1 Example: Single-Dimensional expansion on [-1,1]

Starting with the simplest case, we consider a function of single variable $\zeta = \zeta_1$,

$$f(\zeta) = a + b\zeta, \quad \zeta \in [-1,1], \tag{1}$$

where a and b are arbitrary scalars. We expand $f(\zeta)$ in normalized Legendre polynomials,

$$\begin{aligned} f(\zeta) &= \sum_{i=0}^{\infty} f_i P_i(\zeta), &(2)\\ &= \sum_{i=0}^{2} f_i P_i(\zeta). &(3) \end{aligned}$$

We can truncate the sum at 2 terms because we know a priori $f(\zeta)$ is order 1 in ζ, so it can be exactly represented by Legendre polynomials of up to order 1, and we keep the order 2 term simply for demonstration. Using the orthogonality of the normalized Legendre polynomials, we find the coefficients f_i given by

$$f_i = \int_{-1}^{1} f(\zeta) P_i(\zeta) d\zeta. \tag{4}$$

We can approximate the integral (exactly, as it turns out) with Gauss-Legendre quadrature,

$$\begin{aligned} f_i &= \sum_{\ell=0}^{\infty} w_\ell f(\zeta_\ell) P_i(\zeta_\ell), &(5)\\ &= \sum_{\ell=0}^{2} w_\ell f(\zeta_\ell) P_i(\zeta_\ell), &(6) \end{aligned}$$

where once again, because we know the Legendre polynomial order is no greater than 2 and $f(\zeta)$ is order 1, the integral has maximum order 3 and Legendre quadrature can exactly integrate polynomials of order $2n-1=3$ in our case. It is trivial to insert the values from the Legendre quadrature set and see that the coefficients obtained are

$$f_0 = \frac{2a}{\sqrt{2}}, \quad f_1 = b\sqrt{\frac{2}{5}}, \quad f_2 = 0. \tag{7}$$

If we reconstruct $f(\zeta)$ using these coefficients and the first three normalized Legendre polynomials, we obtain our original function $a + b\zeta$.

2 Arbitrary Domain Uniform Uncertainty

We now consider the same function but of variable ξ that ranges on [3,7] instead of [-1,1]. We can perform a transformation of variables to project it onto the [-1,1] space. We still expand

$$f(\xi) \equiv a + b\xi = \sum_{i=0}^{\infty} f_i P_i(\xi), \quad \xi \in [3,7]. \tag{8}$$

The coefficients f_i are calculated like above, but over the support space [3,5],

$$f_i = \int_{3}^{5} f(\xi) P_i(\xi) d\xi. \tag{9}$$

Since Gauss-Legendre quadrature provides ordinates and weights on [-1,1], we introduce a substitution variable such that

$$\xi = \frac{b-a}{2}y + \frac{a+b}{2}, \tag{10}$$

where y is the new variable. We note that the coefficient to y is the range of uncertainty in ξ and the remaining term is the mean value of ξ. We also note

$$\frac{d\xi}{dy} = \frac{b-a}{2} \rightarrow d\xi = \frac{b-a}{2}dy, \tag{11}$$

and reconstruct our integral as

$$f_i = \int_{\xi=3}^{\xi=5} f\left(\frac{b-a}{2}y + \frac{a+b}{2}\right) P_i\left(\frac{b-a}{2}y + \frac{a+b}{2}\right)\frac{b-a}{2}dy. \tag{12}$$

To complete our transformation, we transform the limits by evaluating $y(\xi)$ at the extrema to obtain

$$f_i = \int_{-1}^{1} f\left(\frac{b-a}{2}y + \frac{a+b}{2}\right) P_i\left(\frac{b-a}{2}y + \frac{a+b}{2}\right)\frac{b-a}{2}dy, \tag{13}$$

conducive to Gauss-Legendre quadrature

$$f_i = \frac{b-a}{2}\sum_{\ell}^{\infty} w_\ell f\left(\frac{b-a}{2}y_\ell + \frac{a+b}{2}\right) P_i\left(\frac{b-a}{2}y_\ell + \frac{a+b}{2}\right), \tag{14}$$

where w_ℓ, y_ℓ are the quadrature weights and abscissas, respectively.

PROBLEM When I do this, I don't get my function back. I fill the following table, using $\tilde{f}(y)$ as the function expanded in polynomials and evaluated on [-1,1] and $f(\xi)$ as the original function expanded on equivalent points [3,7]. I used S2 quadrature and included 2 terms in the polynomial expansion, which should be enough to capture linear behavior.

2.1 Calculations

I followed the following in calculating the coefficients f_0 and f_1, with $f(x) = 2x + 1$.

$$\begin{aligned}
f_0 &= \int_a^b f(x)P_0(x)dx, && (15)\\
&= \int_a^b (2x+1)\frac{1}{\sqrt{2}}dx, && (16)\\
&= \int_{-1}^{1}\left[2\left(\frac{b-a}{2}y + \frac{a+b}{2}\right) + 1\right]\frac{b-a}{2\sqrt{2}}dy, && (17)\\
&= \frac{1}{\sqrt{2}}\left(4\frac{a+b}{2} + 2\right), && (18)
\end{aligned}$$

$$
\begin{aligned}
f_1 &= \int_a^b f(x)P_1(x)dx, && (19)\\
&= \int_a^b (2x^2+x)\sqrt{\frac{3}{2}}dx, && (20)\\
&= \int_{-1}^1 \left[2\left(\frac{b-a}{2}y+\frac{a+b}{2}\right)^2+\frac{b-a}{2}y+\frac{a+b}{2}\right]\frac{b-a}{2}\sqrt{\frac{3}{2}}dy, && (21)\\
&= \frac{b-a}{2}\sqrt{\frac{3}{2}}\left[\frac{4}{3}\left(\frac{b-a}{2}\right)^2+4\left(\frac{a+b}{2}\right)+2\frac{a+b}{2}\right]. && (22)
\end{aligned}
$$

Replacing these in the expansion,

$$
\begin{aligned}
\tilde{f}(y) &= f_0P_0(y)+f_1P_1(y), && (23)\\
&= \left[\frac{(b-a)^3}{4}+\frac{3}{4}(b-a)(b+a)^2+\frac{3}{4}(b^2-a^2)\right]y+a+b+1. && (24)
\end{aligned}
$$

2.2 Results

For sample values I used $a=3, b=7$.

y	ξ	$\tilde{f}(y)$	$f(\xi)$
-1	3	-335	7
-1/2	4	-162	9
0	5	11	11
1/2	6	184	13
1	7	357	15

Where am I going wrong?