24.11.24

Mathematical Analysis
Homework 7
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Q: Ex 1

Compute using Riemann sum:

a) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n}}{n^2}$

$\sqrt[n]{e} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n} = \sum\limits_{i=1}^{n} i \cdot e^{i/n}$

$\dfrac{1}{n^2} \cdot \sum\limits_{i=1}^{n} i \cdot e^{i/n} = \dfrac{1}{n} \sum\limits_{i=1}^{n} i/n \cdot e^{i/n}, \quad x_i = \dfrac{i}{n}$

$\lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{i=1}^{n} x_i \cdot e^{x_i} = \int_0^1 x \cdot e^x dx = x \cdot e^x \Big|_0^1 - \int_0^1 e^x dx = e - e^x \Big|_0^1 = e - (e-1) = 1$

$u = x \quad v' = e^x$

$u' = 1 \quad v = e^x$

b) $\lim\limits_{n\to\infty} \sqrt[n]{\sin\dfrac{\pi}{2n} \sin\dfrac{2\pi}{2n} \ldots \sin\dfrac{(n-1)\pi}{2n}} = y$

$\lim\limits_{n\to\infty} \dfrac{1}{n} \cdot \log\left(\sin\dfrac{\pi}{2n} \cdot \sin\dfrac{2\pi}{2n} \ldots \sin\dfrac{(n-1)\pi}{2n}\right) = \log y$

$\lim\limits_{n\to\infty} \dfrac{1}{n} \cdot \left(\log\sin\dfrac{\pi}{2n} + \log\sin\dfrac{2\pi}{2n} + \ldots + \log\sin\dfrac{(n-1)\pi}{2n}\right) = \log y$

$\lim\limits_{n\to\infty} \dfrac{1}{n} \cdot \sum\limits_{i=1}^{n} \log\sin\dfrac{i\pi}{2n}, \quad \dfrac{i}{n} = x$

$\int_0^1 \log\sin\dfrac{\pi x}{2} dx = \dfrac{2}{\pi} \int_0^{\pi/2} \log\sin t \, dt$

$t = x\pi/2$

$dt = \pi/2 \, dx$

$dx = 2dt/\pi$

$\int_0^{\pi/2} \log\sin t \, dt = \int_0^{\pi/2} \log\sin(\pi/2 - t) \, dt = I$

$2I = \int_0^{\pi/2} \log\sin t \, dt + \int_0^{\pi/2} \log\cos t \, dt = \int_0^{\pi/2} \log(\sin t \cos t) \, dt = \int_0^{\pi/2} \log(2\sin t\cos t) - \log 2 \, dt =$

$= \int_0^{\pi/2} \log(\sin 2t) \, dt - \int_0^{\pi/2} \log 2 \, dt$

$\int_0^{\pi/2} \log(\sin 2t) \, dt = \dfrac{1}{2} \int_0^{\pi} \log(\sin t) \, dt$

$2t = u \qquad f(u) = \log(\sin u)$

$2dt = du \qquad f(2\pi - u) = \log\sin(2\pi - u) = \log\sin u$

$dt = du/2 \qquad f(u) = f(2\pi - u)$

$\dfrac{1}{2} \int_0^{\pi} \log(\sin u) \, du = \dfrac{1}{2} \cdot 2 \int_0^{\pi/2} \log\sin u \, du = \int_0^{\pi/2} \log\sin u \, du$

$2I = \int_0^{\pi/2} \log\sin t \, dt - \int_0^{\pi/2} \log 2 \, dt$

$2I = I - \int_0^{\pi/2} \log 2 \, dt$

$I = -\log 2 \cdot x \Big|_0^{\pi/2} = -\log 2 \cdot \left(\dfrac{\pi}{2} - 0\right) = -\log 2 \cdot \dfrac{\pi}{2}$

$\dfrac{2}{\pi} \cdot \left(-\log 2 \cdot \dfrac{\pi}{2}\right) = -\log 2 = \log y \Rightarrow y = 2^{-1} = \dfrac{1}{2}$

Q. Ex 2

Let $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, for $\alpha > 0$. Prove that $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

and that $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}^*$

$$\int_0^\infty x^{\alpha} \cdot e^{-x} dx = x^{\alpha} \cdot (-e^{-x}) \Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx = \alpha \int_0^\infty x^{\alpha-1} \cdot e^{-x} dx$$

$u = x^{\alpha}$ $\quad v' = e^{-x}$

$u' = \alpha \cdot x^{\alpha-1}$ $\quad v = -e^{-x}$

$\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

Assume $\Gamma(k) = (k-1)!$, for some $k \in \mathbb{N}$. Show that $\Gamma(k+1) = k!$

$\Gamma(k+1) = k\, \Gamma(k) = k \cdot (k-1)! = k! \Rightarrow \Gamma(n) = (n-1)!$, $\forall n \in \mathbb{N}$

```python
import numpy as np

def gaussian(x):
    return np.exp(-x**2)

def trapezoidal_rule(function, a, b, N):
    x = np.linspace(a, b, N + 1)  # N+1 points from a to b
    y = function(x)  # Evaluate the function at these points
    h = (b - a) / N  # Step size
    integral = h * (0.5 * y[0] + np.sum(y[1:-1]) + 0.5 * y[-1])
    return integral

# Compute the integral for increasing a values
results = []
exact_value = np.sqrt(np.pi)
a_values = [1, 2, 3, 5, 10]  # Increasing values of a
N = 10000  # Number of intervals for high accuracy

for a in a_values:
    approx = trapezoidal_rule(gaussian, -a, a, N)
    results.append((a, approx, abs(approx - exact_value)))

for a, approx, error in results:
    print(f"a = {a}, Approximation = {approx}, Absolute Error = {error}")
```

```
C:\Users\Maia\PycharmProjects\analysis.hw7\.venv\Scripts\python.exe C:\Users\Maia\PycharmProjects\analysis.hw7\hw7.py
a = 1, Approximation = 1.493648260719795, Absolute Error = 0.27880559018572093
a = 2, Approximation = 1.764162779571175, Absolute Error = 0.008291071334340927
a = 3, Approximation = 1.772414696474615, Absolute Error = 3.915443090085624e-05
a = 5, Approximation = 1.772453850902791, Absolute Error = 2.7249313916399842e-12
a = 10, Approximation = 1.7724538509055163, Absolute Error = 4.440892098500626e-16

Process finished with exit code 0
```