Analytic Tests regarding Stochastoc Collocation for gPC

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1 Uniform Uncertainty

1.1 $u(Y) = y_1 + y_2$

$$y_1 \sim U(a_1, b_1),$$
$$y_2 \sim U(a_2, b_2),$$
$$R_i \equiv b_i - a_i,$$

1.1.1 Moments

$$\mathbb{E}[u(Y)^r] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Big(y_1 + y_2\Big)^r \rho(y_1)\rho(y_2) dy_2 dy_1, \tag{1}$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{(y_1 + y_2)^r}{R_1 R_2} dy_2 dy_1, \tag{2}$$
$$\tag{3}$$

For $b_i = 5, a_i = 1 \;\forall i \in [1, 2]$,

$$\mathbb{E}[u(Y)] = 6, \tag{4}$$
$$\mathbb{E}[u(Y)^2] = \frac{116}{3} \approx 38.66667, \tag{5}$$
$$\mathbb{E}[u(Y)^3] = 264, \tag{6}$$
$$\mathbb{E}[u(Y)^4] = \frac{28336}{15} \approx 1889.066667. \tag{7}$$

2 Normal Uncertainty

2.1 $u(Y) = y_1 + y_2$

$$y_1 \sim N(\mu_1, \sigma_1),$$
$$y_2 \sim N(\mu_2, \sigma_2),$$

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2.1.1 Moments

$$\mathbb{E}[u(Y)^r] = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \Big(y_1 + y_2\Big)^r \rho(y_1)\rho(y_2)dy_2dy_1, \tag{8}$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{(y_1+y_2)^r}{\sigma_1\sigma_2 2\pi}\exp\Big[-\frac{(y_1-\mu_1)^2}{2\sigma_1^2}\Big]\exp\Big[-\frac{(y_2-\mu_2)^2}{2\sigma_2^2}\Big]dy_2dy_1, \tag{9}$$

$$\tag{10}$$

For $\mu_i = 3, \sigma_i = 2\ \forall i \in [1,2]$,

$$\mathbb{E}[u(Y)] = 6, \tag{11}$$

$$\mathbb{E}[u(Y)^2] = 44, \tag{12}$$

$$\mathbb{E}[u(Y)^3] = 360, \tag{13}$$

$$\mathbb{E}[u(Y)^4] = 3216. \tag{14}$$

3 Gamma Uncertainty

3.1 $u(Y) = y_1 + y_2$

$$y_i \sim G(\min, \alpha)\ \forall i \in [1,2],$$

3.1.1 Moments

$$\mathbb{E}[u(Y)^r] = \int_{\min}^{\infty}\int_{\min}^{\infty} \Big(y_1 + y_2\Big)^r \rho(y_1)\rho(y_2)dy_2dy_1, \tag{15}$$

$$= \int_{\min}^{\infty}\int_{\min}^{\infty} \frac{(y_1+y_2)^r}{(\alpha-1)!(\alpha-1)!} y_1^{\alpha-1}y_2^{\alpha-1}e^{-y_1-y_2}dy_2dy_1. \tag{16}$$

For $\min = 0, \alpha = 2$,

$$\mathbb{E}[u(Y)] = 4, \tag{17}$$

$$\mathbb{E}[u(Y)^2] = 20, \tag{18}$$

$$\mathbb{E}[u(Y)^3] = 120, \tag{19}$$

$$\mathbb{E}[u(Y)^4] = 840. \tag{20}$$

4 Beta Uncertainty

4.1 $u(Y) = y_1 + y_2$

$$y_i \sim B(a, b, \alpha, \beta)\ \forall i \in [1,2],$$

4.1.1 Moments

$$\mathbb{E}[u(Y)^r] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Big(y_1 + y_2\Big)^r \rho(y_1)\rho(y_2)dy_2dy_1, \tag{21}$$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} (y_1 + y_2)^r \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} y_1^{\alpha-1} y_2^{\alpha-1}(1-y_1)^{\beta-1}(1-y_2)^{\beta-1}dy_2dy_1. \tag{22}$$

For $a = 0, b = 1, \alpha = 2, \beta = 5$,

$$\mathbb{E}[u(Y)] = \frac{4}{7} \sim 0.571429, \tag{23}$$

$$\mathbb{E}[u(Y)^2] = \frac{37}{98} \sim 0.377551, \tag{24}$$

$$\mathbb{E}[u(Y)^3] = \frac{41}{147} \sim 0.278912, \tag{25}$$

$$\mathbb{E}[u(Y)^4] = \frac{265}{1176} \sim 0.22534. \tag{26}$$

5 Triangular Uncertainty

5.1 $u(Y) = y_1 + y_2$

$$y_i \sim T(a, b, c)\ \forall i \in [1, 2],$$

5.1.1 Moments

$$\mathbb{E}[u(Y)^r] = \int_a^c\int_a^c (y_1 + y_2)^r \frac{y_1 - a}{c - a}\frac{y_2 - a}{c - a}dy_2dy_1 \tag{27}$$

$$+ 2\int_a^c\int_c^b (y_1 + y_2)^r \frac{y_1 - a}{c - a}\frac{b - y_2}{b - c}dy_2dy_1 \tag{28}$$

$$+ \int_c^b\int_c^b (y_1 + y_2)^r \frac{b - y_1}{b - c}\frac{b - y_2}{b - c}dy_2dy_1. \tag{29}$$

For $a = 1, b = 5, c = 4$,

$$\mathbb{E}[u(Y)] = \frac{20}{3} \sim 6.66667, \tag{30}$$

$$\mathbb{E}[u(Y)^2] = \frac{413}{9} \sim 45.88889, \tag{31}$$

$$\mathbb{E}[u(Y)^3] = \frac{974}{3} \sim 324.66667, \tag{32}$$

$$\mathbb{E}[u(Y)^4] = \frac{23523}{10} \sim 2352.3. \tag{33}$$

6 Exponential Uncertainty

6.1 $u(Y) = y_1 + y_2$

$$y_i \sim E(\lambda)\ \forall i \in [1, 2],$$

6.1.1 Moments

$$\mathbb{E}[u(Y)^r] = \int_0^\infty \int_0^\infty (y_1 + y_2)^r \lambda^2 e^{-\lambda(x+y)} dxdy \tag{34}$$

For $\lambda = 1.5$,

$$\mathbb{E}[u(Y)] = \frac{4}{3} \sim 1.33333, \tag{35}$$

$$\mathbb{E}[u(Y)^2] = \frac{8}{3} \sim 2.66667, \tag{36}$$

$$\mathbb{E}[u(Y)^3] = \frac{64}{9} \sim 7.11111, \tag{37}$$

$$\mathbb{E}[u(Y)^4] = \frac{640}{27} \sim 23.7037. \tag{38}$$