Finite Difference Heat Equation

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1 Introduction

2 Numerical Scheme Derivation in 1D

We begin with the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u is temperature, t is time, ρ is the material density, C_p is the specific heat capacity, k is the conductivity, and x is the one-dimensional axis in space. We implicitly discretize in time steps with uniform spacing Δ_t. This leads to

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta_t}, \tag{2}$$

$$u(x,t) \approx u^{n+1}(x), \qquad n = 1, 2, ... \tag{3}$$

where n is used to denote successive time steps. These approximations are exact in the limit as Δ_t reduces to zero, which restores the continuous time spectrum. Applying this to the heat equation,

$$\frac{u^{n+1}(x) - u^n(x)}{\Delta_t} = \alpha \frac{\partial^2 u^{n+1}(x)}{\partial x^2}. \tag{4}$$

We similarly apply a uniform spatial discretization in x, making use of the second-order central differencing scheme

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\Delta_x^2}, \tag{5}$$

$$u(x,t) \approx u_j(t), \qquad j = 1, 2, .., J, \tag{6}$$

where j indexes the spatial dimension, with cell $j-1$ being to the left of cell j. The heat equation now takes the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta_t} = \alpha \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta_x^2}, \tag{7}$$

$$u_j^{n+1} - u_j^n = \Gamma \left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} \right), \tag{8}$$

$$\Gamma \equiv \frac{\Delta_t \alpha}{\Delta_x^2}. \tag{9}$$

Bringing all the implicit $n+1$ terms to the left and explicit n terms to the right,

$$-\Gamma u_{j+1}^{n+1} + (1+2\Gamma)u_j^{n+1} - \Gamma u_{j-1}^{n+1} = u_j^n. \tag{10}$$

For the boundary cases,

$$j = 1 \rightarrow x_{j-1} = x_L, \tag{11}$$

$$j = J \rightarrow J_{j+1} = x_R, \tag{12}$$

where x_L and x_R are the left and right faces of the problem, respectively. Eqs. (10) through (12) describe a tridiagonal system that can be written as

$$\begin{bmatrix} 1+2\Gamma & -\Gamma & 0 & 0 & ... \\ -\Gamma & 1+2\Gamma & -\Gamma & 0 & ... \\ 0 & -\Gamma & 1+2\Gamma & -\Gamma & ... \\ & & ... & & \\ ... & -\Gamma & 1+2\Gamma & -\Gamma & 0 \\ ... & 0 & -\Gamma & 1+2\Gamma & -\Gamma \\ ... & 0 & 0 & -\Gamma & 1+2\Gamma \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ ... \\ u_{J-2}^{n+1} \\ u_{J-1}^{n+1} \\ u_J^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n + \Gamma u_L \\ u_2^n \\ u_3^n \\ ... \\ u_{J-2}^n \\ u_{J-1}^n \\ u_J^n + \Gamma u_R \end{bmatrix},$$

where $u_L = u(x_L, t) = u(0, t)$ and $u_R = u(x_R, t) = u(L, t)$. This system can be solved using the Thomas algorithm (TDMA) for each time step, obtaining a forward-marching solution for $u(x, t)$.

3 Test Cases

3.1 Fundamental Solution Initial Condition in 1D

We begin with the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{13}$$

$$\alpha \equiv \frac{\rho C_p}{k}, \tag{14}$$

where u is temperature, t is time, ρ is the material density, C_p is the specific heat capacity, k is the conductivity, and x is the one-dimensional axis in space. We apply vacuum boundary conditions and set the initial temperature as a fundamental solution of the time-independent heat equation,

$$u(0,t) = u(L,t) = 0, \tag{15}$$

$$u(x,0) = u_0 \sin\frac{\pi x}{L}, \tag{16}$$

where L is the problem length and u_0 is a constant maximum initial temperature. This second-order homogeneous PDE is separable.

$$u(x,t) \equiv X(x)T(t), \tag{17}$$

$$\frac{1}{\alpha} X(T') = T(X''), \tag{18}$$

$$\frac{T'}{\alpha T} = \frac{X''}{X}. \tag{19}$$

Since the two terms are independent, the two must both be equal to the same constant. Arbitrarily,

$$\frac{T'}{\alpha T} = \frac{X''}{X} = -\lambda^2. \tag{20}$$

Considering the spatial dimension first,

$$X'' = -\lambda^2 X, \tag{21}$$

$$X(x) = c_1 \sin \lambda x + c_2 \cos \lambda x. \tag{22}$$

Applying the left boundary condition,

$$X(0) = 0 = 0 + c_2, \qquad c_2 = 0. \tag{23}$$

Applying the left boundary condition,

$$X(L) = 0 = c_1 \sin \lambda x, \tag{24}$$

which requires either trivially $c_1 = 0$ or

$$\lambda = \frac{n\pi}{L}, \quad n = 0, 1, 2, ... \tag{25}$$

$$X(x) = c_1 \sin \frac{n\pi x}{L}. \tag{26}$$

Considering the temporal dimension,

$$T' + \alpha\lambda^2 T = 0. \tag{27}$$

The root of the characteristic equation is

$$R = \alpha\lambda^2 T, \tag{28}$$

so

$$T(t) = c_3 e^{-\alpha\lambda^2 t}. \tag{29}$$

Combing X and T to obtain the original variable u,

$$u(x,t) = X(x)T(t) = c_4 e^{-\alpha\lambda^2 t} \sin \frac{n\pi x}{L}. \tag{30}$$

The initial condition gives

$$u(x,0) = u_0 \sin \frac{\pi x}{L} = c_4 \sin \frac{n\pi x}{L}. \tag{31}$$

By inspection, the most straightforward solution is

$$c_4 = u_0, \quad n = 1, \tag{32}$$

$$u(x,t) = u_0 e^{-\alpha\lambda^2 t} \sin \frac{\pi x}{L}. \tag{33}$$

4 Results