MATHEMATICS OF DEEP LEARNING

JOAN BRUNA , CIMS + CDS, NYU, SPRING'18

Lecture 3: The Scattering Transform and beyond

LECTURE 3 OVERVIEW

- The Scattering Transform.

- Stability Properties of Scattering Transforms

- Rigid Scattering

- From Scattering to CNNs.

AVERAGE AND UNIQUENESS

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AVERAGE AND UNIQUENESS

➤ The only linear, translation-invariant operator is the average:

$$\forall\ v\ ,\ \Phi(x) = \Phi(\varphi_v x) \Longrightarrow \Phi(x) = \frac{1}{|G|}\int \Phi(\varphi_v x)dv$$

$$\Longrightarrow \Phi(x) = \Phi\left(\frac{1}{|G|}\int \varphi_v x dv\right) = \Phi\left(\frac{1}{|G|}\int x(u)du\right)\ .$$

➤ And a similar argument can be used locally.

FROM AVERAGES TO WAVELETS

➤ Low-pass information is insufficient:

> The SIFT method originally consists in a keypoint detection phase, using a Differences of Gaussians pyramid, followed by a local description around each detected keypoint. The keypoint detection computes local maxima on a scale space generated by isotropic gaussian differences, which induces invariance to translations, rotations and

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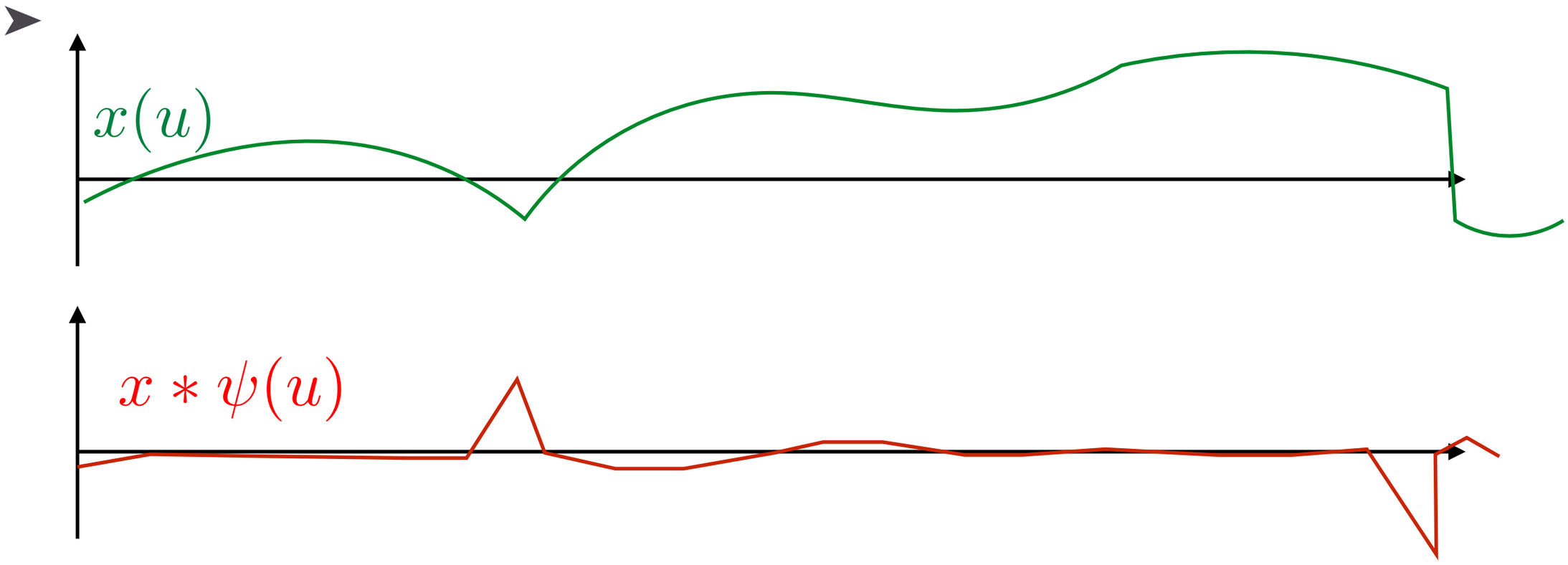
- Thus, we must capture high-frequency.
- These new measurements must involve a non-linearity.
- We want them to preserve stability to deformations.
- And we want them to preserve inter-class variability.

WAVELETS

- ψ: bandpass (ie oscillating) signal, well localized in space and frequency.
- At least one vanishing moment: $\int \psi(u)du = 0$

(we say that ψ has k vanishing moments if $\int \psi(u)u^l du = 0$ for $l < k$)

If $x(u)$ is piece-wise smooth, then $x * \psi(u)$ is mostly zero



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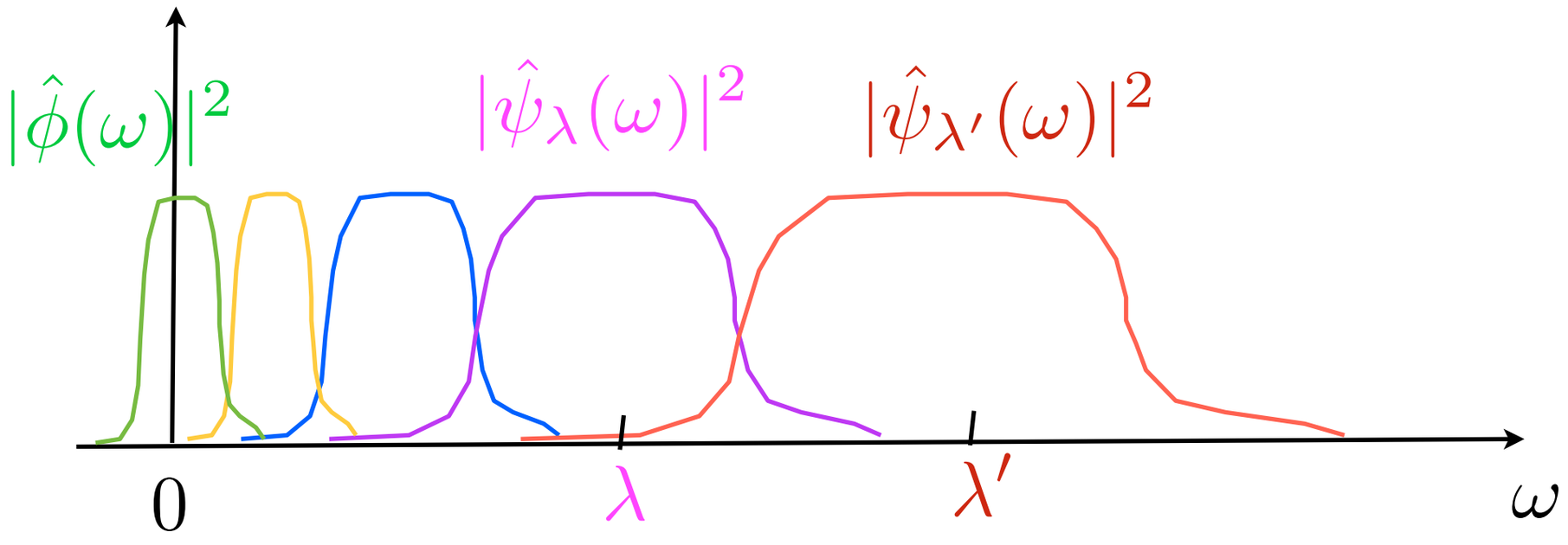
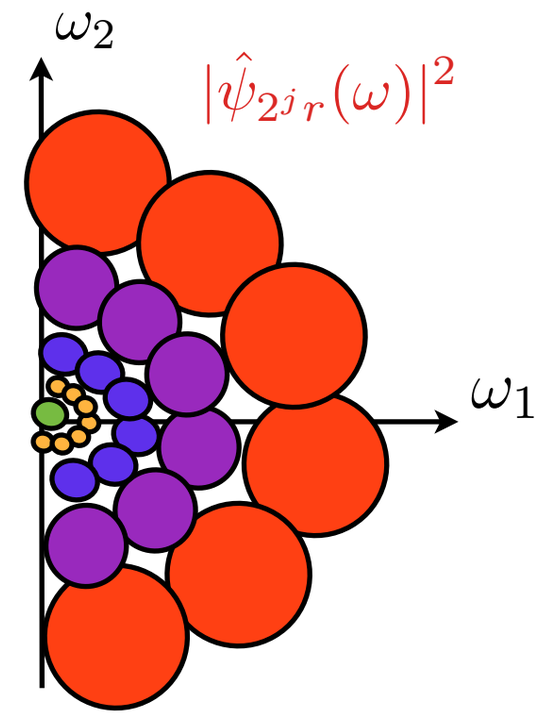
- Dilated wavelets: $\psi_j(u) = 2^{-j}\psi(2^{-j}u)\,,\ j \in \mathbb{Z}$



LITTLEWOOD-PALEY WAVELET FILTER BANKS

- For images, dilated and rotated wavelets:

$$\psi_\lambda(u) = 2^{-j/2}\psi(2^{-j}ru) \ , \ \text{with } \lambda = 2^j r$$



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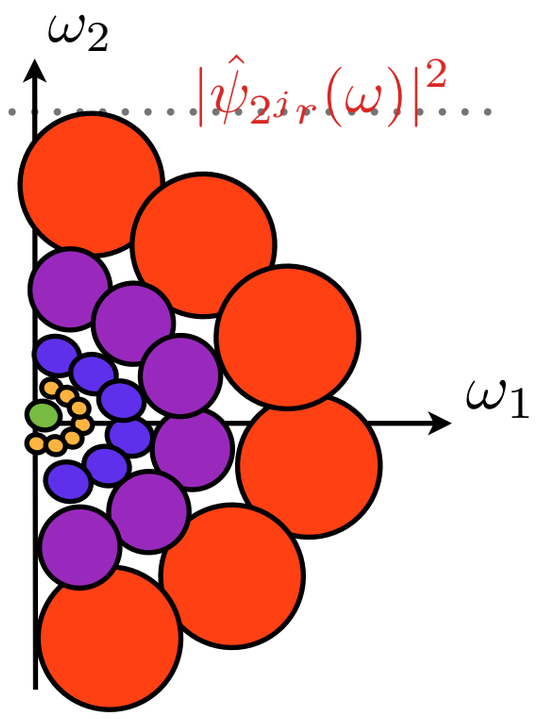


$$Wx = \{x \star \phi(u) \ , \ x \star \psi_\lambda(u)\}_{\lambda \in \Lambda} \qquad x \star \psi(u) = \int x(v)\psi(u-v)dv \ .$$

Theorem (Littlewood-Paley): If there exists $\delta > 0$ such that

$$\forall \omega > 0 \ , \ 1 - \delta \leq |\hat{\phi}(\omega)|^2 + \frac{1}{2}\sum_\lambda |\hat{\psi}(\lambda^{-1}\omega)|^2 \leq 1 \ ,$$

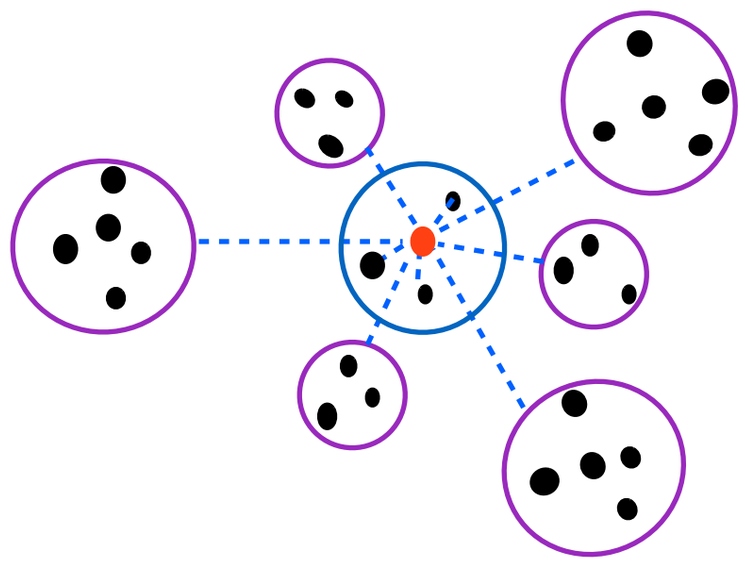
then $\forall\, x \in L^2 \ , \ (1-\delta)\|x\|^2 \leq \|Wx\|^2 \leq \|x\|^2 \ .$

WAVELETS AND SCALE SEPARATION

Interactions of d variables of X: pixels, particules, agents...

- Markov: each variable interacts only with its neighbours and ignores large scale interactions.

Factorisation into multscale interactions



Multiscale regroupments reduce the number of interactions

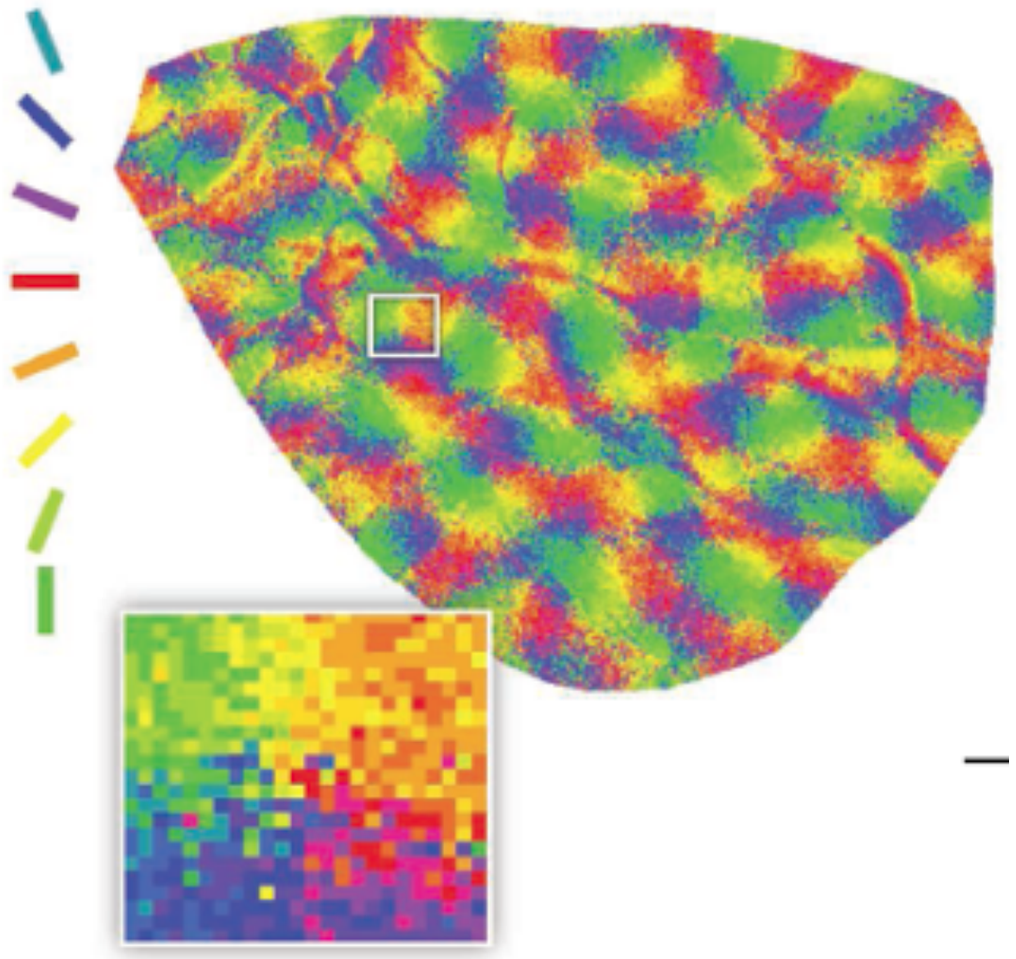
from d to $O(\log d)$

$\Rightarrow$ wavelet transforms and deep convolutional networks

See also the Fast Multipole Method [Greengard and Rokhlin]

WAVELETS IN VISION

- *V1* Model of Simple and Complex cells: First layer of processing is selective in orientation, scale and position.



- cells are organized in *pinwheels*. (more on that later).

WAVELETS AND DEFORMATIONS

- We saw before that a blurring kernel is nearly invariant to deformations:

  Proposition: The local averaging $\Phi(x) = x * \phi_J$ satisfies $\forall\ \|x\| = 1 \in L^2\ ,\ \tau\ ,\ \|\Phi(x) - \Phi(\varphi_\tau x)\| \leq C\|\tau\|\ .$

- What about the wavelet operator $\Phi(x) = \{x * \psi_\lambda\}_\lambda$?

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- What about the wavelet operator $\Phi(x) = \{x * \psi_\lambda\}_\lambda$?
  - We don't have local invariance, but we have a form of local equivariance:

Proposition [Mallat]: For each $\delta > 0$ there exists $C > 0$ such that for all J and all $\tau \in C^2$ with $\|\nabla\tau\|_\infty \leq 1 - \delta$ we have

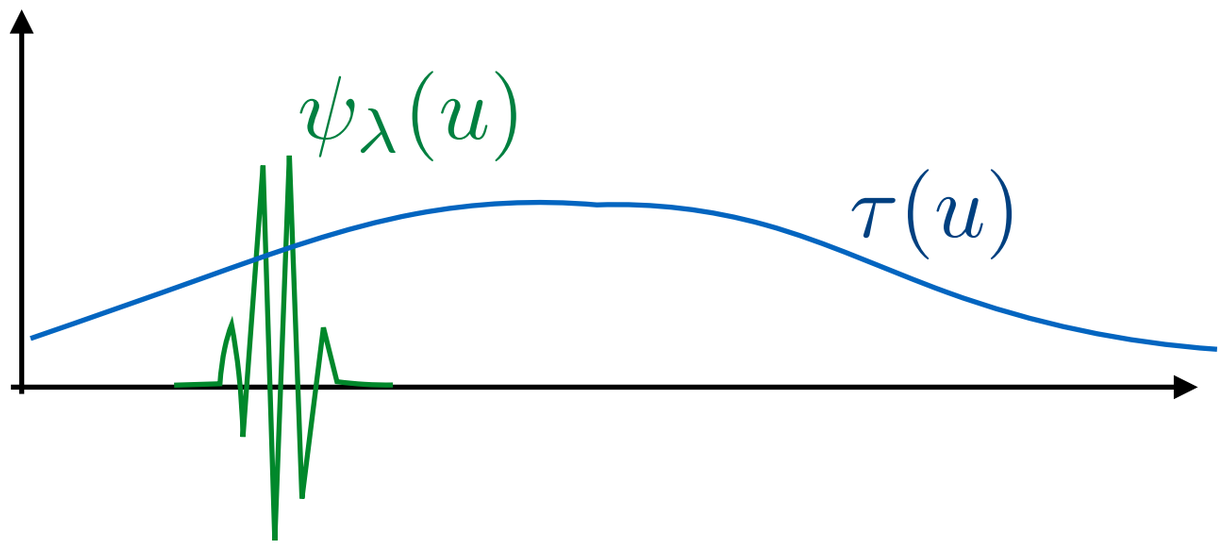
$$\|W_J\varphi_\tau - \varphi_\tau W_J\| \leq C(J\|\nabla\tau\|_\infty + \|H\tau\|_\infty)\ .$$

($H\tau$: Hessian of τ)

WAVELETS AND DEFORMATIONS

➤ Qualitative idea behind this result:

Each ψ_λ only "sees" the part of the deformation τ that intersects its support.



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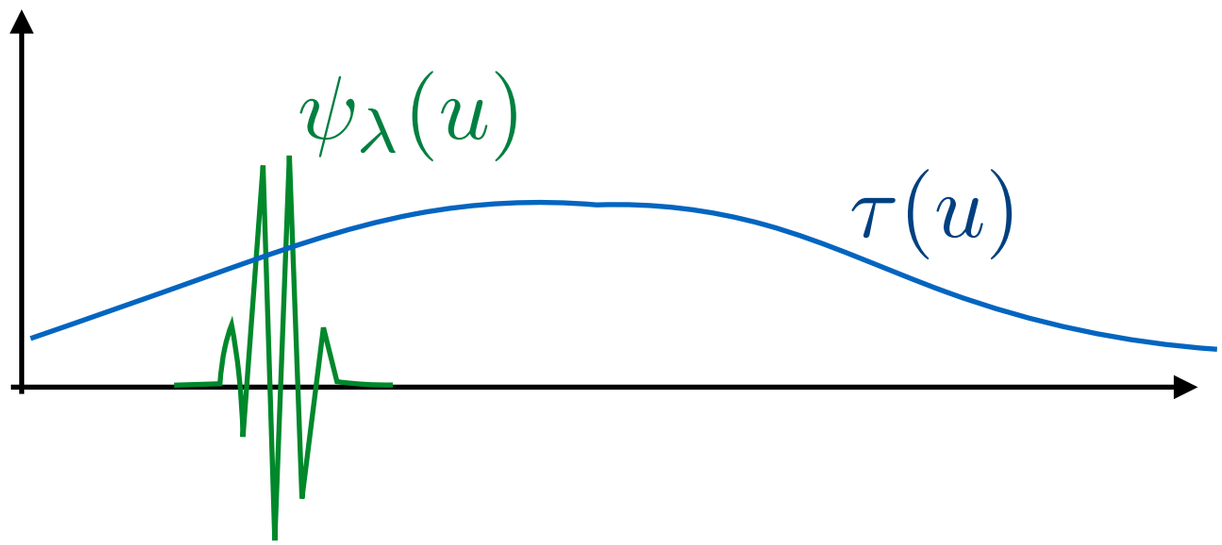

For small scales, ψ_λ has small support, and for u, v within that support, because τ is smooth, $|\tau(v) - \tau(u)| \sim 2^{-j}|\nabla\tau|_\infty$.

Thus $|(\varphi_\tau x) * \psi_\lambda(u) - x * \psi_\lambda(u - \tau(u))| \sim |\nabla\tau|_\infty$.

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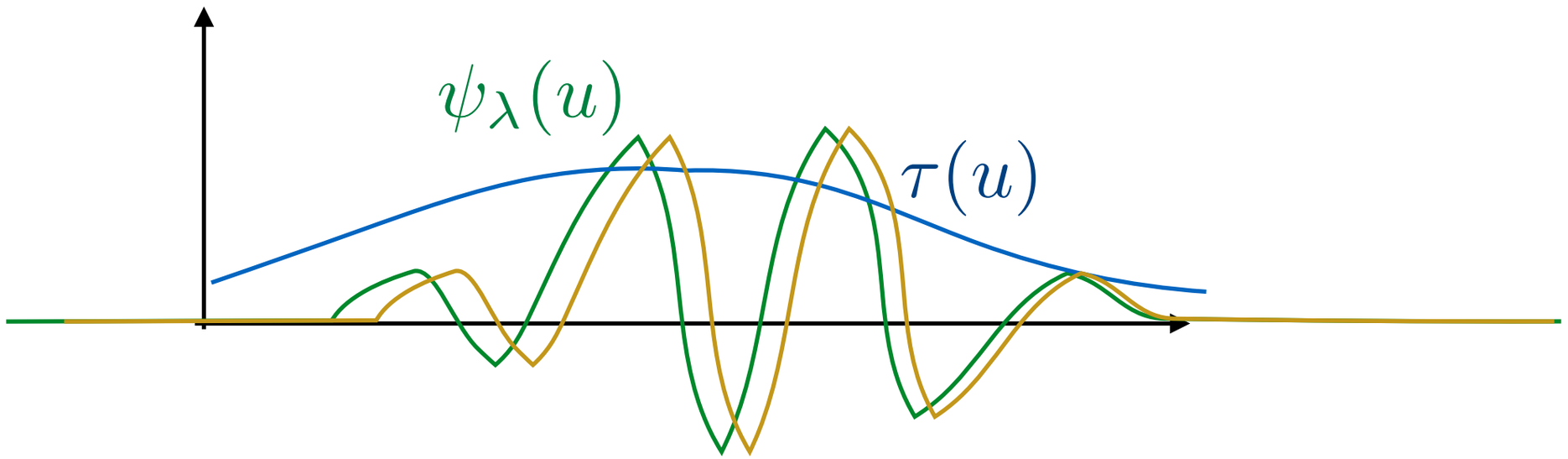

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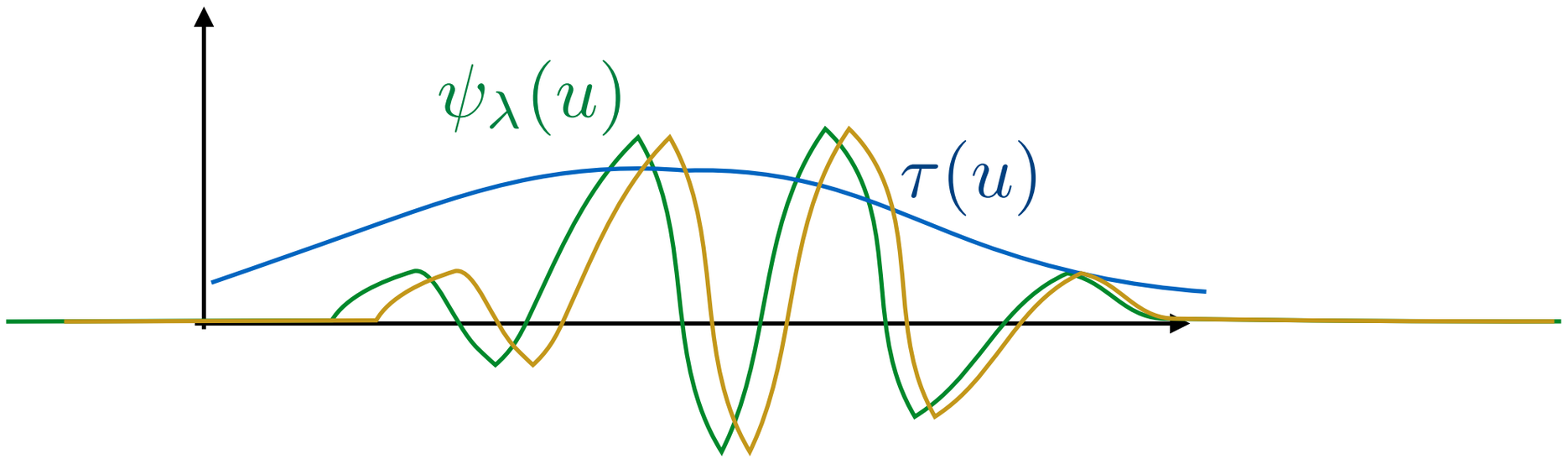

For large scales, ψ_λ is itself smooth, thus $|\varphi_\tau(x * \psi_\lambda) - (\varphi_\tau x) * \psi_\lambda| \sim \|\nabla\tau\|_\infty$.

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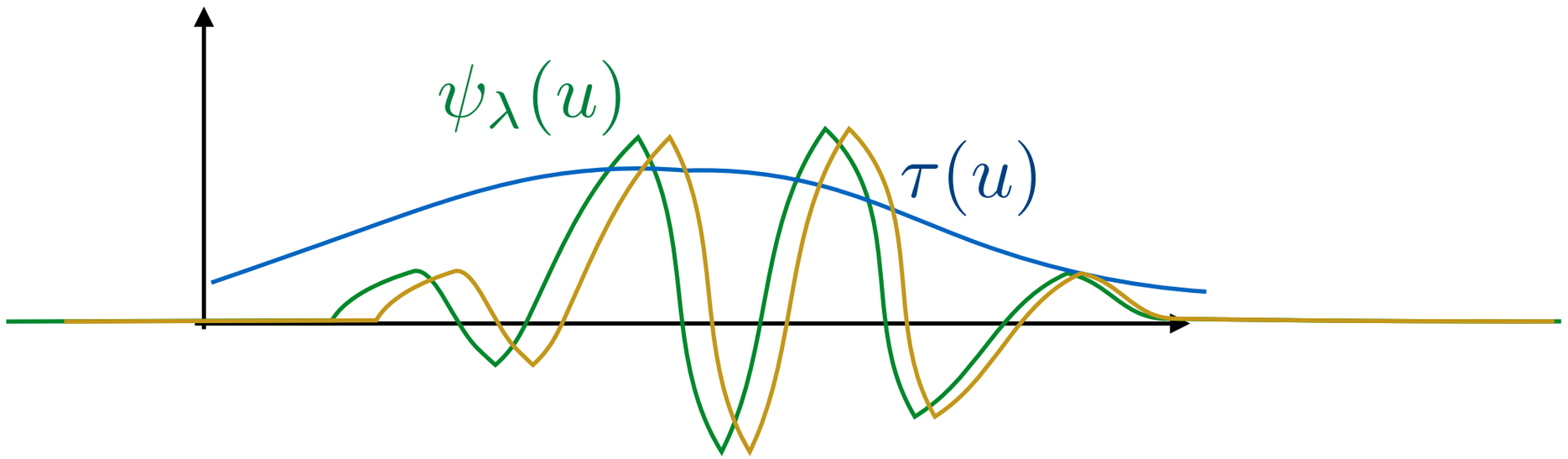
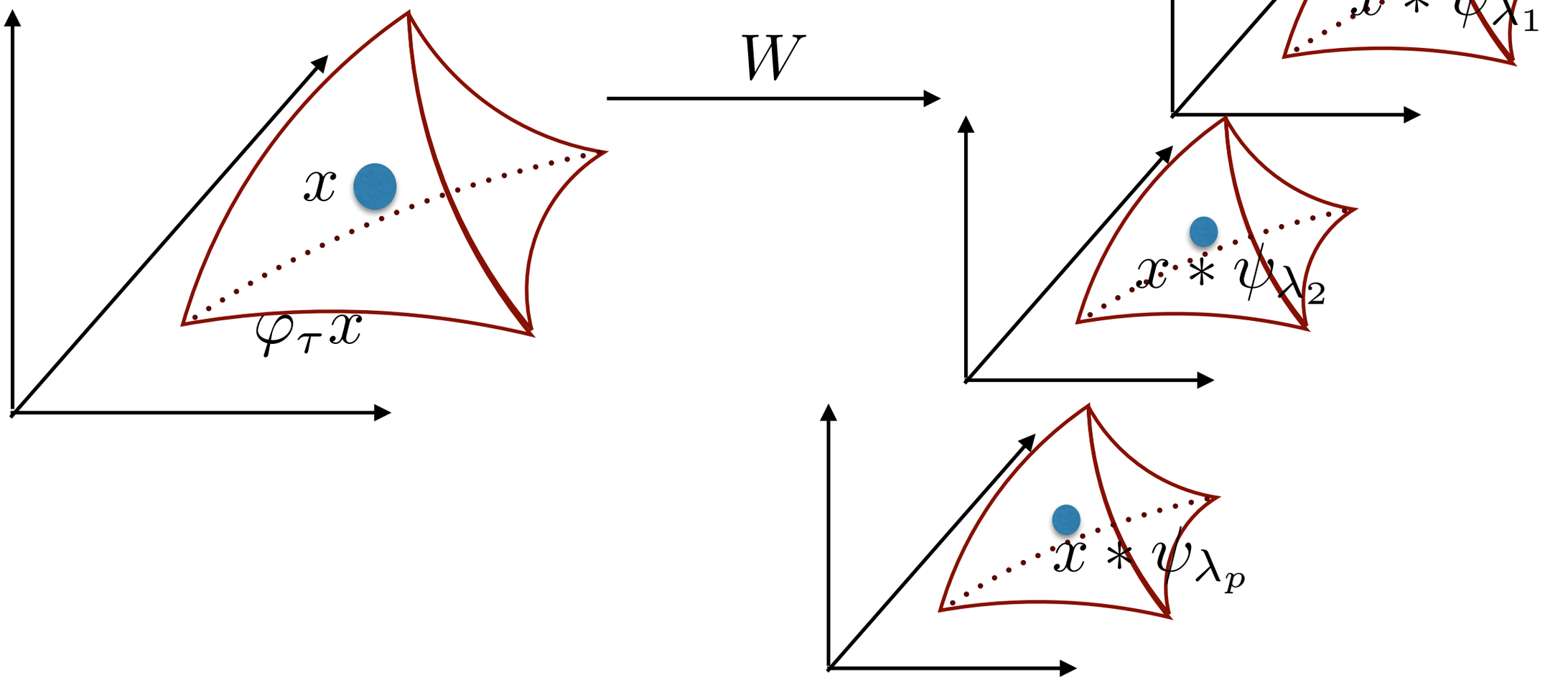


For large scales, ψ_λ is itself smooth, thus
$|\varphi_\tau(x * \psi_\lambda) - (\varphi_\tau x) * \psi_\lambda| \sim \|\nabla\tau\|_\infty$.

And, most importantly, wavelet separates scales
(so errors do not accumulate)

WAVELETS AND NON-LINEARITIES

- The commutation property says that deformations in the input are approximately mapped to deformations in the wavelet domain:



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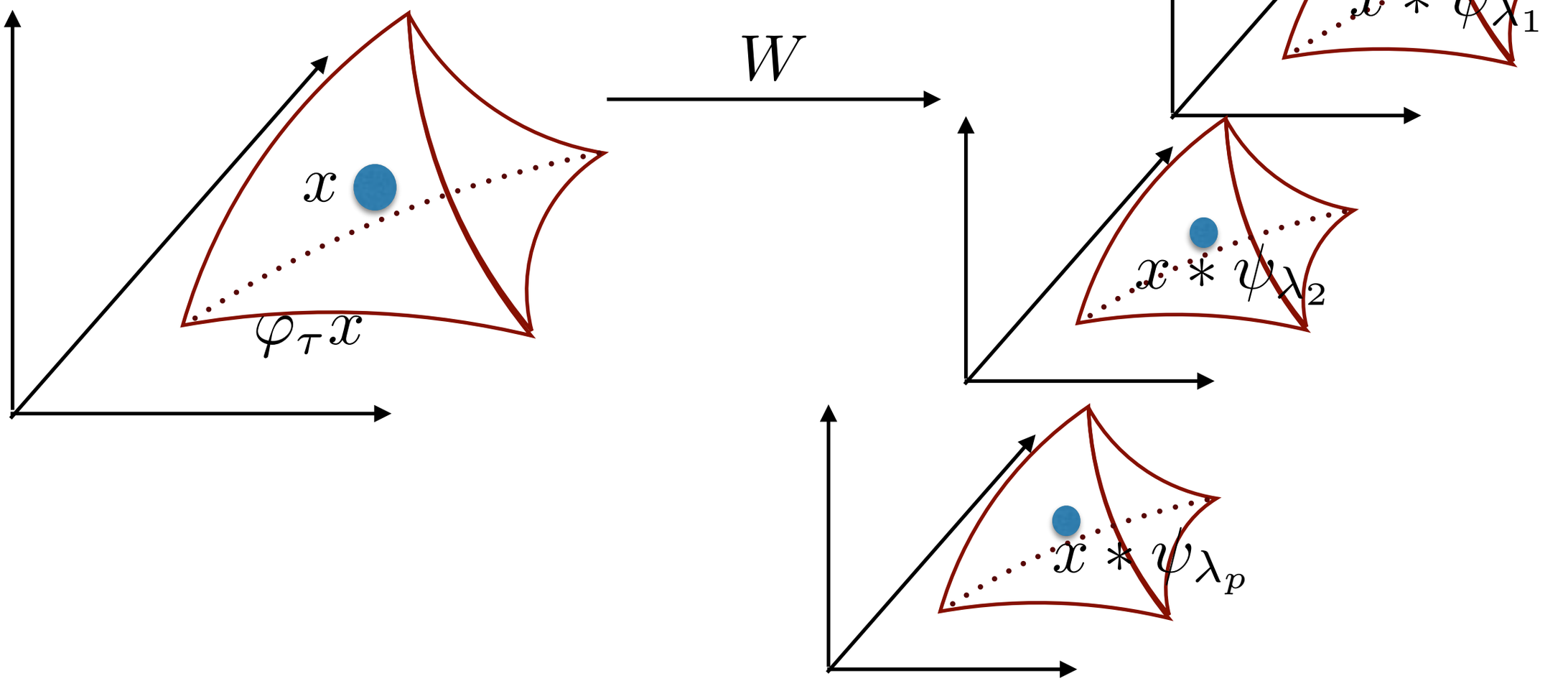

- We want to extract again stable measurements: *need non-linear operator.*

CHARACTERIZATION OF STABLE NON-LINEARITIES

➤Preserve additive stability:

$$\|Mx - Mx'\| \leq \|x - x'\| \ . \qquad M \text{ non-expansive} \ .$$

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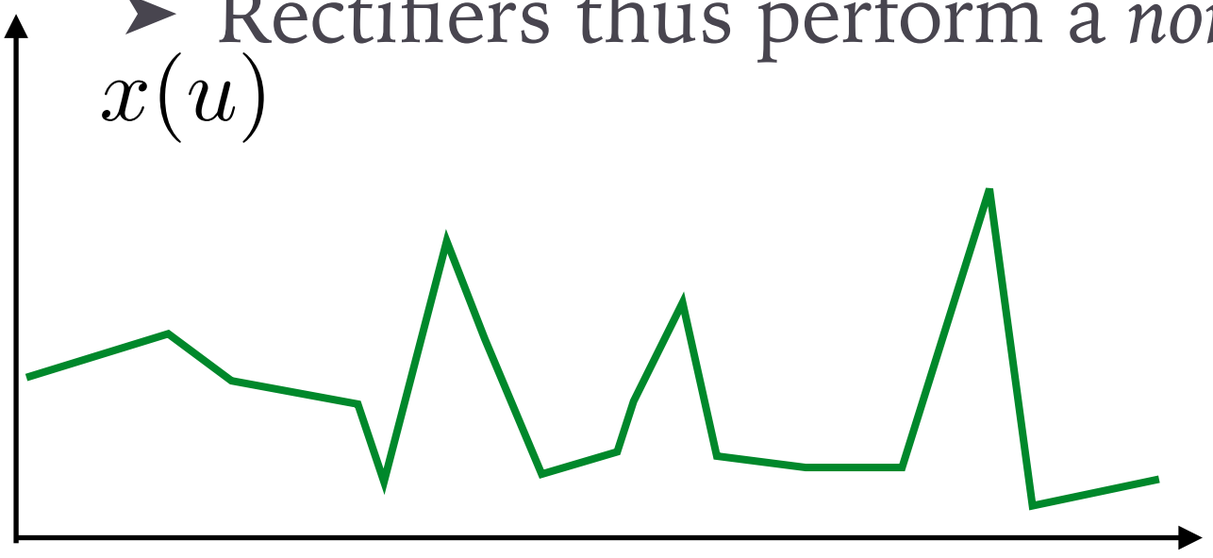
➤Preserve geometric stability: It is sufficient to commute with diffeomorphisms.

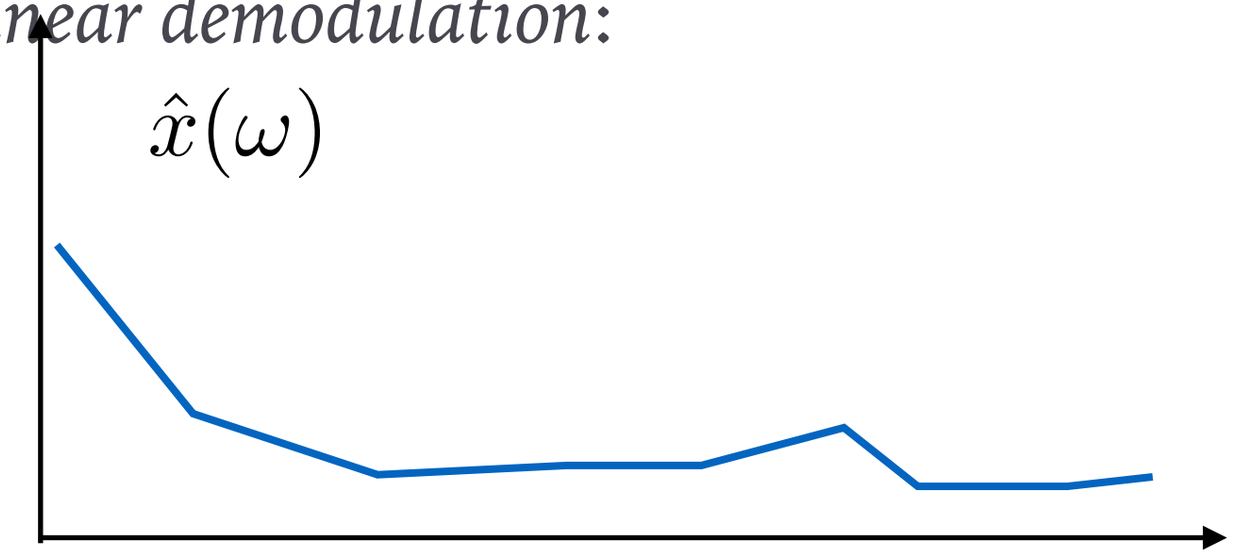
Theorem: If M is non-expansive operator in L^2 such that $\varphi_\tau M = M\varphi_\tau$ for all τ, then M is point-wise:

$$Mx(u) = \rho(x(u)) \ .$$

UNDERSTANDING THE EFFECT OF NONLINEARITIES

➤ Rectifiers thus perform a *non-linear demodulation*:

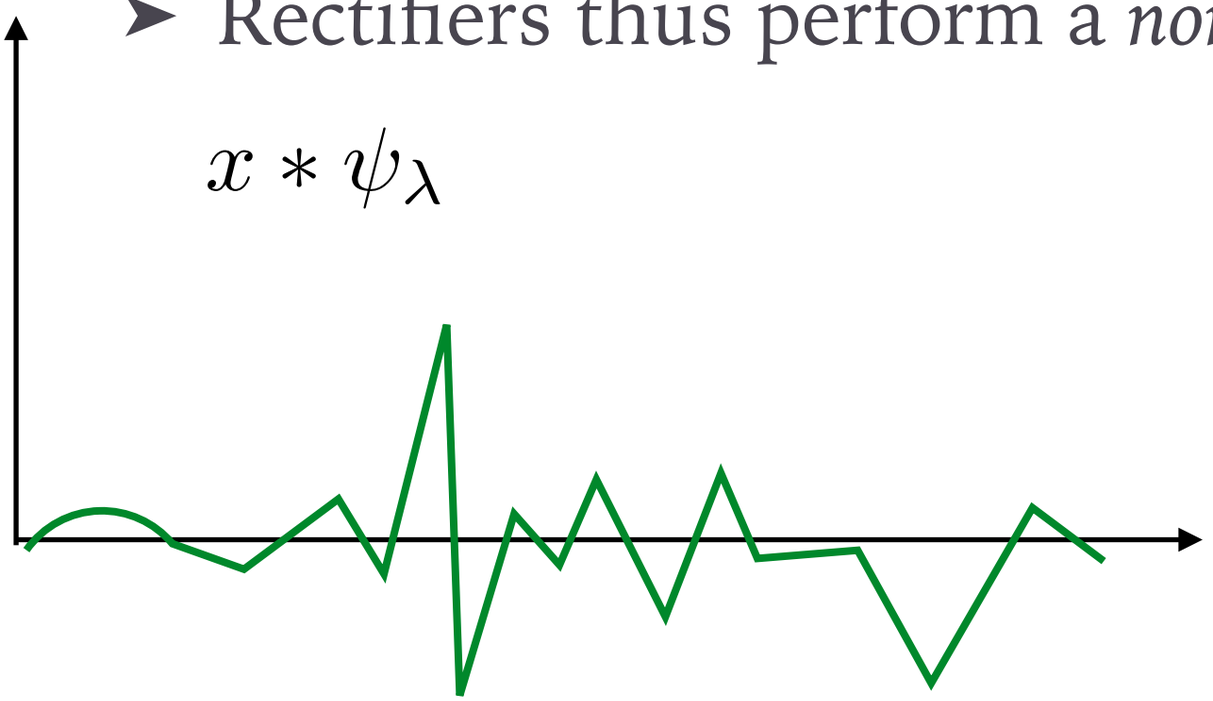




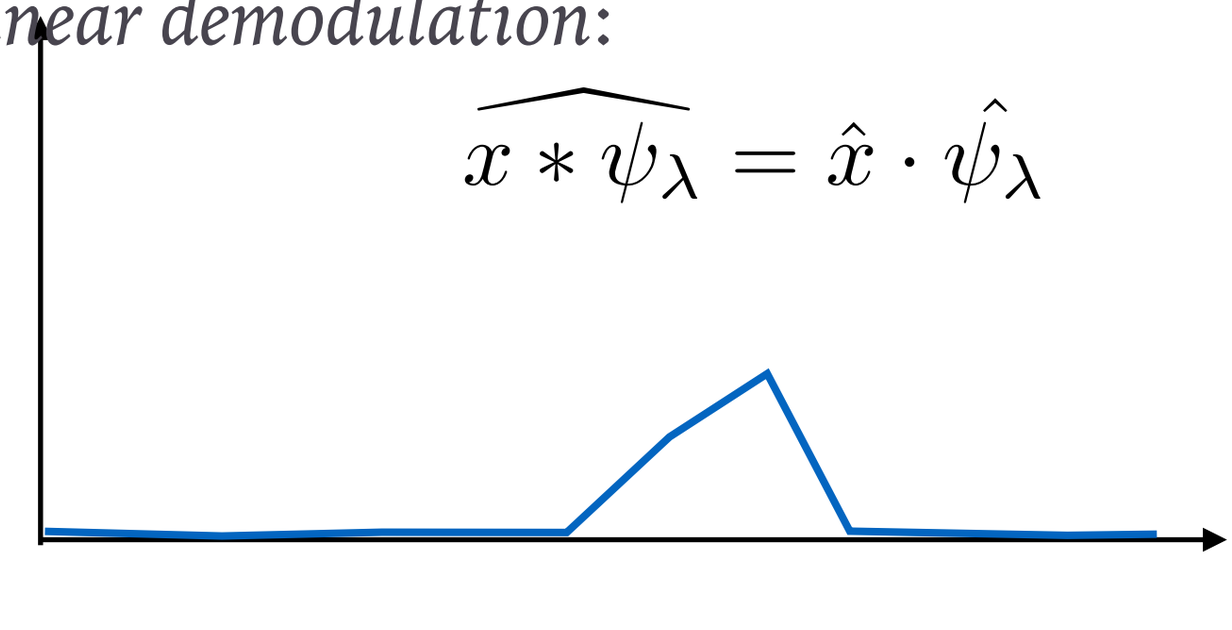
UNDERSTANDING THE EFFECT OF NONLINEARITIES

- Rectifiers thus perform a *non-linear demodulation*:

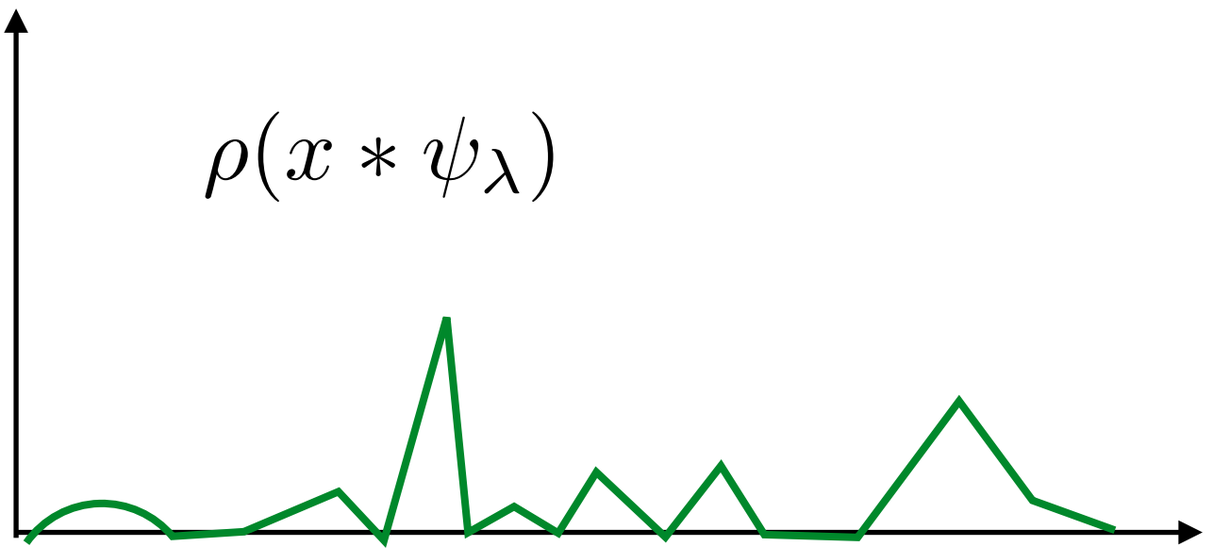
$x * \psi_\lambda$



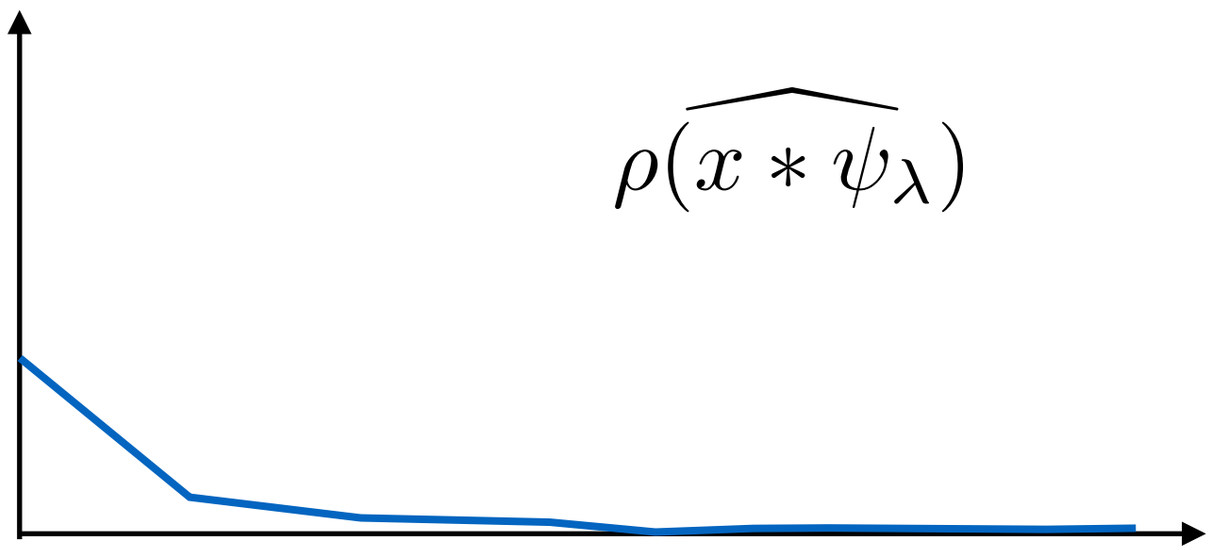
$\widehat{x * \psi_\lambda} = \hat{x} \cdot \hat{\psi}_\lambda$



$\rho(x * \psi_\lambda)$



$\widehat{\rho(x * \psi_\lambda)}$



sometimes called the *envelope*

CHOICE OF POINTWISE NONLINEARITY

- Full rectification $\rho(z) = |z|$ preserves energy:
  - When the wavelet is complex, it produces smoother envelopes (thus more stable features).

- Half rectification (ReLU) $\rho(z) = \max(z, 0)$ captures half the energy, and it also creates *sparsity*.
  - We will see that this is important to perform *detection*.

- Sigmoid nonlinearity $\rho(z) = (1 + e^{-z})^{-1}$.
  - It is not homogeneous
  - Saturating regimes are problematic for learning via back propagation in deep models.
- Other: "Leaky" ReLU [MSR'14]: parametrized half-rectifier, ELU , etc.

SEPARABLE SCATTERING OPERATORS

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- Cascade the *recovery* operator:

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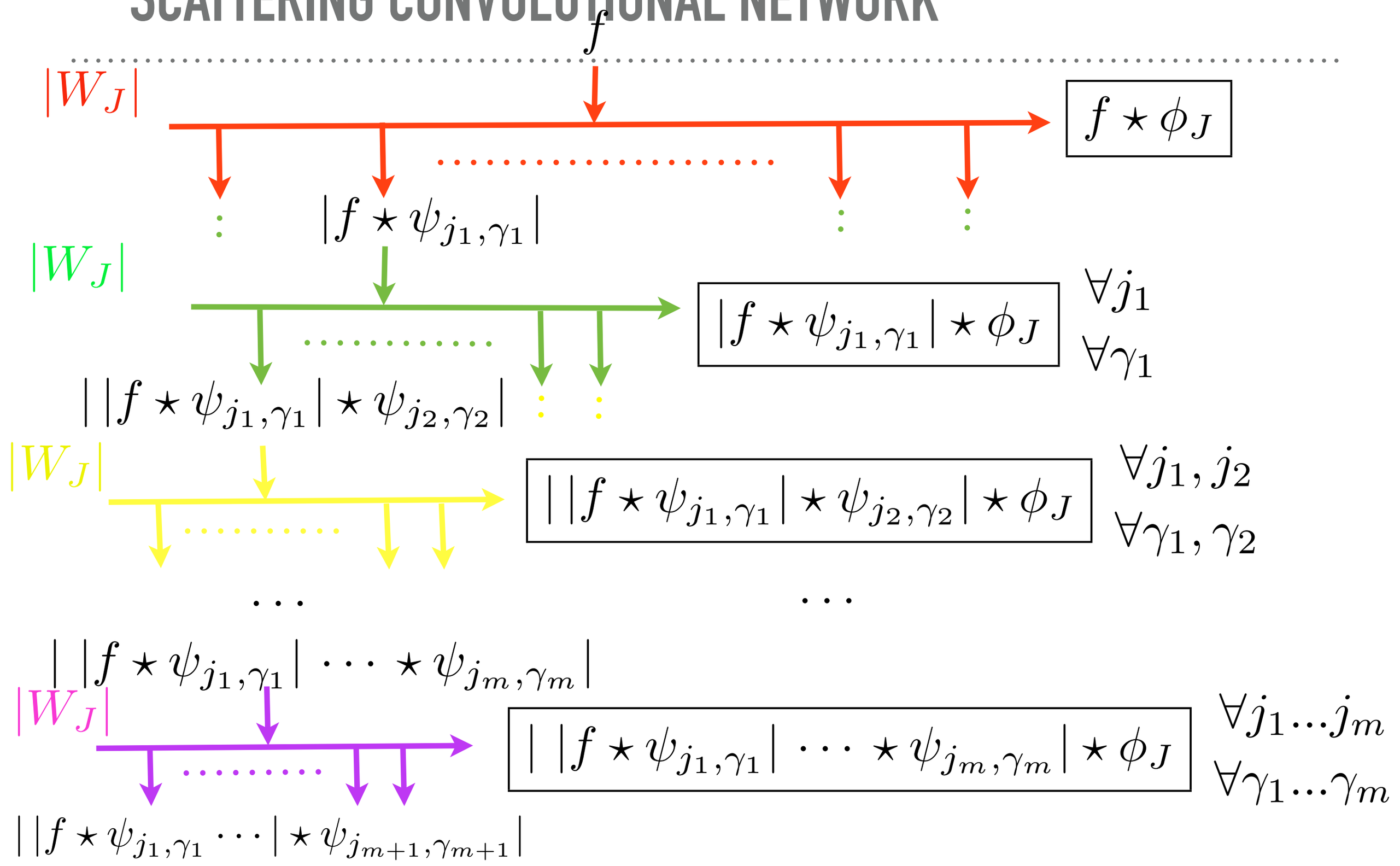
$$p = (\lambda_1, \ldots, \lambda_m) \; :$$

- Scattering coefficient along a path

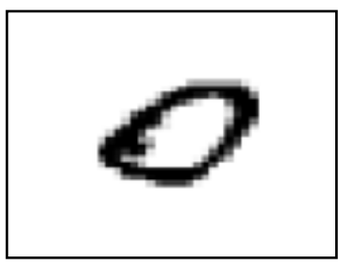
$$S_J[p]x(u) = |||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \ldots | \star \psi_{\lambda_m}| \star \phi_J(u) \, .$$

SCATTERING CONVOLUTIONAL NETWORK



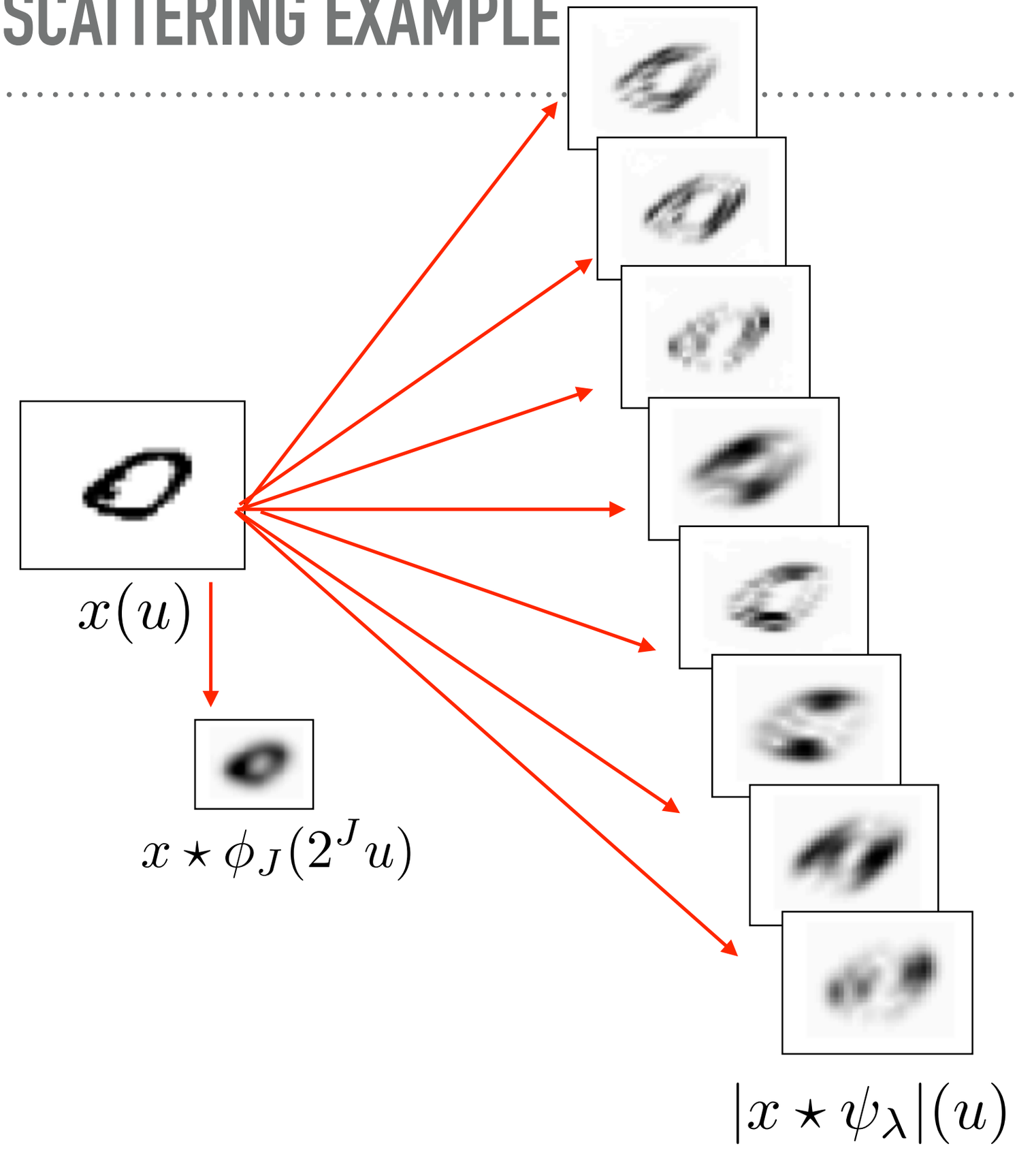
Cascade of contractive operators.

SCATTERING EXAMPLE



$x(u)$

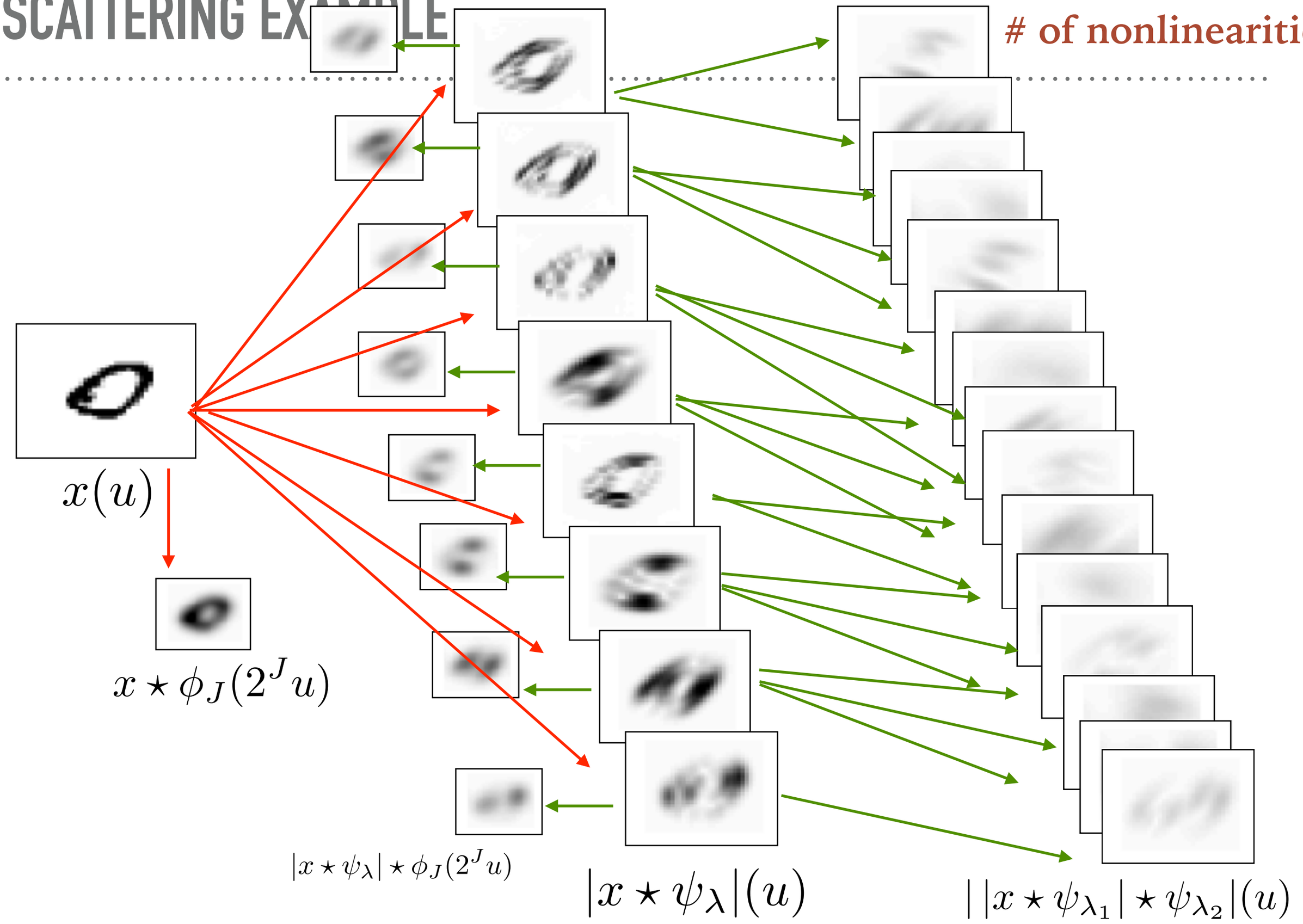
SCATTERING EXAMPLE



SCATTERING EXAMPLE

#layers:

# of nonlinearities



SCATTERING WITH MULTI-RESOLUTION WAVELETS

- We have considered a collection $\psi_{j,\theta}$ of oriented and dilated wavelets, and a translation co-variant wavelet decomposition operator:

$$Wx = \{x \star \phi_J, x \star \psi_{j,\theta}\}$$

- With J scales and L orientations, the redundancy is $(1+JL)$.

MULTI-RESOLUTION WAVELETS

- At each scale j, we consider a low-pass *scaling filter* h and band-pass filters g_θ, $\theta \in [1, \dots, L]$.

- Wavelets and the bluring kernel are obtained at each j by cascading these filters:

$$\phi_j = \phi_{j-1} \star h_j \quad \psi_{j,\theta} = \phi_{j-1} \star g_{j,\theta} \ .$$

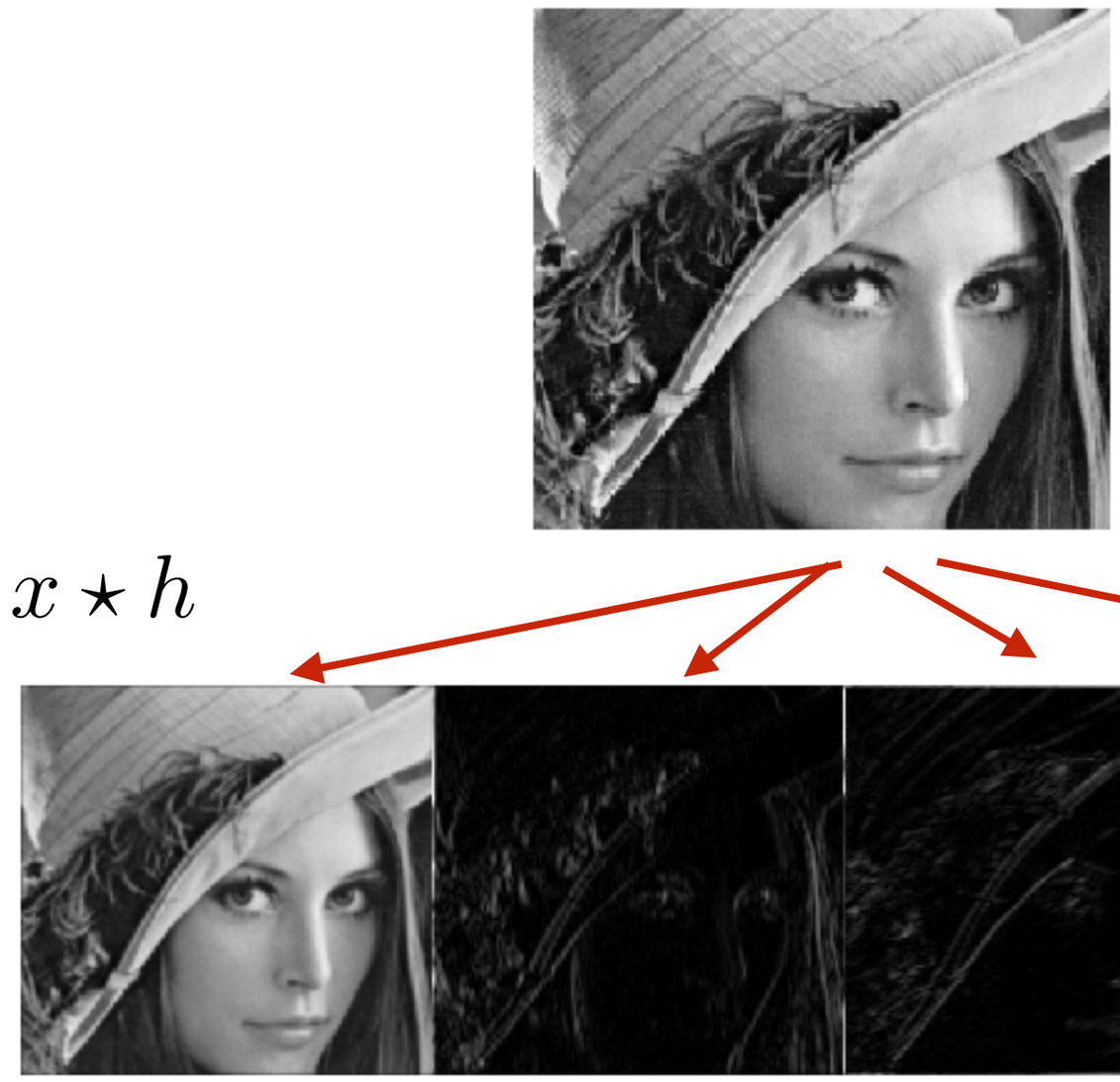
- Decompositions are obtained by cascading fine-to-coarse:

$$x \star \phi_j(u) = (x \star \phi_{j-1}) \star h_j(u) \quad , \quad x \star \psi_{j,\theta}(u) = (x \star \phi_{j-1}) \star g_{j,\theta}(u) \ .$$

- Downsampling (or "*stride*") adaptive to signal smoothness:

$$x \star \phi_j(u) = (x \star \phi_{j-1}) \star h(2u) \quad , \quad x \star \psi_{j,\theta}(u) = (x \star \phi_{j-1}) \star g_\theta(2u) \ .$$

SCATTERING WITH MULTI-RESOLUTION WAVELETS



SCATTERING WITH MULTI-RESOLUTION WAVELETS

x

#layers:

maximum scale

$x \star h$

$|x \star g_\theta|$

SCATTERING CONSERVATION OF ENERGY

➤Additive stability and conservation of energy:

Theorem *(Mallat)*: For appropriate wavelets, the scattering representation is contractive, $\|S_J x - S_J x'\| \leq \|x - x'\|$,
and unitary, $\|S_J x\| = \|x\|$.

$$\|S_J x\|^2 = \sum_{p \in \mathcal{P}_J} \|S_J[p]x\|^2$$

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- In practice, the transform is limited to a finite number of layers m_{max} This result shows residual error converges to 0.

INTERPRETATION

- Unitary Wavelet decomposition preserves energy:

$$\|x\|^2 = \|x \star \phi_J\|^2 + \sum_{j \leq J, \theta} \|x \star \psi_{j,\theta}\|^2 \ .$$

- Repeat formula on each output $|x \star \psi_{j,\theta}|$:

$$\||x \star \psi_{j,\theta}|\|^2 = \||x \star \psi_{j,\theta}| \star \phi_J\|^2 + \sum_{j_2 \leq J, \theta_2} \||x \star \psi_{j,\theta}| \star \psi_{j_2,\theta_2}\|^2 \ .$$

$$\|x\|^2 = \|S_J[0]x\|^2 + \sum_{|p|=1} \|S_J[p]x\|^2 + \sum_{|p|=2} \||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}\|^2$$

$\forall\ m$

$$\|x\|^2 = \sum_{|p|<m} \|S_J[p]x\|^2 + \sum_{|p|=m} \|||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \dots \psi_{\lambda_m}\|^2$$

INTERPRETATION

- Result amounts to proving that

$$\lim_{m\to\infty} \sum_{|p|=m, j_i \leq J} \| ||x \star \psi_{\lambda_1}| \star \ldots | \star \psi_{\lambda_m}|| \|^2 = 0 \ .$$

- *Fact*: Every time we apply the (complex) wavelet modulus, we push energy towards the low frequencies.
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- Fact: Every time we apply the (complex) wavelet modulus, we push energy towards the low frequencies.
- Result is obtained by formally showing this fact.
- Extended to general wavelets in [Waldspurger,'17] and to other frames in [Czaja, Li.16].
  - Decay is in fact exponential for band-limited signals.

SCATTERING GEOMETRIC STABILITY

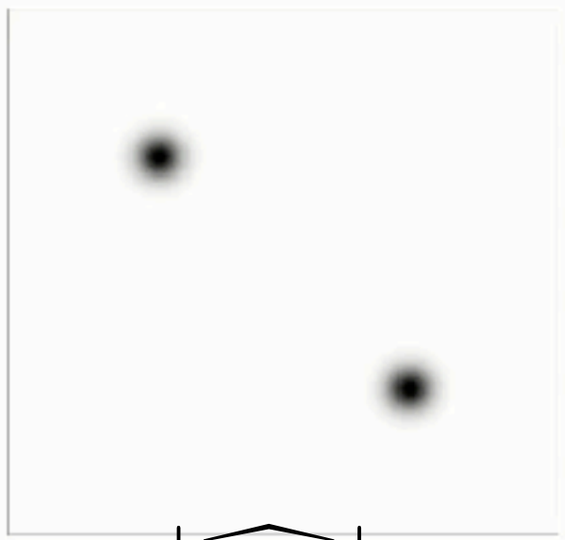
➤Geometric Stability:

$$\|S_J x\|^2 = \sum_{p \in \mathcal{P}_J} \|S_J[p]x\|^2$$

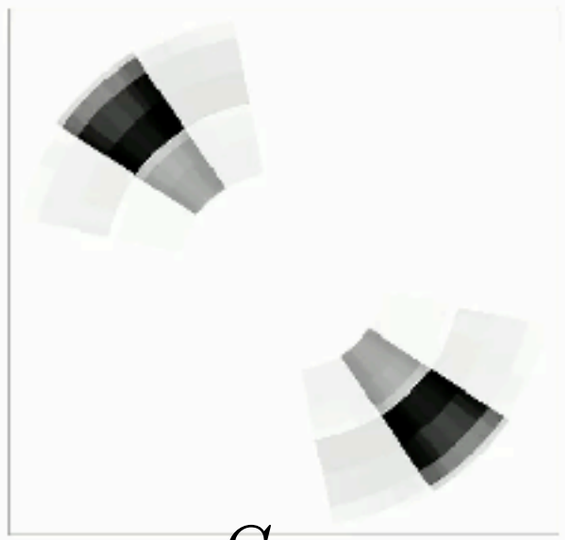
Theorem (Mallat'10): There exists C such that for all $x \in L^2(R^d)$ and all m, the m-th order scattering satisfies

$$\|S_J \varphi_\tau x - S_J x\| \leq Cm\|x\|(2^{-J}|\tau|_\infty + \|\nabla\tau\|_\infty + \|H\tau\|_\infty) \ .$$

$\varphi_\tau x$



$|\widehat{\varphi_\tau x}|$



$S_J \varphi_\tau x$

INTERPRETATION

- Denote

$$A_J x = x \star \phi_J \qquad W_J x = \{x \star \psi_\lambda\}_\lambda \qquad Mx = |x|$$

- We know that

$$\|A_J - A_J \varphi_\tau\| \leq C(2^{-J}|\tau|_\infty + |\nabla \tau|_\infty)$$
$$\|W_J \varphi_\tau - \varphi_\tau W_J\| \leq C(J|\nabla \tau|_\infty + |H\tau|_\infty)$$
$$M\varphi_\tau = \varphi_\tau M$$

$$([A, B] = AB - BA \ : \ \text{Commutator})$$

- $S_J = \{A_J, A_J M W_J, A_J M W_J M W_J, \dots\}$

INTERPRETATION

- Each order contributes separately:

$$\|S_J - S_J\varphi_\tau\|^2 = \|A_J - A_J\varphi_\tau\|^2 + \|A_J MW_J - A_J MW_J\varphi_\tau\|^2 + \dots$$

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➤ Let us inspect a generic term:

$$\|A_J \underbrace{MW_J MW_J \dots MW_J}_{k \text{ times}} - A_J \underbrace{MW_J MW_J \dots MW_J}_{k \text{ times}} \varphi_\tau\|$$

$(U_J = MW_J)$

$$\|A_J U_J^k - A_J U_J^k \varphi_\tau\| \leq \|A_J U_J^k - A_J U_J^{k-1} \varphi_\tau U_J\| + \|A_J U_J^{k-1} \varphi_\tau U_J - A_J U_J^k \varphi_\tau\|$$

INTERPRETATION

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$$\|A_J U_J^k - A_J U_J^k \varphi_\tau\| \leq \|A_J U_J^k - A_J U_J^{k-1}\varphi_\tau U_J\| + \|A_J U_J^{k-1}\varphi_\tau U_J - A_J U_J^k \varphi_\tau\|$$

$$\leq \|A_J U_J^{k-1} - A_J U_J^{k-1}\varphi_\tau\| + \|A_J U_J^{k-1}[\varphi_\tau, U_J]\|$$

$$\leq \|A_J U_J^{k-1} - A_J U_J^{k-1}\varphi_\tau\| + \|[\varphi_\tau, U_J]\|$$

$$\leq k\|[\varphi_\tau, U_J]\| + \|A_J - A_J\varphi_\tau\| \leq k\|[\varphi_\tau, W_J]\| + \|A_J - A_J\varphi_\tau\|$$

DISCRIMINABILITY AND SPARSITY

- Typical non-linearities are contractive:

$$\|\rho(x) - \rho(x')\| \leq \|x - x'\|$$

DISCRIMINABILITY AND SPARSITY

- Typical non-linearities are contractive:

$$\|\rho(x) - \rho(x')\| \leq \|x - x'\|$$

- However, if x, x' are sparse, this inequality is an equality in most of the signal domain.
- Thus sparsity is a means to control and prevent excessive contraction of different signal classes.

SPARSE SIGNAL RECOVERY

Theorem [B,M'14]: Suppose $x_0(t) = \sum_n a_n \delta(t - b_n)$ with $|b_n - b_{n+1}| \geq \Delta$, and $S_J x_0 = S_J x$ with $m = 1$ and $J = \infty$. If ψ has compact support, then

$$x(t) = \sum_n c_n \delta(t - e_n) \ , \ \text{with } |e_n - e_{n+1}| \gtrsim \Delta \ .$$

- Sx essentially identifies sparse measures, up to log spacing factors.

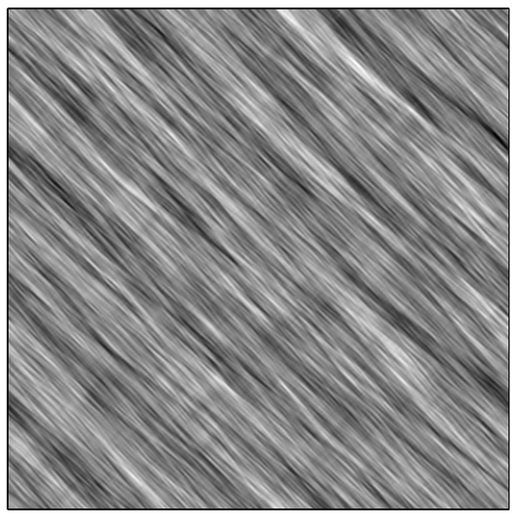
- Here, sparsity is encoded in the measurements themselves, not enforced as in standard CS.

- In 2D, singular measures (ie curves) require m=2 to be well characterized.

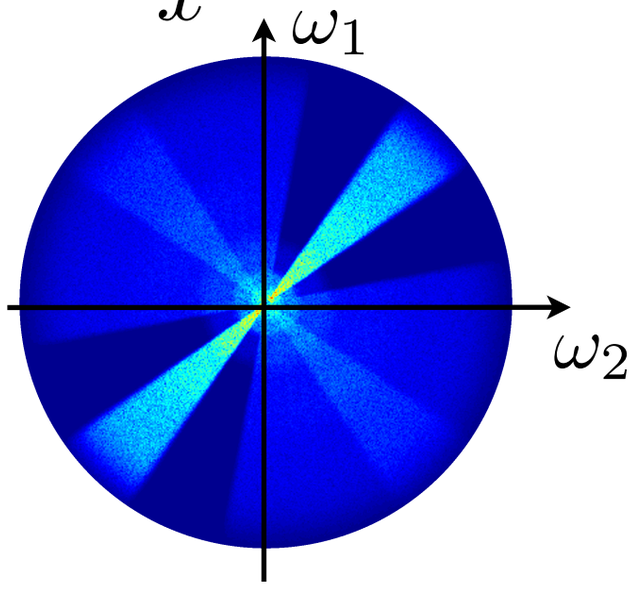
IMAGE EXAMPLES

Images ..... Fourier ..... Wavelet Scattering .....

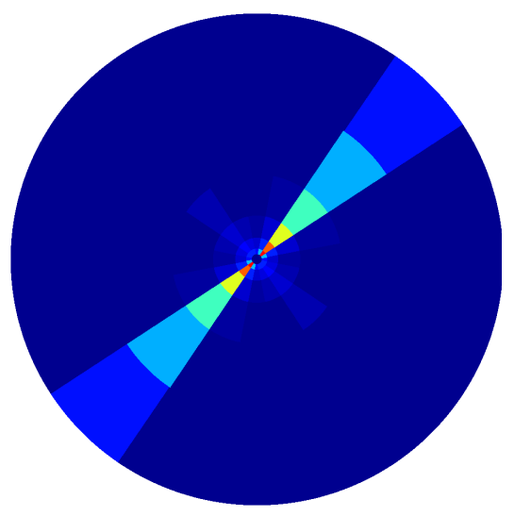
x



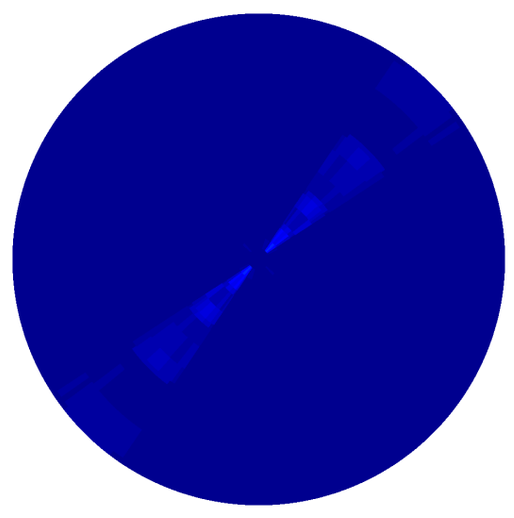
$\hat{x}$

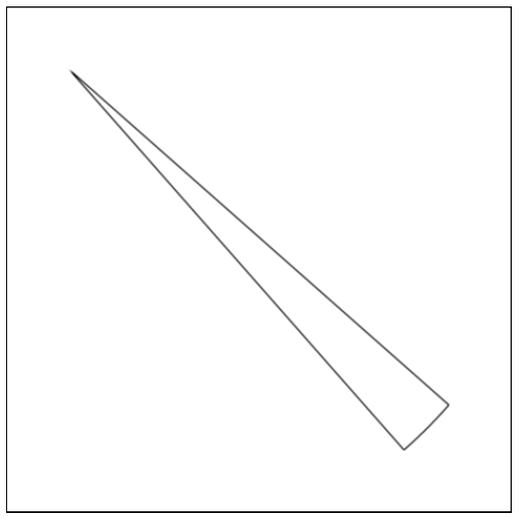


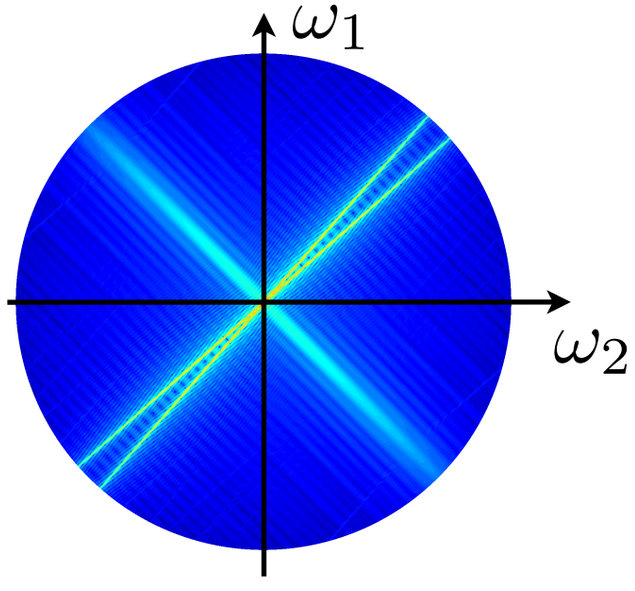
$|x \star \psi_{\lambda_1}| \star \phi_J$

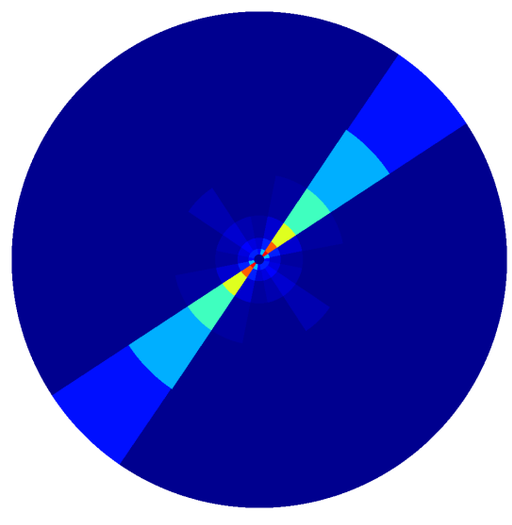


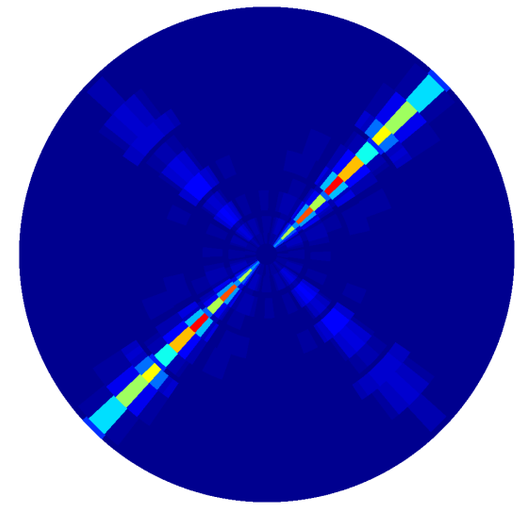
$||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi_J$







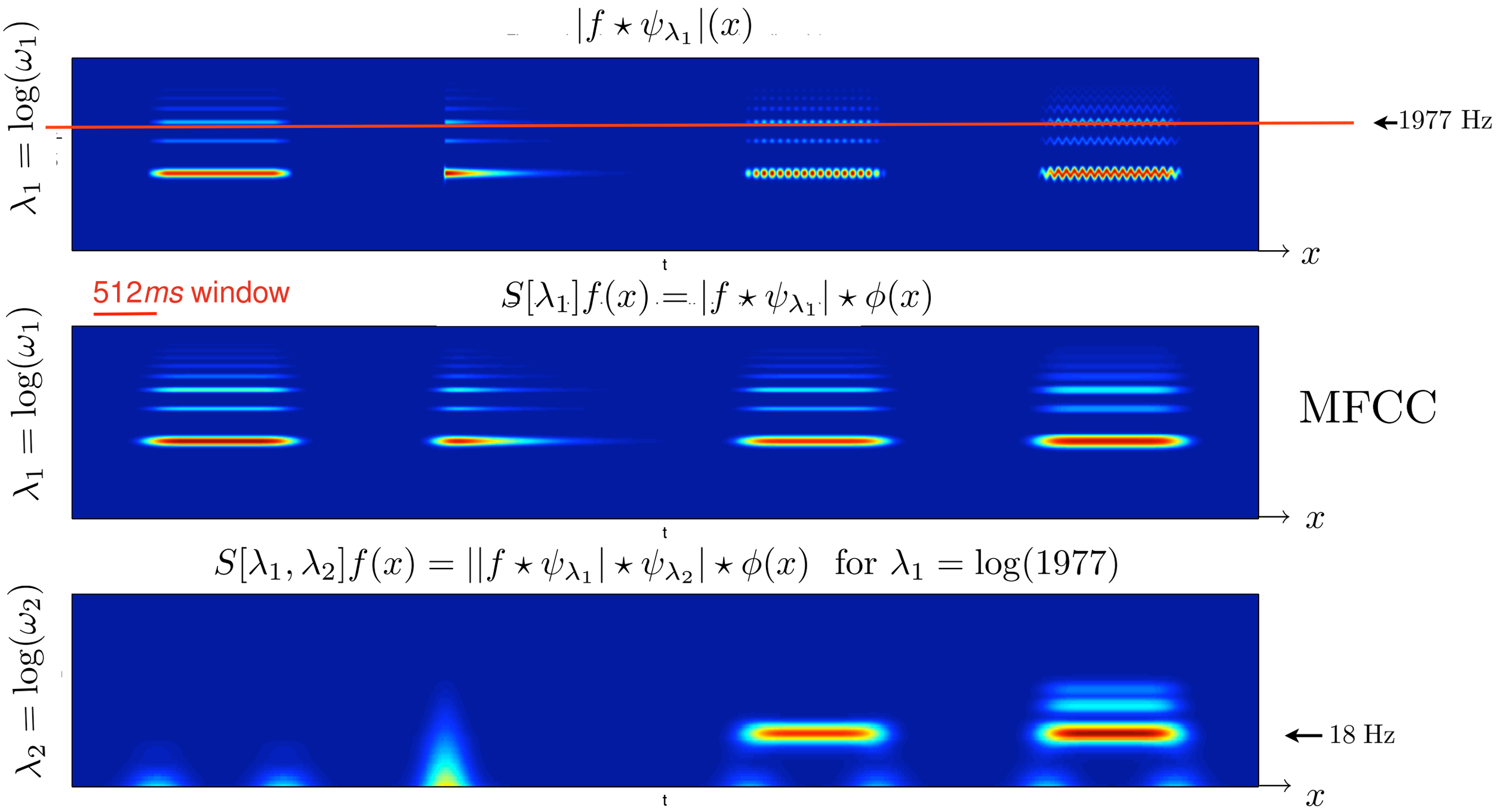




window size = image size

SOUND EXAMPLES

(courtesy J. Anden)



LIMITATIONS OF SEPARABLE SCATTERING

- No feature dimensionality reduction
  - The number of features increases exponentially with depth and polynomially with scale.

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- We are indirectly assuming that each wavelet band is deformed independently
  - We cannot capture the *joint* deformation structure of feature maps
  - Loss of discriminability.
- The deformation model is rigid and non-adaptive
  - We cannot adapt to each class
  - Wavelets are hard to define *a priori* on high-dimensional domains.

JOINT VERSUS SEPARABLE INVARIANCE

- Suppose we simply want stable translation invariance.
- Two-dimensional translation group in a periodic domain:

$$G \cong (\mathbb{R}/([0, N]))^2 = S^1 \times S^1 \cong \mathbb{T}^2$$

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$$G \cong (\mathbb{R}/([0, N]))^2 = S^1 \times S^1 \cong \mathbb{T}^2$$

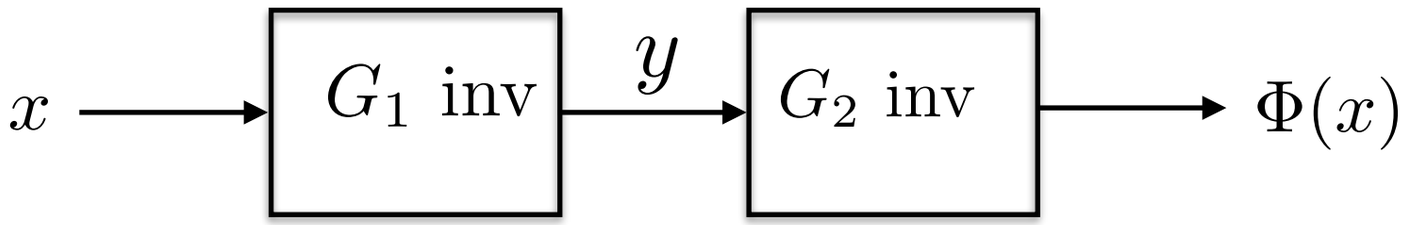
- Each S^1 acts on images along a different coordinate:

$$\varphi_a^1 x(u_1, u_2) = x(u_1 - a, u_2) \ , \ \varphi_a^2 x(u_1, u_2) = x(u_1, u_2 - a)$$

JOINT VERSUS SEPARABLE INVARIANCE

- So we could just consider one-dimensional (stable) translation invariant representations and compose:

$$G = G_1 \times G_2$$



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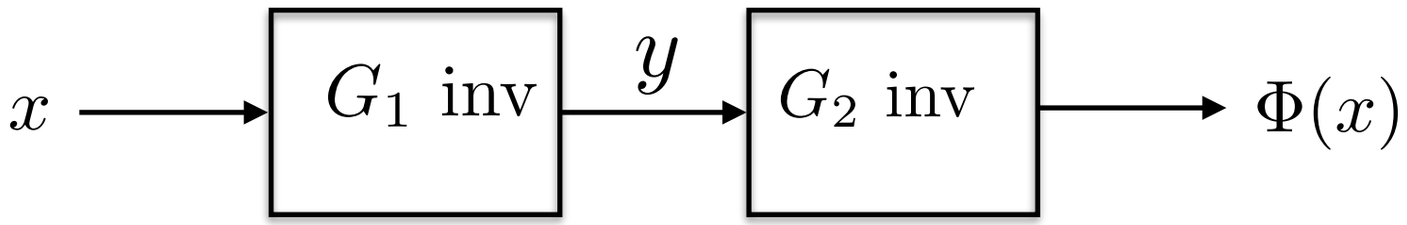

- If for each u_2, $x(\cdot, u_2) \mapsto \Phi_1(x)(\cdot, u_2)$ is G_1 invariant then $\Phi_1(\varphi^1 x) = \Phi_1(x)$ for all x and $\varphi^1 \in G_1$

- If for each λ, $y(\lambda, \cdot) \mapsto \Phi_2(y)(\lambda, \cdot)$ is G_2 invariant then $\Phi_2(\varphi^2 y) = \Phi_2(y)$ for all y and $\varphi^2 \in G_2$

JOINT VERSUS SEPARABLE INVARIANCE

- Thus, if Φ_1 is G_1 invariant and G_2 covariant, and Φ_2 is G_2 invariant, then $\Phi = \Phi_2 \circ \Phi_1$ satisfies

$$\forall\ \varphi \in G,\ \varphi = \varphi^1\varphi^2,\ \varphi^i \in G_i$$

$$\Phi(\varphi x) = \Phi_2\Phi_1(\varphi^1\varphi^2 x) = \Phi_2\Phi_1(\varphi^2 x) = \Phi_2\varphi^2\Phi_1(x) = \Phi_2\Phi_1(x) = \Phi(x)$$

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- So we achieve further invariance by composing partial invariances.
- Is there a problem here?

JOINT VERSUS SEPARABLE INVARIANCE

- The factorization does not capture the joint action of G_1 along the domain (u_1, u_2).
- We are invariant to *too many* things.

WAVELET COVARIANTS

- If we replace input image by first layer output:

$$\rho(x_0 \star \psi_{j,\theta})(u) = x_1(u, j, \theta)$$

Let $\tilde{x}_0 = R_\alpha x_0$ be a rotation of α degrees.

$$\rho(\tilde{x}_0 \star \psi_{j,\theta})(u) = x_1(R_\alpha u, j, \theta + \alpha)$$

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- Similarly, roto-translation acts on x_1 by rotating and translating spatial coordinates and translating orientation coordinates

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- So we can replace convolutions over translation by convolutions over roto-translations.

GROUP CONVOLUTIONS

➤

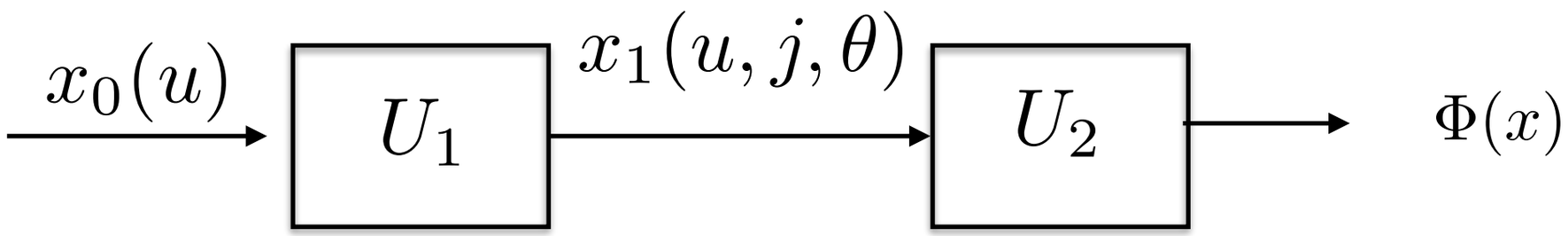
Definition: Let G be a group equipped with a Haar measure $d\mu$, acting on Ω, and $h \in L^1(G)$. The group convolution $x \star_G h$ is defined as

$$x \star_G h(u) = \int_G h(g) x(\varphi_g u) d\mu(g) \quad , \; x \in L^2(\Omega) \; .$$

If $x = x_1(u, j, \theta)$ and G are roto-translations,
these convolutions recombine different orientation channels.

JOINT SCATTERING

- We start by *lifting* the image with spatial wavelet convolutions: stable and covariant to roto-translations.

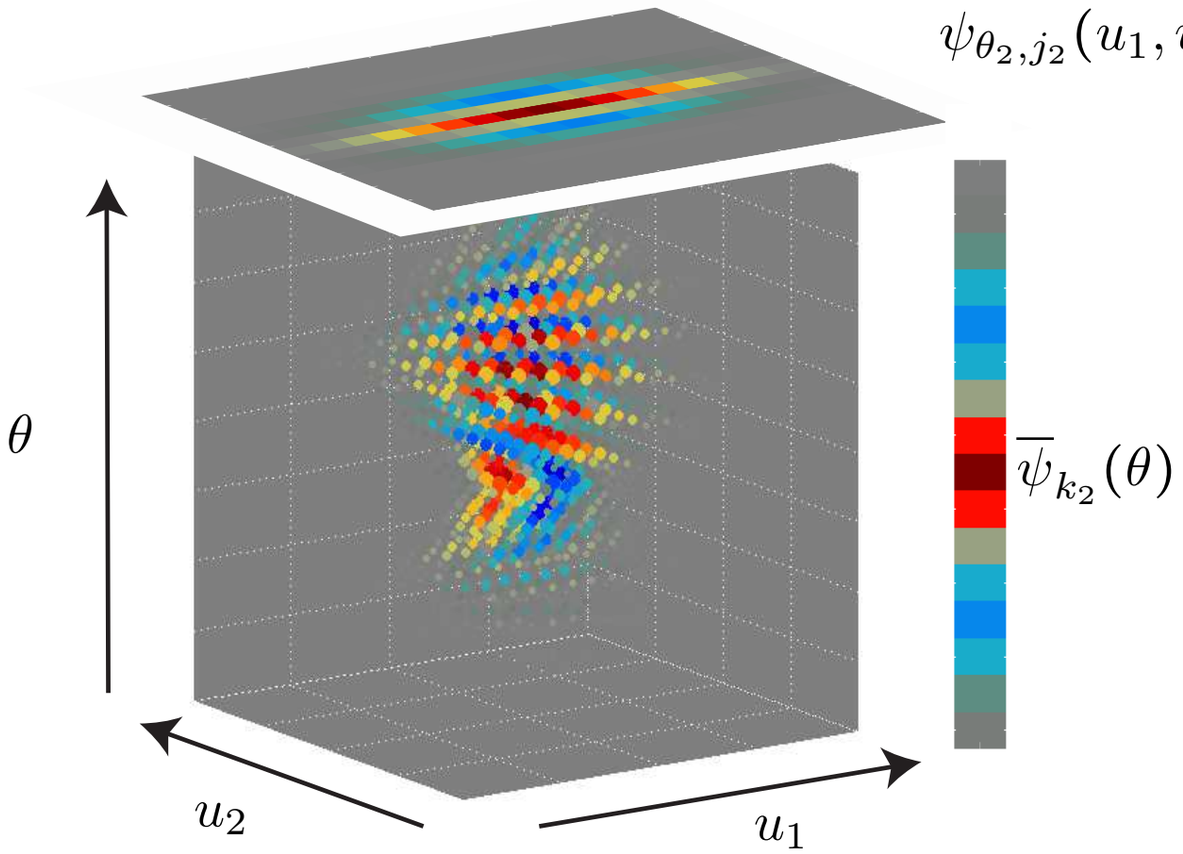


- We then adapt the second wavelet operator to its joint variability structure.
- More discriminability.
- Requires defining wavelets on more complicated domains:

$$\Omega = \mathbb{R}^2 \times S^2.$$

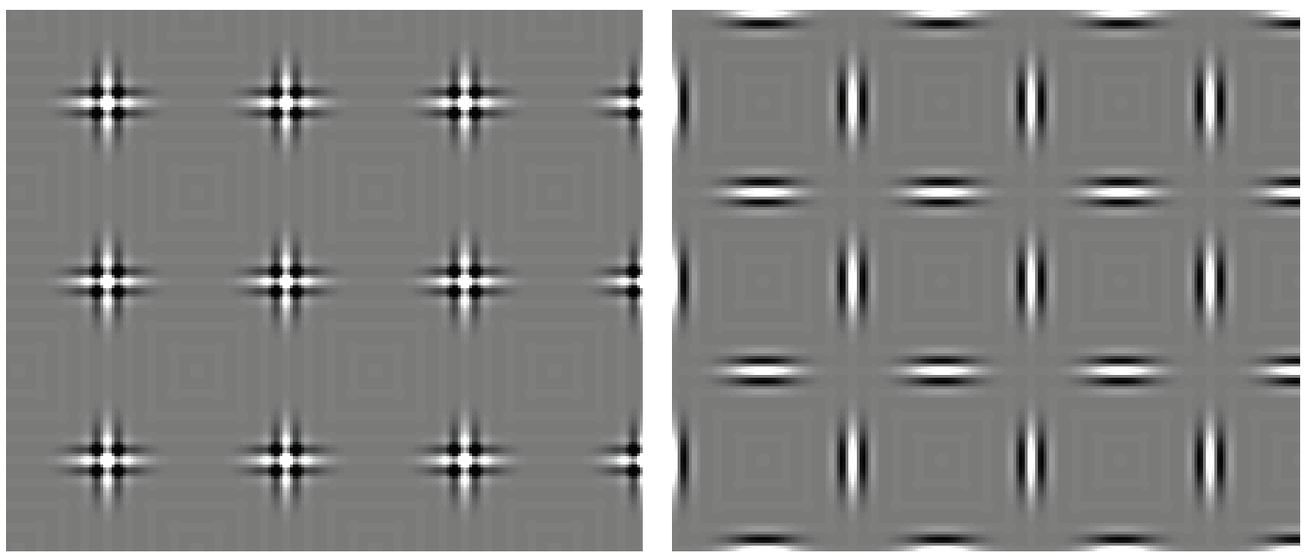
EXAMPLE: ROTO-TRANSLATION SCATTERING

- [Sifre and Mallat'13]



second layer wavelets constructed by a separable product on spatial and rotational wavelets:

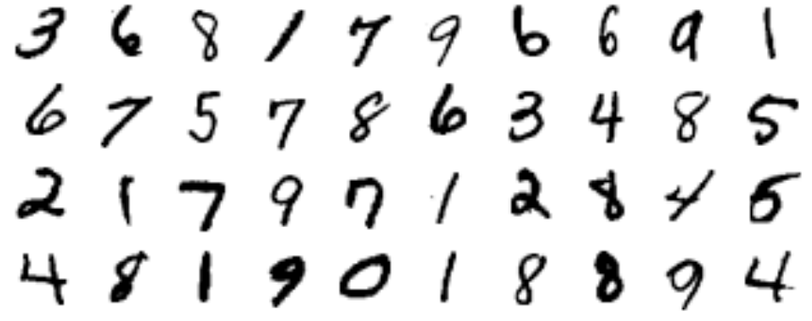
$$\Psi_\lambda(u,\theta) = \psi_{\lambda_1}(u)\psi_{\lambda_2}(\theta)$$



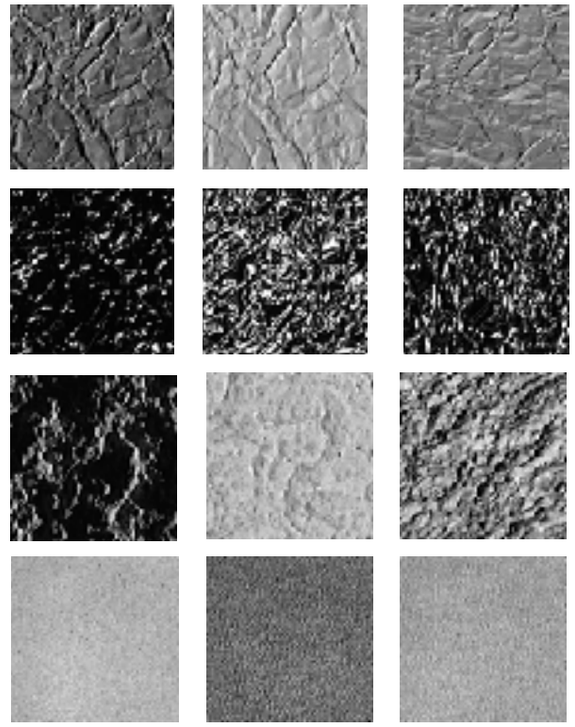
example of patterns that are discriminated by joint scattering but not with separable scattering.

CLASSIFICATION WITH SCATTERING

- State-of-the art on pattern and texture recognition using separable scattering followed by SVM:
  - MNIST, USPS [Pami'13]



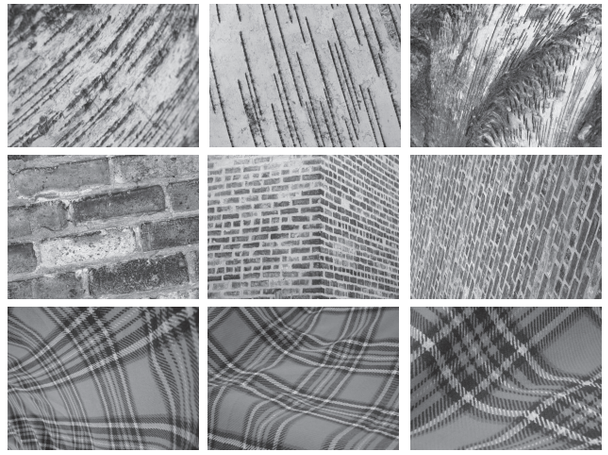
  - Texture (CUREt) [Pami'13]



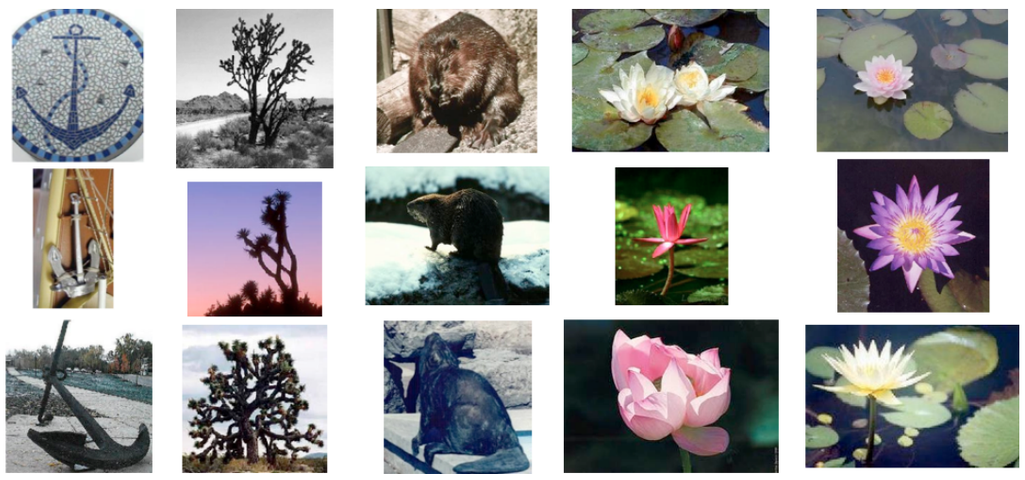
  - Music Genre Classification (GTZAN) [IEEE Acoustic '13]

CLASSIFICATION WITH SCATTERING

- Joint Scattering Improves Performance:
  - More complicated Texture (KTH,UIUC,UMD) [Sifre&Mallat, CVPR'13]



- Small-mid scale Object Recognition (Caltech, CIFAR) [Oyallon&Mallat, CVPR'15]



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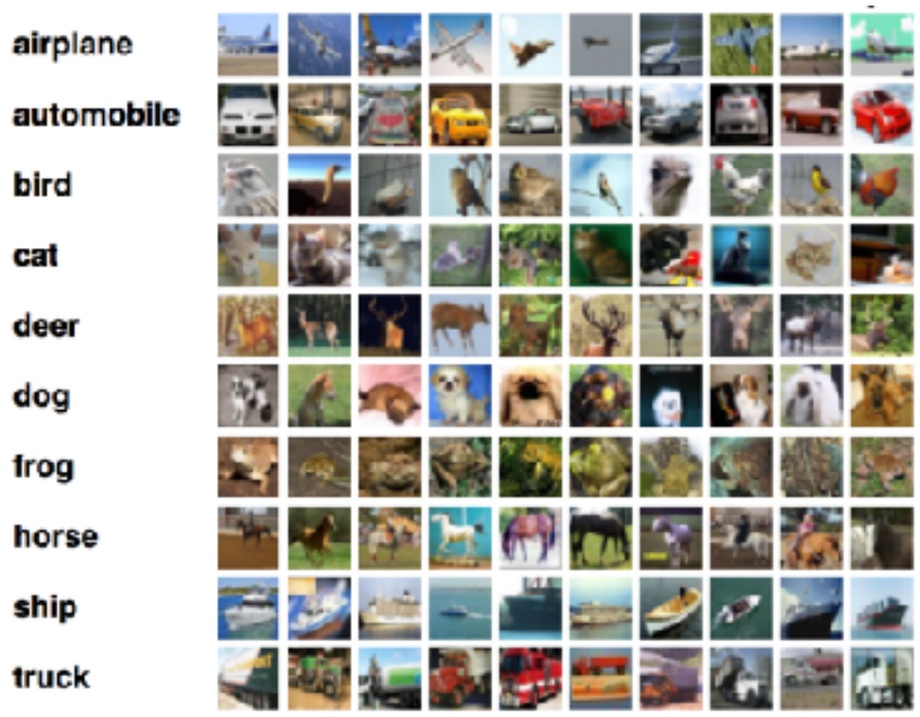


LIMITATIONS OF JOINT SCATTERING

- Variability from physical world expressed in the language of transformation groups and deformations
  - However, there are not many possible groups: essentially the affine group and its subgroups.
- As a new wavelet layer is introduced, we create new coordinates, but we do not destroy existing coordinates
  - Hard to scale: dimensionality reduction is needed.
  - Wavelet design complicated beyond roto-translation groups.
- Beyond physics, many deformations are class-specific and not small.
  - Learning filters from data rather than designing them.

FROM SCATTERING TO CNNS

- Given $x(u, \lambda)$ and a group G acting on both u and λ, we defined wavelet convolutions over G as

$$x \star_G \psi_{\lambda'}(u, \lambda) = \int_v \int_\alpha \psi_\lambda(R_{-\alpha}(u - v))x(v, \alpha)dvd\alpha$$

- In discrete coordinates,

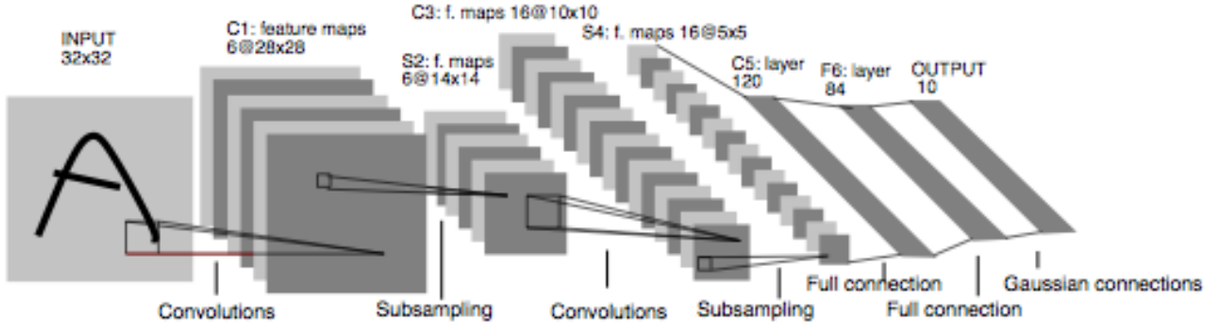
$$x \star_G \psi_{\lambda'}(u, \lambda) = \sum_v \sum_\alpha \overline{\psi}_{\lambda'}(u - v, \alpha, \lambda)x(v, \alpha)$$

- Which in general is a convolutional tensor.

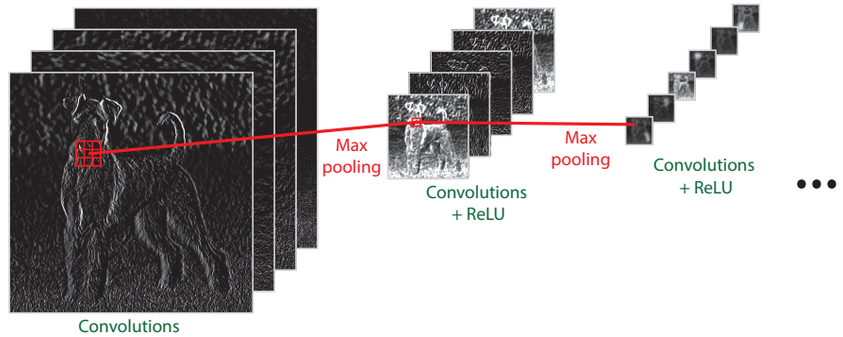
CONVOLUTIONAL NEURAL NETWORKS

[LeCun, 80s,90s]

- Stack multiple layers of **localized** convolutional operators and point-wise contractive non-linearities:




Input image



Input: $x \in L^2(\Omega, \mathbb{R}^p)$. Output: $\tilde{x} \in L^2(\Omega, \mathbb{R}^{\tilde{p}})$.

$$\tilde{x}_{\tilde{j}}(u) = \rho\left(\sum_{j=1}^{p} x_j \star \theta_{j,\tilde{j}}(u)\right) \ , \ \tilde{j} \leq \tilde{p} \ .$$

$\rho(z)$: point-wise nonlinearity (e.g. $\max(0, z)$).

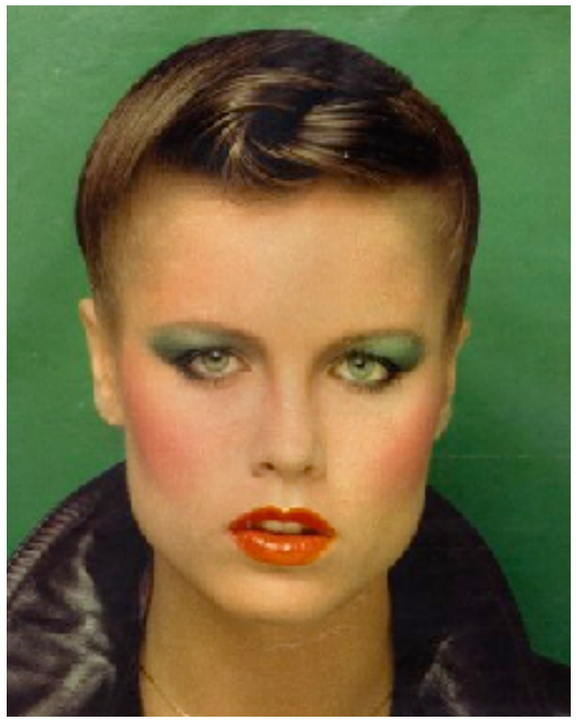
$\Theta = (\theta_{j,\tilde{j}})$: localized convolutional kernel.

- Down-sampling via *pooling* (can be either linear with average, or nonlinear with max) in invariant tasks:

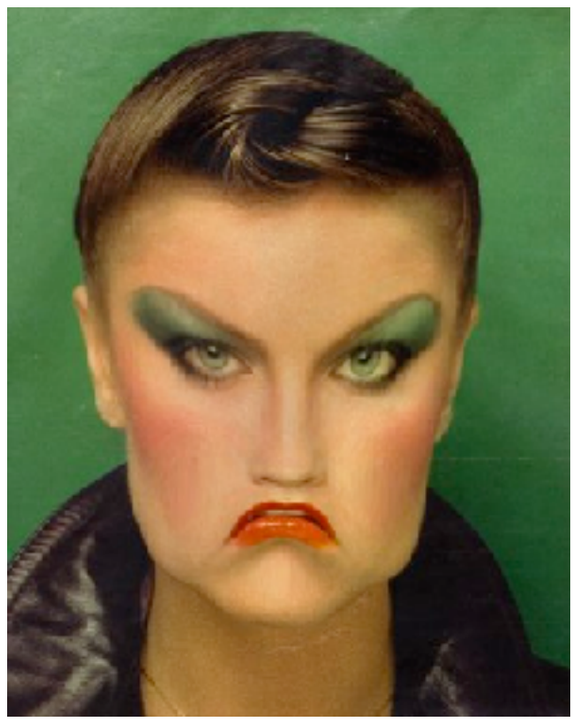
$$\bar{x}_{\tilde{j}}(\bar{u}) = \|\tilde{x}_{\tilde{j}}(\mathcal{N}(u))\| \qquad \mathcal{N}(u)\text{: Neighborhood of } u.$$

CONVOLUTIONAL NEURAL NETWORKS

- Why are CNNs geometrically stable?



$x(u)$



$x_\tau(u)$

$x_{\tau'}(u)$

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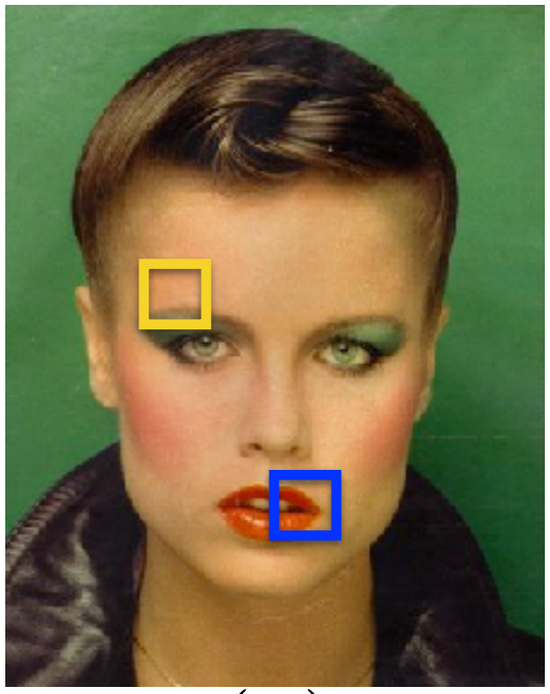
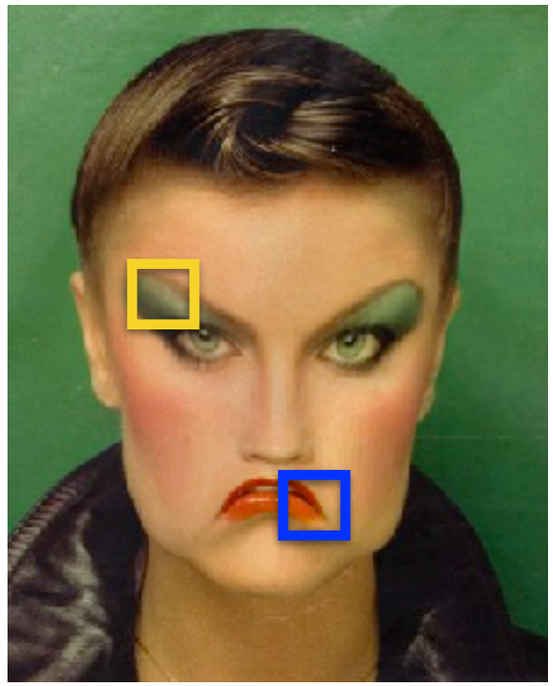
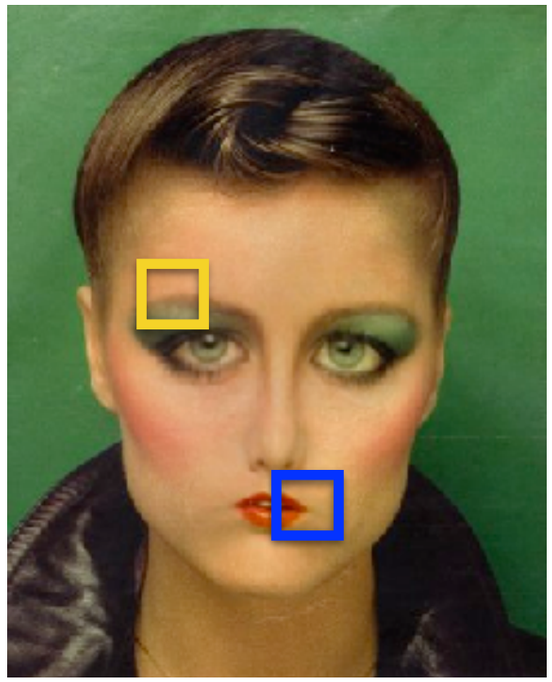


$x(u)$



$x_\tau(u)$



$x_{\tau'}(u)$

- A non-rigid deformation locally looks like a translation if $\|\nabla\tau\|$ small:

$$\Rightarrow x_\tau \star \theta(u) \approx [x \star \theta]_\tau(u)$$

- A point-wise nonlinearity commutes with deformations:

$$\Rightarrow \rho\left(x_\tau \star \theta(u)\right) \approx \rho\left(\left[x \star \theta\right]_\tau(u)\right) = \left[\rho\left(x \star \theta\right)\right]_\tau(u)$$

- Pooling progressively creates invariance to geometric deformations:

$$\|x_\tau(\mathcal{N}(u))\| \approx \|x(\mathcal{N}(u))\| \text{ if } |\tau|\text{small}$$

CONVOLUTIONAL NEURAL NETWORKS

- **Convolutions** to exploit translation invariance/equivariance.
- **Localized** to exploit geometric stability: leads to multi scale architecture.
- These two properties lead to models with $O(\log N)$ trainable parameters.

- Stability is only part of the story. Discriminability via learning/ optimization is another major component for success.