

# PSTAT210CH0

TAO XU

## Review of Real Analysis

Let  $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R} = (-\infty, \infty)$ .

We also work with the **extended real line**

$$\overline{\mathbb{R}} = [-\infty, \infty].$$

## Upper and Lower Limits of a Sequence

### Definition

Let  $(x_n)$  be a sequence in  $\overline{\mathbb{R}}$ .

We call  $x^* \in [-\infty, \infty]$  the **upper limit** (limit superior) of  $(x_n)$  if

$$\limsup_{n \rightarrow \infty} x_n = x^*,$$

meaning:

- (a) For every  $y \in \mathbb{R}$  with  $y > x^*$ ,  
there exists  $N(y)$  such that  $x_n < y$  for all  $n \geq N(y)$ .
- (b) For every  $y < x^*$ ,  
there exist infinitely many  $n$  such that  $x_n > y$ .

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Similarly,  $x_* \in [-\infty, \infty]$  is the **lower limit** (limit inferior) if

$$\liminf_{n \rightarrow \infty} x_n = x_*,$$

meaning:

For every  $y < x_*$ ,  
there exists  $N(y)$  such that  $x_n > y$  for all  $n \geq N(y)$ .

For every  $y > x_*$ ,  
there exist infinitely many  $n$  such that  $x_n < y$ .

### Example

Let  $X = (-1, 1)$ .

Define

$$x_n = \begin{cases} -n, & n \text{ even,} \\ 1/n, & n \text{ odd.} \end{cases}$$

Then

$$\liminf_{n \rightarrow \infty} x_n = -\infty, \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

## Remark

If

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n,$$

then the sequence has a limit, denoted simply by  $\lim_{n \rightarrow \infty} x_n$ .

## Lemma: Existence of limsup and liminf

### Lemma.

Every sequence in  $\mathbb{R}$  has a unique upper (and lower) limit in  $\overline{\mathbb{R}} = [-\infty, \infty]$ .

### Proof sketch.

It suffices to prove the lemma on the compact interval  $[-1, 1]$ .

Define the continuous increasing function

$$f(x) = \frac{2}{\pi} \arctan(x), \quad x \in \mathbb{R},$$

so that  $f(x) \in [-1, 1]$ .

Let

$$y_n = \sup\{x_m : m \geq n\}.$$

Then  $(y_n)$  is decreasing and bounded below, hence convergent.

Let  $y_\infty = \lim_{n \rightarrow \infty} y_n$ . This is the candidate for  $\limsup x_n$ .

### (a) Upper bound property

Fix  $a > y_\infty$ .

Choose  $N(a)$  such that  $y_n \leq a$  for all  $n \geq N(a)$ .

Then  $x_n \leq y_n \leq a$  for all  $n \geq N(a)$ .

### (b) Attainment from below

Fix  $b = y_\infty - \varepsilon < y_\infty$  for some  $\varepsilon > 0$ .

Choose  $n_1$  large enough so that

$$y_{n_1} \in [y_\infty - \varepsilon, y_\infty + \varepsilon].$$

By definition of a supremum, choose  $n_2 \geq n_1$  such that

$$x_{n_2} \in [y_{n_1} - \varepsilon, y_{n_1}].$$

Then

$$x_{n_2} \geq y_\infty - 2\varepsilon,$$

showing infinitely many terms lie arbitrarily close below  $y_\infty$ .

Therefore,

$$\limsup_{n \rightarrow \infty} x_n = y_\infty.$$

## Compactness

A subset  $A$  of a metric space is **compact** if every open cover admits a finite subcover.

If  $\{G_\alpha : \alpha \in A\}$  is an open cover of  $A$ , compactness means:

There exist finitely many indices  $\alpha_1, \dots, \alpha_k$   
such that

$$G_{\alpha_1} \cup \dots \cup G_{\alpha_k} \supseteq A.$$

Suppose  $K \subset \bigcup_{\alpha \in A} G_\alpha$ , where each  $G_\alpha$  is open.

Compactness means:

$$\exists A_0 \subset A \text{ finite such that } K \subset \bigcup_{\alpha \in A_0} G_\alpha.$$

## Theorems on Compactness

### Heine–Borel Theorem

A subset of  $\mathbb{R}^n$  is compact **iff** it is **closed and bounded**.

In  $\mathbb{R}$ , examples of compact sets include closed intervals:

$$[a, b], \quad a, b \in \mathbb{R}.$$

### Sequential Compactness in Complete Metric Spaces

#### Theorem.

Let  $K$  be a subset of a **complete metric space**. Then the following are equivalent:

1.  $K$  is complete.
2. Every sequence in  $K$  has a subsequence converging to a point in  $K$ .

This means:

Completeness of  $K$  guarantees existence of convergent subsequences whose limits stay inside  $K$ .

## Metric Spaces

A **metric space** is a pair  $(X, d)$  where

$d : X \times X \rightarrow [0, \infty)$  satisfies:

For all  $x, y, z \in X$ :

1. **Identity of indiscernibles**

$$d(x, y) = 0 \iff x = y.$$

2. **Symmetry**

$$d(x, y) = d(y, x).$$

3. **Triangle inequality**

$$d(x, z) \leq d(x, y) + d(y, z).$$

These three conditions define the notion of “distance” in abstract spaces.