Irreducible Representations of the Symmetric Group and Other Topics from Group Theory and Combinatorics

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Outline

What is a Group?

A **group** is a set G together with an operator $\cdot$ which satisfy the following properties:

1. For all $a, b \in G,\ a \cdot b \in G$.
2. For all $a, b,$ and $c \in G,\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. There exists a unique element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$.
4. For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

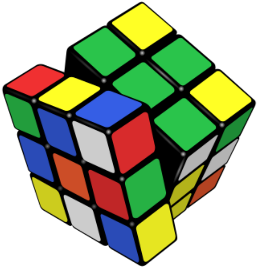
These properties are called the **group axioms**.

What is a Group?

Examples:

- The integers under addition: $(\mathbb{Z}, +)$.
- The real numbers, without zero, under multiplication: $(\mathbb{R}_{\neq 0}, \cdot)$.
- The Rubik's Cube Group: $(G, \cdot)$, where G is the set of cube moves, and $\cdot$ denotes composition.

  G contains 43,252,003,274,489,856,000 elements.



A Rubik's Cube

image source: https://en.wikipedia.org/wiki/Rubik%27s_Cube_group

Permutations and the Symmetric Group

Let A be a finite set with n elements.

A **permutation** σ on A is a bijective map $\sigma : A \to A$.

σ can be thought of as a rearrangement of the elements of A.

The **symmetric group**, denoted S_n, is the group of permutations of n objects under composition.

Permutation Composition Example

$$\sigma = (1432) \qquad\qquad \tau = (1)(24)(3)$$

$$\begin{aligned}\sigma \circ \tau &= (1432)(1)(24)(3)\\ &= (14)(23)\\ \tau \circ \sigma &= (1)(24)(3)(1432)\\ &= (12)(34)\end{aligned}$$

Group Actions and Orbits

A **group action** ρ of G on a set X is a map $\rho : G \times X \to X$ such that

$$\rho(e, x) = x \ \forall x \in X,$$
$$\rho(a \cdot b, x) = \rho(a, \rho(b, x)) \ \forall x \in X, \ \forall a, b \in G.$$

The **orbit** of an element $x \in X$ is

$$G \cdot x = \{\rho(g, x) : g \in G\}$$

The orbits of the elements of X form a partition of X.

Conjugacy Classes

Conjugation is a map $\rho : G \times G \to G$ defined by:

$$\rho(a, b) = a \cdot b \cdot a^{-1} \quad \forall a, b \in G.$$

Conjugation is an example of a group action; here a group is acting on itself. The orbits of this group action are the conjugacy classes.

Two elements of S_n are conjugate if and only if they have the same cycle type.

Homomorphisms

A **homomorphism** ϕ is a map from a group G into a group H such that

$$\phi(a \cdot b) = \phi(a)\phi(b) \quad \forall a, b \in G.$$

If ϕ is bijective, it is an **isomorphism**.

The General Linear Group of A Vector Space

Let V be a vector space.

The **general linear group** of V, denoted $GL(V)$, is the set of invertible linear operators on V, under composition.

Representations on a Vector Space

A **representation** (ϕ, V) of a group G on a vector space V is a homomorphism $\phi : G \to GL(V)$.

Representations allow us to think about the elements of a group as invertible linear operators.

The **dimension** of (ϕ, V) is the dimension of V.

Subrepresentations

Let G be a group, and (ϕ, V) be a representation of G.

$(\phi|_W, W)$ is a **subrepresentation** of (ϕ, V) if

W is a subspace of V, and

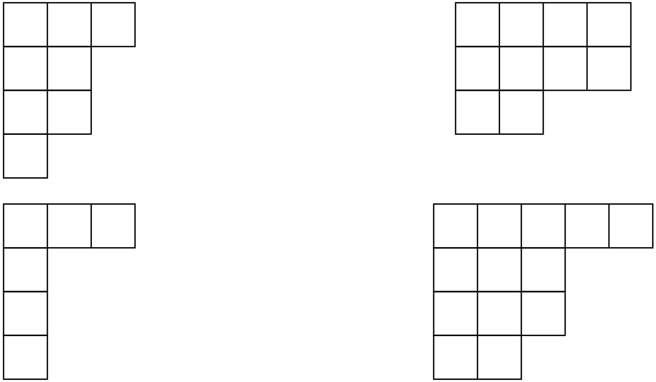
W is closed under the action of G.

Irreducible Representations

A nonzero representation is **irreducible** if it has no proper, nonzero subrepresentations.

Young Diagrams

Young diagrams look like this:



Standard Young Tableaux

A **standard Young tableaux** is a filling of a Young diagram with the integers 1, 2, . . . , n such that the sequence along each row and down each column is increasing.

Here are some examples of standard Young tableaux:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \cline{1-3}\end{array}$$

The Following are Equivalent

For $n \in \mathbb{Z}^+$, the following are equivalent:

- The number of irreducible representations of S_n.
- The number of distinct cycle types of elements of S_n.
- The number of conjugacy classes of S_n.
- The number of Young diagrams with n boxes.

There is no known closed-form expression for this quantity.

Specht Modules

Specht modules are the irreducible representations of the symmetric group, which are indexed by Young diagrams.

Given the symmetric group on n points, for each Young diagram of n boxes, it is possible to compute a basis for that Specht module.

Hook Length Formula

Given a Young diagram λ, the following are equivalent:

- The number of standard Young tableaux with shape λ.
- The dimension of the Specht module indexed by λ.

The **hook length formula** computes this quantity.

Hook Length Formula

For a box x in a Young diagram, the **hook** $h(x)$ is the set of boxes to the right of and below x, including x.

The **hook length** of x is $|h(x)|$.

In the figure to the right, each box displays its hook length.

7	5	2	1
4	2		
3	1		
1			

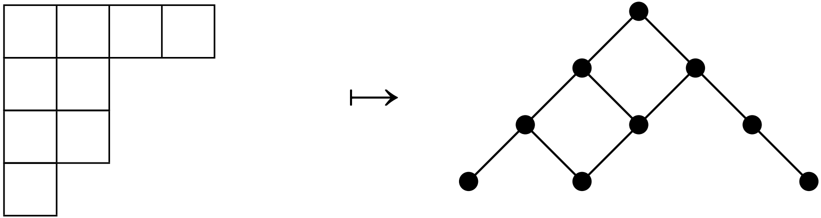
Hook Length Formula

Let λ be a Young diagram with n boxes.

The hook length formula for the number of standard Young tableaux with shape λ, denoted d_λ is:

$$d_\lambda = \frac{n!}{\prod_{x \in \lambda} |h(x)|}$$

General Hook Length Formula



A Young diagram λ corresponds to a partially ordered set (poset). Standard Young tableaux of shape λ correspond to linear extensions of that poset.

Further Study

- D-complete posets.
- Linear Extensions
- General Hook Length Formula

Questions?