# Seminar 2

- (Ratio test) Let $(x_n)$ be a sequence with positive terms s.t. $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell \in \overline{\mathbb{R}}$. If $\ell < 1$, then $\lim_{n\to\infty} x_n = 0$. If $\ell > 1$, then $\lim_{n\to\infty} x_n = \infty$.
- (Stolz-Cesàro) Let $(a_n), (b_n)$ be s.t. (i) $a_n \to 0$ and $b_n \to 0$ with $(b_n)$ decreasing; or (ii) $b_n \to \infty$ with $(b_n)$ increasing. If $\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$.

1. Prove using the $\varepsilon$-definition that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.
2. Study if the sequence $(x_n)$ is bounded, monotone, and convergent, for each of the following:

   (a) $x_n = \sqrt{n+1} - \sqrt{n}$. (b) $x_n = \frac{1}{1\cdot 2} + \ldots + \frac{1}{n(n+1)}$. (c) $x_n = \frac{2^n}{n!}$.

3. Find the limit for each of the following sequences:

   (a) $\sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big)$.
   (b) $\frac{2^n + (-1)^n}{3^n}$.
   (c) $\sqrt[n]{n}$.
   (d) $\frac{(an+1)^2}{4n^2 - 2n + 1}, a \in \mathbb{R}$.
   (e) $\big(a_1^n + a_2^n + \ldots + a_k^n\big)^{\frac{1}{n}}$ with $a_i > 0$.

4. Consider the sequence $(e_n)$ given by $e_n = \big(1 + \dfrac{1}{n}\big)^n$. Prove that $(e_n)$ is increasing and bounded, hence convergent. Its limit is denoted by $e$.
5. Find the limit for each of the following sequences: (a) $\big(\frac{2n+1}{2n-1}\big)^n$. (b) $n\big(\ln(n+2) - \ln(n+1)\big)$.
6. (a) Let $(x_n)$ be convergent. What can you say about the sequence $(a_n)$ of averages,

   $$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}?$$

   Can the averages converge, but the sequence not?

   (b) Compare $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ with $n$ and $\ln n$ by taking the ratio, respectively.

7. Let $(x_n)$ be a sequence such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$. Prove that $\lim_{n\to\infty} \sqrt[n]{x_n} = \ell$.
8. Find the limit for each of the following sequences: (a) $\frac{n}{\sqrt[n]{n!}}$. (b) $\frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}$, $p \in \mathbb{N}$.