

PSTAT210_Ch1_Exercises

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Q1

Suppose $\Omega = \{0, 1\}$ and $\mathcal{C} = \{\{0\}\}$.

Enumerate \mathcal{N} , the class of all σ -fields containing \mathcal{C} .

Q2

Suppose $\Omega = \{0, 1, 2\}$ and $\mathcal{C} = \{\{0\}\}$.

Enumerate \mathcal{N} , the class of all σ -fields containing \mathcal{C} and give $\sigma(\mathcal{C})$.

Q3

Let A_n, A, B_n, B be subsets of Ω .

Show that

$$\limsup_{n \rightarrow \infty} (A_n \cup B_n) = (\limsup_{n \rightarrow \infty} A_n) \cup (\limsup_{n \rightarrow \infty} B_n).$$

If $A_n \rightarrow A$ and $B_n \rightarrow B$, is it true that

$$A_n \cup B_n \rightarrow A \cup B, \quad A_n \cap B_n \rightarrow A \cap B?$$

Q4

Suppose

$$A_n = \left\{ \frac{m}{n} : m \in \mathbb{N} \right\}, \quad n \in \mathbb{N},$$

where \mathbb{N} denotes the set of non-negative integers.

Find

$$\liminf_{n \rightarrow \infty} A_n \quad \text{and} \quad \limsup_{n \rightarrow \infty} A_n.$$

Q5

Let f_n, f be real functions on Ω . Show that

$$\{\omega : f_n(\omega) \neq f(\omega)\} = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| \geq \frac{1}{k}\}.$$

Q6

Suppose $a_n > 0$, $b_n > 1$, and

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} b_n = 1.$$

Define

$$A_n = \{x : a_n \leq x < b_n\}.$$

Find $\limsup_{n \rightarrow \infty} A_n$ and $\liminf_{n \rightarrow \infty} A_n$.

Q7

Let

$$I = \{(x, y) : |x| \leq 1, |y| \leq 1\}$$

be the square with sides of length 2.

Let I_n be the square pinned at $(0, 0)$ rotated through an angle $2\pi n\theta$.

Describe $\limsup_{n \rightarrow \infty} I_n$ and $\liminf_{n \rightarrow \infty} I_n$ when:

1. $\theta = \frac{1}{8}$

2. θ is rational

3. θ is irrational.

(Hint: A theorem of Weyl asserts that $\{e^{2\pi i n\theta}, n \geq 1\}$ is dense in the unit circle when θ is irrational.)

Q8

Let $B \subset \Omega$, $C \subset \Omega$, and define

$$A_n = \begin{cases} B, & n \text{ is odd,} \\ C, & n \text{ is even.} \end{cases}$$

Find $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$.

Q9

Check that

$$A \triangle B = A^c \triangle B^c.$$

Q10

Check that

$$A_n \rightarrow A \quad \text{iff} \quad 1_{A_n} \rightarrow 1_A$$

pointwise.

Q11

Let $0 \leq a_n < \infty$ be a sequence of numbers.
Prove that

$$\sup_{n \geq 1} [0, a_n) = [0, \sup_{n \geq 1} a_n),$$

but

$$\sup_{n \geq 1} [0, \frac{n}{n+1}] \neq [0, \sup_{n \geq 1} \frac{n}{n+1}].$$

Q12

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and let $\mathcal{C} = \{\{2, 4\}, \{6\}\}$.
What is the field generated by \mathcal{C} , and what is the σ -field?

Q13

Suppose $\Omega = \bigcup_{t \in T} C_t$, where $C_s \cap C_t = \emptyset$ for all $s, t \in T$ and $s \neq t$.
Suppose $\widehat{\mathcal{F}}$ is a σ -field on $\widehat{\Omega} = \{C_t, t \in T\}$.
Show that

$$\mathcal{F} := \left\{ A = \bigcup_{C_t \in \widehat{A}} C_t : \widehat{A} \in \widehat{\mathcal{F}} \right\}$$

is a σ -field, and that

$$f : \widehat{A} \mapsto \bigcup_{C_t \in \widehat{A}} C_t$$

is a one-to-one mapping from $\widehat{\mathcal{F}}$ to \mathcal{F} .

Q14

Suppose that \mathcal{A}_n are fields satisfying $\mathcal{A}_n \subset \mathcal{A}_{n+1}$.
Show that $\bigcup_n \mathcal{A}_n$ is a field.
(But see also the next problem.)

Q15

Check that the union of a countable collection of σ -fields \mathcal{B}_j , $j \geq 1$, need not be a σ -field even if $\mathcal{B}_j \subset \mathcal{B}_{j+1}$.

Is a countable union of σ -fields necessarily monotone or a field?

Hint:

Try setting Ω equal to the set of positive integers and let C_j be all subsets of $\{1, \dots, j\}$, and $\mathcal{B}_j = \sigma(C_j)$.

If $\mathcal{B}_1, \mathcal{B}_2$ are two σ -fields, $\mathcal{B}_1 \cup \mathcal{B}_2$ need not be a field.

Q19

For sets A, B , show that

$$1_{A \cup B} = 1_A \vee 1_B, \quad 1_{A \cap B} = 1_A \wedge 1_B.$$

Q20

Suppose \mathcal{C} is a non-empty class of subsets of Ω .

Let $\mathcal{A}(\mathcal{C})$ be the minimal field over \mathcal{C} .

Show that $\mathcal{A}(\mathcal{C})$ consists of sets of the form

$$\bigcup_{i=1}^m \bigcap_{j=1}^{n_i} A_{ij},$$

where for each i, j , either $A_{ij} \in \mathcal{C}$ or $A_{ij}^c \in \mathcal{C}$, and the m sets $\bigcap_{j=1}^{n_i} A_{ij}$, $1 \leq i \leq m$, are disjoint.

Thus, we can explicitly represent the sets in $\mathcal{A}(\mathcal{C})$, even though this is impossible for the σ -field over \mathcal{C} .

Q21

Suppose \mathcal{A} is a field and also has the property that it is closed under countable disjoint unions.

Show that \mathcal{A} is a σ -field.

Q22

Let Ω be a non-empty set and let \mathcal{C} be the collection of all one-point subsets.

Show that

$$\sigma(\mathcal{C}) = \{A \subseteq \Omega : A \text{ is countable}\} \cup \{A \subseteq \Omega : A^c \text{ is countable}\}.$$

Q23

- (a) Suppose on \mathbb{R} that $t_n \downarrow t$.
Show

$$(-\infty, t_n] \downarrow (-\infty, t].$$

- (b) Suppose $t_n \uparrow t$, $t_n < t$.
Show

$$(-\infty, t_n] \uparrow (-\infty, t).$$

Q24

Let $\Omega = \mathbb{N}$, the integers. Define

$$\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ or } A^c \text{ is finite}\}.$$

Show that \mathcal{A} is a field, but not a σ -field.

Q25

Suppose $\Omega = \{e^{i2\pi\theta} : 0 \leq \theta < 1\}$ is the unit circle.

Let \mathcal{A} be the collection of arcs on the unit circle with rational endpoints.

Show that \mathcal{A} is a field but not a σ -field.

Q26

- (a) Suppose \mathcal{C} is a finite partition of Ω ; that is,

$$\mathcal{C} = \{A_1, \dots, A_k\}, \quad \Omega = \bigcup_{i=1}^k A_i, \quad A_i \cap A_j = \emptyset, \quad i \neq j.$$

Show that the minimal algebra (synonym: field) $\mathcal{A}(\mathcal{C})$ generated by \mathcal{C} is the class of unions of subfamilies of \mathcal{C} ; that is,

$$\mathcal{A}(\mathcal{C}) = \left\{ \bigcup_{i \in I} A_i : I \subseteq \{1, \dots, k\} \right\}.$$

(This includes the empty set.)

- (b) What is the σ -field generated by the partition A_1, \dots, A_n ?
 (c) If A_1, A_2, \dots is a countable partition of Ω , what is the induced σ -field?
 (d) If \mathcal{A} is a field of subsets of Ω , we say $A \in \mathcal{A}$ is an *atom* of \mathcal{A} if $A \neq \emptyset$ and if $\emptyset \neq B \subset A$, $B \in \mathcal{A}$ implies $B = A$.
 (So A cannot be split into smaller nonempty sets that are still in \mathcal{A} .)

Example: If $\Omega = \mathbb{R}$ and \mathcal{A} is the field generated by intervals with integer endpoints of the form $(a, b]$ (where a, b are integers), what are the atoms?

As a converse to (a), prove that if \mathcal{A} is a finite field of subsets of Ω , then the atoms of \mathcal{A} constitute a finite partition of Ω that generates \mathcal{A} .

Q27

Show that $\mathcal{B}(\mathbb{R})$ is countably generated;
that is, show that the Borel sets are generated by a countable class \mathcal{C} .

Q28

Show that the periodic sets of \mathbb{R} form a σ -field;
that is, let \mathcal{B} be the class of sets A with the property that $x \in A$ implies $x \pm n \in A$ for all natural numbers n .
Then show that \mathcal{B} is a σ -field.

Q29

Suppose \mathcal{C} is a class of subsets of \mathbb{R} with the property that if $A \in \mathcal{C}$, then A^c is a countable union of elements of \mathcal{C} .
(For instance, the finite intervals in \mathbb{R} have this property.)

Show that $\sigma(\mathcal{C})$ is the smallest class containing \mathcal{C} which is closed under the formation of countable unions and intersections.

Q30

Let \mathcal{B}_i be σ -fields of subsets of Ω for $i = 1, 2$.
Show that the σ -field $\mathcal{B}_1 \vee \mathcal{B}_2$ defined to be the smallest σ -field containing both \mathcal{B}_1 and \mathcal{B}_2 is generated by sets of the form $B_1 \cap B_2$ where $B_i \in \mathcal{B}_i$ for $i = 1, 2$.

Q31

Suppose Ω is uncountable and let \mathcal{G} be the σ -field consisting of sets A such that either A is countable or A^c is countable.
Show that \mathcal{G} is **not countably generated**.

(Hint: If \mathcal{G} were countably generated, it would be generated by a countable collection of one-point sets.)

In fact, if \mathcal{G} is the σ -field of subsets of Ω consisting of the countable and co-countable sets,
then \mathcal{G} is countably generated **iff** Ω is countable.

Q32

Suppose $\mathcal{B}_1, \mathcal{B}_2$ are σ -fields of subsets of Ω such that $\mathcal{B}_1 \subset \mathcal{B}_2$ and \mathcal{B}_2 is countably generated.

Show by example that it is **not** necessarily true that \mathcal{B}_1 is countably generated.

Q33

The **extended real line** is defined as

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}.$$

The Borel sets $\mathcal{B}(\overline{\mathbb{R}})$ are generated by the sets $[-\infty, x]$, $x \in \mathbb{R}$, where $[-\infty, x] = \{-\infty\} \cup (-\infty, x]$.

Show that $\mathcal{B}(\overline{\mathbb{R}})$ is also generated by the following collections of sets:

1. $[-\infty, x)$, $x \in \mathbb{R}$,
2. $(x, \infty]$, $x \in \mathbb{R}$,
3. All finite intervals, together with $\{-\infty\}$ and $\{+\infty\}$.

Now think of $\overline{\mathbb{R}} = [-\infty, \infty]$ as homeomorphic in the topological sense to $[-1, 1]$ under the transformation

$$x \mapsto \frac{x}{1 - |x|}.$$

This transformation maps $[-1, 1]$ to $[-\infty, \infty]$, stretching the finite interval into the infinite interval.

Using the usual topology on $[-1, 1]$, map it onto a topology on $[-\infty, \infty]$.

This defines a collection of open sets on $[-\infty, \infty]$, which can be used to generate a Borel σ -field.

Question: How does this σ -field compare with $\mathcal{B}(\overline{\mathbb{R}})$ described above?

Q34

Suppose \mathcal{B} is a σ -field of subsets of Ω and suppose $A \notin \mathcal{B}$.

Show that $\sigma(\mathcal{B} \cup \{A\})$, the smallest σ -field containing both \mathcal{B} and A , consists of sets of the form

$$AB \cup A^c B', \quad B, B' \in \mathcal{B}.$$

Q35

A σ -field cannot be countably infinite.

Its cardinality is either finite or at least that of the continuum.

Q36

Let $\Omega = \{f, a, n, g\}$ and

$$\mathcal{C} = \{\{f, a, n\}, \{a, n\}\}.$$

Find $\sigma(\mathcal{C})$.

Q37

Suppose $\Omega = \mathbb{Z}$, the integers.
Define for integer k :

$$k\mathbb{Z} = \{kz : z \in \mathbb{Z}\}.$$

Find $\mathcal{B}(\mathcal{C})$ when \mathcal{C} is:

1. $\{3\mathbb{Z}\}$
 2. $\{3\mathbb{Z}, 4\mathbb{Z}\}$
 3. $\{3\mathbb{Z}, 4\mathbb{Z}, 5\mathbb{Z}\}$
 4. $\{3\mathbb{Z}, 4\mathbb{Z}, 5\mathbb{Z}, 6\mathbb{Z}\}$
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Q38

Let $\Omega = \mathbb{R}^\infty$, the space of all sequences of the form

$$\omega = (x_1, x_2, \dots).$$

Let σ be a permutation of $\{1, \dots, n\}$; that is, σ is a one-to-one and onto map of $\{1, \dots, n\}$ onto itself.
If ω is the sequence defined above, define $\sigma\omega$ to be the new sequence

$$(\sigma\omega)_j = \begin{cases} x_{\sigma(j)}, & j \leq n, \\ x_j, & j > n. \end{cases}$$

A **finite permutation** is one that permutes only finitely many coordinates.
A set $\Lambda \subset \Omega$ is **permutable** if

$$\Lambda = \sigma\Lambda := \{\sigma\omega : \omega \in \Lambda\}$$

for all finite permutations σ .

- (i) Let $\{B_n\}_{n \geq 1}$ be a sequence of subsets of \mathbb{R} .
Show that

$$\{\omega = (x_1, x_2, \dots) : \sum_{i=1}^n x_i \in B_n \text{ infinitely often}\}$$

and

$$\{\omega = (x_1, x_2, \dots) : \bigvee_{i=1}^n x_i \in B_n \text{ infinitely often}\}$$

are permutable.

(ii) Show that the permutable sets form a σ -field.

Q39

For a subset $A \subset \mathbb{N}$ of non-negative integers, write $\text{card}(A)$ for the number of elements in A .

A set $A \subset \mathbb{N}$ has **asymptotic density** d if

$$\lim_{n \rightarrow \infty} \frac{\text{card}(A \cap \{1, 2, \dots, n\})}{n} = d.$$

Let \mathcal{A} be the collection of subsets that have an asymptotic density.

Is \mathcal{A} a field? Is it a σ -field?

Hint:

\mathcal{A} is closed under complements, proper differences, and finite disjoint unions, but is **not** closed under countable disjoint unions or finite unions that are not disjoint.

Q40

Show that $\mathcal{B}((0, 1])$ is generated by the following countable collection:

For an integer r ,

$$\{[kr^{-n}, (k+1)r^{-n}) : 0 \leq k < r^n, n = 1, 2, \dots\}.$$

Q41

A **monotone class** \mathcal{M} is a non-empty collection of subsets of Ω closed under monotone limits; that is, if $A_n \uparrow A$ and $A_n \in \mathcal{M}$, then

$$\lim_{n \rightarrow \infty} A_n = \bigcup_n A_n \in \mathcal{M},$$

and if $A_n \downarrow A$ and $A_n \in \mathcal{M}$, then

$$\lim_{n \rightarrow \infty} A_n = \bigcap_n A_n \in \mathcal{M}.$$

Show that a σ -field is a field that is also a monotone class, and conversely, a field that is a monotone class is a σ -field.

Q42

Assume \mathcal{P} is a π -system (that is, \mathcal{P} is closed under finite intersections) and \mathcal{M} is a monotone class (see Exercise 41). Show that

$$\mathcal{P} \subset \mathcal{M} \quad \text{does not imply} \quad \sigma(\mathcal{P}) \subset \mathcal{M}.$$

Q43

Symmetric differences. For subsets A, B, C, D , show

$$1_{A \triangle B} = 1_A + 1_B \pmod{2}.$$

Hence prove:

1. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$,
 2. $(A \triangle B) \triangle (B \triangle C) = A \triangle C$,
 3. $(A \triangle B) \triangle (C \triangle D) = (A \triangle C) \triangle (B \triangle D)$,
 4. $A \triangle B = C \iff A = B \triangle C$,
 5. $A \triangle B = C \triangle D \iff A \triangle C = B \triangle D$.
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Q44

Let \mathcal{A} be a field of subsets of Ω and define

$$\overline{\mathcal{A}} = \{A \subseteq \Omega : \exists A_n \in \mathcal{A}, A_n \rightarrow A\}.$$

Show that $\mathcal{A} \subset \overline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ is a field.