

Homework 4
MATH 541: Abstract Algebra 1
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Sec. 3.1: 2, 14, 22, 24, 36, 40, 41

2

Let $\varphi : G \rightarrow H$ be a homomorphism of groups with kernel K and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the fiber above a and let Y be the fiber above b , i.e., $X = \varphi^{-1}(a), Y = \varphi^{-1}(b)$. Fix an element u of X (so $\varphi(u) = a$). Prove that if $XY = Z$ in the quotient group G/K and w is any member of Z , then there is some $v \in Y$ such that $uv = w$. [Show $u^{-1}w \in Y$]

Proof.

$$Z = XY = \phi^{-1}(a)\phi^{-1}(b) = \phi^{-1}(ab)$$

We know that ϕ is a group homomorphism, so $\phi(xy) = \phi(x)\phi(y)$.

Therefore, $\phi(u^{-1}w) = \phi(u^{-1})\phi(w) = a^{-1}ab = b$, which implies that $u^{-1}w \in Y$. □

14: Consider the additive quotient group \mathbb{Q}/\mathbb{Z}

- a* Show that every coset of \mathbb{Z} in \mathbb{Q} contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.
- b* Show that every element of \mathbb{Q}/\mathbb{Z} has finite order but that there are elements of arbitrarily large order.
- c* Show that \mathbb{Q}/\mathbb{Z} is the torsion subgroup of \mathbb{R}/\mathbb{Z} (cf. Exercise 6, Section 2.1).
- d* Show that \mathbb{Q}/\mathbb{Z} is isomorphic to the multiplicative group of root of unity in \mathbb{C}^\times .

22

Proof. *a*

$$\forall q' \in \mathbb{Q} : q'\mathbb{Z} = \{q' + z : z \in \mathbb{Z}\}, \exists z' \in \mathbb{Z} : q' + z' \in [0, 1), q = q' + z' \implies q\mathbb{Z} = (q' + z')\mathbb{Z}$$

It is easy to see that $(q' + z')\mathbb{Z} = (q')\mathbb{Z}$, which means $q\mathbb{Z} = q'\mathbb{Z}$

- b* Note \mathbb{Q}/\mathbb{Z} is an equivalence classes based on all $q \in [0, 1)$.

$$\forall q \in \mathbb{Q}/\mathbb{Z}, \exists z_1, z_2 \in \mathbb{Z} : q = \frac{z_1}{z_2}$$

Therefore, the order is $\text{lcm}(z_1, z_2)$, which can be arbitrarily large but finite.

- c* It suffices to show \mathbb{Q}/\mathbb{Z} is a subgroup \mathbb{R}/\mathbb{Z} with *b*. We know that the sum of two rational number is a rational number. We also know that the inverse of a rational number is $-q$ which is equivalent to $1 - q$. 0 is clearly a rational number.

d We can just write out the isomorphism.

For all order $z \in \mathbb{Z}$, the root of unity contains exactly z elements such that z is the order of the element.

Therefore, we can just map all element from $\frac{\mathbb{Q}}{\mathbb{Z}}$ to the root of unity with the denominator as the order, and numerator as the index of the element, which is bijective, and vice versa. \square

24:

- a* Prove that if H and K are normal subgroups of a group G then their intersection $H \cap K$ is also a normal subgroup of G .
- b* Prove that the intersection of an arbitrary non-empty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

Proof. *a* Denote $H \cap K = L$

$$\forall l_1, l_2 \in L : l_1 l_2 \in H, l_1 l_2 \in K \implies l_1 l_2 \in L$$

$$\forall l \in L : l^{-1} \in H, l^{-1} \in K \implies l^{-1} \in L$$

$$\forall l \in L : glg^{-1} \in H, glg^{-1} \in K \implies glg^{-1} \in L$$

- b* There're no difference between the two argument, but just suggesting the element is in all normal subgroups instead of H, K . \square

36: Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

40: Prove that if $G/Z(G)$ is cyclic then G is abelian. [If $G/Z(G)$ is cyclic with generator $xZ(G)$, show that every element of G can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.]

41: Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that \overline{x} and \overline{y} commute in \overline{G} if and only if $x^{-1}y^{-1}xy \in N$. (The element $x^{-1}y^{-1}xy$ is called the commutator of x and y and is denoted by $[x, y]$.)

: Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy | x, y \in G \rangle$ is a normal subgroup of G and G/N is abelian (N is called the commutator subgroup of G)