# A Note on Supersymmetric Gauge Theory

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# Contents

# Chapter 1
# Lorentz symmetry

This note is written as a first look of supersymmetric theory for readers who has a brief idea about quantum field theory. To make this note more self-consistent both in contents and conventions. The first chapter is a review of Lorentz. Readers who are familiar with these things can skip to the second chapter.

## 1.1 Lorentz transformation

All the high energy physics theory we are talking about, need to be satisfied Lorentz invariance. So it turns out to be a starting point for a physics theory, which is what we are going to do.

Lorentz transformation is defined to make the spacetime interval invariant. The space time interval is defined as

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \tag{1.1}$$

It is convenient to write it in covariant form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.2}$$

Where $g_{\mu\nu}$ is **Minkowski metric** which in our convention is

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{1.3}$$

A linear transformation

$$x' = \Lambda x + a \tag{1.4}$$

satisfied

$$\mathrm{d}s'^2 = \mathrm{d}s^2 \tag{1.5}$$

is called **Lorentz transformation**, or in some book, **Poincare transformation**. You can easily verify that Lorentz transformation forms a group called **Lorentz group**. If $a = 0$, transformation (1.4) forms a subgroup called **homogeneous Lorentz group**.

In Lorentz invariance theory, not only coordinates but also all the operators should have a transformation rule according to Lorentz transformation, so that we can construct Lorentz invariance formula. That is to say, we need to clarify which representation of Lorentz group [①] they belong to.

To do this, first we need to write down the explicit form of (1.4) with $a = 0$, which is called vector representation. The boost transformation along $x^1$-direction is

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 & 0 & 0 \\ \sinh\phi_1 & \cosh\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = B_1 \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.6}$$

① More especially the homogeneous Lorentz group ($a = 0$), since representation of translation is somehow trivial.

If we write the infinitesimal form of $\Lambda$ as

$$\Lambda(\vec{\theta},\vec{\phi}) = 1 + i\vec{K}^V\cdot\vec{\phi} + i\vec{J}^V\cdot\vec{\theta} + \cdots \tag{1.7}$$

where $\vec{K}^V = (K_1^V, K_2^V, K_3^V)$, $\vec{J}^V = (J_1^V, J_2^V, J_3^V)$ are generators of boost and rotation in vector representation respectively. Then using (1.6), we have

$$K_1^V = \frac{1}{i}\frac{\partial B_1}{\partial \phi_1}\bigg|_{\phi_1=0} = -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{1.8}$$

In the same way, all the other homogeneous Lorentz generators can be obtained:

$$K_2^V = -i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\quad K_3^V = -i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} \tag{1.9}$$

$$J_1^V = -i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad J_2^V = -i\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\quad J_3^V = -i\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix} \tag{1.10}$$

(1.8)–(1.10) are the generators of homogeneous Lorentz group in vector representation. We find that $\vec{K}^V$ are not hermitian matrices. Equation (1.7) tells us that vector representation is not unitary. In fact, any representation of homogeneous Lorentz group is not unitary thus they can not be described as a state vector. This is one reason why we need second quantization for a field theory.

If calculating the commutation relations of (1.8)–(1.10), we can find that they form a closed Lie algebra:

$$[K_i^V, K_j^V] = -i\epsilon_{ijk}J_k^V, \tag{1.11}$$
$$[J_i^V, K_j^V] = i\epsilon_{ijk}K_k^V, \tag{1.12}$$
$$[J_i^V, J_j^V] = i\epsilon_{ijk}J_k^V \tag{1.13}$$

The commutation relations define a Lie algebra and determine the group space of corresponding Lie group near the identity element. Thus the generators for any other representation of homogeneous Lorentz group will still satisfy (1.11)–(1.13).

$$[K_i, K_j] = -i\epsilon_{ijk}J_k, \tag{1.14}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k, \tag{1.15}$$
$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1.16}$$

Conversely, (1.14)–(1.16) can be used to find other representation. A unified form of commutation relation is somehow more useful. Define antisymmetric operator $J_{\mu\nu}$ (where $\mu,\nu = 0,1,2,3$) as

$$J_{ij} = \epsilon_{ijk}J_k \quad (i,j=1,2,3), \qquad J_{0i} = K_i \quad (i=1,2,3) \tag{1.17}$$

The commutation relation can then be written as

$$\left[J_{\mu\nu}, J_{\rho\sigma}\right] = i\left(g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} + g_{\mu\sigma}J_{\nu\rho} - g_{\nu\sigma}J_{\mu\rho}\right) \tag{1.18}$$

which is in a covariant form.

**Exercise 1.1**

*Verify that the Lorentz transformation* (1.4) *forms a group.*

**Exercise 1.2**

*Find the equivalence of* (1.14), (1.15) *and* (1.16) *with* (1.18) *through direct calculation by the redefinition of* (1.17).

## 1.2 Dirac matrices

In this section we introduce an important class of representation which satisfy (1.18) called spinor representation. It is construct by four dimension **Clifford Algebra**[①] called Dirac matrice, defined by

$$\left\{\gamma^{\mu}, \gamma^{\nu}\right\} = 2g^{\mu\nu} \tag{1.19}$$

$$\gamma^{\mu\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0} \tag{1.20}$$

The matrice generator satisfied commutation relation (1.18) is obtained from the Dirac matrice by defining

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right] \tag{1.21}$$

using (1.20), we can obtain that $\Sigma^{ii}$ is hermitian while $\Sigma^{0i}$ is antihermitian, the same as generators in vector representation.

The acting space of such representation is a four dimension spinor (a complex vector).

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \tag{1.22}$$

In analogue of (1.7), we write the infinitesimal form for group element now in covariant way, that is

$$S(\omega) = 1 + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu} + \cdots \tag{1.23}$$

$\omega_{\mu\nu}$ is a antisymmetry parameters. with the degree of freedom is the same as the Lorentz group. Compare with (1.7) and (1.17), we can obtain the relations for the parameters.

$$\theta_k = \frac{1}{2}\omega_{ij}\epsilon_{ijk}, \quad \phi_i = \omega_{0i} \tag{1.24}$$

Write the group element in finite form is

$$S(\omega) = \exp\left\{\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right\} \tag{1.25}$$

Now the Lorentz transformation for 4 dimension Dirac spinor is

$$\Psi(x) \to \Psi'(x') = S(\lambda)\Psi(x) \tag{1.26}$$

① (1.19) is the exact definition of Clifford Algebra, but we still need (1.20) to ensure the hermitian and antihermitian properties of generators.

Since now the group element is not unitary as illustrate before, the dual space of $\Psi(x)$ can not be just complex conjugate. That is to say under Lorentz transformation.

$$\Psi^\dagger\Psi \to \Psi^\dagger S^\dagger S\Psi \neq \Psi^\dagger\Psi \tag{1.27}$$

$\Psi(x)^\dagger\Psi(x)$ is not a scalar. We need to define a dual spinor with respect to $\Psi(x)$.

$$\bar{\Psi}(x) = -\Psi(x)\gamma^0 \tag{1.28}$$

Using the property that $\Sigma^{ij}$s are hermitian, while $\Sigma^{0i}$s are antihermitian, we obtain

$$S^\dagger\gamma^0 = \gamma^0 S^{-1} \tag{1.29}$$

Thus, the Lorentz transformation of $\bar{\Psi}$ can be obtained from (1.26) (1.28), and (1.29).

$$\bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}S^{-1} \tag{1.30}$$

Thus $\bar{\Psi}\Psi$ is a scalar. To define a chiral spinor, we need to introduce

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.31}$$

Using the definition for Dirac matrice (1.19) and (1.20), we can verify that

$$\gamma^5 = (\gamma^5)^{-1} = (\gamma^5)^\dagger \tag{1.32}$$

A spinor is a **left-chiral spinor** $\Psi_L$ if

$$\gamma^5\Psi_L = \Psi_L \tag{1.33}$$

on the other hand, a spinor is a **right-chiral spinor** $\Psi_R$ if

$$\gamma^5\Psi_R = -\Psi_R \tag{1.34}$$

Thus using the property (1.32), we can introduce projective operator to project arbitrary spinor to corresponding left and right chiral $\Psi_L = P_L\Psi, \Psi_R = P_R\Psi$ .

$$P_L = \frac{1+\gamma^5}{2}, \quad P_R = \frac{1-\gamma^5}{2} \tag{1.35}$$

It is very useful to consider a particular representation for the Dirac matrices called **Weyl representation**.

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.36}$$

where $\sigma^i$ are Pauli matrice.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.37}$$

You can check by straight calculation that (1.36) satisfied the definition (1.19) and (1.20). We present Weyl representation here since it provides a way to construct two dimension representation which we will talk about in next section.

I should mention that just like an arbitrary $2\times2$ complex matrice can be obtained by linear combination of three Pauli matrices and an Identity one, $4\times4$ complex matrices can also be constructed by following sixteen independent matrices $1, \gamma^\mu, \Sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5$, which is useful in later analysis.

After the brief introduction of Dirac representation, the generators relation between Dirac representation and vector representation are easily obtained by require $\bar{\Psi}\gamma^\mu\Psi$ transform as a vector under Lorentz transformation.

$$\gamma^\mu \Lambda_\mu{}^\rho = S^{-1}\gamma^\rho S \tag{1.38}$$

**Exercise 1.3**

*Verify that* (1.21) *satisfy the commutation relation* (1.18).

**Exercise 1.4**

*Verify* (1.30).

**Exercise 1.5**

*Verify the spinors obtained by projective operator satisfy the definition of left and right chiral spinor* (1.33) *and* (1.34).

## 1.3 Two-component Weyl spinor

This section we will obtain two-component Weyl spinor from four component one. Two-component representation is important in constructing supersymmetric algebra.

First we need to write (1.36) in a unified form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{1.39}$$

where

$$\sigma^\mu = (\sigma^0, \sigma^i) = (-1, \sigma^i), \quad \bar{\sigma}^\mu = (\bar{\sigma}^0, \bar{\sigma}^i) = (-1, -\sigma^i) \tag{1.40}$$

Using the definition (1.21), we find

$$\begin{aligned}\Sigma^{\mu\nu} &= \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \\ &= i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}\end{aligned} \tag{1.41}$$

with the definition

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \quad \bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) \tag{1.42}$$

The generator now is block diagonal, and so is the group element from(1.25)

$$S(\omega) = \begin{pmatrix} e^{-1/2\omega_{\mu\nu}\sigma^{\mu\nu}} & 0 \\ 0 & e^{-1/2\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}} \end{pmatrix} \tag{1.43}$$

This means that two upper and two lower components of $\Psi$ transform independently under Lorentz transformation. We decoupled the 4-component spinor in the follow form for later convenience. ①

$$\Psi = \begin{pmatrix} \chi \\ \eta^* \end{pmatrix} \tag{1.45}$$

$*$ denote that the lower one is inequivalent to the upper one. And we will see in fact the lower one is equivalent to the complex conjugate of the upper one. From (1.41) for upper two component, the generator is

$$i\sigma^{ij} = -\frac{i}{4}\left[\sigma^i, \sigma^j\right] = \frac{1}{2}\epsilon_{ijk}\sigma^k, \quad i\sigma^{0i} = \frac{i}{2}\sigma^i \tag{1.46}$$

from (1.37), we see that $i\sigma^{ij}$ is hermitian while $i\sigma^{0i}$ is antihermitian. For lower two component

$$i\bar{\sigma}^{ij} = -\frac{i}{4}\left[\sigma^i, \sigma^j\right] = \frac{1}{2}\epsilon_{ijk}\sigma^k, \quad i\bar{\sigma}^{0i} = -\frac{i}{2}\sigma^i \tag{1.47}$$

Using (1.24)(1.46) and (1.43) we can write the Lorentz transformation for two-component spinor (in infinitesimal form)

$$\begin{aligned} \delta\chi &= (\frac{i}{2}\omega_{ij}\frac{1}{2}\epsilon_{ijk}\sigma^k + i\omega_{0i}\frac{i}{2}\sigma^i)\chi = \frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\phi})\chi \\ \delta\eta^* &= (\frac{i}{2}\omega_{ij}\frac{1}{2}\epsilon_{ijk}\sigma^k - i\omega_{0i}\frac{i}{2}\sigma^i)\eta^* = \frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} - i\vec{\phi})\eta^* \end{aligned} \tag{1.48}$$

From the relation above we can immediately write the finite transformation matrice (1.41) as

$$S(\omega) = \begin{pmatrix} s & 0 \\ 0 & s^{-1^\dagger} \end{pmatrix} \tag{1.49}$$

In analogue to Dirac Representation, we want to find the dual part of two-component spinor in order to construct Lorentz invariant scalar. For upper two-component spinor, using (1.48) and the property of Pauli matrice $\sigma^{2^{\mathrm{T}}} = -\sigma^2$ and $\sigma^{i^{\mathrm{T}}}\sigma^2 = -\sigma^2\sigma^i$, which can be calculate straightfully by Pauli matrice we find the Lorentz transformation of $(i\sigma^2\chi)^{\mathrm{T}}$

$$\delta(i\sigma^2\chi)^{\mathrm{T}} = -\frac{i}{2}(\theta_i + i\phi_i)(i\sigma^2\chi)^{\mathrm{T}}\sigma^i \tag{1.50}$$

(1.48) and (1.50) tell us that $(i\sigma^2\chi)^{\mathrm{T}}\chi$ is a scalar under Lorentz transformation. Then $i\sigma_2\chi$ can be the dual part of $\chi$. If we denote $\chi$ by lower index

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{1.51}$$

then we denote its dual part by upper index

$$i\sigma^2\chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \chi_2 \\ -\chi_1 \end{pmatrix} \equiv \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \tag{1.52}$$

① some books's convention is

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{1.44}$$

For the reason that $\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ satisfied (1.33) in Weyl representation is left chiral. So is the right-chiral part. It doesn't mean that $\psi_L$ and $\psi_R$ is chiral in two component representation now the scalar can be rewritten as $\chi^\alpha \chi_\alpha$ with $\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta$. Now $\epsilon^{\alpha\beta} = i\sigma^2$ is antisymmetric matrice playing the role of "metric tensor". The inversed relation is easily obtained $\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta$ with the inverse matrice.

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.53}$$

From (1.50), we get the finite transformation for $\chi^\alpha$, compare with the transformation for $\chi_\alpha$

$$\chi'_\alpha = (s)_\alpha{}^\beta \chi_\beta, \quad \chi'^\alpha = (s^{-1^{\mathrm{T}}})^\alpha{}_\beta \chi^\beta \tag{1.54}$$

$s$ is defined from (1.49). If we write the explicit form for $s$ and $s^{-1^{\mathrm{T}}}$ using (1.48)

$$s = \exp\left\{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\phi})\right\}, \quad s^{-1^{\mathrm{T}}} = \exp\left\{-\frac{i}{2}\vec{\sigma}^{\mathrm{T}}\cdot(\vec{\theta} + i\vec{\phi})\right\} \tag{1.55}$$

Using the property for pauli matrice that $\sigma^2\sigma^{i^{\mathrm{T}}}\sigma^2 = -\sigma^i$ and $(\sigma^2)^2 = 1$ we find that

$$\sigma^2(s^{-1})^{\mathrm{T}}\sigma^2 = \sigma^2(s^{-1})^{\mathrm{T}}(\sigma^2)^{-1} = s \tag{1.56}$$

means that $\chi^\alpha$ and $\chi_\alpha$ are in the equivalent representation.

In the same way, we study the lower two-component spinor. First the dual part of $\eta^*$ is $(-i\sigma^2\eta^*)^{\mathrm{T}}$, since

$$\delta(-i\sigma^2\eta^*)^{\mathrm{T}} = -\frac{i}{2}(\theta_i - i\phi_i)(-i\sigma^2\eta^*)^{\mathrm{T}}\sigma^i \tag{1.57}$$

From (1.48) and (1.57), We see that $(-i\sigma^2\eta^*)^{\mathrm{T}}\eta^*$ transform as a scalar under Lorentz transformation. Thus $(-i\sigma^2\eta^*)$ can be a dual spinor of $\eta^*$. If we denote $\eta$ by upper index

$$\eta^* = \begin{pmatrix} \eta^{*\dot{1}} \\ \eta^{*\dot{2}} \end{pmatrix} \tag{1.58}$$

here dot on the index means the representation now is inequivalent from the upper two component one. The dual part is denoted by lower index.

$$-i\sigma^2\eta^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \eta^{*\dot{1}} \\ \eta^{*\dot{2}} \end{pmatrix} = \begin{pmatrix} -\eta^{*\dot{2}} \\ \eta^{*\dot{1}} \end{pmatrix} = \begin{pmatrix} \eta^*{}_{\dot{1}} \\ \eta^*{}_{\dot{2}} \end{pmatrix} \tag{1.59}$$

Now the Lorentz scalar for lower two-component spinor can be written as

$$\eta^*{}_{\dot{\alpha}}\eta^{*\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\eta^{*\dot{\beta}}\eta^{*\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\eta^*{}_{\dot{\alpha}}\eta^*{}_{\dot{\beta}} \tag{1.60}$$

with the antisymmetric tensor defined as

$$\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.61}$$

From (1.49), and the relation of (1.48) and (1.57), we can write the finite transformation of lower two-component spinor under Lorentz transformation.

$$\eta^{*\prime\dot{\alpha}} = (s^{-1^\dagger})^{\dot{\alpha}}{}_{\dot{\beta}}\eta^{*\dot{\beta}}, \qquad \eta^{*\prime}{}_{\dot{\alpha}} = (s^*)_{\dot{\alpha}}{}^{\dot{\beta}}\eta^*{}_{\dot{\beta}} \tag{1.62}$$

In analogue with (1.56), in the same way we can have

$$s^{-1^\dagger} = \sigma^2 s^* \sigma^2 \tag{1.63}$$

so $\eta^{*\dot{\alpha}}$ and $\eta^*{}_{\dot{\alpha}}$ are equivalent representation. Furthermore, by comparing (1.54) and (1.62), we are justified in identifying dotted spinors with complex conjugate of undotted ones, which is a convenient convention in Majorana condition.

$$\chi^*{}_{\dot{\alpha}} = (\chi_\alpha)^*, \quad \chi^{*\dot{\alpha}} = (\chi^\alpha)^* \tag{1.64}$$

**Exercise 1.6**

*Verify* (1.50) *and* (1.57).

**Exercise 1.7**

*Verify* (1.63).

## 1.4 $SL(2,C)$ Group

In this section we will see a deep relation between $SL(2,C)$ and the connected part of homogeneous Lorentz group, which enable us to write the vector representation in a new way.

To see these, let us first study more details on two-component spinor representation. In analogue to the analysis in Dirac representation, we are now in the position to derive its connection with group element in vector representation. From (1.39) and (1.49), (1.38) becomes

$$\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \Lambda_\mu{}^\rho = \begin{pmatrix} s^{-1} & 0 \\ 0 & s^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^\rho \\ \bar{\sigma}^\rho & 0 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1^\dagger} \end{pmatrix} \tag{1.65}$$

compare each element of matrice we have

$$\Lambda^\rho{}_\mu \sigma^\mu = s\sigma^\rho s^\dagger, \qquad \Lambda^\rho{}_\mu \bar{\sigma}^\mu = s^{-1^\dagger} \bar{\sigma}^\rho s^{-1} \tag{1.66}$$

From (1.54) and (1.62), the indices on $s$ and $s^{-1}$ are undotted, while those on $s^\dagger$ and $s^{\dagger -1}$ are dotted, thus the indice on $\sigma^\mu$ and $\sigma^\mu$ are

$$\sigma^\mu = (\sigma^\mu)_{\alpha\dot{\beta}}, \quad \bar{\sigma}^\rho = (\bar{\sigma}^\rho)^{\dot{\alpha}\beta} \tag{1.67}$$

For example, (1.66) now are

$$\Lambda^\rho{}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} = (s)_\alpha{}^\gamma (\sigma^\rho)_{\gamma\dot{\delta}} (s^\dagger)^{\dot{\delta}}{}_{\dot{\beta}} \tag{1.68}$$

$$\Lambda^\rho{}_\mu (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (s^{-1^\dagger})^{\dot{\alpha}}{}_{\dot{\delta}} (\bar{\sigma}^\rho)^{\dot{\delta}\gamma} (s^{-1})_\gamma{}^\beta \tag{1.69}$$

We can also use antisymmetric tensor to raise and lower the index as a definition. For example

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}} \tag{1.70}$$

There are huge amounts of relations of $\sigma$ and $\bar{\sigma}$ which are listed in Appendix.

A much interesting observation from (1.66) is that if we act a vector $V_\rho$ to both sides of the first equation, we find

$$V'_\mu \sigma^\mu = sV_\rho \sigma^\rho s^\dagger \tag{1.71}$$

If we consider the object $v \equiv V_\mu \sigma^\mu$ as a new representation of Lorentz group. Then (1.71), describes its transformation under Lorentz transformation $V_\mu \to V'_\mu$. Furthermore, the number of bases in acting space is the same in both representation, in fact, you can find one to one correspondence in group acting space by considering

$$\frac{1}{2}\mathrm{tr}[v\bar{\sigma}^\mu] = V^\mu \tag{1.72}$$

these two representations is more likely to be equivalent. We are now in the position to give the definition of $SL(2,C)$ group. Write the explicit form of $v$

$$v = \begin{pmatrix} -V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & -V_0 - V_3 \end{pmatrix} \tag{1.73}$$

Since $V_\mu$ is an arbitrary real four-component vector. $v$ is now an arbitrary hermitian matrix. Thus the transformation must be in the following form, to transform from one hermitian matrix into another.

$$v' = \lambda v \lambda^\dagger \tag{1.74}$$

which is consistent to (1.71). Calculate the determinant

$$\det(v) = V_\mu V^\mu \tag{1.75}$$

To satisfy the Lorentz invariant condition, we need $\det(v)$ to be invariant, thus gives the condition for transformation matrice.

$$|\det(\lambda)| = 1 \tag{1.76}$$

since two $\lambda$s with a difference of phase give the same effect from (1.74), we can conveniently adjust the phase so that

$$\det(\lambda) = 1 \tag{1.77}$$

The $2 \times 2$ complex matrices with unit determinant form a group, known as $SL(2,C)$. The group elements depend on 3 complex parameter, so the degree of freedom is the same as homogeneous Lorentz group. Now we need to ask whether $\lambda$ is the same as $s$, the group element of upper two-component spinor.

Any complex non-singular matrice $\lambda$ may be written in the form

$$\lambda = e^w \tag{1.78}$$

(1.77) gives $\mathrm{tr}(w) = 0$. So we can write $w$ in the basis of Pauli matrices $w = (a_i + ib_i)\sigma^i$ , where $a_i$ and $b_i$ are arbitrary real parameters. since

$$V_0 = -\frac{1}{2}\mathrm{tr}(v) \tag{1.79}$$

is invariant under $\lambda = e^{ib_i\sigma^i}$. So $b_i$s are the three parameters for spacetime rotations, the other three parameters $a_i$s should be the parameters for boosts in consistent with (1.55), which shows $\lambda$ is the same as $s$. So the group space of $SL(2,C)$ and the homogeneous Lorentz group has a corresponding, however since $\lambda$ and $-\lambda$ has the same effect to the transformation $v$ from (1.74). The relation is a double covering. The homogeneous Lorentz group is the same as $SL(2,C)/Z_2$.

From the above analysis, we conclude that $v$ is a representation of Lorentz group equivalent to vector representation, since we find the one to one corresponding of their acting space. And we can also choose a one to one corresponding for their transformation matrice. Thus makes them equivalent in the mapping perspective.

Furthermore from the transformation (1.71) and using the transformation law (1.54) and (1.62), we see that $v$ may be construct from two-component spinor

$$v = \chi_\alpha \chi^*{}_{\dot\beta} = \chi_\alpha (\chi_\beta)^* = \begin{pmatrix} \chi_1\chi_1^* & \chi_2\chi_1^* \\ \chi_1\chi_2^* & \chi_2\chi_2^* \end{pmatrix} \tag{1.80}$$

which is hermitian as required. So we may consider $v$ as a direct product of upper two-component and lower-two component spinor.

**Exercise 1.8**

*Using the property of* (A.1), *verify* (1.72).

## 1.5 $(A, B)$ Representation

We are now going to develop a classification of representation for homogeneous Lorentz group. For an arbitrary symmetry operators $O$, the Lorentz transformation can be written in the following form.

$$U(\Lambda)^{-1} O U(\Lambda) = MO \tag{1.81}$$

with M furnish a representation of the homogeneous Lorentz group. [①] The infinitesimal form of $U(\Lambda)$ is

$$U(\Lambda) = 1 + i\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = 1 + i\theta_k J_k + i\phi_i K_i \tag{1.82}$$

We know that the generator for homogeneous Lorentz group form a closed Lie Algebra (1.14), (1.15), (1.16), but now the unitary condition makes $J$ nad $K$ to be hermitian.

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{1.83}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k \tag{1.84}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1.85}$$

If we define new generators $\vec{A}, \vec{B}$

$$\vec{A} \equiv \frac{1}{2}(\vec{J} + i\vec{K}), \quad \vec{B} \equiv \frac{1}{2}(\vec{J} - i\vec{K}) \tag{1.86}$$

New commutation relation is

$$\left[A_i, A_j\right] = i\epsilon_{ijk}A_k, \quad \left[B_i, B_j\right] = i\epsilon_{ijk}B_k, \quad \left[A_i, B_j\right] = 0 \tag{1.87}$$

The generators now are complexified and decoupled. We see $\vec{A}$ and $\vec{B}$ generate a group of $SU(2)$. This tells us that complexified Lorentz group can be construct by the direct product of two $SU(2)$ group. We know that the representation of $SU(2)$ can be classified by spin(integer or half-integer number). So now we can classify the representation

① $U(\Lambda)$ is in fact the lorentz transformation of a physical state $|\Omega\rangle \to U(\Lambda)|\Omega\rangle$ Thus, quantum mechanically $U(\Lambda)$ must be unitary (sometimes antiunitary if the flow of time direction is changed ) to preserve the possibility. The expectation value of an symmetry operator changes by $\langle\Omega| O |\Omega\rangle \to \langle\Omega| U^{-1}(\Lambda) O U(\Lambda) |\Omega\rangle$ so we can write the transformation for such symmetry operator as $O \to U^{-1}(\Lambda) O U(\lambda)$

of Lorentz group into two spin number (A,B). With the definition of spin representation as follow.

$$\left[\vec{A}, O^{AB}_{ab}\right] = -\sum_{a'} \vec{J}^{(A)}_{aa'} O^{AB}_{a'b}, \quad \left[\vec{B}, O^{AB}_{ab}\right] = -\sum_{b'} \vec{J}^{(B)}_{bb'} O^{AB}_{ab'} \tag{1.88}$$

with $a$ and $b$ run by unit steps from $-A$ to $+A$ and from $-B$ to $+B$. where $\vec{J}^j$ is the spin three-vector matrice for angular momentum $j$

$$(J^{(j)}_1 \pm iJ^{(j)}_2)_{\sigma'\sigma} = \delta_{\sigma',\sigma\pm1}\sqrt{(j \mp \sigma)(j \pm \sigma + 1)}, \quad (J^j_3)_{\sigma'\sigma} = \delta_{\sigma'\sigma}\sigma \tag{1.89}$$

Do the straight calculation we will see

$$\vec{J}^{(0)} = 0, \qquad \vec{J}^{(\frac{1}{2})} = \frac{1}{2}\vec{\sigma} \tag{1.90}$$

Now, it's time to give all the representation we have written before into a classification of $(A, B)$.

First consider a scalar operator which transform under Lorentz transformation is

$$U(\Lambda)^{-1}\Phi U(\Lambda) = \Phi \tag{1.91}$$

Write it in the infinitesimal form, it's easy to see that $\Phi$ commutes with all $K$ and $J$, so that $\Phi$ commutes with all $A$ and $B$. Thus according to (1.88) and (1.89). We see the scalar representation belongs to $(0, 0)$.

Consider the upper two-component spinor operator $\chi$ with transformation

$$U(\Lambda)^{-1}\chi U(\Lambda) = s\chi \tag{1.92}$$

From the (1.55), the infinitesimal form of s is

$$s = 1 + \frac{i}{2}\vec{\sigma} \cdot (\vec{\theta} + i\vec{\phi}) \tag{1.93}$$

Using (1.82), and compare the coefficient of $\theta$ and $\phi$ on both sides of (1.92) we have

$$\left[\vec{J}, \chi\right] = -\frac{1}{2}\vec{\sigma}\chi, \qquad \left[\vec{K}, \chi\right] = -i\frac{1}{2}\vec{\sigma}\chi \tag{1.94}$$

or equivalently

$$\left[\vec{B}, \chi\right] = -\frac{1}{2}\vec{\sigma}\chi, \qquad \left[\vec{A}, \chi\right] = 0 \tag{1.95}$$

So upper two-component spinor $\chi$ is in the representation of $(0, \frac{1}{2})$.

In the same way for the lower two-component spinor $\eta^*$ with transfromation

$$U(\Lambda)^{-1}\eta^* U(\Lambda) = {s^{-1}}^{\dagger}\eta^* \tag{1.96}$$

with the infinitesimal form of ${s^{-1}}^{\dagger}$

$${s^{-1}}^{\dagger} = 1 + \frac{i}{2}\vec{\sigma} \cdot (\vec{\theta} - i\vec{\phi}) \tag{1.97}$$

with (1.82) pluge into (1.96), we have

$$\left[\vec{J}, \eta^*\right] = -\frac{1}{2}\vec{\sigma}\eta^*, \quad \left[\vec{K}, \eta^*\right] = \frac{i}{2}\vec{\sigma}\eta^* \tag{1.98}$$

or equivalently

$$\left[\vec{A}, \eta^*\right] = -\frac{1}{2}\vec{\sigma}\eta^*, \qquad \left[\vec{B}, \eta^*\right] = 0 \tag{1.99}$$

Thus lower two-component spinor $\eta^*$ is in representation of $(\frac{1}{2}, 0)$. Using the fact that vector representation is equivalent to the direct product of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, we can concluded that the vector representation is in $(\frac{1}{2}, \frac{1}{2})$. We can also verify this by using (1.88).

From (1.45), we can easily obtain that the Weyl representation is a direct sum of $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$.

So far, we have developped all the useful representations in the later supersymmetric theory. Especially in the construction of supersymmetric algebra, keep in mind that all the formula should keep in the same pace while transform under Lorentz transformation, then it will be much easier to understand the form of each equation.

**Exercise 1.9**

*Verify* (1.87).

**Exercise 1.10**

*Write the vector representation into $v = V_\mu \sigma^\mu$ and check that it is in $(\frac{1}{2}, \frac{1}{2})$ representation by using the definition* (1.88).

# 1.6 Majorana Spinor

In this section, we will introduce Majorana Spinor, which plays an important rule in supersymmetric field theory construction.

First we need to introduce **charge conjugation matrice** in four-component spinor representation.Since the Clifford algebra has only the one irreducible representation in Dirac representation and $(\gamma^\mu)^{\mathrm{T}}$ also satisfy Clifford algebra (1.19), it must be related by $\gamma^\mu$ by a similarity transformation.

$$C(\gamma^\mu)^{\mathrm{T}} C^{-1} = -\gamma^\mu \tag{1.100}$$

In Weyl representation we can write the charge conjugation matrice in following form.

$$C = i\gamma^2\gamma^0 = i\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \tag{1.101}$$

$i$ is a phase convention choiced to make $C$ real. The explicit form of charge conjugation matrice immediately shows the property for $C$

$$C = -C^{\mathrm{T}} = C^* = -C^\dagger = -C^{-1} \tag{1.102}$$

The antiparticle $\Psi^c$ is defined as

$$\Psi^c = C\bar{\Psi}^{\mathrm{T}} \tag{1.103}$$

From (1.30), we get the transformation for antiparticle spinor as

$$\Psi^c \to \Psi^{c\prime} = CS^{-1^{\mathrm{T}}}\bar{\Psi}^{\mathrm{T}} = S\Psi^c \tag{1.104}$$

where we have used (1.100) in calculation. So we know that the antiparticle spinor transforms in the same way as the usual spinor. If the particle is neutral, in the sence that it is its own antiparticle, we now write the **Majorana condition**.

$$\Psi = \Psi^c \tag{1.105}$$

We then write the Majorana spinor in decoupled way as in (1.45)

$$\Psi^c = i\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} \eta^\alpha \\ \chi^*{}_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \eta_\alpha \\ \chi^{*\dot\alpha} \end{pmatrix} \tag{1.106}$$

where we have used the definition for antisymmetric tensor (1.53) and (1.61). Using the Majorana condition, we find $\eta = \chi$. So the Majorana spinor can be written in decoupled way as

$$\Psi^m = \begin{pmatrix} \chi_\alpha \\ \chi^{*\dot\alpha} \end{pmatrix} \tag{1.107}$$

So the Majorana condition half the degree of freedom of spinor. In supersymmetric theory we often construct a 4-component Majorana spinor from a 2-component spinor. (1.107) tells us that, if we have a $(0, \frac{1}{2})$ spinor $u$, then the Majorana spinor can be construct as

$$\begin{pmatrix} u \\ i\sigma^2 u^* \end{pmatrix} \tag{1.108}$$

If we have a $(\frac{1}{2}, 0)$ spinor $v$, then the Majorana 4-component spinor can be constructed as

$$\begin{pmatrix} -i\sigma^2 v^* \\ v \end{pmatrix} \tag{1.109}$$

From (1.108) The dual part of a Majorana spinor now is

$$\bar\Psi = -\Psi^\dagger\gamma^0 = -\begin{pmatrix} u \\ i\sigma^2 u^* \end{pmatrix}^\dagger \gamma^0 = \begin{pmatrix} u^{\mathrm{T}} & (i\sigma^2 u^*)^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} = \Psi^{\mathrm{T}} C \tag{1.110}$$

We can also derive this relation using Majorana condition (1.105).

Now we need to study the the property of constant Majorana spinor, each component of the spinor should be a number instead of an operator. The number now should be Grassman number. If $a$ and $b$ are Grassman number, then

$$ab = -ba \tag{1.111}$$

in the sense that the lorentz scalar $\chi^\alpha \chi_\alpha$ can be non-trivial. For a pair of Majorana spinors $\theta_1$ , $\theta_2$ and any $4 \times 4$ numerical matrice $M$

$$\bar\theta_1 M \theta_2 = \sum_{\alpha\beta} (\theta_1)_\alpha (CM)_{\alpha\beta} (\theta_2)_\beta = \sum_{\alpha\beta} (\theta_2)_\alpha (M^{\mathrm{T}} C)_{\alpha\beta} (\theta_2)_\beta = \bar\theta_2 C^{-1} M^{\mathrm{T}} C \theta_2 \tag{1.112}$$

As have mentioned that $4 \times 4$ complex matrices can be constructed by following sixteen independent matrix $1, \gamma^\mu, \Sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5$, let $M$ be these bases, using the definition for charge conjugation matrix (1.100), we can obtain

$$M^{\mathrm{T}} = \begin{cases} CMC^{-1} & M = 1, \gamma^\mu\gamma^5, \gamma^5 \\ -CMC^{-1} & M = \gamma^\mu, \Sigma^{\mu\nu} \end{cases} \tag{1.113}$$

It follows that

$$\bar\theta_1 M \theta_2 = \begin{cases} \bar\theta_2 M \theta_1 & M = 1, \gamma^\mu\gamma^5, \gamma^5 \\ -\bar\theta_2 M \theta_1 & M = \gamma^\mu, \Sigma^{\mu\nu} \end{cases} \tag{1.114}$$

In particular, if $\theta_1 = \theta_2$, the bilinear of $M = \gamma^\mu, \Sigma^{\mu\nu}$ vanishes. So the only covariant bilinears are $\bar\theta\theta$ , $\bar\theta\gamma^5\gamma^\mu\theta$ and $\bar\theta\gamma^5\theta$

### Exercise 1.11

*Verify 1.113.*

# Appendix A
# Formula list

## A.1 Pauli matrices

$$\mathrm{tr}[\sigma^{\mu}\bar{\sigma}^{\nu}] = 2g^{\mu\nu} \tag{A.1}$$