Math 542 HW1

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1

Theorem 1.1 *(First Isomorphism Theorem for Modules)* Let M, N be R-modules and let $p : M \to N$be an R-module homomorphism. Then $\ker(\psi)$ is a submodule of M and $M/\ker \cong \psi(M)$.

Proof:

As ψ is a R-module homomorphism,

$$\forall m_1, m_2 \in M, r \in R : \psi(rm_1 + m_2) = r\psi(m_1) + \psi(m_2)$$

For $\ker(\psi)$ to become a submodule, we requires $\forall r \in R, x \in \ker(\psi) : rx \in \ker(\psi)$

We have $r \cdot 0 = 0$, and $\forall x \in \ker(\psi) : \psi(x) = 0$

Then $\forall r \in R, x \in \ker(\psi) : \psi(rx) = r\psi(x) = r \cdot 0 = 0$

Because a module homomorphism must be a group homomorphism. By first isomorphism theorem of group, $M/\ker(\psi) \cong \psi(M)$.

□

Theorem 1.3 *(Second Isomorphism Theorem for Modules)* let A and B be submodules of M. Then $\frac{A+B}{B} \cong \frac{A}{A \cap B}$

Proof:

Construct a map $\psi : A \to \frac{A+B}{B}$ by composing map from $\varphi : A \to A + B$ as a natural map and the canonical projection.

Then we can write $\psi(a)$ as aB, which means its kernel is $A \cap B$.

□

2

2.1

Proof:

We want to show $\forall r \in R, m \in \operatorname{Tor}(M) : rm \in \operatorname{Tor}(M)$

Thus we want to find some r' such that $r'rm = 0$

We know that $\exists r'' : r''m = 0$, then it suffices to find r' such that $r'r = r''$.

As R is an integral domain, we have $r''r = rr'' \Rightarrow (r''r)m = (rr'')m = 0$, and $rr'' \neq 0$ because $r \neq 0 \wedge r'' \neq 0$.

□

2.2

Example 2.2.1

Consider $R = \mathbb{Z}/6\mathbb{Z}$:

$2 \in \operatorname{Tor}(R)$ but $5 \times 2 = 4 \neq \operatorname{Tor}(R)$.

2.3

Proof:

Consider the zero divisor $r_1, r_2 \in R$. We have $r_1, r_2 \neq 0 \wedge r_1 r_2 = 0$. Then consider any non-zero element $m \in M$, $r_2 r_1 m = 0 \Rightarrow r_1 m \in \operatorname{Tor}(M)$. Then it suffices to show that $r_1 m \neq 0$.

However, if $r_1 m = 0$, then $m \in \operatorname{Tor}(M)$, which also satisfy the requirement.

□

2.4

2.4.1

Because $\mathbb{Z}$ is an integral domain, then become a torsion submodules means exist some elements that makes the whole submodule become 0. Then the first entries must be 0.

The second entry is just the whole $\mathbb{Z}/6\mathbb{Z}$ as we always have $6 \in \mathbb{Z}$ that makes every element in $\mathbb{Z}/6\mathbb{Z}$ to be 0.

3

Denote the finite-dimensional $k[x]$-module as V.

To become a submodule V', it must be invariant under the linear transformation represented by x. It suffices to find a polynomial $\chi(A)$ such that $\chi(A) = 0$. By Cayley-Hamilton theorem, this polynomials always exists, which is the charateristic polynomial. Thus every element is a torsion element, and thus the torsion submodules are V itself.

3.1

Proposition 3.1.1 The only finite-dimensional simple $\mathbb{C}[x]$-modules are one-dimensional.

Proof:

As any $n \times n$ matrix with entries in $\mathbb{C}$ has an eigenvector. We know that the span of eigenvector of x will never escape the span, and thus is a submodule.

As long as the dimension of $\mathbb{C}[x]$-modules are not 1, we definately can find a span of eigenvector that has dimensions less than the module.

□

3.2

Proposition 3.2.1 Let $M := \mathbb{C}^2$ be a $\mathbb{C}[x]$-module where the action of x is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find all submodules of M.

Proof:

By (1), we will have the submodules span by the eigenvectors of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x + y = x \\ y = y \end{pmatrix} \Rightarrow ()$$

□

4

4.1

Proof:

For f to be a R-module isomorphism, it must be a isomorphism of the underlying set. Consider f is bijective first.

Therefore, $\exists g : f \circ g = g \circ f = \mathbb{1}$. It suffices to prove that g is a R-module homomorphism.

Because f is a R-module homomorphism

$$\forall m_1, m_2 \in M, r \in R : f(rm_1 + m_2) = rf(m_1) + f(m_2)$$

Then we have $g(f(rm_1 + m_2)) = rm_1 + m_2 = g(rf(m_1) + f(m_2))$.

Because f is bijective, $f(m_1)$ and $f(m_2)$ points to a unique element in M denoted as m_3 and m_4

$$\forall m_3, m_4 \in M : \exists m_1, m_2 \in M : f(m_1) = m_3 \wedge f(m_2) = m_4$$

Therefore, $\forall m_3, m_4 \in M : g(rm_3 + m_4) = rg(m_3) + g(m_4) = rm_1 + m_2$.

Assume the existence of such g that is a R-module homomorphism and $g \circ f = \mathbb{1} = f \circ g$:

We know that the existence of inverse of the underlying set means that f and g is bijective. Then nothing left to be proved.

□

4.2

Proof:

By Theorem 1.1 we have $\ker(f)$ as a submodule of M. However, because M is simple, then $\ker(f)$ is either $\{0\}$ or M.

Then for any non-zero R-module homomorphism $f : M \to M$, we have $\ker(f) = \{0\}$, which means it is injective, and as f maps from M to M, it is subjective, so thus bijective. By (1), we have such g exists.

□

4.3

Proof:

Consider a matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ that has no eigenvector in $\mathbb{R}^2$. Then we have no submodule for this module.

$\operatorname{End}_R(M)$ is all the linear transformation that commute with x, and for this case it is $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and thus we can just send this to $a + bi$.

□

5

5.1

Let A be any $\mathbb{Z}$-module, let a be any element of A and let n be a positive integer. Prove that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\overline{k}) = ka$ is a well defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A \mid na = 0\}$ (so A_n is the annihilator in A of the ideal (n) of $\mathbb{Z}$ — cf. Exercise 10, Section 1).

Proof:

if $na = 0$

$$\varphi(\overline{x} + \overline{y}) = \varphi(x + y \bmod n) = (x + y \bmod a)a$$

$$\varphi(\overline{x}) + \varphi(\overline{y}) = (x + y)a$$

Because na = 0, $(x + y)a = (x + y \bmod n)a$

If φ_a is a valid homomorphism, then

$$\varphi(\overline{x} + \overline{y}) = \varphi(\overline{x}) + \varphi(\overline{y}) \Rightarrow (x + y)a = (x + y \bmod n)a \Rightarrow na = 0$$

□

Proof:

From Previous statement, we have each $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ correspoinded to a set of a such that $na = 0$.

We want to prove $\forall \psi \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) : \exists a \in A : \psi = \varphi_a$

For ψ to be in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$, we will have the property that

$\forall x \in \mathbb{Z}/n\mathbb{Z}, z \in \mathbb{Z} : z\psi(x) = \psi(zx)$

Therefore, $z\psi(x) = \overline{z}\psi(x)$.

Therefore, $\psi(x)$ must have the property that $n\psi(x) = 0$, which fits exactly into the a we have.

□

5.2

Exhibit all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$.

Proof:

By previous exercise, it suffices to find all $a \in \mathbb{Z}/21\mathbb{Z}$ such that $30a = 0 \Rightarrow 9a = 0$.

We have $a_1 = 7, a_2 = 14, a_3 = 0$

□

6 Bonus

Given a ring R, the opposite ring R^{op} is the ring with all the same elements, where addition is defined identically, but for which $x \cdot^{\text{op}} y := y \cdot x$ where $\cdot$ is multiplciation in R and $\cdot^{\text{op}}$ is the multiplication in R^{op}. Take R as a left R-module and show that $\text{Hom}_{R-\text{Mod}}(R, R)$ is isomorphic to R^{op} as a ring.

Proof:

Consider an element f in $\text{Hom}_{R-\text{Mod}}(R, R)$, it must follows the module property.

That is

$$\forall r_1, r_2 \in R : r_1 f(r_2) = f(r_1 r_2) \Rightarrow f(r_1) = f(r_1 \cdot 1) = r_1 f(1)$$

Therefore, f can only have one form $f_r(r') = r'r$.

Then the map $\psi : R \to \text{Hom}_{R-\text{Mod}}(R, R)$ by sending $r \mapsto f_r$. This map is clearly both injective and surjective.

Consider the map $\varphi : \text{Hom}_{R-\text{Mod}}(R, R) \to R^{\text{op}}$ that maps f_r to r in R^{op}. Because it is the inverse of ψ in the underlying set, it is injective and surjective, and thus a isomorphism.

□

Show that if $R = \text{Mat}_{n \times n}(k)$ is the ring of $n \times n$ matrices with entries in a field k, then $R^{\text{op}} \cong R$ where the isomorphism is given by sending a matrix to its transpose.

Proof:

By property of transpose, we have $AB = BA^T$, and thus this is an isomorphism.

□