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Mathematical Analysis
Homework 1
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Q: Ex 1

Let $a, b \in R$ with $a > 0$. If $S \subset R$ is nonempty and bounded above, prove that $\sup_{x \in S}(ax + b) = a \sup(S) + b$

1. Since S is nonempty and bounded above, $\exists \sup(S) \in R$
2. $\forall x \in S$, $x \leq \sup(S) \mid \cdot a$

$ax \leq a \cdot \sup(S) \mid + b$

$ax + b \leq a \cdot \sup(S) + b \Rightarrow a \cdot \sup(S) + b \in ub(ax+b)$ (1)

3. Let $u = \{ub(ax+b) \mid x \in S, a > 0\}$

$ax + b \leq u$, $\forall x \in S \mid -b$

$ax \leq u - b \mid : a$

$x \leq (u-b)/a \Rightarrow (u-b)/a \in ub(S)$

$\sup(S) \leq (u-b)/a$

$a \cdot \sup(S) + b \leq u$ (2)

(1), (2) $\Rightarrow a \cdot \sup(S) + b$ is the least upper bound of $ax + b \Rightarrow$

$\Rightarrow \sup_{x \in S}(ax+b) = a \cdot \sup(S) + b$

Q: Ex 2

Let $a, b \in R$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$

Let $a \neq b$

1. $d_1 \rightarrow$ distance from a to b

$d_1 = |a - b|$

2. Let $\varepsilon = d_1/2 - 1$

$U = (a - d_1/2 + 1, a + d_1/2 - 1)$

$V = (b - d_1/2 + 1, b + d_1/2 - 1)$

3. $d_2 \rightarrow$ distance from U to V

$d_2 = |b - a| - 2\varepsilon = |b - a| - \frac{2|a-b|}{2} + 2 = |b-a| - |a-b| + 2 \Rightarrow$

$\Rightarrow \forall a, b \in R$, $\varepsilon = d_1/2 - 1$, $d_2 \geq 2 \Rightarrow U \cap V = \emptyset$, $\exists U \in \mathcal{V}(a), V \in \mathcal{V}(b)$

Q: Ex 3

Let $A = (0, 1) \cap Q$. Show rigorously (using definitions) that $\inf A = 0$, $\sup A = 1$, $\text{int} A = \emptyset$ and $\text{cl} A = [0, 1]$

1. $\inf(A) = \max(lb(A))$

$lb(A) = (-\infty, 0]$, $lb(A) = \{x \in R \mid x \leq a, \forall a \in A\}$

$\inf(A) = 0$

2. $\sup(A) = \min(ub(A))$

$ub(A) = [1, +\infty)$, $ub(A) = \{x \in R \mid x \geq a, \forall a \in A\}$

$\sup(A) = 1$

3. $\text{int}(A) = \{x \in R \mid \exists V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

$A = (0, 1) \cap Q \in Q$. Every open interval around $\forall x \in A$ contains both rational and irrational numbers. $\Rightarrow \text{int}(A) = \emptyset$

4. $\text{cl}(A) = \{x \in R \mid \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset\}$

$\forall x \in A$, $\exists V \in \mathcal{V}(x)$ s.t. it contains rational numbers from $(0, 1)$. Also, 0 and 1 are also part of $\text{cl}(A)$, because $\forall \varepsilon > 0$, $(0, \varepsilon)$ contains rational numbers from A and $(1 - \varepsilon, 1)$ also contains rational numbers from A. $\Rightarrow \text{cl}(A) = [0, 1]$