PDE_Chapter_2_Exercises

TAO XU

Section 2.1

1. Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = e^x,\ u_t(x,0) = \sin x$.
2. Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = \log(1+x^2),\ u_t(x,0) = 4 + x$.
3. A string of length l, tension T,and mass density ρ is struck at its midpoint by a hammer of diameter $2a$.

   A flea sits at a point $l/4$ from one end.

   If $a < l/4$, how long will it take the disturbance to reach the flea?
4. Show that every solution of the one-dimensional wave equation can be written as

   $u(x,t) = f(x+ct) + g(x-ct)$.
5. **Hammer Blow**:
   Let $\phi \equiv 0$ and

   $$\psi(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$$

   Sketch $u(x,t)$ for $t = a/2c,\ a/c,\ 3a/2c,\ 2a/c,\ 5a/c$.

   Use

   $$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}$$
6. In Exercise 5, write an explicit formula for the maximum displacement

   $\max_x u(x,t)$ as a function of t.
7. Suppose both ϕ and ψ are **odd** functions of x. Show that $u(x,t)$ remains odd in x for all t.
8. **Spherical Waves.** A spherically symmetric solution $u(r,t)$ of the 3-D wave equation satisfies

   $$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$

   (a) Show that for $v = ru$, the function v satisfies $v_{tt} = c^2 v_{rr}$.

   (b) Use the d'Alembert formula to find v and hence u.

   (c) For initial data $u(r,0) = \phi(r),\ u_t(r,0) = \psi(r)$ (with ϕ, ψ even), express $u(r,t)$ explicitly.
9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ with $u(x,0) = x^2,\ u_t(x,0) = e^x$.

   (Hint – factor the operator as in the wave equation.)
10. Solve $u_{xx} + u_x - 20u_{tt} = 0$, with $u(x,0) = \phi(x),\ u_t(x,0) = \psi(x)$.
11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

Section 2.2

1. Use the energy conservation of the wave equation to prove that if the initial data are $\phi \equiv 0$ and $\psi \equiv 0$, then the only solution is $u \equiv 0$.

   (Hint: use the first vanishing theorem in Appendix A.1.)

2. For a solution $u(x,t)$ of the wave equation with $\rho = T = c = 1$, define the energy density $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density $p = u_t u_x$.

   (a) Show $\partial_t e = \partial_x p$ and $\partial_t p = \partial_x e$.

   (b) Show that both $e(x,t)$ and $p(x,t)$ also satisfy the wave equation

3. Show that the wave equation has the following invariances:

   (a) Any translate $u(x-y,t)$ (with y fixed) is also a solution.

   (b) Any derivative of a solution, e.g. u_x, is also a solution.

   (c) The dilated function $u(ax,at)$ is also a solution for any constant a.

4. If $u(x,t)$ satisfies $u_{tt} = u_{xx}$, prove the identity

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$$

   for all x, t, h, k. Sketch the quadrilateral Q whose vertices are the four points in the identity.

5. For the **damped string** (equation (1.3.3)), show that the total energy decreases.

6. Prove that, among all spatial dimensions, **only in three dimensions** can one have distortionless spherical wave propagation with attenuation.

   Specifically, for the n-dimensional spherical wave

$$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right)$$

   consider solutions of the form $u(r,t) = \alpha(r)\, f\big(t - \beta(r)\big)$, where $\alpha(r)$ is the attenuation and $\beta(r)$ the delay.

   Show:

   (a) Substituting the ansatz yields an ODE for f.

   (b) The coefficients of f'', f', and f must vanish.

   (c) Solving gives $n = 1$ or $n = 3$ (unless $u \equiv 0$).

   (d) If $n = 1$, then $\alpha(r)$ is constant (no attenuation).