Numerical Methods for Uncertainty Quantification

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Outline

Discussion Points

- Sources of Uncertainty
- 2 Analytic Methods

3 Numerical Methods





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Uncertainty

- Aleatory (physical)
- Epistemic (measured)





Example Stochastic Problem

$$y_f = y_i + v \sin(\theta)t - \frac{1}{2}gt^2, \tag{1}$$

$$x_f = v \cos(\theta) t. \tag{2}$$

Solution:
$$x_f = \frac{v\cos\theta}{g} \left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i} \right)$$





Example Stochastic Problem

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right) \tag{3}$$

- initial height $y_i = 2 \text{ m}$
- initial velocity v = 50 m/s
- initial trajectory $\theta = 35^{\circ}$
- accel. gravity g = -9.81 m/s/s





Example Stochastic Problem

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right) \tag{4}$$

- initial height $y_i = 1 \pm 1$ m
- initial velocity $v = 50 \pm 5$ m/s
- initial trajectory $\theta = 45 \pm 10^{\circ}$
- accel. gravity $g = 9.7988 \pm 0.0349$ m/s/s





$$x_f = rac{v\cos heta}{g}\left(v\sin heta + \sqrt{v^2\sin^2 heta + 2gy_i}
ight)$$

Min-Max

$$\begin{split} x_{f,\text{min}} &= \frac{(45)(0.5736)}{9.7369} \, \left((45)(0.8192) + \sqrt{(45)^2(0.8192)^2 + 2(9.7369)(0)} \right) = 195.45 \, \text{m} \\ x_{f,\text{max}} &= \frac{(55)(0.8192)}{9.8337} \, \left((55)(0.5736) + \sqrt{(55)^2(0.5736)^2 + 2(9.8337)(2)} \right) = 291.92 \, \text{m} \end{split}$$

Result: $x_f \approx 244 \pm 48 \text{ m}$

Flawed Reasoning

Nonlinear Flight Path





$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right)$$

Min-Max

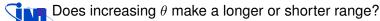
Idaho National Laboratory

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Analytic Uncertainty

$$\sigma_{x_f} = \sqrt{\left(\frac{\partial x_f}{\partial y_i}\right)^2 \sigma_{y_i}^2 + \left(\frac{\partial x_f}{\partial v}\right)^2 \sigma_v^2 + \left(\frac{\partial x_f}{\partial g}\right)^2 \sigma_g^2 + \left(\frac{\partial x_f}{\partial \theta}\right)^2 \sigma_\theta^2}$$

Result: $x_f = 256 \pm 51 \text{ m}$

Works well for simple functions

- Simple derivatives
- Analytic solution





A More Difficult Problem

With Air Resistance:

$$y_f = \frac{v_t}{g} (v \sin \theta + v_t) \left(1 - e^{-gt/v_t} \right) - v_t t, \tag{5}$$

$$x_f = \frac{v v_t \cos \theta}{g} \Big(1 - e^{-gt/v_t} \Big). \tag{6}$$

$$v_t = \frac{mg}{D}, \qquad D = \frac{\rho CA}{2}, \qquad A = \pi r^2$$
 (7)

Solve numerically to get x_f (Forward Euler).





Additional Uncertainty

$$y_t = \frac{v_t}{g} (v \sin \theta + v_t) \left(1 - e^{-gt/v_t} \right) - v_t t, \tag{8}$$

$$x_f = \frac{vv_t \cos \theta}{g} \Big(1 - e^{-gt/v_t} \Big). \tag{9}$$

$$v_t = \frac{mg}{D}, \qquad D = \frac{\rho CA}{2}, \qquad A = \pi r^2$$
 (10)

$$m = 0.145 \pm 0.0725 \text{ kg}, \qquad r = 0.0336 \pm 0.00336 \text{ m}, \quad (11)$$

$$C = 0.5 \pm 0.5,$$
 $\rho_{\text{air}} = 1.2 \pm 0.1 \text{ kg/m}^3.$ (12)





Uncertainty Quantification: Complicated Problems

How do we quantify uncertainty for problems without simple analytic solutions?

- Monte Carlo sampling
- Stochastic Collocation
- High Density Model Reduction (low-order)





Monte Carlo

- Let u(Y) be any system, like $x_f(v, \theta, g, y_i)$
- Randomly sample input parameters, record outputs
- Calculate moments (mean, variance, skew, kurtosis)

$$\mathbb{E}[u^r] \approx \frac{1}{M} \sum_{m=1}^M u \left(Y^{(m)} \right)^r$$

Mean:
$$\bar{u} \approx \frac{1}{M} \sum u(Y^{(m)})$$

(video)



