Homework 1
MATH 541: Abstract Algebra 1
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Section 1.1: 1, 7
Section 1.2: 3, 4, 18
Section 1.3: 1, 5

1.1.1

Clearly

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are satisfying the condtion.

Also, M itself must also satisfy the condition.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Therefore it is not in B.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = \begin{pmatrix} 1 & 1 \\ in1 & 0 \end{pmatrix}$$

$$M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M$$

Therefore, it is not in B.

1.1.7

Proof. To be an equivalence relation, we need three condtions.

$$\begin{cases} a \sim a \\ a \sim b \iff b \sim a \\ a \sim b \wedge b \sim c \implies a \sim c \end{cases}$$

It is very clear that $a \sim a$ because by the definitions of a function f, $f(a) = f(a)$.

If $f(a) = f(b)$ then $f(b) = f(a)$ so $a \sim b \implies b \sim a$.

$f(a) = f(b) \wedge f(b) = f(c) \implies f(a) = f(c)$, so $a \sim b \wedge b \sim c \implies a \sim c$.

We know that the fibers of element y are $\{x \in X : f(x) = y\}$. Therefore, it is very clear that if $a \sim b$, then a,b are in the fibers of $f(a)$. □

1.2.3

Proof. s^k has order 2 because $s^{2k} = (s^2)^k = 1^k = 1$. $sr^ksr^k = sr^{k-1}sr^{-1}r^k = sr^{k-1}sr^{k-1}$ so we can reduce to the case s^2 which is 1.

To prove that s, sr can generate D_{2n}, we need to show that r (s is trivial) can be written as combination of s and sr. This is trivial by compose s and sr which results in r. $\square$

1.2.4

Proof. If $n = 2k$ is even, k is an integer.

For all element that are generated as power of r, it is very clear that it is commute with r^k...

It is very clear that $(r^k)^2 = r^n = 1$. Therefore, $r^kr^{-k} = 1 = r^k \implies r^k = r^{-k}$.

$zs = r^ks = r^{k-1}sr^{-1} = sr^{-k} = sr^k = sz$.

Suppose an element $l = r^is^j$ that commute with all element in D_{2n}.

It suffics to check two thing $(rl = lr, sl = ls)$.

$$lr = rl \implies r^is^jr = rr^is^j \implies s^jr = rs^j$$

$$\begin{cases} j \mod 2 = 0 \implies r = r \\ j \mod 2 = 1 \implies sr = rs = sr^{-1} \implies r = r^{-1}(\text{contradiction}) \end{cases}$$

Therefore $j = 0$.

$$ls = sl \implies r^is^js = sr^is^j \implies r^is = sr^j \implies sr^{-i} = sr^i \implies r^{-i} = r^i \implies r^ir^i = r^ir^{-i} = 1$$

Therefore $i = \frac{n}{2}$ since we assume n is even.

$\square$

1.2.18

1. $v^3 = 1 \implies v^2v = 1 \implies v^2v = v^{-1}v \implies v^2 = v^{-1}$
2. $(v^2u^2)(uv) = (v^2u^2)(v^2u^2) = (uv)(v^2u^2) = uv^3u^2 = u^3$
   $v^2u^3v = u^3 = v^{-1}u^3v = u^3 \implies u^3v = vu^3$
3. $u^4 = 1 \implies u^8 = 1 \implies u^8u = u$. From before, $vu = vu^3u^3u^3 = u^3vu^3u^3 = u^3u^3u^3v = u^9v = uv$
4. $vu = uv = v^2u^2 \implies u = vu^2 \implies 1 = vu = uv$
5. $u^4v^3 = 1 = uvuvuvu = u \implies 1v = 1 \implies v = 1$

1.3.1

$$\sigma = (1\ 3\ 5)(2\ 4)$$

$$\tau = (1\ 5)(2\ 3)$$

$$\sigma^2 = (153)$$

$$\sigma\tau = (1)(2\ 5\ 3\ 4)$$

$$\tau\sigma = (1\ 2\ 4\ 3)(5)$$

$$\tau^2\sigma = \sigma = (1\ 3\ 5)(2\ 4)$$

1.3.5

Proof. The order should be the gcd of the count of elements in each cycle group.

$$\gcd(5, 2, 3, 2) = 30$$

$\square$