Simplicial Complexes and Homology

Tao Hou, University of Oregon

Topological invariant

- Recall that a topological invariant is a type of characteristics for spaces that are preserved by topological equivalence (homeomorphism)

- We shall eventually look at the topological invariant called homology, which people heavily rely on in TDA

- But before looking at that, let's first we look at a simpler invariant called Euler characteristic

Euler characteristics

- Here we consider Polyhedron, which is a 3D object whose building blocks are
  - Polygonal faces (2d)
  - Edges (1d)
  - Vertices (0d)

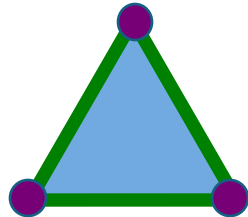image source: Wikipedia
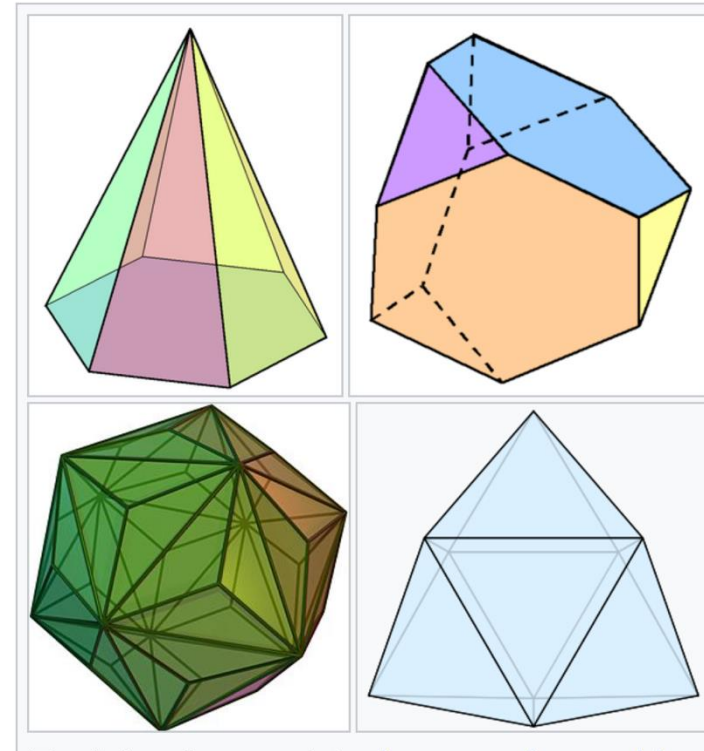
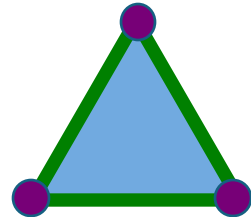We wish to count:

vertex

edge

face

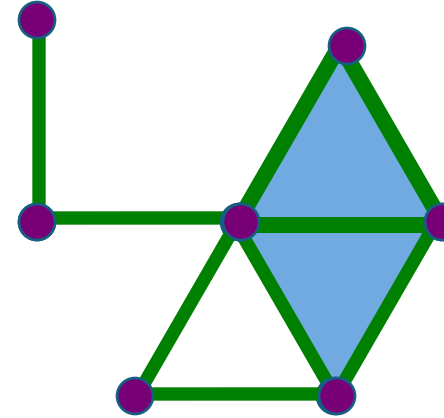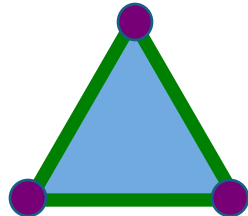We wish to count:

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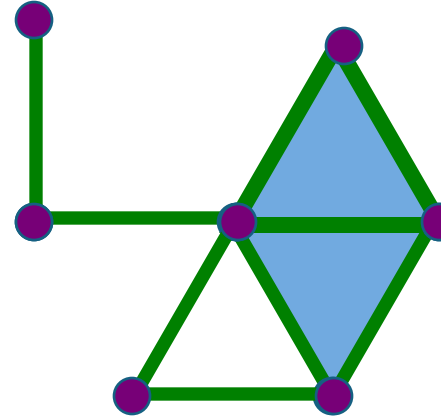face

Example:

We wish to count:

vertex

edge

face

Example:

7 vertices,
9 edges,
2 faces.

Euler characteristic (simple form):

χ = number of vertices – number of edges + number of faces
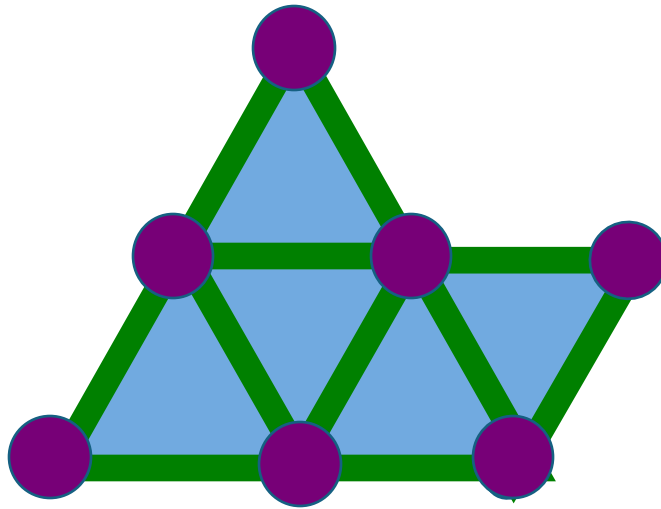
Or in short-hand,

$$\chi = |V| - |E| + |F|$$

where V = set of vertices
E = set of edges
F = set of faces

$$\chi = |V| - |E| + |F|$$

χ = 3 − 3 + 1 = 1
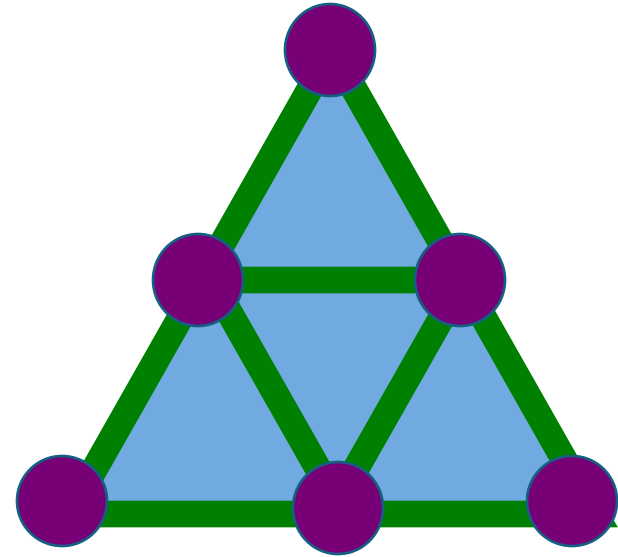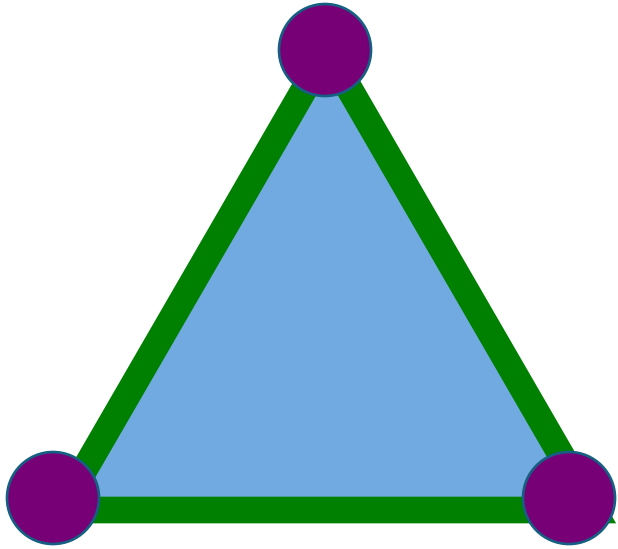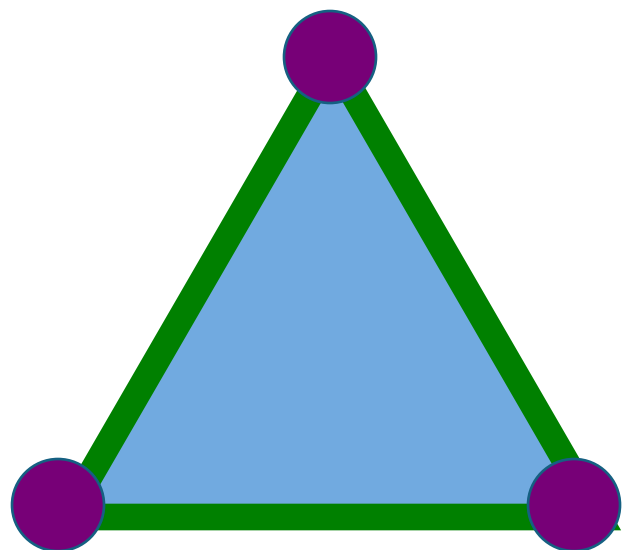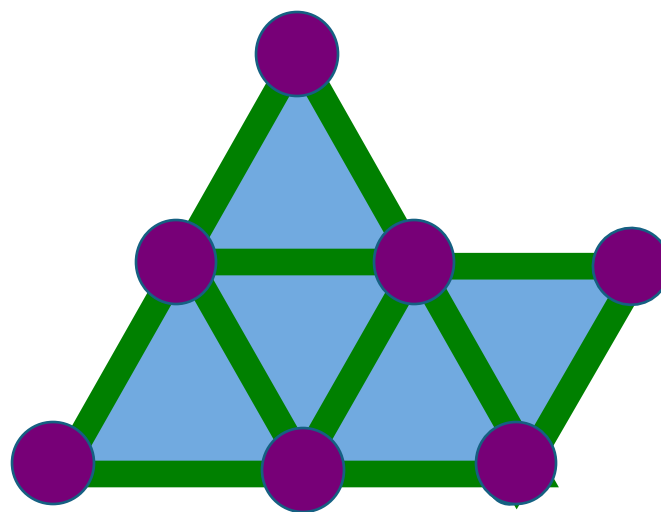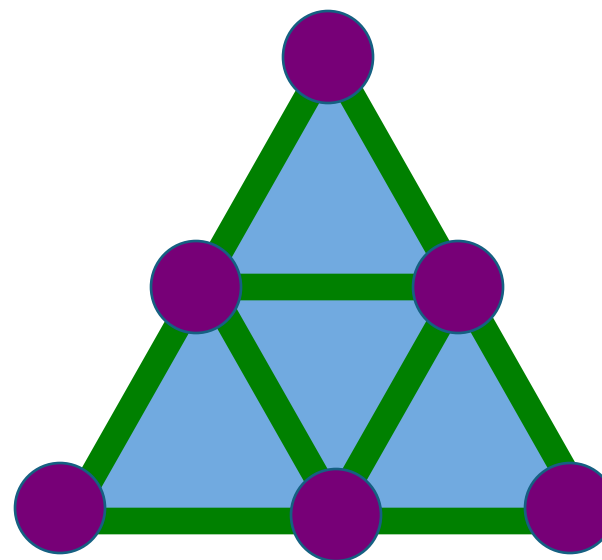
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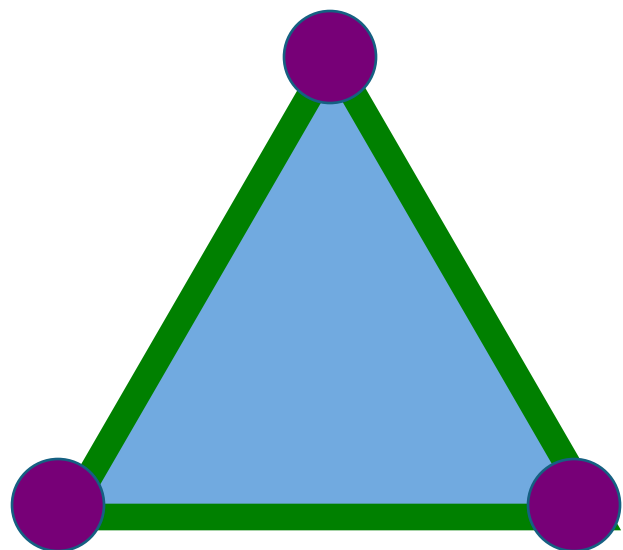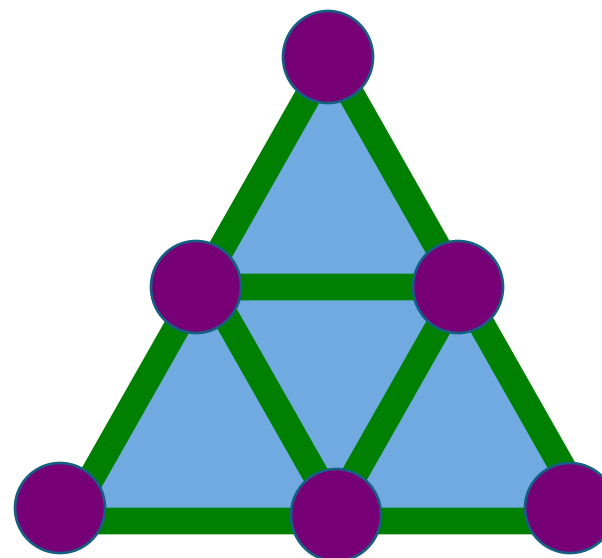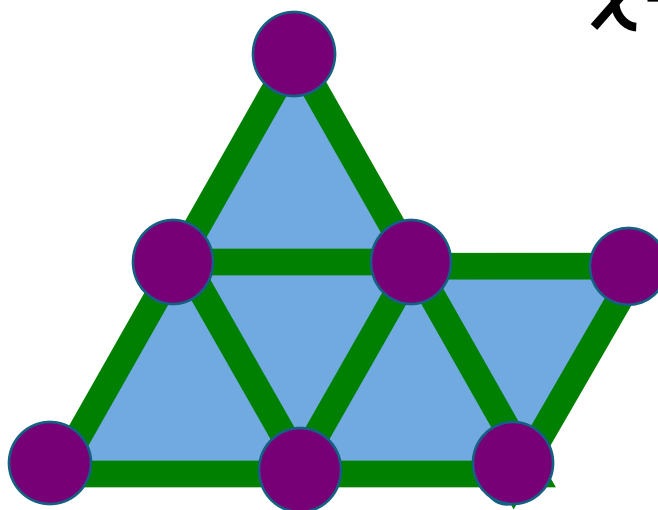
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$\chi = 6 - 9 + 4 = 1$

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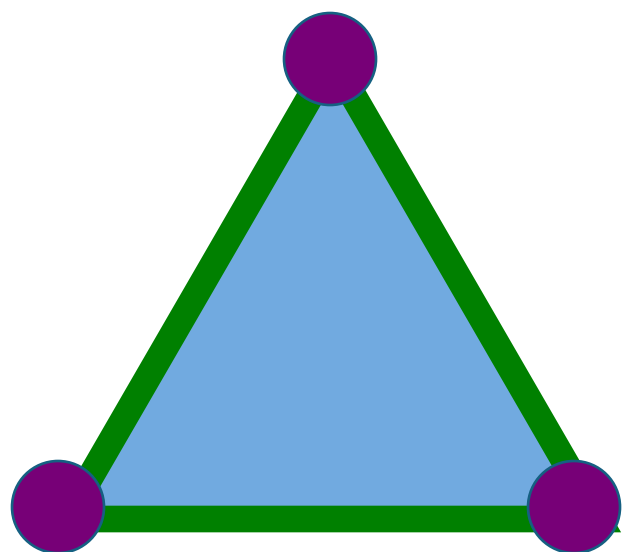
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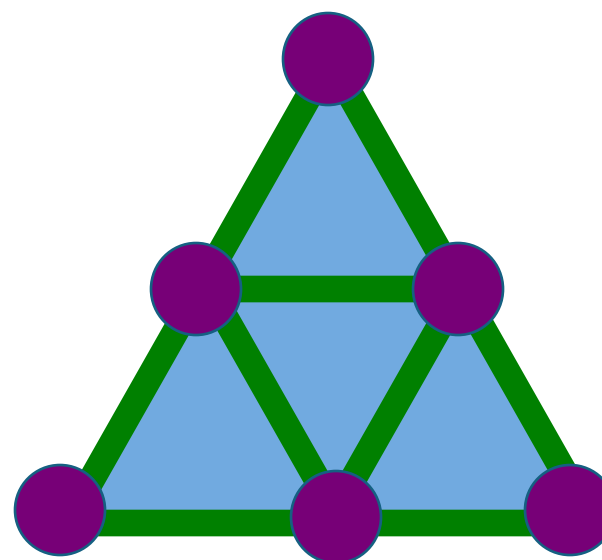
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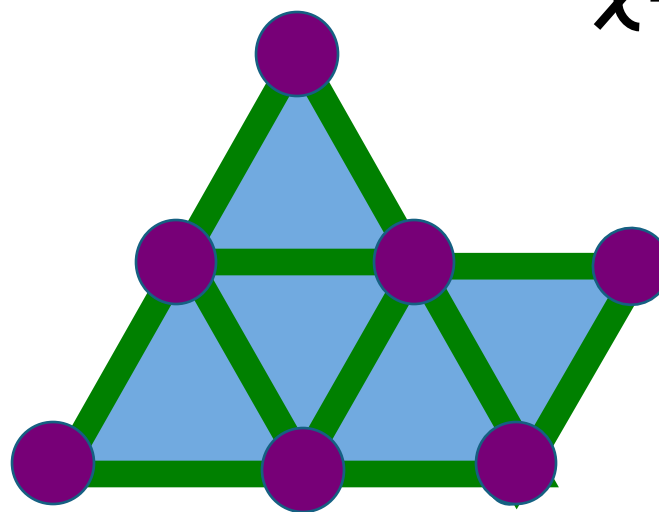
$\chi = 7 - 11 + 5 = 1$

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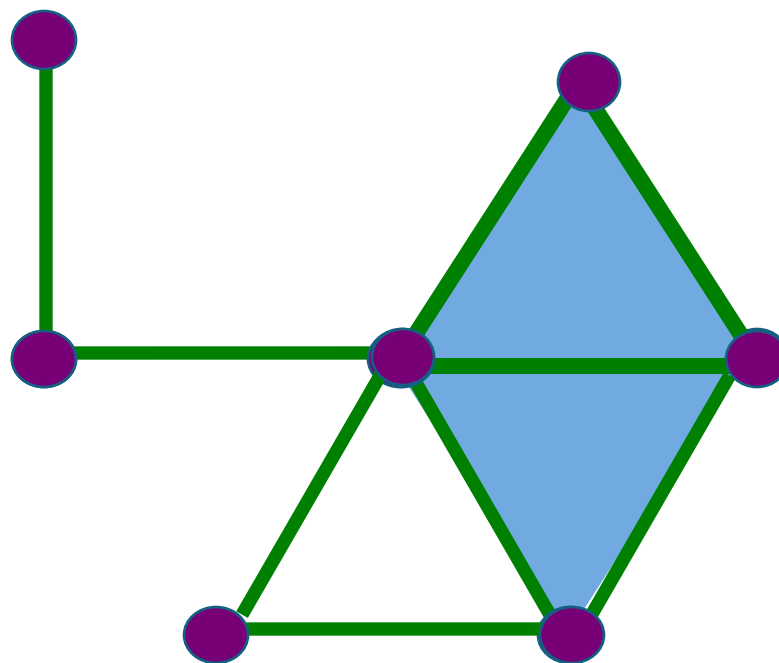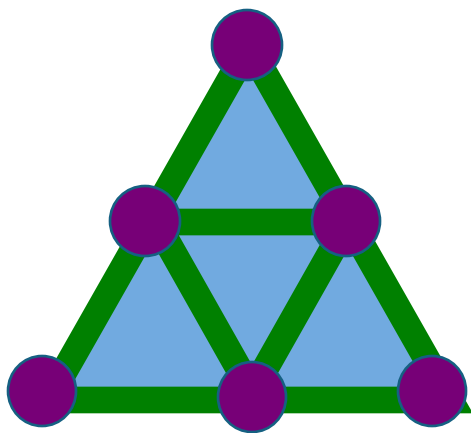
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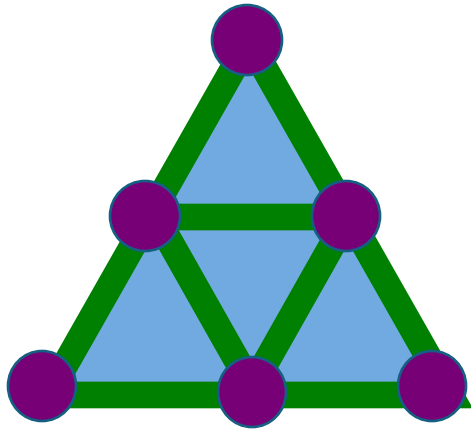
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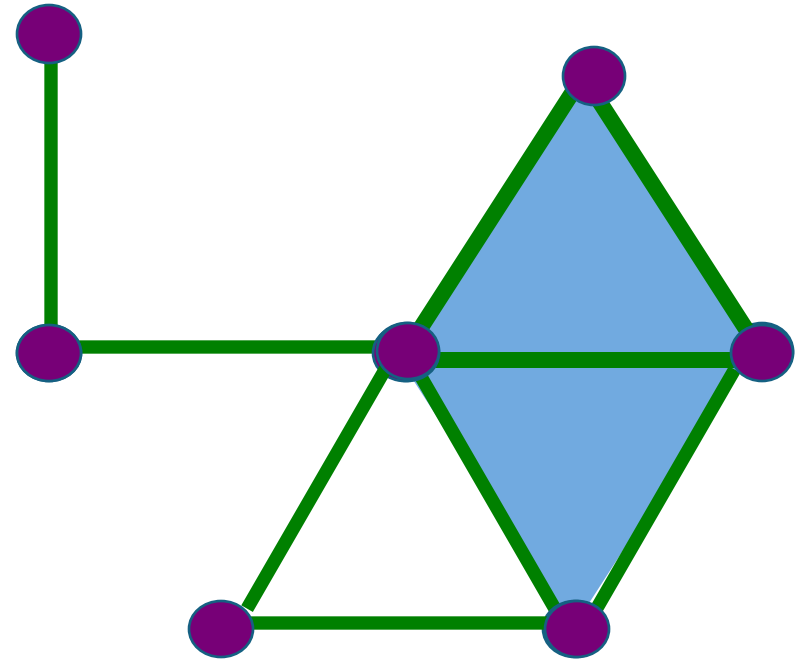
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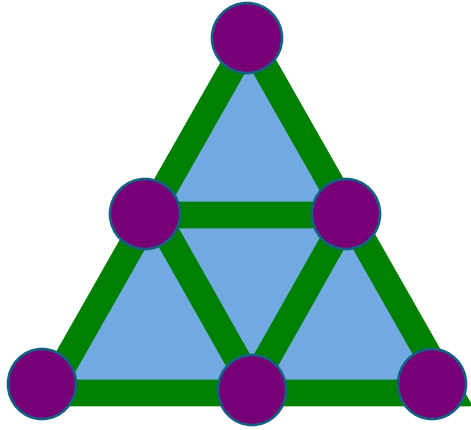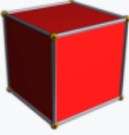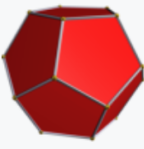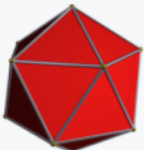
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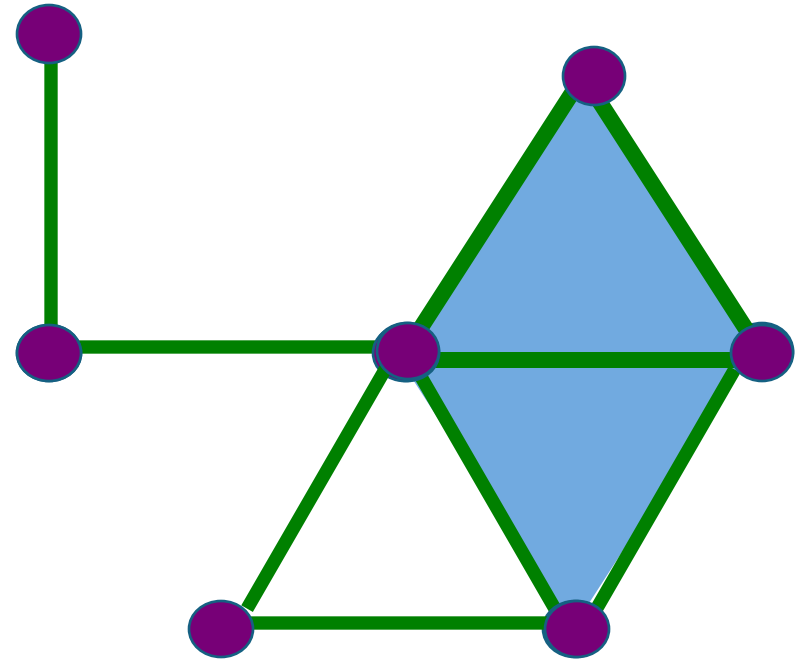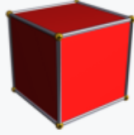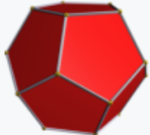
$\chi = 7 - 9 + 2 = 0$
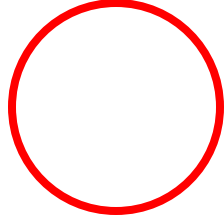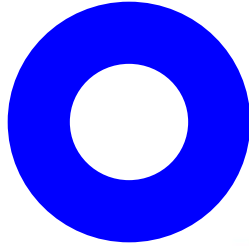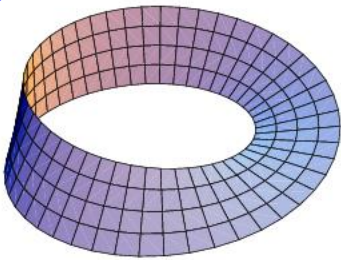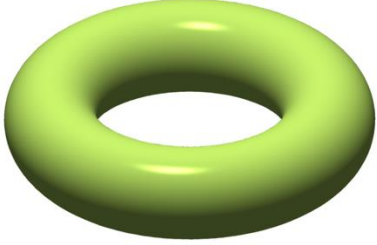
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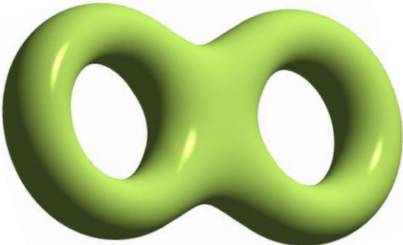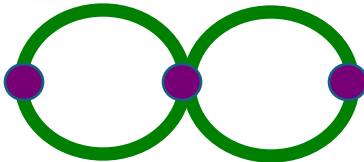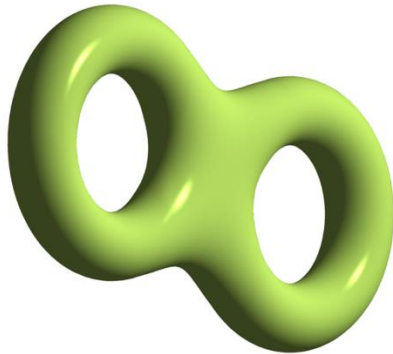
More examples
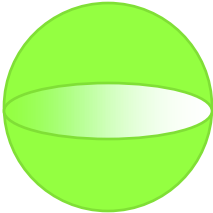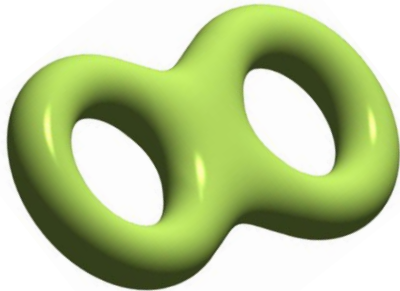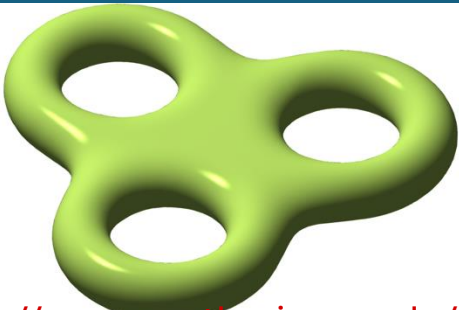
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron					
Hexahedron or cube					
Octahedron					
Dodecahedron					
Icosahedron					

image source: Wikipedia

More examples

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

image source: Wikipedia

Euler characteristic	
0	circle
	Annulus
	Mobius band
	Torus = S^1 x S^1

Euler characteristic	
-1	Solid double torus A graph of two cycles:
-2	Double torus = boundary of solid double torus

Euler characteristic	2-dimensional orientable surface without boundary
2	sphere
0	$S^1 \times S^1$ = torus
-2	genus 2 torus
-4	genus 3 torus

Euler characteristics for graphs

- Graphs: consist of only vertices and edges
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  - **Definition**: A *tree* is a connected graph that does not contain a cycle

Euler characteristics for graphs

- Graphs: consist of only vertices and edges

- So, Euler characteristic becomes: $V - E$

- We can use Euler characteristic to verify whether a graph is a *tree*
    - **Definition**: A *tree* is a connected graph that does not contain a cycle

- **Theorem**: The number of edges in a tree is always equal to the number of vertices $- 1$. So the Euler characteristic becomes

$$V - E = V - (V - 1) = 1.$$

Graphs: Identifying Trees

Defn: A *tree* is a connected graph that does not contain a cycle
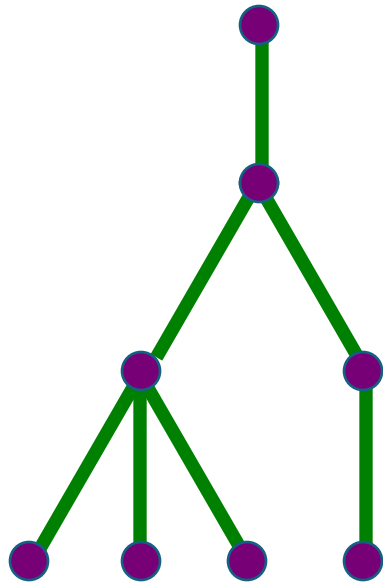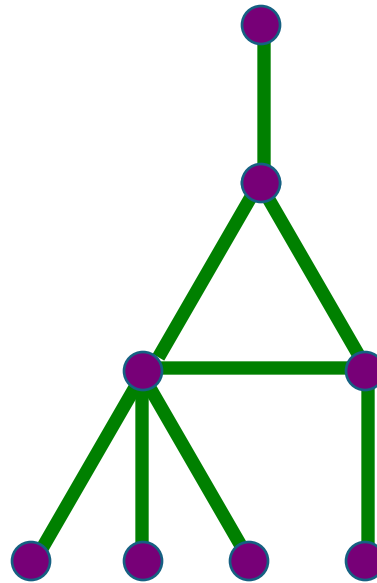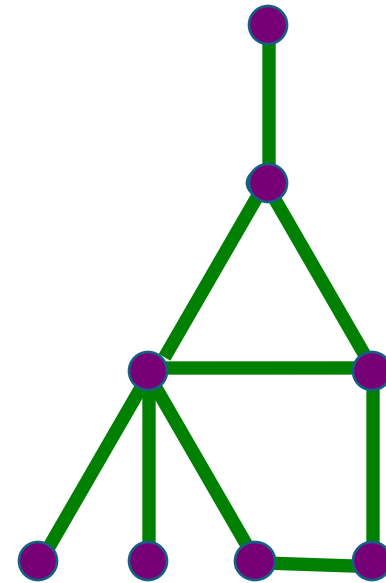
$$\chi = 8 - 7 = 1 \qquad \chi = 8 - 8 = 0 \qquad \chi = 8 - 9 = -1$$

Representation of shapes

- Before moving on to look at the more important invariant, homology

- We need another important definition that will be utilized throughout the course

- This solves a fundamental problem we face when we try to process shapes in computer: we need a way to represent shapes (topological spaces) that is easy for computer to process

- It turns out there have been such an invention in Mathematics already, which is called **simplicial complex**.

Simplicial Complex

- A simplicial complex is a generalization of a polyhedron, with building blocks called *simplices* in different dimensions:
    - 0-simplex: vertex
    - 1-simplex: edge
    - 2-simplex: triangle
    - 3-simplex: tetrahedron
    - ...
    - d-simplex (generalizations)
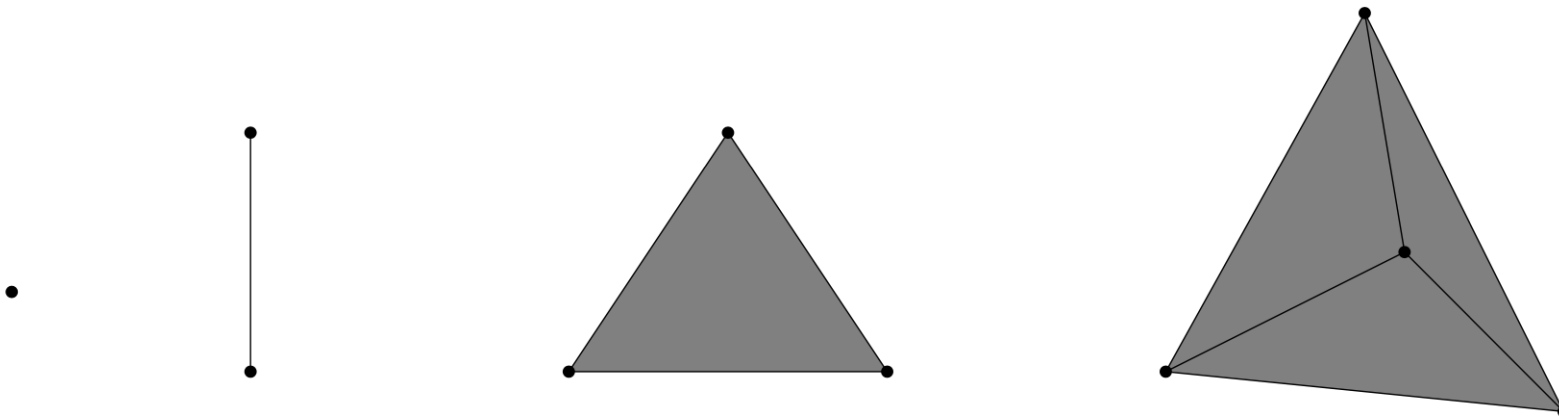
Image source: Patrick Schnider. Introduction to Topological Data Analysis Lecture Notes FS 2023

Simplicial Complex

- The following is a simplicial complex with simplices up to dimension 3:
  - 0-simplices (vertices): 18
  - 1-simplices (edges): 23
  - 2-simplices (triangles): 8
  - 3-simplices (tetrahedra): 1

Image source: Wikipedia
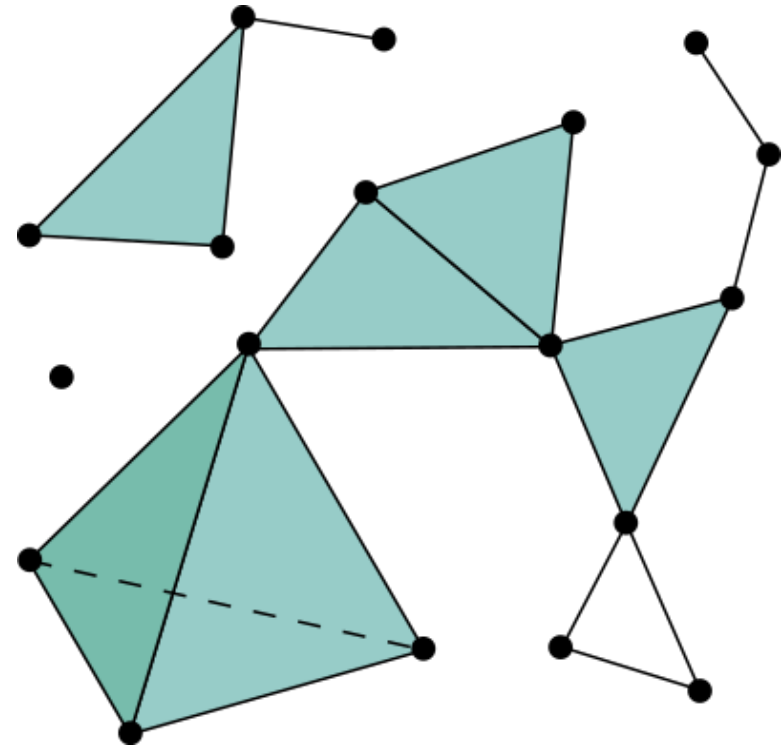
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- **Definition**: The *dimension* of a simplicial complex is the maximum dimension of its simplices

- So the dimension of the left complex is 3

- Note: A simplicial complex is sometimes simply called a *complex*

- A d-dimensional simplicial complex is sometimes simply called a *simplicial d-complex* or *d-complex*

Image source: Wikipedia
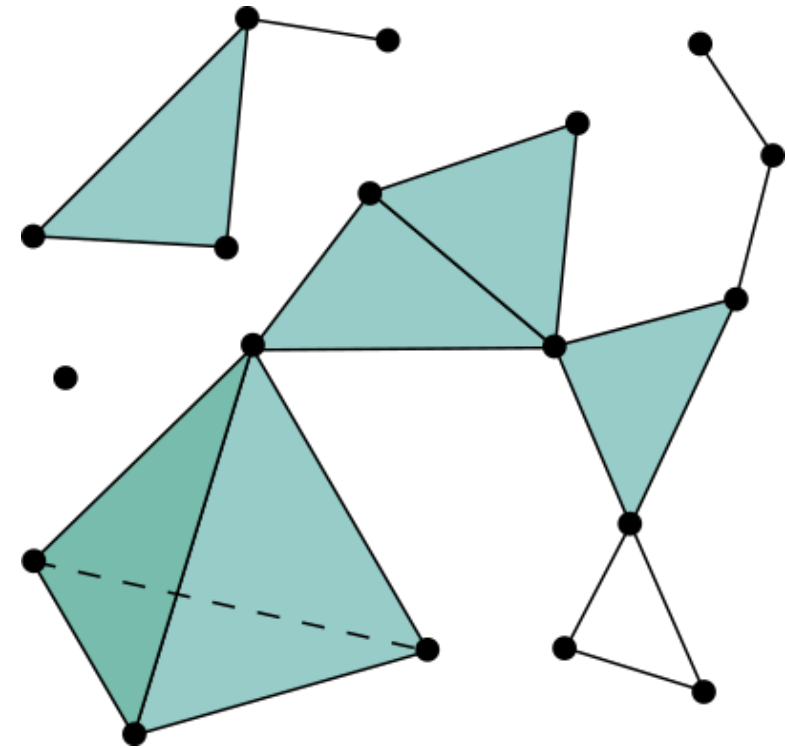
Triangular meshes

- A very common type of simplicial complexes used in computer graphics are triangular meshes (a 3D object whose surface is made up of glueing small triangle patches)

- From a topological point of view, they are nothing but 2-dimensional simplicial complexes
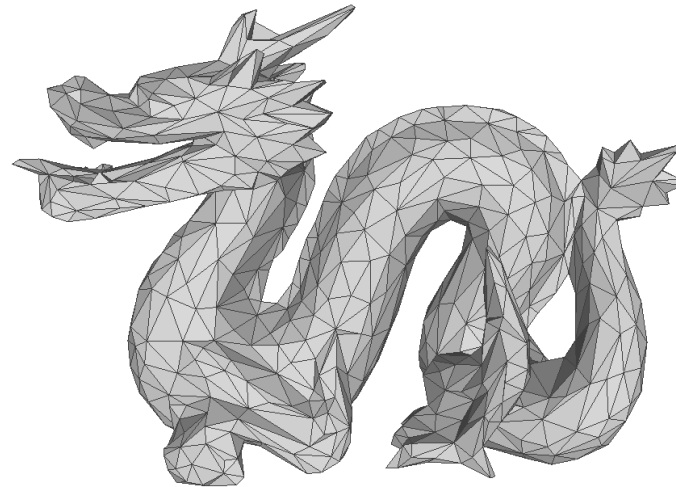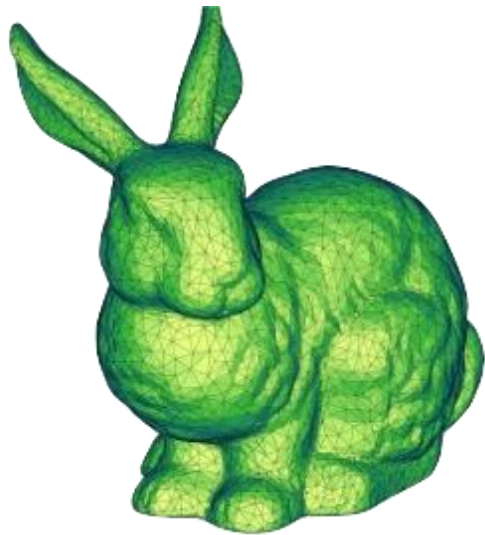
(figure from favpng.com)

Graphs

- Another more common type of simplicial complexes in CS are graphs.
- A graph is a tuple $G = (V, E)$, where V is the set of 0-simplices and E is the set of 1-simplices. So it's a 1-complex.
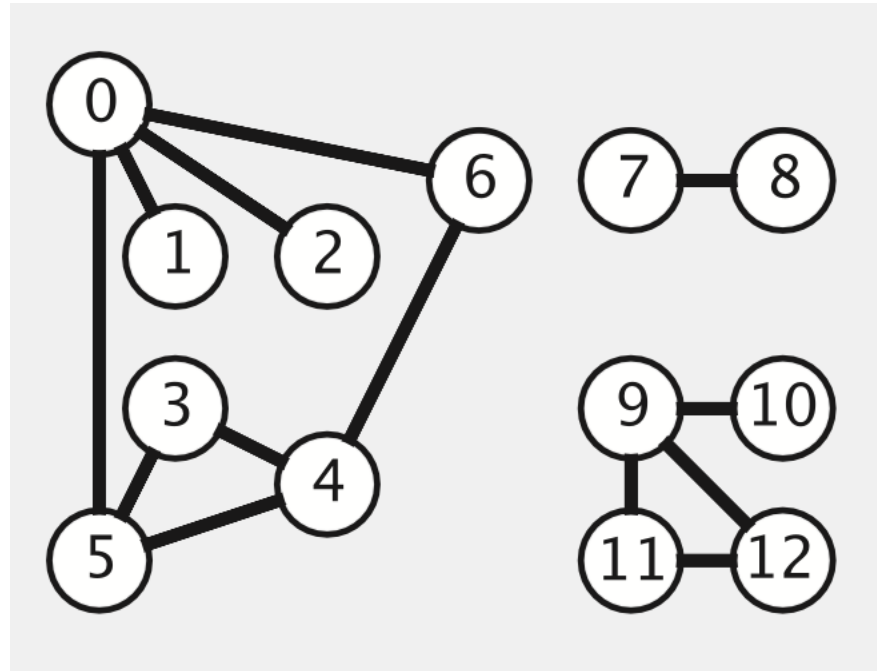
Image source: Sedgewick and Wayne. Algorithms (4th ed)

Faces of a Simplex

- For a simplex σ, we notice that there are other simplices on its boundary, which are called the **faces** of σ.

- If a face τ of σ is a d-dimensional simplex, then we also call it a d-face of σ.

- A convention is that σ is always a face of itself.

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- Ex: A vertex has only one face which is itself

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- Ex: An edge ab has:
    - Two 0-faces: a and b
    - One 1-face: ab

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- Ex: A triangle abc has:
  - Three 0-faces: a, b, and c
  - Three 1-faces: ab, ac, and bc
  - One 2-face: abc

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- Ex: A tetrahedron $abcd$ has:
  - Four 0-faces: a, b, c, and d
  - Six 1-faces: ab, ac, ad, bc, bd, cd
  - Four 2-faces: abc, abd, acd, bcd
  - One 3-face: $abcd$

Simplicial Complex (Formal Definition)

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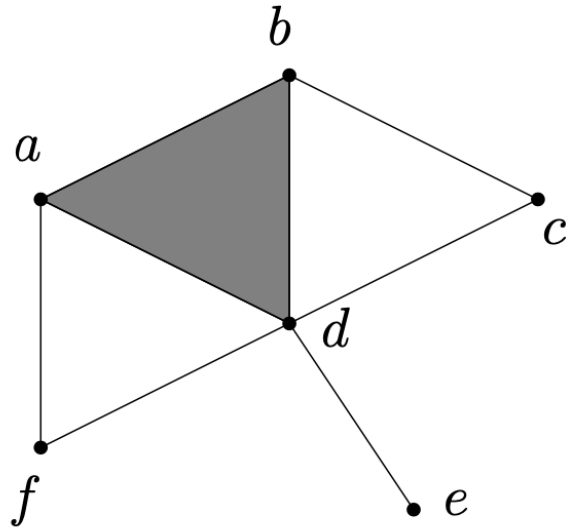
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- Another example: For a graph (1-complex), each edge must join two vertices in the vertex set (a vertex that is a face of an edge must also be in the complex)

Image source: T.K.Dey and Y.Wang. Computational Topology for Data Analysis

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The condition that faces of any simplex in a complex should also be in the complex is very important part in the definition making it mathematically sound

Image source: T.K.Dey and Y.Wang. Computational Topology for Data Analysis

Some remarks

- For denoting a simplicial complex, we typically first assign labels to the vertices of the complex.
    - In class, the labels could be letters but could also be other things
    - In computer programs, the labels are almost always integers $0, \cdots, l-1$.
- Then, each simplex is represented as a set of the vertices on its corner.
- Note: Each d-simplex σ is represented by a set of $d+1$ vertices
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- Also note: A d-simplex is typically represented by a **sorted** array of $d + 1$ vertices (integers) in computer programs, this makes checking the equality of two simplices

Endowing Algebraic Structures to Complexes

- Now we continue towards our goal of defining homology

- There are still *a few steps* before that

- Recall that homology is a "numeric" invariant that computer can handle

- More formally, it's an "algebraic" invariant.

- So, let's give a simplicial complex and its simplices an algebraic structure, so that we could do algebra on it (just like what we used to do 1+1=2 in primary school).

Chains

- We first introduce an algebraic notion called chains.

- A chain is a summation (formal sum) of a bunch of simplices of the same dimension d, and we also call it a d-chain.

- i.e., it is of the general form: $\sum_{i=1}^{k} \sigma_i$, where each σ_i is a d-simplex.

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- 0-chain: $a + d + f$
- 1-chain: $ab + cd + de$
- 2-chain: abd

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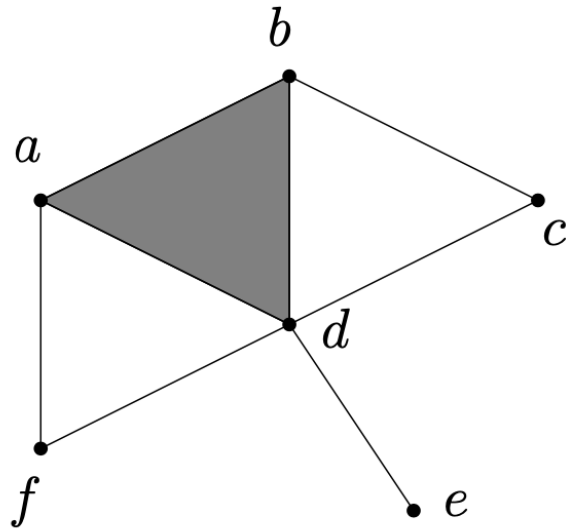
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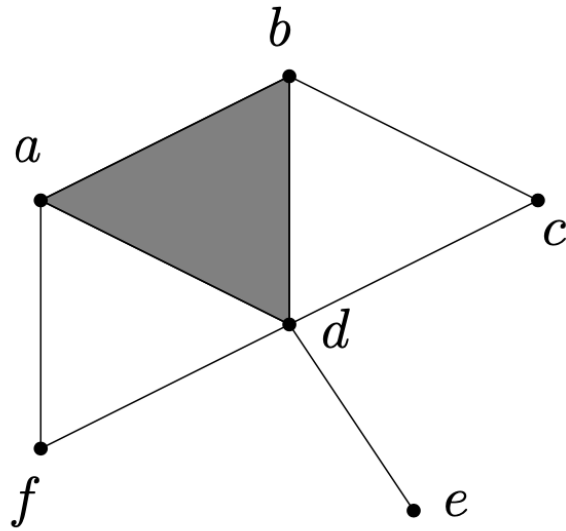
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- Note: we have a special chain '0' which contains no simplices

Image source: T.K.Dey and Y.Wang. Computational Topology for Data Analysis

Summation of Chains

- The summation of two chains is called the "symmetric difference", i.e.,
  - Keep simplices that occurs in exactly one of the chains
  - A simplex occurring in both chains will be cancelled out

image source: Wikipedia
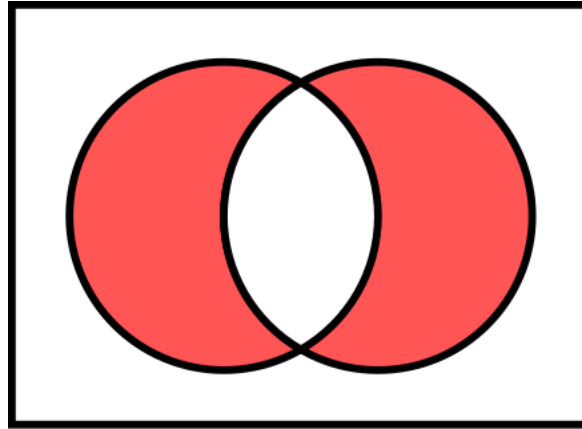
Summation of Chains

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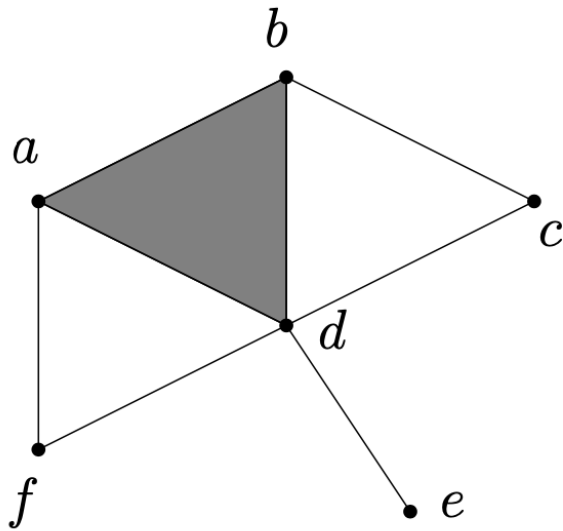
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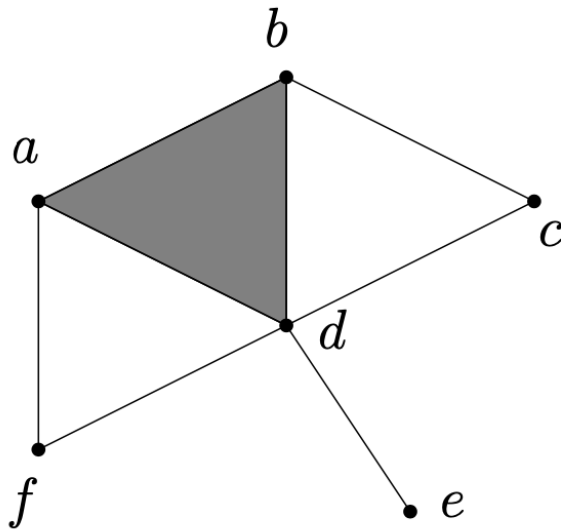
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- $(ab + cd + de) + (bd + cd) = ab + bd + de$

Boundaries of Chains

- Next thing we want to define are boundaries for chains

- But before doing that let's try to define boundary for a region in general

- For a two-dimensional region, the boundary is just the "border" of the region

Image source: https://www.geoapify.com/tutorial/getting-administrative-divisions-boundaries/

Boundaries of Chains

- Question: what is the boundary for a 1-dimensional line segment?

Boundaries of Chains

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- Answer: the two end points (because they are the places where we cannot travel any further within the 1-dimensional region)

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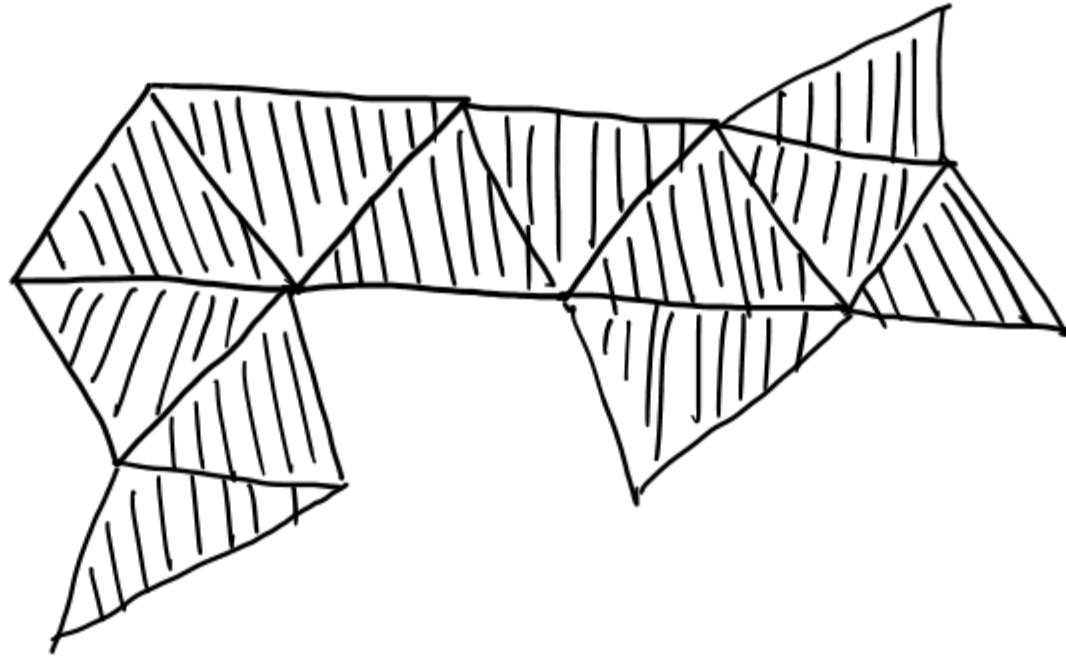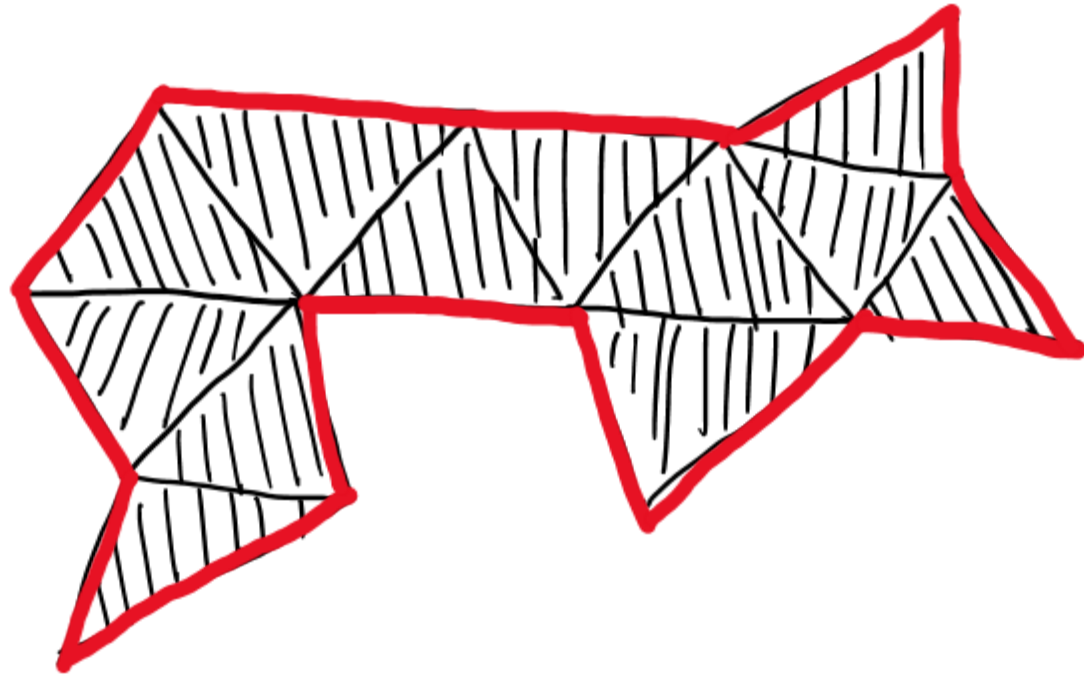
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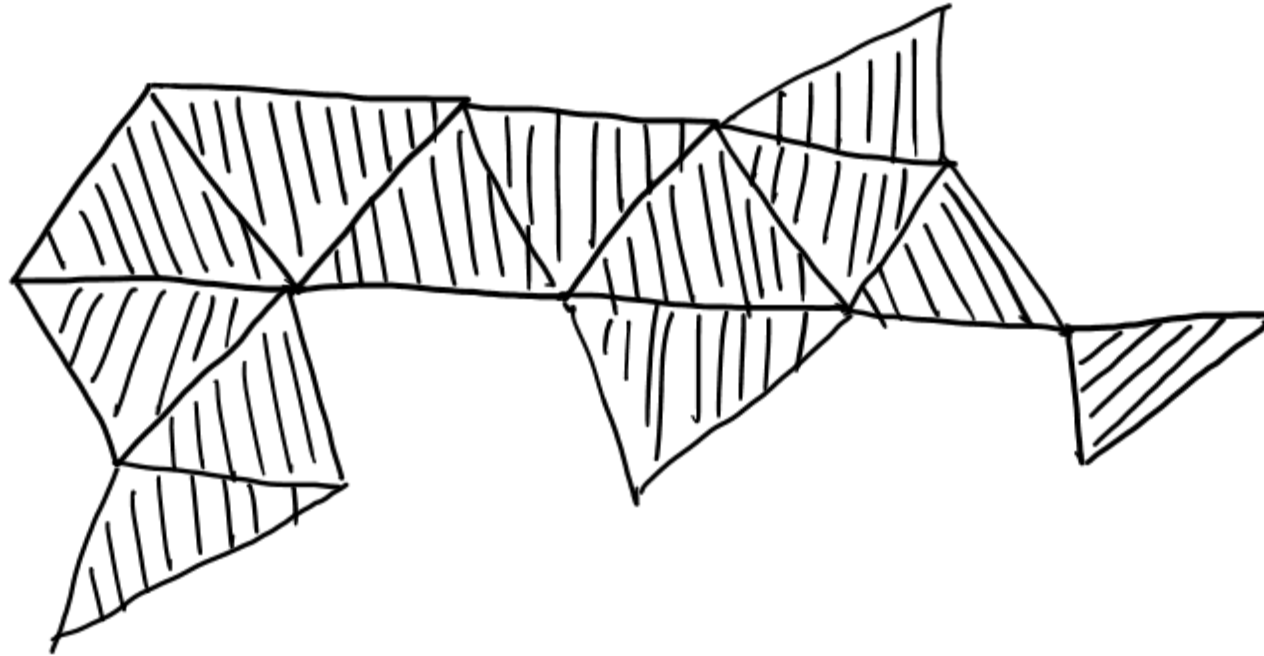
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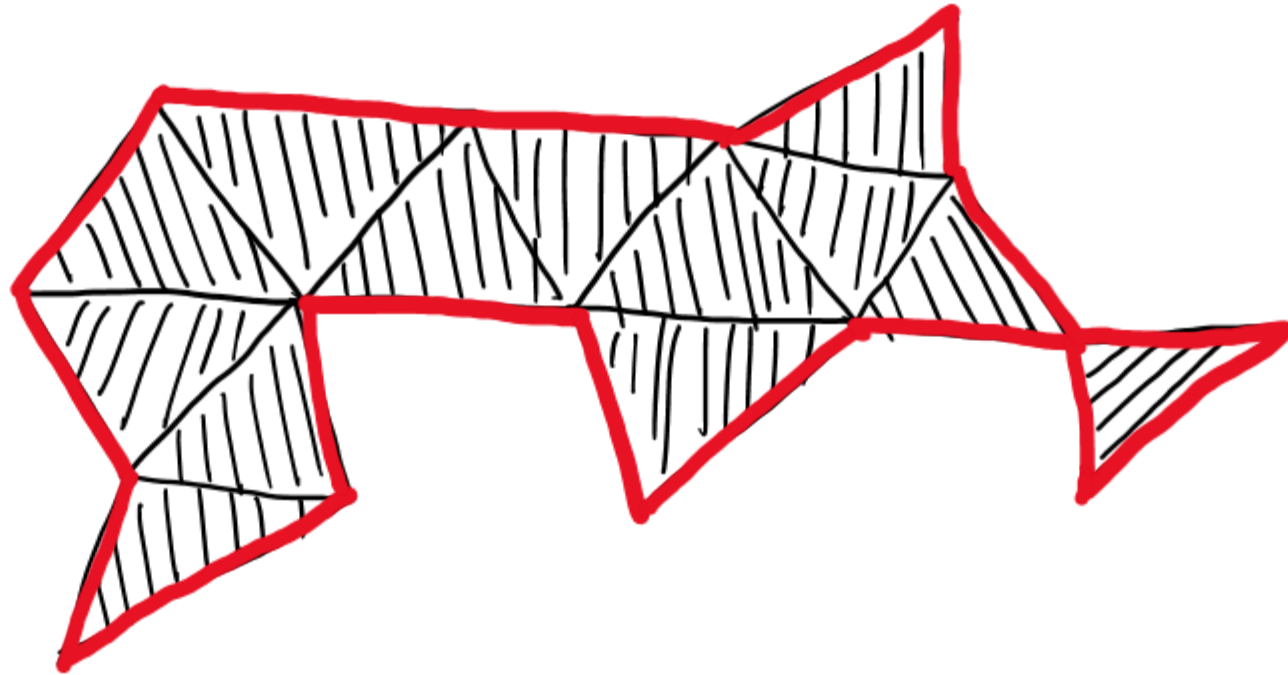
Boundaries of Chains

- **Observation**: the boundary of a d-chain c is also a chain, which is of one dimension lower, so it's a $(d + 1)$-chain. We denote it as $\partial(c)$.

Boundary of a Simplex

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    - The boundary of a tetrahedron $abcd$ is the four triangles it contains:

$$\partial(abcd) = abc + abd + acd + bcd$$

Boundaries of Chains

- **Observation**: The boundary of a chain equals the summation of the boundaries of its simplices, i.e.,

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Quick Question

- What is the boundary of the following 1-chain?
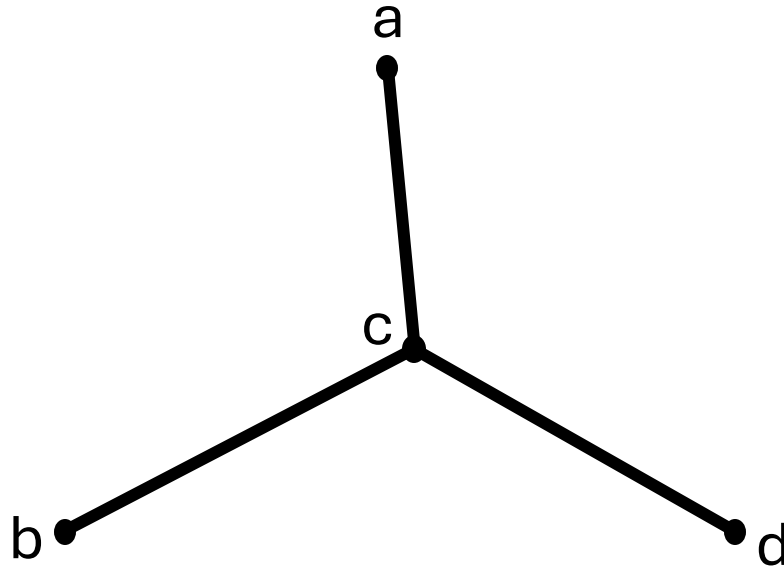
Boundaries of Some Special Chains

- What is the boundary of a circle?
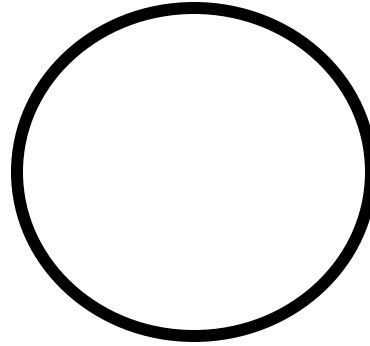
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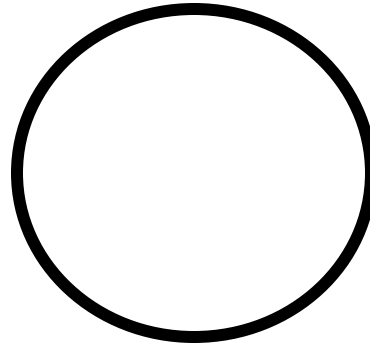
- Answer: 0 (empty)
- **Definition**: We generalize a circle and define a ***d-cycle*** as a d-chain whose boundary is 0.

Cycles

- Example of a 1-cycle (the same as a cycle on graphs):
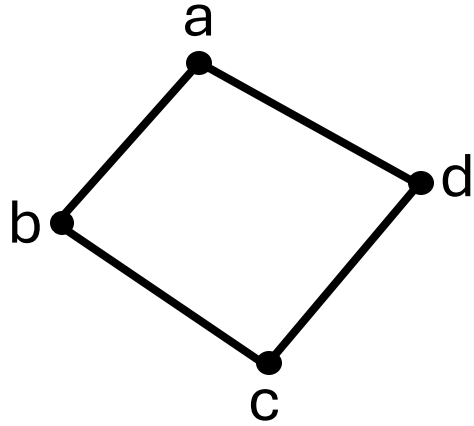
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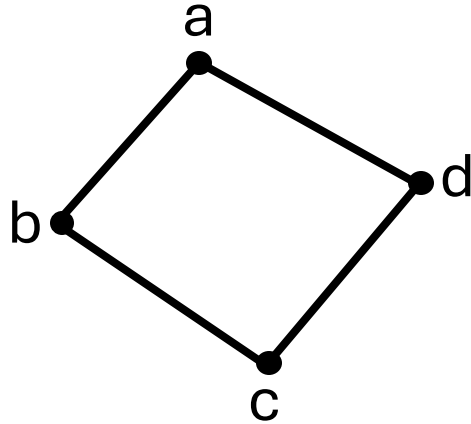
- Example of a 2-cycle (triangulated sphere):
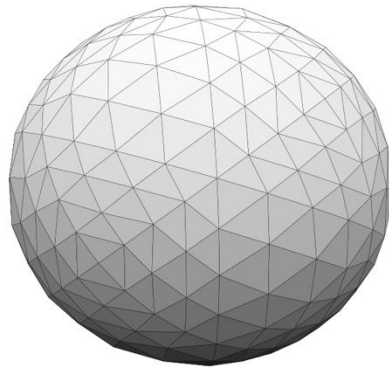
Fundamental theorem of homology

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- The above theorem is a fundamental fact making homology theory possible

- The proof of it and any further algebraic interpretations of it are beyond the scope of this course.

ALGEBRAIC
TOPOLOGY

Allen Hatcher

Elements of
Algebraic Topology

James R. Munkres

ABP

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK
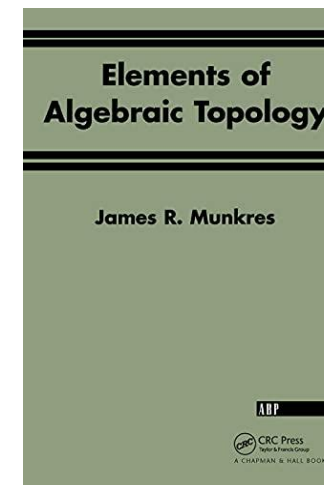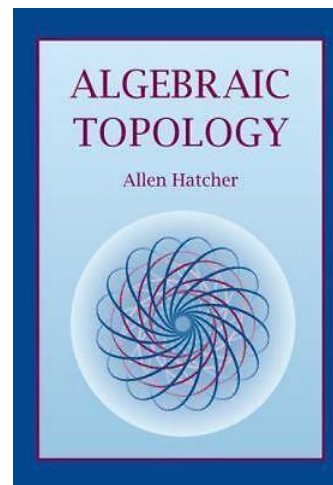
Fundamental theorem of homology

- A simple exercise: calculate the boundary of $\partial(abc + acd + ade)$.
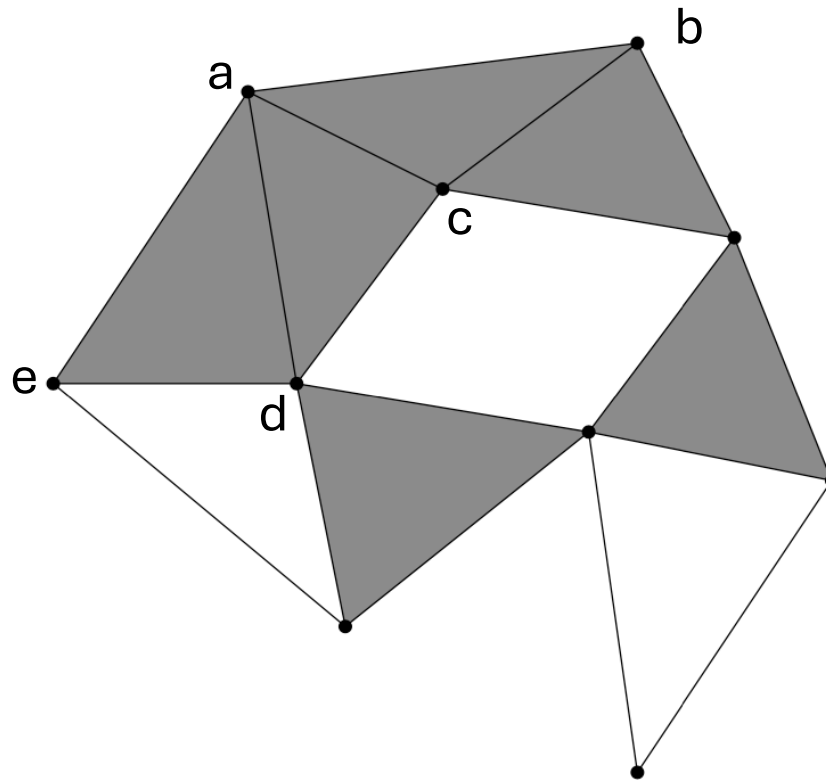
Classification for the set of all cycles

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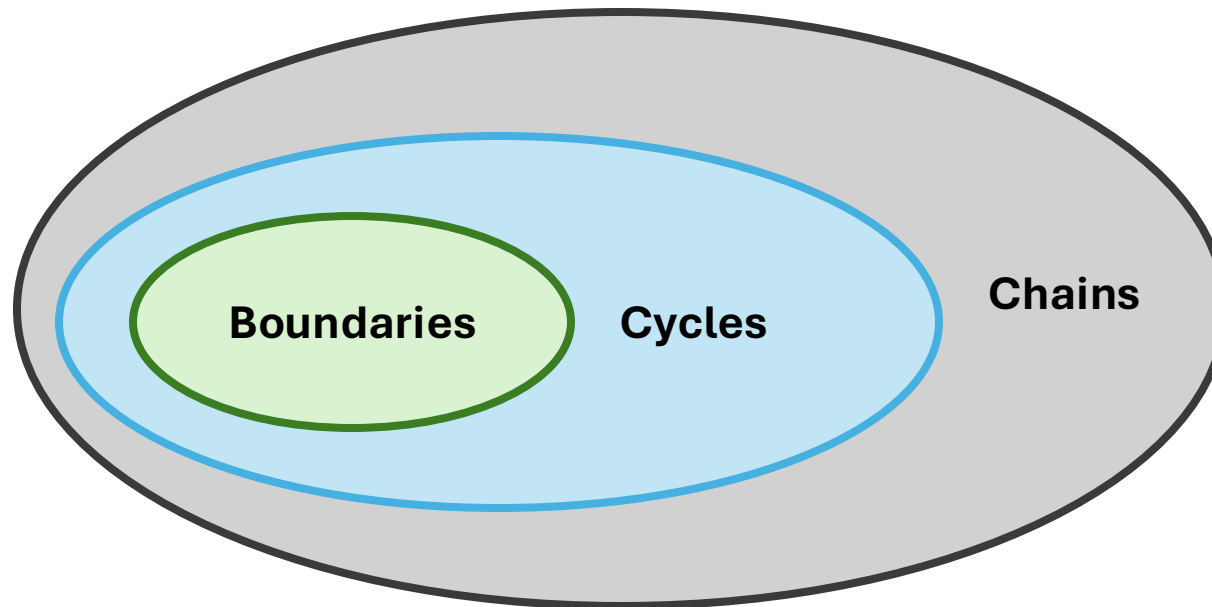
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Boundaries $\subseteq$ Cycles $\subseteq$ Chains

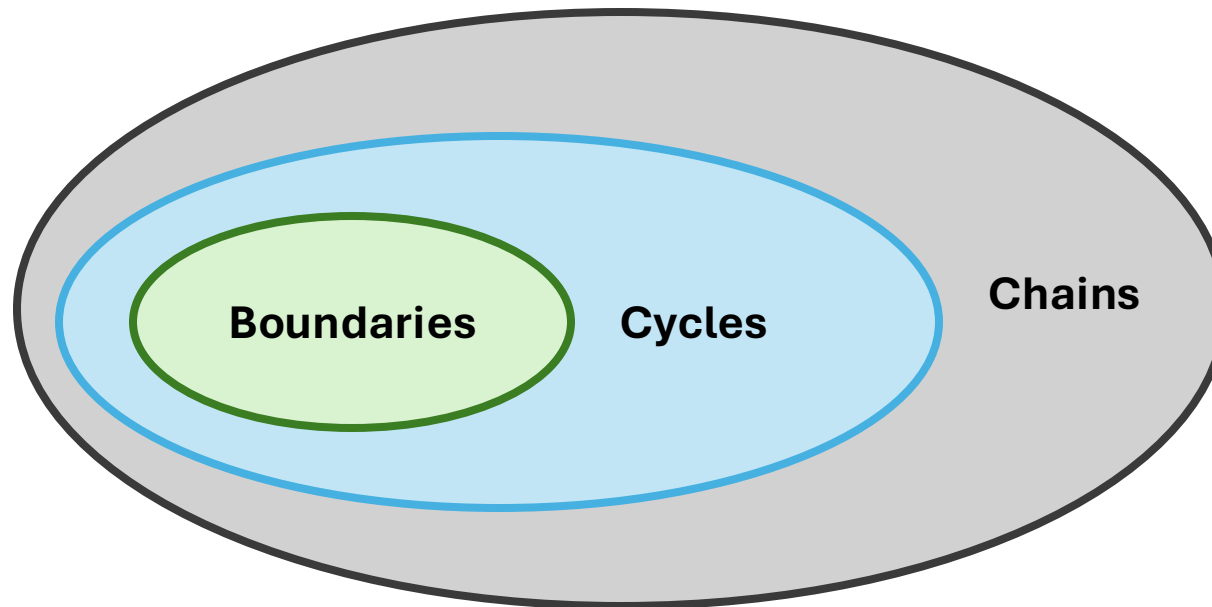# Classification for the set of all cycles

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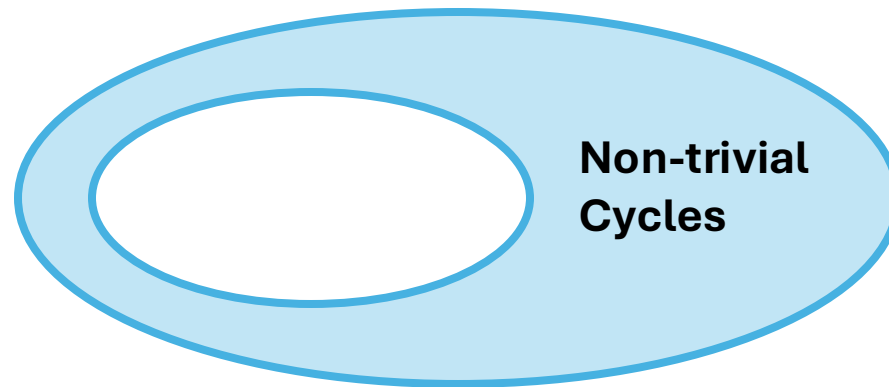
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Homology Studies: Cycles That Are NOT Boundaries
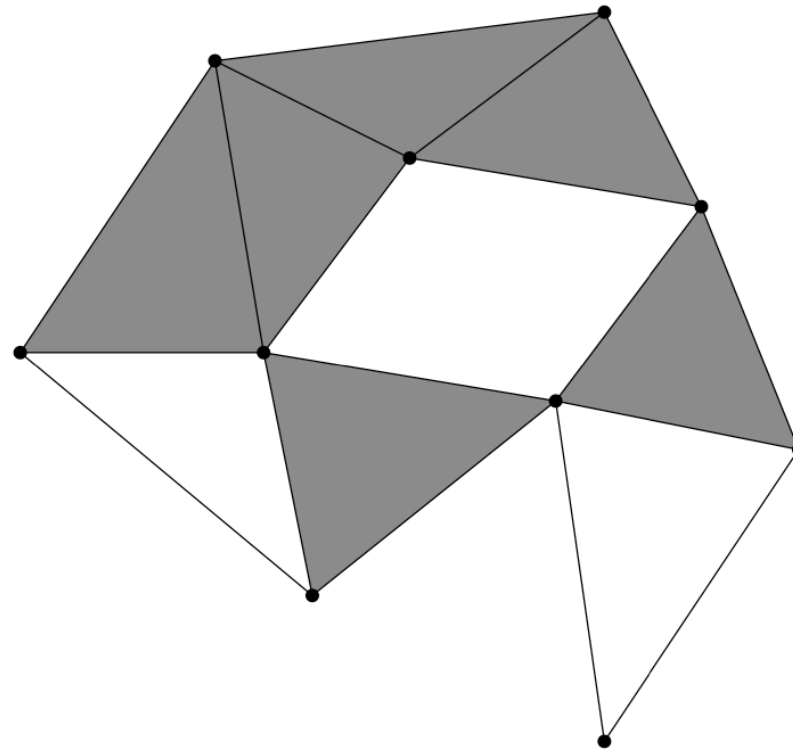
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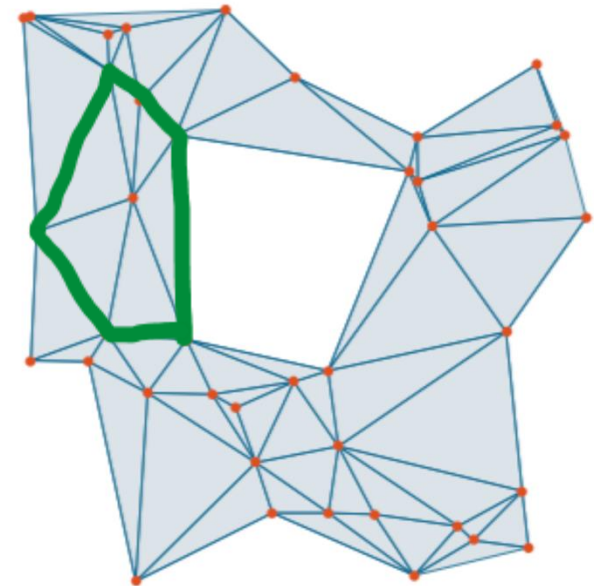
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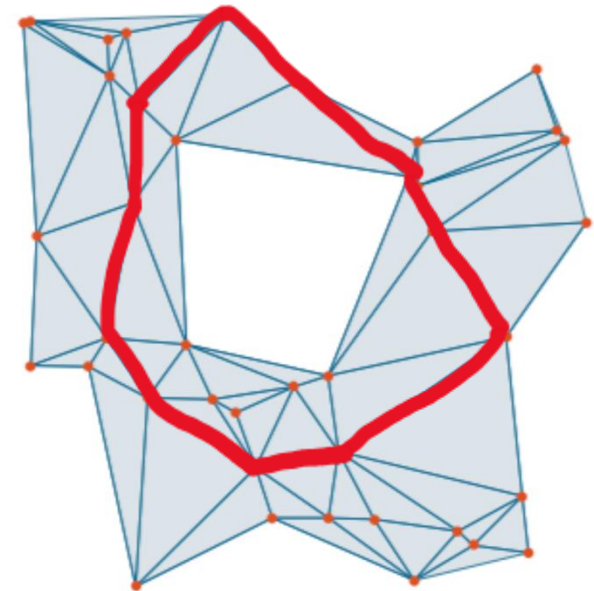
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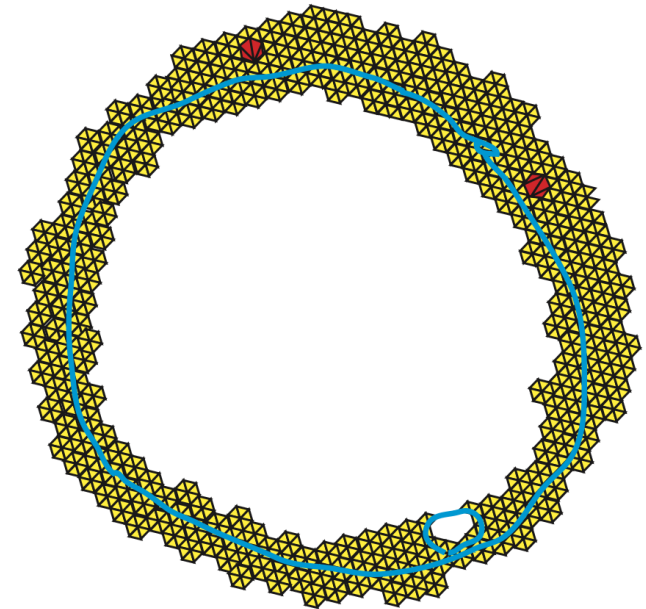Image source: Gunnar Carlsson, Topology and Data

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- Ex: For the right simplicial complex, do you think the red boundary cycle represent the shape, or the large blue non-boundary one?
- **Trying to capture the something like the major blue cycle to represent the shape of data is an aim of TDA and the course!!**

Image source: Gunnar Carlsson, Topology and Data
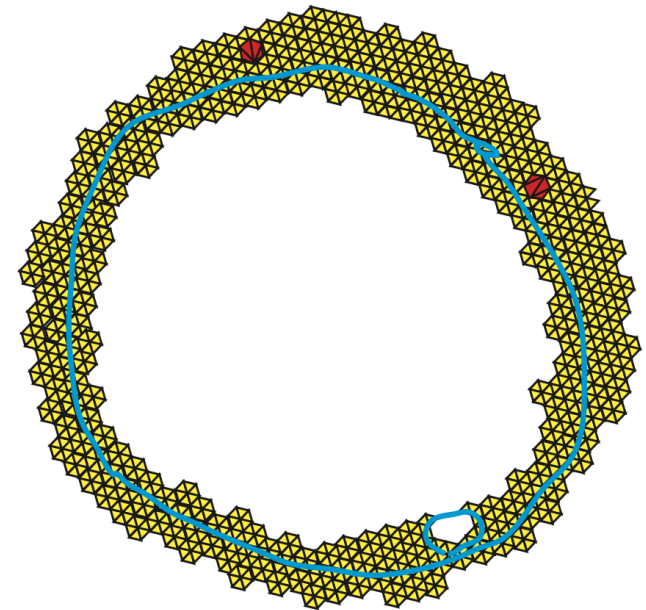
Homology Studies: Cycles That Are NOT Boundaries

- So far we have been showing **1-dimensional non-boundary cycles** (aka. holes) captured by homology theory, partially because it's easy to visualize

- But notice that homology can capture holes in any dimension (≥ 0).

- What about holes in other dimensions?
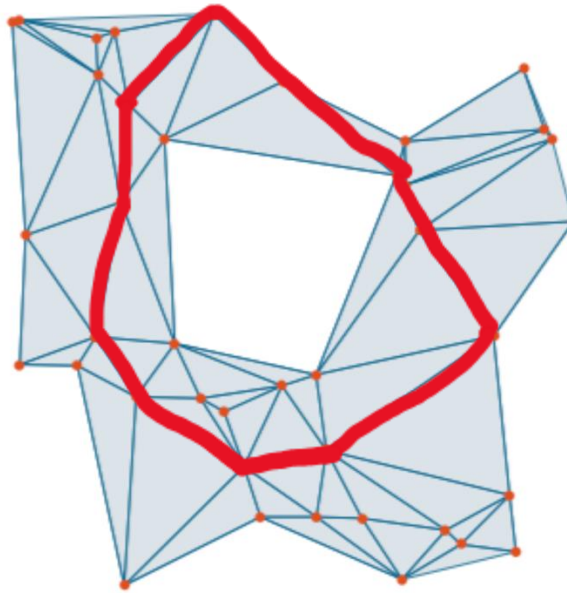
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- **0-dimensional holes** capture "gaps" between different connected components:
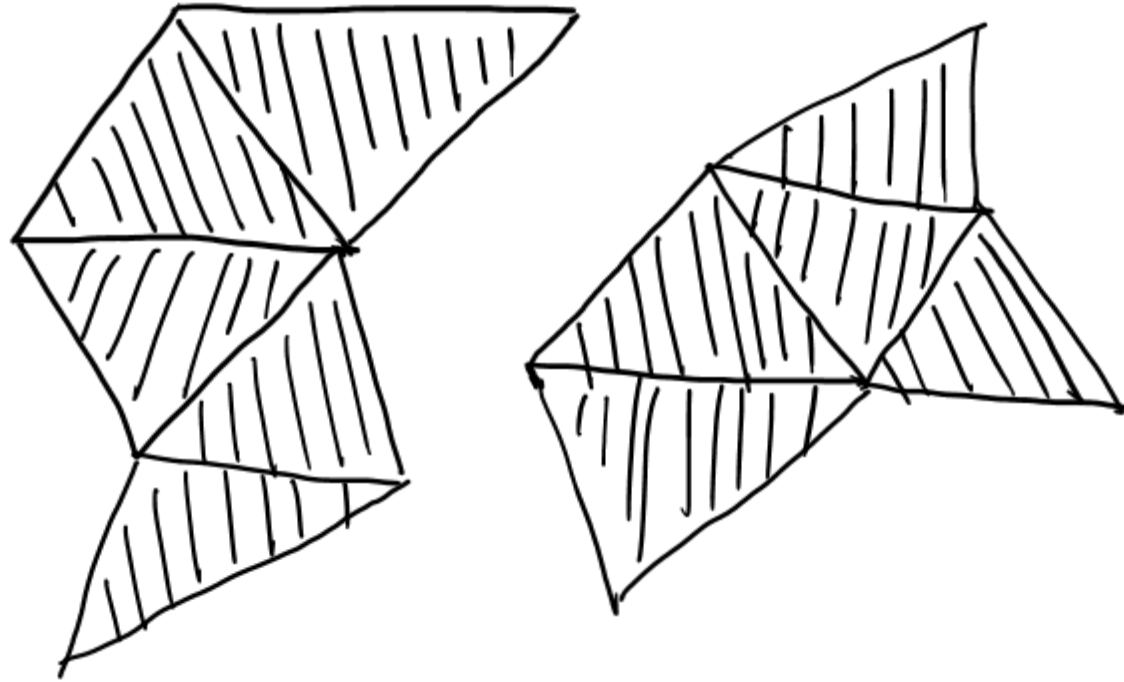
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- **2-dimensional holes** capture "cavity" or "hollowness" inside:
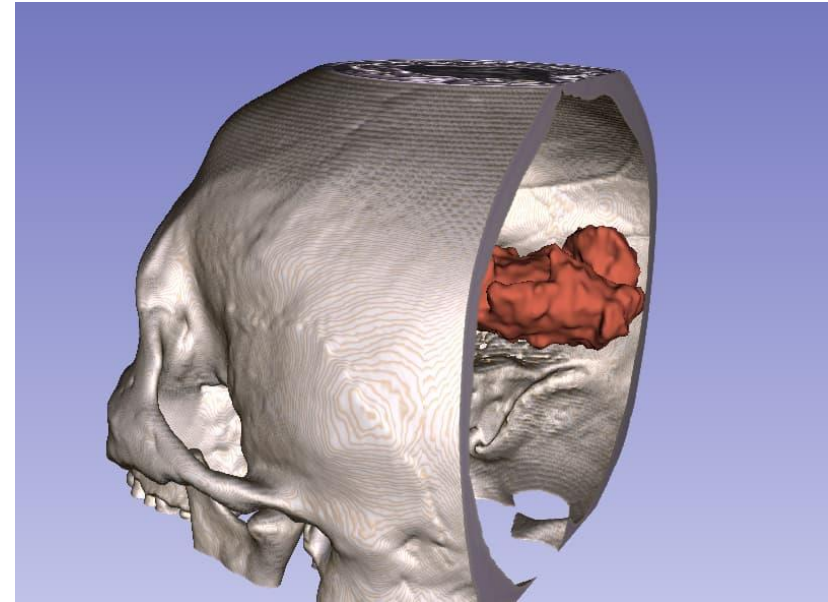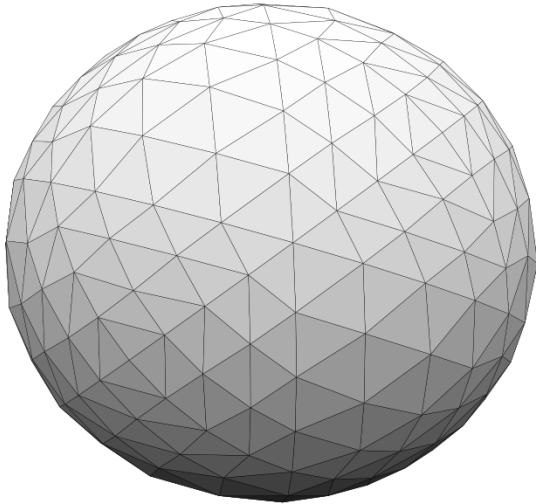
Image source: https://eugene-eeo.github.io/blog/sphere-triangles.html

Homology Basis: Additional Structures on Cycle Space

- Okay, now we know that homology studies cycles that are not boundaries

- But we can observe more algebraic structures on the cycles in a simplicial complex

- **Fact:** Each d-cycle of a simplicial complex is "generated by" a set of non-trivial d-cycles called the ***homology basis***.

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- **Fact:** Each d-cycle of a simplicial complex is "generated by" a set of non-trivial d-cycles called the ***homology basis***.

- Formally, a d-cycle z being "generated by" cycles in the homology basis means that z **can be written as:**

  - A sum of cycles in the basis + a boundary (which is "trivial").

Homology Basis: Additional Structures on Cycle Space

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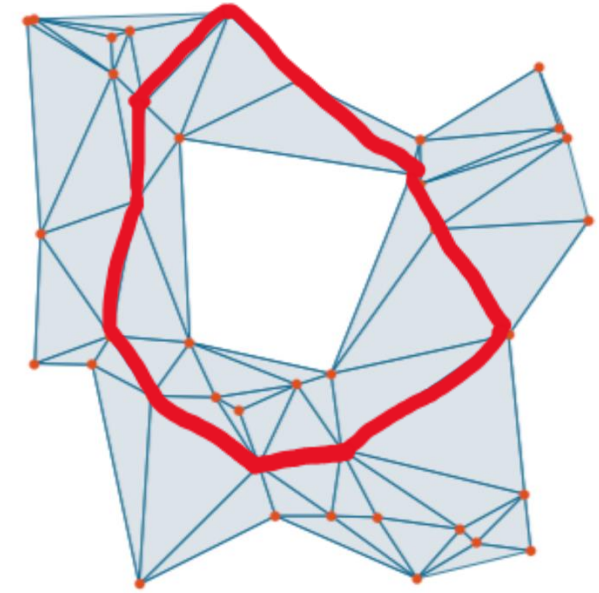
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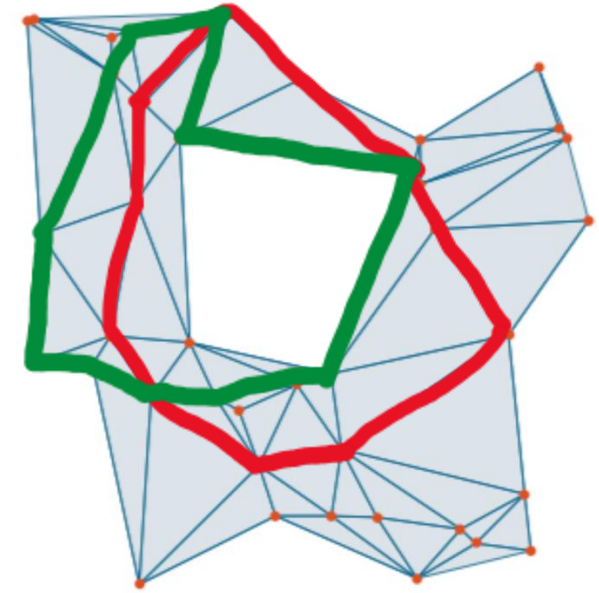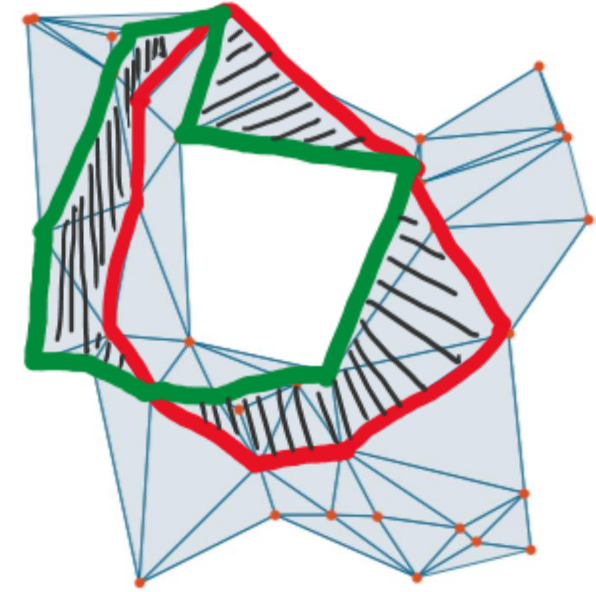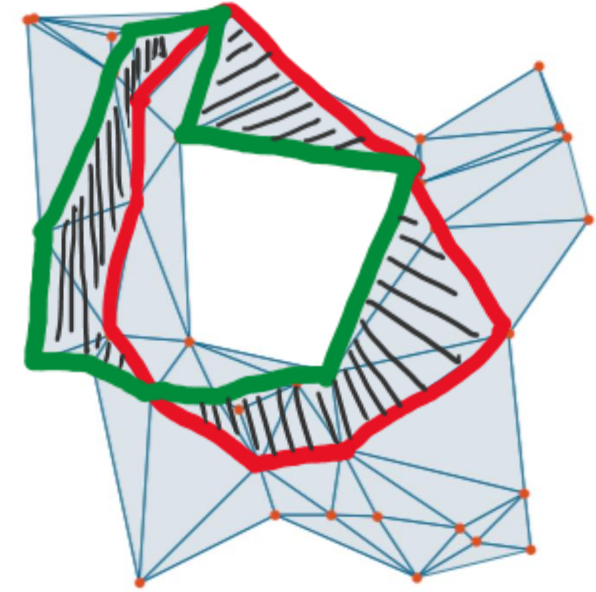
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- In a sense, the red cycle "generates" the green cycle because you can continuously stretch the red one to the green one

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- **Ex**: A homology basis for the 1-cycles in a torus (the surface of a donut) contains two 1-cycles:

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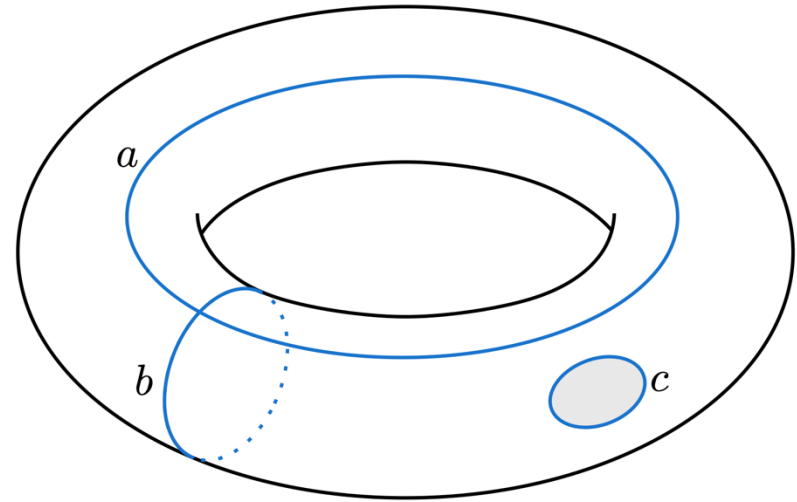
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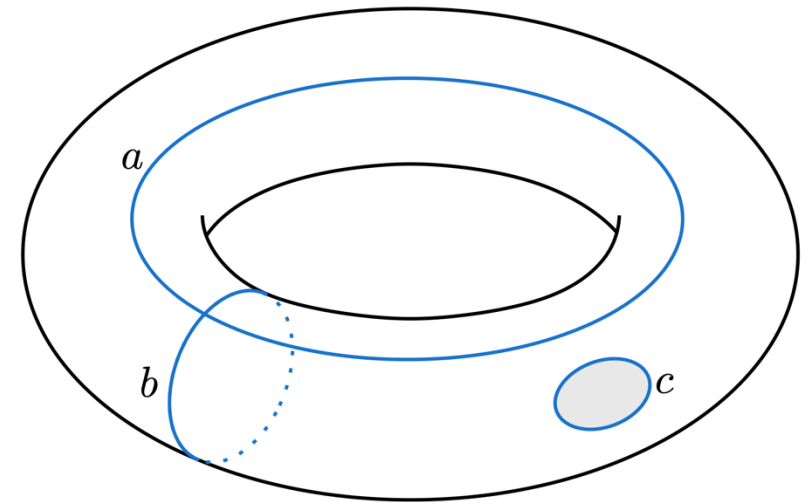 - a non-trivial cycle does not cut the torus into two pieces

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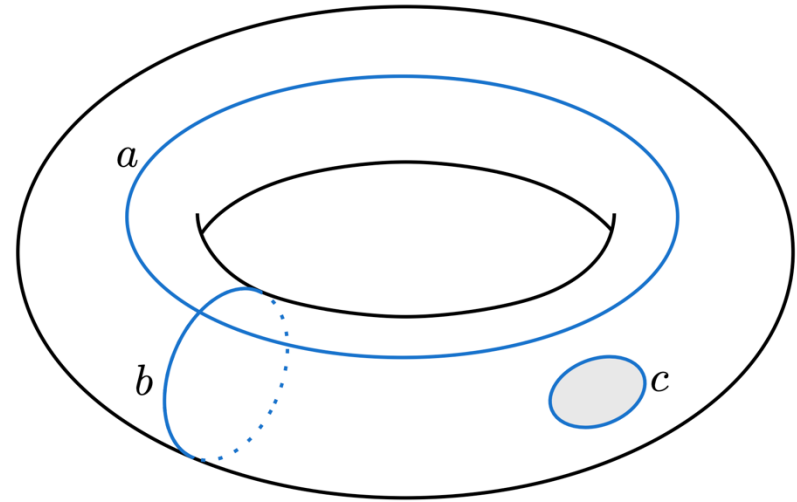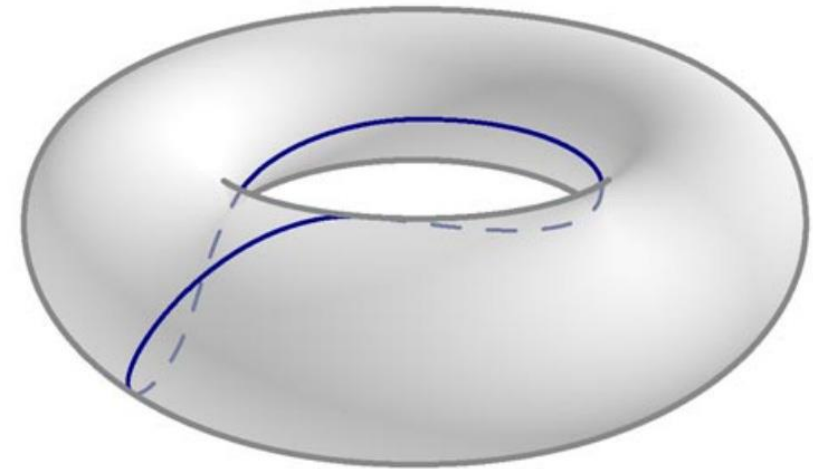 - while a trivial cycle (such as c) does

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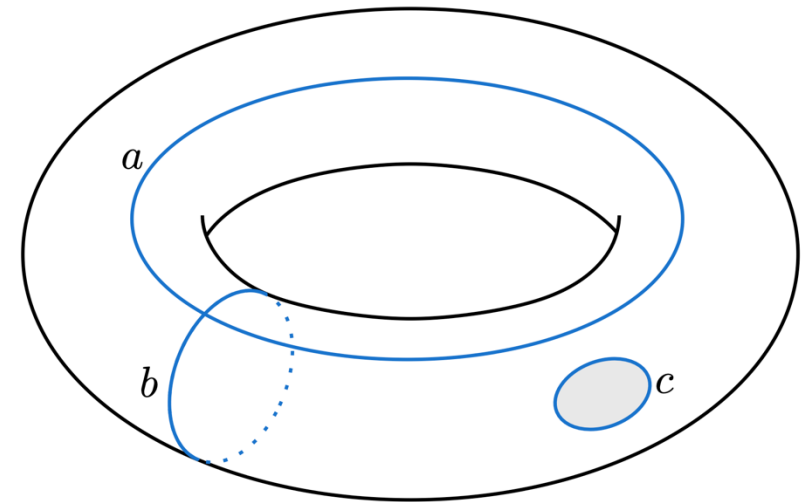Image source: <u>Toroidal Hitomezashi patterns - by Richard Green</u>

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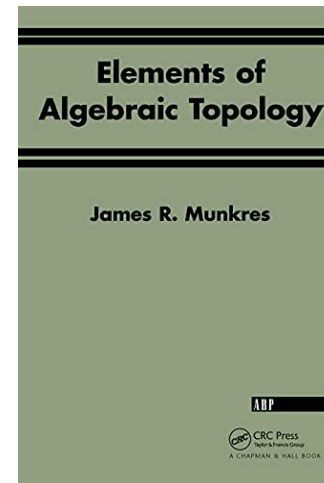- Example of how a and b generate cycles in torus: the cycle in the lower torus equals
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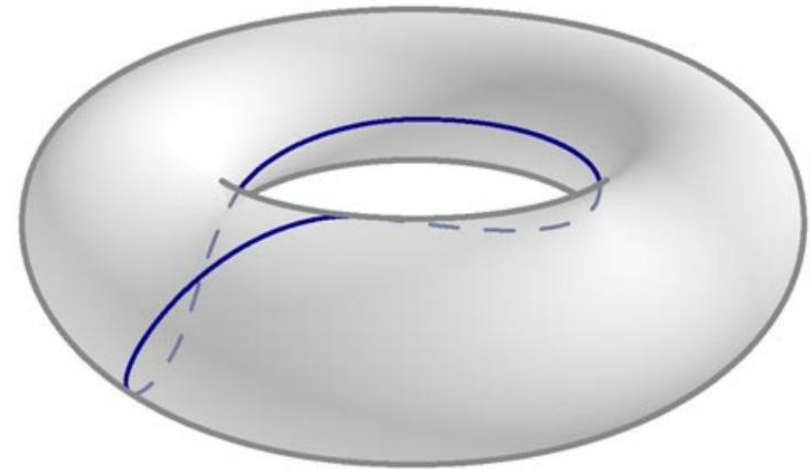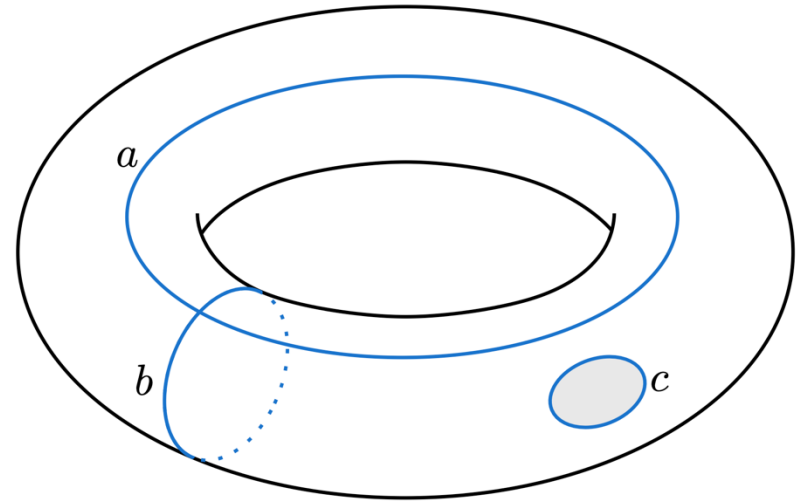
Full Algebraic Structures on Cycle Space

- We will briefly explain the full algebraic structures for the cycles and boundaries we described so far; see the textbooks for detailed formulation

Full Algebraic Structures on Cycle Space

- Recall the following picture.

- We have that all p-chains for simplicial complex K not only form a set, but also form a **vector space** (object studied by linear algebra; see: https://en.wikipedia.org/wiki/Vector_space), denoted by $C_p(K)$

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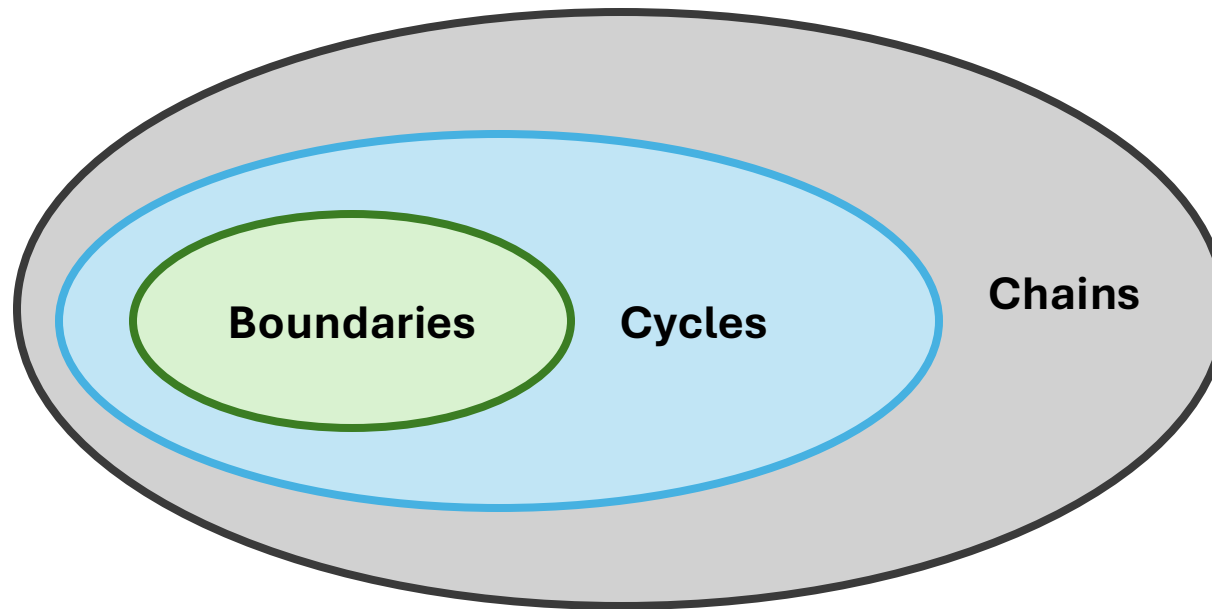
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- Recall the following picture.

- We have that all p-chains for simplicial complex K not only form a set, but also form a **vector space** (object studied by linear algebra; see: https://en.wikipedia.org/wiki/Vector_space), denoted by $C_p(K)$

- Furthermore, boundaries and cycles are not only subsets but also **vector subspaces** of $Z_p(K)$, denoted $B_p(K)$ and $Z_p(K)$ respectively.

Boundaries   **Cycles**   **Chains**
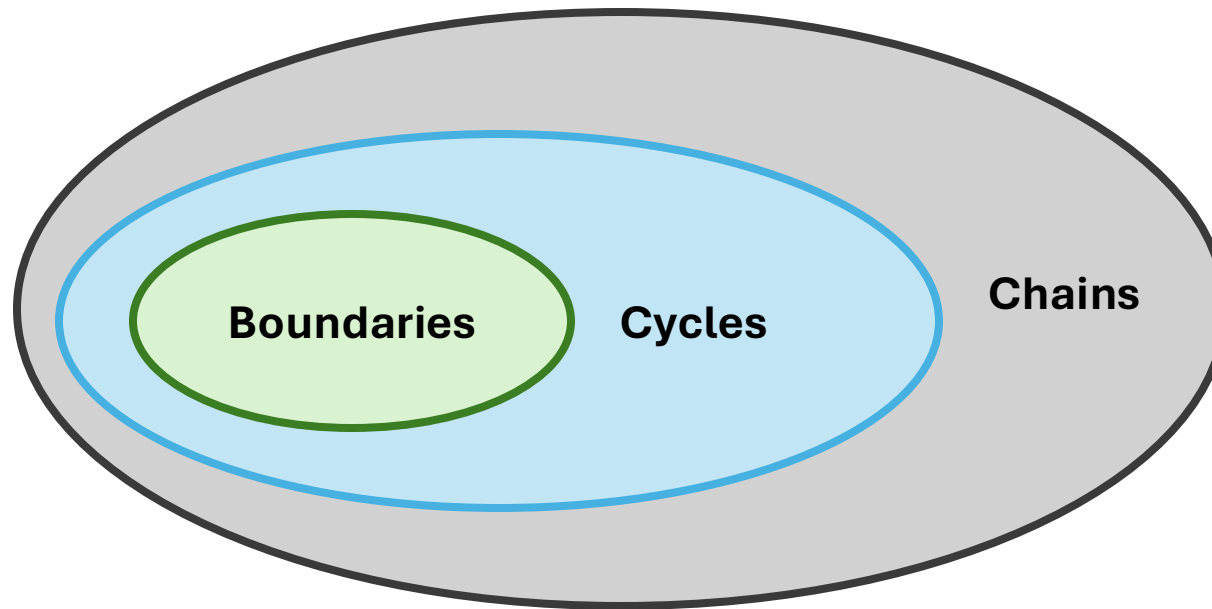
$$B_p(K) \subseteq Z_p(K) \subseteq C_p(K)$$

Homology group

- The p-dimensional homology of a simplicial complex can then be described by the following **quotient group**:

$$H_p(K) = Z_p(K)/B_p(K)$$

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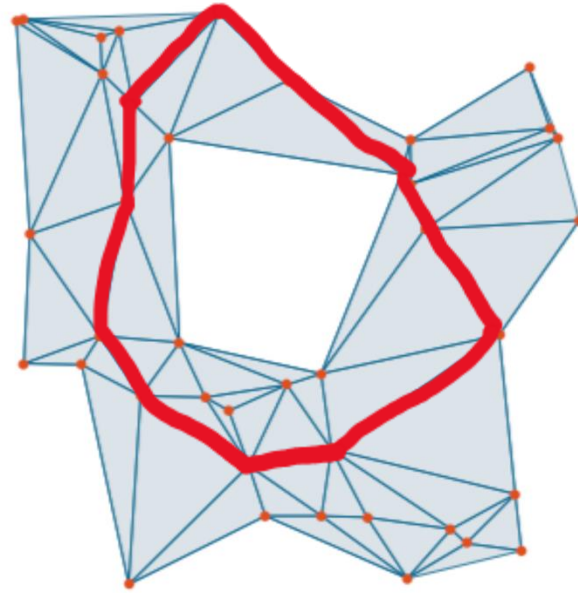   - See: https://en.wikipedia.org/wiki/Basis_(linear_algebra)

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- BTW, the cardinality (number of elements) of the homology basis for the p-dimensional cycles is called the **p-th Betti number**, denoted β_p.

Betti number

- $\beta_1 = 1$

Betti number

- $\beta_1 = 2$