“Simple” Sum Stochastic Collocation Explicit Analytic Demonstration

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1 Introduction

We're using the sparse grid approximation described elsewhere,

$$u(Y) \approx S[u](Y) \equiv \sum_{\vec{i}\in\Lambda(L)} S^i[u](Y), \tag{1}$$

$$S^i[u](Y) = c(\vec{i}) \bigotimes_{n=1}^{N} U[u](Y), \tag{2}$$

$$c(\vec{i}) \equiv \sum_{\vec{j}\in\{0,1\}^N,\ \vec{j}+\vec{i}\in\Lambda} (-1)^{|\vec{j}|_1}, \tag{3}$$

$$\bigotimes_{n=1}^{N} U[u](Y) \equiv \sum_{\vec{k}}^{\vec{i}} u_h(Y^{(\vec{k})}) L_{\vec{k}}(Y). \tag{4}$$

We choose TD index set with expansion level 2, giving us the index set

$$\Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\}. \tag{5}$$

We will attempt to show $S[u](1,1) = u(1,1)$ for $u(Y_1,Y_2) = Y_1+Y_2$, with all Y_n uniformly distributed between 1 and 6. Quadrature points are taken using Gauss-Legendre quadrature with order $m = i+1$.

2 Coefficients

We determine the coefficients $c(\vec{i})$ as

$\vec{i}$	$\vec{j}$	$\lvert\vec{j}\rvert_1$	$\sum(-1)^{\lvert\vec{j}\rvert_1}$	$c(\vec{i})$
(0,0)	(0,0),(0,1),(1,0),(1,1)	0,1,2,1	1-1+1-1	0
(0,1)	(0,0),(0,1),(1,0)	0,1,1	1-1-1	-1
(0,2)	(0,0)	0	1	1
(1,0)	(0,0),(0,1),(1,0)	0,1,1	1-1-1	-1
(1,1)	(0,0)	0	1	1
(2,0)	(0,0)	0	1	1

3 By Index

We calculate the individual terms by index $\vec{i} \in \Lambda$. Final results are shown here. As can be seen, it evaluates to the expected value.

$\vec{i}$	S^i
(0,0)	0
(0,1)	-4.5
(0,2)	4.5
(1,0)	-4.5
(1,1)	2
(2,0)	4.5
TOTAL	2

3.1 (0,0)

Because $c^{0,0} = 0$, there's no need to calculate this term. However,

$$m = i + 1^N = (1,1) \quad \rightarrow \quad Y \in (3.5, 3.5), \tag{6}$$

$$\begin{aligned} S^{0,0} &= c^{0,0} \sum_{k_1=0}^{i_1=0} \sum_{k_2=0}^{i_2=0} u_h(Y^k) \prod_{n=1}^{2} L_{k_n}(Y_n), &(7)\\ &= 0 \cdot u_h(Y_1^0, Y_2^0) L_0(Y_1) L_0(Y_2), &(8)\\ &= 0 \cdot (3.5 + 3.5) = 0. &(9) \end{aligned}$$

3.2 (0,1), equal to (1,0)

$$m = i + 1^N = (1,2) \quad \rightarrow \quad Y \in (3.5, 2.06), (3.5, 4.94), \tag{10}$$

$$\begin{aligned} S^{0,1} &= c^{0,1} \sum_{k_1=0}^{i_1=0} \sum_{k_2=0}^{i_2=0} u_h(Y^k) \prod_{n=1}^{2} L_{k_n}(Y_n), &(11)\\ &= -1 \left[u_h(Y_1^0, Y_2^0) L_0(Y_1) L_0(Y_2) + u_h(Y_1^0, Y_2^1) L_0(Y_1) L_1(Y_2) \right]. &(12) \end{aligned}$$

$$L_0(Y_2) = \prod_{i=0, i\neq 0}^{i=1} \frac{Y_2 - Y_2^i}{Y_2^0 - Y_2^i} = \frac{Y_2 - Y_2^1}{Y_2^0 - Y_2^1} = \frac{1 - 4.94}{2.06 - 4.94} = 1.368, \tag{13}$$

$$L_1(Y_2) = \prod_{i=0, i\neq 1}^{i=1} \frac{Y_2 - Y_2^i}{Y_2^0 - Y_2^i} = \frac{Y_2 - Y_2^0}{Y_2^1 - Y_2^0} = \frac{1 - 2.06}{4.94 - 2.06} = -0.368, \tag{14}$$

$$S^{0,1} = -1 \left[(3.5 + 2.06)(1.368) + (3.5 + 4.94)(-0.368) \right] = -4.5. \tag{15}$$

3.3 (0,2), equal to (0,2)

$$m = (1,3) \quad \rightarrow \quad \vec{Y} \in (3.5, 2.72), (3.5, 3.5), (3.5, 4.27), \tag{16}$$

$$S^{0,2} = 1[u_h(Y_1^0, Y_2^0) L_0(Y_1) L_0(Y_2) + u_h(Y_1^0, Y_2^1) L_0(Y_1) L_1(Y_2) + u_h(Y_1^0, Y_2^2) L_0(Y_1) L_2(Y_2)], \tag{17}$$

$$L_0(Y_2) = \left(\frac{1 - 3.5}{2.72 - 3.5} \right) \left(\frac{1 - 4.27}{2.72 - 4.27} \right) = 6.76, \tag{18}$$

$$L_1(Y_2) = \left(\frac{1 - 2.72}{3.5 - 2.72} \right) \left(\frac{1 - 4.27}{3.5 - 4.27} \right) = -9.36, \tag{19}$$

$$L_2(Y_2) = \left(\frac{1 - 2.72}{4.27 - 2.72} \right) \left(\frac{1 - 3.5}{4.27 - 3.5} \right) = 3.6, \tag{20}$$

$$S^{0,2} = 1 \left[(3.5 + 2.72)6.76 - (3.5 + 3.5)9.36 + (3.5 + 4.27)3.6 \right] = 4.5. \tag{21}$$

3.4 (1,1)

$$m = (2,2) \quad \rightarrow \quad \vec{Y} \in (2.06, 2.06), (2.06, 4.94), (4.94, 2.06), (4.94, 4.94), \tag{22}$$

$$\begin{aligned} S^{1,1} = 1\Big[&u_h(Y_1^0, Y_2^0)L_0(Y_1)L_0(Y_2) + u_h(Y_1^0, Y_2^1)L_0(Y_1)L_1(Y_2) \\ &+ u_h(Y_1^1, Y_2^0)L_1(Y_1)L_0(Y_2) + u_h(Y_1^1, Y_2^1)L_1(Y_1)L_1(Y_2)\Big], \end{aligned} \tag{23}$$

$$L_0(Y_1) = L_0(Y_2) = \frac{1 - 4.94}{2.06 - 4.94} = 1.368, \tag{24}$$

$$L_1(Y_1) = L_1(Y_2) = \frac{1 - 2.06}{4.94 - 2.06} = -0.368, \tag{25}$$

$$S^{1,1} = (2.06 + 2.06)1.368^2 - 2(2.06 + 4.94)(0.368 \cdot 1.368) + (4.94 + 4.94)0.368^2 = 2. \tag{26}$$