Homework 3
MATH 541: Abstract Algebra 1
Spring 2023

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Sec. 0.1: 7
Sec. 0.2: 7
Sec. 0.3: 8
Sec. 1.7: 18, 19, 23
Sec. 2.1: 3

0.1.7

Proof. To be an equivalence relation, we need three condtions.

$$\begin{cases} a \sim a \\ a \sim b \iff b \sim a \\ a \sim b \wedge b \sim c \implies a \sim c \end{cases}$$

It is very clear that $a \sim a$ because by the definitions of a function f, $f(a) = f(a)$.
If $f(a) = f(b)$ then $f(b) = f(a)$ so $a \sim b \implies b \sim a$.
$f(a) = f(b) \wedge f(b) = f(c) \implies f(a) = f(c)$, so $a \sim b \wedge b \sim c \implies a \sim c$.
We know that the fibers of element y are $\{x \in X : f(x) = y\}$. Therefore, it is very clear that if $a \sim b$, then a,b are in the fibers of $f(a)$. □

0.2.7

Proof. Proof by contradiction

Assume there exists an a such that $a^2 = pb^2$, and write $a = \prod_n p_{an}, b = \prod_n p_{bn}$ then we know that $a^2 = (\prod_n p_{an})^2 = pb^2 = p(\prod_n p_{bn})^2$

Because we know that for every integer, there's an unique prime decomposition, so the power of left primes must match the power of right primes.

However, because we know that p is a prime, and all primes component from b will have even power, so the power of p must be odd, which mismatched the power of a's decomposition, which is a contradiction. □

0.3.8

0.3.6

Proof. the square of $\bar{0}^2 = \bar{0}$ Assume we have an element a in $\bar{1}$, write $a = (4b + 1)$ for some integer b.

$$a^2 = aa = (4b + 1)(4b + 1) = 16b + 4b + 4b + 1 \mod 4 = 1$$

For $\bar{2}$

$$(4b + 2)(4b + 2) = 16b + 8b + 8b + 4 \mod 4 = 0$$

For $\bar{3}$

$$(4b+3)(4b+3) \mod 4 = 9 \mod 4 = 1$$

$\square$

0.3.7

Proof. We know that $a^2, b^2 \mod 4 = 0$ or 1, therefore $a^2 + b^2 \mod 4 \leq 2$ $\square$

0.3.8

Proof. Proof by contradiction

Assume the solution exists.

We know that $a^2 + b^2 \mod 4 \neq 3$, so $c^2 \mod 4 \neq 1$, which means $c^2 \mod 4 = 0$

Also we know that $a^2 \mod 4 < 2$, and we know that $c^2 \mod 4 = 0$.

Therefore, $(a^2 + b^2) \mod 4 = 0$

Therefore $a^2 \mod 4 = b^2 \mod 4 = 0$.

$\square$