Seminar 10

- Chain rule: $D(f \circ g)(x) = Df\big(g(x)\big)Dg(x)$.
- Directional derivative: $Df_v(x) := \lim_{h \to 0} \dfrac{f(x+hv) - f(x)}{h} = \nabla f(x) \cdot v$, for $v \in \mathbb{R}^n$.
- Direction of steepest ascent: $\nabla f(x)$. Direction of steepest descent: $-\nabla f(x)$.
- Hessian matrix $H(x) = \big(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\big)_{i,j=\overline{1,n}}$.
- Gradient perpendicular to the level set. Tangent: $\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$.

1. For each of the following functions, compute $\dfrac{\mathrm{d}f}{\mathrm{d}t}$ directly and using the chain rule:

   (a) $f(x, y) = \ln(x^2 + y^2)$,
   $x = t,\ y = t^2$.

   (b) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$,
   $x = \cos t,\ y = \sin t,\ z = t > 0$.

2. Let $f(u, v, w) = u^2 + v^2 - w$ and $u(x, y, z) = x^2 y$, $v(x, y, z) = y^2$, $w(x, y, z) = e^{-xz}$. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ directly and using the chain rule.

3. For $f : \mathbb{R}^2 \to R, f(x, y) = x^2 + xy$ find:

   (a) the gradient of f and the direction of steepest descent at the point $(1, 0)$.

   (b) the directional derivative at the point $(1, 0)$ in the direction of $e_1 + e_2 = (1, 1)$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \frac{1}{2}\|x\|^2$. Find the gradient of f. Find the directional derivative $D_v f(x)$ in two ways: using the definition and using the gradient.

5. Let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a diagonal $n \times n$ matrix and consider the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2} x^T D x$. Prove that $\nabla f(x) = Dx$ and $H(x) = D$.

6. Find the equation of the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at an arbitrary point (x_0, y_0).