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Mathematical Analysis
Homework 3
Țapordei Maia

Q: Ex 1

a) $\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln\frac{(n-1)(n+1)}{n^2} = \sum_{n\geq 2}\left(\ln\left(\frac{n-1}{n}\right)+\ln\left(\frac{n+1}{n}\right)\right) =$

$= \underbrace{\sum_{n\geq 2} \ln\left(\frac{n-1}{n}\right)}_{\ln\frac{1}{2}+\ln\frac{2}{3}+\ln\frac{3}{4}\dots} + \underbrace{\sum_{n\geq 2} \ln\left(\frac{n+1}{n}\right)}_{\ln\frac{3}{2}+\ln\frac{4}{3}+\ln\frac{4}{5}\dots} \Rightarrow \ln\frac{1}{2}+\ln\frac{2}{3}\cdot\frac{3}{2}+\ln\frac{3}{4}\cdot\frac{4}{3}+\dots+0=$

$= \ln\frac{1}{2}$

b) $\sum_{n\geq 1} \frac{n+1}{3^n} = S$

$S = \sum_{n\geq 1}\frac{n}{3^n} + \sum_{n\geq 1}\frac{1}{3^n}$

$\sum_{n\geq 1}\frac{n}{3^n} = \frac{1}{3}\sum_{n\geq 0}\frac{n+1}{3^n} = \frac{1}{3}\left(1+\sum_{n\geq 1}\frac{n+1}{3^n}\right) = \frac{1}{3}+\frac{1}{3}\cdot\sum_{n\geq 1}\frac{n+1}{3^n}$

$\sum_{n\geq 1}\frac{1}{3^n} = \frac{1}{3}\cdot\frac{1}{1-1/3} = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}$

$S = \frac{1}{3}+\frac{S}{3}+\frac{1}{2}$

$S = \frac{5}{6}+\frac{S}{3}$

$18S = 15+6S$

$12S = 15$

$S = \frac{5}{4}$

c) $\sum_{n\geq 1}\frac{n}{n^4+n^2+1} = \sum_{n\geq 1}\frac{n}{(n^2+1)^2-n^2} = \sum_{n\geq 1}\frac{n}{(n^2-n+1)(n^2+n+1)}$

$\frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{A}{n^2-n+1}+\frac{B}{n^2+n+1}$

$n = A(n^2+n+1)+B(n^2-n+1)$

$n = An^2+An+A+Bn^2-Bn+B$

$\left.\begin{array}{l} A+B=0 \\ A-B=1 \end{array}\right\} \Rightarrow \begin{array}{l} 2A=1 \\ A=1/2 \quad B=-1/2 \end{array}$

$\sum_{n\geq 1}\frac{n}{(n^2-n+1)(n^2+n+1)} = \sum_{n\geq 1}\frac{1}{2(n^2-n+1)} - \sum_{n\geq 1}\frac{1}{2(n^2+n+1)} =$

$= \frac{1}{2}\left(\underbrace{\sum_{n\geq 1}\frac{1}{n^2-n+1}}_{1+\frac{1}{3}+\frac{1}{7}+\dots} - \underbrace{\sum_{n\geq 1}\frac{1}{n^2+n+1}}_{\left(\frac{1}{3}+\frac{1}{7}+\dots\right)}\right) = \frac{1}{2}$

Q: Ex 2

a) $\sum_{n\geq 1}\frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{N}$

Ratio test:

$\frac{x^{n+1}}{(n+1)^p}\cdot\frac{n^p}{x^n} = \frac{x\cdot n^p}{(n+1)^p}$

$\lim_{n\to\infty}\frac{x\cdot n^p}{(n+1)^p} = x$, $x>0$

If $x<1 \Rightarrow$ the series converges

If $x>1 \Rightarrow$ the series diverges

If $x=1 \Rightarrow \sum_{n\geq 1}\frac{1}{n^p}$ which converges $\forall p\in\mathbb{N}$

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b) $\sum_{n\geq 2} \frac{1}{(\ln(n))^{\ln(n)}}$

Condensation method:

$\sum_{n\geq 2} \frac{2^n}{(\ln 2^n)^{\ln 2^n}} = \sum_{n\geq 2} \frac{2^n}{(n\ln 2)^{n\ln 2}} = \sum_{n\geq 2} \frac{2^n}{((n\ln 2)^{\ln 2})^n}$

$(n\ln 2)^{\ln 2} \to \infty \Rightarrow \exists n, (n\ln 2)^{\ln 2} \geq 4 \Rightarrow$

$\Rightarrow \sum_{n\geq 2} \frac{2^n}{((n\ln 2)^{\ln 2})^n}$ is bounded above by $\frac{2^n}{4^n} = \frac{1}{2^n} \Rightarrow$

$\Rightarrow$ the series $\sum_{n\geq 2} \frac{1}{(\ln(n))^{\ln(n)}}$ is convergent

c) $\sum_{n\geq 1} (\sqrt[n]{n} - 1)$

Ratio test:

$\lim_{n\to\infty} \frac{\sqrt[n+1]{n+1} - 1}{\sqrt[n]{n} - 1} = \lim_{n\to\infty} \frac{n^{1/(n+1)}}{n^{1/n}}$

$n+1 > n$

$\frac{1}{n+1} < \frac{1}{n}$

$\lim_{n\to\infty} \frac{n^{1/(n+1)}}{n^{1/n}} = 0 < 1 \Rightarrow$ the series is convergent

d) $\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot \ldots \cdot 2n} \cdot \frac{1}{2n+1} = \sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots (2n-1)}{2\cdot 4\cdot \ldots (2n+1)} \cdot \frac{1}{2n}$

$\sum_{n\geq 1} \frac{1}{2n} = \frac{1}{2} \sum_{n\geq 1} \frac{1}{n} \to$ diverges (1)

$a_n = \frac{3\cdot 5\cdot \ldots (2n-1)}{2\cdot 4\cdot \ldots (2n+1)} \cdot \frac{1}{2n} \qquad b_n = \frac{1}{2n}$

Comparison test II:

$\frac{a_n}{b_n} = \frac{3\cdot 5 \ldots (2n-1)}{2\cdot 4\cdot \ldots (2n+1)}$

$\lim_{n\to\infty} \frac{3\cdot 5\cdot \ldots (2n-1)}{2\cdot 4\cdot \ldots (2n+1)} = 1 \in (0,\infty)$ (2)

(1), (2) $\Rightarrow$ the series diverges.

Q: Ex 3

1) Number of sides at iteration n:

At each iteration, each side is increased by 4.

1. 3 2. $3\cdot 4$ 3. $3\cdot 4\cdot 4$ ...

Formula: $3\cdot 4^n$, $n\geq 0$

2) Limit of perimeter

At each iteration, the perimeter increases by 4/3

1. 3 2. $3\cdot 4/3$ 3. $3\cdot 4/3\cdot 4/3$ ...

Formula: $3\cdot (4/3)^n$, $n\geq 0$

$\lim_{n\to\infty} 3\cdot \left(\frac{4}{3}\right)^n = +\infty$

3) Limit of area

$A_0 = \frac{\sqrt{3}}{4}$

Area of each new triangle: $\frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^n}\right)^2 = \frac{\sqrt{3}}{4\cdot 9^n}$

Total area added at each iteration: $3\cdot 4^{n-1} \cdot \frac{\sqrt{3}}{4\cdot 9^n} = A_n$

$A = A_0 + \sum_{n\geq 1} 3\cdot 4^{n-1} \cdot \frac{\sqrt{3}}{4\cdot 9^n}$

$\sum_{n\geq 1} A_n = \frac{3\sqrt{3}}{4} \cdot \frac{1/9}{1-4/9} = \frac{\sqrt{3}}{4} \cdot \frac{3}{5} = \frac{3\sqrt{3}}{20}$

$A = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$

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