MATH201A_Chapter2_Exercises

TAO XU

2.1 (Measurable functions)

Below $(X, \mathcal{M})$ is always a measurable space.

Exercise 1

Let

$$f : X \to \overline{\mathbb{R}}, \qquad Y = f^{-1}(\mathbb{R}).$$

Then f is measurable iff

- $f^{-1}(\{-\infty\}) \in \mathcal{M}$,
- $f^{-1}(\{\infty\}) \in \mathcal{M}$,

and
f is measurable on Y.

Exercise 2

Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.
(a)
The product

$$fg : X \to \overline{\mathbb{R}}, \qquad 0 \cdot (\pm\infty) = 0,$$

is measurable.
(b)
Fix $a \in \overline{\mathbb{R}}$ and define

$$h(x) = \begin{cases} a, & f(x) = -g(x) = \pm\infty, \\ f(x) + g(x), & \text{otherwise.} \end{cases}$$

Then h is measurable.

Exercise 3

If $\{f_n\}$ is a sequence of measurable functions on X, then the set

$$\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is measurable.

Exercise 4

If $f : X \to \mathbb{R}$ and

$$f^{-1}((r, \infty]) \in \mathcal{M} \qquad \forall r \in \mathbb{Q},$$

then f is measurable.

Exercise 5

If $X = A \cup B$ where $A, B \in \mathcal{M}$, then
f is measurable on X iff f is measurable on both A and B.

Exercise 6

The supremum of an uncountable family of measurable $\mathbb{R}$-valued functions on X may fail to be measurable (unless the σ-algebra $\mathcal{M}$ is very special).

Exercise 7

Suppose for each $\alpha \in \mathbb{R}$ we are given sets $E_\alpha \in \mathcal{M}$ such that:

- $\alpha < \beta \Rightarrow E_\alpha \subset E_\beta$,
- $\bigcup_{\alpha \in \mathbb{R}} E_\alpha = X$,
- $\bigcap_{\alpha \in \mathbb{R}} E_\alpha = \emptyset$.

Then there exists a measurable function $f : X \to \mathbb{R}$ such that:

- $f(x) \leq \alpha$ on E_α,
- $f(x) \geq \alpha$ on E_α^c.

(Hint: Use Exercise 4.)

Exercise 8

If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Exercise 9

Let $f : [0,1] \to [0,1]$ be the Cantor function, and define

$$g(x) = f(x) + x.$$

(a)
g is a bijection from $[0,1]$ to $[0,2]$, and

$$h = g^{-1} : [0,2] \to [0,1]$$

is continuous.
(b)
If C is the Cantor set, then

$$m(g(C)) = 1.$$

(c)
By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue non-measurable set.
Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel measurable.

(d) There exist:

- a Lebesgue measurable function
$$F : \mathbb{R} \to \mathbb{R},$$
- and a continuous function
$$G : \mathbb{R} \to \mathbb{R},$$

such that the composition

$$F \circ G$$

is not Lebesgue measurable.

Exercise 10

Prove Proposition 2.11.
(Insert proposition text here if needed — I can also supply a fully typeset R-markdown proof.)

Exercise 11

Suppose

$$f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$$

satisfies:

1. for each fixed $x \in \mathbb{R}$,
$$y \mapsto f(x, y) \quad \text{is Borel measurable;}$$
2. for each fixed $y \in \mathbb{R}^k$,
$$x \mapsto f(x, y) \quad \text{is continuous.}$$

Construction of the approximating functions

For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, define

$$a_i = \frac{i}{n}.$$

For $a_i \le x \le a_{i+1}$, define

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) + f(a_i, y)(a_{i+1} - x)}{a_{i+1} - a_i}.$$

Equivalently, $f_n(x, y)$ is the linear interpolation in the x-variable between the values

$$f(a_i, y), \qquad f(a_{i+1}, y).$$

Claim 1: f_n is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$.

- On each strip $[a_i, a_{i+1}] \times \mathbb{R}^k$,
  f_n is a finite linear combination of
  $f(a_i, y)$ and $f(a_{i+1}, y)$
  with coefficients depending continuously on x only.
- Since $y \mapsto f(a_i, y)$ is Borel measurable for each i,
  f_n is measurable on each strip.
- The union of the strips covers $\mathbb{R} \times \mathbb{R}^k$.

Thus
f_n is Borel measurable.

Claim 2: $f_n(x, y) \to f(x, y)$ pointwise.

For each fixed y:

- $x \mapsto f(x, y)$ is continuous,
- $f_n(\cdot, y)$ is the linear interpolation of $f(\cdot, y)$ on a mesh of size $1/n$.

Hence

$$\lim_{n \to \infty} f_n(x, y) = f(x, y).$$

Conclusion

Since f_n are Borel measurable and

$$f_n \to f \quad \text{pointwise},$$

it follows that
f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$.

Final Statement (Induction)

Conclude by induction on dimension that:
Every function

$$f : \mathbb{R}^n \to \mathbb{R}$$

that is continuous in each variable separately
is Borel measurable.

2.2 (Integration of Nonnegative functions)

Exercise 12

Prove Proposition 2.20. (See Proposition 0.20, where a special case is proved.)

Exercise 13

Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ for all $E \in \mathcal{M}$. However, this need not be true if $\int f = \lim \int f_n = \infty$.

Exercise 14

If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$. (First suppose that g is simple.)

Exercise 15

If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

Exercise 16

If $f \in L^+$ and $\int f < \infty$, for every $\varepsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \varepsilon$.

Exercise 17

Assume Fatou's lemma and deduce the monotone convergence theorem from it.

2.3 (Integration of complex functions)

Exercise 18

Fatou's lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n \geq -g$ where $g \in L^+ \cap L^1$. What is the analogue of Fatou's lemma for nonpositive functions?

Exercise 19

Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.
a. If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.
b. If $\mu(X) = \infty$, the conclusions of (a) can fail. (Find examples on $\mathbb{R}$ with Lebesgue measure.)

Exercise 20

(A Generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$, $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem.)

Exercise 21

Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use Exercise 20.)

Exercise 22

Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

Exercise 23

Given a bounded function $f : [a, b] \to \mathbb{R}$, let

$$H(x) = \lim_{\delta \to 0} \sup_{|y-x| \leq \delta} f(y), \quad h(x) = \lim_{\delta \to 0} \inf_{|y-x| \leq \delta} f(y).$$

Prove Theorem 2.28b by establishing the following lemmas:
a. $H(x) = h(x)$ iff f is continuous at x.
b. In the notation of the proof of Theorem 2.28a, $H = G$ a.e. and $h = g$ a.e. Hence H and h are Lebesgue measurable, and $\int_{[a,b]} H\, dm = \overline{\int_a^b} f$, and $\int_{[a,b]} h\, dm = \underline{\int_a^b} f$.

Exercise 24

Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. Suppose $f : X \to \mathbb{R}$ is bounded. Then f is $\overline{\mathcal{M}}$-measurable (and hence in $L^1(\overline{\mu})$) iff there exist sequences $\{\phi_n\}$ and $\{\psi_n\}$ of $\mathcal{M}$-measurable simple functions such that $\phi_n \leq f \leq \psi_n$ and $\int (\psi_n - \phi_n)\, d\mu < n^{-1}$. In this case, $\lim \int \phi_n\, d\mu = \lim \int \psi_n\, d\mu = \int f\, d\overline{\mu}$.

Exercise 25

Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.
a. $g \in L^1(m)$, and in particular $g < \infty$ a.e.
b. g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.
c. $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

Exercise 26

If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^x f(t)\, dt$, then F is continuous on $\mathbb{R}$.

Exercise 27

Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.
a. $\sum_1^\infty \int_0^\infty |f_n(x)|\, dx = \infty$.
b. $\sum_1^\infty \int_0^\infty f_n(x)\, dx = 0$.
c. $\sum_1^\infty f_n \in L^1([0, \infty), m)$, and $\int_0^\infty \sum_1^\infty f_n(x)\, dx = \log(b/a)$.

Exercise 28

Compute the following limits and justify the calculations:
a. $\lim_{n\to\infty}\int_0^\infty(1+(x/n))^{-n}\sin(x/n)\,dx$.
b. $\lim_{n\to\infty}\int_0^1(1+nx^2)(1+x^2)^{-n}\,dx$.
c. $\lim_{n\to\infty}\int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}\,dx$.
d. $\lim_{n\to\infty}\int_0^\infty n(1+n^2x^2)^{-1}\,dx$. (The answer depends on whether $a>0$, $a=0$, or $a<0$. How does this accord with the various convergence theorems?)

Exercise 29

Show that $\int_0^\infty x^n e^{-x}\,dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx}\,dx = 1/t$. Similarly, show that $\int_{-\infty}^\infty x^{2n}e^{-x^2}\,dx = (2n)!\sqrt{\pi}/4^n n!$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2}\,dx = \sqrt{\pi/t}$ (see Proposition 2.53).

Exercise 30

Show that $\lim_{k\to\infty}\int_0^k x^n(1-k^{-1}x)^k\,dx = n!$.

Exercise 31

Derive the following formulas by expanding part of the integrand into an infinite series and justifying the term-by-term integration. Exercise 29 may be useful. (Note: In (d) and (e), term-by-term integration works, and the resulting series converges, only for $a>1$, but the formulas as stated are actually valid for all $a>0$.)
a. For $a>0$, $\int_{-\infty}^\infty e^{-x^2}\cos ax\,dx = \sqrt{\pi}e^{-a^2/4}$.
b. For $a>-1$, $\int_0^1 x^a(1-x)^{-1}\log x\,dx = \sum_1^\infty(a+k)^{-2}$.
c. For $a>1$, $\int_0^\infty x^{a-1}(e^x-1)^{-1}\,dx = \Gamma(a)\zeta(a)$, where $\zeta(a)=\sum_1^\infty n^{-a}$.
d. For $a>1$, $\int_0^\infty e^{-ax}x^{-1}\sin x\,dx = \arctan(a^{-1})$.
e. For $a>1$, $\int_0^\infty e^{-ax}J_0(x)\,dx = (s^2+1)^{-1/2}$, where $J_0(x)=\sum_0^\infty(-1)^n x^{2n}/4^n(n!)^2$ is the Bessel function of order zero.

2.4 (modes of convergence)

Exercise 32

Suppose $\mu(X)<\infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f,g)=\int\frac{|f-g|}{1+|f-g|}\,d\mu.$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n\to f$ with respect to this metric iff $f_n\to f$ in measure.

Exercise 33

If $f_n>0$ and $f_n\to f$ in measure, then $\int f\le\liminf\int f_n$.

Exercise 34

Suppose $|f_n|\le g\in L^1$ and $f_n\to f$ in measure. a. $\int f=\lim\int f_n$. b. $f_n\to f$ in L^1.

Exercise 35

$f_n\to f$ in measure iff for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $\mu(\{x:|f_n(x)-f(x)|\ge\varepsilon\})<\varepsilon$ for $n\ge N$.

Exercise 36

If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

Exercise 37

Suppose that f_n and f are measurable complex-valued functions and $\phi : \mathbb{C} \to \mathbb{C}$. a. If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e. b. If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly, or in measure, respectively. c. There are counterexamples when the continuity assumptions on ϕ are not satisfied.

Exercise 38

Suppose $f_n \to f$ in measure and $g_n \to g$ in measure. a. $f_n + g_n \to f + g$ in measure. b. $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Exercise 39

If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Exercise 40

In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$".

Exercise 41

If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, \dots \subset X$ such that $\mu((\bigcup_1^\infty E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

Exercise 42

Let μ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Exercise 43

Suppose that $\mu(X) < \infty$ and $f : X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$. a. If $0 < \varepsilon, \delta < 1$ then $E_{\varepsilon,\delta} = \{x : |f(x,y) - f(x,0)| \leq \varepsilon \text{ for all } y < \delta\}$ is measurable. b. For any $\varepsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \varepsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

Exercise 44

(Lusin's Theorem) If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \varepsilon$ and $f|_E$ is continuous. (Use Egoroff's theorem and Theorem 2.26.)

2.5 (Product Measure)

Exercise 45

If $(X_j, \mathcal{M}_j)$ is a measurable space for $j = 1, 2, 3$, then $\bigotimes_{j=1}^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Exercise 46

Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x,z) : x = z \in [0,1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

Exercise 47

Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. (Example: the set of countable ordinals.)

Let $\mathcal{M} = \mathcal{N}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x,y) \in X \times X : y < x\}$. Then E_x and E^y are measurable for all x, y, and $\iint 1_E \, d\mu \, d\nu$ and $\iint 1_E \, d\nu \, d\mu$ exist but are not equal. (If one believes in the continuum hypothesis, one can take $X = [0,1]$ [with a nonstandard ordering] and thus obtain a set $E \subset [0,1]^2$ such that E_x is countable and E^y is co-countable [in particular, Borel] for all x, y, but E is not Lebesgue measurable.)

Exercise 48

Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $n = m+1$, and $f(m,n) = 0$ otherwise. Then $\int |f| \, d(\mu \times \nu) = \infty$, and $\iint f \, d\nu \, d\mu$ and $\iint f \, d\mu \, d\nu$ exist and are unequal.

Exercise 49

Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas. a. If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y. b. If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)

Exercise 50

Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x,y) \in X \times [0,\infty) : y \leq f(x)\}.$$

Then G_f is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \, d\mu$; the same is also true if the inequality $y \leq f(x)$ in the definition of G_f is replaced by $y < f(x)$. (To show measurability of G_f, note that the map $(x,y) \mapsto f(x) - y$ is the composition of $(x,y) \mapsto (f(x), y)$ and $(z,y) \mapsto z - y$.) This is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph."

Exercise 51

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite). a. If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x,y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable. b. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \, d(\mu \times \nu) = \left[\int f \, d\mu\right] \left[\int g \, d\nu\right]$.

Exercise 52

The Fubini-Tonelli theorem is valid when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathcal{N} = \mathcal{P}(Y)$, and ν is counting measure on Y. (Cf. Theorems 2.15 and 2.25.)

2.6 (The n-dim leabesuge integral)

Exercise 53

Fill in the details of the proof of Theorem 2.41.

Exercise 54

How much of Theorem 2.44 remains valid if T is not invertible?

Exercise 55

Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of $\int_E f\, dm^2$, $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$, and $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$ for the following f.

a. $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$.

b. $f(x,y) = (1 - xy)^{-a} (a > 0)$.

c. $f(x,y) = (x - \frac{1}{2})^{-3}$ if $0 < y < |x - \frac{1}{2}|$, $f(x,y) = 0$ otherwise.

Exercise 56

If f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1} f(t)\, dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x)\, dx = \int_0^a f(x)\, dx$.

Exercise 57

Show that $\int_0^\infty e^{-sx} x^{-1} \sin x\, dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy} \sin x$ with respect to x and y. (It may be useful to recall that $\tan(\frac{\pi}{2} - \theta) = (\tan\theta)^{-1}$. Cf. Exercise 31d.)

Exercise 58

Show that $\int e^{-sx} x^{-1} \sin^2 x\, dx = \frac{1}{4} \log(1 + 4s^{-2})$ for $s > 0$ by integrating $e^{-sx} \sin 2xy$ with respect to x and y.

Exercise 59

Let $f(x) = x^{-1} \sin x$.

a. Show that $\int_0^\infty |f(x)|\, dx = \infty$.

b. Show that $\lim_{b\to\infty} \int_0^b f(x)\, dx = \frac{\pi}{2}$ by integrating $e^{-xy} \sin x$ with respect to x and y. (In view of part (a), some care is needed in passing to the limit as $b \to \infty$.)

Exercise 60

$\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ for $x, y > 0$. (Recall that Γ was defined in §2.3. Write $\Gamma(x)\Gamma(y)$ as a double integral and use the argument of the exponential as a new variable of integration.)

Exercise 61

If f is continuous on $[0,\infty)$, for $\alpha > 0$ and $x \geq 0$ let

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt.$$

$I_\alpha f$ is called the αth fractional integral of f.

a. $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$ for all $\alpha, \beta > 0$. (Use Exercise 60.)

b. If $n \in \mathbb{N}$, $I_n f$ is an nth-order antiderivative of f.

2.7 (Integration in polar coordinates)

Exercise 62

The measure σ on S^{n-1} is invariant under rotations.

Exercise 63

The technique used to prove Proposition 2.54 can also be used to integrate any polynomial over S^{n-1}. In fact, suppose $f(x) = \prod_1^n x_j^{\alpha_j}$ ($\alpha_j \in \mathbb{N} \cup \{0\}$) is a monomial. Then $\int f\, d\sigma = 0$ if any α_j is odd, and if all α_j's are even,

$$\int f\, d\sigma = \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1 + \cdots + \beta_n)}, \quad \text{where } \beta_j = \frac{\alpha_j + 1}{2}.$$

Exercise 64

For which real values of a and b is $|x|^a |\log|x||^b$ integrable over $\{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$? Over $\{x \in \mathbb{R}^n : |x| > 2\}$?

Exercise 65

Define $G : \mathbb{R}^n \to \mathbb{R}^n$ by $G(r, \phi_1, \ldots, \phi_{n-2}, \theta) = (x_1, \ldots, x_n)$ where

$$x_1 = r\cos\phi_1, \quad x_2 = r\sin\phi_1\cos\phi_2, \quad x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3, \ldots,$$

$$x_{n-1} = r\sin\phi_1\cdots\sin\phi_{n-2}\cos\theta, \quad x_n = r\sin\phi_1\cdots\sin\phi_{n-2}\sin\theta.$$

a. G maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, and $|G(r, \phi_1, \ldots, \phi_{n-2}, \theta)| = r$.

b. $\det D_{(r,\phi_1,\ldots,\phi_{n-2},\theta)} G = r^{n-1}\sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdots \sin\phi_{n-2}$.

c. Let $\Omega = (0, \infty) \times (0, \pi)^{n-2} \times (0, 2\pi)$. Then $G|\Omega$ is a diffeomorphism and $m(\mathbb{R}^n \setminus G(\Omega)) = 0$.

d. Let $F(\phi_1, \ldots, \phi_{n-2}, \theta) = G(1, \phi_1, \ldots, \phi_{n-2}, \theta)$ and $\Omega' = (0, \pi)^{n-2} \times (0, 2\pi)$. Then $(F|\Omega')^{-1}$ defines a coordinate system on S^{n-1} except on a σ-null set, and the measure σ is given in these coordinates by

$$d\sigma(\phi_1, \ldots, \phi_{n-2}, \theta) = \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdots \sin\phi_{n-2}\, d\phi_1 \cdots d\phi_{n-2} d\theta.$$