Buffon’s Needle Problem

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Buffon’s Needle Problem

Posed in 1777 by Georges-Louis Leclerc, Comte de Buffon

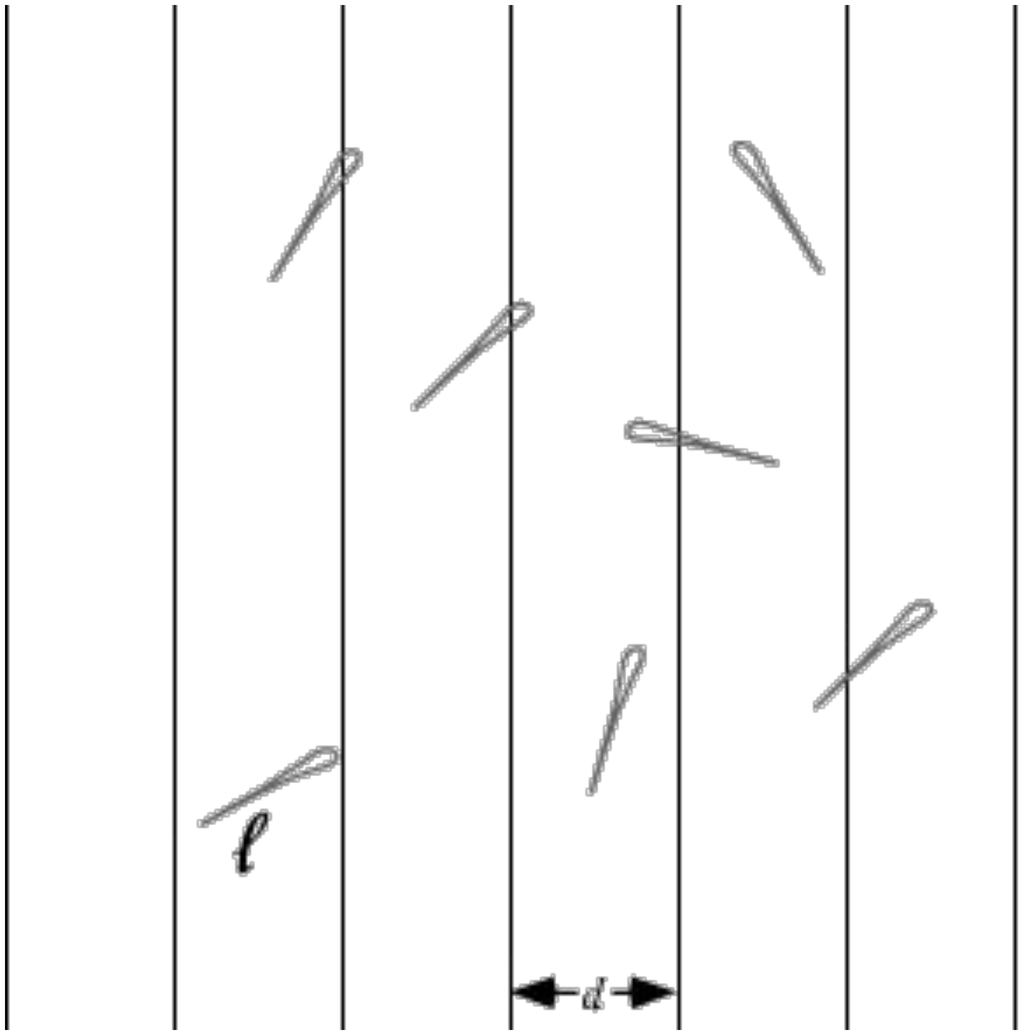
Suppose that you drop a short needle on ruled paper--what is then the probability that the needle comes to lie in a position where it crossed one of the lines?

Result

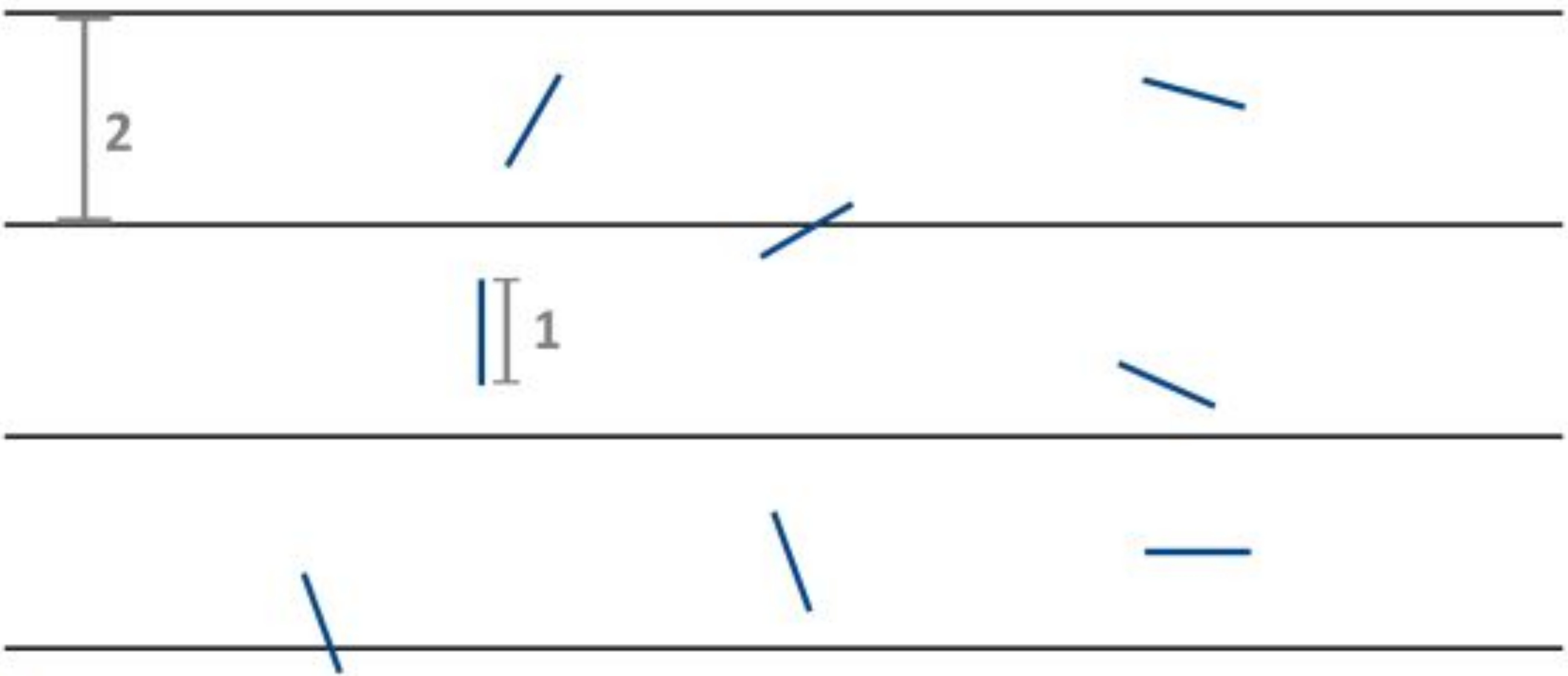
If a *short* needle of length l is dropped on paper that is ruled with equally spaced lines of distance $d \geq l$, then the probability that the needle comes to lie in a position where it crosses one of the lines is exactly:

$$P = 2l/(\pi d)$$

Picture



Buffon's Problem: What is the probability a needle crosses a line?



Mind
Your
Decisions

Solution

Woah we are using LaTeXnow.

We will look at two solutions, one with an integral and one without. How do we get this result?

$$p = \frac{2l}{\pi d}$$

Assumptions

First note these cases have probability zero:

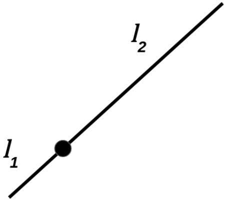
- A needle lands exactly on a line
- A needle lands with one endpoint exactly on a line

We will ignore these cases.

First solution – no integral

Let $E(l) =$ the expected number of line crossings produced by dropping a straight needle of length l.

It is clear that a needle of length
$l = l_1 + l_2$ has probability of crossing a line



E(l$_1 + l_2) = E(l_1) + E(l_2)$

Taking $l_n = l/n$ gives

$$E(l) = E(nl_n) = nE(l_n)$$

First solution

So, we have shown $E(l)$ is **linear** in l!

$$E(al_1 + bl_2) = aE(l_1) + bE(l_2)$$

Clearly $E(l)$ is also **monotonic**:

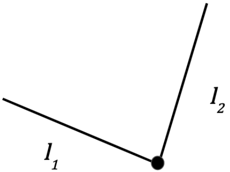
$$l_1 > l_2 \Rightarrow E(l_1) > E(l_2)$$

These two properties mean that the $E(l)$ takes the form

$$E(l) = cl, \quad c > 0$$

First solution

It does not matter if needle segments are joined at an angle.

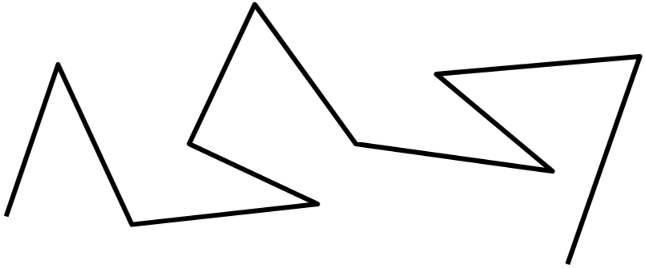


This needle has expected number of crossings

$$E(l) = E(l_1 + l_2) = E(l_1) + E(l_2) = cl_1 + cl_2 = cl$$

First solution

Consider the following **polygonal** needle with total length l

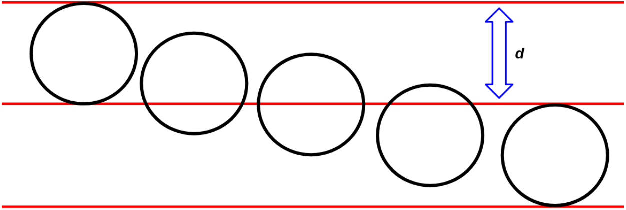


Indeed, this needle has expected number of crossings

$$E(l) = cl$$

First solution

Now let's consider a circular needle C with diameter d.

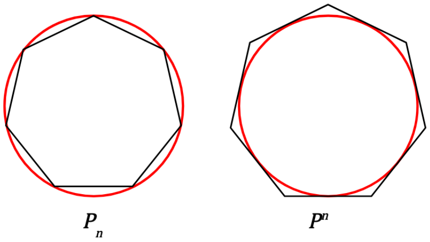


This needle will always produce 2 line crossings.

$$E(C) = 2$$

First solution

We can approximate C with inscribed and circumscribed regular n-gon needles P_n and P^n



We have:

$$E(P_n) \leqslant E(C) \leqslant E(P^n) \text{ for all } n$$

First solution

We have

$$\begin{aligned} E(P_n) \leqslant E(C) &\leqslant E(P^n) \\ \Rightarrow cl(P_n) \leqslant \quad 2 \quad &\leqslant cl(P^n) \end{aligned} \tag{1}$$

And since $P_n, P^n \to C$ as $n \to \infty$

$$\lim_{n\to\infty} l(P_n) = d\pi = \lim_{n\to\infty} l(P^n)$$

Together with (1)

$$cd\pi \leqslant 2 \leqslant cd\pi$$

$$c = \frac{2}{\pi d}$$

First solution

We found:

$$E(l) = \frac{2l}{\pi d}$$

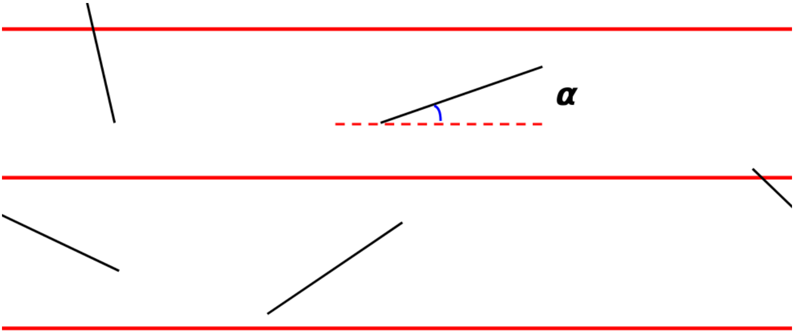
If we impose that our straight needle is *short* $(l \leqslant d)$, then $E(l) =$ the probability that the needle produces a crossing.

$$p = E(l) = \frac{2l}{\pi d}$$

So what is the other solution?

Second Solution

Consider the slope of the needle to the horizontal $0 \leqslant \alpha \leqslant \pi/2$



There is a symmetric argument for negative α.

Second Solution

The probability distribution of α is uniform in $[0, \pi/2]$, so

$$p(\alpha) = \frac{2}{\pi} \qquad \alpha \in [0, \pi/2]$$

A needle of length l with angle α has height $l\sin(\alpha)$. The probability that such a needle produces a crossing is

$$\frac{l\sin(\alpha)}{d}$$

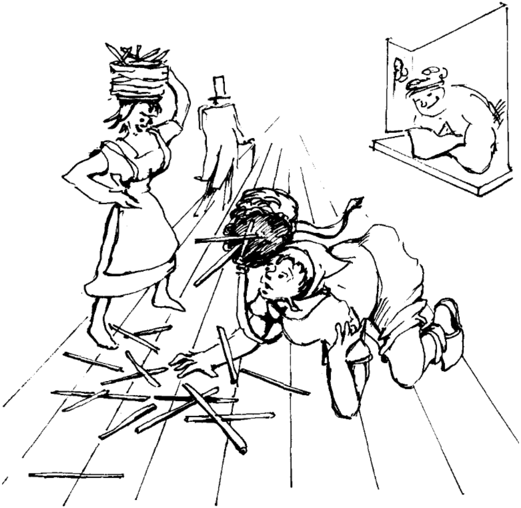
Second Solution

So we get our desired probability by averaging over possible α

$$
\begin{aligned}
p &= \int_0^{\pi/2} \frac{l\sin(\alpha)}{d} p(\alpha)\, d\alpha \\
&= \frac{2}{\pi} \int_0^{\pi/2} \frac{l\sin(\alpha)}{d}\, d\alpha \\
&= \frac{2l}{\pi d} \left[-\cos(\alpha) \right]_0^{\pi/2} \\
&= \frac{2l}{\pi d}
\end{aligned}
$$

Note: the long needle problem ($l \geqslant d$) can also be solved like this, but it's not so pretty.

The end



"Got a problem?"