# Seminar 6

1. Recall the Taylor series for sin and cos:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad \forall x \in \mathbb{R},$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \quad \forall x \in \mathbb{R}.$$

   (a) Prove that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, $\forall x \in \mathbb{R}$.

   (b) Deduce that
$$x - \frac{x^3}{6} < \sin x < x, \ \forall x > 0$$
   and
$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \ \forall x \in \mathbb{R}.$$

   (c) Prove formally Euler's formula $e^{ix} = \cos x + i \sin x$.

2. (a) For $\alpha \in \mathbb{R}$ and $|x| < 1$, prove the generalized binomial expansion (binomial series)
$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{k} x^k, \quad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}, \quad \binom{\alpha}{0} = 1.$$

   (b) Find the first four terms in the binomial series of $\sqrt{1+x}$ and $1/\sqrt{1+x}$.

3. Find the MacLaurin series and its radius of convergence for the following functions:

   (a) $a^x$, $a > 0$.

   (b) $(1+x)\ln(1+x)$.

   (c) $\sin^2(x)$.

   (d) $\arctan x$.

4. For each function $f : \mathbb{R} \to \mathbb{R}$ given below check that $f'(0) = 0$ and find the first $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$. Then, deduce whether 0 is a local extremum point of $f$ or not; in the affirmative, specify if 0 is a global extremum point or just a local one.

   (a) $f(x) = e^x + e^{-x} - x^2$.

   (b) $f(x) = \cos(x^2)$.

   (c) $f(x) = 6\sin x - 6x + x^3$.