PSTAT210_Ch2_Exercises

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Q1

Let Ω be a non-empty set. Let $\mathcal{F}_0$ be the collection of all subsets such that either A or A^c is finite.

(a) Show that $\mathcal{F}_0$ is a field.

Define for $E \in \mathcal{F}_0$ the set function P by

$$P(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ 1, & \text{if } E^c \text{ is finite.} \end{cases}$$

(b) If Ω is countably infinite, show P is finitely additive but not σ-additive on $\mathcal{F}_0$.

(c) If Ω is uncountable, show P is σ-additive on $\mathcal{F}_0$.

Q2

Let $\mathcal{A}$ be the smallest field over the π-system $\mathcal{P}$.
Use the inclusion–exclusion formula to show that probability measures agreeing on $\mathcal{P}$ must also agree on $\mathcal{A}$.

Q3

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Show for events $B_i \subset A_i$ that

$$P\Big(\bigcup_i A_i\Big) - P\Big(\bigcup_i B_i\Big) \leq \sum_i \big(P(A_i) - P(B_i)\big).$$

Q4

Review Exercise 34 in Chapter 1. Suppose P is a probability measure on a σ-field $\mathcal{B}$ and suppose $A \notin \mathcal{B}$.
Let

$$\mathcal{B}_1 = \sigma(\mathcal{B}, A)$$

and show that P has an extension to a probability measure P_1 on $\mathcal{B}_1$.

Q5

Let P be a probability measure on $\mathcal{B}(\mathbb{R})$.
For any $B \in \mathcal{B}(\mathbb{R})$ and $\varepsilon > 0$, show there exists a finite union of intervals A such that

$$P(A \triangle B) < \varepsilon.$$

Hint: Define

$$\mathcal{G} := \big\{B \in \mathcal{B}(\mathbb{R}) : \forall \varepsilon > 0,\ \exists A_\varepsilon \text{ finite union of intervals with } P(A_\varepsilon \triangle B) < \varepsilon\big\}.$$

Q6 Say events $A_1, A_2, \ldots$ are almost disjoint if

$$P(A_i \cap A_j) = 0, \quad i \neq j.$$

Show for such events

$$P\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} P(A_j).$$

Q7

Suppose there are N different types of coupons available when buying cereal; each box contains one coupon and the collector is seeking to collect one of each in order to win a prize. After buying n boxes, what is the probability p_n that the collector has at least one of each type?
(Consider sampling with replacement from a population of N distinct elements. The sample size is $n > N$.)

Q8

We know that $P_1 = P_2$ on $\mathcal{B}$ if $P_1 = P_2$ on $\mathcal{C}$, provided that $\mathcal{C}$ generates $\mathcal{B}$ and is a π-system.
Show this last property cannot be omitted. For example, consider $\Omega = \{a, b, c, d\}$ with

$$P_1(\{a\}) = P_1(\{d\}) = P_2(\{b\}) = P_2(\{c\}) = \frac{1}{6},$$

and

$$P_1(\{b\}) = P_1(\{c\}) = P_2(\{a\}) = P_2(\{d\}) = \frac{1}{3}.$$

Set

$$\mathcal{C} = \{\{a, b\}, \{d, c\}, \{a, c\}, \{b, d\}\}.$$

Q9

Call two sets $A_1, A_2 \in \mathcal{B}$ *equivalent* if $P(A_1 \triangle A_2) = 0$.
For a set $A \in \mathcal{B}$, define the equivalence class

$$A^{\#} = \{B \in \mathcal{B} : P(B \triangle A) = 0\}.$$

This decomposes $\mathcal{B}$ into equivalence classes. Write

$$P^{\#}(A^{\#}) = P(A), \quad \forall A \in A^{\#}.$$

In practice we identify equivalence classes with their members.

An *atom* in a probability space $(\Omega, \mathcal{B}, P)$ is defined as a set $A \in \mathcal{B}$ such that $P(A) > 0$, and if $B \subset A$ and $B \in \mathcal{B}$, then $P(B) = 0$ or $P(A \setminus B) = 0$.
The space is *non-atomic* if it contains no atoms; i.e., for all $A \in \mathcal{B}$ with $P(A) > 0$, there exists $B \subset A$ such that $0 < P(B) < P(A)$.

(a) If $\Omega = \mathbb{R}$ and P is determined by a distribution function $F(x)$, show that the atoms are $\{x : F(x) - F(x^-) > 0\}$.

(b) If $(\Omega, \mathcal{B}, P) = ((0, 1], \mathcal{B}((0, 1]), \lambda)$, where λ is Lebesgue measure, show the probability space is non-atomic.

(c) Show that two distinct atoms have intersection which is the empty set.

(d) A probability space contains at most countably many atoms.
What is the maximum number of atoms the space can contain that have probability at least $1/n$?

(e) If a probability space $(\Omega, \mathcal{B}, P)$ contains no atoms, then for every $a \in (0,1]$ there exists at least one set $A \in \mathcal{B}$ such that $P(A) = a$.

(f) For every probability space $(\Omega, \mathcal{B}, P)$ and any $\varepsilon > 0$, there exists a finite partition of Ω by $\mathcal{B}$ sets, each of which has probability $\leq \varepsilon$ or is an atom with probability $> \varepsilon$.

(g) On the set of equivalence classes define

$$d(A_1^{\#}, A_2^{\#}) = P(A_1 \triangle A_2).$$

Show d is a metric and verify

$$|P(A_1) - P(A_2)| \leq P(A_1 \triangle A_2).$$

Hence $P^{\#}$ is uniformly continuous on the set of equivalence classes.

Q10

Two events A, B on the probability space $(\Omega, \mathcal{B}, P)$ are equivalent (see Exercise 9) if

$$P(A \cap B) = P(A) \vee P(B).$$

Q11

Suppose $\{B_n, n \geq 1\}$ are events with $P(B_n) = 1$ for all n.
Show

$$P\Big(\bigcap_{n=1}^{\infty} B_n\Big) = 1.$$

Q12

Suppose $\mathcal{C}$ is a class of subsets of Ω and suppose $B \subset \Omega$ satisfies $B \in \sigma(\mathcal{C})$.
Show that there exists a countable class $\mathcal{C}_B \subset \mathcal{C}$ such that $B \in \sigma(\mathcal{C}_B)$.

Hint: Define

$$\mathcal{G} := \{B \subset \Omega : \exists \text{ countable } \mathcal{C}_B \subset \mathcal{C} \text{ such that } B \in \sigma(\mathcal{C}_B)\}.$$

Show that $\mathcal{G}$ is a σ-field containing $\mathcal{C}$.

Q13

If $\{B_k\}$ are events such that

$$\sum_{k=1}^{n} P(B_k) > n - 1,$$

then

$$P\Big(\bigcap_{k=1}^{n} B_k\Big) > 0.$$

Q14

If F is a distribution function, then F has at most countably many discontinuities.

Q15

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two semialgebras of subsets of Ω, show that the class

$$\mathcal{S}_1\mathcal{S}_2 := \{A_1A_2 : A_1 \in \mathcal{S}_1, A_2 \in \mathcal{S}_2\}$$

is again a semialgebra of subsets of Ω.
The field (σ-field) generated by $\mathcal{S}_1\mathcal{S}_2$ is identical with that generated by $\mathcal{S}_1 \cup \mathcal{S}_2$.

Q16

Suppose $\mathcal{B}$ is a σ-field of subsets of Ω and suppose $Q : \mathcal{B} \to [0, 1]$ is a set function satisfying

(a) Q is finitely additive on $\mathcal{B}$.

(b) $0 \le Q(A) \le 1$ for all $A \in \mathcal{B}$ and $Q(\Omega) = 1$.

(c) If $A_i \in \mathcal{B}$ are disjoint and $\sum_{i=1}^{\infty} A_i = \Omega$, then $\sum_{i=1}^{\infty} Q(A_i) = 1$.

Show Q is a probability measure; that is, show Q is σ-additive.

Q17

For a distribution function $F(x)$, define

$$F_l^{\leftarrow}(y) = \inf\{t : F(t) \ge y\}, \quad F_r^{\leftarrow}(y) = \inf\{t : F(t) > y\}.$$

We know $F_l^{\leftarrow}(y)$ is left-continuous.
Show $F_r^{\leftarrow}(y)$ is right-continuous and show

$$\lambda\{u \in (0, 1] : F_l^{\leftarrow}(u) \ne F_r^{\leftarrow}(u)\} = 0,$$

where λ is Lebesgue measure. Does it matter which inverse we use?

Q18

Let A, B, C be disjoint events in a probability space with

$$P(A) = 0.6, \quad P(B) = 0.3, \quad P(C) = 0.1.$$

Calculate the probabilities of every event in $\sigma(A, B, C)$.

Q19

Completion.
Let $(\Omega, \mathcal{B}, P)$ be a probability space.
Call a set N *null* if $N \in \mathcal{B}$ and $P(N) = 0$.
Call a set $B \subset \Omega$ *negligible* if there exists a null set N such that $B \subset N$.

Denote the set of all negligible subsets by $\mathcal{N}$.
Call $\mathcal{B}$ *complete* (with respect to P) if every negligible set is null.

If $\mathcal{B}$ is not complete, define

$$\mathcal{B}^* := \{A \cup M : A \in \mathcal{B},\ M \in \mathcal{N}\}.$$

(a) Show $\mathcal{B}^*$ is a σ-field.

(b) If $A_i \in \mathcal{B}$ and $M_i \in \mathcal{N}$ for $i = 1, 2$ and

$$A_1 \cup M_1 = A_2 \cup M_2,$$

then $P(A_1) = P(A_2)$.

(c) Define $P^* : \mathcal{B}^* \to [0, 1]$ by

$$P^*(A \cup M) = P(A), \quad A \in \mathcal{B},\ M \in \mathcal{N}.$$

Show P^* is an extension of P to $\mathcal{B}^*$.

(d) If $B \subset \Omega$ and $A_i \in \mathcal{B}$, $i = 1, 2$, with $A_1 \subset B \subset A_2$ and $P(A_2 \setminus A_1) = 0$,
show $B \in \mathcal{B}^*$.

(e) Show $\mathcal{B}^*$ is complete. Thus every σ-field has a completion.

(f) Suppose $\Omega = \mathbb{R}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R})$.
Let $p_k \geq 0$, $\sum_k p_k = 1$.
Let $\{a_k\}$ be any sequence in $\mathbb{R}$. Define

$$P(\{a_k\}) = p_k, \qquad P(A) = \sum_{a_k \in A} p_k, \quad A \in \mathcal{B}.$$

What is the completion of $\mathcal{B}$?

(g) Say the probability space $(\Omega, \mathcal{B}, P)$ has a complete extension $(\Omega, \mathcal{B}_1, P_1)$ if $\mathcal{B} \subset \mathcal{B}_1$ and $P_1|_{\mathcal{B}} = P$.
Suppose $(\Omega, \mathcal{B}_2, P_2)$ is a second complete extension of $(\Omega, \mathcal{B}, P)$.
Show P_1 and P_2 may not agree on $\mathcal{B}_1 \cap \mathcal{B}_2$.

(h) Is there a *minimal* extension?

Q20

In $(0, 1]$, let $\mathcal{B}$ be the class of sets that either
(a) are of the first category, or
(b) have complement of the first category.

Show that $\mathcal{B}$ is a σ-field.

For $A \in \mathcal{B}$, define $P(A) = 0$ in case (a) and $P(A) = 1$ in case (b).
Is P σ-additive?

Q21

Let $\mathcal{A}$ be a field of subsets of Ω and let μ be a finitely additive probability measure on $\mathcal{A}$ (so $\mu(\Omega) = 1$).

If $A_n \in \mathcal{A}$ and $A_n \downarrow \varnothing$, is it the case that $\mu(A_n) \downarrow 0$?
(Hint: review Problem 2.6.1 with $A_n = \{n, n+1, \ldots\}$.)

Q22

Suppose $F(x)$ is a continuous distribution function on $\mathbb{R}$.
Show F is *uniformly continuous.*

Q23

Multidimensional distribution functions.
For $a, b, x \in \mathcal{B}(\mathbb{R}^k)$, write

$$a \le b \text{ iff } a_i \le b_i,\ \ i = 1, \dots, k;$$
$$(-\infty, x] = \{u \in \mathcal{B}(\mathbb{R}^k) : u \le x\}, \quad (a, b] = \{u \in \mathcal{B}(\mathbb{R}^k) : a < u \le b\}.$$

Let P be a probability measure on $\mathcal{B}(\mathbb{R}^k)$ and define

$$F(x) = P((-\infty, x]).$$

Let $\mathcal{S}_k$ be the semialgebra of k-dimensional rectangles in $\mathbb{R}^k$.

(a) If $a \le b$, show the rectangle $I_k := (a, b]$ can be written as

$$I_k = (-\infty, b] \setminus \bigcup_{(x_1,\dots,x_k)\in\mathcal{V}\setminus\{b\}} (-\infty, (x_1, \dots, x_k)],$$

where $\mathcal{V}$ is the set of vertices of I_k other than b.

(b) Show $\mathcal{B}(\mathbb{R}^k) = \sigma((-\infty, x], x \in \mathbb{R}^k)$.

(c) Check that $\{(-\infty, x], x \in \mathbb{R}^k\}$ is a π-system.

(d) Show P is determined by $F(x)$, $x \in \mathbb{R}^k$.

(e) Show F satisfies:

1. If $x_i \to \infty$ for all $i = 1, \dots, k$, then $F(x) \to 1$.
2. If for some i we have $x_i \to -\infty$, then $F(x) \to 0$.
3. For $\mathcal{S}_k \ni I_k = (a, b]$, use the inclusion–exclusion formula to show

$$P(I_k) = \Delta_{I_k} F,$$

where

$$\Delta_{I_k} F = \sum_{x\in\mathcal{V}} \operatorname{sgn}(x) F(x), \quad \operatorname{sgn}(x) = \begin{cases} +1, & \text{if } \#\{i : x_i = a_i\} \text{ is even,} \\ -1, & \text{if } \#\{i : x_i = a_i\} \text{ is odd.} \end{cases}$$

(f) Show F is continuous from above:

$$\lim_{a \le x \downarrow a} F(x) = F(a).$$

(g) Call $F : \mathbb{R}^k \to [0, 1]$ a *multivariate distribution function* if properties (1), (2), (f), and $\Delta_{I_k} F \ge 0$ hold.
Show that any such F determines a unique probability measure P on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

Q24

Suppose λ_2 is the uniform distribution on the unit square $[0,1]^2$ defined by its distribution function

$$\lambda_2([0,\theta_1]\times[0,\theta_2]) = \theta_1\theta_2, \quad (\theta_1,\theta_2)\in[0,1]^2.$$

(a) Prove that λ_2 assigns 0 probability to the boundary of $[0,1]^2$.

(b) Calculate

$$\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2 : \theta_1\wedge\theta_2 > \tfrac{2}{3}\}.$$

(c) Calculate

$$\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2 : \theta_1\wedge\theta_2 \le x,\ \theta_1\wedge\theta_2 \le y\}.$$

Q25

In the game of bridge, 52 distinguishable cards constituting 4 equal suits are distributed at random among 4 players.
What is the probability that at least one player has a complete suit?

Q26

If $A_1,\dots,A_n$ are events, define

$$S_1 = \sum_{i=1}^{n} P(A_i), \qquad S_2 = \sum_{1\le i<j\le n} P(A_iA_j), \qquad S_3 = \sum_{1\le i<j<k\le n} P(A_iA_jA_k),$$

and so on.

(a) Show the probability (for $1\le m\le n$)

$$p(m) = P\left(\sum_{i=1}^{n} 1_{A_i} = m\right)$$

of exactly m of the events occurring is

$$p(m) = S_m - \binom{m+1}{m}S_{m+1} + \binom{m+2}{m}S_{m+2} - \cdots \pm \binom{n}{m}S_n.$$

Verify that the inclusion–exclusion formula (2.2) is a special case of this result.

(b) Referring to Example 2.1.2, compute the probability of exactly m coincidences.

Q27

Regular measures.
Consider the probability space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), P)$.
A Borel set A is *regular* if

$$P(A) = \inf\{P(G) : G\supset A,\ G \text{ open}\}, \qquad P(A) = \sup\{P(F) : F\subset A,\ F \text{ closed}\}.$$

P is *regular* if all Borel sets are regular.
Let $\mathcal{C}$ denote the collection of regular sets.

(a) Show $\mathbb{R}^k \in \mathcal{C}$ and $\varnothing \in \mathcal{C}$.

(b) Show $\mathcal{C}$ is closed under complements and countable unions.

(c) Let $\mathcal{F}(\mathbb{R}^k)$ be the collection of closed subsets of $\mathbb{R}^k$.
Show $\mathcal{F}(\mathbb{R}^k) \subset \mathcal{C}$.

(d) Show $\mathcal{B}(\mathbb{R}^k) \subset \mathcal{C}$; that is, show regularity.

(e) For any Borel set A,

$$P(A) = \sup\{P(K) : K \subset A,\ K \text{ compact}\}.$$