

**Homework 2**  
MATH 541: Abstract Algebra 1  
Spring 2023

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Sec. 1.4: 10  
Sec. 1.6: 14, 18, 24, 25(a)(b)  
Sec. 1.7: 16, 17

## 1.4

10

1. *Proof.*

$$\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{pmatrix}$$

Because  $a_1, a_2 \neq 0$ , so  $a_1 a_2 \neq 0$ , same for  $c_1, c_2$ . Therefore,  $G$  is closed under matrix mul.  $\square$

2. *Proof.*

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -b/(ac) \\ 0 & 1/c \end{pmatrix}$$

$\square$

Because  $a, c \neq 0$ , so all entries are well defined within  $\mathbb{R}$ , which means it is closed.

3. Because any matrix operation defined in  $GL_2(\mathbb{R})$  is defined in  $G$ , and  $G$  is clearly closed under *addition* and *subtraction*, and  $G$  is a subset of  $GL_2(\mathbb{R})$  where the left lower entry is 0, and  $a, c \neq 0$ , so  $G$  is a subgroup of  $GL_2(\mathbb{R})$ .
4. Follow the similar steps from above, it suffices to check whether the new set  $G'$  is closed under multiplication and inverse.

*Proof.*

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + a_2 b_1 \\ 0 & a_1 a_2 \end{pmatrix}$$

The left top entry and the right bottom entry are the same, which indicates that matrix multiplication is closed in  $G'$ .

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix}$$

which is also inside  $G'$ , therefore  $G'$  is a subgroup of  $G$ .

Therefore,  $G'$  is a subgroup of  $GL_2(\mathbb{R})$ .  $\square$

## 1.6

14

### Kernel is a subgroup

*Proof.* Denote operation on  $G$  as  $\star_G$ , mutatis mutandis for  $\star_H$ .

Consider two element  $x, y \in \text{kernel}(H)$ .

$$\phi(x \star_G y) = \phi(x) \star_H \phi(y) = \mathbb{1} \star_H \mathbb{1} = \mathbb{1}$$

Therefore  $x \star_G y$  is also in  $\text{kernel}(H)$ .

$$\phi(x \star_G x^{-1}) = \phi(x) \star_H \phi(x^{-1}) = \mathbb{1}_H \star_H \phi(x^{-1}) = \mathbb{1}_H \implies \phi(x^{-1}) = \mathbb{1}_H$$

Therefore  $x^{-1}$  is also in  $\text{kernel}(H)$ .

Therefore,  $\text{kernel}(H)$  is a subgroup of  $G$ . □

### injective iff kernel is the identity subgroup of G

*Proof.* First prove that  $\phi$  is injective if the kernel of  $\phi$  is the identity subgroup of  $G$ .

$$\text{kernel}(H) = \mathbb{1}_G$$

Assume  $\phi$  is not injective, i.e. there exists two element  $a, b \in G$  that  $\phi(a) = \phi(b)$  but  $a \neq b$ .

$$\phi(a \star a^{-1} \star b) = \phi(b) = \phi(a) = \phi(a) \star \phi(a^{-1} \star b) \implies \phi(a^{-1} \star b) = \mathbb{1}$$

However, we know that only  $\phi(\mathbb{1}) = \mathbb{1}$ , but  $a \neq b$ , so  $a^{-1} \star b \neq \mathbb{1}$ , which is a contradiction.

Then prove If  $\phi$  is injective, then the kernel is the identity subgroup.

We know that the identity subgroup of  $G$  always map to the identity subgroup of  $H$ , so by injectivity, it is the only subgroup lies in the kernel. □