Math 542 HW2

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1 Chinese Remainder

1.1 10.3.16

For any left ideal I of R define

$$IM = \left\{ \sum_{\text{finite}} a_i m_i \mid a_i \in I, m_i \in M \right\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. This is a submodule of M.

For any ideal I of R let IM be the submodule defined above. Let $A_1, ..., A_k$ be any ideals in the ring R. Prove that the map

$$\varphi : M \to \frac{M}{A_1 M} \times ... \times \frac{M}{A_k M} \text{ defined by } m \mapsto (m + A_1 M, ..., m + A_k M)$$

is an R-module homomorphism with kernel $A_1 M \cap A_2 M \cap ... \cap A_k M$.

Proof: Want to check $\forall x, y \in M : \varphi(x+y) = \varphi(x) + \varphi(y)$ and
$\forall x \in M, r \in R : \varphi(rx) = r\varphi(x)$.

$$\begin{aligned}\forall x, y \in M : \varphi(x+y) &= (x + y + A_1 M, ..., x + y + A_k M) \\ &= (x + A_1 M, ..., x + A_k M) + (y + A_1 M, ..., y + A_k M) \\ &= \varphi(x) + \varphi(y)\end{aligned}$$

$\forall r \in R, x \in M : \varphi(rx) = (rxA_1 M, ..., rxA_k M) = r(xA_1 M, ..., xA_k M)$ because submodule is invariant under the ring R.

To become the kernel, it need to satisfy that $\forall i \in [1, k] : x + A_i M = A_i M$, which implies that $x \in \bigcap_i A_i M$.

□

1.2 10.3.17

In the notation of the Section 1.1, assume further that the ideals $A_1, ..., A_k$ are pairwise comaximal $\left(i.e. \forall i \neq j : A_i + A_j = R\right)$. Prove that

$$\frac{M}{(A_1, ..., A_k)M} \cong \frac{M}{A_1 M} \times ... \times \frac{M}{A_k M}$$

[See proof of the Chinese Remainder Theorem for rings in Section 7.6.]

Proof: Based on the proof of Chinese Remainder Theorem for rings in Section 7.6, it suffices to check the case

□