HOMEWORK 2: DUE FRIDAY SEPTEMBER 22

Problem 1 (Chinese Remainder; 10 points): Dummit and Foote Chapter 10.3 Problems 16 and 17.

Problem 2 (Fractions; 10 points): Suppose that R is an integral domain and let M be an R-module. Let S be a multiplicatively closed subset of R that includes 1 and does not include zero (for instance, complements of prime ideals). Let $S^{-1}M$ be the collection of symbols of the form $\frac{m}{s}$ where $m \in M$ and $s \in S$ and where we insist that $\frac{s' \cdot m}{s's} = \frac{m}{s}$ for any $s' \in S$. This is an abelian group where we define addition by $\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 \cdot m_1 + s_1 \cdot m_2}{s_1 s_2}$ for $m_1, m_2 \in M$ and $s_1, s_2 \in S$. Note that $S^{-1}R$ is a ring if we additionally define multiplication by $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}$ for $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Finally, we note that $S^{-1}M$ is an $S^{-1}R$-module where $\frac{r}{s_1} \cdot \frac{m}{s_2} := \frac{r \cdot m}{s_1 s_2}$.

(1) Show that if $f : M_1 \longrightarrow M_2$ is a homomorphism of R-modules, then the map $S^{-1}f : S^{-1}M_1 \longrightarrow S^{-1}M_2$ sending $\frac{m}{s}$ to $\frac{f(m)}{s}$ is a homomorphism of $S^{-1}R$ modules.

(2) If $S = R - \{0\}$, then note that $S^{-1}R$ is a field. Use this to show that R^n and R^m are not isomorphic if n and m are distinct positive integers.

(3) Let $S = R - \{0\}$ and consider the map $M \longrightarrow (S^{-1}R) \otimes M$ that sends m to $1 \otimes m$. Show that its kernel is the torsion submodule of M.

(4) Show that any linearly independent subset of R^n can be extended to a linearly independent subset of size n. (The bonus problem shows that this result is not true when R is not an integral domain).

Problem 3 (Tensors; 10 points): Do Dummit and Foote Chapter 10.4 Problems 2 and 20.

Problem 4 (Duality; 10 points): Suppose that R is commutative. Let M, N, and U be R-modules. The *dual module of M* is defined to be $M^* := \mathrm{Hom}_R(M, R)$.

(1) Suppose that $(e_1, \ldots, e_n)$ is a basis, i.e. linearly independent spanning set, for M. Define $e_i^* \in M^*$ to be the homomorphism

that sends e_i to 1 and all other e_j ($j \neq i$) to 0. Show that $(e_1^*, \dots, e_n^*)$ is a basis for M^*.

(2) Show that if M is a free R-module of rank n, where n is a positive integer, then $(M^*)^*$ is isomorphic to M. (Hint: Consider the map $M \longrightarrow (M^*)^*$ that sends $m \in M$ to ev_m where $\mathrm{ev}_m : M^* \longrightarrow R$ sends a homomorphism $\phi : M \longrightarrow R$ to $\phi(m)$. To establish that this map is a surjection show that, in the notation of the preceding part, $((e_i)^*)^* = \mathrm{ev}_{e_i}$.)

(3) Show that if R is a field and M and N are finitely generated, then $\mathrm{Hom}_R(M, N) \cong M^* \otimes N$ as R-modules. Show that there is not necessarily such an isomorphism when $R = \mathbb{Z}$ and M and N are finitely generated $\mathbb{Z}$-modules.

Problem 5 (Counterexamples; 10 points): Do **one** of the following two problems: Dummit and Foote Chapter 10.3 Problem 24 (Direct products of free modules need not be free) or Problem 26 (Rank need not be defined for free modules over non-commutative rings).

Bonus Problem 1 (Due to David Speyer; 10 points): This problem will produce a counterexample to the following claim: if R is a commutative ring and n is a positive integer, then any linearly independent subset of R^n can be extended to a linearly independent set of size n. We will build a counterexample with $n = 2$. Let R be the ring of function $f : \mathbb{C}^2 - \{(0,0)\} \longrightarrow \mathbb{C}$ with the property that f agrees with an element of $\mathbb{C}[x, y]$ for all but finitely many values. Consider the R-submodule M of R^2 of elements of the form (rx, ry) where $r \in R$. Prove that M is free and prove that there is no element $(u, v) \in R^2$ so that $\{(x, y), (u, v)\}$ is linearly independent. (Hint: Consider the function $xv - uy$ and show that it vanishes at some point (a, b). (You may use the fact that if an element of $\mathbb{C}[x, y]$ is not a constant, then its zero set contains infinitely many points.) Now find nonzero functions r_1 and r_2 so that $r_1(x, y) + r_2(u, v) = (0, 0)$. You should choose functions r_1 and r_2 that are zero everywhere except for (a, b).).

Honors Problem 1 (Localization and Valuation; 10 points) Suppose that R is an integral domain. R is called *local* if it has a unique maximal ideal $\mathfrak{m}$.

(1) If $\mathfrak{p}$ is a prime ideal in R, then define $S := R - \mathfrak{p}$ and $R_\mathfrak{p} := S^{-1}R$. Show that the set of elements of the form $\frac{a}{s}$ where $s \in S$ and $a \in \mathfrak{p}$ is the unique maximal ideal of $R_\mathfrak{p}$ and hence this ring is local. (For a concrete example, if $R = \mathbb{Z}$ and $\mathfrak{p} = (2)$, then $R_\mathfrak{p}$ is the set of rational numbers with odd denominator.)

(2) A *valuation* on a field K is a surjection $v : K - \{0\} \longrightarrow \mathbb{Z}$ so that $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min(v(x), v(y))$. It is convenient to set $v(0) := +\infty$. The *valuation ring* is then defined to be $\{x \in K : v(x) \geq 0\}$. We will construct an example with $K = \mathbb{Q}$. Since any nonzero rational number can be written as $2^k \frac{a}{b}$ with a and b odd integers, let $v : \mathbb{Q} \longrightarrow \mathbb{Z}$ be the function that sends such a number to k. Show that v is a valuation and that the valuation ring is $\mathbb{Z}_{(2)}$.

(3) Let K be a field and $v : K \longrightarrow \mathbb{Z}$ a valuation. Show that the valuation ring is a local ring whose unique maximal ideal is $\{x \in K : v(x) > 0\}$. Use this to prove every ideal in the valuation ring has the form $\{x \in K : v(x) \geq k\}$ for some integer k. Conclude that the valuation ring is a PID and that for any element $x_0 \in K$ so that $v(x_0) = 1$ every ideal has the form (x_0^k) for some nonnegative integer k.