A Wasserstein Penalty Framework for Stochastic Optimization

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August 2025

1 Introduction

Consider the stochastic optimization problem

$$\min_{\mathbf{B}\in\mathbb{R}^{m\times n}} \mathbb{E}_{\xi\sim\mathbb{P}} f(\mathbf{B},\xi),$$

where $\xi \in \Xi$, write $f(\mathbf{B}) = \mathbb{E}f(\mathbf{B},\xi)$, $\nabla f(\mathbf{B},\xi) = \nabla_{\mathbf{B}} f(\mathbf{B},\xi) = \frac{\partial f(\mathbf{B},\xi)}{\partial \mathbf{B}}$.

Robust optimization has been a very popular topic in the field of optimization.

This article interprets each iteration step in stochastic optimization as a robust decision process.

Algorithm 1 DRO-based Steepest Descent(DROSD)

1: Initialize $\mathbf{B}_0 \leftarrow 0$
2: **for** $t = 0, \ldots, T-1$ **do do**
3: Compute batch gradients $\mathbf{G}_i \leftarrow \nabla_{\mathbf{B}} f(\mathbf{B}_t, \xi_{t,i})$ for $i = 1, \ldots, N_t$
4: Compute average $\overline{\mathbf{G}}_t \leftarrow \frac{1}{N_t}\sum_{i=1}^{N_t} \mathbf{G}_i$
5: Obtain $\mathbf{d}_t$ by solving a DRO subproblem related to the empirical distribution $\mathbb{P}_{t,N_t} = \sum_{i=1}^{N_t} \delta_{\nabla f(\mathbf{B}_t,\xi_{t,i})}$
6: Update parameters $\mathbf{B}_{t+1} \leftarrow \mathbf{B}_t - \mathbf{d}_t$
7: **end for**
8: **return** $\mathbf{B}_T$

Assumption 1 (Lipschitz gradient in $\mathbf{B}$)**.** *There exists $L \geq 0$ such that for all ξ and all $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{R}^{m\times n}$,*

$$\left\|\nabla_{\mathbf{B}} f(\mathbf{B}_1) - \nabla_{\mathbf{B}} f(\mathbf{B}_2)\right\| \leq L\,\|\mathbf{B}_1 - \mathbf{B}_2\|_*,$$

where $\|\cdot\|$ is a chosen matrix norm on $\mathbb{R}^{m\times n}$, with dual norm $\|\|_*$. We will take it to be the nuclear norm in this article, and we will explain why it is optimal to use the nuclear norm. Equivalently, for each fixed ξ, the map $\mathbf{B} \mapsto f(\mathbf{B},\xi)$ is L-smooth with respect to this norm (the constant L does not depend on ξ).

1.1 RELU Formulation

$$\mathbf{d}_t = \arg \min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \sup_{\substack{\mathbb{Q}\in\mathcal{M}(\mathbb{R}^{m\times n}) \\ \mathbb{Q}(\mathbb{R}^{m\times n} \\ W_1(\mathbb{P}_N,\mathbb{Q}))\leq\gamma}} \left\{\mathbb{E}_{G\sim\mathbb{Q}}[\mathrm{RELU}(\langle\mathbf{d}, G\rangle)]\right\}, \tag{Pp}$$

Algorithm 2 Steepest Descent(DROSD)

Initialize $\mathbf{B}_0 \leftarrow 0$, $\eta_0 \leftarrow 0$
for $t = 0, \dots, T-1$ **do do**
  Compute batch gradients $\mathbf{G}_i \leftarrow \nabla_{\mathbf{B}} f(\mathbf{B}_t, \xi_{t,i})$ for $i = 1, \dots, N_t$
  Compute average $\overline{\mathbf{G}}_t \leftarrow \frac{1}{N_t}\sum_{i=1}^{N_t} \mathbf{G}_i$
  $\eta_t \leftarrow \beta\eta_{t-1} + (1-\beta)\|\overline{\mathbf{G}}_t\|$
  $\mathbf{d}_t = \eta_t \frac{\mathbf{G}_t}{\|\mathbf{G}_t\|}$
  Update parameters $\mathbf{B}_{t+1} \leftarrow \mathbf{B}_t - \mathbf{d}_t$
end for
return $\mathbf{B}_T$

Algorithm 3 Steepest Descent

Initialize $\mathbf{B}_0 \leftarrow 0$
for $t = 0, \dots, T-1$ **do do**
  Compute batch gradients $\mathbf{G}_i \leftarrow \nabla_{\mathbf{B}} f(\mathbf{B}_t, \xi_{t,i})$ for $i = 1, \dots, N_t$
  Compute average $\overline{\mathbf{G}}_t \leftarrow \frac{1}{N_t}\sum_{i=1}^{N_t} \mathbf{G}_i$

$$\mathbf{d}_t \; = \; -\max\left\{0, \frac{\|\overline{\mathbf{G}}_t\|_{nuc} - C_t}{2L_w}\right\} U_t V_t^\top,$$

  where

$$U_t \Sigma V_t^\top \text{ is the thin SVD decomposition of } \overline{\mathbf{G}}_t.$$

  Update parameters $\mathbf{B}_{t+1} \leftarrow \mathbf{B}_t - \mathbf{d}_t$
end for
return $\mathbf{B}_T$

2 Formulation Using Moment Ambiguity Set

For simplicity of notation, we write $f(\boldsymbol{x}, \xi) = f(\mathbf{B}, \xi)$, where $\boldsymbol{x} \in \mathbb{R}^{mn}$ is the vectorization of $\mathbf{B}$.

2.1 Linear Formulation

Consider the robust optimization problem proposed by Delange and Ye [2]:

$$\begin{aligned}
\Psi(\boldsymbol{x}, \Delta\boldsymbol{x}, \gamma_1, \gamma_2) = \text{maximize}_{\mu, f_\xi} \quad & \mathbb{E}_{f_\xi}[h(\mathbf{x}, \Delta\boldsymbol{x}, \xi)] \\
\text{subject to} \quad & \mathbb{E}_{f_\xi}[1] = 1 \ , \quad \mathbb{E}_{f_\xi}[\nabla f(\boldsymbol{x}, \xi)] = \mu \\
& \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top] \preceq \gamma_2 \Sigma_0 \\
& \begin{bmatrix} \Sigma_0 & (\mu - \mu_0) \\ (\mu - \mu_0)^\top & \gamma_1 \end{bmatrix} \succeq 0 \\
& f_\xi(\nabla f(\boldsymbol{x}, \xi)) \geq 0 \ , \quad \forall \xi \in \mathcal{S} \ ,
\end{aligned}$$

where we take $h(\boldsymbol{x}, \Delta\mathbf{x}, \xi) = \Delta\boldsymbol{x}^T \nabla f(\boldsymbol{x}, \xi)$.

Claim 1. *Take $\bar{\gamma} = \min\{\gamma_1, \gamma_2\}$, then*

$$\Psi(\boldsymbol{x}, \Delta\boldsymbol{x}, \gamma_1, \gamma_2) = \Delta x^T \mu_0 + \sqrt{\bar{\gamma}}\sqrt{\Delta x^T \Sigma_0 \Delta x}.$$

Proof. In fact, we claim that the maximum is obtained when f_ξ is supported at a single point.

Since,

$$(\mu - \mu_0)(\mu - \mu_0)^T + \mathrm{Cov}(\nabla f(\boldsymbol{x}, \xi)) = \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top],$$

we have that

$$(\mu - \mu_0)(\mu - \mu_0)^T \preceq \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top].$$

Therefore,

$$(\mu - \mu_0)(\mu - \mu_0)^T \preceq \gamma_2 \Sigma_0. \tag{1}$$

If $\boldsymbol{y} \in \mathbb{R}^{mn}$ such that $\Sigma_0 \boldsymbol{y} = 0$, then from (4), we know that $(\mu - \mu_0)^T \boldsymbol{y} = 0$. Therefore,

$$\mu - \mu_0 \in \mathrm{row}\, \Sigma_0 = \mathrm{col}\, \Sigma_0.$$

Since

$$\begin{bmatrix} \Sigma_0 & (\mu - \mu_0) \\ (\mu - \mu_0)^\top & \gamma_1 \end{bmatrix} \succeq 0,$$

$$\Sigma_0 - \frac{1}{\gamma_1}(\mu - \mu_0)(\mu - \mu_0)^T \succeq 0.$$

Therefore,

$$(\mu - \mu_0)(\mu - \mu_0)^T \preceq \gamma_1 \Sigma_0. \tag{2}$$

Combining (4) and (5), we get that

$$(\mu - \mu_0)(\mu - \mu_0)^T \preceq \overline{\gamma} \Sigma_0. \tag{3}$$

What's more,

$$\mathbb{E}_{f_\xi}[h(\boldsymbol{x}, \Delta\boldsymbol{x}, \xi)] = \Delta\boldsymbol{x}^T \mu.$$

The only restriction for μ is (6).

$$
\begin{aligned}
& \Delta \boldsymbol{x}^T\mu \leq \Delta \boldsymbol{x}^T\mu_0 + \Delta \boldsymbol{x}^T(\mu-\mu_0) \\
= \quad & \Delta \boldsymbol{x}^T\mu_0 + (\Sigma_0^{1/2}\Delta \boldsymbol{x})^T\left(\Sigma_0^{\dagger/2}(\mu-\mu_0)\right) \\
\leq \quad & \Delta \boldsymbol{x}^T\mu_0 + \|\Sigma_0^{1/2}\Delta \boldsymbol{x}\|_2\|\Sigma_0^{\dagger/2}(\mu-\mu_0)\|_2 \\
\leq \quad & \Delta \boldsymbol{x}^T\mu_0 + \bar{\gamma}\|\Sigma_0^{1/2}\Delta \boldsymbol{x}\|_2 \\
= \quad & \Delta \boldsymbol{x}^T\mu_0 + \bar{\gamma}\sqrt{\Delta \boldsymbol{x}^T\Sigma_0\Delta \boldsymbol{x}},
\end{aligned}
$$

and exact equality holds when

$$\mu = \mu_0 + \frac{\bar{\gamma}\Sigma_0\Delta \boldsymbol{x}}{\sqrt{\Delta \boldsymbol{x}^T\Sigma_0\Delta \boldsymbol{x}}}.$$

Therefore,

$$\Psi(\boldsymbol{x}, \Delta \boldsymbol{x}, \gamma_1, \gamma_2) = \Delta \boldsymbol{x}^T\mu_0 + \bar{\gamma}\sqrt{\Delta \boldsymbol{x}^T\Sigma_0\Delta \boldsymbol{x}},$$

and the maximum is obtained when f_ξ is supported only on one point. $\square$

2.2 RELU Formulation 1

Consider the robust optimization problem proposed by Delange and Ye [2]:

$$
\begin{aligned}
\Psi(\boldsymbol{x}, \Delta \boldsymbol{x}, \gamma_1, \gamma_2) = \text{maximize}_{\mu, f_\xi} \quad & \mathbb{E}_{f_\xi}[h(\mathbf{x}, \Delta \boldsymbol{x}, \xi)] \\
\text{subject to} \quad & \mathbb{E}_{f_\xi}[1] = 1\ , \quad \mathbb{E}_{f_\xi}[\nabla f(\boldsymbol{x}, \xi)] = \mu \\
& \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top] \preceq \gamma_2\Sigma_0 \\
& \begin{bmatrix} \Sigma_0 & (\mu-\mu_0) \\ (\mu-\mu_0)^\top & \gamma_1 \end{bmatrix} \succeq 0 \\
& f_\xi(\nabla f(\boldsymbol{x}, \xi)) \geq 0\ , \quad \forall \xi \in \mathcal{S}\ ,
\end{aligned}
$$

where we take $h(\boldsymbol{x}, \Delta \mathbf{x}, \xi) = -RELU(-\Delta \boldsymbol{x}^T\nabla f(\boldsymbol{x}, \xi))$.

Claim 2. *Take* $\bar{\gamma} = \min\{\gamma_1, \gamma_2\}$*, then*

$$\Psi(\boldsymbol{x}, \Delta \boldsymbol{x}, \gamma_1, \gamma_2) = \min\{0, \Delta x^T\mu_0 + \sqrt{\bar{\gamma}}\sqrt{\Delta x^T\Sigma_0\Delta x}\}.$$

Proof. In fact, we claim that the maximum is obtained when f_ξ is supported at a single point.

Since,

$$(\mu-\mu_0)(\mu-\mu_0)^T + \mathrm{Cov}(\nabla f(\boldsymbol{x}, \xi)) = \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top],$$

we have that

$$(\mu-\mu_0)(\mu-\mu_0)^T \preceq \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top].$$

Therefore,

$$(\mu-\mu_0)(\mu-\mu_0)^T \preceq \gamma_2\Sigma_0. \tag{4}$$

If $\boldsymbol{y}\in\mathbb{R}^{mn}$ such that $\Sigma_0\boldsymbol{y}=0$, then from (4), we know that $(\mu-\mu_0)^T\boldsymbol{y}=0$. Therefore,

$$\mu-\mu_0 \in \operatorname{row}\Sigma_0 = \operatorname{col}\Sigma_0.$$

Since

$$\begin{bmatrix} \Sigma_0 & (\mu-\mu_0) \\ (\mu-\mu_0)^\top & \gamma_1 \end{bmatrix} \succeq 0,$$

$$\Sigma_0 - \frac{1}{\gamma_1}(\mu-\mu_0)(\mu-\mu_0)^T \succeq 0.$$

Therefore,

$$(\mu-\mu_0)(\mu-\mu_0)^T \preceq \gamma_1\Sigma_0. \tag{5}$$

Combining (4) and (5), we get that

$$(\mu-\mu_0)(\mu-\mu_0)^T \preceq \bar{\gamma}\Sigma_0. \tag{6}$$

What's more, from the concavity of $-\mathrm{RELU}(-\lambda)$,

$$\mathbb{E}_{f_\xi}[h(\boldsymbol{x},\Delta\boldsymbol{x},\xi)] \le \min\{0, \mathbb{E}_{f_\xi}(\Delta\boldsymbol{x}^T\nabla f(\boldsymbol{x},\xi))\} = \min\{0, \Delta\boldsymbol{x}^T\mu\}.$$

The only restriction for μ is (6).

$$\begin{aligned}
& \Delta\boldsymbol{x}^T\mu \le \Delta\boldsymbol{x}^T\mu_0 + \Delta\boldsymbol{x}^T(\mu-\mu_0) \\
= \quad & \Delta\boldsymbol{x}^T\mu_0 + (\Sigma_0^{1/2}\Delta\boldsymbol{x})^T\left(\Sigma_0^{\dagger/2}(\mu-\mu_0)\right) \\
\le \quad & \Delta\boldsymbol{x}^T\mu_0 + \|\Sigma_0^{1/2}\Delta\boldsymbol{x}\|_2\|\Sigma_0^{\dagger/2}(\mu-\mu_0)\|_2 \\
\le \quad & \Delta\boldsymbol{x}^T\mu_0 + \bar{\gamma}\|\Sigma_0^{1/2}\Delta\boldsymbol{x}\|_2 \\
= \quad & \Delta\boldsymbol{x}^T\mu_0 + \bar{\gamma}\sqrt{\Delta\boldsymbol{x}^T\Sigma_0\Delta\boldsymbol{x}},
\end{aligned}$$

and exact equality holds when

$$\mu = \mu_0 + \frac{\bar{\gamma}\Sigma_0\Delta\boldsymbol{x}}{\sqrt{\Delta\boldsymbol{x}^T\Sigma_0\Delta\boldsymbol{x}}}.$$

Therefore,

$$\Psi(\boldsymbol{x},\Delta\boldsymbol{x},\gamma_1,\gamma_2) = \Delta\boldsymbol{x}^T\mu_0 + \bar{\gamma}\sqrt{\Delta\boldsymbol{x}^T\Sigma_0\Delta\boldsymbol{x}},$$

and the maximum is obtained when f_ξ is supported only at one point. $\square$

2.3 RELU Formulation 2

Consider the robust optimization problem proposed by Delange and Ye [2]:

$$\begin{aligned}
\Psi(\boldsymbol{x}, \Delta\boldsymbol{x}, \gamma_1, \gamma_2) = \text{maximize}_{\mu, f_\xi} \quad & \mathbb{E}_{f_\xi}[h(\mathbf{x}, \Delta\boldsymbol{x}, \xi)] \\
\text{subject to} \quad & \mathbb{E}_{f_\xi}[1] = 1\ , \quad \mathbb{E}_{f_\xi}[\nabla f(\boldsymbol{x}, \xi)] = \mu \\
& \mathbb{E}_{f_\xi}[(\nabla f(\boldsymbol{x}, \xi) - \mu_0)(\nabla f(\boldsymbol{x}, \xi) - \mu_0)^\top] \preceq \gamma_2 \Sigma_0 \\
& \begin{bmatrix} \Sigma_0 & (\mu - \mu_0) \\ (\mu - \mu_0)^\top & \gamma_1 \end{bmatrix} \succeq 0 \\
& f_\xi(\nabla f(\boldsymbol{x}, \xi)) \geq 0\ , \quad \forall \xi \in \mathcal{S}\ ,
\end{aligned}$$

where we take $h(\boldsymbol{x}, \Delta\mathbf{x}, \xi) = RELU(\Delta\boldsymbol{x}^T \nabla f(\boldsymbol{x}, \xi))$.

Claim 3. *Take $\bar{\gamma} = \min\{\gamma_1, \gamma_2\}$, then*

$$\Psi(\boldsymbol{x}, \Delta\boldsymbol{x}, \gamma_1, \gamma_2) = \min\{0, \Delta x^T \mu_0 + \sqrt{\bar{\gamma}}\sqrt{\Delta x^T \Sigma_0 \Delta x}\}.$$

3 Formulation Using a Second Order Wasserstein Distance Regularization

Fixing an iteration t in Algorithm 1.1, we solve the following DRO subproblem to obtain d_t:

$$\mathbf{d}_t = \arg\min_{\mathbf{d} \in \mathbb{R}^{m \times n}} \sup_{\substack{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^{m \times n}) \\ \mathbb{Q}(\mathbb{R}^{m \times n}) = 1}} \left\{ \mathbb{E}_{G \sim \mathbb{Q}}[\langle \mathbf{d}, G \rangle] - \frac{1}{2\kappa} W_1^2(\mathbb{P}_{t, N_t}, \mathbb{Q}) \right\}, \tag{P2}$$

where $\langle \mathbf{d}, \mathbf{G} \rangle = \text{trace}(\mathbf{d}^T \mathbf{G})$ and type-p Wasserstein distance $W_p(\mathbb{Q}_1, \mathbb{Q}_2)$ is defined as follows:

Definition 1 (Wasserstein Distance)**.**

$$W_p(\mathbb{Q}_1, \mathbb{Q}_2) = \sqrt[p]{\inf_{\pi \in \Pi(\mathbb{Q}_1, \mathbb{Q}_2)} \int_{\mathbb{R}^m \times \mathbb{R}^m} \|\xi - \xi'\|^p \pi(d\xi, d\xi')},$$

where $\Pi(\mathbb{Q}_1, \mathbb{Q}_2)$ is the collection of distributions on $\mathbb{R}^m \times \mathbb{R}^m$ whose marginal distribution with respect to the first m components is $\mathbb{Q}_1$ and the marginal distribution with respect to the last m components is $\mathbb{Q}_2$.

In the later proofs, we will make use of an important equivalent characterization of Wasserstein distance which is stated below:

Theorem 1 (Kantorovich-Rubinstein[3])**.**

$$W_1(\mathbb{Q}_1, \mathbb{Q}_2) = \sup_{f \text{ is 1-Lipschitz continuous}} \left(\int_{\mathbb{R}^{m \times n}} f(\xi) \mathbb{Q}_1(d\xi) - \int_{\mathbb{R}^{m \times n}} f(\xi') \mathbb{Q}_2(d\xi') \right).$$

Theorem 2. *Problem* (P2) *is equivalent to*

$$\begin{aligned}\mathbf{d}_t &= \arg\min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \left\{\mathbb{E}_{\mathbf{G}\sim\mathbb{P}_{t,N_t}}[\langle\mathbf{d},\mathbf{G}\rangle] + \frac{1}{2}\kappa\|\mathbf{d}\|_*^2\right\} \\ &= \arg\min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \left(\langle\mathbf{d},\overline{\mathbf{G}}_t\rangle + \frac{1}{2}\kappa\|\mathbf{d}\|_*^2\right),\end{aligned} \tag{D2}$$

where

$$\overline{\mathbf{G}}_t = \frac{1}{N_t}\sum_{i=1}^{N_t}\nabla f(\mathbf{B}_t,\xi_{t,i}).$$

Theorem 3. *If* $\|\cdot\| = \|\cdot\|_{nuc}$*, then the optimal solution of* (D2) *is*

$$\mathbf{d}_t = -\frac{1}{\kappa}\|\overline{\mathbf{G}}_t\|_{nuc}\,U_tV_t^\top,$$

where

$$U_t\Sigma V_t^\top \text{ is the thin SVD decomposition of } \overline{\mathbf{G}}_t.$$

4 Formulation Using p-th Order Wasserstein Distance Regularization($p \in (1,+\infty)$)

In order to generalize the results in the last section, we require that

$$\mathbf{d}_t = \arg\min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \sup_{\substack{\mathbb{Q}\in\mathcal{M}(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{G\sim\mathbb{Q}}[\langle\mathbf{d},G\rangle] - \frac{1}{\kappa p}W_1^p(\mathbb{P}_N,\mathbb{Q})\right\}, \tag{Pp}$$

where $p \in (1,+\infty)$.

Theorem 4. *Problem* (Pp) *is equivalent to*

$$\begin{aligned}\mathbf{d}_t &= \arg\min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \left\{\mathbb{E}_{G\sim\mathbb{P}_{t,N_t}}[\langle\mathbf{d},\mathbf{G}\rangle] + \frac{1}{q}\kappa\|\mathbf{d}\|_*^q\right\} \\ &= \arg\min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \left(\langle\mathbf{d},\overline{\mathbf{G}}_t\rangle + \frac{1}{q}\kappa\|\mathbf{d}\|_*^q\right),\end{aligned} \tag{Dp}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\overline{\mathbf{G}}_t = \frac{1}{N_t}\sum_{i=1}^{N_t}\nabla f(\mathbf{B}_t,\xi_{t,i}).$$

Theorem 5. *If* $\|\cdot\| = \|\cdot\|_{nuc}$*, the closed-form solution of* (Dp) *is*

$$\mathbf{d}_t = -\Big(\frac{\|\overline{\mathbf{G}}_t\|_{nuc}}{\kappa}\Big)^{\frac{1}{q-1}}\,U_tV_t^\top,$$

where

$$U_t\Sigma V_t^\top \text{ is the thin SVD decomposition of } \overline{\mathbf{G}}_t.$$

5 Convergence of MUON beyond Uniform Lipschitz Gradient

Assumption 2 (Wasserstein Lipschitz Gradient)**.** *Denote $\mathbb{P}_{\mathbf{B}}$ as the distribution of $\nabla f(\mathbf{B}, \xi)$. We assume that*

$$W_1(\mathbb{P}_{\mathbf{B}}, \mathbb{P}_{\mathbf{B}'}) \leq L_w \|\mathbf{B} - \mathbf{B}'\|_*, \forall\, \mathbf{B}, \mathbf{B}'.$$

where W_1 is the type-1 Wasserstein distance.

There exists $C > 0$ such that $W_1(\mathbb{P}_{t,N}, \mathbb{P}_{\mathbf{B}_t}) \leq C$ with high probability. From Assumption 2, we know that $W_1(\mathbb{P}_{\mathbf{B}_t}, \mathbb{P}_{\mathbf{B}}) \leq L_w \|\mathbf{B}_t - \mathbf{B}\|_*$, $\forall\, \mathbf{B} \in \mathbb{R}^{m \times n}$. Thus, $\forall\, \mathbf{B} \in \mathbb{R}^{m\times n}$, we have that, with high probability,

$$W_1(\mathbb{P}_{t,N}, \mathbb{P}_{\mathbf{B}}) \leq W_1(\mathbb{P}_{t,N}, \mathbb{P}_{\mathbf{B}_t}) + W_1(\mathbb{P}_{\mathbf{B}_t}, \mathbb{P}_{\mathbf{B}}) = C + L_w \|\mathbf{B}_t - \mathbf{B}\|_*.$$

We turn to solve the following subproblem:

$$\mathbf{d}_t = \arg \min_{\mathbf{d} \in \mathbb{R}^{m\times n}} \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^{m\times n}), \mathbb{Q}(\mathbb{R}^{m \times n}) = 1, W(\mathbb{P}_{t,N_t}, \mathbb{Q}) \leq C + L\|\mathbf{d}\|_*} \mathbb{E}_{\mathbf{G}\sim\mathbb{Q}}[\langle \mathbf{d}, \mathbf{G}\rangle]. \quad \text{(P)}$$

Theorem 6. *Problem* (P) *is equivalent to*

$$\mathbf{d}_t = \arg \min_{\mathbf{d} \in \mathbb{R}^{m\times n}} \left(\langle \mathbf{d}, \overline{\mathbf{G}}_t \rangle + C\|\mathbf{d}\|_* + L_w \|\mathbf{d}\|_*^2\right), \quad \text{(D)}$$

where

$$\overline{\mathbf{G}}_t = \frac{1}{N} \sum_{i=1}^{N} \nabla f(\mathbf{B}_t, \xi_{t,i}).$$

Theorem 7. *If $\|\cdot\| = \|\cdot\|_{nuc}$, the closed-form solution of* (D) *is*

$$\mathbf{d}_t \;=\; -\max\left\{0, \frac{\|\overline{\mathbf{G}}_t\|_{nuc} - C}{2L_w}\right\} U_t V_t^\top,$$

where

$U_t \Sigma V_t^\top$ is the thin SVD decomposition of $\overline{\mathbf{G}}_t$.

5.1 Convergence Rate Analysis Beyond Nuclear Norm

Assumption 3 (Light-tail Distribution)**.** *There exists a uniform constant $a > 1$ and $A > 0$ such that*

$$\mathbb{E}_{\mathbf{G}\sim\mathbb{P}_{\mathbf{B}}}[\exp(\|\mathbf{G}\|^a)] \leq A, \forall\, \mathbf{B} \in \mathbb{R}^{m\times n}.$$

Define the event E_t to be $\|\mathbb{E}_{\mathbb{P}_{t,N_t}} \nabla f(\mathbf{B}_t, \xi) - \mathbb{E}_{\mathbb{P}_{\mathbf{B}_t}} \nabla f(\mathbf{B}_t, \xi)\| \leq C_t\}$, then $\{E_t\}_{t=0}^{+\infty}$ is a collection of independent events.

Theorem 8. *Take $T > 0$, $\delta > 0$, choose δ_t such that $\delta > \sum_{t=0}^{T} \delta_t$, then with probability at least $1 - \delta$, we have that*

$$\frac{1}{T}\sum_{t=0}^{T-1} \|\nabla f(\mathbf{B}_t)\|^2 \leq \frac{8L_w(f(\mathbf{B}_0) - f^*)}{T} + \frac{8}{T}\sum_{t=0}^{T-1} C_t^2, \tag{7}$$

and thus,

$$\min_{0\leq t<T} \|\nabla f(\mathbf{B}_t)\|^2 \leq \frac{8L_w(f(\mathbf{B}_0) - f^*)}{T} + \frac{8}{T}\sum_{t=0}^{T-1} C_t^2.$$

Remark 1. *Any unitary invariant [4] cross norm [1] $\|\cdot\|$ shares the same C_t, and when T is sufficiently large, $\sum_{t=0}^{T-1} C_t^2$ becomes the dominant term of the right-hand side. Thus, the right-hand side is approximately invariant among different norms. From proposition 3.12 of [1], the nuclear norm is the largest cross norm over $\|\cdot\|_2$ and $\|\cdot\|_2$. Therefore, when we use a nuclear norm guided steepest descent,* (7) *is the strongest and the descent is better than when other norms are used because $\nabla f(\mathbf{B}_t)$ are usually of high rank and $\|\nabla f(\mathbf{B}_t)\|_{nuc} > \|\nabla f(\mathbf{B}_t)\|$ for most other unitary invariant cross norms.*

References

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A Rigorous Proofs of the Theorems

We give proofs only for Theorem 4, Theorem 5, Theorem 6, and Theorem 7, since Theorem 2 and Theorem 3 are special cases of Theorem 4 and Theorem 5, respectively.

A.1 Proof of Theorem 4

For convenience, we write $N := N_t, \mathbb{P}_N := \mathbb{P}_{t,N_t}$. Take $\mu = \mathbb{Q} - \mathbb{P}_N$, then $\mu \in \mathcal{M}(\mathbb{R}^{m\times n})$ and $\mu(\mathbb{R}^{m\times n}) = 0$.

$$\begin{aligned}
&\sup_{\substack{\mathbb{Q}\in\mathcal{M}_+(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{\mathbf{G}\sim\mathbb{Q}}[\langle \mathbf{d}, \mathbf{G}\rangle] - \frac{1}{\kappa p} W_1^p(\mathbb{P}_N, \mathbb{Q})\right\} \\
= \quad & \mathbb{E}_{\mathbf{G}\sim\mathbb{P}_N}[\langle \mathbf{d}, \mathbf{G}\rangle] + \sup_{\substack{\mathbb{Q}\in\mathcal{M}_+(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{\mathbf{G}\sim\mu}[\langle \mathbf{d}, \mathbf{G}\rangle] - \frac{1}{\kappa p} W_1^p(\mathbb{P}_N, \mathbb{Q})\right\}.
\end{aligned}$$

Consider the normed vector space $Lip(\mathbb{R}^{m\times n})$ with norm $\|\cdot\|_{Lip}$ whose value is the smallest Lipschitz constant of that function. Consider a subset of its dual space $\mathcal{M}(\mathbb{R}^{m\times n})$ that denotes all signed measures on $\mathbb{R}^{m\times n}$, and the dual norm is $\|\cdot\|_{KR}$. Using Hahn Banach theorem, we know that the dual norm of $\|\cdot\|_{KR}$ is also $\|\cdot\|_{Lip}$.

From Theorem 1,

$$\begin{aligned}
W_1(\mathbb{P}_N, \mathbb{Q}) &= \sup_{\|f\|_{Lip}=1} \left(\int_{\mathbb{R}^{m\times n}} f(\xi)\mathbb{P}_N(d\xi) - \int_{\mathbb{R}^{m\times n}} f(\xi')\mathbb{Q}(d\xi')\right) \\
&= \sup_{\|f\|_{Lip}=1} \int_{\mathbb{R}^{m\times n}} f(\xi)\mu(d\xi) = \|\mu\|_{KR}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{\substack{\mathbb{Q}\in\mathcal{M}_+(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{G\sim\mathbb{Q}}[\langle \mathbf{d}, G\rangle] - \frac{1}{\kappa p} W_1^p(\mathbb{P}_N, \mathbb{Q})\right\} \\
= \quad & \mathbb{E}_{\mathbf{G}\sim\mathbb{P}_N}[\langle \mathbf{d}, \mathbf{G}\rangle] + \sup_{\substack{\mathbb{Q}\in\mathcal{M}_+(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{G\sim\mu}[\langle \mathbf{d}, \mathbf{G}\rangle] - \frac{1}{\kappa p} \|\mu\|_{KR}^p\right\}.
\end{aligned}$$

While

$$\sup_{\substack{\mathbb{Q}\in\mathcal{M}_+(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{G\sim\mu}[\langle \mathbf{d}, G\rangle] - \frac{1}{\kappa p} \|\mu\|_{KR}^p\right\}$$

is just the conjugate function of $\frac{1}{\kappa p}\|\cdot\|_{KR}^p$ which equals to $\frac{\kappa}{q}\|\cdot\|_{Lip}^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. We also know that $\frac{\kappa}{q}\|\langle \mathbf{d}, \cdot\rangle\|_{Lip}^q = \frac{\kappa}{q}\|\mathbf{d}\|_{op}^q$.

Therefore, we end up with the conclusion in the theorem.

$$\sup_{\substack{\mathbb{Q}\in\mathcal{M}_+(\mathbb{R}^{m\times n})\\ \mathbb{Q}(\mathbb{R}^{m\times n})=1}} \left\{\mathbb{E}_{G\sim\mathbb{Q}}[\langle\mathbf{d},G\rangle] - \frac{1}{\kappa p}W_1^p(\mathbb{P}_N,\mathbb{Q})\right\}$$
$$= \quad \mathbb{E}_{G\sim\mathbb{P}_N}[\langle\mathbf{d},G\rangle] + \frac{\kappa}{q}\|\mathbf{d}\|_{op}^q.$$

□

B Proof of Theorem 5

Assume that $\mathbf{d} = c\mathbf{M}$, where $\|\mathbf{M}\|_{op} = 1$, then our goal is:

$$\min_{c\geq 0,\mathbf{M}\in\mathbb{R}^{m\times n}} \left(c\langle\mathbf{M},\overline{G}_t\rangle + \frac{1}{q}\kappa c^q\right).$$

By Neumann's Inequality,

$$\min_{\mathbf{d}\in\mathbb{R}^{m\times n}}\langle\mathbf{M},\overline{G}_t\rangle = -\|\overline{G}_t\|_{nuc}.$$

The minimal value is achieved if and only if $\mathbf{M} = -U_tV_t^T$. When the minimal value is achieved, it suffices to minimize

$$-c\|\overline{G}_t\|_{nuc} + \frac{1}{q}\kappa c^q,$$

and the optimal c is $(\frac{\|\overline{G}_t\|_{nuc}}{\kappa})^{1/(q-1)}$ naturally. □

C Proof of Theorem 6

For simplicity, write $N = N_t$, $\mathbb{P}_N = \mathbb{P}_{t,N_t}$. We follow the notations in [5], take $X = (\mathcal{M}(\mathbb{R}^{m\times n})))^N$ to be the space of N-tuples of signed finite measures on $\mathbb{R}^{m\times n}$. Take C as a convex conic subset of X which contains all nonnegative measures. Take $Y = \mathbb{R}^{N+1}$ and $K = \mathbb{R}_{\leq 0}\times\{0\}^N$.
Therefore, $X^* = \mathcal{L}^\infty(\mathbb{R}^{m\times n})$, $C^* = \{f\in C : f\geq 0\}$, $Y^* = \mathbb{R}^{N+1}$, $K^* = \mathbb{R}_{\leq 0}\times\mathbb{R}^N$. We also take $b = (-(C+L\|\mathbf{d}\|_*),-1,\ldots,-1)^T\in\mathbb{R}^{N+1}$.

There exists a nonnegative measure π on $\mathbb{R}^{m\times n}\times\mathbb{R}^{m\times n}$ whose marginal distribution for the first m components is $\mathbb{P}_N$ and the marginal distribution for the second m components is $\mathbb{Q}$, such that

$$\mathbb{E}_{(\xi,\xi')\sim\pi}\|\xi-\xi'\| = W(\mathbb{P}_N,\mathbb{Q}).$$

We denote
$$\mathbf{q} = (\mathbb{Q}^1,\mathbb{Q}^2,\ldots,\mathbb{Q}^N)^T\in X.$$

The first stage problem of (P)

$$\sup_{W(\mathbb{P}_N,\mathbb{Q})\le C+L\|\mathbf{d}\|_*} \mathbb{E}_{\mathbb{Q}}[\langle \mathbf{d}, \nabla_{\mathbf{B}} f_{\mathbf{x}}(\mathbf{B})\rangle]$$

is equivalent to

$$\begin{cases} \sup_{\Pi\in M(\mathbb{R}^{m\times n},\mathbb{R}^{m\times n})} & \int_{\mathbb{R}^{m\times n}} \langle \mathbf{d}, \mathbf{G}\rangle \Pi(dG, \mathbb{R}^{m\times n}), \\ \text{s.t.} & \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{m\times n}} \|\mathbf{G}-\mathbf{G}_i\|\mathbb{Q}^i(dG) \le C + L\|\mathbf{d}\|_{op}, \\ & \int_{\mathbb{R}^{m\times n}} \mathbb{Q}^i(dG) = 1, \end{cases} \tag{8}$$

We write $\mathbf{G} = \nabla_{\mathbf{B}} f_{\mathbf{x}}(\mathbf{B})$, $\mathbf{G}_i = \nabla_{\mathbf{B}} f_{\mathbf{x}_i}(\mathbf{B})$,$c = (\frac{1}{N}\langle \mathbf{d}, \mathbf{G}\rangle)_{i=1}^N$,$A\mathbf{q} = (A^0\mathbf{q}, A^1\mathbf{q}, \ldots, A^N\mathbf{q})^T$, where $A^i\mathbf{q} = \int_{\mathbb{R}^{m\times n}} \mathbb{Q}^i(dG)$, $A^0\mathbf{q} = \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{m\times n}} \|G-G_i\|\mathbb{Q}^i(dG)$.

The (8) is equivalent to

$$\begin{cases} & \sup_{\mathbb{Q}\in C} \langle c, \mathbf{q}\rangle, \\ \text{s.t.} & A\mathbf{q} + b \in K. \end{cases} \tag{9}$$

We now turn to look at its dual problem:

$$\begin{cases} & \sup_{y^*\in K^*} \langle b, y^*\rangle, \\ \text{s.t.} & A^*y^* + c \in C^*. \end{cases} \tag{10}$$

Assume that $y^* = (\lambda, s_1, s_2, \ldots, s_N)$, then

$$\begin{aligned} \langle y^*, A\mathbf{q}\rangle &= \frac{\lambda}{N}\sum_{i=1}^N \int_{\mathbb{R}^{m\times n}} \|\mathbf{G}-\mathbf{G}_i\|\mathbb{Q}^i(dG) + \sum_{i=1}^N s^i \int_{\mathbb{R}^{m\times n}} \mathbb{Q}^i(dG) \\ &= \int_{\mathbb{R}^{m\times n}} \sum_{i=1}^N (\frac{\lambda}{N}\|\mathbf{G}-\mathbf{G}_i\| + s^i)\mathbb{Q}^i(dG) \\ &= \langle A^*y^*, x\rangle. \end{aligned}$$

Hence, $A^*y^* = (\frac{\lambda}{N}\|\mathbf{G}-\mathbf{G}_i\| + s^i)_{i=1}^N$. Since $A^*y^* + c \in C^*$,

$$0 \ \ge \ A^*y^* + c = (\frac{\lambda}{N}\|\mathbf{G}-\mathbf{G}_i\| + s^i + \frac{1}{N}\langle \mathbf{d}, \mathbf{G}\rangle), \forall\, i. \tag{11}$$

(11) is equivalent to that

$$\begin{aligned} -s^i &\ge \frac{1}{N}\sup_{\mathbf{G}}(\langle \mathbf{d}, \mathbf{G}\rangle + \lambda\|\mathbf{G}-\mathbf{G}_i\|) \\ &= \frac{1}{N}\sup_{\mathbf{G}}(\langle \mathbf{d}, \mathbf{G}-\mathbf{G}_i\rangle - |\lambda|\|\mathbf{G}-\mathbf{G}_i\|) + \frac{\langle \mathbf{d}, \mathbf{G}_i\rangle}{N} \\ &= \frac{1}{N}(|\lambda|\|\cdot\|)^*(\mathbf{d}) + \frac{\langle \mathbf{d}, G_i\rangle}{N} \\ &= \frac{\langle \mathbf{d}, \mathbf{G}_i\rangle}{N} + \begin{cases} 0, \|\mathbf{d}\|_* \le -\lambda, \\ +\infty, \text{otherwise.} \end{cases} \end{aligned}$$

Notice that $\langle c, \mathbf{q}\rangle = -\lambda(C + L\|\mathbf{d}\|) - \sum_{i=1}^N s^i$, to minimize this, take $-\lambda = \|\mathbf{d}\|_*$ and $-s^i = \frac{\langle \mathbf{d}, \mathbf{G}_i\rangle}{N}$.

(10) is now written as

$$\min_{\mathbf{d}} \frac{1}{N}\sum_{i=1}^N \langle \mathbf{d}, \mathbf{G}_i\rangle + (C + L\|\mathbf{d}\|_*)\|\mathbf{d}\|_*.$$

To show strong duality in Proposition 3.4 of [4], notice that $A(C) = \mathbb{R}^{N+1}_{\geq 0}$, $A(C) - K = \mathbb{R}^{N+1}_{\geq 0}$, $-b \in \text{int}(A(C) - K)$.

□

D Proof of Theorem 7

The methodology is very similar to the proof of Theorem 5. We repeat the proof here for completeness.

Assume that $\mathbf{d} = c\mathbf{M}$, where $\|\mathbf{M}\|_{op} = 1$, then our goal is:

$$\min_{c\geq 0, \mathbf{M}\in\mathbb{R}^{m\times n}} \left(c\langle \mathbf{M}, \overline{\mathbf{G}}_t\rangle + Cc + Lc^2\right).$$

By Neumann's Inequality,

$$\min_{\mathbf{d}\in\mathbb{R}^{m\times n}} \langle \mathbf{M}, \overline{\mathbf{G}}_t\rangle = -\|\overline{\mathbf{G}}_t\|_{nuc}.$$

The minimal value is achieved if and only if $\mathbf{M} = -U_t V_t^T$. When the minimal value is achieved, it suffices to minimize

$$(C - \|\mathbf{G}_t\|_{nuc})c + Lc^2, c \geq 0.$$

and the minimal value is $\frac{\max\{0, \|G_t\|_{nuc} - C\}}{2L}$ naturally. □

E Proof of Lemma 1

Take any $\xi \in \Xi$, $\mathbf{B} \in \mathbb{R}^{m\times n}$, then

$$\nabla f(\mathbf{B}, \xi) = \frac{\partial f(\mathbf{B}, \xi)}{\partial \mathbf{B}} = \frac{\partial f(\mathbf{B}, \xi)}{\partial \mathbf{a}^l}\mathbf{h}^{l-1^T},$$

is of rank one. □

F Proof of Theorem 8

Lemma 1. *If event E_t happens, then $\forall$ $\mathbf{d} \in \mathbb{R}^{m\times n}$,*

$$f(\mathbf{B}_t + \mathbf{d}) - f(\mathbf{B}_t) \leq \sup_{W(\mathbb{P}_{t,N_t}, \mathbb{Q}) \leq C_t + L_w\|\mathbf{d}\|_*} \mathbb{E}_{\mathbb{Q}}\langle \mathbf{d}, \mathbf{G}\rangle.$$

Proof. By the mean-value problem, there exists $\theta \in (0,1)$, such that

$$f(\mathbf{B}_t + \mathbf{d}) - f(\mathbf{B}_t) = \langle \nabla f(\mathbf{B}_t + \theta \mathbf{d}), \mathbf{d} \rangle = \mathbb{E}_{\mathbf{G} \sim \mathbb{P}_{\mathbf{B}_t + \theta \mathbf{d}}} \langle \mathbf{G}, \mathbf{d} \rangle.$$

Since

$$W(\mathbb{P}_{\mathbf{B}_t + \theta \mathbf{d}}, \mathbb{P}_{t,N_t}) \leq W(\mathbb{P}_{\mathbf{B}_t + \theta \mathbf{d}}, \mathbb{P}_{\mathbf{B}_t}) + W(\mathbb{P}_{t,N_t}, \mathbb{P}_{\mathbf{B}_t}) \leq L_w \|\mathbf{d}\|_* + C_t,$$

we have that,

$$f(\mathbf{B}_t + \mathbf{d}) - f(\mathbf{B}_t) = \mathbb{E}_{\mathbf{G} \sim \mathbb{P}_{\mathbf{B}_t + \theta \mathbf{d}}} \langle \mathbf{G}, \mathbf{d} \rangle \leq \sup_{W(\mathbb{P}_{t,N_t}, \mathbb{Q}) \leq C_t + L_w \|\mathbf{d}\|_*} \mathbb{E}_{\mathbb{Q}} \langle \mathbf{d}, \mathbf{G} \rangle.$$

□

Lemma 2. *On the event E_t, if $\mathbf{B}_{t+1} = \mathbf{B}_t + \mathbf{d}_t$, then*

$$f(\mathbf{B}_{t+1}) - f(\mathbf{B}_t) \leq -\frac{(\|\overline{\mathbf{G}}_t\| - C_t)_+^2}{4L_w}.$$

Proof. Since

$$f(\mathbf{B}_t + \mathbf{d}_t) - f(\mathbf{B}_t) \leq \sup_{W(\mathbb{P}_{t,N_t}, \mathbb{Q}) \leq C_t + L_w \|\mathbf{d}\|_*} \mathbb{E}_{\mathbb{Q}} \langle \mathbf{d}_t, \mathbf{G} \rangle = \langle \mathbf{d}_t, \overline{\mathbf{G}}_t \rangle + C_t \|\mathbf{d}_t\|_* + L_w \|\mathbf{d}_t\|_*^2.$$

Since $\mathbf{d}_t$ is taken to be the infimum of the right-hand side,

$$f(\mathbf{B}_t + \mathbf{d}_t) - f(\mathbf{B}_t) \leq \inf_{c \geq 0} c(-\|\overline{\mathbf{G}}_t\| + C_t) + L_w c^2 = -\frac{(\|\overline{\mathbf{G}}_t\| - C_t)_+^2}{4L_w}.$$

□

Lemma 3.

$$\|\overline{\mathbf{G}}_t - \nabla f(\mathbf{B}_t)\| \leq C_t.$$

Proof. Take arbitrarily $\mathbf{d} \in \mathbb{R}^{m \times n}$, $\|\mathbf{d}\|_* = 1$, we have by Theorem 1 that

$$\begin{aligned} \langle \mathbf{d}, \overline{\mathbf{G}}_t - \nabla f(\mathbf{B}_t) \rangle &= \int_{\mathbb{R}^{m \times n}} \langle \mathbf{d}, \mathbf{G} \rangle \mathbb{P}_{t,N_t}(\mathbf{dG}) - \int_{\mathbb{R}^{m \times n}} \langle \mathbf{d}, \mathbf{G} \rangle \mathbb{P}_{\mathbf{B}_t}(\mathbf{dG}) \\ &\leq W(\mathbb{P}_{t,N_t}, \mathbb{P}_{\mathbf{B}_t}) \\ &\leq C_t. \end{aligned}$$

Therefore, $\|\overline{\mathbf{G}}_t - \nabla f(\mathbf{B}_t)\| \leq C_t$. □

Lemma 4. *If $x, y \geq 0$, then $(x-y)_+^2 \geq \frac{1}{2}x^2 - y^2$.*

Proof. Case 1: $x \geq y$,

In this case, $(x-y)_+ = x - y$, so the left side is $(x-y)^2 = x^2 - 2xy + y^2$. The inequality becomes $x^2 - 2xy + y^2 \geq \frac{1}{2}x^2 - y^2$. Rearranging terms gives $\frac{1}{2}x^2 - 2xy + 2y^2 \geq 0$. Multiplying through by 2 yields $x^2 - 4xy + 4y^2 \geq 0$, or $(x-2y)^2 \geq 0$. This is always true, with equality when $x = 2y$.

Case 2: $x < y$,

In this case, $(x-y)_+ = 0$, so the left side is 0. The inequality becomes $0 \geq \frac{1}{2}x^2 - y^2$, or $y^2 \geq \frac{1}{2}x^2$. Since $x < y$ and both are non-negative, $y^2 > x^2 \geq \frac{1}{2}x^2$, so the inequality holds strictly. ∎

Lemma 5.

$$f(\mathbf{B}_t + \mathbf{d}_t) - f(\mathbf{B}_t) \leq \; - \; \frac{1}{8L_w} \, \|\nabla f(\mathbf{B}_t)\|^2 \; + \; \frac{1}{L_w} \, C_t^2. \tag{12}$$

Proof.

$$\begin{aligned} f(\mathbf{B}_t + \mathbf{d}_t) - f(\mathbf{B}_t) \quad &\leq \quad -\frac{(\|\overline{\mathbf{G}}_t\| - C_t)_+^2}{4L_w} \leq -\frac{(\|\nabla f(\mathbf{B}_t)\| - 2C_t)_+^2}{4L_w} \\ &\leq \quad -\frac{\frac{1}{2}\|\nabla f(\mathbf{B}_t)\|^2 - 4C_t^2}{4L_w} \\ &= \quad -\frac{1}{8L_w}\|\nabla f(\mathbf{B}_t)\|^2 + \frac{1}{L_w}C_t^2. \end{aligned}$$

∎

Now Theorem 8 follows by taking the summation of (12) from 0 to $T-1$, dividing both sides by T, and rearranging.