MATH201A_Chapter1_TEXTBOOK_Question

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2025-10-19

Chapter 1 Exercises

Question 1

A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences (i.e., if $E_1, \dots, E_n \in \mathcal{R}$, then $\bigcup_{j=1}^n E_j \in \mathcal{R}$, and if $E, F \in \mathcal{R}$, then $E \setminus F \in \mathcal{R}$). A ring that is closed under countable unions is called a **σ-ring**.

1.

(a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
(b) If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.
(c) If $\mathcal{R}$ is a σ-ring, then

$$E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}$$

is a σ-algebra.
(d) If $\mathcal{R}$ is a σ-ring, then

$$E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}$$

is a σ-algebra.

Question 2

Complete the proof of **Proposition 1.2**.

Question 3

Let $\mathcal{M}$ be an infinite σ-algebra. (a) Show that $\mathcal{M}$ contains an infinite sequence of disjoint sets. (b) Show that $\mathrm{card}(\mathcal{M}) \geq \mathfrak{c}$.

Question 4

An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $E_j * j = 1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup * j = 1^\infty E_j \in \mathcal{A}$).

Question 5

If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. *(Hint: Show that the latter object is a σ-algebra.)*

Question 6

Complete the proof of **Theorem 1.9**.

Question 7

If $\mu_1, \dots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \dots, a_n \in [0, \infty)$, then $\sum_{j=1}^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Question 8

If $(X, \mathcal{M}, \mu)$ is a measure space and $E_j * j = 1^\infty \subset \mathcal{M}$, then

$$\mu(\liminf E_j) \leq \liminf \mu(E_j),$$

and $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that

$$\mu! \left(\bigcup *j = 1^\infty E_j \right) < \infty.$$

Question 9

If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Question 10

Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define

$$\mu_E(A) = \mu(A \cap E)$$

for $A \in \mathcal{M}$. Then μ_E is a measure.

Question 11

A finitely additive measure μ is a measure iff it is continuous from below as in **Theorem 1.8(c)**. If $\mu(X) < \infty$, then μ is a measure iff it is continuous from above as in **Theorem 1.8(d)**.

Question 12

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a. If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.
b. Say that $E \sim F$ if $\mu(E \triangle F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.
c. For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/! \sim$ of equivalence classes.

Question 13

Every σ-finite measure is **semifinite**.

Question 14

If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

Question 15

Given a measure μ on $(X, \mathcal{M})$, define

$$\mu_0(E) = \sup \mu(F) : F \subset E, ; \mu(F) < \infty.$$

a. μ_0 is a semifinite measure, called the **semifinite part** of μ.
b. If μ is semifinite, then $\mu = \mu_0$. (*Use Exercise 14.*)
c. There is a measure ν on $\mathcal{M}$ (not necessarily unique) which assumes only the values 0 and ∞ such that

$$\mu = \mu_0 + \nu.$$

Question 16

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if

$$E \cap A \in \mathcal{M} \text{ for all } A \in \mathcal{M} \text{ with } \mu(A) < \infty.$$

Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called **saturated**.

a. If μ is σ-finite, then μ is saturated.
b. $\widetilde{\mathcal{M}}$ is a σ-algebra.
c. Define $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the **saturation** of μ.
d. If μ is complete, so is $\tilde{\mu}$.
e. Suppose μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define

$$\overline{\mu}(E) = \sup \mu(A) : A \in \mathcal{M}, A \subset E.$$

Then $\overline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.
f. Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and let $\mathcal{M}$ be the σ-algebra of countable or co-countable sets in X. Let μ_0 be the counting measure on $\mathcal{P}(X_1)$, and define $\mu(E) = \mu_0(E \cap X_1)$ for $E \in \mathcal{M}$. Then μ is a measure on $\mathcal{M}$, $\widetilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e),

$$\tilde{\mu} \neq \overline{\mu}.$$

Question 17

If μ^* is an outer measure on X and $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint μ^*-measurable sets, then

$$\mu^*\left(E \cap \bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$$

for any $E \subset X$.

Question 18

Let $A \subset \mathcal{P}(X)$ be an algebra,
A_σ the collection of countable unions of sets in A,
and $A_{\sigma\delta}$ the collection of countable intersections of sets in A_σ.
Let μ_0 be a premeasure on A and μ^* the induced outer measure.

a. For any $E \subset X$ and $\varepsilon > 0$, there exists $A \in A_\sigma$ with $E \subset A$ and

$$\mu^*(A) \leq \mu^*(E) + \varepsilon.$$

b. If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

c. If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Question 19

Let μ^* be an outer measure on X induced from a finite premeasure μ_0.
If $E \subset X$, define the **inner measure** of E by

$$\mu_*(E) = \mu_0(X) - \mu^*(E^c).$$

Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.
(*Use Exercise 18.*)

Question 20

Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets,
and define the outer and inner measures induced by $\bar{\mu}$ as in (1.12).

a. If $E \subset X$, then $\mu^*(E) \leq \bar{\mu}(E)$, with equality iff there exists $A \in \mathcal{M}^*$ such that $A \supset E$ and $\mu^*(A) = \mu^*(E)$.

b. If μ^* is induced from a premeasure, then $\mu^* = \bar{\mu}$. (*Use Exercise 18a.*)

c. If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \bar{\mu}$.

Question 21

Let μ^* be an outer measure induced from a premeasure and $\bar{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\bar{\mu}$ is **saturated**. (*Use Exercise 18.*)

Question 22

Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to (1.12),
$\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$.

a. If μ is σ-finite, then $\bar{\mu}$ is the completion of μ. (*Use Exercise 18.*)

b. In general, $\bar{\mu}$ is the **saturation of the completion** of μ. (*See Exercises 16 and 21.*)

Question 23

Let A be the collection of finite unions of sets of the form $[a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

a. A is an algebra on $\mathbb{Q}$. (*Use Proposition 1.7.*)

b. The σ-algebra generated by A is $\mathcal{P}(\mathbb{Q})$.

c. Define μ_0 on A by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$.
Then μ_0 is a premeasure on A, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to A is μ_0.

Question 24

Let μ be a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure induced by μ.
Suppose $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not necessarily $E \in \mathcal{M}$).

a. If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

b. Let $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$, and define $\nu(A \cap E) = \mu(A)$.
Then $\mathcal{M}_E$ is a σ-algebra on E, and ν is a measure on $\mathcal{M}_E$.

Question 25

Complete the proof of **Theorem 1.19**.

Question 26

Prove **Proposition 1.20**. (*Use Theorem 1.18.*)

Question 27

Prove **Proposition 1.22(a)**.
(Show that if $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$.)

Question 28

Let F be increasing and right-continuous, and let μ_F be the associated measure.
Then

$$\mu_F(\{a\}) = F(a) - F(a^-), \qquad \mu_F([a,b)) = F(b^-) - F(a^-), \qquad \mu_F([a,b]) = F(b) - F(a^-), \qquad \mu_F((a,b)) = F(b^-) - F(a).$$

Question 29

Let E be a Lebesgue-measurable set.

a. If $E \subset N$, where N is the non-measurable set described in §1.1, then $m(E) = 0$.

b. If $m(E) > 0$, then E contains a non-measurable set.
(It suffices to assume $E \subset [0,1]$. In the notation of §1.1,
$E = \bigcup_{r \in \mathbb{R}} (E \cap N_r)$.)

Question 30

If $E \in \mathcal{L}$ and $m(E) > 0$, then for any $\alpha < 1$ there exists an open interval I such that

$$m(E \cap I) > \alpha\, m(I).$$

Question 31

If $E \in \mathcal{L}$ and $m(E) > 0$, then the set

$$E - E = \{x - y : \, x, y \in E\}$$

contains an interval centered at 0.
(If I is as in Exercise 30 with $\alpha > \frac{3}{4}$, then $E - E$ contains $(-\frac{1}{2}m(I),\ \frac{1}{2}m(I))$.)

Question 32

Suppose $\{\alpha_j\}_{j=1}^{\infty} \subset (0,1)$.

a. $\displaystyle\prod_{j=1}^{\infty}(1 - \alpha_j) > 0$ iff $\displaystyle\sum_{j=1}^{\infty} \alpha_j < \infty$.
(Compare $\sum \log(1 - \alpha_j)$ with $\sum \alpha_j$.)

b. Given $\beta \in (0,1)$, exhibit a sequence $\{\alpha_j\}$ such that

$$\prod_{j=1}^{\infty}(1 - \alpha_j) = \beta.$$

Question 33

There exists a Borel set $A \subset [0,1]$ such that

$$0 < m(A \cap I) < m(I)$$

for every subinterval $I \subset [0,1]$.
(*Hint:* Every subinterval of $[0,1]$ contains Cantor-type sets of positive measure.)