Problem 1

Let $A = \mathbb{Z}/3 \oplus \mathbb{Z}/12$. Find the number of elements in $A \otimes_{\mathbb{Z}} A$.

Solution 1

$$\begin{aligned} A \otimes_{\mathbb{Z}} A &= (\mathbb{Z}/3 \oplus \mathbb{Z}/12) \otimes_{\mathbb{Z}} (\mathbb{Z}/3 \oplus \mathbb{Z}/12) \\ &= (\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/3) \oplus (\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/12) \oplus (\mathbb{Z}/12 \otimes_{\mathbb{Z}} \mathbb{Z}/3) \oplus (\mathbb{Z}/12 \otimes_{\mathbb{Z}} \mathbb{Z}/12) \end{aligned}$$

We know that $\mathbb{Z}/3 \otimes \mathbb{Z}/3$ has 3 elements because there are 3 element in $\mathbb{Z}/3$ that divides 3, and thus this is isomorphic to $\mathbb{Z}/3$.

Similarly, $\mathbb{Z}/12 \otimes \mathbb{Z}/12$ has 12 elements.

Finally, $\mathbb{Z}/3 \otimes \mathbb{Z}/12$ has 3 elements because $\mathbb{Z}/3 \otimes \mathbb{Z}/12 \cong (\mathbb{Z}/3)/(12\mathbb{Z}/3)$, and since 12 divides 3 the $12\mathbb{Z}/3$ is the trivial group.

Since tensor is commutative, we have that $\mathbb{Z}/12 \otimes \mathbb{Z}/3 \cong \mathbb{Z}/3 \otimes \mathbb{Z}/12$, and thus $\mathbb{Z}/12 \otimes \mathbb{Z}/3$ has 3 elements.

Problem 2

Find the number of pairs $(x, y) \in (\mathbb{Z}/30 \times \mathbb{Z}/30)$ so that $4x + 10y = 0$ and $10x + 4y = 0$.

Solution 2

This is equivalent to find the number of solutions for the following equation

$$\begin{pmatrix} 4 & 10 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

And it is equivalent to find the number of solutions for the SNF of the matrix $A = \begin{pmatrix} 4 & 10 \\ 10 & 4 \end{pmatrix}$.

We know that the only invertible element in $\mathbb{Z}$ is ± 1. Thus, the SNF of the matrix is $S\begin{pmatrix} 2 & 0 \\ 0 & 42 \end{pmatrix}T$, where the left top corner must be $\gcd(4, 10, 10, 4)$ and the right bottom corner is the determinant of matrix A.

Since we are in $\mathbb{Z}/30$, $42 = 6 \times 7$, and 7 is invertible in $\mathbb{Z}/30^{\times}$ and thus it is equivalent to 6, and $2 \mid 30$. Thus, the number of solutions is $2 \times 6 = 12$.

Problem 3

A *generating set* of a group G is a set $S \subset G$ so that the smallest subgroup of G containing S is G itself. Prove of disprove: there is a subgroup of $\mathbb{Z}/12^5$ whose smallest generating set has size 7.

Solution 3

Assume such a subgroup H exists. Consider the preimage of H under the map that sends $\mathbb{Z}^5 \to \mathbb{Z}/12^5$.

Observes that the preimage is a subgroup of $\mathbb{Z}^5$ and it must contains a generating set at least the size of the generating set of H. Thus, the preimage must have a generating set of size larger or equal to 7.

Note that a subgroup in $\mathbb{Z}^5$ is a $\mathbb{Z}$-submodule, and since $\mathbb{Z}$ is a PID, all of its submodule must have smaller rank than $\mathbb{Z}^5$. Thus, the preimage must have rank at most 5. However, the rank is equal to the smallest generating set of the module, and thus we have a contradiction.

Problem 4

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Let D be the subgroup of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$. Show that if G is abelian then it is conjugate to a subgroup of D.

Solution 4

We know that C^n is a $C[G]$-module. Further we know that $\mathbb{C}^n$ can be written as sum of one dimensional simple $C[G]$-modules, denoted as V_i.

Then we further know that each V_i will be determined by the action of G, i.e. $\delta_i : G \to \mathbb{C}^\times$, and thus there must exist some B such that $\forall A \in G : BAB^{-1} = \begin{pmatrix} \delta_1(g) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \delta_{n(g)} \end{pmatrix} \in D$

Problem 5

Suppose that (a, b) and (c, d) are two elements of $\mathbb{Z}^2$ and let G be the subgroup of $\mathbb{Z}^2$ that they generate. Suppose that $ad - bc = 12$. Find the number, up to isomorphism, of all possible quotient groups $\mathbb{Z}^2/G$.

Solution 5

It suffices to consider all matrix of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 12$ and $a, b, c, d \in \mathbb{Z}$. Note that $ad - bc = \det(A)$, and $\det(A) = d_1 \times d_2$, where d_1 and d_2 are the invariant factors of A.

Thus it suffices to consider all the possible invariant factor of A.

Since $12 = 3 \times 4$, and thus the only invariant factor combination is $(1, 12)$ and $(2, 6)$, which means the number of quotient group is only 2 up to isomorphism.

Problem 6

Let $A = \mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/3$. Consider the homomorphsm $\varphi : A \to \mathbb{Q} \otimes_{\mathbb{Z}} A$ given by $\varphi(a) = 1 \otimes a$. Find the number of elements in $\ker(\varphi)$.

Solution 6

We know that $\mathbb{Q} \otimes_{\mathbb{Z}} A = \mathbb{Q} \otimes (\mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/3) = \mathbb{Q} \otimes \mathbb{Z} \oplus \mathbb{Q} \otimes \mathbb{Z}/6 \oplus \mathbb{Q} \otimes \mathbb{Z}/3$.

Since this is a direct sum of three tensor product, it will be 0 if and only if all three entries are 0.

Firstly consider how many elements in $\mathbb{Z}/6$ that will be 0 when sending $a \mapsto 1_{\mathbb{Q}} \otimes a$. Since 6 divides 1 in $\mathbb{Q}$, we know that all elements in $\mathbb{Z}/6$ will be 0 when sending $a \mapsto 1_{\mathbb{Q}} \otimes a$.

Similar reasoning yield us that all elements in $\mathbb{Z}/3$ will be 0 when sending $a \mapsto 1_{\mathbb{Q}} \otimes a$.

Finally, we know that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$ is isomorphic to $\mathbb{Q}$ with isomorphism $f : (q \otimes z) \mapsto q \cdot z$, and thus $\mathbb{Q} \otimes \mathbb{Z}$ will be 0 if and only if z is 0, which means we have only 0 maps to 0 in φ. Therefore the number of elements in $\ker(\varphi)$ is $1 \times 3 \times 6 = 18$.

Problem 7

Suppose that A is 4×4 matrix with rational entries and whose characteristic polynomial is $x^2(x^2+1)$. Produce a finite collection S of explicit matrices and show that, for some B in $\mathrm{GL}_4(\mathbb{Q})$, BAB^{-1} belongs to S.

Solution 7

It suffices to find the all possible Rational Canonical Form with characteristic polynomial $x^2(x^2+1)$.

The prime decomposition of $x^2(x^2+1)$ is

$$x, x, (x^2+1)$$

We know that the product of invariant factors are the characteristic polynomial. Thus the possible invariant factors are

$$x^2(x^2+1) = x^4 + x^2$$

$$x, x(x^2+1) = x^3 + x$$

Using the algorithm in the book, we can find the possible Rational Canonical Form are

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since these are the only two possible Rational Canonical Form with charateristic polynomial $x^2(x^2+1)$, we know that A must be conjugate to one of them.