Seminar 4

1. Study the convergence and the absolute convergence of the following series:

   (a) $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$.

   (b) $\displaystyle\sum_{n\geq 1} (-1)^n \sin\frac{1}{n}$.

   (c) $\displaystyle\sum_{n\geq 1} \frac{\sin n}{n^2}$.

2. Prove by differentiating the geometric series that, for $|x| < 1$,

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \qquad \sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

3. Prove by integrating the geometric series that, for $|x| < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \qquad \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = \ln(1+x).$$

4. Prove that $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \arctan x$, for $x \in [-1, 1]$.

5. Find the radius of convergence and the convergence set for each of the following series:

   (a) $\displaystyle\sum_{n\geq 0} \frac{x^n}{2^n}$.

   (b) $\displaystyle\sum_{n\geq 0} 2^n (x-1)^n$.

   (c) $\displaystyle\sum_{n\geq 1} \frac{(x-2)^n}{(n+1)3^n}$.

   (d) $\displaystyle\sum_{n\geq 0} n!x^n$.

   (e) $\displaystyle\sum_{n\geq 1} \frac{(x-1)^n}{n^p}$, $p > 0$.

   (f) $\displaystyle\sum_{n\geq 1} \frac{(-1)^n x^n}{\sqrt{n}}$.