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CONSTRUCTIVE APPROXIMATION

Multivariate Simultaneous Approximation

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Abstract. Theorems of Jackson type are given, for the simultaneous approximation of a function of class C^m and its partial derivatives, by a polynomial and the corresponding partial derivatives.

In this paper we use convolution techniques, the Fourier transform, and complex analysis to prove the following theorem on polynomial approximation of a function of several real variables.

Theorem 1. *Let f be a function of compact support on $\mathbf{R}^N$, of class C^m where $0 \leq m < \infty$, and let K be a compact subset of $\mathbf{R}^N$ which contains the support of f. Then for each nonnegative integer n there is a polynomial p_n of degree at most n on $\mathbf{R}^N$ with the following property: for each multi-index α with $|\alpha| \leq \min\{m, n\}$ we have*

$$\sup_K |D^\alpha(f - p_n)| \leq \frac{C}{n^{m-|\alpha|}} \omega_{f,m}\left(\frac{1}{n}\right), \tag{1}$$

where C is a positive constant depending only on N, m, and the diameter of K.

Here, and later in the paper, we assume that $N \geq 1$, and we use standard multi-index notation [4]. We use the notation

$$\omega_{f,m}(\delta) = \sup_{|\gamma|=m} \left(\sup_{|x-y|\leq\delta} |D^\gamma f(x) - D^\gamma f(y)| \right)$$

if f is a function defined in some Euclidean space, of class C^m, and $\delta > 0$.

Theorem 2. *Let K be a connected compact subset of $\mathbf{R}^N$ such that any two points a and b of K can be joined by a rectifiable arc in K with length no greater than $\sigma|a - b|$, where σ is a positive constant. Let f be a function of class C^m on an open neighborhood*

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of K where $0 \le m < \infty$. Then for each nonnegative integer n there is a polynomial p_n of degree at most n on $\mathbf{R}^N$ with the following property: for each multi-index α with $|\alpha| \le \min\{m, n\}$ we have

$$\sup_K |D^\alpha(f - p_n)| \le \frac{C}{n^{m-|\alpha|}} \sum_{|\gamma| \le m} \sup_K |D^\gamma f|,$$

where C is a positive constant depending only on N, m, and K.

Theorem 2 follows from Theorem 1 and from facts related to the Whitney extension theorem. In detail, the following result follows from the proof of [4, (2.3.17)′]:

If K is a compact subset of $\mathbf{R}^N$ satisfying the condition in the first sentence of Theorem 2, and if m is a nonnegative integer, then there exists a constant $C = C(K, m)$ such that

$$\left| D^\beta \varphi(x) - \sum_{|\beta+\gamma| \le m} \frac{D^{\beta+\gamma}\varphi(a)(x-a)^\gamma}{\gamma!} \right| \le C|x - a|^{m-|\beta|} \sum_{|\gamma|=m} \sup_K |D^\gamma \varphi|$$

for every function φ of class C^m on an open neighborhood of K, every multi-index β with $|\beta| \le m$, and all points a and x in K.

Using this result, and the Whitney extension theorem as given in [4, Theorem 2.3.6] and the paragraph preceding [4, Theorem 2.3.6], one obtains the following version of the Whitney theorem:

If K is a compact subset of $\mathbf{R}^N$ satisfying the condition in the first sentence of Theorem 2, and if m is a nonnegative integer, then there exists a constant $C = C(K, m)$ with the following property: if f is any function of class C^m on an open neighborhood of K, then there exists a function F of class C^m on $\mathbf{R}^N$ such that $D^\alpha F = D^\alpha f$ on K for each multi-index α with $|\alpha| \le m$, and

$$\sum_{|\gamma| \le m} \sup_{\mathbf{R}^N} |D^\gamma F| \le C \sum_{|\gamma| \le m} \sup_K |D^\gamma f|.$$

Now to prove Theorem 2 we let f and K satisfy the hypotheses of that theorem, and let F be associated with f by the version of the Whitney theorem just quoted; then the conclusion of Theorem 2 follows from applying Theorem 1 to the function ΨF and the compact set supp Ψ, where $\Psi \in C_0^\infty(\mathbf{R}^N)$ is a fixed function satisfying $\Psi \equiv 1$ near K, using the fact that

$$\omega_{\Psi F, m} \le 2 \sup_{|\gamma|=m} \sup_{\mathbf{R}^N} |D^\gamma(\Psi F)| \le C \sum_{|\gamma| \le m} \sup_{\mathbf{R}^N} |D^\gamma F|$$

for some positive constant C depending only on Ψ.

Taking $|\alpha| = 0$ in Theorem 2, we obtain a multivariate Jackson theorem for certain compact subsets of $\mathbf{R}^N$. Such a result for smooth, compact submanifolds of $\mathbf{R}^N$ was obtained by Ragozin [7] in 1970. In particular, Ragozin extended some constructive one-variable techniques of Newman and Shapiro [6] to prove a multivariate Jackson theorem for the unit sphere and the unit ball in Euclidean spaces [8]; the general result in [7] then follows from the result for balls, utilizing a Tietze-type extension theorem. Using entirely different methods, including Fourier analysis techniques, Ganzburg [2] in 1981 proved a multivariate Jackson theorem for convex sets in $\mathbf{R}^N$ which are symmetric with respect to the origin and which have sufficiently smooth boundaries.

Regarding simultaneous approximation, very little has been done in the *multivariate* setting. In the univariate case, let T denote the unit circle in the plane and let $f \in C^m(T)$. It is easy to construct a sequence of trigonometric polynomials $\{t_n\}$ with

$$||f^{(j)} - t_n^{(j)}||_T \leq Cn^{j-m}\omega_{f,m}(1/n)$$

for $j = 0, \ldots, m$ where $C = C(m)$; and Kilgore [5] has shown how one can use this result to obtain, with quite a bit of effort, a simultaneous polynomial approximation result on the interval. Our research was motivated by a simultaneous approximation result for the closed unit disk in $\mathbf{R}^2$ found in [1].

The rest of this paper is devoted to the proof of Theorem 1. The proof employs techniques used by H. S. Shapiro, who used convolution methods, the Fourier transform, and complex analysis to study one-variable approximation problems in [9].

We let $\mathcal{S} = \mathcal{S}(\mathbf{R}^N)$ denote the Schwartz class, discussed in [4, Section 7.1], and we use the notation

$$g_{[\varepsilon]}(x) = \frac{1}{\varepsilon^N} g\left(\frac{x}{\varepsilon}\right)$$

if g is any continuous function on $\mathbf{R}^N$. For each $R > 0$ we define the closed box

$$B_R = \{(x_1, x_2, \ldots, x_N) \in \mathbf{R}^N : |x_j| \leq R \text{ for } j = 1, 2, \ldots, N\}$$

and the closed polydisk

$$E_R = \{(z_1, z_2, \ldots, z_N) \in \mathbf{C}^N : |z_j| \leq R \text{ for } j = 1, 2, \ldots, N\}.$$

If $z = (z_1, z_2, \ldots, z_N)$ is any point of $\mathbf{C}^N$, we define

$$\operatorname{Im} z = (\operatorname{Im} z_1, \operatorname{Im} z_2, \ldots, \operatorname{Im} z_N) \in \mathbf{R}^N,$$

and we let $|\operatorname{Im} z|$ denote the Euclidean norm of the vector $\operatorname{Im} z$.

We note that if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a fixed multi-index with $|\alpha| = k$, and if A_α is the number of ways to choose a sequence of k integers $j_1, j_2, \ldots, j_k$ such that $1 \leq j_\ell \leq N$ for each ℓ and $x_{j_1}x_{j_2}\ldots x_{j_k} = x^\alpha$, then

$$A_\alpha = \frac{k!}{\alpha!}. \tag{2}$$

Remark 1. If f is any continuous function on $\mathbf{R}^N$, it is well-known [9] that for any positive numbers C and δ we have

$$\omega_{f,0}(C\delta) \leq (C+1)\omega_{f,0}(\delta).$$

In fact, if n is the smallest integer which is greater than or equal to C, then

$$\omega_{f,0}(C\delta) \leq \omega_{f,0}(n\delta) \leq n\omega_{f,0}(\delta) \leq (C+1)\omega_{f,0}(\delta).$$

Remark 2. If u is a real-valued function of class C^m on $\mathbf{R}$, then for all positive numbers s we have

$$\left| u(s) - \sum_{k=0}^{m} \frac{u^{(k)}(0)}{k!} s^k \right| \leq \frac{s^m \omega_{u,m}(s)}{m!},$$

where $u^{(k)}$ indicates the kth derivative of u. This is proved in [9].

The following lemma and its proof are similar to results of H. S. Shapiro for approximation by trigonometric polynomials in one variable [9, Theorem 20 and Corollary]. Condition (3) appears in the Paley–Wiener–Schwartz theorem [4, Theorem 7.3.1].

Lemma A. *Let δ be a fixed positive constant. Then there is a holomorphic function $G : \mathbf{C}^N \to \mathbf{C}$ satisfying*

$$(3) \qquad |G(z)| \leq A e^{\delta |\operatorname{Im} z|} \quad \text{for all} \quad z \in \mathbf{C}^N,$$

for some positive constant A, such that the restriction $g \equiv G|_{\mathbf{R}^N}$ has the following properties:

(a) $g \in \mathcal{S}(\mathbf{R}^N)$.
(b) *If k is a nonnegative integer, and if*

$$(4) \qquad I_k = \frac{N^k}{k!} \int |w|^k (|w| + 1) |g(w)| \, dw,$$

then for each function f of compact support on $\mathbf{R}^N$, of class C^k, we have

$$|f(x) - (g_{[\varepsilon]} * f)(x)| \leq I_k \varepsilon^k \omega_{f,k}(\varepsilon) \qquad \text{for each } x \in \mathbf{R}^N \text{ and each } \varepsilon > 0.$$

Proof. We let $\Phi \in C_0^\infty(\mathbf{R}^N)$ be a fixed real-valued function such that $0 \leq \Phi \leq 1$ everywhere on $\mathbf{R}^N$:

$$(5) \qquad \Phi \equiv 1 \quad \text{on an open neighborhood of } 0,$$

and the support of Φ is contained in the closed ball of radius δ about the origin. We define

$$G(z_1, z_2, \ldots, z_N) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \Phi(\xi_1, \xi_2, \ldots, \xi_N) e^{-i(\xi_1 z_1 + \xi_2 z_2 + \cdots + \xi_N z_N)} \, d\xi_1 \, d\xi_2 \cdots d\xi_N,$$

for $(z_1, z_2, \ldots, z_N) \in \mathbf{C}^N$. Then G is a holomorphic function on $\mathbf{C}^N$ satisfying (3) with $A = 1/(2\pi)^N \int \Phi(\xi_1, \xi_2, \ldots, \xi_N) \, d\xi_1 \, d\xi_2 \cdots d\xi_N$, and the restriction $g = G|_{\mathbf{R}^N}$ is a function in the class $\mathcal{S}(\mathbf{R}^N)$ such that

$$(6) \quad \Phi(\xi_1, \xi_2, \ldots, \xi_N) = \int_{\mathbf{R}^N} g(x_1, x_2, \ldots, x_N) e^{i(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_N x_N)} \, dx_1 \, dx_2 \cdots dx_N,$$

for $(\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N$. These facts follow from [4, Theorems 7.1.5 and 7.1.14].

Note that by setting $(\xi_1, \xi_2, \ldots, \xi_N) = 0$ in (6), and applying (5), we obtain

$$(7) \qquad \int g(x_1, x_2, \ldots, x_N)\, dx_1\, dx_2 \cdots dx_N = 1.$$

Differentiating (6) repeatedly and then setting $(\xi_1, \xi_2, \ldots, \xi_N) = 0$, and using (5), we see that

$$(8) \qquad \int x_1^{j_1} x_2^{j_2} \cdots x_N^{j_N} g(x_1, x_2, \ldots, x_N)\, dx_1\, dx_2 \cdots dx_N = 0$$

if $j_1, j_2, \ldots, j_N$ are nonnegative integers, not all zero.

To complete the proof of Lemma A we must prove property (b), and for this we let k be a fixed nonnegative integer, and let f be a fixed function of compact support on $\mathbf{R}^N$, of class C^k. We define $R(x, h)$ by the formula

$$f(x+h) = \sum_{|\alpha| \le k} \frac{D^\alpha f(x)}{\alpha!} h^\alpha + R(x, h) \qquad \text{if} \quad x, h \in \mathbf{R}^N,$$

and we prove the following claim:

Claim. *If x, h are points of $\mathbf{R}^N$, then*

$$|R(x, h)| \le \frac{N^k |h|^k \omega_{f,k}(|h|)}{k!}.$$

To prove the claim, we define $u(t) = f(x + th)$ for $t \in \mathbf{R}$. Then u is of class C^k on $\mathbf{R}$; and using the chain rule repeatedly and (2), we see that

$$(9) \qquad u^{(\ell)}(t) = \ell! \sum_{|\alpha| = \ell} \frac{D^\alpha f(x + th)}{\alpha!} h^\alpha \qquad \text{for} \quad t \in \mathbf{R},$$

provided $0 \le \ell \le k$. We now see that

$$(10) \qquad \omega_{u,k}(\delta) \le k!\, |h|^k \omega_{f,k}(\delta |h|) \sum_{|\alpha| = k} \frac{1}{\alpha!} = N^k |h|^k \omega_{f,k}(\delta |h|) \qquad \text{if} \quad \delta > 0,$$

where the inequality follows from (9). The claim now follows from (9), (10), and Remark 2 with $s = 1$. ■

Now if x is any point of $\mathbf{R}^N$ and $\varepsilon > 0$, and all integrations are taken with respect to Lebesgue measure dw over all of $\mathbf{R}^N$, then

$$\begin{aligned} (11) \qquad (g_{[\varepsilon]} * f)(x) - f(x) &= \frac{1}{\varepsilon^N} \int f(x - w) g\left(\frac{w}{\varepsilon}\right) dw - f(x) \\ &= \int [f(x - \varepsilon w) - f(x)] g(w)\, dw \\ &= \int R(x, \varepsilon w) g(w)\, dw, \end{aligned}$$

where the second equality follows from (7) and the change-of-variables formula for multiple integrals, and the third equality follows from (8) and the definition of $R(x, \varepsilon w)$; thus

$$\begin{aligned}|(g_{[\varepsilon]} * f)(x) - f(x)| &\leq \int |R(x, \varepsilon w)|\,|g(w)|\,dw \\ &\leq \frac{N^k \varepsilon^k}{k!}\omega_{f,k}(\varepsilon) \int |w|^k(|w|+1)|g(w)|\,dw,\end{aligned}$$

where the first inequality follows from (11), and the second inequality follows from the Claim and Remark 1. This proves property (b), so the proof of Lemma A is complete. ■

We next discuss an elementary result from the theory of functions of several complex variables. If f is holomorphic on an open neighborhood of a polydisk E_R, with Maclaurin expansion $f(z) = \sum_\alpha a_\alpha z^\alpha$, then for each nonnegative integer n, the nth *Maclaurin polynomial* of f is

$$p_{f,n}(z) \equiv \sum_{|\alpha| \leq n} a_\alpha z^\alpha.$$

Lemma B. *Suppose* $0 < R < S$. *Let* f *be a holomorphic function on an open neighborhood of* E_S *satisfying* $\sup_{E_S} |f| \leq M$. *Then*

$$\sup_{E_R} |f - p_{f,n}| \leq \frac{M}{1 - R/S}\left(\frac{R}{S}\right)^{n+1},$$

for each nonnegative integer n.

Proof. We first give the proof for $N = 1$. Let the Maclaurin series for f be $\sum_{k=0}^\infty a_k z^k$. Applying the hypothesis of the lemma, we have

$$|a_k| = \left|\frac{1}{2\pi i}\oint_{|z|=S} \frac{f(z)\,dz}{z^{k+1}}\right| \leq \frac{M}{R^k}$$

for each $k \geq 0$. Thus for every nonnegative integer n we have

$$\begin{aligned}\sup_{|z|\leq R} |f(z) - p_{f,n}(z)| &= \sup_{|z|\leq R}\left|\sum_{k=n+1}^\infty a_k z^k\right| \\ &\leq M \sum_{k=n+1}^\infty \left(\frac{R}{S}\right)^k \\ &= \frac{M}{1-R/S}\left(\frac{R}{S}\right)^{n+1},\end{aligned}$$

as required.

Now if N is arbitrary, then Lemma B follows from the observation that for each fixed point $Z = (Z_1, Z_2, \ldots, Z_N) \in \mathbf{C}^N$, such that each Z_s is a complex number whose modulus is no greater than one, the nth Maclaurin polynomial for the function $\lambda \to f(\lambda Z)$ is the polynomial $\lambda \to p_{f,n}(\lambda Z)$, and we may apply the result of the preceding paragraph. ∎

Corollary. *Suppose $R > 0$ and $S > R + 1$. Let f be a holomorphic function on an open neighborhood of E_S satisfying $\sup_{E_S} |f| \leq M$. Then*

$$\sup_{E_R} |D^\alpha (f - p_{f,n})| \leq \frac{\alpha!\, M}{1 - R/(S-1)} \left(\frac{R}{S-1}\right)^{n+1}$$

for each multi-index α and each nonnegative integer $n \geq |\alpha|$.

Proof. For each multi-index α we have

$$\sup_{E_{S-1}} |D^\alpha f| \leq \alpha!\, M$$

by the Cauchy inequalities [3, Theorem 2.2.7]; and for each integer $n \geq |\alpha|$, the $(n-|\alpha|)$th Maclaurin polynomial for $D^\alpha f$ is $D^\alpha p_{f,n}$. Thus the corollary follows from Lemma B. ∎

The following elementary remark will be useful in the proof of Theorem 1.

Remark 3. Let f be a function of class C^m on $\mathbf{R}^N$ whose support is contained in the closed box B_R.

(a) $\sup |f| \leq R^m \sup_{|\alpha|=m} \sup |D^\alpha f|$.

In fact, the case $m = 0$ is obvious, the case $m = 1$ is proved by integration along appropriate line segments, and the case $m > 1$ is then proved by applying the case $m = 1$ repeatedly.

(b) If n is any positive integer, then

$$\sup |f| \leq R^m (nR + 1)\omega_{f,m}\left(\frac{1}{n}\right).$$

In fact, we have

$$\sup_{|\alpha|=m} \sup |D^\alpha f| \leq \omega_{f,m}(R) = \omega_{f,m}\left(nR \cdot \frac{1}{n}\right) \leq (nR + 1)\omega_{f,m}\left(\frac{1}{n}\right),$$

where the last inequality follows from Remark 1. From this and (a) we obtain (b).

We now complete the paper by proving Theorem 1. We assume that f is a fixed function of compact support on $\mathbf{R}^N$, of class C^m, and that K is a nonempty compact subset of $\mathbf{R}^N$ which contains the support of f. If the diameter of K is equal to R, then by performing a translation of $\mathbf{R}^N$ we may assume without loss of generality that K is contained in the box B_R. It will be understood that all constants depend on N, m, and R.

We now fix a positive number δ such that

$$\sqrt{N}(2R+1)\delta < \log_e 2. \tag{12}$$

We let G and $g = G|_{\mathbf{R}^N}$ be fixed functions, and let A be a fixed positive constant, with the properties associated with this value of δ in Lemma A; and for each nonnegative integer k we let I_k be given by (4). From (12) we see that for every multi-index α with $|\alpha| \le m$ there is a nonnegative integer $n_0 = n_0(\alpha)$ such that

$$\alpha!\, AR^m(nR+1)(2Rn)^N e^{\sqrt{N}(2R+1)\delta n} 2^{-n} \le \frac{I_{m-|\alpha|}}{n^{m-|\alpha|}} \qquad \text{for all} \quad n \ge n_0,$$

and hence there is a constant $C = C(\alpha) > I_{m-|\alpha|}$ such that

$$\alpha!\, AR^m(nR+1)(2Rn)^N e^{\sqrt{N}(2R+1)\delta n} 2^{-n} \le \frac{C}{n^{m-|\alpha|}} \qquad \text{for all} \quad n \ge 1. \tag{13}$$

For the rest of this proof we let n be a *fixed* positive integer. We use a two-step process to obtain the polynomial p_n of Theorem 1. The first step is to approximate f by the function $g_{[1/n]} * f$, which is the restriction to $\mathbf{R}^N$ of the holomorphic function

$$H(z) = n^N \int_{\mathbf{R}^N} G(n(z-w)) f(w)\, dw \qquad \text{for} \quad z \in \mathbf{C}^N,$$

and the second step is to approximate H by its Maclaurin polynomial $p_{H,n}$. The polynomial $p_{H,n}$ is then the polynomial p_n in the statement of Theorem 1. For later reference we note that

$$\begin{aligned} \sup_{E_{2R+1}} |H| &\le An^N e^{\sqrt{N}(2R+1)\delta n} \int |f(w)|\, dw \\ &\le A(2Rn)^N R^m (nR+1) e^{\sqrt{N}(2R+1)\delta n} \omega_{f,m}\left(\frac{1}{n}\right), \end{aligned} \tag{14}$$

where the first inequality follows from the definition of H and (3), and the second inequality follows from Remark 3(b).

Now let α be a fixed multi-index such that $|\alpha| \le \min\{m, n\}$, and we let $C = C(\alpha)$ be the constant of (13). Then

$$\begin{aligned} \sup_{\mathbf{R}^N} |D^\alpha(f - g_{[1/n]} * f)| &= \sup_{\mathbf{R}^N} |D^\alpha f - g_{[1/n]} * (D^\alpha f)| \\ &\le \frac{C}{n^{m-|\alpha|}} \omega_{D^\alpha f, m-|\alpha|}\left(\frac{1}{n}\right) \\ &\le \frac{C}{n^{m-|\alpha|}} \omega_{f,m}\left(\frac{1}{n}\right), \end{aligned} \tag{15}$$

where the first inequality is by Lemma A(b); and

$$(16)\quad \sup_{B_R} |D^\alpha(H - p_{H,n})| \leq \sup_{E_R} |D^\alpha(H - p_{H,n})|$$
$$\leq \alpha!\, A R^m (nR+1)(2Rn)^N e^{\sqrt{N}(2R+1)\delta n} 2^{-n} \omega_{f,m}\left(\frac{1}{n}\right),$$

where the last inequality follows from (14) and the corollary to Lemma B with $S = 2R + 1$. The estimate (1) for this multi-index α now follows from (13), (15), and (16). ∎

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