HOMEWORK 1: DUE FRIDAY SEPTEMBER 15

Problem 1: (Isomorphism; 10 points) Choose two of the four isomorphism theorems for modules (Theorem 4 in Chapter 10.2) and prove them. (This can be done pretty quickly by using the corresponding isomorphism theorems for groups.)

Problem 2: (Torsion; 10 points) First, do Dummit and Foote Chapter 10.1 Problem 8. Then find the torsion submodules for the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}/6$ and for any finite-dimensional $k[x]$-module where k is a field.

Problem 3: (Simplicity 1; 10 points) A module is called *simple* if its only submodules are $\{0\}$ and itself.

(1) Show that the only simple finite-dimensional $\mathbb{C}[x]$-modules are one-dimensional. (Hint: Use the fact that any $n \times n$ matrix with entries in $\mathbb{C}$ has an eigenvector.)
(2) Let $M := \mathbb{C}^2$ be a $\mathbb{C}[x]$-module where the action of x is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find all submodules of M.

Problem 4: (Simplicity 2; 10 points) Let R be a ring and M an R-module.

(1) Suppose that $f : M \longrightarrow M$ is an R-module homomorphism. Show that f is a bijection if and only if there is an R-module homomorphism $g : M \longrightarrow M$ so that $f \circ g = g \circ f = \mathrm{id}$. (You may use that, if S_1 and S_2 are sets, a map $f : S_1 \longrightarrow S_2$ is a bijection if and only if there is a map $g : S_2 \longrightarrow S_1$ so that $f \circ g = \mathrm{id}_{S_2}$ and $g \circ f = \mathrm{id}_{S_1}$.)
(2) Show that if M is a simple R-module, then $\mathrm{End}_R(M)$ is a division ring, i.e. for every nonzero R-module homomorphism $f : M \longrightarrow M$ there is another R-module homomorphism $g : M \longrightarrow M$ so that $f \circ g = g \circ f = \mathrm{id}$. (Hint: Using the first part, you just need to show that f is a surjection and an injection. Relate this to $\mathrm{im}(f)$ and $\ker(f)$.)
(3) Find an example of a simple two-dimensional $R := \mathbb{R}[x]$-module M. Produce a ring isomorphism from $\mathrm{End}_R(M)$ to $\mathbb{C}$.

Problem 5: (Annihilation; 10 points) Dummit and Foote Chapter 10.2 Problems 4 and 5.

Bonus Problem: (Opposites; 10 points) Given a ring R, the opposite ring R^{op} is the ring with all the same elements, where addition is defined identically, but for which $x \cdot^{op} y := y \cdot x$ where $\cdot$ is multiplication in R and $\cdot^{op}$ is multiplication in R^{op}. Take R as a left R-module and show that $\mathrm{Hom}_{R-Mod}(R, R)$ is isomorphic to R^{op} as a ring. (Hint: Consider the map that sends an element r of R^{op} to the function $\phi : R \longrightarrow R$ given by $\phi(x) = xr$.) Show that if $R = \mathrm{Mat}_{n\times n}(k)$ is the ring of $n \times n$ matrices with entries in a field k, then $R^{op} \cong R$ where the isomorphism is given by sending a matrix to its transpose.

Honors Problem: Let D be a division algebra over $\mathbb{R}$, i.e. D is a ring that contains $\mathbb{R}$ in its center and so that every nonzero element has a multiplicative inverse. Suppose that D is not isomorphic to $\mathbb{R}$.

(1) Let F be a maximal subset of commuting (under multiplication) elements of D. Show that F is isomorphic as a ring to $\mathbb{C}$. Use the fundamental theorem of algebra, which says that the only fields containing $\mathbb{R}$ are $\mathbb{R}$ itself and $\mathbb{C}$.

(2) Consider D as an F-vector space with scalar multiplication given by multiplication on the *left*. Let d be any element of F so that $d^2 = -1$. Let $T : D \longrightarrow D$ be the linear map given by $T(x) = xd$. Show that the only eigenvalues of T are $\pm i$. If D^+ (resp. D^-) is the $+i$ (resp. $-i$) eigenspace, then show that $D = D^+ \oplus D^-$ as an F-vector space. (Hint: Use that for $x \in D$, $x = \frac{x+dxd}{2} + \frac{x-dxd}{2}$.)

(3) Show that $D^+ = F$.

(4) Show that if D^- contains a nonzero element e then, letting $S : D \longrightarrow D$ be given by $S(x) = xe$, S produces a bijection from D^- to D^+. Show, moreover, that, up to scaling by an element of the copy of $\mathbb{R}$ in the center, we may choose e so that $e^2 = -1$. (Hint: Note that $\mathbb{R} + \mathbb{R} \cdot e$ is isomorphic to $\mathbb{C}$ as in part 1.)

(5) Conclude that D is isomorphic as a ring to $\mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ is the algebra of quaternions.