Math 542 HW1

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1

Theorem 1.1 *(First Isomorphism Theorem for Modules)* Let M, N be R-modules and let $p : M \to N$be an R-module homomorphism. Then $\ker(\psi)$ is a submodule of M and $M/\ker \cong \psi(M)$.

Proof As ψ is a R-module homomorphism,

$$\forall m_1, m_2 \in M, r \in R : \psi(rm_1 + m_2) = r\psi(m_1) + \psi(m_2)$$

For $\ker(\psi)$ to become a submodule, we requires $\forall r \in R, x \in \ker(\psi) : rx \in \ker(\psi)$

We have $r \cdot 0 = 0$, and $\forall x \in \ker(\psi) : \psi(x) = 0$

Then $\forall r \in R, x \in \ker(\psi) : \psi(rx) = r\psi(x) = r \cdot 0 = 0$

Because a module homomorphism must be a group homomorphism. By first isomorphism theorem of group, $M/\ker(\psi) \cong \psi(M)$. □

Theorem 1.2 *(Second Isomorphism Theorem for Modules)* let A and B be submodules of M. Then $\frac{A+B}{B} \cong \frac{A}{A \cap B}$

Proof Construct a map $\psi : A \to \frac{A+B}{B}$ by composing map from $\varphi : A \to A + B$ as a natural map and the canonical projection.

Then we can write $\psi(a)$ as aB, which means its kernel is $A \cap B$. □

2

2.1

Proof

We want to show $\forall r \in R, m \in \mathrm{Tor}(M) : rm \in \mathrm{Tor}(M)$

Thus we want to find some r' such that $r'rm = 0$

We know that $\exists r'' : r''m = 0$, then it suffices to find r' such that $r'r = r''$.

As R is an integral domain, we have $r''r = rr'' \Rightarrow (r''r)m = (rr'')m = 0$, and $rr'' \neq 0$ because $r \neq 0 \wedge r'' \neq 0$. □

2.2

Example 2.2.1

Consider $R = \mathbb{Z}/6\mathbb{Z}$:

$2 \in \mathrm{Tor}(R)$ but $5 \times 2 = 4 \neq \mathrm{Tor}(R)$.

2.3

Proof

Consider the zero divisor $r_1, r_2 \in R$. We have $r_1, r_2 \neq 0 \wedge r_1 r_2 = 0$. Then consider any element $m \in M$, $r_2 r_1 m = 0 \Rightarrow r_1 m \in \mathrm{Tor}(M)$. Then it suffices to show that $r_1 m \neq 0$.

However, if $r_1 m = 0$, then $m \in \mathrm{Tor}(M)$, which also satisfy the requirement.

$\square$