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Mathematical Analysis

Homework 6

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Q: Ex 1

Using Taylor series, compute the following limits:

a) $\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{x^5}$

$f(x) = \sin(x) \quad f(0) = 0$

$f'(x) = \cos(x) \quad f'(0) = 1$

$f''(x) = -\sin(x) \quad f''(0) = 0$

$f'''(x) = -\cos(x) \quad f'''(0) = -1$

$f''''(x) = \sin(x) \quad f'''(0) = 0$

$f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot x^2/2! + f'''(0)\cdot x^3/3! + f''''(0)\cdot x^4/4! \ldots$

$f(x) = x - x^3/3! + x^5/5! \ldots$

$\lim_{x\to 0} \frac{(x - x^3/3! + x^5/5! \ldots) - x + x^3/6}{x^5} = \lim_{x\to 0} \frac{(x^5/5! - x^7/7! \ldots)}{x^5} =$

$= \lim_{x\to 0} (1/5! - x^2/7! \ldots) = \frac{1}{120}$

b) $\lim_{x\to 0} \frac{e^{x^2} - \cos(x) - 3x^2/2}{x^4}$

$f(a) = e^{x^2} \quad f(0) = 1$

$f'(a) = 2xe^{x^2} \quad f'(0) = 0$

$f''(a) = 2e^{x^2} + 4x^2e^{x^2} \quad f''(0) = 2$

$f'''(a) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} \quad f'''(0) = 0$

$f''''(a) = 4e^{x^2} + 8x^2e^{x^2} + 8e^{x^2} + 16x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2} \quad f''''(0) = 12$

$f(a) = f(0) + f'(0)\cdot x + f''(0)\cdot x^2/2! + f'''(0)\cdot x^3/3! + f''''(0)\cdot x^4/4!$

$f(a) = 1 + x^2 + x^4/2 \ldots$

$f(b) = \cos(x) \quad f(0) = 1$

$f'(b) = -\sin(x) \quad f'(0) = 0$

$f''(b) = -\cos(x) \quad f''(0) = -1$

$f'''(b) = \sin(x) \quad f'''(0) = 0$

$f''''(b) = \cos(x) \quad f''''(0) = 1$

$f(b) = f(0) + f'(0)\cdot x + f''(0)\cdot x^2/2! + f'''(0)\cdot x^3/3! + f''''(0)\cdot x^4/4! \ldots$

$f(b) = 1 - x^2/2 + x^4/24 \ldots$

$\lim_{x\to 0} \frac{(1 + x^2 + x^4/2 \ldots) - (1 - x^2/2 + x^4/24 \ldots) - 3x^2/2}{x^4} =$

$= \lim_{x\to 0} \frac{(x^4/2 \ldots) - (x^4/24 \ldots)}{x^4} = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$

Q: Ex 2

Prove that the Taylor series of $\ln(1+x)$ around 0 is

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$

$f(x) = \ln(1+x) \quad f(0) = 0$

$f'(x) = 1/(1+x) \quad f'(0) = 1$

$f''(x) = -1/(1+x)^2 \quad f''(0) = -1$

$f'''(x) = 2/(1+x)^3 \quad f'''(0) = 2$

$f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot x^2/2! + f'''(0)\cdot x^3/3! \ldots$

$f(x) = x - x^2/2 + x^3/3 \ldots$

$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \cdot (-1)^{n+1}$

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Q. Ex 3

Using Taylor series, prove that the forward difference $(f(x+h)-f(x))/h$ approximates the derivative $f'(x)$ with an error of order h (first order approximation), i.e. $f'(x) = \frac{f(x+h)-f(x)}{h} + O(h)$, and that the centered difference $(f(x+h)-f(x-h))/2h$ approximates the derivative $f'(x)$ with an error of order h^2 (second order approximation), i.e.
$f'(x) = \frac{f(x+h)-f(x-h)}{2h} + O(h^2)$. Here the Big Oh notation $e_1(h) = O(e_2(h))$ means that $\lim_{h \to 0} \frac{e_1(h)}{e_2(h)} \in (0, \infty)$

1) $f(x+h) = f(x) + f'(x) \cdot h + f''(x) \cdot h^2/2! + f'''(x) \cdot h^3/3! \dots$

$\frac{f(x+h)-f(x)}{h} = \frac{h f'(x) + f''(x) \cdot h^2/2! + f'''(x) \cdot h^3/3!}{h} =$

$= f'(x) + f''(x) \cdot h/2! + f'''(x) \cdot h^2/3! \dots = f'(x) + O(h) \Rightarrow$

$\Rightarrow$ the error is of order h.

2) $f(x-h) = f(x) + f'(x) \cdot (-h) + f''(x) \cdot h^2/2! + f'''(x) \cdot (-h^3)/3! \dots$

$\frac{f(x+h)-f(x-h)}{2h} = \frac{(f(x) + f'(x) \cdot h + f''(x) \cdot h^2/2! \dots) - (f(x) + f'(x) \cdot (-h) + f''(x) \cdot h^2/2!)}{2h} =$

$= \frac{2 f'(x) \cdot h \dots}{2h} = f'(x) + O(h^2) \Rightarrow$ the error is of order h^2.

Explanation of Python code:

When h becomes extremely small, the errors begin to increase due to the loss of numerical significance. The difference $f(x+h) - f(x)$ becomes very small, and it's calculation is affected by truncation errors and the precision of floating-point numbers.

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