CT561: Systems Modelling & Simulation

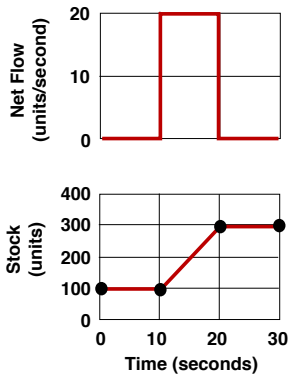
2: Integration – Graphical and Numerical

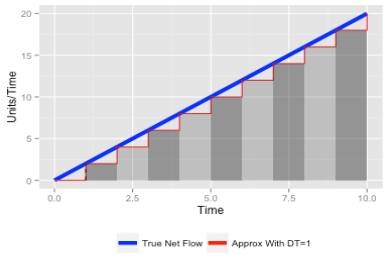
Dr. Jim Duggan,
School of Engineering & Informatics
National University of Ireland Galway.
https://github.com/JimDuggan/SDMR

1

Lecture Topics

- Stocks and Flows Recap
- Integration
  - Graphical
  - Numerical (Euler's Equation)
- Useful flow formulations
  - Fractional increase
  - Fractional decrease

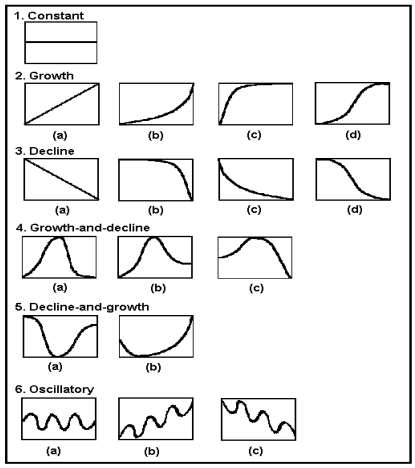




2

Stocks and Flows

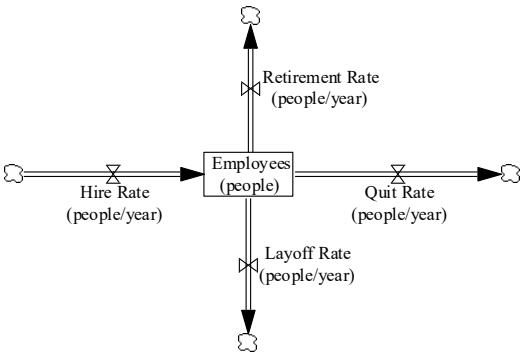
- Enrollments, **Students**, Graduations
- **Account Balance**, Credits, Debits
- Retirements, **Staff**, Recruitments, **Total Staff Retired**
- Absorptions, Emissions, **Carbon in the Atmosphere**
- People Entering, People Leaving, **People in the Store**
- **Water in the Lake**, Rainfall, Evaportions
- Customers Joining, **Customers**, Customers Leaving
- **People Infected**, New Infections, People Recovering, **People Recovered**



3

Stock and Flow Systems

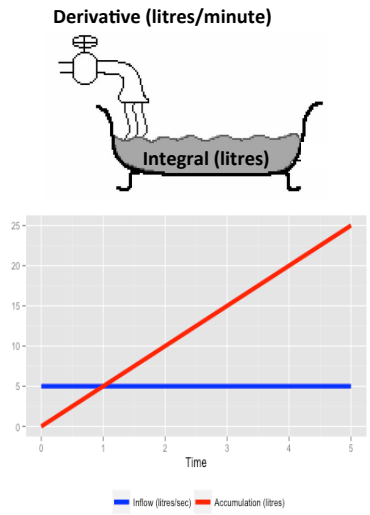
- All stock-flow systems share the same underlying structure.
- The stock **accumulates** its inflows to it, less the outflows from it.
- This is a fundamental concept of calculus (integrals and derivatives)



4

Calculus – Integration

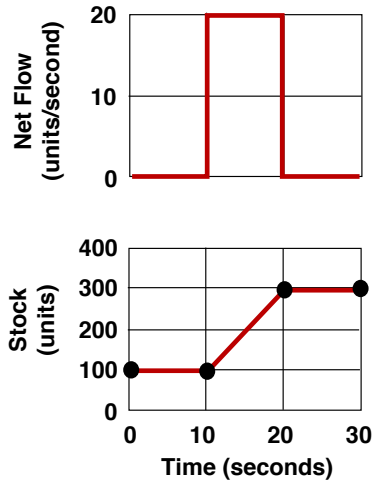
- Calculus is the study of how things **change over time**, and is described by Strogratz (2009) as "*perhaps the greatest idea that humanity has ever had.*"
- **Given the dynamics of the flows, what is the behaviour of the stock?**
- **Integration is the mathematical process of calculating the area under the net flow curve, between initial and final times.**



5

Graphical Integration

- Stocks accumulate their net flow
- The quantity added to a stock over any interval is the area bounded by the graph of the net rate between the start and end of the interval.
- Net Flow = Inflows - Outflows

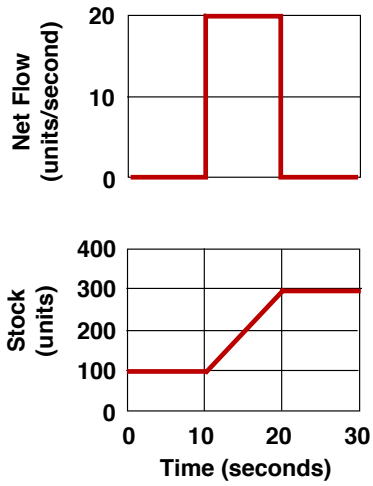


6

Graphical Integration (1/8)

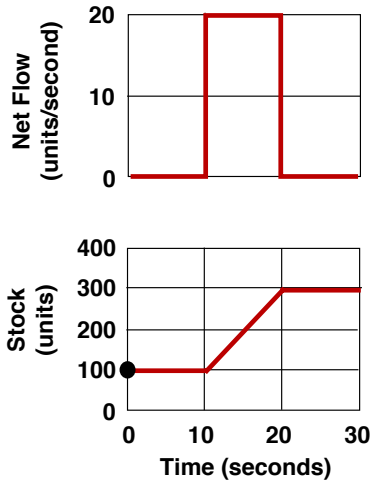
Make a set of axes to graph the stock. Stocks (units) and flows (units per time period) have different units of measure, and must be graphed on different scales.

Make a separate graph for the stock under the graph for the flows, with the time axes lined up.



Graphical Integration (2/8)

Plot the initial value of the stock on the stock graph. The initial value MUST be specified.

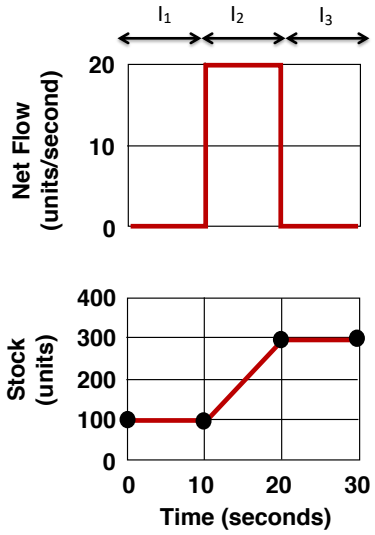


Graphical Integration (3/8)

Break the net flow into intervals with the same behaviour and calculate the amount added to the stock during the interval.

The amount added or subtracted to the stock during an interval is the area under the net rate curve for that same interval.

The total area is then added to the original value of the stock, and this point is then plotted on the stock graph.

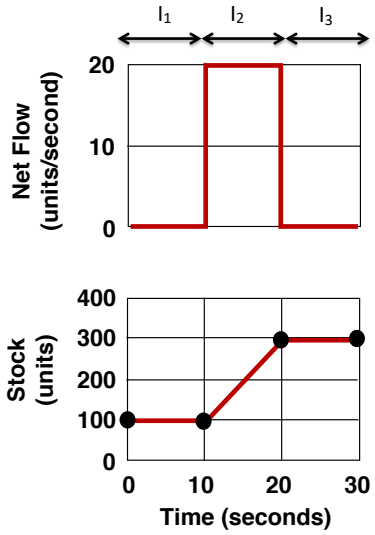


Graphical Integration (4/8)

(6) Sketch the trajectory of the stock between the start and end of each interval.

Find the value of the net rate at the start of the interval.

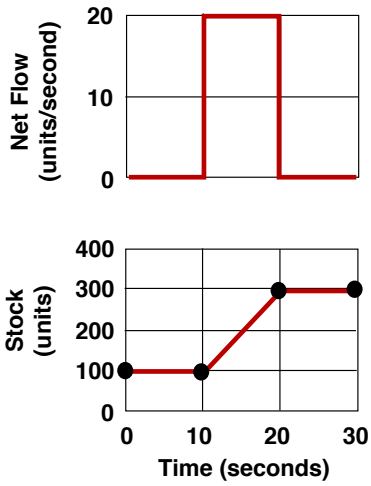
If the net rate is positive, the stock will be increasing at that time, if it is negative, the stock will be decreasing.



Graphical Integration (5/8)

The behaviour of the stock can be inferred from the net flow according to the following rules:

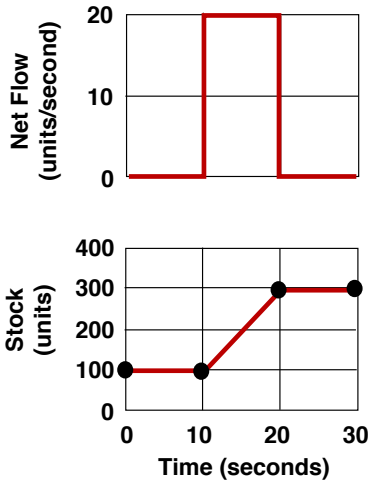
- If the net rate is positive and increasing, the stock increases at an increasing rate (the stock accelerates upwards)
- If the net rate is positive and decreasing, the stock increases at a decreasing rate (the stock is decelerating but still moving upwards)



Graphical Integration (6/8)

The behaviour of the stock can be inferred from the net flow according to the following rules:

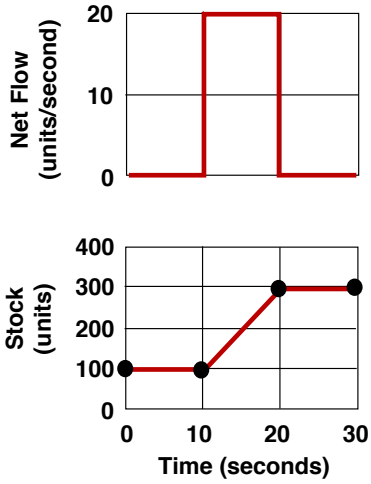
- If the net rate is negative and its magnitude is increasing (the net rate is becoming more negative), the stock decreases at an increasing rate.
- If the net rate is negative and its magnitude is decreasing (becoming less negative), then the stock decreases at a decreasing rate.



Graphical Integration (7/8)

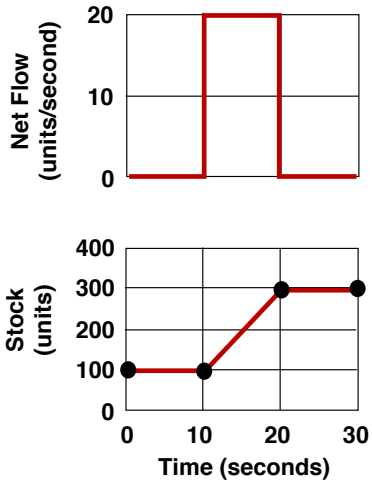
The behaviour of the stock can be inferred from the net flow according to the following rules:

- If the net rate is positive and its magnitude is constant, the stock increases at an constant rate.
- If the net rate is negative and its magnitude is constant, the stock decreases at an constant rate.



Graphical Integration (8/8)

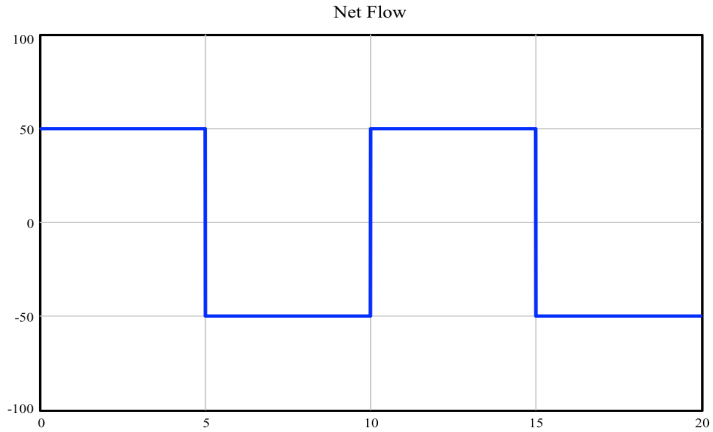
8. Whenever the net rate is zero, the stock is unchanging. Make sure that your graph of the stock shows no change in the stock everywhere the net rate is zero. At points where the net rate changes from positive to negative, the stock reaches a maximum as it ceases to rise and starts to fall. At points where the net rate changes from negative to positive, the stock reaches a minimum as it ceases to fall and starts to rise.
9. Repeat steps 5 through 8 until completion.



Challenge 2.1

Graphically integrate this net flow.

Assume the starting value of the stock is 100.

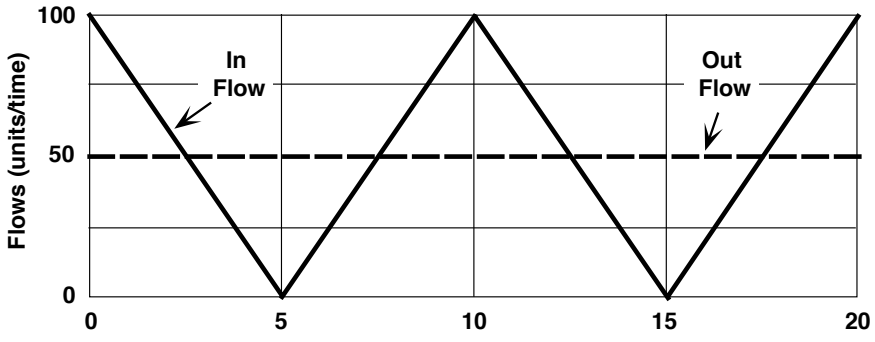


15

Challenge 2.2

Graphically integrate the net flow.

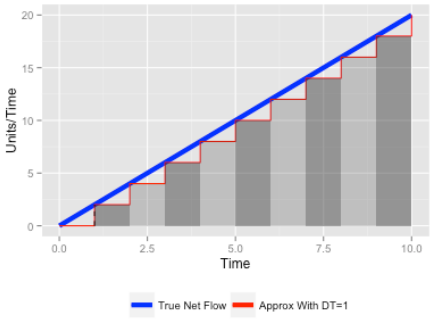
Assume the starting value of the stock is 100.



16

Numerical Integration

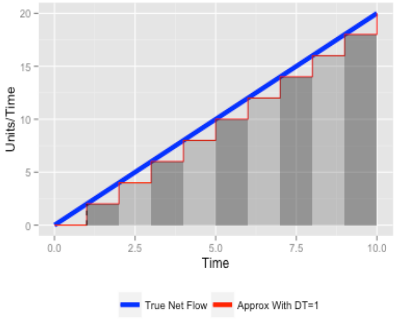
- Euler's Method
- Approximate area under the net flow curve as a summation of rectangles, of width DT
- The smaller DT, the more accurate the result



$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Solution, DT=1

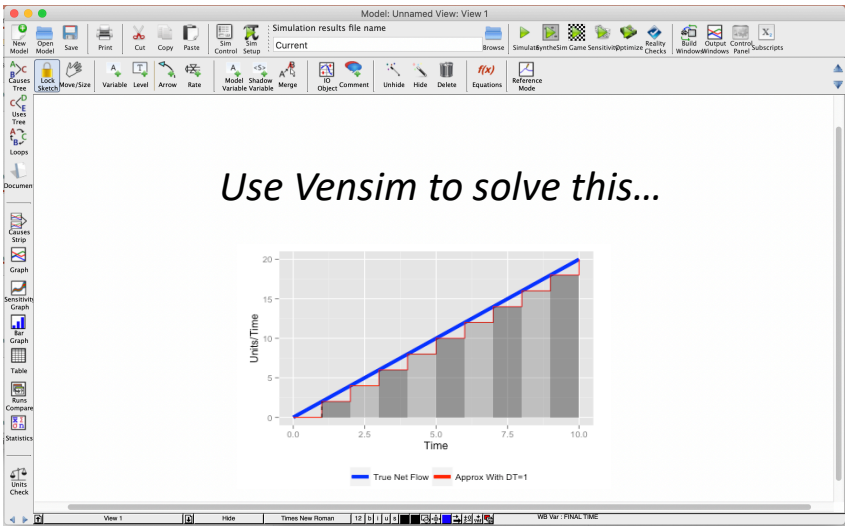
Time	$Stock_t$	Net Flow
0	0	0
1	0+0=0	2
2	0+2=2	4
3	2+4=6	6
4	6+6=12	8
5	12+8=20	10
6	20+10=30	12
7	30+12=42	14
8	42+14=56	16
9	56+16=72	18
10	72+18=90	20



$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Note: Stock only depends on previous stock and net flows

Challenge 2.3

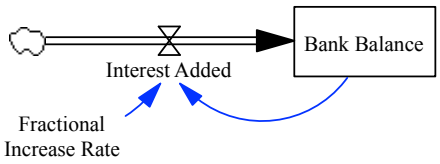


Formulating Flows

- Stocks change over time through the actions of a **flow**.
- Basic flow types:
  - Fractional increase
  - Fractional decrease
- Flow depends on the stock and a constant value (increase or decrease fraction)

Fractional Increase Rate

- Consider a stock *S* with inflow rate R_I
- The inflow is proportional to the size of *S*
- The fractional increase rate is a constant *g*

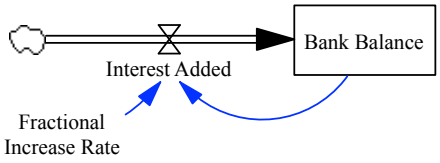


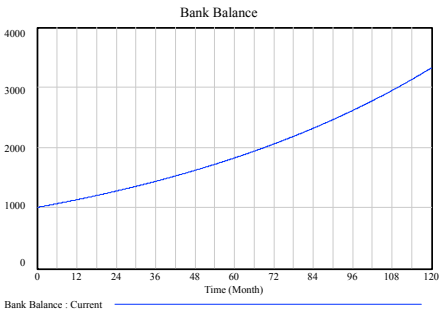
Bank Balance = INTEG(Interest Added , 1000)

Fractional Increase Rate = 0.01

Interest Added = Bank Balance * Fractional Increase Rate

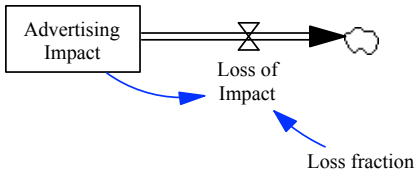
System behaviour
Generates Exponential growth





Fractional Decrease Rate

- Consider a stock *S* with outflow rate R_O
- The outflow is proportional to the size of *S*
- The fractional decrease rate is a constant *d*

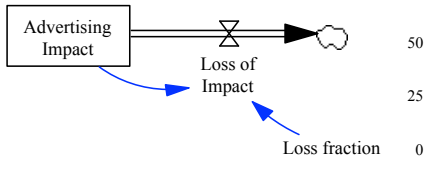


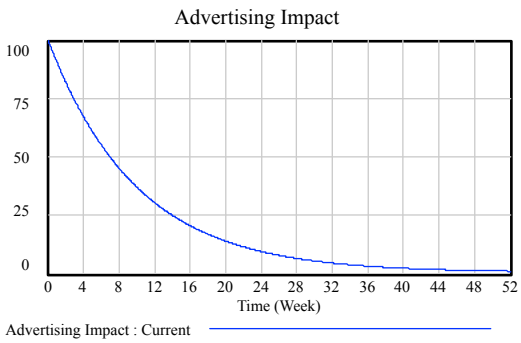
Advertising Impact = INTEG(- Loss of Impact , 100)

Loss fraction = 0.1

Loss of Impact = Loss fraction * Advertising Impact

System behaviour
Generates Exponential decay

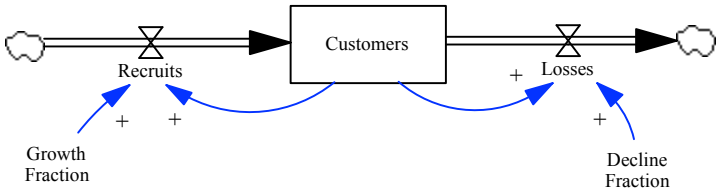




A simple model of Customers

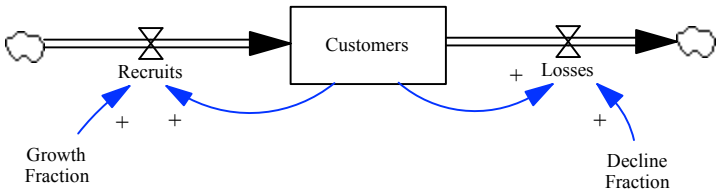
- Given that the customer base is an accumulation, it can be modeled as a stock (assume = 10,000)
- The inflow is recruits, and the outflow are losses, also known as the churn rate.
- The goal of organizations is to limit the losses and maximize the recruits, in order to maintain increasing customers levels, and therefore support company growth.

Stock and Flow Model



Customers= INTEG (Recruits-Losses, 10000)

Flow equations

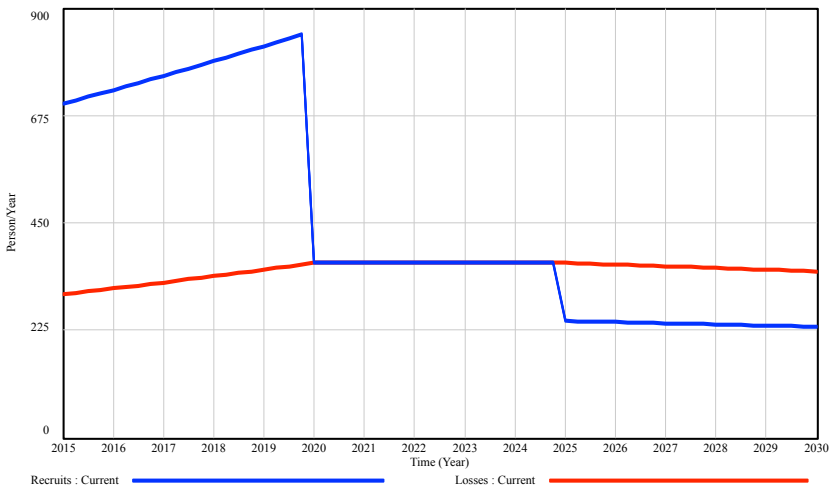


Recruits = Customers*Growth Fraction
Losses = Customers*Decline Fraction

Decline Fraction=0.03
Growth Fraction=0.07-step(0.04,2020)-step(0.01,2025)

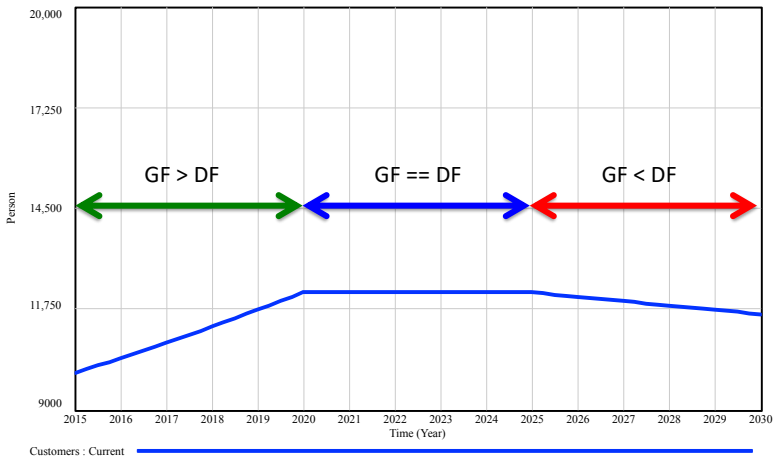
27

Simulation – Flows



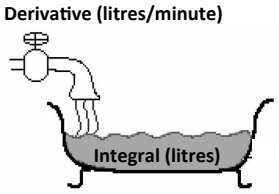
28

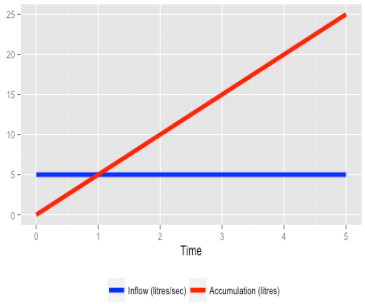
Stock: 3 Phases of behaviour



Calculus – Integration

- Calculus is the study of how things **change over time**, and is described by Strogratz (2009) as "*perhaps the greatest idea that humanity has ever had*."
- **Given the dynamics of the flows, what is the behaviour of the stock?**
- **Integration is the mathematical process of calculating the area under the net flow curve, between initial and final times.**





Challenge 2.4

- A University attracts 30% of its total students as new students each year, and has an initial population of 1000
- It graduates 25% of all students
- For this:
  - Draw a stock and flow model
  - Add the net flow to the model
  - Formulate the equations
  - Build a model in Vensim with DT=0.25
  - Start the model in 2020, and complete in 2030