Seminar 5

1. Find the accumulation points for each of the following sets:

$$[0,1)\cup\{2\},\ \mathbb{Z},\ \mathbb{Q},\ \{0.1, 0.11, \ldots\}.$$

2. Find a function $f:\mathbb{R}\to\mathbb{R}$ that is discontinuous everywhere, with $|f|$ continuous everywhere.

3. Prove that a continuous function $f:[a,b]\to[a,b]$ has at least one fixed point $x^*=f(x^*)$.

4. Study the continuity and the differentiability of the functions f and f', where $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0.\end{cases}$$

5. Prove (from scratch) that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$. Then prove that $(\sin x)'=\cos x$, $(\cos x)'=-\sin x$.

6. Compute the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{\lfloor x\rfloor}{x}$.

(b) $\lim\limits_{x\to\infty} x\big(\ln(x+2)-\ln(x+1)\big)$.

(c) $\lim\limits_{x\to 0}\big(\cos x\big)^{\frac{1}{x^2}}$.

(d) $\lim\limits_{\substack{x\to 0\\ x>0}} x^x$.

(e) $\lim\limits_{\substack{x\to 0\\ x>0}} (\sin x)^x$.

(f) $\lim\limits_{x\to\infty} x\big((1+\frac{1}{x})^x-e\big)$.

7. Find the n^{th} derivative of the following functions:

(a) $f:(-1,\infty)\to\mathbb{R},\ f(x)=\ln(1+x)$.

(b) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin x$.

(c) $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^2\sin x$.

(d) $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{2x}x^3$.