Math 541 Midterm Prime

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Midterm

1) Problem 1

1.a)

Normalizer of N is G

1.b)

We know that N is a normal subgroup of G. So we have

$$\forall g \in G : Ng = gN$$

Then for any $n' \in N \cap H$, we will have $\exists n'' \in N : n'h = hn''$ for all $h \in G$.

However we also know that $\forall n' \in H$, so $hn'' \in H$, which means $hn'' \in N \cap H$.

So $N \cap H \trianglelefteq H$.

2) Problem 2

2.a)

For every complex number $a + bi$, we can write it as $re^{i\theta}$ for some r, θ.

Let's define the map from $\mathbb{R} / \mathbb{Z} \to \mathbb{S}$

$$\overline{x} \mapsto e^{i2\pi x}$$

This is a homomorphic map from R / Z to $\mathbb{S}$ because

$$\varphi(\overline{x + y}) = e^{i2\pi(x+y)} = e^{i2\pi x} re^{i2\pi y} = \varphi(\overline{x}) + \varphi(\overline{y})$$

Also it is obvious that this is both injective and surjective

2.b)

$$\varphi^{-1} = \mathbb{Q} / \mathbb{Z}$$

3) Problem 3

Proof:

Because we are in the modular group, so we can can map n to the $n \bmod q$, and use the group operation to get the result.

By Lagurange theorem, for any element $n \bmod q$ in $(\mathbb{Z} / q\mathbb{Z})^{\times}$, its order divides out the order of the group, which is $p - 1$.

Therefore we will have $n^{q-1} \equiv \mathbb{1} \pmod{q}$, which means $n^q \equiv n \pmod{q}$.

$\square$

4) Problem 4

4.a)

for (gM, gN) to be the identity in $G / M \times G / N$, we need to have $gM = M, gN = N$, which means $g \in M \cap N$.

4.b)

By second isomoprhism theorem

$$\frac{MN}{N} \cong \frac{M}{M \cap N}$$

then by Lagurange theorem

$$|M||N| = |MN||M \cap N|$$

The original question by Lagurange theorem is

$$\frac{|G|}{|M \cap N|} = \frac{|G|}{|M|}\frac{|G|}{|N|} \text{ where } |G| = |MN|$$

By previous statement, $\frac{|MN|}{|M|} = \frac{|N|}{|M \cap N|} \Rightarrow$ the original question is correct by some subsitution.

4.c)

Because it is a natural morphism, and we have that the domain and codomain have the same size, which implies that the kernel is trivial, which means that the morphism is a isomoprhism.

5) Problem 5

Proof:

It suffices to show that both r and s have even order in D_{10}.

For r, the natural morphism from $D_{10} \to S_5$ will map $r \to (5, 4, 3, 2, 1)$, which obviously have even order.

For s, it is obvious that s can be composed by two length 2 cycle, which means that s has even order.

Therefore both r and s have even order, so $D_{10} \leq A_5$.

□

6) Problem 6

it suffices to show that $(1, 2)D(1, 2) \neq D$

Proof:

use r as an example.

$$(1, 2)r(1, 2) = (1, 2, 5, 4, 3)$$

which is obviously not in D_{10}, becauses for any composition with s, it will not be a cycle.

If it is composed by r^n, which is not possible because element wise there is different interval between two elements in $(1, 2, 5, 4, 3)$, which means that it is not in D_{10}.

□