Homework 2
MATH 541: Abstract Algebra 1
Spring 2023

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Sec. 1.4: 10
Sec. 1.6: 14, 18, 24, 25(a)(b)
Sec. 1.7: 16, 17

1.4

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1. *Proof.*
$$\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}$$
Because $a_1, a_2 \neq 0$, so $a_1a_2 \neq 0$, same for c_1, c_2. Therefore, G is closed under matrix mul. □

2. *Proof.*
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -b/(ac) \\ 0 & 1/c \end{pmatrix}$$
□

   Because $a, c \neq 0$, so all entries are well defined within $\mathbb{R}$, which means it is closed.

3. Because any matrix operation defined in $GL_2(\mathbb{R})$ is defined in G, and G is clearly closed under *addition* and *subtraction*, and G is a subset of $GL_2(\mathbb{R})$ where the left lower entry is 0, and $a, c \neq 0$, so G is a subgroup of $GL_2(\mathbb{R})$.

4. Follow the similar steps from above, it suffices to check whether the new set G' is closed under multiplication and inverse.

   Proof.
$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + a_2b_1 \\ 0 & a_1a_2 \end{pmatrix}$$
The left top entry and the right bottom entry are the same, which indicates that matrix multiplication is closed in G'.
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix}$$
which is also inside G', therefore G' is a subgroup of G.

   Therefore, G' is a subgroup of $GL_2(\mathbb{R})$. □

1.6

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Kernel is a subgroup

Proof. Denote operation on G as $\star_G$, mutatis mutandis for $\star_H$.

Consider two element $x, y \in kernel(H)$.

$$\phi(x \star_G y) = \phi(x) \star_H \phi(y) = \mathbb{1} \star_H \mathbb{1} = \mathbb{1}$$

Therefore $x \star_G y$ is also in $kernel(H)$.

$$\phi(x \star_G x^{-1}) = \phi(x) \star_H \phi(x^{-1}) = \mathbb{1}_H \star_H \phi(x^{-1}) = \mathbb{1}_H \implies \phi(x^{-1}) = \mathbb{1}_H$$

Therefore x^{-1} is also in $kernel(H)$.

Therefore, $kernel(H)$ is a subgroup of G. □

injective iff kernel is the identity subgroup of G

Proof. First prove that ϕ is injective if the kernel of ϕ is the identity subgroup of G.

$$kernel(H) = \mathbb{1}_G$$

Assume ϕ is not injective, i.e. there exists two element $a, b \in G$ that $\phi(a) = \phi(b)$ but $a \neq b$.

$$\phi(a \star a^{-1} \star b) = \phi(b) = \phi(a) = \phi(a) \star \phi(a^{-1} \star b) \implies \phi(a^{-1} \star b) = \mathbb{1}$$

However, we know that only $\phi(\mathbb{1}) = \mathbb{1}$, but $a \neq b$, so $a^{-1} \star b \neq \mathbb{1}$, which is a contradiction.

Then prove If ϕ is injective, then the kernel is the identity subgroup.

We know that the identity subgroup of G always map to the identity subgroup of H, so by injectivity, it is the only subgroup lies in the kernel.

□

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Proof. If G is abelian, then $\forall a, b \in G : a \star b = b \star a$.

□