HOMEWORK 3: DUE FRIDAY SEPTEMBER 29

Problem 1 (More tensors; 10 points): Do Dummit and Foote Section 10.4 Problems 10 and 16.

Problem 2 (Short Exact Sequences; 10 points): Let R be a commutative ring. A sequence of R-modules

$$0 \longrightarrow N \xrightarrow{\iota} A \xrightarrow{p} Q \longrightarrow 0$$

is *short exact* if $\iota : N \longrightarrow A$ is an injective R-module homomorphism whose image is the kernel of the surjective R-module homomorphism p. Let B be an R-module. Show that $(p \otimes \mathrm{id}) : A \otimes B \longrightarrow Q \otimes B$ is still surjective and that the image of $\iota \otimes \mathrm{id}$ is still its kernel.

Now consider a short exact sequence of abelian groups $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$ and let $B = \mathbb{Z}/m$ where n and m are positive integers. Show that $\iota \otimes \mathrm{id}$ is injective if and only if $m \mid n$.

Problem 3 (Tensor products of linear maps; 5 points): Suppose that $A : V_1 \longrightarrow V_2$ and $B : W_1 \longrightarrow W_2$ are k-linear maps between k-linear vector spaces. Let $\mathrm{rank}(C)$ be the rank of a linear map C, i.e. the dimension of its image. Show that $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A)\mathrm{rank}(B)$.

Problem 4 (Classifying simple modules; 15 points): Show the following.

(1) For each positive integer n, find all simple $\mathbb{R}[\mathbb{Z}/n\mathbb{Z}]$-modules up to isomorphism and their endomorphism algebras. (Hint: Each simple module will be at most 2-dimensional; think about rotations).

(2) Recall that $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$ whose elements can be written as $a + bi + cj + dk$ for some real numbers a, b, c, d and where

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, ik = -j, kj = -i.$$

Let $Q_8 \subseteq \mathbb{H}$ be the group of order 8 which is generated by i and j. Its elements are $\{\pm 1, \pm i, \pm j, \pm k\}$. Show that there are four non-isomorphic one-dimensional $\mathbb{R}[Q_8]$-modules (note that $Q_8/[Q_8, Q_8] \cong \mathbb{Z}/2 \times \mathbb{Z}/2$). Show that $\mathrm{End}_{\mathbb{R}[Q_8]}(\mathbb{H})$ is isomorphic to $\mathbb{H}$ as a ring. Use this to conclude that, up to isomorphism, $\mathbb{H}$ is the unique simple $\mathbb{R}[Q_8]$-module that is not one-dimensional.

(3) Show that there is a unique simple $\mathbb{C}[Q_8]$-module that is not one-dimensional (Hint: It is two-dimensional, but you don't have to construct it. You can deduce its existence from facts that we have shown and the four one-dimensional $\mathbb{C}[Q_8]$-modules that you found in the previous part).

Problem 5 (Maschke; 10 points): Let k be a field. Let V be a finite-dimensional k-vector space. Suppose that V is a $k[G]$-module where G is a finite group.

(1) A linear map $\pi : V \longrightarrow V$ is called a *projection onto* W if its image is W and if $\pi(w) = w$ for all $w \in W$. Show that, given such a map, V is isomorphic (as a vector space) to $\ker(\pi) \oplus W$.
(2) A linear map $\pi : V \longrightarrow V$ is called G-*equivariant* if $\pi(g \cdot v) = g \cdot \pi(v)$ for all $g \in G$. Show that if π is G-equivariant then its kernel and image are $k[G]$-submodules of V.
(3) Suppose that $\pi : V \longrightarrow V$ is a projection onto a submodule W. Suppose that $|G|$ is invertible in k (this could fail for instance if G has even order and $k = \mathbb{Z}/2$ since 2 is not invertible in k as it coincides with 0). Define a new function $p : V \longrightarrow V$ by $p(v) = \frac{1}{|G|}\sum_{g \in G} g \cdot \pi(g^{-1} \cdot v)$. Show that p is a G-equivariant projection onto W. Conclude that V is isomorphic to $W \oplus \ker(\pi)$ as a $k[G]$-module.
(4) Conclude that V is a direct sum of simple $k[G]$-modules.

Bonus Problem 1 (Characters; 15 points): Let V and W be finite-dimensional $\mathbb{C}[G]$-modules where G is a finite group. Let $V^G := \{v \in V : g \cdot v = v\}$. Let $\chi_V(g)$ be the trace of $g : V \longrightarrow V$.

(1) Show that V^* is a $\mathbb{C}[G]$ module where $g \in G$ acts on the linear map $f : V \longrightarrow \mathbb{C}$ by $(g \cdot f)(v) := f(g^{-1}v)$. Show that this defines a linear group action of G on V^* and that $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$ (for the second equality use that g is diagonalizable since it has finite order.)
(2) Note that G acts on $V \otimes_k W$ by $g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$. Recall that $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes_k W$. Therefore, G acts on $\mathrm{Hom}_{\mathbb{C}}(V, W)$. Show that $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ and $\mathrm{Hom}_{\mathbb{C}}(V, W)^G$ are isomorphic as vector spaces. Also show that $\chi_{V^* \otimes W}(g) = \overline{\chi_V(g)}\chi_W(g)$ for all $g \in G$. (You may use without proving that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$ for a linear map $A \otimes B : V \otimes W \longrightarrow V \otimes W$ induced from linear maps $A : V \longrightarrow V$ and $B : W \longrightarrow W$).

(3) Show that if $\pi : V \longrightarrow V$ is a projection onto a subspace W, then $\dim W = \operatorname{tr}(\pi)$. Show that if we define $\pi : V \longrightarrow V$ by $\pi(v) = \frac{1}{|G|}\sum_{g\in G} g \cdot v$, then π is the projection onto V^G.

(4) Deduce that $\dim \operatorname{Hom}_{\mathbb{C}[G]}(V, W) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g)$. Conclude by Schur's Lemma that if V and W are simple $\mathbb{C}[G]$-modules, then $\frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g)$ is 1 if V and W are isomorphic and 0 otherwise. Show also that if V is a $\mathbb{C}[G]$-module, then it is irreducible if and only if $\frac{1}{|G|}\sum_{g\in G} |\chi(g)|^2 = 1$.

Honors Problem 1 (Artin-Wedderburn; 10 points): Let G be a finite group.

(1) If $R_1, \ldots, R_n$ are rings, show that $(\bigoplus_i R_i)^{op} \cong \bigoplus_i R_i^{op}$.

(2) Suppose that S is a simple finite-dimensional $\mathbb{C}[G]$-module. Let $n > 0$ be an integer. Show that $\operatorname{Hom}_{\mathbb{C}[G]}(S^n, S^n)$ is isomorphic as a ring to $\operatorname{Mat}_{n\times n}(\mathbb{C})$. (Hint: Show that the dimension of these two $\mathbb{C}$-vector spaces coincides and then build a linear injection from one to the other.)

(3) Recall that $\mathbb{C}[G]$ is isomorphic, as a module to $\bigoplus_i S_i^{n_i}$ where S_i is a simple $\mathbb{C}[G]$-module so that S_i is not isomorphic to S_j if $i \neq j$. Using Schur's Lemma and the fact that, for any ring R, $\operatorname{Hom}_R(R, R)$ is isomorphic (as a ring) to R^{op}, show that $\mathbb{C}[G]$ is isomorphic as a ring to $\bigoplus_i \operatorname{Mat}_{n_i\times n_i}(\mathbb{C})$.