16.11.24

Mathematical Analysis
Homework 6
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Q: Ex 1

Using Taylor series, compute the following limits:

a) $\lim_{x \to 0} \frac{\sin(x) - x + x^3/6}{x^5}$

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$f''''(x) = \sin(x)$ $\quad f'''(0) = 0$

$f(x) = f(0) + f'(0) \cdot x + f''(0) \cdot x^2/2! + f'''(0) \cdot x^3/3! + f''''(0) \cdot x^4/4! \ldots$

$f(x) = x - x^3/3! + x^5/5! \ldots$

$\lim_{x \to 0} \frac{(x - x^3/3! + x^5/5! \ldots) - x + x^3/6}{x^5} = \lim_{x \to 0} \frac{(x^5/5! - x^7/7! \ldots)}{x^5} =$

$= \lim_{x \to 0} (1/5! - x^2/7! \ldots) = \frac{1}{120}$

b) $\lim_{x \to 0} \frac{e^{x^2} - \cos(x) - 3x^2/2}{x^4}$

$f(a) = e^{x^2}$ $\quad f(0) = 1$

$f'(a) = 2xe^{x^2}$ $\quad f'(0) = 0$

$f''(a) = 2e^{x^2} + 4x^2e^{x^2}$ $\quad f''(0) = 2$

$f'''(a) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}$ $\quad f'''(0) = 0$

$f''''(a) = 4e^{x^2} + 8x^2e^{x^2} + 8e^{x^2} + 16x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2}$ $\quad f''''(0) = 12$

$f(a) = f(0) + f'(0) \cdot x + f''(0) \cdot x^2/2! + f'''(0) \cdot x^3/3! + f''''(0) \cdot x^4/4!$

$f(a) = 1 + x^2 + x^4/2 \ldots$

$f(b) = \cos(x)$ $\quad f(0) = 1$

$f'(b) = -\sin(x)$ $\quad f'(0) = 0$

$f''(b) = -\cos(x)$ $\quad f''(0) = -1$

$f'''(b) = \sin(x)$ $\quad f'''(0) = 0$

$f''''(b) = \cos(x)$ $\quad f''''(0) = 1$

$f(b) = f(0) + f'(0) \cdot x + f''(0) \cdot x^2/2! + f'''(0) \cdot x^3/3! + f''''(0) \cdot x^4/4! \ldots$

$f(b) = 1 - x^2/2 + x^4/24 \ldots$

$\lim_{x \to 0} \frac{(1 + x^2 + x^4/2 \ldots) - (1 - x^2/2 + x^4/24 \ldots) - 3x^2/2}{x^4} =$

$= \lim_{x \to 0} \frac{(x^4/2 \ldots) - (x^4/24 \ldots)}{x^4} = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$

Q: Ex 2

Prove that the Taylor series of $\ln(1+x)$ around 0 is

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$

$f(x) = \ln(1+x)$ $\quad f(0) = 0$

$f'(x) = 1/(1+x)$ $\quad f'(0) = 1$

$f''(x) = -1/(1+x)^2$ $\quad f''(0) = -1$

$f'''(x) = 2/(1+x)^3$ $\quad f'''(0) = 2$

$f(x) = f(0) + f'(0) \cdot x + f''(0) \cdot x^2/2! + f'''(0) \cdot x^3/3! \ldots$

$f(x) = x - x^2/2 + x^3/3 \ldots$

$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \cdot (-1)^{n+1}$

Q. Ex 3

Using Taylor series, prove that the forward difference $(f(x+h)-f(x))/h$ approximates the derivative $f'(x)$ with an error of order $h$ (first order approximation), i.e. $f'(x) = \frac{f(x+h)-f(x)}{h} + O(h)$, and that the centered difference $(f(x+h)-f(x-h))/2h$ approximates the derivative $f'(x)$ with an error of order $h^2$ (second order approximation), i.e.
$f'(x) = \frac{f(x+h)-f(x-h)}{2h} + O(h^2)$. Here the Big Oh notation $e_1(h) = O(e_2(h))$ means that $\lim_{h \to 0} \frac{e_1(h)}{e_2(h)} \in (0, \infty)$

1) $f(x+h) = f(x) + f'(x)\cdot h + f''(x)\cdot h^2/2! + f'''(x)\cdot h^3/3! \ldots$

$$\frac{f(x+h)-f(x)}{h} = \frac{h f'(x) + f''(x)\cdot h^2/2! + f'''(x)\cdot h^3/3!}{h} =$$

$$= f'(x) + f''(x)\cdot h/2! + f'''(x)\cdot h^2/3! \ldots = f'(x) + O(h) \Rightarrow$$

$\Rightarrow$ the error is of order $h$.

2) $f(x-h) = f(x) + f'(x)\cdot(-h) + f''(x)\cdot h^2/2! + f'''(x)\cdot(-h^3)/3! \ldots$

$$\frac{f(x+h)-f(x-h)}{2h} = \frac{(f(x)+f'(x)\cdot h + f''(x)\cdot h^2/2! \ldots) - (f(x)+f'(x)\cdot(-h)+f''(x)\cdot h^2/2!)}{2h} =$$

$$= \frac{2f'(x)\cdot h \ldots}{2h} = f'(x) + O(h^2) \Rightarrow$$ the error is of order $h^2$.

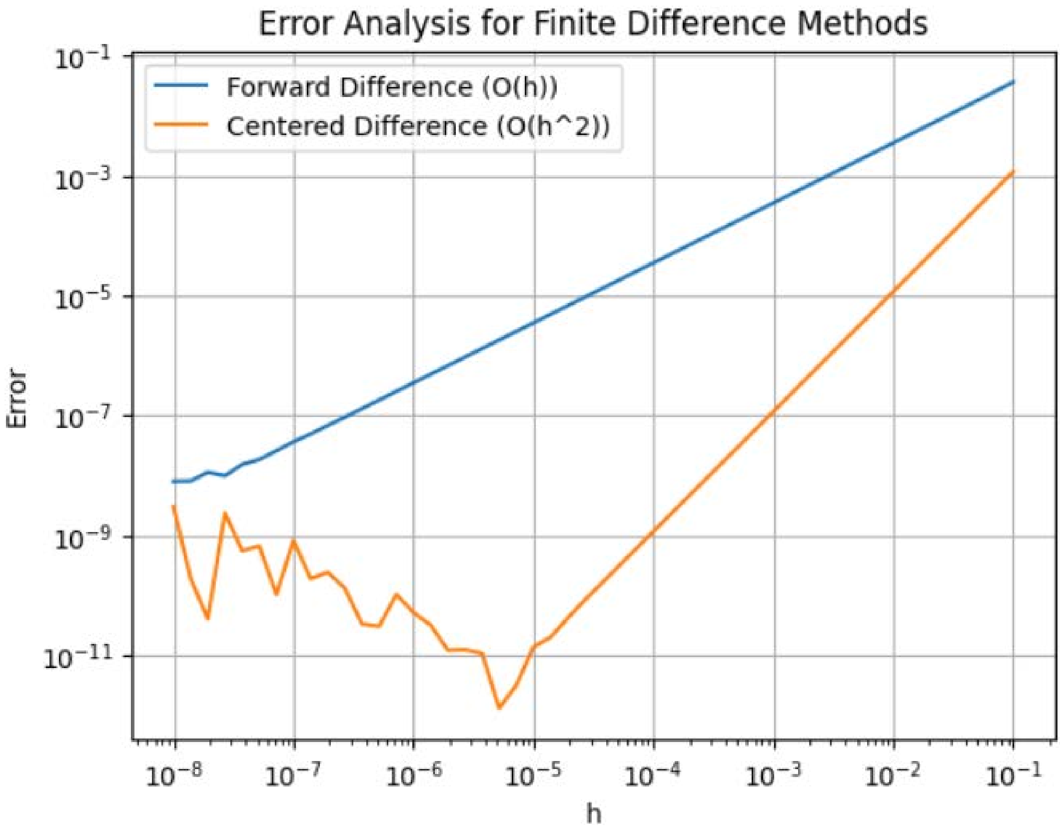
Explanation of Python code:
When $h$ becomes extremely small, the errors begin to increase due to the loss of numerical significance. The difference $f(x+h) - f(x)$ becomes very small, and it's calculation is affected by truncation errors and the precision of floating-point numbers.

Scanat cu CamScanner

hw6.py

```python
import numpy as np
import matplotlib.pyplot as plt

# Definim functia și derivata exactă
def f(x):
    return np.sin(x)

def f_prime_exact(x):
    return np.cos(x)

# Metode de diferentiere
def forward_difference(f, x, h):
    return (f(x + h) - f(x)) / h

def centered_difference(f, x, h):
    return (f(x + h) - f(x - h)) / (2 * h)

# Punctul x și gama de h
x = np.pi / 4
h_values = np.logspace(-8, -1, 50)  # Valori mici ale lui h
errors_forward = []
errors_centered = []

# Calculăm erorile
for h in h_values:
    fd = forward_difference(f, x, h)
    cd = centered_difference(f, x, h)
    exact = f_prime_exact(x)
    errors_forward.append(abs(fd - exact))
    errors_centered.append(abs(cd - exact))
```

```python

# Plotăm erorile
plt.loglog(h_values, errors_forward, label="Forward Difference (O(h))")
plt.loglog(h_values, errors_centered, label="Centered Difference (O(h^2))")
plt.xlabel("h")
plt.ylabel("Error")
plt.legend()
plt.title("Error Analysis for Finite Difference Methods")
plt.grid(True)
plt.show()

```


Error Analysis for Finite Difference Methods
Forward Difference (O(h))
Centered Difference (O(h^2))
Error
h
10^-1
10^-3
10^-5
10^-7
10^-9
10^-11
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1