# Combinatorics and counting

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## *Introduction*

Here is a collection of counting problems. Questions and suggestions are welcome at `per.w.alexandersson@gmail.com`.

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EXCLUSIVE VS. INDEPENDENT CHOICE. Recall that we *add* the counts for exclusive situations, and *multiply* the counts for independent situations. For example, the possible outcomes of a dice throw are exclusive:

$$\text{(Sides of a dice)} = \text{(Even sides)} + \text{(Odd sides)}$$

The different outcomes of selecting a playing card in a deck of cards can be seen as a combination of *independent* choices:

$$\text{(Different cards)} = \text{(Choice of color)} \cdot \text{(Choice of value)}.$$

LABELED VS. UNLABELED SETS is a common cause for confusion. Consider the following two problems:

- Count the number of ways to choose 2 people among 4 people.
- Count the number of ways to partition 4 people into sets of size 2.

In the first example, it is understood that the set of chosen people is a *special* set — it is the *chosen set*. We choose two people, and the other two are not chosen. In the second example, there is no difference between the two couples. The answer to the first question is therefore

$$\binom{4}{2}, \qquad \text{counting the chosen subsets: } \{12, 13, 14, 23, 24, 34\}.$$

The answer to the second question is

$$\frac{1}{2!}\binom{4}{2}, \qquad \text{counting the partitions: } \{12|34, 13|24, 14|23\}.$$

That is, the issue is that there is no way to distinguish the two sets in the partition. However, now consider the following two problems:

- Count the number of ways to choose 2 people among 5 people.
- Count the number of ways to partition 5 people into a set of size 2 and a set of size 3.

In this case, the answer to both questions is $\binom{5}{3}$. The reason for this is that we can distinguish between the two sets in the partition in the second question, for example, *the set of size* 2 is unique.

WE ALWAYS CONSIDER PEOPLE to be unique, and therefore labeled. This means that in a group of $n$ people, we can talk about the first person, the second person, and so on. In a group of $n$ *identical* objects, there is no á priori notion of the first object.

*Overview of formulas*

Every row in the table illustrates a type of counting problem, where the solution is given by the formula. Conversely, every problem is a *combinatorial interpretation* of the formula. In this context, a *group* of things means an unordered set.

| Problem | Type | Formula |
|---|---|---|
| Choose a group of $k$ objects from $n$ different objects | Binomial coefficient | $\binom{n}{k}$ |
| Partition $n$ different objects into $m$ labeled groups, with $k_i$ elements in group $i$ | Multinomial coefficients | $\binom{n}{k_1, \ldots, k_m}$ |
| Partition $n$ different objects into $k$ non-empty groups, where there is no order on the sets | Partitions, Stirling numbers | $S(n,k)$ |
| Partition $n$ different objects into $k$ labeled groups (which could be empty) | Multiplication principle | $k^n$ |
| Partition $n$ identical objects into $m$ labeled groups | Dots and bars | $\binom{n+m-1}{m-1}$ |
| Same, but with non-empty groups | Dots and bars | $\binom{n-1}{m-1}$ |
| Order $n$ different objects | Permutations | $n!$ |
| Choose and order $k$ different objects from $n$ different objects | Permutations | $\frac{n!}{(n-k)!}$ |
| Choose and order $n$ objects, where there are $k_i$ identical objects of type $i$ | Multinomial coefficients | $\binom{n}{k_1, \ldots, k_m}$ |
| Choices for $(X, Y)$ if there are $x$ choices for $X$ and, independently, $y$ choices of $Y$ | Multiplication-principle | $x \cdot y$ |
| Number of elements in $A \cap B \cap C$ | Inclusion-exclusion | $\begin{aligned}&\lvert A \cup B \cup C\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert \\ &\quad - \lvert A \cap B\rvert - \lvert A \cap C\rvert - \lvert B \cap C\rvert \\ &\quad + \lvert A \cap B \cap C\rvert\end{aligned}$ |

### *Binomial- and multinomial coefficients*

Whenever $n \geq 0$ and $0 \leq k \leq n$, we define the *binomial coefficients* as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \qquad \text{(Binomial coefficients)}$$

Sometimes the notation $C(n,k)$ for $\binom{n}{k}$ is used.

The binomial coefficients satisfy the following recursion:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \qquad (1)$$

To choose $k$ objects among $\{1,2,\dots,n\}$, we either exclude $n$, and choose $k$ objects among $\{1,2,\dots,n-1\}$ *or* we include $n$, and choose additional $k-1$ objects among $\{1,2,\dots,n-1\}$.

We have the *binomial theorem*:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \text{(Binomial theorem)}$$

A generalization of the binomial coefficients are the *multinomial coefficients*. Whenever $k_1 + k_2 + \ldots + k_r = n$, they are defined as

$$\binom{n}{k_1, k_2, \dots, k_r} = \frac{n!}{k_1! \cdots k_r!}. \qquad \text{(Multinomial coefficients)}$$

### *Stirling numbers*

The Stirling numbers $S(n,k)$ can be computed recursively via a table, where every row is obtained from the previous via

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k).$$

and using the fact that $S(n,1) = S(n,n) = 1$.

*Proof:* To partition $\{1,2,\cdots,n\}$, into $k$ groups, we either let $n$ be in its own group, and partition $\{1,2,\dots,n-1\}$ into $k-1$ groups, or we partition $\{1,2,\dots,n-1\}$ into $k$ groups and choose which of the $k$ groups $n$ belongs to.

## *Counting problems*

**Problem. 1**

You are creating a 4-digit pin code. How many choices are there in the following cases?

(a) With no restriction.

(b) No digit is repeated.

(c) No digit is repeated, digit number 3 is a 0.

(d) No digit is repeated, and they must appear in increasing order.

(e) No digit is repeated, 2 and 5 must be present.

**Problem. 2**

How many shuffles are there of a deck of cards, such that $A\heartsuit$ is not directly on top of $K\heartsuit$, and $A\spadesuit$ is not directly on top of $K\spadesuit$?

A standard deck has 52 cards, divided into four suits ($\heartsuit$, $\spadesuit$, $\diamondsuit$, $\clubsuit$). There are 13 cards of each suit, $2, 3, \dots, 10$, $J$, $Q$, $K$, $A$, the Jack, Queen, King and Ace

**Problem. 3**

How many different words can be created by rearranging the letters in `SELFIESTICK`?

**Problem. 4**
How many flags can we make with 7 stripes, if we have 2 white, 2 red and 3 green stripes?

**Problem. 5**
We have four different dishes, two dishes of each type. In how many ways can these be distributed among 8 people?

**Problem. 6**
In how many ways can 8 people form couples of two?

**Problem. 7**
We go to a pizza party, and there are 5 types of pizza. We have starved for days, so we can eat 13 slices, but we want to sample each type at least once. In how many ways can we do this? Order does not matter.

**Problem. 8**
How many $r$th order partial derivatives does $f(x_1, \dots, x_n)$ have?

**Problem. 9**
How many integer solutions does $x_1 + x_2 + \cdots + x_n = r$ have, with $x_i \geq 0$?

**Problem. 10**
How many integer solutions does the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

have, if we require that $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 10$ and $x_4 \geq -3$?

**Problem. 11**
How many integer solutions are there to the system of inequalities

$$x_1 + x_2 + x_3 + x_4 \leq 15, \qquad x_1, \dots, x_4 \geq 0?$$

**Problem. 12**
Count the number of non-negative integer solutions to

$$3x_1 + 3x_2 + 3x_3 + 7x_4 = 22.$$

**Problem. 13**
Compute the number of surjections $f : A \to B$ if $|A| = n$ and $|B| = k$.

**Problem. 14**
You are going to an amusement park. There are four attractions, (haunted house, roller coaster, a carousel, water ride). You buy 25 tokens. Each attraction cost 3 tokens each ride, except the roller coaster that costs 5. Obviously, you want to ride each ride at least once, but the order of the rides does not matter.

In how many ways can you spend your tokens? You may have some remaining tokens in the end of the day.

**Problem. 15**
At an amusement park, you pay for attractions using tokens. There are five different attractions which cost 3 tokens each, and one attraction which cost 5 tokens. You have 42 tokens and you want to use all of them. How many different selections of attractions are there?

**Problem. 16**
How many words can you create of length 6, from the letters `a`, `b`, `c` and `d` if

- you must include each letter at least once, and
- `a` must appear exactly once.

**Problem. 17**
Eight different exam questions are to be distributed among three students, such that each student receives at least one question. However, two of the questions are very easy and must be given to different students. In how many ways can this be done?

**Problem. 18**
Prove that $S(n+1, k+1) \geq S(n, k)$ whenever $n \geq 1$ and $1 \leq k \leq n$.

**Problem. 19**
How many words can be made by rearranging `aabbccdd`, such that no `'a'` appears somewhere to the right of some `'c'`?

**Problem. 20**
You have 2 copies of the letter `'A'` and and unlimited supply of the letters `'B'`, `'C'` and `'D'`. How many words of length 10 can you make from these, such that

- all the `A`'s are used
- the third letter is an `A`, and
- there is no `B` appearing between the `A`'s?

## *Mailbox-principle*

**Problem. 21**
What is the maximum number of rooks you can place on an $8 \times 8$ chessboard so that no two rooks can attack each other?

**Problem. 22**

Alice and Bob are dining at a Chinese restaurant, where there are 10 small dishes available. Each dish is priced between 50 and 100 yuan. They decide to order several dishes each, in such a manner that all dishes are different. Moreover, they want to ensure that the price of the dishes Alice chooses have the same total price as the ones Bob selects.

Show that such a choice is possible.

**Problem. 23**

Let $S$ be a subset of $\{1, 2, \ldots, 2n\}$ such that $S$ has $n+1$ elements. Prove that there are different elements $a$, $b$ in $S$ such that $a$ divides $b$.

This is quite a challenging problem!

## *Binomial identities*

In this section, we prove combinatorial identities by giving an interpretation of the different terms and factors involved. The go-to strategy is that the simpler side of the identity tells some story story, and we add some *refinement* or details to the story to get an interpretation of the more complicated side. For example, a poker player can immediately tell you how to prove and interpret the "combinatorial" identity

$$52 = 13 + 13 + 13 + 13.$$

The left hand side count the number of cards. The right hand side *refines* the situation, by counting the number of cards in each suit.

A classical identity is

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}. \tag{2}$$

We know that the left hand side count the number of binary words of length $n$. The right hand side *refines* this count — term $k$ counts the number of binary words with exactly $k$ bits equal to 1.

It is important to emphasize that we add *exclusive* counts, a binary word is counted by exactly one of the binomial coefficients.

**Problem. 24**

Show by using a combinatorial argument that

$$\binom{n}{r} = \sum_{k=0}^{r} \binom{n-m}{k}\binom{m}{r-k} \text{ whenever } 0 \leq m, r \leq n.$$

**Problem. 25**

Prove that

$$\binom{n}{2} 2^{n-2} = \sum_{k=2}^{n} \binom{n}{k}\binom{k}{2} \text{ whenever } n \geq 2.$$

**Problem. 26**
Prove that

$$2n \cdot 3^{n-1} = \sum_{k=1}^{n} k \cdot 2^k \binom{n}{k}.$$

**Problem. 27**
Show that if $a$, $b$ are non-negative integers, we have

$$\binom{a+b}{a}\binom{a+b}{b} = \sum_{k=0}^{a+b} \binom{a+b}{k}\binom{a+b-k}{a-k}\binom{b}{b-k}.$$

**Problem. 28**
Show

$$\binom{n+0}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}.$$

**Problem. 29**
Let $m \geq n$ and show that

Tip: First use $\binom{n}{k} = \binom{n}{n-k}$.

$$\binom{m+n}{n} = \sum_{k=0}^{n} \binom{m}{k}\binom{n}{k}.$$

**Problem. 30**
Let $m \geq n$ and show that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{k}\binom{n}{k-i}.$$

**Problem. 31**
Prove the identity

$$n \cdot 4^{n-1} = \sum_{k=0}^{n} \binom{n}{k} 3^k (n-k).$$

**Problem. 32**
Show that

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}.$$

## *Combinatorial bijections*

Here are a few examples of combinatorial bijections. The bijections in the solutions are just suggestions, there should be plenty of other possible bijections.

**Problem. 33**

Let $\mathrm{S}_n$ be the set of permutations of $\{1, 2, \ldots, n\}$ and let $Q_n$ be the set of integer vectors $\mathbf{w}$ of length $n$, with the property that $1 \leq \mathbf{w}_i \leq i$ for all $i$. Describe a bijection between $\mathrm{S}_n$ and $Q_n$.

**Problem. 34**

Let $B_n$ denote the set of binary words with $n$ digits. Moreover, let $C_n$ be the set of *integer compositions* of $n$. For example,

For example, $B_3$ consists of the $2^3$ words 000, 001, 010, 011, 100, 101, 110, 111.

$$C_4 = \{(4), (1,3), (2,2), (3,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,1)\}$$

Describe a bijection from $B_{n-1}$ to $C_n$.

**Problem. 35**

Consider a labeled tree on $n$ vertices, where 1 is considered the root vertex. We say that the tree is *decreasing* if the labels appear in a decreasing manner on every path from a vertex to the root, see Fig. 1.

Show that the number of such decreasing trees on $n$ vertices is $(n-1)!$, by constructing a bijection from the set of trees, to permutations of $\{1, 2, 3, \ldots, n\}$ ending with a 1.

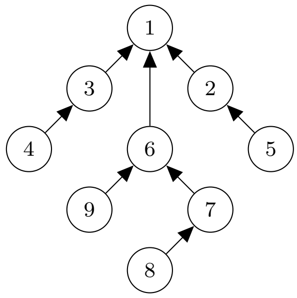

Figure 1: An example of an increasing tree.

## *Inclusion–Exclusion*

**Problem. 36**

There are five people of different height. In how many ways can they stand in a line, so there is no 3 consecutive people with increasing height?

**Problem. 37**

Count the number of decks of cards, where no king is on top of the ace of the same suit.

**Problem. 38**

Count the number of decks of cards, where no king is on top of any ace.

**Problem. 39**

We have a smorgosbord, with 50 dishes, — 5 countries are represented, and there are 10 dishes from each. We want to make a plate with 8 dishes (no duplicates), but make sure that no country is missing. How many ways?

**Problem. 40**

We have $k$ different boxes and $r$ different objects. We want to distribute the objects into the boxes such that at no box is empty. In how many ways can this be done?

Observe that this is the same as the number of surjections from the set of boxes to the set of objects.

**Problem. 41**

There are five people of different height. In how many ways can they stand in a line, so there are no 3 consecutive people appear in order,

In other words, among any three adjacent people, the medium-tallest of them is not standing in the middle.

either increasingly or decreasingly?

## *Reading comprehension*

To see the intricacies in combinatorial reasoning, we now review a variety of counting problems.

Try to identify which of these choices allow for *repetition*, and which are ordered and unordered. Before proceeding, review the difference between *labeled* and *unlabeled* sets.

Words such as line, queue, list and shelf indicate an order, while words such as set, group, pile and bag indicate unordered arrangements. Additionally, people are always considered unique — no two persons are alike and they have names. You need to be aware if there are several sets, queues or groups involved: The two sets

$$\{\{a, b\}, \{c, d\}\} \text{ and } \{\{d, c\}, \{a, b\}\}$$

are considered equal. However, the two arrangements (with *named* sets)

$$A = \{a, b\}, B = \{c, d\} \text{ and } A = \{d, c\}, B = \{a, b\}$$

are considered different. Note that this intricacy can only occur for sets (or lists) of equal sizes.

**Problem. 42**
There are 8 people available. Count the number of ways

(a) to choose 6 of them and arrange them in a line.

(b) to choose 6 of them and place them into lines named $A$ and $B$, with 3 in each.

(c) to choose 6 of them and place them into two equal-sized unlabeled lines.

(d) to choose 6 of them to make a group.

(e) to choose 6 of them and place them into groups named $A$ and $B$, with 3 in each.

(f) to choose 6 of them and make two equal-sized unlabeled groups.

(g) to choose 6 of them and make three equal-sized unlabeled groups.

**Problem. 43**
There are 8 red balls available[1]. Count the number of ways

[1] These are identical!

(a) to choose 6 of them and arrange them in a line.

(b) to choose 6 of them to make a group.

(c) to choose 6 of them and give them to three people, some might not get any.

(d) to choose 6 of them and give them to three people, each person get at least one.

(e) to choose 6 of them and make three non-empty (unlabeled) groups.

(f) to choose 6 of them and divide them into piles.

**Problem. 44**

There are 8 types[2] of cookies available in a store. Count the number of ways

(a) to pick 6 of them and arrange them in a line.

(b) to pick 6 of them and place them into lines named $A$ and $B$, with 3 in each.

(c) to pick 6 of them and place them into two equal-sized unlabeled lines.

(d) to pick 6 of them to make a group.

(e) to pick 6 of them and place them into groups named $A$ and $B$, with 3 in each.

(f) to pick 6 of them and make two equal-sized unlabeled groups.

(g) to pick 6 of them and make three equal-sized unlabeled groups.

(h) For 10 people to choose a cookie type, and each type is selected by at least one person.

[2] This indicates that repetition is allowed — the same type can be used several times

## *Solutions*

**Solution. 1**

(a) There are 4 independent choices, so $10^4$.

(b) $10 \cdot 9 \cdot 8 \cdot 7$.

(c) Choose the remaining three: $9 \cdot 8 \cdot 7$.

(d) $\binom{10}{4}$.

(e) Pick two additional digits and count all permutations: $\binom{8}{2} \times 4!$.

These must be chosen in an unordered fashion, since we later count all $4!$ permutations of the unordered digits.

**Solution. 2**

It is easier to first count the number of forbidden shuffles. We have two different types of forbidden arrangements, see Fig. 2.

The number of decks with $A\heartsuit$ on top of $K\heartsuit$ is $51!$, since we can remove the $A\heartsuit$, shuffle the remaining 51 different cards, and then place the ace of hearts on top of the king of hearts. In the same manner, we have $51!$ forbidden decks involving $A\spadesuit$.

Finally, we need to count the number of elements in the intersection, i.e., decks where both of the forbidden configurations occur. We

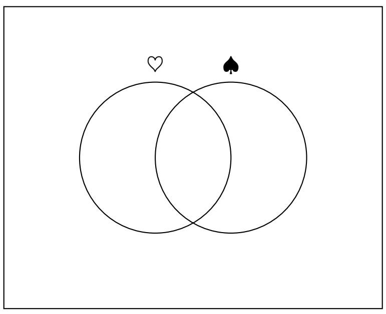

Figure 2: The total deck of cards, $52!$, and the two intersecting forbidden subsets.

remove $A\heartsuit$ and $A\spadesuit$, shuffle the 50 cards, and insert the aces on the respective kings. This gives 50! shuffles. The number of forbidden configurations is therefore $51! + 51! - 50!$, and the total number of good decks is

$$52! - 2 \times 51! + 50!.$$

**Solution. 3**
Since there are duplicates of $E$, $S$ and $I$, there are $11!/2^3$ different words.

**Solution. 4**
It is given by the multinomial coefficient

$$\binom{7}{2,2,3} = \frac{7!}{2! \times 2! \times 3!} = 210.$$

**Solution. 5**
It is given by the multinomial coefficient

$$\binom{8}{2,2,2,2} = \frac{8!}{2^4}$$

It is the same as counting number of different words we can create from `AABBCCDD`. For example, the word `ADCBBCDA` assign dish $A$ to person 1 and person 8.

**Solution. 6**
It is easier to consider the couples as labeled. We first pick 2 people to form couple $A$, then 2 other people to form couple $B$ and so on. The number of ways to create *labeled* couples is

$$\binom{8}{2,2,2,2} = \frac{8!}{2^4}.$$

However, all permutations of the 4 labels produce the same set of couples, so we need to divide this by 4!. The answer is therefore $\frac{8!}{4! \times 2^4}$.

**Solution. 7**
First, we sample the 5 types. That leaves space for 8 more, which we can choose freely, with repetition. The dots-and-bars formula tells us that there are

$$\binom{8+5-1}{5-1} = \binom{12}{4}$$

ways to do this.

**Solution. 8**
There are $n$ types of derivatives and we need to select $r$ of these with repetition allowed. Order does not matter in which we compute derivatives, so dots and bars give

$$\binom{r+n-1}{n-1}.$$

**Solution. 9**
Dots and bars give
$$\binom{r+n-1}{n-1}.$$

**Solution. 10**
Let $y_1 = x_1 - 2$, $y_2 = x_2 - 3$, $y_3 = x_3 - 10$ and $y_4 = x_4 + 3$.
We get a new equation where $y_i \geq 0$ and

$$(y_1 + 2) + (y_2 + 3) + (y_3 + 10) + (y_4 - 3) = 15, \qquad \Leftrightarrow$$
$$y_1 + y_2 + y_3 + y_4 = 3$$

Dots and bars gives $\binom{3+4-1}{4-1}$ integer solutions.

**Solution. 11**
We add one extra variable to turn the inequality to an equality:

$$x_1 + x_2 + x_3 + x_4 + s = 15, \qquad s, x_i \geq 0.$$

This gives $\binom{15+5-1}{5-1}$ integer solutions.

**Solution. 12**
We divide into cases. The only possible values for $x_4$ are $x_4 = 0, 1, 2, 3$, since otherwise the left hand side is too large.

Case $x_4 = 0$: We get $3(x_1 + x_2 + x_3) = 22$. No solutions as the right hand side is not a multiple of 3

Case $x_4 = 1$: We get $3(x_1 + x_2 + x_3) = 15$, so $x_1 + x_2 + x_3 = 5$ which has $\binom{5+2}{5} = 21$ solutions.

Case $x_4 = 2$: We get $3(x_1 + x_2 + x_3) = 8$, no solutions.

Case $x_4 = 3$: We get $3(x_1 + x_2 + x_3) = 1$, no solutions.

Therefore, there are 21 solutions in total.

**Solution. 13**
There are $k!S(n, k)$ surjections — the quantity $k!$ is responsible for the labeling.

**Solution. 14**
We first ride all rides once. That leaves 11 tokens which can be spent as we please. We can ride the roller coaster 0, 1 or 2 times with the remaining tokens:

- 0 **times**. We need to count non-negative integer solutions to $3h + 3c + 3w \leq 11$. This is the same as solving $h + c + w + r = 3$ where $r$ represents the number of remaining tokens. Number of solutions: $C(3 + 4 - 1, 3)$

- 1 **time**. Same strategy gives non-negative solutions to $3h + 3c + 3w \leq 6$, or $h + c + w + r = 2$. This gives $C(2 + 4 - 1, 2)$ number of solutions.

- **2 times**. After riding the coaster 2 times, we have one token left and cannot ride anything else. Only 1 solution.

Total number of ways: $\binom{6}{3} + \binom{5}{2} + 1 = 20 + 10 + 1 = 31$.

**Solution. 15**

This is equivalent with solving $3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 5y = 42$, $x_i, y \geq 0$. We see that $0 \leq y \leq 8$ (we cannot afford 9 rides on the expensive attraction).

Case by case analysis of $y$ show that $y = 0$, $y = 3$, $y = 6$ are the only possible choices for the remaining tokens to be a multiple of 3. This means we can ride cheap rides either 4, 9 or 14 times, so we want to add up the number of solutions to

$$x_1 + \cdots + x_5 = R, \qquad R \in \{4, 9, 14\}.$$

This gives the total answer

$$\binom{4+5-1}{5-1} + \binom{9+5-1}{5-1} + \binom{14+5-1}{5-1}.$$

**Solution. 16**

This problem is not a clear-cut standard problem as in the introduction. However, the fact that each letter appears at least once imposes a lot of restriction, as we only need to decide which two additional letters to add to `abcd`.

This is a strong hint that we need to divide the problem into sub-cases. The fewer cases the better.

- We add two different letters. Thus, the letters appearing are one of `abbccd`, `abbcdd` or `abccdd`. To calculate the number of words we can make from `abbccd`, we use a multinomial coefficient, $\binom{6}{2,2,1,1}$.
- We add the same letter twice. This gives `abbbcd`, `abcccd` or `abcddd`, and each of these options give $\binom{6}{3,1,1,1}$ words.

Expanding the multinomials and putting it all together, there are in total

$$3\frac{6!}{(2!)^2} + 3\frac{6!}{3!}$$

words satisfying the requirements.

**Solution. 17**

Without the extra restriction, the number of ways to do this is $3!S(8,3)$: The Stirling number count ways to distribute the different questions into three non-empty sets and the 3! account for the different ways to distribute the sets among the students.

To construct the forbidden configurations, we can remove one of the easy questions and distribute the remaining 7 questions among the students. The student who gets the easy question is then also given the second easy question. We see that this can be done in $3!S(7,3)$ ways.

The final answer is therefore $3!S(8,3) - 3!S(7,3)$.

**Solution. 18**

The recursion for Stirling numbers tell us that

$$S(n+1,k+1) = (k+1)S(n,k+1) + S(n,k)$$
$$\Leftrightarrow$$
$$S(n+1,k+1) - S(n,k) = (k+1)S(n,k+1)$$

and since $(k+1)S(n,k+1) \geq 0$, the inequality must be true.

**Solution. 19**

We notice that if the positions of the `b`'s and `d`'s are fixed, then the positions of the remaining letters is uniquely determined by the restriction. For example,

$$\texttt{b}\square\square\texttt{d}\square\texttt{b}\square\texttt{d} \quad \Longrightarrow \quad \texttt{baadcbcd}.$$

To create a valid word, it is enough to first choose 2 positions of the 8 available for the `b`'s and then 2 positions for the `d`'s. This can be done in $\binom{8}{2,2,4} = \frac{8!}{2!\times 2!\times 4!}$ ways.

**Solution. 20**

There are two cases to consider, either the `A` in the third position is the first `A` in the word, or it is the second.

If the first `A` appears in the third spot, the word is of the form `XXAY..YA..` with only `C`'s and `D`'s between the `A`'s, and the `X` are one of the letters in `BCD`.

There can be between 0 and 6 letters between the two `A`'s. The number of words with $k$ letters between the `A`'s is given by $2^k \times 3^{8-k}$, since we need to choose either `C` or `D` for the $k$ letters and there are three choices for each of the remaining $8-k$ spots. Summing over the possible values of $k$ gives

$$\sum_{k=0}^{6} 2^k 3^{8-k} = 3^8 \sum_{k=0}^{6} \left(\frac{2}{3}\right)^k.$$

This is a geometric sum and the formula for geometric sum gives

Recall that $\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$.

$$3^8 \times \frac{1-(2/3)^7}{1-(2/3)} = 3^8 \times \frac{\frac{3^7-2^7}{3^7}}{1/3} = 3^2(3^7-2^7).$$

There are now two more cases to consider — words of the forms `AYA...` or `XAA...`. There are $2\times 3^7$ words of the first form and $3^8$ words of the second form.

Because the `Y` has two options, and the remaining open spots have three

Adding the results for all cases give us in total

$$3^9 - 9\times 2^7 + 2\times 3^7 + 3^8.$$

**Solution. 21**

First of all, by putting 8 rooks on one of the diagonals, we see that it is possible to place 8 non-attacking rooks on a board.

However, if we place 9 or more rooks on the board, there will be some row with at least two rooks. These can attack each other. The answer is therefore 8 rooks.

**Solution. 22**

There are $2^{10} = 1024$ different selections of dishes, but the price for each such selection lies between 0 and 1000 yuan. Hence, there must be two different selections[3] of dishes with the same price. Let $S_1$ and $S_2$ be two such selections. It can be the case that there are some common dishes in $S_1$ and $S_2$, but these can be removed from the selections, and we still have that the price of both selections are the same.

[3] At least!

**Solution. 23**

Define the following sets:

$$\begin{aligned} T_1 &= \{1, 2^1, 2^2, 2^3, \dots\}, \\ T_3 &= \{3, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, \dots\}, \\ T_5 &= \{5, 5 \cdot 2^1, 5 \cdot 2^2, 5 \cdot 2^3, \dots\}, \\ &\dots \\ T_{2n-1} &= \{(2n-1), (2n-1) \cdot 2^1, (2n-1) \cdot 2^2, (2n-1) \cdot 2^3, \dots\}, \end{aligned}$$

where we have $n$ such sets in total. Every integer in $I = \{1, 2, \dots, 2n\}$ is of the form $q \cdot 2^k$ for some odd number $q < 2n - 1$, and thus in $T_q$. Therefore, $I \subseteq T_1 \cup T_2 \cup \dots \cup T_{2n-1}$. Furthermore, if we take any two different numbers, $a = q \cdot 2^i$ and $b = q \cdot 2^j$ from $T_q$ with $i < j$ then $a|b$.

Finally, among $n + 1$ numbers from $I$, at least two of these belong to the same $T_q$ since we only have $n$ such sets. But in that case, the above argument showed that the smaller divides the larger.

**Solution. 24**

In the left hand side, we choose $r$ people among $n$ people.

In the right hand side, the first $m$ of the $n$ people are given hats. To choose $r$ among the $n$ people, we can independently pick $k$ of the hat-less people and $r - k$ of hat-people. Summing over all values of $k$ then simply gives an $r$-subset of $n$ people.

This is not a choice! We refine the situation by asking how many of the first $m$ people are chosen. To help you visualize the situation, they are given hats.

**Solution. 25**

Left hand side. There's gonna be a party in a village with $n$ people: 2 out of $n$ people will definitely go and bring pizza. The remaining $n - 2$ people, might, or might not go.

This gives $\binom{n}{2}$.

Right hand side. We sum over total number $k$ of party people.

First choose party subset of size $k$ out of the $n$ people. Then, among the party people, we choose two people who bring pizza.

**Solution. 26**

Left hand side. There are $n$ people in a restaurant choosing among three wines, $A$, $B$ and $C$, where $B$ and $C$ are non-alcoholic. One person is chosen to be a designated driver, and thus has only two options. Hence, the interpretation for the left hand side is

$$\underbrace{n}_{\text{Designated driver}} \times \underbrace{2}_{\text{Drivers wine choice}} \times \underbrace{3^{n-1}}_{\text{Remaining wine choices}}.$$

Right hand side. We sum over the number of people, $k$, who choose either $B$ or $C$. The "sober" people can thus be selected in $\binom{n}{k}$ ways. The designated driver must be chosen among these $k$ people. Finally, the $k$ sober people (including driver) make up their mind about $B$ or $C$. The interpretation is therefore

The ones not selected here must therefore take wine $A$.

$$\sum_{k=1}^{n} \underbrace{\binom{n}{k}}_{\text{People choosing } B/C} \times \underbrace{k}_{\text{Designated driver}} \times \underbrace{2^k}_{\text{Choice of } B/C}.$$

In both sides, we see that the same story is told.

**Solution. 27**

Left hand side. There are $a+b$ people going to party — $a$ of these bring food and $b$ bring drinks. Some might bring both food and drinks, while other people bring nothing.

In other words, the choice of who brings food and who brings drinks is done independently, hence the multiplication.

Right hand side. We refine over the number, $k$, of of people who bring both food and drinks. First, we select these $k$ generous people. Among the remaining $a+b-k$ people, we select the $a-k$ ones who only bring food. This leaves $b$ people so far without a task and we select $b-k$ people of these who bring drinks. The interpretation of the coefficients in the right hand side is therefore

$$\sum_{k=0}^{a+b} \underbrace{\binom{a+b}{k}}_{\text{Brings both}} \times \underbrace{\binom{a+b-k}{a-k}}_{\text{Food only}} \times \underbrace{\binom{b}{b-k}}_{\text{Drinks only}}.$$

**Solution. 28**

Left hand side. There are $n+r+1$ people in total. We first pick a number $k = 0, 1, \ldots, r$, and give hats to the first $n+k$ people. From the people with hats, we select $k$ of the hats and paint them blue. Note that person $n+r+1$ never gets a hat.

Right hand side. Here we have subsets of exactly size $r$ among $n+r+1$ people. Suppose we have such a subset $S$. There are two cases to consider:

- $S$ does not contains person $n+r+1$. Then $S$ is an $r$-subset of the first $n+r$ people. In this case, we give hats to the first $n+r$ people and paint $r$ hats blue according to $S$.

This corresponds to the $\binom{n+r}{r}$ term in the left hand side.

- $S$ do contain person $n+r+1$. In this case there is a smallest integer $i$, $0 \leq i < r$, such that
$$\{n+i+1, n+i+2, \dots, n+r+1\} \subseteq S.$$
With this in mind, the set $T = S \cap \{1, 2, \dots, i-1\}$ contains $i$ elements. We now give hats to the first $n+i$ people, and the subset of these with blue hats is determined by the set $T$.

This corresponds to the $\binom{n+i}{i}$ term.

**Solution. 29**

LEFT HAND SIDE. We have $m+n$ people, $m$ with red hats and $n$ with blue hats, and we count the subsets of size $n$.

RIGHT HAND SIDE. Remember that $\binom{n}{k} = \binom{n}{n-k}$, so that the right hand side is equal to
$$\sum_{k=0}^{n} \binom{m}{k}\binom{n}{n-k}.$$
We refine by the number, $k$, of people in our subset that have red hats. To create an $n$-subset of the $m+n$ people, we select $k$ with red hats, and $n-k$ people blue hats, which is independent choice.

**Solution. 30**

LEFT HAND SIDE. We have $m+n$ people, $m$ with red hats and $n$ with blue hats, and we count the subsets of size $k$.

RIGHT HAND SIDE. We refine by the number, $i$, of people in our subset that have red hats. To create a $k$-subset of the $m+n$ people, we select $i$ with red hats, and $k-i$ people blue hats, which is independent choice.

**Solution. 31**

IN THE LEFT HAND SIDE, there are $n$ people who go on a wine tour. Since they are responsible adults, one person is selected to be the designated driver and he must stay sober. Each of the remaining $n-1$ adults can pick — independently — one out of 4 options from a menu, where the last menu item is the non-alcoholic option.

There are $n$ choices.

IN THE RIGHT HAND SIDE, we first choose a $k$-subset of people who prefers one of the first three options. They then pick which of these they like. The designated driver cannot, of course, be among these $k$ people, and is selected among the remaining $(n-k)$ people — all who also pick the non-alcoholic option. By summing over all possible values of $k$, we see that the situation agrees with the left hand side.

An alternative solution is to expand $(3x+y)^n$ using the binomial theorem, take the derivative with respect to $y$ on both sides, and then put $x = y = 1$.

**Solution. 32**
First multiply both sides with $n+1$. We need to prove

$$\sum_{k=0}^{n} \frac{n+1}{k+1}\binom{n}{k} = 2^{n+1} - 1.$$

Now note that

$$\frac{n+1}{k+1}\binom{n}{k} = \frac{(n+1)n!}{(k+1)k!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}.$$

Thus we need to prove that

$$\sum_{k=0}^{n} \binom{n+1}{k+1} = 2^{n+1} - 1$$
$$\sum_{k=1}^{n+1} \binom{n+1}{k} = 2^{n+1} - 1$$
$$\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^{n+1}$$

but the last line is the binomial theorem.

**Solution. 33**
There are several natural bijections. We shall construct a bijection using recursion over $n$. The base case is $n = 1$, and we let $f$ send the permutation 1 to the vector 1. Suppose now we are given $\pi \in \mathrm{S}_n$, and we want to construct $w \in Q_n$. We let $w_n$ be the position of $n$ in $\pi$. Then, let $\pi'$ be the permutation obtained from $\pi$, with element $n$ removed, so that $\pi' \in \mathrm{S}_{n-1}$. We then let $w' = f(\pi')$ by recursion, and the remaining entries in $w$ are determined by setting $w_i = w'_i$ for $i < n$.

The key property here is to note that given a permutation $\pi' \in \mathrm{S}_{n-1}$, and an integer $w_n$, with $1 \leq w_n \leq n$, we can uniquely recover $\pi$ by inserting $n$ at position $w_n$ in $\pi'$. This shows that every step in the recursion is a bijection, and it follows that $w$ uniquely determines $\pi$.

Here is an example on how we can construct the word $w$ from the permutation $[7, 6, 8, 3, 1, 9, 4, 2, 5]$.

$$\begin{aligned}
[7, 6, 8, 3, 1, 9, 4, 2, 5] &\sim []\\
[7, 6, 8, 3, 1, 4, 2, 5] &\sim [6]\\
[7, 6, 3, 1, 4, 2, 5] &\sim [3, 6]\\
[6, 3, 1, 4, 2, 5] &\sim [1, 3, 6]\\
[3, 1, 4, 2, 5] &\sim [1, 1, 3, 6]\\
[3, 1, 4, 2] &\sim [5, 1, 1, 3, 6]
\end{aligned}$$

$$
\begin{aligned}
[3,1,2] &\sim [3,5,1,1,3,6] \\
[1,2] &\sim [1,3,5,1,1,3,6] \\
[1] &\sim [2,1,3,5,1,1,3,6] \\
[] &\sim [1,2,1,3,5,1,1,3,6]
\end{aligned}
$$

In the first step, we find the largest element (which is 9) in the permutation, and insert its position in $w$. This determines the last element in $w$. In the second step, we find the new largest element (which is 8) in the permutation, and insert its position in $w$. Repeat this process until the permutation is empty.

Below is a Mathematica implementation of the above bijection.

```
(* Base case. *)
f[{1}]:={1};

(* General case. *)
f[pi_List]:=With[{n=Max@pi},
Append[
  (* Apply bijection to pi' to get
     all but the last entry of w *)
  f[DeleteCases[pi,n]]
  ,
  (* The position of n in pi is the
     last entry in w. *)
  Position[pi,n][[1,1]]
]];
```

**Solution. 34**

In order to describe a bijection $f$, we must have a way to construct an element in $C_n$, given some binary word $w \in B_{n-1}$. The easiest way to do this is to give an example. In general, one has to be careful and argue that the construction is indeed a bijection.

Example: $n = 9$, $w = \texttt{00110101}$. We draw 9 balls and put the digits of $w$ between the balls.

$$\bigcirc_0 \;\; \bigcirc_0 \;\; \bigcirc_1 \;\; \bigcirc_1 \;\; \bigcirc_0 \;\; \bigcirc_1 \;\; \bigcirc_0 \;\; \bigcirc_1 \;\; \bigcirc$$

The 1s in the binary word are replaced by separators, and we count the number of balls between the separators.

$$
\begin{array}{ccccccccccccc}
\bigcirc & \bigcirc & \bigcirc & | & \bigcirc & | & \bigcirc & \bigcirc & | & \bigcirc & \bigcirc & | & \bigcirc \\
 & 3 & & & 1 & & 2 & & & 2 & & & 1
\end{array}
$$

The final list of integers, $(3,1,2,2,1)$, has total sum 9 and we have that $f(00110101) = (3,1,2,2,1)$.

Since we start with $n$ balls, we are sure that the method above results in a list with sum $n$. Moreover, since there is at most one separator between two adjacent balls, we are sure that the integers are positive. Hence, the procedure above does give an integer composition of $n$. Given $\alpha \in C_n$, we also fairly easy see how to reverse the steps outlined above and recover a corresponding binary word in $B_{n-1}$. Hence, the map $f$ is a bijection.

**Solution. 35**

Let $P_n$ be the set of permutations of $\{1, 2, 3, \dots, n\}$ ending with a 1, and let $T_n$ be the set of increasing trees on $n$ vertices.

Given a permutation in $P_n$, we construct the edges of a decreasing tree as follows: For each number $i \geq 2$ in $P$, find the first number $j < i$ to the right of $i$. Then we let $i \to j$ be an edge in the tree. Since 1 is the rightmost number in $P$ it is evident that such a $j$ always exists. By construction, we cannot have two edges $i \to j_1$ and $i \to j_2$ where $i > j_1$ and $i > j_2$, and it also follows that every vertex $i > 2$ is connected to a unique smaller vertex. This ensures that the resulting edges really describe a tree in $T_n$.

In the other direction, given a decreasing tree in $T_n$, we can invert the above construction as follows: First start with the permutation $\pi = 1$. We visit the vertices $i = 2, 3, \dots$ in the tree in this order, and if vertex $i$ is a child of $j$, then insert $i$ directly after $j$ in the permutation, until we have processed all vertices. The result is a permutation $\pi \in P_n$.

For example, the tree in the question corresponds to the permutation 524387961.

**Solution. 36**

First number the spots in the line, 12345. Let $A$ be the event that the people at spots 123 are in height order, $B$ be the event that 234 are in order and $C$ the event that 345 are in order.

We seek $5! - |A \cup B \cup C|$ where the last term is computed via inclusion-exclusion. Notice that $|A| = |B| = |C| = \binom{5}{3}2!$ since we choose three out of the five people to be in order, and then put the remaining two people in the two possible ways in the last two spots. In the same manner, it follows that $|A \cap B| = |B \cap C| = \binom{5}{4}$. However, $|A \cap C| = 1$ and $|A \cap B \cap C| = 1$, since this means that the people in all five spots must appear in increasing order. Thus, there are

$$5! - 3\binom{5}{3} \cdot 2! + \left[2\binom{5}{4} + 1\right] - 1$$

different valid lines of people.

**Solution. 37**

Let our sets $A_1$, $A_2$ $A_3$ and $A_4$ be the decks where *king $i$ is on top of ace $i$*.

We computed before that $|A_1| = 51!$, and $|A_1 \cap A_2| = 50!$. Similarly, $|A_1 \cap A_2 \cap A_3| = 49!$ and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 48!$. Therefore, the number of forbidden decks is

$$\binom{4}{1}|A_1| - \binom{4}{2}|A_1 \cap A_2| + \binom{4}{3}|A_1 \cap A_2 \cap A_3| - \binom{4}{4}|A_1 \cap A_2 \cap A_3 \cap A_4|$$

This evaluates to

$$4 \cdot 51! - 6 \cdot 50! + 6 \cdot 49! - 1 \cdot 48!$$

and the number of good decks is then

$$52! - 4 \cdot 51! + 6 \cdot 50! - 6 \cdot 49! + 1 \cdot 48!.$$

**Solution. 38**

Let our sets $A_1$, $A_2$ $A_3$ and $A_4$ be the decks where *king $i$ is on top of an ace*.

Now,

- $|A_1| = 4 \cdot 51!$, since there are 4 aces to put below king 1.
- $|A_1 \cap A_2| = 4 \cdot 3 \cdot 50!$, there are 4 aces for king 1, and 3 remaining ace-choices for king 2.
- $|A_1 \cap A_2 \cap A_3| = 4 \cdot 3 \cdot 2 \cdot 49!$, and
- $|A_1 \cap A_2 \cap A_3 \cap A_4| = 4! \cdot 48!$.

Same reasoning as before gives that the number of good decks is

$$52! - 4(4 \cdot 51!) + 6(4 \cdot 3 \cdot 50!) - 6(4 \cdot 3 \cdot 2 \cdot 49!) + 1(4! \cdot 48!).$$

**Solution. 39**

Let $A_1, \dots, A_5$ be the possible plates where country $i$ *not* represented. We are interested in $|(A_1 \cup \dots \cup A_n)^c|$, which is then given by

$$\binom{50}{8} - 5\binom{40}{8} + \binom{5}{2}\binom{30}{8} - \binom{5}{3}\binom{20}{8} + \binom{5}{4}\binom{10}{8} - 0$$

**Solution. 40**

Let $A_i$ be the events where box $i$ empty. Inclusion-exclusion gives that $|A_1 \cup A_2 \cup \dots \cup A_n|$ is equal to

$$\binom{k}{1}(k-1)^r - \binom{k}{2}(k-2)^r + \binom{k}{3}(k-3)^r + \dots + (-1)^k\binom{k}{k-1}(1)^r$$

by choosing which box that is definitely empty, then two boxes which are definitely empty, and so on. Thus, the number of arrangements where *no* box is empty is given by

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^r.$$

**Solution. 41**

As in the previous exercise of same flavor, let us number the spots, and let $A$ be the event that the people at spots 123 are in height order, $B$ be the event that 234 are in order and $C$ the event that 345 are in order.

We seek $5! - |A \cup B \cup C|$ where the last term is computed via inclusion-exclusion.

First note that

$$|A| = |B| = |C| = \binom{5}{3} \times 2 \times 2$$

as we first pick a 3-subset to of the people to put at the spots, then decide if they should be standing in increasing or decreasing order, and finally the remaining two people have two ways to stand in the remaining spots.

Because of symmetry, $|A \cap B| = |B \cap C|$. To have $A \cap B$, we note that the people standing at 1234 must all be sorted in the same way. Similar to the previous reasoning, this gives

Either 123 and 234 are increasing, or 123 and 234 are both decreasing

$$|A \cap B| = |B \cap C| = \binom{5}{4} \cdot 2.$$

To compute $|A \cap C|$, we need to consider different subcases.

- *$A$ increasing and $C$ increasing.* This forces the tallest guy to stand in the middle. We then choose 2 out of the 4 remaining to stand in front. This uniquely determines the configuration, so there are $\binom{4}{2}$ such configuration.
- *$A$ decreasing and $C$ increasing.* Also $\binom{4}{2}$.
- *$A$ increasing and $C$ increasing.* They must all be standing in increasing order. Only one way.
- *$A$ decreasing and $C$ decreasing.* Only one way.

Adding everything up, $|A \cap C| = 2\binom{4}{2} + 2$.

Finally, $|A \cap B \cap C|$ has only two options — everything increasing or everything decreasing. The final answer is therefore

$$5! - 3 \times 4 \times \binom{5}{3} + \left(2 \cdot 2 \cdot \binom{5}{4} + 2\binom{4}{2} + 2\right) - 2.$$

**Solution. 42**

(a) There are $8 \times 7 \times \cdots \times 3$ ways.

(b) There are $(8 \times 7 \times 6) \times (5 \times 4 \times 3)$ ways.

(c) There are $\frac{1}{2}(8 \times 7 \times \cdots \times 3)$ ways, since the lines two pairs of lines $(abc, def)$ and $(def, abc)$ are considered equal configurations.

(d) There are $\binom{8}{6}$ ways.

(e) We first choose 6 people, and then choose 3 of these to be in group $A$: $\binom{8}{6}\binom{6}{3}$ ways.

(f) There are $\frac{1}{2}\binom{8}{6}\binom{6}{3}$ ways.

(g) There are $\binom{8}{6}\frac{1}{3!}\binom{6}{2,2,2}$ ways.

**Solution. 43**

(a) There is only one way.

(b) There is only one way.

(c) Bars and stars give $\binom{6+2}{2}$ ways.

(d) Bars and stars give $\binom{3+2}{2}$ ways.

(e) This is the same as counting integer partitions of 6 into 3 parts. We get three ways, $4+1+1$, $3+2+1$ and $2+2+2$.

(f) Same as previous question, but we can have as many piles as we like. We get 11 ways,

| | | | |
|---|---|---|---|
| 6 | 5+1 | 4+2 | 3+3 |
| 4+1+1 | 3+2+1 | 2+2+2 | 3+1+1+1 |
| 2+2+1+1 | 2+1+1+1+1 | 1+1+1+1+1+1. | |

**Solution. 44**

(a) Each of the 6 spots has 8 options: $8^6$.

(b) Also $8^6$.

(c) We get $8^3 + \frac{1}{2} \times 8^3(\times 8^3 - 1)$ (the number of cases where groups are identical, plus number of cases where groups are different).

(d) Bars and stars give $\binom{6+8-1}{8-1}$ ways — the bars separate types.

(e) Bars and stars for each labeled group gives $\binom{3+8-1}{8-1}^2$.

(f) Bars and stars, but take into consideration when groups are equal, and not equal:

$$\binom{3+8-1}{8-1} + \frac{1}{2}\binom{3+8-1}{8-1}\left(\binom{3+8-1}{8-1} - 1\right).$$

(g) There are $\binom{2+8-1}{8-1}$ possible groups of size 2 — let this number be $m$. Then there are

$$\binom{m}{3} + m(m-1) + m = \binom{\binom{9}{7}}{3} + \binom{9}{7}\left(\binom{9}{7} - 1\right) + \binom{9}{7}$$

ways to make three equal-sized unlabeled groups. The terms represents the cases when all groups different, two groups equal, and all three groups equal, respectively.

(h) This is the same as counting surjections $f : A \to B$, where $|A| = 10$ and $|B| = 8$, since every person in $A$ picks a type in $B$, and every type is chosen at least once. The number of such surjections is $8!S(10,8)$, where $S(10,8)$ is a Stirling number of the second kind.