# Divide and Conquer

Tao Hou

## The Divide-and-Conquer Paradigm

- **Divide phase:** Divide the problem into subproblems
- **Conquer phase:** Conquer/solve the subproblems (recursively)
- **Combine phase:** Combine the solutions to the subproblems into a solution for the whole problem

## Example: Merge Sort (Review)

- **Divide phase:** Divide the array into two halves from the middle
- **Conquer Phase:** Sort each half recursively
- **Combine phase:** Merge the two sorted halves

## Merge Sort

```
MERGESORT($A$)
1  if $length(A) == 1$
2        return $A$
3  $m = \lfloor length(A)/2 \rfloor$
4  $A_L = $ MERGESORT($A[1\,..\,m]$)
5  $A_R = $ MERGESORT($A[m+1\,..\,length(A)]$)
6  return MERGE($A_L, A_R$)
```

- What the MERGE routine does: given two sorted arrays, return a single sorted array containing all elements of the given two arrays
- The MERGE routine runs in $O(n)$ time where $n$ is the size of the larger given array

## MERGE Algorithm

```
MERGE$(A, B)$
$i, j = 1$
$X = \emptyset$
while $i \leq length(A)$ and $j \leq length(B)$
    if $A[i] \leq B[j]$
        $X = X \circ A[i]$          // appends $A[i]$ to $X$
        $i = i + 1$
    else
        $X = X \circ B[j]$
        $j = j + 1$
while $i \leq length(A)$
    $X = X \circ A[i]$
    $i = i + 1$
while $j \leq length(B)$
    $X = X \circ B[j]$
    $j = j + 1$
return $X$
```

## Run-Time Analysis of Merge Sort

**Input Size:** $n$

$$T(n) = \begin{cases} C_1 & \text{if n=1} \\ 2T(n/2) + n * C_2 & \text{otherwise} \end{cases}$$

Q: How to solve it?

## The Master Theorem

Let $a \geq 1$, $b > 1$, $f(n) = O(n^d)$ where $d \geq 0$, and $c = \log_b a$

$$T(n) = \begin{cases} O(1) & \text{if } n = O(1) \\ aT(n/b) + f(n) & \text{otherwise} \end{cases}$$

1. $c < d$: $T(n) = \Theta(f(n)) = \Theta(n^d)$
2. $c > d$: $T(n) = \Theta(n^c)$
3. $c = d$: $T(n) = \Theta(n^c \log n)$

## The Master Theorem

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3. $c = d$: $T(n) = \Theta(n^c \log n)$

Remark: For case 1, $f(n)$ must also satisfy a *regularity condition* which states that there is some $C < 1$ such that $a \cdot f(n/b) \leq C \cdot f(n)$ for sufficiently large $n$. This regularity condition is almost always true and we will not worry about it.

## Run-Time Analysis of Merge Sort using Master Theorem

$$T(n) = \begin{cases} C_1 & \text{if n=1} \\ 2T(n/2) + C_2 \cdot n & \text{otherwise} \end{cases}$$

Applying the Master Theorem with $a = 2$, $b = 2$, and $d = 1$, we get $c = \log_2 2 = d$ and $T(n) = \Theta(n \log n)$

## Master Theorem: Additional Examples

### Example 1

$$T(n) = T(n/2) + 5n$$

Applying the Master Theorem with $a = 1$, $b = 2$, $d = 1$, we get $c = 0 < d$ and hence $T(n) = \Theta(n)$

## Master Theorem: Additional Examples

### Example 1

$$T(n) = T(n/2) + 5n$$

Applying the Master Theorem with $a = 1$, $b = 2$, $d = 1$, we get $c = 0 < d$ and hence $T(n) = \Theta(n)$

### Example 2

$$T(n) = 4T(n/2) + 2n$$

Applying the Master Theorem with $a = 4$, $b = 2$, $d = 1$, we get $c = 2 > d$ and hence $T(n) = \Theta(n^2)$

## Examples: Using the Master Theorem

### Example 3

$$T(n) = T(n-5) + n$$

- The Master Theorem does not apply here.
- The *iteration method* (briefly reviewed next) can be used to solve this equation

## Run-Time Analysis of Merge Sort (Iteration Method)

We can also solve $T(n)$ using the *Iteration Method* (aka. keep on expanding the formula by applying $T(n)$ to itself, until reaching the base case):

$$\begin{aligned}
(1): T(n) &= 2T(n/2) + C_2 n \\
(2): T(n) &= 2^2 T(n/2^2) + 2C_2 n \\
(3): T(n) &= 2^3 T(n/2^3) + 3C_2 n \\
&\vdots \\
(i): T(n) &= 2^i T(n/2^i) + i \cdot C_2 n
\end{aligned}$$

We stop iterating when $n/2^i = 1$

Setting $n/2^i = 1$ gives a number of iterations $i = \log n$

Plugging the value of $i = \log n$ gives:

$T(n) = 2^i T(n/2^i) + i \cdot C_2 n = 2^{\log n} C_1 + n \cdot \log n = nC_1 + \log n \cdot C_2 \cdot n = \Theta(n \log n)$

## Quicksort (Review)

- ***Divide:*** Partition $A$ into $A[1 \ldots q-1]$ and $A[q+1 \ldots n]$ such that

$$A[1], \ldots, A[q-1] \leq A[q] \leq A[q+1], \ldots, A[n]$$

  - The partition is done by a PARTITION procedure which may change the positions of elements
  - $q$ is returned from the partition procedure and in general we don't have any control over $q$

- ***Conquer:*** Sort $A[1 \ldots q-1]$ and $A[q+1 \ldots n]$ recursively

- ***Combine:*** Nothing to do here

```
QUICKSORT($A$, $begin$, $end$)
1  if $begin < end$
2      $q$ = PARTITION($A$, $begin$, $end$)
3      QUICKSORT($A$, $begin$, $q - 1$)
4      QUICKSORT($A$, $q + 1$, $end$)
```

## Partition

```
PARTITION(A, begin, end)
1  $q = begin$
2  $v = A[end]$
3  for $i = begin$ to $end - 1$
4      if $A[i] < v$
5          swap $A[i]$ and $A[q]$
6          $q = q + 1$
7  swap $A[q]$ and $A[end]$
8  return $q$
```

- Runs in $\Theta(n)$ time
- Further remarks:
  - Assume a *pivot* (center of the partition) $v$ to be at the end
  - Loop invariant (always **true** at the beginning of each iteration):
    $q$ is a separation of $A[begin \dots i-1]$ s.t.

$$A[begin], \dots, A[q-1] < v \text{ and } A[q], \dots, A[i-1] \geq v$$

## Worst-Case Run-Time Analysis of Quick Sort (Review)

**Input Size:** $n$
**Worst Case:** The array partition is very skewed: 0 element on one side, pivot, and the rest on the other side (the pivot is the smallest or largest element)

$$T(n) = \begin{cases} 1 & \text{if n=1} \\ T(n-1) + n & \text{otherwise} \end{cases}$$

We cannot solve $T(n)$ using the master method.

## Using the Iteration Method

We solve $T(n)$ by expanding the recursive formula directly:

$$\begin{aligned}
(1): T(n) &= T(n-1)+n \\
(2): T(n) &= T(n-2)+n-1+n \\
(3): T(n) &= T(n-3)+n-2+n-1+n \\
&. \\
&. \\
(i): T(n) &= T(n-i)+(n-i+1)+(n-i+2)+\cdots+n
\end{aligned}$$

We stop expanding when $n-i=1$
Setting $n-i=1$ gives $i=n-1$
Plugging this value of $i$ in the generic form gives
$T(n) = T(1)+2+3+\cdots+n = 1+2+3+\cdots+n = n(n+1)/2 = \Theta(n^2)$

## Average-Case Run-Time Analysis of Quick Sort (Advanced)

- Idea: count the number of comparisons
- Rename elements (assumed to be distinct) in $A$ as $z_1 < z_2 < \cdots < z_n$
- Define a random variable $X_{ij}$ as:

$$X_{ij} = \begin{cases} 0 & \text{if } z_i \text{ and } z_j \text{ does not compare} \\ 1 & \text{if } z_i \text{ and } z_j \text{ does compare} \end{cases}$$

- The random variable for the number of comparison is:

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

## Average-Case Run-Time Analysis of Quick Sort (Advanced)

- We have

$$\begin{aligned} E[X] &= E\Big[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\Big] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \end{aligned}$$

- By some analysis (we omit),

$$E[X_{ij}] = \frac{2}{j-i+1}$$

## Average-Case Run-Time Analysis of Quick Sort (Advanced)

- Then

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k} \\
&< \sum_{i=1}^{n-1} \left(2 \sum_{k=1}^{\infty} \frac{1}{k}\right) \qquad \text{(inner sum } \textit{harmonic series}\text{)} \\
&= \sum_{i=1}^{n-1} O(\log n) = O(n \log n)
\end{aligned}$$

## Selection

**Problem**

Given an (unsorted) array $A[1 \dots n]$ of numbers and $k \in \mathbb{N}$, find the $k$-th smallest number in $A$

## A First Random Solution

(i) **Divide:** Randomly select a pivot from $A$, partition $A$ into two subarrays $L$ and $R$ s.t. elements in $L \leq$ elements in $R$

(ii) **Conquer:** If $k \leq |L|$, recurse to find the $k$-th smallest element in $L$; otherwise, recurse to find the $(k - |L|)$-th smallest element in $R$

## Random select

```
RandSelect(A, $k$)

1. if $|A| == 1$ then return $A[1]$;
2. $L, R =$ Partition($A$);
3. if $k \leq |L|$ then return RandSelect($L$, $k$);
4. else return RandSelect($R$, $k - |L|$);
```

Figure 1: The algorithm RandSelect

## Random solution: Complexity

(Analysis similar to quicksort)

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- Best case: $O(n)$
- Worst case: $O(n^2)$

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(Analysis similar to quicksort)

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- Worst case: $O(n^2)$
- Average case: $O(n)$

## Linear-Time Selection

Can we have a selection algorithm which runs in linear time in the worst case?

- Observe that the previous random selection runs in quadratic time because sometimes the partition can be unbalanced
- Can we try to choose a “good” pivot for the partition each time so that the partitioned arrays are always balanced?

## Linear-Time Selection

Can we have a selection algorithm which runs in linear time in the worst case?

- Observe that the previous random selection runs in quadratic time because sometimes the partition can be unbalanced
- Can we try to choose a “good” pivot for the partition each time so that the partitioned arrays are always balanced?
- The answer is that we can

## Linear-Time Selection

**Solution:**

(i) Partition the array into $m = \lceil n/5 \rceil$ subarrays, each consisting of 5 (maybe less) consecutive elements

(ii) Find the median of each of the $m$ arrays by brute force

(iii) Recursively find the median $M$ of the $m$ medians

(iv) Using $M$ as pivot, partition $A$ into two subarrays $L$ and $R$

(v) If $k \leq |L|$, recurse to find the $k$-th smallest element in $L$; otherwise, recurse to find the $(k - |L|)$-th smallest element in $R$

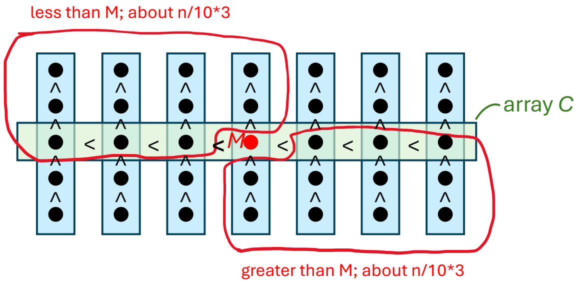
## The Selection Algorithm

```
Select(A, k)
1. if $n \le 25$ then return the $k$-th smallest element in $A$ by brute force;
2. $m = \lceil n/5 \rceil$; create an array $C[1..m]$;
3. for $i = 1$ to $m$ $C[i] :=$ the median of $A[(5i-4)..(5i)]$;
4. $M = \text{Select}(C, m/2)$;
5. Partition $A$ using $M$ as the pivot into $L$ and $R$, where $L$ contains all elements
   that are smaller or equal to $M$ and $R$ contains the rest;
6. if $k \le |L|$ then return $\text{Select}(L, k)$;
7. else return $\text{Select}(R, k - |L|)$;
```

Figure 2: The algorithm Select

## Run-Time Analysis of Select

- Take $n = 35$
- For simplicity, assume all elements are distinct
- Order each small array, and then order the 7 small arrays by their medians



## Run-Time Analysis of Select

In general:

- Ignore the floors and ceilings
- The number of medians in the array $C$ less than $M$ is: $(1/2) \cdot (n/5) = n/10$
- The number of other elements less than $M$ is at least: $2n/10$
- So, at lease $3n/10$ elements is less than $M$
- Similarly, at lease $3n/10$ elements is greater than $M$
- Whether we go to $L$ or $R$ in the algorithm, we drop at least $3n/10$ elements (i.e., keep at most $7n/10$ elements).

## Run-Time Analysis of Select

$$T(n) \leq \begin{cases} O(1) & \text{if } n \leq 25 \\ T(7n/10) + T(n/5) + O(n) & \text{otherwise} \end{cases}$$

We cannot solve $T(n)$ using the master method.

Instead, use the *substitution* method:

1. Guess the solution
2. Plug in the guess and prove the equation to be true based on the assumption that the equation is true for sub-cases

Notice: The substitution method is in some sense a proof by *induction*

## Run-Time Analysis of Select

- Our induction hypothesis:
  - Suppose that $T(i) \leq c \cdot i$ for any $i < n$, where $c$ is a constant
  - Want to prove that $T(n) \leq c \cdot n$, i.e., $T(n) = O(n)$

## Run-Time Analysis of Select

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- We have

$$\begin{aligned} T(n) &\le T(7n/10) + T(n/5) + O(n) \\ &\le c \cdot (7n/10) + c \cdot (n/5) + c'n \\ &= 9cn/10 + c'n = cn \cdot (9/10 + c'/c) \end{aligned}$$

## Run-Time Analysis of Select

- Our induction hypothesis:
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- We have

$$
\begin{aligned}
T(n) &\leq T(7n/10) + T(n/5) + O(n) \\
&\leq c \cdot (7n/10) + c \cdot (n/5) + c'n \\
&= 9cn/10 + c'n = cn \cdot (9/10 + c'/c)
\end{aligned}
$$

- So we only need to choose a $c$ s.t. $c'/c + 9/10 \leq 1$, which is $c \geq 10c'$, so that we will have

$$T(n) \leq cn \cdot (9/10 + c'/c) \leq cn$$

## The Closest Pair of Points

**Problem**

Given a set $S = \{p_1, \ldots, p_n\}$ of points in the plane, where $p_i = (x_i, y_i)$, compute a closest-pair of points in $S$, that is, a pair of distinct points $p_i, p_j \in S$ such that $|p_i p_j| = \min\{|p_r p_s| : \; p_r \neq p_s \in S\}$

Note: we assume the points in $S$ to have *distinct* coordinates; if there are duplicate points in $S$, this is easy to pre-check and the answer is 0

## The Closest-Pair Algorithm: Overview

- **Divide:** Partition the input set $S$ into two sets $S_L$ and $S_R$ of the same size s.t. points in $S_L$ are to the left of points in $S_R$
- **Conquer:** Recursively find the minimum distances of $S_L$ and $S_R$
- **Combine:** Find the minimum distance of point pairs where one is from $S_L$ and the other is from $S_R$; return the minimum of the three minimums

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We aim to achieve $O(n)$ time for both the divide and combine phase so that the entire complexity is $O(n \log n)$

## Preprocessing Step

- Let $X$ be a list containing the points in $S$ sorted w.r.t. their $x$-coordinates, and $Y$ a list containing the points in $S$ sorted w.r.t. their $y$-coordinates. Clearly, $X$ and $Y$ can be obtained in $O(n \log n)$ time (we only do this *once* at the beginning).

So the input to the algorithm, i.e., the set of points, is encoded as a tuple of three arrays $(S, X, Y)$

## Divide Phase

- Partition $S$ into $S_L$ and $S_R$ of equal size s.t. points in $S_L$ are to the *left* of $S_R$ using a central vertical line $D$
- Let $X_L$, $Y_L$ each represent the set of points in $S_L$ sorted by x- and y-coordinates respectively; $X_R$ and $Y_R$ are similarly defined for $S_R$

## Divide Phase: Pseudocode

```
1.  $m = |X|/2$
2.  $D = X[m].x$
3.  $X_L = X[1 \dots m]$
4.  $X_R = X[m+1 \dots |X|]$
5.  **for** $i = 1 \dots |Y|$:
6.      **if** $Y[i].x \leq D$:
7.          append $Y[i]$ to $Y_L$
8.      **else**:
9.          append $Y[i]$ to $Y_R$
10. separate $S$ into $S_L, S_R$ similarly
```

Figure 3: Divide phase

## Conquer Phase

- Recursively call the algorithm on $(S_L, X_L, Y_L)$ to obtain the min-distance $\delta_L$ for $S_L$, and on $(S_R, X_R, Y_R)$ to obtain the min-distance $\delta_R$ for $S_R$.

## Combine Phase

### Idea

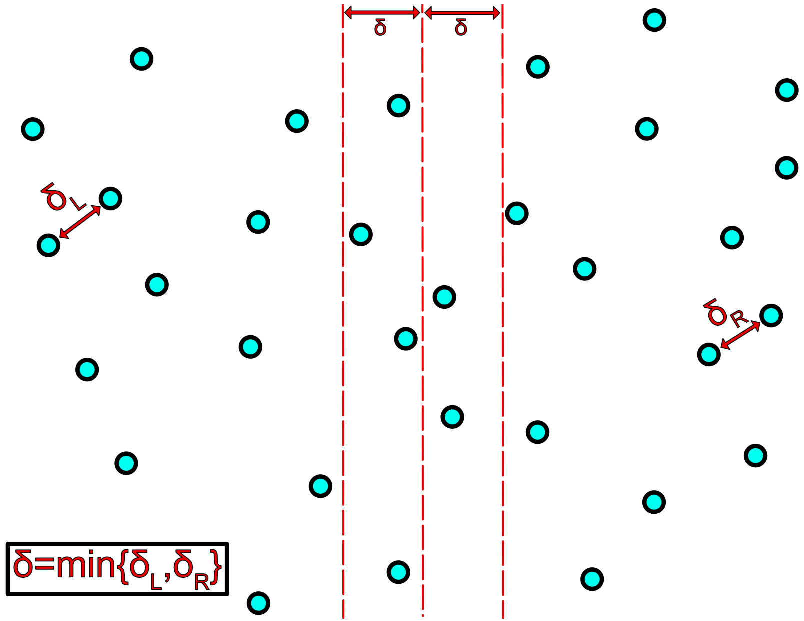
- We have:
  - $\delta_L$: The min dis of pairs in $S_L$
  - $\delta_R$: The min dis of pairs in $S_R$
- Aim of combine phase: Compute the min-dis of the pairs where one point is from $S_L$ and the other is from $S_R$ (i.e., pairs of points from different sides)
- Answer: The minimum of above three minimums

## Details of Combine Phase

**The first observation**

- Let $\delta = \min\{\delta_L, \delta_R\}$
- We only need to consider pairs within a $2\delta$-**wide vertical strip centered around** $D$

## Consider only $2\delta$-wide vertical strip centered around $D$



## Explanation

- We have computed min-dis of points from the same side, which is $\delta$.
- So, to compute the overall min-dis, we can ignore those point pairs whose distances are greater than $\delta$.

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- So, to compute the overall min-dis, we can ignore those point pairs whose distances are greater than $\delta$.

- If two points from different sides are not both from the $2\delta$-wide vertical strip (at least one point is outside the strip), then their distance is greater than $\delta$, and so we can ignore them.

## The Combine Phase

- Let $\delta = \min\{\delta_L, \delta_R\}$
- From $Y$, create $Y_{mid}$ (also sorted by y-coordinates) which is the set of points within the $2\delta$-wide vertical strip centered around $D$

## The Combine Phase

- Let $\delta = \min\{\delta_L, \delta_R\}$
- From $Y$, create $Y_{mid}$ (also sorted by y-coordinates) which is the set of points within the $2\delta$-wide vertical strip centered around $D$
- Go over $Y_{mid}$, and for each point $p$, compute its distance to **at most 7** points in $Y_{mid}$ that follow $p$, and keep track of the min-distance
- Return the smaller of $\delta$ and what we have by scanning $Y_{mid}$

## Combine Phase: Pseudocode

```
1. for $i = 1 \ldots |Y|$:
2.     if $Y[i].x \geq D - \delta$ and $Y[i].x \leq D + \delta$:
3.         append $Y[i]$ to $Y_{mid}$
4. $\bar{\delta} = \infty$
5. for $i = 1 \ldots |Y_{mid}|$:
6.     for $j = 1 \ldots 7$:
7.         if $i + j \leq |Y_{mid}|$ and $\mathrm{dis}(Y_{mid}[i], Y_{mid}[i+j]) < \bar{\delta}$ then
8.             $\bar{\delta} = \mathrm{dis}(Y_{mid}[i], Y_{mid}[i+j])$
9. return $\min\{\delta, \bar{\delta}\}$
```

Figure 4: Combine phase

## Why only scan 7 points?

- For each point $p$ in $Y_{mid}$, we only need to consider other points in $Y_{mid}$ whose distances to $p$ is $< \delta$. This means we only need to consider points within a $2\delta \times 2\delta$ square of $p$.

## Why only scan 7 points?

- For each point $p$ in $Y_{mid}$, we only need to consider other points in $Y_{mid}$ whose distances to $p$ is $< \delta$. This means we only need to consider points within a $2\delta \times 2\delta$ square of $p$.
- **Key observation**: Each $\delta \times \delta$ square contains at most 4 points
    - This square is totally within the left or right side of the vertical separator $D$, meaning that points in the square are either all from $S_L$ or all from $S_R$, so these points are at least $\delta$-distance apart
    - A fact from computational geometry says that such a square cannot fit in more than 4 points

## Why only scan 7 points?

- Therefore, each $2\delta \times \delta$ square contains at most 8 points (including $p$)
- So we only need to scan the 7 points that precede $p$ (ones that are in the upper $2\delta \times \delta$ square) and the 7 points that follow $p$ (ones that are in the lower $2\delta \times \delta$ square) in $Y_{mid}$.
- Further observation: we only need to scan the 7 points that follow $p$, and ignore the 7 points that precede $p$:
  - Suppose there is a point $q$ preceding $p$ in $Y_{mid}$ falling within the upper $2\delta \times \delta$ square for $p$. Then $p$ also falls in the lower $2\delta \times \delta$ square for $q$. So we have checked the pair $p, q$ when we scan $q$.

## The Closest-Pair Algorithm

```
Closest-Pair-Algo
1. if $|S| \leq 3$ return a closes pair $(p_{min}, q_{min})$ in $S$ by brute force;
2. using $X$, compute a vertical line $D$ of equation $x = \ell$ that partitions $S$ into
   $S_L, S_R$ of equal size such that all points in $S_L$ are on $D$ or to the left of it, and
   all points in $S_R$ are on $D$ or to the right of it;
3. using $X$ and $Y$, create the arrays $X_L$, $Y_L$ and $X_R$, $Y_R$;
4. recurse on $S_L, X_L, Y_L$ to compute a closest pair $(p_L, q_L)$; let $\delta_L = |p_L q_L|$;
5. recurse on $S_R, X_R, Y_R$ to compute a closest pair $(p_R, q_R)$; let $\delta_R = |p_R q_R|$;
6. let $\delta = \min\{\delta_L, \delta_R\}$;
7. let $S_{mid}$ be the set of points in $S$ whose $x$-coordinate satisfies $\ell - \delta \leq x \leq x + \delta$;
8. using $Y$, compute the list of points in $S_{mid}$ sorted by their $y$-coordinates;
9. go over $Y_{mid}$ (in the sorted order), and for each point, compute its distance to
   the next (at most) 7 points in $Y_{mid}$ and keep track of the pair of points
   $(p_{mid}, q_{mid})$ of minimum distance;
10. return the closest pair $(p_{min}, q_{min})$ among, $(p_L, q_L), (p_R, q_R)$, and $(p_{mid}, q_{mid})$;
```

Figure 5: The algorithm Closest Pair

## Run-Time Analysis of Closest-Pair

Let $T(n)$ be the running time of Closest-Pair in the worst case on $n$ points.

- Divide phase takes $O(n)$ time.
- Combine phase takes $O(n)$ time.
- Recursive call on $(S_L, X_L, Y_L)$ takes $T(n/2)$ time;
  recursive call on $(S_R, X_R, Y_R)$ takes $T(n/2)$ time.

Therefore, $T(n)$ obeys the following recurrence relation:

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 3 \\ 2T(n/2) + O(n) & \text{otherwise} \end{cases}$$

We can solve $T(n)$ using the Master Theorem to obtain $T(n) = O(n \lg n)$

## Integer Multiplication

**Problem**

Multiply two integers $x, y$ represented as sequences (e.g., arrays) of 0-1 bits where the lengths of the sequences can be **arbitrarily** large (assume the length of the two to be both $n$, with possibly padding 0's)

Notice: This cannot be simply done in constant time: the multiplication of provided by the CPU only supports a *fixed* length on the sequence (e.g., 64).

## An Algorithm Everybody Knows

Solution:

- Compute a “partial product” by multiplying each digit of y separately by x, and then you add up all the partial products.
- Only this time we do the *binary* version, i.e., we multiplying each *bit* of y by x and then add up.

$$
\begin{array}{r}
12 \\
\times\, 13 \\
\hline
36 \\
12\phantom{0} \\
\hline
156
\end{array}
\qquad\qquad
\begin{array}{r}
1100 \\
\times\, 1101 \\
\hline
1100 \\
0000\phantom{0} \\
1100\phantom{00} \\
1100\phantom{000} \\
\hline
10011100
\end{array}
$$

**(a)** **(b)**

(Figure taken from [Kleinberg&Tardos - Algorithm Design])

Time complexity:

## An Algorithm Everybody Knows

Solution:

- Compute a “partial product” by multiplying each digit of y separately by x, and then you add up all the partial products.
- Only this time we do the *binary* version, i.e., we multiplying each *bit* of y by x and then add up.

```
                 1100
               × 1101
               ------
                 1100
   12           0000
 × 13          1100
 ----         1100
   36       --------
  12        10011100
 ----
  156

  (a)          (b)
```

(Figure taken from [Kleinberg&Tardos - Algorithm Design])

Time complexity: $O(n^2)$

## Attempting to Improve the Naive Algorithm

Using, of course, divide and conquer:

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Using, of course, divide and conquer:

- Write $x$ as $x = x_1 \cdot 2^{n/2} + x_0$, where $x_1$ is the “high-order” half bits $x_0$ is the “low-order” half bits
- Similarly write $y$ as $y = y_1 \cdot 2^{n/2} + y_0$

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- Similarly write $y$ as $y = y_1 \cdot 2^{n/2} + y_0$
- Rewrite $xy$ as

$$\begin{aligned} xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\ &= x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0 \end{aligned}$$

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So, to compute $xy$ (multiplying two $n$-sequences), we only need to:

- Recursively compute four multiplications of $n/2$-sequences:

$$x_1 y_1,\ x_1 y_0,\ x_0 y_1, \text{ and } x_0 y_0$$

- Then take the sum $x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0$ (which can be done in $O(n)$ time)

## Pseudocode

```
Recursive-Multiply($x$,$y$)
  1. write $x = x_1 \cdot 2^{n/2} + x_0,\ y = y_1 \cdot 2^{n/2} + y_0$
  2. $x_1 y_1 =$ Recursive-Multiply($x_1, y_1$)
  3. $x_1 y_0 =$ Recursive-Multiply($x_1, y_0$)
  4. $x_0 y_1 =$ Recursive-Multiply($x_0, y_1$)
  5. $x_0 y_0 =$ Recursive-Multiply($x_0, y_0$)
  6. return $x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0$
```

Figure 6: Recursive Multiply

Time complexity:

## Pseudocode

```
Recursive-Multiply($x$,$y$)
  1. write $x = x_1 \cdot 2^{n/2} + x_0,\ y = y_1 \cdot 2^{n/2} + y_0$
  2. $x_1 y_1 = \text{Recursive-Multiply}(x_1, y_1)$
  3. $x_1 y_0 = \text{Recursive-Multiply}(x_1, y_0)$
  4. $x_0 y_1 = \text{Recursive-Multiply}(x_0, y_1)$
  5. $x_0 y_0 = \text{Recursive-Multiply}(x_0, y_0)$
  6. return $x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0$
```

Figure 6: Recursive Multiply

Time complexity:

- $T(n) = 4T(n/2) + O(n)$ which is $O(n^2)$ (no improvement at all!)

## Second Attempt to Improve the Naive Algorithm

- The problem with the previous divide-and-conquer approach is that it involves **four** recursive calls
- If we can reduce the number of recursive calls to **three**, we would have

  $$T(n) = 3T(n/2) + O(n)$$

  which is $O(n^{1.59})$ (quite an improvement!)

## Second Attempt to Improve the Naive Algorithm

- Notice that our goal is to compute the sum

$$xy = x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0 \tag{1}$$

## Second Attempt to Improve the Naive Algorithm

- Notice that our goal is to compute the sum

$$xy = x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0 \tag{1}$$

- Consider another multiplication

$$p = (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$$

  where we observe $x_1 y_0 + x_0 y_1 = p - x_1 y_1 - x_0 y_0$

## Second Attempt to Improve the Naive Algorithm

- Notice that our goal is to compute the sum

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- Consider another multiplication

$$p = (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$$

  where we observe $x_1 y_0 + x_0 y_1 = p - x_1 y_1 - x_0 y_0$
- So, to get the three components in the sum (1), we only need the three multiplications of $n/2$-sequences:

$$x_1 y_1,\ x_0 y_0, \text{ and } p = (x_1 + x_0)(y_1 + y_0)$$

  by letting $x_1 y_0 + x_0 y_1 = p - x_1 y_1 - x_0 y_0$
- And then we can get $xy$ with only three recursive calls!

## Pseudocode (Improved)

```
Recursive-Multiply(x,y)
  1. write $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$
  2. compute $x_1 + x_0$ and $y_1 + y_0$
  3. $p =$ Recursive-Multiply($x_1 + x_0, y_1 + y_0$)
  4. $x_1 y_1 =$ Recursive-Multiply($x_1, y_1$)
  5. $x_0 y_0 =$ Recursive-Multiply($x_0, y_0$)
  6. return $x_1 y_1 \cdot 2^n + (p - x_1 y_1 - x_0 y_0) \cdot 2^{n/2} + x_0 y_0$
```

Figure 7: Recursive Multiply (Improved)

Time complexity:

- $T(n) = 3T(n/2) + O(n)$ which is $O(n^{1.59})$