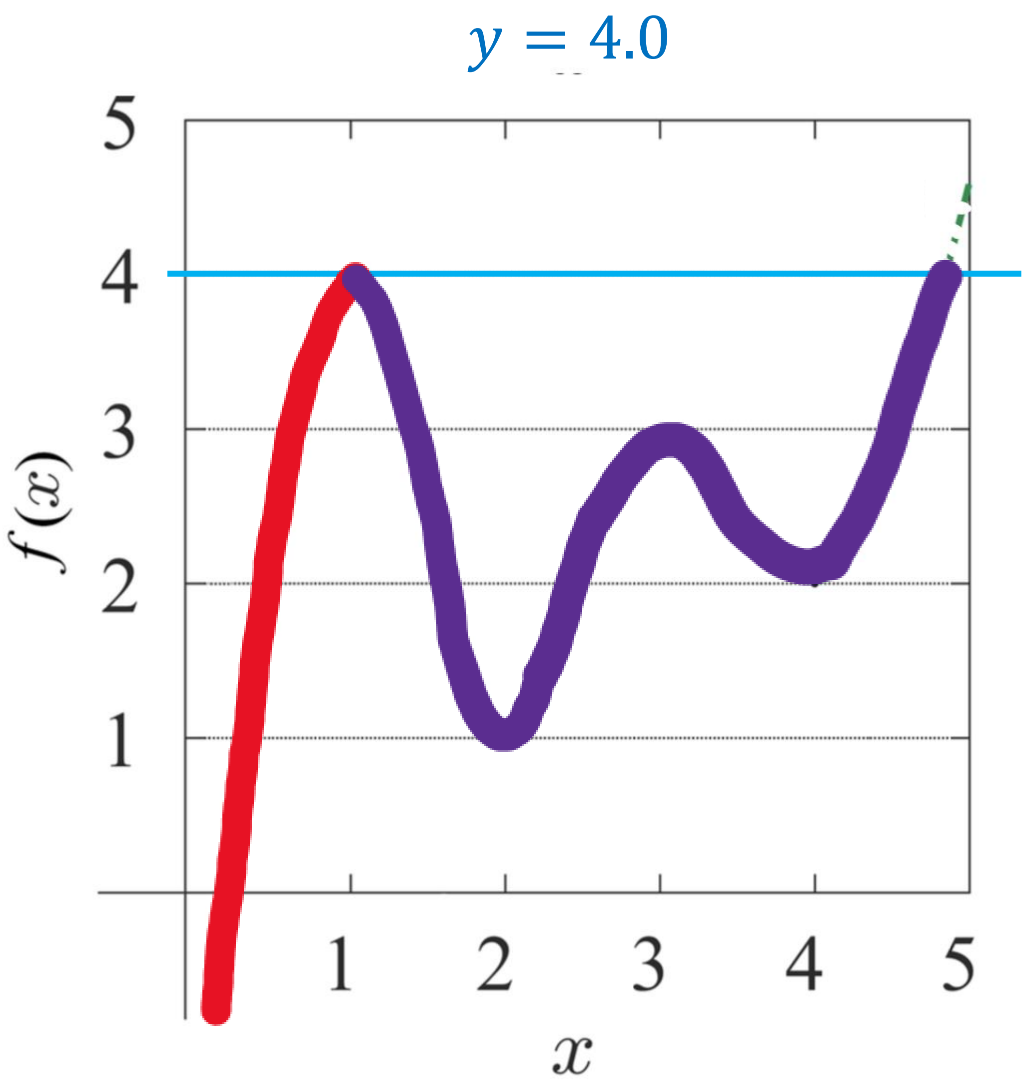
- The red and purple components merge into one (gaps between them disappear)

- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- The red and purple components merge into one (gaps between them disappear)
- A 0-dimensional homology hole disappears (dies)

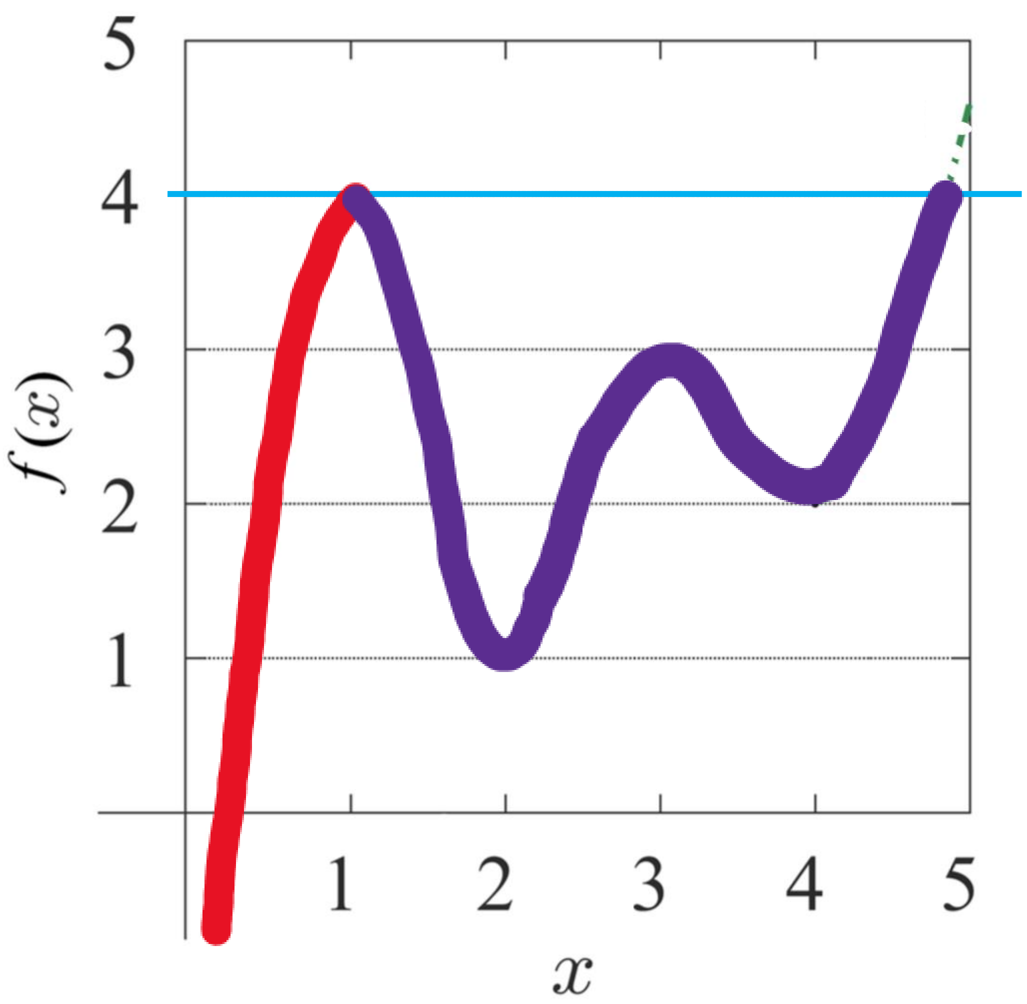
- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

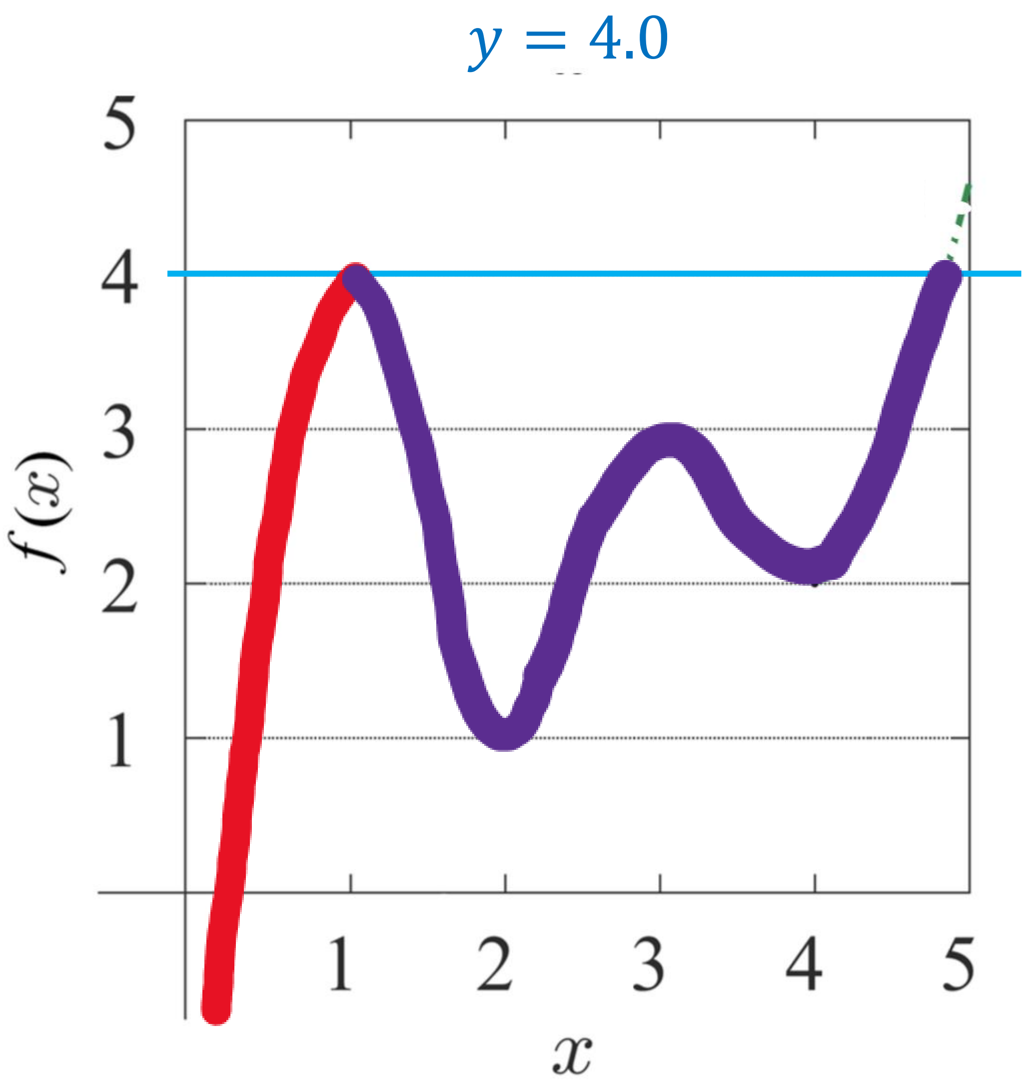
$y = 4.0$



- The red and purple components merge into one (gaps between them disappear)
- A 0-dimensional homology hole disappears (dies)
- The gap between red and purple components appears because of birth of the purple component

• Red: born at -∞	• Purple: born at 1.0	• PD: $(2.0, 3.0)$

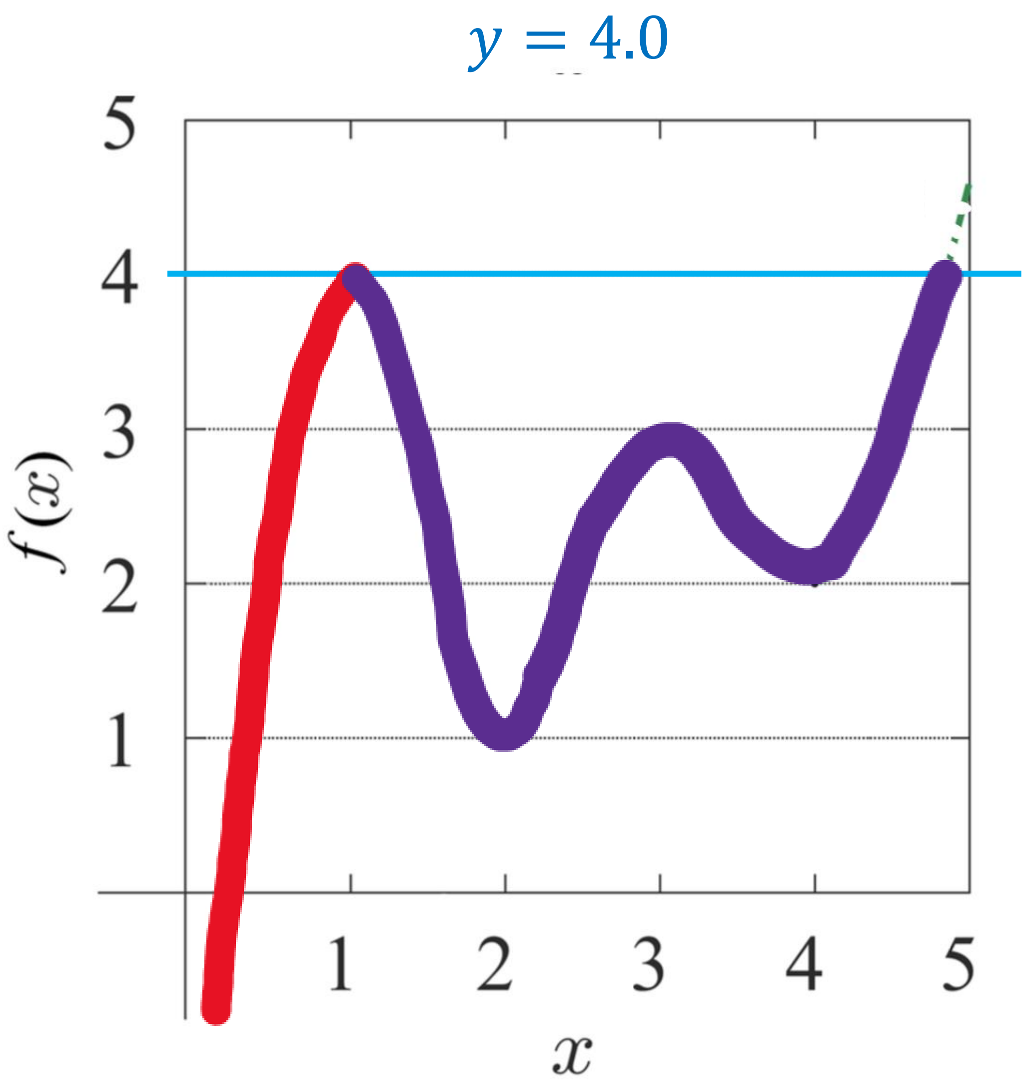
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- The red and purple components merge into one (gaps between them disappear)
- A 0-dimensional homology hole disappears (dies)
- The gap between red and purple components appears because of birth of the purple component
- So the gap is born at 1.0

• Red: born at -∞	• Purple: born at 1.0	• PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



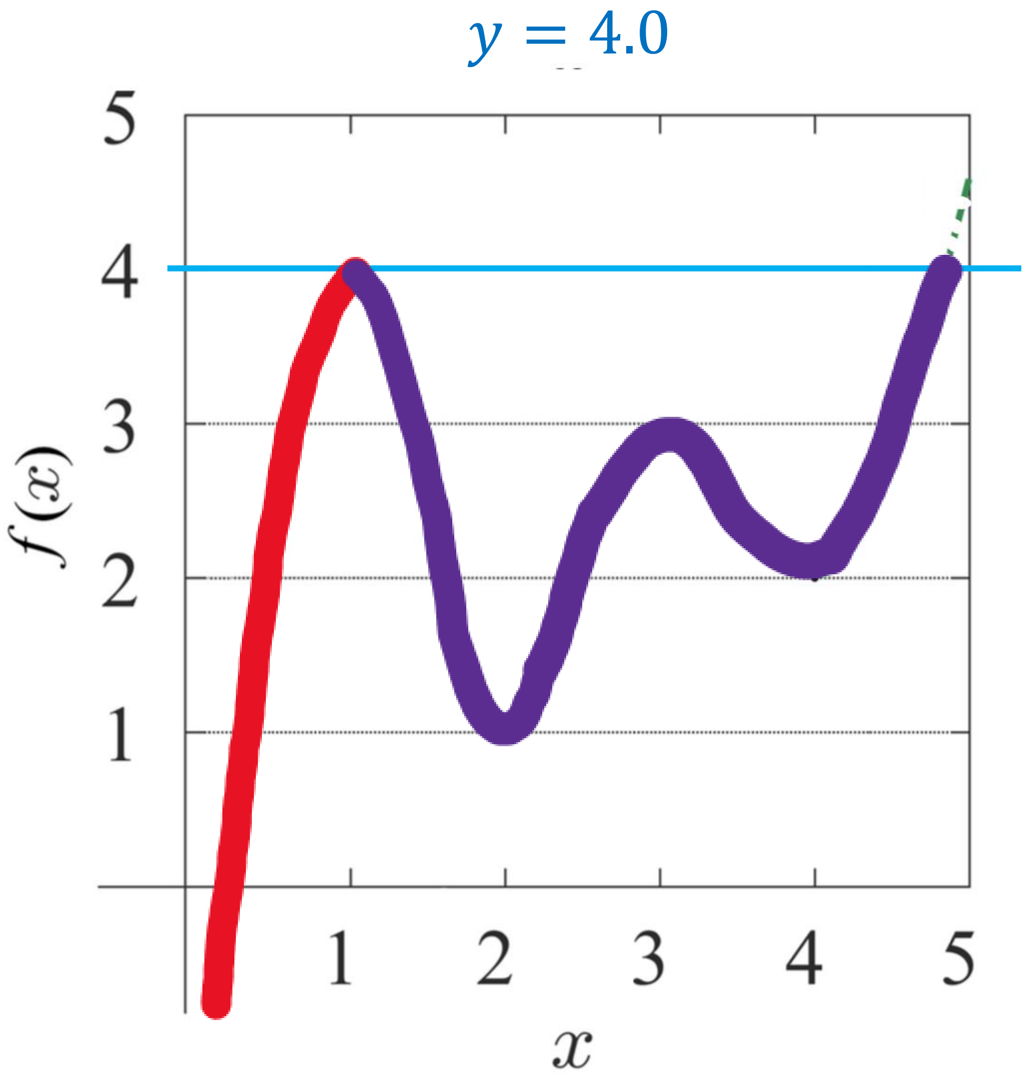
- The red and purple components merge into one (gaps between them disappear)
- A 0-dimensional homology hole disappears (dies)
- The gap between red and purple components appears because of birth of the purple component
- So the gap is born at 1.0
- So we have a 0-dimensional hole born at 1.0 and dies at 4.0

- Red: born at -∞

- Purple: born at 1.0

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



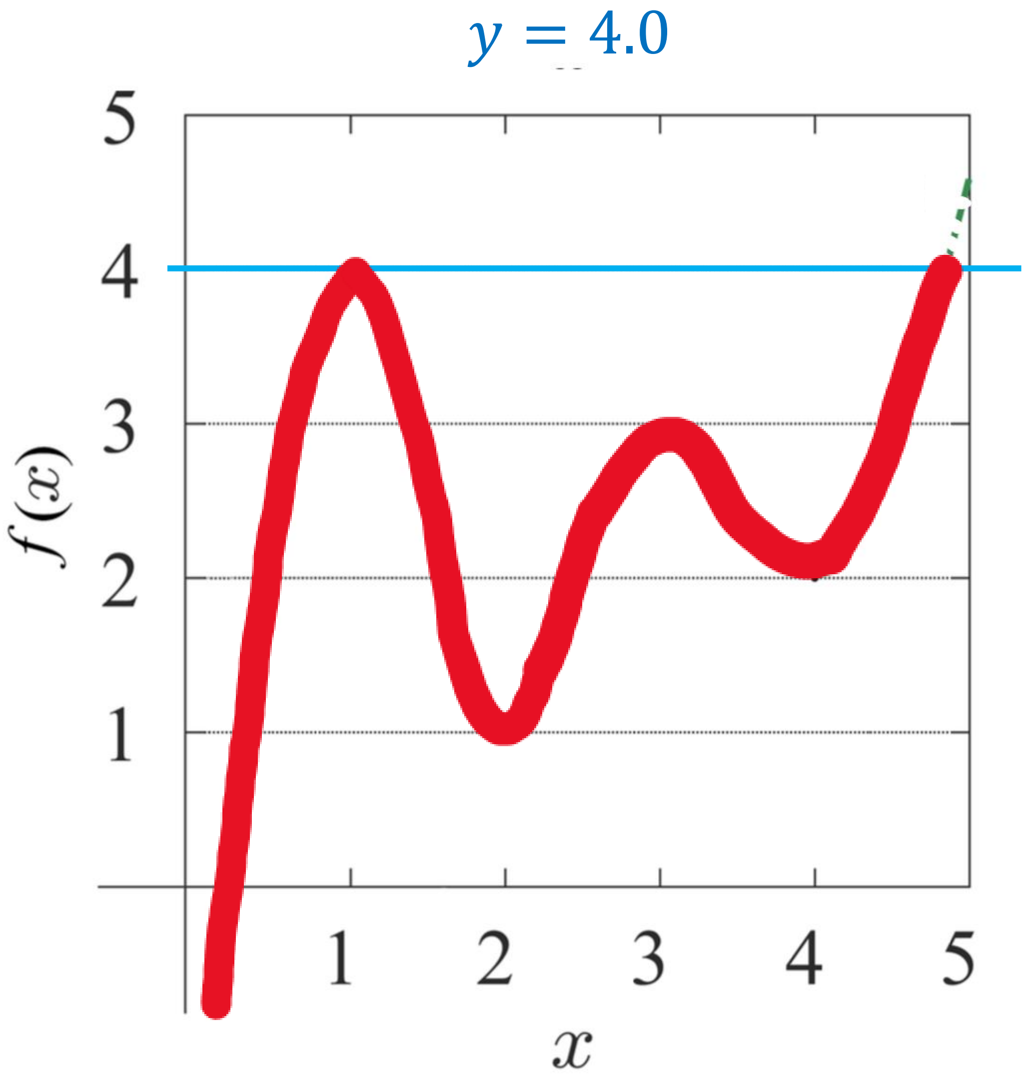
- The red and purple components merge into one (gaps between them disappear)
- A 0-dimensional homology hole disappears (dies)
- The gap between red and purple components appears because of birth of the purple component
- So the gap is born at 1.0
- So we have a 0-dimensional hole born at 1.0 and dies at 4.0

- Red: born at -∞

- PD: $(1.0, 4.0)$

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



- For the merged component, we keep the one born earlier (red), and kill the one born later (purple)

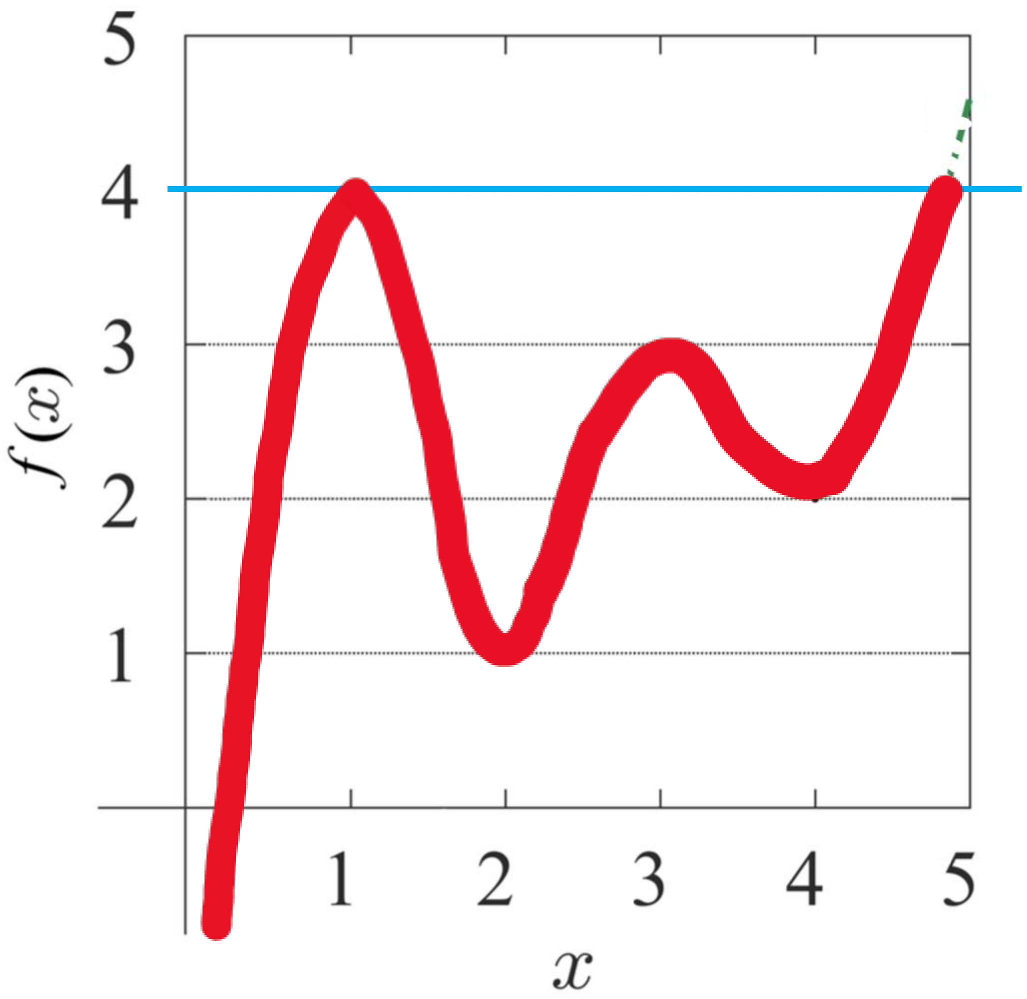
- Red: born at -∞

- PD: $(1.0, 4.0)$

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

$y = 4.0$



- For the merged component, we keep the one born earlier (red), and kill the one born later (purple)
- So we have a single red component born at -∞

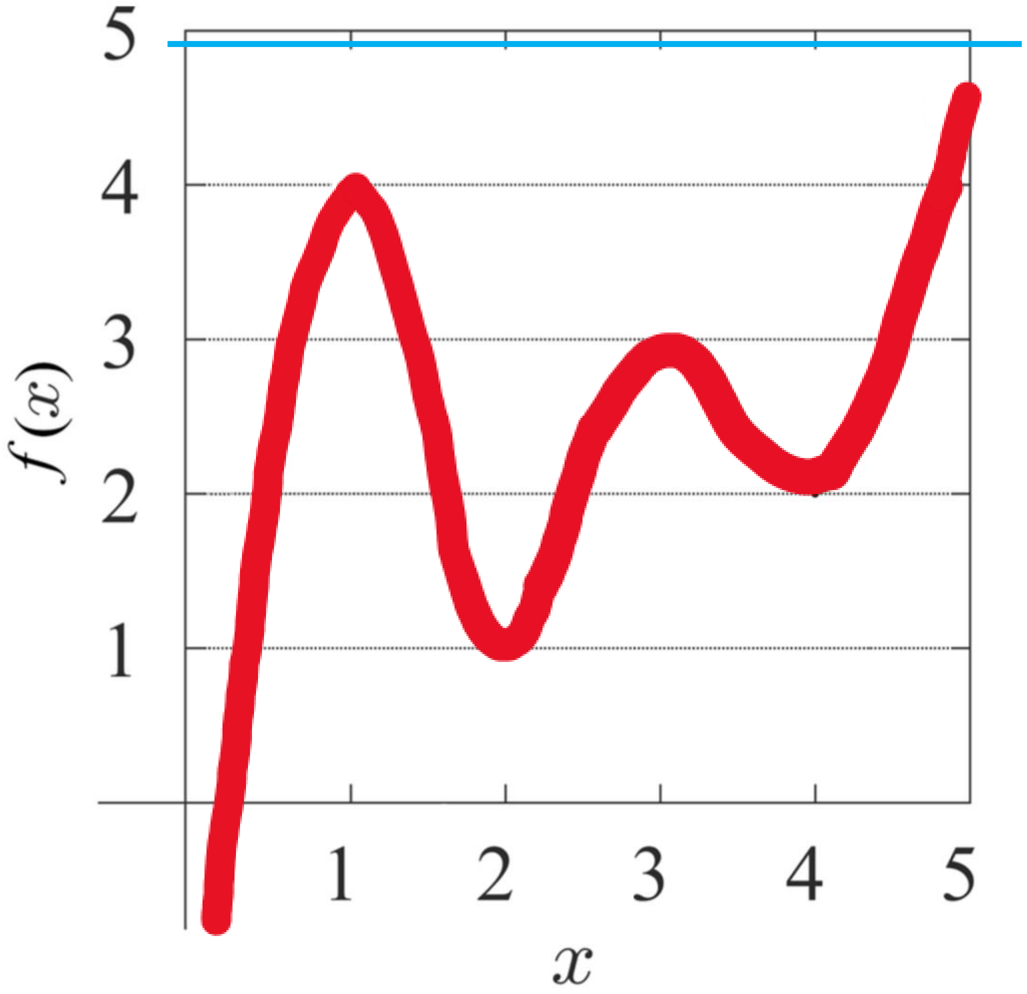
- Red: born at -∞

- PD: $(1.0, 4.0)$

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

α arbitrary large



- As the value for the line keeps on increasing to $+\infty$, the single red component will keep on persisting
- So we have the red component born at $-\infty$ and dies at $+\infty$

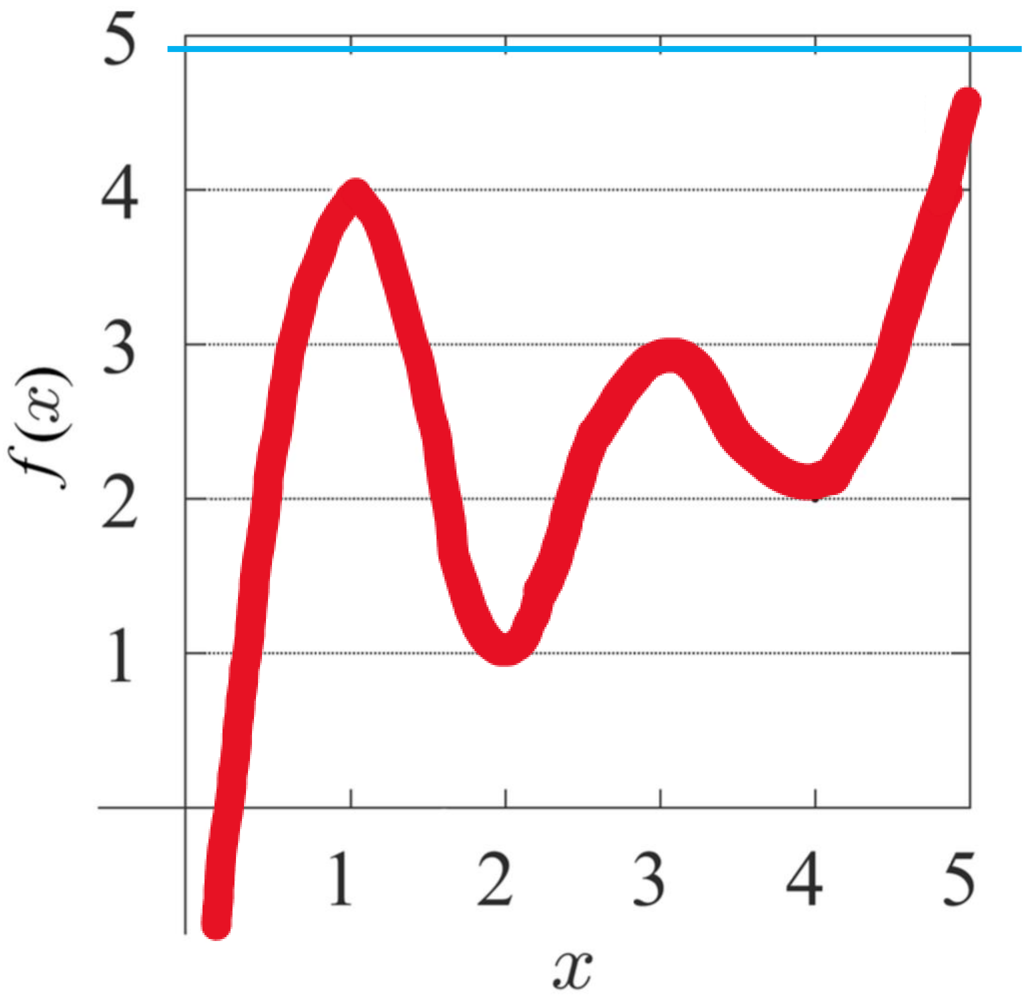
- Red: born at -∞

- PD: $(1.0, 4.0)$

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

α arbitrary large



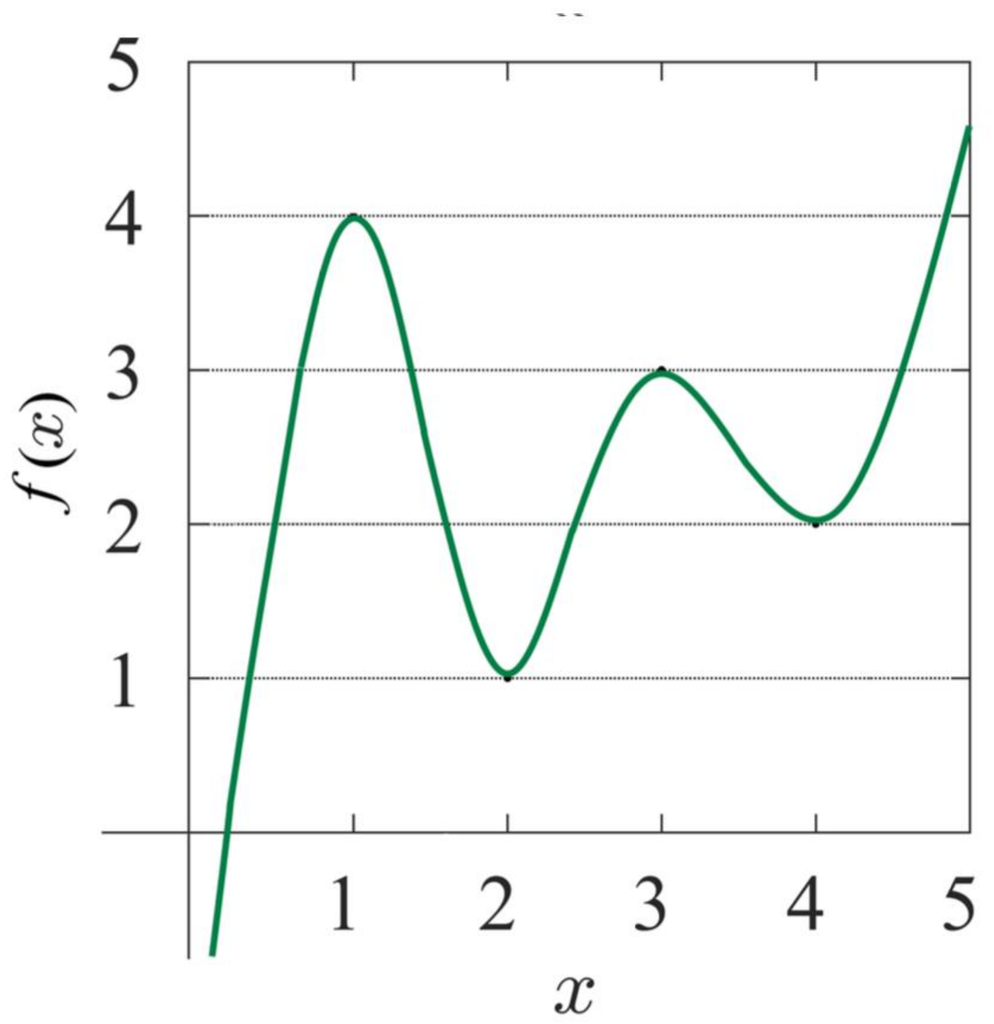
- As the value for the line keeps on increasing to $+\infty$, the single red component will keep on persisting
- So we have the red component born at $-\infty$ and dies at $+\infty$

- PD: $(-\infty, +\infty)$

- PD: $(1.0, 4.0)$

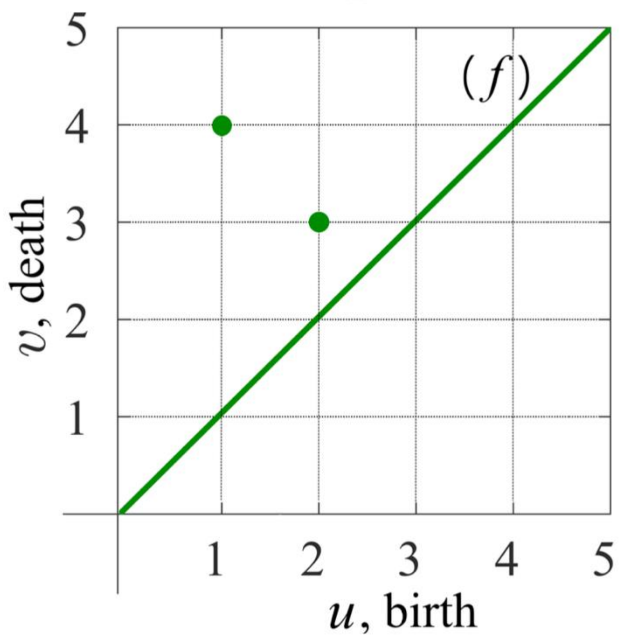
- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



Summary:

- We have three points in the 0-dimension PD
- Each point is tracking the birth and death of a connect component (or gap in between)

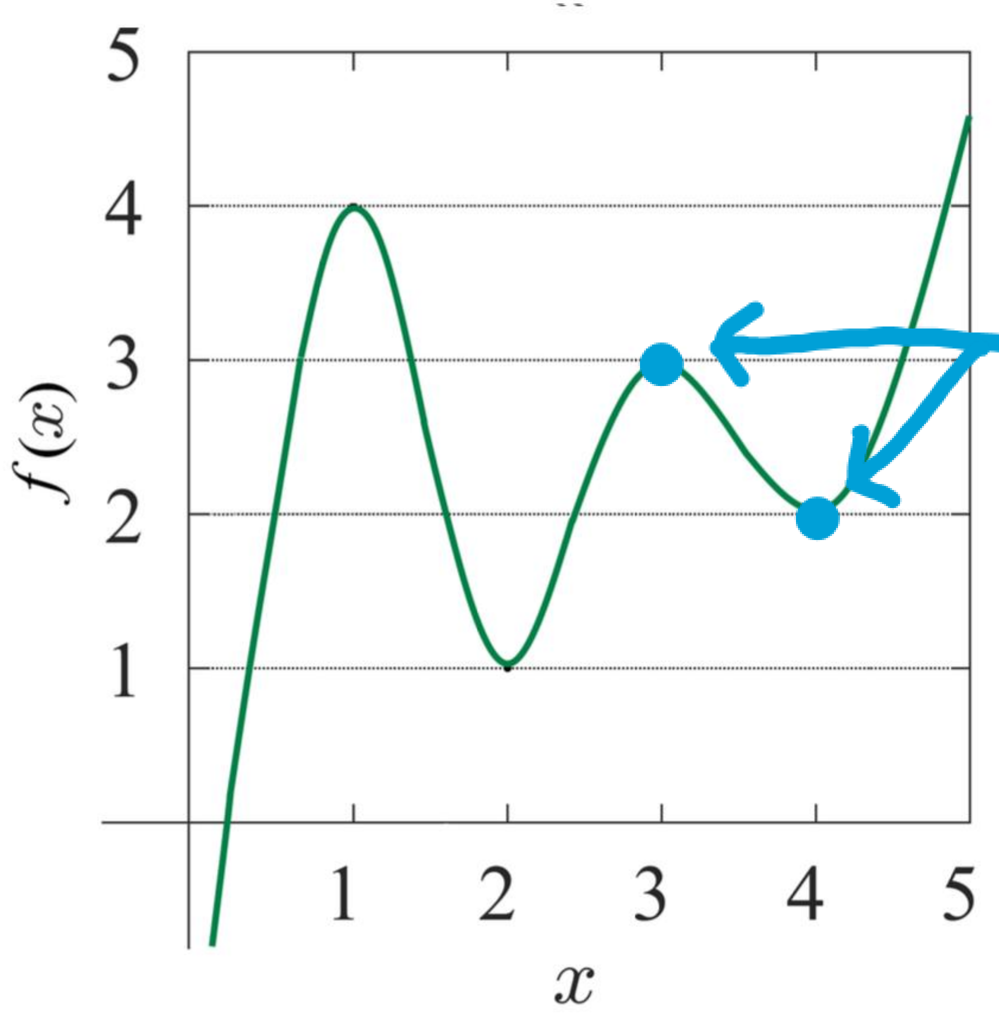


- PD: $(-\infty, +\infty)$

- PD: $(1.0, 4.0)$

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence



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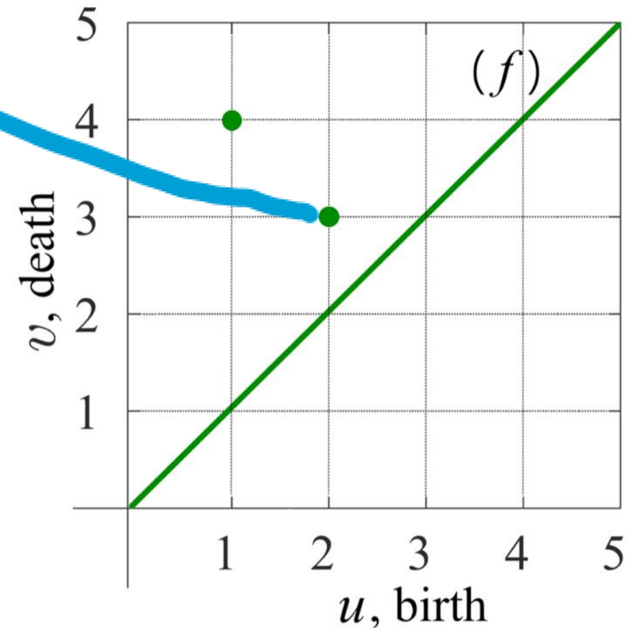

- PD: $(-\infty, +\infty)$

- PD: $(1.0, 4.0)$

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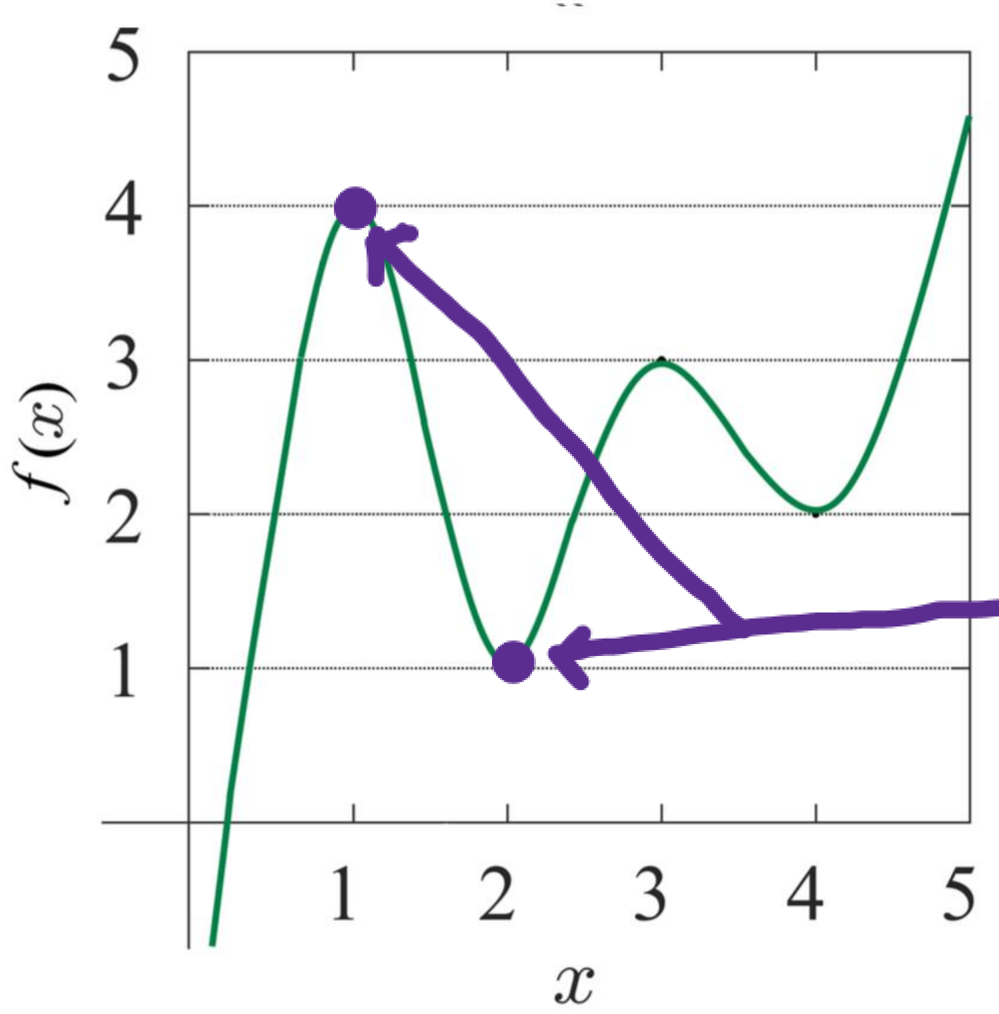
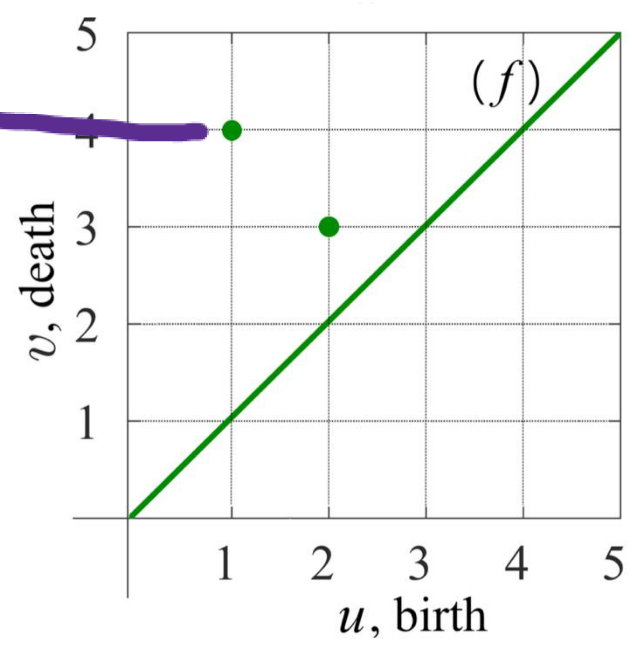




- PD: $(-\infty, +\infty)$

- PD: $(1.0, 4.0)$

- PD: $(2.0, 3.0)$

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Online resources

- A webpage for visualizing 0–th PD: https://gjkoplik.github.io/pers-hom-examples/0d_pers_2d_data_widget.html

A similar but more involved example

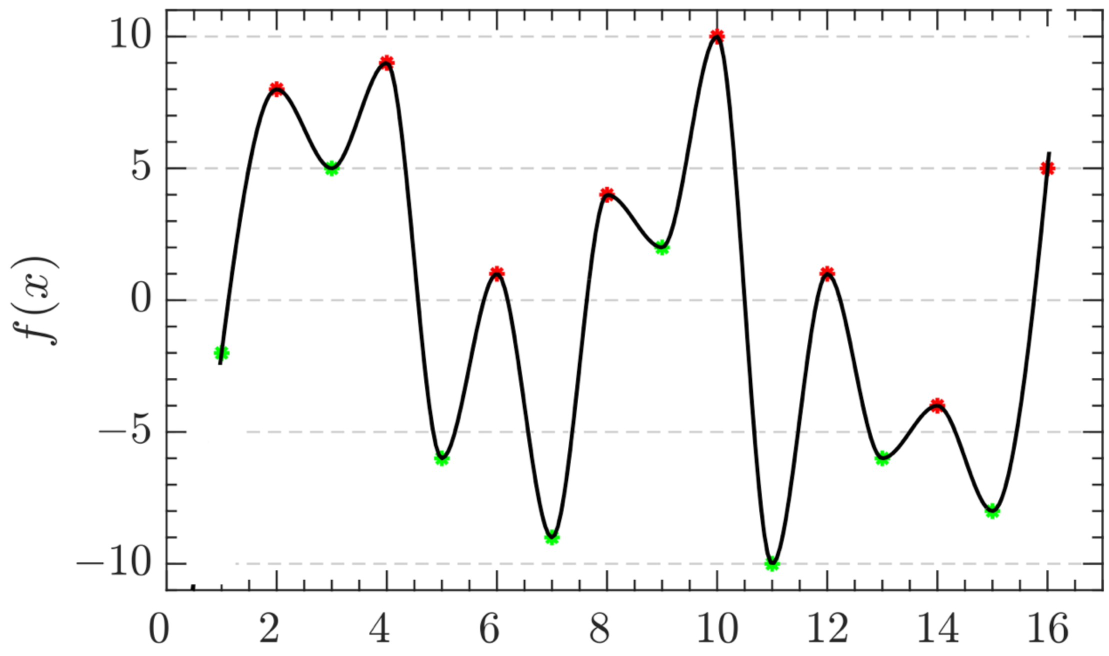
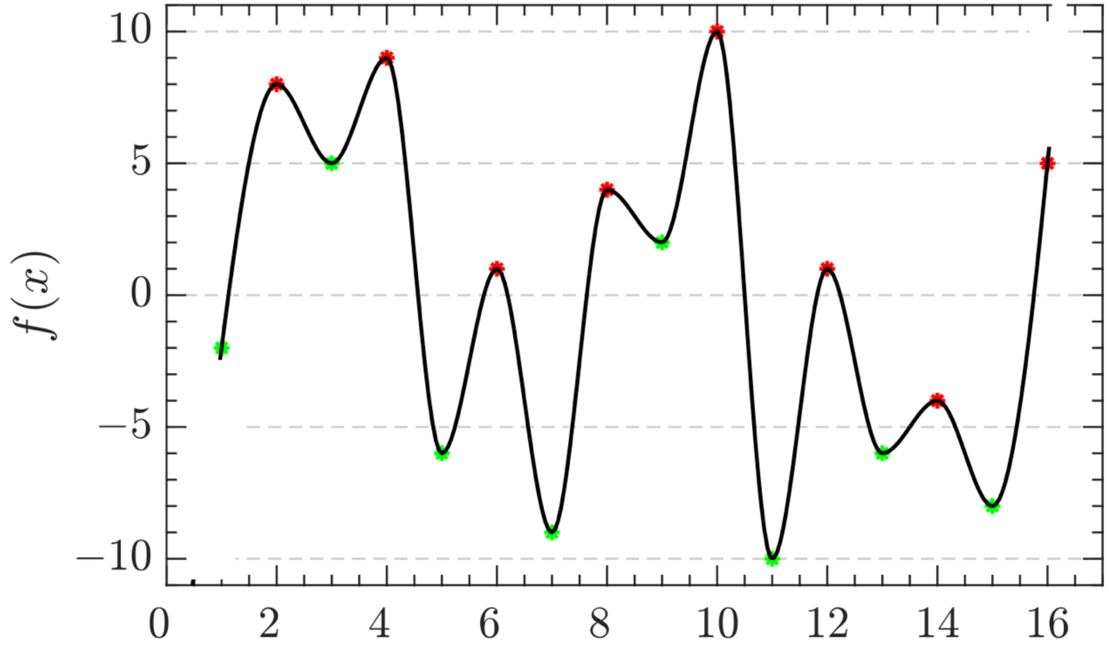


Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

A similar but more involved example



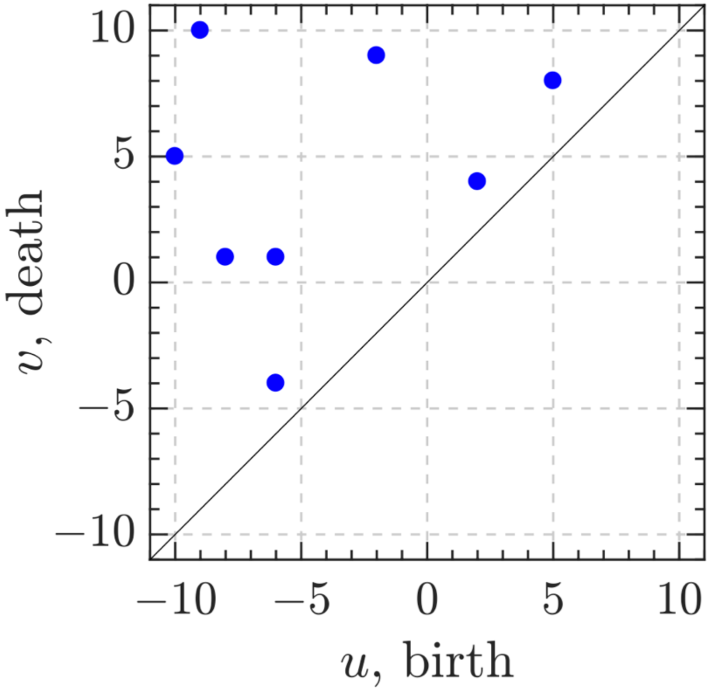


Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Persistent homology on 2D function

- Let’s visualize another example on a 2D function

$$f: \mathbb{R}^2 \to \mathbb{R}$$

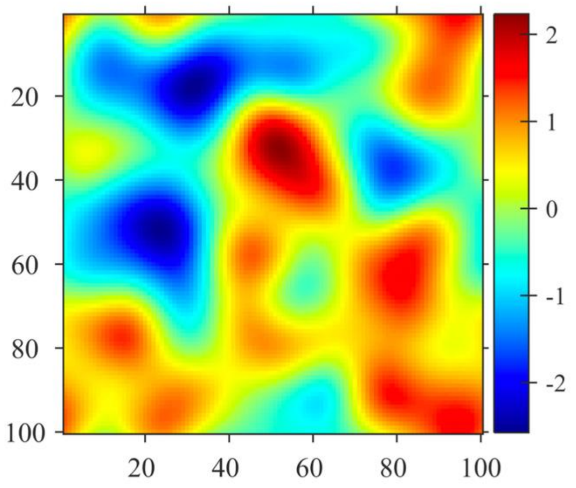
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Persistent homology on 2D function

- Let's visualize another example on a 2D function

$$f: \mathbb{R}^2 \to \mathbb{R}$$

- Right is an example where the value is indicated by color (red for high and blue for low)



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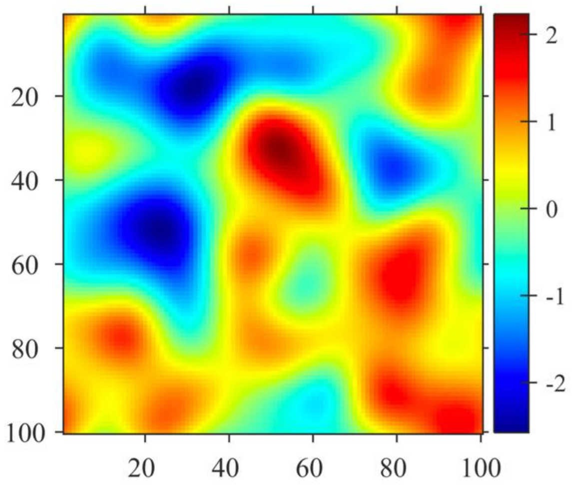
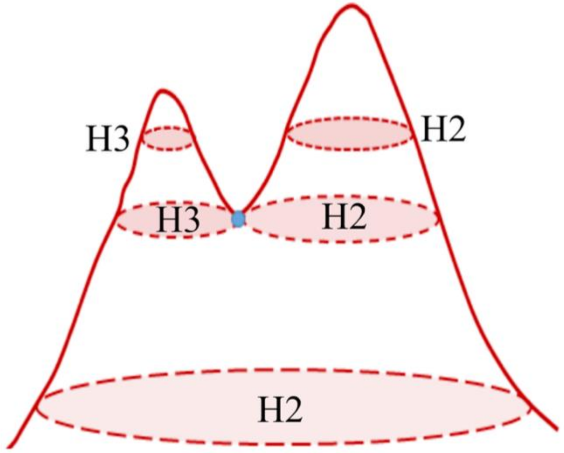
Persistent homology on 2D function

- Let's visualize another example on a 2D function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

- Right is an example where the value is indicated by color (red for high and blue for low)
- You can also treat the value on each point of $\mathbb{R}^2$ as a "height", and plot the function like the bottom one





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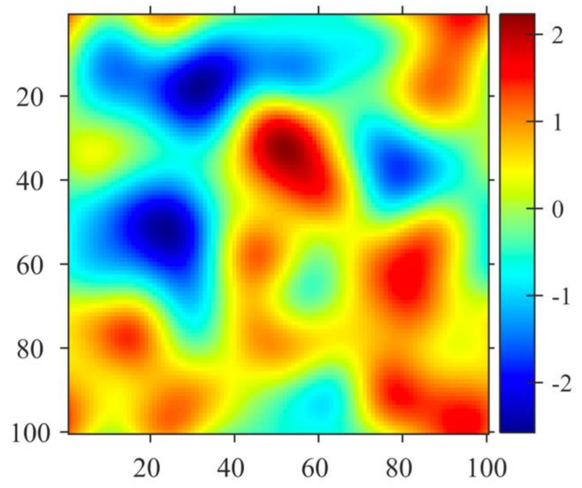
Persistent homology on 2D function

- Let's visualize another example on a 2D function

$$f: \mathbb{R}^2 \to \mathbb{R}$$

- Right is an example where the value is indicated by color (red for high and blue for low)
- You can also treat the value on each point of $\mathbb{R}^2$ as a "height", and plot the function like the bottom one
- Similar to the previous 1D function, as we increase the value α, we consider the part (subset) of the domain $\mathbb{R}^2$ whose values are below α



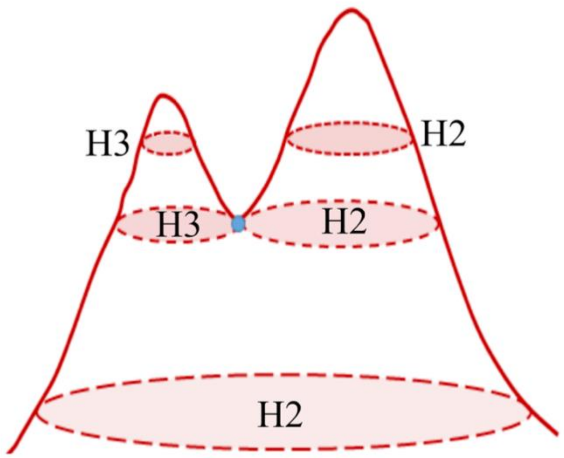


Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Persistent homology on 2D function

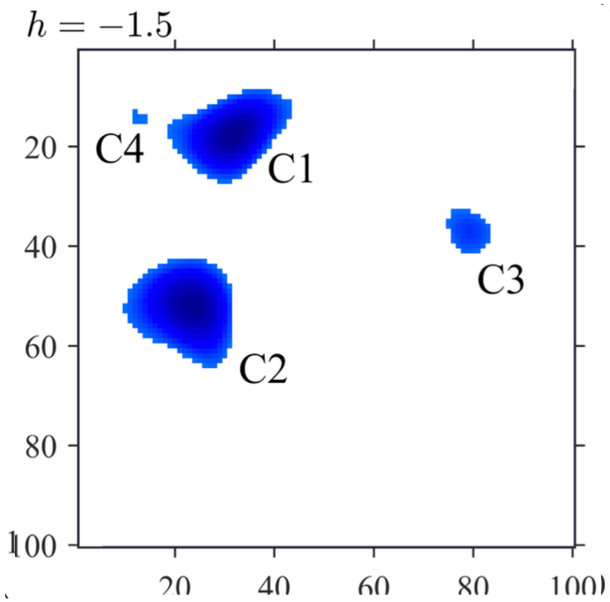


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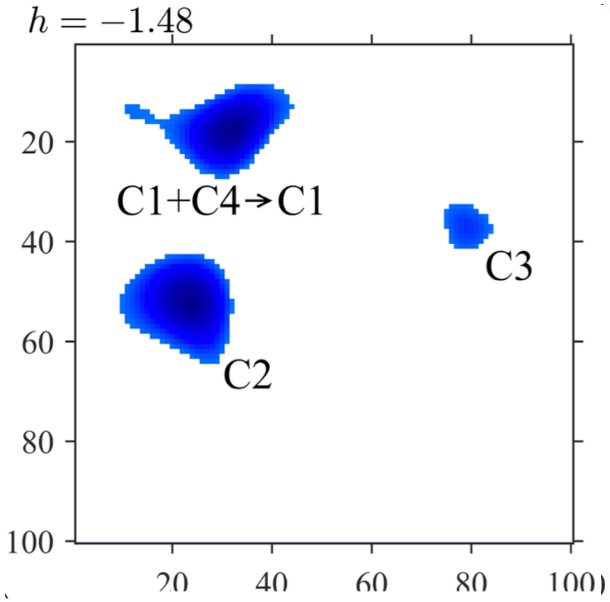
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Persistent homology on 2D function



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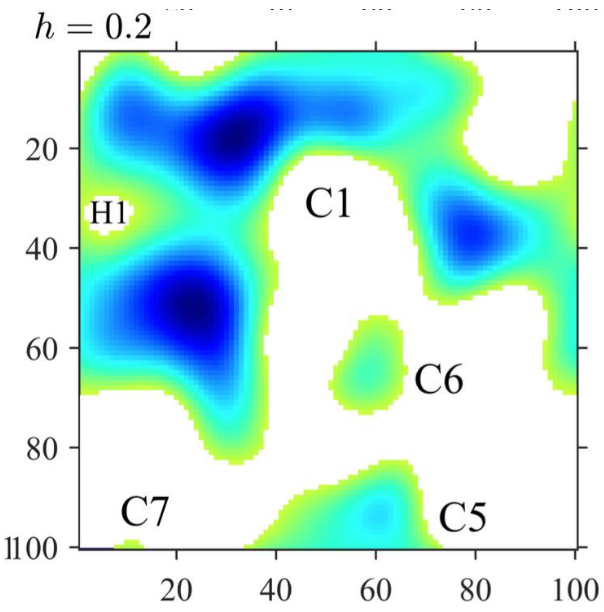
Persistent homology on 2D function



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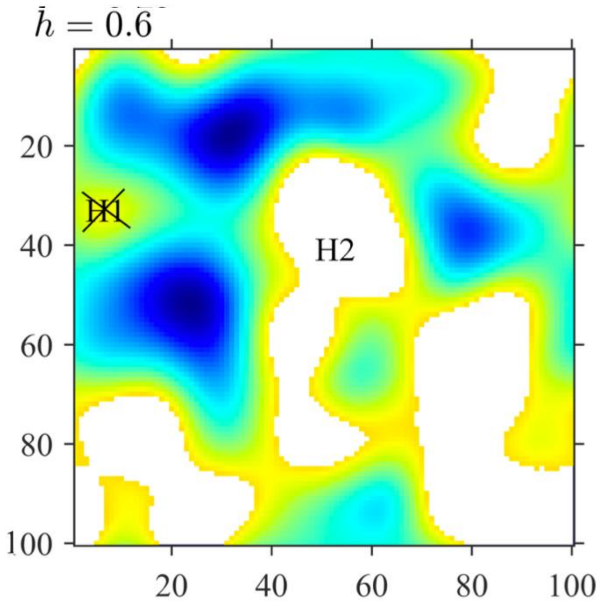
Persistent homology on 2D function



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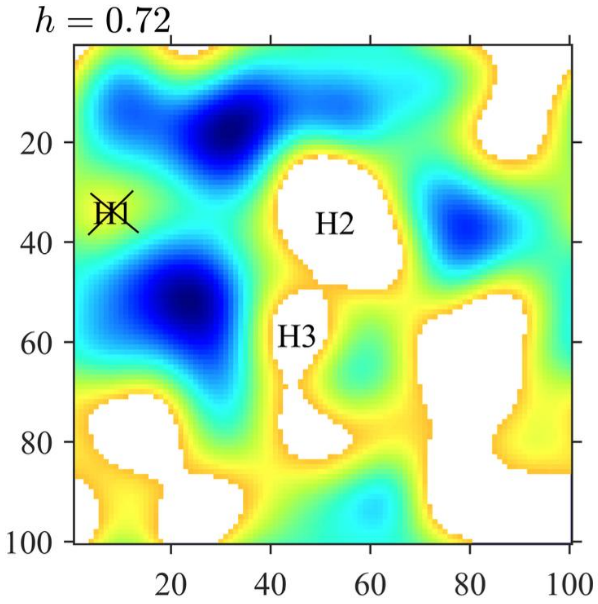

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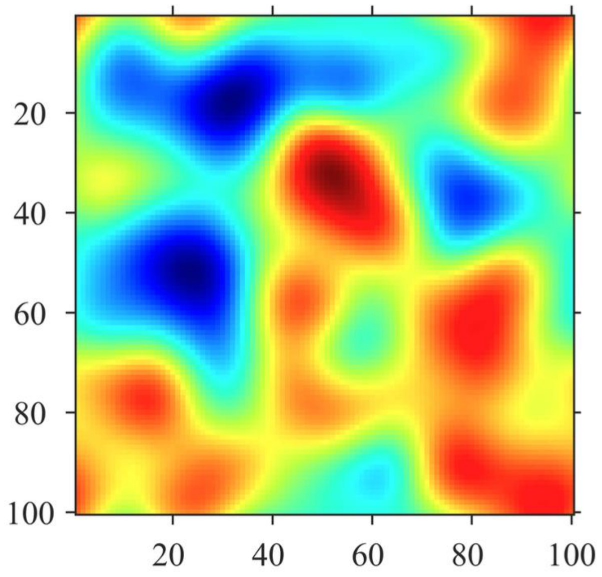

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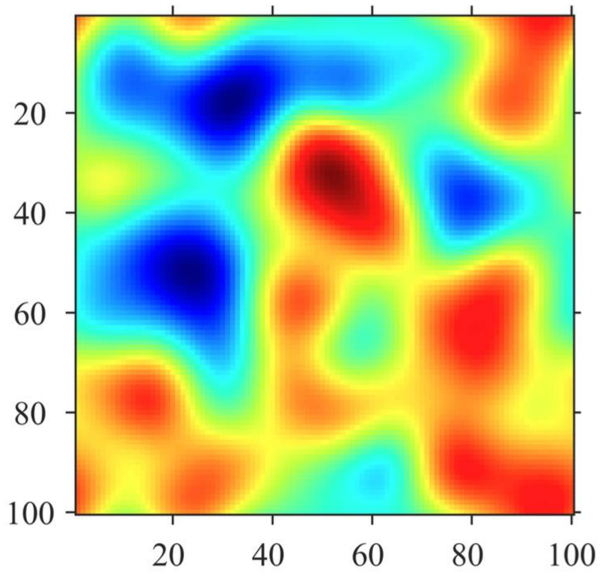
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

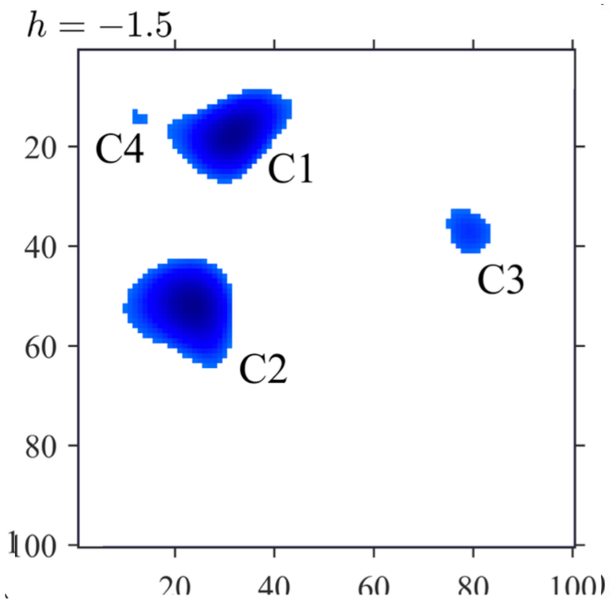
Persistent homology on 2D function



- Similar to the previous 1D function, as we increase the value α, we consider the part (subset) of the domain $\mathbb{R}^2$ whose values are below α
- Now let's track the birth and death of 0D/1D holes

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

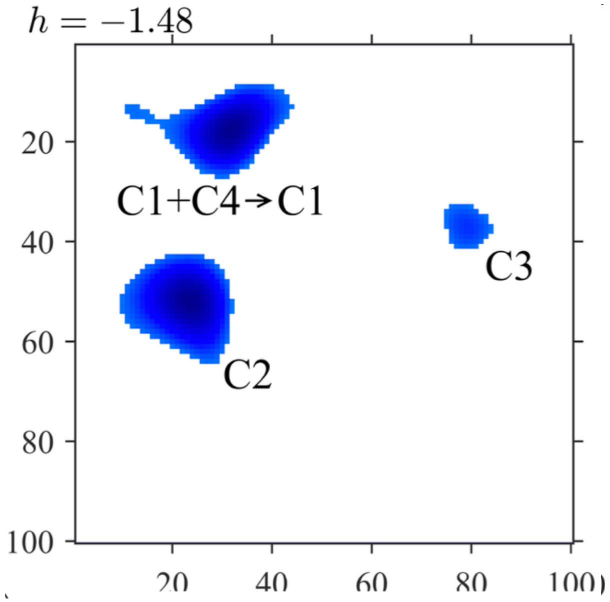
Persistent homology on 2D function



- Four connected components are born at different values
- (Will not display the birth of each component though)

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

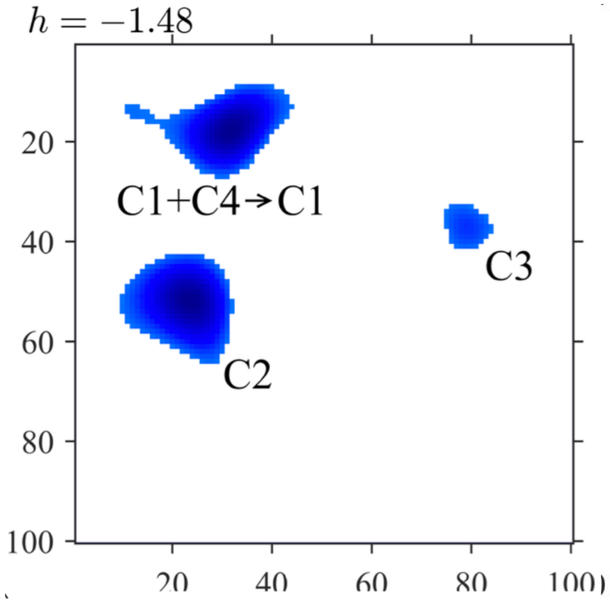
Persistent homology on 2D function



- C1 and C4 merged into the same connected component, thus the gap between them is filled

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Persistent homology on 2D function



- C1 and C4 merged into the same connected component, thus the gap between them is filled
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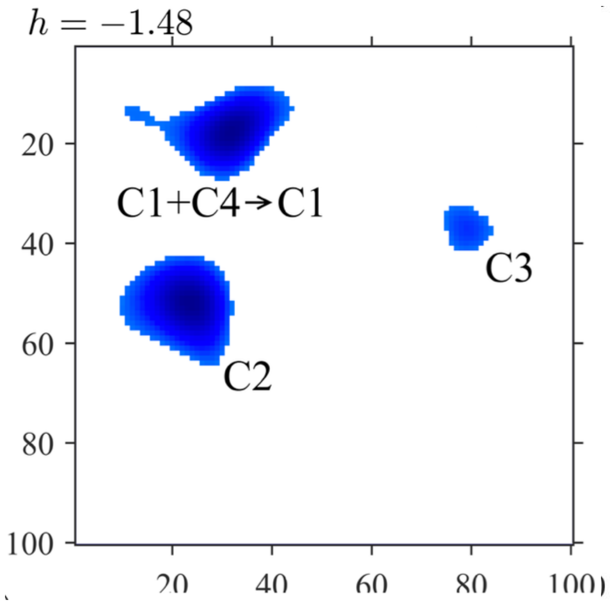
Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

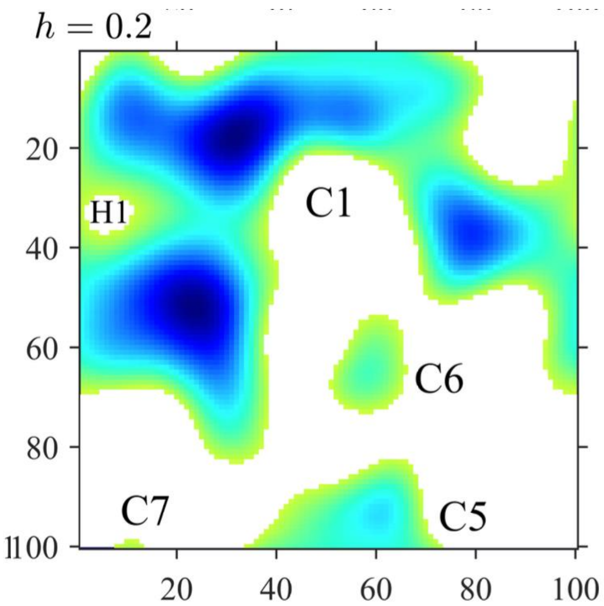
Persistent homology on 2D function



- C1 and C4 merged into the same connected component, thus the gap between them is filled
- Since C1 is born earlier, we keep C1 and kill C4 (the rule adopted by persistent homology)
- We then add a point (b, d) to the 0-d PD where b is the value in which C4 is born and d is current values where C4 dies (merges with other)

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

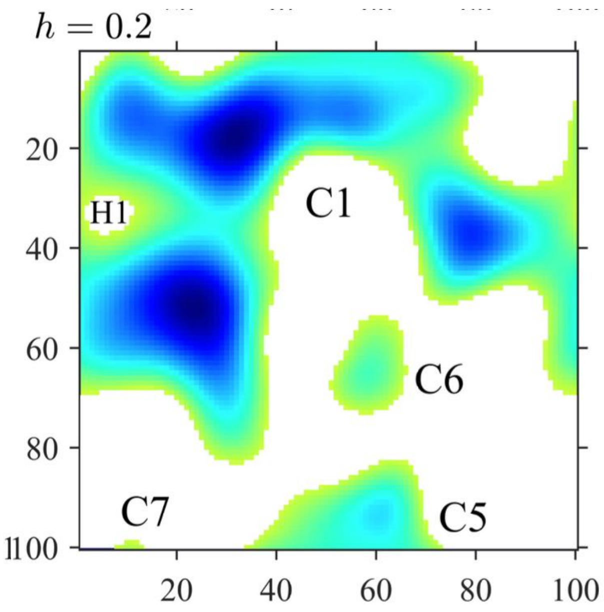
Persistent homology on 2D function



- As the value keep increasing, C1, C2 and C3 merged into the same connected component, producing two additional points in 0-d PD

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

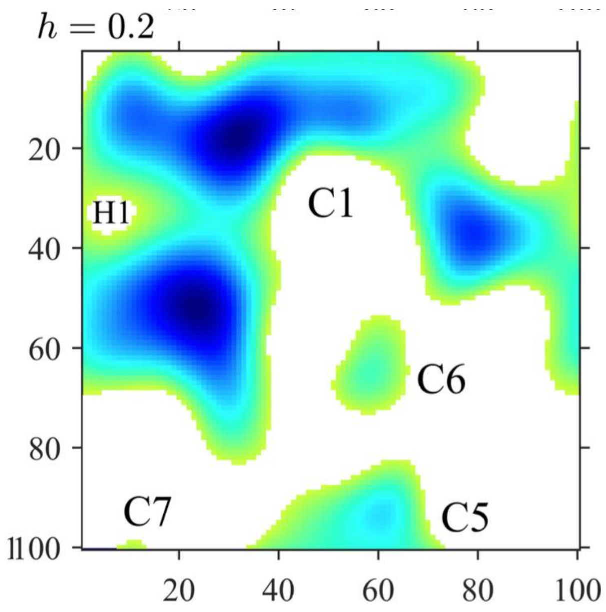
Persistent homology on 2D function



- As the value keep increasing, C1, C2 and C3 merged into the same connected component, producing two additional points in 0-d PD
- Three additional components C5, C6 and C7 are born

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

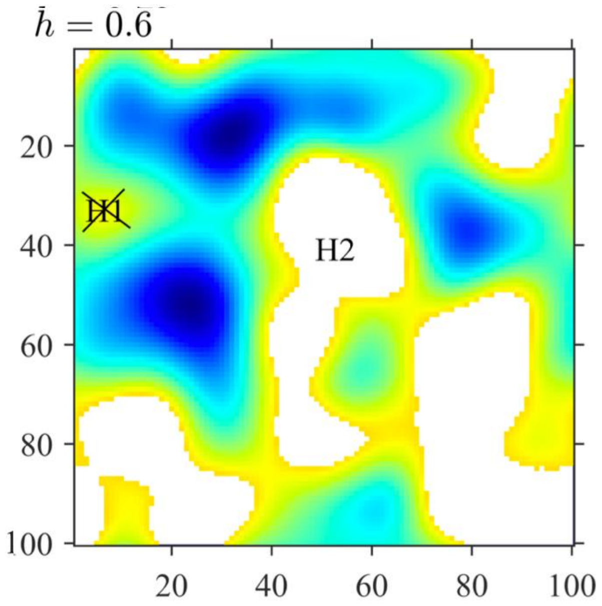
Persistent homology on 2D function



- As the value keep increasing, C1, C2 and C3 merged into the same connected component, producing two additional points in 0-d PD
- Three additional components C5, C6 and C7 are born
- Also, a 1-dimensional hole H1 is born

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

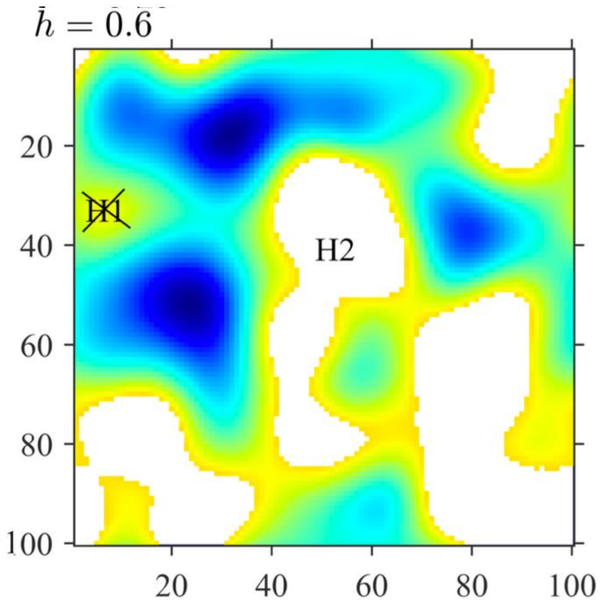
Persistent homology on 2D function



- H1 dies, producing a point in the 1-d PD

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

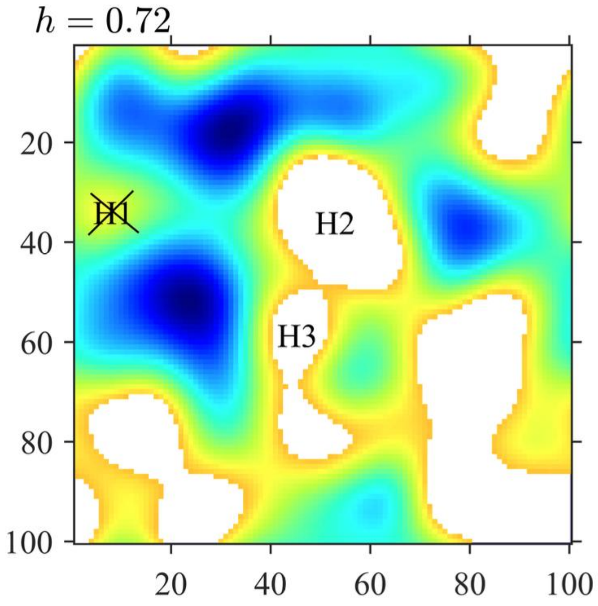
Persistent homology on 2D function



- H1 dies, producing a point in the 1-d PD
- A 1-dimensional hole H2 is born

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

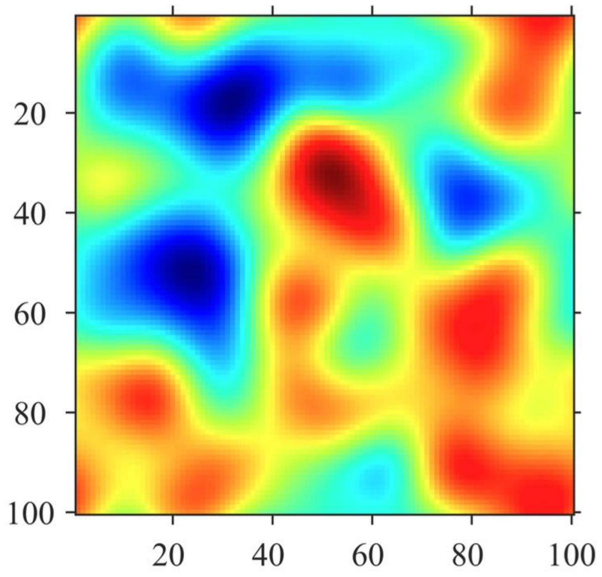
Persistent homology on 2D function



- A 1-dimensional hole H3 is born

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

Persistent homology on 2D function

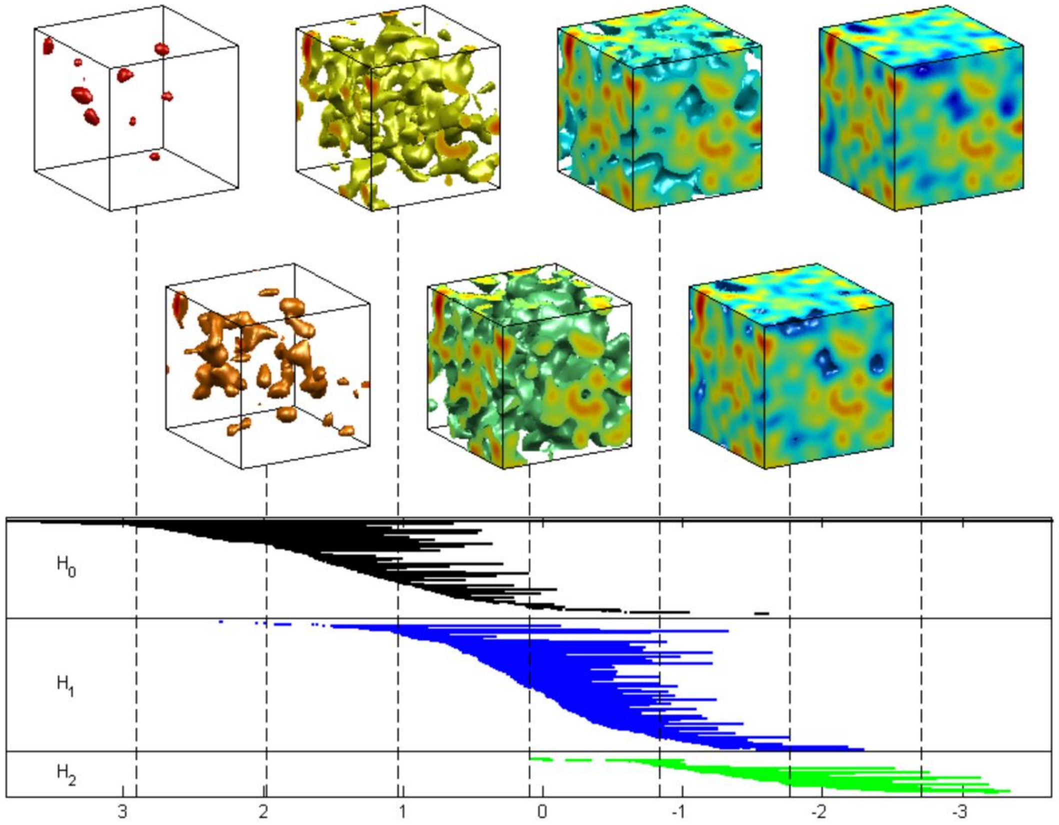


- H2 and H3 die, producing two additional points in the 1-d PD

Image source: Makarenko et al. Topological data analysis and diagnostics of compressible MHD turbulence

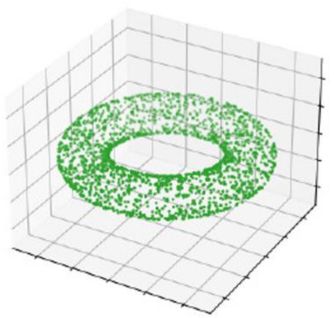
Persistent homology on 3D function

- We can also extend the prev. idea and define persistence on 3D function: $f: \mathbb{R}^3 \to \mathbb{R}$
- Similarly, as we increase the value α, we consider the part (subset) of the domain $\mathbb{R}^3$ (or a cube) whose values are below α

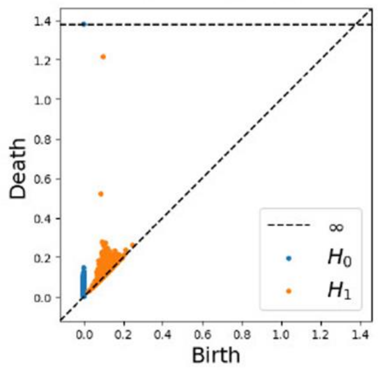


Adler, Robert J., Omer Bobrowski, Matthew S. Borman, Eliran Subag, and Shmuel Weinberger. "Persistent homology for random fields and complexes."

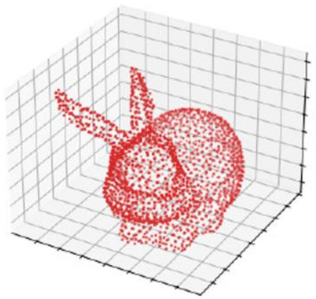
Persistence diagrams for differentiating point clouds



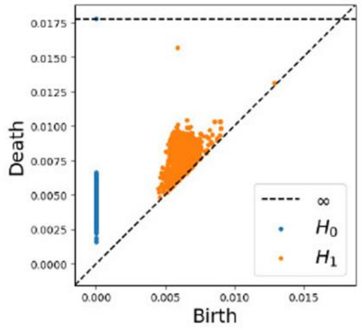
(c) *2000 points on a 3D torus.*



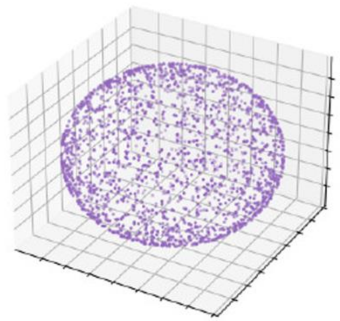
(d) *Corresponding diagram.*



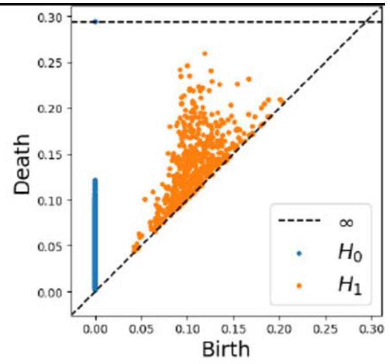
(e) *Stanford bunny with 1889 points.*



(f) *Corresponding diagram.*



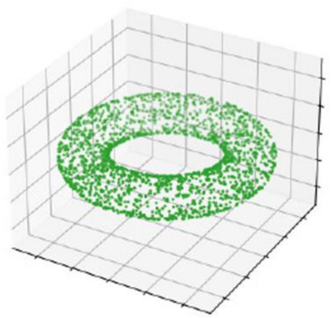
(a) *2000 points on a 3D sphere.*



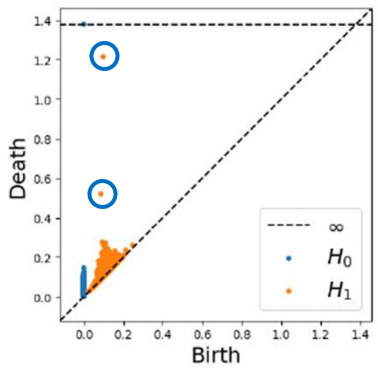
(b) *Corresponding diagram.*

Image source: Wang et al. Stability for Inference with Persistent Homology Rank Functions

Persistence diagrams for differentiating point clouds



(c) *2000 points on a 3D torus.*



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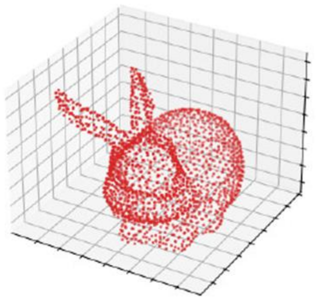

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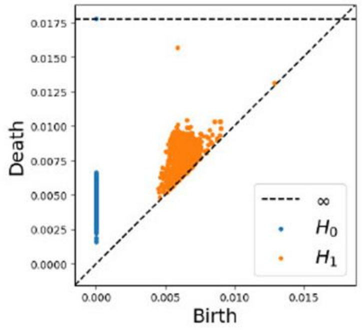
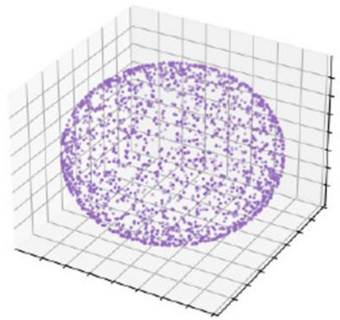


(f) *Corresponding diagram.*

Corresponding to meridian and longitude



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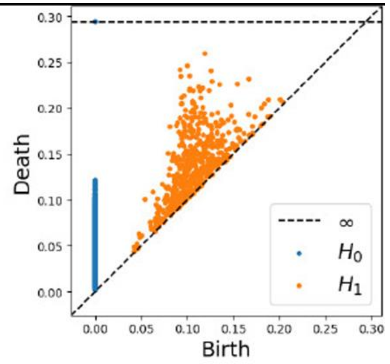
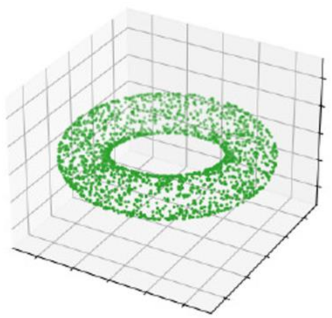


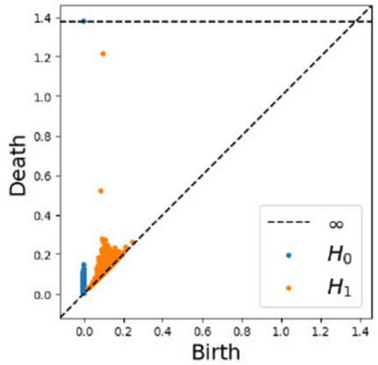
(b) *Corresponding diagram.*

Image source: Wang et al. Stability for Inference with Persistent Homology Rank Functions

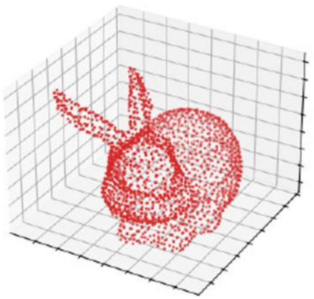
Persistence diagrams for differentiating point clouds



(c) *2000 points on a 3D torus.*



(d) *Corresponding diagram.*



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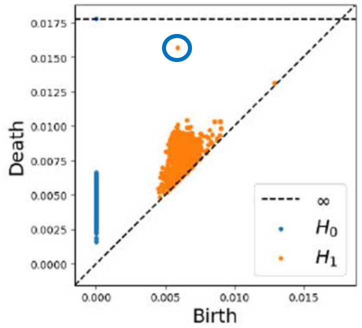
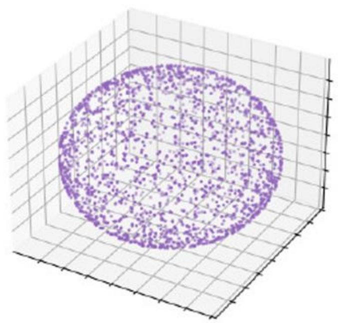


(f) *Corresponding diagram.*

Corresponds to the "crust" of the bunny which is a 2D hole



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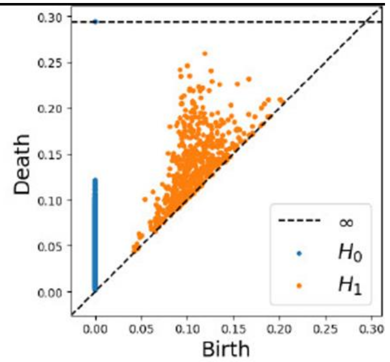

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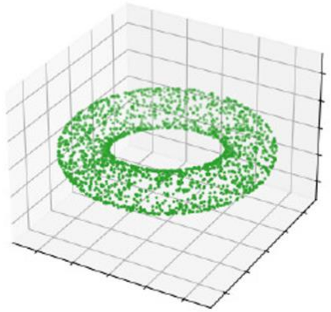
Persistence diagrams for differentiating point clouds



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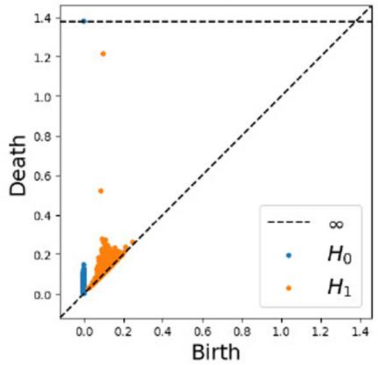
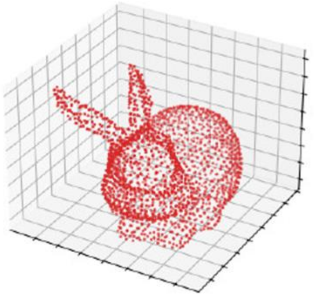


(d) *Corresponding diagram.*



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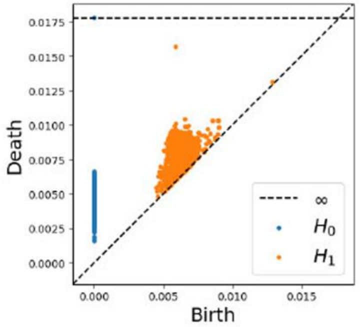
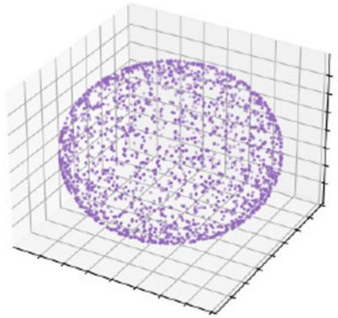


(f) *Corresponding diagram.*



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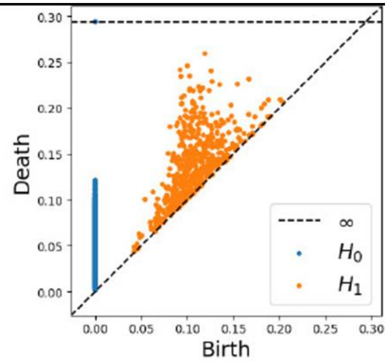

(b) *Corresponding diagram.*

This is a solid ball which has no interesting holes

Image source: Wang et al. Stability for Inference with Persistent Homology Rank Functions

Persistence barcode

Recall:

- **Definition**: A **persistence diagram** (PD) is a set of points on the 2D plane above the diagonal such that for each point (b, d):
  - b indicates birth value (the α value in which the feature is born)
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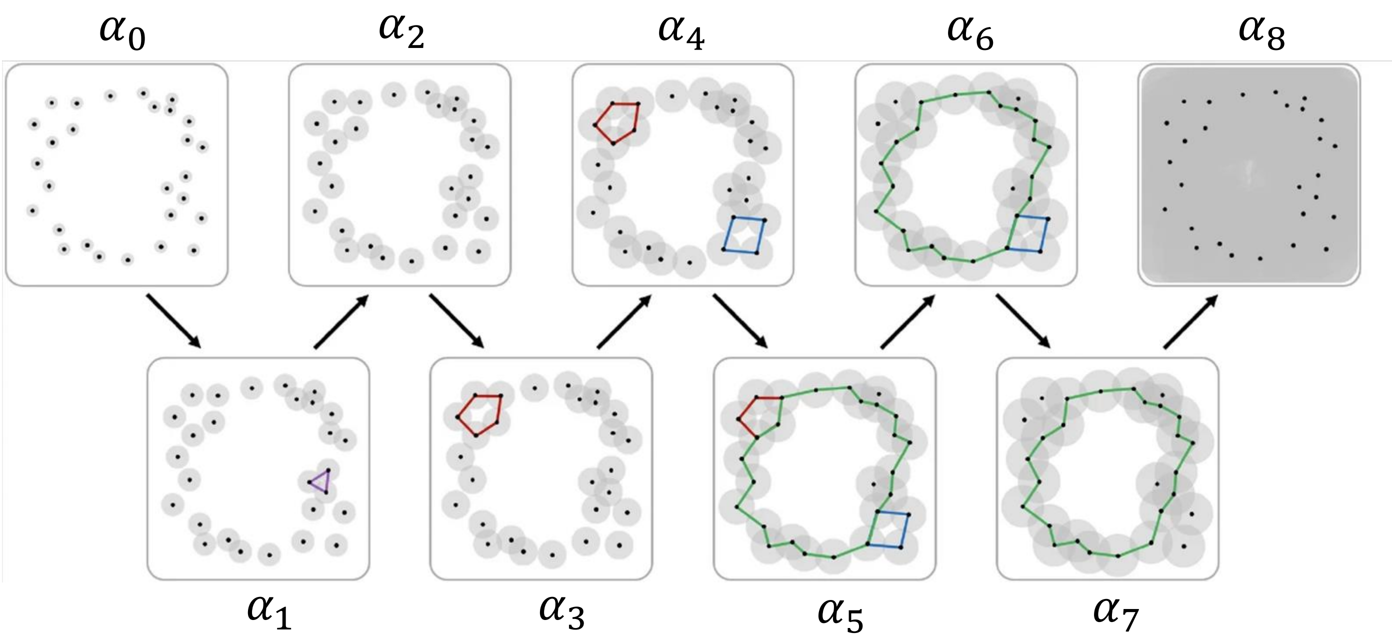
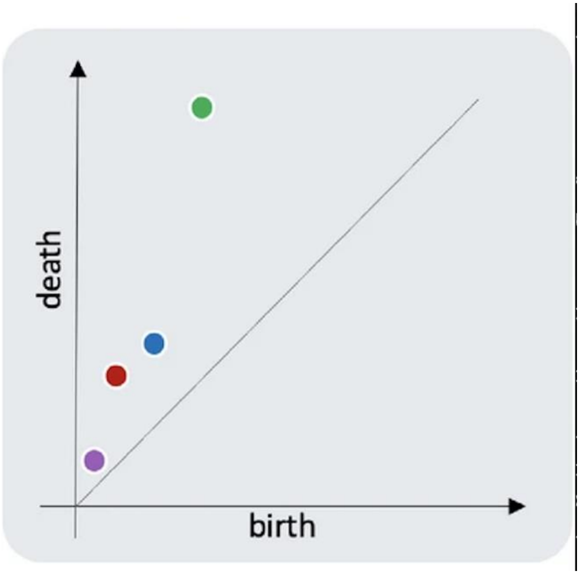
- Sometimes drawing points in PD as interval is a very helpful for visualizing the change of homological features in your data
- Notice that we sometimes use the terms "persistence diagram" and "persistence barcode" interchangeable, i.e., we may call a point in a PD also an interval.

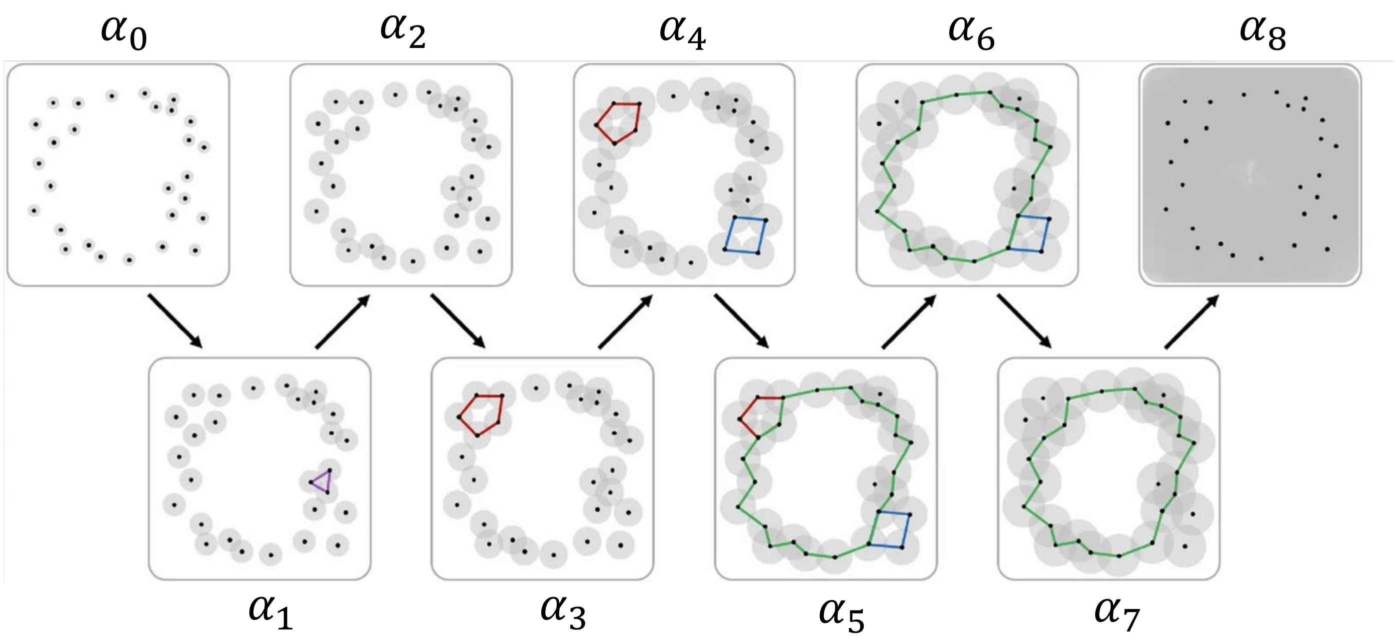
Example

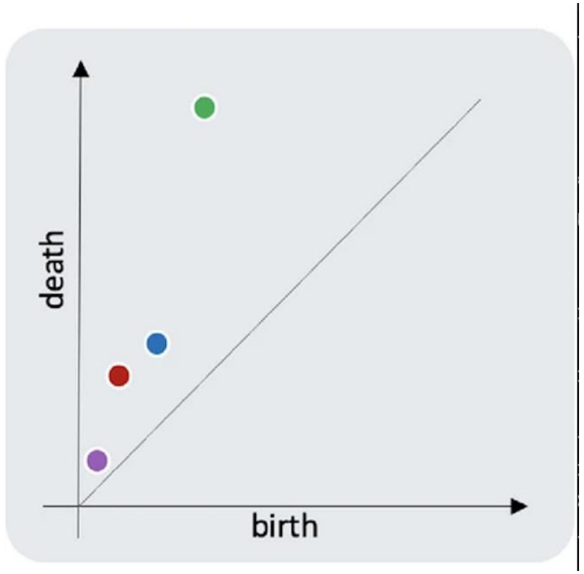




- Corresponding barcode:

Example





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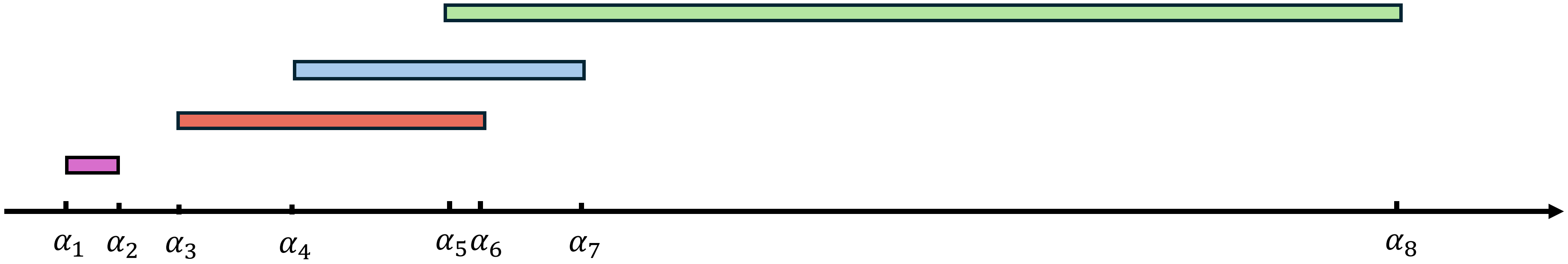

Another example