DEPARTMENT OF CYBERNETICS AND ROBOTICS
ELECTRONICS FACULTY
WROCŁAW UNIVERSITY OF SCIENCE AND TECHNOLOGY

Lecture Notes in Automation and Robotics

Krzysztof Tchoń
Robert Muszyński

Mathematical Methods of Automation and Robotics

Wrocław 2017

Krzysztof Tchoń   Robert Muszyński

Mathematical Methods of Automation and Robotics

Lecture Notes
in Automation and Robotics

Compilation: July 3, 2020

Wrocław 2017

Krzysztof Tchoń, Robert Muszyński
Wrocław 2017

Authors
Krzysztof Tchoń
Robert Muszyński
Department of Cybernetics and Robotics,
Electronics Faculty,
Wrocław University of Science and Technology,
Poland

Computer typesetting
Robert Muszyński
Krzysztof Tchoń

Contents

Nomenclature viii

0 Prelude 1
- 0.1 Basic concepts . . . 1
  - 0.1.1 Set theory . . . 1
  - 0.1.2 Algebra . . . 2
  - 0.1.3 Topology . . . 3
  - 0.1.4 Calculus . . . 4
- 0.2 Linear control systems . . . 5
  - 0.2.1 Controllability . . . 5
  - 0.2.2 Equivalence . . . 7
  - 0.2.3 Classification and normal forms . . . 7
- 0.3 Brunovsky Theorem . . . 10
- 0.4 Basic ideas of this course . . . 13
- 0.5 Proofs . . . 14
  - 0.5.1 Pole Placement Theorem . . . 14
- 0.6 Problems and exercises . . . 16
- 0.7 Bibliographical remarks . . . 17
- Bibliography . . . 17

1 Functions 19
- 1.1 Classes of functions . . . 19
- 1.2 Algebraic structures in the set of functions . . . 21
- 1.3 Inverse Function Theorem . . . 22
- 1.4 Implicit Function Theorem . . . 23
- 1.5 Computation of the inverse function . . . 24
  - 1.5.1 Newton Algorithm . . . 24
  - 1.5.2 Steepest Descent Algorithm . . . 25
- 1.6 Proofs . . . 27
  - 1.6.1 Implicit Function Theorem . . . 27

1.7 Problems and exercises 27
1.8 Bibliographical remarks 29
Bibliography 29

2 Linear functions. Equivalence of functions 30
2.1 Linear functions 30
2.2 Matrices and their norms 31
2.3 LR-equivalence 33
2.4 Submersions and immersions 34
2.5 Proofs 35
2.5.1 Theorem on Submersions 35
2.5.2 Theorem on Immersions 36
2.6 Problems and exercises 36
2.7 Bibliographical remarks 37
Bibliography 37

3 Morse functions. The Fixed Point Theorem 38
3.1 Critical points and values 38
3.2 Morse functions, Morse Theorem 39
3.3 Hadamard's Lemma 41
3.4 Classification of function: Summary 42
3.5 The Fixed Point Theorem 42
3.6 Proofs 43
3.6.1 Hadamard's Lemma 43
3.6.2 Fixed Point Theorem 43
3.7 Problems and exercises 44
3.8 Bibliographical remarks 45
Bibliography 45

4 Time dependent dynamic systems 46
4.1 Differential equations. Theorem on Existence and Uniqueness of Solution 46
4.2 Bellman-Gronwall Lemma, dependence on initial conditions 47
4.3 Time dependent linear systems 48
4.4 Peano-Baker Formula 49
4.5 Ważewski Inequality 50
4.6 Proofs 50
4.6.1 Theorem on Existence and Uniqueness 50
4.6.2 Peano-Baker Formula 52
4.6.3 Ważewski Inequality 52

4.7 Problems and exercises . . . 53
4.8 Bibliographical remarks . . . 54
Bibliography . . . 54

5 Stability 56
5.1 Stability, uniform stability, asymptotic stability . . . 56
5.2 Class K and K_∞ functions . . . 58
5.3 Lyapunov Function Theorem . . . 59
5.4 Barbalat's Lemma . . . 60
5.5 Convergence estimation . . . 61
5.6 Problems and exercises . . . 61
5.7 Bibliographical remarks . . . 62
Bibliography . . . 63

6 Time-independent dynamic systems 64
6.1 System's flow . . . 64
6.2 Equivalence of dynamic systems . . . 65
6.3 Theorem on Differential Equivalence . . . 66
6.4 Straightening Out Theorem . . . 67
6.5 Equilibrium points . . . 68
6.6 Linearisation of dynamic systems . . . 69
6.7 Equivalence of linear systems . . . 69
6.8 Classification of dynamic systems: a summary . . . 70
6.9 Proofs . . . 71
6.9.1 Theorem on Differential Equivalence . . . 71
6.9.2 Straightening Out Theorem . . . 72
6.10 Problems and exercises . . . 73
6.11 Bibliographic remarks . . . 74
Bibliography . . . 74

7 Frobenius Theorem 75
7.1 Vector fields, big adjoint operator . . . 75
7.2 Lie bracket . . . 76
7.3 Lie bracket theorems . . . 77
7.4 Simultaneous Straightening Out Theorem . . . 78
7.5 Distribution and integral manifold . . . 78
7.6 Frobenius Theorem . . . 79
7.7 Proofs . . . 80
7.7.1 Theorem on commutation . . . 80
7.7.2 Simultaneous Straightening Out Theorem . . . 81

7.7.3 Frobenius Theorem . . . 82
7.8 Problems and exercises . . . 83
7.9 Bibliographical remarks . . . 84
Bibliography . . . 85

8 Control systems **86**
8.1 Control affine and driftless systems . . . 87
8.2 Differentiation of the end-point map . . . 89
8.3 Accessibility and controllability . . . 90
8.4 Controllability theorems . . . 92
8.5 Checking controllability . . . 93
8.6 Examples . . . 94
8.7 Problems and exercises . . . 99
8.8 Bibliographical remarks . . . 99
Bibliography . . . 100

9 Equivalence of control systems **101**
9.1 State space and feedback equivalence . . . 101
9.2 State space and feedback linearisation . . . 103
9.3 Equivalence equations . . . 104
9.4 Significance of linearisability for the synthesis of control algorithms . . . 105
9.5 Examples . . . 107
9.6 Problems and exercises . . . 112
9.7 Bibliographical remarks . . . 114
Bibliography . . . 114

10 Input-output decoupling and linearisation **116**
10.1 Differential degree . . . 116
10.2 Decoupling . . . 117
10.3 Dynamics of the decoupled system . . . 118
10.4 Examples . . . 121
10.5 Problems and exercises . . . 123
10.6 Bibliographical remarks . . . 125
Bibliography . . . 125

11 Chained form systems **126**
11.1 Chained form . . . 126
11.2 Murray's Theorem . . . 127
11.3 Integrator backstepping . . . 127

11.4 Examples 129
11.5 Bibliographical remarks 135
Bibliography 135

12 Dynamic feedback, linearisation **136**
12.1 Motivation 136
12.2 Dynamic feedback 139
12.3 Theorems on dynamic linearisation 140
12.3.1 Brunovsky compensator 141
12.4 Differential flatness 142
12.5 Examples 143
12.6 Bibliographical remarks 149
Bibliography 149

13 Limitations of feedback **151**
13.1 Linear systems 151
13.2 Brockett's Theorem 151
13.3 Theorem of Lizárraga 153
13.4 Bibliographical remarks 155
Bibliography 155

Index **156**

List of Figures **160**

List of Theorems **162**

The book is typeset with LaTeX, the document preparation system, originally written by L. Lamport [Lam94], which is an extension of TeX [Knu86a,Knu86b]. The typeface used for mathematics throughout this book, named AMS Euler, is a design by Hermann Zapf [KZ86], commissioned by the American Mathematical Society. The text is set in a typeface called Concrete Roman and Italic, a special version of Knuth's Computer Modern family with weights designed to blend with AMS Euler, prepared to typeset [GKP89].

[GKP89] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematicsa. Addison-Weslay, Reading, 1989.

[Knu86a] D. E. Knuth, The TeXbook, volume A of Computers and Typesetting. Addison-Wesley, Reading, 1986.

[Knu86b] D. E. Knuth, TeX: The Program, volume B of Computers and Typesetting. Addison-Wesley, Reading, 1986.

[KZ86] D. E. Knuth, and H. Zapf, AMS Euler — A new typeface for mathematics. Scholary Publishing, 20:131–157, 1986.

[Lam94] L. Lamport, LaTeX: A Document Preparation System. Addison-Wesley, Reading, 1994.

Nomenclature

Ad	big adjoint operator (75)
ad	small adjoint operator (75)
C^0	class of continuous functions (19)
$\mathcal{C}_f$	set of critical points (38)
C^k	class of differentiable functions (19)
C^∞	class of smooth functions (19)
C^ω	class of analytic functions (19)
e_i	unit vector (78)
f, g	function (1)
L	Lie derivative (77)
$[\,,\,]\mathbb{R}^n$	Lie bracket (75)
$\mathbb{R}^n$	Euclidean space (19)
S^1	unit circle (65)
X, Y, Z	vector field (75)
$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	set (1)

Chapter 0

Prelude

0.1 Basic concepts

At the beginning we shall explain a number of concepts from set theory, algebra, topology and mathematical analysis that will appear later on in these notes. It is assumed that the Reader has had a contact with the language of formal logic and set theoretical operations, and also got some basic knowledge of the calculus, algebra and ordinary differential equations included in the undergraduate teaching curricula at technical universities. We expect that if a certain notion has not been defined in these notes, the Reader is able to find it out in the literature.

0.1.1 Set theory

The concept of a set is treated as a primary concept. Suppose that $\mathcal{X}, \mathcal{Y}$ denote some universa (sets) with elements x, y, z. A subset $\mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$ will be called a binary relation. We say that x is in relation $\mathcal{R}$ with y, $x\mathcal{R}y$, if $(x, y) \in \mathcal{R}$.

Definition 0.1.1 *A relation $\mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$ will be named a function, if*

$$(x, y), (x, z) \in \mathcal{R} \Longrightarrow y = z.$$

The function is written down by the formula $f : \mathcal{X} \longrightarrow \mathcal{Y}$. The set $G_f = \{(x, y)|(x, y) \in f\}$ is referred to as the graph of the function.

Definition 0.1.2 *Given a universum $\mathcal{X}$, the function*

$$f : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$$

will be called a (binary) operation in $\mathcal{X}$.

Definition 0.1.3 *A relation $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ will be called an equivalence relation, if the following conditions hold:*

- *reflexivity:* $x\mathcal{R}x$,
- *symmetry:* $x\mathcal{R}y \Longrightarrow y\mathcal{R}x$,
- *transitivity:* $x\mathcal{R}y \wedge y\mathcal{R}z \Longrightarrow x\mathcal{R}z$.

Every equivalence relation partitions the universum into non-empty and disjoint equivalence classes defined as

$$[x] = \{y \in \mathcal{X} | y\mathcal{R}x\}.$$

This partition is exhaustive, i.e. $\bigcup_{x \in \mathcal{X}} [x] = \mathcal{X}$, so each element of the universum belongs to a certain equivalence class. A classification of elements of a universum consists in the introduction into it of an equivalence relation and the characterisation of every equivalence class by its specific element playing the role of a label. Such an element is called a normal form or, sometimes, a canonical form of elements from this class. Therefore, the objective of a classification is the determination of equivalence classes and ascribing to each of them a normal form. To make the classification effective it is desirable to get a finite number of the equivalence classes. On the other hand, the classification must not be trivial, for example assigning to all the universum's elements a single class. A leitmotif of these notes will be a classification of three universa: functions, dynamic systems, and control systems.

0.1.2 Algebra

Definition 0.1.4 *Let $\mathcal{X}$ denote a universum with a binary operation $\circ$. The system $(\mathcal{X}, \circ)$ is named a group, if there exists in $\mathcal{X}$ a neutral element e, such that $x \circ e = e \circ x = x$ and every element $x \in \mathcal{X}$ has the inverse element $x^{-1} \in \mathcal{X}$ for which $x^{-1} \circ x = x \circ x^{-1} = e$. If the group operation is commutative, $x \circ y = y \circ x$, the group is called commutative (Abelian). When the group operation is associative, $x \circ (y \circ z) = (x \circ y) \circ z$, the group is called associative.*

Definition 0.1.5 *If in the universum $\mathcal{X}$ there are two operations: one $\circ$, with respect to which $\mathcal{X}$ is a group and another $*$, such that they are distributive: $x * (y \circ z) = (x * y) \circ (x * z)$ and also $(y \circ z) * x = (y * x) \circ (z * x)$, then $\mathcal{X}$ will be named a ring. If there exists in the ring an element 1, such that $1 * x = x$, the ring is called a ring with unity.*

Definition 0.1.6 *A universum $\mathcal{X}$ is referred to as a linear space over the set of real numbers* R*, if the group* $(\mathcal{X}, \circ)$ *is commutative and associative, and a multiplication is defined of elements* $\mathcal{X}$ *by numbers* $\alpha, \beta, 1 \in \mathsf{R}$*, having the following properties:* $(\alpha + \beta) \circ x = \alpha x \circ \beta x$*,* $\alpha(x \circ y) = \alpha x \circ \alpha y$*,* $(\alpha\beta) \circ x = \alpha(\beta \circ x)$ *and* $1x = x$*. If, instead of* R *we take a ring with unity then* $\mathcal{X}$ *is called a module over this ring.*

Definition 0.1.7 *A universum* $\mathcal{X}$ *with two operations* $\circ$*,* $*$*, such that* $(\mathcal{X}, \circ)$ *is a linear space over* R *and* $(\mathcal{X}, *)$ *is a ring, while the introduced operations satisfy the conditions* $(x \circ y) * z = (x * z) \circ (y * z)$ *and* $x * (y \circ z) = (x * y) \circ (x * z)$*, we call an algebra.*

0.1.3 Topology

The notion of a topological space will be introduced by means of a family of open sets.

Definition 0.1.8 *Let* $\mathcal{X}$ *denote a universum. Its topology* $\mathcal{X}$ *will be defined as a family* $\mathcal{O}$ *of subsets of* $\mathcal{X}$*, called open sets, with the following properties:*

- *the empty set and the whole universum belong to* $\mathcal{O}$*,*
- *the meet of two open sets is an open set,*
- *the union of arbitrary number of open sets is open.*

The pair $(\mathcal{X}, \mathcal{O})$ *will be called a topological space. By a neighbourhood of a point* $x \in \mathcal{X}$ *we understand any open set* $\mathcal{X}$ *containing* x*.*

Definition 0.1.9 *Let a function* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ *between two topological spaces be given. The function* f *is named continuous, if the counter-image of any open subset of* $\mathcal{Y}$ *is open in* $\mathcal{X}$*. Using the terminology of sequences this implies that for any sequence* $\{x_n\}$ *of elements of the space* $\mathcal{X}$ *there holds*

$$\lim_{n \to +\infty} f(x_n) = f\left(\lim_{n \to +\infty} x_n\right).$$

In what follows we shall exploit topological characteristics of some sets. For this reason we define the following.

Definition 0.1.10 *A closed set is the complement of any open set. An interior* $\operatorname{int} A$ *of a subset* $A \subset \mathcal{X}$ *is defined as the biggest open set contained in* A. *A subset* A *is called a boundary set, if its interior is empty. A subset* A *is dense in* $\mathcal{X}$ *if in every neighbourhood of each point* $x \in \mathcal{X}$ *there are some points from* A. *A set* A *is nowhere dense if it is closed and boundary. A topological space is complete if the limit of every sequence of elements of this space belongs to this space.*

0.1.4 Calculus

A basic tool used in these notes is the differential calculus. A useful scenery for the introduction of the concept of derivative is a Banach space.

Definition 0.1.11 *A topological space* $\mathcal{X}$ *is named a Banach space if it is a linear space (over* R*), normed, and complete. The topology of the Banach space is defined by means of the norm. If* $\|\cdot\|$ *denotes a norm then a neighbourhood of radius* r *of a point* x *in the Banach space takes the form*

$$\{y \in \mathcal{X} | \, \|y - x\| < r\},$$

while a sphere centred at x_0 *with radius* r *is defined as*

$$B_r(x_0) = \{y \in \mathcal{X} | \, \|y - x_0\| = r\}.$$

Definition 0.1.12 *Let* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ *be a transformation of Banach spaces. The Fréchet derivative of the function* f *at a point* x *is a linear function* $Df(x) : \mathcal{X} \longrightarrow \mathcal{Y}$ *that satisfies the condition*

$$f(x + v) = f(x) + Df(x)v + O(v^2),$$

where the Landau symbol $O(\epsilon)$ *denotes terms of order* $\geqslant 2$. *The Gateaux derivative of the function* f *is defined as*

$$Df(x)v = \left.\frac{d}{d\alpha}\right|_{\alpha=0} f(x + \alpha v) = \frac{\partial f(x)}{\partial x} v.$$

The Gateaux derivative is efficiently computable. Its significance results from the fact that if the Gateaux derivative exists and is continuous then it is equal to the Fréchet derivative.

0.2 Linear control systems

By a linear control system we mean a system described by linear differential equations
$$\sigma : \ \dot{x} = Ax(t) + Bu(t), \tag{0.1}$$
where $x \in R^n$ – state variable, $u \in R^m$ – control variable, and A and B are, respectively, the dynamics and control matrices of dimensions $n \times n$ and $n \times m$. The spaces R^n and R^m are called, accordingly, a state space and a control space. For a control system the control problem consists in defining such a control that guarantees the achievement of a control objective.

Every linear control system can be identified with a pair of matrices, $\sigma = (A, B)$, so the set of linear control systems $\Sigma \cong R^{n^2+nm}$. Given a control function $u(t)$ and an initial state x_0, the system's trajectory can be found as a solution of the differential equation (0.1). Invoking the method of variations of constants we get

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds. \tag{0.2}$$

The matrix exponential appearing in the formula (0.2) is defined as a sum of the infinite series
$$e^{tA} = \sum_{i=0}^{\infty} \frac{(tA)^i}{i!}.$$
A number of methods exist allowing for the computation of the matrix exponential without resorting to the summation of the infinite series.

0.2.1 Controllability

A fundamental property of a control system, its *raison d'etre*, is the possibility of reaching any point of the state space using a suitably chosen control. This fundamental property is referred to as controllability. To make this concept precise we adopt the following definition of controllability

Definition 0.2.1 *The system (0.1) is controllable, if for any initial state* x_0 *and any terminal state* x_d *there exists a control* $u(t)$ *and a control time* $T \geqslant 0$*, such that*

$$x(T) = e^{TA}x_0 + \int_0^T e^{(T-s)A}Bu(s)ds = x_d.$$

Because x_0 and x_d are arbitrary, and the matrix e^{TA} is invertible, the property of controllability means that the integral

$$I = \int_0^T e^{-sA} B u(s) ds \tag{0.3}$$

assumes all values from R^n. Having defined the concept of controllability we ask how to check if a linear system is controllable. For linear control systems an answer to this question appears to be relatively simple and leads to effective controllability conditions. Given a state x, let us define a control in the following way

$$u(t) = B^T e^{-tA^T} G_T^{-1} x. \tag{0.4}$$

The matrix

$$G_T = \int_0^T e^{-sA} B B^T e^{-sA^T} ds$$

appearing above is known as the Gram matrix of the system (0.1). It is easily observed that the control (0.4) is well defined on condition that the Gram matrix is invertible. Evidently, a substitution of this control to (0.3) yields $I = x$. On the basis of these observations one can state the following necessary and sufficient controllability condition for a linear system.

Theorem 0.2.1 *The system (0.1) is controllable if and only if for a certain* $T > 0$ *the Gram matrix* $G_T = \int_0^T e^{-sA} B B^T e^{-sA^T} ds$ *is invertible* ($\det G_T \neq 0$). *Furthermore, the control transferring the system from the state* x_0 *to the state* x_d *in time* T *takes the form*

$$u(t) = B^T e^{-tA^T} G_T^{-1} \left(e^{-TA} x_d - x_0 \right).$$

A direct check of conditions stated in Theorem 0.2.1 is not easy, therefore, in order to decide controllability efficiently we use the following Kalman criterion.

Theorem 0.2.2 *For a system* $\sigma = (A, B)$ *described by the formula (0.1) we introduce the Kalman matrix*

$$\Omega = \left[B, AB, \ldots, A^{n-1} B \right].$$

The system (0.1) is controllable if and only if the Kalman matrix has full rank n*,*

$$\operatorname{rank} \Omega = n.$$

0.2.2 Equivalence

Take two linear systems of the form (0.1) given as

$$\begin{aligned} \sigma: \ \dot{x} &= Ax(t) + Bu(t), \\ \sigma': \ \dot{\xi} &= F\xi(t) + G\nu(t), \end{aligned}$$

where $x, \xi \in R^n$ $u, \nu \in R^m$. These control systems will be referred to as equivalent if there exists an unambiguous relationship between their trajectories. More precisely, two kinds of equivalence of linear control systems are distinguished, the S-equivalence and the F-equivalence, defined in the following way.

Definition 0.2.2 *Two linear control systems are* S*-equivalent, i.e.*

$$\sigma \underset{S}{\cong} \sigma' \Longleftrightarrow u = \nu \ \textit{and} \ (\exists P, \det P \neq 0)(\xi = Px, \ \textit{s.t.} \ PA = FP, \ PB = G).$$

Definition 0.2.3 *Two linear control system are* F*-equivalent, i.e.*

$$\begin{aligned} \sigma \underset{F}{\cong} \sigma' \Longleftrightarrow \quad & (\exists P, \det P \neq 0, \ K, \ Q, \ \det Q \neq 0)(\xi = Px, \ u = Kx + Q\nu, \\ & \textit{s.t.} \ PA + PKB = FP, \ PBQ = G). \end{aligned}$$

Both these equivalences are equivalence relation, what means they are reflexive, symmetric and transitive. It is easily seen that the S-equivalence is a specific case of the F-equivalence for $K = 0$ and $Q = I_m$. A relationship between controllability and the system equivalence is revealed by the following

Theorem 0.2.3 *Controllability is an invariant of both these equivalences, i.e. if $\sigma \cong_F \sigma'$ and σ is controllable then also σ' is controllable. A fortiori, the same conclusion is valid for the* S*-equivalence.*

0.2.3 Classification and normal forms

Let a single input linear control system be given

$$\sigma: \ \dot{x} = Ax(t) + bu(t),$$

where $x \in R^n$, $u \in R$, $A \in R^{n^2}$, $b \in R^n$. We shall demonstrate that by a specific choice of the matrix P the system σ can be made S-equivalent to so-called controllability normal form. Since the system σ is controllable, it satisfies the Kalman criterion, so the quadratic matrix

$$\Omega = \left[b, Ab, \ldots, A^{n-1}b\right]$$

is invertible. Take $P = \Omega^{-1}$. We are looking for a matrix F, such that $PA = FP$, i.e. $A\Omega = \Omega F$. We compute

$$A\Omega = \left[Ab, A^2b, \dots, A^n b\right].$$

From the Cayley-Hamilton Theorem we deduce

$$A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \dots - a_0 I_n,$$

where a_i denote coefficients of the characteristic polynomial of the matrix A, $\det(\lambda I_n - A) = 0$. Now, using the condition for S-equivalence, $A\Omega = \Omega F$, we get the equation

$$A\Omega = \begin{bmatrix} Ab & A^2b & \dots & -\sum_{i=0}^{n-1} a_i A^i \end{bmatrix}$$

$$= \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & -a_{n-1} \end{bmatrix} = \Omega F.$$

The control vector g of the normal form results from the identity $Pb = g$, tantamount to $b = \Omega g$, so $g = (1, 0, \dots, 0)^T$ In this way we have proved S-equivalence of the system σ to the the controllability normal form

$$\sigma' : \ \dot{\xi} = F\xi(t) + gu(t), \tag{0.5}$$

containing the matrix F and the vector g given below

$$F = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & -a_{n-1} \end{bmatrix}, \ g = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

An alternative normal form of the system σ, named the controller normal form, can be derived in the following way. We look for a matrix F and a vector g that for a certain matrix P fulfil the relationship $PA = FP$ and $Pb = g$. Let again Ω denote the Kalman matrix. It follows from controllability that this matrix is invertible, therefore there exists the matrix Ω^{-1}. Denote its rows by $v_1^T, \dots, v_n^T$, so that

$$\Omega^{-1} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

By definition the matrix Ω satisfies the condition

$$\Omega^{-1}\Omega = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix} = I_n,$$

that results in the equalities

$$v_n^T b = v_n^T Ab = \cdots = v_n^T A^{n-2} b = 0, \; v_n^T A^{n-1} b = 1.$$

Now we can define the matrix P in the following way

$$P = \begin{bmatrix} v_n^T \\ v_n^T A \\ \vdots \\ v_n^T A^{n-1} \end{bmatrix}$$

The product of matrices is equal to

$$P\Omega = \begin{bmatrix} v_n^T \\ v_n^T A \\ \vdots \\ v_n^T A^{n-1} \end{bmatrix} \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & * \\ \vdots & & & & \vdots \\ 1 & * & \cdots & * & * \end{bmatrix},$$

where asterisks stand for elements whose knowledge is not important. As may be seen, the matrix P is invertible, so it may serve as a basis for introducing S-equivalence. Form the equivalence formula it follows that $FP = PA$; invoking again the Cayley-Hamilton Theorem one shows that this condition is satisfied by the matrix

$$F = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix},$$

where, as before, symbols $a_0, a_1, \dots, a_{n-1}$ refer to the coefficients of the characteristic polynomial of the matrix A. The vector $g = Pb$, so $g = (0, 0, \dots, 0, 1)^T$. In conclusion, we have shown how the linear control system σ can be transformed to the controller normal form

$$\sigma' : \; \dot{\xi} = F\xi(t) + gu(t),$$

founded on the matrix F and the vector g specified above. The controller normal form has found an application at the feedback control synthesis, in particular it allows to prove an important Pole Placement Theorem. Let σ be a linear system (0.1).

Theorem 0.2.4 *Suppose that the system σ is controllable. Then, there exists a feedback $u = Kx$, such that the matrix $A+BK$ of the system with feedback has a prescribed spectrum. Equivalently, for any collection $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers satisfying the symmetry condition $\lambda \in \Lambda \Rightarrow \lambda^* \in \Lambda$, $*-$ conjugation of complex numbers, it holds that*

$$\mathrm{sp}(A + BK) = \Lambda.$$

Obviously, when the spectrum is placed in the left half of the complex plane, we get an asymptotically stable linear system. Thus the stabilisation problem of the system σ consists in finding a feedback control, such that the trajectories of the closed-loop system tend asymptotically to zero. A direct consequence of the Theorem 0.2.4 is then that

Remark 0.2.1 *Every controllable linear system is stabilisable.*

0.3 Brunovsky Theorem

We have shown that a single input linear control system is S-equivalent to the controller normal form (F, g). The explicit equations of this normal form look as follows

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = -a_0\xi_1 - a_1\xi_2 - \cdots - a_{n-1}\xi_n + u \end{cases}.$$

Let us apply to this system the feedback $u = k^T\xi + v$, with $k^T = (a_0,\ a_1,\ \ldots,\ a_{n-1})$. This results in the system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = v \end{cases}.$$

It turns out that this kind of the normal form can be established for every controllable linear system. The corresponding normal form is called the Brunovsky canonical form. We let σ denote a system described by (0.1), with n-dimensional state space and m control inputs, with the control matrix B of rank m. Define for the system σ a string of numbers

$$\begin{cases} \rho_0 = \operatorname{rank} B \\ \rho_1 = \operatorname{rank}\,[B\,AB] - \operatorname{rank} B \\ \quad\vdots \\ \rho_{n-1} = \operatorname{rank}\left[B\,AB\ldots A^{n-1}B\right] - \operatorname{rank}\left[B\,AB\ldots A^{n-2}B\right] \end{cases} .$$

By definition, the numbers ρ_i have two properties:

$$\rho_0 = m \geqslant \rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_{n-1} \geqslant 0$$

and

$$\sum_{i=0}^{n-1} \rho_i = n.$$

One can prove that these numbers are feedback invariants, i.e. systems F-equivalent have identical numbers ρ_i. Moreover, the numbers ρ_i constitute a complete system of feedback invariants, what means that

$$\sigma \underset{F}{\cong} \sigma' \Longleftrightarrow \rho_i(\sigma) = \rho_i(\sigma').$$

It has been demonstrated that instead of n-invariants ρ_i it suffices to take m-invariants $\kappa_1, \kappa_2, \ldots, \kappa_m$ defined in the following way

$$\kappa_i = \#\rho_k | \rho_k \geqslant i, \; i = 1, 2, \ldots, m.$$

The symbol $\#$ denotes the number of elements. The numbers κ_i bear the name of controllability indices of the system σ; they have the following properties:

$$\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_m \geqslant 1$$

and

$$\sum_{i=1}^{m} \kappa_i = n.$$

Similarly as ρ_i, also κ_i form a complete system of feedback invariants. In this context the following result is of fundamental significance.

Theorem 0.3.1 (Brunovsky) *Suppose that a controllable system* $\sigma = (A, B)$ *with* $\operatorname{rank} B = m$ *has controllability indices* $\kappa_1, \kappa_2, \ldots, \kappa_m$. *Then, the system* σ *is* F*-equivalent to the system* $\sigma' = (F, G)$ *in the Brunovsky canonical form with the dynamics matrix*

$$F = \begin{bmatrix} \begin{bmatrix} 0 & I_{\kappa_1 - 1} \\ 0 & 0 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} 0 & I_{\kappa_2 - 1} \\ 0 & 0 \end{bmatrix} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \begin{bmatrix} 0 & I_{\kappa_m - 1} \\ 0 & 0 \end{bmatrix} \end{bmatrix}_{n \times n}$$

and the control matrix

$$G = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{\kappa_1 \times 1} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{\kappa_2 \times 1} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{\kappa_m \times 1} \end{bmatrix}_{n \times m}.$$

It turns out that a system in the Brunovsky canonical form has the structure of m strings of integration, of length $\kappa_1, \kappa_2, \ldots, \kappa_m$, presented schematically in Figure 1. Observe that the subset $\mathcal{B} \subset \Sigma$ of the space $\Sigma \cong R^{n^2+mn}$ of linear control systems that satisfy the conditions stated in the Theorem 0.3.1 includes "almost all" linear systems. More precisely, the systems that do not fulfil these conditions are defined by a number of polynomial equations of the form $\det = 0$, thus they constitute so-called algebraic set, composed of the roots of polynomials depending on the entries of matrices A and B. The algebraic set is closed and boundary (does not contain any open subset). Therefore, its complement that consists of the systems satisfying the

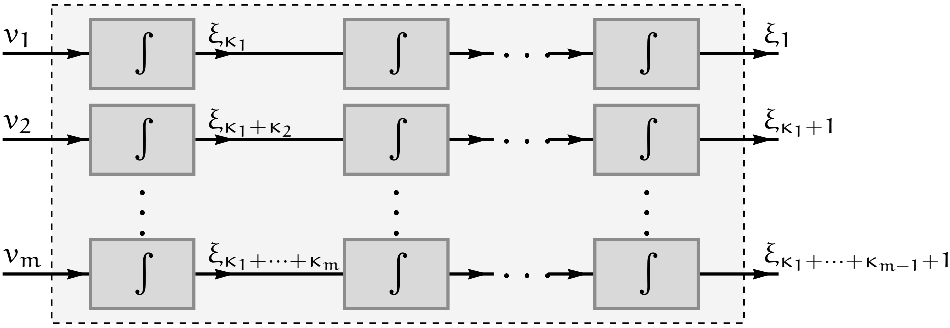

Figure 1: System in the Brunovsky canonical form

Brunovsky conditions is open and dense in Σ. This being so, the Brunovsky Theorem 0.3.1 establishes that the set of "almost all" or "typical" linear control systems can be partitioned into a finite number of classes of systems F-equivalent to a corresponding Brunovsky canonical form. The number of these classes is determined by the number of partitions of the integer n into a sum of m integer components $\geqslant 1$, ordered decreasingly. The number N of these equivalence classes is small for n and m small, but it grows up quickly as n and m increase. Setting $n = km+r$, $r < m$, we get an estimate $p(r) \leqslant N \leqslant p(n-m)$, where $p(r)$ denotes the number of partitions of the integer r, i.e. the number of representations of r in the form of the sum of positive integers. There exists a table of values of $p(r)$ for $r \leqslant 200$, partially displayed below:

r	1	2	3	4	5	6	7	8	9	10	...	200
$p(r)$	1	2	3	5	7	11	15	22	30	42	...	3972999029388

The theorem on Brunovsky canonical forms belongs to the deepest and the most beautiful results of linear control theory.

0.4 Basic ideas of this course

The course's objective is to make the student acquainted with selected mathematical concepts and methods applied in the modern automation and robotics. The guideline of the course relies on a classification of three kinds of mathematical objects: functions, dynamic systems, and control systems. An unrivalled example of such a classification is the Brunovsky Theorem presented in the previous subsection. Following this guideline we shall focus on three so-called pillars of nonlinear analysis, that are

- Inverse Function Theorem,
- Theorem on the Existence and Uniqueness of Solution of a System of Differential Equations,
- Frobenius Theorem on Distributions.

0.5 Proofs

0.5.1 Pole Placement Theorem

The proof of Theorem 0.2.4 exemplifies an application of normal forms of linear systems. We shall restrict the proof to single input systems (A, b), of the form $\dot{x} = Ax(t) + bu(t)$.

Proof: As we have already demonstrated in subsection 0.2.3, controllability of a liner system implies the existence of the controller normal form $\dot{\xi} = F\xi(t) + gu(t)$, such that

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where the numbers $\{a_0, a_1, \dots, a_{n-1}\}$ denote the coefficients of the characteristic polynomial of the matrix A. We recall that the transformation of the system to the controller normal form relies on a matrix

$$P = \begin{bmatrix} v_n^T \\ v_n^T A \\ \vdots \\ v_n^T A^{n-1} \end{bmatrix},$$

in which the row v_n^T comes from the last row of the inverse Kalman matrix Ω^{-1}, such that

$$PA = FP, \quad Pb = g.$$

As the matrices A and F are related by the similarity, their characteristic polynomials, characteristic equations, and spectra are identical. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ stand for eigenvalues of the closed loop system. Using them we define a polynomial

$$\alpha_\gamma(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) = \lambda^n + \gamma_{n-1}\lambda^{n-1} + \dots + \gamma_1\lambda + \gamma_0. \quad (0.6)$$

For the controller normal form with feedback $f = (f_0, f_1, \dots, f_{n-1})$ we consider a matrix

$$F + gf,$$

whose characteristic polynomial is equal to (0.6). Then, we have

$$F + gf = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (f_0, f_1, \dots, f_{n-1})$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \vdots \\ f_0 - a_0 & f_1 - a_1 & f_2 - a_2 & \dots & f_{n-1} - a_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \vdots \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \dots & -\gamma_{n-1} \end{bmatrix}.$$

Observe that above a_i denote the coefficients of the characteristic polynomial of the matrix A, while γ_i are coefficients of the characteristic polynomial of the matrix of the closed loop system. The feedback for the controller normal form can be defined as $f_i = a_i - \gamma_i$. With this choice of the feedback the controller normal form has a prescribed characteristic polynomial $\alpha_\gamma(\lambda)$. Now we return to the original system. Suppose that there exists a feedback $k = (k_0, k_1, \dots, k_{n-1})$ under which there holds

$$P(A + bk) = (F + gf)P.$$

For the reason that $PA = FP$, it must be $Pbk = gfP$, but as $Pb = g$, the above identity will be satisfied provided that

$$k = fP.$$

Taking advantage of the form of the matrix P we obtain

$$k = (f_0, f_1 \dots, f_{n-1}) \begin{bmatrix} v_n^T \\ v_n^T A \\ \vdots \\ v_n^T A^{n-1} \end{bmatrix} = v_n^T (f_{n-1} A^{n-1} + \dots + f_0 I_n)$$

$$= v_n^T \left((a_{n-1} - \gamma_{n-1}) A^{n-1} + \dots + (a_0 - f_0) I_n \right)$$

$$= v_n^T (\underbrace{a_{n-1} A^{n-1} + \dots + a_0 I_n}_{-A^n} - (\gamma_{n-1} A^{n-1} + \dots + \gamma_0 I_n))$$

$$= -v_n^T (A^n + \gamma_{n-1} A^{n-1} + \dots + \gamma_0 I_n) = -v_n^T \alpha_\gamma(A).$$

The last identities result from the Cayley-Hamilton Theorem. The symbol $\alpha_\gamma(A)$ denotes the characteristic polynomial (0.6) determined by the prescribed spectrum, and computed for the matrix A. The formula

$$k = -v_n^T \alpha_\gamma(A)$$

defining the feedback placing the poles in the system (A, b) is referred to as the Ackermann's formula. ∎

0.6 Problems and exercises

Exercise 0.1 Show that similar matrices have the same characteristic polynomials.

Exercise 0.2 Check controllability of the linear control system

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u.$$

Compute e^{tA}.

Exercise 0.3 Check controllability and stability of the linear system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Using the Ackermann's formula find a feedback placing the poles $\{-1, -3\}$.

Exercise 0.4 Check controllability and stability of a model of the inverted pendulum ($\alpha, \delta < 0$, $\beta, \gamma > 0$)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \gamma \\ 0 \\ \delta \end{bmatrix} u.$$

Find a feedback placing the poles $\{-1, -1, -2, -2\}$.

Exercise 0.5 Enumerate possible controllability indices for linear control systems of dimensions $(n, m) = (3, 2)$, $(5, 2)$ and $(7, 2)$.

0.7 Bibliographical remarks

A detailed explanation of basic concepts of set theory, algebra, topology, and mathematical analysis can be found, for instance, in preliminary chapters of the monographs [AMR83, Sas99]. A geometric approach to linear control systems is presented in the book [Won79]. Classic theory of linear control systems is the subject of the textbooks like [Fai98]. Controllability of linear systems in the way similar to ours is exposed in subsection 4.1 of the monograph [Lév09]. The Brunovsky canonical forms have been introduced in the paper [Bru68]; they are also discussed in the mentioned book [Won79]. Complementary information on the action of the feedback group on linear systems are included in the paper[Tch83]. The Cayley-Hamilton Theorem is a basic result of linear algebra, and can be found in the book [Ber05]; from the same source one can also learn on basic properties of the matrix exponential. The Ackermann's formula is dealt with in [Fai98]. The term "pillars of nonlinear analysis" comes from the monograph [AMR83].

Bibliography

[AMR83] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Springer, New York, 1983.

[Ber05] D. S. Bernstein. *Matrix Mathematics.* Princeton University Press, Princeton, 2005.

[Bru68] P. Brunovsky. A classification of linear controllable systems. *Kybernetika*, 6(3):173–188, 1968.

[Fai98] F. W. Fairman. *Linear Control Theory.* Wiley, Chicester, 1998.

[Lév09] J. Lévine. *Analysis and Control of Nonlinear Systems.* Springer, 2009.

[Sas99] S. S. Sastry. *Nonlinear Systems.* Springer, New York, 1999.

[Tch83] K. Tchoń. On generic properties of linear systems: An overview. *Kybernetika*, 19:467–474, 1983.

[Won79] W. M. Wonham. *Linear Multivariable Control.* Springer, Berlin, 1979.

Chapter 1

Functions

1.1 Classes of functions

We shall assume that the notion of the vector space, the definition of the function, the concept of continuity, and the concept of differentiability of functions is known to the Reader. Our interest will be focused on functions (maps, transformations) between real vector spaces

$$f : R^n \longrightarrow R^m, \quad y = f(x). \tag{1.1}$$

This notation means that the components of a vector y are given as

$$\begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ y_2 = f_2(x_1, \dots, x_n) \\ \vdots \\ y_m = f_m(x_1, \dots, x_n) \end{cases}.$$

By default, both these vector spaces R^n and R^m will be equipped with the Euclidean inner product $(\xi, \eta) = \xi^T \eta$. The following classes of functions will be distinguished:

- $C^0(R^n, R^m)$ – the class of continuous functions,
- $C^k(R^n, R^m)$ – the class of functions continuously differentiable up to order k,
- $C^\infty(R^n, R^m)$ – the class of smooth functions,
- $C^\omega(R^n, R^m)$ – the class of analytic functions.

In accordance with this classification, the function $f \in C^k(R^n, R^m)$, if at any point its partial derivatives

$$\frac{\partial^p f_i(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_n^{i_n}}$$

are continuous, where $\sum_{j=1}^n i_j = p$, for all $p \leqslant k$ and all $i = 1, 2, \dots, m$. By a smooth function we understand a function of the class C^k for every k. An analytic function is a smooth function whose every component has a convergent Taylor series. At the point $0 \in R^n$ this means convergence of the series

$$f_i(x) = f_i(0) + Df_i(0)x + \frac{1}{2!}D^2 f_i(0)(x, x) + \dots + \frac{1}{k!}D^k f_i(0)(x, x \dots, x) + \dots,$$

where the symbol D stands for the differentiation. The derivative of a function $f : R^n \longrightarrow R^m$ will be computed in the following way. For a vector $v \in R^n$

$$Df(x)v = \frac{d}{d\alpha}|_{\alpha=0} f(x + \alpha v) = \frac{\partial f(x)}{\partial x} v.$$

The matrix $Df(x)$ is called the Jacobian matrix of the function f at the point x. By definition, the classes of functions distinguished above are related as follows

$$C^\omega \subset C^\infty \subset C^k \subset C^0.$$

Occasionally, further on we shall use more general functions than continuous, such as the piece-wise continuous or piece-wise constant functions. They will be introduced in due time. Given an analytic function, it follows from the definition of analyticity that the values of such a function in the neighbourhood of a point, e.g. zero, are determined by derivatives of this function at the point. A collection of these derivatives is named the jet of the function. The jet of order k at zero has then the form

$$j^k f_i(0) = \left(f_i(0), Df_i(0), D^2 f_i(0), \dots, D^k f_i(0)\right).$$

If, for every component of an analytic function defined on R^n, the jet $j^\infty f_i(0) = 0$ then $f(x)$ is identically equal to 0 on the whole space R^n. In order to better explain the difference between smooth and analytic functions, let's consider the function

$$f(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ e^{-\frac{1}{x}} & \text{for} \quad x > 0 \end{cases},$$

whose plot has been portrayed in Figure (1.1). It is easily checkable that this

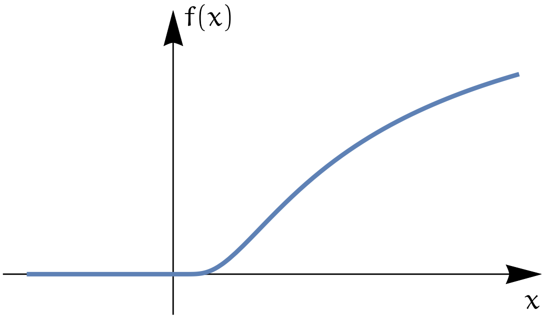

Figure 1.1: Smooth, non-analytic function

function is smooth and its infinite jet at zero vanishes $j^\infty f(0) = 0$. On the other hand, in any neighbourhood of 0 the function $f(x)$ does not vanish. Apparently, the function $f(x)$ is an example of a smooth function that is not analytic. An obvious example of a function that has a finite order of smoothness (it is of the class C^1, but not C^2) is the function

$$f(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ x^2 & \text{for} \quad x > 0 \end{cases} .$$

1.2 Algebraic structures in the set of functions

Consider a pair of continuous functions $f_1, f_2 \in C^0(R^n, R^m)$. They can be added and multiplied by real numbers $\alpha \in R$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (\alpha f_1)(x) = \alpha f_1(x).$$

It follows that continuous functions $C^0(R^n, R^m)$ form a linear space over the set real numbers R. Under assumption that $m = 1$, the continuous functions can also be multiplied by each other

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

A linear space with a multiplication (a product) is called an algebra, so we say that the space $C^0(R^n, R)$ is an algebra. If we focus solely on the operation of multiplication, we shall call the class $C^0(R^n, R)$ a ring. Obviously, smooth functions $C^\infty(R^n, R)$ along with the function multiplication also form a ring. Now, let us choose a smooth function $f \in C^\infty(R^n, R^n)$ and a function $a \in C^\infty(R^n, R)$. The product

$$(af)(x) = a(x) f(x)$$

is a smooth function. This means that $C^\infty(R^n, R^n)$ is a module over the ring of functions $C^\infty(R^n, R)$. Moreover, for two functions $f_1, f_2 \in C^\infty(R^n, R^n)$ one can define another product as

$$[f_1, f_2](x) = Df_2(x)f_1(x) - Df_1(x)f_2(x)$$

that is called the Lie bracket. The linear space $C^\infty(R^n, R^n)$ together with the Lie bracket is referred to as a Lie algebra. This being so, $C^\infty(R^n, R^n)$ is a Lie algebra over real numbers R and simultaneously a module over the ring of smooth functions $C^\infty(R^n, R)$. We shall come back to the notion of the Lie bracket in the section devoted to vector fields.

As another example of the Lie algebra we can take the space of smooth functions $C^\infty(R^{2n}, R)$ with the operation of the Poisson bracket. Suppose that $x = (q, p)$. Then, the Poisson bracket is defined as

$$\{f_1, f_2\}(q, p) = \left(\frac{\partial f_1(x)}{\partial q}\right)^T \frac{\partial f_2(x)}{\partial p} - \left(\frac{\partial f_1(x)}{\partial p}\right)^T \frac{\partial f_2(x)}{\partial q}.$$

The Poisson bracket plays an important role in Hamiltonian mechanics.

As the last example of an algebraic structure in the set of functions let's look at the smooth functions of a single variable $C^\infty(R, R)$. This class is an algebra that is additionally closed with respect to differentiation, i.e. if $f \in C^\infty(R, R)$ then $\dot{f} \in C^\infty(R, R)$. This kind of algebra is called a differential algebra; the differential algebra of functions of time appears in the analysis of differentially flat control systems.

1.3 Inverse Function Theorem

For a pair of continuous functions $f_1, f_2 \in C^0(R^n, R^n)$ one can define an operation called a composition of functions

$$(f_1 \circ f_2)(x) = f_1(f_2(x)),$$

that consists in computing the function f_1 for a value of the function f_2. We introduce the following definition.

Definition 1.3.1 *The function* f_1 *is an inverse function of the function* f_2*, if*

$$(f_1 \circ f_2)(x) = x.$$

The inverse function of f will be denoted by f^{-1}. Elementary examples of functions and their inverses are e^x, and $\ln x$, $\tan x$ and $\arctan x$, $sinx$ and $arcsinx$, etc. For differentiable functions f_1 and f_2 there is the following rule of the differentiation of a composed function (the chain rule)

$$D(f_1 \circ f_2)(x) = Df_1(f_2(x))Df_2(x).$$

The question of existence of the inverse function is answered by the following Inverse Function Theorem, regarded as one of the pillars of nonlinear analysis.

Theorem 1.3.1 (Inverse Function Theorem) *Choose a function* $f \in C^k(R^n, R^n)$ *for a certain* $k \geqslant 1$, *and let* $f(x_0) = y_0$. *Assume that*

$$\text{rank } Df(x_0) = n.$$

Then, in a neighbourhood U *of the point* y_0, *there exists the inverse function* $f^{-1}(y)$, *also of the class* C^k.

It results from the definition of the inverse function $f \circ f^{-1}(x) = x$ and from the chain rule that

$$Df(f^{-1}(x))Df^{-1}(x) = I_n,$$

so

$$Df^{-1}(x) = \left(Df(f^{-1}(x))\right)^{-1}.$$

A function $f \in C^k(R^n, R^n)$ that has the inverse function of the class C^k will be called a diffeomorphism. In the case when f^{-1} exists only locally, the diffeomorphism is named local. The Inverse Function Theorem provides us with a sufficient condition for a local diffeomorphism. We want to admit that there is no necessary and sufficient condition for a function to be a diffeomorphism and each particular case needs to be approached individually.

1.4 Implicit Function Theorem

One of the most significant consequences of the Inverse Function Theorem is the Implicit Function Theorem stated below.

Theorem 1.4.1 (Implicit Function Theorem) *Let a function* $f \in C^k(R^n \times R^m, R^m)$, $w = f(x, y)$, *be given for a certain* $k \geqslant 1$, *such that* $f(x_0, y_0) = w_0$. *Suppose that*

$$\text{rank } \frac{\partial f(x_0, y_0)}{\partial y} = m.$$

Then, there exists a function $y = g(x, w)$ *of the class* C^k*, defined in a neighbourhood of* (x_0, w_0) *and satisfying*

$$f(x, g(x, w)) = w.$$

A proof of this theorem will be provided in Appendix. In order to determine derivatives of the function g we shall reason in the following way. Since $f(x, g(x, w)) = w$ then, by differentiation of both sides of this identity with respect to x, we get

$$\frac{\partial f(x, g(x, w))}{\partial x} + \frac{\partial f(x, g(x, w))}{\partial y}\frac{\partial g(x, w)}{\partial x} = 0,$$

therefore

$$\frac{\partial g(x, w)}{\partial x} = -\left(\frac{\partial f(x, g(x, w))}{\partial y}\right)^{-1}\frac{\partial f(x, g(x, w))}{\partial x}.$$

In a similar way we find

$$\frac{\partial g(x, w)}{\partial w} = -\left(\frac{\partial f(x, g(x, w))}{\partial y}\right)^{-1}.$$

1.5 Computation of the inverse function

In various problems of automation and robotics, as e.g. in the inverse kinematics problem of manipulators, we need to compute the inverse function. Suppose that a function $f \in C^k(R^n, R^n)$ fulfils the conditions of the Inverse Function Theorem, i.e. at any point $\operatorname{rank} Df(x) = n$. Given a terminal point $y_d \in R^n$ we want to determine a point $x_d \in R^n$, such that $f(x_d) = y_d$. Generally this problem is solved numerically. Two algorithms of computing the inverse function will be described below.

1.5.1 Newton Algorithm

According to this algorithm we start from choosing an initial point $x_0 \in R^n$. If our choice is accurate, i.e. $f(x_0) = y_d$, we finish. Otherwise, we perform a "deformation" of the point x_0 to a differentiable curve $x(\theta)$ parametrised by $\theta \in R$, such that $x(0) = x_0$. The error of reaching the terminal point along this curve amounts to

$$e(\theta) = f(x(\theta)) - y_d.$$

Now, we want to pick the curve $x(\theta)$ in such a way that when $\theta \to +\infty$ the error $e(\theta)$ decreases along the curve $x(\theta)$ exponentially. To this objective we require that the error satisfies a differential equation

$$e'(\theta) = -\gamma e(\theta),$$

where $\gamma > 0$ denotes a convergence rate. Suppose that the required curve $x(\theta)$ exists. Having differentiated the error we obtain

$$e'(\theta) = Df(x(\theta))x'(\theta) = -\gamma e(\theta).$$

Due to the invertibility of the matrix $Df(x)$ the above equation means that the curve $x(\theta)$ should solve the differential equation

$$x'(\theta) = -\gamma \left(Df(x(\theta))\right)^{-1} \left(f(x(\theta)) - y_d\right),$$

often attributed to Ważewski-Davidenko, with the initial condition $x(0) = x_0$. Then, the value of the inverse function $x_d = f^{-1}(y_d)$ is obtained as the limit

$$x_d = \lim_{\theta \to +\infty} x(\theta).$$

This algorithm is known as the Newton Algorithm. It follows that in order to compute the inverse function using the Newton Algorithm one needs to solve numerically a certain differential equation, and then pass to the limit of its solution. For computational purposes this algorithm is often presented in a discrete form, e.g. by invoking the Euler scheme, leading to the difference equation

$$x_{k+1} = x_k - \gamma \left(Df(x_k)\right)^{-1} \left(f(x_k)\right) - y_d), \quad k = 0, 1, \dots$$

1.5.2 Steepest Descent Algorithm

Alternatively to the Newton Algorithm one may exploit the following Steepest Descent Algorithm. We begin with guessing a solution x_0, similarly as in the former algorithm. If this is not successful, we define a function $e(x) = f(x) - y_d$. The core idea of this algorithm consists in generating a motion of the point $x \in R^n$ along a curve $x(\theta)$, in the direction of the quickest decrease of the error

$$E(x) = \frac{1}{2} e^T(x) e(x) = \frac{1}{2} ||e(x)||^2.$$

Obviously, this direction is $-\operatorname{grad} E(x)$, therefore the curve $x(\theta)$ needs to obey the equation

$$x'(\theta) = -\gamma \operatorname{grad} E(x(\theta)), \quad \gamma > 0.$$

By definition, the gradient of a function satisfies

$$(\operatorname{grad} E(x), v) = DE(x)v,$$

so, consequently

$$\operatorname{grad} E(x) = (De(x))^T e(x).$$

Eventually, using the definition of $e(x)$, the curve of the steepest descent should solve the differential equation

$$x'(\theta) = -\gamma (Df(x(\theta)))^T (f(x(\theta)) - y_d), \quad x(0) = x_0.$$

Analogously to the Newton Algorithm, the inverse function $x_d = f^{-1}(y_d)$ is computed as the limit

$$x_d = \lim_{\theta \to +\infty} x(\theta)$$

of the trajectory of this differential equation. The discrete version of the Steepest descent Algorithm takes the form

$$x_{k+1} = x_k - \gamma (Df(x_k))^T (f(x_k) - y_d), \quad k = 0, 1, \dots$$

where γ can be interpreted as the step length of the algorithm. A rational way of choosing γ relies on the minimisation of the function

$$E(x_{k+1}) = E(x_k - \gamma \operatorname{grad} E(x_k)).$$

A necessary condition for the minimum is

$$\begin{aligned} \frac{dE(x_{k+1})}{d\gamma} &= -(DE(x_k - \gamma \operatorname{grad} E(x_k)))^T \operatorname{grad} E(x_k) \\ &= -\operatorname{grad}^T E(x_{k+1}) \operatorname{grad} E(x_k) = 0. \end{aligned}$$

It can be seen that with this choice of the coefficient γ the direction of motion in the step $k+1$ is perpendicular to the motion direction in the step k.

1.6 Proofs

1.6.1 Implicit Function Theorem

Proof: The Implicit Function Theorem can be deduced from the Inverse Function Theorem in the following way. Given the function $f(x, y)$ we introduce a function $F : R^n \times R^m \longrightarrow R^n \times R^m$ defined as

$$F(x, y) = (x, f(x, y)) = (x, w).$$

The function F is of the class C^k. Its derivative at the point (x_0, y_0),

$$DF(x_0, y_0) = \begin{bmatrix} I_n & 0 \\ \frac{\partial f(x_0,y_0)}{\partial x} & \frac{\partial f(x_0,y_0)}{\partial y} \end{bmatrix},$$

has rank $n + m$ due to the assumption rank $\frac{\partial f(x_0,y_0)}{\partial y} = m$. Therefore, we can apply to the function F the Inverse Function Theorem that guarantees the existence of the function $G(x, w) = (G_1(x, w), G_2(x, w))$, such that

$$F(G(x, w)) = (G_1(x, w), f(G_1(x, w), G_2(x, w))) = (x, w).$$

The above identity yields

$$G_1(x, w) = x \quad \text{and} \quad f(x, G_2(x, w)) = w,$$

so the function $g(x, w) = G_2(x, w)$. ■

1.7 Problems and exercises

Exercise 1.1 Prove that the functions given below are local diffeomorphisms in a neighbourhood of the point 0:

a) $\varphi : R^3 \longrightarrow R^3$,
$\varphi(x) = (x_3, x_2, x_1 - \sin x_2)^T$,

b) $\varphi : R^4 \longrightarrow R^4$,
$\varphi(x) = (x_1, x_2, -x_3 \sin x_1 + x_4 \cos x_1 - x_2, x_3 \cos x_1 + x_4 \sin x_1)^T$,

c) $\varphi : R^5 \longrightarrow R^5$,
$\varphi(x) = (x_1, \sin x_2, \cos x_2 \sin x_3, x_4, x_5 + x_4^3 - x_1^{10})^T$.

Are these diffeomorphisms global?

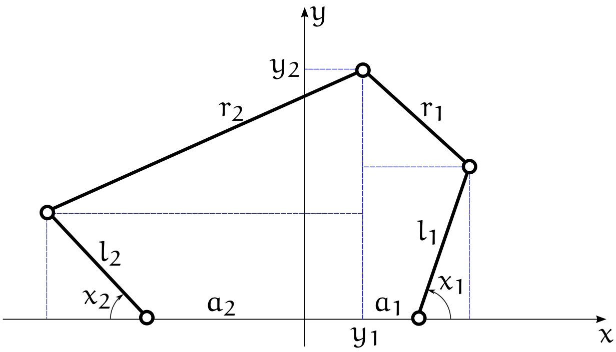

Figure 1.2: Mechanism of the manipulator from exercise 1.5

Exercise 1.2 Show that the following system of equations

$$\begin{cases} x_1 y_1 - x_2 y_2 = 0 \\ x_2 y_1 + x_1 y_2 = 2 \end{cases}$$

defines a function $y = g(x)$. Compute the derivative $Dg(x)$ at the point $x_1 = x_2 = y_1 = y_2 = 1$.

Exercise 1.3 Given the forward kinematics of the robotic manipulator of the type of double pendulum:

$$\begin{cases} y_1 = l_1 \cos x_1 + l_2 \cos(x_1 + x_2) \\ y_2 = l_1 \sin x_1 + l_2 \sin(x_1 + x_2) \end{cases}.$$

show that outside singular configurations there exists a solution of the inverse kinematics problem.

Exercise 1.4 Using the Implicit Function Theorem examine conditions under which the eigenvalues of a matrix $A_{n\times n}$ are functions of the coefficients of its characteristic equation.

Exercise 1.5 Examine the existence of the forward and inverse kinematics of the mechanism presented in Figure 1.2, described by the equations

$$\begin{cases} (a_1 - y_1 + l_1 \cos x_1)^2 + (y_2 - l_1 \sin x_1)^2 = r_1^2 \\ (a_2 + y_1 + l_2 \cos x_2)^2 + (y_2 - l_2 \sin x_2)^2 = r_2^2 \end{cases}.$$

1.8 Bibliographical remarks

Complementary information on functions can be found in the monograph [GG74]. Theorem on the inverse and the implicit functions in Banach spaces have been presented in [AMR83]. An exhaustive exposition of the Newton methods is contained in the book [Deu04]; their application to the motion planning of mobile robots is described in [Tch17, DS03]. The Ważewski-Davidenko equation comes from the papers [Waż47] and [Dav53].

Bibliography

[AMR83] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Springer, New York, 1983.

[Dav53] D. Davidenko. On a new method of numerically integrating a system of nonlinear equations. *Dokl. Akad. Nauk SSSR*, 88, 1053.

[Deu04] P. Deuflhard. *Newton Methods for Nonlinear Problems.* Springer, Berlin, 2004.

[DS03] I. Duleba and J. Sasiadek. Nonholonomic motion planning based on newton algorithm with energy optimization. *IEEE Trans. Control Syst. Tech.*, 11(3), 2003.

[GG74] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities.* Springer, New York, 1974.

[Tch17] K. Tchoń. Endogenous configuration space approach: An intersection of robotics and control theory. In *Nonlinear Systems.* Springer, 2017.

[Waż47] T. Ważewski. Sur l'évaluation du domaine d'existence des fonctions implicites réelles ou complexes. *Ann. Soc. Pol. Math.*, 20:81–120, 1947.

Chapter 2

Linear functions. Equivalence of functions

2.1 Linear functions

A specific class of functions is the class of linear functions. We shall accept the following definition.

Definition 2.1.1 *A function* $f : R^n \longrightarrow R^m$ *is called linear if for every pair of points* $x_1, x_2 \in R^n$ *and every pair of numbers* $\alpha_1, \alpha_2 \in R$ *there holds*

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_2 f(x_1) + \alpha_2 f(x_2).$$

Assume that in R^n and R^m we have chosen bases denoted, respectively by $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_m\}$. Let $y = f(x)$, $x = \sum_{i=1}^n \alpha_i e_i$ and $y = \sum_{j=1}^m \beta_j f_j$. Then, by linearity

$$y = f(x) = f\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{i=1}^n \alpha_i f(e_i).$$

Let the function f transform the basis vectors in the following way

$$f(e_i) = \sum_{j=1}^m a_{ji} f_j.$$

Combining the above calculations we arrive at the identity

$$\beta_j = \sum_{i=1}^n a_{ji} \alpha_i$$

or, for vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$,

$$\beta = A\alpha.$$

The matrix $A = [a_{ij}]$ with m rows and n columns represents the linear function f with respect to the chosen bases. If these bases have been fixed, one can identify linear functions with their matrices. Obviously, the linear functions are analytic.

2.2 Matrices and their norms

In diverse applications we need to compute a norm of the matrix. Recall that the norm in R^n is a function that assumes values greater than or equal to zero,

$$\|\ \| : R^n \longrightarrow R_+,$$

that satisfies the following conditions ($\alpha \in R$, $x_1, x_2 \in R^n$)

$$\|x\| = 0 \Longleftrightarrow x = 0, \quad \|\alpha x\| = |\alpha|\,\|x\|, \quad \|x_1 + x_2\| \leqslant \|x_1\| + \|x_2\|.$$

The last condition is known as the triangle inequality. A well known norm of a vector (in fact this is a family of norms) is the p-norm defined as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p \geqslant 1.$$

Specifically, we distinguish the following p-norms:

- for $p = 1$, 1-norm of a vector x, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$,
- for $p = 2$, 2-norm of a vector x, $\|x\|_2 = \left(\sum_{i=1}^{n} {x_i}^2\right)^{1/2}$,
- for $p = \infty$, ∞-norm of a vector x, $\|x\|_\infty = \max_i |x_i|$

For the reason that $\|x\|_2 = (x, x)^{1/2} = (x^T x)^{1/2}$, 2-norm is identical with the Euclidean norm. It can be shown that the p-norms mentioned above fulfil the inequalities $\|x\|_1 \geqslant \|x\|_2 \geqslant \ldots \geqslant \|x\|_\infty$.

Now, let us pay our attention to matrix norms. Let $A = [a_{ij}]$ denote a matrix of dimension $m \times n$; the set of such matrices will be symbolised as $\mathrm{Mat}(m, n)$. A matrix norm should satisfy three axioms analogous to that for the vector norm, i.e. for $\alpha \in R$ and two matrices $A_1, A_2 \in \mathrm{Mat}(m, n)$ we have

$$\|A\| = 0 \Longleftrightarrow A = 0, \quad \|\alpha A\| = |\alpha|\,\|A\|, \quad \|A_1 + A_2\| \leqslant \|A_1\| + \|A_2\|.$$

Basically, these axioms define the matrix norm, however, for the matrices that can be multiplied by each other, e.g. for $A_1, A_2 \in \mathrm{Mat}(n,n)$, we define an additional property of primary importance, referred to as the ring property,

$$||A_1A_2|| \leqslant ||A_1||\,||A_2||.$$

Having the axioms of the matrix norm we ask, how to define a concrete matrix norm. There are two approaches to this question. First, by listing the entries of the matrix one after another, one can identify a matrix $A \in \mathrm{Mat}(m,n)$ with a vector $A \in R^{mn}$ containing mn components, and then can use a certain p-vector norm. In this context we shall distinguish the 2-norm

$$||A||_F = \left(\sum_{ij} a_{ij}^2\right)^{1/2} = \left(\mathrm{tr}(AA^T)\right)^{1/2},$$

named the Frobenius matrix norm. The matrix norms "inherited" from a vector usually do not have the ring property, however the Frobenius norm does. Second, one can regard the matrix as a kind of operator acting between vector spaces, and interpret the matrix norm as a "measure of amplification" assigned to this operator. The norms devised in the latter way are called operator matrix norms. The operator norm is defined as the biggest ratio of the "amplitude" of the image of a point x to the "amplitude" of this point itself (the original). Formally speaking, this means that

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_{R^m}}{||x||_{R^n}},$$

where we have marked that the original vector and its image may come from different spaces. Due to the property $\frac{||Ax||}{||x||} = ||A\frac{x}{||x||}||$, the operator matrix norm can also be expressed as

$$||A|| = \sup_{||v||=1} ||Av||.$$

By selecting various p-norms in R^n and R^m one can introduce infinitely many matrix norms. Below we shall restrict only to three of them, defined under assumption that the norms of the original and of the image are the same and have the form of either 1 or 2 or ∞ vector norm. The corresponding matrix norms produced in this way will be symbolised by $||A||_1$, $||A||_2$ and $||A||_\infty$. The following result is true

Theorem 2.2.1 *The operator matrix norms are given in as follows:*

$$\begin{aligned}
||A||_1 &= \max_j \textstyle\sum_{i=1}^m |a_{ij}|,\\
||A||_2 &= \bar{\lambda}_{AA^T}^{1/2},\\
||A||_\infty &= \max_i \textstyle\sum_{j=1}^n |a_{ij}|,
\end{aligned}$$

where $\bar{\lambda}_M$ *stands for the biggest eigenvalue of a symmetric matrix* M.

In the face of the multitude of matrix norms, a paramount role is played by the concept of the equivalence of norms.

Definition 2.2.1 *Two matrix norms* $||A||_a$ *and* $||A||_b$ *are equivalent if there exist numbers* $\alpha, \beta > 0$, *such that*

$$\alpha||A||_b \leqslant ||A||_a \leqslant \beta||A||_b.$$

It turns out that the equivalence of norms is an equivalence relation. If two matrix norms are equivalent then the convergence of a sequence of matrices with respect to one of these norms implies the convergence with respect to the other norm.

For invertible square matrix A there holds $1 = ||AA^{-1}||_2 \leqslant ||A||_2||A^{-1}||_2 = \chi(A)$. The number $\chi(A)$ is called the condition number of the matrix A.

2.3 LR-equivalence

In this section we shall introduce a concept of equivalence of functions.

Definition 2.3.1 *Two smooth functions* $f_1, f_2 \in C^\infty(R^n, R^m)$ *are* LR*-equivalent (left-right),* $f_1 \cong_{LR} f_2$, *if there exist diffeomorphisms* $\phi : R^n \longrightarrow R^n$ *and* $\psi : R^m \longrightarrow R^m$, *such that*

$$\psi \circ f_1 = f_2 \circ \phi.$$

In case when the diffeomorphisms ϕ *and* ψ *are defined locally, in some neighbourhoods of the points* x_0 *and* $y_0 = f(x_0)$, *the equivalence is called local,* $f_1 \cong_{LLR} f_2$. *We recall that a local diffeomorphism comes from the Inverse Function Theorem.*

LR equivalence is tantamount to commutativity of a diagram of functions displayed in Figure 2.1.

$$\begin{tikzcd} R^n \arrow[r, "f_1"] \arrow[d, "\phi"] & R^m \arrow[d, "\psi"] \\ R^n \arrow[r, "f_2"] & R^m \end{tikzcd}$$

Figure 2.1: Diagram of LR equivalence

2.4 Submersions and immersions

In this section we shall deal with two classes of functions whose Jacobian matrix has full rank.

Definition 2.4.1 *Let* $f \in C^\infty(R^n, R^m)$, *and let* $m \leqslant n$. *If at any point* $x \in R^n$ *there holds* $\operatorname{rank} Df(x) = m$ *then the function* f *is named a submersion. In the case of* $m \geqslant n$ *and when for every* $x \in R^n$ $\operatorname{rank} Df(x) = n$, *the function* f *is called an immersion. A function* f *that is simultaneously a submersion and an immersion is referred to as a local diffeomorphism.*

Submersions and immersions have non-degenerate linear parts in their Taylor series. Two following theorems establish a normal form of the submersion and the immersion.

Theorem 2.4.1 (On Submersions) *Suppose that* $m \leqslant n$ *and* $f : R^n \longrightarrow R^m$ *is a submersion. Then*

$$f \underset{LLR}{\cong} g,$$

where $g(x) = (x_1, x_2, \ldots, x_m)^T = A_s x, \quad A_s = \begin{bmatrix} I_m & 0 \end{bmatrix}.$

Theorem 2.4.2 (On Immersions) *Let* $m \geqslant n$ *and* $f : R^n \longrightarrow R^m$ *be an immersion. Then*

$$f \underset{LLR}{\cong} g,$$

where $g(x) = (x_1, x_2, \ldots, x_n, 0)^T = A_i x, \quad A_i = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$

Observe that, if f is a submersion then locally it is defined completely by its linear term in the Taylor series

$$f(x) = f(0) + Df(0)x + \frac{1}{2} D^2 f(0)(x, x) + \ldots$$

A similar situation takes place for an immersion. In this sense it can be said that submersions and immersions are 1-determined.

An exciting property of immersions is their genericity. Consider a set of smooth functions $C^\infty(R^n, R^m)$. This set can be endowed with a certain topology that allows us to distinguish subsets of $C^\infty(R^n, R^m)$ that are open, closed, dense, etc. Let $\mathrm{Imm}(R^n, R^m) \subset C^\infty(R^n, R^m)$ denote the set of immersions. Then the following statement is true.

Theorem 2.4.3 (Whitney) *If* $m \geqslant 2n$ *then the set of immersions* $\mathrm{Imm}(R^n, R^m)$ *is open and dense in* $C^\infty(R^n, R^m)$.

This statement means that for $m \geqslant 2n$ every immersion has a neighbourhood consisting solely of immersions, and that in an arbitrarily neighbourhood of a smooth function in $C^\infty(R^n, R^m)$ one can find an immersion. We say that almost every smooth function is an immersion.

2.5 Proofs

2.5.1 Theorem on Submersions

Proof: The proof relies on the construction of a local coordinate changes ϕ and ψ defining the LR equivalence, that satisfy the Inverse Function Theorem. The derivative $Df(x)$ can be written down as a block matrix

$$Df(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_m} & \frac{\partial f_1(x)}{\partial x_{m+1}} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & & & & & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_m} & \frac{\partial f_m(x)}{\partial x_{m+1}} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(x)}{\partial x^m} & \frac{\partial f(x)}{\partial x^{n-m}} \end{bmatrix},$$

where $x^m = (x_1, \ldots, x_m)$ and $x^{n-m} = (x_{m+1}, \ldots, x_n)$. Without any loss of generality we may assume that rank $\frac{\partial f(0)}{\partial x^m} = m$ (otherwise it is enough to re-order the coordinates x). Now, let us define a function $\phi(x) = (f(x), x_{m+1}, \ldots, x_n)^T$. From this definition it follows that ϕ is smooth and that $\phi(0) = 0$. Furthermore, the rank of the Jacobian matrix

$$D\phi(0) = \begin{bmatrix} \frac{\partial f(0)}{\partial x^m} & \frac{\partial f(0)}{\partial x^{n-m}} \\ 0 & I_{n-m} \end{bmatrix}$$

is equal to n, so, by the Inverse Function Theorem, in a certain neighbourhood of the point $0 \in R^n$ the function ϕ is a diffeomorphism. Since $g(x) = x^m$, we get $g \circ \phi(x) = f(x)$, concluding the proof (we take a trivial $\psi(y) = y$). ∎

2.5.2 Theorem on Immersions

Proof: Similarly as in the previous proof we shall use the Inverse Function Theorem. The Jacobian matrix of the function f can be represented in the block form

$$Df(x) = \begin{bmatrix} \frac{\partial f^n(x)}{\partial x} \\ \frac{\partial f^{m-n}(x)}{\partial x} \end{bmatrix},$$

where f^n and f^{m-n} stand for the first n and the remaining $m-n$ components of the function f. Assume that $\operatorname{rank} \frac{\partial f^n(0)}{\partial x} = n$. Let us take $y = (y^n, y^{m-n})$ and define the following change of coordinates $\psi(y) = (f^n(y^n), y^{m-n} + f^{m-n}(y^n))$. The function ψ is smooth and vanishes at zero, $\psi(0) = 0$. Its Jacobian matrix

$$D\psi(0) = \begin{bmatrix} \frac{\partial f^n(0)}{\partial y^n} & 0 \\ * & I_{m-n} \end{bmatrix},$$

where the asterisk denotes a matrix whose form is meaningless. Now, since $\operatorname{rank} D\psi(0) = n$, by the Inverse Function Theorem, in a certain neighbourhood $0 \in R^m$ ψ is a diffeomorphism. Finally, taking $g(x) = (x, 0)$ we have $\psi \circ g(x) = \psi(x, 0) = (f^n(x), 0 + f^{m-n}(x)) = f(x)$, what finishes the proof (now $\phi(x) = x$). ■

2.6 Problems and exercises

Exercise 2.1 For a rotation matrix $R \in SO(3)$ compute the norms $||R||_2$ and $||R||_F$. Find the norm $||R||_1$ for the matrix $R = \mathbf{Rot}(Z, \alpha)$.

Exercise 2.2 Prove that any operator matrix norm has the ring property $||AB|| \leqslant ||A||\,||B||$.

Exercise 2.3 Show that $||A||_F^2 = \operatorname{tr}(AA^T)$, $A_{n\times n}$.

Exercise 2.4 For a matrix $A_{n\times n}$ prove the inequality

$$\frac{1}{\sqrt{n}}||A||_F \leqslant ||A||_1 \leqslant \sqrt{n}||A||_F.$$

Hint: Use the inequality $\left(\sum_{i=1}^n |a_i|\right)^2 \leqslant n \sum_{i=1}^n |a_i|^2$.

Exercise 2.5 Show that the condition number of the matrix A is equal to $\chi(A) = \left(\frac{\overline{\lambda}_{AA^T}}{\underline{\lambda}_{AA^T}}\right)^{1/2}$, where $\overline{\lambda}$ and $\underline{\lambda}$ denote the biggest and the smallest eigenvalue.

Exercise 2.6 For the matrix $A = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ compute the norms $\|A\|_1$, $\|A\|_2$, $\|A\|_F$ and $\|A\|_\infty$, and the condition number $\chi(A)$.

Exercise 2.7 Consider a system of linear equations $Ax = b$, $A_{n\times n}$, with the right hand side perturbed in such a way that $A\hat{x} = b + \varepsilon$. Prove that the relative solution error $\delta x = \frac{\|\hat{x}-x\|}{\|x\|}$ satisfies the estimates

$$\delta b \leqslant \delta x \leqslant \chi(A)\delta b,$$

where $\delta b = \frac{\|\varepsilon\|}{\|b\|}$, and $\chi(A)$ – is the condition number.

Exercise 2.8 Using the Theorems on Submersions and Immersions establish normal forms of the following functions:

a) $f(x) = x_1 + x_2^2$, $x = (x_1, x_2)^T \in R^2$, $f(x) \in R$,

b) $f(x) = (\sin x, \cos x)^T$, $x \in R$, $f(x) \in R^2$,

c) $f(x) = (x, \tan x)^T$, $x \in R$, $f(x) \in R^2$,

d) $f(x) = (x_1 + x_2^2, x_2)^T$, $x = (x_1, x_2)^T \in R^2$, $f(x) \in R^2$.

2.7 Bibliographical remarks

A comprehensive treatment of matrices can be found in the monograph [Ber05]. The exposition of the equivalence of functions, submersions, immersions as well as the Whitney Theorem is based on the classical book [GG74]. Theorems on submersions and immersions in Banach spaces have been presented in the monograph [AMR83]. To a reader interested in singularity theory of functions we recommend the books [GG74, Mar82].

Bibliography

[AMR83] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Springer, New York, 1983.

[Ber05] D. S. Bernstein. *Matrix Mathematics.* Princeton University Press, Princeton, 2005.

[GG74] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities.* Springer, New York, 1974.

[Mar82] J. Martinet. *Singularities of Smooth Functions and Maps.* Cambridge University Press, 1982.

Chapter 3

Morse functions. The Fixed Point Theorem

3.1 Critical points and values

Definition 3.1.1 *Let* $f \in C^\infty(R^n, R^m)$*. A point* $x_0 \in R^n$ *is called a critical point of the function* f *if*

$$\operatorname{rank} Df(x_0) < \min\{m, n\}.$$

A point that is not critical (so $\operatorname{rank} Df(x_0) = \min\{m, n\}$*) will be named a regular point of the function* f*.*

It is easily seen that for $m = 1$ (i.e., for a function $f \in C^\infty(R^n, R)$), critical points are the points at which the derivative $Df(x_0) = 0$. Given a function f, the set of its critical points will be denoted as

$$\mathcal{C}_f = \{x \in R^n | \operatorname{rank} Df(x) < \min\{m, n\}\}.$$

The image $f(\mathcal{C}_f)$ of this set by f is referred to as the set of critical values of the function f. By definition, the set of critical points $\mathcal{C}_f$ is closed in R^n. Example critical points and critical values of a function are presented in Figure 3.1.

It is easily to show that for smooth, but not analytic functions, the set of critical points can be "big", i.e. it can include an open set. A good example is provided by the function from Figure 3.2. The set $\mathcal{C}_f$ coincides in this case with the negative half axis of the real numbers. Contrary to smooth and non-analytic functions, the set of critical points of an analytic function is "small" in the sense that it does not contain any open set (has empty

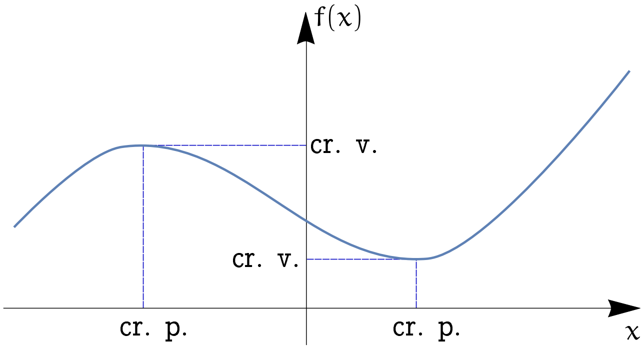

Figure 3.1: Critical points and values

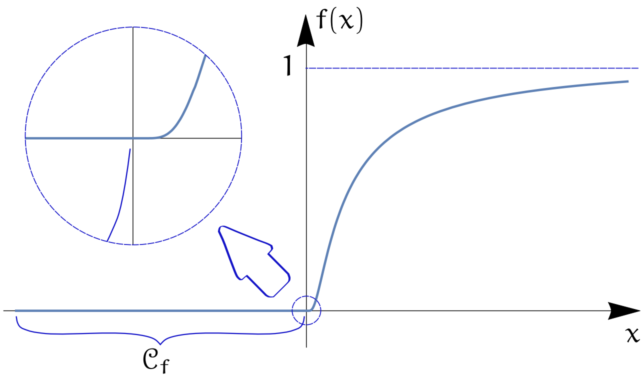

Figure 3.2: A "big" set of critical points $\mathcal{C}_f$

interior), i.e. is a boundary set. Differently to the set of critical points, the set of critical values of smooth functions is always small in the sense specified by the following

Theorem 3.1.1 (Sard) *For any smooth function* $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ *the set of critical values* $f(\mathcal{C}_f)$ *has measure zero in* $\mathbb{R}^m$*.*

This theorem asserts that the set of critical values can be covered by a countable number of open sets (balls) whose total volume is arbitrarily small. Obviously, in case of $m > n$, the image $f(\mathbb{R}^n)$ of the whole space $\mathbb{R}^n$ has measure zero, so, a fortiori, the measure of the set $f(\mathcal{C}_f)$ is also zero.

3.2 Morse functions, Morse Theorem

It follows from the previous chapter that submersions and immersions do not have critical points whatsoever. Being locally equivalent to their linear approximations (the linear portions of their Taylor series) these functions are not tremendously interesting. We feel intuitively that functions that have critical points may be much more interesting. Indeed, this is the case, and the simplest class of functions possessing critical points are the Morse functions.

Definition 3.2.1 *A smooth function* $f \in C^\infty(R^n, R)$ *is called a Morse function if all its critical points are non-degenerate, i.e.*

$$Df(x) = 0 \Longrightarrow \operatorname{rank} D^2 f(x) = n,$$

where $D^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right]$ *denotes the matrix of the second order derivatives of the function* f *(the Hesse matrix).*

In order to better understand the concept of a Morse function f, let us define a function $F = Df : R^n \longrightarrow R^n$. Since f is the Morse function, at each its critical point $Df(x) = 0$ there holds $\operatorname{rank} D(Df)(x) = n$. Invoking the Inverse Function Theorem we conclude that Df is a local diffeomorphism. This being so, if at a point $Df(x_0) = 0$ then in some neighbourhood of the point x_0 it must be $Df(x) \neq 0$, as otherwise Df wouldn't have an inverse function. This observation yields that around a critical point of a Morse function there are no other critical points. We say that the Morse function has isolated critical points. This property allows us to immediately exclude from the class of Morse functions the function displayed in Figure 1.1, because, as we have observed, its critical points occupy an open subset of the real numbers R. Relying on this we may expect that a Morse function $f : R \longrightarrow R$ will have a countable set of extrema. Take as an example the Morse function $f(x) = \sin x$.

The normal forms of the Morse function are characterised by the following

Theorem 3.2.1 (Morse) *Suppose that* $f \in C^\infty(R^n, R)$ *is a Morse function, and let* $f(0) = 0$*,* $Df(0) = 0$*, as well as* $\operatorname{rank} D^2 f(0) = n$*. Then, in a certain neighbourhood of* 0*, it is true that*

$$f \underset{LLR}{\cong} g,$$

where $g(x) = -x_1^2 - x_2^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2$*. The integer* p *denotes the number of negative eigenvalues of the matrix* $D^2 f(0)$*, and is named the index of the critical point* 0*.*

The following functions exemplify the concept of the Morse function:

- $f(x) = x_1^3 + x_1^2 + x_2^2$: The critical point $(0,0)$ has index $p = 0$, so by virtue of the Morse Theorem $f \cong_{LLR} g$, $g(x) = x_1^2 + x_2^2$. Furthermore, the LLR-equivalence is determined by the local diffeomorphism (a substitution of variables) $\phi(x) = (x_1\sqrt{x_1+1}, x_2)$,
- $f(x) = x_1^2 + x_1x_2 - x_2^2$: In this case the index of the critical point $(0,0)$ is equal to $p = 1$, and the Morse Theorem provides the normal form $g(x) = -x_1^2 + x_2^2$. We get $f(x) = g \circ \phi(x)$, where $\phi(x) = (\frac{\sqrt{5}}{2}x_2, x_1 + \frac{1}{2}x_2)$.

3.3 Hadamard's Lemma

The following result can be employed in the proof of the Morse Theorem; it proves also useful outside the context of this theorem.

Theorem 3.3.1 (Hadamard) *Let* $f \in C^\infty(R^n, R)$. *Then, there exist smooth functions* $g_1, g_2, \dots, g_n$, *such that*

$$f(x) = f(0) + \sum_{i=1}^n g_i(x)x_i,$$

where $g_i(x) = \int_0^1 \frac{\partial f(tx)}{\partial x_i} dt$.

One can notice that, having applied this result again, to each function $g_i(x)$, we obtain

$$g_i(x) = g_i(0) + \sum_{j=1}^n h_{ij}(x)x_ix_j,$$

where $g_i(0) = \frac{\partial f(0)}{\partial x_i}$ as well as $h_{ij}(x) = \int_0^1 \frac{\partial g_i(sx)}{\partial x_j} ds = \int_0^1\int_0^1 \frac{\partial^2 f(stx)}{\partial x_i \partial x_j} t dt ds$. In conclusion, we have arrived at the following expression

$$f(x) = f(0) + Df(0)x + \sum_{i,j=1}^n h_{ij}(x)x_ix_j.$$

Following this kind of argument we come up to a sort of Taylor series of the function f.

3.4 Classification of function: Summary

Within the class of functions we have realised our programme of classification of functions and their description by normal forms for three classes of functions: submersions, immersions, and Morse functions. A range of these classifications can be assessed after introducing into the set of smooth functions a certain topology, called the Whitney topology. This is just the topology to which the Whitney Theorem, stated in the previous chapter, refers.

3.5 The Fixed Point Theorem

In this section we shall present one of the most significant theorems of matematics, the Fixed Point Theorem, sometimes referred to as the "carthorse of nonlinear analysis". This name underlines that many fundamental results in analysis can be derived just from this theorem. For the sake of generality we shall formulate this theorem in the framework of Banach spaces. We recall that a space is a Banach space provided that it is a linear, normed, and complete space.

Theorem 3.5.1 (Fixed Point Theorem) *Let $\mathcal{X}$ be a Banach space, equipped with a norm $\|\cdot\|$. Assume that on this space a function*

$$T : \mathcal{X} \longrightarrow \mathcal{X}$$

has been defined, obeying the condition

$$\|T(x_2) - T(x_1)\| \leqslant \rho\|x_2 - x_1\|,$$

Where $0 < \rho < 1$. Then, the function T has a fixed point x^*, such that

$$T(x^*) = x^*.$$

The fixed point is unique, and can be found as the limit $x^* = \lim x_k$ of the sequence

$$x_0, x_1 = T(x_0), \ldots, x_{k+1} = T(x_k), \ldots$$

whose initial element x_0 is an arbitrary point of the space $\mathcal{X}$.

A fundamental assumption made in this theorem is that the function T "shrinks" the distance between points in its domain (such a function is called a contraction). In applications, a useful part is played by a consequence of the Fixed Point Theorem stated as the following.

Theorem 3.5.2 *Suppose that* $S \subset \mathcal{X}$ *is a closed subset of a Banach space on which the function* $T : S \longrightarrow S$ *shrinks. Then,* T *has a unique fixed point in* S.

3.6 Proofs

3.6.1 Hadamard's Lemma

Proof: From the definition of the integral there results immediately that

$$\int_0^1 df(tx) = f(tx)|_0^1 = f(x) - f(0).$$

Exploiting this observation we get

$$f(x) = f(0) + \int_0^1 df(tx) = f(0) + \int_0^1 \sum_0^n \frac{\partial f(tx)}{\partial x_i} x_i dt = f(0) + \sum_0^n g_i(x) x_i,$$

$g_i(x) = \int_0^1 \frac{\partial f(tx)}{\partial x_i} dt$, that finishes the proof. ■

3.6.2 Fixed Point Theorem

Proof: Take a sequence x_0, $x_1 = T(x_0), \ldots, x_{k+1} = T(x_k) \ldots$. The shrinking property implies that

$$\|x_{k+1} - x_k\| = \|T(x_k) - T(x_{k-1})\| \leqslant \rho \|x_k - x_{k-1}\|,$$

therefore

$$\|x_{k+1} - x_k\| \leqslant \rho \|x_k - x_{k-1}\| \leqslant \rho^2 \|x_{k-1} - x_{k-2}\| \leqslant \cdots \leqslant \rho^k \|x_1 - x_0\|.$$

Now, let $m = k + r$. We want to demonstrate that the sequence x_0, $x_1, \ldots$ is a Cauchy sequence, what means that its sufficiently far elements differ from each other as little, as we wish. Indeed, we have a number of inequalities

$$\begin{aligned}
\|x_m - x_k\| &= \|x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{k+1} - x_k\| \\
&\leqslant \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2} + \cdots + \|x_{k+1} - x_k\| \\
&\leqslant \rho^{m-1}\|x_1 - x_0\| + \rho^{m-2}\|x_1 - x_0\| + \cdots + \rho^k \|x_1 - x_0\|
\end{aligned}$$

$$= \rho^k \|x_1 - x_0\| (1 + \rho + \cdot + \rho^{m-k-1}) \leqslant \rho^k \|x_1 - x_0\| (1 + \rho + \cdots) = \frac{\rho^k}{1 - \rho} \|x_1 - x_0\|.$$

From the last inequality it follows that for any $\epsilon > 0$ we can find an integer N, such that for $k \geqslant N$ we get $\|x_m - x_k\| < \epsilon$, i.e. the sequence x_0, x_1,... is Cauchy. Because in a complete space each Cauchy sequence has a limit, we conclude that the limit $x^* = \lim x_{k+1} = \lim T(x_k) = T(x^*)$ exists. In order to show that the limit point x^* is unique, suppose that there are two different fixed points $x^* \neq \hat{x}$ that fulfil the condition $x^* = T(x^*)$ and $T(\hat{x}) = \hat{x}$. We compute

$$\|x^* - \hat{x}\| = \|T(x^* - \hat{x})\| \leqslant \rho\|x^* - \hat{x}\|,$$

that implies that

$$(1 - \rho)\|x^* - \hat{x}\| \leqslant 0.$$

But we have $\rho < 1$, so it must be $x^* = \hat{x}$. In this way the theorem has been proved. ■

3.7 Problems and exercises

Exercise 3.1 Check the existence and (non)degeneracy of critical points of the following functions:

a) $f(x) = x^3$, $x \in R$,

b) $f(x) = x_1^3 - 3x_1^2 x_2$, $x = (x_1, x_2)^T \in R^2$,

c) $f(x) = x_1^2$, $x = (x_1, x_2)^T \in R^2$,

d) $f(x) = x_1 x_2$, $x = (x_1, x_2)^T \in R^2$,

e) $f(x) = x_1^2 \cos x_2 + \sin^2 x_2$, $x = (x_1, x_2)^T \in R^2$.

Exercise 3.2 Without invoking the Morse Theorem show that the function $f(x) = x_1^2 + x_1 x_2 + x_2^2$ is LR-equivalent to the function $g(x) = x_1^2 + x_2^2$.

Exercise 3.3 Similarly as in the problem 3.2 show that the function $f(x) = x_1 x_2 + x_2^2$ is LR-equivalent to $g(x) = x_1^2 + x_2^2$.

Exercise 3.4 Making use of the Morse Theorem find normal forms of the following functions, in a neighbourhood of the point 0:

a) $f(x) = x_1^2 \cos x_2 + \sin^2 x_2$,

b) $f(x) = \cos x_1 - 2x_1 x_2 + \cos x_2 - 2$,

c) $f(x) = x_1 \sin x_2 + x_2 \sin x_1$,

d) $f(x) = x_1^2 \cos x_3 + x_2 x_3 + x_3^2$,

e) $f(x) = \sin x_1 \sin x_2 - x_3^2$,

f) $f(x) = x_1 x_2 + x_2 x_3 - x_1 x_3$.

3.8 Bibliographical remarks

The concepts of critical points, critical values, and Morse functions come from the monograph [GG74]. The Sard Theorem is can be found in [GG74], and also in [AMR83]. The Morse Theorem, together with a proof, has been reported in [GG74]. As a "vehicle" in this proof the Hadamard's Lemma has been used. The Fixed Point Theorem comes from Banach [Ban22]. A proof of the Inverse Function Theorem based on the Fixed Point Theorem is provided in [AMR83].

Bibliography

[AMR83] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Springer, New York, 1983.

[Ban22] S. Banach. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fundamenta Mathematicae*, 3:133–181, 1922.

[GG74] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities.* Springer, New York, 1974.

Chapter 4

Time dependent dynamic systems

4.1 Differential equations. Theorem on Existence and Uniqueness of Solution

In this section we shall study systems of ordinary differential equations, of the form

$$\dot{x} = f(x, t), \quad x \in R^n, \quad f : R^n \times R \longrightarrow R^n, \quad x(0) = x_0. \tag{4.1}$$

By default, the variable t will be interpreted as time. A solution or a trajectory or an integral curve of the system (4.1) is a time function $x(t)$, such that, at any time instant t,

$$\dot{x} = \frac{dx(t)}{dt} = f(x(t), t) \quad \text{and} \quad x(0) = x_0.$$

We say that $x(t)$ satisfies the system (4.1). Obviously, if $x(t)$ satisfies the system of equations then

$$x(t) = x_0 + \int_0^t f(x(\tau), \tau) d\tau.$$

A fundamental question of the analysis of the system (4.1) is the question of existence of the solution $x(t)$, as well as of its uniqueness. The relevant theorem, referred to as the Theorem on Existence and Uniqueness, similarly to the Inverse Function Theorem, creates one of the pillars of nonlinear analysis. This theorem assumes the following form.

Theorem 4.1.1 (On Existence and Uniqueness) *Suppose that the function* $f(x, t)$ *is continuous with respect to* t*, bounded for the initial condition,* $||f(x_0, t)|| \leqslant$

M, *and satisfies the Lipschitz condition with respect to* x, *i.e.*

$$\|f(x_2, t) - f(x_1, t)\| \leqslant L\|x_2 - x_1\|, \quad L > 0,$$

for the points x_1, x_2 *belonging to a certain ball centred at* x_0 *of radius* r, $x_1, x_2 \in B(x_0, r)$. *Then, the system (4.1) has a solution* $x(t)$ *defined on a time interval* $[0, \alpha]$, *starting at* $t = 0$ *from the initial condition* x_0. *Furthermore, this solution is unique.*

The uniqueness of the solution $x(t)$ for $t \in [0, \alpha]$ means that, if there exists another solution $\bar{x}(t)$ defined for $t \in [0, \bar{\alpha}]$ then both these solutions coincide on the common part of their intervals of definiteness, i.e. for $t \in [0, \alpha] \cap [0, \bar{\alpha}]$ there holds $x(t) = \bar{x}(t)$.

A consequence of the Theorem on Existence and Uniqueness is that the solution $x(t)$ is defined locally in time, on an interval $[0, \alpha]$ that depends on the initial condition x_0. In case when $x(t)$ exists for all time instants $t \in R$ and all initial conditions $x_0 \in R^n$, the system (4.1) will be called a time dependent (nonautonomous) dynamic system.

4.2 Bellman-Gronwall Lemma, dependence on initial conditions

An important role in the analysis of systems (4.1) is played by the Bellman-Gronwall Lemma that can be stated in the following form.

Lemma 4.2.1 (Bellman-Gronwall) *Suppose that two functions* $\phi(t), \psi(t) \geqslant 0$ *fulfil the inequality*

$$\phi(t) \leqslant a \int_0^t \phi(s)\psi(s) ds + b, \quad \textit{for} \quad a, b \geqslant 0.$$

Then, it is true that

$$\phi(t) \leqslant b e^{a \int_0^t \psi(s) ds}.$$

As an example application of this lemma we shall demonstrate that the solution of a system of differential equations depends continuously on the initial condition. Let $x_0(t)$ denote such a solution initialised at x_0. Choose another initial condition $x_0 + \eta$, where $\|\eta\| \leqslant \epsilon$, and let the solution starting from $x + \eta$ be denoted as $x_\epsilon(t)$. We ask the following question: assuming that the initial conditions are close to each other (ϵ is small), are the solutions

$x_0(t)$ and $x_\epsilon(t)$ close as well? To answer this question, we compute

$$\begin{aligned}\|x_\epsilon(t) - x_0(t)\| &= \left\| x_0 + \eta + \int_0^t f(x_\epsilon(\tau), \tau) d\tau - x_0 - \int_0^t f(x_0(\tau), \tau) d\tau \right\| \\ &\leqslant \epsilon + \int_0^t \|f(x_\epsilon(\tau), \tau) - f(x_0(\tau), \tau)\| d\tau \leqslant \epsilon + L \int_0^t \|x_\epsilon(\tau) - x_0(\tau)\| d\tau.\end{aligned}$$

To the last expression we apply the lemma 4.2.1. Having substituted $\phi(t) = \|x_\epsilon(t) - x_0(t)\|$, $\psi(t) = 1$, $a = L$, and $b = \epsilon$, we get

$$\|x_\epsilon(t) - x_0(t)\| \leqslant \epsilon e^{L \int_0^t ds} = \epsilon e^{tL}.$$

It follows that for any finite t one can always find such an ϵ that the solution $x_\epsilon(t)$ will be arbitrarily close to $x_0(t)$. This is exactly meant by the continuous dependence of the solution of the initial condition.

4.3 Time dependent linear systems

A specific class of system (4.1) is constituted by linear systems of the form

$$\dot{x} = A(t)x(t), \tag{4.2}$$

where $A(t)$ is a matrix of dimension $n \times n$, depending on time. Invoking the Theorem on Existence and Uniqueness we discover that the premises of this theorem now reduce to a requirement that the matrix function $A(t)$ be continuous and bounded. If this is true, the solution exists for every t and every initial condition x_0, therefore the system (4.2) is an example of a time dependent dynamic system. In the context of dynamic systems more often than "a solution" we shall use "the state trajectory" or just "the trajectory", while the initial condition will be called an initial state of the system. Let $x(s)$ be the trajectory of the system (4.2) for a certain $s \leqslant t$. Then, one can show that

$$x(t) = \Phi(t, s)x(s).$$

The matrix $\Phi(t, s)$ is named the fundamental matrix (the transition matrix) of the system, and solves the equation

$$\frac{\partial \Phi(t, s)}{\partial t} = A(t)\Phi(t, s), \quad \text{on condition that} \quad \Phi(s, s) = I_n.$$

As a matter of fact we have

$$\dot{x} = \frac{\partial \Phi(t, s)}{\partial t} x(s) = A(t)\Phi(t, s)x(s) = A(t)x(t).$$

Furthermore, for three time instants $u \leqslant s \leqslant t$ the following identity holds

$$x(t) = \Phi(t,s)x(s) = \Phi(t,s)\Phi(s,u)x(u),$$

resulting in the so called semigroup property of the fundamental matrix

$$\Phi(t,s)\Phi(s,u) = \Phi(t,u).$$

If one sets $u = t$ then

$$\Phi(t,s)\Phi(s,t) = \Phi(t,t) = I_n,$$

what means that the fundamental matrix is invertible, and $\Phi^{-1}(t,s) = \Phi(s,t)$. In this way we have discovered three important properties of the fundamental matrix

$$\Phi(t,t) = I_n, \quad \Phi^{-1}(t,s) = \Phi(s,t), \quad \Phi(t,s)\Phi(s,u) = \Phi(t,u).$$

4.4 Peano-Baker Formula

If the matrix $A(t)$ does not depend on time, we get a liner dynamic system

$$\dot{x} = Ax(t). \tag{4.3}$$

A feature of this system is that its fundamental matrix can be computed explicitly, namely,

$$\Phi(t,s) = e^{(t-s)A},$$

where the matrix exponential function is defined as the sum of the series $e^{tA} = \sum_{i=1}^{\infty} \frac{(tA)^i}{i!}$. Several methods are known of efficiently computing the matrix exponential, e.g. based on the Cayley-Hamilton theorem. Observe that not only for the system (4.3) the computation of the fundamental matrix is tantamount to the computation of an exponential function; the same is true also for a 1-dimensional time dependent system. Namely, for

$$\dot{x} = a(t)x(t), \quad x, a, x_0 \in R,$$

the trajectory is $x(t) = e^{\int_0^t a(u)du}x_0$. This being so, can one expect that perhaps in general $\Phi(t,s) = e^{\int_s^t A(u)du}$? The answer is negative. In general case the fundamental matrix is expressed by so called Peano-Baker formula that assumes the form of an infinite series

$$\Phi(t,s) = I_n + \int_s^t A(\sigma_1)d\sigma_1 + \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + \cdots$$
$$+ \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)\ldots\int_s^{\sigma_{k-1}} A(\sigma_k)d\sigma_k d\sigma_{k-1}\ldots d\sigma_1 + \cdots$$

A necessity of resorting to this formula is a consequence of non-commutativity of the matrix multiplication. If, for any, t_1, t_2 the matrices $A(t_1)$ and $A(t_2)$ commute, i.e. their commutator

$$[A(t_1), A(t_2)] = A(t_1)A(t_2) - A(t_2)A(t_1) = 0,$$

then the Peano-Baker Formula yields $\Phi(t, s) = e^{\int_s^t A(u)du}$.

4.5 Ważewski Inequality

Consider a time dependent linear system. Its asymptotic behaviour is characterised by the following

Theorem 4.5.1 (Ważewski) *For the system $\dot{x} = A(t)x(t)$ with initial state x_0, let $\bar{A}(t) = \frac{1}{2}(A(t) + A^T(t))$. Then, the norm of the state trajectory fulfils the following Ważewski Inequality*

$$e^{\int_0^t \underline{\lambda}_{\bar{A}}(s)ds}||x_0|| \leqslant ||x(t)|| \leqslant e^{\int_0^t \bar{\lambda}_{\bar{A}}(s)ds}||x_0||,$$

where, for a symmetric matrix M, $\underline{\lambda}_M$ and $\bar{\lambda}_M$ denote, respectively, the smallest and the biggest eigenvalue.

The Ważewski Inequality finds applications in the study of asymptotic stability of linear time dependent dynamic systems.

4.6 Proofs

4.6.1 Theorem on Existence and Uniqueness

Proof: We shall present a proof of this theorem, based on the Fixed Point Theorem. Let for a certain $\alpha > 0$ $C_n^0[0, \alpha]$ denote the space of continuous functions defined on the interval $[0, \alpha]$, with values in R^n. To simplify notation a continuous function belonging to $C_n^0[0, \alpha]$ will be denoted by x. The space $C_n^0[0, \alpha]$ appears to be a Banach space, with the norm

$$||x||_\infty = \sup_{0 \leqslant t \leqslant \alpha} ||x(t)||,$$

where $||x(t)||$ is the Euclidean norm in R^n. We pick a continuous function $x \in C_n^0[0, \alpha]$, and let $z(t) = x_0 + \int_0^t f(x(\tau), \tau)d\tau$. From the premises of the theorem 4.1.1 it follows that $z \in C_n^0[0, \alpha]$, as we can assume that the constant function x_0 belong to this space. Now, we take the ball $B(x_0, r)$, and define a subset $S \subset C_n^0[0,\alpha]$ as

$$S = \left\{ x \in C_n^0[0,\alpha] \,|\, \|x - x_0\|_\infty \leqslant r \right\}.$$

Consider a function

$$P : C_n^0[0,\alpha] \longrightarrow C_n^0[0,\alpha],$$

such that

$$(P(x))(t) = x_0 + \int_0^t f(x(\tau),\tau)d\tau.$$

We shall show that P is a contraction on the set S. To this objective, for two functions $x_1, x_2 \in C_n^0[0,\alpha]$ we compute

$$\|P(x_2) - P(x_1)\|_\infty = \sup_{0\leqslant t\leqslant\alpha} \left\| \int_0^t f(x_2(\tau),\tau)d\tau - \int_0^t f(x_1(\tau),\tau)d\tau \right\|$$
$$\leqslant \sup_{0\leqslant t\leqslant\alpha} \int_0^t \|f(x_2(\tau),\tau) - f(x_1(\tau),\tau)\| d\tau \leqslant L \sup_{0\leqslant t\leqslant\alpha} \int_0^t \|x_2(\tau) - x_1(\tau)\| d\tau,$$

where the last step uses the Lipschitz property. But we have $\|x_2(\tau) - x_1(\tau)\| \leqslant \sup_{0\leqslant t\leqslant\alpha} \|x_2(\tau) - x_1(\tau)\| = \|x_2 - x_1\|_\infty$. Continuing in this way we arrive at the conclusion that

$$\|P(x_2) - P(x_1)\|_\infty \sup_{0\leqslant t\leqslant\alpha} \int_0^t d\tau = L\alpha\|x_2 - x_1\|_\infty.$$

We see that, if only $\rho = L\alpha < 1$ then P is shrinking. Next, we need to check if P takes values in the set S, so if $P : S \longrightarrow S$. Let's choose a function $x \in S$. From the assumptions we deduce

$$\|P(x) - x_0\|_\infty = \sup_{0\leqslant t\leqslant\alpha} \left\| \int_0^t f(x(\tau),\tau)d\tau \right\| \leqslant \sup_{0\leqslant t\leqslant\alpha} \int_0^t \|f(x(\tau),\tau)\| d\tau$$
$$= \sup_{0\leqslant t\leqslant\alpha} \int_0^t \|f(x(\tau),\tau) - f(x_0,\tau) + f(x_0,\tau)\| d\tau$$
$$\leqslant \sup_{0\leqslant t\leqslant\alpha} \left(\int_0^t L\|x(\tau) - x_0\| d\tau + \int_0^t \|f(x_0,\tau)\| d\tau \right)$$
$$\leqslant Lr \sup_{0\leqslant t\leqslant\alpha} \int_0^t d\tau + M \sup_{0\leqslant t\leqslant\alpha} \int_0^t d\tau = (Lr + M)\alpha.$$

Finally, we get that P takes its values in the set S, on condition that $(Lr + M)\alpha \leqslant r$, what means that α should be sufficiently small $\alpha \leqslant \frac{r}{Lr+M}$. Having chosen $\alpha = \min\left\{\frac{\rho}{L}, \frac{r}{Lr+M}\right\}$ we can guarantee that P is a contraction on S. This being so, the theorem 3.5.2 implies that the function P has a fixed point, such that $P(x^*) = x^*$, therefore, for any $t \in [0, \alpha]$,

$$x^*(t) = x_0 + \int_0^t f(x^*(\tau), \tau)d\tau,$$

i.e.

$$\dot{x}^* = f(x^*(t), t), \; x^*(0) = x_0.$$

The theorem has been demonstrated. ■

4.6.2 Peano-Baker Formula

Below we sketch a scheme of deriving the Peano-Baker Formula. We look for a fundamental matrix $\Phi(t, s)$ that fulfils the identity

$$\frac{\partial\Phi(t, s)}{\partial t} = A(t)\Phi(t, s), \quad \text{with the initial condition} \quad \Phi(s, s) = I_n.$$

By integrating this identity from s to t we get

$$\Phi(t, s) = I_n + \int_s^t A(\sigma_1)\Phi(\sigma_1, s)d\sigma_1.$$

Analogously, we compute

$$\Phi(\sigma_1, s) = I_n + \int_s^{\sigma_1} A(\sigma_2)\Phi(\sigma_2, s)d\sigma_2,$$

which, after the substitution to the previous expression, results in

$$\Phi(t, s) = I_n + \int_s^t A(\sigma_1)d\sigma_1 + \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)\Phi(\sigma_2, s)d\sigma_2 d\sigma_1,$$

etc.

4.6.3 Ważewski Inequality

Proof: Suppose that $x(t)$ denotes the trajectory of the linear time-dependent system. We take the square of the norm $||x(t)||^2 = x^T(t)x(t)$, and differentiate it with respect to time

$$\begin{aligned}\frac{d||x(t)||^2}{dt} &= \dot{x}^T(t)x(t) + x^T(t)\dot{x}(t) \\ &= x^T(t)A^T(t)x(t) + x^T(t)A(t)x(t) = 2x^T(t)\bar{A}(t)x(t).\end{aligned}$$

To the last term on the right hand side we apply the Rayleigh-Ritz's inequality, that yields

$$\underline{\lambda}_{\bar{A}}(t)||x(t)||^2 \leqslant x^T(t)\bar{A}(t)x(t) \leqslant \bar{\lambda}_{\bar{A}}(t)||x(t)||^2.$$

In particular, from the right hand side of this inequality, we obtain

$$\frac{d||x(t)||^2}{dt} \leqslant 2\bar{\lambda}_{\bar{A}}(t)||x(t)||^2.$$

The integration of this inequality side-wise results in

$$\int_0^t \frac{d||x(s)||^2}{||x(s)||^2} = \ln\frac{||x(t)||^2}{||x_0||^2} \leqslant 2\int_0^t \bar{\lambda}_{\bar{A}}(s)ds,$$

that directly implies

$$||x(t)||^2 \leqslant ||x_0||^2 e^{2\int_0^t \bar{\lambda}_{\bar{A}}(s)ds}.$$

The above expression is equivalent to the right hand side part of the Ważewski Inequality. The left hand side part can be proved in the same way. ■

4.7 Problems and exercises

Exercise 4.1 Using the Fixed Point Theorem derive a sufficient condition for convergence of the following algorithm of solving a system of linear equations $x = Ax$. Algorithm:

$$x_{k+1} = Ax_k,$$

x_0 – starting point.

Exercise 4.2 Show that the fundamental matrix $\Phi(t, s)$ of the linear system $\dot{x} = A(t)x$ satisfies the equality

$$\frac{\partial\Phi^T(s,t)}{\partial t} = -A^T(t)\Phi^T(s,t).$$

Exercise 4.3 Check that for a constant matrix $A(t) = A$ the Peano-Baker Formula produces the matrix exponential $e^{tA} = \Phi(t, 0)$.

Exercise 4.4 Check that the matrix $M(t) = \int_0^t \Phi(t,s)B(s)B^T(s)\Phi^T(t,s)ds$ obeys the Lyapunov differential equation

$$\dot{M} = B(t)B^T(t) + A(t)M(t) + M(t)A^T(t).$$

Exercise 4.5 Prove the Bellmann-Gronwall Lemma. Hint: Notice that

$$\frac{\phi(t)}{a\int_0^t \phi(s)\psi(s)ds + b} \leqslant 1.$$

Exercise 4.6 Relying on the Ważewski Inequality verify the asymptotic stability of the following linear systems:

a)

$$\begin{cases} \dot{x} = -tx \\ \dot{y} = -y \end{cases},$$

b)

$$\begin{cases} \dot{x} = -x + \frac{2y}{1+t^2} \\ \dot{y} = -y \end{cases},$$

c)

$$\begin{cases} \dot{x} = -2x + 2y\sin t \\ \dot{y} = -2y \end{cases},$$

d)

$$\begin{cases} \dot{x} = -t^2x + y\cos t \\ \dot{y} = -t^2y - x\cos t \end{cases}.$$

4.8 Bibliographical remarks

Basic as well as more advanced knowledge on dynamic systems can be gained from the books [Har64, Arn78]. The proof of the Theorem on Existence and Uniqueness presented in this chapter relies on the monograph [Sas99], also the Bellman-Gronwall Lemma, and the Peano-Baker Formula can be found therein. The Ważewski Inequality comes from the paper [Waż48]. The Rayleigh-Ritz inequality (although without quoting its name) appears in chapter 8.4 of the monograph [Ber05].

Bibliography

[Arn78] V. I. Arnold. *Ordinary differential equations.* MIT Press, Cambridge, Mass., 1978.

[Ber05] D. S. Bernstein. *Matrix Mathematics.* Princeton University Press, Princeton, 2005.

[Har64] Ph. Hartman. *Ordinary Differential Equations.* J. Wiley, New York, 1964.

[Sas99] S. S. Sastry. *Nonlinear Systems.* Springer, New York, 1999.

[Waż48] T. Ważewski. Sur la limitation des intégrales des systémes d'equations différentielles linéaires ordinaires. *Studia Math.*, 10:48–59, 1948.

Chapter 5

Stability

We shall consider time-dependent dynamic systems, of the form

$$\dot{x} = f(x(t), t), \quad x(t_0) = x_0, \tag{5.1}$$

where $f : R^n \times R \longrightarrow R^n$ is a smooth function (C^∞) with respect to the variable x. Observe that from the smoothness, there follows easily the local Lipschitz property. For a proof, it suffices to invoke a vector form of the Hadamard's Lemma 3.3.1, namely, to notice that

$$\begin{aligned} f(x_2, t) - f(x_1, t) &= \int_0^1 df(sx_2 + (1-s)x_1, t)ds \\ &= \int_0^1 \frac{\partial f(sx_2 + (1-s)x_1, t)}{\partial x}(x_2 - x_1)ds = G(x_1, x_2, t)(x_2 - x_1), \end{aligned}$$

where $G(x_1, x_2, t) = \int_0^1 \frac{\partial f(sx_2+(1-s)x_1,t)}{\partial x}ds$. Having computed the norm we get

$$\|f(x_2, t) - f(x_1, t)\| \leqslant \|G(x_1, x_2, t)\| \, \|(x_2 - x_1)\|.$$

Now, since the norm $\|G(x_1, x_2, t)\|$ is a continuous function of its arguments, it is bounded over the compact set $B(x_0, r) \times [0, \alpha]$, i.e. $\|G(x_1, x_2, t)\| \leqslant L$, implying the Lipschitz property of $f(x, t)$. In this way we have established the local existence of the trajectory $x(t)$ of the system (5.1). In what follows we shall assume more, namely that $x(t)$ exists for every time instant t, so that (5.1) is a time dependent smooth dynamic system.

5.1 Stability, uniform stability, asymptotic stability

For a dynamic system we define the equilibrium point.

Definition 5.1.1 *The point $x_0 \in R^n$ is called the equilibrium point of the system (5.1), if for every $t \in R$*

$$f(x_0, t) = 0.$$

Obviously, a linear dynamic system $\dot{x} = A(t)x(t)$ has the equilibrium point $x_0 = 0$. Nevertheless, not every system has such a point, for example the system $\dot{x} = x + t$ has none.

Suppose that $x_0 = 0$ denotes an equilibrium point of the system (5.1). The behaviour of the system's trajectory in a neighbourhood of the equilibrium point is characterised by a property named stability. For time-dependent dynamic systems there exist several concepts of stability of the equilibrium point. They will be presented below. The symbol t_0 denotes the initial time instant.

Definition 5.1.2 *The equilibrium point $x_0 = 0$ of a time-dependent dynamic system is:*

- *stable (S) if*

$$(\forall t_0, \epsilon)(\exists \eta = \eta(t_0, \epsilon))(\forall t \geqslant t_0)(\|x(t_0)\| < \eta \Longrightarrow \|x(t)\| < \epsilon),$$

- *unstable (U) if it is not stable*

$$(\exists t_0, \epsilon)(\forall \eta = \eta(t_0, \epsilon))(\exists t \geqslant t_0)(\|x(t_0)\| < \eta \textit{ and } \|x(t)\| \geqslant \epsilon),$$

- *uniformly stable (US) if it is stable, and η does not depend on t_0, i.e. $\eta = \eta(\epsilon)$,*

- *asymptotically stable (AS) if it is stable, and there exists a number $c = c(t_0)$, such that for $\|x(t_0)\| < c$ the trajectory $x(t) \to x_0$,*

- *uniformly asymptotically stable (UAS) if it is asymptotically stable, and there exists a number c, independent of t_0, such that for $\|x(t_0)\| < c$ the trajectory $x(t)$ approaches x_0 in asymptotically in the uniform way, i.e. there holds that*

$$(\forall \eta > 0)(\exists T = T(\eta))(\forall t \geqslant t_0 + T(\eta))(\|x(t)\| < \eta),$$

- *globally uniformly asymptotically stable (GUAS) if it is UAS and $c = +\infty$.*

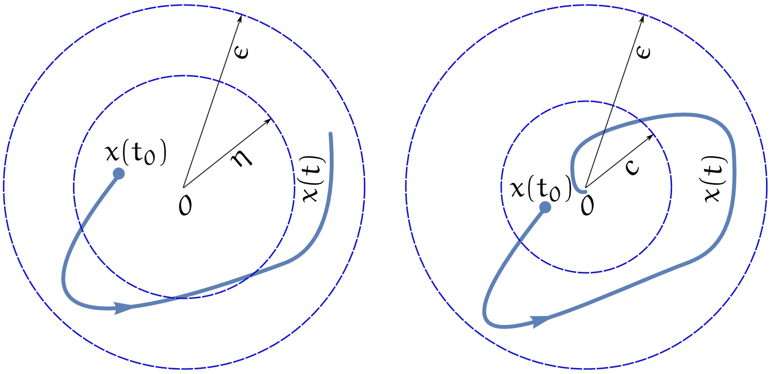

Figure 5.1: Stability and asymptotic stability

The idea of the stability and asymptotic stability is presented in Figure 5.1. For a very simple dynamic system the stability can be deduced directly on the basis of its definition. The following example may serve as an illustration

$$\dot{x} = -\frac{x(t)}{1+t}, \quad x \in R, \quad t > -1.$$

The system's trajectory takes the form $x(t) = \frac{x(t_0)(1+t_0)}{1+t}$, and the point $x_0 = 0$ represents an equilibrium. Now, for the reason that $|x(t)| < |x(t_0)|$, it is enough to pick $\eta = \epsilon$, to discover that the equilibrium point is stable. This point is also asymptotically stable, but it uniformly, because for a given η the requirement that $|x(t_0+T)| = \frac{x(t_0)||(1+t_0)|}{1+t_0+T} < \eta$ leads to the conclusion $T > \frac{|x(t_0)||(1+t_0)|}{\eta} - 1 - t_0 = T(\eta, t_0)$, that means that T depends on the initial time instant t_0.

5.2 Class K and K_∞ functions

Except for some trivial cases, usually the stability of an equilibrium point cannot be decided directly from the definition. Instead, we need some indirect methods. In their statement we shall use so called (comparison) functions of class K, defined in the following way

Definition 5.2.1 *A continuous function $\alpha : [0, a] \longrightarrow R_+$, $a > 0$, is called a class K function if α is strictly increasing and $\alpha(0) = 0$. A function α is a class K_∞ function if $a = +\infty$ and when $r \to +\infty$, the function $\alpha(r) \to +\infty$.*

An example of a class K function, that is not a class K_∞ function is $\alpha(r) = \arctan r$. In contrast, the function $\alpha(r) = r^n$, $n \geqslant 1$, is simultaneously the class K as well as the class K_∞ function.

5.3 Lyapunov Function Theorem

Exploiting the functions of class K one can state the following sufficient conditions of stability.

Theorem 5.3.1 (Lyapunov Function Theorem) *Let a dynamic system (5.1) be given, with an equilibrium point $x_0 = 0$. Suppose that in a region $D \subset R^n$ containing x_0 there exists a C^1 function $V : D \times R \longrightarrow R$. Compute the derivative of V along the trajectory of the dynamic system,*

$$\dot{V}(x,t) = \frac{\partial V(x,t)}{\partial t} + \left(\frac{\partial V(x,t)}{\partial x}\right)^T f(x,t).$$

Then if

- *there exist class K functions α_1, α_2, such that*
$$\alpha_1(||x||) \leqslant V(x,t) \leqslant \alpha_2(||x||) \quad \textit{and} \quad \dot{V}(x,t) \leqslant 0$$
then the point x_0 is uniformly stable,
- *there exist class K functions α_1, α_2 and α_3, such that*
$$\alpha_1(||x||) \leqslant V(x,t) \leqslant \alpha_2(||x||) \quad \textit{and} \quad \dot{V}(x,t) \leqslant -\alpha_3(||x||)$$
then the point x_0 is uniformly asymptotically stable,
- *the condition from the previous item holds for three class K_∞ functions α_1, α_2 and α_3, and $D = R^n$ then the point x_0 is globally uniformly asymptotically stable.*

The function V is called a Lyapunov function, and a stable equilibrium point is often named Lyapunov stable. For an illustration of the Lyapunov Function Theorem consider a time-dependent dynamic system

$$\dot{x} = -\left(1+t^2\right)x^3, \quad x \in R.$$

It is easily seen that the point $x_0 = 0$ is an equilibrium point of this system. We choose $V(x,t) = \frac{1}{2}x^2$. Clearly, for $\alpha_1(r) = \alpha_2(r) = \frac{1}{2}r^2$ it holds that $\alpha_1(|x|) \leqslant V(x,t) \leqslant \alpha_2(|x|)$. The derivative $\dot{V}(x,t) = 1(1+t^2)x^4 \leqslant -x^4$, so for $\alpha_3(r) = r^4$ we have $\dot{V}(x,t) \leqslant -\alpha_3(|x|)$. Since all the functions α_i are class K_∞ functions and $D = R$, Theorem 5.3.1 yields the global uniform asymptotic stability of the point x_0.

5.4 Barbalat's Lemma

In the study of stability of time-dependent systems, besides the Lyapunov Function Theorem, one often uses another result called the Barbalat's Lemma. In order to introduce this lemma, we shall first try to answer the following questions concerning the real functions:

- Suppose that a smooth function $f(t)$ has a limit at $t \to +\infty$. Is it true that $\dot{f}(t) \to 0$?
- Now, let $\dot{f}(t) \to 0$. Does it result in the existence of a limit of function $f(t)$ at $t \to +\infty$?

An answer to both these questions is negative, what can be learnt from the following counter examples: $f(t) = e^{-t} \sin e^{2t}$ and $f(t) = \sin \ln t$. Apparently, in order to answer in positive we need to make an additional assumption about the function $f(t)$. It is included in the following

Theorem 5.4.1 (Barbalat) *Let a function $f \in C^2(R, R)$ be given. If this function has a limit for $t \to +\infty$ and the second order derivative of f is bounded, $|\ddot{f}(t)| \leqslant M$, then $\dot{f}(t) \to 0$.*

The Barbalat's Lemma is often applied to a Lyapunov function in order to either prove the asymptotic stability of a system or to get extra information on the convergence of the system's trajectory. For illustration we shall examine the dynamic system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \sin t \\ \dot{x}_2 = -x_1 \sin t \end{cases} .$$

The point $0 \in R^2$ is an equilibrium point of this system. We choose a function $V(x, t) = x_1^2 + x_2^2$ and compute $\dot{V}(x, t) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2x_1^2$. From the Lyapunov Function Theorem we deduce that the equilibrium point is stable (notice that the function x_1^2 is not a class K function of the norm $||x||$). Could we show more than that? To this objective let's observe that along the trajectory $x(t)$ of the system the function $W(t) = x_1^2(t) + x_2^2(t) \geqslant 0$, while the function $\dot{W}(t) = -2x_1^2(t) \leqslant 0$, what means that $W(t)$ is decreasing (non-increasing) and lower-bounded. This yields the existence of a limit of $W(t)$ at $t \to +\infty$. Compute the second order derivative $\ddot{W}(t) = -4x_1^2\dot{x}_1 = 4x_1^2 - 4x_1x_2 \sin t \leqslant 4x_1^2 + 4|x_1||x_2|$. Since the function $W(t)$ is non-increasing, $W(t) \leqslant W(0)$, it follows that the trajectory

$x(t) = (x_1(t), x_2(t))$ is bounded, ans so is $\ddot{W}(t)$. Finally, from the Barbalat's Lemma we deduce that $\dot{W}(t) = -x_1^2(t) \to 0$, i.e. $x_1(t) \to 0$. In this way we have demonstrated that, besides the (Lyapunov) stability, one of coordinates of the system converges asymptotically to 0.

5.5 Convergence estimation

In the course of analysis of the system's stability based on the Lyapunov Function Theorem, it may happen that the derivative of the Lyapunov function along the trajectory is dependent on the function itself. This situation is very advantageous due to a possibility of estimating the speed of convergence of the trajectory to the equilibrium point. In order to better explain this kind of reasoning we shall consider the system $\dot{x} = -(1+t^2)x^3$. This system has the equilibrium point $0 \in R$. Take the function $V(x,t) = V(x) = \frac{1}{2}x^2$ and let $W(t) = V(x(t))$. It is easily seen that $\dot{W}(t) \leqslant -x^4(t) = -4W^2(t)$. By integration of this inequality sidewise we obtain $W(t) \leqslant \frac{1}{\frac{1}{W(0)}+4t}$, that implies that $|x(t)| \leqslant \left(\frac{2}{\frac{1}{V(0)}+4t}\right)^{1/2}$. As we can see, the system's trajectory approaches 0 with a guaranteed speed of order $t^{-1/2}$. Notice that this estimate is useful, but perhaps not very accurate, as actually the trajectory $x(t) = \frac{1}{\sqrt{x_0^{-2}+2t+\frac{2}{3}t^3}}$ tends to zero quicker, namely as the function $t^{-3/2}$.

5.6 Problems and exercises

Exercise 5.1 Check stability of the following systems:

a)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & e^{\frac{1}{2}t} \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

b)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

c)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

d)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 2\sin t \\ 0 & -(t+1) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Exercise 5.2 Show that the equilibrium point $(0,0)^T$ of the system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + \left(x_1^2 + x_2^2\right)\sin t \\ \dot{x}_2 = -x_1 - x_2 + \left(x_1^2 + x_2^2\right)\cos t \end{cases}$$

is exponentially stable, and define its stability region. Hint: Use $V(x) = x_1^2 + x_2^2$.

Exercise 5.3 Examine stability of the point $(0,0)^T$ of the system

$$\begin{cases} \dot{x}_1 = h(t) - g(t)x_1^3 \\ \dot{x}_1 = -h(t) - g(t)x_2^3 \end{cases},$$

where $g(t)$, $h(t)$ are smooth and upper-bounded, moreover $g(t) \geqslant k > 0$. Hint: Take $V(x) = x_1^2 + x_2^2$.

Exercise 5.4 Let $V(x)$ denote a smooth potential function. Show that the gradient system

$$\dot{x} = -\frac{\partial V(x)}{\partial x} = -DV(x)$$

has no closed orbits.

Exercise 5.5 Let $H(x,y)$, $x, y \in R^n$ denote a smooth Hamilton's function. Prove that the Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial H(x,y)}{\partial y} \\ \dot{y} = -\dfrac{\partial H(x,y)}{\partial x} \end{cases}$$

does not have any asymptotically stable equilibrium point.

5.7 Bibliographical remarks

The exposition of stability theory presented in this chapter relies on the monograph [Kha00]. Also, the chapter 5 of the book [Sas99] is devoted to stability. The concept of the Hamiltonian system and other mechanical analytic concepts can be found in [Arn78].

Bibliography

[Arn78] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, Berlin, 1978.

[Kha00] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, New Jersey, 2000.

[Sas99] S. S. Sastry. *Nonlinear Systems*. Springer, New York, 1999.

Chapter 6

Time-independent dynamic systems

In this chapter we shall be dealing with systems of ordinary differential equations, of the form

$$\dot{x} = f(x(t)), \quad x(0) = x_0, \tag{6.1}$$

where $f \in C^\infty(R^n, R^n)$. Notice that the right hand side of the system (6.1) does not depend on time, therefore the conditions of the Theorem on Existence and Uniqueness reduce to the Lipschitz condition that for the smooth function $f(x)$ is satisfied automatically.

6.1 System's flow

In consequence of what has been said above, the system (6.1) has a solution

$$x(t) = \varphi(t, x), \quad x(0) = x, \quad \frac{d\varphi(t, x)}{dt} = f(\varphi(t, x))$$

defined on a time interval containing the initial time instant. If $\varphi(x, t)$ is defined for every initial state $x \in R^n$ and every time $t \in R$ then (6.1) will be called a time-independent (autonomous) dynamic system, or, simply, a dynamic system. With reference to a dynamic system the function $\varphi(x, t)$ is named the system's flow. The flow depends smoothly on time as well as on the state. It determines a state of the system at time t if its state at time 0 has been x. In order to distinguish from each other the variables x and t we often use the notation $\varphi(x, t) = \varphi_t(x)$. The system's flow has the following properties:

- $\varphi_0(x) = x$ (identity property),

- $\varphi_t \circ \varphi_s(x) = \varphi_{t+s}(x) = \varphi_{s+t}(x) = \varphi_s \circ \varphi_t(x)$ (semigroup property).

Using the above properties, for $s = -t$, we get $\varphi_t \circ \varphi_{-t}(x) = \varphi_{t-t}(x) = \varphi_0(t) = x$, what yields $(\varphi_t)^{-1} = \varphi_{-t}$. Consequently, $\{\varphi_t | t \in R\}$ is a (1-parameter) family of diffeomorphisms of the state space R^n.

Geometrically, the function $f(x)$ appearing on the right hand side of the system (6.1) can be interpreted as a vector field that to every point $x \in R^n$ assigns a direction of motion at this point, such that at any point the system's trajectory must be tangent to the vector defined by the vector field. This being so, it follows that the integration of a differential equation is tantamount to inscribing into the state space curves tangent to the directions defined by a vector field.

Having fixed in the flow the state x and let t change we arrive to the concept of orbit of the dynamic system.

Definition 6.1.1 *The set*

$$\mathcal{O}(x) = \{\varphi_t(x) | t \in R\}$$

is called the orbit of the system, passing through the point x.

Interestingly, there exist only three types of orbits of a dynamic system.

- $\mathcal{O} = \{x\}$ equilibrium point $(\forall t \in R)(\varphi_t(x) = x)$,
- $\mathcal{O} = \{x \in R^n | (\exists t > 0)(\varphi_t(x) = x)\} \cong S^1$ closed orbit,
- $\mathcal{O} \cong R$ open orbit.

Above, the symbol $\cong$ denotes an isomorphism; it can be read out as "looks like". S^1 stands for the unit circle. All three types of orbits can be discovered in the phase portrait of the mathematical pendulum $\ddot{q} = -\sin q$ presented in Figure 6.1. Finally, let us notice that by the Theorem of Existence and Uniqueness the condition $(\exists t > 0)(\varphi_t(x) = x)$ indeed defines a closed orbit. The minimum $T > 0$, such that $\varphi_T = x$ is called the period of the closed orbit. Obviously, $\varphi_{t+T}(x) = \varphi_t \circ \varphi_T(x) = \varphi_t(x)$.

6.2 Equivalence of dynamic systems

Similarly as for functions now we shall introduce a concept of equivalence of dynamic systems. Suppose that two dynamic systems are given, of the form

$$\sigma : \dot{x} = f(x(t)) \quad \text{and} \quad \sigma' : \dot{\xi} = F(\xi(t)), \quad x,\ \xi \in R^n,$$

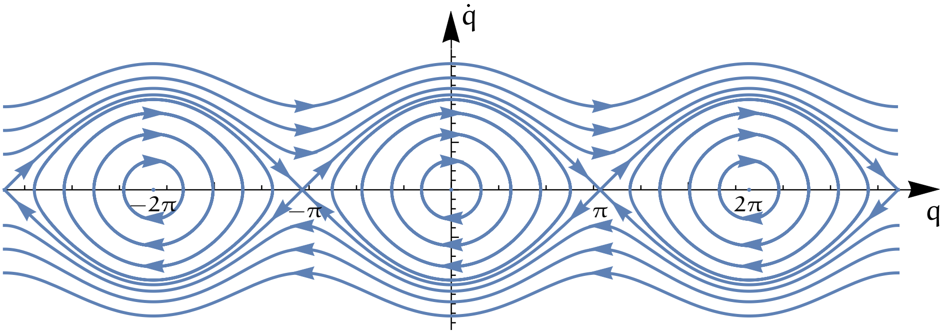

Figure 6.1: Orbits of mathematical pendulum

with flows equal to, respectively, $\varphi_t(x)$ and $\Phi_t(\xi)$. Then, two equivalences can be defined:

Definition 6.2.1 • *Topological equivalence*

$$\sigma \underset{TE}{\cong} \sigma' \iff (\exists \textit{ homeomorphism } \xi = \psi(x))(\psi \circ \varphi_t(x) = \Phi_t \circ \psi(x)).$$

- *Differential equivalence*

$$\sigma \underset{DE}{\cong} \sigma' \iff (\exists \textit{ diffeomorphism } \xi = \psi(x))(\psi \circ \varphi_t(x) = \Phi_t \circ \psi(x)).$$

The concept of diffeomorphism has been introduced in chapter 2. Differently to the diffeomorphism that needs to be continuously differentiable, and have a continuous inverse, the homeomorphism needs to be continuous, invertible, and have a continuous inverse. If the function ψ is defined only locally, we speak of a local equivalence (topological, differentiable), in short LTE and LDE. The essential meaning of the equivalence of dynamic systems is revealed in Figure 6.2.

6.3 Theorem on Differential Equivalence

As follows form the definition, checking both types of equivalences of dynamic systems requires that the systems' flows are known, i.e. that the systems' differential equations have been solved. In most cases this is not possible, so it would be advantageous to have a test of the equivalence that does not involves the flows. It turns out that such a test exists for the Differential Equivalence. In this context the following results is true.

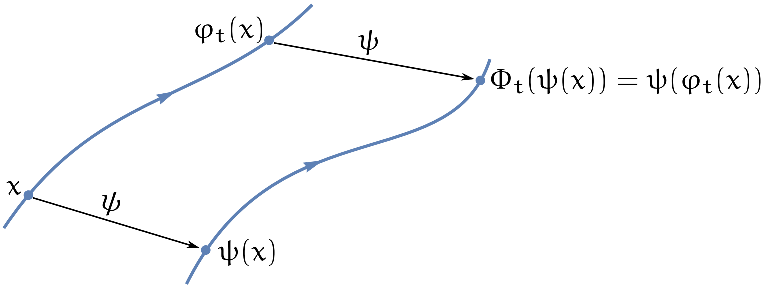

Figure 6.2: Equivalence of dynamic systems

Theorem 6.3.1 (On Differential Equivalence)

$$\sigma \underset{DE}{\cong} \sigma' \iff (\exists \textit{ diffeomorphism } \xi = \psi(x))(D\psi(x)f(x) = F(\psi(x)).$$

6.4 Straightening Out Theorem

For a dynamic system (6.1) the point x_0, at which $f(x_0) = 0$, will be named a singular point or an equilibrium point of this system. In case when $f(x_0) \neq 0$ the point x_0 is referred to as a regular point. The next Theorem on Straightening Out (a vector field) characterises the behaviour of a dynamic system in a neighbourhood of the regular point.

Theorem 6.4.1 (Straightening Out Theorem) *Let* $f(0) \neq 0$. *Then*

$$\sigma \underset{LDE}{\cong} \sigma',$$

for σ' *such that the vector field* $F(\xi) = e_1 = (1, 0, \ldots, 0)^T \in R^n$. *In other words, the system* σ' *assumes the form*

$$\begin{cases} \dot{\xi}_1 = 1 \\ \dot{\xi}_2 = 0 \\ \quad \vdots \\ \dot{\xi}_n = 0 \end{cases},$$

while its flow

$$\Phi_t(\xi) = \xi + te_1.$$

The name and the meaning of this theorem is explained in Figure 6.3. The Straightening Out Theorem implies that, similarly as for functions, the behaviour of dynamic systems around regular (non-singular) points is not very

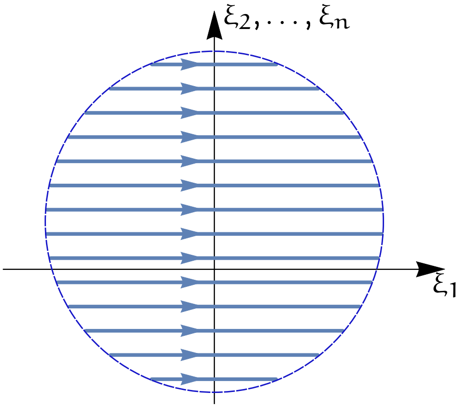

Figure 6.3: Straightening out a vector field

exciting. In a search for more interesting behaviours below we shall focus our attention on the equilibrium (singular) points.

6.5 Equilibrium points

Assume that $x_0 \in R^n$ denotes an equilibrium point of the system (6.1). We take the Taylor series of the vector field $f(x)$ in a neighbourhood of this equilibrium point

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(x) = Df(x_0)(x - x_0) + O(x^2).$$

The matrix $A = Df(x_0)$ is called the matrix of the linear approximation of the system at the point x_0. Further on we shall distinguish two kinds of the equilibrium points.

Definition 6.5.1 *An equilibrium point x_0 is called a hyperbolic, if eigenvalues of the matrix A have non-zero real parts. The point x_0 is named resonant, if eigenvalues λ_i of the matrix A obey the following dependencies: $\lambda_i = \sum_j m_{ij}\lambda_j$ for certain integers $m_{ij} \geqslant 0$, such that $\sum_j m_{ij} \geqslant 2$. The equilibrium point x_0 is referred to as non-resonant if it is not resonant.*

For illustration of the concept of resonant point, let us look at a simple oscillator described as

$$\begin{cases} \dot{x}_1 = \omega x_2 \\ \dot{x}_2 = -\omega x_1 \end{cases}.$$

It is easily checked that its orbits satisfy the identity $x_1^2 + x_2^2 = C$. The equilibrium point $x_0 = 0$ is resonant because the matrix $A = \left[\begin{smallmatrix} 0 & \omega \\ -\omega & 0 \end{smallmatrix}\right]$ of the linear approximation of the oscillator has eigenvalues $\pm i\omega$, therefore $\lambda_1 + \lambda_2 = 0$. It yields $2\lambda_1 + \lambda_2 = \lambda_1$, as required by the definition of the resonant point. An association with the oscillator explains the origin of the name "resonant". Notice that the resonance condition can also hold, when the eigenvalues are real, and sum up to 0. This means that an equilibrium point can simultaneously be resonant and hyperbolic, as in the system $\dot{x}_1 = \alpha x_2$, $\dot{x}_2 = \alpha x_1$.

6.6 Linearisation of dynamic systems

The behaviour of dynamic system in a neighbourhood of a non-resonant equilibrium point is described by the following

Theorem 6.6.1 (Poincaré-Siegel-Sternberg) *Let* $x_0 = 0$ *denote a non-resonant equilibrium point of the dynamic systems* $\sigma : \dot{x} = f(x(t))$. *Then*

$$\sigma \underset{LDE}{\cong} \sigma',$$

where σ'*:* $\dot{\xi} = A\xi$ *and* $A = Df(0)$.

This theorem asserts that around a non-resonant point the dynamic system behaves as its linear approximation at this point. For the hyperbolic point an analogous result is true for the topological equivalence, as stated in

Theorem 6.6.2 (Hartman-Grobman) *Suppose that* $x_0 = 0$ *is a hyperbolic equilibrium point of the dynamic system* $\sigma : \dot{x} = f(x(t))$. *Then*

$$\sigma \underset{LTE}{\cong} \sigma',$$

where σ'*:* $\dot{\xi} = A\xi$ *and* $A = Df(0)$.

6.7 Equivalence of linear systems

In this section we shall consider linear dynamic systems. Let two such systems be given,

$$\sigma : \ \dot{x} = Ax(t) \quad \text{and} \quad \sigma' : \ \dot{\xi} = F\xi(t), \quad x, \xi \in R^n, \ A, F \text{ matrices.}$$

The Differential Equivalence of linear systems means that

$$\sigma \underset{DE}{\cong} \sigma' \iff (\exists\, P - \text{non-singular matrix})\ (PA = FP).$$

It is not hard to observe that the eigenvalues of the matrices A and F are invariants of this equivalence, therefore the equivalent linear systems have matrices with the same eigenvalues. This property means that the corresponding equivalence classes must be very "small", and that there are infinitely many of them; for example two equivalent (identical) diagonal matrices A and F will no longer be equivalent after an arbitrary small perturbation of any of them. For this reason the Differential Equivalence is not a very useful tool for classification of linear systems. If, instead, we use the Topological Equivalence then the following result can be proved.

Theorem 6.7.1 (Kuiper) *Assume that the linear system*

$$\sigma : \ \dot{x} = Ax(t)$$

has a hyperbolic equilibrium point $x_0 = 0$. *Then,*

$$\sigma \underset{TE}{\cong} \sigma'_k, \quad k = 0, 1, \ldots, n,$$

where

$$\sigma'_k : \begin{cases} \dot{\xi}_1 = -\xi_1(t) \\ \quad \vdots \\ \dot{\xi}_k = -\xi_k(t) \\ \dot{\xi}_{k+1} = \xi_{k+1}(t) \\ \quad \vdots \\ \dot{\xi}_n = \xi_n(t) \end{cases} . \tag{6.2}$$

The integer invariant k denotes the number of eigenvalues of the matrix A with negative real parts. In the case of planar systems ($n = 2$), one deduces from (6.2) that there exist three kinds of hyperbolic equilibrium points: the sink points, the source points, and the saddle points. All of them are shown in Figure 6.4

6.8 Classification of dynamic systems: a summary

We have shown that the Differential Equivalence is an efficient tool for describing the behaviour of a dynamic system around the regular points or

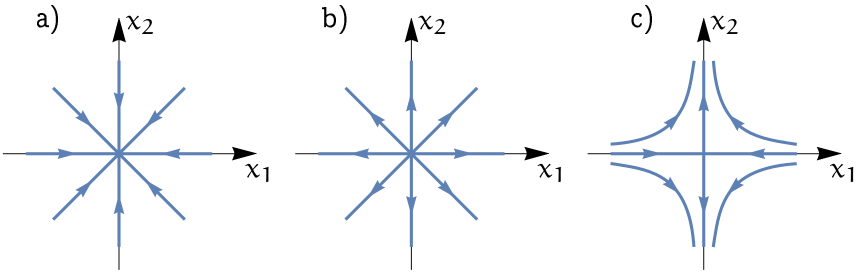

Figure 6.4: Hyperbolic equilibrium points in the plane: a) sink, b) source, c) saddle

the non-resonant singular points. Complementarily, the Topological Equivalence allows to identify the dynamic system with its linear approximation at a hyperbolic equilibrium point. Additional advantages of the Topological Equivalence become visible after combining Theorems 6.6.2 and 6.7.1. In this way we arrive at the following finite classification of dynamic systems.

Theorem 6.8.1 *In a neighbourhood of a hyperbolic equilibrium point a dynamic system is locally topologically equivalent to one from among* $(n+1)$ *normal forms described by the formula (6.2).*

Theorem 6.6.2 also yields that on the basis of the linear approximation at an equilibrium point one can reason about the local stability of this point. This is the essence of so called First Method of Lyapunov of checking stability. In particular, an equilibrium point of the system σ is locally asymptotically stable if the corresponding normal form is σ'_n, and unstable for the remaining normal forms.

6.9 Proofs

6.9.1 Theorem on Differential Equivalence

Proof: We recall that flows of the dynamic systems σ and σ' satisfy the equations

$$\frac{d\varphi_t(x)}{dt} = f(\varphi_t(x)) \quad \text{and} \quad \frac{d\Phi_t(x)}{dt} = F(\Phi_t(x)).$$

- Necessary condition: Suppose that $\psi \circ \varphi_t(x) = \Phi_t \circ \psi(x)$. Since the diffeomorphism ψ is differentiable, we compute the time-derivative of both sides and obtain

$$\frac{d\psi \circ \varphi_t(x)}{dt} = D\psi(\varphi_t(x))\frac{d\varphi_t(x)}{dt} = D\psi(\varphi_t(x))f(\varphi_t(x))$$

and also
$$\frac{d\Phi_t \circ \psi(x)}{dt} = F(\Phi_t(\psi(x)).$$
Having substituted $t = 0$, we deduce from the above identities that
$$D\psi(x)f(x) = F(\psi(x)).$$

- Sufficient condition: We assume that $D\psi(x)f(x) = F(\psi(x))$. Because x is arbitrary, we replace x by the flow $\varphi_t(x)$ that leads to the formula $D\psi(\varphi_t(x))f(\varphi_t(x)) = F(\psi(\varphi_t(x))$. Now, observe that the left hand side of this identity equals $\frac{d\psi\circ\varphi_t(x)}{dt}$, therefore
$$\frac{d\psi \circ \varphi_t(x)}{dt} = F(\psi(\varphi_t(x)).$$
On the other hand, from definition of the system's flow of σ' it follows that
$$\frac{d\Phi_t \circ \psi(x)}{dt} = F(\Phi_t \circ \psi(x)).$$
We have concluded that the functions $\psi \circ \varphi_t(x))$ and $\Phi_t \circ \psi(x)$ satisfy the same differential equation, of the form
$$\frac{d\alpha}{dt} = X(\alpha(t)),$$
with the same initial condition $\Phi_0 \circ \psi(x) = \psi(x)$ and $\psi \circ \varphi_0(x) = \psi(x)$. Finally, from the Theorem on Existence and Uniqueness these solutions coincide,
$$\psi \circ \varphi_t(x) = \Phi_t \circ \psi(x).$$

■

6.9.2 Straightening Out Theorem

Proof: Instead of the diffeomorphism $\xi = \psi(x)$, such that $D\psi(x)f(x) = F(\psi(x))$ we shall device the inverse diffeomorphism $x = \alpha(\xi)$ satisfying the condition $D\alpha(\xi)F(\xi) = f(\alpha(\xi))$. Having assumed $f(0) \neq 0$, perhaps by re-ordering coordinates, we can get $f_1(0) \neq 0$. Under this assumption, using the flow of the system σ, we define
$$\alpha(\xi) = \varphi_{\xi_1}(0, \xi_2, \dots, \xi_n),$$

By the properties of the flow we have that α is smooth and $\alpha(0) = 0$. The derivative

$$D\alpha(\xi) = \left[\frac{\partial \alpha}{\partial \xi_1}, \dots, \frac{\partial \alpha}{\partial \xi_n}\right](\xi)$$
$$= \left[\frac{\partial \varphi_{\xi_1}(0, \xi_2, \dots, \xi_n)}{\partial \xi_1}, \dots, \frac{\partial \varphi_{\xi_1}(0, \xi_2, \dots, \xi_n)}{\partial \xi_n}\right].$$

Now, for $\xi = 0$ we get

$$D\alpha(0) = [f(0), e_2, \dots, e_n],$$

where e_i denotes the i-th basis vector in R^n. Due to the fact that $f_1(0) \neq 0$, the matrix $D\alpha(0)$ has rank n, so, by virtue of the Inverse Function Theorem, in a neighbourhood of 0 the function α is a diffeomorphism. In order to check the equivalence condition, we compute

$$D\alpha(\xi)F(\xi) = D\alpha(\xi)e_1 = \frac{\partial \varphi_{\xi_1}(0, \xi_2, \dots, \xi_n)}{\partial \xi_1} = f(\alpha(\xi)),$$

that finishes the proof. ■

6.10 Problems and exercises

Exercise 6.1 Show that the dynamic system

$$\begin{cases} \dot{x} = -\lambda y + xy \\ \dot{y} = \lambda x + \frac{1}{2}\left(x^2 - y^2\right) \end{cases},$$

$x, y \in R$, $\lambda > 0$, is Hamiltonian. Define its Hamilton's function and draw a phase portrait.

Exercise 6.2 Demonstrate that the dynamic system

$$\begin{cases} \dot{x} = x^2 - y^3 \\ \dot{y} = 2x\left(x^2 - y\right) \end{cases},$$

$x, y \in R$, has the first integral.

Exercise 6.3 Find the first integral of the dynamic system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - 2x^3 \end{cases},$$

$x, y \in R$, and draw its phase portrait.

Exercise 6.4 Find the first integral and draw a phase portrait of the Lotka-Volterra's system

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + bxy \end{cases},$$

$x, y \in R,\ a, b, c > 0$.

Exercise 6.5 Examine stability of the point $(0,0)^T$ of the system

$$\begin{cases} \dot{x} = -y + x\left(x^2 + y^2\right) \sin\sqrt{x^2 + y^2} \\ \dot{y} = x + y\left(x^2 + y^2\right) \sin\sqrt{x^2 + y^2} \end{cases},$$

$x, y \in R$. Hint: Introduce the polar coordinates.

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}.$$

6.11 Bibliographic remarks

Among most recommended texts on dynamic systems there are two monographs by Arnold, [Arn78, Arn83]. The concepts of equivalence of dynamic systems as well the classification theorems included in this chapter come from [Arn83]; they are also the subject of chapter 3 of the book [Lév09]. Topological equivalence of linear systems is dealt with in [Kui75]. The Reader interested in invariant manifolds and bifurcation theory may like the chapter 7 of the monograph [Sas99].

Bibliography

[Arn78] V. I. Arnold. *Ordinary differential equations.* MIT Press, Cambridge, Mass., 1978.

[Arn83] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer, New York, 1983.

[Kui75] N. H. Kuiper. The topology of the solutions of a linear differential equation on R^n. In *Manifolds–Tokyo 1973*, pages 195–203. Tokyo University Press, 1975.

[Lév09] J. Lévine. *Analysis and Control of Nonlinear Systems.* Springer, 2009.

[Sas99] S. S. Sastry. *Nonlinear Systems.* Springer, New York, 1999.

Chapter 7

Frobenius Theorem

7.1 Vector fields, big adjoint operator

In this chapter we shall be busy with a pair of dynamic systems

$$\dot{x} = X(x(t)) \quad \text{and} \quad \dot{y} = Y(y(t)), \quad x, y \in R^n,$$

defined by smooth vector fields $X, Y \in C^\infty(R^n, R^n)$, with flows $\varphi_t(x)$ and $\Phi_t(y)$. By definition, these flows obey the identities

$$\frac{d\varphi_t(x)}{dt} = X(\varphi_t(x)) \quad \text{and} \quad \frac{d\Phi_t(y)}{dt} = Y(\Phi_t(y)).$$

Consider the following action of the vector field X on the vector field Y. Beginning from an initial state x we follow for the time t the trajectory of the vector field X, up to the point $\varphi_t(x)$. Next, at the point $\varphi_t(x)$ we take the vector field Y and move it for the time t along the trajectory of the vector field X, but in the opposite direction. The vector at the point x obtained in this way defines the big adjoint operator

$$\mathrm{Ad}_X^t Y(x) = D\varphi_{-t}(\varphi_t(x))Y(\varphi_t(x)) = (D\varphi_{-t}Y)(\varphi_t(x)).$$

The definition of the operator $\mathrm{Ad}_X^t Y(x)$ is illustrated in Figure 7.1. Observe that for a fixed point x and varying t $\mathrm{Ad}_X^t Y(x)$ becomes a curve in the space R^n. The derivative of this curve determines the Lie bracket of vector fields X and Y,

$$[X, Y](x) = \mathrm{ad}_X Y(x) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_X^t Y(x).$$

We propose to use the name "the big adjoint operator" in order to distinguish Ad from "the small adjoint operator" ad that has been defined above. The following properties of the operator Ad can be derived:

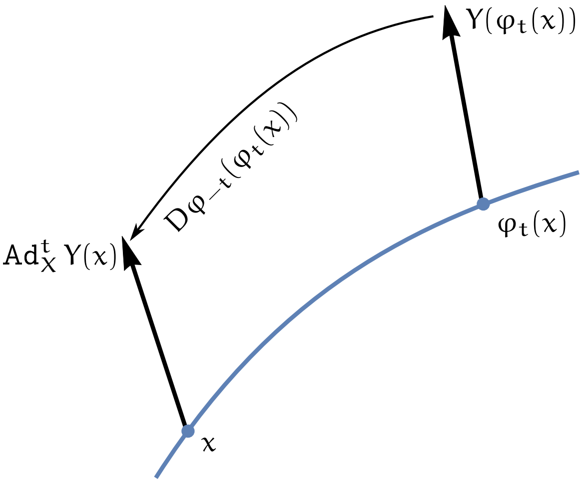

Figure 7.1: Definition of operator $\mathrm{Ad}_X^t Y(x)$

- $\mathrm{Ad}_X^0 Y(x) = Y(x)$,
- $\mathrm{Ad}_X^{t+s} Y(x) = \mathrm{Ad}_X^t \mathrm{Ad}_X^s Y(x)$,
- $\frac{d}{dt} \mathrm{Ad}_X^t Y(x) = \mathrm{Ad}_X^t [X, Y](x)$.

Notice that the last formula implies that if two vector fields X and Y commute, i.e. their Lie bracket $[X, Y](x) = 0$ then $\mathrm{Ad}_X^t Y(x) = \mathrm{Ad}_X^0 Y(x) = Y(x)$.

7.2 Lie bracket

One can show that the definition of the Lie bracket introduced in the previous section coincides with a more classic definition, stated using coordinates, namely

$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

The following properties of the Lie bracket are consequences of the definition:

- $[X, X](x) = 0$ – irreflexivity,
- $[Y, X](x) = -[X, Y](x)$ – antisymmetry,
- $[[X, Y], Z](x) + [[Y, Z], X](x) + [[Z, X], Y](x) = 0$ – Jacobi identity.
- for two numbers $\alpha, \beta \in R$ $[\alpha X + \beta Y, Z](x) = \alpha[X, Z](x) + \beta[Y, Z]$ – bilinearity.

It is easily observed that the Lie bracket assigns to a pair of vector fields another vector field, so it may be regarded as a sort of product of vector fields, resembling the cross product of vectors in R^3. In this context the Jacobi identity results in the non-associativity of the Lie bracket (similarly to the non-associativity of the cross product). This is because

$$[[X, Y], Z] = [X, [Y, Z]] - [Z, [X, Y]] \neq [X, [Y, Z]].$$

As has been mentioned in subsection 1.2, smooth vector fields $V(R^n)$ form a linear space over R and, together with the Lie bracket, constitute a Lie algebra. Moreover, the vector fields also form a module over the smooth functions $C^\infty(R^n, R)$, therefore, for any $f, g \in C^\infty(R^n, R)$ the bracket $[fX, gY] \in V^\infty(R^n)$. The computation of this bracket yields

$$[fX, gY](x) = f(x)g(x)[X, Y](x) + f(x)L_X g(x)Y(x) - g(x)L_Y f(x)X(x),$$

where the symbol $L_X f$ denotes the Lie derivative of the function f with respect to the vector field X defined as $L_X f(x) = df(x)X(x)$.

7.3 Lie bracket theorems

Additional properties of the Lie bracket will be specified in the following two theorems.

Theorem 7.3.1 (On Commutation) *Vector fields commute if and only if the composition of their flows is commutative, i.e.*

$$[X, Y](x) = 0 \Longleftrightarrow \varphi_t \circ \Phi_s(x) = \Phi_s \circ \varphi_t(x).$$

Theorem 7.3.2 *Suppose that $\varphi : R^n \longrightarrow R^n$ defines the Differential Equivalence of dynamic systems determined by the vector fields X_1, Y_1 and X_2, Y_2, so that*

$$D\varphi(x)X_1(x) = Y_1(\varphi(x)) \quad \textit{and} \quad D\varphi(x)X_2(x) = Y_2(\varphi(x)).$$

Then, the Lie brackets of differentially equivalent vector fields are also differentially equivalent,

$$D\varphi(x)[X_1, X_2](x) = [Y_1, Y_2](\varphi(x)).$$

7.4 Simultaneous Straightening Out Theorem

A generalisation of the Straightening Out Theorem proved in section 6 is the Simultaneous Straightening Out Theorem that will be presented below. This theorem finds a direct application in the proof of the fundamental Frobenius Theorem.

Theorem 7.4.1 (Simultaneous Straightening Out) *Given a collection of* $k \geqslant 1$ *vector fields* $X_1, X_2, \ldots, X_k \in V^\infty(R^n)$. *We assume that these vector fields are independent at the point* $0 \in R^n$, *i.e.* $\operatorname{rank}[X_1(0), X_2(0), \ldots, X_k(0)] = k$, *and that in a neighbourhood of zero they commute with each other,* $[X_i, X_j] = 0$, $i, j = 1, 2, \ldots, k$. *This being so, there exists a local diffeomorphism* $\xi = \psi(x)$, *such that*

$$D\psi(x)X_i(x) = e_i(\psi(x)) = e_i, \quad i = 1, 2, \ldots, k,$$

where $e_i \in R^n$ *denotes the* i*-th unit vector field. In other words, the diffeomorphism* ψ *allows to straighten out all the* k *vector fields simultaneously, establishing the equivalences*

$$X_1 \underset{LDE}{\cong} e_1,\ X_2 \underset{LDE}{\cong} e_2, \ldots, X_k \underset{LDE}{\cong} e_k.$$

7.5 Distribution and integral manifold

Given the module $V^\infty(R^n)$ of vector fields over $C^\infty(R^n, R)$, its submodule

$$\mathcal{D} = \underset{C^\infty(R^n,R)}{\operatorname{span}} \{X_1, X_2, \ldots, X_k\}$$

generated by a collection of vector fields $X_i \in V^\infty(R^n)$, independent at any point $x \in R^n$, is called a (vector field) distribution. By definition, at any point x, the distribution $\mathcal{D}$ defines a k-dimensional linear subspace $\mathcal{D}(x) \subset R^n$. Thus, we can speak of a field of subspaces

$$x \mapsto \mathcal{D}(x).$$

For $k = 1$, the distribution $\mathcal{D} = \operatorname{span}_{C^\infty(R^n,R)}\{X_1\}$ generates in R^n straight lines corresponding to the single vector field X_1. The Theorem on Existence and Uniqueness provides condition under which there exists a curve that at every point is tangent to $\mathcal{D}(x)$. This is just the integral curve of the vector field X_1. For a k-dimensional distribution a natural generalisation of the integral curve is the concept of the integral manifold defined as a k-dimensional submanifold of R^n that at any point is tangent to the subspace $\mathcal{D}(x)$. Contrary to the one-dimensional distributions for which the condition of existence of the integral curve are easily satisfied, in the case of $k \geqslant 2$ the integral manifold rarely exists. In the sequel, to avoid technical complications, by an $(n-p)$-dimensional smooth manifold (a submanifold of R^n) we shall understand the subset of R^n defined by p independent equations, so

$$M_{\mathcal{D}} = \{x \in R^n | f_1(x) = 0, f_2(x) = 0, \dots, f_p(x) = 0\},$$

where the functions f_i are smooth. The independence of functions means that $\operatorname{rank}\left[df_1^T(x), df_2^T(x), \dots, df_p^T(x)\right](x) = p$, i.e. their differentials need to be independent for every $x \in M_{\mathcal{D}}$. A distribution that has an integral manifold is referred to as integrable.

7.6 Frobenius Theorem

A necessary and sufficient condition for the existence of an integral manifold is provided by the Frobenius Theorem. It is sometimes called the third pillar of nonlinear analysis. Bellow we shall restrict to the formulation of the sufficient condition.

Theorem 7.6.1 (Frobenius) *Let* $\mathcal{D} = \operatorname{span}_{C^\infty(R^n,R)}\{X_1, X_2, \dots, X_k\}$ *denote a* k*-dimensional distribution defined in a neighbourhood of* $0 \in R^n$*, so* $\operatorname{rank}[X_1(0), X_2(0), \dots, X_k(0)] = k$*. We assume that this distribution is involutive, i.e.* $X, Y \in \mathcal{D} \Longrightarrow [X, Y] \in \mathcal{D}$*. Then, in a certain local coordinate system*

$$\mathcal{D} = \operatorname*{span}_{C^\infty(R^n,R)} \{e_1, e_2, \dots, e_k\}.$$

Equivalently, there exist vector fields $Y_1, Y_2, \dots, Y_k$ *generating the distribution,* $\mathcal{D} = \operatorname{span}_{C^\infty(R^n,R)}\{Y_1, Y_2, \dots, Y_k\}$*, and a local diffeomorphism* $\psi(x) = (\psi_1(x), \psi_2(x), \dots, \psi_n(x))$ *straightening out these vector fields simultaneously. Through any point in a neighbourhood of* $0 \in R^n$ *there passes an integral manifold of the distribution* $\mathcal{D}$*, of dimension* k*, determined by the last* $n-k$ *components of the diffeomorphism* ψ*,*

$$M_{\mathcal{D}} = \{x \in R^n | \psi_{k+1}(x) = 0, \psi_{k+2}(x) = 0, \dots, \psi_n(x) = 0\}.$$

7.7 Proofs

7.7.1 Theorem on commutation

Proof:

- Suppose that $\varphi_t \circ \Phi_s(x) = \Phi_s \circ \varphi_t(x)$. By the differentiation of both sides with respect to s we get

  $$\frac{d}{ds}\varphi_t \circ \Phi_s(x) = D\varphi_t(\Phi_s(x))\frac{d\Phi_s(x)}{ds} = \frac{d}{ds}\Phi_s \circ \varphi_t(x) = Y(\Phi_s \circ \varphi_t(x)).$$

  After a substitution of $s = 0$ and taking $\varphi_{-t}(x)$ instead of x the above expression converts into

  $$D\varphi_t(\varphi_{-t}(x))Y(\varphi_{-t}(x)) = \mathrm{Ad}_X^{-t} Y(x) = Y(x),$$

  implying that

  $$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_X^{-t} Y(x) = [X, Y](x) = 0.$$

- Now, let $[X, Y](x) = 0$. In consequence, $\frac{d}{dt} \mathrm{Ad}_X^{t} Y(x) = 0$, i.e.

  $$\mathrm{Ad}_X^{t} Y(x) = D\varphi_{-t}(\varphi_t(x))Y(\varphi_t(x)) = Y(x).$$

  After replacing x by $\varphi_{-t}(x)$ and then changing $-t$ to t one obtains

  $$D\varphi_t(x)Y(x) = Y(\varphi_t(x)).$$

  Having substituted x for the flow $\Phi_s(x)$ of the vector field Y, we arrive at

  $$D\varphi_t(\Phi_s(x))Y(\Phi_s(x)) = Y(\varphi_t(\Phi_s(x))),$$

  that in turns yields

  $$\frac{d}{ds}\varphi_t \circ \Phi_s(x) = Y(\varphi_t \circ \Phi_s(x))).$$

  However, by definition of the flow on the vector field Y, it follows that

  $$\frac{d}{ds}\Phi_s \circ \varphi_t(x) = Y(\Phi_s \circ \varphi_t(x)).$$

  The last two identities indicate that $\varphi_t \circ \Phi_s(x)$ and $\Phi_s \circ \varphi_t(x)$ satisfy the same differential equation with the same initial condition $\varphi_t \circ \Phi_s(x)|_{s=0} = \varphi_t(x) = \Phi_s \circ \varphi_t(x)|_{s=0}$, therefore, by the Theorem on Existence and Uniqueness it must be

  $$\varphi_t \circ \Phi_s(x) = \Phi_s \circ \varphi_t(x),$$

  what should be demonstrated.

∎

7.7.2 Simultaneous Straightening Out Theorem

Proof: For a proof we shall design a diffeomorphism $x = \alpha(\xi)$, such that $D\alpha(\xi)e_i = X_i(\alpha(\xi))$ for $i = 1, 2, \ldots, k$. Our computations will be accomplished in a neighbourhood of $0 \in R^n$. Without any loss of generality we may assume that the rank condition $\operatorname{rank}[X_1(0), X_2(0), \ldots, X_k(0)] = k$ means independence of the first k rows of the matrix $[X_1(0), X_2(0), \ldots, X_k(0)]$. We propose to use the following function

$$\alpha(\xi) = \varphi_{1\xi_1} \circ \varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n).$$

By definition the function α is smooth and such that $\alpha(0) = 0$. Let's compute its derivative

$$\begin{aligned} D\alpha(\xi) = \Bigg[& \frac{d}{d\xi_1}\varphi_{1\xi_1}(\varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)), \\ & D\varphi_{1\xi_1}(\varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)) \\ & \frac{d}{d\xi_2}(\varphi_{2\xi_2}(\varphi_{3\xi_3} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)), \ldots, \\ & D\varphi_{1\xi_1} \circ \varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)e_{k+1}, \ldots, \\ & D\varphi_{1\xi_1} \circ \varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)e_n \Bigg]. \end{aligned}$$

An analysis of this expression allows us to conclude that

$$\frac{d}{d\xi_1}\varphi_{1\xi_1}(\varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)) = X_1(\alpha(\xi)),$$

and

$$\begin{aligned} & D\varphi_{1\xi_1}(\varphi_{2\xi_2} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)) \\ & \quad \frac{d}{d\xi_2}(\varphi_{2\xi_2}(\varphi_{3\xi_3} \circ \cdots \circ \varphi_{k\xi_k}(0, 0, \ldots, \xi_{k+1}, \ldots, \xi_n)) \\ & \quad = D\varphi_{1\xi_1}(\varphi_{1-\xi_1}(\alpha(\xi)))X_2(\varphi_{1-\xi_1}(\alpha(\xi))) = \mathrm{Ad}_{X_1}^{-\xi_1} X_2(\alpha(\xi)), \end{aligned}$$

and similarly for further components. But, by virtue of the assumption the vector fields commute, so $[X_1, X_2](x) = 0$, yielding

$$\mathrm{Ad}_{X_1}^{-\xi_1} X_2(\alpha(\xi)) = X_2(\alpha(\xi)),$$

etc. In this way we have shown that

$$D\alpha(\xi) = [X_1(\alpha(\xi)), X_2(\alpha(\xi)), \ldots, X_k(\alpha(\xi)), *, \cdots, *],$$

where asterisks stand for the entries that we do not have to know. At the point 0 we have

$$D\alpha(0) = [X_1(0), X_2(0), \dots, X_k(0), e_{k+1}, \dots, e_n].$$

Because $\operatorname{rank} D\alpha(0) = n$, α is a local diffeomorphism around 0. Furthermore, for every $i = 1, 2, \dots, k$, it follows that

$$D\alpha(\xi)e_i = X_i(\alpha(\xi)),$$

therefore the vector fields $X_1, X_2, \dots, X_k$ have been straightened out simultaneously. ■

7.7.3 Frobenius Theorem

Proof: Suppose that the upper $k \times k$ sub-matrix of the matrix

$$[X_1(x), X_2(x), \dots, X_k(x)] = \begin{bmatrix} P(x) \\ Q(x) \end{bmatrix}$$

has rank k in a neighbourhood of $0 \in R^n$. By definition, having multiplied both sides of this identity by the matrix $P^{-1}(x)$, we obtain new generators $Y_1, Y_2, \dots, Y_k$ of the distribution $\mathcal{D}$, of the form

$$Y_i(x) = \begin{pmatrix} e_i \\ * \end{pmatrix},$$

where $e_i \in R^k$ is the i-th unit vector and $*$ symbolises the remaining $n-k$ components of the vector field. We shall show that these new generators commute. To this objective we need to compute their Lie bracket

$$\begin{aligned}[] [Y_i, Y_j](x) &= DY_j(x)Y_i(x) - DY_i(x)Y_j(x) \\ &= \begin{bmatrix} 0 \\ * \end{bmatrix} \begin{pmatrix} e_i \\ * \end{pmatrix} - \begin{bmatrix} 0 \\ * \end{bmatrix} \begin{pmatrix} e_j \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}. \end{aligned}$$

But, by the involutivity of the distribution $\mathcal{D}$, the bracket $[Y_i, Y_j] \in \mathcal{D}$, what means that

$$[Y_i, Y_j](x) = \sum_{r=1}^{k} \alpha_r(x) Y_r(x) = \begin{pmatrix} 0 \\ * \end{pmatrix},$$

i.e. all the coefficient functions $\alpha_i(x) = 0$ as well as $[Y_i, Y_j](x) = 0$. We see that he new generators commute. Invoking the Simultaneous Straightening

Out Theorem we establish the existence of a local diffeomorphism $\psi(x) = (\psi_1(x), \psi_2(x), \dots, \psi_n(x))^T$, such that

$$D\psi(x)Y_j(x) = e_j, \quad j = 1, 2, \dots, k.$$

What remains is to prove the existence of an integral manifold. Let

$$X_i(x) = \sum_{j=1}^{k} \gamma_{ij}(x)Y_j(x)$$

express the vector field X_i in terms of new generators. By multiplying both sides by $D\psi(x)$, and then exploiting the form of vector fields Y_j, we obtain

$$D\psi(x)X_i(x) = \sum_{j=1}^{k} \gamma_{ij}(x)D\psi(x)Y_j(x) = \sum_{j=1}^{k} \gamma_{ij}(x)e_j = \begin{pmatrix} * \\ 0 \end{pmatrix},$$

where 0 refers to the last $n-k$ coordinates. Since

$$D\psi(x)X_i(x) = \begin{bmatrix} D\psi_1(x)X_i(x) \\ \vdots \\ D\psi_k(x)X_i(x) \\ D\psi_{k+1}(x)X_i(x) \\ \vdots \\ D\psi_n(x)X_i(x) \end{bmatrix},$$

it follows that

$$D\psi_{k+1}(x)X_i(x) = 0, \dots, D\psi_n(x)X_i(x) = 0.$$

Concluding, the distribution $\mathcal{D}$ appears to be tangent to the manifold

$$M_{\mathcal{D}} = \{x \in R^n | \psi_{k+1}(x) = 0, \dots, \psi_n(x) = 0\}$$

being its integral manifold. The theorem has been proved. ■

7.8 Problems and exercises

Exercise 7.1 Derive the formula for the Lie bracket of vector fields

$$[X, Y](x) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_X^t Y(x) = DY(x)X(x) - DX(x)Y(x).$$

Exercise 7.2 Prove the theorem 7.3.2. Hint: Show that the right hand side is equal to the left hand side, and observe that $DY_i \circ \varphi D\varphi = D(Y_i \circ \varphi)$.

Exercise 7.3 Show that for every function $f \in C^\infty(R^n, R)$, there holds

$$L_{[X,Y]}f = L_X(L_Y f) - L_Y(L_X f).$$

Exercise 7.4 Making use of the property $[fX, gY] = fg[X, Y] + fL_X gY - gL_Y fX$ show that the distribution $\mathcal{D} = \mathrm{span}_{C^\infty(R^n,R)}\{X_1(x), X_2(x), \ldots, X_k(x)\}$ is involutive if and only if $[X_i, X_j] \in \mathcal{D}$ for $i, j = 1, 2, \ldots, k$.

Exercise 7.5 Check involutivity of the distributions:

a)

$$\mathcal{D}_1 = \underset{C^\infty(R^n,R)}{\mathrm{span}} \left\{ \begin{pmatrix} 1 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

b)

$$\mathcal{D}_1 = \underset{C^\infty(R^n,R)}{\mathrm{span}} \left\{ \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

c)

$$\mathcal{D}_1 = \underset{C^\infty(R^n,R)}{\mathrm{span}} \left\{ \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

7.9 Bibliographical remarks

The concepts of vector fields, Lie brackets, distributions as well the Frobenius Theorem, etc. belong to the field of differential geometry. Necessary basics the Reader can find in diverse monographs concerned with geometric control theory, such as [Isi94, NvdS90, Blo03, Sas99, Lév09]. An advanced exposition of differential geometry is contained in the books [AMR83, Spi79]. The "dynamic" concept of the Lie bracket presented in this chapter has been borrowed from the monograph [AMR83]; in the context of control the adjoint operators have appeared in the paper [Kre85]. The Straightening Out Theorem is a classic result of theory of dynamic systems [Arn83, AMR83]. A generalisation of the Frobenius Theorem can be found in [Sus83].

Bibliography

[AMR83] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer, New York, 1983.

[Arn83] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York, 1983.

[Blo03] A. M. Bloch. *Nonholonomic Mechanics and Control*. Springer, New York, 2003.

[Isi94] A. Isidori. *Nonlinear Control Systems*. Springer, New York, 1994.

[Kre85] A. Krener. $(\mathrm{Ad}_{f,g})$, $(\mathrm{ad}_{f,g})$ and locally $(\mathrm{ad}_{f,g})$ invariant and controllability distributions. *SIAM J. Contr. Opt.*, 23(4):523–549, 1985.

[Lév09] J. Lévine. *Analysis and Control of Nonlinear Systems*. Springer, 2009.

[NvdS90] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, New York, 1990.

[Sas99] S. S. Sastry. *Nonlinear Systems*. Springer, New York, 1999.

[Spi79] M. Spivak. *A Comprehensive Introduction to Differential Geometry*. Publish or Perish, Berkeley, 1979.

[Sus83] H. Sussmann. Lie brackets, real analyticity and geometric control. In *Differential Geometric Control Theory*, pages 1–116. Birkhäuser, Boston, 1983.

Chapter 8

Control systems

A control system is represented by a system of ordinary differential equations dependent on a control variable

$$\begin{cases} \dot{x} = f(x(t), u(t)) \\ y(t) = h(x(t)) \end{cases}, \tag{8.1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$ denote, respectively, the state, the control, and the output variable. We shall assume that the system is smooth, so is the function $f \in C^\infty(R^n \times R^m, R^n)$ describing the system's dynamics as well as the output function $h \in C^\infty(R^n, R^p)$. Usually, we have $m \leqslant n$ and $p \leqslant n$. Notice that with fixed control $u(t)$, the control system becomes a time-dependent dynamic system of ordinary differential equations, of the form

$$\begin{cases} \dot{x} = f(x(t), u(t)) = \bar{f}(x(t), t) \\ y(t) = h(x(t)) \end{cases}.$$

Taking into account the Existence and Uniqueness Theorem one derives the following sufficient conditions under which the trajectory $x(t)$ of the system (8.1) exists

- the function $\bar{f}(x, t)$ depends continuously on time t,
- the function $\bar{f}(x, t)$ satisfies the Lipschitz condition with respect to x,
- the function $\bar{f}(x_0, t)$ is bounded with respect to t.

Observe that the Lipschitz property results from the smoothness of the function $f(x, u)$, while the remaining conditions will follow from the continuity and boundedness of control functions $u(t)$. The control functions will be

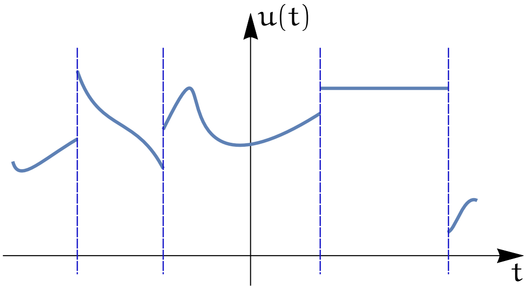

Figure 8.1: Piecewise continuous controls

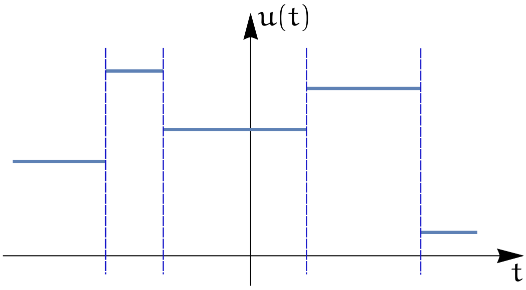

Figure 8.2: Piecewise constant controls

simply referred to as controls. A basic requirement imposed on the admissible control is to guarantee the existence and uniqueness of the trajectory $x(t)$. Therefore, a basic class of admissible control is the class of continuous and bounded functions of time. For some practical, but also theoretic reasons, bounded and piecewise continuous controls are also allowed, including the piecewise constant controls. Examples have been shown in Figures 8.1 and 8.2.

8.1 Control affine and driftless systems

An important subclass of control systems is constituted by control affine systems described by the following equations

$$\begin{cases} \dot{x} = f(x(t)) + g(x(t))u(t)) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) \\ y(t) = h(x(t)) \end{cases} . \tag{8.2}$$

The columns of the matrix $g(x)$ are formed by vector fields $g_1(x), \ldots, g_m(x)$. All vector fields appearing in (8.2) are assumed smooth. Since when the controls are "switched off", $u = 0$, the system's behaviour is determined by the vector field $f(x)$, it is called a drift vector field or just a drift. The significance of control affine systems results from several reasons, like the following:

- Many control systems, including those with Lagrangian dynamics, assume the affine form.
- In case when the controls in the system (8.1) are differentiable, after introducing a new state variable $(x, u) \in R^{n+m}$, one gets a control affine system

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} f(x(t), u(t)) \\ 0 \end{pmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} v(t),$$

whose control is $v = \dot{u}$.

- For u close to zero one can expand the right hand side of the system (8.1) in the Taylor series

$$f(x, u) = f(x, 0) + \frac{\partial f(x, 0)}{\partial u} u + O\left(x, u^2\right),$$

what suggests that the control affine system (8.2) approximates the system (8.1) for small values of controls.

- Linear control system

$$\begin{cases} \dot{x} = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

is control affine.

If in the control affine system the drift $f(x)$ is absent, the equations (8.2) take the form

$$\begin{cases} \dot{x} = g(x(t))u(t)) = \sum_{i=1}^{m} g_i(x(t))u_i(t) \\ y(t) = h(x(t)) \end{cases}. \tag{8.3}$$

The importance of driftless systems is a.o. a consequence of the fact that they represent the kinematics of non-holonomic systems, like the wheeled mobile robots.

8.2 Differentiation of the end-point map

Consider the control system (8.1). Denote by

$$x(t) = \varphi_{x,t}(u), \quad \frac{d\varphi_{x,t}(u)}{dt} = f(\varphi_{x,t}(u), u(t)),$$

the trajectory of this system from the initial state x, subject to a control u. Having fixed the final time T, we can define a function $\mathrm{end}_T : (x, u) \mapsto x(T) = \varphi_{x,T}(u)$ called the end-point map of the system. Under appropriate assumptions imposed on the control system it can be proved that the end-point map is differentiable with respect to the initial state and the control function. We want now to compute both these derivatives.

- For any $t \in R$, $y \in R^n$ it is true that

  $$\frac{\partial \varphi_{x,t}(u)}{\partial x} y = \xi(t) = \frac{d}{d\alpha}\bigg|_{\alpha=0} \varphi_{x+\alpha y,t}(u).$$

  The time derivative

  $$\begin{aligned}\dot{\xi} &= \frac{d}{dt}\frac{d}{d\alpha}\bigg|_{\alpha=0} \varphi_{x+\alpha y,t}(u) = \frac{d}{d\alpha}\bigg|_{\alpha=0} \frac{d}{dt}\varphi_{x+\alpha y,t}(u) \\ &= \frac{d}{d\alpha}\bigg|_{\alpha=0} f(\varphi_{x+\alpha y,t}(u), u(t)) = \frac{\partial f(x(t), u(t))}{\partial x}\xi(t) = A(t)\xi(t).\end{aligned}$$

  After solving the equation $\dot{\xi} = A(t)\xi(t)$ with the initial condition $\xi(0) = y$ one obtains

  $$\frac{\partial\, \mathrm{end}_T(x, u)}{\partial x} y = \xi(T) = \Phi(T, 0)y,$$

  where $\Phi(t, s)$ is the fundamental matrix of the differential equation $\dot{\xi} = A(t)\xi(t)$.

- Analogously, when differentiating with respect to the control function, for $t \in R$ and an admissible control v, one computes

  $$\frac{\partial \varphi_{x,t}(u)}{\partial u} v = \zeta(t) = \frac{d}{d\alpha}|_{\alpha=0} \varphi_{x,t}(u + \alpha v).$$

  Next, the time differentiation results in

  $$\begin{aligned}\dot{\zeta} &= \frac{d}{dt}\frac{d}{d\alpha}\bigg|_{\alpha=0} \varphi_{x,t}(u + \alpha v) = \frac{d}{d\alpha}\bigg|_{\alpha=0} \frac{d}{dt}\varphi_{x,t}(u + \alpha v) \\ &= \frac{d}{d\alpha}\bigg|_{\alpha=0} f(\varphi_{x,t}(u + \alpha v), u(t) + \alpha v(t)) \\ &= \frac{\partial f(x(t), u(t))}{\partial x}\zeta(t) + \frac{\partial f(x(t), u(t))}{\partial u}v(t) = A(t)\zeta(t) + B(t)v(t).\end{aligned}$$

Finally, in order to find the derivative of the end-point map, we need to solve the equation $\dot{\zeta} = A(t)\zeta + B(t)\nu(t)$ with the initial condition $\zeta(0) = 0$, and then substitute

$$\frac{\partial\, \mathrm{end}_T(x,u)}{\partial u}\nu = \zeta(T) = \int_0^T \Phi(T,t)B(t)\nu(t)dt.$$

8.3 Accessibility and controllability

Let σ denote a control affine system (8.2) with piecewise constant admissible controls of the form

$$u_k = \left\{ \left(u^1, t_1\right), \left(u^2, t_2\right), \ldots, \left(u^k, t_k\right) \right\},$$

for a certain $k \in N$ and $u^i \in R^m$. An application of such a control means that over the time interval t_1 the system is controlled by u^1, next, for the time t_2 a constant control u^2 will be applied, etc., finally a constant control u^k acts on the system for the time t_k. After the application of such a control we get a system's trajectory that consists of a segment of the trajectory of the vector field $f(x) + g(x)u^1$ followed by a segment of the trajectory of $f(x) + g(x)u^2$ etc. We can say that the motion of the system σ is determined by a family of associated vector fields

$$\mathcal{F}_\sigma = \{f + gu | u \in R^m\}.$$

In summary, under the control u_k, the motion of σ is defined by vector fields $X_i(x) = f(x) + g(x)u^i$ acting over time intervals t_i, for $i = 1, 2, \ldots, k$. Suppose that $\varphi_{i,t}(x)$ denotes the flow of the vector field $X_i(x)$, and let x_0 be an initial state. Then, acted on by the piece-wise constant control u_k, after the time $\sum_{i=1}^k t_i$, system σ will be transferred from x_0 to the final state

$$x = \varphi_{k,t_k} \circ \varphi_{k-1,t_{k-1}} \circ \cdots \varphi_{1,t_1}(x_0).$$

Choosing various piecewise constant controls u_k, for various k, one obtains a set of states reachable in the system σ from the state x_0 at the time instant t,

$$\mathcal{R}_\sigma(x_0, t) = \left\{ \varphi_{k,t_k} \circ \varphi_{k-1,t_{k-1}} \circ \cdots \varphi_{1,t_1}(x_0) | t_i \geqslant 0, \sum_{i=1}^k t_i = t,\ k \geqslant 0 \right\}.$$

The reachable set from the state x_0 at any time instant is the union

$$\mathcal{R}_\sigma(x_0) = \bigcup_{t \geqslant 0} \mathcal{R}_\sigma(x_0, t).$$

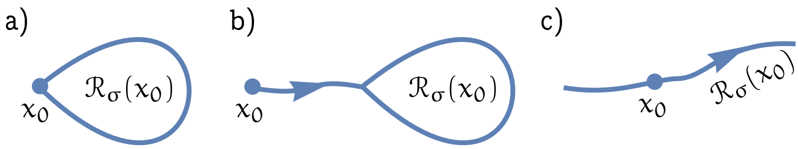

Figure 8.3: Accessibility property: a), b) yes, c) no

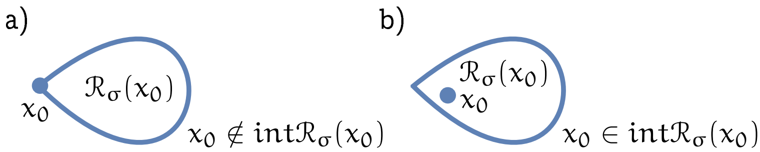

Figure 8.4: Local controllability: a) no, b) yes

The reachable set serves to introduce several controllability-type concepts, specified below.

Definition 8.3.1 *The system σ is called controllable from the point x_0 if $\mathcal{R}_\sigma(x_0) = R^n$, and controllable if $\mathcal{R}_\sigma(x_0) = R^n$ for every x_0.*

Besides the concept of controllability a weaker concept of accessibility is used.

Definition 8.3.2 *The system σ has the accessibility property from the point x_0 if the reachable set from x_0 has non-empty interior,* $\mathrm{int}\,\mathcal{R}_\sigma(x_0) \neq \emptyset$*. The system has the accessibility property if* $\mathrm{int}\,\mathcal{R}_\sigma(x_0) \neq \emptyset$ *for every x_0.*

We recall that the interior of a set is the biggest open set contained in this set. Figure 8.3 illustrates the concept of accessibility.

Intuitively, if a system has the accessibility property from a point x_0, then it is possible to pass from x_0 to a point, from which the system could move in any direction in the state space R^n. However, from the point x_0 itself such an omnidirectional motion may not be possible. The third concept that applies to control affine system is the local controllability.

Definition 8.3.3 *The system σ is locally controllable from the point x_0 if* $x_0 \in \mathrm{int}\,\mathcal{R}_\sigma(x_0)$*, and locally controllable if* $x_0 \in \mathrm{int}\,\mathcal{R}_\sigma(x_0)$ *for every x_0 (see Figure 8.4).*

8.4 Controllability theorems

In order to formulate some controllability conditions, it is advantageous to exploit the big adjoint operator $\mathrm{Ad}_X^t Y(x) = D\varphi_{-t}(\varphi_t(x))Y(\varphi_t(x))$, and the small adjoint operator $\mathrm{ad}_X Y(x) = [X, Y](x)$, defined in the previous chapter. Relying on these operators, to the system σ we assign two distributions

$$\mathcal{D}_\sigma = \langle \mathrm{Ad}_{\mathcal{F}_\sigma} | \mathcal{F}_\sigma \rangle \quad \text{and} \quad d_\sigma = \langle \mathrm{ad}_{\mathcal{F}_\sigma} | \mathcal{F}_\sigma \rangle \,,$$

defined as the smallest distributions containing the family $\mathcal{F}_\sigma$ of vector fields associated with the system σ, both closed with respect to the operators Ad and ad. For $\mathcal{D}_\sigma$ the closeness means that $\mathcal{F}_\sigma \subset \mathcal{D}_\sigma$, and for every $X \in \mathcal{F}_\sigma$, $Y \in \mathcal{D}_\sigma$, the vector field $\mathrm{Ad}_X^t Y \in \mathcal{D}_\sigma$ at any time instant t, and analogously for d_σ. It is easily noticed that $d_\sigma \subset \mathcal{D}_\sigma$. Two following results on controllability can be stated in terms of the introduced distributions.

Theorem 8.4.1 (Chow-Sussmann-Krener) *The control affine system σ has the accessibility property if and only if*

$$\mathcal{D}_\sigma = V^\infty(R^n).$$

Theorem 8.4.2 (Chow-Lobry-Krener) *If $d_\sigma = V^\infty(R^n)$ then the system σ has the accessibility property.*

In both these theorems the vector fields $\mathcal{F}_\sigma$ can be replaced by $F_\sigma = \{f, g_1, g_2, \dots, g_m\}$. The sufficient condition for the accessibility property can be conveniently expressed as the so called Lie Algebra Rank Condition (LARC). To this aim, we need to define the Lie algebra $\mathcal{L}_\sigma$ associated with the system σ, as the smallest Lie algebra that contains the vector fields F_σ. Then, we have the following

Theorem 8.4.3 *If at any point $x \in R^n$*

$$\dim \mathcal{L}_\sigma(x) = n$$

then the system σ has the accessibility property.

It follows that for driftless systems (8.3) the above theorems provide conditions for controllability, namely

Theorem 8.4.4

- *A driftless control system is controllable if and only if $\mathcal{D}_\sigma = V^\infty(R^n)$.*
- *If $d_\sigma = V^\infty(R^n)$ then the driftless system is controllable.*
- *If $\dim \mathcal{L}_\sigma(x) = n$ then the driftless system is controllable.*

8.5 Checking controllability

A conclusion that may be drawn from the study of controllability of nonlinear control systems accomplished above is that general necessary and sufficient controllability conditions are not known. For control affine systems much easily checkable (however much weaker) is the accessibility property. Below we shall pay attention to a few specific cases, when controllability can be established either directly or from the accessibility property.

- Let (u_0, x_0) denote the equilibrium point of the control system (8.1), what means that $f(x_0, u_0) = 0$. We find the linear approximation of the system at this point

$$f(x,u) = f(x_0,u_0) + \frac{\partial f(x_0,u_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,u_0)}{\partial u}(u-u_0) + O((x-x_0)^2,(u-u_0)^2)$$

  and let $A = \frac{\partial f(x_0,u_0)}{\partial x}$ and $B = \frac{\partial f(x_0,u_0)}{\partial u}$, as well as $\xi = x - x_0$, $\nu = u - u_0$. The following theorem holds.

  Theorem 8.5.1 *If the linear system*

$$\dot{\xi} = A\xi(t) + B\nu(t)$$

  is controllable (satisfies the Kalman condition) then the nonlinear system (8.1) is locally controllable from the point x_0.

- The driftless system (8.3) that has the accessibility property is controllable.

- If the drift vector field $f(x)$ of a control affine system, with the flow $\varphi_t(x)$, is Poisson stable then the control affine system having the accessibility property is controllable. We recall that a vector field $X(x)$ is Poisson stable if there exists a dense subset $D \subset R^n$, such that

$$(\forall x \in D)(\forall D \supset U \ni x)(\forall T > 0)(\exists t_1, t_2 \geqslant T) \quad (\varphi_{t_1}(x) \in U \text{ and } \varphi_{-t_2}(x) \in U).$$

  An example of a Poisson stable system is an oscillator.

For the local controllability there exists a sufficient condition established by Sussmann. Below we state this condition in the form applicable to single-input systems. Let a system

$$\dot{x} = f(x(t)) + g(x(t))u, \quad x \in R^n,\ u \in R,$$

be given, with the equilibrium point $(0, x_0)$. We introduce a family of distributions

$$
\begin{aligned}
&\mathcal{S}^1(f,g) = \underset{C^\infty(R^n,R)}{\operatorname{span}} \left\{g, \operatorname{ad}_f g, \dots, \operatorname{ad}_f^i g, \dots\right\}, \\
&\mathcal{S}^2(f,g) = \mathcal{S}^1(f,g) + \underset{C^\infty(R^n,R)}{\operatorname{span}} \left\{ \left[\operatorname{ad}_f^{i_1} g, \operatorname{ad}_f^{i_2} g\right]\Big|_{i_1,i_2 \geqslant 0} \right\}, \\
&\vdots \\
&\mathcal{S}^k(f,g) = \mathcal{S}^{k-1}(f,g) \\
&+ \operatorname{span}_{C^\infty(R^n,R)} \left\{ \left[\operatorname{ad}_f^{i_1} g, \left[\operatorname{ad}_f^{i_2} g, \dots \left[\operatorname{ad}_f^{i_{k-1}} g, \operatorname{ad}_f^{i_k} g\right] \dots\right]\right]\Big|_{i_1,i_2,\dots i_k \geqslant 0} \right\},
\end{aligned}
$$

where $\operatorname{ad}_f^{i+1} g = \left[f, \operatorname{ad}_f^i g\right]$. It is easy to observe that a characteristic feature of the distribution $\mathcal{S}^i(f,g)$ is that the control vector field g appears in it i times. In terms of these distributions the Sussmann's condition can be stated in the following form

Theorem 8.5.2 (Sussmann's Controllability Condition) *Suppose that for a certain integer* k

$$\mathcal{S}^k(f,g)(x_0) = R^n$$

and that, for any odd number $j \leqslant k$*,*

$$\mathcal{S}^j(f,g)(x_0) = \mathcal{S}^{j+1}(f,g)(x_0).$$

Then, the control affine system is locally controllable from the point x_0*.*

The second condition in Theorem 8.5.2 means that the directions of motion at the point x_0 generated by the distribution $\mathcal{S}^k(f,g)$ that contain an even number of appearances of the vector field g should be provided by an odd, smaller by 1, number of appearances of g. The Lie brackets containing an even number of the vector field g are called the "bad Lie brackets". It is easily noticed that if the first condition of Theorem 8.5.2 holds for $k = 1$, then the local controllability results from the controllability of the linear approximation of the system.

8.6 Examples

Example 8.6.1 *Let's examine the controllability of the control affine system*

$$\sigma : \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = 1 \end{cases}$$

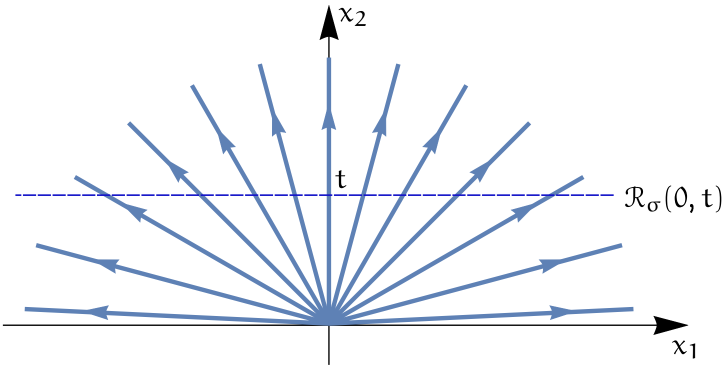

Figure 8.5: Reachable sets $\mathcal{R}_\sigma(0,t)$ and $\mathcal{R}_\sigma(0)$

with vector fields $f(x) = (0,1)^T$ *and* $g(x) = (1,0)^T$, $u \in R$, *from the state* $x_0 = (0,0)^T$. *The system's equations yield that under constant controls*

$$x_2 = \frac{1}{u} x_1.$$

The states reachable from x_0 *(see Figure 8.5) lie on the rays emanating from* x_0, *located in the upper half-plane of the coordinate system, whereas the motion along the* x_2*-axis corresponds to zero control,* $u = 0$, *and the motion along* x_1 *is not possible at all. Therefore, we have* $\mathcal{R}_\sigma(x_0) \subset R^2_+$. *By virtue of definition the system has the accessibility property from* x_0, *however it is neither controllable (the point in the lower half-plane are not reachable at all), nor locally controllable from* x_0 *(*x_0 *does not belong to the interior of the set* $\mathcal{R}_\sigma(x_0)$*).*

Example 8.6.2 *Consider a chained form system*

$$\sigma : \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = u_1 x_2 \end{cases} .$$

The systems is driftless, with two vector fields $g_1(x) = (1,0,x_2)^T$ *and* $g_2(x) = (0,1,0)^T$. *The equilibrium point is given as* $u_0 = 0$, $x_0 = 0$. *We begin with computing the linear approximation*

$$\dot{\xi} = A\xi(t) + Bv(t)$$

of the system σ *at the equilibrium point. This gives* $A = 0$ *and* $B = [g_1(0), g_2(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. *It is obvious that the linear approximation does* *not satisfy the Kalman condition, therefore it is not controllable. It follows that we cannot use the linear approximation to deduce controllability of this system. More generally, it can be noticed that every driftless control system with* $m < n$ *has the linear approximation uncontrollable. Let us now check the Lie algebra rank condition. We have* $F_\sigma = \{g_1, g_2\}$*, so, the Lie algebra of the system* $\mathcal{L}_\sigma$ *contains the vector fields* $g_1, g_2, g_{12} = [g_1, g_2], \ldots$*. We compute* $g_{12}(x) = Dg_2(x)g_1(x) - Dg_1(x)g_2(x) = \frac{\partial g_1(x)}{\partial x_2} = (0, 0, -1)^T$*. Thanks to*

$$\dim \mathcal{L}_\sigma(x) = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 3,$$

the system σ *is controllable.*

Example 8.6.3 *Now, consider the control affine system*

$$\sigma : \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 + u_1 x_2 \end{cases},$$

that results from the previous system after adding the drift vector field $f(x) = (0, 0, x_1)^T$*. The control vector fields* g_1 *and* g_2 *remain unaltered. The equilibrium point is at* $u_0 = 0$*,* $x_0 = 0$*. The linear approximation at this point*

$$\dot{\xi} = A\xi(t) + Bv(t),$$

is determined by matrices $A = Df(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$*. The Kalman matrix*

$$\Omega = \left[B, AB, A^2B\right] = [I_3, *]$$

has rank 3*, therefore the linear approximation is controllable, and the system* σ *is locally controllable from the equilibrium point. Let's check other points using the LARC. We have* $F_\sigma = \{f, g_1, g_2\} \subset \mathcal{L}_\sigma$*. The Lie algebra also contains the vector field* $\mathrm{ad}_f g_1(x) = Dg_1(x)f(x) - Df(x)g_1(x) = (0, 0, -1)^T$ *that makes the LARC to be satisfied,*

$$\dim \mathcal{L}_\sigma(x) = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & -1 \end{bmatrix} = 3,$$

therefore the system σ *has the accessibility property.*

Example 8.6.4 *As the subsequent example we shall study the control affine system in the plane,*

$$\sigma : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \left(1 + x_1^2\right) u \end{cases}$$

with vector fields $f(x) = (x_2, x_1)^T$ *and* $g(x) = \left(0, 1 + x_1^2\right)^T$. *Its equilibrium point is* $u_0 = 0$ *and* $x_0 = 0$, *while its linear approximation is given as*

$$\dot{\xi} = A\xi(t) + Bv(t),$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *The Kalman matrix* $\Omega = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *has rank 2, so the system* σ *is locally controllable from the equilibrium point. The family of vector fields* $F_\sigma = \{f, g\} \subset \mathcal{L}_\sigma$; *besides, the Lie algebra* $\mathcal{L}_\sigma$ *of the system also contains the vector field* $\mathrm{ad}_f\, g(x) = \left(-\left(1 + x_1^2\right), 2x_1x_2\right)^T$. *The LARC*

$$\dim \mathcal{L}_\sigma(x) = \mathrm{rank} \begin{bmatrix} 0 & -\left(1 + x_1^2\right) \\ 1 + x_1^2 & 2x_1x_2 \end{bmatrix} = 2,$$

holds, then the system σ *has the accessibility property. Let us look more carefully at the drift vector field* $f(x)$. *The dynamic system defined by this vector field assumes the form*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}.$$

It is not hard to see that this is an oscillating system whose orbits are circles $x_1^2 + x_2^2 = C$. *This means that the drift vector field is Poison stable, as illustrated in Figure 8.6. In this case the accessibility property implies the controllability of the system* σ.

Example 8.6.5 *Eventually, we shall derive local controllability using the Sussmann's condition. To this objective we take the control affine system*

$$\sigma : \begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = u \end{cases},$$

whose vector fields are $f(x) = \left(x_2^3, 0\right)^T$ *and* $g(x) = (0, 1)^T$. *This system has the equilibrium point for* $u = 0$ *and* $x_0 = 0$. *First we compute the*

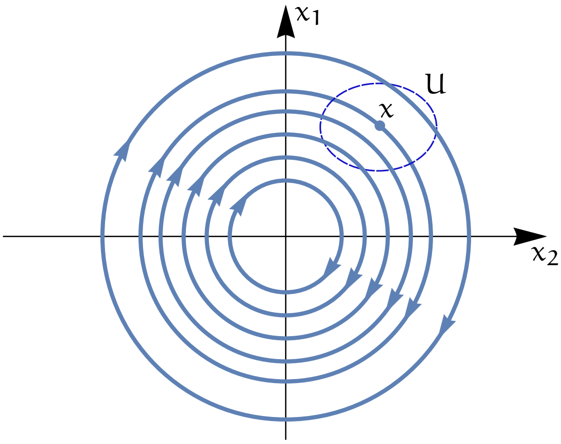

Figure 8.6: Poisson stability

distribution

$$\mathcal{S}^1(f, g) = \operatorname{span}\left\{g, \operatorname{ad}_f g, \ldots, \operatorname{ad}_f^j g, \ldots\right\}.$$

Because $\operatorname{ad}_f g(x) = \left(-3x_2^2, 0\right)^T$ *and for* $j \geqslant 2$ $\operatorname{ad}_f^j g(x) = 0$, *we get*

$$\mathcal{S}^1(f, g)(0) = \operatorname{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

Next, we find

$$\mathcal{S}^2(f, g) = \mathcal{S}^1(f, g) + \operatorname{span}\left\{\left[\operatorname{ad}_f^j g, \operatorname{ad}_f^k g\right]\Big|_{j,k\geqslant 0}\right\}.$$

The bracket $[g, \operatorname{ad}_f g](x) = (-6x_2, 0)^T$, *whereas the remaining brackets, in which the vector field* g *appears twice are equal to zero. This yields*

$$\mathcal{S}^2(f, g)(0) = \operatorname{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathcal{S}^1(f, g)(0).$$

Continuing the computations we find the distribution

$$\mathcal{S}^3(f, g) = \mathcal{S}^2(f, g) + \operatorname{span}\left\{\left[\operatorname{ad}_f^j g, \left[\operatorname{ad}_f^k g, \operatorname{ad}_f^l g\right]\right]\Big|_{j,k,l\geqslant 0}\right\}.$$

Now, taking into consideration that $[g, [g, \operatorname{ad}_f g]](x) = (-6, 0)^T$, *we deduce*

$$\mathcal{S}^3(f, g)(0) = \operatorname{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-6\\0\end{pmatrix}\right\} = R^2.$$

This observation as well as the fact that $\mathcal{S}^2(f, g)(0) = \mathcal{S}^1(f, g)(0)$ allows us to establish on the basis of the Sussmann's condition the local controllability of the system σ from the point 0.

8.7 Problems and exercises

Exercise 8.1 For the linear control system

$$\dot{x} = Ax + Bu,$$

$x \in R^n$, $u \in R^m$, show that the accessibility property implies controllability.

Exercise 8.2 For the control affine system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^2 \end{cases}$$

define the distribution d_σ, and examine the accessibility property, controllability, and local controllability of the system at the point $(0, 0)^T$.

Exercise 8.3 Examine the accessibility property of the controlled Euler equations (a, b, c, d - constant parameters)

$$\begin{cases} \dot{x}_1 = ax_2x_3 + bu \\ \dot{x}_2 = -ax_1x_3 + cu \\ \dot{x}_3 = du \end{cases}.$$

Exercise 8.4 Making use of the result of the exercise 8.3 prove controllability of the controlled Euler equations.

Exercise 8.5 Prove the controllability of the Brockett's integrator

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1u_2 - x_2u_1 \end{cases}.$$

8.8 Bibliographical remarks

A basic knowledge on control system can be gained from the monographs [Isi94, NvdS90, Son98, Kha00, KKK95, Blo03, Sas99, Lév09]. The formula for the derivative of the end-point map comes from [Son98]. Controllability of nonlinear control systems has been addressed in the monographs mentioned above; our presentation agrees with subsection 4.2 of [Lév09], and takes advantage of the results described in the paper [Kre85]. The Sussmann sufficient controllability condition is taken from [Sus83]; its generalisation for multi-input control systems has been published in [Sus87].

Bibliography

[Blo03] A. M. Bloch. *Nonholonomic Mechanics and Control*. Springer, New York, 2003.

[Isi94] A. Isidori. *Nonlinear Control Systems*. Springer, New York, 1994.

[Kha00] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, New Jersey, 2000.

[KKK95] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design*. J. Wiley, New York, 1995.

[Kre85] A. Krener. $(\mathrm{Ad}_{f,g})$, $(\mathrm{ad}_{f,g})$ and locally $(\mathrm{ad}_{f,g})$ invariant and controllability distributions. *SIAM J. Contr. Opt.*, 23(4):523–549, 1985.

[Lév09] J. Lévine. *Analysis and Control of Nonlinear Systems*. Springer, 2009.

[NvdS90] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, New York, 1990.

[Sas99] S. S. Sastry. *Nonlinear Systems*. Springer, New York, 1999.

[Son98] E. D. Sontag. *Mathematical Control Theory*. Springer, New York, 1998.

[Sus83] H. Sussmann. Lie brackets, real analyticity and geometric control. In *Differential Geometric Control Theory*, pages 1–116. Birkhäuser, Boston, 1983.

[Sus87] H. J. Sussmann. A general theorem on local controllability. *SIAM J. Contr. Opt.*, 25:158–194, 1987.

Chapter 9

Equivalence of control systems

9.1 State space and feedback equivalence

Let the following two control-affine systems be given

$$\begin{aligned}\sigma: \ \dot{x} &= f(x(t)) + g(x(t))u(t) = f(x(t)) + \textstyle\sum_{i=1}^{m} g_i(x(t))u_i(t),\\ \sigma': \ \dot{\xi} &= F(\xi(t)) + G(\xi(t))v(t) = F(\xi(t)) + \textstyle\sum_{i=1}^{m} G_i(x(t))v_i(t),\end{aligned}$$

where $x, \xi \in R^n$, $u, v \in R^m$, and all the vector fields are smooth. Similarly as in the case of dynamic systems, we shall define an equivalence of control systems. There are two concepts of such an equivalence: the state space equivalence, called S-equivalence, and the feedback equivalence that will be referred to as F-equivalence.

Definition 9.1.1 *Suppose that* $u = v$. *The systems* σ *and* σ' *will be called* *S-equivalent,*

$$\sigma \underset{S}{\cong} \sigma' \iff (\exists \textit{ diffeomorphism } \xi = \varphi(x))$$
$$(D\varphi(x)f(x) = F(\varphi(x)) \textit{ and } D\varphi(x)g(x) = G(\varphi(x))).$$

The last equality means that $D\varphi(x)g_(x) = G_i(\varphi(x))$ for $i = 1, 2, \ldots, m$. Observe that, as a matter of fact, the S-equivalence of control systems is tantamount to the differential equivalence of the associated vector fields. We do not make any use here of the fact that these systems are control systems. For this reason, a more adequate to control systems is the feedback equivalence.

Definition 9.1.2 *The systems σ and σ' are named F-equivalent,*

$$\sigma \underset{F}{\cong} \sigma' \Longleftrightarrow (\exists \text{ diffeomorphism } \xi = \varphi(x) \text{ and feedback } u = \alpha(x) + \beta(x)v)$$
$$(D\varphi(x)(f(x) + g(x)\alpha(x)) = F(\varphi(x)) \text{ and } D\varphi(x)g(x)\beta(x) = G(\varphi(x))).$$

The function $\alpha(x)$ appearing in the feedback is smooth, while $\beta(x)$ denotes a non-singular matrix of dimension $m \times m$ smoothly depending on x.

It is easily seen that the F-equivalence converts to the S-equivalence when $\alpha(x) = 0$ and $\beta(x) = I_m$. Both these equivalences rely on an assumption that trajectories of the equivalent systems are diffeomorphic, $\xi(t) = \varphi(x(t))$. For the F-equivalence this means that

$$\dot{\xi} = D\varphi(x)\dot{x} = D\varphi(x)(f(x) + g(x)u) = D\varphi(x)(f(x) + g(x)(\alpha(x) + \beta(x)v)$$
$$= D\varphi(x)(f(x) + g(x)\alpha(x)) + D\varphi(x)g(x)\beta(x)v = F(\varphi(x)) + G(\varphi(x))v.$$

Obviously, if the diffeomorphism $\varphi(x)$ is defined locally, we speak of the local S- or F-equivalence, denoted, correspondingly, by the symbols $\cong_{LS}$ and $\cong_{LF}$. The introduced concepts of S- and F-equivalence specify to the corresponding equivalences of linear control systems. Namely, for linear systems

$$\sigma_L : \dot{x} = Ax(t) + Bu(t),$$
$$\sigma'_L : \dot{\xi} = F\xi(t) + Gv(t)$$

we get

$$\sigma_L \underset{F}{\cong} \sigma'_L \Longleftrightarrow (\exists\, P, Q, K)(P(A + BK) = FP \text{ and } PBQ = G),$$

for non-singular matrices P, Q of dimensions $n \times n$ and $m \times m$, and for an arbitrary matrix K of dimension $m \times n$. It follows that for linear systems $\varphi(x) = Px$, $\alpha(x) = Kx$ and $\beta(x) = Q$.

Remark 9.1.1 *In order to establish the F-equivalence between the systems σ and σ' one needs to find the functions $\varphi(x)$, $\alpha(x)$ and $\beta(x)$, for given $f(x)$, $g(x)$, $F(\xi)$ and $G(\xi)$. This boils down to solving the equivalence equations*

$$D\varphi(x)(f(x) + g(x)\alpha(x)) = F(\varphi(x)) \quad \text{and} \quad D\varphi(x)g(x)\beta(x) = G(\varphi(x))).$$

Notice that the matrix $D\varphi(x) = \begin{bmatrix} \frac{\partial \varphi_1(x)}{\partial x_1} & \cdots & \frac{\partial \varphi_1(x)}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial \varphi_n(x)}{\partial x_1} & \cdots & \frac{\partial \varphi_n(x)}{\partial x_n} \end{bmatrix}$ contains partial derivatives of unknown components of the diffeomorphism, so the equivalence equations take the form of nonlinear partial differential equations. In what follows, conditions for the existence of the feedback will be stated for very specific "target" systems σ'. From the viewpoint of control problems, the most important case is when for the system σ' there exist well known control algorithms. Undoubtedly, the linear control systems $\sigma' = \sigma'_L$ belong to this class.

9.2 State space and feedback linearisation

Consider a control-affine system σ. We make the following definitions:

Definition 9.2.1 *The system σ is called state space linearisable (linearisable by a change of coordinates in the state space), in short S-linearisable, if*

$$\sigma \underset{S}{\cong} \sigma'_L.$$

If the S-equivalence holds locally ($\sigma \cong_{LS} \sigma'_L$) then the system σ is referred to as locally S-linearisable.

Similarly, we state the next.

Definition 9.2.2 *The system σ is called feedback linearisable, (F-linearisable), if*

$$\sigma \underset{F}{\cong} \sigma'_L.$$

In the case of a local F-equivalence ($\sigma \cong_{LF} \sigma'_L$), the system σ will be referred to as locally F-linearisable.

Necessary and sufficient conditions for the linearisation will be provided below. Assume that the system σ has in $u = 0 \in R^m$ and $x_0 = 0 \in R^n$ an equilibrium point (so $f(0) = 0$), and let the linear system σ'_L be controllable. For the system σ we define a family of distributions

$$\begin{aligned} \mathcal{D}^0 &= \operatorname{span}\left\{ g_i|_{i=1,\dots,m} \right\}, \\ &\vdots \\ \mathcal{D}^k &= \operatorname{span}\left\{ g_i, \operatorname{ad}_f g_i, \dots, \operatorname{ad}_f^k g_i \Big|_{i=1,\dots,m} \right\}, \end{aligned}$$

for $k \geqslant 0$, where $\operatorname{ad}_f^{k+1} g_i = \left[f, \operatorname{ad}_f^k g_i\right]$. Then we have

Theorem 9.2.1 (Krener-Sussmann-Respondek) *The system σ is locally S-linearisable around 0*

$$\sigma \underset{LS}{\cong} \sigma'_L \Longleftrightarrow \dim \mathcal{D}^{n-1}(0) = n \textit{ and } \left[\mathrm{ad}_f^p g_i, \mathrm{ad}_f^r g_j\right](x) = 0$$

in a certain neighbourhood of 0, *for* $p, r \geqslant 0,\ p + r \leqslant 2n - 1$.

Conditions for F-linearisation are included in the next theorem.

Theorem 9.2.2 (Jakubczyk-Respondek) *The system σ jest is locally* F*-linearisable around* 0*,*

$$\sigma \underset{LF}{\cong} \sigma'_L \Longleftrightarrow \dim \mathcal{D}^{n-1}(0) = n \textit{ and distributions } \mathcal{D}^k \textit{ for } k = 0, 1, \ldots, n-2$$

are in a certain neighbourhood of 0 *of constant dimension and involutive, i.e.*

$$\dim \mathcal{D}^k(x) = \mathrm{const}, \quad \left[\mathcal{D}^k, \mathcal{D}^k\right] \subset \mathcal{D}^k.$$

9.3 Equivalence equations

We shall study in more depth the S- and F-equivalences of control-affine systems

$$\begin{aligned} \sigma: \ \dot{x} &= f(x(t)) + g(x(t))u(t) = f(x(t)) + \textstyle\sum_{i=1}^m g_i(x(t))u_i(t), \\ \sigma': \ \dot{\xi} &= F(\xi(t)) + G(\xi(t))v(t) = F(\xi(t)) + \textstyle\sum_{i=1}^m G_i(\xi(t))v_i(t), \end{aligned}$$

where $x, \xi \in R^n$, $u, v \in R^m$. From the viewpoint of the synthesis of control algorithms, a fundamental problem consists in determining the transformations establishing the equivalence, i.e. a diffeomorphism $\xi = \varphi(x)$ for S-equivalence and a feedback transformations $\xi = \varphi(x)$, $u = \alpha(x) + \beta(x)v$ for the feedback equivalence. To this objective we need to solve a system of partial differential equations called the equivalence equations. We shall now state these equations.

- S-equivalence: $D\varphi(x)f(x) = F(\varphi(x))$, $D\varphi(x)g(x) = G(\varphi(x))$.
- F-equivalence: $D\varphi(x)f(x) + g(x)\alpha(x)) = F(\varphi(x))$, $D\varphi(x)g(x)\beta(x) = G(\varphi(x))$.

In the case when we study the linearisation problem, so when

$$\sigma': \ \dot{\xi} = F\xi(t) + Gv(t),$$

F and G – matrices, the equivalence equations get simplified thanks to a specific choice of the system σ'. To this aim we assume that $u = 0$ and $x_0 = 0$ is an equilibrium point of the system σ. Then, in the problem of S-linearisation the system σ' is taken as the linear approximation of the system σ at the equilibrium point. This means that the matrix $F = \frac{\partial f(0)}{\partial x}$, while the matrix $G = g(0)$. Dealing with the problem of F-linearisation we first find the linear approximation and check its controllability, then compute its controllability indices $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$, and finally define the corresponding Brunovsky canonical form. The matrices F and G need to be chosen in the Brunovsky form. For the linearisation problem the equivalence equations will take the following form:

- S-equivalence: $D\varphi(x)f(x) = F\varphi(x)$, $D\varphi(x)g(x) = G$.
- F-equivalence: $D\varphi(x)(f(x) + g(x)\alpha(x)) = F\varphi(x)$, $D\varphi(x)g(x)\beta(x) = G$.

Solutions of example equivalence equations will be given below, in subsection Examples.

9.4 Significance of linearisability for the synthesis of control algorithms

Consider the control-affine system

$$\sigma: \ \dot{x} = f(x(t)) + g(x(t))u(t)$$

and the feedback equivalent linear system in the Brunovsky form

$$\sigma': \ \dot{\xi} = F\xi(t) + G\nu(t)$$

characterised by controllability indices $(\kappa_1, \kappa_2, \ldots, \kappa_m)$. We assume that $\xi = \varphi(x)$, $u = \alpha(x) + \beta(x)\nu$ denote a feedback linearising the system σ. The result of applying this feedback is shown in Figure 9.1.

Suppose that in the system σ we address the following state trajectory tracking problem: Given a reference trajectory $x_d(t)$, find a control $u(t)$ in the system σ, such that the resulting trajectory $x(t) \to x_d(t)$ for $t \to +\infty$. The linearisability of the system σ will allow us to transform the reference trajectory to the linear system, $\xi_d(t) = \varphi(x_d(t))$, and to formulate the tracking problem in the linear system: Find a control $\nu(t)$, such that the corresponding trajectory $\xi(t) \to \xi_d(t)$. Since the system σ' has the

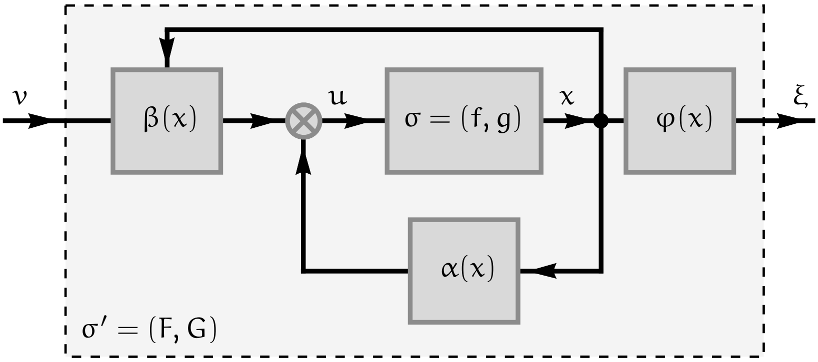

Figure 9.1: Result of applying linearising feedback

Brunovsky form

$$\begin{cases} \dot\xi_1 = \xi_2, \dot\xi_2 = \xi_3, \dots, \dot\xi_{\kappa_1} = \nu_1 \\ \dot\xi_{\kappa_1+1} = \xi_{\kappa_1+2}, \dots, \dot\xi_{\kappa_1+\kappa_2} = \nu_2 \\ \quad\vdots \\ \dot\xi_{\kappa_1+\dots+\kappa_{m-1}+1} = \xi_{\kappa_1+\dots+\kappa_{m-1}+2}, \dots, \dot\xi_{\sum_{i=1}^m \kappa_i = n} = \nu_m \end{cases},$$

what means that $\xi_1^{(\kappa_1)} = \nu_1$, $\xi_{\kappa_1+1}^{(\kappa_2)} = \nu_2$, …,$\xi_{\kappa_1+\dots+\kappa_{m-1}+1}^{(\kappa_m)} = \nu_m$, it is not hard to verify that the tracking algorithm in the system σ' may take the form

$$\lambda : \begin{cases} \nu_1 = \xi_{d1}^{(\kappa_1)} - k_{1\kappa_1-1}(\xi_1 - \xi_{d1})^{(\kappa_1-1)} - \dots - k_{10}(\xi_1 - \xi_{d1}) \\ \quad\vdots \\ \nu_m = \xi_{d\kappa_1+\dots+\kappa_{m-1}+1}^{(\kappa_m)} + \\ \qquad -k_{m\kappa_{m-1}}(\xi_{\kappa_1+\dots+\kappa_{m-1}+1} - \xi_{d\kappa_1+\dots+\kappa_{m-1}+1}^{(\kappa_m-1)}) - \dots + \\ \qquad\qquad -k_{m0}(\xi_{\kappa_1+\dots+\kappa_{m-1}+1} - \xi_{d\kappa_1+\dots+\kappa_{m-1}+1}) \end{cases}. \quad (9.1)$$

With notation $e_i = \xi_i - \xi_{di}$ we derive the tracking error equations in the linear system as follows

$$\begin{cases} e_1^{(\kappa_1)} + k_{1\kappa_1-1}e_1^{(\kappa_1-1)} + \dots + k_{10}e_1 = 0 \\ \quad\vdots \\ e_{\kappa_1+\dots+\kappa_{m-1}+1}^{(\kappa_m)} + k_{m\kappa_m-1}e_{\kappa_1+\dots+\kappa_{m-1}+1}^{(\kappa_m-1)} + \dots + k_{m0}e_{\kappa_1+\dots+\kappa_{m-1}+1} = 0 \end{cases}.$$

If the gains k_{ij} are selected in such a way that the characteristic polynomial of each differential equation is Hurwitz then the tracking problem in

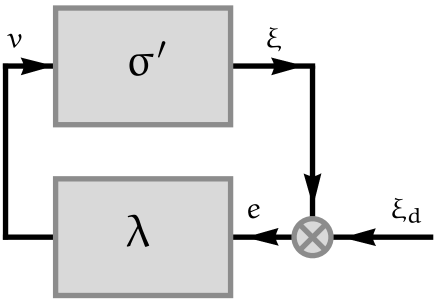

Figure 9.2: Trajectory tracking system

the linear system will be solved by the algorithm (9.1). Having applied the control $\nu(t)$ to the system σ' we find the trajectory $\xi(t)$, then the trajectory $x(t) = \varphi^{-1}(\xi(t))$, and the control $u(t) = \alpha(x(t)) + \beta(x(t))\nu(t)$ in the original system σ. This way of proceeding establishes a synthesis procedure of the tracking control algorithm based on the feedback linearisation. The algorithm is illustrated in Figure 9.2. Similar results can be obtained if, instead of being the linear system, σ' will have another form for which there exists a tracking control algorithm. One of such form will be described in chapter 11.

9.5 Examples

Example 9.5.1 *Let a control-affine system*

$$\sigma : \begin{cases} \dot{x}_1 = x_2\left(1 + x_1^2\right) \\ \dot{x}_2 = \arctan x_1 + u \end{cases}$$

be given, with vector fields $f(x) = \left(x_2\left(1 + x_1^2\right), \arctan x_1\right)^T$, $g(x) = (0, 1)^T$. *We have* $f(0) = 0$. *Assume that the system* σ' *is the linear approximation of the system* σ *at the equilibrium point. This means that*

$$\sigma' : \begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \nu \end{cases} .$$

It is easily checked that the system σ' *is controllable. We shall check whether the system* σ *is S-linearisable. To this objective we compute* $\operatorname{ad}_f g(x) = [f, g](x) = -(1 + x_1^2, 0)^T$, *hence* $\mathcal{D}^1 = \operatorname{span}\{g, \operatorname{ad}_f g\}$ *and*

$$\dim \mathcal{D}^1(0) = 2.$$

Next, we compute the Lie brackets $[g, \mathrm{ad}_f\, g]$, $[g, \mathrm{ad}_f^2\, g]$, $[\mathrm{ad}_f\, g, \mathrm{ad}_f^2\, g]$ *and* $[g, \mathrm{ad}_f^3\, g]$. *We have* $[g, \mathrm{ad}_f\, g] = 0$, *and also* $\mathrm{ad}_f^2\, g = [f, \mathrm{ad}_f\, g] = g$, *what implies* $[g, \mathrm{ad}_f^2\, g] = [g, g] = 0$, $\mathrm{ad}_f^3\, g = [f, \mathrm{ad}_f^2\, g] = [f, g] = \mathrm{ad}_f\, g$ *as well as* $g, \mathrm{ad}_f^3\, g] = [g, \mathrm{ad}_f\, g] = 0$. *Now, relying on the Theorem 9.2.1, we conclude that the system* σ *is locally* S*-linearisable around the equilibrium point.*

Example 9.5.2 *Examine the control-affine system*

$$\sigma : \begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = u \end{cases} ,$$

whose vector fields are $f(x) = (\sin x_2, 0)^T$ *and* $g(x) = (0, 1)^T$. *The vector field* f *vanishes at* 0. *The linear approximation of the system* σ *has the Brunovsky form, so it is controllable. The system* σ' *we also take in the Brunovsky canonical form,*

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \nu \end{cases} .$$

We compute the distributions $\mathcal{D}^0 = \mathrm{span}\{g\}$ *and* $\mathcal{D}^1 = \mathrm{span}\{g, \mathrm{ad}_f\, g\}$. *Because* $\mathrm{ad}_f\, g(x) = [f, g](x) = -(\cos x_2, 0)^T$, *we get* $\dim \mathcal{D}^1(0) = 2$. *The distribution* $\mathcal{D}^0$ *has constant dimension equal to* 1, *and is trivially involutive as a distribution generated by a single vector field. Therefore, the conditions of Theorem (9.2.2) are fulfilled, what implies that in a neighbourhood of the point* 0 *the system* σ *is* F*-linearisable.*

Example 9.5.3 *Now we shall find a diffeomorphism* $\xi = \varphi(x)$ *that realises* S*-linearisation of the system* σ *from Example 9.5.1. We begin with recalling the linear approximation of the system at the point* 0, $F = \frac{\partial f(0)}{\partial x} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ *and* $G = g(0) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. *We are looking for* S*-equivalence between the system* σ *and the linear system* σ' *described by matrices* F, G. *From Example 9.5.1 it follows that the linearising diffeomorphism exists. Suppose that it has the form* $\varphi(x) = (\varphi_1(x), \varphi_2(x))^T$. *Its components need to satisfy the equivalence equations*

$$D\varphi(x)f(x) = F\varphi(x), \quad D\varphi(x)g(x) = G,$$

that yield

$$d\varphi_1(x)f(x) = \varphi_2(x),\ d\varphi_2(x)f(x) = \varphi_1(x),$$
$$d\varphi_1(x)g(x) = 0,\ d\varphi_2(x)g(x) = 1.$$

Due to the form of the vector field $g(x)$ *we get*

$$\frac{\partial\varphi_1(x)}{\partial x_2} = 0 \quad \textit{and} \quad \frac{\partial\varphi_2(x)}{\partial x_2} = 1.$$

The former equality implies that $\varphi_1(x)$ *does not depend on* x_2, *so* $\varphi_1(x) = \varphi_1(x_1)$. *The latter equality is fulfilled, a.o. by* $\varphi_2(x) = x_2$. *Accepting this solution we compute* $\varphi_1(x_1) = d\varphi_1(x)f(x) = \arctan x_1$. *In this way we have found the diffeomorphism* $\xi=(\xi_1,\xi_2)^T=(\arctan x_1,x_2)^T$. *The system equations in new coordinates are*

$$\begin{cases} \dot{\xi}_1 = \frac{1}{1+x_1^2}\dot{x}_1 = x_2 = \xi_2 \\ \dot{\xi}_2 = \dot{x}_2 = \arctan x_1 + u = \xi_1 + u \end{cases}.$$

Example 9.5.4 *In turn, let us establish a feedback that linearises the system analysed in Example 9.5.2. We have already shown that such a feedback exists. We assume that the system* σ' *has the Brunovsky form, so* $F = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $G = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. *Our objective is to find out a diffeomorphism* $\xi = \varphi(x) = (\varphi_1(x), \varphi_2(x))^T$ *and functions* $\alpha(x)$, $\beta(x) \neq 0$ *satisfying the equivalence*

$$d\varphi_1(x)(f(x) + g(x)\alpha(x)) = \varphi_2(x),\ d\varphi_2(x)(f(x) + g(x)\alpha(x)) = 0,$$
$$d\varphi_1(x)g(x)\beta(x) = 0,\ d\varphi_2(x)g(x)\beta(x) = 1.$$

Using the fact that $\beta(x) \neq 0$, *the last two equalities result in the identity* $\frac{\partial\varphi_1(x)}{\partial x_2} = 0$, *so* $\varphi_1(x) = \varphi_1(x_1)$, *and* $\beta(x) = \frac{1}{d\varphi_2(x)g(x)}$. *Taking into account two first equalities, we deduce* $\varphi_2(x) = d\varphi_1(x)f(x)$ *and* $\alpha(x) = -\frac{d\varphi_2(x)f(x)}{d\varphi_2(x)g(x)}$. *For the reason that* φ_1 *depends only on* x_1, *we shall try the simplest solution* $\varphi_1(x) = x_1$. *With this assumption we compute* $\varphi_2(x) = \sin x_2$. *Having exploited the diffeomorphism* $\varphi(x) = (x_1, \sin x_2)^T$, *by suitable substitutions we determine the remaining elements of the linearising feedback,* $\alpha(x) = 0$ *and* $\beta(x) = \frac{1}{\cos x_2}$. *The resulting feedback is well defined in the set* $R \times (-\pi/2, \pi/2)$. *In the new coordinates, after applying the feedback, the system* σ *takes the form*

$$\begin{cases} \dot{\xi}_1 = \dot{x}_1 = x_2 = \xi_2 \\ \dot{\xi}_2 = \cos x_2 u = v \end{cases}.$$

Our choice of the component φ_1 *of the diffeomorphism has been quite arbitrary. This choice is by no means unique, as it is easily verified that the choice* $\varphi_1(x) = \sin x_1$ *leads to* $\varphi_2(x) = \cos x_1 \sin x_2$, $\alpha(x) =$

$\frac{\sin x_1 \sin x_2}{\cos x_1 \cos x_2}$, *and* $\beta(x) = \frac{1}{\cos x_1 \cos x_2}$. *This feedback is well defined in the square* $(-\pi/2, \pi/2)^2$, *and yields the same linear system in the Brunovsky form that the former feedback.*

Example 9.5.5 *Using the Brunovsky canonical form of the single-input linear system we shall now investigate the equivalence equations for a general control-affine system with singe input. Suppose that the system*

$$\sigma: \ \dot{x} = f(x) + g(x)u,$$

is given, where $x \in R^n$ *and* $u \in R$. *Let's take a linear system*

$$\sigma': \ \dot{\xi} = F\xi + Gv$$

in the Brunovsky form, so $F = \left[\begin{smallmatrix} 0 & I_{n-1} \\ 0 & 0 \end{smallmatrix}\right]$ *and* $G = \left[\begin{smallmatrix} 0_{n-1} \\ 1 \end{smallmatrix}\right]$. *We are looking for a diffeomorphism* $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^T$ *and functions* $\alpha(x)$, $\beta(x) \neq 0$. *Under assumption that the system* σ *satisfies conditions of Theorem 9.2.2, the equivalence equations*

$$\begin{aligned} &D\varphi(x)(f(x) + g(x)\alpha(x)) = F\varphi(x) = (\varphi_2(x), \ldots, \varphi_n(x), 0)^T, \\ &D\varphi(x)g(x)\beta(x) = G = (0, \ldots, 0, 1)^T \end{aligned} \tag{9.2}$$

have a solution. It is not hard to notice that the latter group of these equations is of the form

$$d\varphi_1(x)g(x)\beta(x) = \cdots = d\varphi_{n-1}(x)g(x)\beta(x) = 0, \ d\varphi_n(x)g(x)\beta(x) = 1,$$

what results in

$$d\varphi_1(x)g(x) = \cdots = d\varphi_{n-1}(x)g(x) = 0, \ \textit{and } \beta(x) = \frac{1}{d\varphi_n(x)g(x)}.$$

Now, a substitution of the former group of the equivalence equations (9.2) allows one to find the diffeomorphism $\varphi(x)$, *if only its first component is known. Namely,*

$$\varphi_2(x) = d\varphi_1(x)f(x), \ \varphi_3(x) = d\varphi_2(x)f(x), \ldots, \varphi_n(x) = d\varphi_{n-1}(x)f(x).$$

We also obtain $\alpha(x) = -\frac{d\varphi_n(x)f(x)}{d\varphi_n(x)g(x)}$. *The total feedback is then defined by the function* $\varphi_1(x)$. *A geometric meaning of the choice of this function is revealed by the following reasoning, in which, for the sake of conciseness, we shall denote the Lie derivative of a function with respect to a vector field be* $L_f^0\varphi_1(x) = \varphi_1(x)$ *and* $L_f^{k+1}\varphi_1(x) = L_f\left(L_f^k\varphi_1(x)\right)$.

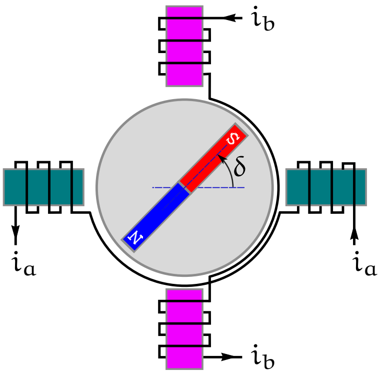

Figure 9.3: Model of induction electric motor

We set $L_{\mathrm{ad}_f\, g}\varphi_1(x) = L_{[f,g]}\varphi_1(x) = L_f L_g \varphi_1(x) - L_g L_f \varphi_1(x)$. *Under such assumptions the equations (9.2) can be written down as*

$$\begin{aligned} &L_g\varphi_1(x) = L_g\varphi_2(x) = \cdots = L_g\varphi_{n-1}(x) = 0,\\ &L_f\varphi_1(x) = \varphi_2(x),\ L_f\varphi_2(x) = \varphi_3(x), \ldots, L_f\varphi_{n-1}(x) = \varphi_n(x). \end{aligned} \tag{9.3}$$

Now, let us compute the Lie derivatives of the function $\varphi_1(x)$ *with respect to the vector fields* $\mathrm{ad}_f\, g$, $\mathrm{ad}_f^2\, g$, $\ldots$, $\mathrm{ad}_f^{n-1}\, g$. *The employment of the relationships (9.3) leads to the following conclusion*

$$\begin{aligned} &L_{\mathrm{ad}_f\, g}\varphi_1(x) = L_f L_g\varphi_1(x) - L_g L_f \varphi_1(x) = -L_g\varphi_2(x) = 0,\\ &L_{\mathrm{ad}_f^2\, g}\varphi_1(x) = L_f L_{\mathrm{ad}_f\, g}\varphi_1(x) - L_{\mathrm{ad}_f\, g} L_f\varphi_1(x) = L_g\varphi_3(x) = 0,\\ &\vdots\\ &L_{\mathrm{ad}_f^{n-2}\, g}\varphi_1(x) = 0,\ L_{\mathrm{ad}_f^{n-1}\, g}\varphi_1(x) = (-1)^{n-1} L_g\varphi_n(x) \neq 0. \end{aligned} \tag{9.4}$$

Geometrically, the equations (9.4) say that the function $\varphi_1(x)$ *needs to be chosen so that at each point the differential* $d\varphi_1(x)$ *be vertical to* $n-1$ *vectors* $g(x)$, $\mathrm{ad}_f\, g(x)$, $\ldots$, $\mathrm{ad}_f^{n-2}\, g(x)$, *while* $d\varphi_1(x)\,\mathrm{ad}_f^{n-1}(x) \neq 0$.

Example 9.5.6 *As a more practical example of a feedback linearisable system we shall examine a model of the induction electric motor displayed schematically in Figure 9.3. The electro-mechanical equations of the motor working without any loading on its shaft can be formulated as follows*

$$\begin{cases} \dot\delta = \omega,\\ \dot\omega = -\frac{k_m}{J} i_a \sin\delta + \frac{k_m}{J} i_b \cos\delta - \frac{F}{J}\omega,\\ \dot i_a = -\frac{R}{L} i_a + \frac{k_m}{L}\omega \sin\delta + \frac{u_a}{L},\\ \dot i_b = -\frac{R}{L} i_b - \frac{k_m}{L}\omega\cos\delta + \frac{u_b}{L}. \end{cases} \tag{9.5}$$

The symbols appearing in these equations have the following meaning: δ *– rotation angle of the rotor,* ω *– angular velocity of the rotor,* i_a, i_b, u_a, u_b *– currents of the stator and the supply voltages,* F, J *– mechanical parameters,* L, R *– electric parameters,* k_m *– electro-mechanical constant. To simplify the notations we introduce new variables* $x_1 = \delta$, $x_2 = \omega$, $x_3 = i_a$, $x_4 = i_b$, $\frac{k_m}{J} = a$, $\frac{F}{J} = b$, $\frac{R}{L} = c$, $\frac{k_m}{L} = d$, $\frac{u_a}{L} = u_1$, $\frac{u_b}{L} = u_2$*. In new variables the system (9.5) becomes control-affine, with the drift* $f(x) = (x_2, ax_3 \sin x_1 + ax_4 \cos x_1 - bx_2, -cx_3 + dx_2 \sin x_1, -cx_4 - dx_2 \cos x_1)^T$*, and control vectors* $g_1(x) = e_3$ *and* $g_2(x) = e_4$*, where* e_i *stands for the* i*-th unit vector in* R^4*. The system (9.5) has an equilibrium point* $x_0 = 0$*, and its linear approximation at this point is the following*

$$A = \frac{\partial f(0)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -b & 0 & a \\ 0 & 0 & -c & 0 \\ 0 & -d & 0 & -c \end{bmatrix}, \quad B = g(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By checking the Kalman condition $\operatorname{rank}\left[B, AB, A^2B\right] = 4$ *we establish that the linear approximation is controllable. For submatrices of the Kalman matrix we get the indices* $\rho_0 = \operatorname{rank} B = 2$, $\rho_1 = \operatorname{rank}[B, AB] - \operatorname{rank} B = 1$ *and* $\rho_2 = \operatorname{rank}[B, AB, A^2B] - \operatorname{rank}[B, AB] = 1$*, so the controllability indices of the linear approximation amount to* $\kappa_1 = 3$ *and* $\kappa_2 = 1$*. In order to check the linearisability conditions of the system in accordance with Theorem 9.2.2, we find the distributions*

$$\begin{aligned}
\mathcal{D}^0 &= \operatorname{span}\{g_1, g_2\} = \operatorname{span}\{e_3, e_4\},\\
\mathcal{D}^1 &= \operatorname{span}\{g_1, g_2, \operatorname{ad}_f g_1, \operatorname{ad}_f g_2\} = \operatorname{span}\{e_2, e_3, e_4\},\\
\mathcal{D}^2 &= \operatorname{span}\{g_1, g_2, \operatorname{ad}_f g_1, \operatorname{ad}_f g_2, \operatorname{ad}_f^2 g_1, \operatorname{ad}_f^2 g_2\} = \operatorname{span}\{e_1, e_2, e_3, e_4\}.
\end{aligned}$$

It is a direct consequence of the form of this distribution that the linearisability conditions are satisfied. The system (9.5) is F*-equivalent to the linear system*

$$\begin{cases} \dot\xi_1 = \xi_2 \\ \dot\xi_2 = \xi_3 \\ \dot\xi_3 = v_1 \\ \dot\xi_4 = v_2 \end{cases}.$$

9.6 Problems and exercises

Exercise 9.1 Examine S-linearisability of the following control systems:

a)

$$\begin{cases} \dot{x}_1 = e^{-x_1}x_2 \\ \dot{x}_2 = e^{x_1} - 1 + u \end{cases},$$

b)

$$\begin{cases} \dot{x}_1 = x_2 \cos^2 x_1 \\ \dot{x}_2 = \tan x_1 + u \end{cases}.$$

Exercise 9.2 Check F-linearisability of the control systems:

a)

$$\begin{cases} \dot{x}_1 = \dfrac{x_2}{\cos x_1} \\ \dot{x}_2 = \sin x_1 + u \end{cases},$$

b)

$$\begin{cases} \dot{x}_1 = x_2 + x_2^2 u \\ \dot{x}_2 = u \end{cases},$$

c)

$$\begin{cases} \dot{x}_1 = x_2 + e^{x_2}x_3 - e^{x_2}x_2^3 \\ \dot{x}_2 = x_3 - x_2^3 \\ \dot{x}_3 = 2x_2^2x_3 - 5x_2^5 + u \end{cases},$$

d)

$$\begin{cases} \dot{x}_1 = x_2 + e^{-x_3}x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases}.$$

Exercise 9.3 Show that the control system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 x_4^2 + \sin x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \end{cases}$$

representing the dynamics of a controlled ball and beam system is not F-linearisable.

Exercise 9.4 For the control system

$$\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = \sin x_3 \\ \dot{x}_3 = x_4^3 + u_1 \\ \dot{x}_4 = x_5 + x_4^3 - x_1^{10} \\ \dot{x}_5 = u_2 \end{cases}$$

write the equivalence equations involving a suitable Brunovsky canonical form, and solve them.

9.7 Bibliographical remarks

The concept of equivalence of control systems has been studied, a.o. in the publications [Kre73, JR80, Sus83, Res85, Jak90], specifically in the context of linearisation. The conditions of S-linearisation come from [Kre73, Res85, Sus83]. Fundamental results concerned with the feedback linearisation can be found in [JR80, HSM83]. An overview of these results is contained, e.g. in chapter 9 of the monograph [Sas99], in chapter 6 book [NvdS90] as well as in chapters 4 and 5 of the monograph [Isi94]. Genericity, or more adequately, non-genericity of linearisability and other properties of nonlinear control system is dealt with in [Tch86]. The linearisation of the model of the induction motor analysed in Example 9.5.6 is shown in [LU89].

Bibliography

[HSM83] L. R. Hunt, R. Su, and G. Meyer. Design for multi-input nonlinear systems. In *Differential Geometric Control Theory*, pages 268–298. Birkhäuser, Boston, 1983.

[Isi94] A. Isidori. *Nonlinear Control Systems*. Springer, New York, 1994.

[Jak90] B. Jakubczyk. Equivalence and invariants of nonlinear control systems. In *Nonlinear Controllability and Optimal Control*, pages 177–218. M. Dekker, New York, 1990.

[JR80] B. Jakubczyk and W. Respondek. On linearization of control systems. *Bull. Ac. Pol. Sci., ser. Sci. Math.*, 20:517–522, 1980.

[Kre73] A. Krener. On the equivalence of of control systems and linearization of nonlinear systems. *SIAM J. Contr. Opt.*, 11:670–676, 1973.

[LU89] A. De Luca and G. Ulivi. Design of exact nonlinear controller for induction motors. *IEEE Trans. Aut. Contr.*, 34(12):1304–1307, 1989.

[NvdS90] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, New York, 1990.

[Res85] W. Respondek. Geometric methods in linearization of control systems. In *Mathematical Control Theory*, pages 453–467. PWN, 1985.

[Sas99] S. S. Sastry. *Nonlinear Systems*. Springer, New York, 1999.

[Sus83] H. Sussmann. Lie brackets, real analyticity and geometric control. In *Differential Geometric Control Theory*, pages 1–116. Birkhäuser, Boston, 1983.

[Tch86] K. Tchoń. Generic properties of linear and nonlinear systems: A comparative study. In *Theory and Applications of Nonlinear Control Systems*, pages 521–532. Elsevier, Amsterdam, 1986.

Chapter 10

Input-output decoupling and linearisation

10.1 Differential degree

We shall deal with a control affine system with output

$$\sigma : \begin{cases} \dot{x} = f(x(t)) + g(x(t))u(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) \\ y = h(x) = (h_1(x), h_2(x), \dots, h_p(x))^T \end{cases}, \qquad (10.1)$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$. Assume that the number of control inputs is equal to the number of outputs, $m = p$. We are interested in the dependence between the j-th output of the system and the control. From the definition of the output function it follows that the control does not influence the output y_j directly. To reveal this influence we differentiate this output with respect to time along the system's trajectory $x(t)$. For the sake of the simplification of notation, we shall use the symbol of the Lie derivative of a function $dh_j(x)f(x) = L_f h_j(x)$, as well as to hide the time argument t,

$$\dot{y}_j = dh_j(x)\dot{x} = dh_j(x)(f(x) + g(x)u) = L_f h_j(x) + L_g h_j(x)u,$$

where $L_g h_j(x) = \left(L_{g_1} h_j(x), L_{g_2} h_j(x), \dots, L_{g_m} h_j(x)\right)^T$. Now, if in a certain neighbourhood of the point x the vector $L_g h_j(x)$ is non-zero, a direct influence of the control on the output y_j has been discovered. Suppose, however, that around x we have $L_g h_j(x) = 0$, what mean that $\dot{y}_j = L_f h_j(x)$. A subsequent differentiation leads to

$$\ddot{y}_j = L_f \dot{h}_j = dL_f h_j(x)\dot{x} = L_f^2 h_j(x) + L_g L_f h_j(x)u.$$

Again a neighbourhood of the point x, it may happen that there holds $L_g L_f h_j(x) \neq 0$. If this is the case, the control influences the second order derivative of the output y_j. But let that around the point x we have $L_g L_f h_j(x) = 0$. If so, we get $\ddot{y}_j = L_f^2 h_j(x)$, and the differentiation may be continued. Finally, suppose that there exist an integer ρ_j, such that all the Lie derivatives $L_g L_f^r h_j(x) = 0$, for $r = 0, 1, \ldots, \rho_j - 2$, but $L_g L_f^{\rho_j - 1} h_j(x) \neq 0$. This being so, the dependence between the control u and the output y_j appears to be the following

$$\begin{cases} y_j = h_j(x) \\ \dot{y}_j = L_f h_j(x) \\ \quad \vdots \\ y_j^{(\rho_j - 1)} = L_f^{\rho_j - 1} h_j(x) \\ y_j^{(\rho_j)} = L^{\rho_j} h_j(x) + L_g L_f^{\rho_j - 1} h_j(x) u \end{cases} . \tag{10.2}$$

The integer ρ_j will be called the differential degree or the relative degree of the output y_j. Repeating this procedure for all outputs we arrive at a collection of differential degrees $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$. One may expect that these differential degrees, if exist, are finite; otherwise certain system's output would not be influenced by any control, what could have indicated a sort of dysfunctionality of the system.

10.2 Decoupling

Given the differential degrees ρ_j, for all the outputs, and using the notation $y^\rho = \left(y_1^{(\rho_1)}, y_2^{(\rho_2)}, \ldots, y_m^{(\rho_m)}\right)^T$, we can write down the input-output relationship in the system as

$$y^\rho = L_f^\rho h(x) + L_g L_f^{\rho - 1} h(x) u = P(x) + D(x) u, \tag{10.3}$$

where

$$P(x) = \left(L_f^{\rho_1} h_1(x), L_f^{\rho_2} h_2(x), \ldots, L_f^{\rho_m} h_m(x)\right)^T \tag{10.4}$$

and

$$D(x) = \begin{bmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x) & \ldots & L_{g_m} L_f^{\rho_1 - 1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\rho_m - 1} h_m(x) & \ldots & L_{g_m} L_f^{\rho_m - 1} h_m(x) \end{bmatrix} . \tag{10.5}$$

The matrix $D(x)$ will be referred to as the decoupling matrix. If for a control affine system there exist the differential degree, and the decoupling matrix

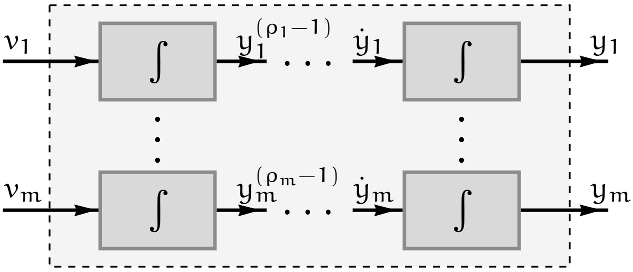

Figure 10.1: Structure of the input-output relationship

is non-singular then, by means of a feedback

$$u = \alpha(x) + \beta(x)v = -D^{-1}(x)P(x) + D^{-1}(x)v$$

the system can be converted to the decoupled form

$$y_j^{(\rho_j)} = v_j, \quad j = 1, 2, \ldots, m. \tag{10.6}$$

As can be easily seen, in the system (10.6) the control number j affects solely the jth output. The structure of the input-output relationship is shown in Figure 10.1.

Summarising, if a system has differential degrees $(\rho_1, \rho_2, \ldots, \rho_m)$, and a non-singular decoupling matrix $D(x)$ then the tracking control problem of an output trajectory $y_d(t)$ in such a system has a natural solution

$$\begin{cases} v_1 = y_{d1}^{(\rho_1)} - k_{1\rho_1-1}(y_1 - y_{d1})^{(\rho_1-1)} + \cdots + k_{10}(y_1 - y_{d1}) \\ \quad \vdots \\ v_m = y_{dm}^{(\rho_m)} - k_{m\rho_m-1}(y_m - y_{dm})^{(\rho_m-1)} + \cdots + k_{m0}(y_m - y_{dm}) \end{cases}.$$

It easy to check that, if $e_j = y_j - y_{dj}$ denotes the tracking error of the jth output then the error equations of the whole system can be represented in the form

$$e_j^{(\rho_j)} + k_{j\rho_j-1}e_j^{(\rho_j-1)} + \cdots + k_{j0}e_j = 0, \quad j = 1, 2, \ldots, m.$$

In order to guarantee the asymptotic stability of the error system the characteristic polynomial of each component error equation needs to be Hurwitz.

10.3 Dynamics of the decoupled system

Let the integers $(\rho_1, \rho_2, \ldots, \rho_m)$ denote differential degrees of the control system (10.1) that is input-output decouplable. Assume that the sum of the differential degrees $\sum_{j=1}^{m} \rho_j = s$, $s \leqslant n$. The feedback

$$u = -D^{-1}(x)P(x) + D^{-1}(x)v,$$

that has enabled to decouple the system transform its dynamics to the following form

$$\begin{cases} \dot{x} = f(x(t)) + g(x(t))D^{-1}(x)P(x) + g(x)D^{-1}(x)v = F(x) + G(x)v \\ y = h(x) \end{cases}.$$

In order to better understand the structure of the dynamic part of the system subject to the decoupling feedback, we introduce new coordinates

$$\xi = \varphi(x) = \begin{pmatrix} h_1(x) \\ L_f h_1(x) \\ \vdots \\ L_f^{\rho_1 - 1} h_1(x) \\ h_2(x) \\ L_f h_2(x) \\ \vdots \\ L_f^{\rho_2 - 1} h_2(x) \\ \vdots \\ h_m(x) \\ L_f h_m(x) \\ \vdots \\ L_f^{\rho_m - 1} h_m(x) \\ x^{n-s} \end{pmatrix},$$

where the first s components have been defined by using the dependences between the inputs and the derivatives of the output of the order ranging from 0 to $\rho_j - 1$, and the remaining coordinates, denoted as $\xi^{n-s} = x^{n-s}$, have been chosen from among $(x_1, x_2, \ldots, x_n)$ in such a way that $\varphi(x)$ be a local diffeomorphism. The independence of the first s coordinates can be proved on the basis of the definition of differential degrees.We observe that after applying the feedback $(\varphi(x), \alpha(x), \beta(x))$ the system's equations take

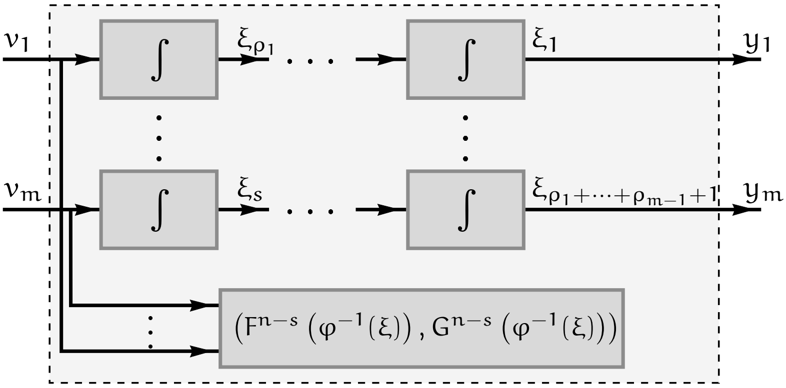

Figure 10.2: Structure of control system (10.7)

the following form

$$\begin{cases} \dot{\xi}_1 = \xi_2, \dot{\xi}_2 = \xi_3, \ldots, \dot{\xi}_{\rho_1} = v_1 \\ \quad \vdots \\ \dot{\xi}_{\rho_1+\cdots+\rho_{m-1}+1} = \xi_{\rho_1+\cdots+\rho_{m-1}+2}, \ldots, \dot{\xi}_s = v_m \\ \dot{\xi}^{n-s} = F^{n-s}\left(\varphi^{-1}(\xi)\right) + G^{n-s}\left(\varphi^{-1}(\xi)\right) v \\ y_1 = \xi_1 \\ y_2 = \xi_{\rho_1+1} \\ \quad \vdots \\ y_m = \xi_{\rho_1+\cdots+\rho_{m-1}+1} \end{cases} . \tag{10.7}$$

Hereabout, by $F^{n-s}(x)$ and $G^{n-s}(x)$ we mean those components of the vector fields $F(x)$ and $G(x)$ that correspond to the coordinates x^{n-s}. The structure of the system (10.7) is presented in Figure 10.2. It turns out that the s-dimensional subsystem of the system (10.7) has been decoupled and linearised. Also, it can be seen that there exists a subsystem described by the coordinates ξ^{n-s} that is controlled by v, but that does not have any influence on the system's output. The evolution of this subsystem should remain under control, and its trajectories be bounded. Suppose that at any time all the outputs of the system (10.7) are equal to zero. Then it follows that all the coordinates $\xi_1, \ldots, \xi_s = 0$, and also $v = 0$. Under such an assumption the dynamics of the coordinates ξ^{n-s} are described by the dynamic system

$$\dot{\xi}^{n-s} = F^{n-s}\left(\varphi^{-1}(0, \xi^{n-s})\right) = \bar{F}\left(\xi^{n-s}\right).$$

These dynamics are named the zero dynamics of the system (10.7). Now, in order to efficiently apply a control algorithm based on the decoupling, the zero dynamics must be stable (Lyapunov stable or asymptotically stable), at least locally. A system with asymptotically stable zero dynamics is called minimal phase. Obviously, if $s = n$ then the zero dynamics are absent, and the method of decoupling provides us with a linearising feedback without solving any equivalence equations. In this context we formulate the following observation.

Remark 10.3.1 *If the differential degrees of the system σ sum up to the the dimension of the state space then the system is feedback linearisable by the feedback of the form*

$$\begin{aligned}
&\varphi(x) = \left(h_1(x), \dots, L_f^{\rho_1 - 1} h_1(x), \dots, h_m(x), \dots, L_f^{\rho_m - 1} h_m(x)\right)^T, \\
&\alpha(x) = -D(x)^{-1} P(x), \\
&\beta(x) = D^{-1}(x),
\end{aligned}$$

where $P(x)$ and $D(x)$ are defined by the expressions (10.4) and (10.5).

10.4 Examples

Example 10.4.1 *Consider the dynamics equations of a non-redundant robotic manipulator with n degrees of freedom, described by the coordinate vector $q \in R^n$, control vector $u \in R^n$, and the vector of task space coordinates $y \in R^n$,*

$$Q(q)\ddot{q} + B(q, \dot{q}) = u,$$

where $Q(q)$ is the inertia matrix, and $B(q, \dot{q})$ denotes the vector of Coriolis, centripetal, and gravitational forces, with the kinematics

$$y = k(q).$$

To express these equations in the form of a control-affine system, we make the substitutions $x = q$ and $\xi = \dot{q}$. It is easily checked that the dynamics will be characterised by the following control-affine system with output

$$\begin{cases}
\dot{x} = \xi, \\
\dot{\xi} = -Q^{-1}(x)B(x, \xi) + Q^{-1}(x)u, \\
y = k(x).
\end{cases}$$

Apparently, the differential degrees of all outputs are the same, so to determine them we can differentiate the whole output vector simultaneously

$$\begin{aligned}\dot{y} &= Dk(x)\dot{x} = J(x)\xi,\\ \ddot{y} &= \dot{J}(x)\xi + J(x)\dot{\xi} = \underbrace{\dot{J}(x)\xi - J(x)Q^{-1}(x)B(x,\xi)}_{P(x,\xi)} + \underbrace{J(x)Q^{-1}(x)}_{D(x)}u.\end{aligned}$$

As a result we have obtained differential degrees $\rho_j = 2$*, and the decoupling matrix* $D(x)$ *that is non-singular outside the singular configurations of the manipulator. Furthermore, the sum of the differential degrees* $s = 2n$ *is equal to the state space dimension, hence the model of dynamics of the manipulator can be decoupled and linearised by feedback. While doing this, we do not need to use any coordinate change. After the application of the feedback*

$$u = -D^{-1}(x)P(x,\xi) + D^{-1}(x)\nu,$$

the input-output relationship takes the simple form

$$\ddot{y} = \nu.$$

To solve the tracking problem of a task space trajectory $y_d(t)$*, it is natural to exploit the PD algorithm with a feedforward term*

$$\nu = \ddot{y}_d - K_1(\dot{y} - \dot{y}_d) - K_0(y - y_d),$$

containing diagonal gain matrices K_0*,* K_1 *with positive entries. The resulting tracking algorithm*

$$u = -D^{-1}(x)P(x,\xi) + D^{-1}(x)(\ddot{y}_d - K_1(\dot{y} - \dot{y}_d) - K_0(y - y_d))$$

is well known in robotics, under the name of the Freund's algorithm.

Example 10.4.2 *Consider a single-input, single-output control system*

$$\begin{cases}\dot{x}_1 = x_3 - x_2^3\\ \dot{x}_2 = -x_2 - u\\ \dot{x}_3 = x_1^2 - x_3 + u\end{cases},$$

with the output function

$$y = h(x) = x_1.$$

The time differentiation of the output yields

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_3 - x_2^3 \\ \ddot{y} &= \underbrace{x_1^2 - x_3 + 3x_2^3}_{P(x)} + \underbrace{(1 + 3x_2^2)}_{D(x)}u.\end{aligned}$$

Thus we have found the differential degree $\rho_1 = 2$ *and the decoupling matrix (more appropriately: the coefficient)* $D(x) \neq 0$. *The feedback* $u = -\frac{P(x)}{D(x)} + \frac{1}{D(x)}\nu$ *leads to the decoupled input-output relationship* $\ddot{y} = \nu$. *For the reason that* $s = 2 < n = 3$, *the zero dynamics appear. In order to determine the zero dynamics we choose new coordinates*

$$\begin{cases}\xi_1 = h(x) = x_1 \\ \xi_2 = L_f h(x) = x_3 - x_2^3 \\ \xi_3 = x_2\end{cases}.$$

It is easily found that in a neighbourhood of the point $0 \in R^3$ *the function* $\xi = \varphi(x)$ *is a diffeomorphism. The system's equations in these new coordinates become*

$$\begin{cases}\dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \nu \\ \dot{\xi}_3 = -x_2 - u = -\xi_3 + \frac{1}{1+3\xi_3^2}\nu - \frac{1}{1+3\xi_3^2}\left(\xi_1^2 - \xi_2 + 2\xi_3^3\right).\end{cases}$$

Setting $y = 0$ *we get the zero dynamics*

$$\dot{\xi}_3 = -\xi_3 - \frac{2\xi_3^3}{1 + 3\xi_3^2} = -\xi_3 k(\xi_3),$$

where $k(\xi_3) \geqslant 1$. *Now, for* $x = \xi_3$, *let us choose the function* $V(x) = {}^1\!/_2 x^2$. *Then we obtain* $\dot{V} = x\dot{x} = -x^2 k(x) \leqslant -x^2 = -2V$. *As a result,* $V(t) \leqslant V(0)e^{-2t}$, *that implies* $|\xi_3(t)| \leqslant |\xi_{30}|e^{-t}$, *hence the zero dynamics are globally asymptotically stable.*

10.5 Problems and exercises

Exercise 10.1 For a non-redundant rigid manipulator described by the equations

$$\begin{cases}Q(q)\ddot{q} + B(q, \dot{q}) = u \\ y = k(q)\end{cases},$$

$q, u, y \in R^n$, relying on the input-output linearisation, derive a tracking algorithm of the trajectory $y_d(t)$ in the task space.

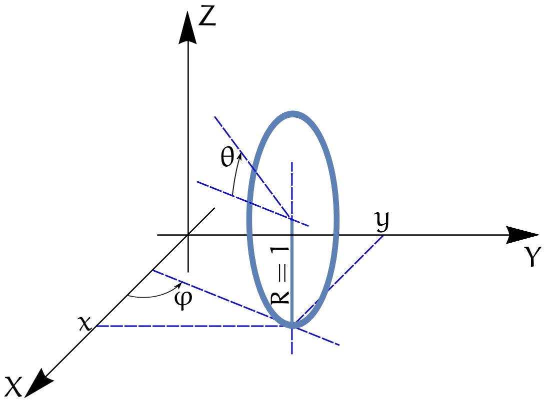

Figure 10.3: Vertical rolling wheel

Exercise 10.2 For the vertical rolling wheel in the plane, shown in Figure 10.3, described as a control-affine system with output

$$\begin{cases} \dot{x} = \eta_1 \cos\varphi \\ \dot{y} = \eta_1 \sin\varphi \\ \dot{\varphi} = \eta_2 \\ \dot{\theta} = \eta_1 \\ \dot{\eta}_1 = u_1 \\ \dot{\eta}_2 = u_2 \\ y_1 = x \\ y_2 = y \end{cases},$$

using the input-output linearisation devise a tracking algorithm of the trajectory $y_d(t) = (y_{d1}(t), y_{d2}(t))^T$. Introduce new coordinates and examine the zero dynamics o the system.

Exercise 10.3 Given a control system of the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sin x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \end{cases}.$$

Invoking the Jakubczyk-Respondek Theorem, and using the input-output linearisation with the output function $y = x_1$, demonstrate that this system is feedback linearisable in a neighborhood of the point $0 \in R^4$

10.6 Bibliographical remarks

The concept of the input-output decoupling and the related notions of the differential degree and the zero dynamics have been described in the monographs [NvdS90, Isi94]. A concise and accessible overview of these issues is also contained in chapter 9 of the book [Sas99]. Robotics aspects are dealt with in the monograph [MZS94].

Bibliography

[Isi94] A. Isidori. *Nonlinear Control Systems.* Springer, New York, 1994.

[MZS94] R. Murray, Li Zexiang, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation.* CRC Press, Boca Raton, 1994.

[NvdS90] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer, New York, 1990.

[Sas99] S. S. Sastry. *Nonlinear Systems.* Springer, New York, 1999.

Chapter 11

Chained form systems

The meaning of feedback for the synthesis of control systems results from the fact that it allows to transform a control problem from a system that is "hard" to analysis to an "easy" system in the normal form, with well known control algorithms. This has been demonstrated in the previous chapter by the example of the feedback linearisation. In this chapter we shall go further in this direction and show another normal form system along with a dedicated control method. The system we think of is the chained form control system.

11.1 Chained form

We shall study a driftless control system with two inputs

$$\sigma:\ \dot{x} = g(x(t))u(t) = g_1(x(t))u_1(t) + g_2(x(t))u_2(t), \quad x \in R^n. \tag{11.1}$$

The control distribution of this system $\mathcal{D} = \mathrm{span}_{C^\infty(R^n,R)}\{g_1, g_2\}$. A control affine system

$$\sigma':\ \dot{\xi} = G(\xi(t))\nu = G_1(\xi(t))\nu_1(t) + G_2(\xi(t))\nu_2(t)$$

is referred to as the chained form control system if it has either of the following two forms

$$\begin{aligned} &\sigma_1':\ \dot{\xi}_1 = \nu_1,\ \dot{\xi}_2 = \nu_2,\ \dot{\xi}_3 = \xi_2\nu_1, \dots, \dot{\xi}_n = \xi_{n-1}\nu_1 \\ &\sigma_2':\ \dot{\xi}_1 = \nu_1,\ \dot{\xi}_2 = \xi_3\nu_1,\ \dot{\xi}_3 = \xi_4\nu_1, \dots, \dot{\xi}_{n-1} = \xi_n\nu_1, \dot{\xi}_n = \nu_2. \end{aligned} \tag{11.2}$$

We look for a feedback $\xi = \varphi(x)$, $u = \beta(x)\nu$, that establishes the F-equivalence of systems σ and σ'. If such a feedback exists, it needs to satisfy the following equivalence equations

$$D\varphi(x)g(x)\beta(x) = G(\varphi(x)).$$

11.2 Murray's Theorem

Given the system (11.1), we define two families of distributions, for $k = 0, 1, \ldots, n-2$

$$\mathcal{D}_0 = \mathcal{D},\ \mathcal{D}_{k+1} = \mathcal{D}_k + [\mathcal{D}_0, \mathcal{D}_k]$$

and

$$\mathcal{D}^0 = \mathcal{D},\ \mathcal{D}^{k+1} = \mathcal{D}^k + \left[\mathcal{D}^k, \mathcal{D}^k\right].$$

The former family is called the small flag, the latter – the big flag of the distribution. The component–distributions of the small and the big flag are nested, i.e. $\mathcal{D}_k \subset \mathcal{D}_{k+1}$, $\mathcal{D}^k \subset \mathcal{D}^{k+1}$, we also have

$$\mathcal{D}_0 = \mathcal{D}^0,\ \mathcal{D}_1 = \mathcal{D}^1,\ \mathcal{D}_k \subset \mathcal{D}^k$$

for $k \geqslant 2$. The last dependence explains the terminology "small" and "big" flag. The following necessary and sufficient condition for a local F-equivalence of systems σ and σ' has been formulated in the language of flags.

Theorem 11.2.1

$$\sigma \underset{LF}{\cong} \sigma' \Longleftrightarrow \dim \mathcal{D}_k(x) = \dim \mathcal{D}^k(x) = k+2$$

for x in a certain open set, and $k = 0, 1, \ldots, n-2$.

11.3 Integrator backstepping

A control method applicable to the chained form systems is the integrator backstepping method. The idea of this method will be explained below, by the example of a single-input control affine system, of the form

$$\begin{cases} \dot{x} = f(x(t)) + g(x(t))\xi(t), \\ \dot{\xi} = u(t), \end{cases} \tag{11.3}$$

having the equilibrium point $u = 0$, $x = 0$, $\xi = 0$, where $x \in R^n$, $\xi, u \in R$. The system (11.3) can represent the error dynamics of a control system. We want to solve the problem of error stabilisation, i.e. to find a control $u(t)$, such that for $t \to +\infty$ the trajectory $(x(t), \xi(t)) \to 0$. The control algorithm will be defined in the form of a state feedback, $u = u(x, \xi)$. To this objective we proceed as follows:

- Consider the subsystem $\dot{x} = f(x(t)) + g(x(t))\xi(t)$, and let us treat the variable ξ as a temporary control. We assume that there exists for this system a stabilising feedback, i.e. a function $\xi = \phi(x)$, $\phi(0) = 0$, and functions $\alpha_1(\|x\|) \leqslant V_1(x) \leqslant \alpha_2(\|x\|)$, $W_1(x) \geqslant 0$, where α_1, α_2 are of class $\mathcal{K}$, such that along the trajectory of the system with feedback $\dot{x} = f(x(t)) + g(x(t))\phi(x)$ there holds

$$\dot{V}_1(x) = dV(x)(f(x) + g(x)\phi(x)) \leqslant -W_1(x) \leqslant 0.$$

- Introduce a new variable $z = \xi - \phi(x)$ and write down the system (11.3) in the form

$$\begin{cases} \dot{x} = \underbrace{f(x(t)) + g(x(t))\phi(x(t))}_{\text{stable}} + g(x(t))z(t), \\ \dot{z} = u(t) - \dot{\phi}(x(t)) = v(t), \end{cases} \tag{11.4}$$

where $\dot{\phi}(x(t))$ symbolises the time differentiation, while v denotes a new control. Now we are looking for a stabilising feedback control for the whole system (11.4). To this aim we choose the function

$$V_2(x, z) = V_1(x) + \frac{1}{2}z^2 \geqslant 0,$$

and compute

$$\dot{V}_2(x, z) = \dot{V}_1(x) + z\dot{z} \leqslant -W_1(x) + dV_1(x)g(x)z + zv = \\ - W_1(x) + (dV_1(x)g(x) + v)z.$$

Notice that after taking

$$dV_1(x)g(x) + v = -kz$$

for a certain $k > 0$, we obtain

$$\dot{V}_2(x, z) \leqslant -W_1(x) - kz^z,$$

that results in the stability of the system (11.4) with the control

$$v = -kz - dV_1(x)g(x).$$

If we assumed that the function $W_1(x) \geqslant \alpha_3(\|x\|)$, for a $\mathcal{K}$-class function α_3 then we would get the asymptotic stability of the system (11.4). Since $\xi = z - \phi(x)$ and $\phi(0) = 0$, the convergence of $(x(t), z(t))$ to 0 implies that $\xi(t)$ also converges to 0. The stabilising control for the system (11.3) has therefore the form

$$u(x, \xi) = -k(\xi - \phi(x)) - dV_1(x)g(x) + d\phi(x)(f(x) + g(x)\xi).$$

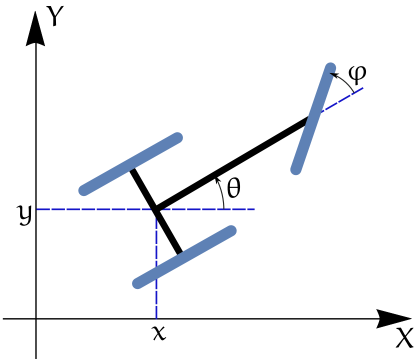

Figure 11.1: Kinematic car

The presented procedure generalises to control systems containing many integrations,

$$\begin{cases} \dot{x} = f(x(t)) + g(x(t))\xi_1(t), \\ \dot{\xi}_1 = \xi_2(t) \\ \vdots \\ \dot{\xi}_{k-1} = \xi_k(t), \\ \dot{\xi}_k = u. \end{cases}$$

11.4 Examples

Example 11.4.1 *The subject of our analysis will be the kinematic car shown in Figure 11.1. Let* $q = (x, y, \theta, \varphi)^T$ *denote the coordinate vector describing the car (see the figure). Under assumption that the lateral slip of the front and the rear wheels is not permitted, the model of kinematics of the car assumes the form of a driftless control system*

$$\begin{cases} \dot{x} = u_1 \cos\theta \cos\varphi \\ \dot{y} = u_1 \sin\theta \cos\varphi \\ \dot{\theta} = u_1 \sin\varphi \\ \dot{\varphi} = u_2 \end{cases} . \tag{11.5}$$

We shall demonstrate that this system is locally F*-equivalent to a chained form system, and more specifically to the system* σ_2' *that has appeared in the formula (11.2). Suppose that the coordinates* θ *and* φ *of the system are bounded to the range* $\pm\pi/2$*, therefore* $|\theta|, |\varphi| < \pi/2$*. With* *such an assumption we can define a preliminary feedback*

$$w = \begin{bmatrix} \cos\theta\cos\varphi & 0 \\ 0 & 1 \end{bmatrix} u,$$

that will allow us to write down the system (11.5) as

$$\begin{cases} \dot{x} = w_1 \\ \dot{y} = w_1 \tan\theta \\ \dot{\theta} = w_1 \dfrac{\tan\varphi}{\cos\theta} \\ \dot{\varphi} = w_2 \end{cases} . \tag{11.6}$$

The system (11.6) is described by two control vector fields

$$g_1(q) = \begin{pmatrix} 1 \\ \tan\theta \\ \dfrac{\tan\varphi}{\cos\theta} \\ 0 \end{pmatrix}, \quad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

so the distribution $\mathcal{D} = \mathrm{span}\{g_1, g_2\}$*. Now we find the small and the big flag. It follows from the Theorem 11.2.1 that it is enough to compute the following distributions:*

$$\begin{aligned} &\mathcal{D}_0 = \mathcal{D}^0 = \mathcal{D} \ni g_1, g_2 \\ &\mathcal{D}_1 = \mathcal{D}^1 = \mathcal{D}_0 + [\mathcal{D}_0, \mathcal{D}_0] \ni g_1, g_2, g_{12} = [g_1, g_2], \\ &\mathcal{D}_2 = \mathcal{D}_1 + [\mathcal{D}_0, \mathcal{D}_1] = \mathcal{D}^2 \ni g_1, g_2, g_{12}, g_{112} = [g_1, g_{12}], g_{212} = [g_2, g_{12}]. \end{aligned}$$

In our case we also have the identity of distributions $\mathcal{D}_2 = \mathcal{D}^2$*; this feature is not general, but results from the fact that the distribution* $\mathcal{D}$ *has two generators. A computation of Lie brackets gives*

$$g_{12}(q) = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{1}{\cos\theta\cos^2\varphi} \\ 0 \end{pmatrix}, \quad g_{112}(q) = \begin{pmatrix} 0 \\ \dfrac{1}{\cos^3\theta\cos^2\varphi} \\ 0 \\ 0 \end{pmatrix}.$$

Now it is easily checked that the distribution $\mathcal{D}_1 = \mathrm{span}\{g_1, g_2, g_{12}\}$*, while the distribution* $\mathcal{D}_2 = \mathrm{span}\{g_1, g_2, g_{12}, g_{112}\}$*, thus at every point* $q \in R \times R \times (-\pi/2, +\pi/2)^2$ *the following conditions hold*

$$\begin{aligned} &\dim \mathcal{D}_0(q) = \dim \mathcal{D}^0(q) = 2, \ \dim \mathcal{D}_1(q) = \dim \mathcal{D}^1(q) = 3, \\ &\dim \mathcal{D}_2(q) = \dim \mathcal{D}^2(q) = 4. \end{aligned}$$

Theorem 11.2.1 yields that the system (11.6) is locally F*-equivalent to the chained form system, and for the reason that the system (11.5) is* F*-equivalent to (11.6) the kinematics of the kinematic car is locally feedback equivalent to the chained form system.*

Example 11.4.2 *As an illustration of the integrator backstepping method we shall derive a stabilisation algorithm of the equilibrium point* $0 \in R^2$ *in the following system*

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = u \end{cases}.$$

In accordance with the scheme of the method we shall first treat the variable x_2 *as a control, and find a stabilising feedback* $x_2 = \phi(x_1)$*. For this purpose we pick a function* $V_1(x_1) = 1/2 x_1^2$*, and compute*

$$\dot{V}_1(x_1) = x_1\dot{x}_1 = x_1^3 - x_1^4 + x_1\phi(x_1) \leqslant x_1^3 + x_1\phi(x_1) = x_1\left(x_1^2 + \phi(x_1)\right).$$

It is easy to observe that the choice $\phi(x_1) = -k_1x_1 - x_1^2$ *for* $k_1 > 0$ *yields* $\dot{V}_1(x_1) \leqslant -k_1x_1^2 = -W_1(x_1)$*, that in turn gives the asymptotic stability of the dynamics of the variable* x_1*. Next, we introduce the variable* $z = x_2 - \phi(x_1)$ *and re-write the equations of the whole system in the form*

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + \phi(x_1) + z \\ \dot{z} = u - \dot{\phi}(x_1) = \nu \end{cases}.$$

For this last system we take the function $V_2(x_1, z) = V_1(x_1) + 1/2 z^2$*. Its derivative along the trajectory amounts to*

$$\dot{V}_2(x_1, z) = \dot{V}_1 + z\dot{z} \leqslant -W_1(x_1) + \left(\frac{dV_1(x_1)}{dx_1} + \nu\right)z.$$

Now if $\frac{dV_1(x_1)}{dx_1} + \nu = -k_2z$ *then the control* $\nu = -k_2z - \frac{dV_1(x_1)}{dx_1}$ *leads to*

$$\dot{V}_2(x_1, z) \leqslant -W_1(x_1) - k_2z^z = -k_1x_1^2 - k_2z^2,$$

so stabilises the system described by the variables (x_1, z)*, ensuring a convergence of the trajectory* $(x_1(t), z(t))$ *to zero. Because* $\phi(0) = 0$*, this implies the convergence to zero of the original trajectory* $(x_1(t), x_2(t))$*. Finally, the stabilising control for the system* (x_1, x_2) *is equal to*

$$u(x_1, x_2) = \nu + \dot{\phi}(x_1) = -k_2(x_2 - \phi(x_1)) - \frac{dV_1(x_1)}{dx_1} + \frac{d\phi(x_1)}{dx_1}(x_1^2 - x_1^3 + x_2),$$

where $V_1(x_1) = 1/2 x_1^2$ *and* $\phi(x_1) = -k_1x_1 - x_1^2$*.*

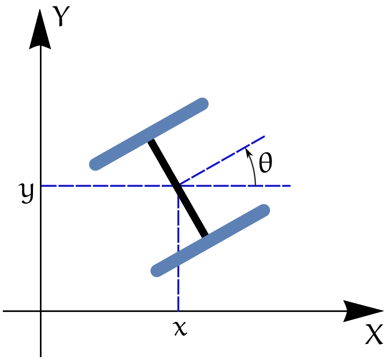

Figure 11.2: Unicycle

Example 11.4.3 *As the next example of the application of the integrator backstepping method we shall consider a kinematics model of the unicycle mobile robot, described in coordinates* $q = (x, y, \theta)^T$*, presented in Figure 11.2. The kinematics equation of the unicycle are the following*

$$\begin{cases} \dot{x} = v \cos\theta(t) \\ \dot{y} = v \sin\theta(t) \\ \dot{\theta} = w \end{cases}.$$

Assume that the control problem of the unicycle consists in the tracking of the reference trajectory $(x_d(t), y_d(t))$*. Let this trajectory be realisable by the unicycle (admissible), what means that there exist a reference control* $(v_d(t), w_d(t))$*, such that* $\dot{x}_d = v_d(t)\cos\theta_d(t)$*,* $\dot{y}_d = v_d(t)\sin\theta_d(t)$*,* $\dot{\theta}_d = w_d(t)$*. We define the tracking errors as* $\bar{x}_e = x_d - x$*,* $\bar{y}_e = y_d - y$*,* $\bar{\theta}_e = \theta_d - \theta$ *and transform this error to the form*

$$\begin{pmatrix} x_e \\ y_e \\ \theta_e \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \bar{x}_e \\ \bar{y}_e \\ \bar{\theta}_e \end{pmatrix}.$$

With this definition of the error, the error dynamics can be expressed by a time-dependent system

$$\begin{cases} \dot{x}_e = w(t)y_e(t) - v(t) + v_d(t)\cos\theta_e(t), \\ \dot{y}_e = -w(t)x_e(t) + v_d(t)\sin\theta_e(t), \\ \dot{\theta}_e = w_d(t) - w(t). \end{cases} \tag{11.7}$$

The synthesis procedure of a control algorithm based on the integrator backstepping method consists of the following steps:

- *Suppose temporarily that in the second equation of the analysed system (11.7) we have $x_e = 0$, and try to stabilise the variable y_e. To this objective we take $\theta_e = -\varphi(y_e v_d)$*[*]*, where $\varphi(z)$ denotes a function having the following properties: $\varphi(0) = 0$, $z\varphi(z) > 0$ for $z \neq 0$, and the derivative $\varphi'(z)$ is bounded. An example of a function that satisfies these requirements is $\varphi(z) = \frac{\sigma z}{1+z^2}$, for a certain $\sigma > 0$.*

- *Compute $\dot{y}_e = -v_d(t)\sin\varphi(y_e(t)v_d(t))$. Taking $V_1(y_e) = {}^1\!/_2 y_e^2$, and using the properties of the function φ, for small values of the function $\varphi(y_e v_d)$ we get the time derivative $\dot{V}_1(y_e) = -y_e v_d \sin\varphi(y_e v_d) < 0$. This yields the uniform asymptotic stability of the variable y_e.*

- *Define the variable $z = \theta_e + \varphi(y_e v_d)$, and compute its time derivative*

$$\dot{z} = \dot{\theta}_e + \varphi'(y_e v_d)(\dot{y}_e v_d + y_e \dot{v}_d)$$
$$= w_d - w + \varphi'(y_e v_d)(-w x_e v_d + v_d^2 \sin\theta_e + y_e \dot{v}_d).$$

- *Take the function*

$$V_2(t, x_e, y_e, z) = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2\gamma}z^2,$$

for some $\gamma > 0$. The differentiation of V_2 along the trajectory of the error system (11.7) results in

$$\dot{V}_2 = x_e\dot{x}_e + y_e\dot{y}_e + \frac{1}{\gamma}z\dot{z} = x_e(-v + v_d\cos\theta_e) + y_e v_d \sin\theta_e$$
$$+ \frac{1}{\gamma}z\left(w_d - w + \varphi'(y_e v_d)\left(-w x_e v_d + v_d^2 \sin\theta_e + y_e \dot{v}_d\right)\right).$$

- *Invoke the Hadamard's Lemma presented in subsection 3.3, in the form*

$$f(x+x_0) = f(x_0) = \int_0^1 df(s(x+x_0)+(1-s)x_0) = f(x_0) + x\int_0^1 f'(sx+x_0)ds.$$

We have $z = \theta_e + \varphi(y_e v_d)$, therefore

$$\sin\theta_e = \sin(z - \varphi(y_e v_d)) = \sin(-\varphi(y_e v_d)) + z\underbrace{\int_0^1 \cos(sz - \varphi(y_e v_d))ds}_{\eta}.$$

[*] the argument of φ is the product of y_e and v_d

- *Utilising the above, compute*

$$\begin{aligned}
\dot{V}_2 &= x_e(-v + v_d\cos\theta_e) - y_e v_d \sin\phi(y_e v_d) + y_e z\eta v_d \\
&\quad + \frac{1}{\gamma} z\left(w_d - w + \varphi'(y_e v_d)\left(-w x_e v_d + v_d^2 \sin\theta_e + y_e \dot{v}_d\right)\right) \\
&= x_e(-v + v_d\cos\theta_e) - y_e v_d \sin\phi(y_e v_d) + \frac{1}{\gamma} z\left(\gamma y_e \eta v_d\right. \\
&\quad \left. + w_d - \left(1 + \varphi'(y_e v_d) x_e v_d\right) w + \varphi'\left(v_d^2 \sin\theta_e + y_e \dot{v}_d\right)\right).
\end{aligned}$$

- *In order to get the derivative* $\dot{V}_2$ *negative, choose the controls* v *and* w *in such a way that satisfy the dependences*

$$\begin{cases} -v + v_d\cos\theta_e = -c_1 x_e \\ -(1+\varphi'(y_e v_d)x_e v_d)w + \gamma y_e \eta v_d + w_d \\ \qquad\qquad + \varphi'\left(v_d^2\sin\theta_e + y_e\dot{v}_d\right) = -c_2 z \end{cases},$$

  for positive coefficients c_1 *and* c_2. *Having made them explicit we obtain*

$$\begin{cases} v = c_1 x_e + v_d\cos\theta_e \\ w = \frac{1}{1+\varphi'(y_e v_d)x_e v_d}\left(c_2 z + \gamma y_e \eta v_d + w_d + \varphi'\left(v_d^2\sin\theta_e + y_e\dot{v}_d\right)\right) \end{cases}$$

  as well as

$$\dot{V}_2 = -c_1 x_e^2 - \underbrace{y_e v_d \sin\phi(y_e v_d)}_{>0} - c_2 z^2 < 0.$$

To finalise our analysis we notice that the inequality $\dot{V}_2 < 0$ *implies a boundedness of the function* V_2*, so also of the variables* x_e, y_e *and* z*. Furthermore, if the reference trajectory* $(v_d(t), w_d(t))$ *is bounded together with its first order derivative then the controls* $v(t)$, $w(t)$ *as well as the derivatives* $\dot{x}_e$, $\dot{y}_e$ *and* $\dot{\theta}_e$ *stay bounded. We then conclude that the second order derivative* $\ddot{V}_2$ *is bounded. Since the function* V_2 *has a limit, and the function* $\ddot{V}_2$ *is bounded, we obtain from Barbalat's Lemma that* $\dot{V}_2 \to 0$*, i.e.* $(x_e(t), y_e(t)v_d(t)\sin\phi(y_e(t)v_d(t)), \theta_e(t)) \to 0$*. Under suitable assumptions imposed on the reference trajectory this allows us to show that also* $y_e(t) \to 0$.

11.5 Bibliographical remarks

The chained form systems play in control theory a particular role, both for purely theoretical reasons (the so called Goursat normal form) as well as with respect to the existence form them control algorithms, see [JN99, MZS94], chapter 8, [Kha00], chapter 14 or [Sas99], chapter 12. The Murray's Theorem can be found in the chapter 8 mentioned above or in chapter 9 of [Sas99]. Example 11.4.1 comes from [MLS94], the Example 11.4.2 has been borrowed from the monograph [Kha00], whereas the Example 11.4.3 is a reconstruction based on [JN97]. To a Reader interested in a more in depth study of the method of backstepping we recommend the monograph [KKK95].

Bibliography

[JN97] Z.-P. Jiang and H. Nijmeijer. Tracking control of mobile robots: A case study in backstepping. *Automatica*, 33(7):1393–1399, 1997.

[JN99] Z.-P. Jiang and H. Nijmeijer. A recursive technique for tracking control of nonholonomic systems in chained form. *IEEE Trans. Automat. Control*, 44(2):265–279, 1999.

[Kha00] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, New Jersey, 2000.

[KKK95] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design*. J. Wiley, New York, 1995.

[MLS94] R. Murray, Z. Li, and S. S. Sastry:. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, Boca Raton, 1994.

[MZS94] R. Murray, Li Zexiang, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, Boca Raton, 1994.

[Sas99] S. S. Sastry. *Nonlinear Systems*. Springer, New York, 1999.

Chapter 12

Dynamic feedback, linearisation

12.1 Motivation

Let us look again at the kinematics of the unicycle

$$\begin{cases} \dot{x} = u_1 \cos\theta \\ \dot{y} = u_1 \sin\theta \\ \dot{\theta} = u_2 \end{cases} .$$

described by the coordinates $q = (x, y, \theta)^T$. Assume that we want to control the end position of the shaft of length d fixed to the unicycle as shown in Figure 12.1. The output function of this system takes the form

$$\begin{cases} y_1 = x + d\cos\theta \\ y_2 = y + d\sin\theta \end{cases} .$$

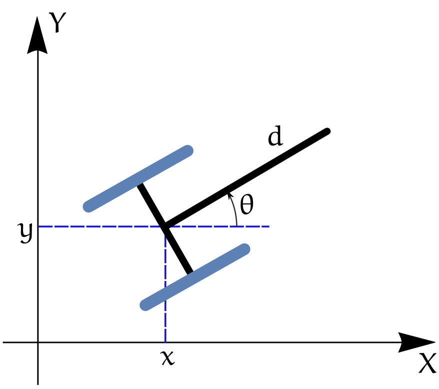

Figure 12.1: Unicycle with shaft

First, let us check, if the system is input-output decouplable. To this aim we differentiate

$$\begin{cases} \dot{y}_1 = \dot{x} - d\dot{\theta}\sin\theta = u_1\cos\theta - u_2 d\sin\theta, \\ \dot{y}_2 = \dot{y} + d\dot{\theta}\cos\theta = u_1\sin\theta + u_2 d\cos\theta, \end{cases}$$

i.e.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & -d\sin\theta \\ \sin\theta & d\cos\theta \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = D(q)u = \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

The differential degrees of both outputs are identical, $\rho_1 = \rho_2 = 1$, and if $\det D(q) = d \neq 0$ then the system is decouplable. Since $\rho_1 + \rho_2 = 2 < 3$, there appear the zero dynamics. To describe them, we shall introduce new coordinates

$$\begin{cases} \xi_1 = y_1 \\ \xi_2 = y_2 \\ \xi_3 = \theta \end{cases}.$$

In these coordinates the system's equations look as the following

$$\begin{cases} \dot{\xi}_1 = \dot{y}_1 = \nu_1 \\ \dot{\xi}_2 = \dot{y}_2 = \nu_2 \\ \dot{\xi}_3 = \dot{\theta} = u_2 = -\frac{1}{d}\nu_1\sin\theta + \frac{1}{d}\nu_2\cos\theta \end{cases}.$$

The assumption that $y_1(t) = 0$ and $y_2(t) = 0$ requires zeroing the coordinates ξ_1, ξ_2, as well as the inputs ν_1 and ν_2. In consequence, the zero dynamics become

$$\dot{\xi}_3 = 0,$$

thus they are bounded. If the control problem consists in the tracking of a prescribed trajectory $(x_d(t), y_d(t))$ the tracking control algorithm may have the form of a proportional (P) regulator with a feedforward term, i.e.

$$\begin{cases} \nu_1 = \dot{x}_d - k_1(x - x_d) \\ \nu_2 = \dot{y}_d - k_2(y - y_d) \end{cases}.$$

As can be seen, the procedure of feedback decoupling and (partial) linearisation of the model of unicycle has been successful, on condition that we want to control a point located at the end of the shaft, in some distance d from the middle point of the rear axle. Now we shall examine in more detail the case of $d = 0$, so of the output function

$$\begin{cases} y_1 = x \\ y_2 = y \end{cases}.$$

In this case we have

$$\begin{cases} \dot{y}_1 = u_1 \cos\theta \\ \dot{y}_2 = u_1 \sin\theta \end{cases},$$

thus the decoupling matrix is singular and the decoupling procedure is not applicable. Not discouraged too much by this fact, we shall differentiate the output function once again under assumption that the controls are differentiable,

$$\begin{cases} \ddot{y}_1 = \dot{u}_1 \cos\theta - u_1\dot{\theta}\sin\theta = \dot{u}_1 \cos\theta - u_1 u_2 \sin\theta \\ \ddot{y}_2 = \dot{u}_1 \sin\theta + u_1\dot{\theta}\cos\theta = \dot{u}_1 \sin\theta + u_1 u_2 \cos\theta \end{cases}.$$

Now, let us assume that in the formulas given above u_1 does not denote a control any more, but an extra state variable. Instead, as the control we shall take $w_1 = \dot{u}_1$ and $w_2 = u_2$. This being so, we get

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & -u_1 \sin\theta \\ \sin\theta & u_1 \cos\theta \end{bmatrix} w = D(q, u_1)w.$$

On condition that $u_1 \neq 0$ the matrix $D(q, u_1)$ becomes a decoupling matrix. Having applied the feedback $v = D(q, u_1)w$ we arrive at a decoupled input-output relationship

$$\begin{cases} \ddot{y}_1 = v_1 \\ \ddot{y}_2 = v_2 \end{cases}.$$

It turns out that, after extending the state space of the unicycle by the variable u_1 and adding to the unicycle's equations the identity $\dot{u}_1 = w_1$, the system

$$\begin{cases} \dot{x} = u_1 \cos\theta \\ \dot{y} = u_1 \sin\theta \\ \dot{\theta} = w_2 \\ \dot{u}_1 = w_1 \end{cases}$$

with output

$$\begin{cases} y_1 = x \\ y_2 = y \end{cases}$$

is decouplable and linearisable by the feedback $\xi = \varphi(x) = (y_1, \dot{y}_1, y_2, \dot{y}_2)^T$, $v = D^{-1}(q, u_1)w$, under which it takes the form

$$\begin{cases} \dot{\xi}_1 = \xi_1 \\ \dot{\xi}_2 = v_1 \\ \dot{\xi}_3 = \xi_4 \\ \dot{\xi}_4 = v_2 \end{cases},$$

valid in the region $R^3 \times R - \{0\}$. The feedback based on an extension of a system by an extra dynamic part is called dynamic. The feedback without such an extension, discussed in section 9.2, is referred to as static. Therefore, the kinematic equations of a moving unicycle ($u_1 \neq 0$) with the zero length of the shaft $d = 0$, are dynamic feedback linearisable, but they are not static feedback linearisable. This shows that the dynamic feedback is a more powerful tool than the static one. In the next section we shall define the concept of the dynamic feedback in a formal way.

12.2 Dynamic feedback

Let a control-affine system

$$\sigma : \ \dot{x} = f(x(t)) + g(x(t))u(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) \tag{12.1}$$

be given, where $x \in R^n$, $u \in R^m$. To this system we add a dynamic compensator

$$\kappa : \begin{cases} \dot{z} = F(x(t)), z(t)) + G(x(t), z(t))w \\ u = H(x, z) + K(x, z)w \end{cases}, \tag{12.2}$$

$z \in R^q$, $w \in R^m$. The variable z is the state variable of the compensator; the dimension of the compensator's state space equals q. A coupling of the system (12.1) and the compensator (12.2) gives the control system

$$(\sigma + \kappa) : \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x(t)) + g(x(t))H(x(t), z(t)) \\ F(x(t), z(t)) \end{pmatrix} + \begin{bmatrix} g(x(t))K(x(t), z(t)) \\ G(x(t), z(t)) \end{bmatrix} w = \Phi(x(t), z(t)) + \Psi(x(t), z(t))w. \tag{12.3}$$

For a control system

$$\sigma' : \ \dot{\xi} = F(\xi(t)) + G(\xi(t))\nu.$$

we introduce the following definition of the dynamic feedback equivalence.

Definition 12.2.1 *The system σ is dynamic feedback equivalent to the system σ', $\sigma \cong_{DF} \sigma'$ if there exists a dynamic compensator κ and a static feedback*

$$\begin{cases} \xi = \varphi(x, z) \\ w = \alpha(x, z) + \beta(x, z)\nu \end{cases},$$

such that

$$(\sigma + \kappa) \underset{F}{\cong} \sigma'.$$

If the diffeomorphism $\varphi(x, z)$ is local, we speak of the local dynamic feedback equivalence.

The system σ is named dynamically feedback linearisable (dynamically linearisable) if σ' is a linear system, and there holds $\sigma \cong_{DF} \sigma'$.

12.3 Theorems on dynamic linearisation

Intuitively, after the analysis of our examples of decoupling and linearisation with extra integrators employed in the control loop, we may believe that the essence of the dynamic feedback consists in a "mutual shifting" of the controls acting on the system. We can assume that such a shifting is achieved by the integration of controls. Therefore, if there is only one input, the dynamic feedback should not be effective. It is indeed the case, as it follows from the next theorem.

Theorem 12.3.1 *A single-input system is dynamic feedback linearisable if and only if it is static feedback linearisable.*

The next result presents a necessary condition for dynamic linearisation. Since a statically linearisable system is *a fortiori* dynamically linearisable, obviously this is also a necessary condition for static linearisation.

Theorem 12.3.2 *If a system is dynamic feedback linearisable in a neighbourhood of an equilibrium point then its linear approximation at this point is controllable.*

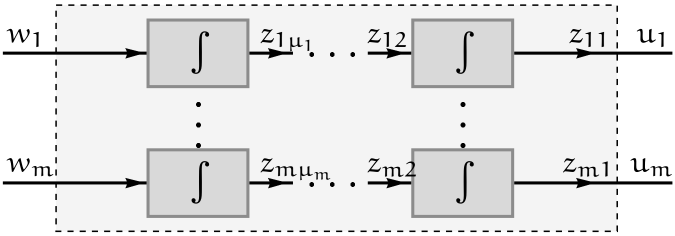

Figure 12.2: Brunovsky compensator

12.3.1 Brunovsky compensator

A glance at the dynamic linearisation problem allows one to expect that the problem is much harder than that of the static linearisation for the reason that in dynamic linearisation we need to design a compensator and then to linearise the system together with the compensator. It turns out that the choice and the linearisation of the compensator can be accomplished in a quite arbitrary manner, by using a linear compensator in the Brunovsky canonical form. This is done in the following way. For the system (12.1) we choose a collection of integers $0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_m$. Their sum defines the state space dimension of the compensator, $q = \sum_{i=1}^{m} \mu_i$. The Brunovsky compensator is a specific instance of the system (12.2), defined in the following way (see Figure 12.2):

$$\begin{cases} \dot{z}_{11} = z_{12},\ \dot{z}_{12} = z_{13}, \ldots, \dot{z}_{1\mu_1} = w_1, & u_1 = z_{11}, \\ \dot{z}_{21} = z_{22},\ \dot{z}_{22} = z_{23}, \ldots, \dot{z}_{2,\mu_2} = w_2, & u_2 = z_{21}, \\ \quad \vdots & \\ \dot{z}_{m1} = z_{m2},\ \dot{z}_{m2} = z_{m3}, \ldots, \dot{z}_{m\mu_m} = w_m, & u_m = z_{m1}. \end{cases}$$

By the definition of the Brunovsky compensator it follows that $u_1^{(\mu_1)} = w_1$, $u_2^{(\mu_2)} = w_2, \ldots, u_m^{(\mu_m)} = w_m$, so the new controls need to be integrated, respectively, $\mu_1, \mu_2, \ldots, \mu_m$ times. We also observe that the dynamics of the Brunovsky compensator do not depend on the state variables of the system (12.1), what makes the compensator a sort of universal. The numbers μ_i are ordered increasingly; thus if a certain $\mu_j = 0$, then all the μ_i preceding μ_j are also equal to zero. $\mu_j = 0$ means that the control input w_j will not be integrated, i.e. $u_j = w_j$. Now, let us take the system (12.1), and choose the Brunovsky compensator described by the integers $(\mu_1, \mu_2, \ldots, \mu_m)$. We define a collection of distributions

$$\Delta_0 = \underset{C^\infty(R^n,R)}{\operatorname{span}} \{g_k | \mu_k = 0\},\ \Delta_{i+1} = \Delta_i + \operatorname{ad}_f \Delta_i + \underset{C^\infty(R^n,R)}{\operatorname{span}} \{g_k | \mu_k = i+1\}.$$

Sufficient conditions for the dynamic linearisation (the linearisation employing the Brunovsky compensator) are included in the following theorem.

Theorem 12.3.3 (Charlet-Lévine-Marino) *The system*

$$\sigma: \ \dot{x} = f(x(t)) + g(x(t))u(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t),$$

$x \in R^n$, $u \in R^m$, *is dynamic feedback linearisable in a neighbourhood of the equilibrium point* $x_0 = 0$, *if*

1. $\dim \Delta_{n+\mu_m-1}(0) = n$,
2. *the distributions* Δ_i, $i = 0, 1, \dots, n+\mu_m-2$ *have around* 0 *constant dimension and are involutive,*
3. *in a certain neighbourhood of zero there holds* $[g_j, \Delta_i] \subset \Delta_{i+1}$, *for* $j = 1, 2, \dots, m$, $\mu_j \geqslant 1$, $i = 0, 1, \dots, n + \mu_m - 2$.

In the case when $i \geqslant \mu_m$, the last component of the distribution Δ_{i+1} is equal to zero. Notice that if $\mu_m = 0$ (therefore all $\mu_j = 0$), the condition for the dynamic linearisability coincides with the condition for the static linearisability presented in Theorem 9.2.2.

12.4 Differential flatness

In the chapter devoted to the input-output decoupling and linearisation we have noticed (Remark 10.3.1) that if the differential degrees of the outputs sum up to the dimension of the system's state space then there exists a feedback (φ, α, β) depending exclusively on the derivatives of the outputs with respect to time, that linearises the system. One can say that both the state variables as well as the controls in the new system depend only on the jets of the outputs. This observation supports the concept of a differentially flat system tat will be defined in the following way.

Definition 12.4.1 *The control-affine system*

$$\sigma: \ \dot{x} = f(x(t)) + g(x(t))u = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t),$$

where $x \in R^n$, $u \in R^m$, *is called differentially flat if there exist functions*

$$y_i = h_i(x), \ i = 1, 2, \dots, m,$$

referred to as the flat outputs, such that almost everywhere, perhaps except at some singular points, the state variables as well as the controls of the system σ can be expressed as some functions of the flat outputs and their time derivatives, i.e.

$$x_i = x_i\left(y, \dot{y}, \ldots, y^{(r_i)}\right), \; i = 1, 2, \ldots, n$$
$$u_j = u_j\left(y, \dot{y}, \ldots, y^{(s_j)}\right), \; j = 1, 2, \ldots, m.$$

A fundamental feature of differentially flat systems is that they are almost everywhere dynamically feedback linearisable. Because we do not have checkable necessary and sufficient conditions for differential flatness, in order to establish the flatness we usually need to resort to the definition. It is known that the kinematics of mobile robots like the unicycle, kinematic car and the tractor pulling trailers are differentially flat. Similarly, we can prove the flatness of the chained form systems. Suppose that we have the chained form system

$$\dot{x}_1 = u_1, \; \dot{x}_2 = u_2, \; \dot{x}_3 = x_2 u_1, \ldots, \dot{x}_n = x_{n-1} u_1.$$

As the flat outputs let us choose $y_1 = x_1$ and $y_2 = x_n$, and then compute

$$x_1 = y_1, \; x_n = y_2, \; u_1 = \dot{x}_1 = \dot{y}_1, \; x_{n-1} = \frac{\dot{x}_n}{u_1} = \frac{\dot{y}_2}{\dot{y}_1},$$
$$x_{n-2} = \frac{\dot{x}_{n-1}}{u_1} = \frac{\ddot{y}_2 \dot{y}_1 - \dot{y}_2 \ddot{y}_1}{\dot{y}_1^3}, \ldots, u_2 = \dot{x}_2.$$

It follows that the chained form system is differentially flat on condition that $\dot{y}_1 = u_1 \neq 0$. An example system that is not differentially flat is the kinematics of the rolling ball. We want to conclude with an observation that, after showing the differential flatness of a system, the design of the linearising dynamic feedback is quite natural. We shall see this when studying Examples 12.5.2 and 12.5.3.

12.5 Examples

Example 12.5.1 *Consider the following control-affine system*

$$\sigma : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = x_3 - x_3 u_2 \end{cases}.$$

This system is described by three vector fields: $f(x) = (x_2, 0, 0, x_3)^T$, $g_1(x) = e_3$, *and* $g_2(x) = e_2 - x_3 e_4$; e_i *denotes the* i*-th unit vector in* R^4. *We shall verify a possibility of linearising this system by either static or dynamic feedback. For completeness we shall begin with checking the necessary condition for the feedback linearisability. It is easily noticed that the point* $u = 0$, $x_0 = 0$ *is an equilibrium point of the* σ. *The linear approximation*

$$A = \frac{\partial f(0)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B = [g_1(0), g_2(0)] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

As can be checked, the rank of the Kalman matrix

$$\operatorname{rank}\left[B, AB, A^2B, A^3B\right] = \operatorname{rank}[B, AB] = 4,$$

so the linear approximation is controllable, and the necessary condition holds. Now we ask the question of the linearisability by the static feedback. We compute

$$\mathcal{D}^0 = \operatorname*{span}_{C^\infty(R^n,R)} \{g_1, g_2\} = \operatorname*{span}_{C^\infty(R^n,R)} \{e_3, e_2 - x_3 e_4\}, \; \dim \mathcal{D}^0(x) = 2.$$

The distribution $\mathcal{D}^0$ *has constant dimension at any point* $x \in R^4$. *Let's find the Lie bracket*

$$g_{12}(x) = [g_1, g_2](x) = Dg_2(x)g_1(x) - Dg_1(x)g_2(x) = -e_4 \notin \mathcal{D}^0.$$

Since the distribution $\mathcal{D}^0$ *is not involutive, the system* σ *is not static feedback linearisable, and we shall try to achieve the dynamic feedback linearisation. Our first step will be the choice of a Brunovsky compensator. Suppose that* $\mu_1 = 0$ *and* $\mu_2 = 1$. *Then we have* $q = 1$ *and* $n = 4$. *We compute the distributions*

$$\Delta_0 = \operatorname*{span}_{C^\infty(R^n,R)} \{g_1\} = \operatorname*{span}_{C^\infty(R^n,R)} \{e_3\}.$$

Obviously, the distribution Δ_0 *has constant dimension* $= 1$ *and is involutive. Next, we find*

$$\Delta_1 = \Delta_0 + ad_f \Delta_0 + \operatorname*{span}_{C^\infty(R^n,R)} \{g_2\}.$$

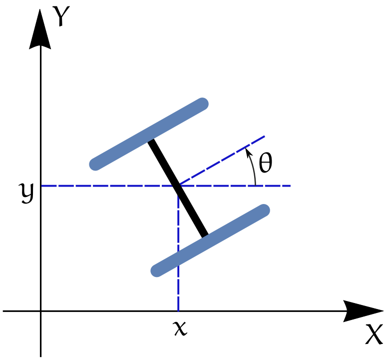

Figure 12.3: Unicycle

Because $\Delta_1 = \text{span}_{C^\infty(R^n,R)}\{e_3, e_4, e_2 - x_3 e_4\} = \text{span}_{C^\infty(R^n,R)}\{e_2, e_3, e_4\}$, *the distribution* Δ_1 *is also constant dimensional (of dimension* 3*) and involutive. In the next step we compute*

$$\Delta_2 = \Delta_1 + ad_f \Delta_1 = \underset{C^\infty(R^n,R)}{\text{span}} \{e_1, e_2, e_3, e_4\}.$$

The distribution Δ_2 *has constant dimension* 4 *that is equal to the dimension of the state space. This implies that the distributions* $\Delta_4 = \Delta_3 = \Delta_2$*, so the conditions number 1 and 2 of the Theorem 12.3.3 are fulfilled. We are left with checking the condition number 3, i.e. showing that*

$$[g_2, \Delta_0] \subset \Delta_1, \; [g_2, \Delta_1] \subset \Delta_2, \; [g_2, \Delta_2] \subset \Delta_3.$$

As a matter of fact, it suffices to check only the first from among these conditions, what follows from the fact that $[g_2, g_1] = e_4 \in \Delta_1$*. Summarising, for all the conditions for the dynamic feedback linearisation are satisfied, the system* σ *is dynamic feedback linearisable. This example reveals that the class of dynamically linearisable systems is larger than the class of systems that can be linearised by means of the static feedback.*

Example 12.5.2 *Let us examine the differential flatness of the kinematics model of the unicycle characterised by coordinates* $q = (x, y, \theta)^T$*, see Figure 12.3,*

$$\begin{cases} \dot{x} = u_1 \cos\theta \\ \dot{y} = u_1 \sin\theta \\ \dot{\theta} = u_2 \end{cases}.$$

To this aim we choose as the (candidate) flat outputs $y_1 = x$, $y_2 = y$, *and compute*

$$x = y_1,\ y = y_2,\ \theta = \arctan\frac{\dot y_2}{\dot y_1},\ u_1 = \pm\sqrt{\dot y_1^2 + \dot y_2^2},\ u_2 = \dot\theta = \frac{\ddot y_2 \dot y_1 - \dot y_2 \ddot y_1}{\dot y_1^2 + \dot y_2^2},$$

that proves the flatness of the unicycle outside the singular point $u_1 = 0$. *In order to design the linearising feedback, let us observe that the state coordinates of the unicycle have been expressed in terms of the flat outputs and their first order time derivatives. For this reason the new coordinates can be defines as*

$$\begin{cases} \xi_1 = y_1 \\ \xi_2 = \dot y_1 \\ \xi_3 = y_2 \\ \xi_4 = \dot y_2 \end{cases}.$$

The unicycle's kinematic equations in these coordinates take the form

$$\begin{cases} \dot\xi_1 = \xi_2 \\ \dot\xi_2 = \ddot y_1 = \dot u_1 \cos\theta - u_1 u_2 \sin\theta = w_1 \cos\theta - u_1 w_2 \sin\theta \\ \dot\xi_3 = \xi_4 \\ \dot\xi_4 = \ddot y_2 = \dot u_1 \sin\theta + u_1 u_2 \cos\theta = w_1 \sin\theta + u_1 w_2 \cos\theta \end{cases}.$$

It follows that to the equations of the unicycle one needs to add the dynamic compensator

$$\begin{cases} \dot u_1 = w_1 \\ u_2 = w_2 \end{cases}$$

and apply the feedback

$$\begin{cases} v_1 = w_1 \cos\theta - u_1 w_2 \sin\theta \\ v_2 = w_1 \sin\theta + u_1 w_2 \cos\theta \end{cases},$$

that converts the kinematic model of the unicycle to the linear system

$$\begin{cases} \dot\xi_1 = \xi_2 \\ \dot\xi_2 = v_1 \\ \dot\xi_3 = \xi_4 \\ \dot\xi_4 = v_2 \end{cases}.$$

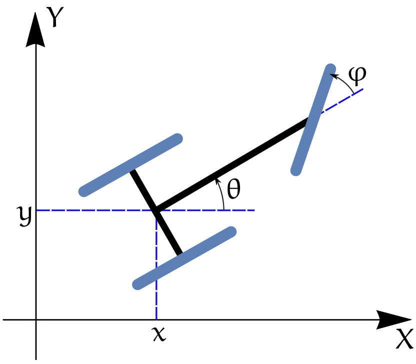

Figure 12.4: Kinematic car

These feedback transformations are well defined in the set of states of the system with compensator $(q, u_1) \in R^3 \times R - \{0\}$*. The state space diffeomorphism assumes the form*

$$\xi = \varphi(q, u_1) = \begin{pmatrix} x \\ u_1 \cos\theta \\ y \\ u_1 \sin\theta \end{pmatrix}.$$

Example 12.5.3 *A slightly more involved is the kinematics model of the kinematic car shown in Figure 12.4. The coordinate vector of the kinematic car* $q = (x, y, \theta, \varphi)^T$*. The kinematics model will be taken in the following form*

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_1 \tan\theta \\ \dot{\theta} = u_1 \frac{\tan\varphi}{\cos\theta} \\ \dot{\varphi} = u_2 \end{cases},$$

that is valid under the condition $|\theta|, |\varphi| < \pi/2$*. As in the previous section, we choose the flat outputs as*

$$\begin{cases} y_1 = x \\ y_2 = y \end{cases}.$$

We compute

$$x = y_1,\ y = y_2,\ u_1 = \dot{y}_1,\ \theta = \arctan\frac{\dot{y}_2}{\dot{y}_1},\ \dot{\theta} = \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{\dot{y}_1^2 + \dot{y}_2^2} = \dot{y}_1\frac{\tan\varphi}{\cos\theta},$$

$$\varphi = \arctan\frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{\dot{y}_1\left(\dot{y}_1^2 + \dot{y}_2^2\right)}\cos\theta(\dot{y}_1, \dot{y}_2),\ u_2 = \dot{\varphi}(\dot{y}_1, \ddot{y}_1, y_1^{(3)}, \dot{y}_2, \ddot{y}_2, y_2^{(3)}).$$

The results of these computations imply that for $u_1 \neq 0$ *the kinematics of the car is differentially flat. Now we shall derive the linearising dynamic feedback. Since* $q = q(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2)$, *we pick new coordinates as*

$$\begin{cases} \xi_1 = y_1 \\ \xi_2 = \dot{y}_1 = u_1 \\ \xi_3 = \ddot{y}_1 = \dot{u}_1 \\ \xi_4 = y_2 \\ \xi_5 = \dot{y}_2 = u_1 \tan\theta \\ \xi_6 = \ddot{y}_2 = \dot{u}_1 \tan\theta + u_1^2 \dfrac{\tan\varphi}{\cos^3\theta} \end{cases}.$$

In these new coordinates the equations of the kinematic car are the following:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = \ddot{u}_1 \\ \dot{\xi}_4 = \xi_5 \\ \dot{\xi}_5 = \xi_6 \\ \dot{\xi}_6 = \ddot{u}_1 \tan\theta + \dot{u}_1 \frac{u_1 \tan\varphi}{\cos^3\theta} + \frac{\left(2\dot{u}_1 u_1 \tan\varphi + u_2 \frac{u_1^2}{\cos^2\varphi}\right)\cos^3\theta + \frac{3}{2}u_1^3 \sin 2\theta \tan^2\varphi}{\cos^6\theta} \end{cases}.$$

This being so, we introduce a two-dimensional dynamic compensator

$$\begin{cases} \dot{u}_1 = \eta \\ \dot{\eta} = w_1 \\ u_2 = w_2 \end{cases},$$

and the feedback

$$\begin{cases} v_1 = w_1 \\ v_2 = \eta \frac{u_1 \tan\varphi}{\cos^3\theta} + \frac{2\eta u_1 \tan\varphi \cos^3\theta + \frac{3}{2}u_1^3 \sin 2\theta \tan^2\varphi}{\cos^3\theta} + w_1 \tan\theta + w_2 \frac{u_1^2}{\cos^2\varphi \cos^3\theta} \end{cases}.$$

Subject to the dynamic linearisation the car's kinematics assume the *form of a linear control system*

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = \nu_1 \\ \dot{\xi}_4 = \xi_5 \\ \dot{\xi}_5 = \xi_6 \\ \dot{\xi}_6 = \nu_2 \end{cases}.$$

This linearisation is justified in the region of state variables of the system with the compensator, given by $(q, u_1, \eta) \in R^2 \times \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)^2 \times (R - \{0\}) \times R$. *The state space diffeomorphism is defined by the formula*

$$\xi = \varphi(q, u_1, \eta) = \begin{pmatrix} x \\ u_1 \\ \eta \\ y \\ u_1 \tan\theta \\ \eta\tan\theta + \frac{u_1^2 \tan\varphi}{\cos^3\theta} \end{pmatrix}.$$

12.6 Bibliographical remarks

Conditions of dynamic feedback linearisability by means of the Brunovsky compensator, and Example 12.5.1 come from the paper [CLM91]. Non-genericity of this kind of linearisation was examined in [Tch94]. The dynamic feedback linearisation of a model of induction motor taking into account the magnetic flow has been described in [Chi93]. The concept of differential flatness is discusses exhaustingly in the paper [FLR95]. The development of the theory and applications of differentially flat systems, mainly in the context of dynamic linearisation, has been described in monographs [SRA04, Lév09].

Bibliography

[Chi93] J. Chiasson. Dynamic feedback libnearization of the induction motor. *IEEE Trans. Aut. Contr.*, 38(10):1588–1594, 1993.

[CLM91] B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic feedback linearization. *SIAM J. Contr. Opt.*, 29(1):38–57, 1991.

[FLR95] M. Fliess, J. Lévine, and P. Rouchon. Flatness and defect of nonlinear systems: Introductory theory and examples. *Int. J. Control*, 61:1327–1361, 1995.

[Lév09] J. Lévine. *Analysis and Control of Nonlinear Systems.* Springer, 2009.

[SRA04] H. Sira-Ramirez and S. K. Agrawal. *Differentially Flat Systems.* M. Dekker, New York, 2004.

[Tch94] K. Tchoń. Towards robustness and genericity of dynamic feedback linearization. *J. Math. Syst., Estim. Control*, 4(2):165–180, 1994.

Chapter 13

Limitations of feedback

In the last chapter of these notes we shall deal with certain limitations of applicability of the feedback to the synthesis of control algorithms for non-linear systems. As usual, the linear control systems will serve us as a point of reference.

13.1 Linear systems

Let a linear system

$$\sigma: \quad \dot{x} = Ax + Bu, \tag{13.1}$$

be given, with m control inputs and n-dimensional state space. We recall that the system σ is feedback stabilisable if there exists a linear function $u = Kx$, such that the linear dynamic system

$$\dot{x} = (A + BK)x$$

has an asymptotically equilibrium point $x_0 = 0$. In chapter 0.2.3 we have stated the Remark 0.2.1, saying that a sufficient condition for stabilisability of a linear system is its controllability. Also, we have shown that this property results from a more general Pole Placement Theorem 0.2.4. Therefore, the controllability of a linear control system guarantees its stabilisability. Apparently, this feature does not generalise to non-linear control systems.

13.2 Brockett's Theorem

Consider a smooth control system

$$\dot{x} = f(x, u), \tag{13.2}$$

and let $u = 0$, $x = x_0$ denote its equilibrium point, i.e. $f(x_0, 0) = 0$. The system (13.2) is called feedback stabilisable if there exists a smooth function $u = \alpha(x)$, $\alpha(x_0) = 0$, such that the point x_0 is an asymptotically stable equilibrium point of the dynamic system

$$\dot{x} = f(x, \alpha(x)).$$

The following theorem establishes a necessary condition for stabilisability of the system (13.2).

Theorem 13.2.1 (Brockett) *Suppose that the system (13.2) is stabilisable, and let $\mathcal{A}$ denote a neighbourhood of x_0. Then, the image of the function*

$$\gamma : \mathcal{A} \times R^m \longrightarrow R^n,\ \gamma(x, u) = f(x, u),$$

is a certain open neighbourhood of the point $0 \in R^n$.

We often say the the Brockett's Theorem defines an obstruction to stabilisability of a non-linear control system. For illustration, take a system (13.2) in the chained form

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \end{cases}.$$

We choose $\mathcal{A} = R^3$ and $\gamma(u, x) = f(u, x) = (u_1, u_2, x_2 u_1)$. In order to get the point $0 \in R^3$, we need to assume that $u_1 = u_2 = 0$. Notice, however, that a point arbitrarily close to zero, of the form $(0, 0, \epsilon) \in R^3$ does not belong to the image of the function f. This means that the chained form system is not feedback stabilisable. Also observe that the chained form system is controllable, and despite that, not feedback stabilisable. A similar conclusion holds for any driftless system, either controllable or not. To this objective, consider the system

$$\dot{x} = g(x)u = \sum_{i=1}^{m} g_i(x) u_i$$

with control vector fields independent at the point x_0. Without any loss of generality we may assume that the matrix $g(x)$ takes the form

$$\begin{bmatrix} \tilde{g}_1(x) \\ \tilde{g}_2(x) \end{bmatrix},$$

such that around the point x_0 rank $\tilde{g}_1(x) = m$. Let

$$\mathcal{A} = \{x \in R^n | \operatorname{rank} \tilde{g}_1(x) = m\}.$$

On the set A there exists the feedback $u = \tilde{g}^{-1}(x)v$ transforming the driftless system to the feedback equivalent form

$$\dot{x} = \begin{bmatrix} I_m \\ h(x) \end{bmatrix} v.$$

We have $\gamma(v, x) = (v, h(x)v)$. In order to reach the point $0 \in R^n$, we need to set $v = 0$. However, as far as $m < n$, no point of the form $(0_m, \epsilon e_i)$, e_i a unit vector in R^{n-m}, belongs to the image of the function γ. Thus, a driftless control system satisfying the condition $m < n$ is not feedback stabilisable. An analogous reasoning results with a conclusion that for a control affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

whose drift vector field belongs to the distribution spanned by the control vector fields, and the number of controls is less than the dimension of the state space, $m < n$, a stabilising feedback does not exist either. The condition provided by the Theorem 13.2.1 is valid also when instead of a smooth one takes a continuous feedback $u = \alpha(x)$.

The Brockett's condition appeared to be one of the milestones in control theory, and initiated an advancement of research on the feedback control methods that would not be impaired by this condition, such as a feedback depending on the state and time or a discontinuous feedback, as well as on the methods of practical stabilisation where instead of the asymptotic error convergence one requires that the system's trajectory approached the equilibrium point in some controlled manner.

13.3 Theorem of Lizárraga

In this section we shall study a result that plays the role of a counterpart of the Brockett's Theorem that applies to the problem of trajectory tracking. In order to state this result, consider a control system of the form

$$\dot{x} = f(x, u), \tag{13.3}$$

containing a continuous function $f : R^n \times R^m \longrightarrow R^n$ such that, for a fixed $u \in R^m$ the vector field $f_u(x) = f(x, u)$ is smooth. We assume that admissible control functions $u(\cdot) \in \mathcal{U}$ are piece-wise continuous, and that for every control function $u(\cdot)$ and every initial state x_0 there exists a trajectory $x_u(t) = \Phi_t(x_0, u(\cdot))$ of the system (13.3). By a reference trajectory for the system (13.3) we shall mean a trajectory $y_\nu(t)$ fulfilling the equation $\dot{y}_\nu = f(y_\nu, \nu)$ for a certain control function $\nu(\cdot) \in \mathcal{U}$. We say that the system (13.3) has a continuous stabiliser if there exists a continuous function

$$u = \alpha(x, y, \nu, t),$$

that satisfies the identity $\alpha(y, y, \nu, t) = \nu$, and is such that for the trajectory $x_\alpha(t)$ of the time-dependent dynamic system

$$\begin{cases} \dot{x}_\alpha = f(x_\alpha, \alpha(x_\alpha, y_\nu, \nu, t)) \\ \dot{y}_\nu = f(y_\nu, \nu) \end{cases}$$

it holds that $x_\alpha(t) \longrightarrow_{t\to+\infty} y_\nu(t)$. The theorem presented below establishes a sufficient condition for the non-existence of a continuous stabiliser.

Theorem 13.3.1 (Lizárraga) *For the decomposition of the control space into a direct sum of two subspaces*

$$R^m = E_1 \oplus E_2$$

let us define two collections of vector fields

$$B_i = \{f_u : R^n \longrightarrow R^n | u \in E_i\}, \quad i = 1, 2,$$

and let $\hat{B}_i(\cdot)$ *denote the smallest Lie algebra of vector fields containing* B_i*. Suppose that for the introduced decomposition there exist submanifolds* $S_1, S_2 \in R^n$ *satisfying the conditions:*

1. S_i *is invariant with respect to* $\hat{B}_i(\cdot)$*, what means that trajectories of vector fields belonging to the Lie algebra* $\hat{B}_i(\cdot)$*, initialised in* S_i *stay within* S_i*,*

2. *dimensions* $\dim \hat{B}_i(p)$ *of the spaces spanned by the vector fields from the Lie algebras* $\hat{B}_i(\cdot)$ *are constant at any point* $p \in S_i$*,*

3. *there exists a point* $q \in S_1 \cap S_2$*, such that the sum of subspaces spanned by the Lie algebras* $\hat{B}_i(\cdot)$ *at this point is equal to the direct sum of these subspaces, and is contained but not equal to the sum of tangent spaces to the submanifolds* S_i

$$\hat{B}_1(q) + \hat{B}_2(q) = \hat{B}_1(q) \oplus \hat{B}_2(q) \subsetneq T_qS_1 + T_qS_2.$$

Then, there is no continuous stabiliser for the system (13.3).

A consequence of the Theorem of Lizárraga is that not every admissible reference trajectory in a non-linear control system can be tracked by means of a continuous feedback depending on the system's state, the reference trajectory, and time. For illustration, take the chained form system

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \end{cases}$$

studied previously. The control space R^2 of this system can be decomposed into a direct sum $R^2 = E_1 \oplus E_2$, where $E_i = \operatorname{span}_R\{e_i\}$, $i = 1, 2$. We have

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} v | v \in R \right\}, \; B_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w | w \in R \right\} = E_2$$

and, obviously, $\hat{B}_i(\cdot) = B_i$. Let's choose the submanifolds $S_1 = S_2 = R^3$. Then the invariance condition is satisfied trivially. Both the subspaces $\hat{B}_i(p)$, $i = 1, 2$ are 1-dimensional, what implies that the dimension condition holds. Eventually, for the point $q = 0 \in S_1 \cap S_2$ we have

$$\hat{B}_1(0) + \hat{B}_2(0) = \hat{B}_1(0) \oplus \hat{B}_2(0) = R^2 \times \{0\} \subsetneq T_0 S_1 + T_0 S_2 = R^3.$$

Since all conditions of the Theorem 13.3.1 are fulfilled, the chained form system does not have a continuous stabiliser.

13.4 Bibliographical remarks

The Brockett's condition has been formulated in [Bro83]. For tens of years it has played a role of the *spiritus movens* of non-linear control theory. The theorem of Lizárraga comes from the paper [Liz04].

Bibliography

[Bro83] R. W. Brockett. *Differential Geometric Control Theory*, chapter Asymptotic stability and feedback stabilization, pages 171–181. Birkhäuser, Boston, 1983.

[Liz04] D. A. Lizárraga. Obstructions to the existence of universal stabilizers for smooth control systems. *Math. Control Signals Syst.*, 16:255–277, 2004.

Index

A

accessibility 91
adjoint operator
  big 75
  small 75
algebra 3
  differential 22
  Lie 22
algorithm
  Newton 24
  stabilisation of the
    equilibrium point 131
  Steepest Descent 25
  tracking by the integrator
    backstepping method 132
  tracking with linearisation 107

B

Banach space 4
big flag 127
bracket
  Lie 22, 76
    properties 76
  Poisson 22

C

control
  closed loop 10
  integrator backstepping method . . 127
control system 86, 89
  affine
    F-equivalence 102
    F-linearisation 103
    S-equivalence 101
    S-linearisation 103
    differentially flat 142
    drift 88
    equivalence equations 104
    equivalence, dynamic feedback . 140
    feedback 118
    feedback equivalence 102
    feedback linearisation 103
    state space equivalence 101
    state space linearisable 103
    trajectory tracking problem 105
  control affine 87
    accessibility 91
    chained form 126
    controllability 91
    decoupling 118
    decoupling matrix 117
    differential degree 117
    driftless 88
    local controllability 91
    relative degree 117
  control-affine
    zero dynamics 121
  feedback stabilisability 152
  linear 5
    F-equivalence 7
    S-equivalence 7
    controllability 5
    controllability normal form 7
    controller normal form 8
    equivalence 7
    feedback equivalence 7
    stabilisability 10
    state space equivalence 7
  minimal phase 121
controllability 5, 91
controls 87

D

degree
  differential 117
  relative 117
derivative
  Fréchet 4
  Gateaux 4
  Lie 77

diffeomorphism . . . 23
  local . . . 23, 34
differential flatness . . . 142
distribution . . . 78
  big flag . . . 127
  integrable . . . 79
  involutivity . . . 79
  small flag . . . 127
drift . . . 88
dynamic compensator . . . 139
  Brunovsky . . . 141
dynamic feedback . . . 139
dynamic system
  autonomous . . . 64
  equilibrium point . . . 67
    hyperbolic . . . 68
    non-resonant . . . 68
    resonant . . . 68
  equivalence
    differential . . . 66
    topological . . . 66
  flow . . . 64
    properties . . . 64
  initial state . . . 48
  integral curve . . . 46
  linear . . . 69
    Differential Equivalence . . . 70
    nonautonomous . . . 48
    time dependent . . . 48
    Topological Equivalence . . . 70
  linearisation . . . 69
  matrix of linear approximation . . . 68
  nonautonomous . . . 47
    linear . . . 48
  orbit . . . 65
  point singular
    non-resonant . . . 68
  regular point . . . 67
  singular point . . . 67
    hyperbolic . . . 68
    resonant . . . 68
  stability . . . 57
    asymptotic . . . 57
    global . . . 57
    uniform . . . 57
  time dependent . . . 47
  time-independent . . . 64
  trajectory . . . 46
dynamic system's flow
  properties . . . 64

E

end-point map . . . 89
equation
  Ważewski-Davidenko . . . 25
equivalence . . . 7
  class . . . 2
  differential . . . 66, 70
  dynamic feedback . . . 140
  feedback . . . 102
  relation . . . 2
  state space . . . 101
  topological . . . 66, 70
equivalence class . . . 2
equivalence relation . . . 2

F

feedback . . . 10, 118
  dynamic . . . 139
  static . . . 139
feedback equivalent . . . 7
feedback synthesis . . . 10
F-equivalence . . . 7, 102
flag
  big . . . 127
  small . . . 127
F-linearisability . . . 103
flow . . . 64
form
  canonical . . . 2
    Brunovsky . . . 11
  chained . . . 126
  normal . . . 2
    linear control system . . . 7
function . . . 1
  analytic . . . 20
  class K . . . 58
  class K_∞ . . . 58
  classes . . . 19
  composition . . . 22
  continuous . . . 3
  differentiable . . . 20
  graph . . . 1
  inverse . . . 22
  jet . . . 20
  linear . . . 30
  LR equivalence . . . 33
  Morse . . . 40

smooth 20

G

graph of a function 1
group 2
associative 2
commutative 2

H

homeomorphism 66

I

immersion 34
index of critical point 40
initial state 48
integral curve 46
integral manifold 79
integrator backstepping 127
interior of a set 4

J

jet 20

K

Kalman
criterion 6
matrix 6
Kalman criterion 6
kinematic car 129
kinematics model
kinematic car 129

L

Landau symbol 4
LaTeX vii
linear, time dependent dynamic system . 48
linearisation 69
feedback 103
state space 103
LR equivalence 33

M

matrix
condition number 33
decoupling 117
fundamental 48
Gram 6
Jacobian 20
Kalman 6
of linear approximation 68
transition 48
model of kinematics
unicycle 132
module 3

N

neighbourhood of a point 3

O

operation 1
orbit 65

P

p-norm 31
Peano-Baker formula 49
point
critical 38
index 40
equilibrium 56, 67
asymptotically stable 57
globally stable 57
hyperbolic 68
non-resonant 68
resonant 68
stable 57
uniformly stable 57
unstable 57
regular 38, 67
singular 67
hyperbolic 68
non-resonant 68
resonant 68
problem
control 5
stabilisation 10
trajectory tracking 105

R

reflexivity 2
ring 2
with unity 2

S

S-equivalence 7, 101
set 1
boundary 4
closed 4
dense 4
interior 4
nowhere dense 4

reachable 90
S-linearisability 103
small flag 127
space
Banach 4
control 5
linear 3
state 5
topological 3
complete 4
stabilisability 10, 152
stability 57
asymptotic 57
global 57
Lyapunov 59
Poisson stable 93
uniform 57
state space equivalent 7
static feedback 139
submersion 34
subset 1
symmetry 2

T
time dependent linear dynamic system . . 48
topology 3
trajectory 46
transitivity 2

U
unicycle 132

V
variable
control 5, 86
state 5, 86
variable
output 86

W
Ważewski Inequality 50

Z
zero dynamics 121

List of Figures

1 System in the Brunovsky canonical form 13

1.1 Smooth, non-analytic function 21
1.2 Mechanism of the manipulator from exercise 1.5 28

2.1 Diagram of LR equivalence 34

3.1 Critical points and values 39
3.2 A "big" set of critical points $\mathcal{C}_f$ 39

5.1 Stability and asymptotic stability 58

6.1 Orbits of mathematical pendulum 66
6.2 Equivalence of dynamic systems 67
6.3 Straightening out a vector field 68
6.4 Hyperbolic equilibrium points in the plane: a) sink, b) source, c) saddle 71

7.1 Definition of operator $\mathrm{Ad}_X^t Y(x)$ 76

8.1 Piecewise continuous controls 87
8.2 Piecewise constant controls 87
8.3 Accessibility property: a), b) yes, c) no 91
8.4 Local controllability: a) no, b) yes 91
8.5 Reachable sets $\mathcal{R}_\sigma(0,t)$ and $\mathcal{R}_\sigma(0)$ 95
8.6 Poisson stability 98

9.1 Result of applying linearising feedback 106
9.2 Trajectory tracking system 107
9.3 Model of induction electric motor 111

10.1 Structure of the input-output relationship 118

10.2 Structure of control system (10.7) 120
10.3 Vertical rolling wheel 124

11.1 Kinematic car 129
11.2 Unicycle 132

12.1 Unicycle with shaft 136
12.2 Brunovsky compensator 141
12.3 Unicycle 145
12.4 Kinematic car 147

List of Theorems

0.3.1 Theorem (Brunovsky) 12

1.3.1 Theorem (Inverse Function Theorem) 23
1.4.1 Theorem (Implicit Function Theorem) 23

2.4.1 Theorem (On Submersions) 34
2.4.2 Theorem (On Immersions) 34
2.4.3 Theorem (Whitney) 35

3.1.1 Theorem (Sard) 39
3.2.1 Theorem (Morse) 40
3.3.1 Theorem (Hadamard) 41
3.5.1 Theorem (Fixed Point Theorem) 42

4.1.1 Theorem (On Existence and Uniqueness) 46
4.5.1 Theorem (Ważewski) 50

5.3.1 Theorem (Lyapunov Function Theorem) 59
5.4.1 Theorem (Barbalat) 60

6.3.1 Theorem (On Differential Equivalence) 67
6.4.1 Theorem (Straightening Out Theorem) 67
6.6.1 Theorem (Poincaré-Siegel-Sternberg) 69
6.6.2 Theorem (Hartman-Grobman) 69
6.7.1 Theorem (Kuiper) 70

7.3.1 Theorem (On Commutation) 77
7.4.1 Theorem (Simultaneous Straightening Out) 78
7.6.1 Theorem (Frobenius) 79

8.4.1 Theorem (Chow-Sussmann-Krener) 92
8.4.2 Theorem (Chow-Lobry-Krener) 92

8.5.2 Theorem (Sussmann's Controllability Condition) 94

9.2.1 Theorem (Krener-Sussmann-Respondek) 104
9.2.2 Theorem (Jakubczyk-Respondek) 104

12.3.3 Theorem (Charlet-Lévine-Marino) 142

13.2.1 Theorem (Brockett) 152
13.3.1 Theorem (Lizárraga) 154