Seminar 1

1. Find the lower and the upper bounds, then sup, inf, max, min for each of the following:

   (a) $[-3, 2) \cup \{3\}$.
   (b) $(-1, 1] \cup (2, \infty)$.
   (c) $(-5, 5) \cap \mathbb{Z}$.
   (d) $\emptyset$.

2. Find the sup, inf, max, min for each of the following sets:

   (a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$.
   (b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$.
   (c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.
   (d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$.

3. Suppose that S is nonempty and bounded above. Show that the set $-S := \{-x \mid x \in S\}$ is bounded below and $\inf(-S) = -\sup(S)$.

4. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that

   $$\inf_{x \in D} \big(f(x) + g(x)\big) \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x) \quad \text{and} \quad \sup_{x \in D} \big(f(x) + g(x)\big) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

   Give examples where the above inequalities are strict.

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

   $$\sup_{x \in S}(ax + b) = a \sup(S) + b.$$

6. Which of the following sets are neighborhoods of 0?

   $$[-1, 1] \cup \{2\}; \quad (-1, 1) \cap \mathbb{Q}; \quad \bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}].$$

7. Let $x \in \mathbb{R}$ and $U, V \in \mathcal{V}(x)$. Prove that $U \cap V \in \mathcal{V}(x)$.

8. ★ Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.

9. Find the interior and the closure for each of the following sets:

   (a) $(1, 2]$.
   (b) $[-3, 2) \cup \{3\}$.
   (c) $(-1, 1] \cup (2, \infty)$.
   (d) $(-5, 5) \cap \mathbb{Z}$.

10. ★ Let $A = (0, 1) \cap \mathbb{Q}$. Show rigorously that $\inf A = 0$, $\sup A = 1$, $\text{int}A = \emptyset$ and $\text{cl}A = [0, 1]$.

Homework questions are marked with ★.
Solutions should be uploaded on Teams in the corresponding Assignment.