Numerical Methods for Uncertainty Quantification

Paul W. Talbot[1]

[1]University of New Mexico

Supported by Idaho National Laboratory

BYU-Idaho Physics Dept. Colloquium, December 2014

Outline

Discussion Points

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1. Sources of Uncertainty
2. Analytic Methods
3. Numerical Methods

Uncertainty

- Aleatory (physical)
- Epistemic (measured)

Example Stochastic Problem

$$y_f = y_i + v\sin(\theta)t - \frac{1}{2}gt^2, \tag{1}$$

$$x_f = v\cos(\theta)t. \tag{2}$$

Solution: $x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right)$

Example Stochastic Problem

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right) \tag{3}$$

- initial height $y_i = 2$ m
- initial velocity $v = 50$ m/s
- initial trajectory $\theta = 35^o$
- accel. gravity $g = -9.81$ m/s/s

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Example Stochastic Problem

$$x_f = \frac{v \cos\theta}{g}\left(v \sin\theta + \sqrt{v^2 \sin^2\theta + 2gy_i}\right) \tag{4}$$

- initial height $y_i = 1 \pm 1$ m
- initial velocity $v = 50 \pm 5$ m/s
- initial trajectory $\theta = 45 \pm 10^o$
- accel. gravity $g = 9.7988 \pm 0.0349$ m/s/s

Uncertainty Quantification

$$x_f = \frac{v \cos \theta}{g} \left(v \sin \theta + \sqrt{v^2 \sin^2 \theta + 2 g y_i} \right)$$

Min-Max

$$x_{f,\text{min}} = \frac{(45)(0.5736)}{9.7369} \left((45)(0.8192) + \sqrt{(45)^2 (0.8192)^2 + 2(9.7369)(0)} \right) = 195.45 \text{ m}$$

$$x_{f,\text{max}} = \frac{(55)(0.8192)}{9.8337} \left((55)(0.5736) + \sqrt{(55)^2 (0.5736)^2 + 2(9.8337)(2)} \right) = 291.92 \text{ m}$$

Result: $x_f \approx 244 \pm 48$ m

Flawed Reasoning

- Nonlinear Flight Path
- Does increasing θ make a longer or shorter range?

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Uncertainty Quantification

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Min-Max

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Uncertainty Quantification

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right)$$

Analytic Uncertainty

$$\sigma_{x_f} = \sqrt{\left(\frac{\partial x_f}{\partial y_i}\right)^2\sigma_{y_i}^2 + \left(\frac{\partial x_f}{\partial v}\right)^2\sigma_v^2 + \left(\frac{\partial x_f}{\partial g}\right)^2\sigma_g^2 + \left(\frac{\partial x_f}{\partial \theta}\right)^2\sigma_\theta^2}$$

Result: $x_f = 256 \pm 51$ m

Works well for simple functions

- Simple derivatives
- Analytic solution

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A More Difficult Problem

With Air Resistance:

$$y_f = \frac{v_t}{g}(v\sin\theta + v_t)\left(1 - e^{-gt/v_t}\right) - v_t t, \tag{5}$$

$$x_f = \frac{v v_t \cos\theta}{g}\left(1 - e^{-gt/v_t}\right). \tag{6}$$

$$v_t = \frac{mg}{D}, \qquad D = \frac{\rho C A}{2}, \qquad A = \pi r^2 \tag{7}$$

Solve numerically to get x_f (Forward Euler).

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Additional Uncertainty

$$y_f = \frac{v_t}{g}(v \sin\theta + v_t)\left(1 - e^{-gt/v_t}\right) - v_t t, \tag{8}$$

$$x_f = \frac{v v_t \cos\theta}{g}\left(1 - e^{-gt/v_t}\right). \tag{9}$$

$$v_t = \frac{mg}{D}, \qquad D = \frac{\rho C A}{2}, \qquad A = \pi r^2 \tag{10}$$

$$m = 0.145 \pm 0.0725 \text{ kg}, \qquad r = 0.0336 \pm 0.00336 \text{ m}, \tag{11}$$

$$C = 0.5 \pm 0.5, \qquad \rho_{\text{air}} = 1.2 \pm 0.1 \text{ kg/m}^3. \tag{12}$$

Uncertainty Quantification: Complicated Problems

How do we quantify uncertainty for problems without simple analytic solutions?

- Monte Carlo sampling
- Stochastic Collocation
- High Density Model Reduction (low-order)

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Uncertainty Quantification

Monte Carlo

- Let $u(Y)$ be any system, like $x_f(v, \theta, g, y_i)$
- Randomly sample input parameters, record outputs
- Calculate moments (mean, variance, skew, kurtosis)

$$\mathbb{E}[u^r] \approx \frac{1}{M}\sum_{m=1}^{M} u\left(Y^{(m)}\right)^r$$

Mean: $\bar{u} \approx \frac{1}{M}\sum u\left(Y^{(m)}\right)$

(video)