We are now ready to prove the case $n = 2k+1$.

Lemma 1. *Let $z_1, \dots, z_n$ be distinct points on the unit circle with*

$$|P'(z_j)| = \prod_{k \neq j} |z_j - z_k| = C \quad \textit{independent of } j.$$

If n is odd, then $z_1, \dots, z_n$ are the vertices of a regular n-gon (up to rotation), and $C = n$.

Proof. Let $z_1, \dots, z_n$ be distinct complex numbers with $|z_j| = 1$, and let

$$P(z) = \prod_{k=1}^{n} (z - z_k)$$

be the monic polynomial with these zeros. Then

$$P'(z_j) = \prod_{k \neq j} (z_j - z_k), \qquad |P'(z_j)| = \prod_{k \neq j} |z_j - z_k|.$$

By hypothesis, $|P'(z_j)| = C > 0$ is independent of j.

Step 1: A conjugation identity at the zeros. Let $S := \prod_{m=1}^{n} z_m$ (so $|S| = 1$). Using $\overline{z} = 1/z$ on the unit circle,

$$\overline{P'(z_j)} = \prod_{k \neq j} (\overline{z_j} - \overline{z_k}) = \prod_{k \neq j} \left(\frac{1}{z_j} - \frac{1}{z_k} \right) = \frac{\prod_{k \neq j} (z_k - z_j)}{z_j^{n-1} \prod_{k \neq j} z_k} = \frac{(-1)^{n-1}}{z_j^{n-2} S} P'(z_j).$$

Set $\beta := (-1)^{n-1}/S$, so $|\beta| = 1$. Then

$$\overline{P'(z_j)} = \beta \, z_j^{-(n-2)} \, P'(z_j) \qquad (j = 1, \dots, n). \tag{1}$$

Step 2: The phases of $P'(z_j)$. Write $P'(z_j) = C \, \eta_j$ with $|\eta_j| = 1$. From (1),

$$\overline{\eta_j} = \frac{\overline{P'(z_j)}}{|P'(z_j)|} = \beta \, z_j^{-(n-2)} \, \eta_j,$$

hence $\eta_j^2 = \beta^{-1} z_j^{n-2}$. Choose σ with $|\sigma| = 1$ and $\sigma^2 = \beta^{-1}$. Then for some $\varepsilon_j \in \{\pm 1\}$,

$$\eta_j = \sigma \, \varepsilon_j \, z_j^{\frac{n-2}{2}}, \qquad \text{so} \qquad \frac{1}{P'(z_j)} = \frac{\overline{\eta_j}}{C} = \frac{\overline{\sigma}}{C} \, \varepsilon_j \, z_j^{-\frac{n-2}{2}}. \tag{2}$$

(Here n is odd, so $(n-2)/2$ is a half-integer; any choice of the square root of z_j is allowed, and the final identities below depend only on *odd* powers, hence are independent of the branch.)

Step 3: Lagrange (barycentric) identities. For a monic P with simple zeros z_j, the classical Lagrange identities give

$$\sum_{j=1}^{n} \frac{z_j^m}{P'(z_j)} = \begin{cases} 0, & m = 0, 1, \dots, n-2, \\ 1, & m = n-1. \end{cases} \tag{3}$$

(For instance, interpolate the monomial z^m at the nodes z_j and compare the z^{n-1}-coefficients.) Using (2) in (3) for $m = 0, 1, \dots, n-2$,

$$\sum_{j=1}^{n} \varepsilon_j \, z_j^{m - \frac{n-2}{2}} = 0.$$

Fix any w_j with $w_j^2 = z_j$, and define $u_j := \varepsilon_j w_j$ (so $|u_j| = 1$ and $u_j^2 = z_j$). Then, because $k := 2m - (n-2)$ is *odd* for $m = 0, 1, \dots, n-2$ (and n odd),

$$\sum_{j=1}^{n} u_j^k = 0 \qquad \text{for all odd } k \in \{-(n-2), -(n-4), \dots, n-2\}. \tag{4}$$

In particular,

$$\sum_{j=1}^{n} u_j^k = 0 \qquad (k = 1, 3, \dots, n-2). \tag{5}$$

Step 4: A polynomial with zeros u_j. Let

$$Q(t) := \prod_{j=1}^{n}(t-u_j) = t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_n,$$

and let $p_k := \sum_{j=1}^{n} u_j^k$ be the power sums. Newton's identities (for a monic polynomial) assert that for $m \geq 1$,

$$p_m + a_1p_{m-1} + a_2p_{m-2} + \cdots + a_{m-1}p_1 + ma_m = 0. \tag{6}$$

From (5) with $m = 1$ we get $p_1 = 0$, hence $a_1 = 0$. Assume inductively that $a_1, a_3, \dots, a_{m-2} = 0$ for some odd $m \leq n-2$. In (6) with this m:

- $p_m = 0$ (by (5));
- if r is even, then p_{m-r} is odd-indexed and hence 0 by (5);
- if r is odd, then $a_r = 0$ by the inductive hypothesis.

Therefore (6) reduces to $ma_m = 0$, so $a_m = 0$. By induction,

$$a_{2r+1} = 0 \qquad \text{for all } r \geq 0 \text{ with } 2r+1 \leq n-2. \tag{7}$$

Step 5: Self-inversive symmetry kills the even coefficients. Since $|u_j| = 1$ for all j,

$$\widetilde{Q}(t) := t^n\,\overline{Q\left(\frac{1}{\bar{t}}\right)}$$

has the same zeros as Q (namely $u_1, \dots, u_n$), hence there is a constant $\kappa \neq 0$ such that $\widetilde{Q} = \kappa Q$. Writing $Q(t) = \sum_{m=0}^{n} c_m t^m$ with $c_n = 1$ and $c_0 = a_n$, a direct computation shows

$$\widetilde{Q}(t) = \sum_{m=0}^{n} \overline{c_{n-m}}\, t^m.$$

Comparing coefficients in $\widetilde{Q} = \kappa Q$ yields

$$\overline{c_{n-m}} = \kappa\, c_m \qquad (m = 0, 1, \dots, n). \tag{8}$$

Since $c_m = a_{n-m}$ for $m < n$ and $c_n = 1$, (8) is equivalently

$$a_m = \mu\, \overline{a_{n-m}} \qquad (m = 0, 1, \dots, n), \tag{9}$$

with $\mu := a_n$ satisfying $|\mu| = 1$ (because $|a_n| = \prod |u_j| = 1$). As n is odd, if $2 \leq m \leq n-1$ is even then $n-m$ is odd, so by (7) we have $a_{n-m} = 0$, and thus $a_m = 0$ by (9). Combining with (7) we obtain

$$a_1 = a_2 = \cdots = a_{n-1} = 0,$$

i.e.

$$Q(t) = t^n + a_n \quad \text{with } |a_n| = 1.$$

Therefore the u_j are the nth roots of $-a_n$, equally spaced on the unit circle.

Step 6: Back to z_j and the value of C. Recall $z_j = u_j^2$. When n is odd, the map $u \mapsto u^2$ permutes the set of nth roots of unity; hence the z_j are equally spaced as well—i.e., the vertices of a regular n-gon (up to rotation). For a regular n-gon one has $P(z) = z^n - \omega^n$ for some unimodular ω, so $P'(z) = nz^{n-1}$ and therefore $|P'(z_j)| = n$. Thus $C = n$. $\square$

Theorem 1 (case $n = 2k+1$). Let $z_1, z_2, \dots, z_{2k+1} \in \mathbb{C}$ with $|z_i| = 1$, and define $s_k := \sum_{i=1}^{2k+1} z_i^k$ for $k \in \mathbb{Z}$. Assume there exist integers $a < b$ such that

$$s_{a+1} = \cdots = s_{a+2k} = s_{b+1} = \cdots = s_{b+2k} = 0.$$

Set $q := b - a$. Upon to reordering,

$$\{z_1, z_2, \dots, z_{2k+1}\} = \{1, \zeta, \dots, \zeta^{2k}\} \qquad (\zeta = e^{2\pi i/(2k+1)}).$$

Proof. By Proposition 1 we already know that $z_1, z_2, \ldots, z_{2k+1}$ are pairwise distinct and $z_i^q = 1$ for $i = 1, 2, \ldots, 2k+1$; in particular $|z_i| = 1$. Let $c_i := z_i^a$. The zero block at a gives

$$\sum_{i=1}^{2k+1} c_i = 0, \ldots \sum_{i=1}^{2k+1} c_i z_i^{2k-1} = 0.$$

This is a set of $2k$ equations with $2k+1$ unknowns. Since the rank of the coefficient matrix is known to be of rank $2k$, the kernel has a dimension 1. Therefore equivalently

$$z_1^{a+1} = \mu \prod_{2 \le i < j \le n} (z_i - z_j), \ldots, \quad z_3^{a+1} = \mu \prod_{1 \le i < j \le n-1} (z_i - z_j).$$

Taking moduli and using $|z_i| = 1$ yields

$$|\mu| |\prod_{2 \le i < j \le n} (z_i - z_j)| = \cdots = |\mu| |\prod_{1 \le i < j \le n-1} (z_i - z_j)| = 1,$$

so

$$\prod_{i \ne 1} |z_1 - z_i| = \cdots = \prod_{i \ne n} |z_n - z_i|.$$

Thus, from lemma 1, the $2k+1$ points $z_1, z_2, \ldots, z_n$ on the unit circle form an regular n-sided polygon, and the conclusion follows. $\square$