Chapter 4
Distribution of Sample Correlation Coefficients

Correlation coefficients measure dependence. Different types of correlation coefficients measure the dependence among variables in different situations.

The simple correlation coefficient, i.e., Karl Pearson's correlation coefficient, measures the degree of linear relationship between two random variables; the multiple correlation coefficient measures the degree of linear relationship between a variable and a set of variables; and the partial correlation coefficient measures the degree of linear relationship between two variables when a set of variables is held fixed.

In the case of joint normal distributions, these correlation coefficients measure the dependence among variables. The sample correlation coefficient is an estimator of the population correlation coefficient. The sample-based correlation coefficients are functions of sufficient statistics, which are invariant under location and scale transformations; the population correlation coefficients are functions of the parameters, which are invariant under these transformations.

Correlation Coefficient of a Bivariate Sample

To understand how to derive the distribution of the correlation coefficient, we consider the simple case of two random variables.

It can be recalled that the correlation coefficients are functions of variance and covariance. So, the covariance matrix of a random vector can be expressed as a function of its correlation coefficients. Thus, considering the PDF of the sample covariance matrix and deriving the distribution of the sample correlation coefficient from it can be helpful.

The distribution of positive definite matrix $A = \sum_{\alpha=1}^{N}(\underset{\sim}{X}_\alpha - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_\alpha - \bar{\underset{\sim}{X}})'$ follows a Wishart distribution $W_p(\Sigma, n)$.

The PDF of the Wishart distribution is

$$f(A) = \frac{|A|^{\frac{n-p-1}{2}} \exp(-\frac{1}{2} tr \Sigma^{-1} A)}{2^{\frac{np}{2}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2}) \pi^{\frac{p(p-1)}{4}} |\Sigma|^{n/2}}$$

which is based on a $p \times 1$ vector $\underset{\sim}{X}$.

We are considering only two univariate random variables X_1 and X_2, so $p = 2$ and subsequently,

$$\underset{\sim}{X} = (X_1, X_2)', \quad \underset{\sim}{x} = (x_1, x_2)', \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$Var(X_1) = \sigma_{11}, \quad Var(X_2) = \sigma_{22}, \quad Cov(X_1, X_2) = \sigma_{12},$$

$$a_{ij} = \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j), \quad i, j = 1, 2.$$

The sample correlation coefficient is defined as

$$r = \frac{a_{12}}{\sqrt{a_{11} a_{22}}}$$

or

$$a_{12} = r\sqrt{a_{11} a_{22}}$$

and the population correlation coefficient is defined as

$$\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11} \sigma_{22}}}$$

or

$$\sigma_{12} = \rho\sqrt{\sigma_{11} \sigma_{22}}.$$

For $p = 2$, $f(A) = f(a_{11}, a_{22}, a_{12})$, it is the joint distribution of a_{11}, a_{22} and a_{12}.

Also

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12}^2 ,$$

$$|\Sigma| = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} = \sigma_{11} \sigma_{22} - \sigma_{12}^2.$$

Using $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2$ and $\sigma_{12} = \rho\sigma_1\sigma_2$, we get

$$|\Sigma| = \sigma_1^2\sigma_2^2(1-\rho^2)$$

and

$$\begin{aligned}
\Sigma^{-1} &= \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \\
&= \frac{1}{(1-\rho^2)}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}.
\end{aligned}$$

So

$$\begin{aligned}
|\Sigma^{-1}| &= \frac{1}{(1-\rho^2)^2}\begin{vmatrix} \frac{1}{\sigma_1^2} & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \frac{1}{\sigma_2^2} \end{vmatrix} \\
&= \frac{1}{(1-\rho^2)^2}\left(\frac{1}{\sigma_1^2\sigma_2^2} - -\frac{\rho^2}{\sigma_1^2\sigma_2^2}\right) \\
&= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}.
\end{aligned}$$

Next we simplify

$$\begin{aligned}
\Sigma^{-1}A &= \frac{1}{(1-\rho^2)}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&= \frac{1}{(1-\rho^2)}\begin{pmatrix} \frac{a_{11}}{\sigma_1^2} - \frac{\rho a_{21}}{\sigma_1\sigma_2} & \frac{a_{12}}{\sigma_1^2} - \frac{\rho a_{22}}{\sigma_2^2} \\ -\frac{\rho a_{11}}{\sigma_1\sigma_2} + \frac{a_{21}}{\sigma_2^2} & -\frac{\rho a_{12}}{\sigma_1\sigma_2} + \frac{a_{22}}{\sigma_2^2} \end{pmatrix}
\end{aligned}$$

and

$$tr(\Sigma^{-1}A) = \frac{1}{(1-\rho^2)}\left(\frac{a_{11}}{\sigma_1^2} + \frac{a_{22}}{\sigma_2^2} - 2\frac{\rho a_{12}}{\sigma_1\sigma_2}\right) = Q, \text{ say.}$$

Now consider the Wishart density for $p = 2$ and simplify it as follows:

$$\begin{aligned}
f(A) &= f(a_{11}, a_{22}, a_{12}) \\
&= \frac{(a_{11}a_{22} - a_{12}^2)^{\frac{n-2-1}{2}}\exp(-\frac{1}{2}Q)}{2^{\frac{(n\cdot 2)}{2}}\prod_{i=0}^{2-1}\Gamma(\frac{n-i}{2})\pi^{\frac{2-1}{4}}|\Sigma|^{n/2}} \\
&= \frac{(a_{11}a_{22} - a_{12}^2)^{\frac{n-3}{2}}\exp(-\frac{1}{2}Q)}{2^n\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}|\Sigma|^{n/2}}.
\end{aligned}$$

Now r is introduced in this expression using the transformations

$$a_{12} = r\sqrt{a_{11}a_{22}}$$

and

$$da_{12} = \sqrt{a_{11}a_{22}}\, dr.$$

The aim is to derive the distribution of r as a marginal distribution from the joint distribution $f(a_{11}, a_{22}, r)$. Thus

$$f(A) = f(a_{11}, a_{22}, r) = f(a_{11}, a_{22}, a_{12})|J|$$

where the Jacobian of transformation is $|J| = \sqrt{a_{11}a_{22}}$. Then

$$f(A) = \frac{(a_{11}a_{22} - r^2 a_{11}a_{22})^{\frac{n-3}{2}}}{2^n\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})|\Sigma|^{n/2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{a_{11}}{\sigma_1^2} + \frac{a_{22}}{\sigma_2^2} - \frac{2\rho r\sqrt{a_{11}a_{22}}}{\sigma_1\sigma_2}\right\}\right]|J|.$$

Now $f(A) = f(a_{11}, a_{22}, r)$. So we simplify

$$\begin{aligned}
f(A) &= \frac{(a_{11}a_{22} - r^2 a_{11}a_{22})^{\frac{n-3}{2}}}{2^n|\Sigma|^{n/2}\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{a_{11}}{\sigma_1^2} + \frac{a_{22}}{\sigma_2^2} - \frac{2\rho r\sqrt{a_{11}a_{22}}}{\sigma_1\sigma_2}\right\}\right]\sqrt{a_{11}a_{22}} \\
&= \frac{(a_{11}a_{22})^{\frac{n-3}{2}}(1-r^2)^{\frac{n-3}{2}}}{2^n|\Sigma|^{n/2}\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})} \exp\left[-\frac{1}{2(1-\rho^2)}\cdot\frac{a_{11}}{\sigma_1^2}\right] \\
&\quad \times \exp\left[-\frac{1}{2(1-\rho^2)}\cdot\frac{a_{22}}{\sigma_2^2}\right]\exp\left[\frac{\rho r\sqrt{a_{11}a_{22}}}{\sigma_1\sigma_2(1-\rho^2)}\right]\sqrt{a_{11}a_{22}} \\
&= \frac{a_{11}^{\frac{n}{2}-1}a_{22}^{\frac{n}{2}-1}(1-r^2)^{\frac{n-3}{2}}}{2^n|\Sigma|^{n/2}\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})} \exp\left[-\frac{1}{2(1-\rho^2)}\cdot\frac{a_{11}}{\sigma_1^2}\right]\exp\left[-\frac{1}{2(1-\rho^2)}\cdot\frac{a_{22}}{\sigma_2^2}\right] \\
&\quad \times \left[\sum_{\alpha=0}^{\infty}\left\{\frac{\rho r\sqrt{a_{11}a_{22}}}{\sigma_1\sigma_2(1-\rho^2)}\right\}^{\alpha}\frac{1}{\alpha!}\right] \quad \text{(The last exponent is expressed as summation)} \\
&= \frac{(1-r^2)^{\frac{n-3}{2}}}{2^n|\Sigma|^{n/2}\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})}\sum_{\alpha=0}^{\infty}\left[\left\{a_{11}^{\frac{n+\alpha}{2}-1}\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}\right)\right\}\right. \\
&\quad \left.\times\left\{a_{22}^{\frac{n+\alpha}{2}-1}\exp\left(-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)\right\}\left\{\frac{\rho r}{\sigma_1\sigma_2(1-\rho^2)}\right\}^{\alpha}\frac{1}{\alpha!}\right].
\end{aligned}$$

Now $f(r) = \int_0^{\infty}\int_0^{\infty} f(a_{11}, a_{22}, r)da_{11}da_{22}$.

Note that

$$\begin{aligned}
\int_0^{\infty} a_{11}^{\frac{n+\alpha}{2}-1}\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}\right)da_{11} &= \Gamma\left(\frac{n+\alpha}{2}\right)\left[2\sigma_1^2(1-\rho^2)\right]^{\frac{n+\alpha}{2}}, \\
\int_0^{\infty} a_{22}^{\frac{n+\alpha}{2}-1}\exp\left(-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)da_{22} &= \Gamma\left(\frac{n+\alpha}{2}\right)\left[2\sigma_2^2(1-\rho^2)\right]^{\frac{n+\alpha}{2}}.
\end{aligned}$$

Substituting these integrals, we get

$$
\begin{aligned}
f(r) &= \frac{(1-r^2)^{\frac{n-3}{2}}}{2^n\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})|\Sigma|^{n/2}} \sum_{\alpha=0}^{\infty}\left[\frac{(\rho r)^\alpha}{\sigma_1^\alpha\sigma_2^\alpha(1-\rho^2)^\alpha \alpha!}\right] \\
&\quad \times \left[\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2 2^{n+\alpha}\sigma_1^{n+\alpha}\sigma_2^{n+\alpha}(1-\rho^2)^{n+\alpha}\right] \\
&= \frac{(1-r^2)^{\frac{n-3}{2}}2^n\sigma_1^n\sigma_2^n(1-\rho^2)^n}{2^n\sqrt{\pi}\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})(\sigma_1^2\sigma_2^2-\rho^2\sigma_1^2\sigma_2^2)^{n/2}} \sum_{\alpha=0}^{\infty}\left[\frac{(\Gamma(\frac{n+\alpha}{2}))^2(\rho r)^\alpha 2^\alpha\sigma_1^\alpha\sigma_2^\alpha(1-\rho^2)^\alpha}{\sigma_1^\alpha\sigma_2^\alpha(1-\rho^2)^\alpha\alpha!}\right] \\
&= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}} \sum_{\alpha=0}^{\infty}\left[\frac{(2\rho r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2\right].
\end{aligned}
$$

Now we simplify it using the duplication formula

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2m)}{2^{2m-1}}.$$

Using $m = \frac{n-1}{2}$ in duplication formula, we get

$$\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right) = \Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right) = \frac{\sqrt{\pi}\Gamma(n-1)}{2^{n-2}}$$

and using it in $f(r)$, we get

$$
\begin{aligned}
f(r) &= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}2^{n-2}}{\sqrt{\pi}\sqrt{\pi}\Gamma(n-1)} \sum_{\alpha=0}^{\infty}\frac{(2\rho r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2 \\
&= \frac{2^{n-2}(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\pi(n-2)!} \sum_{\alpha=0}^{\infty}\frac{(2\rho r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2
\end{aligned}
$$

Here $n = N-1$ and $-1 \le r \le 1$. This is the PDF of the simple correlation coefficient, also known as Karl Pearson's correlation coefficient, r. For $n = 1$, the density reduces to zero.

The main characteristic of $f(r)$ is that the only function of ρ which can be estimated unbiasedly is $\sin^{-1}\rho$, i.e.,

$$E[\sin^{-1} r] = \sin^{-1}\rho.$$

Also, $E(r) \neq \rho$. The large sample approximation of the mean and variance of r is

$$
\begin{aligned}
E(r) &\approx \rho - \frac{\rho(1-\rho^2)}{2n} + O\left(\frac{1}{n^2}\right) \\
Var(r) &\approx \frac{(1-\rho^2)^2}{n} + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

For r, we transform r to $z = \frac{1}{2}\log\frac{1+r}{1-r}$. This is because convergence to normality is more rapid in z transformation.

Fisher (1915) gave the following form of density of r:

$$f(r) = \frac{(1-\rho^2)^{\frac{n}{2}}(1-r^2)^{\frac{n-3}{2}}}{\pi(n-2)!}\left[\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}}\right\}\Bigg|_{x=\rho r}\right].$$

Hotelling (1953) recommended the following form:

$$f(r) = \frac{n-1}{\sqrt{2\pi}}\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}(1-\rho^2)^{\frac{n}{2}}(1-r^2)^{\frac{n-3}{2}}(1-\rho r)^{-n+\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2};n+\frac{1}{2};\frac{1+\rho r}{2}\right)$$

where

$$F(a,b;c,x) = \sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{\Gamma(a)}\cdot\frac{\Gamma(b+j)}{\Gamma(b)}\cdot\frac{\Gamma(c)}{\Gamma(c+j)}\cdot\frac{x^j}{j!}$$

is the Hypergeometric function.

The cumulative density function of r, $P[r \leq r^*] = F[r^*|N,\rho]$ has been tabulated by David (1938) for $\rho = 0.1, 0.2, \ldots, 0.9$, $N = 3, 4, \ldots, 25, 50, 100, 200, 400$. and $r^* = -1. - 0.95, \ldots, 0, 0.5, \ldots, 1$.

Such tables are used for testing the hypotheses related to ρ $H_0 : \rho = \rho_0$ and the decision rules for various alternative hypotheses are as follows:

(i) For $H_1 : \rho > \rho_0$,

Reject H_0 if $r > r^*$ such that $1 - F(r^*|N,\rho_0) = \alpha$.

(ii) For $H_1 : \rho < \rho_0$,

Reject H_0 if $r < r^{**}$, $F(r^{**}|N,\rho_0) = \alpha$.

(iii) For $H_1 : \rho \neq \rho_0$,

Reject H_0 if either $r > r^*$ or $r < r^{**}$, $[1 - F(r^*|N,\rho_0)] + F(r^{**}|N,\rho_0) = \alpha$.

For computations, $N \geq 10$, $|\rho| \leq 0.8$, the test is nearly unbiased.

David (1938) also computed confidence region for ρ, say $f_1(\rho)$ and $f_2(\rho)$, such that

$$P[f_1(\rho) < r < f_2(\rho)|\rho] = 1 - \alpha.$$

Here $f_1(\rho)$ and $f_2(\rho)$ are monotonically increasing functions of ρ and are chosen so

$$1 - F(r^*|N,\rho) = \frac{\alpha}{2} = F(r^{**}|N,\rho).$$

If $\rho = f_i^{-1}(r)$ is inverse of $r = f_i(\rho), i = 1, 2$, then the inequality $f_1(\rho)$ is equivalent to $\rho < f_1^{-1}(r)$ and $r < f_2(\rho)$ is equivalent to $f_2^{-1}(r) < \rho$. Thus

$$P[f_1(\rho) < r < f_2(\rho)|\rho] = 1 - \alpha$$

can be written as

$$P[f_2^{-1}(r) < \rho < f_1^{-1}(r)] = 1 - \alpha.$$

Sample Distribution of Sample Correlation Coefficient when Population Correlation Coefficient is Zero $(\rho = 0)$

We derive now the probability distribution of the sample correlation coefficient from a bivariate normal distribution when the population correlation coefficient is zero. The PDF of r is

$$\begin{aligned} f(r) &= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}} \sum_{\alpha=0}^{\infty}\left[\frac{(2\rho r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2\right] \\ &= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}}\left[\frac{(2\rho r)^0}{0!}\left(\Gamma\left(\frac{n+0}{2}\right)\right)^2 + \sum_{\alpha=1}^{\infty}\left\{\frac{(2\rho r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2\right\}\right] \\ &= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}}\left[1\cdot\left(\Gamma\left(\frac{n}{2}\right)\right)^2 + \sum_{\alpha=1}^{\infty}\left\{\frac{(2\rho r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2\right\}\right]. \end{aligned}$$

When $\rho = 0$,

$$\begin{aligned} f(r) &= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}}\left[\left(\Gamma\left(\frac{n}{2}\right)\right)^2 + \sum_{\alpha=1}^{\infty}\left\{\frac{(2\cdot 0\cdot r)^\alpha}{\alpha!}\left(\Gamma\left(\frac{n+\alpha}{2}\right)\right)^2\right\}\right] \\ &= \frac{(1-r^2)^{\frac{n-3}{2}}(1-\rho^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\sqrt{\pi}}\left[\left(\Gamma\left(\frac{n}{2}\right)\right)^2 + 0\right] \\ &= (1-r^2)^{\frac{n-3}{2}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma\left(\frac{1}{2}\right)} \quad \text{(using } \sqrt{\pi} = \Gamma(1/2)\text{)} \\ &= (1-r^2)^{\frac{N-4}{2}}\frac{\Gamma(\frac{(N-1)}{2})}{\Gamma\left(\frac{1}{2}\right)\Gamma(\frac{N-2}{2})} \quad \text{(using } n = N-1\text{)} \\ &= \frac{(1-r^2)^{\frac{N-4}{2}}}{B(\frac{1}{2}, \frac{N-2}{2})}. \end{aligned}$$

where $B(\cdot,\cdot)$ is a Beta function defined as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Thus we have

$$f(r|\rho = 0) = \frac{1}{B(\frac{1}{2}, \frac{N-2}{2})}(1-r^2)^{\frac{N-4}{2}}; \quad -1 \leq r \leq 1$$

which is a symmetric distribution.

Asymptotic Distribution of a Sample Correlation Coefficient and Fisher's z

As the sample size increases, the sample correlation coefficient tends to follow a normal distribution. The distribution of Fisher's z, which is a function of the sample correlation coefficient, has a variance approximately independent of the population correlation, and Fisher's z tends to normality faster.

If

$$r(n) = \frac{A_{ij}(n)}{\sqrt{A_{ii}(n)A_{jj}(n)}}, \quad i \neq j,$$

is the sample correlation coefficient of a sample of size $N = (n+1$ from bivariate normal distribution with correlation ρ, then

$$\frac{\sqrt{n}\,(r(n)-\rho)}{(1-\rho^2)} \quad \text{or} \quad \frac{\sqrt{N}\,(r(n)-\rho)}{(1-\rho^2)}$$

has the limiting distribution $N(0,1)$.

The Fisher's z is

$$z = \frac{1}{2}\log\frac{1+r}{1-r} = \tanh^{-1} r$$

where

$$r = \tanh z = \frac{\exp(z)-\exp(-z)}{\exp(z)+\exp(-z)}$$

is the correlation coefficient from a bivariate normal distribution with correlation coefficient ρ.

Let

$$\xi = \frac{1}{2}\log\frac{1+\rho}{1-\rho}$$

then $\sqrt{n}(z-\xi)$ has a limiting normal distribution with mean 0 and variance 1.

It can be shown that to a closer approximation

$$E(z) \approx \xi + \frac{\rho}{2n}$$
$$E(z-\xi)^2 \approx \frac{1}{n} + \frac{8-\rho^2}{4n^2}.$$

Distribution of the Set of Correlation Coefficients for a Diagonal Population Matrix

We now find the density of the set of sample correlation coefficients $r_{ij}, (i < j = 1, 2, \ldots, p)$ given $\rho_{ij} = 0$ for all $i \neq j$. In such a case, $\Sigma = diag(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{pp})$ and $|\Sigma| = \prod_{i=1}^{p} \sigma_{ii}$.

We begin with $W_p(\Sigma, n)$ with diagonal Σ as follows.

$$\begin{aligned} f(A) &= \frac{|A|^{\frac{n-p-1}{2}} \exp(-\frac{1}{2} tr \Sigma^{-1} A)}{2^{\frac{np}{2}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2}) \pi^{\frac{p(p-1)}{4}} |\Sigma|^{\frac{n}{2}}} \\ &= \frac{|A|^{\frac{n-p-1}{2}} \exp(-\frac{1}{2} \sum_{i=1}^{p} \frac{a_{ii}}{\sigma_{ii}})}{2^{\frac{np}{2}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2}) \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}}}, \qquad i, j = 1, 2, \ldots, p. \end{aligned}$$

Next, we use the transformations

$$a_{ij} = r_{ij}\sqrt{a_{ii}a_{jj}} \quad , i \neq j$$
$$a_{ii} = a_{ii}$$

and we have

$$\frac{\partial a_{ii}}{\partial r_{ij}} = \sqrt{a_{ii}a_{jj}}$$
$$\frac{\partial a_{ij}}{\partial a_{ii}} = \frac{r_{ij}}{2}\sqrt{\frac{a_{jj}}{a_{ii}}}.$$

The Jacobian of the transformation is

$$
\begin{aligned}
|J| &= \frac{\partial(a_{11}, a_{22}, \ldots, a_{pp}, a_{12}, \ldots, a_{1p}, a_{23}, \ldots, a_{2p}, \ldots, a_{p-1,p})}{\partial(a_{11}, a_{22}, \ldots, a_{pp}, r_{12}, \ldots, r_{1p}, r_{23}, \ldots, r_{2p}, \ldots, r_{p-1,p})} \\
&= \begin{vmatrix}
1 & 0 & \cdots & 0 & \frac{1}{2}\sqrt{\frac{a_{22}}{a_{11}}}r_{12} & \cdots & \frac{1}{2}\sqrt{\frac{a_{pp}}{a_{11}}}\cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \sqrt{a_{11}a_{22}} & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{vmatrix} \\
&= \prod_{i=1}^{p} a_{ii}^{\frac{p-1}{2}}.
\end{aligned}
$$

Substituting $a_{ij} = r_{ij}\sqrt{a_{ii}a_{jj}}$ for $i \neq j$ in $|A|$, we get

$$
\begin{aligned}
|A| &= \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1p} \\
a_{21} & a_{22} & \cdots & a_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
a_{p1} & a_{p2} & \cdots & a_{pp}
\end{vmatrix} \\
&= \begin{vmatrix}
a_{11} & r_{12}\sqrt{a_{11}a_{22}} & r_{13}\sqrt{a_{11}a_{33}} & \cdots & r_{1p}\sqrt{a_{11}a_{pp}} \\
r_{12}\sqrt{a_{11}a_{22}} & a_{22} & r_{23}\sqrt{a_{22}a_{33}} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_{1p}\sqrt{a_{11}a_{pp}} & r_{p2}\sqrt{a_{22}a_{pp}} & \cdots & \cdots & a_{pp}
\end{vmatrix} \\
&= \left(\prod_{i=1}^{p} a_{ii}\right) \begin{vmatrix}
1 & r_{12} & r_{13} & \cdots & r_{1p} \\
r_{21} & 1 & r_{23} & \cdots & r_{2p} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
r_{p1} & r_{p2} & r_{p3} & \cdots & 1
\end{vmatrix} \\
&= |R| \prod_{i=1}^{p} a_{ii}
\end{aligned}
$$

where R is the matrix of correlation coefficients r_{ij}.

The joint density of $a_{11}, a_{22}, \ldots, a_{pp}$ and r_{ij} is now

$$\begin{aligned}
f(a_{11}, \ldots, a_{pp}, r_{ij}) &= f(A)|J| \\
&= \frac{|A|^{\frac{n-p-1}{2}} \exp(-\frac{1}{2}\sum_{i=1}^{p} \frac{a_{ii}}{\sigma_{ii}})}{2^{\frac{np}{2}} \prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \prod_{i=1}^{p} a_{ii}^{\frac{p-1}{2}} \\
&= \frac{|R|^{\frac{n-p-1}{2}} (\prod_{i=1}^{p} a_{ii})^{\frac{n-p-1}{2}} \exp(-\frac{1}{2}\sum_{i=1}^{p} \frac{a_{ii}}{\sigma_{ii}}) \prod_{i=1}^{p} a_{ii}^{\frac{p-1}{2}}}{2^{\frac{np}{2}} \prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \\
&= \frac{|R|^{\frac{n-p-1}{2}}}{\pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \prod_{i=1}^{p} \left[\frac{a_{ii}^{\frac{n}{2}-1} \exp(-\frac{1}{2}\frac{a_{ii}}{\sigma_{ii}})}{2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}}} \right].
\end{aligned}$$

Our aim is to find $f(r_{ij})$ which can be derived as a marginal density from $f(a_{11}, \ldots, a_{pp}, r_{ij})$. Thus

$$\begin{aligned}
f(r_{ij}) &= \int_0^\infty \int_0^\infty \ldots \int_0^\infty f(a_{11}, a_{22}, \ldots, a_{pp}, r_{ij}) da_{11} da_{22} \ldots da_{pp} \\
&= \int_0^\infty \int_0^\infty \ldots \int_0^\infty \frac{|R|^{\frac{n-p-1}{2}}}{\pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \prod_{i=1}^{p} \left[\frac{a_{ii}^{\frac{n}{2}-1} \exp(-\frac{1}{2}\frac{a_{ii}}{\sigma_{ii}})}{2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}}} \right] da_{11} da_{22} \ldots da_{pp}
\end{aligned}$$

Let $\frac{a_{ii}}{2\sigma_{ii}} = u_i$ and we solve the integral using Gamma function $\int_0^\infty x^{n-1} \exp^{-x} dx = \Gamma(n)$.

$$\int_0^\infty a_{ii}^{\frac{n}{2}-1} \exp\left(-\frac{1}{2}\frac{a_{ii}}{\sigma_{ii}}\right) da_{ii} = 2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}} \int_o^\infty u_i^{\frac{n}{2}-1} \exp(-u_i) du_i = \Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}}$$

Also $\frac{a_{ii}}{\sigma_{ii}} \sim \chi^2(n)$. Then

$$\begin{aligned}
f(r_{ij}) &= \frac{|R|^{\frac{n-p-1}{2}}}{\prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}} \cdot 2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{p} \left\{ a_{ii}^{\frac{n}{2}-1} \exp\left(-\frac{1}{2}\frac{a_{ii}}{\sigma_{ii}}\right) da_{ii} \right\} \\
&= \frac{|R|^{\frac{n-p-1}{2}}}{\prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}} \cdot 2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \prod_{i=1}^{p} \left[\Gamma\left(\frac{n}{2}\right) \cdot 2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}} \right] \\
&= \frac{|R|^{\frac{n-p-1}{2}} \cdot 2^{\frac{np}{2}} \left(\prod_{i=1}^{p} \Gamma(\frac{n}{2})\right) \left(\prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}}\right)}{2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2}) \left(\prod_{i=1}^{p} \sigma_{ii}^{\frac{n}{2}}\right)}.
\end{aligned}$$

As $\prod_{i=1}^{p} \Gamma(\frac{n}{2})$ in numerator is independent of i, so $\prod_{i=1}^{p} \Gamma(\frac{n}{2}) = \prod_{i=0}^{p-1} \Gamma(\frac{n}{2})$. Then

$$\begin{aligned}
f(r_{ij}) &= \frac{|R|^{\frac{n-p-1}{2}} \prod_{i=0}^{p-1} \Gamma(\frac{n}{2})}{\pi^{\frac{p(p-1)}{4}} \prod_{i=0}^{p-1} \Gamma(\frac{n-i}{2})} \\
&= \frac{|R|^{\frac{n-p-1}{2}}}{\pi^{\frac{p(p-1)}{4}}} \prod_{i=0}^{p-1} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-i}{2})} \right); \quad -1 \le r_{ij} \le 1
\end{aligned}$$

is the PDF of $f(r_{ij})$ when $\rho_{ij} = 0$ for $i \neq j = 1, 2, \ldots, p$.

Some particular cases:

1. For $p = 2$

For $p = 2$,

$$|R| = \begin{vmatrix} 1 & r_{12} \\ r_{21} & 1 \end{vmatrix} = (1 - r_{12}^2) = (1 - r^2)$$

and thus

$$\begin{aligned} f(r_{ij}) \equiv f(r) &= \frac{|R|^{\frac{n-3}{2}}}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} \\ &= \frac{(1-r^2)^{\frac{n-3}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})} \\ &= \frac{(1-r^2)^{\frac{N-4}{2}}\Gamma(\frac{N-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{N-2}{2})} \quad (n = N-1) \\ &= \frac{(1-r^2)^{\frac{N-4}{2}}}{B(\frac{1}{2}, \frac{N-2}{2})}. \end{aligned}$$

2. For $N = 2$

For $N = 2$, the distribution $f(r_{ij})$ becomes useless because of the factor $\Gamma(\frac{N-2}{2})$ and since $n = 1$.

3. For $N = 4$ For $N = 4$, we have

$$\begin{aligned} f(r) &= \frac{1}{B(\frac{1}{2}, 1)}(1-r^2)^0 \\ &= \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(1)} = \frac{\frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}\cdot 1} = \frac{1}{2} \end{aligned}$$

which is a rectangular distribution.

Estimation of Partial Correlation Coefficient

The partial correlation coefficient measures the correlation between two variables after removing the linear effects of other variables.

Suppose among three variables X_1, X_2, and X_3, the correlation between X_1 and X_2 is partly due to X_3. So if we want to see the effect of X_1 and X_2 on each other after eliminating the linear effect of X_3, we use the concept of the partial correlation coefficient. The correlation coefficient between X_1 and X_2 after eliminating the linear effect of X_3 is called the partial correlation coefficient and is usually denoted as $r_{12.3}$ and defined as

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)}\sqrt{(1-r_{23}^2)}}$$

where r_{ij} is the simple correlation coefficient between X_i and X_j, $i, j = 1, 2, 3$. The range of $r_{12.3}$ is $-1 \leq r_{12.3} \leq 1$.

Estimation of Partial Correlation Coefficient in Multivariate Set-up

Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ and $\underset{\sim}{X}$ is partitioned as $\underset{\sim}{X} = (\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})'$ where $\underset{\sim}{X}^{(1)}$ and $\underset{\sim}{X}^{(2)}$ are $(q \times 1)$ and $(p-q) \times 1$ vectors as $\underset{\sim}{X}^{(1)} = (X_1, X_2, \ldots, X_q)'$ and $\underset{\sim}{X}^{(2)} = (X_{q+1}, X_{q+2}, \ldots, X_p)'$. The parameters $\underset{\sim}{\mu}$ and Σ are suitably partitioned as $\underset{\sim}{\mu} = (\underset{\sim}{\mu}^{(1)}, \underset{\sim}{\mu}^{(2)})'$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$. The conditional distribution of $\underset{\sim}{X}^{(1)}$ given $\underset{\sim}{x}^{(2)}$ is

$$N_q[\underset{\sim}{\mu}^{(1)} + B(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)}), \Sigma_{11.2}]$$

where $B = \Sigma_{12}\Sigma_{22}^{-1}$ and $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.

The partial correlation coefficient between X_i and X_j holding $X_{q+1}, X_{q+2}, \ldots, X_p$ fixed is

$$\rho_{ij.q+1\ldots p} = \frac{\sigma_{ij.q+1\ldots p}}{\sqrt{\sigma_{ii.q+1\ldots p}}\sqrt{\sigma_{jj.q+1\ldots p}}}$$

where $\sigma_{ij.q+1\ldots p}$ is the $(i,j)^{th}$ element of $\Sigma_{11.2}, i \neq j = 1, 2, \ldots, q$.

The next question is how to estimate $\rho_{ij.q+1\ldots p}$? We use the following result to find the estimate.

Result: If $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ are the maximum likelihood estimators of $\theta_1, \theta_2, \ldots, \theta_m$, respectively, then $\phi_1(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m)$, $\phi_2(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m)$, $\ldots$, $\phi_m(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m)$ are the maximum likelihood estimators of $\phi_1(\theta_1,\ldots,\theta_m),\phi_2(\theta_1,\ldots,\theta_m),\ldots,\phi_m(\theta_1,\ldots,\theta_m)$, respectively, if the transformation from $\theta_1,\theta_2,\ldots,\theta_m$ to $\phi_1,\phi_2,\ldots,\phi_m$ are unique.

First we note the one-to-one correspondence between Σ and $(\Sigma_{11.2},B,\Sigma_{22})$ as follows: If $\Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$ is known, then $\Sigma_{11.2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ is known and $B=\Sigma_{12}\Sigma_{22}^{-1}$ is also known. On the other hand, if Σ_{22},B and $\Sigma_{11.2}$ are known, then Σ is known as we can find $\Sigma_{12}=B\Sigma_{22}$ and find $\Sigma_{11}=\Sigma_{11.2}+B\Sigma_{22}B'$.

The maximum likelihood estimate of $\hat{\Sigma}=\frac{A}{N}$ where $A=\sum_{\alpha=1}^{N}(\underset{\sim}{x}_\alpha-\bar{\underset{\sim}{x}})(\underset{\sim}{x}_\alpha-\bar{\underset{\sim}{x}})'$ which can be partitioned using $\underset{\sim}{x}_\alpha=(\underset{\sim}{x}_\alpha^{(1)},\underset{\sim}{x}_\alpha^{(2)})',\alpha=1,2,\ldots,N$ as

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$$

where

$$\begin{aligned}
A_{11} &= \sum_{\alpha=1}^{N}(\underset{\sim}{x}_\alpha^{(1)}-\bar{\underset{\sim}{x}}^{(1)})(\underset{\sim}{x}_\alpha^{(1)}-\bar{\underset{\sim}{x}}^{(1)})'\\
A_{22} &= \sum_{\alpha=1}^{N}(\underset{\sim}{x}_\alpha^{(2)}-\bar{\underset{\sim}{x}}^{(2)})(\underset{\sim}{x}_\alpha^{(2)}-\bar{\underset{\sim}{x}}^{(2)})'\\
A_{12} &= \sum_{\alpha=1}^{N}(\underset{\sim}{x}_\alpha^{(1)}-\bar{\underset{\sim}{x}}^{(1)})(\underset{\sim}{x}_\alpha^{(2)}-\bar{\underset{\sim}{x}}^{(2)})'\\
A_{21} &= \sum_{\alpha=1}^{N}(\underset{\sim}{x}_\alpha^{(2)}-\bar{\underset{\sim}{x}}^{(2)})(\underset{\sim}{x}_\alpha^{(1)}-\bar{\underset{\sim}{x}}^{(1)})'\\
\bar{\underset{\sim}{x}} &= \frac{1}{N}\sum_{\alpha=1}^{N}\underset{\sim}{x}_\alpha=(\bar{\underset{\sim}{x}}^{(1)},\ \bar{\underset{\sim}{x}}^{(2)})'.
\end{aligned}$$

Using the one-to-one correspondence between Σ and $(\Sigma_{11.2},B,\Sigma_{22})$ along with the result, we obtain the maximum likelihood estimates as follows:

$$\hat{\underset{\sim}{\mu}}^{(1)}=\bar{\underset{\sim}{x}}^{(1)},\quad \hat{\underset{\sim}{\mu}}^{(2)}=\bar{\underset{\sim}{x}}^{(2)},$$

$$\hat{\Sigma}_{11}=\frac{A_{11}}{N},\quad \hat{\Sigma}_{12}=\frac{A_{12}}{N},\quad \hat{\Sigma}_{22}=\frac{A_{22}}{N},\quad \hat{\Sigma}_{21}=\frac{A_{21}}{N}$$

and then obtain

$$\hat{B}=A_{12}A_{22}^{-1}$$

$$\hat{\Sigma}_{11.2} = \frac{1}{N}(A_{11} - A_{12}A_{22}^{-1}A_{21}).$$

The maximum likelihood estimator of the partial correlation coefficient

$$\rho_{ij.q+1\ldots p} = \frac{\sigma_{ij.q+1\ldots p}}{\sqrt{\sigma_{ii.q+1\ldots p}}\sqrt{\sigma_{jj.q+1\ldots p}}} \quad i \neq j = 1, \ldots, q$$

is obtained as

$$\hat{\rho}_{ij.q+1\ldots p} = \frac{\hat{\sigma}_{ij.q+1\ldots p}}{\sqrt{\hat{\sigma}_{ii.q+1\ldots p}}\sqrt{\hat{\sigma}_{jj.q+1\ldots p}}} = \frac{a_{ij.q+1\ldots p}}{\sqrt{a_{ii.q+1\ldots p}}\sqrt{a_{jj.q+1\ldots p}}}$$

where $\hat{\sigma}_{ij.q+1\ldots p}$ and $a_{ij.q+1\ldots p}$ are the $(i,j)^{th}$ element of $\hat{\Sigma}_{11.2}$ and $A_{11.2}$, respectively. This result can be formulated in the following theorem:

Theorem 1: Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N$ be a sample of size N from $N_p(\underset{\sim}{\mu}, \Sigma)$. The maximum likelihood estimator of $\rho_{ij.q+1\ldots p}$, which is the partial correlation between the i^{th} and j^{th} variables of the first q components conditional on the last $(p-q)$ components of $\underset{\sim}{x}_\alpha$ $(\alpha = 1, 2, \ldots, n)$ is given by

$$\hat{\rho}_{ij.q+1\ldots p} = \frac{a_{ij.q+1\ldots p}}{\sqrt{a_{ii.q+1\ldots p}}\sqrt{a_{jj.q+1\ldots p}}}$$

where $a_{ij.q+1\ldots p}$ is the $(i,j)^{th}$ element of $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Note that $\hat{\rho}_{ij.q+1\ldots p}$ is the sample partial correlation coefficient between X_i and X_j having taken account of $X_{q+1}, X_{q+2}, \ldots, X_p$ as fixed.

Note that

$$\begin{aligned} A_{11.2} &= \sum_{\alpha=1}^{N} [(\underset{\sim}{x}_\alpha^{(1)} - \bar{\underset{\sim}{x}}^{(1)}) - \hat{B}(\underset{\sim}{x}_\alpha^{(2)} - \bar{\underset{\sim}{x}}^{(2)})][(\underset{\sim}{x}_\alpha^{(1)} - \bar{\underset{\sim}{x}}^{(1)}) - \hat{B}(\underset{\sim}{x}_\alpha^{(2)} - \bar{\underset{\sim}{x}}^{(2)})]' \\ &= A_{11} - \hat{B}A_{22}\hat{B}' \end{aligned}$$

and $[\underset{\sim}{x}_\alpha^{(1)} - \bar{\underset{\sim}{x}}^{(1)} - \hat{B}(\underset{\sim}{x}_\alpha^{(2)} - \bar{\underset{\sim}{x}}^{(2)})]$ is the residual of $\underset{\sim}{x}_\alpha^{(1)}$ from its regression on $\underset{\sim}{x}^{(2)}$ and 1. The partial correlations are the simple correlations between these residuals.

In order to find the distribution of the sample correlation coefficient, we use the Wishart distribution and note the following:

(i) Y_α's, $\alpha = 1, 2, \ldots, N$ are independent,

(ii) $E(\underset{\sim}{Y}_\alpha) = 0$,

(iii) $Var(\underset{\sim}{Y}_\alpha) = \Sigma$,

(iv) $\sum_{\alpha=1}^{N} \underset{\sim}{Y}_\alpha \underset{\sim}{Y}_\alpha' \sim W_p(\Sigma, n)$.

Hence, if we can construct a variable whose covariance matrix is $\Sigma_{11.2}$ and if they are independently distributed, then we can derive a suitable Wishart distribution, and from there we can derive the distribution of the sample correlation coefficient, which will equivalently be the distribution of the sample partial correlation coefficient. Hence, we first consider the following theorem:

Theorem 2: If $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_n$ are independently distributed, if $\underset{\sim}{Y}_\alpha = (\underset{\sim}{Y}_\alpha^{(1)}, \underset{\sim}{Y}_\alpha^{(2)})'$ has the density $f(\underset{\sim}{y}_\alpha)$ and if $f(\underset{\sim}{Y}_\alpha^{(2)} | \underset{\sim}{Y}_\alpha^{(1)} = \underset{\sim}{y}_\alpha^{(1)})$, for $\alpha = 1, 2, \ldots, n$ is the conditional distribution of $\underset{\sim}{Y}_\alpha^{(2)}$ given $\underset{\sim}{Y}_\alpha^{(1)} = \underset{\sim}{y}_\alpha^{(1)}$, then in the conditional distribution of $\underset{\sim}{Y}_1^{(2)}, \underset{\sim}{Y}_2^{(2)}, \ldots, \underset{\sim}{Y}_n^{(2)}$ given $\underset{\sim}{Y}_1^{(1)} = \underset{\sim}{y}_1^{(1)}, \underset{\sim}{Y}_2^{(1)} = \underset{\sim}{y}_2^{(1)}, \ldots, \underset{\sim}{Y}_n^{(1)} = \underset{\sim}{y}_n^{(1)}$, the random vectors $\underset{\sim}{Y}_1^{(2)}, \underset{\sim}{Y}_2^{(2)}, \ldots, \underset{\sim}{Y}_n^{(2)}$ are independent and the density of $\underset{\sim}{Y}_\alpha^{(2)}$ is $f(\underset{\sim}{Y}_\alpha^{(2)} | \underset{\sim}{Y}_\alpha^{(1)} = \underset{\sim}{y}_\alpha^{(1)})$.

Proof: Let $f_1(\underset{\sim}{y}_\alpha^{(1)})$ be the marginal density of $\underset{\sim}{Y}_\alpha^{(1)}$. As $\underset{\sim}{Y}_\alpha$'s $(\alpha = 1, \ldots, n)$ are independently distributed, then the joint marginal density of $\underset{\sim}{Y}_1^{(1)}, \underset{\sim}{Y}_2^{(1)}, \ldots, \underset{\sim}{Y}_n^{(1)}$ is $\prod_{\alpha=1}^{n} f_1(\underset{\sim}{y}_\alpha^{(1)})$. The conditional density of $\underset{\sim}{Y}_1^{(2)}, \underset{\sim}{Y}_2^{(2)}, \ldots, \underset{\sim}{Y}_n^{(2)}$ given $\underset{\sim}{Y}_1^{(1)} = \underset{\sim}{y}_1^{(1)}, \underset{\sim}{Y}_2^{(1)} = \underset{\sim}{y}_2^{(1)}, \ldots, \underset{\sim}{Y}_n^{(1)} = \underset{\sim}{y}_n^{(1)}$ is

$$\frac{\prod_{\alpha=1}^{n} f(\underset{\sim}{y}_\alpha)}{\prod_{\alpha=1}^{n} f_1(\underset{\sim}{y}_\alpha^{(1)})} = \prod_{\alpha=1}^{n} \frac{f(\underset{\sim}{y}_\alpha)}{f_1(\underset{\sim}{y}_\alpha^{(1)})} = \prod_{\alpha=1}^{n} f(\underset{\sim}{y}_\alpha^{(2)} | \underset{\sim}{y}_\alpha^{(1)}).$$

Finding the Distribution of Partial Correlation Coefficient

So now we have found the random vectors $\underset{\sim}{Y}_{1.2_\alpha} \quad \left[\equiv \underset{\sim}{X}_\alpha^{(1)} | \underset{\sim}{x}_\alpha^{(2)} - E(\underset{\sim}{X}_\alpha^{(1)} | \underset{\sim}{x}_\alpha^{(2)})\right]$, $(\alpha = 1, 2, \ldots, N)$ say whose covariance matrix is $\Sigma_{11.2}$ and sample covariance matrix is $A_{11.2} = A_{11} - A_{12} A_{11}^{-1} A_{21}$. We also have established that $A_{11.2}$ is distributed as $\sum_{\alpha=1}^{N-1-(p-q)} \underset{\sim}{Y}_{1.2_\alpha} \underset{\sim}{Y}_{1.2_\alpha}'$, where $\underset{\sim}{Y}_{1.2_\alpha}$ are independent, each according to $N_q(0, \Sigma_{11.2})$.

Also, it follows that if $\Sigma_{12} = 0$, the $A_{11.2}$ is distributed as $\sum_{\alpha=1}^{N-1-(p-q)} \underset{\sim}{Y}_{1.2_\alpha} \underset{\sim}{Y}_{1.2_\alpha}'$ and $A_{12} A_{11}^{-1} A_{21}$ is distributed as $\sum_{\alpha=N-(p-q)}^{N-1} \underset{\sim}{Y}_{1.2_\alpha} \underset{\sim}{Y}_{1.2_\alpha}'$ where $\underset{\sim}{Y}_{1.2_\alpha}$ are independent, each according to $N_q(0, \Sigma_{11.2})$.

Now it follows that the distribution of partial correlation coefficient $r_{ij.q+1,\ldots,p}$ based on N observations is the same as an ordinary correlation coefficient based on $N - (p - q)$

observations with a population correlation coefficient $\rho_{ij.q+1,\ldots,p}$.

So the same steps that were followed in deriving the distribution of the ordinary correlation coefficient can be followed to find the distribution of the sample partial correlation coefficient, by replacing N by $N-(p-q)$ and Σ by $\Sigma_{11.2}$.

Multiple Correlation Coefficient

Consider a $(p\times 1)$ random vector $\underset{\sim}{X}$ with variables $X_1, X_2, \ldots, X_p$ as $\underset{\sim}{X} = (X_1, X_2, \ldots, X_p)'$. Partition $\underset{\sim}{X}$ such that one partition has only one variable X_1 and all other variables $X_2, X_3, \ldots, X_p$ in another subvector $\underset{\sim}{X}_2 = (X_2, X_3, \ldots, X_p)'$. Then $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \underset{\sim}{X}_2 \end{pmatrix}$ and partition Σ accordingly as

$$\Sigma = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\sigma}'_{(1)} \\ \underset{\sim}{\sigma}_{(1)} & \Sigma_{22} \end{pmatrix}$$

where $Var(X_1) = \sigma_{11}$, $Var(\underset{\sim}{X}_2) = \Sigma_{22}$ is of order $(p-1)\times(p-1)$ and $Cov(X_1, \underset{\sim}{X}_2) = \underset{\sim}{\sigma}_{(1)}$ is $(p-1)\times 1$ vector. Based on this setup, we define the multiple correlation coefficient as follows:

Definition: The multiple correlation coefficient between X_1 and $X_2, X_3, \ldots, X_p$ denoted by $R_{1.23\ldots p}$, is the maximum correlation between X_1 and any linear function $\underset{\sim}{\alpha}'\underset{\sim}{X}_2 = \alpha_2 X_2 + \alpha_3 X_3 + \ldots + \alpha_p X_p$ of $X_2, X_3, \ldots, X_p$.

Using this definition, we write

$$\begin{aligned} R_{1.23\ldots p} \; & \max_{\underset{\sim}{\alpha}} \frac{Cov(X_1, \underset{\sim}{\alpha}'\underset{\sim}{X}_2)}{\sqrt{Var(X_1)}\sqrt{Var(\underset{\sim}{\alpha}'\underset{\sim}{X}_2)}} \\ &= \max_{\underset{\sim}{\alpha}} \frac{\underset{\sim}{\alpha}'\underset{\sim}{\sigma}_{(1)}}{\sqrt{\sigma_{11}}\sqrt{\underset{\sim}{\alpha}'\Sigma_{22}\underset{\sim}{\alpha}}}. \end{aligned}$$

We state the Cauchy-Schwarz inequality, which is used in finding this expression.

Cauchy Schwarz inequality: For any two column vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ of real elements,

$$(\underset{\sim}{x}'\underset{\sim}{y})^2 \leq (\underset{\sim}{x}'\underset{\sim}{x})(\underset{\sim}{y}'\underset{\sim}{y})$$

$$\text{or} \qquad \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

The equality being attained when and only when $\lambda \underset{\sim}{x} + \mu \underset{\sim}{y} = 0$ for real scalars λ and μ.

Note that

$$\begin{aligned}
\frac{\underset{\sim}{\alpha}'\underset{\sim}{\sigma}_{(1)}}{\sqrt{\sigma_{11}}\sqrt{\underset{\sim}{\alpha}'\Sigma_{22}\underset{\sim}{\alpha}}} &= \frac{\underset{\sim}{\alpha}'\Sigma_{22}^{1/2}\Sigma_{22}^{-1/2}\underset{\sim}{\sigma}_{(1)}}{\sqrt{\sigma_{11}}\sqrt{\underset{\sim}{\alpha}'\Sigma_{22}\underset{\sim}{\alpha}}} \\
&= \frac{\underset{\sim}{u}'\underset{\sim}{v}}{(\sigma_{11}\underset{\sim}{u}'\underset{\sim}{u})^{1/2}} \quad \left(\text{where } \underset{\sim}{u} = \Sigma_{22}^{1/2}\underset{\sim}{\alpha} \text{ and } \underset{\sim}{v} = \Sigma_{22}^{-1/2}\underset{\sim}{\sigma}_{(1)}\right) \\
&\leq \frac{(\underset{\sim}{u}'\underset{\sim}{u})^{1/2}(\underset{\sim}{v}'\underset{\sim}{v})^{1/2}}{(\sigma_{11}\cdot\underset{\sim}{u}'\underset{\sim}{u})^{1/2}} \quad (\text{using Cauchy Schwarz inequality}) \\
&= \frac{(\underset{\sim}{\alpha}'\Sigma_{22}\underset{\sim}{\alpha})^{1/2}(\underset{\sim}{\sigma}_{(1)}'\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)})^{1/2}}{(\sigma_{11}\cdot\underset{\sim}{\alpha}'\Sigma_{22}\underset{\sim}{\alpha})^{1/2}} \\
&= \left(\frac{\underset{\sim}{\sigma}_{(1)}'\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11}}\right)^{1/2}
\end{aligned}$$

with equality holds when

$$\lambda\Sigma_{22}^{1/2}\underset{\sim}{\alpha} + \mu\Sigma_{22}^{-1/2}\underset{\sim}{\sigma}_{(1)} = 0$$

or

$$\underset{\sim}{\alpha} = -\frac{\mu}{\lambda}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}$$

or

$$\underset{\sim}{\alpha} \propto \Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}.$$

Thus

$$R_{1.23\ldots p} = \frac{\underset{\sim}{\alpha}'\underset{\sim}{\sigma}_{(1)}}{(\sigma_{11}\cdot\underset{\sim}{\alpha}'\Sigma_{22}\underset{\sim}{\alpha})^{1/2}} = \left(\frac{\underset{\sim}{\sigma}_{(1)}'\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11}}\right)^{1/2}.$$

Now we define the multiple correlation in a general setup. Partition

$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}^{(1)} \\ \underset{\sim}{X}^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where $\underset{\sim}{X}^{(1)} = (X_1, X_2, \ldots, X_q)'$ and $\underset{\sim}{X}^{(2)} = (X_{q+1}, X_{q+2}, \ldots, X_p)'$.

Let X_i be a variable in $\underset{\sim}{X}^{(1)}$ with $i = 1, 2, \ldots, q$. The multiple correlation coefficient between X_i and the variables in $\underset{\sim}{X}^{(2)}$ (i.e., $X_{q+1}, X_{q+2}, \ldots, X_p$) is the maximum correlation between X_i and any linear function $\underset{\sim}{\alpha}'\underset{\sim}{X}^{(2)}$, denoted as $R_{i.q+1,\ldots,p}$.

Based on the earlier arguments, it follows that the maximizing value of $\underset{\sim}{\alpha}$ is

$$\underset{\sim}{\alpha} = \Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(i)}$$

where $\underset{\sim}{\sigma}_{(i)}$ is the i^{th} row of Σ_{12}, and hence

$$R_{i.q+1,\ldots,p} = \sqrt{\frac{\underset{\sim}{\sigma}'_{(i)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(i)}}{\sigma_{ii}}}$$

or equivalently

$$R^2_{i.q+1,\ldots,p} = \frac{\sigma_{ii} - \sigma_{ii.q+1\ldots p}}{\sigma_{ii}}$$

where $\sigma_{ii.q+1,\ldots,p}$ is the $(i,i)^{th}$ element, or the i^{th} diagonal element of $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Then using $\sigma_{11.2} = \sigma_{11} - \underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}$, we get

$$\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)} = \sigma_{11} - \sigma_{11.2}.$$

For the sake of convenience, we omit the subscripts and consider the case of multiple correlation coefficient between X_1 and $\underset{\sim}{X}_2 = (X_2, X_3, \ldots, X_p)$. The population multiple correlation coefficient is denoted by $\bar{R}$.

$$\bar{R} = \left(\frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11}}\right)^{1/2} = \left(\frac{B'\Sigma_{22}B}{\sigma_{11}}\right)^{1/2}$$

where $\Sigma = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\sigma}'_{(1)} \\ \underset{\sim}{\sigma}_{(1)} & \Sigma_{22} \end{pmatrix}$ and $B = \Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}$.

Next, we define the multiple correlation coefficient based on a sample $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N$, $(N > p)$. Then

$$\begin{aligned}
\hat{\Sigma} &= \frac{A}{N} = \frac{1}{N}\sum_{\alpha=1}^{N}(\underset{\sim}{x}_\alpha - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_\alpha - \bar{\underset{\sim}{x}})' \\
&= \begin{pmatrix} \hat{\sigma}_{11} & \hat{\underset{\sim}{\sigma}}'_{(1)} \\ \hat{\underset{\sim}{\sigma}}_{(1)} & \hat{\Sigma}_{22} \end{pmatrix} \\
&= \frac{1}{N}\begin{pmatrix} a_{11} & \underset{\sim}{a}'_{(1)} \\ \underset{\sim}{a}_{(1)} & A_{22} \end{pmatrix} \\
\hat{B} &= \hat{\Sigma}_{22}^{-1}\hat{\sigma}_{(1)}.
\end{aligned}$$

The sample multiple correlation coefficient is given as

$$
\begin{aligned}
R &= \left(\frac{\hat{B}'\hat{\Sigma}_{22}\hat{B}}{\hat{\sigma}`}\right)^{1/2} \\
&= \left(\frac{\hat{\underset{\sim}{\sigma}}'\hat{\Sigma}_{22}^{-1}\hat{\underset{\sim}{\sigma}}}{\hat{\sigma}_{11}}\right)^{1/2} \\
&= \left(\frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11}}\right)^{1/2}.
\end{aligned}
$$

To find the maximum likelihood estimator of the population multiple correlation coefficient, we note that there is a one-to-one transformation between $(\bar{R}, \underset{\sim}{\sigma}_{(1)}, \Sigma_{22})$ and Σ. This is seen as follows:

Given $\Sigma = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\sigma}'_{(1)} \\ \underset{\sim}{\sigma}_{(1)} & \Sigma_{22} \end{pmatrix}$, we can find $\bar{R} = \left(\frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11}}\right)^{1/2}$.

Given $\bar{R}, \underset{\sim}{\sigma}_{(1)}$ and Σ_{22}, we can find $\Sigma = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\sigma}'_{(1)} \\ \underset{\sim}{\sigma}_{(1)} & \Sigma_{22} \end{pmatrix}$. Note that $\underset{\sim}{\sigma}_{(1)}$ and Σ_{22} are known, so σ_{11} is found as $\sigma_{11} = \frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\bar{R}^2}$.

Thus R is the maximum likelihood estimator of $\bar{R}$.

Distribution of the Sample Multiple Correlation Coefficient when the Population Multiple Correlation Coefficient is Zero

The sample multiple correlation coefficient is

$$
\begin{aligned}
R^2 &= \frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11}}, \\
\text{so} \qquad 1 - R^2 &= 1 - \frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11}} = \frac{a_{11} - \underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11}} = \frac{a_{11.2}}{a_{11}}, \\
\text{and then} \qquad \frac{R^2}{1-R^2} &= \frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11.2}}.
\end{aligned}
$$

We know from earlier derived results from Wishart distribution that $A_{11.2} \sim W_q(\Sigma_{11.2}, n-p+q)$. Substituting $p = 1$ in Wishart distribution, we get the χ^2-distribution, so $a_{11.2} \sim \chi^2_{(n-p+q)}$. Then, with $q = 1$, we have $\frac{a_{11.2}}{\sigma_{11.2}} \sim \chi^2(n-p+1)$ where $\sigma_{11.2} = \sigma_{11} - \underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}$.

Next, we use the following theorem:

Theorem 3: If $A \sim W_p(\Sigma, n)$. Further, A and Σ are partitioned as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, then the conditional distribution of A_{12} given A_{22} is $N(\Sigma_{12}\Sigma_{22}^{-1}A_{22}, \Sigma_{11.2} \otimes A_{22})$ where $\otimes$ is the Kronecker product.

Using this result in our case, the conditional distribution of $\underset{\sim}{a}_{(1)}$ given A_{22} is

$$f(\underset{\sim}{a}_{(1)}|A_{22}) \sim N(\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}A_{22}, \sigma_{11.2}A_{22})$$

and thus,

$$\frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{\sigma_{11.2}} \sim \chi^2_{p-1}.$$

Now we use the following theorem from Wishart distribution that $A_{11.2} \sim W_1(\sigma_{11.2}, n - p + q)$ and is independent of A_{12} and A_{22}, we conclude that $a_{11.2}$ is independent of $\underset{\sim}{a}_{(1)}$ and A_{22}.

Thus we find that

$$\begin{aligned}
\frac{R^2}{1-R^2} \cdot \frac{n-p+1}{p-1} &= \frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}/\sigma_{11.2}}{a_{11.2}/\sigma_{11.2}} \cdot \frac{n-p+1}{p-1} \\
&\sim \frac{\chi^2_{p-1}/(p-1)}{\chi^2_{n-p+1}/(n-p+1)} \\
&\sim F_{p-1,n-p+1} \;\text{ or }\; F_{p-1,N-p}.
\end{aligned}$$

Thus $\frac{R^2}{1-R^2} \cdot \frac{N-p}{p-1}$ follows an F-distribution with $(p-1)$ and $(N-p)$ degrees of freedom.

Next, we find the probability density function of R using the F-distribution.

The probability density function of $Y \sim F_{p-1,N-p}$ is

$$f_Y(y) = \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{N-p}{2})} \left(\frac{p-1}{N-p}\right)^{\frac{p-1}{2}} y^{\frac{p-1}{2}-1} \left(1 + \frac{p-1}{N-p}\right)^{-\frac{N+1}{2}}$$

where $y \equiv \frac{n-p+1}{p-1}\frac{R^2}{1-R^2}$ in our case. So

$$\frac{N-p}{p-1} \cdot \frac{R^2}{1-R^2} \sim F_{p-1,N-p}.$$

Let

$$y = \frac{N-p}{p-1} \cdot \frac{R^2}{1-R^2}.$$

Then

$$\begin{aligned}
\left(\frac{p-1}{N-p}\right)\left|\frac{dy}{dR}\right| &= \left[\frac{2R}{1-R^2} + \frac{2R^3}{(1-R^2)^2}\right] \\
&= \frac{2R}{(1-R^2)}\left(1 + \frac{R^2}{1-R^2}\right) \\
&= \frac{2R}{(1-R^2)^2}\ .
\end{aligned}$$

The Jacobian of transformation is

$$|J| = \left(\frac{N-p}{p-1}\right)\frac{2R}{(1-R^2)^2}$$

and the density function of R is

$$\begin{aligned}
g(R) &= f_Y(y)|J| \\
&= \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(\frac{p-1}{2})\Gamma(\frac{N-p}{2})}\left(\frac{p-1}{N-p}\right)^{\frac{p-1}{2}}\left(\frac{N-p}{p-1}\cdot\frac{R^2}{1-R^2}\right)^{\frac{p-1}{2}-1} \\
&\quad \times \left[1 + \frac{p-1}{N-p}\cdot\frac{N-p}{p-1}\cdot\frac{R^2}{1-R^2}\right]^{-\frac{N-1}{2}} \cdot \frac{2R}{(1-R^2)^2} \\
&= \frac{2\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(\frac{p-1}{2})\Gamma(\frac{N-p}{2})}\left(\frac{R^2}{1-R^2}\right)^{\frac{p-1}{2}-1} \cdot \left(1 + \frac{R^2}{1-R^2}\right)^{-\frac{N+1}{2}} \cdot \frac{R}{(1-R^2)^2} \\
&= \frac{2\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(\frac{p-1}{2})\Gamma(\frac{N-p}{2})} R^{p-3+1}\left(\frac{1}{1-R^2}\right)^{\frac{p-1}{2}-1-\frac{N-1}{2}+1} \\
&= \frac{2\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(\frac{p-1}{2})\Gamma(\frac{N-p}{2})} R^{p-2}(1-R^2)^{\frac{N-p}{2}-1}, \qquad 0 \le R \le 1.
\end{aligned}$$

The moments of R^2 when $\bar{R}^2 = 0$ are

$$E(R^{2k}) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+k)} \cdot \frac{\Gamma(\frac{p-1}{2}+k)}{\Gamma(\frac{p-1}{2})} \quad \text{where } \ n = N-1.$$

In particular,

$$\begin{aligned}
E(R^2) &= \frac{p-1}{N-1} \\
Var(R^2) &= \frac{2(N-p)(p-1)}{(N^2-1)(N-1)}.
\end{aligned}$$

Alternative Proof of Distribution of Simple Correlation Coefficient when Population Correlation is Zero

Consider a bivariate normal population $f_{X_1,X_2}(x_1,x_2) \sim N_2(0,0,\sigma_1^2,\sigma_2^2,0)$ where $E(X_1) = 0$, $E(X_2)=0$, $Var(X_1)=\sigma_1^2$, $Var(X_2)=\sigma_2^2$ and population correlation coefficient $\rho = 0$.

A sample of size n on $\underset{\sim}{X} = (X_1, X_2)'$ is drawn and the observations are $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$. The sample correlation coefficient is

$$r = \frac{\sum_{i=1}^n (x_{1i}-\bar{x}_1)(x_{2i}-\bar{x}_2)}{\sqrt{\sum_{i=1}^n (x_{1i}-\bar{x}_1)^2}\sqrt{\sum_{i=1}^n (x_{2i}-\bar{x}_2)^2}}.$$

Consider an orthogonal linear transformation $\underset{\sim}{Y} = C\underset{\sim}{X}$ where C is an orthogonal matrix such that $CC' = I$.

Choose $C_{1\alpha} = \frac{1}{\sqrt{n}}$ and then

$$y_1 = \frac{1}{\sqrt{n}}(x_{11}+x_{12}+\cdots+x_{1n}) = \sqrt{n}\bar{x}_1.$$

Since $\underset{\sim}{Y}'\underset{\sim}{Y} = \underset{\sim}{X}'\underset{\sim}{X}$, so in this case,

$$\begin{aligned}
\underset{\sim}{X}'\underset{\sim}{X} &= \sum_{\alpha=1}^n x_{1\alpha}^2 = \sum_{\alpha=1}^n [(x_{1\alpha}-\bar{x}_1)+\bar{x}_1]^2 \\
&= \sum_{\alpha=1}^n (x_{1\alpha}-\bar{x}_1)^2 + n\bar{x}_1^2 \\
&= \sum_{\alpha=1}^n (x_{1\alpha}-\bar{x}_1)^2 + y_1^2
\end{aligned}$$

and

$$\underset{\sim}{Y}'\underset{\sim}{Y} = \sum_{i=1}^n y_i^2 = \sum_{i=2}^n y_i^2 + y_1^2$$

or,

$$\sum_{i=1}^n y_i^2 - y_1^2 = \sum_{i=2}^n y_i^2.$$

Since $\sum_{i=1}^n y_i^2 = \sum_{i=1}^n x_{1\alpha}^2$, so we get

$$\sum_{i=2}^n y_i^2 = \sum_{\alpha=1}^n (x_{1\alpha}-\bar{x}_1)^2.$$

Also,

$$\begin{aligned}
E(y_i) &= \sum_{\alpha=1}^{n} C_{i\alpha} E(x_{1\alpha}) = 0 \\
E(y_i^2) &= \sum_{\alpha=1}^{n} C_{i\alpha}^2 E(x_{1\alpha}^2) + \sum_{i \neq j}\sum C_{i\alpha} C_{j\alpha'} E(x_{1\alpha} x_{1\alpha'}) \\
&= \sigma_1^2
\end{aligned}$$

as $E(x_{1\alpha} x_{1\alpha'}) = 0$ and $\sum_{\alpha=1}^{n} C_{1\alpha}^2 = 1$ due to orthogonal matrix. Since y_i is a linear combination of normally distributed random variables, so y_i is also normally distributed. Hence $y_2, y_3, \ldots, y_n$ are independent and $y_i \sim N(0, \sigma_1^2)$, $i = 2, 3, \ldots, n$.

Consequently

$$\frac{\sum_{i=2}^{n} y_i^2}{\sigma_1^2} \sim \chi_{n-1}^2.$$

Note that $\sum_{\alpha=1}^{n} C_{2\alpha} = 0$, $\sum_{\alpha=1}^{n} C_{2\alpha}^2 = 1$ and $\sum_{\alpha=1}^{n} C_{2\alpha} C_{1\alpha} = 0$ due to orthogonal matrix C.

Now consider

$$\begin{aligned}
y_2 &= \sum_{\alpha=1}^{n} C_{2\alpha} x_{1\alpha} \\
&= \sum_{\alpha=1}^{n} \frac{(x_{2\alpha} - \bar{x}_2)}{\sqrt{\sum_{\alpha=1}^{n} (x_{2\alpha} - \bar{x}_2)^2}} \cdot x_{1\alpha} \\
&= \frac{\sum_{\alpha=1}^{n} (x_{2\alpha} - \bar{x}_2)(x_{1\alpha} - \bar{x}_1)}{\sqrt{\sum_{\alpha=1}^{n} (x_{2\alpha} - \bar{x}_2)^2}} \\
&= r \sqrt{\sum_{\alpha=1}^{n} (x_{1\alpha} - \bar{x}_1)^2} \\
&= r \sqrt{\sum_{i=2}^{n} y_i^2} \ .
\end{aligned}$$

Hence

$$\sum_{i=3}^{n} y_i^2 = \sum_{i=2}^{n} y_i^2 - y_2^2 = (1 - r^2) \sum_{\alpha=2}^{n} (x_{1\alpha} - \bar{x}_1)^2.$$

Further,

$$\frac{\sum_{i=3}^{n} y_i^2}{\sigma_1^2} = \frac{(1-r^2)\sum_{\alpha=1}^{n}(x_{1\alpha}-\bar{x}_1)^2}{\sigma_1^2} \sim \chi^2_{(n-2)},$$
$$\frac{y_2^2}{\sigma_1^2} = \frac{r^2\sum_{\alpha=1}^{n}(x_{1\alpha}-\bar{x}_1)^2}{\sigma_1^2} \sim \chi^2_1,$$

and $\frac{\sum_{i=3}^{n} y_i^2}{\sigma_1^2}$ and $\frac{y_2^2}{\sigma_1^2}$ are independent.

Thus r^2 is distributed as the ratio of χ_1^2 and $(\chi_1^2 + \chi_{n-2}^2)$ random variables,

$$r^2 \sim \frac{\chi_1^2}{\chi_1^2 + \chi_{n-2}^2}$$

which is distributed as Beta $(\frac{1}{2}, \frac{n-2}{2})$. So

$$f(r^2) = \frac{1}{B(\frac{1}{2}, \frac{n-2}{2})}(r^2)^{-\frac{1}{2}}(1-r^2)^{\frac{n-4}{2}}; \quad 0 \le r^2 \le 1.$$

Since $dr^2 = 2rdr$, hence $|J| = \left|\frac{dr^2}{dr}\right|$ is the Jacobian of transformation and so

$$\begin{aligned} f(r) &= f(r^2)|J| \\ f(r) &= \begin{cases} \frac{1}{B(\frac{1}{2}, \frac{n-2}{2})}(1-r^2)^{\frac{n-4}{2}}; & -1 \le r \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Distribution of Sample Multiple Correlation Coefficient when the Population Multiple Correlation Coefficient is Not Zero

Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_p)'$ be a $p \times 1$ vector from multivariate normal distribution $N_p(\underset{\sim}{\mu}, \Sigma)$. with $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$ and $Cov(\underset{\sim}{X}) = \Sigma$. Partition $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \underset{\sim}{X}_2 \end{pmatrix}$, $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}$, and $\Sigma = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\sigma}'_{(1)} \\ \underset{\sim}{\sigma}_{(1)} & \Sigma_{22} \end{pmatrix}$ where X_1 is univariate and $\underset{\sim}{X}_2$ is of order $(p-1)\times 1$. Also $E(X_1) = \mu_1$, $E(\underset{\sim}{X}_2) = \underset{\sim}{\mu}_2$, $Var(X_1) = \sigma_{11}$, $Cov(\underset{\sim}{X}_2) = \Sigma_{22}$ which is of order $(p-1)\times(p-1)$ and $\underset{\sim}{\sigma}_{(1)}$ is of order $(p-1)\times 1$. The population multiple correlation coefficient is

$$\bar{R} = \sqrt{\frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11}}}$$

is nonzero.

The distribution of sample multiple correlation coefficient between X_1 and $\underset{\sim}{X}_2$, denoted by R, is

$$f(R^2) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{n-p+1}{2})}(R^2)^{\frac{p-3}{2}}(1-R^2)^{\frac{n-p-1}{2}}(1-\bar{R}^2)^{\frac{n}{2}}\ {}_2F_1\left(\frac{n}{2},\frac{n}{2};\frac{p-1}{2};\bar{R}^2R^2\right), \quad 0 \le R^2 \le 1$$

where $n = N-1$, the generalized Hypergeometric function is given by

$$ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_q)_k} \cdot \frac{z^k}{k!}$$

and the and Pochhammer symbol is $(a)_k = a(a+1)\ldots(a+k-1)$.

Proof: Since $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$, the maximum likelihood estimators of $\underset{\sim}{\mu}$ and Σ are $\hat{\underset{\sim}{\mu}} = \bar{\underset{\sim}{x}}$ and $\hat{\Sigma} = \frac{A}{N}$, respectively. Also, $A \sim W_p(\Sigma, n)$. As $\underset{\sim}{X} = (X_1, \underset{\sim}{X}_2)'$, so partition A accordingly as

$$A = \begin{pmatrix} a_{11} & \underset{\sim}{a}'_{(1)} \\ \underset{\sim}{a}_{(1)} & A_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} R^2 &= \frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11}}, \\ 1-R^2 &= \frac{a_{11} - \underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11}} = \frac{a_{11.2}}{a_{11}}. \\ \frac{R^2}{1-R^2} &= \frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{a_{11.2}}, \end{aligned}$$

and $\frac{a_{11.2}}{\sigma_{11.2}} \sim \chi^2_{n-p+1}$ where $\sigma_{11.2} = \sigma_{11} - \underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}$.

Using the result

"If $A \sim W_p(\Sigma, n)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, then the conditional distribution of A_{12} given A_{22} is $N(\Sigma_{12}\Sigma_{22}^{-1}A_{22}, \Sigma_{11.2} \otimes A_{22})$" where $\otimes$ denotes the Kronecker product operator, we can derive that

$$f(\underset{\sim}{a}_{(1)}/A_{22}) \sim N_{p-1}(\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}A_{22}, \sigma_{11.2}A_{22})$$

and

$$\frac{\underset{\sim}{a}'_{(1)}A_{22}^{-1}\underset{\sim}{a}_{(1)}}{\sigma_{11.2}} \sim \chi^2_{p-1}(\delta),$$

i.e., χ^2 distribution with $(p-1)$ degrees of freedom and noncentrality parameter

$$\delta = \frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}A_{22}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11.2}}.$$

Hence the conditional density on A_{22} or equivalently δ is

$$Z \equiv \frac{n-p+1}{p-1} \cdot \frac{R^2}{1-R^2} \sim \frac{\chi^2_{p-1}(\delta)/(p-1)}{\chi^2_{n-p+1}/(n-p+1)} \sim F_{p-1,n-p+1}(\delta),$$

i.e., noncentral F-distribution with $p-1$ and $(n-p+1)$ degrees of freedom and noncentrality parameter δ, which is defined using the following result:

Result: Let $Z_1 \sim \chi^2_{n_1}(\delta)$, $Z_2 \sim \chi^2_{n_2}$, and Z_1 and Z_2 are independent. Then

$$F = \frac{Z_1/n_1}{Z_2/n_2}$$

follows a noncentral F distribution with noncentrality parameter δ, denoted as $F_{n_1,n_2}(\delta)$, and its density is given by

$$\exp\left(-\frac{\delta}{2}\right) \,_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{\frac{n_1}{2n_2}\cdot\delta\cdot f}{1+\frac{n_1}{n_2}f}\right) \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \frac{f^{\frac{n_1}{2}-1}\cdot(\frac{n_1}{n_2})^{\frac{n_1}{2}}}{(1+\frac{n_1}{n_2}f)^{\frac{n_1+n_2}{2}}}.$$

The conditional density function of Z is noncentral-F with the density given by

$$\exp\left(-\frac{\delta}{2}\right) \,_1F_1\left(\frac{n}{2}; \frac{p-1}{2}; \frac{\frac{p-1}{2(n-p+1)}\delta z}{1+\frac{p-1}{n-p+1}z}\right) \cdot \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{n-p+1}{2})} \frac{z^{\frac{p-3}{2}}(\frac{p-1}{n-p+1})^{\frac{p-1}{2}}}{\left[1+\frac{p-1}{n-p+1}z\right]^{\frac{n}{2}}} \quad \text{for } \; z>0.$$

Now convert Z to R^2 using

$$Z = \frac{n-p+1}{p-1} \cdot \frac{R^2}{1-R^2}$$

$$|J| = \left|\frac{dZ}{dR^2}\right| = \left(\frac{n-p+1}{p-1}\right) \cdot \frac{1}{(1-R^2)^3}$$

and the conditional density of R^2 given δ is

$$f(R^2|\delta) = \exp\left(-\frac{\delta}{2}\right) \,_1F_1\left(\frac{n}{2}; \frac{p-1}{2}; \frac{\delta}{2}R^2\right) \cdot \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{n-p+1}{2})}(R^2)^{\frac{p-3}{2}}(1-R^2)^{\frac{n-p-1}{2}}$$

for $0 < R^2 < 1$.

To obtain the unconditional density function of R^2, first obtain the joint density function of R^2 and δ by multiplying $f(R^2|\delta)$ by the PDF of δ. Then, in the next step, obtain the marginal density of R^2 by integrating out other variables. So, first we find the PDF of δ as follows:

PDF of δ:

Here $A_{22} \sim W(\Sigma_{22}, n)$. So using the result that "if $A \sim W_p(\Sigma, n)$ and M is an $m \times p$ nonsingular matrix then $M'AM \sim W(M\Sigma M', n)$", it implies that as $\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}$ is a vector, so

$$\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}A_{22}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)} \sim W(\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}, n).$$

Substituting $M = \underset{\sim}{Y}'(\equiv \underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1})$ as a $(1 \times m)$ vector, we have $\underset{\sim}{Y}'A\underset{\sim}{Y} \sim W(\underset{\sim}{Y}'\Sigma\underset{\sim}{Y}, n)$. This is further expressed as

$$\frac{\underset{\sim}{Y}'A\underset{\sim}{Y}}{\underset{\sim}{Y}'\Sigma\underset{\sim}{Y}} \sim \chi^2_n.$$

Thus

$$v \equiv \frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}A_{22}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}} \sim \chi^2_n.$$

If we define

$$\theta = \frac{\bar{R}^2}{1-\bar{R}^2} = \frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11.2}},$$

then

$$\delta = \frac{\underset{\sim}{\sigma}'_{(1)}\Sigma_{22}^{-1}A_{22}\Sigma_{22}^{-1}\underset{\sim}{\sigma}_{(1)}}{\sigma_{11.2}} = \theta v.$$

Now we consider the first step and obtain the joint density of R^2 and v by multiplying $f(R^2|v)$ by $\delta = \theta v$ where $v \sim \chi^2_n$, as follows:

$$\begin{aligned} f(R^2, v) = & \frac{\exp\left(-\frac{v}{2}\right) v^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \exp\left(\frac{-\theta v}{2}\right) \\ & \times\ {}_1F_1\left(\frac{n}{2}; \frac{p-1}{2}; \frac{\theta v R^2}{2}\right) \cdot \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{n-p+1}{2})}(R^2)^{\frac{p-3}{2}}(1-R^2)^{\frac{n-p-1}{2}}. \end{aligned}$$

Next the density $f(R^2)$ is obtained as marginal density from $f(R^2, v)$ as

$$f(R^2) = \int_0^\infty f(R^2, v)dv.$$

First, we solve the integral as follows:

$$\begin{aligned}
G =& \frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty \exp\left(-\frac{\theta v}{2}\right)\exp\left(-\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\frac{n}{2}-1} {}_1F_1\left(\frac{n}{2},\frac{p-1}{2},\frac{\theta v R^2}{2}\right) d\left(\frac{v}{2}\right)\\
=& \frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty \exp\left(-\frac{v}{2}(1+\theta)\right)\left(\frac{v}{2}\right)^{\frac{n}{2}-1} {}_1F_1\left(\frac{n}{2},\frac{p-1}{2},\frac{\theta v R^2}{2}\right) d\left(\frac{v}{2}\right)\\
=& \frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty \exp\left(-\frac{v}{2(1-\bar{R}^2)}\right)\left(\frac{v}{2}\right)^{\frac{n}{2}-1} {}_1F_1\left(\frac{n}{2},\frac{p-1}{2},\frac{\theta v R^2}{2}\right) d\left(\frac{v}{2}\right).
\end{aligned}$$

Substituting $u=\frac{v}{2(1-\bar{R}^2)}$ and using $\theta=\frac{\bar{R}^2}{1-\bar{R}^2}$ gives $d\left(\frac{v}{2}\right)=(1-\bar{R}^2)du$ and $\frac{v}{2}=\frac{u\bar{R}^2}{2}$. Also $\frac{\theta v R^2}{2}=\frac{u\bar{R}^2}{\theta}$. Continuing the integral, we have

$$\begin{aligned}
G =& \frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty \exp(-u)[u(1-\bar{R}^2)]^{\frac{n}{2}-1} {}_1F_1\left(\frac{n}{2},\frac{p-1}{2},uR^2\bar{R}^2\right)(1-\bar{R}^2)du\\
=& \frac{(1-\bar{R}^2)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^\infty \exp(-u)u^{\frac{n}{2}-1} {}_1F_1\left(\frac{n}{2},\frac{p-1}{2},uR^2\bar{R}^2\right)du.
\end{aligned}$$

Using the Hypergeometric function

$$\begin{aligned}
{}_1F_1\left(\frac{n}{2},\frac{p-1}{2},\bar{R}^2R^2u\right) &= \sum_{k=0}^\infty \frac{(\frac{n}{2})_k}{(\frac{p-1}{2})_k}\frac{(\bar{R}^2R^2u)^k}{k!}\\
&= \sum_{k=0}^\infty \frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)\cdots(\frac{p-1}{2}+k-1)}(\bar{R}^2R^2)^k\frac{u^k}{k!}\\
&\equiv \sum_{k=0}^\infty A_k\frac{u^k}{k!}
\end{aligned}$$

where

$$A_k=\frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k+1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)\cdots(\frac{p-1}{2}+k-1)}R^{2k}\bar{R}^{2k}.$$

Thus, the integral G is expressed as

$$G=\int_0^\infty e^{-u}u^{\frac{n}{2}-1}\left(\sum_{k=0}^\infty A_k\frac{u^k}{k!}\right)du.$$

The k^{th} term of this integral G is

$$\begin{aligned}
&\int_0^\infty \frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)\cdots(\frac{p-1}{2}+k-1)}(\bar{R}^2R^2)^k\frac{e^{-u}u^{\frac{n}{2}+k-1}}{k!}du\\
=& \frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)\cdots(\frac{p-1}{2}+k-1)}\frac{(\bar{R}^2R^2)^k}{k!}\Gamma\left(\frac{n}{2}+k\right)\\
\equiv& W_k, \text{ say.}
\end{aligned}$$

Observe that for

$$
\begin{aligned}
k=0, &\quad W_0=\Gamma\left(\frac{n}{2}\right),\\
k=1, &\quad W_1=\frac{\frac{n}{2}}{(\frac{p-1}{2})}\frac{(\bar{R}^2R^2)}{1!}\Gamma\left(\frac{n}{2}+1\right)\\
k=2, &\quad W_2=\frac{\frac{n}{2}(\frac{n}{2}+1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)}\frac{(\bar{R}^2R^2)^2}{2!}\Gamma\left(\frac{n}{2}+2\right)\\
k=3, &\quad W_3=\frac{\frac{n}{2}(\frac{n}{2}+1)(\frac{n}{2}+2)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)(\frac{p-1}{2}+2)}\frac{(\bar{R}^2R^2)^3}{3!}\Gamma\left(\frac{n}{2}+3\right)
\end{aligned}
$$

and so on, where

$$
\begin{aligned}
\Gamma\left(\frac{n}{2}+1\right) &= \frac{n}{2}\Gamma\left(\frac{n}{2}\right),\\
\Gamma\left(\frac{n}{2}+2\right) &= \left(\frac{n}{2}+1\right)\frac{n}{2}\Gamma\left(\frac{n}{2}\right),\\
\Gamma\left(\frac{n}{2}+3\right) &= \left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)\frac{n}{2}\Gamma\left(\frac{n}{2}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& W_0+W_1+W_2+W_3+\cdots\\
&=\Gamma\left(\frac{n}{2}\right)\Bigg[1+\frac{\frac{n}{2}}{(\frac{p-1}{2})}\frac{\bar{R}^2R^2}{1!}\cdot\frac{n}{2}+\frac{\frac{n}{2}(\frac{n}{2}+1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)}\frac{n}{2}\left(\frac{n}{2}+1\right)\frac{(\bar{R}^2R^2)^2}{2!}\\
&\quad+\frac{\frac{n}{2}(\frac{n}{2}+1)(\frac{n}{2}+2)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)(\frac{p-1}{2}+2)}\frac{n}{2}\cdot\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)\frac{(\bar{R}^2\bar{R}^2)^3}{3!}\\
&\quad+\cdots\\
&\quad+\frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)}{(\frac{p-1}{2})(\frac{p-1}{2}+1)\cdots(\frac{p-1}{2}+k-1)}\cdot\frac{n}{2}\left(\frac{n}{2}+1\right)\cdots\left(\frac{n}{2}+k-1\right)\frac{(\bar{R}^2R^2)^k}{k!}+\cdots\Bigg]\\
&=\Gamma\left(\frac{n}{2}\right)\ {}_2F_1\left(\frac{n}{2},\frac{n}{2},\frac{p-1}{2},R^2\bar{R}^2\right).
\end{aligned}
$$

So we have a general result

$$
\begin{aligned}
&\int_0^\infty \exp(-zt)t^{a-1}\ {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;kt)dt\\
&=\Gamma(a)z^{-a}\ {}_{p+1}F_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;kz^{-1})\ \text{ for }\ p\le q.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
G &= \frac{(1-\bar{R}^2)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} \exp(-u) u^{\frac{n}{2}-1} \, {}_1F_1\left(\frac{n}{2}, \frac{p-1}{2}, uR^2\bar{R}^2\right) du \\
&= (1-\bar{R}^2)^{\frac{n}{2}} \, {}_2F_1\left(\frac{n}{2}, \frac{n}{2}, \frac{p-1}{2}, \bar{R}^2 R^2\right).
\end{aligned}
$$

Substituting G in $f(R^2) = \int_0^{\infty} f(R^2, v) dv$, we get

$$
f(R^2) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{n-p+1}{2}\right)} (R^2)^{\frac{p-3}{2}} (1-R^2)^{\frac{n-p-1}{2}} \, {}_2F_1\left(\frac{n}{2}, \frac{n}{2}, \frac{p-1}{2}; R^2\bar{R}^2\right) \quad ; \quad 0 \leq R^2 \leq 1
$$

Moments of R^2 when $\bar{R}^2 \neq 0$

The g^{th} moment of R^2 when $\bar{R}^2 \neq 0$ is

$$
\begin{aligned}
E\left[(1-R^2)^g\right] &= \sum_{k=0}^{\infty} C_k \frac{\Gamma\left(\frac{n-p+1}{2}+g\right)}{\Gamma\left(\frac{n-p+1}{2}\right)} \cdot \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}+k+g\right)} \\
&= \frac{\left(\frac{n-p+1}{2}\right)_g}{\left(\frac{n}{2}\right)_g} (1-\bar{R}^2)^g \, {}_2F_1\left(g, g; \frac{n}{2}+g; \bar{R}^2\right)
\end{aligned}
$$

where

$$
C_k = (-1)^k \binom{-\frac{n}{2}}{k} (1-\bar{R}^2)^{\frac{n}{2}} (\bar{R}^2)^k.
$$

In particular,

$$
\begin{aligned}
E(R^2) &= 1 - \left(\frac{n-p+1}{n}\right)(1-\bar{R}^2) \, {}_2F_1\left(1,1; \frac{n}{2}+1; \bar{R}^2\right) \\
&= \bar{R}^2 + \left(\frac{p-1}{n}\right)(1-\bar{R}^2) + \left(\frac{2}{n-2}\right)\bar{R}^2(1-\bar{R}^2) + O\left(\frac{1}{n^2}\right) \\
Var(R^2) &= \frac{\left(\frac{n-p+1}{2}\right)_2}{\left(\frac{n}{2}\right)_2} (1-\bar{R}^2)^2 \, {}_2F_1\left(2,2; \frac{n}{2}+2; \bar{R}^2\right) \\
&\quad - \left[\left(\frac{n-p+1}{n}\right)(1-\bar{R}^2) \, {}_2F_1\left(1,1; \frac{n}{2}+1; \bar{R}^2\right)\right]^2 \\
&= \frac{n-p+1}{n^2(n+2)}(1-\bar{R}^2)^2 \left[2(p-1) + 4\bar{R}^2\left\{\frac{4(p-1)+n(n-p+1)}{n+4}\right\}\right] + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Thus R^2 is a biased estimator of $\bar{R}^2$. In fact, $E(R^2) > \bar{R}^2$, so R^2 overestimates $\bar{R}^2$. An unbiased estimator of R^2 is

$$\begin{aligned} T(R^2) &= 1 - \left(\frac{n-2}{n-p+1}\right)(1-R^2)\, {}_2F_1\left(1,1;\frac{n-p+3}{2};(1-R^2)\right) \\ &= R^2 - \left(\frac{n-2}{n-p+1}\right)(1-R^2) - \frac{2(n-2)(1-R^2)^2}{(n-p+1)(n-p+3)} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Note that $t(R^2) < 0$ for R^2 near zero.