Vector Calculus for Engineers

Lecture Notes for coursera

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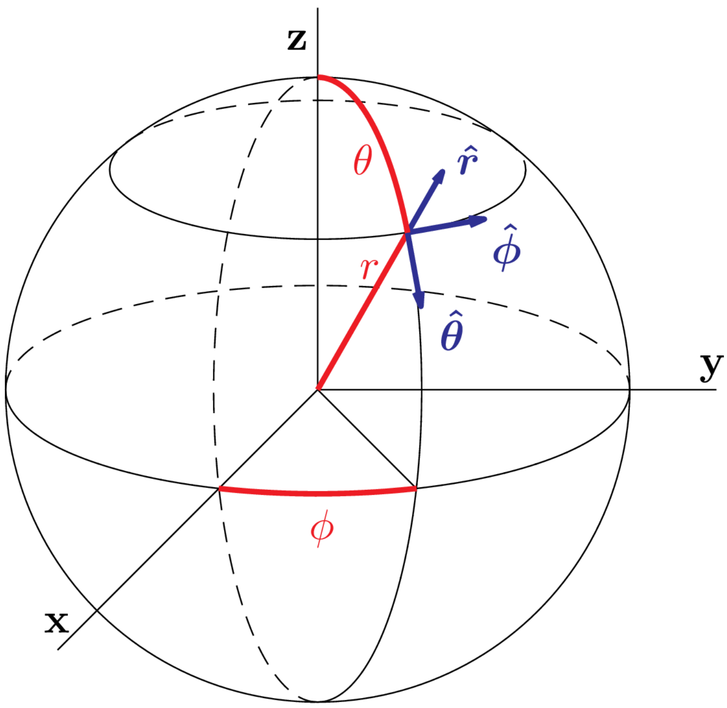

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Preface

View the promotional video on YouTube

These are the lecture notes for my online Coursera course, Vector Calculus for Engineers. Students who take this course are expected to already know single-variable differential and integral calculus to the level of an introductory college calculus course. Students should also be familiar with matrices, and be able to compute a three-by-three determinant.

I have divided these notes into chapters called Lectures, with each Lecture corresponding to a video on Coursera. I have also uploaded all my Coursera videos to YouTube, and links are placed at the top of each Lecture.

There are some problems at the end of each lecture chapter. These problems are designed to exemplify the main ideas of the lecture. Students taking a formal university course in multivariable calculus will usually be assigned many more problems, some of them quite difficult, but here I follow the philosophy that less is more. I give enough problems for students to solidify their understanding of the material, but not so many that students feel overwhelmed. I do encourage students to attempt the given problems, but, if they get stuck, full solutions can be found in the Appendix. I have also included practice quizzes as an additional source of problems, with solutions also given.

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Hong Kong
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Contents

Week I

Vectors

In this week's lectures, we learn about vectors. Vectors are line segments with both length and direction, and are fundamental to engineering mathematics. We will define vectors, how to add and subtract them, and how to multiply them using the scalar and vector products (dot and cross products). We use vectors to learn some analytical geometry of lines and planes, and introduce the Kronecker delta and the Levi-Civita symbol to prove vector identities. The important concepts of scalar and vector fields are discussed.

Lecture 1

Vectors

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We define a vector in three-dimensional Euclidean space as having a length (or magnitude) and a direction. A vector is depicted as an arrow starting at one point in space and ending at another point. All vectors that have the same length and point in the same direction are considered equal, no matter where they are located in space. (Variables that are vectors will be denoted in print by boldface, and in hand by an arrow drawn over the symbol.) In contrast, scalars have magnitude but no direction. Zero can either be a scalar or a vector and has zero magnitude. The negative of a vector reverses its direction. Examples of vectors are velocity and acceleration; examples of scalars are mass and charge.

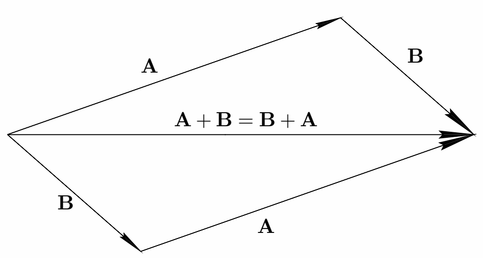
Vectors can be added to each other and multiplied by scalars. A simple example is a mass m acted on by two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. Newton's equation then takes the form $m\boldsymbol{a} = \boldsymbol{F}_1 + \boldsymbol{F}_2$, where $\boldsymbol{a}$ is the acceleration vector of the mass. Vector addition is commutative and associative:

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}, \qquad (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C});$$

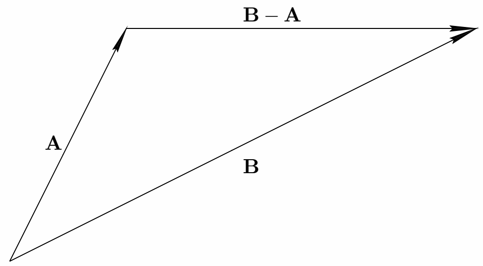
and scalar multiplication is distributive:

$$k(\boldsymbol{A} + \boldsymbol{B}) = k\boldsymbol{A} + k\boldsymbol{B}.$$

Multiplication of a vector by a positive scalar changes the length of the vector but not its direction. Vector addition can be represented graphically by placing the tail of one of the vectors on the head of the other. Vector subtraction adds the first vector to the negative of the second. Notice that when the tail of $\boldsymbol{A}$ and $\boldsymbol{B}$ are placed at the same point, the vector $\boldsymbol{B} - \boldsymbol{A}$ points from the head of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$, or equivalently, the tail of $-\boldsymbol{A}$ to the head of $\boldsymbol{B}$.



Vector addition



Vector subtraction

Problems for Lecture 1

1. Show graphically that vector addition is associative, that is, $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$.

2. Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solutions to the Problems

Lecture 2

Cartesian coordinates

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To solve a physical problem, we usually impose a coordinate system. The familiar three-dimensional x-y-z coordinate system is called the Cartesian coordinate system. Three mutually perpendicular lines called axes intersect at a point called the origin, denoted as $(0,0,0)$. All other points in three-dimensional space are identified by their coordinates as (x,y,z) in the standard way. The positive directions of the axes are usually chosen to form a right-handed coordinate system. When one points the right hand in the direction of the positive x-axis, and curls the fingers in the direction of the positive y-axis, the thumb should point in the direction of the positive z-axis.

A vector has a length and a direction. If we impose a Cartesian coordinate system and place the tail of a vector at the origin, then the head points to a specific point. For example, if the vector $\boldsymbol{A}$ has head pointing to (A_1, A_2, A_3), we say that the x-component of $\boldsymbol{A}$ is A_1, the y-component is A_2, and the z-component is A_3. The length of the vector $\boldsymbol{A}$, denoted by $|\boldsymbol{A}|$, is a scalar and is independent of the orientation of the coordinate system. Application of the Pythagorean theorem in three dimensions results in

$$|\boldsymbol{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}.$$

We can define standard unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, to be vectors of length one that point along the positive directions of the x-, y-, and z-coordinate axes, respectively. Using these unit vectors, we write

$$\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}.$$

With also $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$, vector addition and scalar multiplication can be expressed component-wise and is given by

$$\boldsymbol{A} + \boldsymbol{B} = (A_1 + B_1)\boldsymbol{i} + (A_2 + B_2)\boldsymbol{j} + (A_3 + B_3)\boldsymbol{k}, \qquad c\boldsymbol{A} = cA_1\boldsymbol{i} + cA_2\boldsymbol{j} + cA_3\boldsymbol{k}.$$

The position vector, $\boldsymbol{r}$, is defined as the vector that points from the origin to the point (x,y,z), and is used to locate a specific point in space. It can be written in terms of the standard unit vectors as

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

A displacement vector is the difference between two position vectors. For position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, the displacement vector that points from the head of $\boldsymbol{r}_1$ to the head of $\boldsymbol{r}_2$ is given by

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}.$$

Problems for Lecture 2

1. *a*) Given a Cartesian coordinate system with standard unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, let the mass m_1 be at position $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and the mass m_2 be at position $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$. In terms of the standard unit vectors, determine the unit vector that points from m_1 to m_2.

b) Newton's law of universal gravitation states that two point masses attract each other along the line connecting them, with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. The magnitude of the force acting on each mass is therefore

$$F = G\frac{m_1 m_2}{r^2},$$

where m_1 and m_2 are the two masses, r is the distance between them, and G is the gravitational constant. Let the masses m_1 and m_2 be located at the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Write down the vector form for the force acting on m_1 due to its gravitational attraction to m_2.

Solutions to the Problems

Lecture 3

Dot product

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We define the dot product (or scalar product) between two vectors $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$ and $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$ as

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_1B_1 + A_2B_2 + A_3B_3.$$

One can prove that the dot product is commutative, distributive over addition, and associative with respect to scalar multiplication; that is,

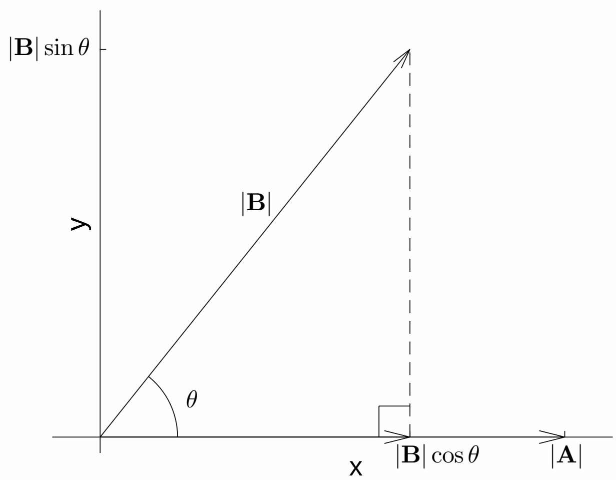
$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}, \qquad \boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}, \qquad \boldsymbol{A} \cdot (c\boldsymbol{B}) = (c\boldsymbol{A}) \cdot \boldsymbol{B} = c(\boldsymbol{A} \cdot \boldsymbol{B}).$$

A geometric interpretation of the dot product is also possible. Given any two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, place the vectors tail-to-tail, and impose a coordinate system with origin at the tails such that $\boldsymbol{A}$ is parallel to the x-axis and $\boldsymbol{B}$ lies in the x-y plane, as shown in the figure. The angle between the two vectors is denoted as θ.

Then in this coordinate system, $\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{i}$, $\boldsymbol{B} = |\boldsymbol{B}|\cos\theta\boldsymbol{i} + |\boldsymbol{B}|\sin\theta\boldsymbol{j}$, and

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta,$$

a result independent of the choice of coordinate system. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel, then $\theta = 0$ and $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|$ and in particular, $\boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|^2$. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular, then $\theta = \pi/2$ and $\boldsymbol{A} \cdot \boldsymbol{B} = 0$.



Problems for Lecture 3

1. Using the definition of the dot product $\boldsymbol{A} \cdot \boldsymbol{B} = A_1B_1 + A_2B_2 + A_3B_3$, prove that

a) $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$;

b) $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$;

c) $\boldsymbol{A} \cdot (k\boldsymbol{B}) = (k\boldsymbol{A}) \cdot \boldsymbol{B} = k(\boldsymbol{A} \cdot \boldsymbol{B})$.

2. Determine all the combinations of dot products between the standard unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

3. Let $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$. Calculate the dot product of $\boldsymbol{C}$ with itself and thus derive the law of cosines.

Solutions to the Problems

Lecture 4

Cross product

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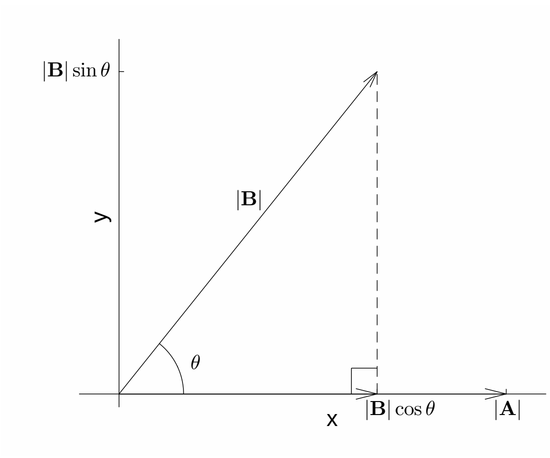
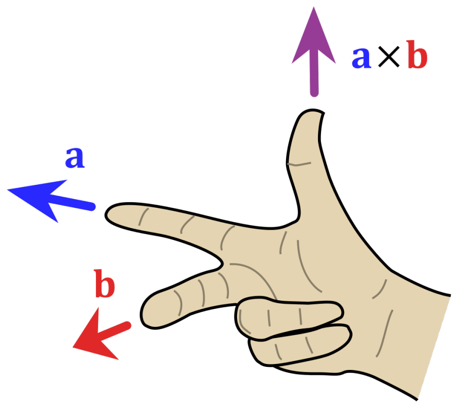
We define the cross product (or vector product) between two vectors $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$ and $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$ as

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2B_3 - A_3B_2)\boldsymbol{i} + (A_3B_1 - A_1B_3)\boldsymbol{j} + (A_1B_2 - A_2B_1)\boldsymbol{k}.$$

Use of the three-by-three determinant is a useful mnemonic to remember the formula. One can prove that the cross product is anticommutative, distributive over addition, and associative with respect to scalar multiplication; that is

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}, \qquad \boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}, \qquad \boldsymbol{A} \times (c\boldsymbol{B}) = (c\boldsymbol{A}) \times \boldsymbol{B} = c(\boldsymbol{A} \times \boldsymbol{B}).$$





Right-hand rule (from Wikipedia)

A geometric interpretation of the cross product is also possible. Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ with angle θ between them, impose a coordinate system so that $\boldsymbol{A}$ is parallel to the x-axis and $\boldsymbol{B}$ lies in the x-y plane. Then $\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{i}$, $\boldsymbol{B} = |\boldsymbol{B}|\cos\theta\boldsymbol{i} + |\boldsymbol{B}|\sin\theta\boldsymbol{j}$, and $\boldsymbol{A} \times \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\sin\theta\boldsymbol{k}$. The coordinate-independent relationship is

$$|\boldsymbol{A} \times \boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}|\sin\theta,$$

where θ lies between zero and 180°. Furthermore, the vector $\boldsymbol{A} \times \boldsymbol{B}$ points in the direction perpendicular to the plane formed by $\boldsymbol{A}$ and $\boldsymbol{B}$, and its sign is determined by the right-hand rule.

Problems for Lecture 4

1. Using properties of the determinant, prove that

a) $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$;

b) $\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$;

c) $\boldsymbol{A} \times (k\boldsymbol{B}) = (k\boldsymbol{A}) \times \boldsymbol{B} = k(\boldsymbol{A} \times \boldsymbol{B})$.

2. Determine all the combinations of cross products between the standard unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

3. Show that the cross product is not in general associative. That is, find an example using unit vectors such that

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}.$$

Solutions to the Problems

Practice quiz: Vectors

1. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be any vectors. Which of the following statements is false?

a) $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$

b) $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$

c) $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$

d) $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$

2. Let $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{B} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. Then $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{j}$ is equal to

a) $a_2b_3 - a_3b_2$

b) $a_3b_1 - a_1b_3$

c) $a_1b_2 - a_2b_1$

d) $a_1b_3 - a_3b_1$

3. Which vector is not equal to zero?

a) $\boldsymbol{i} \times (\boldsymbol{j} \times \boldsymbol{k})$

b) $(\boldsymbol{i} \times \boldsymbol{j}) \times \boldsymbol{k}$

c) $(\boldsymbol{i} \times \boldsymbol{i}) \times \boldsymbol{j}$

d) $\boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j})$

Solutions to the Practice quiz

Lecture 5

Analytic geometry of lines

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In two dimensions, the equation for a line in slope-intercept form is $y = mx + b$, and in point-slope form is $y - y_1 = m(x - x_1)$. In three dimensions, a line is most commonly expressed as a parametric equation.

Suppose that a line passes through a point with position vector $\boldsymbol{r}_0$ and in a direction parallel to the vector $\boldsymbol{u}$. Then, from the definition of vector addition, we can specify the position vector $\boldsymbol{r}$ for any point on the line by

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{u}t,$$

where t is a parameter that can take on any real value.

This parametric equation for a line has clear physical meaning. If $\boldsymbol{r}$ is the position vector of a particle, then $\boldsymbol{u}$ is the velocity vector, and t is the time. In particular, differentiating $\boldsymbol{r} = \boldsymbol{r}(t)$ with respect to time results in $d\boldsymbol{r}/dt = \boldsymbol{u}$.

A nonparametric equation for the line can be obtained by eliminating t from the equations for the components. The component equations are

$$x = x_0 + u_1 t, \qquad y = y_0 + u_2 t, \qquad z = z_0 + u_3 t;$$

and eliminating t results in

$$\frac{x - x_0}{u_1} = \frac{y - y_0}{u_2} = \frac{z - z_0}{u_3}.$$

Example: Find the parametric equation for a line that passes through the points $(1, 2, 3)$ and $(3, 2, 1)$. Determine the intersection point of the line with the $z = 0$ plane.

To find a vector parallel to the direction of the line, we first compute the displacement vector between the two given points:

$$\boldsymbol{u} = (3 - 1)\boldsymbol{i} + (2 - 2)\boldsymbol{j} + (1 - 3)\boldsymbol{k} = 2\boldsymbol{i} - 2\boldsymbol{k}.$$

Choosing a point on the line with position vector $\boldsymbol{r}_0 = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, the parametric equation for the line is given by

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{u}t = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} - 2\boldsymbol{k}) = (1 + 2t)\boldsymbol{i} + 2\boldsymbol{j} + (3 - 2t)\boldsymbol{k}.$$

The line crosses the $z = 0$ plane when $3 - 2t = 0$, or $t = 3/2$, and $(x, y) = (4, 2)$.

Problems for Lecture 5

1. Find the parametric equation for a line that passes through the points $(1, 1, 1)$ and $(2, 3, 2)$. Determine the intersection point of the line with the $x = 0$ plane, $y = 0$ plane and $z = 0$ plane.

Solutions to the Problems

Lecture 6

Analytic geometry of planes

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A plane in three-dimensional space is determined by three non-collinear points. Two linearly independent displacement vectors with direction parallel to the plane can be formed from these three points, and the cross-product of these two displacement vectors will be a vector that is orthogonal to the plane. We can use the dot product to express this orthogonality.

So let three points that define a plane be located by the position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$, and construct any two displacement vectors, such as $\boldsymbol{s}_1 = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and $\boldsymbol{s}_2 = \boldsymbol{r}_3 - \boldsymbol{r}_2$. A vector normal to the plane is given by $\boldsymbol{N} = \boldsymbol{s}_1 \times \boldsymbol{s}_2$, and for any point in the plane with position vector $\boldsymbol{r}$, and for any one of the given position vectors $\boldsymbol{r}_i$, we have $\boldsymbol{N} \cdot (\boldsymbol{r} - \boldsymbol{r}_i) = 0$. With $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $\boldsymbol{N} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and $d = \boldsymbol{N} \cdot \boldsymbol{r}_i$, the equation for the plane can be written as $\boldsymbol{N} \cdot \boldsymbol{r} = \boldsymbol{N} \cdot \boldsymbol{r}_i$, or

$$ax + by + cz = d.$$

Notice that the coefficients of x, y and z are the components of the normal vector to the plane.
Example: Find an equation for the plane defined by the three points $(2, 1, 1)$, $(1, 2, 1)$, *and* $(1, 1, 2)$. *Determine the equation for the line in the x-y plane formed by the intersection of this plane with the* $z = 0$ *plane.*

To find two vectors parallel to the plane, we compute two displacement vectors from the three points:

$$\boldsymbol{s}_1 = (1-2)\boldsymbol{i} + (2-1)\boldsymbol{j} + (1-1)\boldsymbol{k} = -\boldsymbol{i} + \boldsymbol{j}, \qquad \boldsymbol{s}_2 = (1-1)\boldsymbol{i} + (1-2)\boldsymbol{j} + (2-1)\boldsymbol{k} = -\boldsymbol{j} + \boldsymbol{k}.$$

A normal vector to the plane is then found from

$$\boldsymbol{N} = \boldsymbol{s}_1 \times \boldsymbol{s}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}.$$

And the equation for the plane can be found from $\boldsymbol{N} \cdot \boldsymbol{r} = \boldsymbol{N} \cdot \boldsymbol{r}_1$, or

$$(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \cdot (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}), \quad \text{or} \quad x + y + z = 4.$$

The intersection of this plane with the $z = 0$ plane forms the line given by $y = -x + 4$.

Problems for Lecture 6

1. Find an equation for the plane defined by the three points $(-1,-1,-1)$, $(1,1,1)$ and $(1,-1,0)$. Determine the equation for the line in the x-y plane formed by the intersection of this plane with the $z=0$ plane.

Solutions to the Problems

Practice quiz: Analytic geometry

1. The line that passes through the points $(0, 1, 1)$ and $(1, 0, -1)$ has parametric equation given by

a) $t\boldsymbol{i} + (1+t)\boldsymbol{j} + (1+2t)\boldsymbol{k}$

b) $t\boldsymbol{i} + (1-t)\boldsymbol{j} + (1+2t)\boldsymbol{k}$

c) $t\boldsymbol{i} + (1+t)\boldsymbol{j} + (1-2t)\boldsymbol{k}$

d) $t\boldsymbol{i} + (1-t)\boldsymbol{j} + (1-2t)\boldsymbol{k}$

2. The line of Question 1 intersects the $z = 0$ plane at the point

a) $(\frac{1}{2}, \frac{1}{2}, 0)$

b) $(-\frac{1}{2}, \frac{1}{2}, 0)$

c) $(\frac{1}{2}, -\frac{1}{2}, 0)$

d) $(-\frac{1}{2}, -\frac{1}{2}, 0)$

3. The equation for the line in the x-y plane formed by the intersection of the plane defined by the points $(1, 1, 1)$, $(1, 1, 2)$ and $(2, 1, 1)$ and the $z = 0$ plane is given by

a) $y = x$

b) $y = x + 1$

c) $y = x - 1$

d) $y = 1$

Solutions to the Practice quiz

Lecture 7

Kronecker delta and Levi-Civita symbol

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We define the Kronecker delta δ_{ij} to be $+1$ if $i = j$ and 0 otherwise, and the Levi-Civita symbol ϵ_{ijk} to be $+1$ if i, j, and k are a cyclic permutation of $(1, 2, 3)$ (that is, one of $(1, 2, 3)$, $(2, 3, 1)$ or $(3, 1, 2)$); -1 if an anticyclic permutation of $(1, 2, 3)$ (that is, one of $(3, 2, 1)$, $(2, 1, 3)$ or $(1, 3, 2)$); and 0 if any two indices are equal. More formally,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j; \end{cases}$$

and

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2); \\ -1, & \text{if } (i, j, k) \text{ is } (3, 2, 1), (2, 1, 3) \text{ or } (1, 3, 2); \\ 0, & \text{if } i = j, \text{ or } j = k, \text{ or } k = i. \end{cases}$$

For convenience, we will use the Einstein summation convention when working with these symbols, where a repeated index implies summation over that index. For example, $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$; and $\epsilon_{ijk}\epsilon_{ijk} = 6$, where we have summed over i, j, and k. This latter expression contains a total of $3^3 = 27$ terms in the sum, where six of the terms are equal to one and the remaining terms are equal to zero.

There is a remarkable relationship between the product of Levi-Civita symbols and the Kronecker delta, given by the determinant

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).$$

Important identities that follow from this relationship are

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \qquad \epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}.$$

The Kronecker delta, Levi-Civita symbol, and the Einstein summation convention are used to derive some common vector identities. The dot-product is written as $\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i$, and the Levi-Civita symbol is used to write the ith component of the cross-product as $[\boldsymbol{A} \times \boldsymbol{B}]_i = \epsilon_{ijk} A_j B_k$, which can be made self-evident by explicitly writing out the three components. The Kronecker delta finds use as $\delta_{ij} A_j = A_i$.

Problems for Lecture 7

1. Prove the following cyclic and anticyclic permutation identities:

a) $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$;

b) $\epsilon_{ijk} = -\epsilon_{jik}, \quad \epsilon_{ijk} = -\epsilon_{kji}, \quad \epsilon_{ijk} = -\epsilon_{ikj}$.

2. Verify the cross-product relation $[\boldsymbol{A} \times \boldsymbol{B}]_i = \epsilon_{ijk} A_j B_k$ by considering $i = 1, 2, 3$.

3. Prove the following Kronecker-delta identities:

a) $\delta_{ij} A_j = A_i$;

b) $\delta_{ik}\delta_{kj} = \delta_{ij}$.

4. Given the most general identity relating the Levi-Civita symbol to the Kronecker delta,

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}),$$

prove the following simpler and more useful relations:

a) $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$;

b) $\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}$.

Solutions to the Problems

Lecture 8

Vector identities

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Three useful vector identities are

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}), \tag{8.1}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} \tag{8.2}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}) \tag{8.3}$$

Parentheses are optional when expressions have only one possible interpretation, but for clarity they are often written. Proofs of these vector identities make use of the following Kronecker delta and Levi-Civita identities: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$; $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$; and $\delta_{ij}A_j = A_i$.

The first identity, called the scalar triple product, can be proved using the cyclic property of the Levi-Civita tensor:

$$A_i\epsilon_{ijk}B_jC_k = B_j\epsilon_{jki}C_kA_i = C_k\epsilon_{kij}A_iB_j.$$

Another proof writes the scalar triple product as the three-by-three determinant

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix},$$

and uses the property that the determinant changes sign under row interchange. The scalar triple product is also the volume of the parallelepiped defined by the three vectors.

The second identity, called the vector triple product, can be proved by writing the i^{th} component as

$$\begin{aligned}
[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})]_i &= \epsilon_{ijk}A_j[\boldsymbol{B} \times \boldsymbol{C}]_k = \epsilon_{ijk}A_j\epsilon_{klm}B_lC_m = \epsilon_{kij}\epsilon_{klm}A_jB_lC_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m = A_jC_jB_i - A_jB_jC_i = [(\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}]_i.
\end{aligned}$$

The third identity, called the scalar quadruple product, has proof

$$\begin{aligned}
(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) &= [\boldsymbol{A} \times \boldsymbol{B}]_i[\boldsymbol{C} \times \boldsymbol{D}]_i = \epsilon_{ijk}A_jB_k\epsilon_{ilm}C_lD_m = \epsilon_{ijk}\epsilon_{ilm}A_jB_kC_lD_m \\
&= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m = A_jC_jB_kD_k - A_jD_jB_kC_k \\
&= (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}).
\end{aligned}$$

Problems for Lecture 8

1. Prove the Jacobi identity:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = 0.$$

2. Prove Lagrange's identity in three dimensions:

$$|\boldsymbol{A} \times \boldsymbol{B}|^2 = |\boldsymbol{A}|^2|\boldsymbol{B}|^2 - (\boldsymbol{A} \cdot \boldsymbol{B})^2.$$

Solutions to the Problems

Practice quiz: Vector algebra

1. The expression $\epsilon_{ijk}\epsilon_{ljm}$ is equal to

a) $\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

b) $\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}$

c) $\delta_{km}\delta_{il} - \delta_{kl}\delta_{im}$

d) $\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}$

2. The expression $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$ is always equal to

a) $\boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A})$

b) $\boldsymbol{A} \times (\boldsymbol{C} \times \boldsymbol{B})$

c) $(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$

d) $(\boldsymbol{C} \times \boldsymbol{B}) \times \boldsymbol{A}$

3. Which of the following expressions may not be zero?

a) $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{B})$

b) $\boldsymbol{A} \cdot (\boldsymbol{A} \times \boldsymbol{B})$

c) $\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{B})$

d) $\boldsymbol{B} \cdot (\boldsymbol{A} \times \boldsymbol{B})$

Solutions to the Practice quiz

Lecture 9

Scalar and vector fields

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In some physical problems scalars and vectors can be functions of both space and time. We call these types of variables fields. For example, the temperature in some spatial domain is a scalar field, and we can write

$$T(\boldsymbol{r}, t) = T(x, y, z; t),$$

where we use the common notation of a semicolon on the right-hand-side to separate the space and time dependence. Notice that the position vector $\boldsymbol{r}$ is used to locate the temperature in space. As another example, the velocity vector $\boldsymbol{u}$ of a flowing fluid is a vector field, and we can write

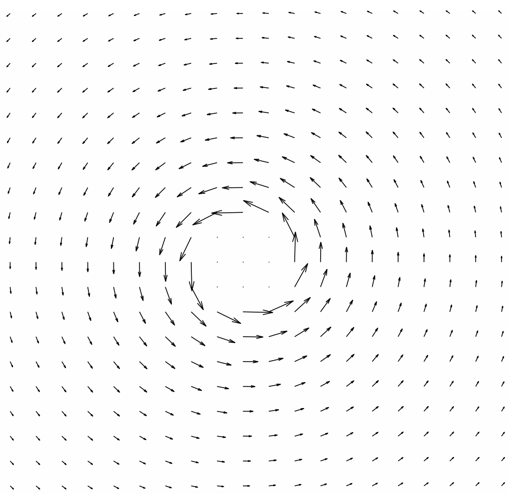
$$\boldsymbol{u}(\boldsymbol{r}, t) = u_1(x, y, z; t)\boldsymbol{i} + u_2(x, y, z; t)\boldsymbol{j} + u_3(x, y, z; t)\boldsymbol{k}.$$

The equations governing a field are sometimes called the field equations, and these equations commonly take the form of partial differential equations. For example, the equations for the electric and magnetic vector fields are the famous Maxwell's equations, and the equation for the velocity vector field is called the Navier-Stokes equation. The equation for the scalar field (called the wave function) in non-relativistic quantum mechanics is called the Schrödinger equation.

There are many ways to visualize scalar and vector fields, and one tries to make plots as informative as possible. On the right is a simple visualization of the two-dimensional vector field given by

$$\boldsymbol{B}(x, y) = \frac{-y\,\boldsymbol{i} + x\,\boldsymbol{j}}{x^2 + y^2},$$

where the vectors at each point are represented by arrows.



Problems for Lecture 9

1. Give some physical examples of scalar and vector fields.

Solutions to the Problems

Week II

Differentiation

In this week's lectures, we learn about the derivatives of scalar and vector fields. We define the partial derivative and derive the method of least squares as a minimization problem. We learn how to use the chain rule for a function of several variables, and derive the triple product rule used in chemistry. From the *del* differential operator, we define the gradient, divergence, curl and Laplacian. We learn some useful vector derivative identities and how to derive them using the Kronecker delta and Levi-Civita symbol. Vector identities are then used to derive the electromagnetic wave equation from Maxwell's equations in free space. Electromagnetic waves are fundamental to all modern communication technologies.

Lecture 10

Partial derivatives

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For a function $f = f(x,y)$ of two variables, we define the partial derivative of f with respect to x as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

and similarly for the partial derivative of f with respect to y. To take a partial derivative with respect to a variable, take the derivative with respect to that variable treating all other variables as constants. As an example, consider

$$f(x,y) = 2x^3y^2 + y^3.$$

We have

$$\frac{\partial f}{\partial x} = 6x^2y^2, \quad \frac{\partial f}{\partial y} = 4x^3y + 3y^2.$$

Second derivatives are defined as the derivatives of the first derivatives, so we have

$$\frac{\partial^2 f}{\partial x^2} = 12xy^2, \quad \frac{\partial^2 f}{\partial y^2} = 4x^3 + 6y;$$

and for continuous differentiable functions, the mixed second partial derivatives are independent of the order in which the derivatives are taken,

$$\frac{\partial^2 f}{\partial x \partial y} = 12x^2y = \frac{\partial^2 f}{\partial y \partial x}.$$

To simplify notation, we introduce the standard subscript notation for partial derivatives,

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad \text{etc.}$$

The Taylor series of $f(x,y)$ about the origin is developed by expanding the function in a multivariable power series that agrees with the function value and all its partial derivatives at the origin. We have

$$f(x,y) = f + f_x x + f_y y + \frac{1}{2!}\left(f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2\right) + \dots,$$

where the function and all its partial derivatives on the right-hand side are evaluated at the origin and are constants.

Problems for Lecture 10

1. Compute the three partial derivatives of

$$f(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^n}.$$

2. Given the function $f = f(t, x)$, find the Taylor series expansion of the expression

$$f(t + \alpha\Delta t, x + \beta\Delta t f(t, x))$$

to first order in Δt.

Solutions to the Problems

Lecture 11

The method of least squares

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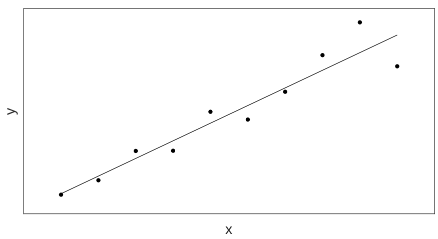
Local maxima and minima of a multivariable function can be found by computing the zeros of the partial derivatives. These zeros are called the critical points of the function. A critical point need not be a maximum or minimum, for example it might be a minimum in one direction and a maximum in another (called a saddle point), but in many problems maxima or minima may be assumed to exist. Here, we illustrate the procedure for minimizing a function by solving the least-squares problem.

Suppose there is some experimental data that you want to fit by a straight line (illustrated on the right). In general, let the data consist of a set of n points given by $(x_1, y_1), \ldots, (x_n, y_n)$. Here, we assume that the x values are exact, and the y values are noisy. We further assume that the best fit line to the data takes the form $y = \beta_0 + \beta_1 x$. Although we know that the line can not go through all the data points, we can try to find the line that minimizes the sum of the squares of the vertical distances between the line and the points.



Define this function of the sum of the squares to be

$$f(\beta_0, \beta_1) = \sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)^2 .$$

Here, the data is assumed given and the unknowns are the fitting parameters β_0 and β_1. It should be clear from the problem specification, that there must be values of β_0 and β_1 that minimize the function $f = f(\beta_0, \beta_1)$. To determine, these values, we set $\partial f/\partial \beta_0 = \partial f/\partial \beta_1 = 0$. This results in the equations

$$\sum_{i=1}^{n}(\beta_0 + \beta_1 x_i - y_i) = 0, \qquad \sum_{i=1}^{n} x_i (\beta_0 + \beta_1 x_i - y_i) = 0.$$

We can write these equations as a linear system for β_0 and β_1 as

$$\beta_0 n + \beta_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \qquad \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i.$$

The solution for β_0 and β_1 in terms of the data is given by

$$\beta_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}, \quad \beta_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2},$$

where the summations are from $i = 1$ to n.

Problems for Lecture 11

1. By minimizing the sum of the squares of the vertical distance between the line and the points, determine the least-squares line through the data points $(1,1)$, $(2,3)$ and $(3,2)$.

Solutions to the Problems

Lecture 12

Chain rule

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Partial derivatives are used in applying the chain rule to a function of several variables. Consider a two-dimensional scalar field $f = f(x, y)$, and define the total differential of f to be

$$df = f(x + dx, y + dy) - f(x, y).$$

We can write df as

$$\begin{aligned} df &= [f(x + dx, y + dy) - f(x, y + dy)] + [f(x, y + dy) - f(x, y)] \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \end{aligned}$$

If one has $f = f(x(t), y(t))$, say, then division of df by dt results in

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

And if one has $f = f(x(r, \theta), y(r, \theta))$, say, then the corresponding chain rule is given by

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

Example: Consider the differential equation $\frac{dx}{dt} = u(t, x(t))$. *Determine a formula for* $\frac{d^2x}{dt^2}$ *in terms of* u *and its partial derivatives.*

Applying the chain rule, we have at time t,

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} \\ &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}. \end{aligned}$$

The above formula is called the material derivative and in three dimensions forms a part of the Navier-Stokes equation for fluid flow.

Problems for Lecture 12

1. Let $f(x, y) = e^{xy}$, with $x = r\cos\theta$ and $y = r\sin\theta$. Compute the partial derivatives $\partial f/\partial r$ and $\partial f/\partial\theta$ in two ways:

a) Use the chain rule on $f = f(x(r, \theta), y(r, \theta))$;

b) Eliminate x and y in favor of r and θ and compute the partial derivatives directly.

Solutions to the Problems

Lecture 13

Triple product rule

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Suppose that three variables x, y and z are related by the equation $f(x, y, z) = 0$, and that it is possible to write $x = x(y, z)$ and $z = z(x, y)$. Taking differentials of x and y, we have

$$dx = \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz, \qquad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

We can make use of the second equation to eliminate dz in the first equation to obtain

$$dx = \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right);$$

or collecting terms,

$$\left(1 - \frac{\partial x}{\partial z}\frac{\partial z}{\partial x}\right)dx = \left(\frac{\partial x}{\partial y} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial y}\right)dy.$$

Since dx and dy are independent variations, the terms in parenthesis must be zero. The left-hand-side results in the reciprocity relation

$$\frac{\partial x}{\partial z}\frac{\partial z}{\partial x} = 1,$$

which states the intuitive result that $\partial z/\partial x$ and $\partial x/\partial z$ are multiplicative inverses of each other. The right-hand-side results in

$$\frac{\partial x}{\partial y} = -\frac{\partial x}{\partial z}\frac{\partial z}{\partial y},$$

which when making use of the reciprocity relation, yields the counterintuitive triple product rule,

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

Lecture 14

Triple product rule: example

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Example: Demonstrate the triple product rule using the ideal gas law.

The ideal gas law states that

$$PV = nRT,$$

where P is the pressure, V is the volume, T is the absolute temperature, n is the number of moles of the gas, and R is the ideal gas constant. We say P, V and T are the state variables, and the ideal gas law is a relation of the form

$$f(P, V, T) = PV - nRT = 0.$$

We can write $P = P(V, T)$, $V = V(P, T)$ and $T = T(P, V)$, that is,

$$P = \frac{nRT}{V}, \qquad V = \frac{nRT}{P}, \qquad T = \frac{PV}{nR};$$

and the partial derivatives are given by

$$\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}, \qquad \frac{\partial V}{\partial T} = \frac{nR}{P}, \qquad \frac{\partial T}{\partial P} = \frac{V}{nR}.$$

The triple product results in

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -\left(\frac{nRT}{V^2}\right)\left(\frac{nR}{P}\right)\left(\frac{V}{nR}\right) = -\frac{nRT}{PV} = -1,$$

where we make use of the ideal gas law in the last equality.

Problems for Lecture 13

1. Suppose the three variables x, y and z are related by the linear expression $ax + by + cz = 0$. Show that x, y and z satisfy the triple product rule.

2. Suppose the four variables x, y, z and t are related by the linear expression $ax + by + cz + dt = 0$. Determine a corresponding quadruple product rule for these variables.

Solutions to the Problems

Practice quiz: Partial derivatives

1. Let $f(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$. The mixed second partial derivative $\dfrac{\partial^2 f}{\partial x \partial y}$ is equal to

a) $\dfrac{2(x + y)}{(x^2 + y^2 + z^2)^{5/2}}$

b) $\dfrac{(x + y)^2}{(x^2 + y^2 + z^2)^{5/2}}$

c) $\dfrac{x^2 + y^2}{(x^2 + y^2 + z^2)^{5/2}}$

d) $\dfrac{3xy}{(x^2 + y^2 + z^2)^{5/2}}$

2. The least-squares line through the data points $(0, 1)$, $(1, 3)$, $(2, 3)$ and $(3, 4)$ is given by

a) $y = \dfrac{7}{5} + \dfrac{9x}{10}$

b) $y = \dfrac{5}{7} + \dfrac{9x}{10}$

c) $y = \dfrac{7}{5} + \dfrac{10x}{9}$

d) $y = \dfrac{5}{7} + \dfrac{10x}{9}$

3. Let $f = f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$. Which of the following is true?

a) $\dfrac{\partial f}{\partial \theta} = x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$

b) $\dfrac{\partial f}{\partial \theta} = -x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$

c) $\dfrac{\partial f}{\partial \theta} = y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y}$

d) $\dfrac{\partial f}{\partial \theta} = -y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y}$

Solutions to the Practice quiz

Lecture 15

Gradient

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Consider the three-dimensional scalar field $f = f(x, y, z)$, and the differential df, given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Using the dot product, we can write this in vector form as

$$df = \left(\frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j} + \frac{\partial f}{\partial z} \boldsymbol{k} \right) \cdot (dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}) = \boldsymbol{\nabla} f \cdot d\boldsymbol{r},$$

where in Cartesian coordinates,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j} + \frac{\partial f}{\partial z} \boldsymbol{k}$$

is called the gradient of f. The nabla symbol $\boldsymbol{\nabla}$ is pronounced "del" and $\boldsymbol{\nabla} f$ is pronounced "del-f". Another useful way to view the gradient is to consider $\boldsymbol{\nabla}$ as a vector differential operator which has the form

$$\boldsymbol{\nabla} = \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z}.$$

Because of the properties of the dot product, the differential df is maximum when the infinitesimal displacement vector $d\boldsymbol{r}$ is along the direction of the gradient $\boldsymbol{\nabla} f$. We then say that $\boldsymbol{\nabla} f$ points in the direction of maximally increasing f, and whose magnitude gives the slope (or gradient) of f in that direction.

Example: Compute the gradient of $f(x, y, z) = xyz$.

The partial derivatives are easily calculated, and we have

$$\boldsymbol{\nabla} f = yz\,\boldsymbol{i} + xz\,\boldsymbol{j} + xy\,\boldsymbol{k}.$$

Problems for Lecture 15

1. Let the position vector be given by $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, with $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$. Compute the gradient of the following scalar fields and write the result in terms of $\boldsymbol{r}$ and r.

a) $\phi(x, y, z) = x^2 + y^2 + z^2$;

b) $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

Solutions to the Problems

Lecture 16

Divergence

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Consider in Cartesian coordinates the three-dimensional vector field, $\boldsymbol{u} = u_1(x,y,z)\boldsymbol{i} + u_2(x,y,z)\boldsymbol{j} + u_3(x,y,z)\boldsymbol{k}$. The divergence of $\boldsymbol{u}$, denoted as $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ and pronounced "del-dot-u", is defined as the scalar field given by

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{u} &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \\ &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}.\end{aligned}$$

Here, the dot product is used between a vector differential operator $\boldsymbol{\nabla}$ and a vector field $\boldsymbol{u}$. The divergence measures how much a vector field spreads out, or diverges, from a point. A more math-based description will be given later.

Example: Let the position vector be given by $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Find $\boldsymbol{\nabla} \cdot \boldsymbol{r}$.

A direct calculation gives

$$\boldsymbol{\nabla} \cdot \boldsymbol{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3.$$

Example: Let $\boldsymbol{F} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$ for all $\boldsymbol{r} \neq 0$. Find $\boldsymbol{\nabla} \cdot \boldsymbol{F}$.

Writing out the components of $\boldsymbol{F}$, we have

$$\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{k}.$$

Using the quotient rule for the derivative, we have

$$\frac{\partial F_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{1}{|\boldsymbol{r}|^3} - \frac{3x^2}{|\boldsymbol{r}|^5},$$

and analogous results for $\partial F_2/\partial y$ and $\partial F_3/\partial z$. Adding the three derivatives results in

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{3}{|\boldsymbol{r}|^3} - \frac{3(x^2+y^2+z^2)}{|\boldsymbol{r}|^5} = \frac{3}{|\boldsymbol{r}|^3} - \frac{3}{|\boldsymbol{r}|^3} = 0,$$

valid as long as $|\boldsymbol{r}| \neq 0$, where $\boldsymbol{F}$ diverges. In the study of electrostatics, $\boldsymbol{F}$ is proportional to the electric field of a point charge located at the origin.

Problems for Lecture 16

1. Find the divergence of the following vector fields:

a) $\boldsymbol{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$;

b) $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$.

Solutions to the Problems

Lecture 17

Curl

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Consider in Cartesian coordinates the three-dimensional vector field $\boldsymbol{u} = u_1(x,y,z)\boldsymbol{i} + u_2(x,y,z)\boldsymbol{j} + u_3(x,y,z)\boldsymbol{k}$. The curl of $\boldsymbol{u}$, denoted as $\boldsymbol{\nabla} \times \boldsymbol{u}$ and pronounced "del-cross-u", is defined as the vector field given by

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\boldsymbol{k}.$$

Here, the cross product is used between a vector differential operator and a vector field. The curl measures how much a vector field rotates, or curls, around a point. A more math-based description will be given later.

Example: Show that the curl of a gradient is zero, that is, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$.

We have

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) &= \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{pmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\boldsymbol{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\boldsymbol{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\boldsymbol{k} = 0, \end{aligned}$$

using the equality of mixed partials.

Example: Show that the divergence of a curl is zero, that is, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0$.

We have

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) &= \frac{\partial}{\partial x}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) \\ &= \left(\frac{\partial^2 u_1}{\partial y \partial z} - \frac{\partial^2 u_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 u_2}{\partial z \partial x} - \frac{\partial^2 u_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 u_3}{\partial x \partial y} - \frac{\partial^2 u_3}{\partial y \partial x}\right) = 0, \end{aligned}$$

again using the equality of mixed partials.

Problems for Lecture 17

1. Find the curl of the following vector fields:

a) $\boldsymbol{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$;

b) $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$.

Solutions to the Problems

Lecture 18

Laplacian

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The Laplacian is the differential operator $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \nabla^2$, given in Cartisian coordinates as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The Laplacian can be applied to either a scalar field or a vector field. The Laplacian applied to a scalar field, $f = f(x, y, z)$, can be written as the divergence of the gradient, that is,

$$\nabla^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Laplacian applied to a vector field, acts on each component of the vector field separately. With $\boldsymbol{u} = u_1(x, y, z)\boldsymbol{i} + u_2(x, y, z)\boldsymbol{j} + u_3(x, y, z)\boldsymbol{k}$, we have

$$\nabla^2 \boldsymbol{u} = \nabla^2 u_1 \boldsymbol{i} + \nabla^2 u_2 \boldsymbol{j} + \nabla^2 u_3 \boldsymbol{k}.$$

The Laplacian appears in some classic partial differential equations. The Laplace equation, wave equation, and diffusion equation all contain the Laplacian and are given, respectively, by

$$\nabla^2 \Phi = 0, \qquad \frac{\partial^2 \Phi}{\partial t^2} = c^2 \nabla^2 \Phi, \qquad \frac{\partial \Phi}{\partial t} = D \nabla^2 \Phi.$$

Example: Find the Laplacian of $f(x, y, z) = x^2 + y^2 + z^2$.

We have $\nabla^2 f = 2 + 2 + 2 = 6$.

Problems for Lecture 18

1. Compute $\nabla^2 \left(\frac{1}{r}\right)$ for $r \neq 0$. Here, $r = \sqrt{x^2 + y^2 + z^2}$.

Solutions to the Problems

Practice quiz: The del operator

1. Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$. What is the value of $\boldsymbol{\nabla}\left(\frac{1}{r^2}\right)$?

a) $-\frac{\boldsymbol{r}}{r^3}$

b) $-\frac{2\boldsymbol{r}}{r^3}$

c) $-\frac{\boldsymbol{r}}{r^4}$

d) $-\frac{2\boldsymbol{r}}{r^4}$

2. Let $\boldsymbol{F} = \frac{\boldsymbol{r}}{r}$. The divergence $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ is equal to

a) $\frac{1}{r}$

b) $\frac{2}{r}$

c) $\frac{1}{r^2}$

d) $\frac{2}{r^2}$

3. The curl of the position vector, $\boldsymbol{\nabla} \times \boldsymbol{r}$, is equal to

a) $\frac{\boldsymbol{r}}{r}$

b) 0

c) $\frac{(\boldsymbol{\nabla} \cdot \boldsymbol{r})\boldsymbol{r}}{r}$

d) $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

Solutions to the Practice quiz

Lecture 19

Vector derivative identities

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Let $f = f(\boldsymbol{r})$ be a scalar field and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r})$ and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r})$ be vector fields, where $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$, and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. Here, we will change notation and define the position vector to be $\boldsymbol{r} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$ (instead of using the coordinates x, y and z). We have already shown that the curl of a gradient is zero, and the divergence of a curl is zero, that is,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0, \qquad \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0.$$

Other sometimes useful vector derivative identities include

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}, \\
\boldsymbol{\nabla} \cdot (f\boldsymbol{u}) &= \boldsymbol{u} \cdot \boldsymbol{\nabla} f + f \boldsymbol{\nabla} \cdot \boldsymbol{u}, \\
\boldsymbol{\nabla} \times (f\boldsymbol{u}) &= \boldsymbol{\nabla} f \times \boldsymbol{u} + f \boldsymbol{\nabla} \times \boldsymbol{u}, \\
\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{v}) &= (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{u}), \\
\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}), \\
\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v}) &= \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) - \boldsymbol{v}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v}.
\end{aligned}$$

One use of the del operator that we haven't yet seen is

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3},$$

which acts on a scalar field as

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} f = u_1 \frac{\partial f}{\partial x_1} + u_2 \frac{\partial f}{\partial x_2} + u_3 \frac{\partial f}{\partial x_3};$$

or acts on a vector field as

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (\boldsymbol{u} \cdot \boldsymbol{\nabla} v_1)\,\boldsymbol{i} + (\boldsymbol{u} \cdot \boldsymbol{\nabla} v_2)\,\boldsymbol{j} + (\boldsymbol{u} \cdot \boldsymbol{\nabla} v_3)\,\boldsymbol{k}.$$

In some of these identities, the parentheses are optional when the expression has only one possible interpretation. For example, it is common to see $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ written as $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{v}$. The parentheses are mandatory when the expression can be interpreted in more than one way, for example $\boldsymbol{\nabla} \times \boldsymbol{u} \times \boldsymbol{v}$ could mean either $\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v})$ or $(\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{v}$, and these two expressions are usually not equal.

Proof of all of these identities is most readily done by manipulating the Kronecker delta and Levi-Civita symbols, and I give an example in the next lecture.

Lecture 20

Vector derivative identities (proof)

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To prove the vector derivative identities, we use component notation, the Einstein summation convention, the Levi-Civita symbol and the Kronecker delta. The ith component of the curl of a vector field is written using the Levi-Civita symbol as

$$(\boldsymbol{\nabla} \times \boldsymbol{u})_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j};$$

and the divergence of a vector field is written as

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u_i}{\partial x_i}.$$

The contraction of the Kronecker delta with a vector field is given by $\delta_{ij} u_j = u_i$, and the Levi-Civita symbol is invariant under cyclical permutation of its indices, that is, $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$. If only two indices are interchanged, then the symbol changes sign, for example, $\epsilon_{ijk} = -\epsilon_{jik}$. Furthermore, a useful identity when a vector derivative identity contains two cross products is

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

As one example, I prove here the vector derivative identity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}).$$

We have

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= \frac{\partial}{\partial x_i}\left(\epsilon_{ijk} u_j v_k\right) \\
&= \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} v_k + \epsilon_{ijk} u_j \frac{\partial v_k}{\partial x_i} \\
&= v_k \epsilon_{kij} \frac{\partial u_j}{\partial x_i} - u_j \epsilon_{jik} \frac{\partial v_k}{\partial x_i} \\
&= \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}).
\end{aligned}$$

The crucial step in the proof is the use of the product rule for the derivative. The rest of the proof just requires facility with the notation and the manipulation of the indices of the Levi-Civita symbol.

Problems for Lecture 20

1. Use the Kronecker delta, the Levi-Civita symbol and the Einstein summation convention, and the identities

$$\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij} a_i b_j, \quad (\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k, \quad \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl},$$

to prove the following identities:

a) $\boldsymbol{\nabla} \cdot (f\boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} f + f \boldsymbol{\nabla} \cdot \boldsymbol{u}$;

b) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

2. Consider the vector differential equation

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{u}(t, \boldsymbol{r}(t)),$$

where

$$\boldsymbol{r} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}, \qquad \boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}.$$

a) Write down the differential equations for dx_1/dt, dx_2/dt and dx_3/dt;

b) Use the chain rule to determine formulas for d^2x_1/dt^2, d^2x_2/dt^2 and d^3x_3/dt^2;

c) Write your solution for $d^2\boldsymbol{r}/dt^2$ as a vector equation using the $\boldsymbol{\nabla}$ differential operator.

Solutions to the Problems

Lecture 21

Electromagnetic waves

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Maxwell's equations in free space are most simply written using the del operator, and are given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

Here I use the so-called SI units familiar to engineering students, where the constants ϵ_0 and μ_0 are called the permittivity and permeability of free space, respectively.

From the four Maxwell's equations, we would like to obtain a single equation for the electric field $\boldsymbol{E}$. To do so, we can make use of the curl of the curl identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}.$$

To obtain an equation for $\boldsymbol{E}$, we take the curl of the third Maxwell's equation and commute the time and space derivatives

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \boldsymbol{B}).$$

We apply the curl of the curl identity to obtain

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \boldsymbol{B}),$$

and then apply the first Maxwell's equation to the left-hand-side, and the fourth Maxwell's equation to the right-hand-side. Rearranging terms, we obtain the three-dimensional wave equation given by

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = c^2 \nabla^2 \boldsymbol{E},$$

with c the wave speed given by $c = 1/\sqrt{\mu_0 \epsilon_0} \approx 3 \times 10^8$ m/s. This is, of course, the speed of light in vacuum.

Problems for Lecture 21

1. Derive the wave equation for the magnetic field $\boldsymbol{B}$.

Solutions to the Problems

Practice quiz: Vector calculus algebra

1. Using the vector derivative identity $\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{v}) = (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) + \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$, determine which of the following identities is valid?

a) $\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$

b) $\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$

c) $\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$

d) $\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$

2. Which of the following expressions is not always zero?

a) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)$

b) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})$

c) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f)$

d) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))$

3. Suppose the electric field is given by $\boldsymbol{E}(\boldsymbol{r},t) = \sin(z-ct)\boldsymbol{i}$. Then which of the following is a valid free-space solution for the magnetic field $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r},t)$?

a) $\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\sin(z-ct)\boldsymbol{i}$

b) $\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\sin(z-ct)\boldsymbol{j}$

c) $\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\sin(x-ct)\boldsymbol{i}$

d) $\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\sin(x-ct)\boldsymbol{j}$

Solutions to the Practice quiz

Week III

Integration and Curvilinear Coordinates

In this week's lectures, we learn about integrating scalar and vector fields. Double and triple integrals of scalar fields are taught, as are line integrals and surface integrals of vector fields. The important technique of using curvilinear coordinates, namely polar coordinates in two dimensions, and cylindrical and spherical coordinates in three dimensions, is used to simplify problems with cylindrical or spherical symmetry. The change of variables formula for multidimensional integrals using the Jacobian of the transformation is explained.

Lecture 22

Double and triple integrals

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Double and triple integrals, written as

$$\int_A f\,dA = \iint_A f(x,y)\,dx\,dy, \qquad \int_V f\,dV = \iiint_V f(x,y,z)\,dx\,dy\,dz,$$

are the limits of the sums of $\Delta x\Delta y$ (or $\Delta x\Delta y\Delta z$) multiplied by the integrand. A single integral is the area under a curve $y = f(x)$; a double integral is the volume under a surface $z = f(x,y)$. A triple integral is used, for example, to find the mass of an object by integrating over its density.

To perform a double or triple integral, the correct limits of the integral needs to be determined, and the integral is performed as two (or three) single integrals. For example, an integration over a rectangle in the x-y plane can be written as either

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy \qquad \text{or} \qquad \int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx.$$

In the first double integral, the x integration is done first (holding y fixed), and the y integral is done second. In the second, the order of integration is reversed. Either order of integration will give the same result.

Example: Compute the volume of the surface $z = x^2y$ above the x-y plane with base given by a unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

To find the volume, we integrate $z = x^2y$ over its base. The integral over the unit square is given by either of the double integrals

$$\int_0^1\int_0^1 x^2y\,dx\,dy \qquad \text{or} \qquad \int_0^1\int_0^1 x^2y\,dy\,dx.$$

The respective calculations are

$$\int_0^1\int_0^1 x^2y\,dx\,dy = \int_0^1 \left(\frac{x^3y}{3}\bigg|_{x=0}^{x=1}\right) dy = \frac{1}{3}\int_0^1 y\,dy = \frac{1}{3}\frac{y^2}{2}\bigg|_{y=0}^{y=1} = \frac{1}{6};$$

$$\int_0^1\int_0^1 x^2y\,dy\,dx = \int_0^1 \left(\frac{x^2y^2}{2}\bigg|_{y=0}^{y=1}\right) dx = \frac{1}{2}\int_0^1 x^2\,dy = \frac{1}{2}\frac{x^3}{3}\bigg|_{x=0}^{x=1} = \frac{1}{6}.$$

In this case, an even simpler integration method separates the x and y dependence and writes

$$\int_0^1\int_0^1 x^2y\,dx\,dy = \int_0^1 x^2\,dx \int_0^1 y\,dy = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}.$$

Problems for Lecture 22

1. A cube has edge length L, with mass density increasing linearly from ρ_1 to ρ_2 from one face of the cube to the opposite face. By solving a triple integral, compute the mass of the cube in terms of L, ρ_1 and ρ_2.

Solutions to the Problems

Lecture 23

Example: Double integral with triangle base

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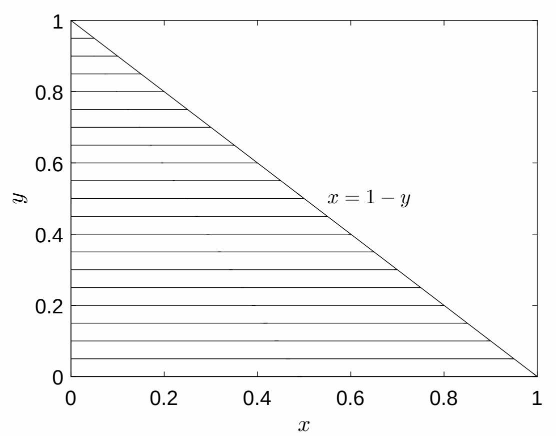
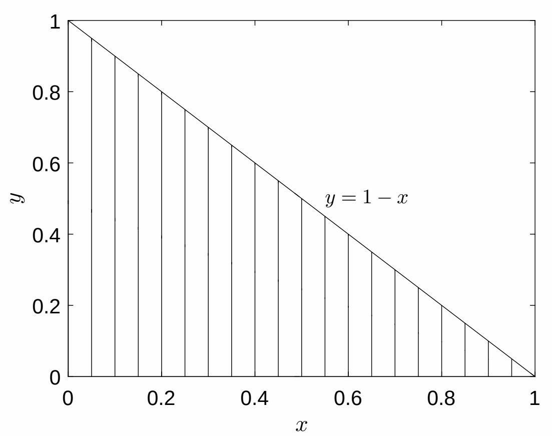
Example: Compute the volume of the surface $z = x^2y$ above the x-y plane with base given by a right triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.





The integral over the triangle (see the figures) is given by either one of the double integrals

$$\int_0^1 \int_0^{1-y} x^2 y \, dx \, dy, \quad \text{or} \quad \int_0^1 \int_0^{1-x} x^2 y \, dy \, dx.$$

The respective calculations are

$$\begin{aligned}\int_0^1 \int_0^{1-y} x^2 y \, dx \, dy &= \int_0^1 \left(\left. \frac{x^3 y}{3} \right|_{x=0}^{x=1-y} \right) dy = \frac{1}{3} \int_0^1 (1-y)^3 y \, dy = \frac{1}{3} \int_0^1 (y - 3y^2 + 3y^3 - y^4) \, dy \\ &= \frac{1}{3} \left(\frac{y^2}{2} - y^3 + \frac{3y^4}{4} - \frac{y^5}{5} \right) \Big|_0^1 = \frac{1}{3} \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) = \frac{1}{60};\end{aligned}$$

$$\begin{aligned}\int_0^1 \int_0^{1-y} x^2 y \, dy \, dx &= \int_0^1 \left(\left. \frac{x^2 y^2}{2} \right|_{y=0}^{y=1-x} \right) dx = \frac{1}{2} \int_0^1 x^2 (1-x)^2 \, dx = \frac{1}{2} \int_0^1 (x^2 - 2x^3 + x^4) \, dx \\ &= \frac{1}{2} \left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{1}{60};\end{aligned}$$

Problems for Lecture 23

1. Compute the volume of the surface $z = x^2y$ above the x-y plane with base given by a parallelogram with vertices $(0,0)$, $(1,0)$, $(4/3,1)$ and $(1/3,1)$.

Solutions to the Problems

Practice quiz: Multidimensional integration

1. The volume of the surface $z = xy$ above the x-y plane with base given by a unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ is equal to

a) $\frac{1}{5}$

b) $\frac{1}{4}$

c) $\frac{1}{3}$

d) $\frac{1}{2}$

2. A cube has edge length of 1 cm, with mass density increasing linearly from $1\,\text{g/cm}^3$ to $2\,\text{g/cm}^3$ from one face of the cube to the opposite face. The mass of the cube is given by

a) 3.0 g

b) 1.5 g

c) 1.33 g

d) 1.0 g

3. The volume of the surface $z = xy$ above the x-y plane with base given by the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$ is equal to

a) $\frac{1}{6}$

b) $\frac{1}{5}$

c) $\frac{1}{4}$

d) $\frac{1}{3}$

Solutions to the Practice quiz

Lecture 24

Polar coordinates

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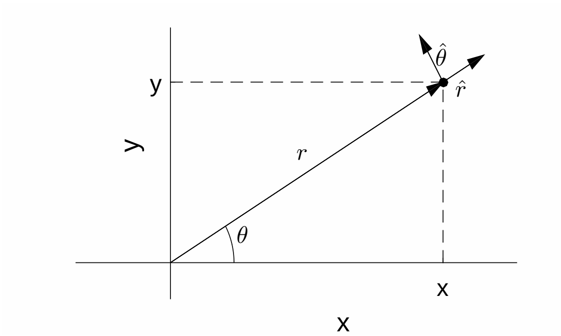

In two dimensions, polar coordinates is the most commonly used curvilinear coordinate system. The relationship between the Cartesian coordinates and polar coordinates is given by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

One defines unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ to be orthogonal and in the direction of increasing r and θ, respectively (see above figure). The $\hat{\boldsymbol{r}}$-$\hat{\boldsymbol{\theta}}$ unit vectors are rotated an angle θ from the $\boldsymbol{i}$-$\boldsymbol{j}$ unit vectors. Simple trigonometry shows that

$$\hat{\boldsymbol{r}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}.$$

It is important to remember that the direction of the unit vectors in curvilinear coordinates are not fixed, but depend on their location. Here, $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\theta)$, and by differentiating the polar unit vectors, one can show that

$$\frac{d\hat{\boldsymbol{r}}}{d\theta} = \hat{\boldsymbol{\theta}}, \qquad \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\boldsymbol{r}}.$$

The Cartesian partial derivatives can be transformed into polar form using the chain rule. Using the relationship between the Cartesian coordinates and polar coordinates, we have for a scalar field $f = f(x(r,\theta), y(r,\theta))$,

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}, \\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y};
\end{aligned}$$

and the inverse relationship is given by

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}, \qquad \frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}.$$

Problems for Lecture 24

1. The inverse of a two-by-two matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

a) Given

$$\hat{\boldsymbol{r}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j},$$

invert a two-by-two matrix to solve for $\boldsymbol{i}$ and $\boldsymbol{j}$.

b) Given

$$\frac{\partial f}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}, \qquad \frac{\partial f}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y},$$

invert a two-by-two matrix to solve for $\partial f/\partial x$ and $\partial f/\partial y$.

2. Determine expressions for $r\hat{\boldsymbol{r}}$ and $r\hat{\boldsymbol{\theta}}$ in Cartesian coordinates.

Solutions to the Problems

Lecture 25

Example: Central force

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A central force is a force acting on a point mass and pointing directly towards a fixed point in space. The origin of the coordinate system is chosen at this fixed point, and the axis orientated such that the initial position and velocity of the mass lies in the x-y plane. The subsequent motion of the mass is then two dimensional, and polar coordinates can be employed.

The position vector of the point mass in polar coordinates is given by

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}.$$

The velocity of the point mass is obtained by differentiating $\mathbf{r}$ with respect to time. The added difficulty here is that the unit vectors $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}(\theta(t))$ and $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\theta(t))$ are also functions of time. When differentiating, we will need to use the chain rule in the form

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d\hat{\boldsymbol{r}}}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}, \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\frac{d\theta}{dt}\hat{\boldsymbol{r}}.$$

As is customary, here we will use the dot notation for the time derivative. For example, $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

The velocity of the point mass is then given by

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}};$$

and the acceleration is given by

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\frac{d\hat{\boldsymbol{r}}}{dt} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}. \end{aligned}$$

A central force can be written as $\boldsymbol{F} = -f\hat{\boldsymbol{r}}$, where usually $f = f(r)$. Newton's equation $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ then becomes the two equations

$$m(\ddot{r} - r\dot{\theta}^2) = -f, \qquad m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0.$$

The second equation is usually expressed as conservation of angular momentum, and after multiplication by r, is written in the form

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0, \quad \text{or} \quad mr^2\dot{\theta} = \text{constant}.$$

Problems for Lecture 25

1. The angular momentum $\boldsymbol{l}$ of a point mass m relative to an origin is defined as

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p},$$

where $\boldsymbol{r}$ is the position vector of the mass and $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ is the momentum of the mass. Show that

$$|\boldsymbol{l}| = mr^2|\dot{\theta}|.$$

Solutions to the Problems

Lecture 26

Change of variables (single integral)

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A double (or triple) integral written in Cartesian coordinates, can sometimes be more easily computed by changing coordinate systems. To do so, we need to derive a change of variables formula.

It is illuminating to first revisit the change-of-variables formula for single integrals. Consider the integral

$$I = \int_{x_0}^{x_f} f(x)\,dx.$$

Let $u(s)$ be a differentiable and invertible function. We can change variables in this integral by letting $x = u(s)$ so that $dx = u'(s)\,ds$. The integral in the new variable s then becomes

$$I = \int_{u^{-1}(x_0)}^{u^{-1}(x_f)} f(u(s))u'(s)\,ds.$$

The key piece for us here is the transformation of the infinitesimal length $dx = u'(s)\,ds$.

We can be more concrete by examining a specific transformation. Consider the calculation of the area of a circle of radius R, given by the integral

$$A = 4\int_0^R \sqrt{R^2 - x^2}\,dx.$$

To more easily perform this integral, we can let $x = R\cos\theta$ so that $dx = -R\sin\theta\,d\theta$. The integral then becomes

$$A = 4R^2 \int_0^{\pi/2} \sqrt{1-\cos^2\theta}\sin\theta\,d\theta = \int_0^{\pi/2} \sin^2\theta\,d\theta,$$

which can be done using the double-angle formula to yield $A = \pi R^2$.

Lecture 27

Change of variables (double integral)

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We consider the double integral

$$I = \iint_A f(x, y)\, dx\, dy.$$

We would like to change variables from (x, y) to (s, t). For simplicity, we will write this change of variables as $x = x(s, t)$ and $y = y(s, t)$. The region A in the x-y domain transforms into a region A' in the s-t domain, and the integrand becomes a function of the new variables s and t by the substitution $f(x, y) = f(x(s, t), y(s, t))$. We now consider how to transform the infinitesimal area $dx\, dy$.

The transformation of $dx\, dy$ is obtained by considering how an infinitesimal rectangle is transformed into an infinitesimal parallelogram, and how the area of the two are related by the absolute value of a determinant. The main result, which we do not derive here, is given by

$$dx\, dy = |\det(J)|\, ds\, dt,$$

where J is called the Jacobian of the transformation, and is defined as

$$J = \frac{\partial(x, y)}{\partial(s, t)} = \begin{pmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{pmatrix}.$$

To be more concrete, we again calculate the area of a circle. Here, using a two-dimensional integral, the area of a circle can be written as

$$A = \iint_A dx\, dy,$$

where the integral subscript A denotes the region in the x-y plane that defines the circle. To perform this integral, we can change from Cartesian to polar coordinates. Let

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We have

$$dx\, dy = \left|\det\begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{pmatrix}\right| dr\, d\theta = \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| dr\, d\theta = r\, dr\, d\theta.$$

The region in the r-θ plane that defines the circle is $0 \le r \le R$ and $0 \le \theta \le 2\pi$. The integral then becomes

$$A = \int_0^{2\pi} \int_0^R r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^R r\, dr = \pi R^2.$$

Problems for Lecture 27

1. The mass density of a flat object can be specified by $\sigma = \sigma(x, y)$, with units of mass per unit area. The total mass of the object is found from the double integral

$$M = \int\int_A \sigma(x, y)\, dx\, dy.$$

Suppose a circular disk of radius R has mass density ρ_0 at its center and ρ_1 at its edge, and its density is a linear function of the distance from the center. Find the total mass of the disk.

2. Compute the Gaussian integral given by $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$. Use the well-known trick

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$$

Solutions to the Problems

Practice quiz: Polar coordinates

1. $r\hat{\boldsymbol{\theta}}$ is equal to

a) $x\boldsymbol{i} + y\boldsymbol{j}$

b) $x\boldsymbol{i} - y\boldsymbol{j}$

c) $y\boldsymbol{i} + x\boldsymbol{j}$

d) $-y\boldsymbol{i} + x\boldsymbol{j}$

2. $\dfrac{d\hat{\boldsymbol{\theta}}}{d\theta}$ is equal to

a) $\hat{\boldsymbol{r}}$

b) $-\hat{\boldsymbol{r}}$

c) $\hat{\boldsymbol{\theta}}$

d) $-\hat{\boldsymbol{\theta}}$

3. Suppose a circular disk of radius $1\,\text{cm}$ has mass density $10\,\text{g}/\text{cm}^2$ at its center, and $1\,\text{g}/\text{cm}^2$ at its edge, and its density is a linear function of the distance from the center. The total mass of the disk is equal to

a) $8.80\,\text{g}$

b) $10.21\,\text{g}$

c) $12.57\,\text{g}$

d) $17.23\,\text{g}$

Solutions to the Practice quiz

Lecture 28

Cylindrical coordinates

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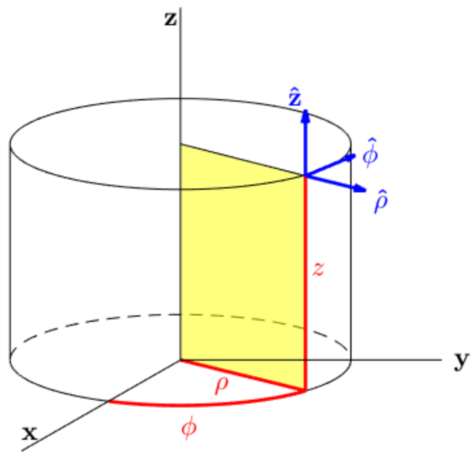
Cylindrical coordinates extends polar coordinates to three dimensions by adding a Cartesian coordinate along the z-axis. To conform to standard usage, we change notation and define the relationship between the Cartesian and the cylindrical coordinates to be

$$x = \rho \cos\phi, \qquad y = \rho \sin\phi, \qquad z = z;$$

with inverse relation

$$\rho = \sqrt{x^2 + y^2}, \qquad \tan\phi = y/x.$$

A spatial point (x, y, z) in Cartesian coordinates is now specified by (ρ, ϕ, z) in cylindrical coordinates (see figure).



The orthogonal unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ point in the direction of increasing ρ, ϕ and z, respectively. The position vector is given by $\boldsymbol{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}$. The volume element transforms as $dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$.

The del operator can be found using the polar form of the Cartesian derivatives (see the problems). The result is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}.$$

The Laplacian, $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$, is computed taking care to differentiate the unit vectors with respect to ϕ:

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.\end{aligned}$$

The divergence and curl of a vector field, $\boldsymbol{A} = A_\rho(\rho,\phi,z)\hat{\boldsymbol{\rho}} + A_\phi(\rho,\phi,z)\hat{\boldsymbol{\phi}} + A_z(\rho,\phi,z)\hat{\boldsymbol{z}}$, are given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}, \\ \boldsymbol{\nabla}\times\boldsymbol{A} &= \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\boldsymbol{z}}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right).\end{aligned}$$

Problems for Lecture 28

1. Determine the del operator $\boldsymbol{\nabla}$ in cylindrical coordinates. There are several ways to do this, but a straightforward, though algebraically lengthy one, is to transform from Cartesian coordinates using

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z},$$

$$\hat{\boldsymbol{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}, \qquad \hat{\boldsymbol{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}},$$

and

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}, \qquad \frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}.$$

2. Compute $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}}$ in two ways:

a) With the divergence in cylindrical coordinates;

b) By transforming to Cartesian coordinates.

3. Using cylindrical coordinates, compute $\boldsymbol{\nabla}\times\hat{\boldsymbol{\rho}}$, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}$ and $\boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}}$.

Solutions to the Problems

Lecture 29

Spherical coordinates (Part A)

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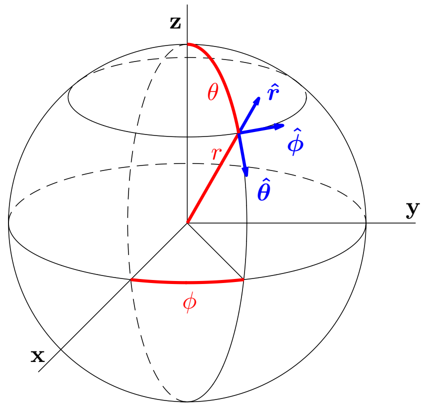
Spherical coordinates are useful for problems with spherical symmetry. The relationship between the Cartesian and the spherical coordinates is given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. A spatial point (x, y, z) in Cartesian coordinates is now specified by (r, θ, ϕ) in spherical coordinates (see figure). The orthogonal unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ point in the direction of increasing r, θ and ϕ, respectively.



The position vector is given by

$$\boldsymbol{r} = r\hat{\boldsymbol{r}};$$

and the volume element transforms as

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

The spherical coordinate unit vectors can be written in terms of the Cartesian unit vectors by

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k},\\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k},\\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j};
\end{aligned}$$

with inverse relation

$$\begin{aligned}
\boldsymbol{i} &= \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}},\\
\boldsymbol{j} &= \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}},\\
\boldsymbol{k} &= \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.
\end{aligned}$$

By differentiating the unit vectors, we can derive the sometimes useful identities

$$\begin{aligned}
\frac{\partial\hat{\boldsymbol{r}}}{\partial\theta} &= \hat{\boldsymbol{\theta}}, & \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} &= -\hat{\boldsymbol{r}}, & \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta} &= 0;\\
\frac{\partial\hat{\boldsymbol{r}}}{\partial\phi} &= \sin\theta\hat{\boldsymbol{\phi}}, & \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} &= \cos\theta\hat{\boldsymbol{\phi}}, & \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} &= -\sin\theta\hat{\boldsymbol{r}} - \cos\theta\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Problems for Lecture 29

1. Write the relationship between the spherical coordinate unit vectors and the Cartesian unit vectors in matrix form. Notice that the transformation matrix Q is orthogonal and invert the relationship using $\mathrm{Q}^{-1} = \mathrm{Q}^{\mathrm{T}}$.

2. Use the Jacobian change-of-variables formula for triple integrals, given by

$$dx\,dy\,dz = \left| \det \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial \phi \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \phi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \phi \end{pmatrix} \right| dr\,d\theta\,d\phi,$$

to derive $dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi$.

3. Consider a scalar field $f = f(r)$ that depends only on the distance from the origin. Using $dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi$, and an integration region V inside a sphere of radius R centered at the origin, show that

$$\int_V f\,dV = 4\pi \int_0^R r^2 f(r)\,dr.$$

4. Suppose a sphere of radius R has mass density ρ_0 at its center, and ρ_1 at its surface, and its density is a linear function of the distance from the center. Find the total mass of the sphere. What is the average density of the sphere?

Solutions to the Problems

Lecture 30

Spherical coordinates (Part B)

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First, we determine the gradient in spherical coordinates. Consider the scalar field $f = f(r, \theta, \phi)$. Our definition of a total differential is

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi = \boldsymbol{\nabla} f \cdot d\boldsymbol{r}.$$

In spherical coordinates,

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}(\theta, \phi),$$

and using

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} = \sin\theta\,\hat{\boldsymbol{\phi}},$$

we have

$$d\boldsymbol{r} = dr\,\hat{\boldsymbol{r}} + r\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} d\theta + r\frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} d\phi = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}.$$

Using the orthonormality of the unit vectors, we can therefore write df as

$$df = \left(\frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}\right) \cdot (dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}),$$

showing that

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Some messy algebra will yield for the Laplacian

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2};$$

and for the divergence and curl of a vector field, $\boldsymbol{A} = A_r(r,\theta,\phi)\hat{\boldsymbol{r}} + A_\theta(r,\theta,\phi)\hat{\boldsymbol{\theta}} + A_\phi(r,\theta,\phi)\hat{\boldsymbol{\phi}}$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi},$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\hat{\boldsymbol{r}}}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right].$$

Problems for Lecture 30

1. Using the formulas for the spherical coordinate unit vectors in terms of the Cartesian unit vectors, prove that

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} = \sin\theta\, \hat{\boldsymbol{\phi}}.$$

2. Compute the divergence and curl of the spherical coordinate unit vectors.

3. Using spherical coordinates, show that for $r \neq 0$,

$$\nabla^2 \left(\frac{1}{r}\right) = 0.$$

Solutions to the Problems

Practice quiz: Cylindrical and spherical coordinates

1. With $\rho = \sqrt{x^2 + y^2}$, the function $\nabla^2 \left(\frac{1}{\rho}\right)$ is equal to

a) 0

b) $\frac{1}{\rho^3}$

c) $\frac{2}{\rho^3}$

d) $\frac{3}{\rho^3}$

2. Let $\boldsymbol{r} = x\boldsymbol{i}$. Then $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ is equal to

a) $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$

b) $(\boldsymbol{i}, \boldsymbol{k}, \boldsymbol{j})$

c) $(\boldsymbol{i}, -\boldsymbol{k}, \boldsymbol{j})$

d) $(\boldsymbol{i}, \boldsymbol{k}, -\boldsymbol{j})$

3. Suppose a sphere of radius 5 cm has mass density $10\,\mathrm{g/cm^3}$ at its center, and $5\mathrm{g/cm^3}$ at its surface, and its density is a linear function of the distance from the center. The total mass of the sphere is given by

a) 3927 g

b) 3491 g

c) 3272 g

d) 3142 g

Solutions to the Practice quiz

Lecture 31

Line integral of a vector field

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The line integral of a vector field $\boldsymbol{u}$ along a directed curve C is written as one of

$$\int_C \boldsymbol{u} \cdot d\boldsymbol{r} \qquad \text{or} \qquad \oint_C \boldsymbol{u} \cdot d\boldsymbol{r},$$

where the latter form is used when the curve C is closed, with ending point equal to starting point. To define the line integral, we break the curve into small displacement vectors, take the dot product of the average value of $\boldsymbol{u}$ on each piece of the curve with its displacement vector, and sum over all these scalar values.

A general method to calculate the line integral is to parameterize the curve. Let the curve be parameterized by the function $\boldsymbol{r} = \boldsymbol{r}(t)$ as t goes from t_0 to t_f. Using $d\boldsymbol{r} = \dfrac{d\boldsymbol{r}}{dt}\,dt$, the line integral becomes

$$\int_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_{t_0}^{t_f} \boldsymbol{u}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}\,dt.$$

Sometimes the curve is simple enough that $d\boldsymbol{r}$ can be computed directly. No matter, line integrals are always done by converting them into single-variable integrals.

Example: Compute the line integral of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ in the x-y plane along two curves from the origin to the point $(x, y) = (1, 1)$. The first curve C_1 consists of two line segments, the first from $(0, 0)$ to $(1, 0)$, and the second from $(1, 0)$ to $(1, 1)$. The second curve C_2 is a straight line from the origin to $(1, 1)$.

The computation along the first curve requires two separate integrations. For the curve along the x-axis, we use $d\boldsymbol{r} = dx\boldsymbol{i}$ and for the curve at $x = 1$ in the direction of $\boldsymbol{j}$, we use $d\boldsymbol{r} = dy\boldsymbol{j}$. The line integral is therefore given by

$$\int_{C_1} \boldsymbol{r} \cdot d\boldsymbol{r} = \int_0^1 x\,dx + \int_0^1 y\,dy = 1.$$

For the second curve, we parameterize the line by $\boldsymbol{r}(t) = t(\boldsymbol{i} + \boldsymbol{j})$ as t goes from 0 to 1, so that $d\boldsymbol{r} = dt(\boldsymbol{i} + \boldsymbol{j})$, and the integral becomes

$$\int_{C_2} \boldsymbol{r} \cdot d\boldsymbol{r} = \int_0^1 2t\,dt = 1.$$

The two line integrals are equal, and for this case only depend on the starting and ending points of the curve.

Problems for Lecture 31

1. In the x-y plane, calculate the line integral of the vector field $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ counterclockwise around a square with vertices $(0, 0)$, $(L, 0)$, (L, L), and $(0, L)$.

2. In the x-y plane, calculate the line integral of the vector field $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ counterclockwise around the circle of radius R centered at the origin.

Solutions to the Problems

Lecture 32

Surface integral of a vector field

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The surface integral of the normal component of a vector field $\boldsymbol{u}$ computed over a surface S is written as one of

$$\int_S \boldsymbol{u} \cdot d\boldsymbol{S} \qquad \text{or} \qquad \oint_S \boldsymbol{u} \cdot d\boldsymbol{S},$$

where the latter is used when the surface S is closed. One can write $d\boldsymbol{S} = \hat{\boldsymbol{n}} dS$, where $\hat{\boldsymbol{n}}$ is a normal unit vector to the surface. To define the surface integral, we break the surface into little areas, take the dot product of the average value of $\boldsymbol{u}$ on each area with the normal unit vector times the area itself, and then sum. For an open surface, the direction of the normal vectors needs to be specified as there are two choices, but for a closed surface, by convention the direction is always chosen to point outward.

This surface integral is often called a flux integral. If $\boldsymbol{u}$ is the fluid velocity (length divided by time), and ρ is the fluid density (mass divided by volume), then the surface integral

$$\int_S \rho \boldsymbol{u} \cdot d\mathbf{S}$$

computes the mass flux, that is, the mass passing through the surface S per unit time.

Example: Compute the flux integral of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ over a square box centered at the origin with sides parallel to the axes and side lengths equal to L .

From symmetry, we need only compute the surface integral through one face and them multiply by six. Consider the face located at $x = L/2$. The normal vector to this face is $\boldsymbol{i}$, and the infinitesimal surface area is $dy\,dz$. The integral over this surface, say S_1, is therefore given by

$$\int_{S_1} \boldsymbol{r} \cdot d\boldsymbol{S} = \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left(\frac{L}{2}\right) dy\,dz = \frac{L^3}{2}.$$

Multiplying by six, we obtain

$$\oint_S \boldsymbol{r} \cdot d\boldsymbol{S} = 3L^3.$$

Problems for Lecture 32

1. Compute the flux integral of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ over a sphere of radius R centered at the origin. Use $d\boldsymbol{S} = \hat{\boldsymbol{r}}\, R^2 \sin\theta\, d\theta\, d\phi$.

Solutions to the Problems

Practice quiz: Vector integration

1. The line integral of the vector field $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ counterclockwise around a triangle with vertices $(0,0)$, $(L,0)$, and $(0,L)$ is equal to

a) 0

b) $\frac{1}{2}L^2$

c) L^2

d) $2L^2$

2. Consider a right circular cylinder of radius R and length L centered on the z-axis. The surface integral of $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ over the cylinder is given by

a) 0

b) $\pi R^2 L$

c) $2\pi R^2 L$

d) $4\pi R^2 L$

3. The flux integral of $\boldsymbol{u} = z\boldsymbol{k}$ over the upper hemisphere of a sphere of radius R centered at the origin with normal vector $\hat{\boldsymbol{r}}$ is given by

a) $\frac{2\pi}{3}R^3$

b) $\frac{4\pi}{3}R^3$

c) $2\pi R^3$

d) $4\pi R^3$

Solutions to the Practice quiz

Week IV

Fundamental Theorems

In this week's lectures, we learn about the fundamental theorems of vector calculus. These include the gradient theorem, the divergence theorem, and Stokes' theorem. We show how these theorems are used to derive continuity equations, define the divergence and curl in coordinate-free form, and convert the integral version of Maxwell's equations to differential form.

Lecture 33

Gradient theorem

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The gradient theorem is a generalization of the fundamental theorem of calculus for line integrals. Let $\boldsymbol{\nabla}\phi$ be the gradient of a scalar field ϕ, and let C be a directed curve that begins at the point $\boldsymbol{r}_1$ and ends at $\boldsymbol{r}_2$. Suppose we can parameterize the curve C by $\boldsymbol{r} = \boldsymbol{r}(t)$, where $t_1 \le t \le t_2$, $\boldsymbol{r}(t_1) = \boldsymbol{r}_1$, and $\boldsymbol{r}(t_2) = \boldsymbol{r}_2$. Then using the chain rule in the form

$$\frac{d}{dt}\phi(\boldsymbol{r}) = \boldsymbol{\nabla}\phi(\boldsymbol{r}) \cdot \frac{d\boldsymbol{r}}{dt},$$

and the standard fundamental theorem of calculus, we have

$$\begin{aligned}\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} &= \int_{t_1}^{t_2} \boldsymbol{\nabla}\phi(\boldsymbol{r}) \cdot \frac{d\boldsymbol{r}}{dt}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\phi(\boldsymbol{r})\, dt \\ &= \phi(\boldsymbol{r}(t_2)) - \phi(\boldsymbol{r}(t_1)) = \phi(\boldsymbol{r}_2) - \phi(\boldsymbol{r}_1).\end{aligned}$$

A more direct way to derive this result is to write the differential

$$d\phi = \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r},$$

so that

$$\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \int_C d\phi = \phi(\boldsymbol{r}_2) - \phi(\boldsymbol{r}_1).$$

We have thus shown that the line integral of the gradient of a function is path independent, depending only on the endpoints of the curve. In particular, we have the general result that

$$\oint_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = 0$$

for any closed curve C.

Example: Compute the line integral of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ in the x-y plane from the origin to the point $(1,1)$.

We have $\boldsymbol{r} = \frac{1}{2}\boldsymbol{\nabla}(x^2 + y^2)$, so that the line integral is path independent. Therefore,

$$\int_C \boldsymbol{r} \cdot d\boldsymbol{r} = \frac{1}{2}\int_C \boldsymbol{\nabla}(x^2 + y^2) \cdot d\boldsymbol{r} = \frac{1}{2}(x^2 + y^2)\Big|_{(0,0)}^{(1,1)} = 1.$$

Problems for Lecture 33

1. Let $\phi(\boldsymbol{r}) = x^2y + xy^2 + z$.

a) Compute $\boldsymbol{\nabla}\phi$.

b) Compute $\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r}$ from $(0,0,0)$ to $(1,1,1)$ using the gradient theorem.

c) Compute $\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r}$ along the lines segments $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

Solutions to the Problems

Lecture 34

Conservative vector fields

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For a vector field $\boldsymbol{u}$ defined on $\mathcal{R}^3$, except perhaps at isolated singularities, the following conditions are equivalent:

1. $\boldsymbol{\nabla} \times \boldsymbol{u} = 0$;

2. $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ for some scalar field ϕ;

3. $\displaystyle\int_C \boldsymbol{u} \cdot d\boldsymbol{r}$ is path independent for any curve C;

4. $\displaystyle\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = 0$ for any closed curve C.

When these conditions hold, we say that $\boldsymbol{u}$ is a conservative vector field.

Example: Let $\boldsymbol{u}(x, y) = x^2(1 + y^3)\boldsymbol{i} + y^2(1 + x^3)\boldsymbol{j}$. Show that $\boldsymbol{u}$ is a conservative vector field, and determine $\phi = \phi(x, y)$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$.

To show that $\boldsymbol{u}$ is a conservative vector field, we can prove $\boldsymbol{\nabla} \times \boldsymbol{u} = 0$:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2(1+y^3) & y^2(1+x^3) & 0 \end{vmatrix} = (3x^2y^2 - 3x^2y^2)\boldsymbol{k} = 0.$$

To find the scalar field ϕ, we solve

$$\frac{\partial \phi}{\partial x} = x^2(1 + y^3), \qquad \frac{\partial \phi}{\partial y} = y^2(1 + x^3).$$

Integrating the first equation with respect to x holding y fixed, we find

$$\phi = \int x^2(1 + y^3)\,dx = \frac{1}{3}x^3(1 + y^3) + f(y),$$

where $f = f(y)$ is a function that depends only on y. Differentiating ϕ with respect to y and using the second equation, we obtain

$$x^3y^2 + f'(y) = y^2(1 + x^3) \qquad \text{or} \qquad f'(y) = y^2.$$

One more integration results in $f(y) = y^3/3 + c$, with c constant, and the scalar field is given by

$$\phi(x, y) = \frac{1}{3}(x^3 + x^3y^3 + y^3) + c.$$

Problems for Lecture 34

1. Let $\boldsymbol{u} = (2xy + z^2)\boldsymbol{i} + (2yz + x^2)\boldsymbol{j} + (2zx + y^2)\boldsymbol{k}$.

a) Show that $\boldsymbol{u}$ is a conservative vector field.

b) Calculate the scalar field ϕ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$.

Solutions to the Problems

Practice quiz: Gradient theorem

1. Let $\phi(\boldsymbol{r}) = xyz$. The value of $\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ is equal to

a) 0

b) 1

c) 2

d) 3

2. Let $\boldsymbol{u} = y\boldsymbol{i} + x\boldsymbol{j}$. The value of $\oint_C \boldsymbol{u} \cdot d\boldsymbol{r}$, where C is the unit circle centered at the origin, is given by

a) 0

b) 1

c) 2

d) 3

3. Let $\boldsymbol{u} = (2x + y)\boldsymbol{i} + (2y + x)\boldsymbol{j} + \boldsymbol{k}$. If $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, then ϕ can be equal to

a) $(x + y)^2 + z$

b) $(x - y)^2 + z$

c) $x^2 + xy + y^2 + z$

d) $x^2 - xy + y^2 + z$

Solutions to the Practice quiz

Lecture 35

Divergence theorem

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Let $\boldsymbol{u}$ be a differentiable vector field defined inside and on a smooth closed surface S enclosing a volume V. The divergence theorem states that the integral of the divergence of $\boldsymbol{u}$ over the enclosed volume is equal to the flux of $\boldsymbol{u}$ through the bounding surface; that is,

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV = \oint_S \boldsymbol{u} \cdot d\boldsymbol{S}.$$

We first prove the divergence theorem for a rectangular solid with sides parallel to the axes. Let the rectangular solid be defined by $a \le x \le b$, $c \le y \le d$, and $e \le z \le f$. With $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$, the volume integral over V becomes

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV = \int_e^f \int_c^d \int_a^b \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) dx\, dy\, dz.$$

The three terms in the integral can be integrated separately using the fundamental theorem of calculus. Each term in succession is integrated as

$$\int_e^f \int_c^d \left(\int_a^b \frac{\partial u_1}{\partial x}\, dx \right) dy\, dz = \int_e^f \int_c^d \left(u_1(b, y, z) - u_1(a, y, z) \right) dy\, dz;$$

$$\int_e^f \int_a^b \left(\int_c^d \frac{\partial u_2}{\partial y}\, dy \right) dx\, dz = \int_e^f \int_a^b \left(u_2(x, d, z) - u_2(x, c, z) \right) dx\, dz;$$

$$\int_c^d \int_a^b \left(\int_e^f \frac{\partial u_3}{\partial z}\, dz \right) dx\, dy = \int_c^d \int_a^b \left(u_3(x, y, f) - u_3(x, y, e) \right) dx\, dy.$$

The integrals on the right-hand-sides correspond exactly to flux integrals over opposite sides of the rectangular solid. For example, the side located at $x = b$ corresponds with $d\boldsymbol{S} = \boldsymbol{i}\, dy\, dz$ and the side located at $x = a$ corresponds with $d\boldsymbol{S} = -\boldsymbol{i}\, dy\, dz$. Summing all three integrals yields the divergence theorem for the rectangular solid.

Now, given any volume enclosed by a smooth surface, we can subdivide the volume by a very fine three-dimensional rectangular grid and apply the above result to each rectangular solid in the grid. All the volume integrals over the rectangular solids add. The internal rectangular solids, however, share connecting side faces through which the flux integrals cancel, and the only flux integrals that remain are those from the rectangular solids on the boundary of the volume with outward facing surfaces. The result is the divergence theorem for any volume V enclosed by a smooth surface S.

Problems for Lecture 35

1. Prove the divergence theorem for a sphere of radius R centered at the origin. Use spherical coordinates.

Solutions to the Problems

Lecture 36

Divergence theorem: Example I

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The divergence theorem is given by

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV = \oint_S \boldsymbol{u} \cdot d\boldsymbol{S}.$$

Test the divergence theorem using $\boldsymbol{u} = xy\,\boldsymbol{i} + yz\,\boldsymbol{j} + zx\,\boldsymbol{k}$ *for a cube of side* L *lying in the first octant with a vertex at the origin.*

Here, Cartesian coordinates are appropriate and we use $\boldsymbol{\nabla} \cdot \boldsymbol{u} = y + z + x$. We have for the left-hand side of the divergence theorem,

$$\begin{aligned}
\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV &= \int_0^L \int_0^L \int_0^L (x + y + z)\, dx\, dy\, dz \\
&= L^4/2 + L^4/2 + L^4/2 \\
&= 3L^4/2.
\end{aligned}$$

For the right-hand side of the divergence theorem, the flux integral only has nonzero contributions from the three sides located at $x = L$, $y = L$ and $z = L$. The corresponding unit normal vectors are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, and the corresponding integrals are

$$\begin{aligned}
\oint_S \boldsymbol{u} \cdot d\boldsymbol{S} &= \int_0^L \int_0^L Ly\, dy\, dz + \int_0^L \int_0^L Lz\, dx\, dz + \int_0^L \int_0^L Lx\, dx\, dy \\
&= L^4/2 + L^4/2 + L^4/2 \\
&= 3L^4/2.
\end{aligned}$$

Problems for Lecture 36

1. Test the divergence theorem using $\boldsymbol{u} = x^2y\,\boldsymbol{i} + y^2z\,\boldsymbol{j} + z^2x\,\boldsymbol{k}$ for a cube of side L lying in the first octant with a vertex at the origin.

2. Compute the flux integral of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ over a square box with side lengths equal to L by applying the divergence theorem to convert the flux integral into a volume integral.

Solutions to the Problems

Lecture 37

Divergence theorem: Example II

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The divergence theorem is given by

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV = \oint_S \boldsymbol{u} \cdot d\boldsymbol{S}.$$

Test the divergence theorem using $\boldsymbol{u} = r^2\hat{\boldsymbol{r}}$ for a sphere of radius R centered at the origin.

To compute the left-hand-side of the divergence theorem, we recall the formula for the divergence of a vector field $\boldsymbol{u}$ in spherical coordinates:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}.$$

Here, $u_r = r^2$ is the only nonzero component of $\boldsymbol{u}$, and we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{d}{dr}\left(r^4\right) = 4r.$$

Therefore, using $dV = r^2\sin\theta\, dr\, d\theta\, d\phi$, we have

$$\begin{aligned}
\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 4r^3 \sin\theta\, dr\, d\theta\, d\phi \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{R} 4r^3\, dr = 4\pi R^4.
\end{aligned}$$

For the right-hand-side of the divergence theorem, we have $\boldsymbol{u} = R^2\hat{\boldsymbol{r}}$ and $d\boldsymbol{S} = \hat{\boldsymbol{r}}R^2\sin\theta\, d\theta\, d\phi$, so that

$$\oint_S \boldsymbol{u} \cdot d\boldsymbol{S} = \int_0^{2\pi}\int_0^{\pi} R^4 \sin\theta\, d\theta\, d\phi = 4\pi R^4.$$

Problems for Lecture 37

1. Test the divergence theorem using $\boldsymbol{u} = \hat{\boldsymbol{r}}/r$ for a sphere of radius R centered at the origin.

2. Compute the flux integral of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ over a sphere of radius R by applying the divergence theorem to convert the flux integral into a volume integral.

3. Consider the volume integral

$$\int_V \nabla^2 \left(\frac{1}{r}\right) dV.$$

By computing $\nabla^2(1/r)$ when $r \neq 0$, show that the volume integral equals zero when the volume V does not include the origin. When V includes the orgin, compute the integral by using the divergence theorem to convert the volume integral into a flux integral.

Solutions to the Problems

Lecture 38

Continuity equation

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The divergence theorem is often used to derive a continuity equation, which expresses the local conservation of some physical quantity such as mass or electric charge. Here, we derive the continuity equation for a compressible fluid such as a gas. Let $\rho(\boldsymbol{r}, t)$ be the fluid density at position $\boldsymbol{r}$ and time t, and $\boldsymbol{u}(\boldsymbol{r}, t)$ be the fluid velocity. We will assume no sources or sinks of fluid. We place a small test volume V in the fluid flow and consider the change in the fluid mass M inside V.

The fluid mass M in V varys because of the mass flux through the surface S surrounding V, and one has

$$\frac{dM}{dt} = -\oint_S \rho\boldsymbol{u} \cdot d\boldsymbol{S}.$$

Now the fluid mass is given in terms of the fluid density by

$$M = \int_V \rho\, dV,$$

and application of the divergence theorem to the surface integral results in

$$\frac{d}{dt}\int_V \rho\, dV = -\int_V \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u})\, dV.$$

Taking the time derivative inside the integral on the left-hand side, and combining the two integrals yields

$$\int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u}) \right) dV = 0.$$

Since this integral vanishes for any test volume placed in the fluid, the integrand itself must be zero, and we have derived the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u}) = 0.$$

For an incompressible fluid for which the density ρ is uniform and constant, the continuity equation reduces to

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0.$$

A vector field with zero divergence is called incompressible or solenoidal.

Problems for Lecture 38

1. Show that the continuity equation can be written as

$$\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho + \rho\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0.$$

2. The electric charge density (charge per unit volume) is give by $\rho(\boldsymbol{r}, t)$ and the volume current density (current per unit area) is given by $\boldsymbol{J}(\boldsymbol{r}, t)$. Local conservation of charge states that the time rate of change of the total charge within a volume is equal to the negative of the charge flowing out of that volume, resulting in the equation

$$\frac{d}{dt}\int_V \rho(\boldsymbol{r}, t)\, dV = -\oint_S \boldsymbol{J}\cdot d\boldsymbol{S}.$$

From this law of charge conservation, derive the electrodynamics continuity equation.

Solutions to the Problems

Practice quiz: Divergence theorem

1. The integral of $\boldsymbol{u} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$ over the closed surface of a right circular cone with radius R and length L and base in the x-y plane is given by

a) 0

b) $\pi RL\sqrt{R^2 + L^2}$

c) $2\pi RL\sqrt{R^2 + L^2}$

d) $3\pi RL\sqrt{R^2 + L^2}$

2. The surface integral $\oint_S \boldsymbol{r} \cdot d\boldsymbol{S}$ over a right circular cylinder of radius R and length L is equal to

a) 0

b) $\pi R^2 L$

c) $2\pi R^2 L$

d) $3\pi R^2 L$

3. Which velocity field is not incompressible ($\boldsymbol{\nabla} \cdot \boldsymbol{u} \neq 0$)?

a) $\boldsymbol{u} = xy\boldsymbol{i} - \frac{1}{2}y^2\boldsymbol{j}$

b) $\boldsymbol{u} = (1 + x)\boldsymbol{i} + (1 - y)\boldsymbol{j}$

c) $\boldsymbol{u} = (x^2 - xy)\boldsymbol{i} + \left(\frac{1}{2}y^2 - 2xy\right)\boldsymbol{j}$

d) $\boldsymbol{u} = (x + y)^2\boldsymbol{i} + (x - y)^2\boldsymbol{j}$

Solutions to the Practice quiz

Lecture 39

Green's theorem

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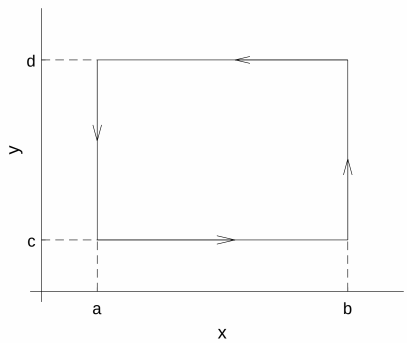
Green's theorem is a two-dimensional version of Stokes' theorem, and serves as a simpler introduction. Let $\boldsymbol{u} = u_1(x,y)\boldsymbol{i} + u_2(x,y)\boldsymbol{j}$ be a differentiable two-dimensional vector field defined on the x-y plane. Green's theorem relates an area integral over S in the plane to a line integral around C surrounding this area, and is given by

$$\int_S \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) dS = \oint_C (u_1\,dx + u_2\,dy).$$



We will first prove Green's theorem for a rectangle with sides parallel to the axes. Let the rectangle be defined by $a <= x <= b$ and $c <= y <= d$, as pictured here. The area integral is given by

$$\int_S \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) dS = \int_c^d \int_a^b \frac{\partial u_2}{\partial x} dx\,dy - \int_a^b \int_c^d \frac{\partial u_1}{\partial y} dy\,dx.$$

The inner integrals can be done using the fundamental theorem of calculus, and we obtain

$$\int_S \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) dS = \int_c^d [u_2(b,y) - u_2(a,y)]\;dy + \int_a^b [u_1(x,c) - u_1(x,d)]\;dx = \oint_C (u_1\,dx + u_2\,dy).$$

Note that the line integral is done so that the bounded area is always to the left, which means counterclockwise.

Now, given any closed smooth curve in the x-y plane enclosing an area, we can subdivide the area by a very fine two-dimensional rectangular grid and apply the above result to each rectangle in the grid. All the area integrals over the internal rectangles add. The internal rectangles share connecting sides over which the line integrals cancel, and the only line integrals that remain are those that approximate the given bounding curve. The result is Green's theorem for any area S in the plane bounded by a curve C.

Problems for Lecture 39

1. Test Green's theorem using $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ for a square of side L lying in the first quadrant with vertex at the origin.

2. Test Green's theorem using $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ for a circle of radius R centered at the origin.

Solutions to the Problems

Lecture 40

Stokes' theorem

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Green's theorem for a differentiable two-dimensional vector field $\boldsymbol{u} = u_1(x,y)\,\boldsymbol{i} + u_2(x,y)\,\boldsymbol{j}$ and a smooth curve C in the x-y plane surrounding an area S is given by

$$\int_S \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dS = \oint_C (u_1\,dx + u_2\,dy).$$

Green's theorem can be extended to three dimensions. With $\boldsymbol{u} = u_1(x,y,z)\,\boldsymbol{i} + u_2(x,y,z)\,\boldsymbol{j}$, we see that

$$\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{k};$$

and with $d\boldsymbol{S} = \boldsymbol{k}\,dS$ and $u_1\,dx + u_2\,dy = \boldsymbol{u} \cdot d\boldsymbol{r}$, Green's theorem can be rewritten in the form

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{u} \cdot d\boldsymbol{r}.$$

This three-dimensional extension of Green's theorem is called Stokes' theorem. Here, S is a general three-dimensional surface bounded by a closed spatial curve C. A simple example would be a hemisphere located anywhere in space bounded by a circle. The orientation of the closed curve and the normal vector to the surface should follow the right-hand rule. If your fingers of your right hand point in the direction of the line integral, your thumb should point in the direction of the normal vector to the surface.

Problems for Lecture 40

1. From Stokes' theorem, determine the form of Green's theorem for a curve lying in the

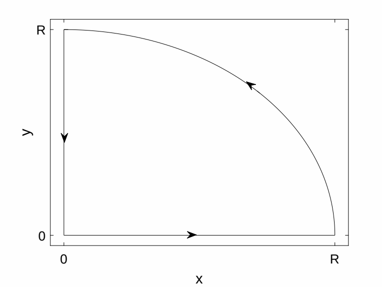
a) y-z plane;

b) z-x plane.

2. Test Stokes' theorem using $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ for a hemisphere of radius R with $z > 0$ bounded by a circle of radius R lying in the x-y plane with center at the origin.

Solutions to the Problems

Practice quiz: Stokes' Theorem



1. Let $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$. Compute $\oint_C \boldsymbol{u} \cdot d\boldsymbol{r}$ for the quarter circle of radius R as illustrated. Here, it is simpler to apply Stokes' theorem to compute an area integral. The answer is

a) 0

b) $\frac{1}{2}\pi R^2$

c) πR^2

d) $2\pi R^2$

2. Let $\boldsymbol{u} = \frac{-y}{x^2+y^2}\boldsymbol{i} + \frac{x}{x^2+y^2}\boldsymbol{j}$. Compute the value of $\int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot d\boldsymbol{S}$ over a circle of radius R centered at the origin in the x-y plane with normal vector $\boldsymbol{k}$. Here, because $\boldsymbol{u}$ is singular at $\boldsymbol{r} = 0$, it is necessary to apply Stokes' theorem and compute a line integral. The answer is

a) 0

b) π

c) 2π

d) 4π

3. Let $\boldsymbol{u} = -x^2y\boldsymbol{i} + xy^2\boldsymbol{j}$. Compute $\oint_C \boldsymbol{u} \cdot d\boldsymbol{r}$ for a unit square in the first quadrant with vertex at the origin. Here, it is simpler to compute an area integral. The answer is

a) 0

b) $\frac{1}{3}$

c) $\frac{2}{3}$

d) 1

Solutions to the Practice quiz

Lecture 41

Meaning of the divergence and the curl

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With $\boldsymbol{u}$ a differentiable vector field defined inside and on a smooth closed surface S enclosing a volume V, the divergence theorem states

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{u}\, dV = \oint_S \boldsymbol{u} \cdot d\boldsymbol{S}.$$

We can limit this expression by shrinking the integration volume down to a point to obtain a coordinate-free representation of the divergence as

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \lim_{V \to 0} \frac{1}{V} \oint_S \boldsymbol{u} \cdot d\boldsymbol{S}.$$

Picture V as the volume of a small sphere with surface S and $\boldsymbol{u}$ as the velocity field of some fluid of constant density. Then if the flow of fluid into the sphere is equal to the flow of fluid out of the sphere, the surface integral will be zero and $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. However, if more fluid flows out of the sphere than in, then $\boldsymbol{\nabla} \cdot \boldsymbol{u} > 0$ and if more fluid flows in than out, $\boldsymbol{\nabla} \cdot \boldsymbol{u} < 0$. Positive divergence indicates a source of fluid and negative divergence indicates a sink of fluid.

Now consider a surface S bounded by a curve C. Stokes' theorem states that

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{u} \cdot d\boldsymbol{r}.$$

We can limit this expression by shrinking the integration surface down to a point. With $\boldsymbol{n}$ a unit normal vector to the surface, with direction given by the right-hand rule, we obtain

$$(\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} = \lim_{S \to 0} \frac{1}{S} \oint_C \boldsymbol{u} \cdot d\boldsymbol{r}.$$

Picture S as the area of a small disk bounded by a circle C and again picture $\boldsymbol{u}$ as the velocity field of a fluid. The line integral of $\boldsymbol{u} \cdot d\boldsymbol{r}$ around the circle C is called the flow's circulation and measures the swirl of the fluid around the center of the circle. The vector field $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is called the vorticity of the fluid. The vorticity is most decidedly nonzero in a wirling (say, turbulent) fluid, composed of eddies of all different sizes.

Problems for Lecture 41

1. The incompressible Navier-Stokes equation governing fluid flow is given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u},$$

with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Here, ρ and ν are fluid density and viscosity.

a) By taking the divergence of the Navier-Stokes equation, derive the following equation for the pressure in terms of the velocity field:

$$\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

b) By taking the curl of the Navier-Stokes equation, and defining the vorticity as $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$, derive the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

You can use all the vector identities presented in these lecture notes, but you will need to prove that

$$\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}.$$

Solutions to the Problems

Lecture 42

Maxwell's equations

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Maxwell's equations in SI units and in integral form are given by

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{q_{\text{enc}}}{\varepsilon_0}, \qquad \text{(Gauss's law for electric fields)}$$

$$\oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0, \qquad \text{(Gauss's law for magnetic fields)}$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{S}, \qquad \text{(Faraday's law)}$$

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 \left(I_{\text{enc}} + \varepsilon_0 \frac{d}{dt}\int_S \boldsymbol{E} \cdot d\boldsymbol{S} \right), \qquad \text{(Ampère-Maxwell law)}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, q_{enc} and I_{enc} are the charge or current enclosed by the bounding surface or curve, and ε_0 and μ_0 are dimensional constants called the permittivity and permeability of free space.

The transformation from integral to differential form is a straightforward application of both the divergence and Stokes' theorem. The charge q and the current I are related to the charge density ρ and the current density $\boldsymbol{J}$ by

$$q = \int_V \rho \, dV, \qquad I = \int_S \boldsymbol{J} \cdot d\boldsymbol{S}.$$

We apply the divergence theorem to the surface integrals and Stokes' theorem to the line integrals, replace q_{enc} and I_{enc} by integrals, and combine results into single integrals to obtain

$$\int_V \left(\boldsymbol{\nabla} \cdot \boldsymbol{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0, \qquad \int_S \left(\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} \right) \cdot d\boldsymbol{S} = 0,$$

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{B}) \, dV = 0, \qquad \int_S \left(\boldsymbol{\nabla} \times \boldsymbol{B} - \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \right) \cdot d\boldsymbol{S} = 0.$$

Since the integration volumes and surfaces are of arbitrary size and shape, the integrands must vanish and we obtain the aesthetically appealing differential forms for Maxwell's equations given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right).$$

Problems for Lecture 42

1. Using Gauss's law given by

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{q_{\text{enc}}}{\varepsilon_0},$$

determine the electric field of a point charge q at the origin. Assume the electric field is spherically symmetric.

2. Using Ampère's law given by

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 I_{\text{enc}},$$

determine the magnetic field of a current carrying infinite wire placed on the z-axis. Assume the magnetic field has cylindrical symmetry.

Solutions to the Problems

Appendix

Appendix A

Matrix addition and multiplication

View this lecture on YouTube

Two-by-two matrices A and B, with two rows and two columns, can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

The first row of matrix A has elements a_{11} and a_{12}; the second row has elements a_{21} and a_{22}. The first column has elements a_{11} and a_{21}; the second column has elements a_{12} and a_{22}. Matrices can be multiplied by scalars and added. This is done element-by-element as follows:

$$kA = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}, \qquad A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}.$$

Matrices can also be multiplied. Matrix multiplication does not commute, and two matrices can be multiplied only if the number of columns of the matrix on the left equals the number of rows of the matrix on the right. One multiplies matrices by going across the rows of the first matrix and down the columns of the second matrix. The two-by-two example is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

Making use of the definition of matrix multiplication, a system of linear equations can be written in matrix form. For instance, a general system with two equations and two unknowns is given by

$$a_{11}x_1 + a_{12}x_2 = b_1, \qquad a_{21}x_1 + a_{22}x_2 = b_2;$$

and the matrix form of this equation is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

In short, this matrix equation is commonly written as

$$Ax = b.$$

Appendix B

Matrix determinants and inverses

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We denote the inverse of an n-by-n matrix A as A^{-1}, where

$$AA^{-1} = A^{-1}A = I,$$

and where I is the n-by-n identity matrix satisfying $IA = AI = A$. In particular, if A is an invertible matrix, then the unique solution to the matrix equation $Ax = b$ is given by $x = A^{-1}b$.

It can be shown that a matrix A is invertible if and only if its determinant is not zero. Here, we only need two-by-two and three-by-three determinants. The two-by-two determinant, using the vertical bar notation, is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21};$$

that is, multiply the diagonal elements and subtract the product of the off-diagonal elements.

The three-by-three determinant is given in terms of two-by-two determinants as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The rule here is to go across the first row of the matrix, multiplying each element in the row by the determinant of the matrix obtained by crossing out that element's row and column, and adding the results with alternating signs.

We will need to invert two-by-two and three-by-three matrices, but this will mainly be simple because our matrices will be orthogonal. The rows (or columns) of an orthogonal matrix, considered as components of a vector, are orthonormal. For example, the following two matrices are orthogonal matrices:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}.$$

For the first matrix, the row vectors $\boldsymbol{\hat{r}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{\hat{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$ have unit length and are orthogonal, and the same can be said for the rows of the second matrix.

The inverse of an orthogonal matrix is simply given by its transpose, obtained by interchanging the matrices rows and columns. For example,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For more general two-by-two matrices, the inverse can be found from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

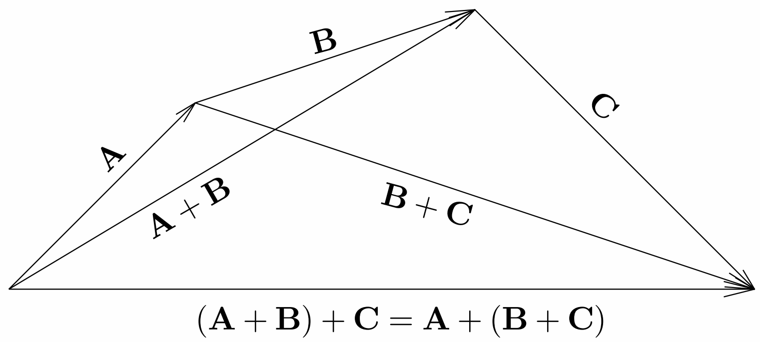
In words, switch the diagonal elements, negate the off-diagonal elements, and divide by the determinant.

Appendix C

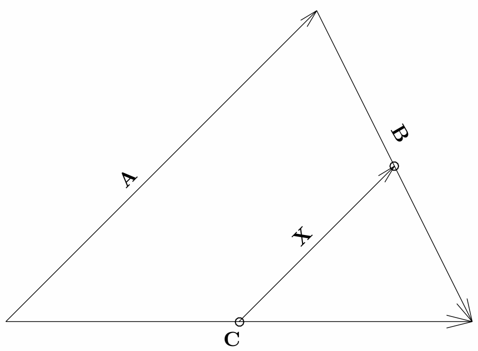
Problem solutions

Solutions to the Problems for Lecture 1

1. We show the associative law graphically:



2. Draw a triangle with sides composed of the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, with $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. Then draw the vector $\boldsymbol{X}$ pointing from the midpoint of $\boldsymbol{C}$ to the midpoint of $\boldsymbol{B}$.



From the figure, we see that

$$\frac{1}{2}\boldsymbol{C} + \boldsymbol{X} = \boldsymbol{A} + \frac{1}{2}\boldsymbol{B}.$$

Using $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$, this equation becomes

$$\frac{1}{2}(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{X} = \boldsymbol{A} + \frac{1}{2}\boldsymbol{B},$$

and solving for $\boldsymbol{X}$ yields $\boldsymbol{X} = \frac{1}{2}\boldsymbol{A}$. Therefore $\boldsymbol{X}$ is parallel to $\boldsymbol{A}$ and one-half its length.

Solutions to the Problems for Lecture 2

1.

a) The unit vector that points from m_1 to m_2 is given by

$$\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} = \frac{(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

b) The force acting on m_1 with position vector $\boldsymbol{r}_1$ due to the mass m_2 with position vector $\boldsymbol{r}_2$ is written as

$$\boldsymbol{F} = Gm_1m_2\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} = Gm_1m_2\frac{(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}}.$$

Solutions to the Problems for Lecture 3

1.

a) $\boldsymbol{A} \cdot \boldsymbol{B} = A_1B_1 + A_2B_2 + A_3B_3 = B_1A_1 + B_2A_2 + B_3A_3 = \boldsymbol{B} \cdot \boldsymbol{A}$;

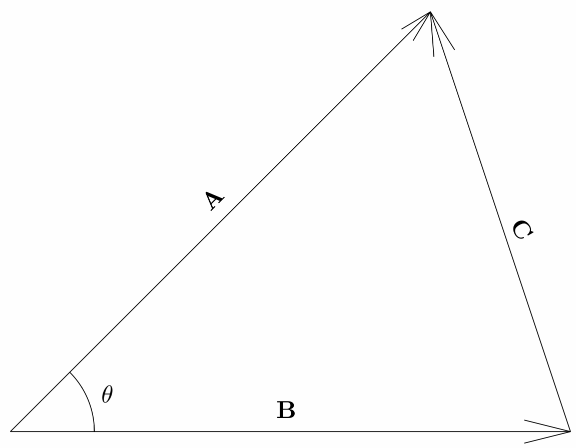
b) $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = A_1(B_1 + C_1) + A_2(B_2 + C_2) + A_3(B_3 + C_3) = A_1B_1 + A_1C_1 + A_2B_2 + A_2C_2 + A_3B_3 + A_3C_3 = (A_1B_1 + A_2B_2 + A_3B_3) + (A_1C_1 + A_2C_2 + A_3C_3) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$;

c) $\boldsymbol{A} \cdot (k\boldsymbol{B}) = A_1(kB_1) + A_2(kB_2) + A_3(kB_3) = (kA_1)B_1 + (kA_2)B_2 + (kA_3)B_3 = k(A_1B_1) + k(A_2B_2) + k(A_3B_3) = (k\boldsymbol{A}) \cdot \boldsymbol{B} = k(\boldsymbol{A} \cdot \boldsymbol{B})$

2. The dot product of a unit vector with itself is one, and the dot product of a unit vector with one perpendicular to itself is zero. That is,

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1; \qquad \boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0; \qquad \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{k} \cdot \boldsymbol{j} = 0.$$

3. Consider the triangle composed of three vectors pictured below.



With $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$, we have

$$|\boldsymbol{C}|^2 = \boldsymbol{C} \cdot \boldsymbol{C} = (\boldsymbol{A} - \boldsymbol{B}) \cdot (\boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{B} - 2\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|^2 + |\boldsymbol{B}|^2 - 2|\boldsymbol{A}||\boldsymbol{B}| \cos\theta,$$

where θ is the angle between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. In the usual notation, if A, B and C are the lengths of the sides of a triangle, and θ is the angle opposite side C, then

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

Solutions to the Problems for Lecture 4

1.

a)

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = -\mathbf{B} \times \mathbf{A}.$$

b)

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 + C_1 & B_2 + C_2 & B_3 + C_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}. \end{aligned}$$

c)

$$\begin{aligned} \mathbf{A} \times (k\mathbf{B}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ kB_1 & kB_2 & kB_3 \end{vmatrix} = k \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ kA_1 & kA_2 & kA_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= k(\mathbf{A} \times \mathbf{B}) = (k\mathbf{A}) \times \mathbf{B}. \end{aligned}$$

2. The cross product of a unit vector with itself is equal to the zero vector, the cross product of a unit vector with another (keeping the order cyclical in $\mathbf{i}, \mathbf{j}, \mathbf{k}$) is equal to the third unit vector, and reversing the order of multiplication changes the sign. That is,

$$\mathbf{i} \times \mathbf{i} = 0, \qquad \mathbf{j} \times \mathbf{j} = 0, \qquad \mathbf{k} \times \mathbf{k} = 0;$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j};$$

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

3. One such example is

$$\begin{aligned} &\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}, \\ &(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = 0 \times \mathbf{k} = 0. \end{aligned}$$

Solutions to the Practice quiz: Vectors

1. c. As an example, $\boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j}) \neq (\boldsymbol{i} \times \boldsymbol{i}) \times \boldsymbol{j}$.

2. b.

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{j} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \boldsymbol{j} = a_3 b_1 - a_1 b_3.$$

3. d.

$$\boldsymbol{i} \times (\boldsymbol{j} \times \boldsymbol{k}) = \boldsymbol{i} \times \boldsymbol{i} = 0, \quad (\boldsymbol{i} \times \boldsymbol{j}) \times \boldsymbol{k} = \boldsymbol{k} \times \boldsymbol{k} = 0, \quad (\boldsymbol{i} \times \boldsymbol{i}) \times \boldsymbol{j} = 0 \times \boldsymbol{j} = 0, \quad \boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j}) = \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}.$$

Solutions to the Problems for Lecture 5

1. We first compute the displacement vector between $(1, 1, 1)$ and $(2, 3, 2)$:

$$\boldsymbol{u} = (2-1)\boldsymbol{i} + (3-1)\boldsymbol{j} + (2-1)\boldsymbol{k} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}.$$

Choosing a point on the line to be $(1, 1, 1)$, the parametric equation for the line is given by

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{u}t = (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})t = (1+t)\boldsymbol{i} + (1+2t)\boldsymbol{j} + (1+t)\boldsymbol{k}.$$

The line crosses the $x = 0$ and $z = 0$ planes when $t = -1$ at the intersection point $(0, -1, 0)$, and crosses the $y = 0$ plane when $t = -1/2$ at the intersection point $(1/2, 0, 1/2)$.

Solutions to the Problems for Lecture 6

1. We find two vectors parallel to the plane defined by the three points, $(-1,-1,-1)$, $(1,1,1)$, and $(1,-1,0)$:

$$\boldsymbol{s}_1 = (1+1)\boldsymbol{i} + (1+1)\boldsymbol{j} + (1+1)\boldsymbol{k} = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}, \qquad \boldsymbol{s}_2 = (1-1)\boldsymbol{i} + (-1-1)\boldsymbol{j} + (0-1)\boldsymbol{k} = -2\boldsymbol{j} - \boldsymbol{k}.$$

We can divide $\boldsymbol{s}_1$ by 2 to construct a normal vector from

$$\boldsymbol{N} = \frac{1}{2}\boldsymbol{s}_1 \times \boldsymbol{s}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 0 & -2 & -1 \end{vmatrix} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}.$$

The equation for the plane can be found from $\boldsymbol{N} \cdot (\boldsymbol{r} - \boldsymbol{r}_2) = 0$, or $\boldsymbol{N} \cdot \boldsymbol{r} = \boldsymbol{N} \cdot \boldsymbol{r}_2$, or

$$(\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = (\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}), \quad \text{or} \quad x + y - 2z = 0.$$

The intersection of this plane with the $z = 0$ plane forms the line given by $y = -x$.

Solutions to the Practice quiz: Analytic geometry

1. d. Write the parametric equation as $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{u}t$. Using the point $(0, 1, 1)$, we take $\boldsymbol{r}_0 = \boldsymbol{j} + \boldsymbol{k}$ and from both points $(0, 1, 1)$ and $(1, 0, -1)$, we have $\boldsymbol{u} = (1-0)\boldsymbol{i} + (0-1)\boldsymbol{j} + (-1-1)\boldsymbol{k} = \boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$. Therefore $\boldsymbol{r} = \boldsymbol{j} + \boldsymbol{k} + (\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})t = t\boldsymbol{i} + (1-t)\boldsymbol{j} + (1-2t)\boldsymbol{k}$.

2. a. The line is parameterized as $\boldsymbol{r} = t\boldsymbol{i} + (1-t)\boldsymbol{j} + (1-2t)\boldsymbol{k}$. The intersection with the $z = 0$ plane occurs when $t = 1/2$ so that $\boldsymbol{r} = \frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$. The intersection point is therefore $(\frac{1}{2}, \frac{1}{2}, 0)$.

3. d. We first find the parametric equation for the plane. From the points $(1, 1, 1)$, $(1, 1, 2)$ and $(2, 1, 1)$, we construct the two displacement vectors

$$\boldsymbol{s}_1 = (1-1)\boldsymbol{i} + (1-1)\boldsymbol{j} + (2-1)\boldsymbol{k} = \boldsymbol{k}$$
$$\boldsymbol{s}_2 = (2-1)\boldsymbol{i} + (1-1)\boldsymbol{j} + (1-2)\boldsymbol{k} = \boldsymbol{i} - \boldsymbol{k}.$$

The normal vector to the plane can be found from

$$\boldsymbol{N} = \boldsymbol{s}_1 \times \boldsymbol{s}_2 = \boldsymbol{k} \times (\boldsymbol{i} - \boldsymbol{k}) = \boldsymbol{k} \times \boldsymbol{i} - \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{j}.$$

Therefore, the parametric equation for the plane, given by $\boldsymbol{N} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$, is determined to be

$$\boldsymbol{j} \cdot ((x-1)\boldsymbol{i} + (y-1)\boldsymbol{j} + (z-1)\boldsymbol{k}) = 0,$$

or $y = 1$. This plane is parallel to the x-z plane and when $z = 0$ is simply the line $y = 1$ for all values of x. Note now that we could have guessed this result because all three points defining the plane are located at $y = 1$.

Solutions to the Problems for Lecture 7

1.

a) If ijk is a cyclic permutation of $(1,2,3)$, then $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = 1$. If ijk is an anticyclic permutation of $(1,2,3)$, then $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -1$. And if any two indices are equal, then $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = 0$. The use is that we can cyclically permute the indices of the Levi-Civita tensor without changing its value.

b) If ijk is a cyclic permutation of $(1,2,3)$, then $\epsilon_{ijk} = 1$ and $\epsilon_{jik} = \epsilon_{kji} = \epsilon_{ikj} = -1$. If ijk is an anticyclic permutation of $(1,2,3)$, then $\epsilon_{ijk} = -1$ and $\epsilon_{jik} = \epsilon_{kji} = \epsilon_{ikj} = 1$. And if any two indices are equal, then $\epsilon_{ijk} = \epsilon_{jik} = \epsilon_{ikj} = 0$. The use is that we can swap any two indices of the Levi-Civita symbol if we change its sign.

2. We have

$$\begin{aligned}
\epsilon_{1jk}A_jB_k &= \epsilon_{123}A_2B_3 + \epsilon_{132}A_3B_2 = A_2B_3 - A_3B_2 = [\boldsymbol{A} \times \boldsymbol{B}]_1, \\
\epsilon_{2jk}A_jB_k &= \epsilon_{231}A_3B_1 + \epsilon_{213}A_1B_3 = A_3B_1 - A_1B_3 = [\boldsymbol{A} \times \boldsymbol{B}]_2, \\
\epsilon_{3jk}A_jB_k &= \epsilon_{312}A_1B_2 + \epsilon_{321}A_2B_1 = A_1B_2 - A_2B_1 = [\boldsymbol{A} \times \boldsymbol{B}]_3.
\end{aligned}$$

3.

a) Now, $\delta_{ij}A_j = \delta_{i1}A_1 + \delta_{i2}A_2 + \delta_{i3}A_3$. The only nonzero term has the index of A equal to i, therefore $\delta_{ij}A_j = A_i$.

b) Now, $\delta_{ik}\delta_{kj} = \delta_{i1}\delta_{1j} + \delta_{i2}\delta_{2j} + \delta_{i3}\delta_{3j}$. If $i \neq j$, then every term in the sum is zero. If $i = j$, then only one term is nonzero and equal to one. Therefore, $\delta_{ik}\delta_{kj} = \delta_{ij}$. This result could also be viewed as an application of Part (a).

4. We make use of the identities $\delta_{ii} = 3$ and $\delta_{ik}\delta_{jk} = \delta_{ij}$. For the Kronecker delta, the order of the indices doesn't matter. We also use

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).$$

a)

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{imn} &= \delta_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}) + \delta_{in}(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}) \\
&= 3(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}) \\
&= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.
\end{aligned}$$

b) We use the result of a) and find

$$\epsilon_{ijk}\epsilon_{ijn} = \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} = 3\delta_{kn} - \delta_{kn} = 2\delta_{kn}.$$

Solutions to the Problems for Lecture 8

1.

$$
\begin{aligned}
&\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) \\
&= [(\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}] + [(\boldsymbol{B} \cdot \boldsymbol{A})\boldsymbol{C} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}] + [(\boldsymbol{C} \cdot \boldsymbol{B})\boldsymbol{A} - (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B}] \\
&= [(\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B}] + [(\boldsymbol{B} \cdot \boldsymbol{A})\boldsymbol{C} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}] + [(\boldsymbol{C} \cdot \boldsymbol{B})\boldsymbol{A} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}] \\
&= 0.
\end{aligned}
$$

2. We can prove using the identity $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$. We have

$$|\boldsymbol{A} \times \boldsymbol{B}|^2 = (\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{A} \cdot \boldsymbol{A})(\boldsymbol{B} \cdot \boldsymbol{B}) - (\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{B} \cdot \boldsymbol{A}) = |\boldsymbol{A}|^2|\boldsymbol{B}|^2 - (\boldsymbol{A} \cdot \boldsymbol{B})^2.$$

An alternative proof uses

$$|\boldsymbol{A} \times \boldsymbol{B}|^2 = |\boldsymbol{A}|^2|\boldsymbol{B}|^2 \sin^2\theta = |\boldsymbol{A}|^2|\boldsymbol{B}|^2(1 - \cos^2\theta) = |\boldsymbol{A}|^2|\boldsymbol{B}|^2 - |\boldsymbol{A}|^2|\boldsymbol{B}|^2\cos^2\theta = |\boldsymbol{A}|^2|\boldsymbol{B}|^2 - (\boldsymbol{A} \cdot \boldsymbol{B})^2.$$

Solutions to the Practice quiz: Vector algebra

1. c. The relevant formula from the lecture is $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$. To apply this formula, we need to rearrange the indices keeping cyclical order:

$$\epsilon_{ijk}\epsilon_{ljm} = \epsilon_{jki}\epsilon_{jml} = \delta_{km}\delta_{il} - \delta_{kl}\delta_{im}.$$

2. d. The other expressions can be shown to be false using $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$ and, in general, $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$.

3. c. Use the facts that $\boldsymbol{A} \times \boldsymbol{B}$ is orthogonal to both $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \cdot \boldsymbol{B}$ is zero if $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal, and $\boldsymbol{A} \times \boldsymbol{B}$ is zero if $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel.

Solutions to the Problems for Lecture 9

1. *Scalar fields*: electrostatic potential, gravitational potential, temperature, humidity, concentration, density, pressure, wavefunction of quantum mechanics.
Vector fields: electric and magnetic fields, magnetic vector potential, velocity, force fields such as gravity.

Solutions to the Problems for Lecture 10

1. Using the chain rule,

$$\frac{\partial f}{\partial x} = \frac{-2nx}{(x^2+y^2+z^2)^{n+1}}, \quad \frac{\partial f}{\partial y} = \frac{-2ny}{(x^2+y^2+z^2)^{n+1}}, \quad \frac{\partial f}{\partial z} = \frac{-2nz}{(x^2+y^2+z^2)^{n+1}}.$$

2. Define

$$f(t+\epsilon, x+\delta) = g(\epsilon, \delta).$$

Then the first-order Taylor series expansion of g is given by

$$g(\epsilon, \delta) = g(0,0) + \epsilon g_t(0,0) + \delta g_x(0,0),$$

which in terms of f becomes

$$f(t+\epsilon, x+\delta) = f(t,x) + \epsilon f_t(t,x) + \delta f_x(t,x).$$

Applying this expansion to $f(t+\alpha\Delta t, x+\beta\Delta t f(t,x))$, we have to first-order in Δt,

$$f(t+\alpha\Delta t, x+\beta\Delta t f(t,x)) = f(t,x) + \alpha\Delta t f_t(t,x) + \beta\Delta t f(t,x) f_x(t,x).$$

Solutions to the Problems for Lecture 11

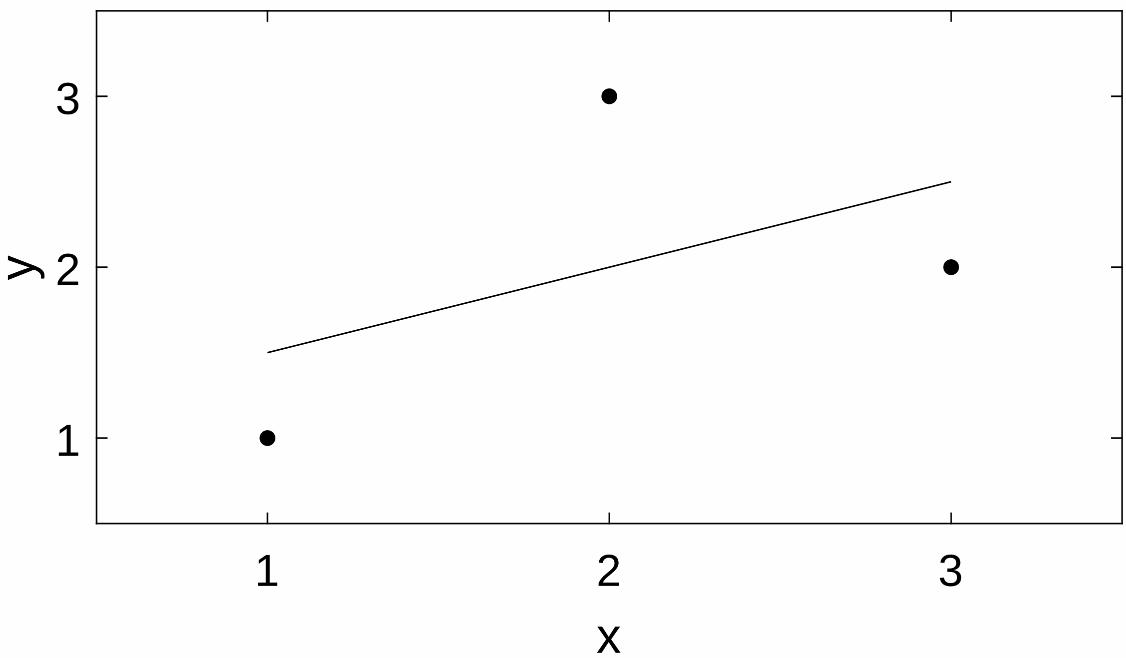
1. The formulas derived in the text are

$$\beta_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}, \quad \beta_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2},$$

where the sum is from $i = 1$ to 3. Here, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $y_1 = 1$, $y_2 = 3$ and $y_3 = 2$. We have

$$\beta_0 = \frac{(14)(6) - (13)(6)}{(3)(14) - (6)^2} = 1, \qquad \beta_1 = \frac{(3)(13) - (6)(6)}{(3)(14) - (6)^2} = 1/2.$$

The best fit line is therefore $y = 1 + x/2$. The graph of the data and the line are shown below.



Solutions to the Problems for Lecture 12

1.

a) With $f(x,y)=e^{xy}$, $x=r\cos\theta$, and $y=r\sin\theta$, application of the chain rule results in

$$\begin{aligned}\frac{\partial f}{\partial r}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\\&=ye^{xy}\cos\theta+xe^{xy}\sin\theta\\&=r\sin\theta\cos\theta e^{r^2\cos\theta\sin\theta}+r\sin\theta\cos\theta e^{r^2\cos\theta\sin\theta}\\&=2r\sin\theta\cos\theta e^{r^2\cos\theta\sin\theta},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial \theta}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\\&=ye^{xy}(-r\sin\theta)+xe^{xy}(r\cos\theta)\\&=-r^2\sin^2\theta e^{r^2\cos\theta\sin\theta}+r^2\cos^2\theta e^{r^2\cos\theta\sin\theta}\\&=r^2(\cos^2\theta-\sin^2\theta)e^{r^2\cos\theta\sin\theta}.\end{aligned}$$

b) Substituting for x and y, we have $f=e^{r^2\cos\theta\sin\theta}$. Then

$$\begin{aligned}\frac{\partial f}{\partial r}&=2r\cos\theta\sin\theta e^{r^2\cos\theta\sin\theta},\\\frac{\partial f}{\partial \theta}&=r^2(\cos^2\theta-\sin^2\theta)e^{r^2\cos\theta\sin\theta}.\end{aligned}$$

Solutions to the Problems for Lecture 14

1. Suppose $ax + by + cz = 0$. We have the relations

$$x = \frac{-by - cz}{a}, \qquad y = \frac{-ax - cz}{b}, \qquad z = \frac{-ax - by}{c}.$$

The partial derivatives are

$$\frac{\partial x}{\partial y} = -\frac{b}{a}, \qquad \frac{\partial y}{\partial z} = -\frac{c}{b}, \qquad \frac{\partial z}{\partial x} = -\frac{a}{c};$$

and the triple product is

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = \left(-\frac{b}{a}\right)\left(-\frac{c}{b}\right)\left(-\frac{a}{c}\right) = -1.$$

2. Suppose $ax + by + cz + dt = 0$. We have the relations

$$x = \frac{-by - cz - dt}{a}, \qquad y = \frac{-ax - cz - dt}{b}, \qquad z = \frac{-ax - by - dt}{c}, \qquad t = \frac{-ax - by - cz}{d}.$$

The partial derivatives are

$$\frac{\partial x}{\partial y} = -\frac{b}{a}, \qquad \frac{\partial y}{\partial z} = -\frac{c}{b}, \qquad \frac{\partial z}{\partial t} = -\frac{d}{c}, \qquad \frac{\partial t}{\partial x} = -\frac{a}{d};$$

and the quadruple product is

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial t}\frac{\partial t}{\partial x} = \left(-\frac{b}{a}\right)\left(-\frac{c}{b}\right)\left(-\frac{d}{c}\right)\left(-\frac{a}{d}\right) = 1.$$

Apparently an odd number of products yields -1 and an even number of products yields $+1$.

Solutions to the Practice quiz: Partial derivatives

1. d. The partial derivative with respect to x is given by

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}};$$

and the mixed second partial derivative is then given by

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{3xy}{(x^2+y^2+z^2)^{5/2}}$$

2. a. From the data points $(0,1)$, $(1,3)$, $(2,3)$ and $(3,4)$, we compute

$$\sum x_i = 6, \quad \sum x_i^2 = 14, \quad \sum y_i = 11, \quad \sum x_i y_i = 21.$$

Then using

$$\beta_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2}, \quad \beta_1 = \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2},$$

we have

$$\begin{aligned}
\beta_0 &= \frac{(14)(11)-(21)(6)}{(4)(14)-(6)^2} = \frac{154-126}{56-36} = \frac{28}{20} = \frac{7}{5}, \\
\beta_1 &= \frac{(4)(21)-(6)(11)}{(4)(14)-(6)^2} = \frac{84-66}{56-36} = \frac{18}{20} = \frac{9}{10}.
\end{aligned}$$

The least-squares line is therefore given by $y = 7/5 + 9x/10$.

3. d. Let $f = f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$. Then application of the chain rule results in

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -r\sin\theta \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y} \\
&= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.
\end{aligned}$$

Solutions to the Problems for Lecture 15

1.

a) Let $\phi(x, y, z) = x^2 + y^2 + z^2$. The gradient is given by

$$\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}(x^2 + y^2 + z^2) = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}.$$

In terms of the position vector, we have

$$\boldsymbol{\nabla}(r^2) = 2\boldsymbol{r}.$$

b) Let $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$. The gradient is given by

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \boldsymbol{\nabla}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)\\ &= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\boldsymbol{i} - \frac{y}{(x^2 + y^2 + z^2)^{3/2}}\boldsymbol{j} - \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\boldsymbol{k}.\end{aligned}$$

In terms of the position vector, we have

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\boldsymbol{r}}{r^3}.$$

Solutions to the Problems for Lecture 16

1.

a) With $\boldsymbol{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$, we have

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{F} &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx) \\ &= y + z + x = x + y + z.\end{aligned}$$

b) With $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$, we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0.$$

Solutions to the Problems for Lecture 17

1.

a) With $\boldsymbol{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$, we have

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix} = -y\boldsymbol{i} - z\boldsymbol{j} - x\boldsymbol{k}.$$

b) With $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$, we have

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x - x)\boldsymbol{i} + (y - y)\boldsymbol{j} + (z - z)\boldsymbol{k} = 0.$$

Solutions to the Problems for Lecture 18

1. We have

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right).$$

We can compute the derivatives with respect to x and use symmetry to find the other two terms. We have

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-x}{(x^2+y^2+z^2)^{3/2}};$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) &= \frac{-(x^2+y^2+z^2)^{3/2} + 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\\ &= -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

It is easy to guess the derivative with respect to y and z, and we have

$$\begin{aligned}\nabla^2\left(\frac{1}{r}\right) &= -\frac{3}{(x^2+y^2+z^2)^{3/2}} + \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\\ &= -\frac{3}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{(x^2+y^2+z^2)^{3/2}} = 0.\end{aligned}$$

Solutions to the Practice quiz: The del operator

1. d. We have

$$\begin{aligned}\boldsymbol{\nabla}\left(\frac{1}{r^2}\right) &= \boldsymbol{\nabla}\left(\frac{1}{x^2+y^2+z^2}\right)\\ &= \frac{-2x}{(x^2+y^2+z^2)^2}\boldsymbol{i} + \frac{-2y}{(x^2+y^2+z^2)^2}\boldsymbol{j} + \frac{-2z}{(x^2+y^2+z^2)^2}\boldsymbol{k}\\ &= -\frac{2\boldsymbol{r}}{r^4}.\end{aligned}$$

2. b. We use

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \boldsymbol{\nabla}\cdot\left(\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}}\right).$$

Now,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) &= \frac{\sqrt{x^2+y^2+z^2} - x^2(x^2+y^2+z^2)^{-1/2}}{x^2+y^2+z^2}\\ &= \frac{1}{r} - \frac{x^2}{r^3},\end{aligned}$$

and similarly for the partial derivatives with respect to y and z. Adding all three partial derivatives results in

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{3}{r} - \frac{x^2+y^2+z^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

3.

b. We have

$$\boldsymbol{\nabla}\times\boldsymbol{r} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix} = 0.$$

Solutions to the Problems for Lecture 20

1.

a) We compute:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(f\boldsymbol{u}) &= \frac{\partial}{\partial x_i}(fu_i)\\ &= \frac{\partial f}{\partial x_i}u_i + f\frac{\partial u_i}{\partial x_i}\\ &= \boldsymbol{u}\cdot\boldsymbol{\nabla}f + f\boldsymbol{\nabla}\cdot\boldsymbol{u}.\end{aligned}$$

b) We compute the ith component:

$$\begin{aligned}[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\frac{\partial u_m}{\partial x_l}\right)\\ &= \epsilon_{ijk}\epsilon_{klm}\frac{\partial^2 u_m}{\partial x_j\partial x_l}\\ &= \epsilon_{kij}\epsilon_{klm}\frac{\partial^2 u_m}{\partial x_j\partial x_l}\\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\frac{\partial^2 u_m}{\partial x_j\partial x_l}\\ &= \frac{\partial^2 u_j}{\partial x_j\partial x_i} - \frac{\partial^2 u_i}{\partial x_j\partial x_j}\\ &= [\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})]_i - [\nabla^2\boldsymbol{u}]_i.\end{aligned}$$

Therefore, $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$.

2.

a) With $d\boldsymbol{r}/dt = \boldsymbol{u}(t,\boldsymbol{r}(t))$, the component equations are given by

$$\frac{dx_1}{dt} = u_1(t;x_1,x_2,x_3), \qquad \frac{dx_2}{dt} = u_2(t;x_1,x_2,x_3), \qquad \frac{dx_3}{dt} = u_3(t;x_1,x_2,x_3).$$

b) Using the chain rule, we can compute the second derivative of x_1 as

$$\begin{aligned}\frac{d^2x_1}{dt^2} &= \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u_1}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial u_1}{\partial x_3}\frac{dx_3}{dt}\\ &= \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3}.\end{aligned}$$

Similarly for x_2 and x_3, we have

$$\frac{d^2x_2}{dt^2} = \frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3}, \qquad \frac{d^2x_3}{dt^2} = \frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3}.$$

c) Using the operator

$$\boldsymbol{u}\cdot\boldsymbol{\nabla} = u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3},$$

the three components can be combined into the vector expression

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}.$$

This expression is called the material acceleration, and is found in the Navier-Stokes equation of fluid mechanics.

Solutions to the Problems for Lecture 21

1. Start with Maxwell's equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\qquad \boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}.$$

Take the curl of the fourth Maxwell's equation, and commute the time and space derivatives to obtain

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=\mu_0\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{E}),$$

or after applying the curl of the curl identity,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B})-\nabla^2\boldsymbol{B}=\mu_0\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{E}).$$

Apply the second Maxwell's equation to the left-hand-side, and the third Maxwell's equation to the right-hand-side. Rearranging terms, we obtain the three-dimensional wave equation given by

$$\frac{\partial^2\boldsymbol{B}}{\partial t^2}=c^2\nabla^2\boldsymbol{B},$$

where $c=1/\sqrt{\mu_0\epsilon_0}$.

Solutions to the Practice quiz: Vector calculus algebra

1. a. We make use of the vector identity

$$\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{v}) = (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) + \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{u}).$$

Setting $\boldsymbol{v} = \boldsymbol{u}$, we have

$$\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = 2(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + 2\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}).$$

Therefore,

$$\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}.$$

2. c. The curl of a gradient (a. and d.) and the divergence of a curl (b.) are zero. The divergence of a gradient (c) is the Laplacian and is not always zero.

3. b. With $\boldsymbol{E}(\boldsymbol{r}, t) = \sin(z - ct)\boldsymbol{i}$, we have $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$, and

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(z-ct) & 0 & 0 \end{vmatrix} = \cos(z - ct)\boldsymbol{j}.$$

Maxwell's equation $\boldsymbol{\nabla}\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t}$ then results in

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\cos(z - ct)\boldsymbol{j},$$

which can be integrated (setting the constant to zero) to obtain

$$\boldsymbol{B} = \frac{1}{c}\sin(z - ct)\boldsymbol{j}.$$

Solutions to the Problems for Lecture 22

1. In general, the mass of a solid with mass density $\rho = \rho(x, y, z)$ is given by

$$M = \iiint_V \rho(x, y, z)\, dx\, dy\, dz.$$

To determine the mass of the cube, we place our coordinate system so that one corner of the cube is at the origin and the adjacent corners are on the positive x, y and z axes. We assume that the density of the cube is only a function of z, with

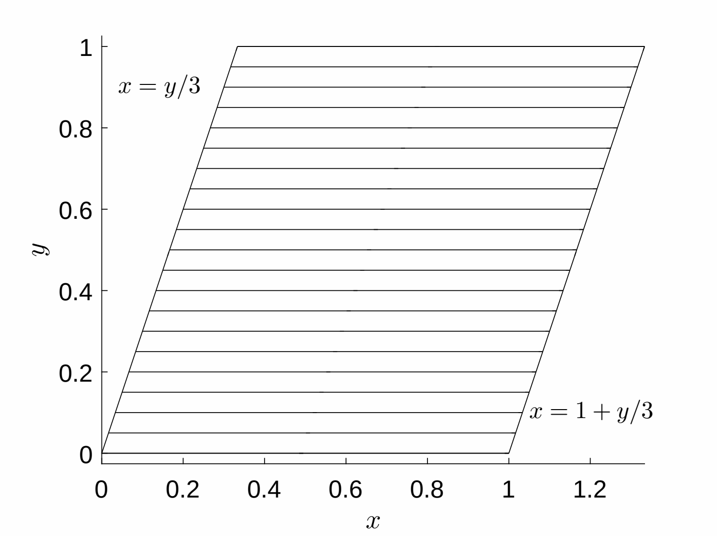
$$\rho(z) = \rho_1 + \frac{z}{L}(\rho_2 - \rho_1).$$

The mass of the cube is then given by

$$\begin{aligned} M &= \int_0^L \int_0^L \int_0^L \left[\rho_1 + \frac{z}{L}(\rho_2 - \rho_1)\right] dx\, dy\, dz = \int_0^L dx \int_0^L dy \int_0^L \left[\rho_1 + \frac{z}{L}(\rho_2 - \rho_1)\right] dz \\ &= L^2 \left[\rho_1 z + \frac{z^2}{2L}(\rho_2 - \rho_1)\right]_0^L = L^3 \left[\rho_1 + \frac{1}{2}(\rho_2 - \rho_1)\right] = \frac{1}{2} L^3(\rho_1 + \rho_2). \end{aligned}$$

Solutions to the Problems for Lecture 23

1.



The integral over the parallelogram (see the figure) is given by

$$
\begin{aligned}
\int_0^1 \int_{y/3}^{1+y/3} x^2 y \, dx \, dy &= \int_0^1 \left. \frac{x^3 y}{3} \right|_{x=y/3}^{x=1+y/3} dy \\
&= \frac{1}{3} \int_0^1 y \left(\left(1 + \frac{1}{3}y\right)^3 - \left(\frac{1}{3}y\right)^3 \right) dy \\
&= \frac{1}{3} \int_0^1 y \left(1 + y + \frac{1}{3}y^2 \right) dy \\
&= \frac{1}{3} \left(\frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{12}y^4 \right) \Big|_0^1 \\
&= \frac{1}{3} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{12} \right) = \frac{11}{36}.
\end{aligned}
$$

Solutions to the Practice quiz: Multidimensional integration

1. b. To find the volume, we integrate $z = xy$ over its base. We have

$$\int_0^1 \int_0^1 xy\,dx\,dy = \int_0^1 x\,dx \int_0^1 y\,dy = \left(\int_0^1 x\,dx\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

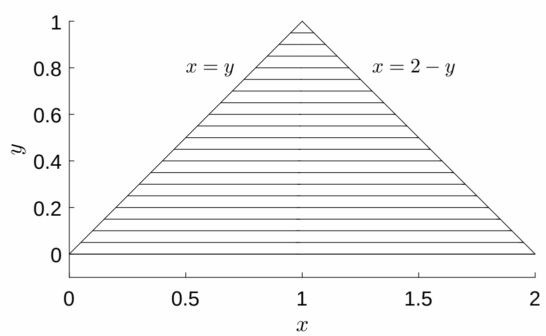
2. b. To determine the mass of the cube, we place our coordinate system so that one corner of the cube is at the origin and the adjacent corners are on the positive x, y and z axes. We assume that the density of the cube is only a function of z, with

$$\rho(z) = (1+z)\,\mathrm{g/cm}^3.$$

The mass of the cube in grams is then given by

$$M = \int_0^1 \int_0^1 \int_0^1 (1+z)\,dx\,dy\,dz = \int_0^1 dx \int_0^1 dy \int_0^1 (1+z)\,dz = (z + \frac{1}{2}z^2)\big|_0^1 = 1.5\,\mathrm{g}.$$

3. d. We draw a picture of the triangle and illustrate the chosen direction of integration.



Integrating first along x and then along y, the volume is given by

$$\begin{aligned}\int_0^1 \int_y^{2-y} xy\,dx\,dy &= \int_0^1 \frac{1}{2}x^2\Big|_y^{2-y} y\,dy = \frac{1}{2}\int_0^1 y\left[(2-y)^2 - y^2\right] dy \\ &= 2\int_0^1 (y - y^2)\,dy = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.\end{aligned}$$

Solutions to the Problems for Lecture 24

1.

a) The matrix form for the relationship between $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{i}$, $\boldsymbol{j}$ is given by

$$\begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{pmatrix}.$$

Inverting the two-by-two matrix, we have

$$\begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}.$$

Therefore,

$$\boldsymbol{i} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}, \qquad \boldsymbol{j} = \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}.$$

b) The matrix form for the relationship between $\partial f/\partial r$, $\partial f/\partial\theta$ and $\partial f/\partial x$, $\partial f/\partial y$ is given by

$$\begin{pmatrix} \partial f/\partial r \\ \partial f/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}.$$

Inverting the two-by-two matrix, we have

$$\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial f/\partial r \\ \partial f/\partial\theta \end{pmatrix}.$$

Therefore,

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}, \qquad \frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}.$$

2. We have

$$r\hat{\boldsymbol{r}} = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j},$$

and

$$r\hat{\boldsymbol{\theta}} = -r\sin\theta\,\boldsymbol{i} + r\cos\theta\,\boldsymbol{j} = -y\boldsymbol{i} + x\boldsymbol{j}.$$

Solutions to the Problems for Lecture 25

1. We have

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m\dot{\boldsymbol{r}}) = mr\hat{\boldsymbol{r}} \times (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}(\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}).$$

Now, $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are perpendicular unit vectors so that $|\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}| = 1$, and

$$|\boldsymbol{l}| = mr^2|\dot{\theta}||\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}| = mr^2|\dot{\theta}|.$$

Solutions to the Problems for Lecture 27

1. The mass density of the disk is given by

$$\sigma(r) = \rho_0 + (\rho_1 - \rho_0)(r/R).$$

Integrating the mass density in polar coordinates to find the total mass of the disk, we have

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^R \left[\rho_0 + (\rho_1 - \rho_0)(r/R)\right] r\, dr\, d\theta \\ &= 2\pi \left[\frac{\rho_0 r^2}{2} + \frac{(\rho_1 - \rho_0) r^3}{3R} \right]_{r=0}^{r=R} \\ &= \frac{1}{3}\pi R^2 (\rho_0 + 2\rho_1). \end{aligned}$$

2. It is simplest to do this integral by transforming to polar coordinates. With $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$, we have

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^\infty r e^{-r^2}\, dr = 2\pi \int_0^\infty r e^{-r^2}\, dr.$$

Let $u = r^2$ and $du = 2r\, dr$. Then the integral transforms to

$$I^2 = \pi \int_0^\infty e^{-u}\, du = -\pi e^{-u} \Big|_0^\infty = \pi.$$

Therefore,

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.$$

Solutions to the Practice quiz: Polar coordinates

1. d. Using $\hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$, we have

$$r\hat{\boldsymbol{\theta}} = -r\sin\theta\boldsymbol{i} + r\cos\theta\boldsymbol{j} = -y\boldsymbol{i} + x\boldsymbol{j}.$$

2. b. With $\hat{\boldsymbol{r}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$, we have

$$\frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j} = -\hat{\boldsymbol{r}}.$$

3. The mass density of the disk is given in polar coordinates by

$$\sigma = \sigma(r) = (10 - 9r)\,\mathrm{g/cm^2}.$$

The mass is found by integrating in polar coordinates using $dx\,dy = r\,dr\,d\theta$. Calculating in grams, we have

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^1 (10 - 9r)r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (10 - 9r)r\,dr \\ &= 2\pi(5r^2 - 3r^3)\Big|_0^1 = 4\pi \approx 12.57\,\mathrm{g}. \end{aligned}$$

Solutions to the Problems for Lecture 28

1. We have

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= (\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}})\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right) + (\sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}})\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{\rho}}\left(\cos^2\phi\frac{\partial}{\partial\rho} - \frac{\cos\phi\sin\phi}{\rho}\frac{\partial}{\partial\phi} + \sin^2\phi\frac{\partial}{\partial\rho} + \frac{\sin\phi\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right) \\
&\quad + \hat{\boldsymbol{\phi}}\left(-\sin\phi\cos\phi\frac{\partial}{\partial\rho} + \frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\phi} + \cos\phi\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos^2\phi}{\rho}\frac{\partial}{\partial\phi}\right) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}.
\end{aligned}$$

2. The calculations are

a)

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho) = \frac{1}{\rho};$$

b)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}} &= \boldsymbol{\nabla}\cdot(\cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}) \\
&= \boldsymbol{\nabla}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= \frac{\sqrt{x^2+y^2} - x^2(x^2+y^2)^{-1/2}}{x^2+y^2} + \frac{\sqrt{x^2+y^2} - y^2(x^2+y^2)^{-1/2}}{x^2+y^2} \\
&= \frac{2\sqrt{x^2+y^2} - \sqrt{x^2+y^2}}{x^2+y^2} \\
&= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{\rho}.
\end{aligned}$$

3. $\boldsymbol{\nabla}\times\hat{\boldsymbol{\rho}} = 0$, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}} = 0$ and

$$\boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho) = \frac{1}{\rho}\hat{\boldsymbol{z}}.$$

Solutions to the Problems for Lecture 29

1. The spherical coordinate unit vectors can be written in terms of the Cartesian unit vectors by

$$\begin{aligned}\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k},\\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k},\\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}.\end{aligned}$$

In matrix form, this relationship is written as

$$\begin{pmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{i}\\ \boldsymbol{j}\\ \boldsymbol{k}\end{pmatrix}.$$

The columns (and rows) of the transforming matrix Q are observed to be orthonormal so that Q is an orthogonal matrix. We have $\mathrm{Q}^{-1} = \mathrm{Q}^{\mathrm{T}}$ so that

$$\begin{pmatrix}\boldsymbol{i}\\ \boldsymbol{j}\\ \boldsymbol{k}\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{pmatrix};$$

or in expanded form

$$\begin{aligned}\boldsymbol{i} &= \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}},\\ \boldsymbol{j} &= \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}},\\ \boldsymbol{k} &= \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.\end{aligned}$$

2. We need the relationship between the Cartesian and the spherical coordinates, given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

The Jacobian to compute is

$$\begin{aligned}\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi\\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi\\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi\end{vmatrix} &= \begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}\\ &= r^2\sin\theta\begin{vmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{vmatrix}\\ &= r^2\sin\theta\left(\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi\right)\\ &= r^2\sin\theta\left(\sin^2\theta + \cos^2\theta\right)\left(\sin^2\phi + \cos^2\phi\right)\\ &= r^2\sin\theta.\end{aligned}$$

Therefore, $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

3. We have

$$\begin{aligned}\int_V f\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^R f(r)r^2\sin\theta\,dr\,d\theta\,d\phi \\ &= \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^R r^2 f(r)\,dr \\ &= 4\pi\int_0^R r^2 f(r)\,dr,\end{aligned}$$

where we have used $\int_0^{2\pi} d\phi = 2\pi$ and $\int_0^{\pi}\sin\theta\,d\theta = -\cos\theta\big|_0^{\pi} = 2$.

4. To find the mass, we use the result

$$M = \iiint_V \rho(x,y,z)\,dx\,dy\,dz,$$

where ρ is the object's mass density. Here, the density ρ is given by

$$\rho(r) = \rho_0 + (\rho_1 - \rho_0)(r/R),$$

and the total mass of the sphere is given by

$$\begin{aligned}M &= \int_0^{2\pi}\int_0^{\pi}\int_0^R \left[\rho_0 + (\rho_1-\rho_0)(r/R)\right] r^2\sin\theta\,dr\,d\theta\,d\phi \\ &= 4\pi\int_0^R\left[\rho_0 r^2 + (\rho_1-\rho_0)\frac{r^3}{R}\right]dr = 4\pi\left[\frac{\rho_0 R^3}{3} + \frac{(\rho_1-\rho_0)R^3}{4}\right] \\ &= \frac{4}{3}\pi R^3\left(\frac{1}{4}\rho_0 + \frac{3}{4}\rho_1\right).\end{aligned}$$

The average density of the sphere is its mass divided by its volume, given by

$$\bar{\rho} = \frac{1}{4}\rho_0 + \frac{3}{4}\rho_1.$$

Solutions to the Problems for Lecture 30

1. We begin with

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k},\\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k},\\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}.
\end{aligned}$$

Differentiating,

$$\frac{\partial\hat{\boldsymbol{r}}}{\partial\theta} = \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k} = \hat{\boldsymbol{\theta}};$$

and

$$\frac{\partial\hat{\boldsymbol{r}}}{\partial\phi} = -\sin\theta\sin\phi\,\boldsymbol{i} + \sin\theta\cos\phi\,\boldsymbol{j} = \sin\theta\,\hat{\boldsymbol{\phi}}.$$

2. The computations are

$$\begin{aligned}
&\boldsymbol{\nabla}\cdot\hat{\boldsymbol{r}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{r}, \qquad \boldsymbol{\nabla}\times\hat{\boldsymbol{r}} = 0;\\
&\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\theta}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta) = \frac{\cos\theta}{r\sin\theta}, \qquad \boldsymbol{\nabla}\times\hat{\boldsymbol{\theta}} = \frac{\hat{\boldsymbol{\phi}}}{r}\frac{\partial}{\partial r}(r) = \frac{\hat{\boldsymbol{\phi}}}{r};\\
&\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}} = 0, \qquad \boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}} = \frac{\hat{\boldsymbol{r}}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta) - \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial r}(r) = \frac{\hat{\boldsymbol{r}}\cos\theta}{r\sin\theta} - \frac{\hat{\boldsymbol{\theta}}}{r}.
\end{aligned}$$

3. Using spherical coordinates, for $r \neq 0$ for which $1/r$ diverges, we have

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-1\right) = 0.$$

Solutions to the Practice quiz: Cylindrical and spherical coordinates

1. b. We compute using the Laplacian in cylindrical coordinates:

$$\nabla^2\left(\frac{1}{\rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(-\frac{1}{\rho^2}\right) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right) = \frac{1}{\rho^3}.$$

2. c. When $\boldsymbol{r} = x\boldsymbol{i}$, the position vector points along the x-axis. Then $\hat{\boldsymbol{r}}$ also points along the x-axis, $\hat{\boldsymbol{\theta}}$ points along the negative z-axis and $\hat{\boldsymbol{\phi}}$ points along the y-axis. We have $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}) = (\boldsymbol{i}, -\boldsymbol{k}, \boldsymbol{j})$.

3. c. To find the mass, we use the result

$$M = \iiint_V \rho(x, y, z)\, dx\, dy\, dz,$$

where ρ is the object's mass density. Here, with the density ρ in units of $\mathrm{g/cm^3}$, we have

$$\rho = \rho(r) = 10 - r.$$

The integral is easiest to do in spherical coordinates, and using $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$, and computing in grams, we have

$$\begin{aligned}
M &= \int_0^{2\pi}\int_0^{\pi}\int_0^{5} (10 - r)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\
&= 4\pi \int_0^5 (10r^2 - r^3)\, dr \\
&= 4\pi\left(\frac{10}{3}r^3 - \frac{1}{4}r^4\right)\Big|_0^5 = \frac{3125\pi}{3}\ \mathrm{g} \\
&\approx 3272\,\mathrm{g} \approx 3.3\,\mathrm{kg}.
\end{aligned}$$

Solutions to the Problems for Lecture 31

1. We integrate $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ counterclockwise around the square. We write

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{u} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{u} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{u} \cdot d\boldsymbol{r} + \int_{C_4} \boldsymbol{u} \cdot d\boldsymbol{r},$$

where the curves C_i represent the four sides of the square. On C_1 from $(0,0)$ to $(L,0)$, we have $y = 0$ and $d\boldsymbol{r} = dx\boldsymbol{i}$ so that $\int_{C_1} \boldsymbol{u} \cdot d\boldsymbol{r} = 0$. On C_2 from $(L,0)$ to (L,L), we have $x = L$ and $d\boldsymbol{r} = dy\boldsymbol{j}$ so that $\int_{C_2} \boldsymbol{u} \cdot d\boldsymbol{r} = \int_0^L Ldy = L^2$. On C_3 from (L,L) to $(0,L)$, we have $y = L$ and $d\boldsymbol{r} = dx\boldsymbol{i}$ so that $\int_{C_3} \boldsymbol{u} \cdot d\boldsymbol{r} = \int_L^0 -Ldx = L^2$. The sign of this term is tricky, but notice that the curve is going in the -$\boldsymbol{i}$ direction and so is the x-component of the vector field so the dot product should be positive. On C_4 from $(0,L)$ to $(0,0)$, we have $x = 0$ and $d\boldsymbol{r} = dy\boldsymbol{j}$ so that $\int_{C_4} \boldsymbol{u} \cdot d\boldsymbol{r} = 0$. Summing the four contributions, we found

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = 2L^2,$$

which is twice the area of the square.

2. We integrate $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ counterclockwise around a unit circle. To parameterize a circle with radius R, we write

$$x = R\cos\theta, \qquad y = R\sin\theta.$$

Therefore, $\boldsymbol{u} = -R\sin\theta\boldsymbol{i} + R\cos\theta\boldsymbol{j}$ and $d\boldsymbol{r} = (-R\sin\theta\boldsymbol{i} + R\cos\theta\boldsymbol{j})d\theta$. We have $\boldsymbol{u} \cdot d\boldsymbol{r} = R^2 d\theta$ and

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_0^{2\pi} R^2\, d\theta = 2\pi R^2,$$

which is twice the area of the circle.

Solutions to the Problems for Lecture 32

1. In spherical coordinates, on the surface of a sphere of radius R centered at the origin, we have $\boldsymbol{r} = R\hat{\boldsymbol{r}}$ and $d\boldsymbol{S} = \hat{\boldsymbol{r}}\, R^2 \sin\theta\, d\theta\, d\phi$. Therefore,

$$\oint_S \boldsymbol{r} \cdot d\boldsymbol{S} = \int_0^{2\pi} \int_0^{\pi} R^3 \sin\theta\, d\theta\, d\phi = 4\pi R^3.$$

Solutions to the Practice quiz: Vector integration

1. c. We integrate $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$ counterclockwise around the right triangle. We write

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{u} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{u} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{u} \cdot d\boldsymbol{r},$$

where the curves C_i represent the three sides of the triangle. On C_1 from $(0,0)$ to $(L,0)$, we have $y = 0$ and $d\boldsymbol{r} = dx\boldsymbol{i}$ so that $\int_{C_1} \boldsymbol{u} \cdot d\boldsymbol{r} = 0$. On C_2 from $(L,0)$ to $(0,L)$, we parameterize the line segment by $\boldsymbol{r} = (L-s)\boldsymbol{i} + s\boldsymbol{j}$ as s goes from zero to L so that $d\boldsymbol{r} = -ds\boldsymbol{i} + ds\boldsymbol{j}$. Therefore, on this line segment, $\boldsymbol{u} \cdot d\boldsymbol{r} = (x+y)ds = L\,ds$. We have $\int_{C_2} \boldsymbol{u} \cdot d\boldsymbol{r} = \int_0^L L\,ds = L^2$. On C_4 from $(0,L)$ to $(0,0)$, we have $x = 0$ and $d\boldsymbol{r} = dy\boldsymbol{j}$ so that $\int_{C_4} \boldsymbol{u} \cdot d\boldsymbol{r} = 0$. Summing the four contributions, we found

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = L^2,$$

which is twice the area of the triangle.

2. c. In cylindrical coordinates, $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} = \rho\hat{\boldsymbol{\rho}}$. The cylinder ends have normal vectors $\hat{\boldsymbol{z}}$ and $-\hat{\boldsymbol{z}}$, which are perpendicular to $\boldsymbol{u}$. On the side of the cylinder, we have $\boldsymbol{u} = R\hat{\boldsymbol{\rho}}$ and $d\boldsymbol{S} = \hat{\boldsymbol{\rho}}dS$, so that

$$\oint_S \boldsymbol{u} \cdot d\boldsymbol{S} = R\int dS = R(2\pi RL) = 2\pi R^2 L.$$

3. a. We perform the flux integral in spherical coordinates. On the surface of the sphere of radius R, we have

$$\boldsymbol{u} = z\boldsymbol{k} = (R\cos\theta)(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}),$$

and

$$d\boldsymbol{S} = \hat{\boldsymbol{r}}\,R^2\sin\theta\,d\theta\,d\phi.$$

Therefore, the surface integral over the upper hemisphere becomes

$$\begin{aligned}\int_S \boldsymbol{u} \cdot d\boldsymbol{S} &= \int_0^{2\pi}\int_0^{\pi/2} R^3\cos^2\theta\sin\theta\,d\theta\,d\phi \\ &= 2\pi R^3\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta = 2\pi R^3\int_0^1 w^2\,dw = \frac{2\pi}{3}R^3.\end{aligned}$$

Solutions to the Problems for Lecture 33

1. With $\phi(\boldsymbol{r}) = x^2y + xy^2 + z$:

a) $\boldsymbol{\nabla}\phi = (2xy + y^2)\boldsymbol{i} + (x^2 + 2xy)\boldsymbol{j} + \boldsymbol{k}$

b) Using the gradient theorem, $\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \phi(1,1,1) - \phi(0,0,0) = 3$.

c) Integrating over the three directed line segments given by (1) $(0,0,0)$ to $(1,0,0)$; (2) $(1,0,0)$ to $(1,1,0)$, and; (3) $(1,1,0)$ to $(1,1,1)$:

$$\begin{aligned}
\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} &= \int_{C_1} \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} \\
&= 0 + \int_0^1 (1 + 2y)\, dy + \int_0^1 dz \\
&= 3.
\end{aligned}$$

Solutions to the Problems for Lecture 34

1. With $\boldsymbol{u} = (2xy + z^2)\boldsymbol{i} + (2yz + x^2)\boldsymbol{j} + (2zx + y^2)\boldsymbol{k}$:

a)

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{u} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy + z^2 & 2yz + x^2 & 2zx + y^2 \end{vmatrix} \\
&= (2y - 2y)\boldsymbol{i} + (2z - 2z)\boldsymbol{j} + (2x - 2x)\boldsymbol{k} \\
&= 0.
\end{aligned}$$

b) We need to satisfy

$$\frac{\partial \phi}{\partial x} = 2xy + z^2, \qquad \frac{\partial \phi}{\partial y} = 2yz + x^2, \qquad \frac{\partial \phi}{\partial z} = 2zx + y^2.$$

Integrate the first equation to get

$$\phi = \int (2xy + z^2)\, dx = x^2 y + xz^2 + f(y, z).$$

Take the derivative with respect to y and satsify the second equation:

$$x^2 + \frac{\partial f}{\partial y} = 2yz + x^2 \qquad \text{or} \qquad \frac{\partial f}{\partial y} = 2yz.$$

Integrate this equation for f to get

$$f = \int 2yz\, dy = y^2 z + g(z).$$

Take the derivative of $\phi = x^2 y + xz^2 + y^2 z + g(z)$ with respect to z and satisfy the last gradient equation:

$$2xz + y^2 + g'(z) = 2zx + y^2 \qquad \text{or} \qquad g'(z) = 0.$$

Therefore, $g(z) = c$ where c is a constant, and $\phi = x^2 y + y^2 z + z^2 x + c$.

Solutions to the Practice quiz: Gradient theorem

1. b. $\displaystyle\int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \phi(1,1,1) - \phi(0,0,0) = 1.$

2. a. Since $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\nabla} \times (y\boldsymbol{i} + x\boldsymbol{j}) = 0$, the line integral $\boldsymbol{u}$ around any closed curve is zero. By inspection, we can also observe that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, where $\phi = xy$.

3. c. To solve the multiple choice question, we can always take the gradients of the four choices. Without the advantage of multiple choice, however, we need to compute ϕ and we do so here. We solve

$$\frac{\partial\phi}{\partial x} = 2x + y, \qquad \frac{\partial\phi}{\partial y} = 2y + x, \qquad \frac{\partial\phi}{\partial z} = 1.$$

Integrating the first equation with respect to x holding y and z fixed, we find

$$\phi = \int (2x + y)\, dx = x^2 + xy + f(y,z).$$

Differentiating ϕ with respect to y and using the second equation, we obtain

$$x + \frac{\partial f}{\partial y} = 2y + x \qquad \text{or} \qquad \frac{\partial f}{\partial y} = 2y.$$

Another integration results in $f(y,z) = y^2 + g(z)$. Finally, differentiating ϕ with respect to z yields $g'(z) = 1$, or $g(z) = z + c$. The final solution is

$$\phi(x,y,z) = x^2 + xy + y^2 + z + c.$$

Answer c. is correct with the constant $c = 0$.

Solutions to the Problems for Lecture 35

1. Using spherical coordinates, let $\boldsymbol{u} = u_r(r,\theta,\phi)\hat{\boldsymbol{r}} + u_\theta(r,\theta,\phi)\hat{\boldsymbol{\theta}} + u_\phi(r,\theta,\phi)\hat{\boldsymbol{\phi}}$. Then the volume integral becomes

$$\int_V (\boldsymbol{\nabla}\cdot\boldsymbol{u})\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right) r^2\sin\theta\,dr\,d\theta\,d\phi.$$

Each term in the integrand can be integrated once. The first term is integrated as

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^R \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)\right) r^2\sin\theta\,dr\,d\theta\,d\phi &= \int_0^{2\pi}\int_0^{\pi}\left(\int_0^R \frac{\partial}{\partial r}(r^2 u_r)\,dr\right)\sin\theta\,d\theta\,d\phi\\ &= \int_0^{2\pi}\int_0^{\pi} u_r(R,\theta,\phi)R^2\sin\theta\,d\theta\,d\phi.\end{aligned}$$

The second term is integrated as

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^R \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta)\right) r^2\sin\theta\,dr\,d\theta\,d\phi &= \int_0^{2\pi}\int_0^R\left(\int_0^{\pi}\frac{\partial}{\partial\theta}(\sin\theta\,u_\theta)\,d\theta\right) r\,dr\,d\phi\\ &= \int_0^{2\pi}\int_0^R \left(\sin(\pi)\,u_\theta(r,\pi,\phi) - \sin(0)\,u_\theta(r,0,\phi)\right) r\,dr\,d\phi\\ &= 0,\end{aligned}$$

since $\sin(\pi) = \sin(0) = 0$. Similarly, the third term is integrated as

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^R \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right) r^2\sin\theta\,dr\,d\theta\,d\phi &= \int_0^{\pi}\int_0^R\left(\int_0^{2\pi}\frac{\partial u_\phi}{\partial\phi}\,d\phi\right) r\,dr\,d\theta\\ &= \int_0^{\pi}\int_0^R \left(u_\phi(r,\theta,2\pi) - u_\phi(r,\theta,0)\right) r\,dr\,d\theta\\ &= 0,\end{aligned}$$

since $u_\phi(r,\theta,2\pi) = u_\phi(r,\theta,0)$ because ϕ is a periodic variable with same physical location at 0 and 2π.

Therefore, we have

$$\int_V (\boldsymbol{\nabla}\cdot\boldsymbol{u})\,dV = \int_0^{2\pi}\int_0^{\pi} u_r(R,\theta,\phi)R^2\sin\theta\,d\theta\,d\phi = \oint_S \boldsymbol{u}\cdot d\boldsymbol{S},$$

where S is a sphere of radius R located at the origin, with unit normal vector given by $\hat{\boldsymbol{r}}$, and infinitesimal surface area given by $dS = R^2\sin\theta\,d\theta\,d\phi$.

Solutions to the Problems for Lecture 36

1. With $\boldsymbol{u} = x^2y\,\boldsymbol{i} + y^2z\,\boldsymbol{j} + z^2x\,\boldsymbol{k}$, we use $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 2xy + 2yz + 2zx$. We have for the left-hand side of the divergence theorem,

$$\begin{aligned}
\int_V (\boldsymbol{\nabla}\cdot\boldsymbol{u})\,dV &= 2\int_0^L\int_0^L\int_0^L (xy + yz + zx)\,dx\,dy\,dz \\
&= 2\left[\int_0^L x\,dx\int_0^L y\,dy\int_0^L dz + \int_0^L dx\int_0^L y\,dy\int_0^L z\,dz + \int_0^L x\,dx\int_0^L dy\int_0^L z\,dz\right] \\
&= 2(L^5/4 + L^5/4 + L^5/4) \\
&= 3L^5/2.
\end{aligned}$$

For the right-hand side of the divergence theorem, the flux integral only has nonzero contributions from the three sides located at $x = L$, $y = L$ and $z = L$. The corresponding unit normal vectors are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, and the corresponding integrals are

$$\begin{aligned}
\oint_S \boldsymbol{u}\cdot d\boldsymbol{S} &= \int_0^L\int_0^L L^2y\,dy\,dz + \int_0^L\int_0^L L^2z\,dx\,dz + \int_0^L\int_0^L L^2x\,dx\,dy \\
&= L^5/2 + L^5/2 + L^5/2 \\
&= 3L^5/2.
\end{aligned}$$

2. With $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, we have $\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3$. Therefore, from the divergence theorem we have

$$\int_S \boldsymbol{r}\cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{r}\,dV = 3\int_V dV = 3L^3.$$

Note that the integral is equal to three times the volume of the box and is independent of the placement and orientation of the coordinate system.

Solutions to the Problems for Lecture 37

1. With $\boldsymbol{u} = \hat{\boldsymbol{r}}/r$, we use spherical coordinates to compute

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{d}{dr}(r) = \frac{1}{r^2}.$$

Therefore, for the left-hand side of the divergence theorem we have

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV = 4\pi \int_0^R \left(\frac{1}{r^2}\right) r^2\, dr = 4\pi R.$$

For the right-hand side of the divergence theorem, we have for a sphere of radius R centered at the origin, $d\boldsymbol{S} = \hat{\boldsymbol{r}}\, dS$ and

$$\oint_S \boldsymbol{u} \cdot d\boldsymbol{S} = \oint_S \frac{1}{R}\, dS = \frac{4\pi R^2}{R} = 4\pi R.$$

2. With $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, we have $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$. Therefore, from the divergence theorem we have

$$\int_S \boldsymbol{r} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{r}\, dV = 3\int_V dV = 3\left(\frac{4}{3}\pi R^3\right) = 4\pi R^3.$$

Note that the integral is equal to three times the volume of the sphere and is independent of the placement and orientation of the coordinate system.

3. Computing in spherical coordinates, the Laplacian of $1/r$ is given by

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right) = \frac{1}{r^2}\frac{d}{dr}(-1) = 0,$$

provided $r \neq 0$ where ϕ is singular. Therefore, if the volume V does not contain the origin, then

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = 0, \qquad (0,0,0) \notin V.$$

However, if V contains the origin, we need only integrate over a small sphere of volume $V' \in V$ centered at the origin, since $\nabla^2(1/r) = 0$ outside of V'. We therefore have from the divergence theorem

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = \int_{V'} \nabla^2\left(\frac{1}{r}\right) dV = \oint_{S'} \boldsymbol{\nabla}\left(\frac{1}{r}\right) \cdot d\boldsymbol{S},$$

where the surface S' is now the surface of a sphere of radius R, say, centered at the origin. Using spherical coordinates,

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = \hat{\boldsymbol{r}}\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{\hat{\boldsymbol{r}}}{r^2},$$

and since $d\boldsymbol{S} = \hat{\boldsymbol{r}} dS$, we have

$$\oint_{S'} \boldsymbol{\nabla}\left(\frac{1}{r}\right) \cdot d\boldsymbol{S} = -\frac{1}{R^2}\oint_{S'} dS = -4\pi,$$

since the surface area of the sphere is $4\pi R^2$. Therefore,

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = \begin{cases} 0, & (0,0,0) \notin V; \\ -4\pi, & (0,0,0) \in V. \end{cases}$$

For those of you familar with the Dirac delta function, say from my course *Differential Equations for Engineers*, what we have here is

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\boldsymbol{r}),$$

where $\delta(\boldsymbol{r})$ is the three-dimensional Dirac delta function satisfying

$$\delta(\boldsymbol{r}) = 0, \qquad \text{when } \boldsymbol{r} \neq 0,$$

and

$$\int_V \delta(\boldsymbol{r})\, dV = 1, \qquad \text{provided the origin is in } V.$$

Solutions to the Problems for Lecture 38

1. The continuity equation as derived in the lecture is given by

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0.$$

Using the vector identity $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u}$, the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0.$$

2. We begin with

$$\frac{d}{dt} \int_V \rho(\boldsymbol{r}, t)\, dV = -\oint_S \boldsymbol{J} \cdot d\boldsymbol{S}.$$

The divergence theorem applied to the right-hand side results in

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{J}\, dV;$$

and combining both sides of the equation and bringing the time derivative inside the integral results in

$$\int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} \right) dV = 0.$$

Since the integral is zero for any volume V, we obtain the electrodynamics continuity equation given by

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0.$$

Solutions to the Practice quiz: Divergence theorem

1. a. With $\boldsymbol{u} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$, we have $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Therefore,

$$\oint_S \boldsymbol{u} \cdot d\boldsymbol{S} = \int_V (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, dV = 0.$$

2. d.

$$\oint_S \boldsymbol{r} \cdot d\boldsymbol{S} = \int_V (\boldsymbol{\nabla} \cdot \boldsymbol{r})\, dV = 3\int_V dV = 3\pi R^2 L.$$

3. d. Computing the divergences, we have

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \left[xy\boldsymbol{i} - \frac{1}{2}y^2\boldsymbol{j}\right] = y - y = 0, \\
&\boldsymbol{\nabla} \cdot [(1+x)\boldsymbol{i} + (1-y)\boldsymbol{j}] = 1 - 1 = 0, \\
&\boldsymbol{\nabla} \cdot \left[(x^2 - xy)\boldsymbol{i} + \left(\frac{1}{2}y^2 - 2xy\right)\boldsymbol{j}\right] = (2x - y) + (y - 2x) = 0, \\
&\boldsymbol{\nabla} \cdot \left[(x+y)^2\boldsymbol{i} + (x-y)^2\boldsymbol{j}\right] = 2(x+y) - 2(x-y) = 4y.
\end{aligned}$$

Solutions to the Problems for Lecture 39

1. With $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$, we use $\partial u_2/\partial x - \partial u_1/\partial y = 2$. For a square of side L, we have for the left-hand side of Green's theorem

$$\int_A \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dA = 2 \int_A dA = 2L^2.$$

When the square lies in the first quadrant with vertex at the origin, we have for the right-hand side of Green's theorem,

$$\oint_C (u_1\, dx + u_2\, dy) = \int_0^L 0\, dx + \int_L^0 (-L)\, dx + \int_L^0 0\, dy + \int_0^L L\, dy = 2L^2.$$

2. With $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$, we use $\partial u_2/\partial x - \partial u_1/\partial y = 2$. For a circle of radius R, we have for the left-hand side of Green's theorem,

$$\int_A \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dA = 2 \int_A dA = 2\pi R^2.$$

For a circle of radius R centered at the origin, we change variables to $x = R\cos\theta$ and $y = R\sin\theta$. Then $dx = -R\sin\theta$ and $dy = R\cos\theta$, and we have for the right-hand side of Green's theorem,

$$\oint_C (u_1\, dx + u_2\, dy) = \oint_C (-y\, dx + x\, dy) = \int_0^{2\pi} (R^2 \sin^2\theta + R^2 \cos^2\theta) d\theta = 2\pi R^2.$$

Solutions to the Problems for Lecture 40

1. Let $\boldsymbol{u} = u_1(x,y,z)\,\boldsymbol{i} + u_2(x,y,z)\,\boldsymbol{j} + u_3(x,y,z)\,\boldsymbol{k}$. Then

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\boldsymbol{k}.$$

a) For an area lying in the y-z plane bounded by a curve C, the normal vector to the area is $\boldsymbol{i}$. Therefore, Green's theorem is given by

$$\int_A \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) dA = \oint_C (u_2 dy + u_3 dz);$$

b) For an area lying in the z-x plane bounded by a curve C, the normal vector to the area is $\boldsymbol{j}$. Therefore, Green's theorem is given by

$$\int_A \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) dA = \oint_C (u_3 dz + u_1 dx);$$

The correct orientation of the curves are determined by the right-hand rule, using a right-handed coordinate system.

2. We have $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$. The right-hand side of Stokes' theorem was computed in an earlier problem on Green's theorem and we repeat the solution here. For a circle of radius R lying in the x-y plane with center at the origin, we change variables to $x = R\cos\phi$ and $y = R\sin\phi$. Then $dx = -R\sin\phi$ and $dy = R\cos\phi$, and we have for the right-hand side of Stokes' theorem,

$$\oint_C \boldsymbol{u}\cdot d\boldsymbol{r} = \oint_C (u_1\,dx + u_2\,dy) = \oint_C (-y\,dx + x\,dy) = \int_0^{2\pi} (R^2\sin^2\phi + R^2\cos^2\phi)d\theta = 2\pi R^2.$$

The left-hand side of Stokes' theorem uses

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\boldsymbol{k};$$

so that with $d\boldsymbol{S} = \hat{\boldsymbol{r}}R^2\sin\theta\,d\theta\,d\phi$, we have

$$\int_S (\boldsymbol{\nabla}\times\boldsymbol{u})\cdot d\boldsymbol{S} = 2R^2\int_0^{2\pi}\int_0^{\pi/2} \boldsymbol{k}\cdot\hat{\boldsymbol{r}}\sin\theta\,d\theta\,d\phi.$$

With

$$\boldsymbol{k} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}},$$

we have

$$\boldsymbol{k}\cdot\hat{\boldsymbol{r}} = \cos\theta;$$

and

$$\begin{aligned}\int_S (\boldsymbol{\nabla}\times\boldsymbol{u})\cdot d\boldsymbol{S} &= 2R^2\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta\sin\theta\,d\theta \\ &= 2\pi R^2 \sin^2\theta\big|_0^{\pi/2} = 2\pi R^2.\end{aligned}$$

Solutions to the Practice quiz: Stokes' theorem

1. b. With $\boldsymbol{u} = -y\boldsymbol{i} + x\boldsymbol{j}$, we have $\boldsymbol{\nabla} \times \boldsymbol{u} = 2\boldsymbol{k}$ and we use Stokes' theorem to write

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot d\boldsymbol{S} = 2 \int dA = \frac{1}{2}\pi R^2,$$

where we have used $d\boldsymbol{S} = \boldsymbol{k}\, dA$ and the area of the quarter circle is $\frac{1}{4}\pi R^2$.

2. c. With $\boldsymbol{u} = \frac{-y}{x^2+y^2}\boldsymbol{i} + \frac{x}{x^2+y^2}\boldsymbol{j}$, one can show by differentiating that $\boldsymbol{\nabla} \times \boldsymbol{u} = 0$ provided $(x, y) \neq (0, 0)$. However, the integration region contains the origin so the integral is best done by applying Stokes' theorem. We use cylindrical coordinates to write

$$\boldsymbol{u} = \frac{-y}{x^2+y^2}\boldsymbol{i} + \frac{x}{x^2+y^2}\boldsymbol{j} = \frac{\hat{\boldsymbol{\phi}}}{\rho}.$$

Then,

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_0^{2\pi} \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) \cdot (\hat{\boldsymbol{\phi}}\rho\, d\phi) = \int_0^{2\pi} d\phi = 2\pi.$$

3. c. With $\boldsymbol{u} = -x^2y\boldsymbol{i} + xy^2\boldsymbol{j}$, we have $\boldsymbol{\nabla} \times \boldsymbol{u} = (x^2 + y^2)\boldsymbol{k}$. Therefore,

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{r} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot d\boldsymbol{S} = \int_0^1 \int_0^1 (x^2 + y^2)\, dx\, dy = \int_0^1 x^2\, dx \int_0^1 dy + \int_0^1 dx \int_0^1 y^2\, dy = \frac{2}{3},$$

where we have used $d\boldsymbol{S} = \boldsymbol{k}\, dx\, dy$.

Solutions to the Problems for Lecture 41

1.

a) The Navier-Stokes equation and the continuity equation are given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0.$$

Taking the divergence of the Navier-Stokes equation and using the continuity equation results in

$$\boldsymbol{\nabla} \cdot ((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}) = -\frac{1}{\rho}\nabla^2 p.$$

Now,

$$\boldsymbol{\nabla} \cdot ((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}) = \frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

Therefore,

$$\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

b) Taking the curl of the Navier-Stokes equation, and using $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} p = 0$, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \nu \nabla^2 \boldsymbol{\omega}.$$

To simplify the second term, we first prove the identity

$$\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}.$$

We prove by considering the ith component of the left-hand side:

$$\begin{aligned}
[\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u})]_i &= \epsilon_{ijk} u_j \epsilon_{klm} \frac{\partial u_m}{\partial x_l} = \epsilon_{kij}\epsilon_{klm} u_j \frac{\partial u_m}{\partial x_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \frac{\partial u_m}{\partial x_l} = u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} \\
&= \frac{1}{2}\frac{\partial}{\partial x_i}(u_j u_j) - u_j \frac{\partial u_i}{\partial x_j} \\
&= \left[\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u})\right]_i - [(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}]_i \,.
\end{aligned}$$

Therefore, using that the curl of a gradient and the divergence of a curl is equal to zero, and $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, we have

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} &= \boldsymbol{\nabla} \times \left(\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u})\right) \\
&= -\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) \\
&= -\left[\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega}\right] \\
&= -(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega}.
\end{aligned}$$

Putting it all together gives us the vorticity equation, given by

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

Solutions to the Problems for Lecture 42

1. The electric field from a point charge at the origin should be spherically symmetric. We therefore write using spherical coordinates, $\boldsymbol{E}(\boldsymbol{r}) = E(r)\hat{\boldsymbol{r}}$. Integrating Gauss's law over a spherical shell of radius r, we have

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = E(r) \oint_S dS = 4\pi r^2 E(r) = \frac{q}{\varepsilon_0}.$$

Therefore, the electric field is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}.$$

2. The magnetic field from a current carrying infinite wire should have cylindrical symmetry. We therefore write using cylindrical coordinates, $\boldsymbol{B}(\boldsymbol{r}) = B(\rho)\hat{\boldsymbol{\phi}}$. Integrating Ampère's law over a circle of radius ρ in the x-y plane in the counterclockwise direction, we obtain

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = B(\rho) \oint_C dr = 2\pi\rho B(\rho) = \mu_0 I,$$

where I is the current in the wire. Therefore,

$$B(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}}.$$