Math 542 HW2

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1 Q1

1.1 10.4.10

Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.

1.1.1

For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^{n} m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^{n} m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.

Proof: $N \cong R^n \Rightarrow M \otimes N \cong M \otimes R^n \cong \underbrace{(M \otimes R) \oplus \ldots \oplus (M \otimes R)}_{n \text{ times}}$

Because the construction is a free module, so every entries are linearly independent, so summing them is equivalent to the cartisian product.

Therefore, if $\sum_{i=1}^{n} m_i \otimes e_i = 0$, every entries must equal to 0, then either $m_i = 0$ or $e_i = 0$ for all i. However, we know that $e_i \neq 0$, then $m_i = 0$.

□

1.1.2

Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0.

Proof: Consider the $1 \otimes 2$ in the proposed Hint.

In $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}$, we have proved that $1 \otimes 2 = 0$ in last homework, but this is a counterexample for the claim above as $1 \neq 0 \in \mathbb{Z}/2\mathbb{Z}$.

□

1.2 10.4.16

Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are natrually R-modules.

1.2.1

Prove that every element of $\frac{R}{I} \otimes_R \frac{R}{J}$ can be written as simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

Proof: Assume we have $a \otimes b + c \otimes d \in \frac{R}{I} \otimes \frac{R}{J}$. By the property of tensor product, it must satisfy $a \otimes b + c \otimes d = a(1 \otimes b) + c(1 \otimes d) = (1 \otimes ab) + (1 \otimes cd) = 1 \otimes (ab + cd)$. This is doable because $\frac{R}{I}$ and $\frac{R}{J}$ are subring of R.

□

1.2.2

Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I+J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I+J)$.

Proof: To prove surjectivity, we already have every element of $\frac{R}{I} \otimes \frac{R}{J}$ can be written as $(1 \bmod I) \otimes (r \bmod J)$, and thus implies that $1r' \bmod (I+J)$ is everything in $R/(I+J)$.

To prove injectivity, we have the only element maps to 0 to be the element when $r = 0$ or $r' = 0$. However, each will mean they are 0 in the tensor product.

□

2 Short Exact Sequences

Let R be a commutative ring. A sequence of R-modules

$$0 \to N \xrightarrow{\iota} A \xrightarrow{p} Q \longrightarrow 0$$

is *short exact* if $\iota : N \to A$ is an injective R-module homomorphism whose image is the kernel of the surjective R-module homomorphism p. Let B be an R-module.

2.1

Show that $p \otimes \mathbb{1} : A \otimes B \to Q \otimes B$ is still surjective and that the image of $\iota \otimes \mathbb{1}$ is still its kernel.

Proof: We know that both p and $\mathbb{1}$ are surjective. Then we can prove that $p \otimes \mathbb{1}$ is surjective by the property of tensor product.

Consider the following diagram:

$$\begin{tikzcd}
A \oplus B \arrow[r, "f"] \arrow[d, "p \oplus \mathbb{1}"'] & A \otimes B \arrow[d, dashed, "p \otimes \mathbb{1}"'] \\
Q \oplus B \arrow[r, "\tilde{f}"'] & Q \otimes B
\end{tikzcd}$$

The kernel of $p \otimes \mathbb{1}$ will be also be the kernel of $p \oplus \mathbb{1}$ as only 0 be sent to 0 and the diagram commute. Because $p \oplus \mathbb{1}$ is surjective, and thus $p \otimes \mathbb{1}$ is also surjective.

With a similar reasoning, the image of ι, which is the kernel of p also lies in the kernel of $p \otimes \mathbb{1}$.

□

2.2

Now consider a short exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ and let $B = \mathbb{Z}/m$ where n and m are positive integers. Show that $\iota \otimes \mathbb{1}$ is injective if and only if $\gcd(m, n) = 1$.

Proof: Consider the following graph:

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z}/m \arrow[r, hook, "\iota \oplus \mathbb{1}"] & \mathbb{Z} \oplus \mathbb{Z}/m \arrow[r, "f"] \arrow[d, "p \oplus \mathbb{1}"'] & \mathbb{Z} \otimes \mathbb{Z}/m \cong \mathbb{Z}/m \arrow[d, dashed, "p \otimes \mathbb{1}"'] & \mathbb{Z} \otimes \mathbb{Z}/m \arrow[l, "\iota \otimes \mathbb{1}"] \\
& \mathbb{Z}/n \oplus \mathbb{Z}/m \arrow[r, "\tilde{f}"'] & \mathbb{Z}/n \otimes \mathbb{Z}/m &
\end{tikzcd}$$

If $\iota \oplus \mathbb{1}$ is injective, then the kernel of $p \oplus \mathbb{1}$ are the whole $\mathbb{Z} \oplus \mathbb{Z}/m$. Thus, the kernel of $p \otimes \mathbb{1}$ is also $\mathbb{Z} \otimes \mathbb{Z}/m$.

We have $A \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong A/nA \Rightarrow \mathbb{Z}/n \otimes \mathbb{Z}/m = \mathbb{Z}/(n(m\mathbb{Z}))$.

If $\gcd(m, n) = 1$, $\mathbb{Z}/(nm\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z} \otimes \mathbb{Z}/m$, thus $\mathbb{Z}/m \otimes \mathbb{Z}/n$ is the trivial group, and thus kernel of $p \otimes \mathbb{1}$ is the whole $\mathbb{Z}/m$, whcih means $\iota \otimes \mathbb{1}$ is injective.

On the other hand, $\iota \otimes \mathbb{1}$ is injective menas that $\mathbb{Z}/n \otimes \mathbb{Z}/m$ is the trivial group, and thus implies that $\gcd(m, n) = 1$.

□

3 Tensor Proudcts of Linear Maps

Suppose that $A : V_1 \to V_2$ and $B : W_1 \to W_2$ are k-linear maps between k-linear vectors spaces. Let $\operatorname{rank}(C)$ be the rank of a linear map C, i.e. the dimension of its image. Show that $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \operatorname{rank}(B)$.

Proof: This can be done by consider the basis of image of A and image of B. Then the image of $A \otimes B$ is the tensor product of the basis of the image of A and B, which will contains $\operatorname{rank}(A)\operatorname{rank}(B)$ elements.

□

4 Classifying Simple Modules

4.1

For each positive integer n, find all simple $\mathbb{R}[\mathbb{Z}/n\mathbb{Z}]$-modules up to isomorphism and their endomorphism algebras. (Hint: Each simple module will be at most 2-dimensional; think about rotations)

Proof: We know that rotation matrix is simple under $\mathbb{R}[\mathbb{Z}/n\mathbb{Z}]$. Then for $\mathbb{Z}/n\mathbb{Z}$, we just consider the rotation matrix with angle $\frac{2\pi}{n}$.

Because we have the theorem $|\mathbb{Z}/n\mathbb{Z}| = \sum \frac{\dim(S_i)^2}{\operatorname{End}(S_i)}$. Because we have $\dim(S_i) = 2$ for all i, then the dimension of the endomorphism rings must be 4.

□

4.2

Recall that $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$ whose elements can be written as $a + bi + cj + dk$ for some real numbers a, b, c, d where $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ji = -k, ik = -j, kj = -i$. Let $Q_8 \subset \mathbb{H}$ be the group of order 8 which is generated by i, j. Its elements are $\{\pm 1, \pm i, \pm j, \pm k\}$.

4.2.1

Show that there are four non-isomorphic one-dimensional $\mathbb{R}[Q_8]$-modules. (Note that $Q_8/[Q_8, Q_8] \cong \mathbb{Z}/2 \times \mathbb{Z}/2$).

Proof: An one dimensional $\mathbb{R}[Q_8]$ module are in the following form:

$$\{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

For one dimensional module, we are essentially consider the homomorphism between $Q_8 \to \mathbb{R}^\times$, but $\mathbb{R}^\times$ is commutative, which means it suffices to consider $Q_8/[Q_8, Q_8] \to \mathbb{R}^\times$, which is essentially $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{R}^\times$, which is $\operatorname{Hom}(\mathbb{Z}/2, \mathbb{R}^\times)^2$, which has size 4.

□

4.2.2

Show that $\mathrm{End}_{\mathbb{R}[Q_8]}(\mathbb{H})$ is isomorphic to $\mathbb{H}$ as a ring.

Proof: We can see that $\mathbb{R}[Q_8]$ is isomorphic to $\mathbb{H}$ as a ring, and thus $\mathrm{End}_{\mathbb{R}[Q_8]}(\mathbb{H}) \cong \mathbb{H}$.

□

4.2.3

Use previous one to conclude, up to isomorphism, $\mathbb{H}$ is the unique simple $\mathbb{R}[Q_8]$-module that is not one-dimensional.

Proof: Since we have 4 one dimensional simple module and one 4 dimensional simple module, then we have $|Q_8| = \sum \frac{\dim(S_i)^2}{\mathrm{End}(S_i)} = 4 + 4 = 8$.

Then we know $\mathbb{H}$ is the unique simple $\mathbb{R}[Q_8]$-module that is not one dimensional.

□

4.3

Show that there is a unique simple $\mathbb{C}[Q_8]$-module that is not one-dimensional. (Hint: It is two-dimensional, but you don't have to construct it. You can deduce its existence from fact we have shown and the four one-dimensional $\mathbb{C}[Q_8]$-modules that you found in the previous part.)

Proof: We know that $|Q_8| = 8$, and we have four one dimensional simple module, so the last one must be two dimensional by the formula.

□

5 Maschke

Let k be a field. Let V be a finite-dimensional k-vector space. Suppose that V is a $k[G]$-module where G is a finite group.

5.1

A linear map $\pi : V \to V$ is called a *projection onto* W if its image is W and if $\pi(w) = w$ for all $w \in W$. Show that, given such a map, V is isomorphic (as a vector space) to $\ker(\pi) \oplus W$.

Proof: By rank-nullity theorem, we have $\ker(\pi) = |V| - |W|$. Then we can see that $V = \ker(\pi) \oplus W$ as long as $\ker(\pi)$ and W are disjoint except $\{0\}$. However this is clear given $\pi(w) = w$.

□

5.2

A linear map $\pi : V \to V$ is called G-equivariant if $\pi(g \cdot v) = g \cdot \pi(v)$ for all $g \in G$. Show that if π is G-equivariant then its kernel and image are $k[G]$-submodules of V.

Proof: We know that π is G-equivariant, then $\forall g \in G : \pi(g \cdot v) = g \cdot \pi(v)$. Thus it means that $\pi \in \mathrm{Hom}_{k[G]}(V, V)$, which suggests that its kernel and image are $k[G]$-submodules of V.

□

5.3

Suppose that $\pi : V \to V$ is a projection onto a submodule W. Suppose $|G|$ is invertible in k (this could fail for instance if G has even order and $k = \mathbb{Z}/2$ since 2 is not invertible in k as it coincides with 0). Define a new function $p : V \to V$ by $p(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v)$. Show that p is a G-equaiariant projection onto W. Conclude that V is isomorphic to $W \oplus \ker(\pi)$ as a $k[G]$-module.

Proof:

$$p(g'w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}g'w) = \frac{1}{|G|} \sum_{g \in G} g'g'^{-1}g(g^{-1}g'w) = g' \sum_{g'' \in G} g''\pi\left(g''^{-1}w\right) = g'\pi(w)$$

Then p is G-equivarant projection, then W and $\ker(\pi)$ are all $k[G]$-submodule, which implies that V is isomorphic to $W \oplus \ker(\pi)$ as a $k[G]$-module.

□

5.4

Conclude that V is a direct sum of simple $k[G]$-modules.

Proof: We can make W a simple $k[G]$-module, and W is a direct sum of a simple module and its kernel. Then we do the same thing to the kernel. Then we can conclude that V is a direct sum of simple $k[G]$-modules.

□