

Homework 5
MATH 541: Abstract Algebra 1
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Section 3.3: 3, 4, 7, 9, 10

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3: Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

1. $K \leq H$ or
2. $G = HK$ and $|K : K \cap H| = p$.

Solution: Assume that $K \not\leq H$

By second isomorphism theorem,

$$\frac{K}{K \cap H} \cong \frac{HK}{H} \implies |K : K \cap H| = |HK : H|$$

If $K \leq H$,

$$|K : K \cap H| = |K : K| = 1 = |HK : H|$$

Otherwise,

$$\exists z > 1 \in \mathbb{Z}^+ : |HK : H| = z$$

$$|G : H| = p = |G : HK| |HK : H|$$

Because $HK \leq G$ and $H \leq HK$, both upper side and lower side is integer.

We also know that $HK > H \iff K \not\leq H$, so $|HK : H| \neq 1$, which $\implies |G : HK| = 1 \implies G = HK \dots$

Then $|K : K \cap H| = |HK : H| = p$.

□

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4: Let C be a normal subgroup of the group A and let D be a normal subgroup of B . Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Solution: Proof of finite cases.

It is easy to see that $(C \times D) \trianglelefteq (A \times B)$, by definition of normal subgroup.

$$\forall (a, b) \in (A \times B) : \forall c_1, d_1 \in C, D, \exists c_2, d_2 \in C, D : (c_1 a, d_1 b) \in (C \times D) = (c_2 a, d_2 b) \in (C \times D)$$

If A, B, C, D is finite, the second statement follows directly from the Lagrange theorem.

Proof of infinite cases.

It follows from definition of how we construct quotient group.

We can write element in $(A \times B)/(C \times D)$ as $(a, b)(C \times D)$.

Then we have

Denote element in A as a or a_i , and mutatis mutandis for B, C, D .

Denote element in A/C as \bar{a} and mutatis mutandis for B/D , and $(A \times B)/(C \times D)$.

Claim: map

$$\phi(\overline{(a, b)}) = (\bar{a}, \bar{b})$$

is a bijection.

Proof of claim.

It is a bijection because it is a function from a set to itself.

$$\begin{aligned} & \forall (a, b) \in (A \times B) : \forall c_1, d_1 \in C, D, \exists c_2, d_2 : (c_1 a, d_1 b) \in (C \times D) = (c_2 a, d_2 b) \\ & \implies \forall (a, b) \in (A \times B) : \forall c_1, d_1 \in C, D, \exists c_2, d_2 : (c_1, d_1)(a, b) = (c_2, d_2)(a, b) \end{aligned}$$

□

7: Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

Solution: It suffices to show that $(M \cap N)$ is the kernel of a morphism from $G \rightarrow (G/M) \times (G/N)$, by send $g \in G$ to (gM, gN) , which is clear.

It is also clear that $M \cap N$ is the kernel of the morphism from G to $G/(M \cap N)$.

Because we know that both morphisms are surjective, we know that they are isomorphic.

□

9: Let p be a prime and let G be a group of order $p^a m$, where p does not divide m . Assume P is a subgroup of G of order p^a , and N is a normal subgroup of G of order $p^b n$, where p does not divide n . Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$.

Solution: By second isomorphism theorem,

$$\frac{PN}{N} \cong \frac{P}{P \cap N} \implies \frac{(|PN|)}{(|N|)} = \frac{(|P|)}{(|P \cap N|)}$$

We also know that $P \cap N \leq N, P$

Therefore,

$$\begin{aligned} \exists z_1 \in Z^+ &:= \frac{(|N|)}{(|P \cap N|)} \\ \exists z_2 \in Z^+ &:= \frac{(|P|)}{(|P \cap N|)} \end{aligned}$$

Therefore,

$$\begin{aligned} |N| &= |P \cap N| \cdot z_1 \text{ and } |P| = |P \cap N| \cdot z_2 \\ \implies \frac{|P|}{|P \cap N|} &= \frac{|PN|}{z_1 |P \cap N|} \implies z_1 = |PN|/|P| = \frac{|PN|}{p^a} \end{aligned}$$

We know that $PN \leq G$ so $|PN| \leq |G|$ and $|PN| \mid |G|$.

Therefore,

$$\exists x \in Z^+ : |PN| = p^a x \implies z_1 = x$$

We also know that $N \leq PN \implies x \mid n$. However, $|N| \mid x \implies n \mid x \implies n = x$.

Therefore, $|PN| = p^a n$.

Then, by second isomorphism theorem,

$$\frac{|PN|}{|N|} = \frac{|P|}{|P \cap N|} = p^{a-b} \implies |P \cap N| = p^b$$

□

10: Generalize the preceding exercise as follows.

A subgroup H of a finite group G is called a *Hall subgroup* of G if its index in G is relatively prime to the order of H (i.e. $\gcd(|G:H|, |H|) = 1$). Prove that if H is a Hall subgroup of G and N is a normal subgroup of G , then $H \cap N$ is a Hall subgroup of N , and HN/N is a Hall subgroup of G/N .

Solution: In the previous question, we only use the fact that p^a is relatively prime to m and n .

Therefore, it suffices to write the order of G as xy , where x and y are relatively prime, and $|H| = x$, and $|N| = an$ where $\gcd(a, n) = 1$ and $x \nmid a, y \nmid n$.

Then, we know that $|G:H| = y$.

By previous question we know that $|H \cap N| = \frac{x}{a}$, and $|HN/N| = \frac{x}{a}$.

Then it is clear that $H \cap N \leq N$ is a Hall subgroup of N .

Similarly, $|G/N| = \frac{x}{a} \frac{y}{n}$, which means that HN/N is a Hall subgroup of G/N because $\frac{y}{n}$ will also be relatively prime to $\frac{x}{a}$. \square