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Mathematical Analysis
Homework 2
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Q: Ex 1

Prove using the ε-definition that $\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

$\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$ if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon, \forall n \geq N_\varepsilon$

$\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon$

$\left|\frac{2n+2-2n-3}{4n+6}\right| < \varepsilon$

$\left|-\frac{1}{4n+6}\right| < \varepsilon$

$\frac{1}{4n+6} < \varepsilon$

$\frac{1}{\varepsilon} < 4n+6$

$\frac{1-6\varepsilon}{4\varepsilon} < n \Rightarrow \left(\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}\right)$ is True $\forall n \geq N_\varepsilon$, $N_\varepsilon = \frac{1-6\varepsilon}{4\varepsilon}$

Q: Ex 2

a) $\lim\limits_{n\to\infty} (1+2+\ldots+n)^{1/n} = \lim\limits_{n\to\infty} \left(\frac{n(n+1)}{2}\right)^{1/n} = \lim\limits_{n\to\infty} e^{\frac{\ln\frac{n(n+1)}{2}}{n}} =$

$= e^{\lim\limits_{n\to\infty} \frac{\ln\frac{(n(n+1))}{2}}{n}} = 1$

$e^n > n(n+1)/2$

$\ln e^n > \ln(n(n+1)/2)$

$n > \ln(n(n+1)/2)$

b) $\lim\limits_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^n$

$\frac{(\ln(n+1))'}{(\ln(n))'} \approx \frac{1/n+1}{1/n} \approx \frac{n}{n+1} \to 1$

$\lim\limits_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^n \approx \lim\limits_{n\to\infty} 1^\infty$

$\lim\limits_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^n = \lim\limits_{n\to\infty} \left(1 + \frac{\ln(n+1) - \ln(n)}{\ln(n)}\right)^n = e^{\lim\limits_{n\to\infty} \frac{n \cdot \ln((n+1)/n)}{\ln(n)}}$

$\ln((n+1)/n) \approx \ln 1 \to 0$

$e^0 = 1$

c) $\lim\limits_{n\to\infty} \frac{n^n}{1+2^2+3^3+\ldots+n^n} = \lim\limits_{n\to\infty} \frac{n^n}{n^n(1/n^n + 2^2/n^n + 3^3/n^n + \ldots + 1)} =$

$= \lim\limits_{n\to\infty} \frac{1}{1/n^n + 2^2/n^n + \ldots + 1}$

$\frac{1}{n^n} + \frac{2^2}{n^n} + \ldots + \frac{(n-1)^{n-1}}{n^n} \to 0$

$\lim\limits_{n\to\infty} \frac{1}{1/n^n + 2^2/n^n + \ldots + 1} = 1$

Q: Ex 3

For $x_n = \frac{\sin(n)}{n}$, study if x_n is bounded, monotone and convergent. Find its limit.

1. Boundedness:

$-1 \leq \sin(n) \leq 1$

$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \Rightarrow x_n$ is bounded

2. Monotonicity

$\sin(n) \to$ NOT monotone, it oscilates between -1 and 1 $\Rightarrow$

$\Rightarrow \frac{\sin(n)}{n} \to$ NOT monotone

3. Convergence

$\lim_{n\to\infty} \frac{\sin(n)}{n}$

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

$$\rightarrow 0 \leftarrow$$

Sandwich theorem: $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0 \Rightarrow x_n$ is convergent

Q: Ex 4

Prove that x_n, $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n)$ is decreasing and bounded, hence convergent. Its limit is typically denoted by γ and is known as the Euler-Mascheroni constant.

$x_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \qquad x_{n+1} = \sum_{k=1}^{n} \frac{1}{k+1} - \ln(n+1)$

$x_{n+1} - x_n = \sum_{k=1}^{n} \frac{1}{k+1} - \ln(n+1) - \sum_{k=1}^{n} \frac{1}{k} + \ln(n) =$

$= \left(\sum_{k=1}^{n} \frac{1}{k+1} - \sum_{k=1}^{n} \frac{1}{k}\right) - (\ln(n+1) - \ln(n)) = \frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right) =$

$= \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right)$

$\ln\left(1 + \frac{1}{n}\right) \approx \frac{1}{n} \quad (1)$

$n+1 \geq n$

$\frac{1}{n+1} \leq \frac{1}{n} \quad (2)$

$(1), (2) \Rightarrow \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right) \leq 0 \Rightarrow x_{n+1} \leq x_n \Rightarrow x_n$ is decreasing (3)

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(n) = \gamma$ (Euler-Mascheroni constant) $\Rightarrow$

$x_n \to \gamma \Rightarrow$ it is bounded from below by a constant (4)

$(3), (4) \Rightarrow x_n$ is convergent