Introduction To SemiLinear Reps of Infinite Symmetric Groups

Part 1: Definition and motivation

<u>Def</u>: Let G be a group, K is a $\underbrace{\text{field with a } G\text{-action}}_{G\text{-field}}$ (i.e. $G \to \mathrm{Aut}(K)$) Let $K\langle G\rangle$ be the (twisted) group ring. i.e. it consists of elements $\sum_{g\in G} a_g [g]$ with $a_g \in K$ and the multiplication is given by

$$(a[g])\cdot(b[h]) = a\, \underset{\substack{\uparrow \\ g \text{ acts on } b}}{b^g}\, [gh]$$

A K-semilinear rep of G is a K-vector space V with a $K\langle G\rangle$-module structure.

In other words, take $a \in K$, $v \in V$, $g \in V$

we have $\quad g.(av) = a^g (g.v)$

<u>Eg</u>: (Trivial one) Take G gp, K a G-field

S is a G-set, then $K[S]$ has a natural $K\langle G\rangle$-module structure by diagonal action

$$g.(as) = a^g (g.s)$$

<u>Motivation</u>: 1) Semi-linear reps arise naturally in algebraic geometry (however deep, I won't talk about this since I know quite little)

2) We can simplify the picture by the Galöis descent:

Take G a finite group, acts faithfully on a field K, i.e. $G \hookrightarrow \mathrm{Aut}(K)$, let $F = K^G$.

there is a quasi-equivalence of categories

$$\{K\text{-semilinear reps of } G\} \cong \{F\text{-vector spaces}\}$$

$$M \longmapsto M^G \quad \text{"Galöis descent"}$$

$$K \otimes_F V \longleftarrow V$$

In particular, any K-semilinear rep of G is isomorphic to direct sum of trivial semilinear rep K (being image of F)

Def: (algebraic Smooth rep) Let G be a topological gp (or algebraic gp), V is a G-set (in general will be a rep), we say $v \in V$ is smooth if $\exists$ compact open subgp $H \subset G$ such that $Hv = v$.

If any $v \in V$ is smooth, we say V is Smooth.

From now on, we mainly care about the category of smooth semilinear rep of G, $\mathrm{Sm}_K(G)$

Prop:

It admits a simple generator if and only if G is precompact, i.e any open subgp of G is of finite index. (For example, $\mathbb{Z}_p$, $\mathrm{Gal}(L|K)$ ← infinite profinite groups ⇒ Galöis gp with Krull topology.)

Part 2: Some Structure of $Sm_K(\mathfrak{S}_\Psi)$

Let Ψ be a countably infinite set and $\mathfrak{S}_\Psi$ be the permutation group of Ψ, i.e. the infinite symmetric group.

Rmk: It's clear $\mathfrak{S}_\Psi$ does not contain any 'infinite' element. In other words, for any $g \in \mathfrak{S}_\Psi$, there $\exists$ a finite subset $\Phi \subset \Psi$ such that $g\Phi \subset \Phi$ and g fixes any element in $\Psi - \Phi$.

This is one way to define the infinite symmetric gp. There $\exists$ 'other kind' of infinite symmetric gp, which will not be discussed here.

There is a choice of K, we adopt the most natural one in this section: $K = k(x_i \mid i \in \Psi)$ "the infinite rational function field"

First result is that Projectives are not important in $Sm_K(\mathfrak{S}_\Psi)$

Thm (Rovinsky 2022)

There is no non-zero projective in $Sm_K(\mathfrak{S}_\Psi)$ $\square$

Next we may think about the injectives.

Indeed we may introduce a new topo spectrum:

Def (Gabriel Spectrum) Let C be a Grothendieck cat (The reader can omit the meaning of it since any cat of modules over associate ring is a grothendieck cat. By def, a g-cat is an ab cat such that arbitrary coproduct exists and there exists a generator Y and colimit is exact)

in this case it means $\forall X \in C$ $\exists \bigoplus Y \twoheadrightarrow X$.

the Gabrie Spectrum $Spec(C)$ of C is a topological space with points of indecomposable injectives.

The topology is given by the following:

for $\forall X \in C$, Let $[X]$ be the set of pts E such that $Hom_C(X, E) = 0$.

take base of opens to be $[X]$ when X runs over all (finite presentable) objects
compact
[i.e. colim commute with $Hom(X, -)$]

Since $[X] \cap [Y] = [X \oplus Y]$, this is well-defined

Question: Why it is called spectrum?

Answer:

Prop: Take R noetherian unital commutative ring $C = Mod(R)$, then Gabriel spectrum $Spec(C)$ is homeomorphic to $Spec(R)$, the prime spectrum.

We care about the structure of $Spec(Sm_k(G_\mathbb{Z}))$

Thm 1 (Rovinsky, 2022)

any element

[Assume k is a non-trivial $\mathfrak{S}_\Psi$-field.] Fix $x \in \Psi$

For any integer $s \geq 0$, the injective hull of objects $k\langle\binom{\Phi}{s}\rangle$ of $Sm_k(\mathfrak{S}_\Psi)$ represent distinct pts P_s in $\mathrm{Spec}(Sm_k(\mathfrak{S}_\Psi))$. Any set containing infinitely many P_s is dense.

they're all the pts

Turn to Cat $Sm_k(\mathfrak{S}\ )$, we have the following fact :

It's locally noetherian, i.e. $\exists$ a (small) generating objects such that they're notherian, namely any ascending chain of subojects ~~of~~ is stationary

Part 3. $k(\mathfrak{X}) = k(x_i \mid i \in \mathfrak{X})$, $G = G_{\mathfrak{X}}$

It's natural to ask a simple question,

$$\boxed{\text{For each } n \geq 1, \text{ can we construct a simple object of dim } n \text{ in } Sm_{k(\mathfrak{X})}(G)}$$

This question is easy to answer. Since there are only 1-dimensional simples in $Sm_{k(\mathfrak{X})}(G)$

Thm: $K \subset k(\mathfrak{X})$ G-subfield, any smooth K-smooth semilinear rep of G can be embedded into a direct product of copies of $k(\mathfrak{X})$; any $k(\mathfrak{X})$-semilinear rep of G of finite length is isomorphic to a direct sum of copies of $k(\mathfrak{X})$

Next, we may want to consider some G-subfield K of $k(\mathfrak{X})$,

$$\boxed{\text{For each } n \geq 1, \text{ can we construct a simple object of dim } n \text{ in } Sm_{K}(G)}$$

[Galöis Case]

$\boxed{\text{Case 1}}$ $K := k(\mathfrak{X})^H$ where H is a finite subgp of $\mathrm{Aut}_G(k(\mathfrak{X})) = \mathrm{Aut}(k(x)|k) \cong PGL(2,k)$

$\boxed{\text{Case 2}}$ $K := L^H$ where $k(\mathfrak{X})|L$ is transcendental and H is a finite subgp of $\mathrm{Aut}_G(L)$

(Key method) Make use of linear reps of H over k

For each irr (linear) rep ρ, Let $V_\rho := \mathrm{Hom}_{k[H]}(\rho, k(\mathfrak{X}))$

then $V_\rho \in Sm_K(G)$

Moreover, V_ρ is irreducible and $\dim_K V_\rho = \dim_k \rho$
simple

In Case 1, we know all the finite subgps of $PGL(2,k)$

they're $\mathbb{Z}/r\mathbb{Z}$, D_r, A_4, S_4, A_5
cyclic dihedral

$\mathbb{Z}/r\mathbb{Z}$: since any irr of ab gp is 1-dim
we get 1-dim

D_r : simple reps are 1-dim and 2-dim

A_4 : Irr reps are of dim 1, 1, 1, 3
$\downarrow$

S_4 : $[S_4 : A_4] = 2$ so no new results than A_4

A_5 : dim 1, 3, 3, 4, 5

Indeed we can produce two 3-dim dual to each other
1 4-dim self dual
1 5-dim self dual

However, it's too limited in this case.

Only at most dim 5.

Let's look at $\boxed{\text{Case 2}}$

Thm
Let $k \subsetneq K \subset k(\mathfrak{X})$, K alg closed, $G_{\mathfrak{X}}$-field.

then $\mathrm{tr.deg}(k(\mathfrak{X}) | K) \leq 3$.

we can give an explicity characterization

1) if $\mathrm{tr.deg}(k(\mathfrak{X}) | K) = 1$, either $\exists\ R(X) \in k(X) - k$

s.t. $K = k(R(x) - R(y) \mid x, y \in \mathfrak{X}) = K_1$

or $\exists\ R(X) \in k(X) - k$

s.t. $K = k(\frac{R(x)}{R(y)} \mid x, y \in \mathfrak{X}) = K_2$

2) if $\mathrm{tr.deg}(k(\mathfrak{X}) | K) = 2$, there $\exists\ R(X) \in k(X) - k$

s.t. $K = k(\frac{R(x) - R(y)}{R(x) - R(z)} \mid x, y, z \text{ pairwise different in } \mathfrak{X})$

3) if $\mathrm{tr.deg}(k(\mathfrak{X}) | K) = 3$, then there $\exists\ R(X) \in k(X) - k$

s.t. $K = k(\frac{R(x) - R(y)}{R(x) - R(w)} \frac{R(z) - R(w)}{R(z) - R(y)} \mid x, y, z, w \text{ pairwise different in } \mathfrak{X})$

Our aim is to find $K' \underset{G_{\mathbb{Z}}}{\subset} K$ $K \mid K'$ Galöis

see whether there $\exists$ higher dim example

in $Sm_{K'}(G_{\mathbb{Z}})$

Thm (Qin ~)

1) $Aut_{G_{\mathbb{Z}}}(K_1) \cong k^*$

$Aut_{G_{\mathbb{Z}}}(K_2) \cong \mathbb{Z}/2\mathbb{Z}$

2) $Aut_{G_{\mathbb{Z}}}(K)$ is trivial

3) $Aut_{G_{\mathbb{Z}}}(K)$ is trivial.

So all the naive ways of constructing them fail. We must consider new method.

For example, either we can consider non-Galöis case, or we may try change the group even.