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Mathematical Analysis
Homework 7
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Q: Ex 1

Compute using Riemann sum:

a) $\lim_{n\to\infty} \frac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \dots + n\sqrt[n]{e^n}}{n^2}$

$\sqrt[n]{e} + 2\sqrt[n]{e^2} + \dots + n\sqrt[n]{e^n} = \sum_{i=1}^{n} i \cdot e^{i/n}$

$\frac{1}{n^2} \cdot \sum_{i=1}^{n} i \cdot e^{i/n} = \frac{1}{n} \sum_{i=1}^{n} i/n \cdot e^{i/n}, \quad x_i = \frac{i}{n}$

$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \cdot e^{x_i} = \int_0^1 x \cdot e^x dx = x \cdot e^x \Big|_0^1 - \int_0^1 e^x dx = e - e^x \Big|_0^1 = e - (e-1) = 1$

$u = x \quad v' = e^x$
$u' = 1 \quad v = e^x$

b) $\lim_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \sin\frac{2\pi}{2n} \dots \sin\frac{(n-1)\pi}{2n}} = y$

$\lim_{n\to\infty} \frac{1}{n} \cdot \log\left(\sin\frac{\pi}{2n} \cdot \sin\frac{2\pi}{2n} \dots \sin\frac{(n-1)\pi}{2n}\right) = \log y$

$\lim_{n\to\infty} \frac{1}{n} \cdot \left(\log\sin\frac{\pi}{2n} + \log\sin\frac{2\pi}{2n} + \dots + \log\sin\frac{(n-1)\pi}{2n}\right) = \log y$

$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} \log\sin\frac{i\pi}{2n}, \quad \frac{i}{n} = x$

$\int_0^1 \log\sin\frac{\pi x}{2} dx = \frac{2}{\pi} \int_0^{\pi/2} \log\sin t \, dt$

$t = x\pi/2$
$dt = \pi/2 \, dx$
$dx = 2dt/\pi$

$\int_0^{\pi/2} \log\sin t \, dt = \int_0^{\pi/2} \log\sin(\pi/2 - t) \, dt = I$

$2I = \int_0^{\pi/2} \log\sin t \, dt + \int_0^{\pi/2} \log\cos t \, dt = \int_0^{\pi/2} \log(\sin t \cos t) \, dt = \int_0^{\pi/2} \log(2\sin t\cos t) - \log 2 \, dt =$

$= \int_0^{\pi/2} \log(\sin 2t) \, dt - \int_0^{\pi/2} \log 2 \, dt$

$\int_0^{\pi/2} \log(\sin 2t) \, dt = \frac{1}{2} \int_0^{\pi} \log(\sin t) \, dt$

$2t = u \qquad f(u) = \log(\sin u)$
$2dt = du \qquad f(2\pi - u) = \log\sin(2\pi - u) = \log\sin u$
$dt = du/2 \qquad f(u) = f(2\pi - u)$

$\frac{1}{2} \int_0^{\pi} \log(\sin u) \, du = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log\sin u \, du = \int_0^{\pi/2} \log\sin u \, du$

$2I = \int_0^{\pi/2} \log\sin t \, dt - \int_0^{\pi/2} \log 2 \, dt$

$2I = I - \int_0^{\pi/2} \log 2 \, dt$

$I = -\log 2 \cdot x \Big|_0^{\pi/2} = -\log 2 \cdot \left(\frac{\pi}{2} - 0\right) = -\log 2 \cdot \frac{\pi}{2}$

$\frac{2}{\pi} \cdot \left(-\log 2 \cdot \frac{\pi}{2}\right) = -\log 2 = \log y \Rightarrow y = 2^{-1} = \frac{1}{2}$

Q. Ex 2

Let $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, for $\alpha > 0$. Prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

and that $\Gamma(n) = (n-1)!$, $n \in N^*$

$\int_0^\infty x^\alpha \cdot e^{-x} dx = x^\alpha \cdot (-e^{-x}) \Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx = \alpha \int_0^\infty x^{\alpha-1} \cdot e^{-x} dx$

$u = x^\alpha$ $\quad v' = e^{-x}$

$u' = \alpha \cdot x^{\alpha-1}$ $\quad v = -e^{-x}$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

Assume $\Gamma(k) = (k-1)!$, for some $k \in N$. Show that $\Gamma(k+1) = k!$

$\Gamma(k+1) = k\,\Gamma(k) = k \cdot (k-1)! = k! \Rightarrow \Gamma(n) = (n-1)!,\ \forall n \in N$

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$\cdot dt =$