# Distributions and Quadratures

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## 1 General Syntax

- Probability Measure: $f(y)$
- Normalization Factor: $A$
- Probability Distribution: $Af(y)$
- Generic function: $h(y)$
- Lower bound $L$
- Upper bound $R$

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## 2 Distributions

### 2.1 Uniform

Continuous values $y \in [L, R]$.

$$\mu = \frac{R+L}{2}, \tag{1}$$

$$\sigma = \frac{R-L}{2} \tag{2}$$

$$f(y) = \frac{1}{2\sigma}, \tag{3}$$

$$A = 1, \tag{4}$$

$$1 = \frac{1}{2\sigma} \int_L^R dy, \tag{5}$$

### 2.2 Normal

Continuous values $y \in (-\infty, \infty)$.

$$f(y) = \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right), \tag{6}$$

$$A = \frac{1}{\sigma\sqrt{2\pi}} \tag{7}$$

$$1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy, \tag{8}$$

### 2.3 Gamma

Continuous values $y \in [L, \infty)$.

$$f(y) = y^{\alpha-1} e^{-\beta y}, \tag{9}$$

$$A = \frac{\beta^\alpha}{\Gamma(\alpha)}, \tag{10}$$

$$1 = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_L^\infty (y-L)^{\alpha-1} e^{-\beta(y-L)} dy, \tag{11}$$

### 2.4 Beta

Continuous values $y \in [L, R]$.

$$f(y) = y^{\alpha-1}(1-y)^{\beta-1}, \tag{12}$$

$$A = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{13}$$

$$1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_L^R y^{\alpha-1}(1-y)^{\beta-1} dy. \tag{14}$$

### 2.5 Triangular

Continuous values $y \in [L, R]$. $c$ is the $y$-value where the apex sits, and has a value of $\frac{2}{R-L}$.

$$f(y) = \begin{cases} \frac{y-L}{c-L} & L \leq y < c, \\ 1 & y = c, \\ \frac{R-y}{R-c} & c < y \leq R, \\ 0 & \text{else.} \end{cases} \tag{15}$$

$$A = \frac{2}{R-L}, \tag{16}$$

$$1 = \int_L^c \frac{2}{(R-L)} \frac{y-L}{(c-L)} dy + \int_c^R \frac{2}{(R-L)} \frac{R-y}{(R-c)} dy \tag{17}$$

### 2.6 Poisson

Discrete values $y \in \mathbb{N}_0$.

$$f(y) = \frac{\lambda^y}{y!}, \tag{18}$$

$$A = e^{-\lambda}, \tag{19}$$

$$1 = e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}, \quad y \in \mathbb{N}_0. \tag{20}$$

### 2.7 Binomial

Discrete values $y \in \mathbb{N}_{\nvdash}, y < n$,

$$f(y) = p^y (1-p)^{n-y}, \tag{21}$$

$$A = \frac{n!}{y!(n-y)!}, \tag{22}$$

$$1 = \frac{n!}{y!(n-y)!} \sum_{y=0}^{n} p^y (1-p)^{n-y}. \tag{23}$$

### 2.8 Bernoulli

Boolean values $y \in [0, 1]$.

$$f(y) = \begin{cases} 1-p & y = 0, \\ p & y = 1, \end{cases} \tag{24}$$

$$A = 1, \tag{25}$$

$$1 = \sum_{y=1}^{2} f(y) = p + (1-p). \tag{26}$$

### 2.9 Logistic

Continuous values $y \in (-\infty, \infty)$.

$$f(y) = \operatorname{sech}^2\left(\frac{y-\mu}{2s}\right), \tag{27}$$

$$A = \frac{1}{4s}, \tag{28}$$

$$1 = \frac{1}{4s}\int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{y-\mu}{2s}\right) dy. \tag{29}$$

### 2.10 Exponential

Continuous values $y \in [0, \infty)$ TODO make L to infinity

$$f(y) = e^{-\lambda x}, \tag{30}$$

$$A = \lambda, \tag{31}$$

$$1 = \int_0^{\infty} \lambda e^{-\lambda x}. \tag{32}$$

### 2.11 Arbitrary

Continuous values $-\infty < L \leq y \leq R < \infty$

$$f(y) = f(y), \tag{33}$$

$$F(y) = \int_L^y f(y')dy', \tag{34}$$

$$1 = \int_L^R f(y')dy'. \tag{35}$$

## 3 Quadrature

### 3.1 Legendre

$$\int_{-1}^{1} h(x)d(x) = \sum_{\ell=1}^{\infty} w_\ell h(x_\ell) \tag{36}$$

### 3.2 Hermite

$$\int_{-\infty}^{\infty} h(x)\exp\left(\frac{-x^2}{2}\right)dy = \sum_{h=1}^{\infty} w_h h(x_h) \tag{37}$$

### 3.3 Laguerre

$$\int_0^{\infty} h(x)x^\alpha e^{-x}dx = \sum_{\mathcal{L}=1}^{\infty} w_{\mathcal{L}} h(x_{\mathcal{L}}) \tag{38}$$

### 3.4 Jacobi

$$\int_{-1}^{1} h(x)(1-x)^{\alpha}(1+x)^{\beta}dx = \sum_{j=1}^{\infty} w_j h(x_j) \tag{39}$$

### 3.5 Clenshaw-Curtis

$$\int_{-1}^{1} h(x)dx = \sum w_{cc} h(x_{cc}) \tag{40}$$

## 4 Conversions

### 4.1 Uniform and Legendre

$$y = \sigma x + \mu, \tag{41}$$

$$x = \frac{y-\mu}{\sigma}, \tag{42}$$

$$\int_{a}^{b} h(y) f_{\ell}(y) dy = \frac{1}{2} \sum_{\ell=1}^{\infty} w_{\ell} h(\sigma x_{\ell} + \mu) \tag{43}$$

### 4.2 Normal and Hermite

$$y = \sigma x + \mu, \tag{44}$$

$$x = \frac{y-\mu}{\sigma}, \tag{45}$$

$$\int_{-\infty}^{\infty} h(y) f_h(y) dy = \frac{1}{\sqrt{2\pi}} \sum_{h=1}^{\infty} w_h h(\sigma x_h + \mu) \tag{46}$$

### 4.3 Gamma and Laguerre

$$y = \frac{x}{\beta} + L, \tag{47}$$

$$x = (y-L)\beta, \tag{48}$$

$$\int_{L}^{\infty} h(y) f_g(y) dy = \frac{1}{(\alpha-1)!} \sum_{g=1}^{\infty} w_g h\left(\frac{x_g}{\beta} + L\right) \tag{49}$$

Points $x_g$ and weights $w_g$ must be obtained from Laguerre quadrature as

```
pts,wts = laguerre_generator(alpha-1).
```

### 4.4 Beta and Jacobi

General Beta:

$$1 = \frac{1}{\mathrm{B}(\alpha,\beta)(R-L)} \int_L^R \left(\frac{y-L}{R-L}\right)^{\alpha-1} \left(1 - \frac{y-L}{R-L}\right)^{\beta-1} dy, \tag{50}$$

To convert to standard Beta:

$$z = \frac{y-L}{R-L}, \quad y = (R-L)z + L, \quad dy = (R-L)dz, \tag{51}$$

$$1 = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} dz, \tag{52}$$

To convert to same form as Jacobi:

$$z = \frac{1+x}{2}, \quad x = 2z-1, \quad dz = \frac{1}{2}dx, \tag{53}$$

$$1 = \frac{1}{2^{\alpha+\beta-1}\mathrm{B}(\alpha,\beta)} \int_{-1}^1 (1+x)^{\alpha-1}(1-x)^{\beta-1} dx, \tag{54}$$

Combined:

$$y = \frac{R-L}{2}x + \frac{R+L}{2}, \quad x = \left(y - \frac{R+L}{2}\right)\left(\frac{2}{R-L}\right) \tag{55}$$

Especially note the naming convention

$$\alpha_{\text{Jacobi}} = \beta_{\text{Beta}} - 1, \qquad \beta_{\text{Jacobi}} = \alpha_{\text{Beta}} - 1, \tag{56}$$

So,

$$\int_L^R h(y) f_B(y) dy = \frac{1}{2^{\alpha_B+\beta_B-1}\mathrm{B}(\alpha_B,\beta_B)} \sum_{b=1}^{\infty} w_b h\left(\frac{R-L}{2}x_b + \frac{R+L}{2}\right). \tag{57}$$

Points $x_j$ and weights $w_j$ must be obtained from Jacobi quadrature as

```
pts,wts = jacobi_generator(beta-1, alpha-1).
```

### 4.5 Arbitrary and Legendre

Let $u \in [0,1]$, and note $F(y) \in [0,1]$.

$$du = dF(y) = f(y)dy, \tag{58}$$

$$F(y) = u \quad \therefore \quad y = F^{-1}(u), \tag{59}$$

$$dy = \frac{1}{f(y)}du, \tag{60}$$

$$\int_L^R h(y) f(y) dy = \int_0^1 h\big(F^{-1}(u)\big) f\big(F^{-1}(u)\big) \frac{1}{f(F^{-1}(u))} du, \tag{61}$$

$$= \int_0^1 h\big(F^{-1}(u)\big) du. \tag{62}$$

$$x = \frac{u-\mu}{\sigma} \therefore u = \sigma x + \mu, \tag{63}$$

$$u = \frac{R-L}{2}x + \frac{R+L}{2}, \quad R=1, L=0, \tag{64}$$

$$u = \frac{1}{2}(x+1), \tag{65}$$

$$\int_L^R h(y)f(y)dy = \int_0^1 h\big(F^{-1}(u)\big)du, \tag{66}$$

$$= \frac{1}{2}\sum_{\ell=1}^{\infty} w_\ell h\left(F^{-1}\left(\frac{1}{2}(x+1)\right)\right)du. \tag{67}$$