Seminar 10

1. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for \mathbb{R}^3 .

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of \mathbb{R}^3 , $B' = (v'_1, v'_2) = ((1, 1), (1, -2))$ of \mathbb{R}^2 and let $E' = (e'_1, e'_2)$ be the canonical basis of \mathbb{R}^2 . Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

3. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ be defined by

$$f(e_1) = (1, 2, 3, 4), f(e_2) = (4, 3, 2, 1), f(e_3) = (-2, 1, 4, 1)$$

on the elements of the canonical basis of \mathbb{R}^3 . Determine:

- (i) f(v) for every $v \in \mathbb{R}^3$.
- (ii) the matrix of f in the canonical bases.
- (iii) a basis and the dimension of Ker f and Im f.
- **4.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of \mathbb{R}^4 :

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

- (i) Show that $v = (1, 4, 1, -1) \in Ker f$ and $v' = (2, -2, 4, 2) \in Im f$.
- (ii) Determine a basis and the dimension of Ker f and Im f.
- (iii) Define f.
- **5.** Consider the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ and its bases $E = (1, X, X^2)$ and $B = (1, X 1, X^2 + 1)$. Consider $\varphi \in End_{\mathbb{R}}(\mathbb{R}_2[X])$ defined by

$$\varphi(a_0 + a_1X + a_2X^2) = (a_0 + a_1) + (a_1 + a_2)X + (a_0 + a_2)X^2.$$

Determine the matrices $[\varphi]_E$ and $[\varphi]_B$.

- **6.** In the real vector space \mathbb{R}^2 consider the bases $B=(v_1,v_2)=((1,2),(1,3))$ and $B'=(v_1',v_2')=((1,0),(2,1))$ and let $f,g\in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B=\begin{pmatrix} 1 & 2\\ -1 & -1 \end{pmatrix}$ and $[g]_{B'}=\begin{pmatrix} -7 & -13\\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f\circ g]_{B'}$.
 - 7. Consider the endomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of \mathbb{R}^2 and show that f is an automorphism.

8. Let V be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine |V|, $|End_K(V)|$ and $|Aut_K(V)|$.

[Hint: use the isomorphism between $End_K(V)$ and $M_n(K)$, where $dim_K(V) = n$.]