Math 542 HW4

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1 Simple wedges

Let V be a finite dimensional k-vector space where k is a field.

1.1

Say that $\alpha \in \Lambda^n V$ is *divisible* by $v \in V$ if there is some $\beta \in \Lambda^{n-1} V$ so that $\alpha = \beta \wedge v$. Show that α is divisible by $v \in V$ if and only if $\alpha \wedge v = 0$. Conclude that the set of vectors $v \in V$ that divide α is a subspace.

Solution 1.1.1: If α is divisible by $v \in V$, then exists β such that $\alpha = \beta \wedge v$. Thus $\alpha \wedge v = \beta \wedge v \wedge v = 0$

If $\alpha \wedge v = 0$,

We can write α as linear combination of basis in $\Lambda^n V$, where $\alpha \wedge v = 0$ implies $v \wedge e_i = 0$ for all e_i. Since all basis are simple, we can find some β such that $e_i = v \wedge \beta$.

$$e_i \wedge v = 0 \Leftrightarrow (a_1 e_1 \wedge \dots \wedge a_n e_n) \wedge (b_1 e_1 + \dots + b_k e_k) = 0$$
$$\Leftrightarrow a_1 e_1 \wedge \dots \wedge a_n e_n \wedge b_1 e_1 + \dots + a_1 e_1 \wedge \dots \wedge a_n e_n \wedge b_k e_k = 0$$

Since $e_i \wedge e_i = 0$

$$a_1 e_1 \wedge \dots \wedge a_n e_n \wedge b_1 e_1 + \dots + a_1 e_1 \wedge \dots \wedge a_n e_n \wedge b_k e_k = 0$$
$$\Leftrightarrow a_1 e_1 \wedge \dots \wedge a_n e_n \wedge b_{n+1} e_{n+1} + \dots + a_1 e_1 \wedge \dots \wedge a_n e_n \wedge b_k e_k = 0$$

Because these wedges are linearly independent, we have $a_1 a_2 \dots a_n b(n+1), a_1 a_2 \dots a_n b_k = 0$. Since $a_1 a_2 \dots a_n$ is not zero, we have $b_{n+1} = \dots = b_k = 0$. Thus $\alpha = \beta \wedge v$ where $\beta_i = \frac{1}{\beta_{n-1}} a_1 e_1 \wedge \dots \wedge a_{n-1} e_{n-1}$. By taking linear combination of β_i we will get the desired β.

The conclusion follows if we consider the map φ_α, where the set divide α is its kernel. This map is automatically linear and thus its kernel must be a subspace.

1.2

Given nonzero $\alpha \in \Lambda^n V$ consider the map $\varphi_\alpha : V \to \Lambda^{n+1} V$ where $\varphi_{\alpha(v)} = v \wedge \alpha$. The element α is called simple if there are vectors $v_1, \dots, v_n \in V$ so that $\alpha = v_1 \wedge \dots \wedge v_n$. Show that α is simple if and only if the kernel φ_α has dimension n.

Solution 1.2.1: If α is simple, then the argument in the last question shows that the kernel of φ_α has dimension n.

If the kernel of φ_α has dimension n, then there exists $v_1, ..., v_n$ such that $v_1 \wedge ... \wedge v_n$ is in the kernel of φ_α. Consider the kernel of φ_α.

$$v \wedge v_1 \wedge ... \wedge v_n = 0$$

From the previous quesiton we can see that this implies v is divisible by $v_1 \wedge ... \wedge v_n$. Thus exists some $\beta \in \Lambda^{n-1} V$ such that $\alpha = \beta \wedge v$.

1.3

Let $d \coloneqq \dim V$. Show that every element of $\Lambda^{d-1} V$ is simple.

Solution 1.3.1: Because we only have one element in $\Lambda^d V$, which means that the homomorphism φ_α must has kernel of dimension $d - 1$. Thus α is simple.

2 Plucker

2.1

Proof: If a is simple, then $a = e_i \wedge e_j \Rightarrow e_i \wedge e_j \wedge e_i \wedge e_j = 0$.

Assume $\alpha \wedge \alpha = 0$, we will have $\alpha = \sum c_{ij} e_i \wedge e_j$.

Then we have $\alpha \wedge \alpha = \sum c_{ij} c_{kl} e_i \wedge e_j \wedge e_k \wedge e_l = 0$.

By the expansion in part 2, we have $c_{ij} c_{kl} - c_{ik} c_{jl} + c_{il} c_{jk} = 0$.

Consider a simple vector $v_1 \wedge v_2 = \sum a_{ij} e_i \wedge \sum b_{ij} e_i = \sum_{1 \leq i < j \leq d} (a_i b_j - b_i a_j) e_i \wedge e_j$.

Matching c_{ij} with $a_i b_j - b_i a_j$, we have $c_{ij} = a_i b_j - b_i a_j$.

Since $(a_i b_j - a_j b_i)(a_k b_l - a_l b_k) - (a_i b_k - b_i a_k)(a_j b_l - a_l b_j) + (a_i b_l - a_l b_i)(a_j b_k - a_k b_j) = 0$, and $c_{ij} c_{kl} - c_{ik} c_{jl} + c_{il} c_{jk} = 0$, we are able to find v_1, v_2 such that $\alpha = v_1 \wedge v_2$. □

2.2

Proof: α is simple if and only if $\alpha \wedge \alpha = 0$.

We have

$$\left(\sum_{1 \leq i < j < d} c_{ij} e_i \wedge e_j \right) \wedge \left(\sum_{1 \leq i < j < d} c_{ij} e_i \wedge e_j \right) = 0$$

Then we consider the coefficient of $e_i \wedge e_j \wedge e_k \wedge e_l$.

We will have $c_{ij} c_{kl} - c_{ik} c_{jl} + c_{il} c_{jk} = 0$. Since the basis are linearly independent, we need the coefficient to be 0 to make $\alpha \wedge \alpha = 0$.

The converse follows the same logic. □

2.3

Proof :

$$\begin{aligned}
v_1 \wedge v_2 &= (v_{11}e_1 + v_{12}e_2 + \ldots + v_{1d}e_d) \wedge (v_{21}e_1 + v_{22}e_2 + \ldots + v_{2d}e_d) \\
&= (v_{11}v_{22} - v_{21}v_{12})e_1 \wedge e_2 + (v_{11}v_{23} - v_{13}v_{32})e_1 \wedge e_3 + \ldots + (v_{1,d-1}v_{2,d} - v_{2,d-1}v_{1d})e_{d-1} \wedge e_d \\
&= \sum_{i<j} A_{ij} e_i \wedge e_j
\end{aligned}$$

As $v_1 \wedge v_2$ is simple, the claim follows. □

3 Dicyclic groups

3.1

Solution 3.1.1: $a^{2n} = e^{\frac{2\pi i}{2n}} = \mathbb{1}$

$j^4 = (-1)^4 = \mathbb{1}$

$a^n j^{-2} = e^{\pi i} j^{-2} = -1 \cdot j^{-2} = -1 \cdot j^2 = -1 \cdot -1 = 1$

$$\begin{aligned}
j^{-1}aja &= j^3\left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)i\right)j\left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)i\right) \\
&= \left(j^3\cos\left(\frac{\pi}{n}\right) - j^2 ij \sin\left(\frac{\pi}{n}\right)\right)\left(j\cos\left(\frac{\pi}{n}\right) - ij\sin\left(\frac{\pi}{n}\right)\right) \\
&= \left(\cos\left(\frac{\pi}{n}\right)^2 - j^2ij^2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) - j^3ij\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right) + j^2ijij\sin\left(\frac{\pi}{n}\right)^2\right) \\
&= \cos\left(\frac{\pi}{n}\right)^2 - \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)(j^2ij^2 + j^3ij) + \sin\left(\frac{\pi}{n}\right)^2 j^2ijij \\
&= \cos\left(\frac{\pi}{n}\right)^2 - \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)(i - i) + \sin\left(\frac{\pi}{n}\right)^2 j^2ijij \\
&= \cos\left(\frac{\pi}{n}\right)^2 + \sin\left(\frac{\pi}{n}\right)^2 j^2ijij \\
&= \cos\left(\frac{\pi}{n}\right)^2 + \sin\left(\frac{\pi}{n}\right)^2 \\
&= 1
\end{aligned}$$

3.2

Solution 3.2.1:

Assume we have a submodule $M \subset \mathbb{H}$.

Then M must be invariant under $\mathbb{R}[\mathrm{Dic}_{4n}]$.

However, i, j, k all lies in Dic_{4n}, which means that M must be invariant under $\mathbb{R}[i, j, k]$. Thus M must be the whole $\mathbb{H}$.

Thus $\mathbb{H}$ is simple.

The $\mathrm{End}(\mathbb{H})$ is all the $\mathbb{R}[\mathrm{Dic}_{4n}]$-homomorphism from $\mathbb{H}$ to $\mathbb{H}$.

$$\forall r \in \mathbb{R}[\mathrm{Dic}_{4n}] : f \in \mathrm{End}(\mathbb{H}) : f(rh) = rf(h)$$

Thus the homomorphism must be invariant under multiplication of i, j, k, which means as an homomorphism it needs to commute with multiplication of i, j, k, which means it is isomorphisc to the center of $\mathbb{H}$.

3.3

Solution 3.3.1: After quotient out $\langle \pm 1 \rangle$, $-1 = 1$ in the resulting group.

So $j^2 = 1$, $a^n = 1$, $(ja)^2 = \mathbb{1}$. Thus by mapping $a \mapsto r, j \mapsto s$, we have a isomorphism.

3.4

Solution 3.4.1: Note that the same argument in part (2) showing $\mathbb{H}$ is simple also applies here, because for all odd k, i, j, k (the quaternion k) lies in the image of ϕ_k.

Analogous to why the rotations send $\mathbb{R}[D_{2n}]$ by sending $r \mapsto r^k$ is non-isomorphic, we have the same non-isomorphism here.

3.5

Solution 3.5.1: Since we have the formula to classify simple modules

$$|\mathrm{Dic}_{4n}| = \sum \frac{\dim(\mathbb{H}_k)}{\dim(\mathrm{End}(\mathbb{H}_k))^2}$$

and each $\mathbb{H}_k$ is 4 dimensional, we have

$$4n = \sum \frac{4}{\dim(\mathrm{End}(\mathbb{H}_k))} = \sum \frac{4}{1} = 4n$$

which means we have found all the simple modules.