HDMR and SC on SG for an Analytic Problem

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1 Approximations

We use the sparse grid approximation for stochastic collocation described elsewhere,

$$u(Y) \approx S[u](Y) \equiv \sum_{\vec{i}\in\Lambda(L)} S^i[u](Y), \tag{1}$$

$$S^i[u](Y) = c(\vec{i}) \bigotimes_{n=1}^{N} U[u](Y), \tag{2}$$

$$c(\vec{i}) \equiv \sum_{\vec{j}\in\{0,1\}^N,\ \vec{j}+\vec{i}\in\Lambda} (-1)^{|\vec{j}|_1}, \tag{3}$$

$$\bigotimes_{n=1}^{N} U[u](Y) \equiv \sum_{\vec{k}}^{\vec{i}} u_h(Y^{(\vec{k})}) L_{\vec{k}}(Y). \tag{4}$$

Further, we apply a Cut-HDMR representation, where

$$u(Y) = u_0 + \sum_{i}^{N} u_i + \sum_{i}^{N}\sum_{j}^{i} u_{i,j} + ..., \tag{5}$$

$$u_0 = u(\bar{Y}) \tag{6}$$

$$u_i \equiv u(Y_i, \bar{Y}), \tag{7}$$

$$u_{i,j} \equiv u(Y_i, Y_j, \bar{Y}), \tag{8}$$

and so on. Here N is the total number of uncertain inputs, and $\bar{Y}$ indicates holding any variables not explicitly listed at a reference value (in this case, the mean value). This cut-HDMR representation allows us to approximate by truncating at a particular interaction level H. For instance, an $H2$ approximation would only include the terms

$$u(Y) \approx H_2[u](Y) = u_0 + \sum_{i}^{N} u_i + \sum_{i}^{N}\sum_{j}^{i-1} u_{i,j}. \tag{9}$$

We combine the two by using SC on SG to evaluate the HDMR terms. For example,

$$u(Y) \approx H_2[u](Y) \approx S_0 + \sum_{i}^{N} S_i + \sum_{i}^{N}\sum_{j}^{i-1} S_{i,j}, \tag{10}$$

$$S_0 \equiv S[u](\bar{Y}), \tag{11}$$

$$S_i \equiv S[u](Y_i, \bar{Y}), \tag{12}$$

$$S_{i,j} \equiv S[u](Y_i, Y_j, \bar{Y}). \tag{13}$$

2 Problem

The problem we are applying uncertainty to is equivalent to the flux of particles emitted through multiple equally-spaced purely-absorbing materials with a source on the opposite side. The general solution to such a system is

$$u(\Sigma) = A \prod_i^N e^{-\Sigma_i x_i}, \tag{14}$$

where A is the initial source strength, Σ_i is the macroscopic interaction cross section, and x_i is the width of the material. Assuming all materials are equal in size and the initial source is unity, we can normalize to obtain the general solution

$$u(Y) = \prod_i^N e^{-Y_i} = e^{-\sum_i^N Y_i}. \tag{15}$$

The expected value of $u(Y)$ is

$$\langle u(Y) \rangle = \int_a^b u(Y) P(Y) dY, \tag{16}$$

where $P(Y)$ is the joint-pdf of the combined uncertainty space of Y. In our case, we will assume Y_n are distributed uniformly from 1 to 6. The joint PDF for this independent inputs is

$$P(Y) = 5^{-N}. \tag{17}$$

2.1 Cases

We consider four distinct cases, varying the number of uncertain variable inputs as $N \in (5, 10, 15, 30)$. For each case, we will compare analog Monte Carlo uncertainty quantification with direct stochastic collocation on sparse grids as well as cut-HDMR for truncation levels $H \in (1, 2, 3)$. For each case, we increase the level L of the sparse grid expansion and plot the resulting error in the expected value of $u(Y)$ as a function of total deterministic solver runs. This effectively produces a comparison of error obtained by computational cost. For clarity, the approximations made are

- L, the "level" of sparse grid approximation, used to produce quadrature points;
- H, the HDMR truncation level,
- TD, HDMR, or MC for (total degree) stochastic collocation, HDMR, or MC sampling methods.

2.2 Results

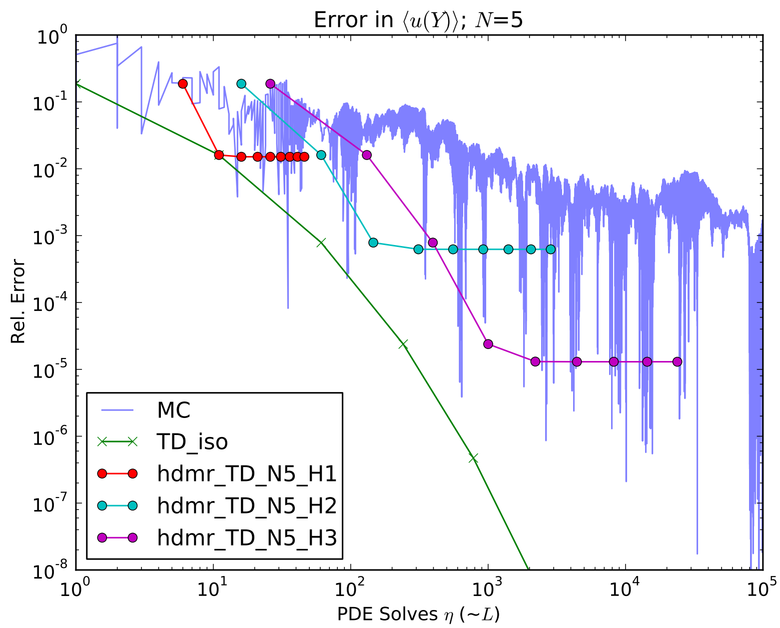

Figure 1: $N = 5$

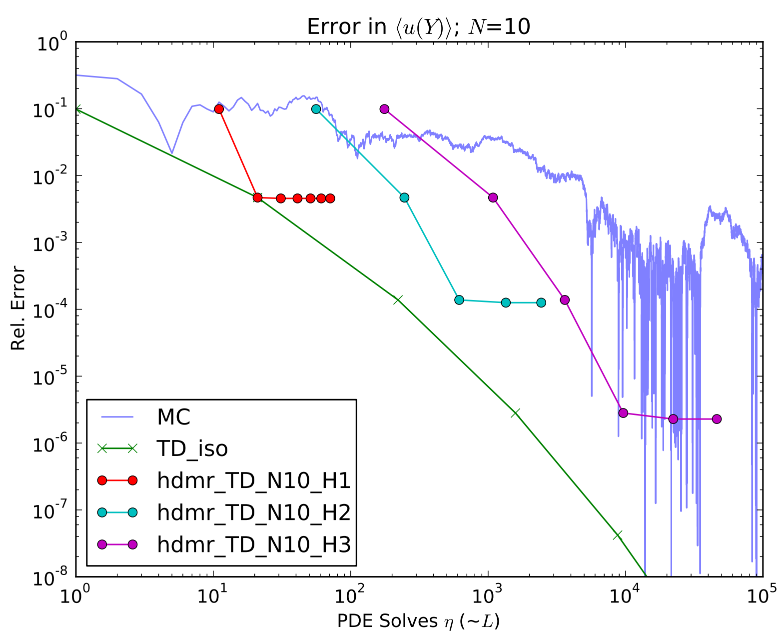

Figure 2: $N = 10$

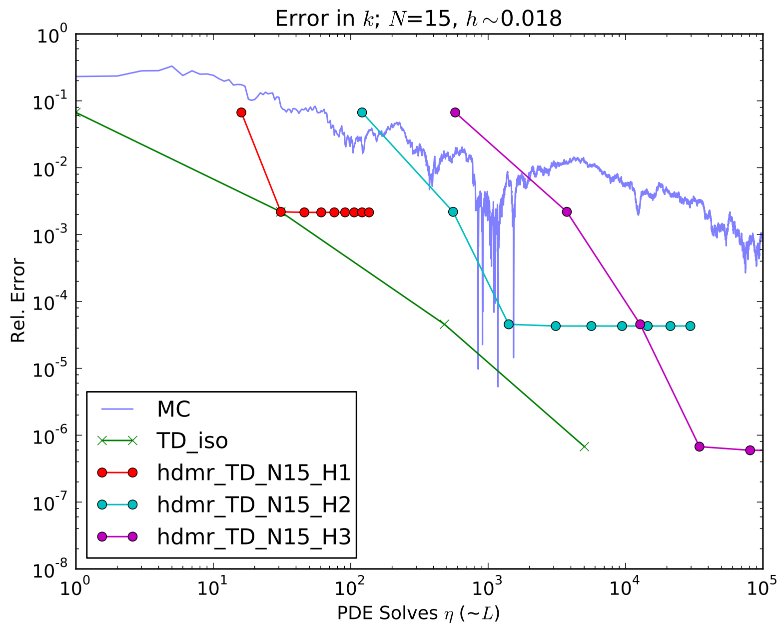

Figure 3: $N = 15$

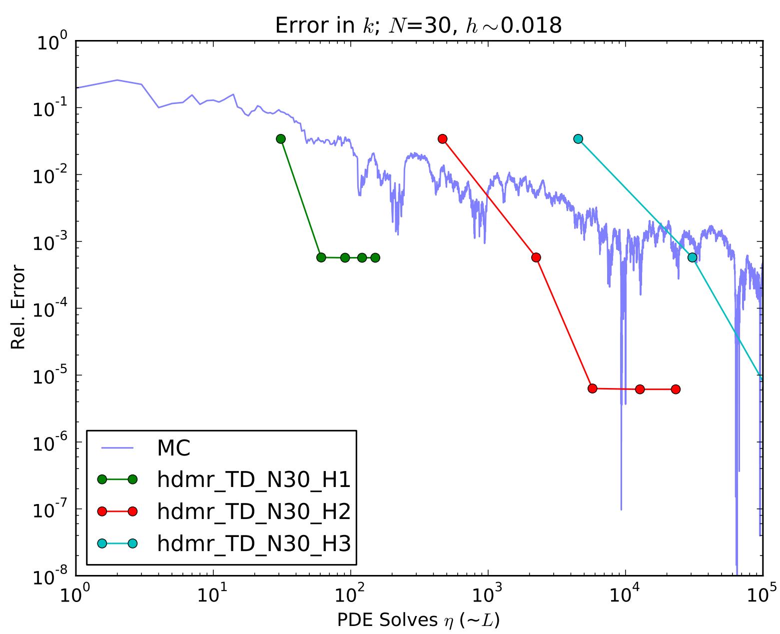

Figure 4: $N = 30$