Homework 5
MATH 541: Abstract Algebra 1
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Section 3.3: 3, 4, 7, 9, 10

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3: Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

1. $K \leqslant H$ or
2. $G = HK$ and $|K : K \cap H| = p$.

Solution: By Lagrange theorem,

$$|G : H| = \frac{|G|}{|H|} = p$$

Therefore, we know that p is a factor of $|G|$.
By Lagrange theorem again,

$$K \leq G \implies \exists k \in \mathbb{Z} \text{ s.t. } |G| = k|K|$$

Therefore, we can have two cases:

1. $\exists z \in \mathbb{Z}^+ : zp = k$
2. $\exists z \in \mathbb{Z}^+ : zp = |K|$.

For 1,

$$zp|K| = |G| = p|H| \implies z|K| = |H| \implies \frac{|H|}{|K|} = z \in \mathbb{Z}^+$$

We then have $|G : K| = |G : H|z$, which means that

$\square$

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4: Let C be a normal subgroup of the group A and let D be a normal subgroup of B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.