12.1

11

We can see that $p^{k-1}M$ and $p^k M$ is a R-module.

Consider the ideal $I = \left(p^{k-1}, a\right)$ and $J = \left(p^k, a\right)$.

Consider the homomorhism of $I \to p^{k-1}M$ which is the restriction to I of the homomorphism $R \to \frac{R}{a} = M$.

Thus, we have $\frac{I}{J} \cong \frac{p^{k-1}M}{p^k M}$ as R-module by isomorphism theorem. Then we can see that $\frac{I}{J}$ will be $R/(p)$ if $k \leq n$ and (0) if $k > n$ as $I = \left(\gcd\left(p^{k-1}, a\right)\right), J = \left(\gcd\left(p^k, a\right)\right)$.

12

By previous exercise we can see that $p^{k-1}M/p^k M \cong p^{k-1}(M_1 \oplus \ldots \oplus M_n)/p^k(M_1 \oplus \ldots \oplus M_n)$, where M_i is generated by the elementary divisors of M. By previous exercise, we can see that each $p^{k-1}\frac{M_i}{p^k}M_i \cong R/(p)$ if the i-th elementary divisor is power of p^a with $a \geq k$.

Since $M_1 \cong M_2$, then they both can be written as some direct sum of $R/(a)$. Then follows the previous part, we can conclude they have the same number of elementary divisor p^a with $a \geq k$. Since this is true for all elementary divisors in R, we can conclude that M_1 and M_2 have the same set of elementary divisors.

12.2

10

It suffices to find all invariant factor where the largest one is $(x+2)^2(x-1)$.

We can see that we have

$$
\begin{gathered}
(x+2), (x+2)(x-1), (x+2)^2(x-1) \\
(x+2), (x+2)^2, (x+2)^2(x-1) \\
(x+2), (x+2), (x+2), (x+2)^2(x-1) \\
(x-1), (x-1), (x-1), (x+2)^2(x-1) \\
(x-1), (x+2)(x-1), (x+2)^2(x-1) \\
(x+2)^2(x-1), (x+2)^2(x-1)
\end{gathered}
$$

Each will have a representator that can be written based on the rational canonical form construction.

$$
A_p = \begin{pmatrix}
0 & 0 & \ldots & 0 & -\frac{a_0}{a_n} \\
1 & 0 & \ldots & 0 & -\frac{a_1}{a_n} \\
\vdots & 1 & \ddots & & \vdots \\
\vdots & \vdots & \ddots & & \vdots \\
0 & 0 & \ldots & 1 & -\frac{a_{n-1}}{a_n}
\end{pmatrix}
$$

12

Given that the matrix satisfy $A^6 = I$, we have the minimal polynomial satisfying $x^6 - 1 = 0$.

$$x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1) = (x+1)^2(x^2+x+1)^2$$

Thus the minimal polynomial needs to be a factor of $(x+1)^2(x^2+x+1)^2$.

We have choice $x+1, (x+1)^2, (x^2+x+1), (x+1)(x^2+x+1)$.

Thus results are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

For 4×4 matrices satifying $B^{20} = I$. We have a similar expression.

$$x^{20} - 1 = 0$$
$$(x-1)^2(x^4+x^3+x^2+x+1)^2(x^{10}+1) = 0$$

For the minimal polynomial, we have

$$(x-1)^i \quad \forall i \leq 4$$
$$(x^4+x^3+x^2+x+1)$$

Thus we can construct similar matrix based on these minimal polynomials as before.

12.3

18

By factoring out the elementary divisors

$$(x-2)^3, (x-3)^2$$
$$(x-2)^2, (x-2), (x-3)^2$$
$$(x-2), (x-2), (x-2), (x-3)^2$$
$$(x-2)^3, (x-3), (x-3)$$
$$(x-2)^2, (x-2), (x-3), (x-3)$$
$$(x-2), (x-2), (x-2), (x-3), (x-3)$$

Then constructing the Jordon Canonical form based on that

$$(x-2)^3, (x-3)^2$$
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Others are similar.

22

This suggests that we have the minimal polynomial a factor of $x^3 - x = 0 = (x+1)(x-1)x$. Then we can see that the minimal polynomial have no repeated root, which means it is diagnolizable.

This is not true over any field F as we can have $(x+1) = (x-1)$ in $\mathbb{F}_2$.

22

$$\varphi(\nu_j) = -a_{1j}v_1 - ... - a_{j-1j}v_{j-1} + (x - a_{jj})v_j - a_{j+1j}v_{j+1} - ...a_{nj}v_n$$

By applying the definition of $x(v_j)$, we have this equal to 0.

Assume we have some v lies in $\ker(\varphi)$.

Then

$$\varphi(v_j) = \varphi\left(\sum a_i\xi_i\right) = \sum \varphi(a_i\xi_i) = \sum a_i\varphi(\xi_i) = \sum a_i v_i$$

23

We have the homomorphism $\varphi : F[x]^n \to V$.

$$\varphi(x\xi_j) = x\varphi(\xi_j) = xv_j = a_{1j}v_1 + ... + a_{nj}v_n$$

Thus consider the preimage of such an element, which will be in some form of $\sum f_i$ plus the kernel of φ, which is ν_j.

We have $F[x]\xi_i = F\xi_i + F'\xi_i$ where F' denotes all polynomial with constant term 0. Since $F'\xi_i = F[x]x(\xi_j) = F[x](\nu_j + f_j)$ the claim follows.

By previous claim we have

$$\sum F[x]\xi_i = \sum F[x]\nu_i + \sum F\xi_i$$

For an element to be in the kernel of φ, the constant term must be 0. Thus we have any element in $\ker(\varphi)$ can be written as $\sum F[x]\nu_i$.

This claim follows by the definition of ν_i. We have

$$\nu_j = -a_{1j}\xi_1 - ... - a_{j-1j}\xi_{j-1} + (x - a_{jj})\xi_j - a_{j+1j}\xi_{j+1} - ... - a_{nj}\xi_n$$

Thus the j-th column of the relations matrix will be as $\begin{pmatrix} -a_{1j} \\ \vdots \\ x-a_{jj} \\ \vdots \\ -a_{nj} \end{pmatrix}$.

Since this is a matrix represented by a set of elements generate the kernel of φ, we can change the basis of the kernel to the basis of the relations matrix, and thus results a diagonal matrix. The transpose won't change the diagonal property, and thus we can prove the claim.