**Homework 4**
MATH 541: Abstract Algebra 1
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Sec. 3.1: 2, 14, 22, 24, 36, 40, 41

# 2

Let $\varphi : G \to H$ be a homomorphism of groups with kernel $K$ and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the fiber above $a$ and let $Y$ be the fiber above $b$, i.e., $X = \varphi^{-1}(a), Y = \varphi^{-1}(b)$. Fix an element $u$ of $X$ (so $\varphi(u) = a$). Prove that if $XY = Z$ in the quotient group $G/K$ and $w$ is any member of $Z$, then there is some $v \in Y$ such that $uv = w$. [Show $u^{-1}w \in Y$]

*Proof.*

$$Z = XY = \phi^{-1}(a)\phi^{-1}(b) = \phi^{-1}(ab)$$

We know that $\phi$ is a group homomorphism, so $\phi(xy) = \phi(x)\phi(y)$.
Therefore, $\phi(u^{-1}w) = \phi(u^{-1})\phi(w) = a^{-1}ab = b$, which implies that $u^{-1}w \in Y$. □

**14:** Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$

- $a$ Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leqslant q < 1$.
- $b$ Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.
- $c$ Show that $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$ (cf. Exercise 6, Section 2.1).
- $d$ Show that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of root of unity in $\mathbb{C}^\times$.

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*Proof.* $a$

$$\forall q' \in \mathbb{Q} : q'\mathbb{Z} = \{q' + z : z \in \mathbb{Z}\}, \exists z' \in \mathbb{Z} : q' + z' \in [0, 1), q = q' + z' \implies q\mathbb{Z} = (q' + z')\mathbb{Z}$$

It is easy to see that $(q' + z')\mathbb{Z} = (q')\mathbb{Z}$, which means $q\mathbb{Z} = q'\mathbb{Z}$

$b$ Note $\mathbb{Q}/\mathbb{Z}$ is an equivalence classes based on all $q \in [0, 1)$.

$$\forall q \in \mathbb{Q}/\mathbb{Z}, \exists z_1, z_2 \in \mathbb{Z} : q = \frac{z_1}{z_2}$$

Therefore, the order is $\operatorname{lcm}(z_1, z_2)$, which can be arbitrarily large but finite.

$c$ It suffices to show $\mathbb{Q}/\mathbb{Z}$ is a subgroup $\mathbb{R}/\mathbb{Z}$ with $b$. We know that the sum of two rational number is a rational number. We also know that the inverse of a rational number is $-q$ which is equivalent to $1 - q$. 0 is clearly a rational number.

*d* We can just write out the isomorphism.

For all order $z \in \mathbb{Z}$, the root of unity contains exactly $z$ elements such that $z$ is the order of the element.

Therefore, we can just map all element from $\frac{\mathbb{Q}}{\mathbb{Z}}$ to the root of unity with the denominator as the order, and numerator as the index of the element, which is bijective, and vice versa.

$\square$

**22:**

*a* Prove that if $H$ and $K$ are normal subgroups of a group $G$ then their intersection $H \cap K$ is also a normal subgroup of $G$.

*b* Prove that the intersection of an arbitrary non-empty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

*Proof.* *a* Denote $H \cap K = L$

$$\forall l_1, l_2 \in L : l_1 l_2 \in H, l_1 l_2 \in K \implies l_1 l_2 \in L$$

$$\forall l \in L : l^{-1} \in H, l^{-1} \in K \implies l^{-1} \in L$$

$$\forall l \in L : glg^{-1} \in H, glg^{-1} \in K \implies glg^{-1} \in L$$

*b* There're no difference between the two argument, but just suggesting the element is in all normal subgroups instead of $H, K$.

$\square$

**24:** Prove that if $N \trianglelefteq G$ and $H$ is any subgroup of $G$ then $N \cap H \trianglelefteq H$.

*Proof.* By definition: $\forall g \in G : gN = Ng \implies \forall h \in H : hN = Nh$. Because $H$ is always normal under $H$. Then, follow last question, we can conclude that $N \cap H \trianglelefteq H$. $\square$

**36:** Prove that if $G/Z(G)$ is cyclic then $G$ is abelian. [If $G/Z(G)$ is cyclic with generator $xZ(G)$, show that every element of $G$ can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.]

*Proof.* By definition, $Z(G) = \{z \in G | zg = gz\}$.

If $G/Z(G)$ is cyclic, we can write it with one generator $xZ(G)$, which means we can write every element as $x^a Z(G)^a = x^a Z(G)$. By definition of quotient group equivalent class,

$$\forall g \in G : \exists a \in \mathbb{Z}, z \in Z(G) : g = x^a z$$

$$\forall g_1, g_2 \in G : g_1 g_2 = x^{a_1} z_1 x^{a_2} z_2 = z_1 x^{a_1} x^{a_2} z_2 = z_1 x^{a_1 + a_2} z_2 = z_1 z_2 x^{a_1 + a_2} = z_2 z_1 x^{a_2} x^{a_1} = z_2 x^{a_2} z_1 x^{a_1} = g_2 g_1$$

$\square$

**40:** Let $G$ be a group, let $N$ be a normal subgroup of $G$ and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$. (The element $x^{-1}y^{-1}xy$ is called the commutator of $x$ and $y$ and is denoted by $[x,y]$.)

*Proof.* $\Longleftarrow$

$$x^{-1}y^{-1}xy \in N \implies x \sim y^{-1}xy \implies yy^{-1}xy = yx = xy$$

$\Longrightarrow$
If $\overline{xy} = \overline{yx}$

$$\forall x, y \in \overline{xy} : xy = yx \implies x^{-1}y^{-1}xy = \mathbb{1} \implies x^{-1}y^{-1}xy \in N$$

□

**41:** Let $G$ be a group. Prove that $N = \langle x^{-1}y^{-1}xy | x, y \in G\rangle$ is a normal subgroup of $G$ and $G/N$ is abelian ($N$ is called the commutator subgroup of $G$)

*Proof.* Subgroup:

$$x_1^{-1}y_1^{-1}x_1y_1x_2^{-1}y_2^{-1}x_2y_2 = (x_1y_1)^{-1}(x_1y_1)^{-1^{-1}}((x_2y_2)^{-1})(x_2y_2)$$

Therefore, it is closed under multiplicative. Inverse is trivial.
Therefore, $N$ is a subgroup of $G$.
Normal:

$$\forall g \in G : \forall x, y \in G : gx^{-1}y^{-1}xyg^{-1} = (g^{-1}x)^{-1}y^{-1}x(yg^{-1}) \in N$$

G/N is abelian:
This is obvious given question 40. □