

Seminar 8

1. Prove that for any $x, y \in \mathbb{R}^n$ the following identities hold:
 - (a) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (the parallelogram identity).
 - (b) $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$.
2. Find the orthogonal projection of a vector $v \in \mathbb{R}^2$ onto a vector $a \in \mathbb{R}^2$.
3. Show that $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is a rotation matrix with angle θ in \mathbb{R}^2 .
4. Consider the p -norm $\|x\|_p := (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$, $p \geq 1$ and $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$. Draw the unit ball in \mathbb{R}^2 for the p -norm with $p \in \{1, 2, \infty\}$.
5. Find the interior, the closure and the boundary for each of the following sets:
 - (a) $[0, 1) \times (1, 2]$.
 - (b) $\{(x, y) \in \mathbb{R}^2 \mid |x| < |y|\}$.
 - (c) $\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$.
 - (d) $\{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 \leq 1, x \leq 1\}$.
6. Draw the level sets $L_c = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$ and sketch the graph for the following:
 - (a) $f(x, y) = x + y$ and $c \in \{0, \pm 1\}$.
 - (b) $f(x, y) = 2x^2 + y^2$ and $c \in \{0, 1, 4\}$.
 - (c) $f(x, y) = \sqrt{x^2 + y^2}$ and $c \in \{0, 1, 2\}$.
 - (d) $f(x, y) = x^2 - y^2$ and $c \in \{0, \pm 1\}$.