# Stochastic Collocation

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## 1 Introduction

The concept behind using stochastic collocation for propagating uncertainty is that the uncertainty space can be spanned by quadrature rules deterministically instead of using stochastic sampling.

## 2 Uncertainty

TODO -realizations

### 2.1 Uncertain Variables

We consider a given variable $x$ with some uncertainty in value. We define $\bar{x}$ as the average value of this variable. The uncertainty in $x$ can be given in several ways. In this work we consider two types of uncertainty: uniform and normal.

#### 2.1.1 Uniformly-Uncertain Variables

A uniformly-uncertain variable, hereafter referred to as uniform variable, has an equal probability of being realized at all its potential values. For example, for a six-sided die, each face is equally probable to end facing up when the die is rolled. For a continuous uniform variable, the range of realizations are well-described by an average value $\bar{x}$, minimum value $\bar{x}-\sigma$, and maximum value $\bar{x}+\sigma$. We note the following characteristics of uniform variables:

- The possible values of $x$ range linearly from $\bar{x}-\sigma$ to $\bar{x}+\sigma$.
- The possible values of $x$ as a function of $w$ are found along $x(w)=\sigma w+\bar{x}$.
- The probability of any realization $w$ is given by a horizontal distribution $P(w)$ which has a value of $1/2$ from $-1$ to $1$ and is zero everywhere else.

### 2.2 Uncertain Processes

TODO

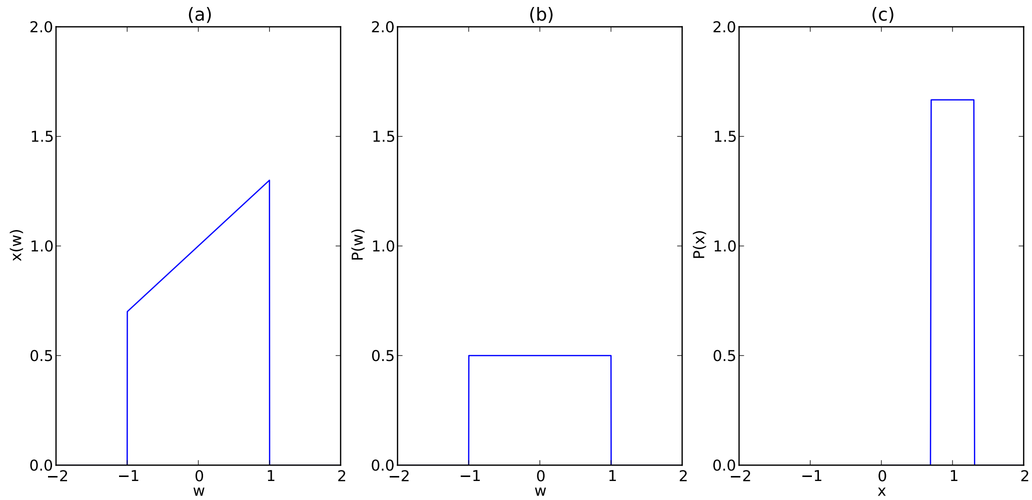


Figure 1: (a) Possible values of $x(w)$ as a function of $w$. (b) Probability distribution of $w$. (c) Probability distribution of $x(w)$.

# 3 Quadrature

Gaussian quadrature makes the following assumption:

$$\int_a^b f(x)dx = \int_a^b g(x)P(x)dx \approx \sum_n^N w_n f(x_n), \tag{1}$$

where $f(x)$ is any function continuous over the interval $a \to b$, $P(x)$ is a probability distribution function for $x$, $N$ is the order of quadrature chosen and $w_n$ and $x_n$ are given by the choice of quadrature and its order. In general $x_n$ are referred to as the abscissas of the quadrature, and $w_n$ as the associated weights. In the limit that $N$ approaches infinity, the quadrature is exact.

There are many different Gaussian quadrature sets that may are selected based on their abscissa distribution and the limits $a$ and $b$ of the integral in question.

## 3.1 Implementing Integration with Quadrature

Here we pause to make a note of the methods surrounding evaluating integrals using quadrature sets. We consider in particular an implemented function `integrate(f(x))` that accepts a function and integrates it using quadrature. In general, as above, a quadrature evaluates an integral using some weighting function $P(x)$, not to be confused with the quadrature weights $w_n$. The Legendre quadrature is singular because its weighting function is 1; this means that $f(x) = g(x)$ for Legendre quadrature, and a function `integrateLegendre(f(x))` that uses Legendre quadrature will integrate $f(x)$ exactly as expected.

For other quadrature sets, however, $P(x) \neq 1$ and a function such as `integrateHermite(f(x))` will not accurately return $\int f(x)dx$. Use of these quadrature sets require some knowledge of the user, since the integration routine will actually integrate $\int f(x)P(x)dx$. In the case of Hermite quadrature, $P(x) = \exp(-x^2)$, so `integrateHermite(f(x))` would return $\int f(x)\exp(-x^2)dx$. This is useful for a user that wants to integrate a function times a weighting factor by only passing the function.

### 3.2 Gauss-Legendre

Gauss-Legendre quadrature spans $[-1, 1]$ with $P(x) = 1$, and is perhaps the most commonly used quadrature in particle transport. The abscissa are given by finding the roots of the Legendre polynomials

$$P_N(x) = \frac{1}{2^N N!} \frac{d^N}{dx^N}[(x^2-1)^N], \tag{2}$$

and the weights are given by

$$w_n = \frac{2}{(1-x_n^2)[P'_N(x_n)]^2}. \tag{3}$$

Gauss-Legendre quadrature is well-suited for spanning the uncertainty space of uniformly-distributed uncertainty.

### 3.3 Gauss-Hermite

Gauss-Hermite quadrature spans $[-\infty, \infty]$ and $W(x) = e^{-x^2}$. The abscissa come from the roots of the physicist's Hermite polynomial

$$H_N(x) = (-1)^N e^{x^2} \frac{d^N}{dx^N} e^{-x^2}, \tag{4}$$

and the weights are given by

$$w_n = \frac{2^{N-1} N! \sqrt{\pi}}{N^2 [H_{N-1}(x_n)]^2}. \tag{5}$$

Gauss-Hermite quadrature is well-suited for spanning the uncertainty space of normally-distributed uncertainty; however, careful consideration must be made of the probability function associated with choosing the Hermite quadrature. Since we are most interested in using Hermite quadrature to span the uncertainty space of a variable with normal (Gaussian)-distributed uncertainty, we desire the probability distribution to be normal; that is, we desire the probability distribution to be $P(x) = e^{-x^2/2}$, noting the half in the numerator. We can adjust our integral to agree well with the normal distribution probability function by defining

$$b \equiv \frac{a}{\sqrt{2}}, \qquad a = \sqrt{2} b. \tag{6}$$

The integral we desire to approximate using the normal distribution is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(a) e^{a^2/2} da \approx \sum_i w_i f(a_i). \tag{7}$$

Replacing $b\sqrt{2}$ for $a$ everywhere restores the Hermite form of the integral,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(b\sqrt{2}) e^{b^2} \sqrt{2} db \approx \sum_i \frac{w_i}{\sqrt{\pi}} f(b_i \sqrt{2}). \tag{8}$$