MATH124A_Midterm_practice

TAO XU

Section 1.1

1. Verify the linearity and nonlinearity of the eight example PDEs given in the text by checking whether or not the conditions
   $L(u + v) = Lu + Lv$ and $L(cu) = cLu$ hold.
2. Which of the following operators are linear?
   (a) $Lu = u_x + xu_y$
   (b) $Lu = u_x + uu_y$
   (c) $Lu = u_x + u_y^2$
   (d) $Lu = u_x + u_y + 1$
   (e) $Lu = \sqrt{1 + x^2}\,(\cos y)\, u_x + u_{xy} - [\arctan(x/y)]u$
3. For each of the following equations, state its order and classify as nonlinear, linear inhomogeneous, or linear homogeneous. Give reasons.
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$
   (g) $u_x + e^y u_y = 0$
   (h) $u_t + u_{xxxx} + \sqrt{1 + u} = 0$
4. Show that the difference of two solutions of an inhomogeneous linear equation $Lu = g$ with the same g is a solution of the homogeneous equation $Lu = 0$.
5. Which of the following collections of 3-vectors $[a, b, c]$ are vector spaces? Provide reasons.
   (a) Vectors with $b = 0$.
   (b) Vectors with $b = 1$.

(c) Vectors with $ab = 0$.

(d) All linear combinations of $[1,1,0]$ and $[2,0,1]$.

(e) All vectors satisfying $c - a = 2b$.

6. Determine whether the three vectors $[1,2,3]$, $[-2,0,1]$, and $[1,10,17]$ are linearly dependent or independent. Do they span $\mathbb{R}^3$?
7. Are the functions $1+x$, $1-x$, and $1+x+x^2$ linearly dependent or independent? Explain why.
8. Find a vector that, together with $[1,1,1]$ and $[1,2,1]$, forms a basis of $\mathbb{R}^3$.
9. Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis and its dimension.
10. Show that the solutions of $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis for it.
11. Verify that $u(x,y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all (differentiable) functions f and g of one variable.
12. Verify by direct substitution that

$$u_n(x,y) = \sin(nx)\,\sinh(ny)$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

Section 1.2

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.
2. Solve the equation $3u_y + u_{xy} = 0$. (Hint: Let $v = u_y$.)
3. Solve $(1+x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.
4. Check that the general solution $u(x,y) = f(e^{-xy})$ indeed solves $u_x + yu_y = 0$.
5. Solve $xu_x + yu_y = 0$.
6. Solve $\sqrt{1-x^2}\; u_x + u_y = 0$ with the condition $u(0,y) = y$.
7. (a) Solve $yu_x + xu_y = 0$ with $u(0,y) = e^{-y^2}$.

   (b) In which region of the xy-plane is the solution uniquely determined?
8. Solve $au_x + bu_y + cu = 0$.
9. Solve $u_x + u_y = 1$.
10. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.
11. Solve $au_x + bu_y = f(x,y)$ (given f).
    If $a \neq 0$, write your solution in the form

$$u(x,y) = (a^2+b^2)^{-1/2} \int_L f\, ds + g(bx - ay),$$

where g is an arbitrary function of one variable, and L is the characteristic line segment from the y-axis to (x,y) (line integral).

12. Show that the new coordinate axes defined by

$$x' = ax + by, \qquad y' = bx - ay$$

are orthogonal.

13. Use the coordinate method to solve

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

Section 1.3

1. Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity.

2. A flexible chain of length l is hanging from one end $x = 0$ but oscillates horizontally.

   Let the x-axis point downward and the u-axis point to the right.

   Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain.

   Assume that the oscillations are small.

   Find the PDE satisfied by the chain.

3. On the sides of a thin rod, heat exchange takes place (obeying Newton's law of cooling—flux proportional to temperature difference) with a medium of constant temperature T_0.

   What is the equation satisfied by the temperature $u(x, t)$, neglecting its variation across the rod?

4. Suppose that some particles which are suspended in a liquid medium would be pulled down at the constant velocity $V > 0$ by gravity in the absence of diffusion.

   Taking account of the diffusion, find the equation for the concentration of particles.

   Assume homogeneity in the horizontal directions x and y.

   Let the z-axis point upwards.

5. Derive the equation of one-dimensional diffusion in a medium that is moving along the x-axis to the right at constant speed V.

6. Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder.

   Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate.

   From the three-dimensional heat equation derive the equation

   $$u_t = k\,(u_{rr} + u_r/r)$$

7. Solve Exercise 6 in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2 + y^2 + z^2}$.

   Derive the equation

   $$u_t = k\,(u_{rr} + 2u_r/r)$$

8. For the hydrogen atom, if $\int |u|^2\,dx = 1$ at $t = 0$, show that the same is true at all later times.

   (Hint: Differentiate the integral with respect to t, taking care about the solution being complex valued. Assume that u and $\nabla u \to 0$ fast enough as $|x| \to \infty$.)

9. This is an exercise on the divergence theorem

   $$\iiint_D \nabla \cdot F\,dx = \iint_{\partial D} F \cdot n\,dS$$

   valid for any bounded domain D in space with boundary surface ∂D and unit outward normal vector n.

Verify it in the following case by calculating both sides separately:

$$F = r^2 x, \quad x = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}, \quad r^2 = x^2 + y^2 + z^2$$

and D is the ball of radius a and center at the origin.

10. If $f(x)$ is continuous and $|f(x)| \le 1/(|x|^3 + 1)$ for all x, show that

$$\iiint_{\text{all space}} \nabla \cdot f\, dx = 0$$

(Hint: Take D to be a large ball, apply the divergence theorem, and let its radius tend to infinity.)

11. If $\operatorname{curl} v = 0$ in all of three-dimensional space, show that there exists a scalar function $\phi(x, y, z)$ such that $v = \nabla\phi$.

Section 1.4

1. By trial and error, find a solution of the diffusion equation

$$u_t = u_{xx}$$

with the initial condition

$$u(x, 0) = x^2$$

2.

(a) Show that the temperature of a metal rod, insulated at the end $x = 0$, satisfies the boundary condition

$$\frac{\partial u}{\partial x} = 0$$

(Use Fourier's law.)

(b) Do the same for the diffusion of gas along a tube that is closed off at the end $x = 0$. (Use Fick's law.)

(c) Show that the three-dimensional version of (a) (insulated solid) or (b) (impermeable container) leads to the boundary condition

$$\frac{\partial u}{\partial n} = 0$$

3. A homogeneous body occupying the solid region D is completely insulated. Its initial temperature is $f(x)$. Find the **steady-state temperature** that it reaches after a long time. *(Hint: No heat is gained or lost.)*

4. A rod occupying the interval $0 \le x \le l$ is subject to the heat source

$$f(x) = 0 \text{ for } 0 < x < \frac{l}{2}, \qquad f(x) = H \text{ for } \frac{l}{2} < x < l$$

where $H > 0$.

The rod has constants $c = \rho = \kappa = 1$, and its ends are kept at zero temperature.

(a) Find the **steady-state temperature** of the rod.

(b) Which point is the hottest, and what is the temperature there?

5. In Exercise 1.3.4, find the **boundary condition** if the particles lie above an impermeable horizontal plane $z = a$.

6. Two homogeneous rods have the same cross section, specific heat c, and density ρ,

   but different conductivities κ_1, κ_2 and lengths L_1, L_2.

   Let $k_j = \kappa_j/(c\rho)$ be their diffusion constants.

   They are welded together so that temperature u and heat flux κu_x are continuous.

   The left rod has its left end maintained at temperature 0, the right rod at temperature T.

   (a) Find the **equilibrium temperature distribution** in the composite rod.

   (b) Sketch it as a function of x for $k_1 = 2,\ k_2 = 1,\ L_1 = 3,\ L_2 = 2,\ T = 10$.

7. In linearized gas dynamics (sound), verify the following:

   (a) If $\operatorname{curl} v = 0$ at $t = 0$, then $\operatorname{curl} v = 0$ for all later times.

   (b) Each component of v and ρ satisfies the wave equation.