Numerical Methods for Uncertainty Quantification

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Outline

Discussion Points

1. Uncertainty

Uncertainty

- Aleatory (physical)
- Epistemic (measured)

Example Stochastic Problem

$$y_f = y_i + v\sin(\theta)t - \frac{1}{2}gt^2, \tag{1}$$

$$x_f = v\cos(\theta)t. \tag{2}$$

Solution: $x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right)$

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Example Stochastic Problem

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right) \tag{3}$$

- initial height $y_i = 2$ m
- initial velocity $v = 50$ m/s
- initial trajectory $\theta = 35^o$
- accel. gravity $g = -9.81$ m/s/s

Example Stochastic Problem

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right) \tag{4}$$

- initial height $y_i = 2 \pm 0.1$ m
- initial velocity $v = 50 \pm 5$ m/s
- initial trajectory $\theta = 35 \pm 5^o$
- accel. gravity $g = 9.81 \pm 0.01$ m/s/s

Uncertainty Quantification

$$x_f = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gy_i}\right)$$

Min-Max

- $x_{f,\min} = \left(()() + \sqrt{()^2()^2 + 2()()}\right) = \text{m}$
- $x_{f,\max} = \left(()() + \sqrt{()^2()^2 + 2()()}\right) = \text{m}$

Result: $y_f \approx 7.16 \pm 1.12\text{m}$

Flawed Reasoning

- Nonlinear Flight Path
- Does increasing θ make a longer or shorter range?

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Uncertainty Quantification

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Analytic Uncertainty

$$\sigma_{x_f} = \sqrt{\left(\frac{\partial x_f}{\partial y_i}\right)^2 \sigma_{y_i}^2 + \left(\frac{\partial x_f}{\partial v}\right)^2 \sigma_v^2 + \left(\frac{\partial x_f}{\partial g}\right)^2 \sigma_g^2 + \left(\frac{\partial x_f}{\partial \theta}\right)^2 \sigma_\theta^2}$$

Result: $y_f = 0 \pm 0\text{m}$

Works well for simple functions

- Simple derivatives
- Analytic solution

Uncertainty Quantification

No Air Resistance:

$$y_f = v\sin(\theta)t - \frac{1}{2}gt^2, \tag{5}$$

$$x_f = v\cos(\theta)t. \tag{6}$$

With Air Resistance:

$$y_f = \frac{v_t}{g}(v\sin\theta + v_t)\left(1 - e^{-gt/v_t}\right) - v_t t, \tag{7}$$

$$x_f = \frac{v v_t \cos\theta}{g}\left(1 - e^{-gt/v_t}\right). \tag{8}$$

Solve numerically to get x_f (Forward Euler).

Uncertainty Quantification: Complicated Problems

How do we quantify uncertainty for problems without simple analytic solutions?

- Monte Carlo sampling
- Stochastic Collocation
- High Density Model Reduction (low-order)

Uncertainty Quantification

Monte Carlo

- Let $u(Y)$ be any system, like $x_f(v, \theta, g, y_i)$
- Randomly sample input parameters, record outputs
- Calculate moments (mean, variance, skew, kurtosis)

$$\mathbb{E}[u^r] \approx \frac{1}{M} \sum_{m=1}^{M} u\left(Y^{(m)}\right)^r$$

$$\text{Mean: } \bar{u} \approx \tfrac{1}{M} \sum u\left(Y^{(m)}\right)$$

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