Math 542 HW4

Hongtao Zhang

1 Adjugates and Laplace

1.1

Solution 1.1.1

We know that $\det(A)$ is defined by the coefficient of $Ae_1 \wedge ... Ae_n = v_1 \wedge ... \wedge v_n = \det(A) e_1 \wedge ... \wedge e_n$.

Then we consider A_{ij} by seeing that it is the coefficient of $e_1 \wedge ... \wedge e_n$ after we replace $v_1 \wedge ... v_i \wedge ... \wedge v_n$ with $v_1 \wedge ... \wedge e_j \wedge ... \wedge v_n$.

Consider a matrix B_{ij} which is the matrix after removing the i-th row and j-th column of A.

Now we consider the exterior basis removing e_i.

$$B_{ij} e_1 \wedge ... \wedge B_{ij} e_{j-1} \wedge B_{ij} e_{j+1} \wedge ... \wedge e_n = A_{ij} e_1 \wedge ... \wedge e_{j-1} \wedge e_{j+1} \wedge ... \wedge e_n$$
$$= w_1 \wedge ... \wedge w_{j-1} \wedge w_{j+1} \wedge ... \wedge w_n$$

where $w_i = v_i$ projecting to the subspace orthogonal to e_i.

Note that

$$e_j \wedge \left(w_1 \wedge ... \wedge w_{j-1} \wedge w_{j+1} \wedge ... \wedge w_n\right) = e_j \wedge \left(\left(v_1 - \operatorname{proj}_{e_i} v_i\right) \wedge ...\right)$$

Then we can see the projection terms just become 0 after wedging with e_j.

$$e_j \wedge \left(w_1 \wedge ... \wedge w_{j-1} \wedge w_{j+1} \wedge ... \wedge w_n\right) = e_j \wedge v_1 \wedge ... \wedge v_{j-1} \wedge v_{j+1} \wedge ... \wedge v_n$$

which is what we want to show.

1.2

Solution 1.2.1

This can be done easily by seeing that $v_i = \sum_j a_{ij} e_j$, then by lienarity of exterior product the result follows.

1.3

Solution 1.3.1

Consider the first row times first column of $A \operatorname{adj}(A)$. We can see that it is $\sum (-1)^{1+j} a_{1j} A_{1j} = \det(A)$ from part (2).

Consider the multiplication at row 2 and column 1.

The result is

$$\sum_j (-1)^{2+j} a_{2j} A_{1j}$$

This is 0 because with (1) we have showed that A_{ij} is the coefficient of $e_1 \wedge ... \wedge e_n$, but when adding the linear combination from another row, we essentially recover one of the vector. Therefore the wedge product become 0.

Therefore $A \operatorname{adj}(A) = \det(A) I$. The reason for the commutativity is due to that we only have non-zero scalar multiplication and addition in the diagonal, which is commutative.

2 Cayley-Hamilton

Solution 2.1

Consider how B looks like

$$B = \begin{pmatrix} A - a_{11} & -a_{12} & \dots & -a(1n) \\ -a_{21} & A - a_{22} & \dots & -a(2n) \\ \vdots & \ddots & & \\ -a_{n_1} & -a_{n_2} & \dots & A - a_{n_n} \end{pmatrix}$$

We can see that $\sum_{j=1}^{n} b_{ij} e_j = A e_i - \sum_{j=1}^{n} a_{1n} e_j = 0$.

Consider the adjugate matrix C of B.

We know that $CB = \det(B) I$, which means $\forall k : \sum_{i=1}^{n} c_{ki} b_{ij} = \delta_{kj} \det(B)$.

Consider

$$\sum_j \sum_j c_{ki} b_{ij} e_j = \sum_j \delta_{kj} \det(B) e_j = \sum_i c_{ki} 0 = 0$$

Thus $\det(B) = 0$.

We can see that a matrix defined by entries i and j with $\delta_{ij} x - a_{ij}$ will be sent to B in R'.

3 Free Module

Solution 3.1

It suffices to consider $R^{m+1} \to R^m$. Assume $\varphi : R^{m+1} \to R^m$ is the injection, and let $\psi : R^m \to R^{m+1}$ be the projection.

Consider $A = \psi \circ \varphi$.

Consider the minimal degree nonzero polynomial such that $p(A) = 0$. Since A is injective, we have the n-th degree non-zero when applying A as x. Thus, by Cayley-Hamilton we have the constant term non-zero.

However, consider applying $p(A)$ to $(0, ..., 0, 1) \in R^m + 1$, the result will not be 0 because the final term is always 0, while the constant is non-zero, which means the result is non-zero, which is a contradiction.

4 Commutator subgroups of matrix group

4.1

Solution 4.1.1

This is a direct result of smith normal form.

4.2

Solution 4.2.1

Because all elementary row/column operations matrix can be written as a $E_{ij}(a)$. Since every matrix in $\mathrm{SL}(n, k)$ can be written as identity times some $E_{ij}(a)$, we can conclude that $\mathrm{SL}(n, k)$ is product of $E_{ij}(a)$.

4.3

Solution 4.3.1

Consider the commutator of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & -b + a^2 b \\ 0 & 1 \end{pmatrix}$$

Then for any elementary matrix we just need to have $-b + a^2 b = a'$ so $E_{ij}(a') = ABA^{-1}B^{-1}$ for some A, B.

4.4

This can be done by induction on n with the previous question.

5 Determinats of exterior and tensor powers