Homework 2
MATH 541: Abstract Algebra 1
Spring 2023

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Sec. 1.4: 10
Sec. 1.6: 14, 18, 24, 25(a)(b)
Sec. 1.7: 16, 17

1.4

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1. *Proof.*
$$\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}$$
Because $a_1, a_2 \neq 0$, so $a_1a_2 \neq 0$, same for c_1, c_2. Therefore, G is closed under matrix mul. □

2. *Proof.*
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -b/(ac) \\ 0 & 1/c \end{pmatrix}$$
□
Because $a, c \neq 0$, so all entries are well defined within $\mathbb{R}$, which means it is closed.

3. Because any matrix operation defined in $GL_2(\mathbb{R})$ is defined in G, and G is clearly closed under *addition* and *subtraction*, and G is a subset of $GL_2(\mathbb{R})$ where the left lower entry is 0, and $a, c \neq 0$, so G is a subgroup of $GL_2(\mathbb{R})$.

4. Follow the similar steps from above, it suffices to check whether the new set G' is closed under multiplication and inverse.

   Proof.
$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + a_2b_1 \\ 0 & a_1a_2 \end{pmatrix}$$
The left top entry and the right bottom entry are the same, which indicates that matrix multiplication is closed in G'.
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix}$$
which is also inside G', therefore G' is a subgroup of G.

   Therefore, G' is a subgroup of $GL_2(\mathbb{R})$. □

1.6

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Kernel is a subgroup

Proof. Denote operation on G as $\star_G$, mutatis mutandis for $\star_H$.

Consider two element $x, y \in kernel(H)$.

$$\phi(x \star_G y) = \phi(x) \star_H \phi(y) = \mathbb{1} \star_H \mathbb{1} = \mathbb{1}$$

Therefore $x \star_G y$ is also in $kernel(H)$.

$$\phi(x \star_G x^{-1}) = \phi(x) \star_H \phi(x^{-1}) = \mathbb{1}_H \star_H \phi(x^{-1}) = \mathbb{1}_H \implies \phi(x^{-1}) = \mathbb{1}_H$$

Therefore x^{-1} is also in $kernel(H)$.

Therefore, $kernel(H)$ is a subgroup of G. □

injective iff kernel is the identity subgroup of G

Proof. First prove that ϕ is injective if the kernel of ϕ is the identity subgroup of G.

$$kernel(H) = \mathbb{1}_G$$

Assume ϕ is not injective, i.e. there exists two element $a, b \in G$ that $\phi(a) = \phi(b)$ but $a \neq b$.

$$\phi(a \star a^{-1} \star b) = \phi(b) = \phi(a) = \phi(a) \star \phi(a^{-1} \star b) \implies \phi(a^{-1} \star b) = \mathbb{1}$$

However, we know that only $\phi(\mathbb{1}) = \mathbb{1}$, but $a \neq b$, so $a^{-1} \star b \neq \mathbb{1}$, which is a contradiction.

Then prove If ϕ is injective, then the kernel is the identity subgroup.

We know that the identity subgroup of G always map to the identity subgroup of H, so by injectivity, it is the only subgroup lies in the kernel.

□

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Proof. If G is abelian, then $\forall a, b \in G : a \star b = b \star a$.

Denote the map as ϕ

$$\phi(a \star b) = (a \star b) \star (a \star b) = a \star a \star b \star b = \phi(a) \star \phi(b)$$

If $\phi(a \star b) = \phi(a) \star \phi(b)$

$$(a \star b) \star (a \star b) = a \star a \star b \star b \implies b \star a = a \star b$$

, which means $\star$ is commutative under G.

□

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We can write G as

$$G = \{x, y, xy, yx, (xy)^2, x(yx)^2, ...\}$$

We can show that yx also have order n

$$(yx)^{n+1} = y(xy)^n x = yx \implies (yx)^n = \mathbb{1}$$

Therefore there's $n-1$ elements that is power of xy, $n-1$ elements that is with a x and a power of yx, and x, y, so the over all $|G| = 2n$.

We have proved in the last homework that D_{2n} can be generated by s and sr, which both have order 2.

Clearly G is not abelian, and D_{2n} is not abelian.

Therefore, if we construct a mapping from $G \to D_{2n}$ that maps x to s, and y to sr, based on the isomophism test, it is isomorphic.

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1. *Proof.*

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

It suffics to check the two basis.

For $(1, 0)$, after applying the matrix, it becomes $(\cos\theta, \sin\theta)$, which is true by definition.

For $(0, 1)$, after applying the matrix, it becomes $(-\sin\theta, \cos\theta)$, which is true by rotating the axis by 90 degree.

□

2. We know that $\theta = \frac{2\pi}{n}$ so

$$\phi(r^n) = \phi(r)^n = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^n = \mathbb{1}$$

$$\phi(s^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \mathbb{1}$$

$$\begin{aligned}
\phi(rs) &= \phi(r) \star \phi(s) \\
&= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} \\
&= \phi(sr^{-1})
\end{aligned}$$

1.7

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Proof. 1. $(gg) \cdot a = ggag^{-2} = g \cdot (gag^{-1}) = g \cdot (g \cdot a)$

2. $\mathbb{1} \cdot a = \mathbb{1}a\mathbb{1}^{-1} = a$

$\square$

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Proof. We can find the inverse of the mapping easily that is simply $x \mapsto g^{-1}xg$, which means it is bijective because it is a map from G to G.

Assume $x^n = \mathbb{1}$, so $(gxg^{-1})^n = gx^ng^{-1} = \mathbb{1}$. Also because $|x| = n$, so any power less than n is not identity.

We know that the mapping is a isomorphic mapping, so it is injective. Therefore, it is clear that $|A| = |gAg^{-1}|$.

$\square$