UQ

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We use standard orthonormal Legendre polynomials and quadrature to expand $f(\zeta)$. Since these polynomials are orthogonormal over [-1,1], I have some uncertainty on how this should work, and I can see a couple cases. First, a review of the expansion.

1 Standard Case

$$f(\xi) \approx \sum_{i=0}^{I} f_i P_i(\xi). \tag{1}$$

The coefficients are given because of the orthonormal Legendre polynomials,

$$\int_{-1}^{1} \sum_{i=0}^{I} f_i P_i(\xi) d\xi = f_i = \int_{-1}^{1} f(\xi) P_i(\xi) d\xi, \tag{2}$$

applying Gauss-Legendre quadrature,

$$f_i = \sum_{\ell=1}^{L} w_\ell f(\xi_\ell) P_i(\xi_\ell). \tag{3}$$

To demonstrate a simple case, we consider the simple linear function

$$f(\xi) = a + b\xi, \quad \xi \in (-1,1), \tag{4}$$
$$= \sum_{i=0}^{I=1} f_i P_i(\xi). \tag{5}$$

$$f_i = \sum_{\ell=1}^{L=2} w_\ell f(\xi_\ell) P_i(\xi_\ell), \tag{6}$$
$$= w_1 f(\xi_1) P_i(\xi_1) + w_2 f(\xi_2) P_i(\xi_2). \tag{7}$$

Using the weights $w_\ell = (1,1)$ and Gauss points $\xi_\ell = \pm 1/\sqrt{3}$, as well as the orthonormal Legendre polynomials

$$P_0(x) = \frac{1}{\sqrt{2}}, \tag{8}$$

$$P_1(x) = \sqrt{\frac{3}{2}} x, \tag{9}$$

we can find our coefficients,

$$f_0 = (1)\left(a + b\left(\frac{-1}{\sqrt{3}}\right)\right)\frac{1}{\sqrt{2}} + (1)\left(a + b\left(\frac{1}{\sqrt{3}}\right)\right)\frac{1}{\sqrt{2}}, \tag{10}$$
$$= \frac{1}{\sqrt{2}}\left[a - \frac{b}{\sqrt{3}} + a + \frac{b}{\sqrt{3}}\right], \tag{11}$$
$$= a\sqrt{2}. \tag{12}$$

$$f_1 = (1)\left(a + b\left(\frac{-1}{\sqrt{3}}\right)\right)\sqrt{\frac{3}{2}}\frac{(-1)}{\sqrt{3}} + (1)\left(a + b\left(\frac{1}{\sqrt{3}}\right)\right)\sqrt{\frac{3}{2}}\frac{1}{\sqrt{3}}, \tag{13}$$
$$= b\sqrt{\frac{2}{3}}. \tag{14}$$

Reconstructing the original equation,

$$f(\xi) = \sum_{i=0}^{I} f_i P_i(\xi) = f_0 P_0(\xi) + f_1 P_1(x), \tag{15}$$
$$= \frac{a\sqrt{2}}{\sqrt{2}} + b\sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}}\xi, \tag{16}$$
$$= a + b\xi. \tag{17}$$

2 Adjusted Range

The problem at hand is when f is a function of a variable that isn't distributed in the standard way. For instance, we take the same function, but of $\zeta \in (3, 5)$,

$$f(\zeta) = a + b\zeta, \quad \zeta \in (3, 5). \tag{18}$$

We still expand $f(\zeta)$ as before,

$$f(\zeta) = \sum_{i=0}^{I=1} f_i P_i(\zeta). \tag{19}$$

The problem, however, is in how to calculate the coefficients. $P_i(x)$ are only orthogonal over (-1,1), but the integration range is (a, b):

$$f_i = \int_a^b f(\zeta) P_i(\zeta) d\zeta. \tag{20}$$

An integral over arbitrary range (a, b) can be shited to $(-1, 1)$ as

$$\int_a^b g(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}\xi + \frac{a+b}{2}\right) dz, \quad x \in (a, b), \xi \in (-1, 1), \tag{21}$$

or, defining

$$\sigma \equiv \frac{b-a}{2}, \qquad \mu \equiv \frac{b+a}{2}, \tag{22}$$

$$\int_a^b g(x)dx = \sigma \int_{-1}^1 f\left(\sigma z + \mu\right) dz, \quad x \in (a,b), z \in (-1,1). \tag{23}$$

Applying this to finding the coefficients,

$$f_i = \int_a^b f(\zeta) P_i(\zeta) d\zeta. \tag{24}$$

The question lies how to handle $P_i(\zeta)$, covered in the sections below as "Brute Force" and "Hold Polys".

3 Brute Force

The first approach is simply to apply the shifted integral rule,

$$f_i = \int_a^b f(\zeta) P_i(\zeta) d\zeta = \sigma \int_{-1}^1 f(\sigma\xi + \mu) P(\sigma\xi + \mu) d\xi. \tag{25}$$

Applying quadrature,

$$f_i = \sigma \sum_\ell w_\ell f(\sigma\xi_\ell + \mu) P_i(\sigma\xi_\ell + \mu). \tag{26}$$

Expanding the first coefficient,

$$\begin{aligned}
f_0 &= \sigma w_1 f(\sigma\xi_1 + \mu) P_0(\sigma\xi_1 + \mu) + \sigma w_2 f(\sigma\xi_2 + \mu) P_0(\sigma\xi_2 + \mu), && (27)\\
&= \sigma \left(\left[\left(a + b\left[\sigma\xi_1 + \mu\right]\right) \frac{1}{\sqrt{2}} \right] + \left[\left(a + b\left[\sigma\xi_2 + \mu\right]\right) \frac{1}{\sqrt{2}} \right] \right), && (28)\\
&= \frac{\sigma}{\sqrt{2}} \left[\left(a + b\left[\frac{-\sigma}{\sqrt{3}} + \mu\right] \right) + \left(a + b\left[\frac{\sigma}{\sqrt{3}} + \mu\right] \right) \right], && (29)\\
&= \sigma\sqrt{2}\left(a + \mu\right). && (30)
\end{aligned}$$

However, on expanding this expression, we do not receive the expected result $f_0 P_0(\zeta){=}a$.

4 Hold Polys

Since the Legendre polynomials aren't orthogonal over (a,b), I thought perhaps we invent alternative functions $P_i^*(\zeta)$ such that

$$\int_a^b P_i^*(\zeta) d\zeta = \sigma \int_{-1}^1 P_i(x) dx, \tag{31}$$

$$\sigma \equiv \frac{b-a}{2} \text{ range}, \qquad \mu \equiv \frac{a+b}{2} \text{ (mean)}. \tag{32}$$

$$\begin{aligned}
f(\xi) &= a + b\xi, \quad \xi \in [a,b], && (33)\\
&= \sum_i f_i P_i(x), && (34)
\end{aligned}$$

$$
\begin{aligned}
f_i &= \int_a^b f(\xi) P_i^*(\xi) d\xi, && (35)\\
&= \sigma \int_{-1}^{1} f(\sigma x + \mu) P_i(x) dx, && (36)\\
&\approx \sigma \sum_\ell w_\ell f(\sigma x_\ell + \mu) P_i(x_\ell), && (37)\\
&= \sigma w_1 f(\sigma x_1 + \mu) P_i(x_1) + w_2 f(\sigma x_2 + \mu) P_i(x_2). && (38)
\end{aligned}
$$

The first coefficient is

$$
\begin{aligned}
f_0 &= \sigma \left[(1) \left(1 + 2 \left(\frac{-\sigma}{\sqrt{3}} + \mu \right) \right) \frac{1}{\sqrt{2}} + (1) \left(1 + 2 \left(\frac{\sigma}{\sqrt{3}} + \mu \right) \right) \frac{1}{\sqrt{2}} \right], && (39)\\
&= \frac{b-a}{2\sqrt{2}} \left[1 + 2 \left(\frac{-(b-a)}{2\sqrt{3}} + \frac{b+a}{2} \right) + 1 + 2 \left(\frac{(b-a)}{2\sqrt{3}} + \frac{b+a}{2} \right) \right], && (40)\\
&= \frac{b-a}{2\sqrt{2}} \left[2 - \frac{(b-a)}{\sqrt{3}} + 2b + 2a + \frac{(b-a)}{\sqrt{3}} \right], && (41)\\
&= \frac{b-a}{\sqrt{2}} (1 + b + a). && (42)
\end{aligned}
$$

Unfortunately, this also does not give the expected $f_0 P_0(\zeta) = a$.