Homework 5
MATH 541: Abstract Algebra 1
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Section 3.3: 3, 4, 7, 9, 10

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3: Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

1. $K \leqslant H$ or
2. $G = HK$ and $|K : K \cap H| = p$.

Solution: Assume that $K \nleq H$

By second isomorphism theorem,

$$\frac{K}{K \cap H} \cong \frac{HK}{H} \implies |K : K \cap H| = |HK : H|$$

If $K \leq H$,

$$|K : K \cap H| = |K : K| = 1 = |HK : H|$$

Otherwise,

$$\exists z > 1 \in Z^{+} : |HK : H| = z$$

$$|G : H| = p = |G : HK||HK : H|$$

Because $HK \leq G$ and $H \leq HK$, both upper side and lower side is integer.

We also know that $HK > H \impliedby K \nleq H$, so $|HK : H| \neq 1$, which $\implies |G : HK| = 1 \implies G = HK\ldots$

Then $|K : K \cap H| = |HK : H| = p$.

$\square$

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4: Let C be a normal subgroup of the group A and let D be a normal subgroup of B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Solution: Proof of finite cases.

It is easy to see that $(C \times D) \trianglelefteq (A \times B)$, by definition of normal subgroup.

$$\forall (a, b) \in (A \times B) : \forall c_1, d_1 \in C, D, \exists c_2, d_2 \in C, D : (c_1 a, d_1 b) \in (C \times D) = (c_2 a, d_2 b) \in (C \times D)$$

If A, B, C, D is finite, the second statement follows directly from the Lagrange theorem.
Proof of infinite cases.
It follows from definition of how we construct quotient group.
We can write element in $(A \times B)/(C \times D)$ as $(a, b)(C \times D)$.
Then we have
Denote element in A as a or a_i, and mutatis mutandis for B, C, D.
Denote element in A/C as $\overline{a}$ and mutatis mutandis for B/D, and $(A \times B)/(C \times D)$.
Claim: map

$$\phi(\overline{(a, b)}) = (\overline{a}, \overline{b})$$

is a bijection.
Proof of claim.
It is a bijection because it is a function from a set to itself.

$$\begin{aligned} &\forall (a, b) \in (A \times B) : \forall c_1, d_1 \in C, D, \exists c_2, d_2 : (c_1 a, d_1 b) \in (C \times D) = (c_2 a, d_2 b) \\ \implies &\forall (a, b) \in (A \times B) : \forall c_1, d_1 \in C, D, \exists c_2, d_2 : (c_1, d_1)(a, b) = (c_2, d_2)(a, b) \end{aligned}$$

$\square$

7: Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

Solution: It suffices to show that $(M \cap N)$ is the kernel of a morphism from $G \to (G/M) \times (G/N)$, by firstly send $g \in G$ to (gM, gN).

Then it is very clear that $M \cap N$ is the kernel of this map.

It is also clear that $M \cap N$ is the kernel of the morphism from G to $G/(M \cap N)$.

Because we know that both morphisms are surjective, we know that they are isomorphic.

$\square$

Theorem 1. *1.* $A \leq B \iff \overline{A} \leq \overline{B}$

2. $A \leq B \implies |B : A| = |\overline{B} : \overline{A}|$

3. $\overline{\langle A,\, B\rangle} = \langle \overline{A},\, \overline{B}\rangle$

4. $\overline{A \cap B} = \overline{A} \cap \overline{B}$

5. $A \trianglelefteq G \iff \overline{A} \trianglelefteq \overline{G}.$

9: Let p be a prime and let G be a group of order $p^a m$, where p does not divide m. Assume P is a subgroup of G of order p^a, and N is a normal subgroup of G of order $p^b n$, where p does not divide n. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$.

Solution: By second isomorphism theorem,

$$\frac{PN}{N} \cong \frac{P}{P \cap N} \implies \frac{(|PN|)}{(|N|)} = \frac{(|P|)}{(|P \cap N|)}$$

We also know that $P \cap N \leq N, P$
Therefore,

$$\exists z_1 \in Z^+ := \frac{(|N|)}{|(P \cap N)|}$$
$$\exists z_2 \in Z^+ := \frac{(|P|)}{|(P \cap N)|}$$

Therefore,

$$|N| = |P \cap N| \cdot z_1 \text{ and } |P| = |P \cap N| \cdot z_2$$
$$\implies \frac{|P|}{|P \cap N|} = \frac{|PN|}{z_1 |P \cap N|} \implies z_1 = |PN|/|P| = \frac{|PN|}{p^a}$$

We know that $PN \leqslant G$ so $|PN| \leqslant |G|$ and $|PN| \backslash |G|$.
Therefore,

$$\exists x \in Z^+ : |PN| = p^a x \implies z_1 = x$$

We also know that $N \leqslant PN \implies x \backslash n$. However, $|N| \backslash x \implies n \backslash x \implies n = x$.
Therefore, $|PN| = p^a n$.
Then, by second isomorphism theorem,

$$\frac{|PN|}{|N|} = \frac{|P|}{|P \cap N|} = p^{a-b} \implies |P \cap N| = p^b$$

□

10: Generalize the preceding exercise as follows.

A subgroup H of a finite group G is called a *Hall subgroup* of G if its index in G is relatively prime to the order of H (i.e. $\gcd(|G : H|, |H|) = 1$). Prove that if H is a Hall subgroup of G and N is a normal subgroup of G, then $H \cap N$ is a Hall subgroup of N, and HN/N is a Hall subgroup of G/N.

Solution: In the previous question, we only use the fact that p^a is relatively prime to m and n.

Therefore, it suffices to write the order of G as $x^a y$, where x and y are relatively prime, and $|H| = x^a$, and $|N| = x^b n$.

Then, we know that $|G : H| = y$.

By previous question we know that $|H \cap N| = x^b$, and $|HN/N| = x^{a-b}$.

Then it is clear that $H \cap N \leqslant N$ is a Hall subgroup of N.

Similarly, $|G/N| = x^{a-b}\frac{y}{n}$, which means that HN/N is a Hall subgroup of G/N because $\frac{y}{n}$ will also be relatively prime to x^{a-b}. $\square$