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QUADRATURE AND INTERPOLATION FORMULAS FOR TENSOR PRODUCTS OF CERTAIN CLASSES OF FUNCTIONS

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Let K be a normed linear space and let $\tau_i, \tau_i^{(j)}$ be certain of its elements. Let K^s be the tensor product of K by itself s times, i. e., the space of formal finite sums of the form $\sum_{1\le p\le N} \lambda_p \tau_p^{(1)} \otimes \cdots \otimes \tau_p^{(s)}$ (where the λ_p are numbers), for which addition and multiplication by numbers is defined in a trivial manner and which has been factored with respect to all relations of the form

$$\Big(\sum_{p=1}^{N_1} \lambda_p^{(1)}\tau_p^{(1)}\Big) \otimes \cdots \otimes \Big(\sum_{p=1}^{N_s} \lambda_p^{(s)}\tau_p^{(s)}\Big) = \sum_{p_1=1}^{N_1} \cdots \sum_{p_s=1}^{N_s} \lambda_{p_1}^{(1)} \cdots \lambda_{p_s}^{(s)} \tau_{p_1}^{(1)} \otimes \cdots \otimes \tau_{p_s}^{(s)}.$$

We shall consider those K^s for which it is possible to introduce a norm such that $\|\tau_1 \otimes \cdots \otimes \tau_s\| = \|\tau_1\| \cdots \|\tau_s\|$. Then we have the following

Theorem. *Let* $\vartheta_\nu (\nu = 0, 1, \cdots, q)$ *and* I *be elements of* K *such that for* $\alpha > 0$

$$\|I\| \leqslant B, \quad \|\vartheta_\nu\| \leqslant B, \quad \|I - \vartheta_\nu\| \leqslant A \cdot 2^{-\nu\alpha},$$

$$\theta_0 = \vartheta_0, \quad \theta_\nu = \vartheta_\nu - \vartheta_{\nu-1} \quad (\nu \geqslant 1).$$

Then

$$\|I \otimes \cdots \otimes I - \sum_{\nu_1 + \cdots + \nu_s \leqslant q} \theta_{\nu_1} \otimes \cdots \otimes \theta_{\nu_s}\| \leqslant C(A, B, s, \alpha) \frac{q^{s-1}}{2^{\alpha q}}.$$

Moreover, the indicated sum is in fact a linear combination of terms $\vartheta_{\nu_1} \otimes \cdots \otimes \vartheta_{\nu_s}$ *with* $q - s \le \nu_1 + \cdots + \nu_s \le q$.

The proof of this theorem is entirely trivial in case K and K^s are spaces of numbers, the operation of tensor multiplication coincides with ordinary multiplication, and by norm of a number is understood its modulus. The proof in the general case is completely analogous.

From this simple theorem may be deduced a series of interesting corollaries concerning quadrature and interpolation formulas for certain classes of functions, for instance W_s^α [1], E_s^α [2], H_s^α [3,4].

Let K be, for instance, the space of continuous linear functionals on the class W_1^α [1]—the class of 1-periodic functions expandable in a Fourier series $f(x) = \sum_{-\infty}^{\infty} a_m e^{2\pi i m x}$ with norm $\|f\|^2 = \sum_{-\infty}^{\infty} |c_m|^2 \bar{m}^{2\alpha}$.* Then the elements of K may be identified with sequences $\tau = (\cdots, c_m, \cdots)$ where $\|\tau\|^2 = \sum_{-\infty}^{\infty} |c_m|^2 \bar{m}^{-2\alpha}$. Let $\tau^{(i)} = (\cdots, c_m^{(i)}, \cdots)$. Define $\tau^{(1)} \otimes \cdots \otimes \tau^{(s)}$ as the infinite tensor of rank s

$$T = \begin{pmatrix} & \vdots & \cdot^{\cdot^{\cdot}} \\ \cdots & c_{m_1 \ldots m_s} & \cdots \\ \cdot^{\cdot^{\cdot}} & \vdots & \end{pmatrix}, \tag{1}$$

where $c_{m_1 \cdots m_s} = c_{m_1}^{(1)} \cdots c_{m_s}^{(s)}$. We shall understand addition of tensors and their multiplication by numbers in the usual sense and define the norm of tensor (1) by the formula

*Here and in the following $\bar{m} = \max(1, |m|)$.

$$\| T \|^2 = \sum_{m_1, \ldots, m_s} | c_{m_1 \ldots m_s} |^2 (\bar{m}_1 \ldots \bar{m}_s)^{-2\alpha}.$$

Then K^s will be, generally speaking, a subspace of the normed linear space of continuous linear functionals (c.l.f.) on the class W_s^α and the norms in the two classes will coincide. For I we take the c.l.f. of integration on W_1^α: $(I, f(x)) = \int_0^1 f(x)\, dx$. Then $I \otimes \cdots \otimes I$ will be the c.l.f. of s-fold integration in W_s^α. For ϑ_ν we take the c.l.f. corresponding to some good quadrature formula in W_1^α with 2^ν nodes of integration, for instance: $(\vartheta_\nu, f(x)) = (1/2^\nu) \sum_{k=1}^{2\nu} f(k/2^\nu)$. It is easily verified that for $\alpha > 1/2$ we have $\| I \| = 1$, $\| \vartheta_\nu \| \le B(\alpha)$, $\| I - \vartheta_\nu \| \le A(\alpha) \cdot 2^{-\nu\alpha}$.

In order to clarify the meaning of the assertion of the theorem in this case, we make the following remark. If $\vartheta(\xi) \in K$ is defined by the equation $(\delta(\xi), f(x)) = f(\xi)$, then this c.l.f. is identified with the sequence $(\cdots, e^{2\pi i m \xi}, \cdots)$, and then the c.l.f. $\delta(\xi_1, \cdots, \xi_s) = \delta(\xi_1) \otimes \cdots \otimes \delta(\xi_s)$ is identified with the tensor (1) where $c_{m_1 \cdots m_s} = e^{2\pi i (m_1 \xi_1 + \cdots + m_s \xi_s)}$. Therefore, $(\delta(\xi_1, \cdots, \xi_s), f(x_1, \cdots, x_s)) = f(\xi_1, \cdots, \xi_s)$ for $f \in W_s^\alpha$. If $\tau^{(i)}$ is a linear combination of $\delta(\xi_1^{(i)}), \cdots, \delta(\xi_{N_i}^{(i)})$, then $\tau^{(1)} \otimes \cdots \otimes \tau^{(s)}$ is a linear combination of $\delta(\xi_{n_1}^{(1)}, \cdots, \xi_{n_s}^{(s)})$ $(1 \le n_i \le N_i)$. Since in our case ϑ_ν, and thus θ_ν, are linear combinations of $\delta(1/2^\nu), \cdots, \delta(2^\nu/2^\nu), \cdots$, therefore $\theta_{\nu_1} \otimes \cdots \otimes \theta_{\nu_s}$ is a linear combination of the terms $\delta(n_1/2^{\nu_1}), \cdots, (n_s/2^{\nu_s})$. Consequently, the c.l.f. $\sum_{\nu_1 + \cdots + \nu_s \le q} \theta_{\nu_1} \otimes \cdots \otimes \theta_{\nu_s}$, applied to any function from W_s^α will give a linear combination of its values at the nodes of the grid

$$\left(\frac{n_1}{2^{\nu_1}}, \ldots, \frac{n_s}{2^{\nu_s}} \right), \quad 1 \leqslant n_i \leqslant 2^{\nu_i}, \quad \nu_1 + \ldots + \nu_s \leqslant q,$$

$$\left(\sum_{\nu_1 + \ldots + \nu_s \leqslant q} \theta_{\nu_1} \otimes \ldots \otimes \theta_{\nu_s}, f(x_1, \ldots, x_s) \right) = \tag{2}$$

$$= \sum_{\nu_1 + \ldots + \nu_s \leqslant q} \; \sum_{1 \leqslant n_i \leqslant 2^{\nu_i}} p_{\nu_1 \ldots \nu_s}^{n_1 \ldots n_s} f\left(\frac{n_1}{2^{\nu_1}}, \ldots, \frac{n_s}{2^{\nu_s}} \right).$$

The coefficients $p_{\nu_1 \cdots \nu_s}^{n_1 \cdots n_s}$ may be calculated in $C(s)$ operations while, according to the theorem, all of the coefficients with $\nu_1 + \cdots + \nu_s < q - s$ will be equal to zero. It is easily seen, moreover, that we will use values of the function at $O(q^{s-1} \cdot 2^q)$ points. According to our theorem and by the definition of the norm of a c.l.f.

$$\left| \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_s)\, dx_1 \ldots dx_s - \sum_{\nu_1 + \ldots + \nu_s \leqslant q} \; \sum_{1 \leqslant n_i \leqslant 2^{\nu_i}} p_{\nu_1 \ldots \nu_s}^{n_1 \ldots n_s} f\left(\frac{n_1}{2^{\nu_1}}, \ldots, \frac{n_s}{2^{\nu_s}} \right) \right| \leqslant$$

$$\leqslant C(\alpha, s) \frac{q^{s-1}}{2^{\alpha q}} \| f \|_{W_s^\alpha}.$$

If we denote the number of nodes of integration by N, then $N = O(q^{s-1} \cdot 2^q)$ and the right side of the last inequality will be $O(N^{-\alpha} \log^{(\alpha+1)(s-1)} N)$. Taking the class E_1^α instead of W_1^α, i.e., defining the norm $\|\tau\| = \sum_{-\infty}^{\infty} |c_m| \bar{m}^{-\alpha}$, in K, we will obtain analogously that the same quadrature formula gives in turn the same error on the class E_s^α. Taking in place of W_1^α the class E_2^α, in place of I the c.l.f. of double integration over the unit square, setting

$$\vartheta_\nu = \frac{1}{u_\nu} \sum_{k=1}^{u_\nu} \delta\left(\frac{k}{u_\nu}, \frac{k u_{\nu-2}}{u_\nu}\right) \quad (u_1 = 1,\ u_2 = 2, \ldots, u_n = u_{n-1} + u_{n-2})$$

and using the result of the paper [5] in the form $\|I - \vartheta_\nu\| \leq A(\alpha)\, \nu / u_\nu^\alpha$, we arrive analogously at the quadrature formula in E_{2s}^α with $O(q^{s-1} u_q)$ nodes. As can be seen by modifying slightly the proof of the fundamental theorem, the corresponding error will be $O(q^{2s-1}/u_q^\alpha)$ or, in terms of the number of nodes, $O((\log^{(s-1)(\alpha+2)+1} N)/N^\alpha)$.

By modifying the proof a little more, we can obtain an estimate of the error of the quadrature formula on the class E_{2s-1}^α in the form $O((\log^{(s-1)(\alpha+2)} N)/N^\alpha)$. For $\alpha \geq 2$ these results are more precise than the error estimates of quadrature formulas over parallelepiped grids obtained in the papers [2,5].

Now take

$$I = \delta(\xi), \quad \vartheta_\nu = \frac{1}{2^\nu} \sum_{k=1}^{2^\nu} V_{2^\nu+1}\left(\xi - \frac{k}{2^\nu}\right) \delta\left(\frac{k}{2^\nu}\right),$$

in the theorem, where $V_n(x) = (\sin^2 2\pi n x - \sin^2 \pi n x)/n \sin^2 \pi x$ is the Vallée-Poussin operator [6]. Using the boundedness of the integral and interpolating norm of V_n [7] and estimates of the best approximations by trigonometric polynomials of periodic functions from the classes E_1^α, W_1^α and $\widetilde{H}_1^\alpha$ (the last class is denoted $W_*^{(r)} H^{(\beta)}$ in [7], where $r = [\alpha]$, $\beta = \{\alpha\}$), we can obtain

$$\|I - \vartheta_\nu\|_{E_1^\alpha} = O(2^{-\nu(\alpha-1)}), \quad \|I - \vartheta_\nu\|_{W_1^\alpha} = O(2^{-\nu(\alpha-1/2)}) \quad \|I - \vartheta_\nu\|_{\widetilde{H}_1^\alpha} = O(2^{-\nu\alpha}).$$

Then we obtain from our theorem an interpolation formula with nodes (2) which yields accuracy (in the sense of the maximum of the modulus of the error) $O((\log^{\alpha(s-1)} N)/N^{\alpha-1})$ on E_s^α, $O((\log^{(\alpha+1/2)(s-1)} N)/N^{\alpha-1/2})$ on W_s^α, and $O((\log^{(\alpha+1)(s-1)} N)/N^\alpha)$ on $\widetilde{H}_s^\alpha$.

On the other hand, it can be shown that no interpolation formula with N nodes can give accuracy better than $O(1/N^{\alpha-1})$ on E_s^α, better than $O(1/N^{\alpha-1/2})$ on W_s^α, or better than $O(1/N^\alpha)$ on the class $\widetilde{H}_s^\alpha$. More accurate lower estimates have been obtained at the present time by I. F. Šarygin [13].

Interpolation over parallelepiped grids is considered in the papers [1, 4, 9, 10]. However, as is shown in [1, 10], the best estimate in this case for all of the classes considered can not be better than $O(1/N^{\alpha/2})$, so that the present result is the most accurate one existing and can be improved only with respect to the logarithms.

In the case of the classes of nonperiodic functions H_s^α [3, 4] or $W^{(r)} H^{(\beta)}$[7], instead of the c.l.f. ϑ_ν that we have considered we must take others corresponding to good quadrature or interpolation formulas on these classes, for instance the quadrature formulas of Sard or Nikol'skiĭ [8] or the interpolation formulas of Newton. Error estimates in the nonperiodic case will in turn be the same as in the periodic case (for the corresponding classes of course) and will be more precise than for the quadrature and interpolation formulas considered in [3, 4].

The proposed construction has been applied to estimates of the diameters of the classes $\widetilde{H}_s^\alpha$ and of those similar to them. However, the method indicated in this paper only yields another proof of the theorem of K. I. Babenko [11]. On the basis of the quadrature and interpolation formulas that have been described one can also construct methods for numerical integration of integral and differential equations, etc. Random elements of K may be chosen for the ϑ_ν, allowing us to obtain estimates in the mean of the remainder terms of the quadrature formulas with a random choice either of the nodes of integration themselves or of the number of these nodes if the corresponding results are known in the one-dimensional case. The results obtained do not differ essentially from the results of the paper [12]. For instance, the mathematical expectation of the modulus of the error of the quadrature formula in W_s^{α} for $\alpha > 1/2$ will be $O\left(\frac{\log^{(\alpha+3/2)(s-1)}N}{N^{\alpha+1/2}}\right)$, while in the paper [12] it is estimated by $O\left(\frac{\log^{s+1}N}{N^{\alpha+1/2}}\right)$.

The applicability of our method to the classes E_s^{α}, W_s^{α}, H_s^{α} depends on the fact that the spaces of c.l.f. on them contain tensor products of spaces of c.l.f. on corresponding one-dimensional spaces.

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